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Alexander J. Zaslavski

Optimization on Solution Sets of Common Fixed Point Problems

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Aims and Scope

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Optimization on Solution Sets of Common Fixed Point Problems

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Preface

In this book, we study the subgradient projection algorithm and its modifications for minimization of convex functions on solution sets of common fixed point problems and on solution sets of convex feasibility problems, under the presence of computational errors. Usually the problem, studied in the literature, is described by an objective function and a set of feasible points. For this algorithm, each iteration consists of two steps. The first step is a calculation of a subgradient of the objective function, while in the second one, we calculate a projection on the feasible set. In each of these two steps there is a computational error. In general, these two computational errors are different. In our recent research presented in [93, 95, 96] we show that the algorithm generates a good approximate solution, if all the computational errors are bounded from above by a small positive constant. Moreover, if we know computational errors for the two steps of our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this. It should be mentioned that in [93, 95] analogous results were obtained for many others important algorithms in optimization and in the game theory.

In our study in [93, 95] we considered optimization problems defined on a set of feasibility points, which is given explicitly as the fixed point set of an operator. It was used the fact that we can calculate a projection operator on a set of feasibility points with small computational errors. Of course, this is possible only when the feasibility set is simple, like a simplex or a half-space. In practice, the situation is more complicated. In real world applications, the feasibility set is an intersection of a finite family of simple closed convex sets. Calculating the projection on their intersection is impossible, and instead of this, one has to work with projections on these simple sets which determine the feasibility set as their intersection, considering the products of these projections (the iterative algorithm), their convex combinations (the Cimmino algorithm), and a more recent and advanced dynamic string-averaging algorithm which was first introduced by Y. Censor, T. Elfving, and G. T. Herman in [23] for solving a convex feasibility problem, when a given collection of sets is divided into blocks and the algorithms operate in such a manner that all the blocks are processed in parallel. In our book [94] we studied approximate solutions of common fixed point problems for a finite family of operators and approximate solutions of convex feasibility problems, taking into account computational errors. The goal was to find a point which is close enough for each element of a given finite family of sets. Optimization problems were not considered in [94]. In the present book, we deal with a problem, which is much more difficult and complicated than the problems studied in [93–95]: to find a point which is close enough for each element of a given finite family of sets and such that the value of a given objective function at this point is close to the infimum of this function on the feasibility set.

In this book our goal is to find approximate minimizers of convex functions on solution sets of common fixed point problems and on solution sets of convex feasibility problems, under the presence of computational errors. We show that our algorithms generate a good approximate solution, if all the computational errors are bounded from above by a small positive constant. If we know computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this.

Analysis of the behavior of an optimization algorithm is based on the choice of an appropriate estimation which holds for each of its iterations. First, this estimation holds for an initial iteration. It is shown that if the estimation is true for a current iteration t, then it is also true for the next iteration $t + 1$. Thus, we conclude that the estimation is true for all iterations of the algorithm. Using this estimation, it is shown that after a certain number of iterations, we obtain an approximate solution of our problems. In [93, 95], an estimation was used which allows us after a certain number of iterations to obtain a point where the value of given objective function is close to the infimum of this function on the feasibility set. We should not worry that this point is close to the feasibility set, because this is guaranteed by the algorithm. In [94], where the feasibility set is an intersection of a finite family of sets, we use another estimation. This estimation allows us after a certain number of iterations to obtain a point which is close to every element of the family of sets. Here, we have to find another estimation. Using this new estimation it is shown that after a certain number of iterations we obtain a point which is close to every element of the family of sets, whose intersection is our feasibility set, and where the value of given objective function is close to the infimum of this function on the feasibility set.

It should be mentioned that the subgradient projection algorithm is used for many problems arising in real world applications. The results of our book allow us to use this algorithm for problems with complicated sets of feasible points arising in engineering and, in particular, in computed tomography and radiation therapy planning.

The book contains 10 chapters. Chapter 1 is an introduction. In Chapter 2, we consider a minimization of a convex function on a common fixed point set of a finite family of quasinonexpansive mappings in a Hilbert space. We use the Cimmino subgradient algorithm, the iterative subgradient algorithm, and the dynamic string-averaging subgradient algorithm. In Chapter 3, we consider a minimization of a convex function on an intersection of two sets in a Hilbert space. One of them is a common fixed point set of a finite family of quasi-nonexpansive mappings while the second one is a common zero point set of finite family of maximal monotone operators. We use the Cimmino proximal point subgradient algorithm, the iterative proximal point subgradient algorithm, and the dynamic string-averaging proximal point subgradient algorithm and show that each of them generates a good approximate solution. In Chapters 4–6, we study a minimization of a convex function on a solution set of a convex feasibility problem in a general Hilbert space. The solution set is an intersection of a finite family of convex closed sets such that every set is a collection of points where the values of the corresponding convex constraint function does not exceed zero. In Chapter 4, we study the Cimmino subgradient projection algorithm, in Chapter 5 we analyze the iterative subgradient projection algorithm, while in Chapter 6 the dynamic string-averaging subgradient projection algorithm is discussed. In Chapters 7 and 8, we study minimization problems with smooth objective functions using a fixed point gradient projection algorithm and a Cimmino gradient projection algorithm, respectively. In Chapter 9, we study the convergence of the projected subgradient method for a class of constrained optimization problems in a Hilbert space. For this class of problems, an objective function is assumed to be convex, but a set of admissible points is not necessarily convex. Our goal is to obtain an ϵ-approximate solution in the presence of computational errors, where ϵ is a given positive number. An extension of the projected subgradient method for zero-sum games with two players under the presence of computational errors is given in Chapter 10.

All the results presented in the book are new. The author believes that this book will be useful for researches interested in the optimization theory and its applications.

Rishon LeZion, Israel
February 25, 2021

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Contents

Chapter 1
Introduction

In this book we study optimization on solution sets of common fixed point problems. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We show that an algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a small constant. Moreover, if we known computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this. In this section we discuss algorithms which are studied in the book.

1.1 Subgradient Projection Method

In this book we use the following notation. For every $z \in R^1$ denote by $\lfloor z \rfloor$ the largest integer which does not exceed z:

$$\lfloor z \rfloor = \max\{i \in R^1 : i \text{ is an integer and } i \leq z\}.$$

For every nonempty set D, every function $f : D \to R^1$ and every nonempty set $C \subset D$ we set

$$\inf(f, C) = \inf\{f(x) : x \in C\}$$

and

$$\operatorname{argmin}(f, C) = \operatorname{argmin}\{f(x) : x \in C\} = \{x \in C : f(x) = \inf(f, C)\}.$$

Let X be a Hilbert space equipped with an inner product denoted by $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$. For each $x \in X$ and each $r > 0$ set

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$$B_X(x,r) = \{y \in X : \ \|x - y\| \le r\}$$

and set

$$B(x,r) = B_X(x,r)$$

if the space X is understood.

For each $x \in X$ and each nonempty set $E \subset X$ set

$$d(x,E) = \inf\{\|x - y\| : \ y \in E\}.$$

For each nonempty open convex set $U \subset X$ and each convex function $f : U \to R^1$, for every $x \in U$ set

$$\partial f(x) = \{l \in X : \ f(y) - f(x) \ge \langle l, y - x\rangle \text{ for all } y \in U\}$$

which is called the subdifferential of the function f at the point x [61, 62, 75]. Denote by Card(A) the cardinality of a set A. We suppose that the sum over an empty set is zero.

In this book we study the subgradient algorithm and its modifications for minimization of convex functions, under the presence of computational errors. It should be mentioned that the subgradient projection algorithm is one of the most important tools in the optimization theory [1, 14, 15, 22, 31, 37, 46, 48, 50, 52–54, 65, 72, 73, 86, 90, 93, 97], nonlinear analysis [11, 16, 17, 40, 51, 68, 78, 83, 91, 92] and their applications. Usually the problem, studied in the literature, is described by an objective function and a set of feasible points. For this algorithm each iteration consists of two steps. The first step is a calculation of a subgradient of the objective function while in the second one we calculate a projection on the feasible set. In each of these two steps there is a computational error. In general, these two computational errors are different. In our recent research [93, 95, 96] we show that the algorithm generate a good approximate solution, if all the computational errors are bounded from above by a small positive constant. Moreover, if we known computational errors for the two steps of our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this. It should be mentioned that in [93, 95] analogous results were obtained for many others important algorithms in optimization and in the game theory.

We use the subgradient projection algorithm for constrained minimization problems in Hilbert spaces equipped with an inner product denoted by $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$. It should be mentioned that optimization problems in infinite-dimensional Banach and Hilbert spaces are studied in [2, 3, 7, 24, 33, 64] while the subgradient projection algorithm is analyzed in [3, 12, 36, 44, 47, 57, 67, 74, 79, 81, 82].

Let C be a nonempty closed convex subset of X, U be an open convex subset of X such that $C \subset U$ and let $f : U \to R^1$ be a convex function.

Suppose that there exist $L > 0$, $M_0 > 0$ such that

$$C \subset B_X(0, M_0),$$

$$|f(x) - f(y)| \leq L\|x - y\| \text{ for all } x, y \in U.$$

It is not difficult to see that for each $x \in U$,

$$\emptyset \neq \partial f(x) \subset B_X(0, L).$$

For every nonempty closed convex set $D \subset X$ and every $x \in X$ there is a unique point $P_D(x) \in D$ satisfying

$$\|x - P_D(x)\| = \inf\{\|x - y\| : \ y \in D\}.$$

We consider the minimization problem

$$f(z) \to \min, \ z \in C.$$

Suppose that $\{a_k\}_{k=0}^{\infty} \subset (0, \infty)$. Let us describe our algorithm.

Subgradient Projection Algorithm
Initialization: select an arbitrary $x_0 \in U$.
Iterative step: given a current iteration vector $x_t \in U$ calculate

$$\xi_t \in \partial f(x_t)$$

and the next iteration vector $x_{t+1} = P_C(x_t - a_t\xi_t)$.

In [93] we study this algorithm under the presence of computational errors. Namely, in [93] we suppose that $\delta \in (0, 1]$ is a computational error produced by our computer system, and study the following algorithm.

Subgradient Projection Algorithm with Computational Errors
Initialization: select an arbitrary $x_0 \in U$.
Iterative step: given a current iteration vector $x_t \in U$ calculate

$$\xi_t \in \partial f(x_t) + B_X(0, \delta)$$

and the next iteration vector $x_{t+1} \in U$ such that

$$\|x_{t+1} - P_C(x_t - a_t\xi_t)\| \leq \delta.$$

In Chapter 2 of [95] we consider more complicated, but more realistic, version of this algorithm. Clearly, for the algorithm each iteration consists of two steps. The first step is a calculation of a subgradient of the objective function f while in the second one we calculate a projection on the set C. In each of these two steps there is a computational error produced by our computer system. In general, these two computational errors are different. This fact is taken into account in the following projection algorithm studied in Chapter 2 of [95].

Suppose that $\{a_k\}_{k=0}^{\infty} \subset (0, \infty)$ and $\delta_f, \delta_C \in (0, 1]$.

Initialization: select an arbitrary $x_0 \in U$.

Iterative step: given a current iteration vector $x_t \in U$ calculate

$$\xi_t \in \partial f(x_t) + B_X(0, \delta_f)$$

and the next iteration vector $x_{t+1} \in U$ such that

$$\|x_{t+1} - P_C(x_t - a_t\xi_t)\| \le \delta_C.$$

Note that in practice for some problems the set C is simple but the function f is complicated. In this case δ_C is essentially smaller than δ_f. On the other hand, there are cases when f is simple but the set C is complicated and therefore δ_f is much smaller than δ_C.

In Chapter 2 of [95] we proved the following result (see Theorem 2.4).

Theorem 1.1 *Let* $\delta_f, \delta_C \in (0, 1]$, $\{a_k\}_{k=0}^{\infty} \subset (0, \infty)$ *and let*

$$x_* \in C$$

satisfy

$$f(x_*) \le f(x) \text{ for all } x \in C.$$

Assume that $\{x_t\}_{t=0}^{\infty} \subset U$, $\{\xi_t\}_{t=0}^{\infty} \subset X$,

$$\|x_0\| \le M_0 + 1$$

and that for each integer $t \ge 0$,

$$\xi_t \in \partial f(x_t) + B_X(0, \delta_f)$$

and

$$\|x_{t+1} - P_C(x_t - a_t\xi_t)\| \le \delta_C.$$

Then for each natural number T,

$$\sum_{t=0}^{T} a_t(f(x_t) - f(x_*))$$

$$\le 2^{-1}\|x_* - x_0\|^2 + \delta_C(T + 1)(4M_0 + 1)$$

$$+\delta_f(2M_0+1)\sum_{t=0}^{T} a_t + 2^{-1}(L+1)^2\sum_{t=0}^{T} a_t^2.$$

Moreover, for each natural number T,

$$f((\sum_{t=0}^{T} a_t)^{-1}\sum_{t=0}^{T} a_t x_t) - f(x_*),\ \min\{f(x_t):\ t=0,\dots,T\} - f(x_*)$$

$$\le 2^{-1}(\sum_{t=0}^{T} a_t)^{-1}\|x_* - x_0\|^2 + (\sum_{t=0}^{T} a_t)^{-1}\delta_C(T+1)(4M_0+1)$$

$$+\delta_f(2M_0+1) + 2^{-1}(\sum_{t=0}^{T} a_t)^{-1}(L+1)^2\sum_{t=0}^{T} a_t^2.$$

Theorem 1.1 a generalization of Theorem 2.4 of [93] proved in the case when $\delta_f = \delta_C$.

We are interested in an optimal choice of a_t, $t = 0, 1, \dots$. Let T be a natural number and $A_T = \sum_{t=0}^{T} a_t$ be given. It is shown in [95] that the best choice is $a_t = (T+1)^{-1}A_T$, $t = 0, \dots, T$.

Let T be a natural number and $a_t = a > 0$, $t = 0, \dots, T$. It is shown in [95] that the best choice of a is

$$a = (2\delta_C(4M_0+1))^{1/2}(L+1)^{-1}.$$

Now we can think about the best choice of T. It is not difficult to see that it should be at the same order as $\lfloor \delta_C^{-1} \rfloor$.

In [96] we generalize the results obtained in [95] for the subgradient projection algorithm in the case when instead of the projection operator on C it is used a quasi-nonexpansive retraction on C.

1.2 Fixed Point Subgradient Algorithms

In the previous section we study the subgradient projection algorithm for minimization of a convex function f on a convex closed set C in a Hilbert space. In our analysis it was used the fact that we can calculate a projection operator P_C with small computational errors. Of course, this is possible only when the C is simple, like a simplex or a half-space. In practice the situation is more complicated. In real world applications the set C is an intersection of a finite family of simple closed convex sets C_i, $i = 1, \dots, C_m$. To calculate the mapping P_C is impossible and instead of it one has to work with projections P_{C_i}, $i = 1, \dots, m$ on the simple sets

$C_1, \dots, C_m$ considering the products $\prod_{i=1}^m P_{C_i}$ (the iterative algorithm), convex combination of P_{C_i}, $i = 1, \dots, m$ (the Cimmino algorithm) and a more recent and advanced dynamic string-averaging algorithm. The dynamic string-averaging methods were were first introduced by Y. Censor, T. Elfving, and G. T. Herman in [23] for solving a convex feasibility problem, when a given collection of sets is divided into blocks and the algorithms operate in such a manner that all the blocks are processed in parallel. Iterative methods for solving common fixed point problems is a special case of dynamic string-averaging methods with only one block. Iterative methods and dynamic string-averaging methods are important tools for solving convex feasibility problems and common fixed point problems in a Hilbert space [5, 6, 8, 16, 17, 19–21, 25, 27, 28, 30, 45, 66, 76, 77, 84, 85, 94].

In Chapter 2 of the book we consider a minimization of a convex function on a common fixed point set of a finite family of quasi-nonexpansive mappings in a Hilbert space. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We use the Cimmino subgradient algorithm, the iterative subgradient algorithm and the dynamic string-averaging subgradient algorithm and show that each of them generates a good approximate solution, if the sequence of computational errors is bounded from above by a small constant. Moreover, if we known computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this.

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

Suppose that m is a natural number, $\bar{c} \in (0, 1]$, $P_i : X \to X$, $i = 1, \dots, m$, for every integer $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset$$

and that the inequality

$$\|z - x\|^2 \geq \|z - P_i(x)\|^2 + \bar{c}\|x - P_i(x)\|^2$$

holds for every for every integer $i \in \{1, \dots, m\}$, every point $x \in X$ and every point $z \in \mathrm{Fix}(P_i)$. Set

$$F = \cap_{i=1}^m \mathrm{Fix}(P_i).$$

For every positive number ϵ and every integer $i \in \{1, \dots, m\}$ set

$$F_\epsilon(P_i) = \{x \in X : \ \|x - P_i(x)\| \leq \epsilon\},$$

$$\tilde{F}_\epsilon(P_i) = F_\epsilon(P_i) + B(0, \epsilon),$$

$$F_\epsilon = \cap_{i=1}^m F_\epsilon(P_i),$$

$$\tilde{F}_\epsilon = \cap_{i=1}^m \tilde{F}_\epsilon(P_i)$$

and

$$\widehat{F}_\epsilon = F_\epsilon + B(0, \epsilon).$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

Let $M_* > 0$ satisfy

$$F \cap B(0, M_*).$$

and let $f : X \to R^1$ be a convex continuous function. In Chapter 2 we consider the minimization problem

$$f(x) \to \min, \; x \in F.$$

Assume that

$$\inf(f, F) = \inf(f, F \cap B(0, M_*)).$$

Fix $\alpha > 0$. Let us describe our first algorithm.

Cimmino Subgradient Algorithm

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$l_k \in \partial f(x_k),$$

pick $w_{k+1} = (w_{k+1}(1), \dots, w_{k+1}(m)) \in R^m$ such that

$$w_{k+1}(i) \geq 0, \; i = 1, \dots, m,$$

$$\sum_{i=1}^m w_{k+1}(i) = 1$$

and define the next iteration vector

$$x_{k+1} = \sum_{i=1}^m w_{k+1}(i) P_i(x_k - \alpha l_k).$$

In Chapter 2 this algorithm is studied under the presence of computational errors and two convergence results are obtained. Fix

$$\Delta \in (0, m^{-1}).$$

We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f while $\delta_p \in [0, 1]$ is a computational error produced by our computer system, when we calculate the operators P_i, $i = 1, \dots, m$. Let $\alpha > 0$ be a step size.

Cimmino Subgradient Algorithm with Computational Errors

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$\xi_k \in \partial f(x_k) + B(0, \delta_f),$$

pick $w_{k+1} = (w_{k+1}(1), \dots, w_{k+1}(m)) \in R^m$ such that

$$w_{k+1}(i) \geq \Delta, \ i = 1, \dots, m,$$

$$\sum_{i=1}^{m} w_{k+1}(i) = 1,$$

calculate

$$y_{k,i} \in B(P_i(x_k - \alpha \xi_k), \delta_p), \ i = 1, \dots, m$$

and the next iteration vector and $x_{t+1} \in X$ such that

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t,i}\| \leq \delta_p.$$

In this algorithm, as well for other algorithms considered in the book, we assume that the step size does not depend on the number of iterative step k. The same analysis can be done when step sizes depend on k. On the other hand, as it was shown in [93, 95], in the case of computational errors the best choice of step sizes is step sizes which do not depend on iterative step numbers.

In the following result obtained in Chapter 2 (Theorem 2.9) we assume the objective function f satisfies a coercivity growth condition.

Theorem 1.2 *Let the function f be Lipschitz on bounded subsets of X,*

$$\lim_{\|x\| \to \infty} f(x) = \infty,$$

$M \geq 2M_* + 8$, $L_0 \geq 1$,

$$M_1 > \sup\{|f(u)| : \ u \in B(0, M_* + 4)\} + 4,$$

$$f(u) > M_1 + 4 \text{ for all } u \in X \setminus B(0, 2^{-1}M),$$

$$|f(z_1) - f(z_2)| \le L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(0, 3M + 4),$$

$\delta_f, \delta_p \in [0, 1], \alpha > 0$ *satisfy*

$$\alpha \le L_0^{-2},\ \alpha \ge \delta_f(6M + L_0 + 2),\ \alpha \ge 2\delta_p(6M + 2), \tag{1.1}$$

T *be a natural number and let*

$$\gamma_T = \max\{\alpha(L_0 + 1),\ (\Delta\bar{c})^{-1/2}(4M^2T^{-1} + \alpha(L_0 + 1)(12M + 4))^{1/2} + \delta_p\}.$$

Assume that $\{x_t\}_{t=0}^{T} \subset X$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$(w_t(1), \dots, w_t(m)) \in R^m,\ t = 1, \dots, T,$$

$$\sum_{i=1}^{m} w_t(i) = 1,\ t = 1, \dots, T,$$

$$w_t(i) \ge \Delta,\ i = 1, \dots, m,\ t = 1, \dots, T,$$

$$x_0 \in B(0, M)$$

and that for all integers $t \in \{0, \dots, T - 1\}$,

$$B(\xi_t, \delta_f) \cap \partial f(x_t) \ne \emptyset,$$

$$y_{t,i} \in B(P_i(x_t - \alpha\xi_t), \delta_p),\ i = 1, \dots, m,$$

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i)y_{t,i}\| \le \delta_p.$$

Then

$$\|x_t\| \le 2M + M_*,\ t = 0, \dots, T$$

and

$$\min\{\max\{\Delta\bar{c}\sum_{i=0}^{m}\|x_t - \alpha\xi_t - y_{t,i}\|^2 - \alpha(L_0 + 1)(12M + 4),$$

$$2\alpha(f(x_t) - \inf(f, F)) - 4\delta_p(6M + 3) - \alpha^2L_0^2 - 2\alpha(6M + L_1 + 1)\} :$$

$$t = 0, \dots, T-1\} \le 4M^2T^{-1}.$$

Moreover, if $t \in \{0, \dots, T-1\}$ *and*

$$\max\{\Delta\bar{c}\sum_{i=0}^{m}\|x_t - \alpha\xi_t - y_{t,i}\|^2 - \alpha(L_0+1)(12M+4),$$

$$2\alpha(f(x_t) - \inf(f, F)) - 4\delta_p(6M+3)$$

$$-\alpha^2L_0^2 - 2\alpha\delta_f(6M+L_0+1)\} \le 4M^2T^{-1}, \tag{1.2}$$

then

$$f(x_t) \le \inf(f, F) + 2M^2(T\alpha)^{-1}$$

$$+2\alpha^{-1}\delta_p(6M+3) + 2^{-1}\alpha L_0^2 + \delta_f(6M+L_0+3) \tag{1.3}$$

and

$$x_t \in \widehat{F}_{\gamma_T}.$$

In Chapter 2 we also obtain an extension of this result (Theorem 2.10) when instead of assuming that f satisfies the growth condition we suppose that there exists $r_0 \in (0, 1]$ such that the set F_{r_0} is bounded.

In Theorem 1.2 the computational errors δ_f, δ_p are fixed. Assume that they are positive. Let us choose α, T. First, we choose α in order to minimize the right-hand side of (1.3). Since T can be an arbitrary large we need to minimize the function

$$2\alpha^{-1}\delta_p(6M+3) + 2^{-1}\alpha L_0^2,\ \alpha > 0.$$

Its minimizer is

$$\alpha = 2L_0^{-1}(\delta_p(6M+3))^{1/2}.$$

Since α satisfies (1.1) we obtain the following restrictions on δ_f, δ_p:

$$\delta_f \le 2L_0^{-1}\delta_p^{1/2}(6M+3)^{1/2}(6M+L_0+2)^{-1},$$

$$\delta_p \le 4^{-1}L_0^{-2}(6M+3)^{-1}.$$

In this case

$$\gamma_T = \max\{2L_0^{-1}(\delta_p(6M+3))^{1/2}(L_0+1),$$

$$(\Delta\bar{c})^{-1/2}(4M^2T^{-1} + 2L_0^{-1}(\delta_p(6M+3))^{1/2}(L_0+1)(12M+4))^{1/2} + \delta_p\}.$$

We choose T with the same order as δ_p^{-1}. For example, $T = \lfloor \delta_p^{-1} \rfloor$. In this case in view of Theorem 1.2, there exists $t \in \{0, \ldots, T-1\}$ such that then

$$f(x_t) \le \inf(f, F) + c_1\delta_p^{1/2} + \delta_f(6M + L_0 + 3)$$

and

$$x_t \in \widehat{F}_{c_2\delta_p^{1/4}}$$

where c_1, c_2 are positive constants which depend on $M, L_0, \Delta, \bar{c}$.

Let us explain how we can obtain t satisfying (1.2). Set

$$E = \{t \in \{0, \ldots, T-1\} : \ \Delta\bar{c}\sum_{i=0}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$\le \alpha(L_0+1)(12M+4) + 4M^2T\alpha^{-1}\}$$

and find $t_* \in E$ such that $f(x_{t_*}) \le f(x_t)$ for all $t \in E$. This t satisfies (1.2).

In Chapter 2 we also establish analogs of Theorem 1.2 for the iterative subgradient algorithm and the dynamic string-averaging subgradient algorithm.

1.3 Proximal Point Subgradient Algorithm

In Chapter 3 we consider a minimization of a convex function on an intersection of two sets in a Hilbert space. One of them is a common fixed point set of a finite family of quasi-nonexpansive mappings while the second one is a common zero point set of finite family of maximal monotone operators. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We use the Cimmino proximal point subgradient algorithm, the iterative proximal point subgradient algorithm and the dynamic string-averaging proximal point subgradient algorithm and show that each of them generates a good approximate solution, if the sequence of computational errors is bounded from above by a small constant. Moreover, if we known computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this.

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ which induces a complete norm $\|\cdot\|$.

A multifunction $T : X \to 2^X$ is called a monotone operator if and only if

$$\langle z - z', w - w'\rangle \ \ge 0 \quad \forall z, z', w, w' \in X$$

$$\text{such that } w \in T(z) \text{ and } w' \in T(z').$$

It is called maximal monotone if, in addition, the graph

$$\{(z, w) \in X \times X : \ w \in T(z)\}$$

is not properly contained in the graph of any other monotone operator $T' : X \to 2^X$. A fundamental problem consists in determining an element z such that $0 \in T(z)$. For example, if T is the subdifferential ∂f of a lower semicontinuous convex function $f : X \to (-\infty, \infty]$, which is not identically infinity, then T is maximal monotone (see [60, 63]), and the relation $0 \in T(z)$ means that z is a minimizer of f.

Let $T : X \to 2^X$ be a maximal monotone operator. The proximal point algorithm generates, for any given sequence of positive real numbers and any starting point in the space, a sequence of points and the goal is to show the convergence of this sequence. Note that in a general infinite-dimensional Hilbert space this convergence is usually weak. The proximal algorithm for solving the inclusion $0 \in T(z)$ is based on the fact established by Minty [59], who showed that, for each $z \in X$ and each $c > 0$, there is a unique $u \in X$ such that

$$z \in (I + cT)(u),$$

where $I : X \to X$ is the identity operator ($Ix = x$ for all $x \in X$).

The operator

$$P_{c,T} := (I + cT)^{-1}$$

is therefore single-valued from all of X onto X (where c is any positive number). It is also nonexpansive:

$$\|P_{c,T}(z) - P_{c,T}(z')\| \le \|z - z'\| \text{ for all } z, z' \in X$$

and

$$P_{c,T}(z) = z \text{ if and only if } 0 \in T(z).$$

Following the terminology of Moreau [63] $P_{c,T}$ is called the proximal mapping associated with cT.

The proximal point algorithm generates, for any given sequence $\{c_k\}_{k=0}^{\infty}$ of positive real numbers and any starting point $z^0 \in X$, a sequence $\{z^k\}_{k=0}^{\infty} \subset X$, where

$$z^{k+1} := P_{c_k,T}(z^k), \ k = 0, 1, \ldots$$

It is not difficult to see that the

$$\text{graph}(T) := \{(x, w) \in X \times X : \ w \in T(x)\}$$

is closed in the norm topology of $X \times X$.

Set

$$F(T) = \{z \in X : \ 0 \in T(z)\}.$$

Proximal point method is an important tool in solving optimization problems [32, 34, 35, 38, 43, 55, 70, 87, 88]. It is also used for solving variational inequalities with monotone operators [9, 18, 41, 42, 71, 80, 89, 94] which is an important topic of nonlinear analysis and optimization [4, 10, 13]. Usually algorithms considering in the literature generate sequences which converge weakly to an element of $F(T)$.

Let $\mathcal{L}_1$ be a finite set of maximal monotone operators $T : X \to 2^X$ and $\mathcal{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Let $\bar{c} \in (0, 1]$ and let $\bar{c} = 1$, if $\mathcal{L}_2 = \emptyset$.

We suppose that

$$F(T) = \{z \in X : \ 0 \in T(z)\} \neq \emptyset \text{ for any } T \in \mathcal{L}_1$$

and that for every mapping $T \in \mathcal{L}_2$,

$$\text{Fix}(T) := \{z \in X : \ T(z) = z\} \neq \emptyset,$$

$$\|z - x\|^2 \geq \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2$$

$$\text{for all } x \in X \text{ and all } z \in \text{Fix}(T).$$

Let $M_* > 0$,

$$F := (\cap_{T \in \mathcal{L}_1} F(T)) \cap (\cap_{Q \in \mathcal{L}_2} \text{Fix}(Q)) \neq \emptyset$$

and

$$F \cap B(0, M_*) \neq \emptyset.$$

Let $\epsilon > 0$. For every monotone operator $T \in \mathcal{L}_1$ define

$$F_\epsilon(T) = \{x \in X : \ T(x) \cap B(0, \epsilon) \neq \emptyset\}$$

and for every mapping $T \in \mathcal{L}_2$ set

$$\text{Fix}_\epsilon(T) = \{x \in X : \ \|T(x) - x\| \leq \epsilon\}.$$

Define

$$F_\epsilon = (\cap_{T\in\mathcal{L}_1} F_\epsilon(T)) \cap (\cap_{Q\in\mathcal{L}_2}\mathrm{Fix}_\epsilon(Q)),$$

$$\tilde{F}_\epsilon = (\cap_{T\in\mathcal{L}_1}\{x \in X :\ d(x, F_\epsilon(T)) \le \epsilon\})$$

$$\cap(\cap_{Q\in\mathcal{L}_2}\{x \in X :\ d(x, \mathrm{Fix}_\epsilon(Q)) \le \epsilon\}).$$

Let $f : X \to R^1$ be a convex continuous function. We consider the minimization problem

$$f(x) \to \min,\ \ x \in F.$$

Assume that

$$\inf(f, F) = \inf(f, F \cap B(0, M_*)).$$

Let $\bar{\lambda} > 0$ and let $\bar{\lambda} = \infty$ and $\bar{\lambda}^{-1} = 0$, if $\mathcal{L}_1 = \emptyset$.

Recall that the sum over an empty set is zero.

Fix $\alpha > 0$.

Let us describe our algorithm.

Cimmino Proximal Point Subgradient Algorithm

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ pick $c(T) \ge \bar{\lambda}$, $T \in \mathcal{L}_1$ and $w : \mathcal{L}_1 \cup \mathcal{L}_2 \to (0, \infty)$ such that

$$\sum\{w(S) :\ S \in \mathcal{L}_1 \cup \mathcal{L}_2\} = 1,$$

$$l_k \in \partial f(x_k)$$

and define the next iteration vector

$$x_{k+1} = \sum_{S\in\mathcal{L}_2} w(S)S(x_k - \alpha l_k) + \sum_{S\in\mathcal{L}_1} w(S)P_{c(S),S}(x_k - \alpha l_k).$$

In Chapter 3 this algorithm is studied under the presence of computational errors. Fix

$$\Delta \in (0, \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2)^{-1}].$$

We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f while $\delta_p \in [0, 1]$ is a computational error produced by our computer system, when we calculate the operators $P_{c,S}$, $S \in \mathcal{L}_1$, $c \ge \bar{\lambda}$ and $S \in \mathcal{L}_2$. Let $\alpha > 0$ be a step size.

Cimmino Proximal Point Subgradient Algorithm with Computational Errors

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ pick $c(T) \ge \bar{\lambda}$, $T \in \mathcal{L}_1$ and $w : \mathcal{L}_1 \cup \mathcal{L}_2 \to [\Delta, \infty)$ such that

$$\sum\{w(S) : \ S \in \mathcal{L}_1 \cup \mathcal{L}_2\} = 1,$$

calculate

$$l_k \in \partial f(x_k) + B(0, \delta_f)$$

and

$$y_{k,S} \in B(S(x_k - \alpha\xi_k), \delta_p), \ \ S \in \mathcal{L}_2,$$

$$y_{k,S} \in B(P_{c_k(S),S}(x_k - \alpha\xi_k), \delta_p), \ \ S \in \mathcal{L}_1$$

and calculate the next iteration vector $x_{k+1} \in X$ satisfying

$$\|x_{k+1} - \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} w(S) y_{k,S}\| \le \delta_p.$$

The following result is established in Chapter 3 (Theorem 3.6).

Theorem 1.3 *Let the function f be Lipschitz on bounded subsets of X,*

$$\lim_{\|x\| \to \infty} f(x) = \infty,$$

$M \ge 2M_* + 6$, $L_0 \ge 1$,

$$M_1 > \sup\{|f(u)| : \ u \in B(0, M_* + 4)\} + 4,$$

$$f(u) > M_1 + 4 \textit{ for all } u \in X \setminus B(0, 2^{-1}M),$$

$$|f(z_1) - f(z_2)| \le L_0\|z_1 - z_2\| \textit{ for all } z_1, z_2 \in B(0, 3M + 4),$$

$\delta_f, \delta_p \in [0, 1], \alpha > 0$ *satisfy*

$$\alpha \le \min\{L_0^{-2}, (L_0 + 1)^{-1}\}, \ \ \alpha \ge 2\delta_p(6M + 3),$$

$$\delta_f \le (6M + L_0 + 1)^{-1},$$

T be a natural number and let

$$\gamma_T = (4M^2T^{-1} + \alpha(L_0 + 1)(12M + 1) + \delta_p(12M + 13)(\Delta\bar{c})^{-1})^{1/2} + \delta_p.$$

Assume that for all $t = 1, \dots, T$,

$$w_t : \mathcal{L}_1 \cup \mathcal{L}_2 \to [\Delta, \infty),$$

$$\sum \{w_t(S) : \ S \in \mathcal{L}_1 \cup \mathcal{L}_2\} = 1,$$

$$c(T) \geq \bar{\lambda}, \ T \in \mathcal{L}_1,$$

$\{x_t\}_{t=0}^{T} \subset X$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$x_0 \in B(0, M)$$

and that for all integers $t \in \{0, \dots, T-1\}$,

$$B(\xi_t, \delta_f) \cap \partial f(x_t) \neq \emptyset,$$

$$y_{t,S} \in B(S(x_t - \alpha\xi_t), \delta_p), \ S \in \mathcal{L}_2,$$

$$y_{t,S} \in B(P_{c_t(S),S}(x_t - \alpha\xi_t), \delta_p), \ S \in \mathcal{L}_1,$$

$$\|x_{t+1} - \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} w_{t+1}(S) y_{t,S}\| \leq \delta_p.$$

Then

$$\min\{\max\{\Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(12M + 13),$$

$$2\alpha(f(x_t) - \inf(f, F))$$

$$-\delta_p(6M + 3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_f(6M + L_1 + 1)\} :$$

$$t = 0, \dots, T-1\} \leq 4M^2 T^{-1}.$$

Moreover, if $t \in \{0, \dots, T-1\}$ *and*

$$\max\{\Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(12M + 13),$$

$$2\alpha(f(x_t) - \inf(f, F))$$

$$-\delta_p(6M+3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_f(6M+L_1+1)\} \le 4M^2T^{-1}$$

then

$$f(x_t) \le \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p(3M+2) + 4^{-1}\alpha L_0^2 + \delta_f(6M+L_0+1)$$

and

$$x_t \in \tilde{F}_{\max\{\alpha(L_0+1)+\gamma_T, \bar{\lambda}^{-1}\gamma_T\}}.$$

In Chapter 3 we also obtain an extension of this result (Theorem 3.7) when instead of assuming that f satisfies the growth condition we suppose that there exists $r_0 \in (0, 1]$ such that the set $\tilde{F}_{r_0}$ is bounded.

As in the case of Theorem 1.2 we choose α, T and an approximate solution of our problem after T iterations. In Chapter 3 we also establish analogs of Theorem 1.3 for the iterative proximal point subgradient algorithm and the dynamic string-averaging proximal point subgradient algorithm.

1.4 Cimmino Subgradient Projection Algorithm

In Chapter 4 we consider a minimization of a convex function on a solution set of a convex feasibility problem in a general Hilbert space using the Cimmino subgradient projection algorithm. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We show that an algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a small constant. Moreover, if we know computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this.

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

We recall the following useful facts on convex functions.

Let $f : X \to R^1$ be a continuous convex function such that

$$\{x \in X : f(x) \le 0\} \ne \emptyset.$$

Let $y_0 \in X$. For every $l \in \partial f(y_0)$ it is easy to see that

$$\{x \in X : f(x) \le 0\} \subset \{x \in X : f(y_0) + \langle l, x - y_0 \rangle \le 0\}.$$

It is well-known that the following lemma holds (see Lemma 11.1 of [94]).

Lemma 1.4 *Let $y_0 \in X$, $f(y_0) > 0$, $l \in \partial f(y_0)$ and let*

$$D = \{x \in X : \ f(y_0) + \langle l, x - y_0 \rangle \le 0\}.$$

The $l \neq 0$ and

$$P_D(y_0) = y_0 - f(y_0)\|l\|^{-2} l.$$

Let us now describe the convex feasibility problem and the Cimmino subgradient projection algorithm which is studied in Chapter 4.

Let m be a natural number and $f_i : X \to R^1$, $i = 1, \dots, m$ be convex continuous functions.

For every integer $i = 1, \dots, m$ put

$$C_i = \{x \in X : \ f_i(x) \le 0\},$$

$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : \ f_i(x) \le 0\}.$$

We suppose that

$$C \neq \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given positive number ϵ a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if

$$f_i(x) \le \epsilon \text{ for all } i = 1, \dots, m.$$

Let $M_* > 0$ and

$$C \cap B(0, M_*) \neq \emptyset.$$

Let $f : X \to R^1$ be a continuous function. We consider the minimization problem

$$f(x) \to \min, \ x \in C.$$

Assume that

$$\inf(f, C) = \inf(f, C \cap B(0, M_*)).$$

Fix

$$\bar{\Delta} \in (0, m^{-1}].$$

Let us describe our algorithm.

Cimmino Subgradient Projection Algorithm Fix $\alpha > 0$.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$l_k \in \partial f(x_k),$$

pick $w_{k+1} = (w_{k+1}(1), \dots, w_{k+1}(m)) \in R^m$ such that

$$w_{k+1}(i) \ge \bar{\Delta},\ i = 1, \dots, m,$$

$$\sum_{i=1}^{m} w_{k+1}(i) = 1,$$

for each $i \in \{1, \dots, m\}$,

$$\text{if } f_i(x_k - \alpha l_k) \le 0 \text{ then } x_{k,i} = x_k - \alpha l_k,\ l_{k,i} = 0$$

and if $f_i(x_k - \alpha l_k) > 0$ then

$$l_{k,i} \in \partial f_i(x_k - \alpha l_k),$$

$$x_{k,i} = x_k - \alpha l_k - f_i(x_k - \alpha l_k)\|l_{k,i}\|^{-2} l_{k,i}$$

and define the next iteration vector

$$x_{k+1} = \sum_{i=1}^{m} w_{k+1}(i) x_{k,i}.$$

In Chapter 4 this algorithm is studied under the presence of computational errors.

Cimmino Subgradient Projection Algorithm with Computational Errors

We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f, $\delta_C \in [0, 1]$ is a computational error produced by our computer system, when we calculate subgradients of the constraint functions f_i, $i = 1, \dots, m$ and $\bar{\delta}_C$ is a computational error produced by our computer system, when we calculate auxiliary projection operators. Let $\alpha > 0$ be a step size and $\Delta \in (0, 1]$.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_t \in X$ calculate

$$l_t \in \partial f(x_t) + B(0, \delta_f),$$

pick $w_{t+1} = (w_{t+1}(1), \dots, w_{t+1}(m)) \in R^m$ such that

$$w_{t+1}(i) \geq \bar{\Delta},\ i = 1, \dots, m,$$

$$\sum_{i=1}^{m} w_{t+1}(i) = 1,$$

for each $i \in \{1, \dots, m\}$,

$$\text{if } f_i(x_t - \alpha l_t) \leq \Delta,\ \text{then } y_{t+1,i} = x_t - \alpha l_t,\ l_{t,i} = 0,$$

if $f_i(x_t - \alpha l_t) > \Delta$, then we calculate

$$l_{t,i} \in \partial f_i(x_t - \alpha l_t) + B(0, \delta_C),$$

(this implies that $l_{t,i} \neq 0$),

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2} l_{t,i}, \bar{\delta}_C)$$

and the next iteration vector

$$x_{t+1} \in B(\sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}, \bar{\delta}_C).$$

Let $\Delta \in (0, 1]$, $\delta_f, \delta_C, \bar{\delta}_C \in [0, 1]$, $\alpha \in (0, 1]$, $\tilde{M} \geq M_*$, $M_0 \geq \max\{1, \tilde{M}\}$, $M_1 > 2$, $L_0 \geq 1$,

$$f_i(B(0, 3\tilde{M} + 4)) \subset [-M_0.M_0],\ i = 1, \dots, m,$$

$$|f_i(u) - f_i(v)| \leq (M_1 - 2)\|u - v\|$$

$$\text{for all } u, v \in (0, 3\tilde{M} + 2) \text{ and all } i = 1, \dots, m,$$

$$|f(u) - f(v)| \leq L_0\|u - v\| \text{ for all } u, v \in (0, 3\tilde{M} + 4),$$

$$\alpha \leq 2^{-1}(L_0 + 1)^{-1}(6\tilde{M} + 5)^{-1},\ \delta_C \leq 32^{-1}\Delta^2(6\tilde{M} + 5)^{-2}(M_0 + 5)^{-1},$$

$$\alpha \leq \min\{8^{-1}(L_0 + 1)^{-1}\Delta M_1^{-1}, L_0^{-2}\},$$

$$\bar{\delta}_C \leq 8^{-1}\Delta M_1^{-1}, \delta_C \leq 2^{-7}\Delta^3 M_1^{-1}(6\tilde{M} + 1)^{-2},$$

$$\alpha \leq 96^{-1}(L_0 + 1)^{-1}(6\tilde{M} + 5)^{-1}\bar{\Delta}\Delta^2 M_1^{-2},$$

$$\bar{\delta}_C < 32^{-1}\bar{\Delta}\Delta^2 M_1^{-2}(6\tilde{M} + 5)^{-1},$$

$$\delta_f(6\tilde{M} + L_0 + 1) \le 1, \ \delta_C < 2^{-9}\Delta^4\bar{\Delta}M_1^{-2}(6\tilde{M} + 5)^{-3}.$$

In Chapter 4 we obtain the following result (Theorem 4.6).

Theorem 1.5 *Let* $M_{*,0} > 0$*,*

$$|f(u)| \le M_{*,0} \text{ for all } u \in B(0, M_*),$$

$\tilde{M} \ge 2M_* + 2,$

$$\alpha \ge \max\{64\delta_C(3\tilde{M} + 1)\Delta^{-2}(6\tilde{M} + 1)^2, \ \bar{\delta}_C(4\tilde{M} + 5)\},$$

$$f(u) > M_{*,0} + 8 \text{ for all } u \in X \setminus B(0, 2^{-1}\tilde{M}),$$

T be a natural number satisfying

$$T \ge 128\tilde{M}^2\bar{\Delta}^{-1}\Delta^{-2}M_1^2,$$

$\{x_t\}_{t=0}^T \subset X$, $\{l_t\}_{t=0}^{T-1} \subset X$, $l_{t,i} \in X$, $t = 0, \dots, T-1$, $i = 1, \dots, m$,

$$\|x_0\| \le \tilde{M},$$

$$w_t = (w_t(1), \dots, w_t(m)) \in R^m, \ t = 1, \dots, T,$$

$$w_t(i) \ge \bar{\Delta}, \ i = 1, \dots, m, \ t = 1, \dots, T,$$

$$\sum_{i=1}^m w_t(i) = 1, \ t = 1, \dots, T$$

and $y_{t,i} \in X$, $t = 1, \dots, T$, $i = 1, \dots, m$.

Assume that for all integers $t \in \{0, \dots, T-1\}$ *and all integers* $i \in \{1, \dots, m\}$*,*

$$B(l_t, \delta_f) \cap \partial f(x_t) \neq \emptyset,$$

$$\text{if } f_i(x_t - \alpha l_t) \le \Delta, \text{ then } y_{t+1,i} = x_t - \alpha l_t, \ l_{t,i} = 0,$$

if $f_i(x_t - \alpha l_t) > \Delta$*, then*

$$B(l_{t,i}, \delta_C) \cap \partial f_i(x_t - \alpha l_t) \neq \emptyset$$

(this implies that $l_{t,i} \neq 0$*),*

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2}l_{t,i}, \delta_C)$$

and that

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}\| \le \bar{\delta}_C.$$

Then

$$\|x_t\| \le 3\tilde{M},\ t = 0, \dots, T,$$

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \delta_C(4\tilde{M} + 5),$$

$$\bar{\Delta} \sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$-\bar{\delta}_C(6\tilde{M} + 5) - 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3\} : t = 0, \dots, T - 1\} \le 4M^2 T^{-1}.$$

Moreover, if $t \in \{0, \dots, T - 1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \delta_C(4\tilde{M} + 5),$$

$$\bar{\Delta} \sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$-\bar{\delta}_C(6\tilde{M} + 5) - 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3\} \le 4M^2 T^{-1}$$

then

$$f(x_t) \le \inf(f, C)) + 2M^2(T\alpha)^{-1} + \alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1)$$

$$+32(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 + \alpha^{-1}\bar{\delta}_C(4\tilde{M} + 5),$$

$$f_i(x_t) \le \Delta + M_2\alpha(L_0 + 1),\ i = 1, \dots, m.$$

In Chapter 4 we also obtain an extension of this result (Theorem 4.7) when instead of assuming that f satisfies the growth condition we suppose that there exists $r_0 \in (0, 1]$ such that the set

$$\{x \in X :\ f_i(x) \le r_0,\ i = 1, \dots, m\}$$

is bounded. Chapter 4 also contains Theorems 4.9 and 4.10 which are extensions of Theorems 4.6 and 4.7 obtained for a modification of the Cimmino subgradient projection algorithm.

As in the case of Theorem 1.2 we choose α, T and an approximate solution of our problem after T iterations.

In Chapters 5 and 6 we continue to study the optimization problem considered in Chapter 4. In Chapter 5 we analyze the iterative subgradient projection algorithm while in Chapter 6 the dynamic string-averaging subgradient projection algorithm is used. The analogs of the theorem above are established for these two algorithms.

In Chapters 7 and 8 we study minimization problems with smooth objective functions using a fixed point gradient projection algorithm and a Cimmino gradient projection algorithm respectively.

1.5 Examples

In this section we consider several examples arising in the real world applications. They belong to the class of problems considered in the book and all our results can be applied to them.

Example 1.6 In [22] it was studied a problem of computerized tomography image reconstruction, posed as a constrained minimization problem aiming at finding a constraint-compatible solution that has a reduced value of the total variation of the reconstructed image.

The fully-discretized model in the series expansion approach to the image reconstruction problem of x-ray computerized tomography (CT) is formulated in the following manner. A Cartesian grid of square picture elements, called pixels, is introduced into the region of interest so that it covers the whole picture that has to be reconstructed. The pixels are numbered in some agreed manner, say from 1 (top left corner pixel) to J (bottom right corner pixel). The x-ray attenuation function is assumed to take a constant value x_j throughout the jth pixel, for $j = 1, 2, ..., J$. Sources and detectors are assumed to be points and the rays between them are assumed to be lines. Further, assume that the length of intersection of the ith ray with the jth pixel, denoted by a_j^i, for $i = 1, 2, ..., I$, $j = 1, 2, ..., J$, represents the weight of the contribution of the jth pixel to the total attenuation along the ith ray.

The physical measurement of the total attenuation along the ith ray, denoted by b_i, represents the line integral of the unknown attenuation function along the path of the ray. Therefore, in this fully-discretized model, the line integral turns out to be a finite sum and the model is described by a system of linear equations

$$\sum_{j=1}^{J} x_j a_j^i = b_i,\ i = 1, \ldots, I.$$

In matrix notation we rewrite this system of linear equations as

$$Ax = b,$$

where $b \in R^I$ is the measurement vector, $x \in R^J$ is the image vector, and the $I \times J$ matrix $A = (a^i_j)$ is the projection matrix.

In [22] the image reconstruction problem is represented by the optimization problem

$$\text{minimize } \{f(x) : Ax = b \text{ and } 0 \le x \le 1\},$$

where the function $f(x)$ is the total variation (TV) of the image vector x.

Example 1.7 In [39] the CT reconstruction problem is formulated as a constrained optimization problem of the following kind:

$$\text{Find } x^* = \operatorname{argmin} f(x) \text{ subject to } \|y - Ax\|_2 \le \epsilon,$$

where the positive constant ϵ, the vector y and the matrix A are given and $\|\cdot\|_2$ is the Euclidean norm. As it was pointed out in [39], there are many possible choices for the regularizing convex function f. A popular option is a total variation.

Example 1.8 In [26] string-averaging algorithmic structures are used for handling a family of operators in situations where the algorithm needs to employ the operators in a specific order. String-averaging allows to use strings of indices taken from the index set of all operators, to apply the operators along these strings, and to combine their end-points in some agreed manner to yield the next iterate of the algorithm.

It is considered a Hilbert space X with the inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\|\cdot\|$ and a finite family of mappings $T_i : X \to X, i = 1, \ldots, m$. For a given $u \in X$ it is studied the problem

$$\|u - x\| \to \min, \ \ x \in D,$$

where D is the set of common fixed points of the mappings $T_i, i = 1, \ldots, m$.

Example 1.9 In [29] the development of radiation therapy treatment planning is considered from a mathematical point of view as the following optimization problem.

Let $f(d, x)$ be a given objective convex function and let $c_m(d, x)$ be given convex constraint functions, for $m = 1, \ldots, M$. Let a_{ij} be given for $j = 1, \ldots, J$ and $i = 1, \ldots, I$, and let l_m and u_m be lower and upper bounds for the constraints c_m, for $m = 1, \ldots, M$, respectively. The problem is to find a radiation intensity vector $x^* \in R^I$ and a corresponding dose vector $d^* \in R^J$ that solve the problem:

$$f(d,x) \to \min,$$

such that

$$\langle a^j, x\rangle = d_j,\ \ j = 1,\ldots,J,$$

$$l_m \le c_m(d,x) \le u_m,\ \ m = 1,\ldots,M,$$

$$x_i \ge 0,\ \ i = 1,\ldots,I.$$

In practice, the objective function f and the constraints are typically chosen to be convex so that the subgradient projection algorithm is applicable. Traditionally, a widely used objective function is the 2-Norm of the difference of dose d and a desired dose b.

Example 1.10 In [49] it is analyzed total variation (TV) minimization for semi-supervised learning from partially-labeled network-structured data. Its approach exploits an intrinsic duality between TV minimization and network flow problems.

Consider a dataset of N data points that can be represented as supported at the nodes of a simple undirected weighted graph $G = (V, E, W)$, where V are nodes, E are edges and W are edge weights. It is assumed that labels x_i are known at only a few nodes $i \in V$ of a (small) training set $M \subset V$. The goal is to learn the unknown labels x_i for all data points $i \in V \setminus M$ outside the training set. This learning problem is formulated as the optimization problem

$$\sum_{(i,j)\in E} W_{i,j}|\tilde{x}_j - \tilde{x}_i| \to \min$$

$$\text{subject to } \tilde{x} \in R^N,\ \ \tilde{x}_i = x_i,\ \ i \in M.$$

Example 1.11 The following problem of adaptive filtering and equalization is considered in [58]:

$$\langle w, f\rangle \to \max$$

$$\text{subject to } |\langle w, g^{(i)}\rangle| \le 1,\ \ i = 1,\ldots,m.$$

Here f is a direction associated with the desired signal, while $g^{(i)}$ are directions associated with interference or noise signals.

Example 1.12 This is an example of a resource allocation or resource sharing problem considered in [58], where the resource to be allocated is the bandwidth over each of a set of links. Consider a network with m edges or links, labeled $1,\ldots,m$, and n flows, labeled $1,\ldots,n$. Each flow has an associated non-negative flow rate f_j; each edge or link has an associated positive capacity c_i. Each flow passes over a fixed set of links (its route); the total traffic t_i on link i is the sum of the flow rates over all flows that pass through link i. The flow routes are described by a routing matrix $A \in \{0, 1\}^{m \times n}$ defined as $A_{ij} = 1$ if flow j passes through link i and $A_{ij} = 0$ otherwise. Thus, the vector of link traffic, $t \in R^m$, is given by $t = Af$. The link capacity constraints can be expressed as $Af \leq c$. With a given flow vector f, we associate a total utility

$$U(f) = U_1(f_1) + \cdots + U_n(f_n),$$

where U_i is the utility for flow i, which we assume is concave and nondecreasing. We will choose flow rates that maximize total utility, in other words, that are solutions of the problem

$$U(f) \to \max$$

$$\text{subject to } Af \leq c, \;\; f \geq 0.$$

This is the network utility maximization problem.

Example 1.13 Many important engineering problems can be represented in the form of the following quadratically constrained quadratic program [56]:

$$\langle x, P_0 x\rangle + \langle q_0, x\rangle \to \min$$

$$\text{subject to } \langle x, P_i x\rangle + \langle q_i, x\rangle + r_i \leq 0, \;\; i = 1, \ldots, m,$$

where $x, q_i \in R^k$, $r_i \in R^1$ and P_i is a symmetric matrix of rank $k \times k$ which is positive semi-definite.

Chapter 2
Fixed Point Subgradient Algorithm

In this chapter we consider a minimization of a convex function on a common fixed point set of a finite family of quasi-nonexpansive mappings in a Hilbert space. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We use the Cimmino subgradient algorithm, the iterative subgradient algorithm and the dynamic string-averaging subgradient algorithm and show that each of them generates a good approximate solution, if the sequence of computational errors is bounded from above by a small constant. Moreover, if we known computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this.

2.1 Common Fixed Point Problems

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

For each $x \in X$ and each nonempty set $E \subset X$ put

$$d(x, E) = \inf\{\|x - y\| : \ y \in E\}.$$

For every point $x \in X$ and every positive number $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

Suppose that m is a natural number, $\bar{c} \in (0, 1]$, $P_i : X \to X$, $i = 1, \dots, m$, for every integer $i \in \{1, \dots, m\}$,

$$\mathrm{Fix}(P_i) := \{z \in X : \ P_i(z) = z\} \neq \emptyset$$

A. J. Zaslavski, *Optimization on Solution Sets of Common Fixed Point Problems*, Springer Optimization and Its Applications 178,
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and that the inequality

$$\|z - x\|^2 \geq \|z - P_i(x)\|^2 + \bar{c}\|x - P_i(x)\|^2 \tag{2.1}$$

holds for every for every integer $i \in \{1, \dots, m\}$, every point $x \in X$ and every point $z \in \mathrm{Fix}(P_i)$. Set

$$F = \cap_{i=1}^{m} \mathrm{Fix}(P_i). \tag{2.2}$$

For every positive number ϵ and every integer $i \in \{1, \dots, m\}$ set

$$F_\epsilon(P_i) = \{x \in X : \|x - P_i(x)\| \leq \epsilon\}, \tag{2.3}$$

$$\tilde{F}_\epsilon(P_i) = F_\epsilon(P_i) + B(0, \epsilon), \tag{2.4}$$

$$F_\epsilon = \cap_{i=1}^{m} F_\epsilon(P_i), \tag{2.5}$$

$$\tilde{F}_\epsilon = \cap_{i=1}^{m} \tilde{F}_\epsilon(P_i) \tag{2.6}$$

and

$$\widehat{F}_\epsilon = F_\epsilon + B(0, \epsilon). \tag{2.7}$$

A point belonging to the set F is a solution of our common fixed point problem while a point which belongs to the set $\tilde{F}_\epsilon$ is its ϵ-approximate solution.

Let $M_* > 0$ satisfy

$$F \cap B(0, M_*) \neq \emptyset. \tag{2.8}$$

Proposition 2.1 *Let* $\epsilon > 0$, $i \in \{1, \dots, m\}$ *and let*

$$\|P_i(x) - P_i(y)\| \leq \|x - y\| \text{ for all } x, y \in X. \tag{2.9}$$

Then $\tilde{F}_\epsilon(P_i) \subset F_{3\epsilon}(P_i)$.

Proof Let $x \in \tilde{F}_\epsilon(P_i)$. By (2.4), there exists

$$y \in F_\epsilon(P_i)$$

such that $\|x - y\| \leq \epsilon$. In view of (2.3) and (2.9),

$$\|y - P_i(y)\| \leq \epsilon,$$

$$\|P_i(x) - P_i(y)\| \leq \epsilon,$$

$$\|x - P_i(x)\| \leq \|x - y\| + \|y - P_i(y)\| + \|P_(y) - P_i(x)\| \leq 3\epsilon$$

and $x \in F_{3\epsilon}(P_i)$. Proposition 2.1 is proved.

Corollary 2.2 *Assume that $\epsilon > 0$ and that for all $i \in \{1, \dots, m\}$,*

$$\|P_i(x) - P_i(y)\| \leq \|x - y\| \textit{ for all } x, y \in X.$$

Then $\tilde{F}_\epsilon \subset F_{3\epsilon}$.

Proposition 2.3 *Let $\epsilon > 0$, $i \in \{1, \dots, m\}$ and let*

$$Fix(P_i) = P_i(X). \tag{2.10}$$

Then

$$\tilde{F}_\epsilon(P_i) \subset Fix(P_i) + B(0, 2\epsilon).$$

Proof Let $x \in \tilde{F}_\epsilon(P_i)$. By (2.3) and (2.4), there exists $y \in X$ such that

$$\|x - y\| \leq \epsilon, \ \|y - P_i(y)\| \leq \epsilon.$$

In view of the relations above and (2.10),

$$\|x - P_i(y)\| \leq 2\epsilon,$$

$$x \in \mathrm{Fix}(P_i) + B(0, 2\epsilon).$$

Proposition 2.3 is proved.

Corollary 2.4 *Assume that $\epsilon > 0$ and that (2.10) holds for all $i \in \{1, \dots, m\}$, Then*

$$\tilde{F}_\epsilon \subset \cap_{i=1}^m (Fix(P_i) + B(0, 2\epsilon)).$$

Example 2.5 ([8, 93]) Let D be a nonempty closed convex subset of X. Then for each $x \in X$ there is a unique point $P_D(x) \in D$ satisfying

$$\|x - P_D(x)\| = \inf\{\|x - y\| : \ y \in D\}.$$

Moreover,

$$\|P_D(x) - P_D(y)|| \leq \|x - y\| \text{ for all } x, y \in X$$

and for each $x \in X$ and each $z \in D$,

$$\langle z - P_D(x), x - P_D(x)\rangle \leq 0,$$

$$\|z - P_D(x)\|^2 + \|x - P_D(x)\|^2 \le \|z - x\|^2.$$

Example 2.6 Denote by I the identity self-mapping of X: $I(x) = x$, $x \in X$. A mapping $T : X \to X$ is called firmly nonexpansive [8] if for all $x, y \in X$,

$$\|T(x) - T(y)\|^2 + \|(I - T)(x) - (I - T)(y)\|^2 \le \|x - y\|^2.$$

It is easy to see that if a mapping $T : X \to X$ is firmly nonexpansive and $z \in X$ satisfies $z = T(z)$, then for all $y \in X$,

$$\|z - T(y)\|^2 + \|y - T(y)\|^2 \le \|z - y\|^2.$$

2.2 The Cimmino Subgradient Algorithm

Let $f : X \to R^1$ be a convex continuous function. We consider the minimization problem

$$f(x) \to \min, \ x \in F.$$

Assume that

$$\inf(f, F) = \inf(f, F \cap B(0, M_*)). \tag{2.11}$$

Fix $\alpha > 0$. Let us describe our algorithm.

Cimmino Subgradient Algorithm

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$l_k \in \partial f(x_k),$$

pick $w_{k+1} = (w_{k+1}(1), \dots, w_{k+1}(m)) \in R^m$ such that

$$w_{k+1}(i) \ge 0, \ i = 1, \dots, m,$$

$$\sum_{i=1}^{m} w_{k+1}(i) = 1$$

and define the next iteration vector

$$x_{k+1} = \sum_{i=1}^{m} w_{k+1}(i) P_i(x_k - \alpha l_k).$$

In this chapter this algorithm is studied under the presence of computational errors and two convergence results are obtained. Fix

$$\Delta \in (0, m^{-1}). \tag{2.12}$$

We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f while $\delta_p \in [0, 1]$ is a computational error produced by our computer system, when we calculate the operators P_i, $i = 1, \dots, m$. Let $\alpha > 0$ be a step size.

Cimmino Subgradient Algorithm with Computational Errors

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$\xi_k \in \partial f(x_k) + B(0, \delta_f),$$

pick $w_{k+1} = (w_{k+1}(1), \dots, w_{k+1}(m)) \in R^m$ such that

$$w_{k+1}(i) \geq \Delta,\ i = 1, \dots, m,$$

$$\sum_{i=1}^{m} w_{k+1}(i) = 1,$$

calculate

$$y_{k,i} \in B(P_i(x_k - \alpha\xi_k), \delta_p),\ i = 1, \dots, m$$

and the next iteration vector and $x_{k+1} \in X$ such that

$$\|x_{k+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t,i}\| \leq \delta_p.$$

In this algorithm, as well for other algorithms considered in the book, we assume that the step size does not depend on the number of iterative step k. The same analysis can be done when step sizes depend on k. On the other hand, as it was shown in [93, 95], in the case of computational errors, the best choice of step sizes is step sizes which do not depend on iterative step numbers.

2.3 Two Auxiliary Results

Our study of the algorithm is based on the following auxiliary results.

Lemma 2.7 *Let $F_0 \subset X$ be nonempty, $M_0 > 0$, $L_0 \geq 1$,*

$$\|f(z_1 - f(z_2)\| \le L_0\|z_1 - z_2\| \textit{ for all } z_1, z_2 \in B(0, M_0 + 4), \tag{2.13}$$

a mapping $Q : X \to X$ *satisfy*

$$Q(z) = z,\ \ z \in F_0, \tag{2.14}$$

$$\|Q(u) - z\| \le \|u - z\| \textit{ for all } u \in X \textit{ and all } z \in F_0 \tag{2.15}$$

and let $\delta_1, \delta_2 \in [0, 1]$, $\alpha \in (0, 1]$. *Assume that*

$$z \in F_0 \cap B(0, M_0), \tag{2.16}$$

$$x \in B(0, M_0), \tag{2.17}$$

$$\xi \in \partial f(x) + B(0, \delta_1) \tag{2.18}$$

and that

$$u \in X \tag{2.19}$$

satisfies

$$\|u - Q(x - \alpha\xi)\| \le \delta_2. \tag{2.20}$$

Then

$$\alpha(f(x) - f(z))$$

$$\le 2^{-1}\|x - z\|^2 - 2^{-1}\|u - z\|^2 + \delta_2(2M_0 + 2 + \alpha L_0)$$

$$+2^{-1}\alpha^2 L_0^2 + \alpha\delta_1(2M_0 + L_0 + 1).$$

Proof By (2.18), there exists

$$l \in \partial f(x) \tag{2.21}$$

such that

$$\|l - \xi\| \le \delta_1. \tag{2.22}$$

In view of (2.13) and (2.17),

$$\partial f(x) \subset B(0, L_0). \tag{2.23}$$

In view of (2.21),

$$f(z) - f(x) \geq \langle l, z - x \rangle. \tag{2.24}$$

It follows from (2.22) and (2.23) that

$$\|x - \alpha\xi - z\|^2 = \|x - \alpha l + (\alpha l - \alpha\xi) - z\|^2$$

$$= \|x - \alpha l - z\|^2 + \alpha^2\|l - \xi\|^2 + 2\alpha\|l - \xi\|\|x - \alpha l - z\|$$

$$\leq \|x - \alpha l - z\|^2 + \alpha^2\delta_1^2 + 2\alpha\delta_1(2M_0 + \alpha L_0). \tag{2.25}$$

By (2.23) and (2.24),

$$\|x - \alpha l - z\|^2 = \|x - z\|^2 - 2\alpha\langle l, x - z\rangle + \alpha^2\|l\|^2$$

$$\leq \|x - z\|^2 + 2\alpha(f(z) - f(x)) + \alpha^2 L_0^2. \tag{2.26}$$

In view of (2.25) and (2.26),

$$\|x - \alpha\xi - z\|^2 \leq \|x - z\|^2 + 2\alpha(f(z) - f(x)) + \alpha^2 L_0^2$$

$$+ \alpha^2\delta_1^2 + 2\alpha\delta_1(2M_0 + L_0). \tag{2.27}$$

It follows from (2.15)–(2.17), (2.20), (2.22), (2.23) and (2.27) that

$$\|u - z\|^2 = \|u - Q(x - \alpha\xi) + Q(x - \alpha\xi) - z\|^2$$

$$\leq \|u - Q(x - \alpha\xi)\|^2 + 2\|u - Q(x - \alpha\xi)\|\|Q(x - \alpha\xi) - z\| + \|Q(x - \alpha\xi) - z\|^2$$

$$\leq \delta_2^2 + 2\delta_2\|x - \alpha\xi - z\| + \|x - \alpha\xi - z\|^2$$

$$\leq \delta_2^2 + 2\delta_2(2M_0 + \alpha(L_0 + 1))$$

$$+\|x - z\|^2 + 2\alpha(f(z) - f(x)) + \alpha^2 L_0^2$$

$$+\alpha^2\delta_1^2 + 2\alpha\delta_1(2M_0 + L_0).$$

This relation implies that

$$2\alpha(f(x) - f(z))$$

$$\leq \|x - z\|^2 - \|u - z\|^2 + \delta_2^2 + 2\delta_2(2M_0 + \alpha(L_0 + 1))$$

$$+\alpha^2 L_0^2 + \alpha^2\delta_1^2 + 2\alpha\delta_1(2M_0 + L_0)$$

$$\leq \|x - z\|^2 - \|u - z\|^2 + 2\delta_2(2M_0 + 2 + \alpha L_0)$$

$$+\alpha^2 L_0^2 + 2\alpha\delta_1(2M_0 + L_0 + 1).$$

Lemma 2.7 is proved.

Lemma 2.8 *Let* $M_0 \geq M_*$, $\delta_1, \delta_2 \in [0, 1]$,

$$w(i) \geq \Delta,\ i = 1, \dots, m, \tag{2.28}$$

$$\sum_{i=1}^{m} w(i) = 1, \tag{2.29}$$

$$z \in F \cap B(0, M_0), \tag{2.30}$$

$$x \in B(0, M_0), \tag{2.31}$$

$$x_0 \in B(x, \delta_1), \tag{2.32}$$

for all $i = 1, \dots, m$, $y_i \in X$ *satisfy*

$$\|y_i - P_i(x_0)\| \leq \delta_2 \tag{2.33}$$

and let

$$y \in B(\sum_{i=1}^{m} w(i)y_i, \delta_2). \tag{2.34}$$

Then

$$\|z - x\|^2 - \|z - y\|^2 \geq \Delta\bar{c}\sum_{i=1}^{m} \|x_0 - y_i\|^2$$

$$-\delta_1(4M_0 + 1) - 2\delta_2(4M_0 + 4) - \Delta\bar{c}m\delta_2((4M_0 + 2)\bar{c}^{-1/2} + 1).$$

Proof In view of (2.1) and (2.30), for $i = 1, \dots, m$,

$$\|z - P_i(x_0)\|^2 + \bar{c}\|x_0 - P_i(x_0)\|^2 \leq \|z - x_0\|^2, \tag{2.35}$$

$$\|z - P_i(x_0)\| \leq \|z - x_0\|. \tag{2.36}$$

Since the function $u \to \|u - z\|^2$, $u \in X$ is convex it follows from (2.28) and (2.29) that

$$\|z - \sum_{i=1}^{m} w(i)P_i(x_0)\|^2 \le \sum_{i=1}^{m} w(i)\|z - P_i(x_0)\|^2. \tag{2.37}$$

By (2.28), (2.29), (2.35) and (2.37),

$$\begin{aligned}
&\|z - x_0\|^2 - \|z - \sum_{i=1}^{m} w(i)P_i(x_0)\|^2 \\
&\ge \|z - x_0\|^2 - \sum_{i=1}^{m} w(i)\|z - P_i(x_0)\|^2 \\
&\ge \sum_{i=1}^{m} w(i)(\|z - x_0\|^2 - \|z - P_i(x_0)\|^2) \\
&\ge \Delta \sum_{i=1}^{m} (\|z - x_0\|^2 - \|z - P_i(x_0)\|^2) \\
&\ge \Delta \bar{c} \sum_{i=1}^{m} \|x_0 - P_i(x_0)\|^2.
\end{aligned} \tag{2.38}$$

In view of (2.30)–(2.32),

$$\|z - x\| \le 2M_0, \ \|z - x_0\| \le 2M_0 + 1. \tag{2.39}$$

By (2.32) and (2.39),

$$\begin{aligned}
&|\|z - x\|^2 - \|z - x_0\|^2| \\
&\le |\|z - x\| - \|z - x_0\||(\|z - x\| + \|z - x_0\|) \le \delta_1(4M_0 + 1).
\end{aligned} \tag{2.40}$$

It follows from (2.28), (2.29), (2.36) and (2.39) that

$$\|z - \sum_{i=1}^{m} w(i)P_i(x_0)\| \le \sum_{i=1}^{m} w(i)\|z - P_i(x_0)\| \le \|z - x_0\| \le 2M_0 + 1. \tag{2.41}$$

In view of (2.33) and (2.34),

$$|\|z - y\| - \|z - \sum_{i=1}^{m} w(i)P_i(x_0)\|| \le \|y - \sum_{i=1}^{m} w(i)P_i(x_0)\| \le 2\delta_2. \tag{2.42}$$

Equations (2.41) and (2.42) imply that

$$|\|z - y\|^2 - \|z - \sum_{i=1}^{m} w(i)P_i(x_0)\|^2|$$

$$\leq 2\delta_2(\|z - y\| + \|z - \sum_{i=1}^{m} w(i)P_i(x_0)\|) \leq 2\delta_2(4M_0 + 4). \tag{2.43}$$

By (2.38), (2.40) and (2.43),

$$\|z - x\|^2 - \|z - y\|^2$$

$$\geq \|z - x_0\|^2 - \delta_1(4M_0 + 1)$$

$$-\|z - \sum_{i=1}^{m} w(i)P_i(x_0)\|^2 - 2\delta_2(4M_0 + 4)$$

$$\geq \Delta\bar{c}\sum_{i=1}^{m}\|x_0 - P_i(x_0)\|^2$$

$$-\delta_1(4M_0 + 1) - 2\delta_2(4M_0 + 4). \tag{2.44}$$

It follows from (2.3), (2.33) and (2.35) that for $i = 1, \dots, m$,

$$|\|x_0 - y_i\|^2 - \|x_0 - P_i(x_0)\|^2|$$

$$\leq \|y_i - P_i(x_0)\|(2\|x_0 - P_i(x_0)\| + \|y_i - P_i(x_0)\|)$$

$$\leq \delta_2(2\|x_0 - P_i(x_0)\| + \delta_2)$$

$$\leq \delta_2(2\|z - x_0\|\bar{c}^{-1/2} + 1) \leq \delta_2((4M_0 + 2)\bar{c}^{-1/2} + 1). \tag{2.45}$$

By (2.44) and (2.45),

$$\|z - x\|^2 - \|z - y\|^2$$

$$\geq \Delta\bar{c}\sum_{i=1}^{m}\|x_0 - y_i\|^2 - \Delta\bar{c}\delta_2((4M_0 + 2)\bar{c}^{-1/2} + 1)m$$

$$-\delta_1(4M_0 + 1) - 2\delta_2(4M_0 + 4).$$

Lemma 2.8 is proved.

2.4 The First Result for the Cimmino Subgradient Algorithm

In the following result we assume the objective function f satisfies the coercivity growth condition.

Theorem 2.9 *Let the function f be Lipschitz on bounded subsets of X,*

$$\lim_{\|x\|\to\infty} f(x) = \infty,$$

$M \geq 2M_* + 8$, $L_0 \geq 1$,

$$M_1 > \sup\{|f(u)| : u \in B(0, M_* + 4)\} + 4, \tag{2.46}$$

$$f(u) > M_1 + 4 \textit{ for all } u \in X \setminus B(0, 2^{-1}M), \tag{2.47}$$

$$|f(z_1) - f(z_2)| \leq L_0\|z_1 - z_2\| \textit{ for all } z_1, z_2 \in B(0, 3M + 4), \tag{2.48}$$

$\delta_f, \delta_p \in [0, 1], \alpha > 0$ *satisfy*

$$\alpha \leq L_0^{-2},\ \alpha \geq \delta_f(6M + L_0 + 2),\ \alpha \geq 2\delta_p(6M + 2), \tag{2.49}$$

T be a natural number and let

$$\gamma_T = \max\{\alpha(L_0 + 1),\ (\Delta\bar{c})^{-1/2}(4M^2T^{-1} + \alpha(L_0 + 1)(12M + 4))^{1/2} + \delta_p\}. \tag{2.50}$$

Assume that $\{x_t\}_{t=0}^{T} \subset X$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$(w_t(1), \ldots, w_t(m)) \in R^m,\ t = 1, \ldots, T,$$

$$\sum_{i=1}^{m} w_t(i) = 1,\ t = 1, \ldots, T, \tag{2.51}$$

$$w_t(i) \geq \Delta,\ i = 1, \ldots, m,\ t = 1, \ldots, T, \tag{2.52}$$

$$x_0 \in B(0, M) \tag{2.53}$$

and that for all integers $t \in \{0, \ldots, T - 1\}$,

$$B(\xi_t, \delta_f) \cap \partial f(x_t) \neq \emptyset, \tag{2.54}$$

$$y_{t,i} \in B(P_i(x_t - \alpha\xi_t), \delta_p),\ i = 1, \ldots, m, \tag{2.55}$$

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t,i}\| \le \delta_p. \quad (2.56)$$

Then

$$\|x_t\| \le 2M + M_*, \ t = 0, \dots, T$$

and

$$\min\{\max\{\Delta\bar{c} \sum_{i=0}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2 - \alpha(L_0 + 1)(12M + 4),$$

$$2\alpha(f(x_t) - \inf(f, F)) - 4\delta_p(6M + 3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1)\} :$$

$$t = 0, \dots, T - 1\} \le 4M^2 T^{-1}.$$

Moreover, if $t \in \{0, \dots, T - 1\}$ *and*

$$\max\{\Delta\bar{c} \sum_{i=0}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2 - \alpha(L_0 + 1)(12M + 4),$$

$$2\alpha(f(x_t) - \inf(f, F)) - 4\delta_p(6M + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1)\} \le 4M^2 T^{-1},$$

then

$$f(x_t) \le \inf(f, F) + 2M^2(T\alpha)^{-1}$$

$$+2\alpha^{-1}\delta_p(6M + 3) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 3)$$

and

$$x_t \in \widehat{F}_{\gamma_T}.$$

Proof In view of (2.8), there exists

$$z \in B(0, M_*) \cap F. \quad (2.57)$$

By (2.53) and (2.57),

$$\|z - x_0\| \le 2M. \quad (2.58)$$

We show that for all $t = 0, \dots, T$,

$$\|z - x_t\| \le 2M. \tag{2.59}$$

In view of (2.58), (2.59) is true for $t = 0$.

Assume that there exists an integer $k \in \{0, \dots, T\}$ such that

$$\|z - x_k\| > 2M. \tag{2.60}$$

By (2.58) and (2.60), $k > 0$. We may assume without loss of generality that (2.59) holds for all integers $t = 0, \dots, k-1$. In particular,

$$\|z - x_{k-1}\| \le 2M. \tag{2.61}$$

By (2.1), (2.2), (2.51), (2.52), (2.54)–(2.56) and (2.61), we apply Lemma 2.7 with

$$\delta_1 = \delta_f,\ \delta_2 = 2\delta_p,\ F_0 = F,\ M_0 = 3M,$$

$$Q = \sum_{i=1}^{m} w_k(i) P_i,\ x = x_{k-1},\ \xi = \xi_{k-1},\ u = x_k$$

and obtain that

$$\alpha(f(x_{k-1}) - f(z))$$

$$\le 2^{-1}\|x_{k-1} - z\|^2 - 2^{-1}\|x_k - z\|^2$$

$$+ 2\delta_p(6M + 2) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{2.62}$$

There are two cases:

$$\|z - x_k\| \le \|z - x_{k-1}\| \tag{2.63}$$

$$\|z - x_k\| > \|z - x_{k-1}\|. \tag{2.64}$$

Assume that (2.63) holds. Then in view of (2.61),

$$\|x_k - z\| \le 2M.$$

Assume that (2.64) is true. By (2.62) and (2.64),

$$\alpha(f(x_{k-1}) - f(z))$$

$$\le 2\delta_p(6M + 2) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{2.65}$$

It follows from (2.46), (2.49), (2.57) and (2.65) that

$$f(x_{k-1})$$

$$< M_1 + 2\alpha^{-1}\delta_p(6M+2) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1) \le M_1 + 3. \quad (2.66)$$

In view of (2.47), (2.57) and (2.66),

$$\|x_{k-1}\| \le M/2, \ \|x_{k-1} - z\| \le M_* + 2^{-1}M. \quad (2.67)$$

By (2.1), (2.2), (2.51), (2.52), (2.55)–(2.57) and (2.67),

$$\|x_k - z\| \le 2\delta_p + \|z - \sum_{i=1}^{m} w_k(i)P_i(x_{k-1} - \alpha\xi_{k-1})\|$$

$$\le 2\delta_p + \|z - (x_{k-1} - \alpha\xi_{k-1})\|$$

$$\le 2\delta_p + \|z - x_{k-1}\| + \alpha\|\xi_{k-1}\|$$

$$\le 2\delta_p + M_* + M/2 + \alpha\|\xi_{k-1}\|. \quad (2.68)$$

Equations (2.47) and (2.48) imply that

$$\partial f(x_{k-1}) \subset B(0, L_0). \quad (2.69)$$

By (2.54) and (2.69),

$$\|\xi_{k-1}\| \le L_0 + 1. \quad (2.70)$$

In view of (2.68) and (2.70),

$$\|x_k - z\| \le 2 + M_* + 2^{-1}M + \alpha(L_0 + 1) \le 4 + M_* + 2^{-1}M \le 2M.$$

Thus in the both cases

$$\|x_k - z\| \le 2M.$$

This contradicts (2.60). The contradiction we have reached proves that (2.59) is true for all $t = 0, \dots, T$. Together with (2.57) this implies that

$$\|x_t\| \le 2M + M_*, \ t = 0, \dots, T. \quad (2.71)$$

Let $t \in \{0, \dots, T-1\}$. By (2.48) and (2.71),

$$\partial f(x_t) \subset B(0, L_0). \tag{2.72}$$

In view of (2.54) and (2.72),

$$\|\xi_t\| \le L_0 + 1. \tag{2.73}$$

By (2.1), (2.2), (2.48), (2.51), (2.52) and (2.54)–(2.56), we apply Lemma 2.7 with

$$\delta_1 = \delta_f,\ \delta_2 = 2\delta_p,\ F_0 = F,\ M_0 = 3M,$$

$$Q = \sum_{i=1}^{m} w_{t+1}(i) P_i,\ x = x_t,\ \xi = \xi_t,\ u = x_{t+1}$$

and obtain that

$$\alpha(f(x_t) - f(z))$$
$$\le 2^{-1}\|x_t - z\|^2 - 2^{-1}\|x_{t+1} - z\|^2$$
$$+ 2\delta_p(6M + 2 + \alpha L_0) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{2.74}$$

In view of (2.73),

$$\|x_t - (x_t - \alpha\xi_t)\| = \alpha\|\xi_t\| \le \alpha(L_0 + 1). \tag{2.75}$$

By (2.51), (2.52), (2.55)–(2.57), (2.71) and (2.73) we apply Lemma 2.8 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ M_0 = 3M,$$

$$w(i) = w_{t+1}(i),\ i = 1, \dots, m,\ y_i = y_{t,i},\ i = 1, \dots, m,$$

$$x = x_t,\ x_0 = x - \alpha\xi_t,\ y = x_{t+1}$$

and obtain that

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2 \ge \Delta\bar{c}\sum_{i=1}^{m}\|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-\alpha(L_0+1)(12M+1) - 2\delta_p(12M+4) - \Delta\bar{c}m\delta_p((12M+2)\bar{c}^{-1/2}+1), \tag{2.76}$$

$$\Delta\bar{c}\sum_{i=1}^{m}\|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$\leq \|z - x_t\|^2 - \|z - x_{t+1}\|^2$$

$$+\alpha(L_0 + 1)(12M + 1) + 2\delta_p(12M + 4) + \Delta\bar{c}m\delta_p((12M + 2)\bar{c}^{-1/2} + 1).$$

It follows from (2.74) and (2.76) that

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2 \geq \max\{\Delta\bar{c}\sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - 2\delta_p(12M + 4) - \Delta\bar{c}m\delta_p((12M + 2)\bar{c}^{-1/2} + 1),$$

$$2\alpha(f(x_t) - f(z))$$

$$- 4\delta_p(6M + 2 + \alpha L_0) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1)\}. \tag{2.77}$$

By (2.53), (2.57) and (2.77),

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1}(\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\geq \sum_{t=0}^{T-1}(\max\{\Delta\bar{c}\sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - 2\delta_p(12M + 4) - \Delta\bar{c}m\delta_p((12M + 2)\bar{c}^{-1/2} + 1),$$

$$2\alpha(f(x_t) - f(z))$$

$$- 4\delta_p(6M + 2 + \alpha L_0) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1)\}). \tag{2.78}$$

Since z is an arbitrary element of $F \cap B(0, M_*)$ it follows from (2.11), (2.49) and (2.78) that

$$4M^2 T^{-1} \geq \min\{\max\{\Delta\bar{c}\sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - 2\delta_p(12M + 4) - \Delta\bar{c}m\delta_p((12M + 2)\bar{c}^{-1/2} + 1),$$

$$2\alpha(f(x_t) - \inf(f, F))$$

$$-4\delta_p(6M + 2 + \alpha L_0) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1)\} : \ t = 0, \ldots, T - 1\}$$

$$\geq \min\{\max\{\Delta\bar{c}\sum_{i=1}^{m}\|x_t-\alpha\xi_t-y_{t,i}\|^2$$

$$-\alpha(L_0+1)(12M+4),\ 2\alpha(f(x_t)-\inf(f,F))$$

$$-4\delta_p(6M+3)-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1)\}:\ t=0,\dots,T-1\}. \quad (2.79)$$

Assume that $t\in\{0,\dots,T-1\}$ and that

$$\max\{\Delta\bar{c}\sum_{i=1}^{m}\|x_t-\alpha\xi_t-y_{t,i}\|^2$$

$$-\alpha(L_0+1)(12M+4),\ 2\alpha(f(x_t)-\inf(f,F))$$

$$-4\delta_p(6M+3)-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1)\}\leq 4M^2T^{-1}. \quad (2.80)$$

By (2.80),

$$\sum_{i=1}^{m}\|x_t-\alpha\xi_t-y_{t,i}\|^2$$

$$\leq(\Delta\bar{c})^{-1}(4M^2T^{-1}+\alpha(L_0+1)(12M+4)), \quad (2.81)$$

$$f(x_t)\leq\inf(f,F))$$

$$+2M^2(T\alpha)^{-1}+2\alpha^{-1}\delta_p(6M+3)+2^{-1}\alpha L_0^2+\delta_f(6M+L_0+1). \quad (2.82)$$

In view of (2.55) and (2.81), for all $i=1,\dots,m$,

$$\|x_t-\alpha\xi_t-P_i(x_t-\alpha\xi_t)\|\leq\delta_p+\|x_t-\alpha\xi_t-y_{t,i}\|$$

$$\leq\delta_p+(\Delta\bar{c})^{-1/2}(4M^2T^{-1}+\alpha(L_0+1)(12M+4))^{1/2}. \quad (2.83)$$

It follows from (2.50), (2.73) and (2.83) that

$$x_t\in\widehat{F}_{\gamma_T}.$$

Theorem 2.9 is proved.

In Theorem 2.9 the computational errors δ_f,δ_p are fixed. Assume that they are positive. Let us choose α, T. First, we choose α in order to minimize the right-hand side of (2.82). Since T can be an arbitrary large we need to minimize the function

$$2\alpha^{-1}\delta_p(6M+3)+2^{-1}\alpha L_0^2,\ \alpha>0.$$

Its minimizer is

$$\alpha=2L_0^{-1}(\delta_p(6M+3))^{1/2}.$$

Since α satisfies (2.49) we obtain the following restrictions on δ_f, δ_p:

$$\delta_f \le 2L_0^{-1}\delta_p^{1/2}(6M+3)^{1/2}(6M+L_0+2)^{-1},$$

$$\delta_p \le 4^{-1}L_0^{-2}(6M+3)^{-1}.$$

In this case

$$\gamma_T=\max\{2L_0^{-1}(\delta_p(6M+3))^{1/2}(L_0+1),$$

$$(\Delta\bar{c})^{-1/2}(4M^2T^{-1}+2L_0^{-1}(\delta_p(6M+3))^{1/2}(L_0+1)(12M+4))^{1/2}+\delta_p\}.$$

We choose T with the same order as δ_p^{-1} For example, $T=\lfloor\delta_p^{-1}\rfloor$. In this case in view of Theorem 2.9, there exists $t\in\{0,\dots,T-1\}$ such that then

$$f(x_t)\le \inf(f,F)+c_1\delta_p^{1/2}+\delta_f(6M+L_0+3)$$

and

$$x_t\in \widehat{F}_{c_2\delta_p^{1/4}}$$

where c_1, c_2 are positive constants which depend on $M, L_0, \Delta, \bar{c}$.

Let us explain how we can obtain t satisfying (2.80). Set

$$E=\{t\in\{0,\dots,T-1\}:\ \Delta\bar{c}\sum_{i=0}^{m}\|x_t-\alpha\xi_t-y_{t,i}\|^2$$

$$\le \alpha(L_0+1)(12M+4)+4M^2T\alpha^{-1}\}$$

and find $t_*\in E$ such that $f(x_{t_*})\le f(x_t)$ for all $t\in E$. This t satisfies (2.80).

2.5 The Second Result for the Cimmino Subgradient Algorithm

In the following theorem we assume that the set F is bounded.

Theorem 2.10 *Let* $r_0 \in (0, 1]$,

$$F_{r_0} \subset B(0, M_*), \tag{2.84}$$

$M \geq M_* + 4$, $L_0 \geq 1$,

$$|f(z_1) - f(z_2)| \leq L_0\|z_1 - z_2\| \textit{ for all } z_1, z_2 \in B(0, 3M + 4), \tag{2.85}$$

$\delta_f, \delta_p \in [0, 1], \alpha \in (0, 1]$ *satisfy*

$$\alpha \leq 16^{-1}\Delta\bar{c}r_0^2(L_0+1)^{-1}(12M+1)^{-1}, \ \delta_p \leq 16^{-1}\Delta\bar{c}r_0^2(362M+2)^{-1}, \tag{2.86}$$

T *be a natural number and let*

$$\gamma_T = \max\{\alpha(L_0 + 1), \ (\Delta\bar{c})^{-1/2}(4M^2T^{-1}$$

$$+ \alpha(L_0 + 4)(2M + 1) + \delta_p(16M + 12))^{1/2} + \delta_p\}. \tag{2.87}$$

Assume that $\{x_t\}_{t=0}^T \subset X$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$(w_t(1), \ldots, w_t(m)) \in R^m, \ t = 1, \ldots, T,$$

$$w_t(i) \geq \Delta, \ i = 1, \ldots, m, \ t = 1, \ldots, T, \tag{2.88}$$

$$\sum_{i=1}^{m} w_t(i) = 1, \ t = 1, \ldots, T, \tag{2.89}$$

$$x_0 \in B(0, M) \tag{2.90}$$

and that for all integers $t \in \{0, \ldots, T - 1\}$,

$$B(\xi_t, \delta_f) \cap \partial f(x_t) \neq \emptyset, \tag{2.91}$$

$$y_{t,i} \in B(P_i(x_t - \alpha\xi_t), \delta_p), \ i = 1, \ldots, m, \tag{2.92}$$

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i)y_{t,i}\| \leq \delta_p. \tag{2.93}$$

Then

$$\|x_t\| \leq 2M + M_*, \ t = 0, \ldots, T$$

and

$$\min\{\max\{\Delta\bar{c}\sum_{i=1}^{m}\|x_t-\alpha\xi_t-y_{t,i}\|^2-\alpha(L_0+1)(12M+1)-\delta_p(16M+12),$$

$$2\alpha(f(x_t)-\inf(f,F))$$

$$-4\delta_p(6M+L_0+2)-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1)\}:$$

$$t=0,\dots,T-1\}\le 4M^2T^{-1}.$$

Moreover, if $t\in\{0,\dots,T-1\}$ *and*

$$\max\{\Delta\bar{c}\sum_{i=1}^{m}\|x_t-\alpha\xi_t-y_{t,i}\|^2-\alpha(L_0+1)(12M+1)-\delta_p(16M+12),$$

$$2\alpha(f(x_t)-\inf(f,F))$$

$$-4\delta_p(6M+L_0+2)-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1)\}\le 4M^2T^{-1},$$

then

$$f(x_t)\le\inf(f,F)+2M^2(T\alpha)^{-1}$$

$$+2\alpha^{-1}\delta_p(6M+L_0+2)+2^{-1}\alpha L_0^2+\delta_f(6M+L_0+1)$$

and

$$x_t\in\widehat{F}_{\gamma_T}.$$

Proof In view of (2.8), there exists

$$z\in B(0,M_*)\cap F. \tag{2.94}$$

By (2.90) and (2.94),

$$\|z-x_0\|\le 2M. \tag{2.95}$$

We show that for all $t=0,\dots,T$,

$$\|z-x_t\|\le 2M. \tag{2.96}$$

In view of (2.95), (2.96) is true for $t=0$.

Assume that there exists an integer $k\in\{0,\dots,T\}$ such that

$$\|z - x_k\| > 2M. \tag{2.97}$$

By (2.95) and (2.97), $k > 0$. We may assume without loss of generality that (2.96) holds for all integers $t = 0, \ldots, k-1$. In particular,

$$\|z - x_{k-1}\| \le 2M. \tag{2.98}$$

In view of (2.94) and (2.98),

$$\|x_{k-1}\| \le 2M + M_*. \tag{2.99}$$

Equations (2.85), (2.91) and (2.99) imply that

$$\partial f(x_{k-1}) \subset B(0, L_0),$$

$$\|\xi_{k-1}\| \le L_0 + 1. \tag{2.100}$$

In view of (2.100),

$$\|x_k - (x_k - \alpha\xi_{k-1})\| = \alpha\|\xi_{k-1}\| \le \alpha(L_0 + 1). \tag{2.101}$$

By (2.92), (2.93) and (2.101) we apply Lemma 2.8 with

$$\delta_1 = \alpha(L_0 + 1),\ \ \delta_2 = \delta_p,\ \ M_0 = 3M,$$

$$w(i) = w_k(i),\ \ i = 1, \ldots, m,\ \ y_i = y_{k-1,i},\ \ i = 1, \ldots, m,$$

$$x = x_{k-1},\ \ x_0 = x_{k-1} - \alpha\xi_{k-1},\ \ y = x_k$$

and obtain that

$$\|z - x_{k-1}\|^2 - \|z - x_k\|^2 \ge \Delta\bar{c}\sum_{i=1}^{m}\|x_{k-1} - \alpha\xi_{k-1} - y_{k-1,i}\|^2$$

$$-\alpha(L_0+1)(12M+1) - 2\delta_p(12M+4) - \Delta\bar{c}m\delta_p((12M+2)\bar{c}^{-1/2}+1). \tag{2.102}$$

There are two cases:

$$\|z - x_k\| \le \|z - x_{k-1}\| \tag{2.103}$$

$$\|z - x_k\| > \|z - x_{k-1}\|. \tag{2.104}$$

Assume that (2.103) holds. Then in view of (2.98) and (2.103),

$$\|x_k - z\| \le 2M.$$

This contradicts (2.97). The contradiction we have reached proves (2.104). It follows from (2.102) and (2.104) that

$$\Delta\bar{c}\sum_{i=1}^{m}\|x_{k-1} - \alpha\xi_{k-1} - y_{k-1,i}\|^2$$

$$\le \alpha(L_0+1)(12M+1)+2\delta_p(12M+4)+\Delta\bar{c}m\delta_p((12M+2)\bar{c}^{-1/2}+1). \tag{2.105}$$

By (2.12), (2.86), (2.92) and (2.105), for all $i = 1, \dots, m$,

$$\|x_{k-1} - \alpha\xi_{k-1} - P_i(x_{k-1} - \alpha\xi_{k-1})\|$$

$$\le (\Delta\bar{c})^{-1/2}(\alpha(L_0+1)(12M+1) + 2\delta_p(12M+4)$$

$$+\Delta\bar{c}m\delta_p((12M+2)\bar{c}^{-1/2}+1))^{1/2} + \delta_p$$

$$\le (\Delta\bar{c})^{-1/2}(\alpha(L_0+1)(12M+1) + \delta_p(36M+12))^{1/2} + \delta_p \le r_0.$$

Together with (2.84) this implies that

$$x_{k-1} - \alpha\xi_{k-1} \in F_{r_0} \subset B(0, M_*). \tag{2.106}$$

By (2.86), (2.100) and (2.106),

$$\|x_{k-1}\| \le M_* + 1. \tag{2.107}$$

In view of (2.94) and (2.107),

$$\|z - x_{k-1}\| \le 2M_* + 1. \tag{2.108}$$

It follows from (2.1), (2.88), (2.89) and (2.92)–(2.94) that

$$\|x_k - z\| \le 2\delta_p + \|z - \sum_{i=1}^{m} w_k(i)P_i(x_{k-1} - \alpha\xi_{k-1})\|$$

$$\le 2\delta_p + \|z - (x_{k-1} - \alpha\xi_{k-1})\|$$

$$\le 2\delta_p + \|z - x_{k-1}\| + \alpha\|\xi_{k-1}\|. \tag{2.109}$$

By (2.100) and (2.109),

$$\|x_k - z\| \le 2 + 2M_* + 1 + \alpha(L_0 + 1) \le 2M_* + 4 \le 2M.$$

This contradicts (2.97). The contradiction we have reached proves that (2.96) is true for all $t = 0, \dots, T$. Together with (2.94) this implies that

$$\|x_t\| \le 2M + M_*, \ t = 0, \dots, T. \tag{2.110}$$

Let $t \in \{0, \dots, T-1\}$. By (2.85) and (2.110),

$$\partial f(x_t) \subset B(0, L_0). \tag{2.111}$$

In view of (2.91) and (2.111),

$$\|\xi_t\| \le L_0 + 1. \tag{2.112}$$

By (2.1), (2.2), (2.85), (2.88), (2.89) and (2.92)–(2.94), we apply Lemma 2.7 with

$$\delta_1 = \delta_f, \ \delta_2 = 2\delta_p, \ F_0 = F, \ M_0 = 3M,$$

$$Q = \sum_{i=1}^{m} w_{t+1}(i) P_i, \ x = x_t, \ \xi = \xi_t, \ u = x_{t+1}$$

and obtain that

$$\alpha(f(x_t) - f(z))$$
$$\le 2^{-1}\|x_t - z\|^2 - 2^{-1}\|x_{t+1} - z\|^2$$
$$+ 2\delta_p(6M + 2 + \alpha L_0) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_1(6M + L_0 + 1). \tag{2.113}$$

In view of (2.88), (2.89), (2.110) and (2.112),

$$\|x_t - (x_t - \alpha\xi_t)\| = \alpha\|\xi_t\| \le \alpha(L_0 + 1). \tag{2.114}$$

By Lemma 2.8 applied with

$$\delta_1 = \alpha(L_0 + 1), \ \delta_2 = \delta_p, \ M_0 = 3M,$$

$$w(i) = w_{t+1}(i), \ i = 1, \dots, m, \ y_i = y_{t,i}, \ i = 1, \dots, m,$$

$$x = x_t, \ x_0 = x_t - \alpha\xi_t, \ y = x_{t+1}$$

we obtain that

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2 \ge \Delta\bar{c} \sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-\alpha(L_0+1)(12M+1) - 2\delta_p(12M+4) - \Delta\bar{c}m\delta_p((12M+2)\bar{c}^{-1/2}+1). \tag{2.115}$$

It follows from (2.12), (2.113) and (2.115) that

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2 \ge \max\{\Delta\bar{c} \sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - 2\delta_p(12M + 4) - \delta_p(12M + 3),$$

$$2\alpha(f(x_t) - f(z))$$

$$-4\delta_p(6M + 2 + L_0) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1)\}. \tag{2.116}$$

By (2.95) and (2.116),

$$4M^2 \ge \|z - x_0\|^2 \ge \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1} (\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\ge \sum_{t=0}^{T-1} (\max\{\Delta\bar{c} \sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(36M + 12),$$

$$2\alpha(f(x_t) - f(z))$$

$$-4\delta_p(6M + 2 + L_0) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1)\}). \tag{2.117}$$

Since z is an arbitrary element of $F \cap B(0, M_*)$ it follows from (2.11) and (2.117) that

$$4M^2 T^{-1} \ge \min\{\max\{\Delta\bar{c} \sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(36M + 12),$$

$$2\alpha(f(x_t) - \inf(f, F))$$

$$-4\delta_p(6M+2+L_0)-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1)\} : \ t=0,\dots,T-1\}. \quad (2.118)$$

Assume that $t \in \{0, \dots, T-1\}$ and that

$$\max\{\Delta\bar{c}\sum_{i=1}^{m}\|x_t-\alpha\xi_t-y_{t,i}\|^2$$

$$-\alpha(L_0+1)(12M+1)-\delta_p(36M+12), \ 2\alpha(f(x_t)-\inf(f,F))$$

$$-4\delta_p(6M+2+L_0)-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1)\} \le 4M^2T^{-1}. \quad (2.119)$$

By (2.119),

$$\sum_{i=1}^{m}\|x_t-\alpha\xi_t-y_{t,i}\|^2$$

$$\le (\Delta\bar{c})^{-1}(4M^2T^{-1}+\alpha(L_0+1)(12M+1)+\delta_p(36M+12)), \quad (2.120)$$

$$2\alpha(f(x_t)-\inf(f,F))$$

$$-4\delta_p(6M+2+L_0)-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1) \le 4M^2T^{-1},$$

$$f(x_t) \le \inf(f,F))$$

$$+2M^2(T\alpha)^{-1}+2\alpha^{-1}\delta_p(6M+2+L_0)+2^{-1}\alpha L_0^2+\delta_f(6M+L_0+1). \quad (2.121)$$

In view of (2.92) and (2.120), for all $i = 1, \dots, m$,

$$\|x_t-\alpha\xi_t-P_i(x_t-\alpha\xi_t)\| \le \delta_p+\|x_t-\alpha\xi_t-y_{t,i}\|$$

$$\le \delta_p+(\Delta\bar{c})^{-1/2}(4M^2T^{-1}+\alpha(L_0+1)(12M+1)+\delta_p(36M+12))^{1/2}.$$

It follows from the relation above, (2.12), (2.87) and (2.121) that

$$x_t \in \widehat{F}_{\gamma_T}.$$

Theorem 2.10 is proved.

In Theorem 2.10 the computational errors δ_f, δ_p are fixed. Assume that they are positive. Let us choose α, T. First, we choose α in order to minimize the right-hand side of (2.121). Since T can be an arbitrary large we need to minimize the function

$$2\alpha^{-1}\delta_p(6M+2+L_0)+2^{-1}\alpha L_0^2, \ \alpha > 0.$$

Its minimizer is

$$\alpha = 2L_0^{-1}(\delta_p(6M + 2 + L_0))^{1/2}.$$

Since α satisfies (2.86) we obtain the following restriction on δ_p:

$$\delta_p \le 32^{-2}(\Delta\bar{c})^2 r_0^4 (12M + 1)^{-2}(6M + 2 + L_0)^{-1}.$$

In this case

$$\gamma_T = \max\{(L_0 + 1)L_0^{-1}(4\delta_p(6M + 2 + L_0))^{1/2},$$

$$(\Delta\bar{c})^{-1/2}(4M^2T^{-1} + (L_0 + 4)L_0^{-1}(2M + 1)(4\delta_p(6M + 2 + L_0))^{1/2}$$

$$+\delta_p(16M + 12))^{1/2} + \delta_p\}.$$

We choose T with the same order as δ_p^{-1}. For example, $T = \lfloor \delta_p^{-1} \rfloor$. In this case in view of Theorem 2.10, there exists $t \in \{0, \dots, T - 1\}$ such that then

$$f(x_t) \le \inf(f, F) + c_1\delta_p^{1/2} + \delta_f(6M + L_0 + 1)$$

and

$$x_t \in \widehat{F}_{c_2\delta_p^{1/4}}$$

where c_1, c_2 are positive constants which depend on $M, L_0, \Delta, \bar{c}$.

As in the case of Theorem 2.9 we choose an approximate solution after T iterations.

2.6 The Iterative Subgradient Algorithm

We continue to consider the minimization problem

$$f(x) \to \min,\ x \in F$$

considered in Sections 2.1 and 2.2 using the notation and definitions introduced there. We also suppose that all the assumptions introduced there hold.

Fix an integer

$$\bar{N} \ge m.$$

Denote by $\mathcal{R}$ the set of all mappings $r : \{1, 2, \dots\} \to \{1, \dots, m\}$ such that for every integer $j \geq 0$,

$$\{1, \dots, m\} \subset \{r(j\bar{N} + 1), \dots, r(j + 1)\bar{N}\}. \tag{2.122}$$

Let $\alpha > 0$ and $r \in \mathcal{R}$.

Let us describe our algorithm.

Iterative Subgradient Algorithm

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_{k\bar{N}} \in X$ calculate

$$l_k \in \partial f(x_{k\bar{N}}),$$

and define the next iteration vector

$$x_{(k+1)\bar{N}} = (\prod_{i=k\bar{N}+1}^{(k+1)\bar{N}} P_i)(x_{k\bar{N}} - \alpha l_k).$$

In this chapter the iterative subgradient method is studied under the presence of computational errors. We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f while $\delta_p \in [0, 1]$ is a computational error produced by our computer system, when we calculate the operators P_i, $i = 1, \dots, m$. Let $\alpha > 0$ be a step size.

Iterative Subgradient Algorithm with Computational Errors

Initialization: select an arbitrary $x_0 \in X$ and $r \in \mathcal{R}$.

Iterative step: given a current iteration vector $x_{k\bar{N}} \in X$ calculate

$$\xi_k \in \partial f(x_{k\bar{N}}) + B(0, \delta_f),$$

$$x_{k\bar{N}+1} \in B(P_{r(k\bar{N}+1)}(x_{k\bar{N}} - \alpha\xi_k), \delta_p)$$

and for every $t \in \{k\bar{N} + 1, \dots, (k + 1)\bar{N}\} \setminus \{k\bar{N} + 1\}$, calculate

$$x_t \in B(P_{r(t)}(x_{t-1}), \delta_p).$$

In this algorithm, as well for other algorithms considered in the book, we assume that the step size does not depend on the number of iterative step k. The same analysis can be done when step sizes depend on k. On the other hand, as it was shown in [93, 95], in the case of computational errors the best choice of step sizes is step sizes which do not depend on iterative step numbers.

2.7 Auxiliary Results

Our study of the algorithm is based on the following auxiliary results.

Lemma 2.11 *Let $F_0 \subset X$ be nonempty, $M_0 \geq M_*$, $L_0 \geq 1$,*

$$\|f(z_1 - f(z_2)\| \leq L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(0, M_0 + 4), \tag{2.123}$$

$p \geq 1$ be an integer, mappings $Q_i : X \to X$, $i = 1, \dots, p$ satisfy for every $i \in \{1, \dots, p\}$,

$$Q_i(z) = z, \ z \in F_0, \tag{2.124}$$

$$\|Q_i(u) - z\| \leq \|u - z\| \text{ for all } u \in X \text{ and all } z \in F_0 \tag{2.125}$$

and let $\delta_1, \delta_2 \in [0, 1]$, $\alpha \in (0, 1]$. Assume that

$$z \in F_0 \cap B(0, M_0), \tag{2.126}$$

$$x \in B(0, M_0), \tag{2.127}$$

$$\xi \in \partial f(x) + B(0, \delta_1) \tag{2.128}$$

and that

$$\{u_t\}_{t=0}^{p} \subset\in X$$

satisfy

$$u_0 = x - \alpha\xi, \tag{2.129}$$

$$\|u_i - Q_i(u_{i-1})\| \leq \delta_2, \ i = 1, \dots, p. \tag{2.130}$$

Then

$$\alpha(f(x) - f(z))$$

$$\leq 2^{-1}\|x - z\|^2 - 2^{-1}\|u_p - z\|^2 + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_1(2M_0 + L_0 + 1)$$

$$+\delta_2(2M_0 + 2 + \alpha L_0 + M_0(p - 1) + \delta_2 p(p - 1)/2).$$

Proof By (2.123)–(2.130) and Lemma 2.7,

$$\alpha(f(x) - f(z))$$

$$\leq 2^{-1}\|x - z\|^2 - 2^{-1}\|u_1 - z\|^2$$

$$+ \delta_2(2M_0 + 2 + \alpha L_0) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_1(2M_0 + L_0 + 1). \tag{2.131}$$

In view of (2.123) and (2.127),

$$\partial f(x) \subset B(0, L_0). \tag{2.132}$$

Equations (2.128) and (2.132) imply that

$$\|l\| \leq L_0 + 1. \tag{2.133}$$

It follows from (2.120), (2.126), (2.127) and (2.133) that

$$\|u_0 - z\| = \|x - \alpha\xi - z\| \leq 2M_0 + \alpha(L_0 + 1). \tag{2.134}$$

Let $i \in \{1, \dots, p\}$. By (2.125), (2.126) and (2.130),

$$\|z - u_i\| \leq \|z - Q_i(u_{i-1})\| + \|Q_i(u_{i-1}) - u_i\|$$

$$\leq \|z - u_{i-1}\| + \delta_2. \tag{2.135}$$

In view of (2.135),

$$\|z - u_i\| \leq \|z - u_0\| + i\delta_2. \tag{2.136}$$

By (2.126), (2.127), (2.135) and (2.136),

$$\|z - u_{i-1}\|^2 - \|z - u_i\|^2$$

$$= (\|z - u_{i-1}\| - \|z - u_i\|)(\|z - u_{i-1}\| + \|z - u_i\|)$$

$$\geq -\delta_2(\|z - u_{i-1}\| + \|z - u_i\|)$$

$$\geq -\delta_2(2\|z - u_0\| + p\delta_2) \geq -\delta_2(2M_0 + p\delta_2). \tag{2.137}$$

By (2.137),

$$\|z - u_1\|^2 - \|z - u_p\|^2$$

$$= \sum\{\|z - u_i\|^2 - \|z - u_{i+1}\|^2 : \ i \in \{1, \dots, p\} \setminus \{p\}\}$$

$$\geq -\delta_2(2M_0 + p\delta_2)(p - 1). \tag{2.138}$$

It follows from (2.131) and (2.138) that

$$\alpha(f(x) - f(z))$$

$$\leq 2^{-1}\|x - z\|^2 - 2^{-1}\|u_1 - z\|^2 + \delta_2(2M_0 + 2 + \alpha L_0)$$

$$+2^{-1}\alpha^2 L_0^2 + \alpha\delta_1(2M_0 + L_0 + 1)+$$

$$+2^{-1}\|z - u_1\|^2 - 2^{-1}\|z - u_p\|^2 + 2^{-1}\delta_2(2M_0 + p\delta_2)(p - 1)$$

$$= 2^{-1}\|x - z\|^2 - 2^{-1}\|u_p - z\|^2 + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_1(2M_0 + L_0 + 1)$$

$$+\delta_2(2M_0 + 2 + \alpha L_0 + M_0(p - 1) + \delta_2 p(p - 1)/2).$$

Lemma 2.11 is proved.

Lemma 2.12 *Let $F_0 \subset X$ be nonempty, $M_0 \geq 0$, $\delta_1, \delta_2 \geq 0$, $p \geq 1$ be an integer, mappings $Q_i : X \to X$, $i = 1, \dots, p$ satisfy for every $i \in \{1, \dots, p\}$,*

$$Q_i(z) = z, \ z \in F_0, \tag{2.139}$$

$$\|u - z\|^2 \geq \|Q_i(u) - z\|^2 + \bar{c}\|u - Q_i(u)\|$$

$$\textit{for all } u \in X \textit{ and all } z \in F_0 \tag{2.140}$$

$$z \in F \cap B(0, M_0), \tag{2.141}$$

$$x \in B(0, M_0), \tag{2.142}$$

$$\{u_t\}_{t=0}^{p} \subset X$$

satisfy

$$\|u_0 - x\| \leq \delta_1, \tag{2.143}$$

$$\|u_t - Q_t(u_{t-1})\| \leq \delta_2, \ t = 1, \dots, p. \tag{2.144}$$

Then

$$\|z - x\|^2 - \|z - u_p\|^2 \geq \bar{c}\sum_{i=1}^{m}\|u_{t-1} - u_t\|^2$$

$$-\delta_1(2M_0 + \delta_1) - 2\delta_2 p(2M_0 + \delta_1 + p\delta_2) - p\delta_2(4M_0 + 2\delta_1 + \delta_2(2p + 1)).$$

Proof In view of (2.141)–(2.143),

$$|\|z-x\|^2 - \|z-u_0\|^2| \le \|x-u_0\|(\|z-x\| + \|z-u_0\|)$$

$$\le \delta_1(2M_0+\delta_1), \tag{2.145}$$

$$\|z-u_0\| \le \|z-x\| + \delta_1 \le 2M_0 + \delta_1. \tag{2.146}$$

Let $t \in \{1,\dots,p\}$. By (2.140), (2.141) and (2.144),

$$\|z-u_t\| \le \|z-Q_t(u_{t-1})\| + \|Q_t(u_{t-1})-u_t\| \le \|z-u_{t-1}\| + \delta_2. \tag{2.147}$$

Equations (2.146) and (2.147) imply that

$$\|z-u_t\| \le \|z-u_0\| + t\delta_2 \le 2M_0 + \delta_1 + p\delta_2. \tag{2.148}$$

It follows from (2.140), (2.141), (2.144) and (2.148) that

$$\|z-u_{t-1}\|^2 - \|z-u_t\|^2$$

$$= \|z-u_{t-1}\|^2 - \|z-Q_t(u_{t-1})\|^2 + \|z-Q_t(u_{t-1})\|^2 - \|z-u_t\|^2$$

$$\ge \bar{c}\|u_{t-1}-Q_t(u_{t-1})\|^2 + (\|z-Q_t(u_{t-1})\| - \|z-u_t\|)(\|z-Q_t(u_{t-1})\| + \|z-u_t\|)$$

$$\ge \bar{c}\|u_{t-1}-Q_t(u_{t-1})\|^2 - 2\delta_2\|z-u_t\|$$

$$\ge \bar{c}\|u_{t-1}-Q_t(u_{t-1})\|^2 - 2\delta_2(2M_0+\delta_1+p\delta_2). \tag{2.149}$$

By (2.145) and (2.149),

$$\|z-x\|^2 - \|z-u_p\|^2$$

$$\ge \|z-u_0\|^2 - \|z-u_p\|^2 - \delta_1(2M_0+\delta_1)$$

$$-\delta_1(2M_0+\delta_1) + \sum_{t=1}^{p}(\|z-u_{t-1}\|^2 - \|z-u_t\|^2)$$

$$\ge -\delta_1(2M_0+\delta_1) + \bar{c}\sum_{t=1}^{p}(\|u_{t-1}-Q_t(u_{t-1})\|^2 - 2\delta_2 p(2M_0+\delta_1+p\delta_2). \tag{2.150}$$

It follows from (2.140), (2.141), (2.144), (2.146) and (2.148) that for $t = 1,\dots,p$,

$$|\|u_{t-1}-u_t\|^2 - \|u_{t-1}-Q_t(u_{t-1})\|^2|$$

$$\leq \|u_t - Q_t(u_{t-1})\|(\|u_{t-1} - Q_t(u_{t-1})\| + \|u_{t-1} - u_t\|)$$

$$\leq \delta_2(2\|u_{t-1} - Q_t(u_{t-1})\| + \delta_2) \leq \delta_2(2\|z - u_{t-1}\| + \delta_2)$$

$$\leq \delta_2(4M_0 + 2\delta_1 + (2p+1)\delta_2). \tag{2.151}$$

In view of (2.150) and (2.151),

$$\|z - x\|^2 - \|z - u_p\|^2$$

$$\geq -\delta_1(2M_0 + \delta_1) - 2\delta_2 p(2M_0 + \delta_1 + p\delta_2)$$

$$+\bar{c}\sum_{t=1}^{p}(\|u_{t-1} - u_t\|^2 - p\delta_2(4M_0 + 2\delta_1 + \delta_2(2p+1)).$$

Lemma 2.12 is proved.

2.8 Convergence Results for the Iterative Subgradient Algorithm

In our first result we assume that the objective function f satisfies the coercivity growth condition.

Theorem 2.13 *Let the function f be Lipschitz on bounded subsets of X,*

$$\lim_{\|x\| \to \infty} f(x) = \infty,$$

$M \geq 2M_* + 6$, $L_0 \geq 1$,

$$M_1 \geq \sup\{|f(u)| : u \in B(0, M_* + 4)\}, \tag{2.152}$$

$$f(u) > M_1 + 4 \textit{ for all } u \in X \setminus B(0, 2^{-1}M), \tag{2.153}$$

$$|f(z_1) - f(z_2)| \leq L_0\|z_1 - z_2\| \textit{ for all } z_1, z_2 \in B(0, 3M + 4), \tag{2.154}$$

$\delta_f, \delta_p \in [0, 1], \alpha > 0$ *satisfy*

$$\alpha \leq (L_0 + 1)^{-2},\ \alpha \geq \delta_p(3M(\bar{N} + 2) + 3),\ \delta_p \leq \bar{N}^{-2},\ \delta_f \leq (6M + L_0 + 1)^{-1}, \tag{2.155}$$

$r \in \mathcal{R}$, *T be a natural number and let*

$$\gamma_T = \alpha(L_0 + 1) + \bar{N}(4M^2 T^{-1}\bar{c}^{-1}$$

$$+\bar{c}^{-1}(\alpha(L_0+1)(6M+1)+2\bar{c}^{-1}\delta_p\bar{N}(6M+2))^{1/2}. \tag{2.156}$$

Assume that $\{x_t\}_{t=0}^{T\bar{N}} \subset X$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$x_0 \in B(0, M) \tag{2.157}$$

and that for all integers $k \in \{0, \dots, T-1\}$,

$$B(\xi_k, \delta_f) \cap \partial f(x_{k\bar{N}}) \neq \emptyset, \tag{2.158}$$

$$\|x_{k\bar{N}+1} - P_{r(k\bar{N}+1)}(x_{k\bar{N}} - \alpha\xi_k)\| \leq \delta_p \tag{2.159}$$

and for every $t \in \{k\bar{N}+1, \dots, (k+1)\bar{N}\} \setminus \{k\bar{N}+1\}$,

$$\|x_t - P_{r(t)}(x_{t-1})\| \leq \delta_p. \tag{2.160}$$

Then

$$\|x_t\| \leq 3M+3,\ t = 0, \dots, T\bar{N}$$

$$\|x_{k\bar{N}}\| \leq 3M,\ k = 0, \dots, T$$

and

$$\min\{\max\{2\alpha(f(x_{k\bar{N}}) - \inf(f, F))$$

$$-2\alpha\delta_f(6M+L_0+1) - \alpha^2 L_0^2 - 2\alpha_p(6M\bar{N}+3),$$

$$\bar{c}\{\|x_{k\bar{N}} - \alpha\xi_k - x_{k\bar{N}+1}\|^2$$

$$+\bar{c}\sum\{\|x_{k\bar{N}+t-1} - x_{k\bar{N}+t}\|^2 :\ t \in \{1, \dots, \bar{N}\} \setminus \{1\}\}$$

$$-\alpha(L_0+1)(6M+1) - 4\delta_p(6M+4)\bar{N}\}$$

$$\leq 4M^2T^{-1}.$$

Moreover, if $k \in \{0, \dots, T-1\}$ *and*

$$\max\{2\alpha(f(x_{k\bar{N}}) - \inf(f, F))$$

$$-2\delta_p(6M\bar{N}+3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M+L_0+1),$$

$$\bar{c}\|x_{k\bar{N}} - \alpha\xi_k - x_{k\bar{N}+1}\|^2$$

$$+\bar{c}\sum\{\|x_{k\bar{N}+t-1}-x_{k\bar{N}+t}\|^2:\ t\in\{1,\dots,\bar{N}\}\setminus\{1\}\}$$

$$-\alpha(L_0+1)(6M+1)-2\delta_p\bar{N}(6M+4)\}\le 4M^2T^{-1}$$

then

$$f(x_{k\bar{N}})\le \inf(f,F)+2M^2(T\alpha)^{-1}$$

$$+\alpha^{-1}\delta_p(6M\bar{N}+3)+2^{-1}\alpha L_0^2+\delta_f(6M+L_0+1)$$

and

$$x_{k\bar{N}}\in \tilde{F}_{\gamma_T}.$$

Proof In view of (2.8), there exists

$$z\in B(0,M_*)\cap F. \tag{2.161}$$

By (2.157) and (2.161),

$$\|z-x_0\|\le M+M_*\le 2M. \tag{2.162}$$

Assume that $k\in\{0,\dots,T-1\}$ and that

$$\|z-x_{k\bar{N}}\|\le 2M. \tag{2.163}$$

It follows from (2.1), (2.159), (2.161) and (2.163) that

$$\|z-x_{k\bar{N}+1}\|\le \|z-P_{r(k\bar{N}+1)}(x_{k\bar{N}}-\alpha\xi_k)\|$$

$$+\|P_{r(k\bar{N}+1)}(x_{k\bar{N}}-\alpha\xi_k)-x_{k\bar{N}+1}\|\le \|z-x_{k\bar{N}}+\alpha\xi_k\|+\delta_p$$

$$\le 2M+\delta_p+\alpha\|\xi_k\|. \tag{2.164}$$

In view of (2.161) and (2.163),

$$\|x_{k\bar{N}}\|\le 3M. \tag{2.165}$$

By (2.154) and (2.165),

$$\partial f(x_{k\bar{N}})\subset B(0,L_0). \tag{2.166}$$

Equations (2.158) and (2.166) imply that

$$\|\xi_k\| \le L_0 + 1. \tag{2.167}$$

It follows from (2.155), (2.164) and (2.167) that

$$\|z - x_{k\bar{N}+1}\| \le 2M + \delta_p + \alpha(L_0 + 1) \le 2M + 2. \tag{2.168}$$

By (2.1), (2.161) and (2.168), for every $t \in \{k\bar{N} + 1, \dots, (k + 1)\bar{N}\} \setminus \{k\bar{N} + 1\}$,

$$\|z - x_t\| \le \|z - P_{r(t)}(x_{t-1})\| + \|P_{r(t)}(x_{t-1}) - x_t\| \le \|z - x_{t-1}\| + \delta_p, \tag{2.169}$$

$$\|z - x_t\| \le \|z - x_{k\bar{N}+1}\| + \bar{N}\delta_p \le 2M + 2 + \bar{N}\delta_p \le 2M + 3.$$

Thus we have shown that the following property holds:

(i) is $k \in \{0, \dots, T - 1\}$ and $\|z - x_{k\bar{N}}\| \le 2M$, then

$$\|z - x_t\| \le 2M + 3,\ t = k\bar{N} + 1, \dots, (k + 1)\bar{N}$$

(2.169) is true for every $t \in \{k\bar{N} + 1, \dots, (k + 1)\bar{N}\} \setminus \{k\bar{N} + 1\}$.

We show that for all $k \in \{0, \dots, T\}$, (2.163) holds. In view of (2.162), equation (2.163) holds for $k = 0$. Assume that there exists an integer $q \in \{0, \dots, T\}$ such that

$$\|z - x_{q\bar{N}}\| > 2M. \tag{2.170}$$

By (2.162) and (2.170), $q > 0$. We may assume without loss of generality that (2.163) holds for all integers $k = 0, \dots, q - 1$. In particular,

$$\|z - x_{(q-1)\bar{N}}\| \le 2M. \tag{2.171}$$

In view of (2.161) and (2.171),

$$\|x_{(q-1)\bar{N}}\| \le 2M + M_*. \tag{2.172}$$

By (2.1), (2.2), (2.154), (2.158)–(2.161) and (2.172), we apply Lemma 2.11 with

$$\delta_1 = \delta_f,\ \delta_2 = \delta_p,\ F_0 = F,\ M_0 = 3M,\ p = \bar{N},$$

$$Q_i = P_{r(i+(q-1)\bar{N})},\ i = 1, \dots, \bar{N},\ x = x_{(q-1)\bar{N}},\ \xi = \xi_{q-1},$$

$$u_0 = x_{(q-1)\bar{N}} - \alpha_{q-1}\xi_{q-1},\ u_t = x_{(q-1)\bar{N}+t},\ t = 1, \dots, \bar{N}$$

and obtain that

$$\alpha(f(x_{(q-1)\bar{N}}) - f(z))$$

$$\leq 2^{-1}\|x_{(q-1)\bar{N}} - z\|^2 - 2^{-1}\|x_{q\bar{N}} - z\|^2 + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1)$$

$$+ \delta_p(6M + 2 + \alpha L_0 + 3M(\bar{N} - 1) + \delta_p \bar{N}(\bar{N} - 1)/2). \tag{2.173}$$

There are two cases:

$$\|z - x_{q\bar{N}}\| \leq \|z - x_{(q-1)\bar{N}}\| \tag{2.174}$$

$$\|z - x_{q\bar{N}}\| > \|z - x_{(q-1)\bar{N}}\|. \tag{2.175}$$

Assume that (2.174) holds. Then in view of (2.171),

$$\|x_{q\bar{N}} - z\| \leq 2M.$$

Assume that (2.175) is true. By (2.173) and (2.175),

$$\alpha(f(x_{(q-1)\bar{N}}) - f(z))$$

$$\leq 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1)$$

$$+\delta_p(6M + 2 + \alpha L_0 + 3M(\bar{N} - 1) + \delta_p \bar{N}(\bar{N} - 1)/2).$$

Together with (2.152), (2.155) and (2.161) this implies that

$$f(x_{(q-1)\bar{N}}) \leq M_1 + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1)$$

$$+\alpha^{-1}\delta_p(6M + 3 + 3M((\bar{N} - 1) + 1))) \leq M_1 + 3.$$

Combined with (2.153) this implies that

$$\|x_{(q-1)\bar{N}}\| \leq M/2. \tag{2.176}$$

Equations (2.154) and (2.176) imply that

$$\partial f(x_{(q-1)\bar{N}}) \subset B(0, L_0). \tag{2.177}$$

By (2.158) and (2.177),

$$\|\xi_k\| \leq L_0 + 1. \tag{2.178}$$

In view of (2.1), (2.2), (2.155), (2.159) and (2.161),

$$\|z - x_{(q-1)\bar{N}+1}\|$$

$$\leq \|z - P_{r((q-1)\bar{N})+1)}(x_{(q-1)\bar{N}} - \alpha\xi_{q-1})\| + \delta_p$$

$$\leq \delta_p + \|z - x_{(q-1)\bar{N}}\| + \alpha(L_0 + 1) \leq M_* + M/2 + 2. \tag{2.179}$$

Property (i), (2.155), (2.160), (2.171) and (2.179) imply that

$$\|z - x_{q\bar{N}}\| \leq \|z - x_{(q-1)\bar{N}+1}\| + \bar{N}\delta_p \leq M_* + M/2 + 3 < M.$$

Thus in the both cases

$$\|z - x_{q\bar{N}}\| \leq 2M.$$

This contradicts (2.170). The contradiction we have reached proves that

$$\|z - x_{k\bar{N}}\| \leq 2M, \ k = 0, \dots, T. \tag{2.180}$$

Property (i) and (2.180) imply that for all $k \in \{0, \dots, T-1\}$ and all $t \in \{k\bar{N} + 1, \dots, (k+1)\bar{N}\}$,

$$\|z - x_t\| \leq 2M + 3.$$

Therefore

$$\|z - x_t\| \leq 2M + 3, \ t = 0, \dots, \bar{N}T \tag{2.181}$$

and in view of (2.161),

$$\|x_{k\bar{N}}\| \leq 3M, \ k = 0, \dots, T, \tag{2.182}$$

$$\|x_t\| \leq 3M + 3, \ t = 0, \dots, \bar{N}T. \tag{2.183}$$

Equations (2.154) and (2.183) imply that for all $t = 0, \dots, \bar{N}T$,

$$\partial f(x_t) \subset B(0, L_0). \tag{2.184}$$

By (2.158) and (2.184),

$$\|\xi_k\| \leq L_0 + 1, \ k = 0, \dots, T-1. \tag{2.185}$$

Let $k \in \{0, \dots, T-1\}$. By (2.1), (2.2), (2.154), (2.155) and (2.161), we apply Lemma 2.11 with

$$\delta_1 = \delta_f, \ \delta_2 = \delta_p, \ F_0 = F, \ M_0 = 3M, \ p = \bar{N},$$

$$Q_i = P_{r(i+k\bar{N})},\ i = 1, \dots, \bar{N},\ x = x_{k\bar{N}},\ \xi = \xi_k,$$

$$u_0 = x_{k\bar{N}} - \alpha_k \xi_k,\ u_t = x_{k\bar{N}+t},\ t = 1, \dots, \bar{N}$$

and obtain that

$$\begin{aligned}
&\alpha(f(x_{k\bar{N}}) - f(z)) \\
&\le 2^{-1}\|x_{k\bar{N}} - z\|^2 - 2^{-1}\|x_{(k+1)\bar{N}} - z\|^2 \\
&+2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1) \\
&+\delta_p(6M + 2 + \alpha L_0 + 3M(\bar{N} - 1) + \delta_p \bar{N}(\bar{N} - 1)/2) \\
&\le 2^{-1}\|x_{k\bar{N}} - z\|^2 - 2^{-1}\|x_{(k+1)\bar{N}} - z\|^2 \\
&+2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1) \\
&+ \delta_p(3M(\bar{N} + 1) + 1). \qquad (2.186)
\end{aligned}$$

By (2.1), (2.2), (2.155), (2.161) (2.182) and (2.185), we apply Lemma 2.12 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ F_0 = F,\ M_0 = 3M,\ p = \bar{N},$$

$$Q_i = P_{r(i+k\bar{N})},\ i = 1, \dots, \bar{N},\ x = x_{k\bar{N}},$$

$$u_0 = x_{k\bar{N}} - \alpha\xi_k,\ u_t = x_{k\bar{N}+t},\ t = 1, \dots, \bar{N}$$

and obtain that

$$\begin{aligned}
&\|z - x_{k\bar{N}}\|^2 - \|z - x_{(k+1)\bar{N}}\|^2 \\
&\ge \bar{c}\|x_{k\bar{N}+1} - (x_{k\bar{N}} - \alpha\xi_k)\|^2 \\
&+\bar{c}\sum\{\|x_{k\bar{N}+t} - x_{k\bar{N}+t-1}\|^2 :\ t \in \{1, \dots, \bar{N}\} \setminus \{1\}\} \\
&-\alpha(6M + \alpha(L_0 + 1))(L_0 + 1) - 2\delta_p \bar{N}(6M + \alpha(L_0 + 1) + \bar{N}\delta_p) \\
&-\bar{N}\delta_p(12M + 2\alpha(L_0 + 1) + \delta_p(2\bar{N} + 1)) \\
&\ge \bar{c}\|x_{k\bar{N}+1} - (x_{k\bar{N}} - \alpha\xi_k)\|^2
\end{aligned}$$

$$+\bar{c}\sum\{\|x_{k\bar{N}+t}-x_{k\bar{N}+t-1}\|^2:\ t\in\{1,\dots,\bar{N}\}\setminus\{1\}\}$$

$$-\alpha(6M+1)(L_0+1)-2\delta_p\bar{N}(6M+2)-\bar{N}\delta_p(12M+6). \tag{2.187}$$

By (2.155), (2.157), (2.161), (2.186) and (2.187),

$$4M^2\geq\|z-x_0\|^2\geq\|z-x_0\|^2-\|z-x_{T\bar{N}}\|^2$$

$$=\sum_{k=0}^{T-1}(\|z-x_{k\bar{N}}\|^2-\|z-x_{(k+1)\bar{N}}\|^2)$$

$$\geq\sum_{k=0}^{T-1}\max\{2\alpha(f(x_{k\bar{N}})-f(z))$$

$$-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1)-2\delta_p(6M\bar{N}+3),$$

$$\bar{c}\|x_{k\bar{N}+1}-(x_{k\bar{N}}-\alpha\xi_k)\|^2$$

$$+\bar{c}\sum\{\|x_{k\bar{N}+t}-x_{k\bar{N}+t-1}\|^2:\ t\in\{1,\dots,\bar{N}\}\setminus\{1\}\}$$

$$-\alpha(6M+1)(L_0+1)-4\delta_p\bar{N}(6M+4)\}.$$

Since z is an arbitrary element of $B(0,M_*)\cap F$ it follows from the relation above and (2.11) that

$$4M^2\geq T\min\{\max\{2\alpha(f(x_{k\bar{N}})-\inf(f,F))$$

$$-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1)-2\delta_p(6M\bar{N}+3),$$

$$\bar{c}\|x_{k\bar{N}+1}-(x_{k\bar{N}}-\alpha\xi_k)\|^2+\bar{c}\sum\{\|x_{k\bar{N}+t}-x_{k\bar{N}+t-1}\|^2:\ t\in\{1,\dots,\bar{N}\}\setminus\{1\}\}$$

$$-\alpha(6M+1)(L_0+1)-4\delta_p\bar{N}(6M+4)\}:\ k=0,\dots,T-1\}.$$

Let $k\in\{0,\dots,T-1\}$ and

$$4M^2T^{-1}\geq\max\{2\alpha(f(x_{k\bar{N}})-\inf(f,F))$$

$$-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1)-2\delta_p(6M\bar{N}+3),$$

$$\bar{c}\|x_{k\bar{N}+1}-(x_{k\bar{N}}-\alpha\xi_k)\|^2$$

$$+\bar{c}\sum\{\|x_{k\bar{N}+t}-x_{k\bar{N}+t-1}\|^2:\ t\in\{1,\dots,\bar{N}\}\setminus\{1\}\}$$

$$-\alpha(6M+1)(L_0+1)-4\delta_p\bar{N}(6M+4)\}.$$

This relation implies that

$$f(x_{k\bar{N}})-\inf(f,F)$$

$$\leq 2M^2(T\alpha)^{-1}+\alpha^{-1}\delta_p(6M\bar{N}+3)+2^{-1}\alpha L_0^2+\delta_f(6M+L_0+1)$$

and that for all $t\in\{1,\dots,\bar{N}\}\setminus\{1\}$,

$$\|x_{k\bar{N}+1}-(x_{k\bar{N}}-\alpha\xi_k)\|,\ \|x_{k\bar{N}+t}-x_{k\bar{N}+t-1}\|$$

$$\leq (4T^{-1}M^2+\bar{c}^{-1}+\bar{c}^{-1}\alpha(6M+1)(L_0+1)+4\bar{c}^{-1}\delta_p\bar{N}(6M+4))^{1/2}. \tag{2.188}$$

Set

$$\Delta=\bar{c}^{-1}\alpha(6M+1)(L_0+1)+4\bar{c}^{-1}\delta_p\bar{N}(6M+4). \tag{2.189}$$

By (2.185), (2.188) and (2.189),

$$\|x_{k\bar{N}}-x_{k\bar{N}+1}\|$$

$$\leq \|x_{k\bar{N}}-(x_{k\bar{N}}-\alpha\xi_k)\|+\|x_{k\bar{N}+1}-(x_{k\bar{N}}-\alpha\xi_k)\|$$

$$\leq \alpha(L_0+1)+(4M^2T^{-1}\bar{c}^{-1}+\Delta)^{1/2} \tag{2.190}$$

and that for all $t\in\{1,\dots,\bar{N}\}\setminus\{1\}$,

$$\|x_{k\bar{N}+t}-x_{k\bar{N}+t-1}\|\leq (4M^2T^{-1}\bar{c}^{-1}+\Delta)^{1/2} \tag{2.191}$$

In view of (2.190) and (2.191), for all $t\in\{1,\dots,\bar{N}\}$,

$$\|x_{k\bar{N}}-x_{k\bar{N}+t}\|\leq \alpha(L_0+1)+(4M^2T^{-1}\bar{c}^{-1}+\Delta)^{1/2}\bar{N}. \tag{2.192}$$

It follows from (2.159), (2.188) and (2.189) that

$$\|x_{k\bar{N}}-\alpha\xi_k-P_{r(k\bar{N}+1)}(x_{k\bar{N}}-\alpha\xi_k)\|$$

$$\leq \|x_{k\bar{N}}-\alpha\xi_k-x_{k\bar{N}+1}\|+\delta_p$$

$$\leq (4M^2T^{-1}\bar{c}^{-1}+\Delta)^{1/2}+\delta_p. \tag{2.193}$$

By (2.160) and (2.191), for all $t \in \{1, \dots, \bar{N}\} \setminus \{1\}$,

$$\|x_{k\bar{N}+t-1} - P_{r(k\bar{N}+t)}(x_{k\bar{N}+t-1})\| \le \|x_{k\bar{N}+t-1} - x_{k\bar{N}+t}\| + \delta_p$$

$$\le (4M^2T^{-1}\bar{c}^{-1} + \Delta)^{1/2} + \delta_p. \tag{2.194}$$

It follows from (2.155), (2.156), (2.189), (2.193) and (2.194) that

$$x_{k\bar{N}} \in \tilde{F}_{\alpha(L_0+1)+(4M^2T^{-1}\bar{c}^{-1}+\Delta)^{1/2}\bar{N}} = \tilde{F}_{\gamma_T}.$$

Theorem 2.13 is proved.

In Theorem 2.13 the computational errors δ_f, δ_p are fixed. Assume that they are positive. Let us choose α, T. First, we choose α. Since T can be an arbitrary large we need to minimize the function

$$2\alpha^{-1}\delta_p(6M\bar{N} + 3) + 2^{-1}\alpha L_0^2, \ \alpha > 0.$$

Its minimizer is

$$\alpha = 2L_0^{-1}(\delta_p(6M\bar{N} + 3))^{1/2}.$$

Since α satisfies (2.155) we obtain the following restrictions on δ_p:

$$\delta_p \le 4^{-1}L_0^2(1 + L_0)^{-4}(6M\bar{N} + 3)^{-1}$$

$$\text{and } \delta_p \le 4L_0^{-2}(3M(\bar{N} + 2) + 3)^{-2}(6M\bar{N} + 3).$$

In this case

$$\gamma_T = 2(L_0 + 1)L_0^{-1}(\delta_p(6M\bar{N} + 3))^{1/2} + \bar{N}(4M^2T^{-1}\bar{c}^{-1}$$

$$+2\bar{c}^{-1}L_0^{-1}(\delta_p(6M\bar{N} + 3))^{1/2}(L_0 + 1)(6M + 1) + 2\bar{c}^{-1}\delta_p\bar{N}^2(6M + 2))^{1/2}.$$

We choose T with the same order as δ_p^{-1} For example, $T = \lfloor \delta_p^{-1} \rfloor$. In this case in view of Theorem 2.13, there exists $k \in \{0, \dots, T - 1\}$ such that then

$$f(x_{k\bar{N}}) \le \inf(f, F) + c_1\delta_p^{1/2} + \delta_f(6M + L_0 + 1)$$

and

$$x_{k\bar{N}} \in \widehat{F}_{c_2\delta_p^{1/4}}$$

where c_1, c_2 are positive constants which depend on $M, L_0, \bar{N}, \bar{c}$.

Analogously to Theorem 2.9 we choose an approximate solution of our problem after T iterations.

In the next theorem we assume that the set F is bounded.

Theorem 2.14 *Let* $r_0 \in (0, 1]$,

$$\tilde{F}_{r_0} \subset B(0, M_*), \tag{2.195}$$

$M \geq M_* + 2$, $L_0 \geq 1$,

$$|f(z_1) - f(z_2)| \leq L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(0, 3M + 4), \tag{2.196}$$

$\delta_f, \delta_p \in [0, 1], \alpha \in (0, 1]$ *satisfy*

$$\alpha \leq 4^{-1}\bar{c}r_0^2(L_0 + 1)^{-1}(6M + 1)^{-1}(\bar{N} + 1)^{-2},$$

$$\delta_p \leq 8^{-1}\bar{c}r_0^2(6M + 4)^{-1}(\bar{N} + 1)^{-2}\bar{N}^{-1}, \tag{2.197}$$

$$r \in \mathcal{R}, \tag{2.198}$$

T *be a natural number and let*

$$\gamma_T = (\alpha(L_0 + 1) + \bar{N}\bar{c}^{-1/2}(4M^2T^{-1} + \alpha(L_0 + 1)(6M + 1) + 4\delta_p\bar{N}(6M + 4))^{1/2}. \tag{2.199}$$

Assume that $\{x_t\}_{t=0}^{T\bar{N}} \subset X$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$x_0 \in B(0, M) \tag{2.200}$$

and that for all integers $k \in \{0, \dots, T - 1\}$,

$$B(\xi_k, \delta_f) \cap \partial f(x_{k\bar{N}}) \neq \emptyset, \tag{2.201}$$

$$\|x_{k\bar{N}+1} - P_{r(k\bar{N}+1)}(x_{k\bar{N}} - \alpha\xi_k)\| \leq \delta_p \tag{2.202}$$

and for every $t \in \{k\bar{N} + 1, \dots, (k + 1)\bar{N}\} \setminus \{k\bar{N} + 1\}$,

$$\|x_t - P_{r(t)}(x_{t-1})\| \leq \delta_p. \tag{2.203}$$

Then

$$\|x_{k\bar{N}}\| \leq 3M, \ k = 0, \dots, T$$

$$\|x_t\| \leq 3M + 3, \ t = 0, \dots, T\bar{N}$$

and

$$\min\{\max\{2\alpha(f(x_{k\bar{N}})-\inf(f,F))$$
$$-\delta_p(6M\bar{N}+3)-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1),$$
$$\bar{c}\|x_{k\bar{N}}-\alpha\xi_k-x_{k\bar{N}+1}\|^2$$
$$+\bar{c}\sum\{\|x_{k\bar{N}+t+1}-x_{k\bar{N}+t}\|^2:\ t\in\{1,\dots,\bar{N}\}\setminus\{1\}\}$$
$$-\alpha(L_0+1)(6M+1)-4\delta_p(6M+4)\bar{N}\}:\ k=0,\dots,T-1\}$$
$$\leq 4M^2T^{-1}.$$

Moreover, if $k\in\{0,\dots,T-1\}$ *and*

$$\max\{2\alpha(f(x_{k\bar{N}})-\inf(f,F))-\delta_p(6M\bar{N}+3)-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1),$$
$$\bar{c}\|x_{k\bar{N}}-\alpha\xi_k-x_{k\bar{N}+1}\|^2$$
$$+\bar{c}\sum\{\|x_{k\bar{N}+t-1}-x_{k\bar{N}+t}\|^2:\ t\in\{1,\dots,\bar{N}\}\setminus\{1\}\}$$
$$-\alpha(L_0+1)(6M+1)-4\delta_p(6M+4)\bar{N}\}$$
$$\leq 4M^2T^{-1}$$

then

$$f(x_{k\bar{N}})\leq\inf(f,F)+2M^2(T\alpha)^{-1}$$
$$+2^{-1}\alpha^{-1}\delta_p(6M\bar{N}+3)+2^{-1}\alpha L_0^2+\delta_f(6M+L_0+1)$$

and

$$x_{k\bar{N}}\in\tilde{F}_{\gamma_T}.$$

Proof In view of (2.8), there exists

$$z\in B(0,M_*)\cap F. \tag{2.204}$$

By (2.18) and (2.200),

$$\|z-x_0\|\leq M+M_*\leq 2M. \tag{2.205}$$

Assume that $k\in\{0,\dots,T-1\}$ and that

$$\|z - x_{k\bar{N}}\| \le 2M. \tag{2.206}$$

It follows from (2.1), (2.2), (2.202), (2.204) and (2.206) that

$$\|z - x_{k\bar{N}+1}\| \le \|z - P_{r(k\bar{N}+1)}(x_{k\bar{N}} - \alpha\xi_k)\|$$

$$+\|P_{r(k\bar{N}+1)}(x_{k\bar{N}} - \alpha\xi_k) - x_{k\bar{N}+1}\| \le \|z - x_{k\bar{N}} + \alpha\xi_k\| + \delta_p$$

$$\le 2M + \delta_p + \alpha\|\xi_k\|. \tag{2.207}$$

In view of (2.204) and (2.206),

$$\|x_{k\bar{N}}\| \le 3M. \tag{2.208}$$

By (2.196) and (2.208),

$$\partial f(x_{k\bar{N}}) \subset B(0, L_0). \tag{2.209}$$

Equations (2.201) and (2.209) imply that

$$\|\xi_k\| \le L_0 + 1. \tag{2.210}$$

It follows from (2.197), (2.207) and (2.210) that

$$\|z - x_{k\bar{N}+1}\| \le 2M + \delta_p + \alpha(L_0 + 1) \le 2M + 2. \tag{2.211}$$

By (2.1), (2.2), (2.197), (2.204) and (2.211), for every $t \in \{k\bar{N}+1, \dots, (k+1)\bar{N}\} \setminus \{k\bar{N} + 1\}$,

$$\|z - x_t\| \le \|z - P_{r(t)}(x_{t-1})\| + \|P_{r(t)}(x_{t-1}) - x_t\| \le \|z - x_{t-1}\| + \delta_p, \tag{2.212}$$

$$\|z - x_t\| \le \|z - x_{k\bar{N}+1}\| + \bar{N}\delta_p \le 2M + 2 + \bar{N}\delta_p \le 2M + 3. \tag{2.213}$$

Thus we have shown that the following property holds:

(i) if $k \in \{0, \dots, T-1\}$ and $\|z - x_{k\bar{N}}\| \le 2M$, then (2.211) holds and for every $t \in \{k\bar{N} + 1, \dots, (k+1)\bar{N}\} \setminus \{k\bar{N} + 1\}$, (2.212) and (2.213) are true.

We show that for all $k \in \{0, \dots, T\}$, (2.206) holds. In view of (2.205), (2.206) holds for $k = 0$. Assume that there exists an integer $q \in \{0, \dots, T\}$ such that

$$\|z - x_{q\bar{N}}\| > 2M. \tag{2.214}$$

By (2.205) and (2.214), $q > 0$. We may assume without loss of generality that (2.206) holds for all integers $k = 0, \dots, q-1$. In particular,

$$\|z - x_{(q-1)\bar{N}}\| \leq 2M. \tag{2.215}$$

In view of (2.8) and (2.215),

$$\|x_{(q-1)\bar{N}}\| \leq 2M + M_*. \tag{2.216}$$

Equations (2.116) and (2.196) imply that

$$\partial f(x_{(q-1)\bar{N}}) \subset B(0, L_0). \tag{2.217}$$

By (2.201) and (2.217),

$$\|\xi_k\| \leq L_0 + 1. \tag{2.218}$$

In view of (2.218),

$$\|x_{(q-1)\bar{N}} - (x_{(q-1)\bar{N}} - \alpha\xi_{q-1})\| \leq \alpha\|\xi_{q-1}\| \leq \alpha(L_0 + 1). \tag{2.219}$$

By (2.1), (2.2), (2.216), (2.219) and (2.204), we apply Lemma 2.12 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ F_0 = F,\ M_0 = 3M,\ p = \bar{N},$$

$$Q_i = P_{r(i+(q-1)\bar{N})},\ i = 1, \ldots, \bar{N},\ x = x_{(q-1)\bar{N}},$$

$$u_0 = x_{(q-1)\bar{N}} - \alpha\xi_{q-1},\ u_t = x_{(q-1)\bar{N}+t},\ t = 1, \ldots, \bar{N}$$

and obtain that

$$\begin{gathered}
\|z - x_{(q-1)\bar{N}}\|^2 - \|z - x_{q\bar{N}}\|^2 \\
\geq \bar{c}\|x_{(q-1)\bar{N}+1} - (x_{(q-1)\bar{N}} - \alpha\xi_{q-1})\|^2 \\
+\bar{c}\sum\{\|x_{(q-1)\bar{N}+t} - x_{(q-1)\bar{N}+t-1}\|^2 :\ t \in \{1, \ldots, \bar{N}\} \setminus \{1\}\} \\
-\alpha(6M + \alpha(L_0 + 1))(L_0 + 1) - 2\delta_p\bar{N}(6M + \alpha(L_0 + 1) + \bar{N}\delta_p) \\
- \bar{N}\delta_p(12M + 2\alpha(L_0 + 1) + \delta_p(2\bar{N} + 1)).
\end{gathered} \tag{2.220}$$

There are two cases:

$$\|z - x_{q\bar{N}}\| \leq \|z - x_{(q-1)\bar{N}}\| \tag{2.221}$$

$$\|z - x_{q\bar{N}}\| > \|z - x_{(q-1)\bar{N}}\|. \tag{2.222}$$

Assume that (2.221) holds. Then in view of (2.215),

$$\|x_{q\bar{N}} - z\| \leq 2M.$$

Assume that (2.222) is true. Set

$$\Delta_0 = \alpha(6M+1)(L_0+1) + 4\delta_p \bar{N}(6M+4). \tag{2.223}$$

By (2.197), (2.220), (2.222) and (2.223), for all $t \in \{1, \ldots, \bar{N}\} \setminus \{1\}$,

$$\|x_{(q-1)\bar{N}+1} - (x_{(q-1)\bar{N}} - \alpha\xi_{q-1})\|, \ \|x_{(q-1)\bar{N}+t} - x_{(q-1)\bar{N}+t-1}\|$$
$$\leq (\bar{c}^{-1}\Delta_0)^{1/2}. \tag{2.224}$$

In view of (2.218),

$$\|x_{(q-1)\bar{N}} - (x_{(q-1)\bar{N}} - \alpha\xi_{q-1})\| \leq \alpha(L_0+1). \tag{2.225}$$

Equations (2.224) and (2.25) imply that

$$\|x_{(q-1)\bar{N}} - x_{(q-1)\bar{N}+1}\|$$
$$\leq \|x_{(q-1)\bar{N}} - (x_{(q-1)\bar{N}} - \alpha\xi_{q-1})\| + \|x_{(q-1)\bar{N}} - \alpha\xi_{q-1} - x_{(q-1)\bar{N}+1}\|$$
$$\leq \alpha(L_0+1) + (\bar{c}^{-1}\Delta_0)^{1/2}. \tag{2.226}$$

It follows from (2.202) and (2.224) that

$$\|x_{(q-1)\bar{N}} - \alpha\xi_{q-1} - P_{r((q-1)\bar{N}+1)}(x_{(q-1)\bar{N}} - \alpha\xi_{q-1})\| \leq \delta_p + (\bar{c}^{-1}\Delta_0)^{1/2}. \tag{2.227}$$

By (2.203) and (2.224), for all $t \in \{1, \ldots, \bar{N}\} \setminus \{1\}$,

$$\|x_{(q-1)\bar{N}+t-1} - P_{r((q-1)\bar{N}+t)}(x_{(q-1)\bar{N}+t-1})\|$$
$$\leq (\bar{c}^{-1}\Delta_0)^{1/2} + \|P_{r((q-1)\bar{N}+t)}(x_{(q-1)\bar{N}+t-1}) - x_{(q-1)\bar{N}+t}\|$$
$$\leq (\bar{c}^{-1}\Delta_0)^{1/2} + \delta_p. \tag{2.228}$$

In view of (2.197), (2.223), (2.224) and (2.26), for every $t = 1, \ldots, \bar{N}$,

$$\|x_{(q-1)\bar{N}} - x_{(q-1)\bar{N}+t}\| \leq (\bar{c}^{-1}\Delta_0)^{1/2}\bar{N} + \alpha(L_0+1) \leq r_0. \tag{2.229}$$

By (2.225) and (2.227)–(2.229),

$$x_{(q-1)\bar{N}} \in \tilde{F}_{r_0}.$$

Together with (2.195) this implies

$$\|x_{(q-1)\bar{N}}\| \le M_*. \tag{2.230}$$

By (2.229) and (2.230),

$$\|x_{q\bar{N}}\| \le M_* + r_0 \le M_* + 1 < M$$

and in view of (2.204),

$$\|x_{q\bar{N}} - z\| < M + M_*.$$

This contradicts (2.214). The contradiction we have reached proves that (2.206) is true for all $k = 0, \dots, T$. Property (i), (2.204) and (2.206) imply that

$$\|x_{k\bar{N}}\| \le 3M, \ k = 0, \dots, T \tag{2.231}$$

$$\|x_t\| \le 3M + 3, \ t = 0, \dots, T\bar{N}. \tag{2.232}$$

By (2.196) and (2.232), for all $t \in \{0, \dots, T\bar{N}\}$,

$$\partial f(x_t) \subset B(0, L_0). \tag{2.233}$$

In view of (2.201) and (2.233),

$$\|\xi_k\| \le L_0 + 1, \ k = 0, \dots, T - 1. \tag{2.234}$$

Let $k \in \{0, \dots, T - 1\}$. By (2.1), (2.2), (2.196), (2.197) and (2.201)–(2.204), we apply Lemma 2.11 with

$$\delta_1 = \delta_f, \ \delta_2 = \delta_p, \ F_0 = F, \ M_0 = 3M, \ p = \bar{N},$$

$$Q_i = P_{r(i+k\bar{N})}, \ i = 1, \dots, \bar{N}, \ x = x_{k\bar{N}}, \ \xi = \xi_k,$$

$$u_0 = x_{k\bar{N}} - \alpha\xi_k, \ u_t = x_{k\bar{N}+t}, \ t = 1, \dots, \bar{N}$$

and obtain that

$$\alpha(f(x_{k\bar{N}}) - f(z))$$

$$\le 2^{-1}\|x_{k\bar{N}} - z\|^2 - 2^{-1}\|x_{(k+1)\bar{N}} - z\|^2$$

$$+2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1)$$

$$+\delta_p(6M + 2 + \alpha L_0 + 3M(\bar{N} - 1)) + \delta_p \bar{N}(\bar{N} - 1)/2$$

$$\leq 2^{-1}\|x_{k\bar{N}} - z\|^2 - 2^{-1}\|x_{(k+1)\bar{N}} - z\|^2 + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1)$$

$$+ \delta_p(6M\bar{N} + 3). \tag{2.235}$$

By (2.1), (2.2), (2.196), (2.197), (2.202), (2.203) and (2.234), we apply Lemma 2.12 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ F_0 = F,\ M_0 = 3M,\ p = \bar{N},$$

$$Q_i = P_{r(i+k\bar{N})},\ i = 1, \ldots, \bar{N},\ x = x_{k\bar{N}},$$

$$u_0 = x_{k\bar{N}} - \alpha\xi_k,\ u_t = x_{k\bar{N}+t},\ t = 1, \ldots, \bar{N}$$

and obtain that

$$\|z - x_{k\bar{N}}\|^2 - \|z - x_{(k+1)\bar{N}}\|^2$$

$$\geq \bar{c}\|x_{k\bar{N}+1} - (x_{k\bar{N}} - \alpha\xi_k)\|^2$$

$$+\bar{c}\sum\{\|x_{k\bar{N}+t} - x_{k\bar{N}+t-1}\|^2 :\ t \in \{1, \ldots, \bar{N}\} \setminus \{1\}\}$$

$$-\alpha(6M + \alpha(L_0 + 1))(L_0 + 1) - 2\delta_p\bar{N}(6M + \alpha(L_0 + 1) + \bar{N}\delta_p)$$

$$-\bar{N}\delta_p(12M + 2\alpha(L_0 + 1) + \delta_p(2\bar{N} + 1))$$

$$\geq \bar{c}\|x_{k\bar{N}+1} - (x_{k\bar{N}} - \alpha\xi_k)\|^2$$

$$+\bar{c}\sum\{\|x_{k\bar{N}+t} - x_{k\bar{N}+t-1}\|^2 :\ t \in \{1, \ldots, \bar{N}\} \setminus \{1\}\}$$

$$- \alpha(6M + 1)(L_0 + 1) - 4\delta_p\bar{N}(6M + 4). \tag{2.236}$$

By (2.200), (2.204), (2.235) and (2.236),

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_{T\bar{N}}\|^2$$

$$= \sum_{k=0}^{T-1}(\|z - x_{k\bar{N}}\|^2 - \|z - x_{(k+1)\bar{N}}\|^2)$$

$$\geq \sum_{k=0}^{T-1} \max\{2\alpha(f(x_{k\bar{N}}) - f(z))$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1) - \delta_p(6M\bar{N} + 3),$$

$$\bar{c}\|x_{k\bar{N}+1} - (x_{k\bar{N}} - \alpha\xi_k)\|^2$$

$$+\bar{c}\sum\{\|x_{k\bar{N}+t} - x_{k\bar{N}+t-1}\|^2 : \ t \in \{1, \dots, \bar{N}\} \setminus \{1\}\}$$

$$-\alpha(6M + 1)(L_0 + 1) - 4\delta_p\bar{N}(6M + 4)\}.$$

Since z is an arbitrary element of $B(0, M_*) \cap F$ it follows from the relation above and (2.11) that

$$4M^2 \geq T \min\{\max\{2\alpha(f(x_{k\bar{N}}) - \inf(f, F))$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1) - 2\delta_p(6M\bar{N} + 3),$$

$$\bar{c}\|x_{k\bar{N}+1} - (x_{k\bar{N}} - \alpha\xi_k)\|^2$$

$$+\bar{c}\sum\{\|x_{k\bar{N}+t} - x_{k\bar{N}+t-1}\|^2 : \ t \in \{1, \dots, \bar{N}\} \setminus \{1\}\}$$

$$-\alpha(6M + 1)(L_0 + 1) - 4\delta_p\bar{N}(6M + 4)\} : \ k = 0, \dots, T - 1\}.$$

Set

$$\Delta = \alpha(6M + 1)(L_0 + 1) + 4\delta_p\bar{N}(6M + 4). \tag{2.237}$$

Let $k \in \{0, \dots, T - 1\}$ and

$$4M^2T^{-1} \geq \max\{2\alpha(f(x_{k\bar{N}}) - \inf(f, F))$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1) - \delta_p(6M\bar{N} + 3),$$

$$\bar{c}\|x_{k\bar{N}+1} - (x_{k\bar{N}} - \alpha\xi_k)\|^2$$

$$+\bar{c}\sum\{\|x_{k\bar{N}+t} - x_{k\bar{N}+t-1}\|^2 : \ t \in \{1, \dots, \bar{N}\} \setminus \{1\}\}$$

$$-\alpha(6M + 1)(L_0 + 1) - 4\delta_p\bar{N}(6M + 4)\}. \tag{2.238}$$

In view of (2.238),

$$f(x_{k\bar{N}}) - \inf(f, F)$$

$$\leq 2M^2(T\alpha)^{-1} + 2^{-1}\alpha^{-1}\delta_p(6M\bar{N} + 3) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1)$$

and that for all $t \in \{1, \dots, \bar{N}\} \setminus \{1\}$,

$$\|x_{k\bar{N}+1} - (x_{k\bar{N}} - \alpha\xi_k)\|, \ \|x_{k\bar{N}+t} - x_{k\bar{N}+t-1}\|$$

$$\leq (4T^{-1}M^2 + \alpha(6M + 1)(L_0 + 1) + 4\delta_p\bar{N}(6M + 4))^{1/2}\bar{c}^{-1/2}$$

$$= (4T^{-1}M^2 + \Delta)^{1/2}\bar{c}^{-1/2}. \tag{2.239}$$

By (2.202), (2.203) and (2.239),

$$\|x_{k\bar{N}} - \alpha\xi_k - P_{r(k\bar{N}+1)}(x_{k\bar{N}} - \alpha\xi_k)\|, \ \|x_{k\bar{N}+t-1} - P_{r(k\bar{N}+t)}(x_{k\bar{N}+t-1})\|$$

$$\leq (4T^{-1}M^2 + \Delta)^{1/2}\bar{c}^{-1/2} + \delta_p, \ t \in \{1, \dots, \bar{N}\} \setminus \{1\}. \tag{2.240}$$

It follows from (2.234) and (2.239) that

$$\|x_{k\bar{N}} - x_{k\bar{N}+1}\|$$

$$\leq \|x_{k\bar{N}} - (x_{k\bar{N}} - \alpha\xi_k)\| + \|x_{k\bar{N}+1} - (x_{k\bar{N}} - \alpha\xi_k)\|$$

$$\leq \alpha(L_0 + 1) + (4M^2T^{-1} + \Delta)^{1/2}\bar{c}^{-1/2}. \tag{2.241}$$

In view of (2.239) and (2.241), for all $t \in \{1, \dots, \bar{N}\}$,

$$\|x_{k\bar{N}} - x_{k\bar{N}+t}\| \leq \alpha(L_0 + 1) + (4M^2T^{-1} + \Delta)^{1/2}\bar{c}^{-1/2}\bar{N}. \tag{2.242}$$

It follows from (2.199), (2.237), (2.240) and (2.242) that

$$x_{k\bar{N}} \in \tilde{F}_{\alpha(L_0+1)+(4M^2T^{-1}+\Delta)^{1/2}\bar{c}^{-1}\bar{N}} \subset \tilde{F}_{\gamma_T}.$$

Theorem 2.14 is proved.

Analogously to Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations.

2.9 Dynamic String-Averaging Subgradient Algorithm

We continue to consider the minimization problem

$$f(x) \to \min, \ x \in F$$

introduced in Sections 2.1 and 2.2 using the notation and definitions introduced there. We also suppose that all the assumptions introduced there hold.

We apply a dynamic string-averaging method with variable strings and weights in order to obtain a good approximative solution of our problem.

Next we describe the dynamic string-averaging method with variable strings and weights.

By an index vector, we a mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q, \ P[t] = P_{t_q} \cdots P_{t_1}. \tag{2.243}$$

It is not difficult to see that for each index vector t

$$P[t](x) = x \text{ for all } x \in F, \tag{2.244}$$

$$\|P[t](x) - P[t](y)\| = \|x - P[t](y)\| \le \|x - y\| \tag{2.245}$$

for every point $x \in F$ and every point $y \in X$.

Fix a number

$$\Delta \in (0, m^{-1}] \tag{2.246}$$

and an integer

$$\bar{q} \ge m. \tag{2.247}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ satisfies } \sum_{t \in \Omega} w(t) = 1 \tag{2.248}$$

such that

$$p(t) \le \bar{q} \ \text{ for all } t \in \Omega, \tag{2.249}$$

$$w(t) \ge \Delta \text{ for all } t \in \Omega, \tag{2.250}$$

$$\cup_{t \in \Omega} (\{t_1, \dots, t_{p(t)}\}) = \{1, \dots, m\}. \tag{2.251}$$

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{t\in\Omega} w(t)P[t](x),\ \ x \in X. \tag{2.252}$$

It is easy to see that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F,$$

$$\|P_{\Omega,w}(x) - P_{\Omega,w}(y)\| = \|x - P_{\Omega,w}(y)\| \le \|x - y\|$$

for every point $x \in F$ and every point $y \in X$.

The dynamic string-averaging subgradient (DSAS) method with variable strings and variable weights can now be described by the following algorithm.

The Dynamic String-Averaging Subgradient Algorithm

Let $\alpha > 0$.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$\xi_k \in \partial f(x_k),$$

pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1},w_{k+1}}(x_k - \alpha\xi_k).$$

In this chapter we study this algorithm taking into account computational errors.

In order to proceed we need the following definitions.

Let $\delta \ge 0$, $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector. Define

$$A_0(x, t, \delta) = \{y \in X :\ \ \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that}$$

$$y_0 = x \text{ and for all } i = 1, \dots, p(t),$$

$$\|y_i - P_{t_i}(y_{i-1})\| \le \delta,$$

$$y = y_{p(t)}\}. \tag{2.253}$$

Let $\delta \ge 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$A(x, (\Omega, w), \delta) = \{y \in X :\ \ \text{there exist}$$

$$y_t \in A_0(x, t, \delta),\ \ t \in \Omega \text{ such that}$$

$$\|y - \sum_{t \in \Omega} w(t) y_t\| \le \delta\}. \tag{2.254}$$

In this chapter we study the dynamic string-averaging subgradient algorithm taking into account computational errors. We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f while $\delta_p \in [0, 1]$ is a computational error produced by our computer system, when we calculate the operators P_i, $i = 1, \dots, m$. Let $\alpha > 0$ be a step size.

The Dynamic String-Averaging Subgradient Algorithm with Computational Errors

Let $\alpha > 0$.

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$\xi_k \in \partial f(x_k) + B(0, \delta_f),$$

pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate

$$x_{k+1} \in A(x_k - \alpha \xi_k, (\Omega_{k+1}, w_{k+1}), \delta_p).$$

2.10 Auxiliary Results

In order to study our algorithm we need the following auxiliary results.

Lemma 2.15 *Let* $M_0 \ge M_* + 2$, $L_0 \ge 1$, $(\Omega, w) \in \mathcal{M}$,

$$\|f(z_1 - f(z_2)\| \le L_0 \|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(0, M_0 + 4), \tag{2.255}$$

and let $\delta_1, \delta_2 \in [0, 1]$, $\alpha \in (0, 1]$ *satisfy*

$$\delta_2 \le \bar{q}^{-1}, \ \alpha \le (L_0 + 1)^{-1}. \tag{2.256}$$

Assume that

$$z \in F \cap B(0, M_*), \tag{2.257}$$

$$x \in B(0, M_0), \tag{2.258}$$

$$\xi \in \partial f(x) + B(0, \delta_1) \tag{2.259}$$

and that

$$y \in A(x - \alpha\xi, (\Omega, w), \delta_2). \tag{2.260}$$

Then

$$\alpha(f(x) - f(z))$$

$$\leq 2^{-1}\|x - z\|^2 - 2^{-1}\|y - z\|^2 + \delta_2(6M_0 + 4)$$

$$+2\delta_2\bar{q}M_0 + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_1(2M_0 + L_0 + 1).$$

Proof By (2.254) and (2.260), for each $t \in \Omega$, there exists

$$y_t \in A_0(x - \alpha\xi, t, \delta_2) \tag{2.261}$$

such that

$$\|y - \sum_{t \in \Omega} w(t) y_t\| \leq \delta_2. \tag{2.262}$$

In view of (2.253) and (2.261), for every $t \in \Omega$ there exist

$$y_i^{(t)} \in X, \ i = 0, \dots, p(t)$$

such that

$$y_0^{(t)} = x - \alpha\xi, \tag{2.263}$$

for all $i = 1, \dots, p(t)$,

$$\|y_i^{(t)} - P_{t_i}(y_{i-1}^{(t)})\| \leq \delta_2, \tag{2.264}$$

$$y_t = y_{p(t)}^{(t)}. \tag{2.265}$$

In view of (2.255) and (2.258),

$$\partial f(x) \subset B(0, L_0). \tag{2.266}$$

Equations (2.259) and (2.266) imply that

$$\|\xi\| \leq L_0 + 1. \tag{2.267}$$

Let $t = (t_1, \ldots, t_{p(t)}) \in \Omega$. By (2.128), (2.255), (2.257)–(2.259), (2.263), (2.264) and Lemma 2.11 applied with $F_0 = F$, $u_i = y_i^{(t)}$, $i = 0, \ldots, p(t)$, $Q_i = P_{t_i}$, $i = 1, \ldots, p(t)$,

$$\alpha(f(x) - f(z))$$
$$\le 2^{-1}\|x - z\|^2 - 2^{-1}\|y_t - z\|^2 + \delta_2(2M_0 + 2 + \alpha L_0) + 2^{-1}\alpha^2 L_0^2$$
$$+ \alpha\delta_1(2M_0 + L_0 + 1) + \delta_2(M_0(\bar{q} - 1) + \delta_2\bar{q}(\bar{q} - 1)/2). \tag{2.268}$$

Equations (2.258), (2.263) and (2.267) imply that

$$\|y_0^{(t)}\| \le M_0 + \alpha(L_0 + 1). \tag{2.269}$$

By (2.1), (2.2), (2.256), (2.257), (2.264) and (2.269),

$$\|z - y_1^{(t)}\| \le \|z - P_{t_1}(y_0^{(t)})\| + \|P_{t_1}(y_0^{(t)}) - y_1^{(t)}\|$$
$$\le \|z - y_0^{(t)}\| + \delta_2$$
$$\le M_0 + M_* + \delta_2 + \alpha(L_0 + 1) \le 2M_0. \tag{2.270}$$

It follows from (2.1), (2.2), (2.257) and (2.264) that for all $t = 1, \ldots, p(t)$,

$$\|z - y_i^{(t)}\| \le \|z - P_{t_i}(y_{i-1}^{(t)})\| + \|P_{t_i}(y_{i-1}^{(t)}) - y_i^{(t)}\|$$
$$\le \|z - y_{i-1}^{(t)}\| + \delta_2. \tag{2.271}$$

In view of (2.256), (2.270) and (2.271),

$$\|z - y_{p(t)}^{(t)}\| \le \|z - y_1^{(t)}\| + \delta_2(p(t) - 1)$$
$$\le M_* + M_0 + \alpha(L_0 + 1) + \delta_2\bar{q} \le M_* + M_0 + 2 \le 2M_0. \tag{2.272}$$

By (2.259), (2.268) and the convexity of the function $u \to \|u - z\|^2$, $u \in X$,

$$\alpha(f(x) - f(z)) - 2^{-1}\|x - z\|^2$$
$$-2\delta_2\bar{q}M_0 - \delta_2(2M_0 + 3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_1(2M_0 + L_0 + 1)$$
$$\le -2^{-1}\sum_{t\in\Omega} w(t)\|y_t - z\|^2$$

$$\leq -2^{-1}\|\sum_{t\in\Omega} w(t)y_t - z\|^2. \tag{2.273}$$

By (2.265), (2.272) and the convexity of the norm,

$$\|z - \sum_{t\in\Omega} w(t)y_t\| \leq \sum_{t\in\Omega} w(t)\|z - y_t\| \leq 2M_0. \tag{2.274}$$

In view of (2.262),

$$|\|z - y\| - \|z - \sum_{t\in\Omega} w(t)y_t\|| \leq \delta_2. \tag{2.275}$$

It follows from (2.274) and (2.275) that

$$|\|z - y\|^2 - \|z - \sum_{t\in\Omega} w(t)y^{(t)}_{p(t)}\|^2|$$

$$\leq \delta_2(|\|z - y\| + \|z - \sum_{t\in\Omega} w(t)y^{(t)}_{p(t)}\|) \leq \delta_2(4M_0 + 1). \tag{2.276}$$

Equations (2.273) and (2.276) imply that

$$\alpha(f(x) - f(z)) - 2^{-1}\|x - z\|^2$$

$$-2\delta_2\bar{q}M_0 - \delta_2(2M_0 + 3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_1(2M_0 + L_0 + 1)$$

$$\leq -2^{-1}\|z - y\|^2 + \delta_2(4M_0 + 1).$$

This completes the proof of Lemma 2.15.

Lemma 2.16 *Let* $M_0 \geq M_* + 2$, $(\Omega, w) \in \mathcal{M}$, *and let* $\delta_1, \delta_2 \in [0, 1]$ *satisfy*

$$\delta_2 \leq (\bar{q} + 1)^{-1}. \tag{2.277}$$

Assume that

$$z \in F \cap B(0, M_*), \tag{2.278}$$

$$x \in B(0, M_0), \tag{2.279}$$

$$x_0 \in B(x, \delta_1), \tag{2.280}$$

$$y \in A(x_0, (\Omega, w), \delta_2), \tag{2.281}$$

$$y_t \in A_0(x_0, t, \delta_2),\ \ t \in \Omega \tag{2.282}$$

satisfy

$$\|y - \sum_{t \in \Omega} w(t) y_t\| \le \delta_2, \tag{2.283}$$

for every $t \in \Omega$*,*

$$y_i^{(t)} \in X,\ \ i = 0, \ldots, p(t)$$

satisfy

$$y_0^{(t)} = x_0, \tag{2.284}$$

for all $i = 1, \ldots, p(t)$*,*

$$\|y_i^{(t)} - P_{t_i}(y_{i-1}^{(t)})\| \le \delta_2, \tag{2.285}$$

$$y_t = y_{p(t)}^{(t)}. \tag{2.286}$$

Then

$$\|z - x\|^2 - \|z - y\|^2$$

$$\ge \bar{c}\Delta \sum_{t \in \Omega} \sum_{i=1}^{p(t)} \|y_{i-1}^{(t)} - P_{t_i}(y_{i-1}^{(t)})\|^2 - 4\delta_1 M_0 - 4M_0\delta(\bar{q} + 1),$$

$$\|z - x\|^2 - \|z - y\|^2$$

$$\ge \bar{c}\Delta \sum_{t \in \Omega} \sum_{i=1}^{p(t)} \|y_{i-1}^{(t)} - y_i^{(t)})\|^2 - 4\delta_1 M_0 - 8M_0\delta(\bar{q} + 1).$$

Proof Let $t = (t_1, \ldots, t_{p(t)}) \in \Omega$. By (2.278)–(2.280) and (2.284),

$$\|z - y_0^{(t)}\| = \|z - x_0\| \le \|z - x\| + \|x - x_0\| \le M_0 + M_* + \delta_1. \tag{2.287}$$

It follows from (2.278)–(2.280), (2.284) and (2.287) that

$$|\|z - x\|^2 - \|z - y_0^{(t)}\|^2| \le \delta_1(2M_0 + 2M_* + \delta_1). \tag{2.288}$$

Equations (2.1), (2.2) and (2.278) imply that for all $i = 1, \ldots, p(t)$,

$$\|z - y_i^{(t)}\| \le \|z - P_{t_i}(y_{i-1}^{(t)})\| + \|P_{t_i}(y_{i-1}^{(t)}) - y_i^{(t)}\|$$

$$\le \|z - y_{i-1}^{(t)}\| + \delta_2 \tag{2.289}$$

and in view of (2.287),

$$\|z - y_i^{(t)}\| \le \|z - y_0^{(t)}\| + i\delta_2$$

$$\le M_* + M_0 + \delta_2 i + \delta_1 \le M_* + M_0 + \delta_2 \bar{q} + 1. \tag{2.290}$$

Let $i \in \{1, \dots, p(t)\}$. By (2.1), (2.277), (2.285), (2.287) and (2.290),

$$\|z - y_{i-1}^{(t)}\|^2 - \|z - y_i^{(t)}\|^2$$

$$= \|z - y_{i-1}^{(t)}\|^2 - \|z - P_{t_i}(y_{i-1}^{(t)})\|^2 + \|z - P_{t_i}(y_{i-1}^{(t)})\|^2 - \|z - y_i^{(t)}\|^2$$

$$\ge \bar{c}\|y_{i-1}^{(t)} - P_{t_i}(y_{i-1}^{(t)})\|^2$$

$$-(\|z - P_{t_i}(y_{i-1}^{(t)})\| - \|z - y_i^{(t)}\|)(\|z - P_{t_i}(y_{i-1}^{(t)})\| + \|z - y_i^{(t)}\|)$$

$$\ge \bar{c}\|y_{i-1}^{(t)} - P_{t_i}(y_{i-1}^{(t)})\|^2 - \delta_2(\|z - y_{i-1}^{(t)})\| + \|z - y_i^{(t)}\|)$$

$$\ge \bar{c}\|y_{i-1}^{(t)} - P_{t_i}(y_{i-1}^{(t)})\|^2 - 4\delta_2 M_0. \tag{2.291}$$

By (2.284), (2.288) and (2.291),

$$\|z - x\|^2 - \|z - y_{p(t)}^{(t)}\|^2$$

$$= \|z - x\|^2 - \|z - x_0\|^2 + \|z - y_0^{(t)}\|^2 - \|z - y_{p(t)}^{(t)}\|^2$$

$$\ge -4\delta_1 M_0 + \sum_{i=1}^{p(t)} (\|z - y_{i-1}^{(t)}\|^2 - \|z - y_i^{(t)}\|^2)$$

$$\ge -4\delta_1 M_0 + \bar{c} \sum_{i=1}^{p(t)} \|y_{i-1}^{(t)} - P_{t_i}(y_{i-1}^{(t)})\|^2 - 4\delta_2 M_0 \bar{q}. \tag{2.292}$$

It follows from (2.248), (2.250), (2.292) and the convexity of the function $\|u - z\|^2$, $u \in X$ that

$$\|z - x\|^2 - \|z - \sum_{t\in\Omega} w(t) y^{(t)}_{p(t)}\|^2$$

$$\geq \|z - x\|^2 - \sum_{t\in\Omega} w(t)\|z - y^{(t)}_{p(t)}\|^2$$

$$\geq \sum_{t\in\Omega} w(t)(\bar{c}\sum_{i=1}^{p(t)} \|y^{(t)}_{i-1} - P_{t_i}(y^{(t)}_{i-1})\|^2 - 4\delta_1 M_0 - 4\delta_2 M_0 \bar{q})$$

$$\geq \bar{c}\Delta \sum_{i=1}^{p(t)} \|y^{(t)}_{i-1} - P_{t_i}(y^{(t)}_{i-1})\|^2 - 4\delta_1 M_0 - 4\delta_2 M_0 \bar{q}. \tag{2.293}$$

By (2.248), (2.271), (2.290) and the convexity of the norm,

$$\|z-\sum_{t\in\Omega} w(t) y^{(t)}_{p(t)}\| \leq \sum_{t\in\Omega} w(t)\|z-y^{(t)}_{p(t)}\| \leq M_0+M_*+\delta_2\bar{q}+1 \leq 2M_0. \tag{2.294}$$

In view of (2.283),

$$|\|z - y\| - \|z - \sum_{t\in\Omega} w(t) y^{(t)}_{p(t)}\|| \leq \delta_2. \tag{2.295}$$

Equations (2.277), (2.294) and (2.295) imply that

$$\|z - y\| \leq M_* + M_0 + 1 + \delta_2(\bar{q} + 1) \leq M_* + M_0 + 2 \leq 2M_0. \tag{2.296}$$

It follows from (2.283), (2.286), (2.294) and (2.296) that

$$|\|z - y\|^2 - \|z - \sum_{t\in\Omega} w(t) y^{(t)}_{p(t)}\|^2|$$

$$\leq \|y - \sum_{t\in\Omega} w(t) y^{(t)}_{p(t)}\|(\|z - y\| + \|z - \sum_{t\in\Omega} w(t) y^{(t)}_{p(t)}\|) \leq 4\delta_2 M_0. \tag{2.297}$$

In view of (2.293) and (2.297),

$$\|z - x\|^2 - \|z - y\|^2$$

$$= \|z - x\|^2 - \|z - \sum_{t\in\Omega} w(t) y^{(t)}_{p(t)}\|^2 + \|z - \sum_{t\in\Omega} w(t) y^{(t)}_{p(t)}\|^2 - \|z - y\|^2$$

$$\geq \bar{c}\Delta \sum_{t\in\Omega}\sum_{i=1}^{p(t)} \|y_{i-1}^{(t)} - P_{t_i}(y_{i-1}^{(t)})\|^2 - 4\delta_1 M_0 - 4\delta_2 M_0 \bar{q} - 4M_0\delta_2. \tag{2.298}$$

By (2.1), (2.277), (2.285), (2.287) and (2.290), for every $t \in \Omega$ and every $i \in \{1, \ldots, p(t)\}$,

$$|\|y_{i-1}^{(t)} - P_{t_i}(y_{i-1}^{(t)})\|^2 - \|y_{i-1}^{(t)} - y_i^{(t)}\|^2|$$

$$\leq \|y_i^{(t)} - P_{t_i}(y_{i-1}^{(t)})\|(\|y_{i-1}^{(t)} - y_i^{(t)}\| + \|y_{i-1}^{(t)} - P_{t_i}(y_{i-1}^{(t)})\|)$$

$$\leq \delta_2(2\|y_{i-1}^{(t)} - P_{t_i}(y_{i-1}^{(t)})\| + \delta_2) \leq \delta_2(2\|z - y_{i-1}^{(t)}\| + \delta_2)$$

$$\leq \delta_2(2(M_0 + M_* + \delta_2\bar{q} + 1) + \delta_2) \leq 4\delta_2 M_0. \tag{2.299}$$

It follows from (2.249), (2.250), (2.298) and (2.299) that

$$\|z - x\|^2 - \|z - y\|^2$$

$$\geq \bar{c}\Delta \sum_{t\in\Omega}\sum_{i=1}^{p(t)} \|y_{i-1}^{(t)} - y_i^{(t)}\|^2$$

$$-4\delta_1 M_0 - 4\delta_2 M_0(\bar{q} + 1) - \bar{c}\Delta \sum_{t\in\Omega}\sum_{i=1}^{p(t)} 4\delta_2 M_0$$

$$\geq \bar{c}\Delta \sum_{t\in\Omega}\sum_{i=1}^{p(t)} \|y_{i-1}^{(t)} - y_i^{(t)}\|^2$$

$$-4\delta_1 M_0 - 4\delta_2 M_0(\bar{q} + 1) - \bar{c}(\sum_{t\in\Omega} w(t))4\delta_2 M_0\bar{q}$$

$$\geq \bar{c}\Delta \sum_{t\in\Omega}\sum_{i=1}^{p(t)} \|y_{i-1}^{(t)} - y_i^{(t)}\|^2$$

$$-4\delta_1 M_0 - 8\delta_2 M_0(\bar{q} + 1).$$

Lemma 2.16 is proved.

2.11 The First Theorem for the Dynamic String-Averaging Subgradient Algorithm

In the following theorem we assume that the objective function satisfies the coercivity growth condition.

Theorem 2.17 *Let the function f be Lipschitz on bounded subsets of X,*

$$\lim_{\|x\|\to\infty} f(x) = \infty,$$

$M \geq 2M_* + 6$, $L_0 \geq 1$,

$$M_1 \geq \sup\{|f(u)| : u \in B(0, M_* + 4)\}, \tag{2.300}$$

$$f(u) > M_1 + 4 \text{ for all } u \in X \setminus B(0, 2^{-1}M), \tag{2.301}$$

$$|f(z_1) - f(z_2)| \leq L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(0, 3M + 4), \tag{2.302}$$

$\delta_f, \delta_p \in [0, 1], \alpha > 0$ *satisfy*

$$\alpha \leq \min\{(L_0 + 1)^{-1}, L^{-2}\},\ \alpha \geq M\delta_p(6\bar{q} + 19),$$

$$\delta_p \leq (\bar{q} + 1)^{-1},\ \delta_f \leq (6M + L_0 + 1)^{-1}, \tag{2.303}$$

T *be a natural number and let*

$$\gamma_T = (4M^2T^{-1} + 12M\alpha(L_0 + 1)$$

$$+ 24M\delta_p(\bar{q} + 1)\bar{c}^{-1}\Delta^{-1})^{1/2} + \alpha(L_0 + 1). \tag{2.304}$$

Assume that $(\Omega_k, w_k) \in \mathcal{M}$, $k = 1, \ldots, T$, $\{x_k\}_{t=0}^{k} \subset X$, $\{\xi_k\}_{k=0}^{T-1} \subset X$,

$$x_0 \in B(0, M) \tag{2.305}$$

and that for all integers $k \in \{1, \ldots, T\}$,

$$B(\xi_{k-1}, \delta_f) \cap \partial f(x_{k-1}) \neq \emptyset, \tag{2.306}$$

$$x_k \in A(x_{k-1} - \alpha\xi_{k-1}, (\Omega_k, w_k), \delta_p), \tag{2.307}$$

$$y_{k,t} \in A_0(x_{k-1} - \alpha\xi_{k-1}, t, \delta_p),\ t \in \Omega_k, \tag{2.308}$$

$$\|x_k - \sum_{t\in\Omega_k} w_k(t)y_{k,t}\| \leq \delta_p \tag{2.309}$$

and that for every $t \in \Omega_k$, $y_i^{(k,t)} \in X$, $i = 0, \dots, p(t)$ satisfy

$$y_0^{(k,t)} = x_{k-1} - \alpha \xi_{k-1}, \tag{2.310}$$

$$\|y_i^{(k,t)} - P_{t_i}(y_{i-1}^{(k,t)})\| \le \delta_p,\ i = 1, \dots, p(t) \tag{2.311}$$

and

$$y_{k,t} = y_{p(t)}^{(k,t)}. \tag{2.312}$$

Then

$$\|x_t\| \le 3M,\ t = 0, \dots, T$$

and

$$\min\{\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p M(6\bar{q} + 19) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{t \in \Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2 - 12M\alpha(L_0 + 1)$$

$$-24M\delta_p(\bar{q} + 1)\} :\ k = 0, \dots, T-1\} \le 4M^2T^{-1}.$$

Moreover, if $k \in \{0, \dots, T-1\}$ and

$$\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p M(6\bar{q} + 19) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{t \in \Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2 - 12M\alpha(L_0 + 1)$$

$$-24M\delta_p(\bar{q} + 1)\} \le 4M^2T^{-1},$$

then

$$f(x_k) \le \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p M(6\bar{q} + 19) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1)$$

and

$$x_k \in \tilde{F}_{\gamma_T}.$$

Proof In view of (2.8), there exists

$$z \in B(0, M_*) \cap F. \tag{2.313}$$

By (2.305) and (2.313),

$$\|z - x_0\| \le M + M_* \le 2M. \tag{2.314}$$

Assume that $k \in \{0, \dots, T-1\}$ and that

$$\|z - x_k\| \le 2M. \tag{2.315}$$

By (2.302), (2.313) and (2.315),

$$\partial f(x_k) \subset B(0, L_0). \tag{2.316}$$

Equations (2.306) and (2.316) imply that

$$\|\xi_k\| \le L_0 + 1. \tag{2.317}$$

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$. By (2.310), (2.315) and (2.317),

$$\|z - y_0^{(k+1,t)}\| = \|z - x_k + \alpha\xi_k\| \le 2M + \alpha(L_0 + 1). \tag{2.318}$$

It follows from (2.1), (2.2), (2.311) and (2.313) that for all $i = 1, \dots, p(t)$,

$$\|z - y_i^{(k,t)}\| \le \|z - P_{t_i}(y_{i-1}^{(k+1,t)})\|$$

$$+\|P_{t_i}(y_{i-1}^{(k+1,t)}) - y_i^{(k+1,t)}\| \le \|z - y_{i-1}^{(k+1,t)}\| + \delta_p.$$

Thus we have shown that the following property holds:

(i) if $k \in \{0, \dots, T-1\}$ and (2.315) holds, then for all $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$ (2.318) is true and for all $i = 1, \dots, p(t)$,

$$\|z - y_0^{(k+1,t)}\| \le 2M + \alpha(L_0 + 1),$$

$$\|z - y_i^{(k+1,t)}\| \le \|z - y_{i-1}^{(k+1,t)}\| + \delta_p,$$

$$\|z - y_i^{(k+1,t)}\| \le \|z - y_0^{(k+1,t)}\| + \bar{q}\delta_p$$

$$\leq 2M + \alpha(L_0 + 1) + \delta_p \bar{q} \leq 2M + 2$$

(see (2.303)).

We show that for all $k \in \{0, \dots, T\}$, (2.315) holds. Assume that there exists an integer $n \in \{0, \dots, T\}$ such that

$$\|z - x_n\| > 2M. \tag{2.319}$$

By (2.314) and (2.319), $n > 0$. We may assume without loss of generality that (2.315) holds for all integers $k = 0, \dots, n-1$. In particular,

$$\|z - x_{n-1}\| \leq 2M. \tag{2.320}$$

In view of (2.313) and (2.320),

$$\|x_{n-1}\| \leq 2M + M_*. \tag{2.321}$$

By (2.302), (2.303), (2.307), (2.313) and (2.321), we apply Lemma 2.15 with

$$\delta_1 = \delta_f, \ \delta_2 = \delta_p, \ M_0 = 3M, \ (\Omega, w) = (\Omega_n, w_n),$$

$$x = x_{n-1}, \ \xi = \xi_{n-1}, \ y = x_n,$$

and obtain that

$$\alpha(f(x_{n-1}) - f(z))$$

$$\leq 2^{-1}\|x_{n-1} - z\|^2 - 2^{-1}\|x_n - z\|^2 + \delta_p(12M + 1)$$

$$+ 6\delta_p \bar{q} M + \delta_p(6M + 3) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{2.322}$$

There are two cases:

$$\|z - x_n\| \leq \|z - x_{n-1}\| \tag{2.323}$$

$$\|z - x_n\| > \|z - x_{n-1}\|. \tag{2.324}$$

Assume that (2.323) holds. Then in view of (2.320) and (2.323),

$$\|x_n - z\| \leq 2M.$$

This contradicts (2.319). The contradiction we have reached proves that (2.324) is true. By (2.300), (2.303), (2.313), (2.322) and (2.324),

$$\alpha(f(x_{n-1}) - f(z))$$

$$\leq \delta_p(18M + 4 + 6\bar{q}M) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1)$$

and

$$f(x_{n-1})$$

$$\leq M_1 + \alpha^{-1}\delta_p M(6\bar{q} + 19) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1) \leq M_1 + 3. \tag{2.325}$$

In view of (2.301) and (2.325),

$$\|x_{n-1}\| \leq M/2. \tag{2.326}$$

Property (i), (2.302), (2.303), (2.306), (2.310), (2.313) and (2.326) imply that

$$\|z - x_{n-1}\| \leq 2^{-1}M + M_*$$

and that for all $t \in (\Omega_n, w_n)$,

$$\|z - y_0^{(n,t)}\| \leq \|z - x_{n-1} - \alpha\xi_{n-1}\| \leq 2^{-1}M + M_* + \alpha(L_0 + 1),$$

$$\|z - y_{p(t)}^{(n,t)}\| \leq \|z - y_0^{(n,t)}\| + \bar{q}\delta_p \leq 2^{-1}M + M_* + 1 + \bar{q}\delta_p \leq 2^{-1}M + M_* + 2. \tag{2.327}$$

By (2.309), (2.312) and the convexity of the norm,

$$\|z - x_n\| \leq \|z - \sum_{t \in \Omega_n} w_n(t) y_{p(t)}^{(n,t)}\| + \|\sum_{t \in \Omega_n} w_n(t) y_{p(t)}^{(n,t)} - x_n\|$$

$$\leq \sum_{t \in \Omega_n} w_n(t)\|z - y_{p(t)}^{(n,t)}\| + \delta_p$$

$$\leq 2^{-1}M + M_* + 2 + \delta_p \leq 2M.$$

This contradicts (2.319). The contradiction we have reached proves that

$$\|z - x_t\| \leq 2M, \; t = 0, \ldots, T. \tag{2.328}$$

In view of (2.313) and (2.328),

$$\|x_t\| \leq 3M, \; t = 0, \ldots, T. \tag{2.329}$$

By (2.302), (2.306) and (2.329),

$$\partial f(x_t) \subset B(0, L_0), \; \|\xi_t\| \leq L_0 + 1, \; t = 0, \ldots, T - 1. \tag{2.330}$$

Let $k \in \{0, \dots, T-1\}$. By (2.302), (2.303), (2.306), (2.313) and (2.329), we apply Lemma 2.15 with

$$\delta_1 = \delta_f,\ \delta_2 = \delta_p,\ M_0 = 3M,\ (\Omega, w) = (\Omega_{k+1}, w_{k+1}),$$

$$x = x_k,\ \xi = \xi_k,\ y = x_{k+1},$$

and obtain that

$$\alpha(f(x_k) - f(z))$$
$$\leq 2^{-1}\|x_k - z\|^2 - 2^{-1}\|x_{k+1} - z\|^2 + \delta_p(12M + 1)$$
$$+ 6\delta_p \bar{q} M + \delta_p(6M + 3) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{2.331}$$

By (2.307), (2.313), (2.329) and (2.330), we apply Lemma 2.16 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ M_0 = 3M,\ (\Omega, w) = (\Omega_{k+1}, w_{k+1}),$$

$$x = x_k,\ x_0 = x_k - \alpha\xi_k,\ y = x_{k+1},\ y_t = y_{k+1,t},\ t \in \Omega_{k+1},$$

$$y_i^{(t)} = y_i^{(k+1,t)},\ t \in \Omega_{k+1},\ i = 0, \dots, p(t)$$

and obtain that

$$\|z - x_k\|^2 - \|z - x_{k+1}\|^2$$
$$\geq \bar{c}\Delta \sum_{t \in \Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2$$
$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1). \tag{2.332}$$

In view of (2.331),

$$\|x_k - z\|^2 - \|x_{k+1} - z\|^2$$
$$\geq 2\alpha(f(x_k) - f(z))$$
$$-2\delta_p(18M + 4 + 6\bar{q}M) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1).$$

Together with (2.314) and (2.332) this implies that

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{k=0}^{T-1} (\|z - x_k\|^2 - \|z - x_{k+1}\|^2)$$

$$\geq \sum_{k=0}^{T-1} \max\{2\alpha(f(x_k) - f(z))$$

$$-2\delta_p M(6\bar{q} + 19) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{t \in \Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2$$

$$-12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\}.$$

Since z is an arbitrary element of $B(0, M_*) \cap F$ (see (2.313)) it follows from the relation above and (2.11) that

$$4M^2 T^{-1} \geq \min\{\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p M(6\bar{q} + 19) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{t \in \Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2$$

$$-12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\} : \ k \in \{0, \ldots, T - 1\}\}.$$

Let $k \in \{0, \ldots, T - 1\}$ and

$$\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p M(6\bar{q} + 19) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{t \in \Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2$$

$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\} \leq 4M^2 T^{-1}. \tag{2.333}$$

In view of (2.333),

$$f(x_k) \leq \inf(f, F)$$

$$+ 2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p M(6\bar{q} + 19) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1) \quad (2.334)$$

By (2.333), for all $t \in \Omega_{k+1}$ and all $i \in \{1, \ldots, p(t)\}$,

$$\|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|$$

$$\leq ((4M^2T^{-1} + 12M\alpha(L_0 + 1) + 24M\delta_p(\bar{q} + 1))\bar{c}^{-1}\Delta^{-1})^{1/2} \quad (2.335)$$

and in view of (2.310) and (2.330),

$$\|y_i^{(k+1,t)} - x_k\| \leq \|y_i^{(k+1,t)} - y_0^{(k+1,t)}\| + \|y_0^{(k+1,t)} - x_k\|$$

$$\leq \alpha(L_0 + 1) + \bar{q}(4M^2T^{-1} + 12M\alpha(L_0 + 1) + 24M\delta_p(\bar{q} + 1)\bar{c}^{-1}\Delta^{-1})^{1/2}. \quad (2.336)$$

It follows from (2.311) and (2.355) that for all $t \in \Omega_{k+1}$ and all $i \in \{1, \ldots, p(t)\}$,

$$\|y_{i-1}^{(k+1,t)} - P_{t_i}(y_{i-1}^{(k+1,t)})\| \leq \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\| + \|y_i^{(k+1,t)} - P_{t_i}(y_{i-1}^{(k+1,t)})\|$$

$$\leq (4M^2T^{-1} + 12M\alpha(L_0 + 1) + 24M\delta_p(\bar{q} + 1)\bar{c}^{-1}\Delta^{-1})^{1/2} + \delta_p. \quad (2.337)$$

By (2.303), (2.304), (2.336) and (2.337),

$$x_k \in \tilde{F}_{\gamma_T}.$$

Theorem 2.17 is proved.

Let $\delta_f, \delta_p > 0$. In Theorem 2.17 they are fixed. Let us choose α, T. First, we choose α. Since T can be an arbitrary large we need to minimize the function

$$\alpha^{-1}\delta_p M(6\bar{q} + 19) + 2^{-1}\alpha L_0^2, \ \alpha > 0.$$

Its minimizer is

$$\alpha = L_0^{-1}(2\delta_p M(6\bar{q} + 19))^{1/2}.$$

Since α satisfies (2.303) we obtain the following restrictions on δ_p:

$$\delta_p \leq M^{-1}(6\bar{q} + 19)^{-1} \min\{(2L_0^2)^{-1}, 8^{-1}\}.$$

In this case

$$\gamma_T = \max\{(4M^2T^{-1} + 12M(L_0 + 1)L_0^{-1}(2\delta_p M(6\bar{q} + 19))^{1/2}$$

$$+24M\delta_p(\bar{q} + 1)\bar{c}^{-1}\Delta^{-1})^{1/2}\bar{q}$$

$$+(L_0+1)L_0^{-1}(2\delta_p M(6\bar{q}+19))^{1/2}\}.$$

We choose T with the same order as δ_p^{-1}. For example, $T = \lfloor \delta_p^{-1} \rfloor$. In this case in view of Theorem 2.17, there exists $k \in \{0, \ldots, T-1\}$ such that then

$$f(x_k) \le \inf(f, F) + c_1\delta_p^{1/2} + \delta_f(6M + L_0 + 1)$$

and

$$x_t \in \widehat{F}_{c_2\delta_p^{1/4}}$$

where c_1, c_2 are positive constants which depend on $M, L_0, \Delta, \bar{c}$.

Analogously to Theorem 2.9 we choose an approximate solution of our problem after T iterations.

2.12 The Second Theorem for the Dynamic String-Averaging Subgradient Algorithm

In the following theorem we assume that the set F is bounded.

Theorem 2.18 *Let* $r_0 \in (0, 1]$,

$$\tilde{F}_{r_0} \subset B(0, M_*), \tag{2.338}$$

$M \ge 2M_* + 2$, $L_0 \ge 1$,

$$|f(z_1) - f(z_2)| \le L_0\|z_1 - z_2\| \textit{ for all } z_1, z_2 \in B(0, 3M+4), \tag{2.339}$$

$\delta_f, \delta_p \in [0, 1]$, $\alpha > 0$ *satisfy*

$$\alpha \le (96\bar{q}^2)^{-1}(L_0+1)^{-1}r_0^2\bar{c}\Delta M^{-1},\ \delta_p \le 48^{-1}r_0^2\bar{c}\Delta M^{-1}(\bar{q}+1)^{-1}(2\bar{q})^{-2}, \tag{2.340}$$

T be a natural number and let

$$\gamma_T = \alpha(L_0+1) + \delta_p$$

$$+\bar{q}(4M^2T^{-1} + 12M\alpha(L_0+1) + 24M\delta_p(\bar{q}+1)\bar{c}^{-1}\Delta^{-1})^{1/2}. \tag{2.341}$$

Assume that $(\Omega_k, w_k) \in \mathcal{M}$, $k = 1, \ldots, T$, $\{x_k\}_{t=0}^{k} \subset X$, $\{\xi_k\}_{k=0}^{T-1} \subset X$,

$$x_0 \in B(0, M), \tag{2.342}$$

for all integers $k \in \{1, \dots, T\}$,

$$B(\xi_{k-1}, \delta_f) \cap \partial f(x_{k-1}) \neq \emptyset,$$

$$x_k \in A(x_{k-1} - \alpha\xi_{k-1}, (\Omega_k, w_k), \delta_p), \tag{2.343}$$

$$y_{k,t} \in A_0(x_{k-1} - \alpha\xi_{k-1}, t, \delta_p),\ t \in \Omega_k, \tag{2.344}$$

$$\|x_k - \sum_{t\in\Omega_k} w_k(t)y_{k,t}\| \le \delta_p \tag{2.345}$$

and that for every $t \in \Omega_k$, $y_i^{(k,t)} \in X$, $i = 0, \dots, p(t)$ *satisfy*

$$y_0^{(k,t)} = x_{k-1} - \alpha\xi_{k-1}, \tag{2.346}$$

$$\|y_i^{(k,t)} - P_{t_i}(y_{i-1}^{(k,t)})\| \le \delta_p,\ i = 1, \dots, p(t) \tag{2.347}$$

and

$$y_{k,t} = y_{p(t)}^{(k,t)}. \tag{2.348}$$

Then

$$\|x_t\| \le 3M,\ t = 0, \dots, T$$

and

$$\min\{\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p M(6\bar{q} + 19) - \alpha^2 L_0^2 - 2\alpha(6M + L_1 + 1),$$

$$\bar{c}\Delta \sum_{t\in\Omega_k+1} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2 - 12M\alpha(L_0 + 1)$$

$$-24M\delta_p(\bar{q} + 1)\} :\ k = 0, \dots, T - 1\} \le 4M^2T^{-1}.$$

Moreover, if $k \in \{0, \dots, T - 1\}$ *and*

$$\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p M(6\bar{q} + 19) - \alpha^2 L_0^2 - 2\alpha(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{t\in\Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2 - 12M\alpha(L_0+1)$$

$$-12M\delta_p(\bar{q}+1)\} \le 4M^2T^{-1},$$

then

$$f(x_k) \le \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p M(6\bar{q}+19) + 2^{-1}\alpha L_0^2 + \delta_f(6M+L_0+1)$$

and

$$x_k \in \tilde{F}_{\gamma_T}.$$

Proof In view of (2.8), there exists

$$z \in B(0, M_*) \cap F. \tag{2.349}$$

By (2.342) and (2.349),

$$\|z - x_0\| \le M + M_* \le 2M. \tag{2.350}$$

Assume that $k \in \{0, \dots, T-1\}$ and that

$$\|z - x_k\| \le 2M. \tag{2.351}$$

By (2.339), (2.349) and (2.351),

$$\partial f(x_k) \subset B(0, L_0). \tag{2.352}$$

Equations (2.343) and (2.352) imply that

$$\|\xi_k\| \le L_0 + 1. \tag{2.353}$$

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$. By (2.346), (2.351) and (2.353),

$$\|z - y_0^{(k+1,t)}\| = \|z - x_k + \alpha\xi_k\| \le 2M + \alpha(L_0+1). \tag{2.354}$$

It follows from (2.1), (2.2), (2.347) and (2.349) that for all $i = 1, \dots, p(t)$,

$$\|z - y_i^{(k+1,t)}\| \le \|z - P_{t_i}(y_{i-1}^{(k+1,t)})\|$$

$$+\|P_{t_i}(y_{i-1}^{(k+1,t)}) - y_i^{(k+1,t)}\| \le \|z - y_{i-1}^{(k+1,t)}\| + \delta_p.$$

Thus we have shown that the following property holds:

(i) is $k \in \{0, \ldots, T-1\}$ and (2.351) holds, then (2.352) and (2.353) are true, for all $t = (t_1, \ldots, t_{p(t)}) \in \Omega_{k+1}$ (2.354) is true and for all $i = 1, \ldots, p(t)$,

$$\|z - y_i^{(k+1,t)}\| \le \|z - y_{i-1}^{(k+1,t)}\| + \delta_p,$$

$$\|z - y_i^{(k+1,t)}\| \le \|z - y_0^{(k+1,t)}\| + \bar{q}\delta_p.$$

We show that for all $k \in \{0, \ldots, T\}$, (2.351) holds. Assume that there exists an integer $n \in \{0, \ldots, T\}$ such that

$$\|z - x_n\| > 2M. \tag{2.355}$$

By (2.350) and (2.355), $n > 0$. We may assume without loss of generality that (2.351) holds for all integers $k = 0, \ldots, n-1$. In particular,

$$\|z - x_{n-1}\| \le 2M. \tag{2.356}$$

By property (i), (2.340), (2.349) and (2.356), we apply Lemma 2.16 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ M_0 = 3M,\ (\Omega, w) = (\Omega_n, w_n),$$

$$x = x_{n-1},\ x_0 = x_{n-1} - \alpha\xi_{n-1},\ y_n = x_n,\ y_t = y_{n,t},\ t \in \Omega_n,$$

$$y_i^{(t)} = y_i^{(n,t)},\ t \in \Omega_n,\ i = 0, \ldots, p(t)$$

and obtain that

$$\|z - x_{n-1}\|^2 - \|z - x_n\|^2$$

$$\ge \bar{c}\Delta \sum_{t \in \Omega_n} \sum_{i=1}^{p(t)} \|y_{i-1}^{(n,t)} - y_i^{(n,t)}\|^2$$

$$-12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1). \tag{2.357}$$

If

$$\|z - x_n\| \le \|z - x_{n-1}\|,$$

then in view of (2.356),

$$\|z - x_n\| \le 2M.$$

This contradicts (2.355). The contradiction we have reached proves that

$$\|z - x_{n-1}\| < \|z - x_n\|.$$

Together with (2.340), (2.347) and (2.357) this implies that for all $t \in \Omega_n$ and all $i \in \{1, \dots, p(t)\}$,

$$\|y_{i-1}^{(n,t)} - y_i^{(n,t)}\| \le ((12M\alpha(L_0 + 1) + 24M\delta_p(\bar{q} + 1))(\bar{c}\Delta)^{-1})^{1/2} \le r_0(2\bar{q}), \tag{2.358}$$

$$\|y_{i-1}^{(n,t)} - P_{t_i}(y_{i-1}^{(n,t)})\|$$

$$\le \|y_{i-1}^{(n,t)} - y_i^{(n,t)}\| + P_{t_i}(y_{i-1}^{(n,t)}) - y_i^{(n,t)}\| \le r_0(2\bar{q}) + \delta_p. \tag{2.359}$$

Property (i), (2.340), (2.346), (2.353) and (2.359) imply that for all $t \in \Omega_n$ and all $i \in \{0, 1, \dots, p(t)\}$,

$$\|x_{n-1} - y_i^{(n,t)}\| \le \|x_{n-1} - y_0^{(n,t)}\| + \|y_0^{(n,t)} - y_i^{(n,t)}\|$$

$$\le \alpha(L_0 + 1) + i(r_0(2\bar{q}) + \delta_p)$$

$$\le \alpha(L_0 + 1) + r_0/2 + \bar{q}\delta_p \le r_0. \tag{2.360}$$

In view of (2.359) and (2.360),

$$x_{n-1} \in \tilde{F}_{r_0}. \tag{2.361}$$

By (2.338) and (2.361),

$$\|x_{n-1}\| \le M_*. \tag{2.362}$$

It follows from (2.340), (2.345), (2.348) and (2.360) that

$$\|x_{n-1} - x_n\| \le \|x_{n-1} - \sum_{t \in \Omega_n} w_n(t) y_{n,t}\| + \delta_p$$

$$\le \sum_{t \in \Omega_k} w_n(t)\|x_{n-1} - y_{n,t}\| + \delta_p$$

$$\le \alpha(L_0 + 1) + r_0/2 + \bar{q}\delta_p + \delta_p \le r_0 \le 1.$$

Combined with (2.338) and (2.349) this implies that

$$\|x_n\| \le M_* + 1,$$

$$\|z - x_n\| \le M_* + 1 + M_* < M.$$

This contradicts (2.355). The contradiction we have reached proves that (2.351) holds for all $t = 0, \dots, T$. Together with (2.349) this implies that

$$\|x_t\| \le 3M, \ k = 0, \dots, T. \tag{2.363}$$

Let $k \in \{0, \dots, T-1\}$. By (2.339), (2.340), (2.343), (2.349) and (2.363), we apply Lemma 2.15 with

$$\delta_1 = \delta_f, \ \delta_2 = \delta_p, \ M_0 = 3M, \ (\Omega, w) = (\Omega_k, w_k),$$

$$x = x_k, \ \xi = \xi_k, \ y = x_{k+1},$$

and obtain that

$$\alpha(f(x_k) - f(z))$$

$$\le 2^{-1}\|x_k - z\|^2 - 2^{-1}\|x_{k+1} - z\|^2 + \delta_p(12M + 1)$$

$$+ 6\delta_p \bar{q} M + \delta_p(6M + 3) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{2.364}$$

By (2.340), (2.343)–(2.345), (2.348), (2.349), (2.353) and (2.363), we apply Lemma 2.16 with

$$\delta_1 = \alpha(L_0 + 1), \ \delta_2 = \delta_p, \ M_0 = 3M, \ (\Omega, w) = (\Omega_{k+1}, w_{k+1}),$$

$$x = x_k, \ x_0 = x_k - \alpha\xi_k, \ y_t = x_{k+1}, \ y_t = y_{k+1,t}, \ t \in \Omega_{k+1},$$

$$y_i^{(t)} = y_i^{(k+1,t)}, \ t \in \Omega_{k+1}, \ i = 0, \dots, p(t)$$

and obtain that

$$\|z - x_k\|^2 - \|z - x_{k+1}\|^2$$

$$\ge \bar{c}\Delta \sum_{t \in \Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2$$

$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1). \tag{2.365}$$

In view of (2.364),

$$\|x_k - z\|^2 - \|x_{k+1} - z\|^2$$

$$\geq 2\alpha(f(x_k) - f(z))$$

$$-2\delta_p(18M + 4 + 6\bar{q}M) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1). \tag{2.366}$$

It follows from (2.350), (2.365) and (2.366) that

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{k=0}^{T-1} (\|z - x_k\|^2 - \|z - x_{k+1}\|^2)$$

$$\geq \sum_{k=0}^{T-1} \max\{2\alpha(f(x_k) - f(z))$$

$$-2\delta_p M(6\bar{q} + 19) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{t \in \Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2$$

$$-12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\}. \tag{2.367}$$

Since z is an arbitrary element of $B(0, M_*) \cap F$ it follows from (2.11) and (2.367) that

$$4M^2 T^{-1} \geq \min\{\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p M(6\bar{q} + 19) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{t \in \Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2$$

$$-12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\} : \ k \in \{0, \dots, T - 1\}.$$

Let $k \in \{0, \dots, T - 1\}$ and

$$\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p M(6\bar{q} + 19) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{t\in\Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2$$

$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\} \le 4M^2T^{-1}. \tag{2.368}$$

In view of (2.368),

$$f(x_k) \le \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p M(6\bar{q} + 19) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1).$$

By (2.346), (2.353), (2.363) and (2.368), for all $t \in \Omega_{k+1}$ and all $i \in \{1, \dots, p(t)\}$,

$$\|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|$$

$$\le (4M^2T^{-1} + 12M\alpha(L_0 + 1) + 24M\delta_p(\bar{q} + 1)\bar{c}^{-1}\Delta^{-1})^{1/2} \tag{2.369}$$

and

$$\|y_{i-1}^{(k+1,t)} - x_k\| \le \|y_i^{(k+1,t)} - y_0^{(k+1,t)}\| + \|y_0^{(k+1,t)} - x_k\|$$

$$\le \alpha(L_0 + 1) + \bar{q}(4M^2T^{-1} + 12M\alpha(L_0 + 1)$$

$$+ 24M\delta_p(\bar{q} + 1)\bar{c}^{-1}\Delta^{-1})^{1/2}. \tag{2.370}$$

It follows from (2.347) and (2.369) that for all $t \in \Omega_{k+1}$ and all $i \in \{1, \dots, p(t)\}$,

$$\|y_{i-1}^{(k+1,t)} - P_{t_i}(y_{i-1}^{(k+1,t)})\| \le \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\| + \|y_i^{(k+1,t)} - P_{t_i}y_{i-1}^{(k+1,t)}\|$$

$$\le (4M^2T^{-1} + 12M\alpha(L_0 + 1) + 24M\delta_p(\bar{q} + 1)\bar{c}^{-1}\Delta^{-1})^{1/2} + \delta_p. \tag{2.371}$$

By (2.341), (2.370) and (2.341),

$$x_k \in \tilde{F}_{\gamma_T}.$$

Theorem 2.18 is proved.

Let $\delta_f, \delta_p > 0$. In Theorem 2.18 they are fixed. As in the case of Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations.

Chapter 3
Proximal Point Subgradient Algorithm

In this chapter we consider a minimization of a convex function on an intersection of two sets in a Hilbert space. One of them is a common fixed point set of a finite family of quasi-nonexpansive mappings while the second one is a common zero point set of finite family of maximal monotone operators. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We use the Cimmino proximal point subgradient algorithm, the iterative proximal point subgradient algorithm and the dynamic string-averaging proximal point subgradient algorithm and show that each of them generates a good approximate solution, if the sequence of computational errors is bounded from above by a small constant. Moreover, if we known computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this.

3.1 Preliminaries

Let $(X, \langle\cdot,\cdot\rangle)$ be a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ which induces a complete norm $\|\cdot\|$.

A multifunction $T : X \to 2^X$ is called a monotone operator if and only f

$$\langle z - z', w - w'\rangle \ \geq 0 \quad \forall z, z', w, w' \in X$$

$$\text{such that } w \in T(z) \text{ and } w' \in T(z'). \tag{3.1}$$

It is called maximal monotone if, in addition, the graph

$$\{(z, w) \in X \times X : \ w \in T(z)\}$$

A. J. Zaslavski, *Optimization on Solution Sets of Common Fixed Point Problems*,
Springer Optimization and Its Applications 178,
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is not properly contained in the graph of any other monotone operator $T' : X \to 2^X$. A fundamental problem consists in determining an element z such that $0 \in T(z)$. For example, if T is the subdifferential ∂f of a lower semicontinuous convex function $f : X \to (-\infty, \infty]$, which is not identically infinity, then T is maximal monotone (see [60, 63]), and the relation $0 \in T(z)$ means that z is a minimizer of f.

Let $T : X \to 2^X$ be a maximal monotone operator. The proximal point algorithm generates, for any given sequence of positive real numbers and any starting point in the space, a sequence of points and the goal is to show the convergence of this sequence. Note that in a general infinite-dimensional Hilbert space this convergence is usually weak. The proximal algorithm for solving the inclusion $0 \in T(z)$ is based on the fact established by Minty [59], who showed that, for each $z \in X$ and each $c > 0$, there is a unique $u \in X$ such that

$$z \in (I + cT)(u),$$

where $I : X \to X$ is the identity operator ($Ix = x$ for all $x \in X$).

The operator

$$P_{c,T} := (I + cT)^{-1} \tag{3.2}$$

is therefore single-valued from all of X onto X (where c is any positive number). It is also nonexpansive:

$$\|P_{c,T}(z) - P_{c,T}(z')\| \le \|z - z'\| \text{ for all } z, z' \in X \tag{3.3}$$

and

$$P_{c,T}(z) = z \text{ if and only if } 0 \in T(z). \tag{3.4}$$

Following the terminology of Moreau [63] $P_{c,T}$ is called the proximal mapping associated with cT.

The proximal point algorithm generates, for any given sequence $\{c_k\}_{k=0}^{\infty}$ of positive real numbers and any starting point $z^0 \in X$, a sequence $\{z^k\}_{k=0}^{\infty} \subset X$, where

$$z^{k+1} := P_{c_k,T}(z^k),\ k = 0, 1, \ldots$$

It is not difficult to see that the

$$\text{graph}(T) := \{(x, w) \in X \times X :\ w \in T(x)\}$$

is closed in the norm topology of $X \times X$.

Set

$$F(T) = \{z \in X : \ 0 \in T(z)\}.$$

Usually algorithms considering in the literature generate sequences which converge weakly to an element of $F(T)$.

For every point $x \in X$ and every nonempty set $A \subset X$ define

$$d(x, A) := \inf\{\|x - y\| : \ y \in A\}.$$

For every point $x \in X$ and every positive number r put

$$B(x, r) = \{y \in X : \ \|x - y\| \leq r\}.$$

We denote by $\mathrm{Card}(A)$ the cardinality of the set A.

Let $\mathcal{L}_1$ be a finite set of maximal monotone operators $T : X \to 2^X$ and $\mathcal{L}_2$ be a finite set of mappings $T : X \to X$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

Let $\bar{c} \in (0, 1]$ and let $\bar{c} = 1$, if $\mathcal{L}_2 = \emptyset$.

We suppose that

$$F(T) = \{z \in X : \ 0 \in T(z)\} \neq \emptyset \text{ for any } T \in \mathcal{L}_1 \tag{3.5}$$

and that for every mapping $T \in \mathcal{L}_2$,

$$\mathrm{Fix}(T) := \{z \in X : \ T(z) = z\} \neq \emptyset, \tag{3.6}$$

$$\|z - x\|^2 \geq \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2 \tag{3.7}$$

$$\text{for all } x \in X \text{ and all } z \in \mathrm{Fix}(T).$$

Let $M_* > 0$,

$$F := (\cap_{T \in \mathcal{L}_1} F(T)) \cap (\cap_{Q \in \mathcal{L}_2} \mathrm{Fix}(Q)) \neq \emptyset \tag{3.8}$$

and

$$F \cap B(0, M_*) \neq \emptyset. \tag{3.9}$$

Let $\epsilon > 0$. For every monotone operator $T \in \mathcal{L}_1$ define

$$F_\epsilon(T) = \{x \in X : \ T(x) \cap B(0, \epsilon) \neq \emptyset\} \tag{3.10}$$

and for every mapping $T \in \mathcal{L}_2$ set

$$\mathrm{Fix}_\epsilon(T) = \{x \in X : \ \|T(x) - x\| \leq \epsilon\}. \tag{3.11}$$

Define

$$F_\epsilon = (\cap_{T\in\mathcal{L}_1} F_\epsilon(T)) \cap (\cap_{Q\in\mathcal{L}_2}\mathrm{Fix}_\epsilon(Q)), \tag{3.12}$$

$$\tilde{F}_\epsilon = (\cap_{T\in\mathcal{L}_1}\{x \in X:\ d(x, F_\epsilon(T)) \le \epsilon\})$$

$$\cap\, (\cap_{Q\in\mathcal{L}_2}\{x \in X:\ d(x, \mathrm{Fix}_\epsilon(Q)) \le \epsilon\}). \tag{3.13}$$

Let $f : X \to R^1$ be a convex continuous function. We consider the minimization problem

$$f(x) \to \min,\ x \in F.$$

Assume that

$$\inf(f, F) = \inf(f, F \cap B(0, M_*)). \tag{3.14}$$

Let $\bar\lambda > 0$ and let $\bar\lambda = \infty$ and $\bar\lambda^{-1} = 0$, if $\mathcal{L}_1 = \emptyset$.
Recall that the sum over an empty set is zero.
Fix $\alpha > 0$.
Let us describe our algorithm.

Cimmino Proximal Point Subgradient Algorithm
Initialization: select an arbitrary $x_0 \in X$.
Iterative step: given a current iteration vector $x_k \in X$ pick $c(T) \ge \bar\lambda$, $T \in \mathcal{L}_1$ and $w : \mathcal{L}_1 \cup \mathcal{L}_2 \to (0, \infty)$ such that

$$\sum\{w(S):\ S \in \mathcal{L}_1 \cup \mathcal{L}_2\} = 1,$$

$$l_k \in \partial f(x_k)$$

and define the next iteration vector

$$x_{k+1} = \sum_{S\in\mathcal{L}_2} w(S)S(x_k - \alpha l_k) + \sum_{S\in\mathcal{L}_1} w(S)P_{c(S),S}(x_k - \alpha l_k).$$

Fix

$$\Delta \in (0, \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2)^{-1}]. \tag{3.15}$$

In this chapter the Cimmino proximal point subgradient algorithm is studied under the presence of computational errors.

We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f while $\delta_p \in$

[0, 1] is a computational error produced by our computer system, when we calculate the operators $P_{c,S}$, $S \in \mathcal{L}_1$, $c \geq \bar{\lambda}$ and $S \in \mathcal{L}_2$. Let $\alpha > 0$ be a step size.

Cimmino Proximal Point subgradient Algorithm with Computational Errors

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ pick $c(T) \geq \bar{\lambda}$, $T \in \mathcal{L}_1$ and $w : \mathcal{L}_1 \cup \mathcal{L}_2 \to [\Delta, \infty)$ such that

$$\sum\{w(S) : S \in \mathcal{L}_1 \cup \mathcal{L}_2\} = 1,$$

calculate

$$l_k \in \partial f(x_k) + B(0, \delta_f)$$

and

$$y_{k,S} \in B(S(x_k - \alpha\xi_k), \delta_p), \; S \in \mathcal{L}_2,$$

$$y_{k,S} \in B(P_{c_k(S),S}(x_k - \alpha\xi_k), \delta_p), \; S \in \mathcal{L}_1$$

and calculate the next iteration vector $x_{k+1} \in X$ satisfying

$$\|x_{k+1} - \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} w(S) y_{k,S}\| \leq \delta_p.$$

3.2 Auxiliary Results

In order to study the behavior of our algorithm we need auxiliary results presented in this section.

It is easy to see that the following lemma holds.

Lemma 3.1 *Let $z, x_0, x_1 \in X$. Then*

$$2^{-1}\|z - x_0\|^2 - 2^{-1}\|z - x_1\|^2 - 2^{-1}\|x_0 - x_1\|^2 = \langle x_0 - x_1, x_1 - z\rangle.$$

Lemma 3.2 *Assume that*

$$z \in F, \tag{3.16}$$

the integers p, q satisfy $0 \leq p < q$,

$$\{S_k\}_{k=p}^{q-1} \subset \mathcal{L}_2 \cup \{P_{c,T} : c \geq \bar{\lambda}, \; T \in \mathcal{L}_1\},$$

$$\{\epsilon_k\}_{k=p}^{q-1} \subset (0, \infty), \ \{x_k\}_{k=p}^{q} \subset X$$

and that for all integers $k \in \{p, \dots, q-1\}$,

$$\|x_{k+1} - S_k(k)(x_k)\| \le \epsilon_k. \tag{3.17}$$

Then, for every integer $k \in \{p+1 \dots, q\}$ *the following inequality holds:*

$$\|z - x_k\| \le \|z - x_p\| + \sum_{i=p}^{k-1} \epsilon_i.$$

Proof Let an integer $k \in \{p, \dots, q-1\}$. By (3.3)–(3.8), (3.16) and (3.17),

$$\|z - x_{k+1}\| \le \|z - S_k(x_k)\| + \|S_k(x_k) - x_{k+1}\| \le \|z - x_k\| + \epsilon_k.$$

This implies the validity of Lemma 3.2.

Lemma 3.3 *Let*

$$A \in \mathcal{L}_2 \cup \{P_{c,T} : \ T \in \mathcal{L}_1, \ c \in [\bar{\lambda}, \infty)\}, \ x \in X, \tag{3.18}$$

$$x \in F. \tag{3.19}$$

Then

$$\|z - x\|^2 - \|z - A(x)\|^2 - \bar{c}\|x - A(x)\|^2 \ge 0. \tag{3.20}$$

Proof There are two cases:

(i) $T \in \mathcal{L}_2$;
(ii) there exist a mapping $T \in \mathcal{L}_1$,a number $c \in [\bar{\lambda}, \infty)$ such that $A = P_{c,T}$.

If (i) holds, then (3.20) follows from (3.7) and (3.17). Assume that (ii) holds. Then by Lemma 3.2,

$$2^{-1}\|z - x\|^2 - 2^{-1}\|z - A(x)\|^2 - 2^{-1}\|x - A(x)\|^2$$
$$= \langle x - A(x), A(x) - z\rangle. \tag{3.21}$$

By (ii) and (3.2),

$$A(x) = P_{c,T}(x) \text{ and } x \in (I + cT)(A(x)),$$

$$x - A(x) \in cT(A(x)). \tag{3.22}$$

By (3.1), (3.18) and (3.22), and Equation (3.20) holds. Lemma 3.3 is proved.

Lemma 3.4 *Let* $x \in X$, $T \in \mathcal{L}_1$, $c \geq \bar{\lambda}$, $\gamma > 0$ *and*

$$\|x - P_{c,T}(x)\| \leq \gamma. \tag{3.23}$$

Then $B(x, \gamma) \cap F_{\gamma\bar{\lambda}^{-1}}(T) \neq \emptyset$.

Proof Set $A = P_{c,T}$. Then we have

$$A(x) = P_{c,T}(x) = (I + cT)^{-1}(x),$$

$$x \in (I + cT)(A(x)),$$

$$x - A(x) \in cT(A(x)),$$

$$c^{-1}(x - A(x)) \in T(A(x)). \tag{3.24}$$

In view of (3.22),

$$\|c^{-1}(x - A(x))\| \leq \gamma\bar{\lambda}^{-1}. \tag{3.25}$$

By (3.24) and (3.25),

$$A(x) \in F_{\gamma\bar{\lambda}^{-1}}(T).$$

Lemma 3.4 is proved.

Lemma 3.5 *Let* $M_0 \geq M_*$, $\delta_1, \delta_2 \in [0, 1]$,

$$w : \mathcal{L}_1 \cup \mathcal{L}_2 \to [\Delta, \infty), \tag{3.26}$$

$$\sum\{w(S) : \ S \in \mathcal{L}_1 \cup \mathcal{L}_2\} = 1, \tag{3.27}$$

$$c(T) \geq \bar{l}, \ T \in \mathcal{L}_1, \tag{3.28}$$

$$z \in F \cap B(0, M_0), \tag{3.29}$$

$$x \in B(0, M_0), \tag{3.30}$$

$$x_0 \in B(x, \delta_1), \tag{3.31}$$

$y_S \in X$, $S \in \mathcal{L}_1 \cup \mathcal{L}_2$ *satisfy*

$$\|y_S - S(x_0)\| \leq \delta_2, \ S \in \mathcal{L}_2, \tag{3.32}$$

$$\|y_S - P_{c(S),S}(x_0)\| \leq \delta_2, \ S \in \mathcal{L}_1, \tag{3.33}$$

and let

$$y \in B(\sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} w(S) y_S, \delta_2). \tag{3.34}$$

Then

$$\|z - x\|^2 - \|z - y\|^2 \geq \Delta \bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_0 - y_S\|^2$$

$$-\delta_1(4M_0 + 1) - 2\delta_2(4M_0 + 4) - \delta_2(8M_0 + 5).$$

Proof In view of (3.28) and Lemma 3.3, for all

$$A \in \mathcal{L}_2 \cup \{P_{c,T} : T \in \mathcal{L}_1, \; c \in [\bar{\lambda}, \infty)\},$$

we have

$$\|z - x_0\|^2 \geq \|z - A(x_0)\|^2 - \bar{c}\|x_0 - A(x_0)\|^2, \tag{3.35}$$

$$\|z - A(x_0)\| \leq \|z - x_0\|. \tag{3.36}$$

Since the function $u \to \|u - z\|^2$, $u \in X$ is convex we have

$$\|z - (\sum_{S \in \mathcal{L}_2} w(S) S(x_0) + \sum_{S \in \mathcal{L}_1} w(S) P_{c(S),S}(x_0))\|^2$$

$$\leq \sum_{S \in \mathcal{L}_2} w(S)\|z - S(x_0)\|^2 + \sum_{S \in \mathcal{L}_1} w(S)\|z - P_{c(S),S}(x_0)\|^2. \tag{3.37}$$

By (3.26), (3.27) and (3.35)–(3.37),

$$\|z - x_0\|^2 - \|z - (\sum_{S \in \mathcal{L}_2} w(S) S(x_0) + \sum_{S \in \mathcal{L}_1} w(S) P_{c(S),S}(x_0))\|^2$$

$$\geq \sum_{S \in \mathcal{L}_2} (\|z - x_0\|^2 - \|z - S(x_0)\|^2) w(S)$$

$$+ \sum_{S \in \mathcal{L}_1} (\|z - x_0\|^2 - \|z - P_{c(S),S}(x_0)\|^2) w(S)$$

$$\geq \Delta \bar{c} \sum_{S \in \mathcal{L}_2} \|x_0 - S(x_0)\|^2 + \Delta \bar{c} \sum_{S \in \mathcal{L}_1} \|x_0 - P_{c(S),S}(x_0)\|^2. \tag{3.38}$$

In view of (3.28), (3.29) and (3.31),

$$\|z - x\| \le 2M_0, \ \|z - x_0\| \le 2M_0 + 1. \tag{3.39}$$

By (3.31) and (3.39),

$$\begin{aligned} &|\|z - x\|^2 - \|z - x_0\|^2| \\ &\le |\|z - x\| - \|z - x_0\||(\|z - x\| + \|z - x_0\|) \le \delta_1(4M_0 + 1). \end{aligned} \tag{3.40}$$

It follows from the convexity of the norm, (3.26), (3.27), (3.35) and (3.39) that

$$\begin{aligned} &\|z - (\sum_{S \in \mathcal{L}_2} w(S)S(x_0) + \sum_{S \in \mathcal{L}_1} w(S)P_{c(S),S}(x_0))\| \\ &\le \sum_{S \in \mathcal{L}_2} \|z - S(x_0)\|w(S) + \sum_{S \in \mathcal{L}_1} \|z - P_{c(S),S}(x_0)\|w(S) \\ &\le \|z - x_0\| \le 2M_0 + 1. \end{aligned} \tag{3.41}$$

In view of (3.26), (3.27), (3.32) and (3.34),

$$|\|z - y\| - \|z - (\sum_{S \in \mathcal{L}_2} w(S)S(x_0) + \sum_{S \in \mathcal{L}_1} w(S)P_{c(S),S}(x_0))\|| \le 2\delta_2. \tag{3.42}$$

Equations (3.41) and (3.42) imply that

$$|\|z - y\|^2 - \|z - (\sum_{S \in \mathcal{L}_2} w(S)S(x_0) + \sum_{S \in \mathcal{L}_1} w(S)P_{c(S),S}(x_0))\|^2 \le 2\delta_2(4M_0 + 4). \tag{3.43}$$

By (3.38), (3.40) and (3.43),

$$\begin{aligned} &\|z - x\|^2 - \|z - y\|^2 \\ &\ge \|z - x_0\|^2 - \delta_1(4M_0 + 1) \\ &-\|z - \sum_{S \in \mathcal{L}_2} w(S)S(x_0) - \sum_{S \in \mathcal{L}_1} w(S)P_{c(S),S}(x_0)\|^2 - 2\delta_2(4M_0 + 4) \\ &\ge \Delta\bar{c} \sum_{S \in \mathcal{L}_2} \|x_0 - S(x_0)\|^2 + \Delta\bar{c} \sum_{S \in \mathcal{L}_1} \|x_0 - P_{c(S),S}(x_0)\|^2 \\ &- \delta_1(4M_0 + 1) - 2\delta_2(4M_0 + 4). \end{aligned} \tag{3.44}$$

It follows from (3.36) and (3.39) that for every $S \in \mathcal{L}_2$,

$$\|z - S(x_0)\| \le \|z - x_0\| \le 2M_0 + 1,$$

$$\|x_0 - S(x_0)\| \le 4M_0 + 2, \tag{3.45}$$

and that for every $S \in \mathcal{L}_1$,

$$\|z - P_{c(S),S}(x_0)\| \le \|z - x_0\| \le 2M_0 + 1,$$

$$\|x_0 - P_{c(S),S}(x_0)\| \le 4M_0 + 2. \tag{3.46}$$

By (3.32), (3.33), (3.45) and (3.46), for every $S \in \mathcal{L}_2$,

$$||\|x_0 - S(x_0)\|^2 - \|x_0 - y_S\|^2| \le \delta_2(\|x_0 - S(x_0)\| + \|x_0 - y_S\|)$$

$$\le \delta_2(2\|x_0 - S(x_0)\| + \delta_2) \le \delta_2(8M_0 + 5) \tag{3.47}$$

and for every $S \in \mathcal{L}_1$,

$$||\|x_0 - P_{c(S),S}(x_0)\|^2 - \|x_0 - y_S\|^2| \le \delta_2(\|x_0 - P_{c(S),S}(x_0)\| + \|x_0 - y_S\|)$$

$$\le \delta_2(2\|x_0 - P_{c(S),S}(x_0)\| + \delta_2) \le \delta_2(8M_0 + 5). \tag{3.48}$$

It follows from (3.26), (3.27), (3.44), (3.47) and (3.48) that

$$\|z - x\|^2 - \|z - y\|^2$$

$$\ge \Delta\bar{c} \sum_{S \in \mathcal{L}_2} \|x_0 - y_S\|^2 + \Delta\bar{c} \sum_{S \in \mathcal{L}_1} \|x_0 - y_S\|^2$$

$$-\delta_1(4M_0 + 1) - 2\delta_2(4M_0 + 4) - \bar{c}\delta_2 \sum_{S \in \mathcal{L}_2} w(S)(8M_0 + 5)$$

$$-\bar{c}\delta_2 \sum_{S \in \mathcal{L}_1} w(S)(8M_0 + 5)$$

$$= \Delta\bar{c} \sum_{S \in \mathcal{L}_2 \cup \mathcal{L}_1} \|x_0 - y_S\|^2$$

$$-\delta_1(4M_0 + 1) - 2\delta_2(4M_0 + 4) - \bar{c}\delta_2(8M_0 + 5).$$

Lemma 3.5 is proved.

3.3 The First Result for the Cimmino Proximal Point Subgradient Algorithm

In the following theorem we assume that the objective function f satisfies the coercivity growth condition.

Theorem 3.6 *Let the function f be Lipschitz on bounded subsets of X,*

$$\lim_{\|x\| \to \infty} f(x) = \infty,$$

$M \geq 2M_* + 6$, $L_0 \geq 1$,

$$M_1 > \sup\{|f(u)| : u \in B(0, M_* + 4)\} + 4, \tag{3.49}$$

$$f(u) > M_1 + 4 \textit{ for all } u \in X \setminus B(0, 2^{-1}M), \tag{3.50}$$

$$|f(z_1) - f(z_2)| \leq L_0 \|z_1 - z_2\| \textit{ for all } z_1, z_2 \in B(0, 3M + 4), \tag{3.51}$$

$\delta_f, \delta_p \in [0, 1]$, $\alpha > 0$ *satisfy*

$$\alpha \leq \min\{L_0^{-2}, (L_0 + 1)^{-1}\}, \ \alpha \geq 2\delta_p(6M + 3),$$

$$\delta_f \leq (6M + L_0 + 1)^{-1}, \tag{3.52}$$

T be a natural number and let

$$\gamma_T = (4M^2T^{-1} + \alpha(L_0 + 1)(12M + 1) + \delta_p(12M + 13)(\Delta\bar{c})^{-1})^{1/2} + \delta_p. \tag{3.53}$$

Assume that for all $t = 1, \ldots, T$,

$$w_t : \mathcal{L}_1 \cup \mathcal{L}_2 \to [\Delta, \infty), \tag{3.54}$$

$$\sum\{w_t(S) : S \in \mathcal{L}_1 \cup \mathcal{L}_2\} = 1, \tag{3.55}$$

$$c(S) \geq \bar{\lambda}, \ S \in \mathcal{L}_1, \tag{3.56}$$

$\{x_t\}_{t=0}^T \subset X$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$x_0 \in B(0, M) \tag{3.57}$$

and that for all integers $t \in \{0, \ldots, T-1\}$,

$$B(\xi_t, \delta_f) \cap \partial f(x_t) \neq \emptyset, \tag{3.58}$$

$$y_{t,S} \in B(S(x_t - \alpha\xi_t), \delta_p),\ \ S \in \mathcal{L}_2, \tag{3.59}$$

$$y_{t,S} \in B(P_{c_t(S),S}(x_t - \alpha\xi_t), \delta_p),\ \ S \in \mathcal{L}_1, \tag{3.60}$$

$$\|x_{t+1} - \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} w_{t+1}(S) y_{t,S}\| \le \delta_p. \tag{3.61}$$

Then

$$\min\{\max\{\Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(12M + 13),$$

$$2\alpha(f(x_t) - \inf(f, F))$$

$$-\delta_p(6M + 3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_f(6M + L_1 + 1)\} :$$

$$t = 0, \ldots, T - 1\} \le 4M^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T - 1\}$ *and*

$$\max\{\Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(12M + 13),$$

$$2\alpha(f(x_t) - \inf(f, F))$$

$$-\delta_p(6M + 3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_f(6M + L_1 + 1)\} \le 4M^2 T^{-1}$$

then

$$f(x_t) \le \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p(3M + 2) + 4^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1)$$

and

$$x_t \in \tilde{F}_{\max\{\alpha(L_0+1)+\gamma_T, \bar{\lambda}^{-1}\gamma_T\}}.$$

Proof In view of (3.9), there exists

$$z \in B(0, M_*) \cap F. \tag{3.62}$$

By (3.57) and (3.62),

$$\|z - x_0\| \le 2M. \tag{3.63}$$

We show that for all $t = 0, \dots, T$,

$$\|z - x_t\| \le 2M. \tag{3.64}$$

In view of (3.63) and (3.64) is true for $t = 0$.

Assume that there exists an integer $k \in \{0, \dots, T\}$ such that

$$\|z - x_k\| > 2M. \tag{3.65}$$

By (3.63) and (3.65), $k > 0$. We may assume without loss of generality that (3.64) holds for all integers $t = 0, \dots, k - 1$. In particular,

$$\|z - x_{k-1}\| \le 2M. \tag{3.66}$$

By (3.51), (3.58), (3.62) and (3.66), we apply Lemma 3.3 with

$$\delta_1 = \delta_f,\ \delta_2 = 2\delta_p,\ F_0 = F,\ M_0 = 3M,$$

$$Q = \sum_{S \in \mathcal{L}_2} w_k(S)S + \sum_{S \in \mathcal{L}_1} w_k(S)P_{c(S),S},$$

$$x = x_{k-1},\ \xi = \xi_{k-1},\ u = x_k$$

and obtain that

$$\alpha(f(x_{k-1}) - f(z))$$

$$\le 2^{-1}\|x_{k-1} - z\|^2 - 2^{-1}\|x_k - z\|^2$$

$$+ 2\delta_p(6M + 2 + \alpha L_0) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_1(6M + L_0 + 1). \tag{3.67}$$

If

$$\|z - x_k\| \le \|z - x_{k-1}\|,$$

then in view of (3.66),

$$\|x_k - z\| \le 2M.$$

This contradicts (3.65). Therefore

$$\|z - x_k\| > \|z - x_{k-1}\|. \tag{3.68}$$

By (3.52), (3.67) and (3.68),

$$\alpha(f(x_{k-1}) - f(z))$$
$$< 2\delta_p(6M + 3) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{3.69}$$

It follows from (3.49), (3.52), (3.62) and (3.69) that

$$f(x_{k-1})$$
$$< M_1 + 2\alpha^{-1}\delta_p(6M + 3) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1) \le M_1 + 3. \tag{3.70}$$

In view of (3.50) and (3.70),

$$\|x_{k-1}\| \le M/2. \tag{3.71}$$

Equations (3.62) and (3.71) imply that

$$\|x_{k-1} - z\| \le M_* + 2^{-1}M. \tag{3.72}$$

Equations (3.51), (3.58) and (3.72) imply that

$$\partial f(x_{k-1}) \subset B(0, L_0), \ \ \|\xi_{k-1}\| \le L_0 + 1. \tag{3.73}$$

By (3.54), (3.55), (3.60)–(3.62), (3.72), (3.73) and Lemma 3.3,

$$\|x_k - z\|$$
$$\le 2\delta_p + \|z - \sum_{S \in \mathcal{L}_2} w_k(S)S(x_{k-1} - \alpha\xi_{k-1}) - \sum_{S \in \mathcal{L}_1} w_k(S)P_{c_k(S),S}(x_{k-1} - \alpha\xi_{k-1})\|$$
$$\le 2\delta_p + \|z - (x_{k-1} - \alpha\xi_{k-1})\|$$
$$\le 2\delta_p + \|z - x_{k-1}\| + \alpha(L_0 + 1)$$
$$\le M_* + M/2 + 3 < 2M.$$

This contradicts (3.65). The contradiction we have reached proves that (3.64) is true for all $t = 0, \ldots, T$. In view of (3.62) and (3.64),

$$\|x_t\| \le 2M + M_*, \ \ t = 0, \ldots, T. \tag{3.74}$$

Let $t \in \{0, \ldots, T - 1\}$. By (3.51), (3.58) and (3.74),

$$\partial f(x_t) \subset B(0, L_0), \ \|\xi_t\| \le L_0 + 1. \tag{3.75}$$

By (3.51), (3.58), (3.60)–(3.62), (3.74) and Lemma 3.3, we apply Lemma 2.7 with

$$\delta_1 = \delta_f, \ \delta_2 = 2\delta_p, \ F_0 = F, \ M_0 = 3M,$$

$$Q = \sum_{S \in \mathcal{L}_2} w_k(S)S + \sum_{S \in \mathcal{L}_1} w_k(S)P_{c_k(S),S},$$

$$x = x_t, \ \xi = \xi_t, \ u = x_{k+1}$$

and obtain that

$$\alpha(f(x_t) - f(z))$$

$$\le 2^{-1}\|x_t - z\|^2 - 2^{-1}\|x_{t+1} - z\|^2$$

$$+ 2\delta_p(6M + 2 + \alpha L_0) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{3.76}$$

By (3.59), (3.60), (3.62), (3.74) and (3.75) we apply Lemma 3.5 with

$$\delta_1 = \alpha(L_0 + 1), \ \delta_2 = \delta_p, \ M_0 = 3M,$$

$$x = x_t, \ x_0 = x_t - \alpha\xi_t, \ y = x_{t+1},$$

$$w(S) = w_{t+1}(S), \ S \in \mathcal{L}_1 \cup \mathcal{L}_2,$$

$$c(S) = c_{t+1}(S), \ S \in \mathcal{L}_1,$$

$$y_S = y_{t,S}, \ S \in \mathcal{L}_1 \cup \mathcal{L}_2$$

and obtain that

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2 \ge \Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$- \alpha(L_0 + 1)(12M + 1) - \delta_p(48M + 13). \tag{3.77}$$

It follows from (3.52), (3.76) and (3.77) that

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2 \ge \max\{\Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(48M + 13),$$

$$2\alpha(f(x_t) - f(z))$$

$$- 2\delta_p(6M + 3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_f(6M + L_0 + 1)\}. \quad (3.78)$$

By (3.63) and (3.78),

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1}(\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\geq \sum_{t=0}^{T-1}(\max\{\Delta\bar{c} \sum_{S\in\mathcal{L}_1\cup\mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(48M + 13),$$

$$2\alpha(f(x_t) - f(z))$$

$$- 2\delta_p(6M + 3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_f(6M + L_0 + 1)\}). \quad (3.79)$$

Since z is an arbitrary element of $F \cap B(0, M_*)$ it follows from (3.14) and (3.79) that

$$4M^2 T^{-1} \geq \min\{\max\{\Delta\bar{c} \sum_{S\in\mathcal{L}_1\cup\mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(48M + 13),$$

$$2\alpha(f(x_t) - \inf(f, F))$$

$$-\delta_p(6M + 3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_f(6M + L_0 + 1)\} : \ t = 0, \ldots, T - 1\}.$$

Assume that $t \in \{0, \ldots, T - 1\}$ and that

$$\max\{\Delta\bar{c} \sum_{S\in\mathcal{L}_1\cup\mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(12M + 13),$$

$$2\alpha(f(x_t) - \inf(f, F))$$

$$- \delta_p(6M + 3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_f(6M + L_0 + 1)\} \leq 4M^2 T^{-1}. \quad (3.80)$$

By (3.80),

$$f(x_t) \le \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p(3M+2) + 4^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1).$$

In view of (3.59), (3.60) and (3.80), for every $S \in \mathcal{L}_2$,

$$\|x_t - \alpha\xi_t - S(x_t - \alpha\xi_t)\| \le \delta_p + \|x_t - \alpha\xi_t - y_{t,S}\|$$

$$\le \delta_p + \Delta^{-1}\bar{c}^{-1}(4M^2T^{-1} + \alpha(L_0+1)(12M+1) + \delta_p(12M+13))^{1/2} \le \gamma_T$$

and for every $S \in \mathcal{L}_1$,

$$\|x_t - \alpha\xi_t - P_{c_t(S),S}(x_t - \alpha\xi_t)\| \le \delta_p + \|x_t - \alpha\xi_t - y_{t,S}\|$$

$$\le \delta_p + \Delta^{-1}\bar{c}^{-1}(4M^2T^{-1} + \alpha(L_0+1)(12M+1) + \delta_p(12M+13))^{1/2} \le \gamma_T.$$

Together with (3.10), (3.11), (3.14) and Lemma 3.4 this implies that

$$x_t - \alpha\xi_t \in \mathrm{Fix}_{\gamma_T}(S),\ \ S \in \mathcal{L}_2,$$

$$B(x_t - \alpha\xi_t, \gamma_T) \cap F_{\gamma_T\bar{\lambda}^{-1}}(S) \ne \emptyset,\ S \in \mathcal{L}_1.$$

Combined with (3.75) this implies that

$$x_t \in \tilde{F}_{\max\{\alpha(L_0+1)+\gamma_T,\bar{\lambda}^{-1}\gamma_T\}}.$$

Theorem 3.6 is proved.

Let $\delta_f, \delta_p > 0$ are fixed. As in the case of Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations.

3.4 The Second Result for the Cimmino Proximal Point Subgradient Algorithm

In the following theorem the set F is bounded.

Theorem 3.7 *Let* $r_0 \in (0, 1]$,

$$\tilde{F}_{r_0 \max\{1,\bar{\lambda}^{-1}\}} \subset B(0, M_*), \tag{3.81}$$

$M \ge M_* + 2$, $L_0 \ge 1$,

$$|f(z_1) - f(z_2)| \le L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(0, 3M + 4), \tag{3.82}$$

$\delta_f, \delta_p \in [0, 1], \alpha > 0$ *satisfy*

$$\alpha \le 8^{-1}\Delta\bar{c}r_0^2(L_0 + 1)^{-1}(12M + 1)^{-1}\min\{1, \bar{\lambda}^{-1}\}^2,$$

$$\delta_p \le 32^{-1}\Delta\bar{c}r_0^2(12M + 4)^{-1}\min\{1, \bar{\lambda}^{-1}\}^2, \tag{3.83}$$

T be a natural number and let

$$\gamma_T = (4M^2T^{-1} + \alpha(L_0+1)(12M+1) + 4\delta_p(12M+4)(\Delta\bar{c})^{-1})^{1/2} + \delta_p. \tag{3.84}$$

Assume that for all $t = 1, \dots, T$,

$$w_t : \mathcal{L}_1 \cup \mathcal{L}_2 \to [\Delta, \infty), \tag{3.85}$$

$$c_t(S) \ge \bar{\lambda}, \ S \in \mathcal{L}_1, \tag{3.86}$$

$$\sum\{w_t(S) : \ S \in \mathcal{L}_1 \cup \mathcal{L}_2\} = 1, \tag{3.87}$$

$\{x_t\}_{t=0}^T \subset X$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$x_0 \in B(0, M) \tag{3.88}$$

and that for all integers $t \in \{0, \dots, T - 1\}$,

$$B(\xi_t, \delta_f) \cap \partial f(x_t) \ne \emptyset, \tag{3.89}$$

$$y_{t,S} \in B(S(x_t - \alpha\xi_t), \delta_p), \ S \in \mathcal{L}_2, \tag{3.90}$$

$$y_{t,S} \in B(P_{c_t(S),S}(x_t - \alpha\xi_t), \delta_p), \ S \in \mathcal{L}_1, \tag{3.91}$$

$$\|x_{t+1} - \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} w_{t+1}(S)y_{t,S}\| \le \delta_p. \tag{3.92}$$

Then

$$\|x_t\| \le 3M, \ t = 0 \dots, T,$$

$$\min\{\max\{\Delta\bar{c}\sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2}\|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - 4\delta_p(12M + 4),$$

$$2\alpha(f(x_t) - \inf(f, F)) - 2\delta_p(6M+3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_1 + 1)\} :$$

$$t = 0, \dots, T-1\} \le 4M^2T^{-1}.$$

Moreover, if $t \in \{0, \dots, T-1\}$ *and*

$$\max\{\Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0+1)(12M+1) - 4\delta_p(12M+4),$$

$$2\alpha(f(x_t) - \inf(f, F))$$

$$-2\delta_p(6M+3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_1 + 1)\} \le 4M^2T^{-1}$$

then

$$f(x_t) \le \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p(6M+3) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1)$$

and

$$x_t \in \tilde{F}_{\gamma_T \max\{1, \bar{\lambda}^{-1}\}}.$$

Proof In view of (3.9), there exists

$$z \in B(0, M_*) \cap F. \tag{3.93}$$

By (3.88) and (3.93),

$$\|z - x_0\| \le 2M. \tag{3.94}$$

We show that for all $t = 0, \dots, T$,

$$\|z - x_t\| \le 2M. \tag{3.95}$$

In view of (3.94) and (3.95) is true for $t = 0$.

Assume that there exists an integer $k \in \{0, \dots, T\}$ such that

$$\|z - x_k\| > 2M. \tag{3.96}$$

By (3.94) and (3.96), $k > 0$. We may assume without loss of generality that (3.95) holds for all integers $t = 0, \dots, k-1$. In particular,

$$\|z - x_{k-1}\| \le 2M. \tag{3.97}$$

Equations (3.93) and (3.97) imply that

$$\|x_{k-1}\| \le 2M + M_*/2, \tag{3.98}$$

By (3.82) and (3.98),

$$\partial f(x_{k-1}) \subset B(0, L_0). \tag{3.99}$$

It follows from (3.89) and (3.99) that

$$\|\xi_{k-1}\| \le L_0 + 1. \tag{3.100}$$

By (3.85)–(3.87), (3.90)–(3.93), (3.98) and (3.100) we apply Lemma 3.5 with

$$\delta_1 = \alpha(L_0 + 1),\ \ \delta_2 = \delta_p,\ \ M_0 = 3M,$$

$$x = x_{k-1},\ \ x_0 = x_{k-1} - \alpha\xi_{k-1},\ \ y = x_k,$$

$$w(S) = w_k(S),\ \ S \in \mathcal{L}_1 \cup \mathcal{L}_2,$$

$$c(S) = c_k(S),\ \ S \in \mathcal{L}_1,$$

$$y_S = y_{k-1,S},\ \ S \in \mathcal{L}_1 \cup \mathcal{L}_2$$

and obtain that

$$\|z - x_{k-1}\|^2 - \|z - x_k\|^2 \ge \Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_{k-1} - \alpha\xi_{k-1} - y_{k-1,S}\|^2$$

$$- \alpha(L_0 + 1)(12M + 1) - \delta_p(48M + 16). \tag{3.101}$$

If

$$\|z - x_k\| \le \|z - x_{k-1}\|,$$

then in view of (3.97),

$$\|x_k - z\| \le 2M.$$

This contradicts (3.96). Therefore

$$\|z - x_k\| > \|z - x_{k-1}\|. \tag{3.102}$$

Set

$$\tilde{r} = ((\bar{c}\Delta)^{-1}(L_0 + 1)\alpha(12M + 1) + 4\delta_p(12M + 4))^{1/2} + \delta_p. \tag{3.103}$$

By (3.101) and (3.102),

$$\sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_{k-1} - \alpha\xi_{k-1} - y_{k-1,S}\|^2$$

$$\leq (\bar{c}\Delta)^{-1}(\alpha(L_0 + 1)(12M + 1) + 4\delta_p(12M + 4)). \tag{3.104}$$

It follows from (3.83), (3.90), (3.103) and (3.104) that for every $S \in \mathcal{L}_2$,

$$\|x_{k-1} - \alpha\xi_{k-1} - S(x_{k-1} - \alpha\xi_{k-1})\|$$

$$\leq \delta_p + ((\bar{c}\Delta)^{-1}(\alpha(L_0 + 1)(12M + 1) + 4\delta_p(12M + 4)))^{1/2} = \tilde{r}. \tag{3.105}$$

It follows from (3.83), (3.91), (3.103) and (3.104) that for every $S \in \mathcal{L}_1$,

$$\|x_{k-1} - \alpha\xi_{k-1} - P_{c_{k-1}(S),S}(x_{k-1} - \alpha\xi_{k-1})\|$$

$$\leq \delta_p + ((\bar{c}\Delta)^{-1}(\alpha(L_0 + 1)(12M + 1) + 4\delta_p(12M + 4)))^{1/2} = \tilde{r} \leq r_0. \tag{3.106}$$

Lemma 3.4 and (3.106) imply that

$$B(x_{k-1} - \alpha\xi_{k-1}, \tilde{r}) \cap F_{\tilde{r}\bar{\lambda}^{-1}}(S) \neq \emptyset,\ S \in \mathcal{L}_1. \tag{3.107}$$

In view of (3.13), (3.81), (3.105) and (3.107),

$$x_{k-1} - \alpha\xi_{k-1} \in \tilde{F}_{\tilde{r}\max\{1,\bar{\lambda}^{-1}\}} \subset \tilde{F}_{r_0\max\{1,\bar{\lambda}^{-1}\}} \subset B(0, M_*). \tag{3.108}$$

Equations (3.92), (3.103), (3.104) and (3.108) imply that

$$\|x_k\| \leq \delta_p + \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} w_k(S)\|y_{k,S}\| \leq \|x_{k-1} - \alpha\xi_{k-1}\| + \tilde{r} \leq M_* + 1. \tag{3.109}$$

In view of (3.93) and (3.109),

$$\|z - x_k\| \leq 2M_* + 1 < 2M.$$

This contradicts (3.96). The contradiction we have reached proves that (3.95) is true for all $t = 0, \ldots, T$. Together with (3.93) and (3.95) this implies that

$$\|x_t\| \leq 2M + M_*,\ t = 0, \ldots, T. \tag{3.110}$$

Let $t \in \{0, \dots, T-1\}$. By (3.82) and (3.110),

$$\partial f(x_t) \subset B(0, L_0). \tag{3.111}$$

In view of (3.89)–(3.92) and (3.111),

$$\|\xi_t\| \le L_0 + 1. \tag{3.112}$$

By (3.82), (3.93), (3.110) and Lemma 3.3 we apply Lemma 2.7 with

$$\delta_1 = \delta_f,\ \delta_2 = 2\delta_p,\ F_0 = F,\ M_0 = 3M,$$

$$Q = \sum_{S \in \mathcal{L}_2} w_{t+1}(S)S + \sum_{S \in \mathcal{L}_1} w_{t+1}(S)P_{c_{t+1}(S),S},$$

$$x = x_t,\ \xi = \xi_t,\ u = x_{t+1}$$

and obtain that

$$\alpha(f(x_t) - f(z))$$

$$\le 2^{-1}\|x_t - z\|^2 - 2^{-1}\|x_{t+1} - z\|^2$$

$$+ 2\delta_p(6M + 2 + \alpha L_0) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{3.113}$$

By (3.85)–(3.87), (3.90)–(3.93), (3.110) and (3.112) we apply Lemma 3.5 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ M_0 = 3M,$$

$$x = x_t,\ x_0 = x_t - \alpha\xi_t,\ y = x_{t+1},$$

$$w(S) = w_{t+1}(S),\ S \in \mathcal{L}_1 \cup \mathcal{L}_2,$$

$$c(S) = c_{t+1}(S),\ S \in \mathcal{L}_1,$$

$$y_S = y_{t,S},\ S \in \mathcal{L}_1 \cup \mathcal{L}_2$$

and obtain that

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2 \ge \Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$- \alpha(L_0 + 1)(12M + 1) - \delta_p(48M + 16). \tag{3.114}$$

It follows from (3.113) and (3.114) that

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2 \geq \max\{\Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(48M + 16),$$

$$2\alpha(f(x_t) - f(z))$$

$$-4\delta_p(6M + 3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1)\}. \tag{3.115}$$

By (3.94) and (3.115),

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1} (\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\geq \sum_{t=0}^{T-1} (\max\{\Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(48M + 16),$$

$$2\alpha(f(x_t) - f(z))$$

$$-4\delta_p(6M + 3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1)\}). \tag{3.116}$$

Since z is an arbitrary element of $F \cap B(0, M_*)$ it follows from (3.14) and (3.116) that

$$4M^2 T^{-1} \geq \min\{\max\{\Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - \delta_p(48M + 16),$$

$$2\alpha(f(x_t) - \inf(f, F))$$

$$-4\delta_p(6M + 3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1)\} : \ t = 0, \ldots, T - 1\}.$$

Assume that $t \in \{0, \ldots, T - 1\}$ and that

$$\max\{\Delta\bar{c} \sum_{S \in \mathcal{L}_1 \cup \mathcal{L}_2} \|x_t - \alpha\xi_t - y_{t,S}\|^2$$

$$-\alpha(L_0 + 1)(12M + 1) - 4\delta_p(12M + 4),$$

$$2\alpha(f(x_t) - \inf(f, F))$$

$$- 2\delta_p(6M + 3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1)\} \le 4M^2T^{-1}. \tag{3.117}$$

By (3.117),

$$f(x_t) \le \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p(6M + 3) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1).$$

In view of (3.84), (3.90), (3.91) and (3.117), for every $S \in \mathcal{L}_2$,

$$\|x_t - \alpha\xi_t - S(x_t - \alpha\xi_t)\|$$

$$\le \delta_p + (\Delta^{-1}\bar{c}^{-1}(4M^2T^{-1} + \alpha(L_0 + 1)(12M + 1) + 4\delta_p(12M + 4)))^{1/2} \le \gamma_T \tag{3.118}$$

and for every $S \in \mathcal{L}_1$,

$$\|x_t - \alpha\xi_t - P_{c_t(S),S}(x_t - \alpha\xi_t)\|$$

$$\le \delta_p + (\Delta^{-1}\bar{c}^{-1}(4M^2T^{-1} + \alpha(L_0 + 1)(12M + 1) + 4\delta_p(12M + 4)))^{1/2} \le \gamma_T. \tag{3.119}$$

By (3.119) and Lemma 3.4,

$$B(x_t - \alpha\xi_t, \gamma_T) \cap F_{\gamma_T\bar{\lambda}^{-1}}(S) \ne \emptyset, \ S \in \mathcal{L}_1. \tag{3.120}$$

It follows from (3.84), (3.112), (3.119) and (3.120) that

$$x_t \in \tilde{F}_{\gamma_T \max\{1,\bar{\lambda}^{-1}\}}.$$

Theorem 3.7 is proved.

Let $\delta_f, \delta_p > 0$ are fixed. As in the case of Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations.

3.5 The Iterative Proximal Point Subgradient Algorithm

We continue to consider the minimization problem

$$f(x) \to \min, \ x \in F$$

introduced in Sections 3.1 using the notation and definitions and assumptions introduced there. We also suppose that all the assumptions introduced there hold.

Fix an integer

$$\bar{N} \geq \mathrm{Card}(\mathcal{L}_1 \cup \mathcal{L}_2).$$

Denote by $\mathcal{R}$ the set of all mappings

$$S : \{1, 2, \dots\} \to \mathcal{L}_2 \cup \{P_{c,Q} : \ c \geq \bar{\lambda}, \ \ Q \in \mathcal{L}_1\}$$

such that for every integer $k \geq 0$, the following properties hold:

(i) for each $Q \in \mathcal{L}_2$, there exists $i \in \{k\bar{N} + 1, \dots, (k+1)\bar{N}\}$ such that $S(i) = Q$,
(ii) for each $Q \in \mathcal{L}_1$, there exist $i \in \{k\bar{N} + 1, \dots, (k + 1)\bar{N}\}$ and $c \geq \bar{\lambda}$ such that $S(i) = P_{c,Q}$.

Fix $\alpha > 0$ and $S \in \mathcal{R}$.

Let us describe our algorithm.

The Iterative Proximal Point (IPP) Subgradient Algorithm

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$l_k \in \partial f(x_k),$$

and define the next iteration vector

$$x_{k+1} = \prod_{i=k\bar{N}+1}^{(k+1)\bar{N}} S(i)(x_{k\bar{N}} - \alpha l_k).$$

In this chapter we analyze this algorithm under the presence of computational errors.

We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f while $\delta_p \in [0, 1]$ is a computational error produced by our computer system, when we calculate the operators $P_{c,S}$, $S \in \mathcal{L}_1, \ c \geq \bar{\lambda}$ and $S \in \mathcal{L}_2$. Let $\alpha > 0$ be a step size.

The Iterative Proximal Point (IPP) Subgradient Algorithm with Computational Errors

Initialization: select an arbitrary $x_0 \in X$ and $S \in \mathcal{R}$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$\xi_k \in \partial f(x_k) + (0, \delta_f),$$

calculate

$$y_{k+1,0} = x_k - \alpha \xi_k,$$

$$y_{k+1,t} \in B(S(k\bar{N}+t)y_{k+1,t-1}, \delta_p),\ t = 1, \dots, \bar{N}$$

and define

$$x_{k+1} = y_{k+1,\bar{N}}.$$

3.6 The First Theorem for the Iterative Proximal Point Subgradient Algorithm

In the following theorem we assume that the objective function f satisfies the coercivity growth condition.

Theorem 3.8 *Let the function f be Lipschitz on bounded subsets of X,*

$$\lim_{\|x\|\to\infty} f(x) = \infty,$$

$M \ge 2M_* + 6$, $L_0 \ge 1$,

$$M_1 \ge \sup\{|f(u)| :\ u \in B(0, M_* + 4)\}, \tag{3.121}$$

$$f(u) > M_1 + 6 \text{ for all } u \in X \setminus B(0, 2^{-1}M), \tag{3.122}$$

$$|f(z_1) - f(z_2)| \le L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(0, 3M + 4), \tag{3.123}$$

$\delta_f, \delta_p \in [0, 1], \alpha > 0$ *satisfy*

$$\alpha \le (L_0 + 1)^{-1},\ \alpha \ge \delta_p(3M(\bar{N} + 1) + 4),$$

$$\delta_p \le \min\{\bar{N}^{-2}, (3M(\bar{N} + 3))^{-1}\},\ \delta_f \le (6M + L_0 + 1)^{-1}, \tag{3.124}$$

$S \in \mathcal{R}$, *T be a natural number and let*

$$\gamma_T = (\bar{c}^{-1}(4M^2T^{-1} + 2\delta_p\bar{N}(6M + 2)$$

$$+ \alpha(L_0 + 1)(6M + 1)))^{1/2}(\bar{N} + 1) + \delta_p + \alpha(L_0 + 1). \tag{3.125}$$

Assume that $\{x_t\}_{t=0}^{T} \subset X$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$x_0 \in B(0, M), \tag{3.126}$$

for every integer $k \in \{1, \dots, T\}$,

$$B(\xi_{k-1}, \delta_f) \cap \partial f(x_{k-1}) \neq \emptyset, \tag{3.127}$$

$y_{k,t} \in X,\ t = 0, \ldots, \bar{N},$

$$y_{k,0} = x_{k-1} - \alpha \xi_{k-1}, \tag{3.128}$$

$$\|y_{k,T} - S((k-1)\bar{N} + t) y_{k,t-1}\| \leq \delta_p,\ t = 1, \ldots, \bar{N}, \tag{3.129}$$

$$x_k = y_{k,\bar{N}}. \tag{3.130}$$

Then

$$\|x_t\| \leq 3M,\ t = 0, \ldots, T$$

and

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, F))$$

$$-\delta_p(3M(\bar{N}+1)+4) - \alpha^2 L_0^2 - \alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\sum_{i=1}^{\bar{N}}\{\|y_{t+1,i-1} - y_{t+1,i}\|^2$$

$$-\alpha(L_0+1)(6M+1) - 2\delta_p \bar{N}(6M+3)\} :\ t = 0, \ldots, T-1\}$$

$$\leq 4M^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T-1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, F))$$

$$-\delta_p(3M(\bar{N}+1)+4) - \alpha^2 L_0^2 - \alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\sum_{i=1}^{\bar{N}}\|y_{t+1,i-1} - y_{t+1,i}\|^2$$

$$-\alpha(L_0+1)(6M+1) - 2\delta_p \bar{N}(6M+3)\} \leq 4M^2 T^{-1},$$

then

$$f(x_t) \leq \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p(3M(\bar{N}+1)+4) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1)$$

and

$$x_t \in \tilde{F}_{\gamma_T \max\{1,\bar{\lambda}^{-1}\}}.$$

Proof In view of (3.9), there exists

$$z \in B(0, M_*) \cap F. \tag{3.131}$$

By (3.126) and (3.131),

$$\|z - x_0\| \le 2M. \tag{3.132}$$

We show that for all $t = 0, \dots, T$,

$$\|z - x_t\| \le 2M. \tag{3.133}$$

In view of (3.132), (3.133) is true for $t = 0$. Assume that $k \in \{0, \dots, T\}$ and that

$$\|z - x_k\| > 2M. \tag{3.134}$$

Equations (3.132) and (3.134) imply that $k > 0$. We may assume without loss of generality that (3.133) holds for all integers $t = 0, \dots, k-1$. In particular,

$$\|z - x_{k-1}\| \le 2M. \tag{3.135}$$

In view of (3.131) and (3.135),

$$\|x_{k-1}\| \le 3M. \tag{3.136}$$

By (3.123), (3.127)–(3.131), (3.136) and Lemma 3.3, we apply Lemma 2.11 with

$$\delta_1 = \delta_f,\ \delta_2 = \delta_p,\ F_0 = F,\ M_0 = 3M,\ p = \bar{N},$$

$$Q_i = S((k-1)\bar{N} + i),\ i = 1, \dots, \bar{N},$$

$$x = x_{k-1},\ \xi = \xi_{k-1},$$

$$u_0 = x_{k-1} - \alpha\xi_{k-1},\ u_i = y_{k,i},\ i = 0, \dots, \bar{N}$$

and obtain that

$$\alpha(f(x_{k-1}) - f(z))$$

$$\leq 2^{-1}\|x_{k-1}-z\|^2 - 2^{-1}\|x_k - z\|^2 + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1)$$

$$+\delta_p(6M + 2 + \alpha L_0 + 3M(\bar{N}-1) + \delta_p\bar{N}(\bar{N}-1)/2). \tag{3.137}$$

If

$$\|z - x_k\| \leq \|z - x_{k-1}\|,$$

then in view of (3.135),

$$\|x_k - z\| \leq 2M.$$

This contradicts (3.134). The contradiction we have reached proves that

$$\|z - x_{k-1}\| < \|z - x_k\|. \tag{3.138}$$

By (3.121), (3.124), (3.131), (3.137) and (3.138),

$$f(x_{k-1}) \leq M_1 + \delta_f(6M + L_0 + 1)$$

$$+\alpha^{-1}\delta_p(6M + 3 + 3M(\bar{N}-1) + 1) \leq M_1 + 3.$$

Together with (3.122) implies that

$$\|x_{k-1}\| \leq M/2. \tag{3.139}$$

In view of (3.131) and (3.139),

$$\|x_{k-1} - z\| \leq M_* + 2^{-1}M. \tag{3.140}$$

Equations (3.123) and (3.140) imply that

$$\partial f(x_{k-1}) \subset B(0, L_0). \tag{3.141}$$

By (3.127) and (3.141),

$$\|\xi_{k-1}\| \leq L_0 + 1. \tag{3.142}$$

It follows from (3.124), (3.128), (3.140) and (3.142) that

$$\|z - y_{k,0}\| = \|z - x_{k-1} + \alpha\xi_{k-1}\|$$

$$\leq M_* + 2^{-1}M + \alpha(L_0 + 1) \leq M_* + 2^{-1}M + 1. \tag{3.143}$$

Lemma 3.3, (3.124), (3.128)–(3.131) and (3.143) imply that

$$\|z - y_{k,t}\| \le \|z - S((k-1)\bar{N} + t)(y_{k,t-1})\| + \delta_p$$

$$\le \|z - y_{k-1,t}\| + \delta_p,$$

$$\|z - y_{k,t}\| \le t\delta_p + \|z - (x_{k-1} - \alpha\xi_{k-1})\|$$

$$\le \bar{N}\delta_p + M_* + 2^{-1}M + 1,$$

$$\|z - x_k\| \le 2 + M_* + 2^{-1}M < M.$$

This contradicts (3.124). The contradiction we have reached proves that (3.133) holds for all $k = 0, \dots, T$. By (3.131) and (3.133),

$$\|x_t\| \le 3M, \ t = 0, \dots, T. \tag{3.144}$$

Equations (3.123), (3.127) and (3.144) imply that for all $k = 0, \dots, T-1$,

$$\partial f(x_k) \subset B(0, L_0), \tag{3.145}$$

$$\|\xi_k\| \le L_0 + 1. \tag{3.146}$$

Let $k \in \{1, \dots, T\}$. In view of (3.124), (3.128), (3.133) and (3.146),

$$\|z - y_{k,0}\| = \|z - (x_{k-1} - \alpha\xi_{k-1})\| \le 2M + \alpha(L_0 + 1) \le 2M + 1. \tag{3.147}$$

Lemma 3.3, (3.124), (3.129), (3.131) and (3.147) imply that for all $t = 1, \dots, \bar{N}$,

$$\|z - y_{k,t}\| \le \|z - y_{k-1,t}\| + \delta_p,$$

$$\|z - y_{k,t}\| \le 2M + 1 + \bar{N}\delta_p \le 2M + 2. \tag{3.148}$$

Equations (3.131) and (3.148) imply that

$$\|y_{k,t}\| \le 3M + 2, \ k = 1, \dots, T, \ t = 0, \dots, \bar{N}. \tag{3.149}$$

Let $t \in \{0, \dots, T-1\}$. By (3.123), (3.128)–(3.131), (3.144) and Lemma 3.3, we apply Lemma 2.11 with

$$\delta_1 = \delta_f, \ \delta_2 = \delta_p, \ F_0 = F, \ M_0 = 3M, \ p = \bar{N},$$

$$Q_i = S(t\bar{N} + i), \ i = 1, \dots, \bar{N},$$

$$x = x_t,\ \xi = \xi_t,$$

$$u_0 = x_t - \alpha\xi_t,\ u_i = y_{t+1,i},\ i = 0, \dots, \bar{N}$$

and obtain that

$$\alpha(f(x_t) - f(z))$$

$$\leq 2^{-1}\|x_t - z\|^2 - 2^{-1}\|x_{t+1} - z\|^2$$

$$+2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1)$$

$$+ \delta_p(6M + 2 + \alpha L_0 + 3M(\bar{N} - 1) + \delta_p\bar{N}(\bar{N} - 1)/2). \tag{3.150}$$

By (3.128)–(3.131), (3.144) and Lemma 3.3, we apply Lemma 2.12 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ F_0 = F,\ M_0 = 3M,\ p = \bar{N},$$

$$Q_i = S(t\bar{N} + i),\ i = 1, \dots, \bar{N},$$

$$x = x_t,\ u_0 = x_t - \alpha\xi_t = y_{t+1,0},$$

$$u_i = y_{t+1,i},\ i = 1, \dots, \bar{N}, x_{t+1} = y_{t+1,\bar{N}}$$

and obtain that

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2$$

$$\geq \bar{c}\sum_{i=1}^{\bar{N}} \|y_{t+1,i-1} - y_{t+1,i}\|^2$$

$$-2\delta_p\bar{N}(6M + \alpha(L_0 + 1) + \bar{N}\delta_p + 1) - \alpha(6M + \alpha(L_0 + 1))(L_0 + 1). \tag{3.151}$$

By (3.124), (3.150) and (3.151),

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2$$

$$\geq \max\{2\alpha(f(x_t) - f(z))$$

$$-\alpha^2 L_0^2 - \alpha\delta_f(6M + L_0 + 1) - \delta_p(6M + 3 + 3M(\bar{N} - 1) + 1),$$

$$\bar{c}\sum_{i=1}^{\bar{N}} \|y_{t+1,i-1} - y_{t+1,i}\|^2$$

$$- 2\delta_p \bar{N}(6M + 3) + \alpha(6M + 1)(L_0 + 1)\}. \tag{3.152}$$

It follows from (3.132) and (3.152) that

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1} (\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\geq \sum_{t=0}^{T-1} \max\{2\alpha(f(x_t) - f(z))$$

$$-\alpha^2 L_0^2 - \alpha\delta_f(6M + L_0 + 1) - \delta_p(3M(\bar{N} + 1) + 4),$$

$$\bar{c} \sum_{i=1}^{\bar{N}} \|y_{t+1,i-1} - y_{t+1,i}\|^2$$

$$- 2\delta_p \bar{N}(6M + 3) - \alpha(6M + 1)(L_0 + 1)\}. \tag{3.153}$$

Since z is an arbitrary element of $B(0, M_*) \cap F$ it follows from (3.14) and (3.153) that

$$4M^2 T^{-1} \geq \min\{\max\{2\alpha(f(x_t) - \inf(f, F))$$

$$-\alpha^2 L_0^2 - \alpha\delta_f(6M + L_0 + 1) - \delta_p(3M(\bar{N} + 1) + 4),$$

$$\bar{c} \sum_{i=1}^{\bar{N}} \|y_{t+1,i-1} - y_{t+1,i}\|^2$$

$$-2\delta_p \bar{N}(6M + 3) - \alpha(6M + 1)(L_0 + 1)\} : \ t = 0, \ldots, T - 1\}.$$

Let $t \in \{0, \ldots, T - 1\}$ and

$$4M^2 T^{-1} \geq \max\{2\alpha(f(x_t) - \inf(f, F))$$

$$-\alpha^2 L_0^2 - \alpha\delta_f(6M + L_0 + 1) - \delta_p(3M(\bar{N} + 1) + 4),$$

$$\bar{c} \sum_{i=1}^{\bar{N}} \|y_{t+1,i-1} - y_{t+1,i}\|^2$$

$$-2\delta_p\bar{N}(6M+3)-\alpha(6M+1)(L_0+1)\}. \tag{3.154}$$

By (3.154),

$$f(x_t)\le \inf(f,F)$$

$$+2M^2(T\alpha)^{-1}+\alpha^{-1}\delta_p(3M(\bar{N}+1)+4)+2^{-1}\alpha L_0^2+\delta_f(6M+L_0+1),$$

$$\bar{c}\sum_{i=1}^{\bar{N}}\|y_{t+1,i-1}-y_{t+1,i}\|^2$$

$$\le 4M^2T^{-1}+2\delta_p\bar{N}(6M+2)+\alpha(6M+1)(L_0+1). \tag{3.155}$$

In view of (3.125), (3.129) and(3.155), for all $i\in\{1,\dots,\bar{N}\}$,

$$\|y_{t+1,i-1}-S(t\bar{N}+i)(y_{t+1,i-1})\|$$

$$\le \delta_p+(\bar{c}^{-1}(4M^2T^{-1}+2\delta_p\bar{N}(6M+2)$$

$$+\alpha(6M+1)(L_0+1))^{1/2}\le\gamma_T. \tag{3.156}$$

It follows from (3.125), (3.128), (3.146) and (3.155) that for all $i\in\{1,\dots,\bar{N}\}$,

$$\|x_t-y_{t+1,0}\|\le\alpha(L_0+1),$$

$$\|x_t-y_{t+1,i}\|\le\|x_t-y_{t+1,0}\|$$

$$+i(\bar{c}^{-1}(4M^2T^{-1}+2\delta_p\bar{N}(6M+2)+\alpha(6M+1)(L_0+1)))^{1/2}$$

$$\le\alpha(L_0+1)+\bar{N}(\bar{c}^{-1}(4M^2T^{-1}+2\delta_p\bar{N}(6M+2)+\alpha(6M+1)(L_0+1)))^{1/2}\le\gamma_T. \tag{3.157}$$

By (3.156), (3.157) and Lemma 3.4,

$$x_t\in\tilde{F}_{\gamma_T\max\{1,\bar{\lambda}^{-1}\}}.$$

Theorem 3.8 is proved.

Let $\delta_f,\delta_p>0$ are fixed. As in the case of Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations.

3.7 The Second Theorem for the Iterative Proximal Point Subgradient Algorithm

In the following theorem the set F is bounded.

Theorem 3.9 *Let* $r_0 \in (0, 1]$,

$$\tilde{F}_{r_0} \subset B(0, M_*), \tag{3.158}$$

$M \ge 2M_*$, $L_0 \ge 1$,

$$|f(z_1) - f(z_2)| \le L_0 \|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(0, 3M + 4), \tag{3.159}$$

$\delta_f, \delta_p \in [0, 1]$, $\alpha > 0$ *satisfy*

$$\alpha \le 8^{-1}\bar{c}r_0^2(L_0 + 1)^{-1}(6M + 1)^{-1}(\bar{N} + 1)^{-2} \min\{1, \bar{\lambda}\}^2$$

$$\delta_p \le 16^{-1}\bar{c}r_0^2\bar{N}^{-1}(\bar{N} + 1)^{-2}(6M + 2)^{-1} \min\{1, \bar{\lambda}\}^2, \tag{3.160}$$

$S \in \mathcal{R}$, T *be a natural number and let*

$$\gamma_T = \delta_p + \bar{c}^{-1}(4M^2T^{-1} + 4\delta_p\bar{N}(6M + 2) + \alpha(L_0 + 1(6M + 1)))^{1/2}. \tag{3.161}$$

Assume that $\{x_t\}_{t=0}^T \subset X$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$x_0 \in B(0, M), \tag{3.162}$$

for every integer $k \in \{1, \dots, T\}$,

$$B(\xi_{k-1}, \delta_f) \cap \partial f(x_{k-1}) \ne \emptyset, \tag{3.163}$$

$y_{k,t} \in X,\ t = 0, \dots, \bar{N}$,

$$y_{k,0} = x_{k-1} - \alpha\xi_{k-1}, \tag{3.164}$$

$$\|y_{k,T} - S((k - 1)\bar{N} + t)y_{k,t-1}\| \le \delta_p,\ t = 1, \dots, \bar{N}, \tag{3.165}$$

$$x_k = y_{k,\bar{N}}. \tag{3.166}$$

Then

$$\|x_t\| \le 3M,\ t = 0, \dots, T$$

and

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, F))$$

$$-2\delta_p(3M(\bar{N}+3)+4) - \alpha^2 L_0^2 - 2\alpha(6M + L_0 + 1),$$

$$\bar{c}\sum_{i=1}^{\bar{N}}\{\|y_{t+1,i-1} - y_{t+1,i}\|^2$$

$$-\alpha(L_0+1)(6M+1) - 4\delta_p\bar{N}(6M+2)\} : \ t = 0, \dots, T-1\} \le 4M^2T^{-1}.$$

Moreover, if $t \in \{0, \dots, T-1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, F))$$

$$-2\delta_p(3M(\bar{N}+3)+4) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\sum_{i=1}^{\bar{N}}\|y_{t+1,i-1} - y_{t+1,i}\|^2$$

$$-\alpha(L_0+1)(6M+1) - 4\delta_p\bar{N}(6M+2)\} \le 4M^2T^{-1},$$

then

$$f(x_t) \le \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p(3M(\bar{N}+3)+4) + 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1)$$

and

$$x_t \in \tilde{F}_{\max\{1,\bar{\lambda}^{-1}\}(\alpha(L_0+1)+(\bar{N}+1)(\gamma_T+\delta_p))}.$$

Proof In view of (3.9), there exists

$$z \in B(0, M_*) \cap F. \tag{3.167}$$

By (3.162) and (3.167),

$$\|z - x_0\| \le 2M. \tag{3.168}$$

We show that for all $t = 0, \dots, T$,

$$\|z - x_t\| \le 2M. \tag{3.169}$$

In view of (3.168) and (3.169) is true for $t = 0$. Assume that $k \in \{0, \dots, T\}$ and that

$$\|z - x_k\| > 2M. \tag{3.170}$$

Equations (3.168) and (3.170) imply that $k > 0$. We may assume without loss of generality that (3.169) holds for all integers $t = 0, \dots, k-1$. In particular,

$$\|z - x_{k-1}\| \le 2M. \tag{3.171}$$

In view of (3.167) and (3.171),

$$\|x_{k-1}\| \le 2M + M_*. \tag{3.172}$$

Equations (3.159) and (3.172) imply that

$$\partial f(x_{k-1}) \subset B(0, L_0). \tag{3.173}$$

By (3.163) and (3.173),

$$\|\xi_{k-1}\| \le L_0 + 1. \tag{3.174}$$

By (3.160), (3.164)–(3.167), (3.172), (3.174) and Lemma 3.3, we apply Lemma 2.12 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ F_0 = F,\ M_0 = 3M,\ p = \bar{N},$$

$$Q_i = S((k-1)\bar{N} + i),\ i = 1, \dots, \bar{N},$$

$$x = x_{k-1},\ u_0 = x_{k-1} - \alpha\xi_{k-1} = y_{k,0},$$

$$u_i = y_{k,i},\ i = 1, \dots, \bar{N}, x_k = y_{k,\bar{N}}$$

and obtain that

$$\|z - x_{k-1}\|^2 - \|z - x_k\|^2$$

$$\ge \bar{c} \sum_{i=1}^{\bar{N}} \|y_{k,i-1} - y_{k,i}\|^2$$

$$- 2\delta_p \bar{N}(6M + \alpha(L_0 + 1) + (1 + \bar{N})\delta_p) - \alpha(6M + 1)(L_0 + 1). \tag{3.175}$$

In view of (3.170) and (3.171),

$$\|z - x_{k-1}\| < \|z - x_k\|.$$

Combined with (3.160) and (3.175) this implies that

$$\sum_{i=1}^{\bar{N}} \|y_{k,i-1} - y_{k,i}\|^2$$

$$\leq \bar{c}^{-1}(2\delta_p \bar{N}(6M + 2)) + \bar{c}^{-1}\alpha(L_0 + 1)(6M + 1). \tag{3.176}$$

Set

$$\gamma = (\bar{c}^{-1}(2\delta_p \bar{N}(6M + 2)) + \bar{c}^{-1}\alpha(L_0 + 1)(6M + 1))^{1/2}. \tag{3.177}$$

By (3.165), (3.176) and (3.177), for $i = 1, \dots, \bar{N}$,

$$\|y_{k,i-1} - y_{k,i}\| \leq \gamma, \tag{3.178}$$

$$\|y_{k,i-1} - S((k-1)\bar{N} + i)(y_{k,i-1})\| \leq \gamma + \delta_p. \tag{3.179}$$

Equations (3.164) and (3.174) imply that

$$\|x_{k-1} - y_{k,0}\| \leq \alpha(L_0 + 1). \tag{3.180}$$

In view of (3.178) and (3.180), for all $i = 1, \dots, \bar{N}$,

$$\|x_{k-1} - y_{k,i}\| \leq \alpha(L_0 + 1) + \bar{N}\gamma. \tag{3.181}$$

Let $Q \in \mathcal{L}_2$. By the definition of $\mathcal{R}$, there exists $i \in \{1, \dots, \bar{N}\}$ such that

$$S((k-1)\bar{N} + i) = Q.$$

Together with (3.179) this implies that

$$y_{k,i-1} \in \text{Fix}_{\gamma+\delta_p}(Q).$$

Combined with (3.180) and (3.181) this implies that

$$d(x_{k-1}, \text{Fix}_{\gamma+\delta_p}(Q)) \leq \alpha(L_0 + 1) + \bar{N}\gamma. \tag{3.182}$$

Let $Q \in \mathcal{L}_1$. By the definition of $\mathcal{R}$ (see (ii)), there exist $c(Q) \geq \bar{\lambda}$, $i \in \{1, \dots, \bar{N}\}$ such that

$$S((k-1)\bar{N} + i) = P_{c(Q),Q}.$$

Together with (3.179) and Lemma 3.4 this implies that

$$B(y_{k,i-1}, \gamma + \delta_p) \cap F_{\bar{\lambda}^{-1}(\delta_p+\gamma)}(Q) \neq \emptyset.$$

Combined with (3.180) and (3.181) this implies that

$$d(x_{k-1}, F_{\bar{\lambda}^{-1}(\delta_p+\gamma)}(Q)) \leq \alpha(L_0 + 1) + (\bar{N} + 1)\gamma + \delta_p. \tag{3.183}$$

By (3.182) and (3.183),

$$x_{k-1} \in \tilde{F}_{\max\{1,\bar{\lambda}^{-1}\}(\alpha(L_0+1)+(\bar{N}+1)\gamma+\delta_p)}. \tag{3.184}$$

In view of (3.160) and (3.177),

$$\max\{1, \bar{\lambda}^{-1}\}\alpha(L_0 + 1) \leq r_0/4, \tag{3.185}$$

$$\max\{1, \bar{\lambda}^{-1}\}(\bar{N} + 1)\delta_p \leq r_0/4, \tag{3.186}$$

$$2\bar{c}^{-1}\delta_p \bar{N}(6M + 2) \leq 8^{-1} r_0^2 (\bar{N} + 1)^{-2} \min\{1, \bar{\lambda}\}^2,$$

$$\bar{c}^{-1}\alpha(L_0 + 1)(6M + 1) \leq 8^{-1} r_0^2 (\bar{N} + 1)^{-2} \min\{1, \bar{\lambda}\}^2,$$

$$\gamma \leq 2^{-1} r_0 (\bar{N} + 1)^{-1} \min\{1, \bar{\lambda}\}. \tag{3.187}$$

By (3.184)–(3.187),

$$x_{k-1} \in \tilde{F}_{r_0}. \tag{3.188}$$

Equations (3.158) and (3.188) imply that

$$x_{k-1} \in B(0, M_*). \tag{3.189}$$

By (3.166), (3.181), (3.185), (3.187) and (3.189),

$$\|x_{k-1} - x_k\| \leq \|x_{k-1} - y_{k,\bar{N}}\| \leq \alpha(L_0 + 1) + \bar{N}\gamma \leq 1. \tag{3.190}$$

In view of (3.189) and (3.190),

$$\|x_k\| \leq M_* + 1.$$

Together with (3.167) this implies that

$$\|z - x_k\| \leq 2M_* + 1 < 2M.$$

This contradicts (3.170). The contradiction we have reached proves that (3.169) holds for all $k = 0, \dots, T$. By (3.161) and (3.169),

$$\|x_t\| \le 3M, \ t = 0, \dots, T. \tag{3.191}$$

Equations (3.159), (3.163) and (3.191) imply that for all $t = 0, \dots, T-1$,

$$\partial f(x_t) \subset B(0, L_0), \ \|\xi_t\| \le L_0 + 1. \tag{3.192}$$

Let $k \in \{1, \dots, T\}$. In view of (3.160), (3.164) and (3.169),

$$\|z - y_{k,0}\| = \|z - (x_{k-1} - \alpha\xi_{k-1})\| \le 2M + \alpha(L_0 + 1) \le 2M + 1. \tag{3.193}$$

Lemma 3.3, (3.160), (3.165), (3.167) and (3.193) imply that for all $t = 1, \dots, \bar{N}$,

$$\|z - y_{k,t}\| \le \|z - S((k-1)\bar{N} + t)y_{k-1,t}\| + \delta_p$$

$$\le \|z - y_{k,t-1}\| + \delta_p,$$

$$\|z - y_{k,t}\| \le \|z - y_{k,0}\| + \delta_p\bar{N} \le 2M + 1 + \bar{N}\delta_p \le 2M + 2. \tag{3.194}$$

Let $t \in \{0, \dots, T-1\}$. By (3.159), (3.163), (3.165), (3.166), (3.191) and Lemma 3.3, we apply Lemma 2.11 with

$$\delta_1 = \delta_f, \ \delta_2 = \delta_p, \ F_0 = F, \ M_0 = 3M, \ p = \bar{N},$$

$$Q_i = S(t\bar{N} + i), \ i = 1, \dots, \bar{N},$$

$$x = x_t, \ \xi = \xi_t,$$

$$u_0 = x_t - \alpha\xi_t, \ u_i = y_{t,i}, \ i = 1, \dots, \bar{N},$$

$$x_{t+1} = y_{t,\bar{N}}$$

and obtain that

$$\alpha(f(x_t) - f(z))$$

$$\le 2^{-1}\|x_t - z\|^2 - 2^{-1}\|x_{t+1} - z\|^2 + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1)$$

$$+ \delta_p(6M + 2 + \alpha L_0 + 3M(\bar{N} - 1) + \delta_p\bar{N}(\bar{N} - 1)/2). \tag{3.195}$$

By (3.160), (3.164)–(3.167), (3.191), (3.192) and Lemma 3.3, we apply Lemma 2.12 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ F_0 = F,\ M_0 = 3M,\ p = \bar{N},$$

$$Q_i = S(t\bar{N} + i),\ i = 1, \dots, \bar{N},$$

$$x = x_t,\ u_0 = x_t - \alpha\xi_t = y_{t+1,0},$$

$$u_i = y_{t+1,i},\ i = 1, \dots, \bar{N}, x_{t+1} = y_{t+1,\bar{N}}$$

and obtain that

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2$$

$$\geq \bar{c} \sum_{i=1}^{\bar{N}} \|y_{t+1,i-1} - y_{t+1,i}\|^2$$

$$- 4\delta_p \bar{N}(6M + \alpha(L_0 + 1) + (\bar{N} + 1)\delta_p) - \alpha(6M + 1)(L_0 + 1). \tag{3.196}$$

By (3.160), (3.195) and (3.196),

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2$$

$$\geq \max\{2\alpha(f(x_t) - f(z))$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1) - 2\delta_p(3M(\bar{N} + 1) + 4),$$

$$\bar{c} \sum_{i=1}^{\bar{N}} \|y_{t+1,i-1} - y_{t+1,i}\|^2$$

$$- 4\delta_p \bar{N}(6M + 2) - \alpha(6M + 1)(L_0 + 1)\}. \tag{3.197}$$

It follows from (3.168) and (3.197) that

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1} (\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\geq \sum_{t=0}^{T-1} \max\{2\alpha(f(x_t) - f(z))$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1) - 2\delta_p(3M(\bar{N} + 3) + 4),$$

$$\bar{c}\sum_{i=1}^{\bar{N}}\|y_{t+1,i-1}-y_{t+1,i}\|^2$$

$$-\,4\delta_p\bar{N}(6M+2)-\alpha(6M+1)(L_0+1)\}. \tag{3.198}$$

Since z is an arbitrary element of $B(0, M_*) \cap F$ it follows from (3.14) and (3.198) that

$$4M^2T^{-1} \geq \min\{\max\{2\alpha(f(x_t)-\inf(f, F))$$

$$-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1)-2\delta_p(3M(\bar{N}+3)+4),$$

$$\bar{c}\sum_{i=1}^{\bar{N}}\|y_{t+1,i-1}-y_{t+1,i}\|^2$$

$$-4\delta_p\bar{N}(6M+2)-\alpha(6M+1)(L_0+1)\}:\ t=0,\dots,T-1\}.$$

Let $t \in \{0, \dots, T-1\}$ and

$$4M^2T^{-1} \geq \max\{2\alpha(f(x_t)-\inf(f, F))$$

$$-\alpha^2L_0^2-2\alpha\delta_f(6M+L_0+1)-2\delta_p(3M(\bar{N}+3)+4),$$

$$\bar{c}\sum_{i=1}^{\bar{N}}\|y_{t+1,i-1}-y_{t+1,i}\|^2$$

$$-\,4\delta_p\bar{N}(6M+2)-\alpha(6M+1)(L_0+1)\}. \tag{3.199}$$

By (3.199),

$$f(x_t) \leq \inf(f, F)$$

$$+2M^2(T\alpha)^{-1}+\alpha^{-1}\delta_p(3M(\bar{N}+3)+4)+2^{-1}\alpha L_0^2+\delta_f(6M+L_0+1),$$

$$\sum_{i=1}^{\bar{N}}\|y_{t+1,i-1}-y_{t+1,i}\|^2$$

$$\leq \bar{c}^{-1}(4M^2T^{-1}+4\delta_p\bar{N}(6M+2)+\alpha(6M+1)(L_0+1)). \tag{3.200}$$

In view of (3.165) and (3.200), for all $i \in \{1, \dots, \bar{N}\}$,

$$\|y_{t+1,i-1} - y_{t+1,i}\|$$
$$\leq (\bar{c}^{-1}(4M^2T^{-1} + 4\delta_p\bar{N}(6M+2) + \alpha(6M+1)(L_0+1)))^{1/2}, \tag{3.201}$$

$$\|y_{t+1,i-1} - S(t\bar{N}+i)(y_{t+1,i-1})\|$$
$$\leq \delta_p + (\bar{c}^{-1}(4M^2T^{-1} + 4\delta_p\bar{N}(6M+2) + \alpha(6M+1)(L_0+1)))^{1/2}. \tag{3.202}$$

Recall that (see (3.161)) that

$$\gamma_T = \delta_p + \bar{c}^{-1}(4M^2T^{-1} + 4\delta_p\bar{N}(6M+2) + \alpha(L_0+1)(6M+1))^{1/2}. \tag{3.203}$$

It follows from (3.164), (3.192) and (3.201) that for all $i \in \{1, \dots, \bar{N}\}$,

$$\|x_t - y_{t+1,0}\| \leq \alpha(L_0+1), \tag{3.204}$$

$$\|x_t - y_{t+1,i}\| \leq \alpha(L_0+1)$$
$$+ \bar{N}(\bar{c}^{-1}(4M^2T^{-1} + 4\delta_p\bar{N}(6M+2) + \alpha(6M+1)(L_0+1)))^{1/2}. \tag{3.205}$$

Let $Q \in \mathcal{L}_2$. By the definition of $\mathcal{R}$, there exists $i \in \{1, \dots, \bar{N}\}$ such that

$$S(t\bar{N}+i) = Q.$$

By (3.202) and (3.203) this implies that

$$y_{t+1,i-1} \in \mathrm{Fix}_{\gamma_T}(Q).$$

Combined with (3.205) this implies that

$$d(x_t, \mathrm{Fix}_{\gamma_T}(Q)) \leq \alpha(L_0+1) + \bar{N}\gamma_T. \tag{3.206}$$

Let $Q \in \mathcal{L}_1$. By the definition of $\mathcal{R}$, there exist $c(Q) \geq \bar{\lambda}$, $i \in \{1, \dots, \bar{N}\}$ such that

$$S(t\bar{N}+i) = P_{c(Q),Q}.$$

In view of (3.202), (3.203) and Lemma 3.4,

$$B(y_{t+1,i-1}, \gamma_T) \cap F_{\bar{\lambda}^{-1}\gamma_T}(Q) \neq \emptyset.$$

Combined with (3.204) and (3.205) this implies that

$$d(x_t, F_{\bar{\lambda}^{-1}\gamma_T}(Q)) \leq \alpha(L_0+1) + (\bar{N}+1)\gamma_T. \tag{3.207}$$

By (3.206) and (3.207),

$$x_t \in \tilde{F}_{\max\{1,\bar{\lambda}^{-1}\}(\alpha(L_0+1)+(\bar{N}+1)\gamma_T)}.$$

Theorem 3.9 is proved.

Let $\delta_f, \delta_p > 0$ are fixed. As in the case of Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations.

3.8 Dynamic String-Averaging Proximal Point Subgradient Algorithm

We continue to consider the minimization problem

$$f(x) \to \min, \ x \in F$$

introduced in Sections 3.1 using the notation and definitions introduced there. We also suppose that all the assumptions introduced there hold.

We apply a dynamic string-averaging (DSA) method with variable strings and weights in order to obtain a good approximative solution of our problem.

Set

$$\mathcal{L} = \mathcal{L}_2 \cup \{P_{c,Q} : \ Q \in \mathcal{L}_1, \ c \in [\bar{\lambda}, \infty)\}. \tag{3.208}$$

Next we describe the dynamic string-averaging method with variable strings and weights.

By a mapping vector, we a mean a vector $S = (S_1, \ldots, S_p)$ such that $S_i \in \mathcal{L}$ for all $i = 1, \ldots, p$.

For a mapping vector $S = (S_1, \ldots, S_q)$ set

$$p(S) = q, \ P[S] = S_q \cdots S_1. \tag{3.209}$$

Lemma 3.3 and (3.209) imply that for each mapping vector $S = (S_1, \ldots, S_p)$

$$P[S](x) = x \text{ for all } x \in F, \tag{3.210}$$

$$\|P[S](x) - P[S](y)\| = \|x - P[S](y)\| \leq \|x - y\| \tag{3.211}$$

for every point $x \in F$ and every point $y \in X$.

Fix a number

$$\Delta \in (0, \mathrm{Card}(\mathcal{L})^{-1}] \tag{3.212}$$

and an integer

$$\bar{q} \geq \mathrm{Card}(\mathcal{L}). \tag{3.213}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of mapping vectors and

$$w : \Omega \to [\Delta, \infty) \text{ satisfies } \sum_{t \in \Omega} w(t) = 1 \tag{3.214}$$

such that

$$p(S) \leq \bar{q} \ \text{ for all } S \in \Omega, \tag{3.215}$$

$$\mathcal{L}_2 \subset \cup_{S \in \Omega} \{S_1, \dots, S_{p(S)}\} \tag{3.216}$$

and that the following property holds:

(i) for each $Q \in \mathcal{L}_1$, there exist $S = (S_1, \dots, S_{p(S)}) \in \Omega$ and $c(Q) \geq \bar{\lambda}$ such that $P_{c(Q),Q} \in \{S_1, \dots, S_{p(S)}\}$.

Let $(\Omega, w) \in \mathcal{M}$. Define

$$P_{\Omega,w}(x) = \sum_{S \in \Omega} w(S) P[S](x), \ \ x \in X. \tag{3.217}$$

It follows from (3.210), (3.211), (3.214) and (3.216) that

$$P_{\Omega,w}(x) = x \text{ for all } x \in F, \tag{3.218}$$

$$\|x - P_{\Omega,w}(y)\| \leq \|x - y\| \tag{3.219}$$

for every point $x \in F$ and every point $y \in X$.

The dynamic string-averaging subgradient method with variable strings and variable weights can now be described by the following algorithm.

Fix $\alpha > 0$.

The Dynamic String-Averaging Proximal Point Subgradient Algorithm

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$\xi_k \in \partial f(x_k),$$

pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} = P_{\Omega_{k+1},w_{k+1}}(x_k - \alpha\xi_k).$$

In order to proceed we need the following definitions.

Let $\delta \geq 0$, $x \in X$ and let $S = (S_1, \dots, S_{p(S)})$ be a mapping vector. Define

$$A_0(x, S, \delta) = \{y \in X : \text{ there is a sequence } \{y_i\}_{i=0}^{p(S)} \subset X \text{ such that}$$

$$y_0 = x \text{ and for all } i = 1, \dots, p(S),$$

$$\|y_i - S_i(y_{i-1})\| \leq \delta,$$

$$y = y_{p(S)}\}. \tag{3.220}$$

Let $\delta \geq 0$, $x \in X$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$A(x, (\Omega, w), \delta) = \{y \in X : \text{ there exist}$$

$$y_S \in A_0(x, S, \delta),\ S \in \Omega \text{ such that}$$

$$\|y - \sum_{S \in \Omega} w(S)y_S\| \leq \delta\}. \tag{3.221}$$

In this chapter we analyze this algorithm under the presence of computational errors. We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f while $\delta_p \in [0, 1]$ is a computational error produced by our computer system, when we calculate the operators $P_{c,S}$, $S \in \mathcal{L}_1$, $c \geq \bar{\lambda}$ and $S \in \mathcal{L}_2$. Let $\alpha > 0$ be a step size.

The Dynamic String-Averaging Proximal Point Subgradient Algorithm with Computational Errors

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$\xi_k \in \partial f(x_k) + B(0, \delta_f),$$

pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration vector x_{k+1} by

$$x_{k+1} \in A(x_k - \alpha\xi_k, (\Omega_{k+1}, w_{k+1}), \delta_p).$$

3.9 Auxiliary Results

In our study of the algorithm we use the following auxiliary results.

Lemma 3.10 *Let $M_0 \geq M_* + 2$, $L_0 \geq 1$, $(\Omega, w) \in \mathcal{M}$,*

$$\|f(z_1 - f(z_2)\| \leq L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(0, M_0 + 4), \tag{3.222}$$

and let $\delta_1, \delta_2 \in [0, 1]$, $\alpha \in (0, 1]$ satisfy

$$\delta_2 \leq \bar{q}^{-1},\ \alpha \leq (L_0 + 1)^{-1}. \tag{3.223}$$

Assume that

$$z \in F \cap B(0, M_*), \tag{3.224}$$

$$x \in B(0, M_0), \tag{3.225}$$

$$\xi \in \partial f(x) + B(0, \delta_1) \tag{3.226}$$

and that

$$y \in A(x - \alpha\xi, (\Omega, w), \delta_2). \tag{3.227}$$

Then

$$\alpha(f(x) - f(z))$$

$$\leq 2^{-1}\|x - z\|^2 - 2^{-1}\|y - z\|^2$$

$$+\delta_2(M_0(2\bar{q} + 4) + 4) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_1(2M_0 + L_0 + 1).$$

Proof By (3.221) and (3.227), for each $S = (S_1, \dots, S_{p(S)}) \in \Omega$, there exists

$$y_S \in A_0(x - \alpha\xi, S, \delta_2) \tag{3.228}$$

such that

$$\|y - \sum_{t \in \Omega} w(S)y_S\| \leq \delta_2. \tag{3.229}$$

In view of (3.220) and (3.228), for every $S = (S_1, \dots, S_{p(S)}) \in \Omega$ there exist

$$y_i^{(S)} \in X,\ i = 0, \dots, p(S)$$

such that

$$y_0^{(S)} = x - \alpha\xi, \tag{3.230}$$

for all $i = 1, \dots, p(S)$,

$$\|y_i^{(S)} - S_i(y_{i-1}^{(S)})\| \le \delta_2, \tag{3.231}$$

$$y_S = y_{p(S)}^{(S)}. \tag{3.232}$$

In view of (3.20) and (3.225),

$$\partial f(x) \subset B(0, L_0). \tag{3.233}$$

Equations (3.226) and (3.233) imply that

$$\|\xi\| \le L_0 + 1. \tag{3.234}$$

Let $S = (S_1, \dots, S_{p(S)}) \in \Omega$. By (3.222), (3.224)–(3.226), (3.230), (3.231), Lemma 3.3 and Lemma 2.7 applied with $F_0 = F$ and $Q = S_1$,

$$\begin{aligned}
&\alpha(f(x) - f(z)) \\
&\le 2^{-1}\|x - z\|^2 - 2^{-1}\|y_1^{(S)} - z\|^2 + \delta_2(2M_0 + 2 + \alpha L_0) + 2^{-1}\alpha^2 L_0^2 \\
&\quad + \alpha\delta_1(2M_0 + L_0 + 1).
\end{aligned} \tag{3.235}$$

Equations (3.225), (3.230) and (3.234) imply that

$$\|y_0^{(S)}\| \le M_0 + \alpha(L_0 + 1). \tag{3.236}$$

By (3.223), (3.224), (3.231), (3.236) and Lemma 3.3,

$$\begin{aligned}
\|z - y_1^{(S)}\| &\le \|z - S_1(y_0^{(S)})\| + \|S_1(y_0^{(S)}) - y_1^{(S)}\| \\
&\le \|z - y_0^{(S)}\| + \delta_2 \\
&\le M_0 + M_* + \delta_2 + \alpha(L_0 + 1) \le 2M_0.
\end{aligned} \tag{3.237}$$

It follows from (3.224), (3.231) and Lemma 3.3 that for all $i = 1, \dots, p(S)$,

$$\|z - y_i^{(S)}\| \le \|z - S_i(y_{i-1}^{(S)})\| + \|S_i(y_{i-1}^{(S)}) - y_i^{(S)}\|$$

$$\leq \|z - y_{i-1}^{(S)}\| + \delta_2. \tag{3.238}$$

In view of (3.223), (3.237) and (3.238),

$$\|z - y_{p(S)}^{(S)}\| \leq \|z - y_1^{(S)}\| + \delta_2(p(S) - 1)$$

$$\leq M_* + M_0 + \alpha(L_0 + 1) + \delta_2 \bar{q} \leq M_* + M_0 + 2 \leq 2M_0. \tag{3.239}$$

Equations (3.227) and (3.229) imply that

$$\|z - y_1^{(S)}\|^2 - \|z - y_{p(S)}^{(S)}\|^2$$

$$= (\|z - y_1^{(S)}\| - \|z - y_{p(S)}^{(S)}\|)(\|z - y_1^{(S)}\| + \|z - y_{p(S)}^{(S)}\|)$$

$$\geq -\delta_2 p(S)(\|z - y_1^{(S)}\| + \|z - y_{p(S)}^{(S)}\|) \geq -4M_0\delta_2\bar{q}. \tag{3.240}$$

It follows from (3.223), (3.235) and (3.239) that

$$\alpha(f(x) - f(z))$$

$$\leq 2^{-1}\|x - z\|^2 - 2^{-1}\|y_1^{(S)} - z\|^2$$

$$+\delta_2(2M_0 + 2 + \alpha L_0) + 2^{-1}\alpha^2 L_0^2$$

$$+\alpha\delta_1(2M_0 + L_0 + 1) + 2^{-1}\|z - y_1^{(S)}\|^2 - 2^{-1}\|z - y_{p(S)}^{(S)}\|^2 + 2\delta_2\bar{q}M_0$$

$$= 2^{-1}\|x - z\|^2 - 2^{-1}\|y_{p(S)}^{(S)} - z\|^2$$

$$+\delta_2(M_0(2\bar{q} + 2) + 3) + 2^{-1}\alpha^2 L_0^2$$

$$+ \alpha\delta_1(2M_0 + L_0 + 1). \tag{3.241}$$

By (3.241) and the convexity of the function $u \to \|u - z\|^2$, $u \in X$,

$$\alpha(f(x) - f(z)) - 2^{-1}\|x - z\|^2$$

$$-\delta_2(M_0(2\bar{q} + 2) + 3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_1(2M_0 + L_0 + 1)$$

$$\leq -2^{-1} \sum_{S \in \Omega} w(S)\|y_{p(S)}^{(S)} - z\|^2$$

$$\leq -2^{-1}\|\sum_{S\in\Omega} w(S)y^{(S)}_{p(S)} - z\|^2. \tag{3.242}$$

By (3.239) and the convexity of the norm,

$$\|z - \sum_{S\in\Omega} w(S)y^{(S)}_{p(S)}\| \leq \sum_{S\in\Omega} w(S)\|z - y^{(S)}_{p(S)}\| \leq 2M_0. \tag{3.243}$$

In view of (3.229) and (3.232),

$$|\|z - y\| - \|z - \sum_{S\in\Omega} w(S)y^{(S)}_{p(S)}\|| \leq \delta_2. \tag{3.244}$$

It follows from (3.243) and (3.244) that

$$|\|z - y\|^2 - \|z - \sum_{S\in\Omega} w(S)y^{(S)}_{p(S)}\|^2|$$

$$\leq \delta_2(|\|z - y\| + \|z - \sum_{S\in\Omega} w(S)y^{(S)}_{p(S)}\|) \leq \delta_2(4M_0 + 1). \tag{3.245}$$

Equations (3.242) and (3.245) imply that

$$\alpha(f(x) - f(z)) - 2^{-1}\|x - z\|^2$$

$$-\delta_2(M_0(2\bar{q} + 2) + 3) - 2^{-1}\alpha^2 L_0^2 - \alpha\delta_1(2M_0 + L_0 + 1)$$

$$\leq -2^{-1}\|z - y\|^2 + 2^{-1}\delta_2(4M_0 + 1).$$

This completes the proof of Lemma 3.10.

Lemma 3.11 *Let $M_0 \geq M_* + 2$, $(\Omega, w) \in \mathcal{M}$,*

$$z \in F \cap B(0, M_*), \tag{3.246}$$

and let $\delta_1, \delta_2 \in [0, 1]$ satisfy

$$\delta_2 \leq (\bar{q} + 1)^{-1}. \tag{3.247}$$

Assume that

$$x \in B(0, M_0), \tag{3.248}$$

$$x_0 \in B(x, \delta_1), \tag{3.249}$$

$$y \in A(x_0, (\Omega, w), \delta_2), \tag{3.250}$$

$$y_S \in A_0(x_0, S, \delta_2), \ \ S \in \Omega \tag{3.251}$$

satisfy

$$\|y - \sum_{S \in \Omega} w(S) y_S\| \le \delta_2, \tag{3.252}$$

for every $S \in \Omega$,

$$y_i^{(S)} \in X, \ \ i = 0, \dots, p(S)$$

satisfy

$$y_0^{(S)} = x_0, \tag{3.253}$$

for all $i = 1, \dots, p(S)$,

$$\|y_i^{(S)} - S_i(y_{i-1}^{(S)})\| \le \delta_2, \tag{3.254}$$

$$y_S = y_{p(S)}^{(S)}. \tag{3.255}$$

Then

$$\|z - x\|^2 - \|z - y\|^2$$

$$\ge \bar{c}\Delta \sum_{S \in \Omega} \sum_{i=1}^{p(S)} \|y_{i-1}^{(S)} - y_i^{(S)}\|^2 - 4\delta_1 M_0 - 8M_0\delta_2(\bar{q} + 1).$$

Proof Let $S = (S_1, \dots, S_{p(S)}) \in \Omega$. By (3.246), (3.248), (3.249) and (3.253),

$$\|z - y_0^{(S)}\| = \|z - x_0\| \le \|z - x\| + \|x - x_0\| \le M_0 + M_* + \delta_1. \tag{3.256}$$

It follows from (3.246)–(3.249), (3.253) and (3.256) that

$$|\|z - x\|^2 - \|z - y_0^{(S)}\|^2| \le \delta_1(2M_0 + 2M_* + \delta_1). \tag{3.257}$$

Equations (3.246), (3.254), (3.256) and Lemma 3.3 imply that for all $i = 1, \dots, p(S)$,

$$\|z - y_i^{(S)}\| \le \|z - S_i(y_{i-1}^{(S)})\| + \|S_i(y_{i-1}^{(S)}) - y_i^{(S)}\|$$

$$\leq \|z - y_{i-1}^{(S)}\| + \delta_2,$$

$$\|z - y_i^{(S)}\| \leq \|z - y_0^{(S)}\| + i\delta_2$$

$$\leq M_* + M_0 + \delta_1 + i\delta_2$$

$$\leq M_* + M_0 + \delta_2\bar{q} + \delta_1. \tag{3.258}$$

Let $i \in \{1, \dots, p(S)\}$. By (3.246), (3.247), (3.256), (3.258) and Lemma 3.3,

$$\|z - y_{i-1}^{(S)}\|^2 - \|z - y_i^{(S)}\|^2$$

$$= \|z - y_{i-1}^{(S)}\|^2 - \|z - S_i(y_{i-1}^{(S)})\|^2 + \|z - S_i(y_{i-1}^{(S)})\|^2 - \|z - y_i^{(S)}\|^2$$

$$\geq \bar{c}\|y_{i-1}^{(S)} - S_i(y_{i-1}^{(S)})\|^2$$

$$-|\|z - S_i(y_{i-1}^{(S)})\| - \|z - y_i^{(S)}\||(\|z - S_i(y_{i-1}^{(S)})\| + \|z - y_i^{(S)}\|)$$

$$\geq \bar{c}\|y_{i-1}^{(S)} - S_i(y_{i-1}^{(S)})\|^2 - \delta_2(\|z - y_{i-1}^{(S)})\| + \|z - y_i^{(S)}\|)$$

$$\geq \bar{c}\|y_{i-1}^{(S)} - S_i(y_{i-1}^{(S)})\|^2 - 4\delta_2 M_0. \tag{3.259}$$

By (3.253), (3.257) and (3.259),

$$\|z - x\|^2 - \|z - y_{p(S)}\|^2$$

$$= \|z - x\|^2 - \|z - x_0\|^2 + \|z - y_0^{(S)}\|^2 - \|z - y_{p(S)}^{(S)}\|^2$$

$$\geq -4\delta_1 M_0 + \sum_{i=1}^{p(S)}(\|z - y_{i-1}^{(S)}\|^2 - \|z - y_i^{(S)}\|^2)$$

$$\geq -4\delta_1 M_0 + \bar{c}\sum_{i=1}^{p(S)}\|y_{i-1}^{(S)} - S_i(y_{i-1}^{(S)})\|^2 - 4\delta_2 M_0\bar{q}. \tag{3.260}$$

It follows from (3.214), (3.255), (3.260) and the convexity of the function $\|u - z\|^2$, $u \in X$ that

$$\|z - x\|^2 - \|z - \sum_{S \in \Omega} w(S) y_{p(S)}^{(S)}\|^2$$

$$\geq \|z - x\|^2 - \sum_{S \in \Omega} w(S)\|z - y_{p(S)}^{(S)}\|^2$$

$$\geq \sum_{S \in \Omega} w(S)(\bar{c} \sum_{i=1}^{p(S)} \|y_{i-1}^{(S)} - S_i(y_{i-1}^{(S)})\|^2 - 4\delta_1 M_0 - 4\delta_2 M_0 \bar{q})$$

$$\geq \bar{c}\Delta \sum_{S \in \Omega} \sum_{i=1}^{p(S)} \|y_{i-1}^{(S)} - S_i(y_{i-1}^{(S)})\|^2 - 4\delta_1 M_0 - 4\delta_2 M_0 \bar{q}. \tag{3.261}$$

By (3.214), (3.247), (3.258), (3.261) and the convexity of the norm,

$$\|z - \sum_{S \in \Omega} w(S) y_{p(S)}^{(S)}\| \leq \sum_{S \in \Omega} w(S)\|z - y_{p(S)}^{(S)}\|$$

$$\leq M_0 + M_* + \delta_2 \bar{q} + 1 \leq 2M_0. \tag{3.262}$$

In view of (3.252) and (3.255),

$$|\|z - y\| - \|z - \sum_{S \in \Omega} w(S) y_{p(S)}^{(S)}\|| \leq \delta_2. \tag{3.263}$$

Equations (3.247), (3.262) and (3.263) imply that

$$\|z - y\| \leq M_* + M_0 + 1 + \delta_2(\bar{q} + 1) \leq M_* + M_0 + 2 \leq 2M_0. \tag{3.264}$$

It follows from (3.252), (3.255), (3.262) and (3.264) that

$$|\|z - y\|^2 - \|z - \sum_{S \in \Omega} w(S) y_{p(S)}^{(S)}\|^2|$$

$$\leq \|y - \sum_{S \in \Omega} w(S) y_{p(S)}^{(S)}\|(\|z - y\| + \|z - \sum_{S \in \Omega} w(S) y_{p(S)}^{(S)}\|) \leq 4\delta_2 M_0. \tag{3.265}$$

In view of (3.261) and (3.265),

$$\|z - x\|^2 - \|z - y\|^2$$

$$= \|z - x\|^2 - \|z - \sum_{S \in \Omega} w(S) y_{p(S)}^{(S)}\|^2 + \|z - \sum_{S \in \Omega} w(S) y_{p(S)}^{(S)}\|^2 - \|z - y\|^2$$

$$\geq \bar{c}\Delta \sum_{S \in \Omega} \sum_{i=1}^{p(S)} \|y_{i-1}^{(S)} - S_i(y_{i-1}^{(S)})\|^2 - 4\delta_1 M_0 - 4\delta_2 M_0 \bar{q} - 4M_0\delta_2. \tag{3.266}$$

By (3.246), (3.247), (3.254), (3.258) and Lemma 3.3, for every $S \in \Omega$ and every $i \in \{1, \dots, p(S)\}$,

$$\|y_{i-1}^{(S)} - S_i(y_{i-1}^{(S)})\| \le \|z - y_{i-1}^{(S)}\|,$$

$$|\|y_{i-1}^{(S)} - S_i(y_{i-1}^{(S)})\|^2 - \|y_{i-1}^{(S)} - y_i^{(S)})\|^2|$$

$$\le \delta_2(2\|y_{i-1}^{(S)} - S_i(y_{i-1}^{(S)})\| + \delta_2)$$

$$\le \delta_2(2\|z - y_{i-1}^{(S)}\| + \delta_2)$$

$$\le 2\delta_2(M_0 + M_* + \delta_2(\bar{q} + 1) + \delta_1) \le 4\delta_2 M_0. \tag{3.267}$$

It follows from (3.266) and (3.267) that

$$\|z - x\|^2 - \|z - y\|^2$$

$$\ge \bar{c}\Delta \sum_{S \in \Omega} \sum_{i=1}^{p(S)} \|y_{i-1}^{(S)} - y_i^{(S)}\|^2$$

$$-4\delta_1 M_0 - 4\delta_2 M_0(\bar{q} + 1) - \bar{c}\Delta \sum_{S \in \Omega} \sum_{i=1}^{p(S)} 4\delta_2 M_0$$

$$\ge \bar{c}\Delta \sum_{S \in \Omega} \sum_{i=1}^{p(S)} \|y_{i-1}^{(S)} - y_i^{(S)}\|^2$$

$$-4\delta_1 M_0 - 4\delta_2 M_0(\bar{q} + 1) - 4\bar{c}\delta_2 M_0 \bar{q}.$$

Lemma 3.11 is proved.

3.10 The First Theorem for the DSA Proximal Point Subgradient Algorithm

In the following theorem we assume that the objective function f satisfies the coercivity growth condition.

Theorem 3.12 *Let the function f be Lipschitz on bounded subsets of X,*

$$\lim_{\|x\| \to \infty} f(x) = \infty,$$

$M \geq 2M_* + 6$, $L_0 \geq 1$,

$$M_1 \geq \sup\{|f(u)| : \ u \in B(0, M_* + 4)\}, \tag{3.268}$$

$$f(u) > M_1 + 4 \textit{ for all } u \in X \setminus B(0, 2^{-1}M), \tag{3.269}$$

$$|f(z_1) - f(z_2)| \leq L_0\|z_1 - z_2\| \textit{ for all } z_1, z_2 \in B(0, 3M + 4), \tag{3.270}$$

$\delta_f, \delta_p \in [0, 1], \alpha > 0$ *satisfy*

$$\alpha \leq \min\{(L_0 + 1)^{-1}, L^{-2}\}, \ \alpha \geq 2M\delta_p(6\bar{q} + 6),$$

$$\delta_p \leq (\bar{q} + 1)^{-1}, \ \delta_f \leq (6M + L_0 + 1)^{-1}, \tag{3.271}$$

T *be a natural number and let*

$$\gamma_T = \max\{1, \bar{\lambda}^{-1}\}(\alpha(L_0 + 1)$$

$$+(\bar{q}+1)(\delta_p+((\bar{c}\Delta)^{-1}(4M^2T^{-1}+12M\alpha(L_0+1)+24\delta_p(\bar{q}+1)))^{1/2}). \tag{3.272}$$

Assume that $(\Omega_k, w_k) \in \mathcal{M}$, $k = 1, \dots, T$, $\{x_k\}_{t=0}^{k} \subset X$, $\{\xi_k\}_{k=0}^{T-1} \subset X$,

$$x_0 \in B(0, M) \tag{3.273}$$

and that for all integers $k \in \{1, \dots, T\}$,

$$B(\xi_{k-1}, \delta_f) \cap \partial f(x_{k-1}) \neq \emptyset, \tag{3.274}$$

$$x_k \in A(x_{k-1} - \alpha\xi_{k-1}, (\Omega_k, w_k), \delta_p), \tag{3.275}$$

$$y_{k,t} \in A_0(x_{k-1} - \alpha\xi_{k-1}, t, \delta_p), \ t \in \Omega_k, \tag{3.276}$$

$$\|x_k - \sum_{t \in \Omega_k} w_k(t) y_{k,t}\| \leq \delta_p \tag{3.277}$$

and that for every $S = (S_1, \dots, S_{p(S)}) \in \Omega_k$, $y_i^{(k,S)} \in X$, $i = 0, \dots, p(S)$ *satisfy*

$$y_0^{(k,S)} = x_{k-1} - \alpha\xi_{k-1}, \tag{3.278}$$

$$\|y_i^{(k,S)} - S_i(y_{i-1}^{(k,S)})\| \leq \delta_p, \ i = 1, \dots, p(S) \tag{3.279}$$

and

$$y_{k,S} = y^{(k,S)}_{p(S)}. \tag{3.280}$$

Then

$$\|x_t\| \le 3M,\ t = 0, \dots, T$$

and

$$\min\{\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p(6M(\bar{q}+2)+4) - \alpha^2 L_0^2 - 2\alpha\delta_f(3M + L_1 + 1),$$

$$\bar{c}\Delta \sum_{S\in\Omega_{k+1}} \sum_{i=1}^{p(S)} \|y^{(k+1,S)}_{i-1} - y^{(k+1,S)}_i\|^2 - 12M\alpha(L_0+1)$$

$$-24M\delta_p(\bar{q}+1)\}:\ k = 0, \dots, T-1\} \le 4M^2T^{-1}.$$

Moreover, if $k \in \{0, \dots, T-1\}$ *and*

$$\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p(6M\bar{q} + 12M + 4) - \alpha^2 L_0^2 - 2\alpha\delta_f(3M + L_1 + 1),$$

$$\bar{c}\Delta \sum_{S\in\Omega_{k+1}} \sum_{i=1}^{p(S)} \|y^{(k+1,S)}_{i-1} - y^{(k+1,S)}_i\|^2 - 12M\alpha(L_0+1)$$

$$-24\delta_p(\bar{q}+1)\} \le 4M^2T^{-1},$$

then

$$f(x_k) \le \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p(6M\bar{q} + 12M + 4) + 2^{-1}\alpha L_0^2 + \delta_f(3M + L_0 + 1)$$

and

$$x_k \in \tilde{F}_{\gamma_T}.$$

Proof In view of (3.9), there exists

$$z \in B(0, M_*) \cap F. \tag{3.281}$$

By (3.273) and (3.281),

$$\|z - x_0\| \le M + M_* \le 2M. \tag{3.282}$$

Assume that $k \in \{0, \dots, T-1\}$ and that

$$\|z - x_k\| \le 2M. \tag{3.283}$$

By (3.270), (3.278) and (3.283),

$$\partial f(x_k) \subset B(0, L_0). \tag{3.284}$$

Equations (3.274) and (3.284) imply that

$$\|\xi_k\| \le L_0 + 1. \tag{3.285}$$

Let $S = (S_1, \dots, S_{p(S)}) \in \Omega_{k+1}$. By (3.278), (3.283) and (3.285),

$$\|z - y_0^{(k+1,S)}\| = \|z - x_k + \alpha\xi_k\| \le 2M + \alpha(L_0 + 1). \tag{3.286}$$

It follows from (3.279), (3.281), (3.286) and Lemma 3.3 that for all $i = 1, \dots, p(S)$,

$$\|z - y_i^{(k+1,S)}\| \le \|z - S_i(y_{i-1}^{(k+1,S)})\|$$

$$+\|S_i(y_{i-1}^{(k+1,S)}) - y_i^{(k+1,S)}\| \le \|z - y_{i-1}^{(k+1,S)}\| + \delta_p.$$

Thus we have shown that the following property holds:

(i) is $k \in \{0, \dots, T-1\}$ and (3.283) holds, then (3.285) is true for all $S = (S_1, \dots, S_{p(S)}) \in \Omega_{k+1}$ and for all $i = 1, \dots, p(S)$,

$$\|z - y_0^{(k+1,S)}\| \le 2M + \alpha(L_0 + 1),$$

$$\|z - y_i^{(k+1,S)}\| \le \|z - y_{i-1}^{(k+1,S)}\| + \delta_p,$$

$$\|z - y_i^{(k+1,S)}\| \le \|z - y_0^{(k+1,S)}\|$$

$$+\bar{q}\delta_p \le 2M + \alpha(L_0 + 1) + \bar{q}\delta_p \le 2M + 2.$$

We show that for all $k \in \{0, \dots, T\}$ (3.283) holds. Assume that there exists an integer $n \in \{0, \dots, T\}$ such that

$$\|z - x_n\| > 2M. \tag{3.287}$$

By (3.283) and (3.287), $n > 0$. We may assume without loss of generality that (3.283) holds for all integers $k = 0, \dots, n-1$. In particular,

$$\|z - x_{n-1}\| \le 2M. \tag{3.288}$$

In view of (3.281) and (3.288),

$$\|x_{n-1}\| \le 2M + M_*. \tag{3.289}$$

By (3.270), (3.271), (3.274), (3.275), (3.281) and (3.289), we apply Lemma 3.10 with

$$\delta_1 = \delta_f,\ \delta_2 = \delta_p,\ M_0 = M,\ (\Omega, w) = (\Omega_n, w_n),$$

$$x = x_{n-1},\ \xi = \xi_{n-1},\ y = x_n,$$

and obtain that

$$\alpha(f(x_{n-1}) - f(z))$$

$$\le 2^{-1}\|x_{n-1} - z\|^2 - 2^{-1}\|x_n - z\|^2 + \delta_p(12M\bar{q} + 6M + 3)$$

$$+ \delta_p(6M + 1) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{3.290}$$

In view of (3.287) and (3.288),

$$\|z - x_n\| > \|z - x_{n-1}\|. \tag{3.291}$$

By (3.290) and (3.291),

$$\alpha(f(x_{n-1}) - f(z))$$

$$\le \delta_p(6M + 4 + 12\bar{q}M) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{3.292}$$

It follows from (3.268), (3.271), (3.281) and (3.292),

$$f(x_{n-1}) \le M_1 + \alpha^{-1}\delta_p M(12\bar{q} + 12)$$

$$+ 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1) \le M_1 + 3. \tag{3.293}$$

In view of (3.269) and (3.293),

$$\|x_{n-1}\| \le M/2. \tag{3.294}$$

In view of (3.281) and (3.294),

$$\|z - x_{n-1}\| \le 2^{-1}M + M_*. \tag{3.295}$$

Property (i) and equations (3.271), (3.278) and (3.295) imply that for all $S = (S_1, \dots, S_{p(S)}) \in \Omega_n$,

$$\|z - y_0^{(n,S)}\| \le \|z - x_{n-1} - \alpha\xi_{n-1}\| \le 2^{-1}M + M_* + \alpha(L_0 + 1),$$

$$\|z - y_{p(S)}^{(n,S)}\| \le \|z - y_0^{(n,S)}\|$$

$$+ \bar{q}\delta_p \le 2^{-1}M + M_* + 1 + \delta_p\bar{q} \le 2^{-1}M + M_* + 2. \tag{3.296}$$

By (3.277), (3.281) and (3.296),

$$\|z - x_n\| \le \|z - \sum_{S\in\Omega_n} w_n(S)y_{p(S)}^{(n,S)}\| + \|\sum_{S\in\Omega_n} w_n(S)y_{p(S)}^{(n,S)} - x_n\|$$

$$\le \sum_{S\in\Omega_n} w_n(S)\|z - y_{p(S)}^{(n,S)}\| + \delta_p$$

$$\le 2^{-1}M + M_* + 2 + \delta_p \le 2M.$$

This contradicts (3.287). The contradiction we have reached proves that

$$\|z - x_t\| \le 2M,\ t = 0, \dots, T. \tag{3.297}$$

In view of (3.281) and (3.297),

$$\|x_t\| \le 3M,\ t = 0, \dots, T. \tag{3.298}$$

By (3.270) and (3.298),

$$\partial f(x_t) \subset B(0, L_0),\ t = 0, \dots, T - 1. \tag{3.299}$$

In view of (3.274) and (3.299),

$$\|\xi_k\| \le L_0 + 1,\ k = 0, \dots, T - 1. \tag{3.300}$$

Let $k \in \{0, \dots, T - 1\}$. By (3.270), (3.274), (3.275), (3.281) and (3.298), we apply Lemma 3.10 with

$$\delta_1 = \delta_f,\ \delta_2 = \delta_p,\ M_0 = 3M,\ (\Omega, w) = (\Omega_{k+1}, w_{k+1}),$$

$$x = x_k,\ \xi = \xi_k,\ y = x_{k+1},$$

and obtain that

$$\alpha(f(x_k) - f(z))$$
$$\leq 2^{-1}\|x_k - z\|^2 - 2^{-1}\|x_{k+1} - z\|^2 + \delta_p(12M + 4)$$
$$+ 6\delta_p \bar{q} M + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \tag{3.301}$$

By (3.275)–(3.281), (3.298) and (3.300), we apply Lemma 3.11 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ M_0 = 3M,\ (\Omega, w) = (\Omega_{k+1}, w_{k+1}),$$
$$x = x_k,\ x_0 = x_k - \alpha\xi_k,\ y = x_{k+1},\ y_S = y_{k+1,S},\ S \in \Omega_{k+1},$$
$$y_i^{(S)} = y_i^{(k+1,S)},\ S \in \Omega_{k+1},\ i = 0, \dots, p(S)$$

and obtain that

$$\|z - x_k\|^2 - \|z - x_{k+1}\|^2$$
$$\geq \bar{c}\Delta \sum_{S \in \Omega_{k+1}} \sum_{i=1}^{p(S)} \|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\|^2$$
$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1). \tag{3.302}$$

In view of (3.301),

$$\|x_k - z\|^2 - \|x_{k+1} - z\|^2$$
$$\geq 2\alpha(f(x_k) - f(z))$$
$$- 2\delta_p(12M + 4 + 6\bar{q}M) - \alpha^2 L_0^2 - 2\alpha\delta_f(3M + L_0 + 1). \tag{3.303}$$

It follows from (3.282), (3.302) and (3.303) that

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$
$$= \sum_{k=0}^{T-1} (\|z - x_k\|^2 - \|z - x_{k+1}\|^2)$$

$$\geq \sum_{k=0}^{T-1} \max\{2\alpha(f(x_k) - f(z))$$

$$-2\delta_p M(6\bar{q} + 12) - 8\delta_p - \alpha^2 L_0^2 - 2\alpha\delta_f(3M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{S \in \Omega_{k+1}} \sum_{i=1}^{p(S)} \|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\|^2$$

$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\}. \tag{3.304}$$

Since z is an arbitrary element of $B(0, M_*) \cap F$ (see (3.281)) it follows from (3.14) and (3.304) that

$$4M^2T^{-1} \geq \min\{\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p M(6\bar{q} + 12) - 8\delta_p - \alpha^2 L_0^2 - 2\alpha\delta_f(3M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{S \in \Omega_{k+1}} \sum_{i=1}^{p(S)} \|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\|^2$$

$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\} : \ k \in \{0, \dots, T - 1\}. \tag{3.305}$$

Let $k \in \{0, \dots, T - 1\}$ and

$$\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p M(6\bar{q} + 12) - 8\delta_p - \alpha^2 L_0^2 - 2\alpha\delta_f(3M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{S \in \Omega_{k+1}} \sum_{i=1}^{p(S)} \|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\|^2$$

$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\} \leq 4M^2T^{-1}. \tag{3.306}$$

In view of (3.306),

$$f(x_k) \leq \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p(6\bar{q}M + 12M + 4) + 2^{-1}\alpha L_0^2 + \delta_f(3M + L_0 + 1).$$

Equation (3.306) implies that

$$\sum_{S\in\Omega_{k+1}}\sum_{i=1}^{p(S)}\|y_{i-1}^{(k+1,S)}-y_i^{(k+1,S)}\|^2$$

$$\le(\bar{c}\Delta)^{-1}(12M\alpha(L_0+1)+24M\delta_p(\bar{q}+1)+4M^2T^{-1}). \tag{3.307}$$

Set

$$\gamma_0=((\bar{c}\Delta)^{-1}(12M\alpha(L_0+1)+24M\delta_p(\bar{q}+1)+4M^2T^{-1}))^{1/2}. \tag{3.308}$$

By (3.279), (3.307) and (3.308), for all $S\in\Omega_{k+1}$ and all $i\in\{1,\dots,p(S)\}$,

$$\|y_{i-1}^{(k+1,S)}-y_i^{(k+1,S)}\|\le\gamma_0, \tag{3.309}$$

$$\|y_{i-1}^{(k+1,S)}-S_i(y_{i-1}^{(k+1,S)})\|\le\gamma_0+\delta_p. \tag{2.308}$$

In view of (3.278), (3.300) and (3.309), for all $S\in\Omega_{k+1}$ and all $i\in\{1,\dots,p(S)\}$,

$$\|y_i^{(k+1,S)}-x_k\|\le\|y_0^{(k+1,S)}-x_k\|+\gamma_0 i$$

$$\le\alpha(L_0+1)+\bar{q}\gamma_0. \tag{3.310}$$

Let $Q\in\mathcal{L}_2$. By (3.216), there exists $S\in\Omega$ and $i\in\{1,\dots,p(S)\}$ such that

$$S_i=Q.$$

Combined with (3.308) this implies that

$$d(x_k,\mathrm{Fix}_{\gamma_0+\delta_p}(Q))\le\alpha(L_0+1)+\bar{q}\gamma_0. \tag{3.311}$$

Let $Q\in\mathcal{L}_1$. By property (i), there exist $c\ge\bar{\lambda}$, $S\in\Omega_{k+1}$ and $i\in\{1,\dots,p(S)\}$ such that

$$S_i=P_{c,Q}.$$

Together with (3.308) and Lemma 3.4 this implies that

$$B(y_{i-1}^{(k+1)},\gamma_0+\delta_p)\cap F_{\bar{\lambda}^{-1}(\gamma_0+\delta_p)}(Q)\ne\emptyset. \tag{3.312}$$

It follows from (3.310) and (3.312) that

$$d(x_k,F_{\bar{\lambda}^{-1}(\gamma_0+\delta_p)}(Q))\le\|x_k-y_{i-1}^{k+1}\|+\gamma_0+\delta_p$$

$$\le\alpha(L_0+1)+(\bar{q}+1)\gamma_0+\delta_p. \tag{3.313}$$

By (3.272), (3.308), (3.310) and (3.313),

$$x_t \in \tilde{F}_{\gamma_T}.$$

Theorem 3.12 is proved.

Let $\delta_f, \delta_p > 0$ are fixed. As in the case of Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations.

3.11 The Second Theorem for the DSA Subgradient Algorithm

In the following theorem the set F is bounded.

Theorem 3.13 *Let* $r_0 \in (0, 1]$,

$$\tilde{F}_{r_0} \subset B(0, M_*), \tag{3.314}$$

$M \geq 2M_* + 2$, $L_0 \geq 1$,

$$|f(z_1) - f(z_2)| \leq L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B(0, 3M + 4), \tag{3.315}$$

$\delta_f, \delta_p \in [0, 1]$, $\alpha > 0$ *satisfy*

$$\alpha \leq 96^{-1}(1 + \bar{q})^{-2}(L_0 + 1)^{-1}r_0^2\bar{c}\Delta M^{-1}\min\{1, \bar{\lambda}\}^2,$$

$$\delta_p \leq 96^{-1}r_0^2\bar{c}\Delta M^{-1}(\bar{q} + 1)^{-3}\min\{1, \bar{\lambda}\}^2, \tag{3.316}$$

T be a natural number and let

$$\gamma_T = \max\{1, \bar{\lambda}^{-1}\}(\alpha(L_0 + 1)$$

$$+(\bar{q}+1)(\delta_p+((\bar{c}\Delta)^{-1}(4M^2T^{-1}+12M\alpha(L_0+1)+24\delta_p(\bar{q}+1))^{1/2}). \tag{3.317}$$

Assume that $(\Omega_k, w_k) \in \mathcal{M}$, $k = 1, \ldots, T$, $\{x_k\}_{t=0}^{k} \subset X$, $\{\xi_k\}_{k=0}^{T-1} \subset X$,

$$x_0 \in B(0, M) \tag{3.318}$$

and that for all integers $k \in \{1, \ldots, T\}$,

$$B(\xi_{k-1}, \delta_f) \cap \partial f(x_{k-1}) \neq \emptyset, \tag{3.319}$$

$$x_k \in A(x_{k-1} - \alpha\xi_{k-1}, (\Omega_k, w_k), \delta_p), \tag{3.320}$$

$$y_{k,t} \in A_0(x_{k-1} - \alpha\xi_{k-1}, t, \delta_p), \ t \in \Omega_k, \tag{3.321}$$

$$\|x_k - \sum_{t \in \Omega_k} w_k(t) y_{k,t}\| \le \delta_p \tag{3.322}$$

and that for every $S = (S_1, \dots, S_{p(S)}) \in \Omega_k$, $y_i^{(k,S)} \in X$, $i = 0, \dots, p(S)$ *satisfy*

$$y_0^{(k,S)} = x_{k-1} - \alpha\xi_{k-1}, \tag{3.323}$$

$$\|y_i^{(k,S)} - S_i(y_{i-1}^{(k,S)})\| \le \delta_p, \ i = 1, \dots, p(S) \tag{3.324}$$

and

$$y_{k,S} = y_{p(S)}^{(k,S)}. \tag{3.325}$$

Then

$$\|x_t\| \le 3M, \ t = 0, \dots, T$$

and

$$\min\{\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p(M(6\bar{q} + 12) + 4) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{S \in \Omega_{k+1}} \sum_{i=1}^{p(S)} \|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\|^2 - 12M\alpha(L_0 + 1)$$

$$-24M\delta_p(\bar{q} + 1)\} : \ k = 0, \dots, T - 1\} \le 4M^2 T^{-1}.$$

Moreover, if $k \in \{0, \dots, T - 1\}$ *and*

$$\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p(M(6\bar{q} + 12) + 4) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{S \in \Omega_{k+1}} \sum_{i=1}^{p(S)} \|y_{i-1}^{(k+1,t)} - y_i^{(k+1,t)}\|^2 - 12M\alpha(L_0 + 1)$$

$$-24M\delta_p(\bar{q} + 1)\} \le 4M^2 T^{-1},$$

then

$$f(x_k) \le \inf(f, F) + 2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p(6M\bar{q} + 12M + 4)$$

$$+2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1)$$

and

$$x_k \in \tilde{F}_{\gamma_T}.$$

Proof In view of (3.9), there exists

$$z \in B(0, M_*) \cap F. \tag{3.326}$$

By (3.318) and (3.326),

$$\|z - x_0\| \le 2M. \tag{3.327}$$

Assume that $k \in \{0, \dots, T-1\}$ and that

$$\|z - x_k\| \le 2M. \tag{3.328}$$

By (3.315), (3.326) and (3.328),

$$\partial f(x_k) \subset B(0, L_0). \tag{3.329}$$

Equations (3.319) and (3.329) imply that

$$\|\xi_k\| \le L_0 + 1. \tag{3.330}$$

Let $S = (S_1, \dots, S_{p(S)}) \in \Omega_{k+1}$. By (3.323), (3.328) and (3.330),

$$\|z - y_0^{(k+1,S)}\| = \|z - x_k + \alpha\xi_k\| \le 2M + \alpha(L_0 + 1). \tag{3.331}$$

It follows from (3.324), (3.326) and Lemma 3.3 that for all $i = 1, \dots, p(S)$,

$$\|z - y_i^{(k+1,S)}\| \le \|z - S_i(y_{i-1}^{(k+1,S)})\|$$

$$+\|S_i(y_{i-1}^{(k+1,S)}) - y_i^{(k+1,S)}\| \le \|z - y_{i-1}^{(k+1,S)}\| + \delta_p.$$

Thus we have shown that the following property holds:

(i) is $k \in \{0, \dots, T-1\}$ and (3.328) holds, then (3.330) is true and for all $S = (S_1, \dots, S_{p(S)}) \in \Omega_{k+1}$ and for all $i = 1, \dots, p(S)$,

$$\|z - y_0^{(k+1,S)}\| \le 2M + \alpha(L_0 + 1),$$

$$\|z - y_i^{(k+1,S)}\| \le \|z - y_0^{(k+1,S)}\| + \bar{q}\delta_p$$

$$\le 2M + \alpha(L_0 + 1) + \bar{q}\delta_p \le 2M + 2$$

(see (3.316)).

We show that for all $k \in \{0, \dots, T\}$ Equation (3.328) holds. Assume that there exists an integer $n \in \{0, \dots, T\}$ such that

$$\|z - x_n\| > 2M. \tag{3.332}$$

By (3.327) and (3.332), $n > 0$. We may assume without loss of generality that (3.328) holds for all integers $k = 0, \dots, n-1$. In particular,

$$\|z - x_{n-1}\| \le 2M. \tag{3.333}$$

In view of (3.326) and (3.333),

$$\|x_{n-1}\| \le 2M + M_*. \tag{3.334}$$

By property (i), (3.316), (3.320)–(3.326), (3.330) and (3.334), we apply Lemma 3.11 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ M_0 = 3M,\ (\Omega, w) = (\Omega_n, w_n),$$

$$x = x_{n-1},\ \xi = \xi_{n-1},\ x_0 = x_{n-1} - \alpha\xi_{n-1},\ y = x_n,\ y_S = y_{n,S},\ S \in \Omega_n,$$

$$y_i^{(S)} = y_i^{(n,S)},\ S \in \Omega_n,\ i = 0, \dots, p(S)$$

and obtain that

$$\|z - x_{n-1}\|^2 - \|z - x_n\|^2$$

$$\ge \bar{c}\Delta \sum_{S \in \Omega_n} \sum_{i=1}^{p(S)} \|y_{i-1}^{(n,S)} - y_i^{(n,S)}\|^2$$

$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1). \tag{3.335}$$

In view of (3.332) and (3.333),

$$\|z - x_n\| > \|z - x_{n-1}\|. \tag{3.336}$$

By (3.335) and (3.336), for all $S \in \Omega_n$ and all $i \in \{1, \dots, p(S)\}$,

$$\|y_{i-1}^{(n,S)} - y_i^{(n,S)}\| \le ((12M\alpha(L_0+1) + 24M\delta_p(\bar{q}+1))(\bar{c}\Delta)^{-1})^{1/2}. \tag{3.337}$$

It follows from (3.324) and (3.337) that for all $S \in \Omega_n$ and all $i \in \{1, \dots, p(S)\}$,

$$\|y_{i-1}^{(n,S)} - S_i(y_{i-1}^{(n,S)})\| \le \|y_{i-1}^{(n,S)} - y_i^{(n,S)}\| + \|S_i(y_{i-1}^{(n,S)}) - y_i^{(n,S)}\|$$

$$\le ((12M\alpha(L_0+1) + 24M\delta_p(\bar{q}+1))(\bar{c}\Delta)^{-1})^{1/2} + \delta_p. \tag{3.338}$$

Property (i), (3.224), (3.330), (3.333) and (3.337) imply that for all $S \in \Omega_n$ and all $i \in \{0, 1, \dots, p(S)\}$,

$$\|x_{n-1} - y_i^{(n,S)}\| \le \|x_{n-1} - y_0^{(n,S)}\| + \|y_0^{(n,S)} - y_i^{(n,S)}\|$$

$$\le \alpha(L_0+1) + \bar{q}((12M\alpha(L_0+1) + 24M\delta_p(\bar{q}+1))(\bar{c}\Delta)^{-1})^{1/2}. \tag{3.339}$$

Set

$$\tilde{r} = ((12M\alpha(L_0+1) + 24M\delta_p(\bar{q}+1))(\bar{c}\Delta)^{-1})^{1/2}. \tag{3.340}$$

Let $Q \in \mathcal{L}_2$. By (3.216), there exists $S \in \Omega_n$ and $i \in \{1, \dots, p(S)\}$ such that

$$S_i = Q.$$

Combined with (3.338)–(3.340) this implies that

$$d(x_{n-1}, \mathrm{Fix}_{\tilde{r}+\delta_p}(Q)) \le \alpha(L_0+1) + \bar{q}(\tilde{r}+\delta_p),\ \ Q \in \mathcal{L}_2. \tag{3.341}$$

Let $Q \in \mathcal{L}_1$. By property (i), there exist $c \ge \bar{\lambda}$, $S \in \Omega_n$ and $i \in \{1, \dots, p(S)\}$ such that

$$S_i = P_{c,Q}. \tag{3.342}$$

Lemma 3.4, (3.338), (3.340) and (3.342) this implies that

$$B(y_{i-1}^{(n,S)}, \tilde{r}+\delta_p) \cap F_{\bar{\lambda}^{-1}(\tilde{r}+\delta_p)}(Q) \ne \emptyset. \tag{3.343}$$

It follows from (3.241), (3.339) and (3.343) that

$$d(x_{n-1}, F_{\bar{\lambda}^{-1}(\tilde{r}+\delta_p)}(Q)) \le \alpha(L_0+1) + (\bar{q}+1)\tilde{r} + \delta_p. \tag{3.344}$$

By (3.316), (3.344) and (3.347),

$$x_{n-1} \in \tilde{F}_{\alpha(L_0+1)+\max\{1,\bar{\lambda}\}^{-1}(\bar{q}+1)\tilde{r}+\delta_p)}. \tag{3.345}$$

In view of (3.314), (3.326) and (3.345),

$$\|x_{n-1}\| \le M_*, \tag{3.346}$$

$$\|z - x_{n-1}\| \le 2M_*. \tag{3.347}$$

Property (i), (3.323), (3.330) and (3.333) imply that for all $S \in \Omega_n$,

$$\|z - y_{p(S)}^{(n,S)}\| \le \|z - y_0^{(n,S)}\| + \bar{q}\delta_p$$

$$\le \|z - x_{n-1}\| + \alpha(L_0 + 1) + \bar{q}\delta_p \le 2M_* + 2. \tag{3.348}$$

By (3.325), (3.348) and the convexity of the norm,

$$\|z - x_n\| \le \|z - \sum_{S \in \Omega_n} w_n(S) y_{p(S)}^{(n,S)}\| + \|\sum_{S \in \Omega_n} w_n(S) y_{p(S)}^{(n,S} - x_n\|$$

$$\le \sum_{S \in \Omega_n} w_n(S) \|z - y_{p(S)}^{(n,S)}\| + \delta_p$$

$$\le 2M_* + 3 < 2M.$$

This contradicts (3.332). The contradiction we have reached proves that

$$\|z - x_t\| \le 2M, \ t = 0, \dots, T. \tag{3.349}$$

In view of (3.326) and (3.349),

$$\|x_t\| \le 3M, \ t = 0, \dots, T. \tag{3.350}$$

By (3.315), (3.319) and (3.350),

$$\partial f(x_t) \subset B(0, L_0), \ \|\xi_t\| \le L_0 + 1, \ t = 0, \dots, T - 1. \tag{3.351}$$

Let $k \in \{0, \dots, T - 1\}$. By (3.315), (3.316), (3.319), (3.320), (3.323), (3.326) and (3.350) we apply Lemma 3.10 with

$$\delta_1 = \delta_f, \ \delta_2 = \delta_p, \ M_0 = 3M, \ (\Omega, w) = (\Omega_{k+1}, w_{k+1}),$$

$$x = x_k, \ \xi = \xi_k, \ y = x_{k+1},$$

and obtain that

$$\alpha(f(x_k) - f(z))$$

$$\leq 2^{-1}\|x_k - z\|^2 - 2^{-1}\|x_{k+1} - z\|^2 + \delta_p(12M + 4)$$

$$+ 6\delta_p \bar{q} M + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6M + L_0 + 1). \quad (3.352)$$

By (3.316), (3.320)–(3.326), (3.350) and (3.351), we apply Lemma 3.11 with

$$\delta_1 = \alpha(L_0 + 1),\ \delta_2 = \delta_p,\ M_0 = 3M,\ (\Omega, w) = (\Omega_{k+1}, w_{k+1}),$$

$$x = x_k,\ x_0 = x_k - \alpha\xi_k,\ y = x_{k+1},\ y_S = y_{k+1,S},\ S \in \Omega_{k+1},$$

$$y_i^{(S)} = y_i^{(k+1,S)},\ S \in \Omega_{k+1},\ i = 0, \dots, p(S)$$

and obtain that

$$\|z - x_k\|^2 - \|z - x_{k+1}\|^2$$

$$\geq \bar{c}\Delta \sum_{S \in \Omega_{k+1}} \sum_{i=1}^{p(S)} \|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\|^2$$

$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1). \quad (3.353)$$

In view of (3.352),

$$\|x_k - z\|^2 - \|x_{k+1} - z\|^2$$

$$\geq 2\alpha(f(x_k) - f(z))$$

$$- 2\delta_p(12M + 4 + 6\bar{q}M) - \alpha^2 L_0^2 - 2\alpha\delta_f(3M + L_0 + 1). \quad (3.354)$$

It follows from (3.327), (3.353) and (3.354) that

$$4M^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{k=0}^{T-1} (\|z - x_k\|^2 - \|z - x_{k+1}\|^2)$$

$$\geq \sum_{k=0}^{T-1} \max\{2\alpha(f(x_k) - f(z))$$

$$-2\delta_p(12M + 4 + 6M\bar{q}) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{S \in \Omega_{k+1}} \sum_{i=1}^{p(S)} \|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\|^2$$

$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\}. \tag{3.355}$$

Since z is an arbitrary element of $B(0, M_*) \cap F$ it follows from (3.14) and (3.355) that

$$4M^2T^{-1} \geq \min\{\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p(12M + 4 + 6M\bar{q}) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{S \in \Omega_{k+1}} \sum_{i=1}^{p(S)} \|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\|^2$$

$$-12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\} : \ k \in \{0, \dots, T - 1\}.$$

Let $k \in \{0, \dots, T - 1\}$ and

$$\max\{2\alpha(f(x_k) - \inf(f, F))$$

$$-2\delta_p(6M\bar{q} + 12M + 4) - \alpha^2 L_0^2 - 2\alpha\delta_f(6M + L_0 + 1),$$

$$\bar{c}\Delta \sum_{S \in \Omega_{k+1}} \sum_{i=1}^{p(S)} \|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\|^2$$

$$- 12M\alpha(L_0 + 1) - 24M\delta_p(\bar{q} + 1)\} \leq 4M^2T^{-1}. \tag{3.356}$$

In view of (3.356),

$$f(x_k) \leq \inf(f, F)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\delta_p(6M\bar{q} + 12M + 4)$$

$$+ 2^{-1}\alpha L_0^2 + \delta_f(6M + L_0 + 1). \tag{3.357}$$

and

$$\sum_{S \in \Omega_{k+1}} \sum_{i=1}^{p(S)} \|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\|^2$$

$$\leq (\bar{c}\Delta)^{-1}(12M\alpha(L_0+1)+24M\delta_p(\bar{q}+1)+4M^2T^{-1}). \tag{3.358}$$

Set

$$\gamma_0 = ((\bar{c}\Delta)^{-1}(12M\alpha(L_0+1)+24M\delta_p(\bar{q}+1)+4M^2T^{-1}))^{1/2}. \tag{3.359}$$

By (3.358) and (3.359), for all $S \in \Omega_{k+1}$ and all $i \in \{1, \ldots, p(S)\}$,

$$\|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\| \leq \gamma_0. \tag{3.360}$$

It follows from (3.324) and (3.360) that for all $S \in \Omega_{k+1}$ and all $i \in \{1, \ldots, p(S)\}$,

$$\|y_{i-1}^{(k+1,S)} - S_i(y_{i-1}^{(k+1,S)})\| \leq \|y_{i-1}^{(k+1,S)} - y_i^{(k+1,S)}\| + \|S_i(y_{i-1}^{(k+1,S)}) - y_i^{(k+1,S)}\|$$

$$\leq \gamma_0 + \delta_p. \tag{3.361}$$

In view of (3.323), (3.360) and (3.361), for all $S \in \Omega_{k+1}$ and all $i \in \{1, \ldots, p(S)\}$,

$$\|y_i^{(k+1,S)} - x_k\| \leq \|y_0^{(k+1,S)} - x_k\| + \gamma_0 i$$

$$\leq \alpha(L_0+1) + \bar{q}\gamma_0. \tag{3.362}$$

Let $Q \in \mathcal{L}_2$. By (3.216), there exists $S \in \Omega_{k+1}$ and $i \in \{1, \ldots, p(S)\}$ such that

$$S_i = Q.$$

Combined with (3.361) and (3.362) this implies that

$$d(x_k, \mathrm{Fix}_{\gamma_0+\delta_p}(Q)) \leq \alpha(L_0+1) + \bar{q}\gamma_0. \tag{3.363}$$

Let $Q \in \mathcal{L}_1$. By property (i), there exist $c \geq \bar{\lambda}$, $S \in \Omega_{k+1}$ and $i \in \{1, \ldots, p(S)\}$ such that

$$S_i = P_{c,Q}. \tag{3.364}$$

In view of (3.361), (3.364) and Lemma 3.4,

$$B(y_{i-1}^{(k+1)}, \gamma_0 + \delta_p) \cap F_{\bar{\lambda}^{-1}(\gamma_0+\delta_p)}(Q) \neq \emptyset.$$

Combined with (3.362) this implies that

$$d(x_k, F_{\bar{\lambda}^{-1}(\gamma_0+\delta_p)}(Q)) \leq \|x_k - y_{i-1}^{(k+1)}\| + \gamma_0 + \delta_p$$

$$\alpha(L_0+1) + (\bar{q}+1)\gamma_0 + \delta_p. \tag{3.365}$$

By (3.317), (3.363) and (3.365),

$$x_k \in \tilde{F}_{\alpha(L_0+1)+\bar{q}\gamma_0+(\gamma_0+\delta_p)\max\{1,\bar{\lambda}^{-1}\}} \subset \tilde{F}_{\gamma_T}.$$

Theorem 3.13 is proved.

Let $\delta_f, \delta_p > 0$ are fixed. As in the case of Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations.

Chapter 4
Cimmino Subgradient Projection Algorithm

In this chapter we consider a minimization of a convex function on a solution set of a convex feasibility problem in a general Hilbert space using the Cimmino subgradient projection algorithm. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We show that an algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a small constant. Moreover, if we known computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this.

4.1 Preliminaries

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

For every point $x \in X$ and every nonempty set $A \subset X$ define

$$d(x, A) = \inf\{\|x - y\| : \ y \in A\}.$$

For every point $x \in X$ and every positive number r put

$$B(x, r) = \{y \in X : \ \|x - y\| \le r\}.$$

It is well-known [8, 93] that for every nonempty closed convex subset C of the space X and every point $x \in X$ there exists a unique point $P_C(x) \in C$ which satisfies

$$\|x - P_C(x)\| = \inf\{\|x - y\| : \ y \in C\}, \tag{4.1}$$

A. J. Zaslavski, *Optimization on Solution Sets of Common Fixed Point Problems*, Springer Optimization and Its Applications 178,
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$$\|P_C(x) - P_C(y)\| \leq \|x - y\| \text{ for all } x, y \in X \tag{4.2}$$

and for every point $x \in X$ and every point $z \in C$,

$$\langle z - P_C(x), x - P_C(x)\rangle \leq 0, \tag{4.3}$$

$$\|z - P_C(x)\|^2 + \|x - P_C(x)\|^2 \leq \|z - x\|^2. \tag{4.4}$$

Denote by $\mathcal{N}$ be a set of all nonnegative integers. We recall the following useful facts on convex functions.

Let $f : X \to R^1$ be a continuous convex function such that

$$\{x \in X : \; f(x) \leq 0\} \neq \emptyset. \tag{4.5}$$

Let $y_0 \in X$. Then

$$\partial f(y_0) = \{l \in X : \; f(y) - f(y_0) \geq \langle l, y - y_0\rangle \text{ for all } y \in X\} \tag{4.6}$$

is the subdifferential of f at the point y_0 [61, 62, 75].

For every $l \in \partial f(y_0)$ it follows from (4.6) that

$$\{x \in X : \; f(x) \leq 0\} \subset \{x \in X : \; f(y_0) + \langle l, x - y_0\rangle \leq 0\}. \tag{4.7}$$

It is well-known that the following lemma holds (see Lemma 11.1 of [94]).

Lemma 4.1 *Let $y_0 \in X$, $f(y_0) > 0$, $l \in \partial f(y_0)$ and let*

$$D = \{x \in X : \; f(y_0) + \langle l, x - y_0\rangle \leq 0\}.$$

The $l \neq 0$ and

$$P_D(y_0) = y_0 - f(y_0)\|l\|^{-2}l.$$

Let us now describe the convex feasibility problem and the Cimmino subgradient projection algorithm which will studied in this chapter.

Let m be a natural number and $f_i : X \to R^1$, $i = 1, \ldots, m$ be convex continuous functions.

For every integer $i = 1, \ldots, m$ put

$$C_i = \{x \in X : \; f_i(x) \leq 0\}, \tag{4.8}$$

$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : \; f_i(x) \leq 0\}. \tag{4.9}$$

We suppose that

$$C \neq \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given positive number ϵ a point $x \in X$ is called an ϵ-approximate solution of the feasibility problem if

$$f_i(x) \leq \epsilon \text{ for all } i = 1, \dots, m.$$

Let $M_* > 0$ and

$$C \cap B(0, M_*) \neq \emptyset. \tag{4.10}$$

Let $f : X \to R^1$ be a continuous function. We consider the minimization problem

$$f(x) \to \min, \ x \in C.$$

Assume that

$$\inf(f, C) = \inf(f, C \cap B(0, M_*)). \tag{4.11}$$

4.2 Cimmino Subgradient Projection Algorithm

Fix

$$\bar{\Delta} \in (0, m^{-1}]. \tag{4.12}$$

Let us describe our algorithm.

Cimmino Subgradient Projection Algorithm Fix $\alpha > 0$.
Initialization: select an arbitrary $x_0 \in X$.
Iterative step: given a current iteration vector $x_k \in X$ calculate

$$l_k \in \partial f(x_k),$$

pick $w_{k+1} = (w_{k+1}(1), \dots, w_{k+1}(m)) \in R^m$ such that

$$w_{k+1}(i) \geq \bar{\Delta}, \ i = 1, \dots, m,$$

$$\sum_{i=1}^{m} w_{k+1}(i) = 1,$$

for each $i \in \{1, \dots, m\}$,

$$\text{if } f_i(x_k - \alpha l_k) \le 0 \text{ then } x_{k,i} = x_k - \alpha l_k,\ l_{k,i} = 0$$

and if $f_i(x_k - \alpha l_k) > 0$ then

$$l_{k,i} \in \partial f_i(x_k - \alpha l_k),$$

$$x_{k,i} = x_k - \alpha l_k - f_i(x_k - \alpha l_k)\|l_{k,i}\|^{-2} l_{k,i}$$

and define the next iteration vector

$$x_{k+1} = \sum_{i=1}^{m} w_{k+1}(i) x_{k,i}.$$

In this chapter we analyze this algorithm under the presence of computational errors. We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f, $\delta_C \in [0, 1]$ is a computational error produced by our computer system, when we calculate subgradients of the constraint functions f_i, $i = 1, \dots, m$ and $\bar{\delta}_C$ is a computational error produced by our computer system, when we calculate auxiliary projection operators. Let $\alpha > 0$ be a step size and $\Delta \ge 0$.

Cimmino Subgradient Projection Algorithm with Computational Errors

Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_t \in X$ calculate

$$l_t \in \partial f(x_t) + B(0, \delta_f),$$

pick $w_{t+1} = (w_{t+1}(1), \dots, w_{t+1}(m)) \in R^m$ such that

$$w_{t+1}(i) \ge \bar{\Delta},\ i = 1, \dots, m,$$

$$\sum_{i=1}^{m} w_{t+1}(i) = 1,$$

for each $i \in \{1, \dots, m\}$,

$$\text{if } f_i(x_t - \alpha l_t) \le \Delta, \text{ then } y_{t+1,i} = x_t - \alpha l_t,\ l_{t,i} = 0,$$

if $f_i(x_t - \alpha l_t) > \Delta$, then we calculate

$$l_{t,i} \in \partial f_i(x_t - \alpha l_t) + B(0, \delta_C),$$

(this implies that $l_{t,i} \ne 0$),

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2}l_{t,i}, \bar{\delta}_C)$$

and the next iteration vector

$$x_{t+1} \in B(\sum_{i=1}^{m} w_{t+1}(i)y_{t+1,i}, \bar{\delta}_C).$$

Let $\delta_f, \delta_C, \bar{\delta}_C \in [0, 1]$ and $\alpha > 0$. In order to study our algorithm we need the following auxiliary results.

Lemma 4.2 *Let $\Delta \in (0, 1]$, $M \geq M_*$, $M_0 \geq 1$, $M_1 > 2$, $\delta_1, \delta_2, \bar{\delta}_2 \in [0, 1]$,*

$$f_i(B(0, M + 1)) \subset [-M_0.M_0],\ i = 1, \dots, m, \tag{4.13}$$

$$|f_i(u) - f_i(v)| \leq (M_1 - 2)\|u - v\|$$

$$\textit{for all } u, v \in (0, M + 2) \textit{ and all } i = 1, \dots, m, \tag{4.14}$$

$$\delta_2 \leq \min\{2^{-1}\Delta(2M + 1)^{-1},\ 16^{-1}M_0^{-1}(2M + 1)^{-2}\Delta^2\}, \tag{4.15}$$

$$z \in B(0, M) \cap C, \tag{4.16}$$

$$x \in B(0, M), \tag{4.17}$$

$$x_0 \in B(x, \delta_1), \tag{4.18}$$

$j \in \{1, \dots, m\}$,

$$f_j(x_0) > 0 \tag{4.19}$$

and let at least one of the following conditions hold:

(a) for each $i \in \{1, \dots, m\}$,

$$B(0, M_*) \cap \{v \in X :\ f_i(x) \leq -\Delta\} \neq \emptyset; \tag{4.20}$$

(b)

$$f_j(x_0) > \Delta. \tag{4.21}$$

Assume that

$$\xi \in \partial f_j(x_0), \tag{4.22}$$

$$l \in B(\xi, \delta_2) \tag{4.23}$$

(this implies that $\xi \neq 0,\ l \neq 0$*),*

$$u \in B(x_0 - f_j(x_0)\|l\|^{-2}l, \bar{\delta}_2). \tag{4.24}$$

Then

$$\|\xi\| \geq \Delta(2M+1)^{-1},$$

$$\|l\| \geq 2^{-1}\Delta(2M+1)^{-1},$$

$$\|z-u\| \leq \|z-x\| + \delta_1 + \bar{\delta}_2 + 16M_0\delta_2\Delta^{-2}(2M+1)^2,$$

$$\|u-(x_0 - f_j(x_0)\|\xi\|^{-2}\xi)\| \leq \bar{\delta}_2 + 16M_0\delta_2\Delta^{-2}(2M+1)^2,$$

$$\|z-x\|^2 - \|x-u\|^2 - \|z-u\|^2$$

$$\geq -\delta_1(2M+1) - 16\delta_2\Delta^{-2}(2M+4)^3 - (\delta_1+\bar{\delta}_2)(2M+5),$$

$$\|x-u\| \geq -\delta_1 - \bar{\delta}_2 - 16\delta_2 M_0\Delta^{-2}(2M+1)^2 + M_1^{-1}f_j(x_0).$$

Proof Set

$$D = \{v \in X : \ f_j(x_0) + \langle \xi, v - x_0\rangle \leq 0\}. \tag{4.25}$$

In view of (4.9), (4.19) and (4.22),

$$\xi \neq 0.$$

Set

$$y = x_0 - f_j(x_0)\|\xi\|^{-2}\xi. \tag{4.26}$$

Lemma 4.1, (4.19), (4.20), (4.22) and (4.25) imply that

$$P_D(x_0) = y. \tag{4.27}$$

By (4.4), (4.16), (4.25) and (4.27),

$$\|z-y\|^2 = \|z - P_D(x_0)\| \leq \|z-x_0\|^2 - \|x_0-y\|^2. \tag{4.28}$$

Equations (4.13), (4.15), (4.17) and (4.18) imply that

$$f_j(x_0) \leq M_0. \tag{4.29}$$

In view of (4.14), (4.15), (4.18) and (4.22),

$$\|\xi\| \le M_1 - 2. \tag{4.30}$$

By (4.15), (4.23) and (4.30),

$$\|l\| \le M_1 - 1. \tag{4.31}$$

If (b) holds set

$$z_j = z. \tag{4.32}$$

If (a) holds, then in view of (4.20), there exists

$$z_j \in B(0, M_*) \text{ such that } f_j(z_j) \le -\Delta. \tag{4.33}$$

By (4.16), (4.17)–(4.22), (4.32) and (4.33),

$$-\Delta \ge f_j(z_j) - f_j(x_0) \ge \langle \xi, z_j - x_0 \rangle$$

$$\ge -\|\xi\|\|z_j - x_0\| \ge -\|\xi\|(2M + 1),$$

$$\|\xi\| \ge \Delta(2M + 1)^{-1}. \tag{4.34}$$

It follows from (4.15), (4.23) and (4.34) that

$$\|l\| \ge \|\xi\| - \delta_2 \ge \Delta(2M + 1)^{-1} - \delta_2 \ge 2^{-1}\Delta(2M + 1)^{-1} \tag{4.35}$$

and

$$l \ne 0.$$

By (4.24), (4.26) and (4.29),

$$\|u - y\| \le \bar{\delta}_2 + \|x_0 - f_j(x_0)\|l\|^{-2}l - y\|$$

$$\le \bar{\delta}_2 + \|f_j(x_0)\|\xi\|^{-2}\xi - f_j(x_0)\|l\|^{-2}l\|$$

$$\le \bar{\delta}_2 + M_0\|\|\xi\|^{-2}\xi - \|l\|^{-2}l\|. \tag{4.36}$$

In view of (4.15), (4.23), (4.34) and (4.35),

$$\|\|\xi\|^{-2}\xi - \|l\|^{-2}l\|$$

$$\le \|l\|^{-2}\|l - \xi\| + \|\xi\||\|\xi\|^{-2} - \|l\|^{-2}|$$

$$\leq \|l\|^{-2}\|l-\xi\| + \|l\|^{-2}\|\xi\|^{-1}|\|\xi\|^2 - \|l\|^2|$$

$$\leq \|l\|^{-2}(\delta_2 + \|\xi\|^{-1}\delta_2(\|l\| + \|\xi\|))$$

$$\leq \|l\|^{-2}\delta_2(1 + \|\xi\|^{-1}(\delta_2 + 2\|\xi\|))$$

$$\leq 4\delta_2\|l\|^{-2} \leq 16\delta_2\Delta^{-2}(2M+1)^2. \tag{4.37}$$

Equations (4.36) and (4.37) imply that

$$\|u-y\| \leq \bar{\delta}_2 + 16M_0\Delta^{-2}(2M+1)^2. \tag{4.38}$$

By (4.18), (4.28) and (4.38),

$$\|z-u\| \leq \|z-y\| + \|u-y\|$$

$$\leq \|z-x_0\| + \bar{\delta}_2 + 16M_0\Delta^{-2}(2M+1)^2$$

$$\leq \|z-x\| + \delta_1 + \bar{\delta}_2 + 16M_0\Delta^{-2}(2M+1)^2\delta_2. \tag{4.39}$$

It follows from (4.16)–(4.18) that

$$|\|z-x_0\|^2 - \|z-x\|^2\||$$

$$\leq \|x-x_0\|(\|z-x_0\| + \|z-x\|)$$

$$\leq \delta_1(\|z-x\| + \|z-x\| + \delta_1) \leq \delta_1(2M+1). \tag{4.40}$$

By (4.15)–(4.18), (4.28) and (4.38),

$$|\|x-u\|^2 - \|x_0-y\|^2|$$

$$\leq \|(x-u)-(x_0-y)\|(\|x_0-y\| + \|x-u\|)$$

$$\leq (\|x-x_0\| + \|u-y\|)(\|x_0-y\| + \|x-u\|)$$

$$\leq (\delta_1 + \bar{\delta}_2 + 16M_0\Delta^{-2}(2M+1)^2\delta_2)(\|x_0-y\|$$

$$+\|x_0-y\| + \|x-x_0\| + \|u-y\|)$$

$$\leq (\delta_1 + \bar{\delta}_2 + 16M_0\Delta^{-2}(2M+1)^2\delta_2)(2\|z-x_0\| + \|x-x_0\| + \|u-y\|)$$

$$\leq (\delta_1 + \bar{\delta}_2 + 16M_0\Delta^{-2}(2M+1)^2\delta_2)(2M + 3\delta_1$$

$$+\bar{\delta}_2 + 16M_0\Delta^{-2}(2M+1)^2\delta_2)$$

$$\leq (\delta_1 + \bar{\delta}_2 + 16M_0\Delta^{-2}(2M+1)^2\delta_2)(2M+5). \tag{4.41}$$

By (4.15)–(4.18), (4.28) and (4.38),

$$|\|z-y\|^2 - \|z-u\|^2| \leq \|y-u\|(\|z-y\| + \|z-u\|)$$

$$\leq \|y-u\|(2\|z-y\| + \|y-u\|)$$

$$\leq (\bar{\delta}_2 + 16M_0\Delta^{-2}(2M+1)^2\delta_2)(2M$$

$$+16M_0\Delta^{-2}(2M+1)^2\delta_2 + 2 + \delta_2)$$

$$\leq (\bar{\delta}_2 + 16M_0\Delta^{-2}(2M+1)^2\delta_2)(2M+4). \tag{4.42}$$

In view of (4.28) and (4.40)–(4.42),

$$\|z-x\|^2 - \|x-u\|^2 - \|z-u\|^2$$

$$= \|z-x_0\|^2 - \|z-y\|^2 - \|x_0-y\|^2$$

$$(\|z-x\|^2 - \|z-x_0\|^2) + (\|z-y\|^2 - \|z-u\|^2)$$

$$+(\|x_0-y\|^2 - \|x-u\|^2)$$

$$\geq -\delta_1(2M+1) - 16\Delta^{-2}(2M+4)^3\delta_2$$

$$-(\delta_1 + \bar{\delta}_2 + 16M_0\Delta^{-2}(2M+1)^2\delta_2)(2M+5) - \bar{\delta}_2(2M+4).$$

It follows from (4.18), (4.19), (4.26), (4.30) and (4.38) that

$$\|x-u\| \geq -\|x_0-x\| - \|u-y\| + \|x_0-y\|$$

$$\geq -\delta_1 - \bar{\delta}_2 - 16M_0\Delta^{-2}(2M+1)^2\delta_2 + \|x_0-y\|$$

$$\geq -\delta_1 - \bar{\delta}_2 - 16M_0\Delta^{-2}(2M+1)^2\delta_2 + M_1^{-1}f_j(x_0).$$

Lemma 4.2 is proved.

Lemma 4.3 *Let* $\delta_1 \in [0, 1]$, $M \geq M_*$,

$$x \in B(0, M),\ x_0 \in B(x, \delta_1)$$

and $z \in B(0, M) \cap C$. *Then*

$$\|z - x_0\| \le \|z - x\| + \delta_1 \le 2M + \delta_1, \ \|x - x_0\| \le \delta_1,$$

$$\|z - x\|^2 - \|z - x_0\|^2 - \|x - x_0\|^2 \ge -\delta_1(4M + 2).$$

Proof Clearly,

$$\|z - x\|^2 - \|z - x_0\|^2 - \|x - x_0\|^2$$

$$\ge (\|z - x\| - \|z - x_0\|)(\|z - x\| + \|z - x_0\|) - \delta_1^2$$

$$\ge \delta_1(4M + \delta_1) - \delta_1^2 \ge -\delta_1(4M + 4).$$

Lemma 4.3 is proved.

Let $\Delta \in (0, 1]$, $\delta_f, \delta_C, \bar{\delta}_C \in [0, 1]$, $\alpha \in (0, 1]$, $\tilde{M} \ge M_*$, $M_0 \ge \max\{1, \tilde{M}\}$, $M_1 > 2$, $L_0 \ge 1$,

$$f_i(B(0, 3\tilde{M} + 4)) \subset [-M_0, M_0], \ i = 1, \dots, m, \tag{4.43}$$

$$|f_i(u) - f_i(v)| \le (M_1 - 2)\|u - v\|$$

$$\text{for all } u, v \in (0, 3\tilde{M} + 2) \text{ and all } i = 1, \dots, m, \tag{4.44}$$

$$|f(u) - f(v)| \le L_0\|u - v\| \text{ for all } u, v \in (0, 3\tilde{M} + 4), \tag{4.45}$$

$$\alpha \le 2^{-1}(L_0 + 1)^{-1}(6\tilde{M} + 5)^{-1},$$

$$\delta_C \le 32^{-1}\Delta^2(6\tilde{M} + 5)^{-2}(M_0 + 5)^{-2}. \tag{4.46}$$

Lemma 4.4 *Let $\Delta_0 \ge 0$ and let at least one of the following conditions hold:*

(a) $\Delta_0 = \Delta$;
(b) for each $i \in \{1, \dots, m\}$,

$$B(0, M_*) \cap \{x \in X : \ f_i(x) \le -\Delta\} \ne \emptyset.$$

Assume that

$$z \in B(0, M_*) \cap C, \tag{4.47}$$

$$x \in B(z, 2\tilde{M}), \tag{4.48}$$

$l \in X$ satisfies

$$B(l, \delta_f) \cap \partial f(x) \ne \emptyset, \tag{4.49}$$

$$w(i) \geq \bar{\Delta},\ i = 1, \dots, m, \tag{4.50}$$

$$\sum_{i=1}^{m} w(i) = 1, \tag{4.51}$$

$l_i, y_i \in X,\ i = 1, \dots, m$, for every $j \in \{1, \dots, m\}$,

$$\textit{if } f_j(x - \alpha l) \leq \Delta_0,\ \textit{then } y_j = x - \alpha l,\ l_j = 0, \tag{4.52}$$

if $f_j(x - \alpha l) > \Delta_0$, then

$$B(l_j, \delta_C) \cap \partial f(x - \alpha l) \neq \emptyset \tag{4.53}$$

(this implies that $l_j \neq 0$),

$$y_j \in B(x - \alpha l - f_j(x - \alpha l)\|l_j\|^{-2} l_j, \bar{\delta}_C) \tag{4.54}$$

and that

$$\|y - \sum_{i=1}^{m} w(i) y_i\| \leq \bar{\delta}_C. \tag{4.55}$$

Then

$$\|x - z\|^2 - \|y - z\|^2$$

$$\geq 2\alpha(f(x) - f(z)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \bar{\delta}_C(4\tilde{M} + 5),$$

$$\|x - z\|^2 - \|y - z\|^2$$

$$\geq \bar{\Delta} \sum_{j=1}^{m} \|x - y_j\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5) - \bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C \Delta^{-2}(6\tilde{M} + 4)^3$$

and for each $j \in \{1, \dots, m\}$, if $f_j(x - \alpha l) \leq \Delta_0$, then

$$\|x - y_j\| \leq \alpha(L_0 + 1)$$

and if $f_j(x - \alpha l) > \Delta_0$, then

$$\|x - y_j\|$$

$$\geq -\alpha(L_0 + 1) - \bar{\delta}_C - 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2 + M_1^{-1} f_j(x - \alpha l).$$

Proof Let $j \in \{1, \ldots, m\}$. Assume that

$$f_j(x - \alpha l) \leq \Delta_0. \tag{4.56}$$

In view of (4.52) and (4.56),

$$y = x - \alpha l. \tag{4.57}$$

By (4.45), (4.47) and (4.48),

$$\partial f(x) \subset B(0, L_0). \tag{4.58}$$

Equations (4.49), (4.57) and (4.58),

$$\|l\| \leq L_0 + 1, \;\; \|x - y_j\| \leq \alpha(L_0 + 1). \tag{4.59}$$

By (4.45), (4.47)–(4.49), (4.57) and Lemma 2.7 applied with $F_0 = X$, $M_0 = \tilde{M}$, $\delta_1 = \delta_f$, $Q = I$ (the identity mapping in X), $\xi = l$, $u = y_j$ and an arbitrary $\delta_2 \in (0, 1]$,

$$\alpha(f(x) - f(z)) \leq 2^{-1}\|x - z\|^2 - 2^{-1}\|y_j - z\|^2$$

$$+ \alpha^2 L_0^2 + \alpha\delta_f(6\tilde{M} + L_0 + 1). \tag{4.60}$$

Lemma 4.3 applied with $\delta = \alpha(L_0 + 1)$, $M = 3\tilde{M}$, $x_0 = y_j$, (4.47), (4.48) and (4.59) imply that

$$\|z - x\|^2 - \|z - y_j\|^2 - \|x - y_j\|^2 \geq -2\alpha(L_0 + 1)(6\tilde{M} + 1). \tag{4.61}$$

Assume that

$$f_j(x - \alpha l) > \Delta_0. \tag{4.62}$$

In view of (4.53) and (4.62), there exists

$$\xi_j \in \partial f(x - \alpha l) \tag{4.63}$$

such that

$$\|\xi_j - l_j\| \leq \delta_C. \tag{4.64}$$

Equations (4.49) and (4.63) imply that

$$\partial f(x) \subset B(0, L_0), \ \|l\| \le L_0 + 1. \tag{4.65}$$

By (4.24), (4.43)–(4.49), (4.54), (4.62), (4.64) and Lemma 4.2 applied with $\delta_1 = \alpha(L_0+1)$, $\delta_2 = \delta_C$, $\bar{\delta}_2 = \bar{\delta}_C$, $M = 3\tilde{M}$, $x_0 = x - \alpha l$, $\xi = \xi_j$, $l = l_j$, $u = y_j$, we have

$$\xi_j \ne 0, \ l_j \ne 0, \|\xi_j\| \ge \Delta(6\tilde{M}+1)^{-1}, \tag{4.66}$$

$$\|x - y_j\| \ge -\alpha(L_0+1) - \bar{\delta}_C$$

$$-16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2 + M_1^{-1} f_j(x - \alpha l), \tag{4.67}$$

$$\|z - x\|^2 - \|z - y_j\|^2$$

$$\ge \|x - y_j\|^2 - \alpha(L_0+1)(6\tilde{M}+1)$$

$$-16\delta_C \Delta^{-2}(6\tilde{M}+4)^3 - (\alpha(L_0+1) + \bar{\delta}_C)(6\tilde{M}+5), \tag{4.68}$$

$$\|y_j - (x - \alpha l - f_j(x-\alpha l)\|\xi_j\|^{-2}\xi_j)\| \le \bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2. \tag{4.69}$$

By (4.45)–(4.49), (4.53), (4.62), (4.63), (4.69) and Lemma 2.7 with

$$F_0 = C, \ M_0 = 3\tilde{M}, \ \delta_1 = \delta_f, \ \xi = l,$$

$$u = y_j, \ \delta_2 = \bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2, \ Q = P_D,$$

where

$$D = \{v \in X : \ f_j(x - \alpha l) + \langle \xi_j, v - (x - \alpha l)\rangle \le 0\},$$

we have

$$\alpha(f(x) - f(z)) \le 2^{-1}\|x - z\|^2 - 2^{-1}\|y_j - z\|^2$$

$$+\alpha^2 L_0^2 + \alpha\delta_f(6\tilde{M} + L_0 + 1) + 32(6\tilde{M}+2)\delta_C\Delta^{-2}(6\tilde{M}+1)^2. \tag{4.70}$$

In both cases, in view of (4.60) and (4.70),

$$\|x - z\|^2 - \|y_j - z\|^2$$

$$\ge 2\alpha(f(x) - f(z)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$- 64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2. \quad (4.71)$$

In both cases, by (4.61) and (4.68),

$$\|z - x\|^2 - \|z - y_j\|^2$$
$$\geq \|x - y_j\|^2 - 2\alpha(L_0 + 1)(6\tilde{M} + 1) - 16\delta_C \Delta^{-2}(6\tilde{M} + 4)^3$$
$$- (\alpha(L_0 + 1) + \bar{\delta}_C)(6\tilde{M} + 5). \quad (4.72)$$

It follows from (4.59) and (4.68), if $f_j(x - \alpha l) \leq \Delta_0$, then

$$\|x - y_j\| \leq \alpha(L_0 + 1) \quad (4.73)$$

and if $f_j(x - \alpha l) > \Delta_0$, then

$$\|x - y_j\| \geq -\alpha(L_0+1) - \bar{\delta}_C - 16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2 + M_1^{-1} f_j(x - \alpha l). \quad (4.74)$$

Since the function $v \to \|v - z\|^2$, $v \in X$ is convex it follows from (4.50), (4.51) and (4.71) that

$$\|x - z\|^2 - \|\sum_{j=1}^{m} w_j y_j - z\|^2$$
$$\geq \|x - z\|^2 - \sum_{j=1}^{m} w_j \|y_j - z\|^2$$
$$\geq \sum_{j=1}^{m} (w_j(\|x - z\|^2 - \|y_j - z\|^2)$$
$$\geq 2\alpha(f(x) - f(z)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$
$$- 64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2. \quad (4.75)$$

By (4.50), (4.51) and (4.55),

$$|\|z - y\|^2 - \|\sum_{j=1}^{m} w_j y_j - z\|^2|$$

$$\leq \|y - \sum_{j=1}^{m} w_j y_j\|(\|z - y\| + \|\sum_{j=1}^{m} w_j y_j - z\|)$$

$$\leq \bar{\delta}_C(2\|\sum_{j=1}^{m} w_j y_j - z\| + 1) \leq \bar{\delta}_C(2\sum_{j=1}^{m} w_j\|y_j - z\| + 1). \quad (4.76)$$

In view of (4.46), (4.48) and (4.68), for every $j \in \{1, \ldots, m\}$,

$$\|z - y_j\|^2 \leq \|z - x\|^2 + 2 \leq (\|z - x\| + 2)^2 \leq (2\tilde{M} + 2)^2,$$

$$\|z - y_j\| \leq 2\tilde{M} + 2. \quad (4.77)$$

Equations (4.76) and (4.77) imply that

$$|\|z - y\|^2 - \|\sum_{j=1}^{m} w_j y_j - z\|^2| \leq \bar{\delta}_C(4\tilde{M} + 5). \quad (4.78)$$

It follows from (4.76) and (4.78) that

$$\|x - z\| - \|z - y\|^2$$

$$\geq \|x - z\|^2 - \|\sum_{j=1}^{m} w_j y_j - z\|^2 - \bar{\delta}_C(4\tilde{M} + 5)$$

$$\geq 2\alpha(f(x) - f(z)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \bar{\delta}_C(4\tilde{M} + 5).$$

Since the function $v \to \|v - z\|^2$, $v \in X$ is convex it follows from (4.50), (4.51), (4.72) and (4.78) that

$$\|x - z\|^2 - \|z - y\|^2$$

$$\geq \|x - z\|^2 - \|\sum_{j=1}^{m} w_j y_j - z\|^2 - \bar{\delta}_C(4\tilde{M} + 5)$$

$$\geq -\bar{\delta}_C(4\tilde{M} + 5) + \|x - z\|^2 - \sum_{j=1}^{m} w_j\|y_j - z\|^2$$

$$\geq -\bar{\delta}_C(4\tilde{M}+5) + \sum_{j=1}^{m} w_j(\|x-z\|^2 - \|y_j - z\|^2)$$

$$\geq -\bar{\delta}_C(4\tilde{M}+5) + \sum_{j=1}^{m} w_j\|x-y_j\|^2$$

$$-2\alpha(L_0+1)(6\tilde{M}+12) - 16\delta_C\Delta^{-2}(6\tilde{M}+4)^3$$

$$-(\alpha(L_0+1)+\bar{\delta}_C)(6\tilde{M}+5)$$

$$\geq \bar{\Delta}\sum_{j=1}^{m} w_j\|x-y_j\|^2 - 3\alpha(L_0+1)(6\tilde{M}+5)$$

$$-\bar{\delta}_C(6\tilde{M}+5) - 16\delta_C\Delta^{-2}(6\tilde{M}+4)^3.$$

Lemma 4.4 is proved.

4.3 Two Convergence Results

We use the notation and definitions introduced in Sections 4.1 and 4.2 and suppose that all the assumptions made there hold (in particular, see (4.43)–(4.46)).

Let

$$\alpha \leq \min\{8^{-1}(L_0+1)^{-1}\Delta M_1^{-1}, L_0^{-2}\},$$

$$\bar{\delta}_C \leq 8^{-1}\Delta M_1^{-1}, \delta_C \leq 2^{-7}\Delta^3 M_1^{-1}(6\tilde{M}+1)^{-2}, \tag{4.79}$$

$$\alpha \leq 96^{-1}(L_0+1)^{-1}(6\tilde{M}+5)^{-1}\bar{\Delta}\Delta^2 M_1^{-2},$$

$$\bar{\delta}_C < 32^{-1}\bar{\Delta}\Delta^2 M_1^{-2}(6\tilde{M}+5)^{-1}, \tag{4.80}$$

$$\delta_f(6\tilde{M}+L_0+1) \leq 1,\ \ \delta_C < 2^{-9}\Delta^4\bar{\Delta}M_1^{-2}(6\tilde{M}+5)^{-3}. \tag{4.81}$$

Proposition 4.5 *Let T be a natural number satisfying*

$$T \geq 128\tilde{M}^2\bar{\Delta}^{-1}\Delta^{-2}M_1^2, \tag{4.82}$$

$\{x_t\}_{t=0}^{T} \subset X$, $\{l_t\}_{t=0}^{T-1} \subset X$, $l_{t,i} \in X$, $t = 0, \ldots, T-1$, $i = 1, \ldots, m$,

$$x_t \in B(z, 2\tilde{M}) \text{ for all } t = 0, \dots, T \text{ and every } z \in B(0, M_*) \cap C, \tag{4.83}$$

$$w_t = (w_t(1), \dots, w_t(m)) \in R^m, \ t = 1, \dots, T,$$

$$w_t(i) \geq \bar{\Delta}, \ i = 1, \dots, m, \ t = 1, \dots, T, \tag{4.84}$$

$$\sum_{i=1}^{m} w_t(i) = 1, \ t = 1, \dots, T \tag{4.85}$$

and $y_{t,i} \in X$, $t = 1, \dots, T$, $i = 1, \dots, m$.

Assume that for all integers $t \in \{0, \dots, T-1\}$ *and all integers* $i \in \{1, \dots, m\}$,

$$B(l_t, \delta_f) \cap \partial f(x_t) \neq \emptyset, \tag{4.86}$$

$$\text{if } f_i(x_t - \alpha l_t) \leq \Delta, \text{ then } y_{t+1,i} = x_t - \alpha l_t, \ l_{t,i} = 0, \tag{4.87}$$

if $f_i(x_t - \alpha l_t) > \Delta$, *then*

$$B(l_{t,i}, \delta_C) \cap \partial f_i(x_t - \alpha l_t) \neq \emptyset \tag{4.88}$$

(this implies that $l_{t,i} \neq 0$*),*

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2} l_{t,i}, \bar{\delta}_C) \tag{4.89}$$

and that

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}\| \leq \bar{\delta}_C. \tag{4.90}$$

Then

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \delta_C(4\tilde{M} + 5),$$

$$\bar{\Delta} \sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$-\bar{\delta}_C(6\tilde{M} + 5) - 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3\} : t = 0, \dots, T-1\} \leq 4M^2 T^{-1}.$$

Moreover, if $t \in \{0, \dots, T-1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M}+2)\delta_C \Delta^{-2}(6\tilde{M}+1)^2 - \bar{\delta}_C(4\tilde{M}+5),$$

$$\bar{\Delta}\sum_{i=1}^{m}\|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0+1)(6\tilde{M}+5)$$

$$-\bar{\delta}_C(6\tilde{M}+5) - 16\delta_C \Delta^{-2}(6\tilde{M}+5)^3\} \le 4M^2T^{-1}$$

then

$$f(x_t) \le \inf(f, C) + 2M^2(T\alpha)^{-1} + \alpha L_0^2 + \delta_f(6\tilde{M}+L_0+1)$$

$$+32(6\tilde{M}+2)\delta_C \Delta^{-2}(6\tilde{M}+1)^2\alpha^{-1} + \alpha^{-1}\bar{\delta}_C(4\tilde{M}+5),$$

$$f_i(x_t) \le \Delta + M_2\alpha(L_0+1),\ i=1,\dots,m.$$

Proof In view of (4.10), there exists

$$z \in B(0, M_*) \cap C. \tag{4.91}$$

Let $t \in \{0, \dots, T-1\}$. By (4.83)–(4.86), (4.91) and Lemma 4.4 applied with

$$\Delta_0 = \Delta \text{ (see condition (a))},\ x = x_t,\ l = l_t,$$

$$w(i) = w_{t+1}(i),\ l_i = l_{t,i},\ i=1,\dots,m,\ y_i = y_{t+1,i},\ i=1,\dots,m,\ y = x_{t+1}$$

we obtain that

$$\|x_t - z\|^2 - \|x_{t+1} - z\|^2$$

$$\ge 2\alpha(f(x_t) - f(z)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M}+L_0+1)$$

$$-64(6\tilde{M}+2)\delta_C\Delta^{-2}(6\tilde{M}+1)^2 - \bar{\delta}_C(4\tilde{M}+5), \tag{4.92}$$

$$\|x_t - z\|^2 - \|x_{t+1} - z\|^2$$

$$\ge \bar{\Delta}\sum_{i=1}^{m}\|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0+1)(6\tilde{M}+5) - \bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\Delta^{-2}(6\tilde{M}+5)^3, \tag{4.93}$$

and that for every $j \in \{1, \dots, m\}$, if $f_j(x_t - \alpha l_t) \le \Delta$, then

$$\|x_t - y_{t+1,j}\| \le \alpha(L_0+1) \tag{4.94}$$

and if $f_j(x_t - \alpha l_t) > \Delta$, then in view of (4.79),

$$\|x_t - y_{t+1,j}\|$$

$$\geq -\alpha(L_0+1) - \bar{\delta}_C - 16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2 + M_1^{-1}\Delta \geq 2^{-1}\Delta M_1^{-1}. \tag{4.95}$$

By (4.83), (4.92) and (4.93),

$$4\tilde{M}^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1} (\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\geq \sum_{t=0}^{T-1} \max\{2\alpha(f(x_t) - f(z)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M}+2)\delta_C \Delta^{-2}(6\tilde{M}+1)^2 - \bar{\delta}_C(4\tilde{M}+5),$$

$$\bar{\Delta}\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0+1)(6\tilde{M}+5) - \bar{\delta}_C(6\tilde{M}+5)$$

$$- 16\delta_C \Delta^{-2}(6\tilde{M}+5)^3\}. \tag{4.96}$$

Since z is an arbitrary element of $B(0, M_*) \cap C$ it follows from (4.11) and (4.96) that

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M}+2)\delta_C \Delta^{-2}(6\tilde{M}+1)^2 - \delta_C(4\tilde{M}+5),$$

$$\Delta\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0+1)(6\tilde{M}+5)$$

$$-\bar{\delta}_C(6\tilde{M}+5) - 16\delta_C \Delta^{-2}(6\tilde{M}+5)^3\} : t = 0, \dots, T-1\} \leq 4\tilde{M}^2 T^{-1}.$$

Let $t \in \{0, \dots, T-1\}$ and

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M}+2)\delta_C \Delta^{-2}(6\tilde{M}+1)^2 - \delta_C(4\tilde{M}+5),$$

$$\Delta \sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0+1)(6\tilde{M}+5)$$

$$-\bar{\delta}_C(6\tilde{M}+5) - 16\delta_C\Delta^{-2}(6\tilde{M}+5)^3\} \le 4\tilde{M}^2T^{-1}. \quad (4.97)$$

In view of (4.97),

$$f(x_t) \le \inf(f, C) + 2\tilde{M}^2(T\alpha)^{-1} + \alpha L_0^2 + \delta_f(6\tilde{M}+L_0+1)$$

$$+32(6\tilde{M}+2)\alpha^{-1}\delta_C\Delta^{-2}(6\tilde{M}+1)^2 + \alpha^{-1}\bar{\delta}_C(4\tilde{M}+5).$$

By (4.79), (4.80) and (4.97), for every $i \in \{1, \dots, m\}$,

$$\|x_t - y_{t+1,i}\|$$

$$\le (\bar{\Delta}^{-1}(3\alpha(L_0+1)(6\tilde{M}+5) + \bar{\delta}_C(6\tilde{M}+5)$$

$$+16\delta_C\Delta^{-2}(6\tilde{M}+5)^3 + 4\tilde{M}^2T^{-1}))^{1/2}$$

$$< 2^{-1}\Delta M_1^{-1}. \quad (4.98)$$

Equations (4.95) and (4.98) imply that for every $i \in \{1, \dots, m\}$,

$$f_i(x_t - \alpha l_t) \le \Delta. \quad (4.99)$$

It follows from (4.83) and (4.91) that

$$\|x_t\| \le 3\tilde{M}. \quad (4.100)$$

By (4.45), (4.86) and (4.100),

$$\|\xi_t\| \le L_0, \ \|l_t\| \le L_0 + 1. \quad (4.101)$$

In view of (4.79), (4.100) and (4.101),

$$\|x_t - \alpha l_t\| \le \alpha(L_0+1) + 3\tilde{M} \le 3\tilde{M} + 1.$$

By (4.44), (4.99)–(4.101) and the relation above, for $i = 1, \dots, m$,

$$f_i(x_t) \le f_i(x_t - \alpha l_t) + |f_i(x_t) - f_i(x_t - \alpha l_t)|$$

$$\le \Delta + (M_1 - 2)\alpha\|l_t\| \le \Delta + (M_1 - 2)\alpha(L_0 + 1).$$

Proposition 4.5 is proved.

Analogously to Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations.

In our first convergence result we assume that the objective function f satisfies some growth condition.

Theorem 4.6 *Let* $M_{*,0} > 0$,

$$|f(u)| \le M_{*,0} \text{ for all } u \in B(0, M_*),$$

$\tilde{M} \ge 2M_* + 2$,

$$\alpha \ge \max\{64\delta_C(3\tilde{M}+1)\Delta^{-2}(6\tilde{M}+1)^2,\ \bar{\delta}_C(4\tilde{M}+5)\}, \tag{4.102}$$

$$f(u) > M_{*,0} + 8 \text{ for all } u \in X \setminus B(0, 2^{-1}\tilde{M}), \tag{4.103}$$

T be a natural number satisfying

$$T \ge 128\tilde{M}^2\bar{\Delta}^{-1}\Delta^{-2}M_1^2, \tag{4.104}$$

$\{x_t\}_{t=0}^{T} \subset X$, $\{l_t\}_{t=0}^{T-1} \subset X$, $l_{t,i} \in X$, $t = 0, \dots, T-1$, $i = 1, \dots, m$,

$$\|x_0\| \le \tilde{M}, \tag{4.105}$$

$$w_t = (w_t(1), \dots, w_t(m)) \in R^m,\ t = 1, \dots, T, \tag{4.106}$$

$$w_t(i) \ge \bar{\Delta},\ i = 1, \dots, m,\ t = 1, \dots, T, \tag{4.107}$$

$$\sum_{i=1}^{m} w_t(i) = 1,\ t = 1, \dots, T \tag{4.108}$$

and $y_{t,i} \in X$, $t = 1, \dots, T$, $i = 1, \dots, m$.

Assume that for all integers $t \in \{0, \dots, T-1\}$ *and all integers* $i \in \{1, \dots, m\}$,

$$B(l_t, \delta_f) \cap \partial f(x_t) \neq \emptyset, \tag{4.109}$$

$$\text{if } f_i(x_t - \alpha l_t) \le \Delta,\ \text{then } y_{t+1,i} = x_t - \alpha l_t,\ l_{t,i} = 0, \tag{4.110}$$

if $f_i(x_t - \alpha l_t) > \Delta$, *then*

$$B(l_{t,i}, \delta_C) \cap \partial f_i(x_t - \alpha l_t) \neq \emptyset \tag{4.111}$$

(this implies that $l_{t,i} \neq 0$*),*

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2} l_{t,i}, \delta_C) \tag{4.112}$$

and that

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}\| \leq \bar{\delta}_C. \tag{4.113}$$

Then

$$\|x_t\| \leq 3\tilde{M},\ t = 0, \dots, T,$$

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \bar{\delta}_C(4\tilde{M} + 5),$$

$$\bar{\Delta}\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$-\bar{\delta}_C(6\tilde{M} + 5) - 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3\} : t = 0, \dots, T - 1\} \leq 4M^2 T^{-1}.$$

Moreover, if $t \in \{0, \dots, T - 1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \delta_C(4\tilde{M} + 5),$$

$$\bar{\Delta}\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$-\bar{\delta}_C(6\tilde{M} + 5) - 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3\} \leq 4M^2 T^{-1}$$

then

$$f(x_t) \leq \inf(f, C) + 2M^2(T\alpha)^{-1} + \alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1)$$

$$+32(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2\alpha^{-1} + \alpha^{-1}\bar{\delta}_C(4\tilde{M} + 5),$$

$$f_i(x_t) \leq \Delta + M_1\alpha(L_0 + 1),\ i = 1, \dots, m.$$

Proof By Proposition 4.5, in order to prove the theorem it is sufficient to show that for all $z \in B(0, M_*) \cap C$,

$$x_t \in B(z, 2\tilde{M}),\ t = 0, \dots, T.$$

Let

$$z \in B(0, M_*) \cap C. \tag{4.114}$$

By (4.105) and (4.114),

$$\|z - x_0\| \le M_* + \tilde{M} < 2\tilde{M}. \tag{4.115}$$

Assume that there exists an integer $k \in \{0, \dots, T\}$ such that

$$\|z - x_k\| > 2\tilde{M}. \tag{4.116}$$

By (4.115) and (4.116), $k > 0$. We may assume without loss of generality that

$$\|z - x_t\| \le 2\tilde{M}, \ t = 0, \dots, k-1$$

and in particular,

$$\|z - x_{k-1}\| \le 2\tilde{M}. \tag{4.117}$$

By (4.80), (4.81), (4.107)–(4.114), (4.117) and Lemma 4.4 applied with

$$\Delta_0 = \Delta \text{ (see condition (a))}, \ x = x_{k-1},$$

$$l = l_{k-1}, \ w(i) = w_k(i), \ l_i = l_{k-1,i}, \ i = 1, \dots, m,$$

$$y_i = y_{k,i}, \ i = 1, \dots, m, \ y = x_k$$

we obtain that

$$\begin{aligned} &\|x_{k-1} - z\|^2 - \|x_k - z\|^2 \\ &\ge 2\alpha(f(x_{k-1}) - f(z)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1) \\ &- 64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \bar{\delta}_C(4\tilde{M} + 5), \end{aligned} \tag{4.118}$$

$$\begin{aligned} &\|x_{k-1} - z\|^2 - \|x_k - z\|^2 \\ &\ge -3\alpha(L_0 + 1)(6\tilde{M} + 5) - \bar{\delta}_C(6\tilde{M} + 5) \\ &- 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3. \end{aligned} \tag{4.119}$$

In view of (4.116) and (4.117),

$$\|x_{k-1} - z\| < \|x_k - z\|. \tag{4.120}$$

It follows from (4.79), (4.80), (4.102), (4.114), (4.118) and (4.120) that

$$f(x_{k-1}) \le f(z) + \alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1)$$

$$+64\alpha^{-1}(3\tilde{M} + 1)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 + 2^{-1}\alpha^{-1}\bar{\delta}_C(4\tilde{M} + 5)$$

$$\le M_{*,0} + 4. \tag{4.121}$$

By (4.119),

$$\|x_k - z\|^2 \le \|x_{k-1} - z\|^2 + 1 \le (\|x_{k-1} - z\| + 1)^2. \tag{4.122}$$

Equations (4.103) and (4.121) imply that

$$\|x_{k-1}\| \le \tilde{M}/2. \tag{4.123}$$

It follows from (4.102), (4.114), (4.122) and (4.123) that

$$\|x_k - z\| \le \tilde{M}/2 + M_* + 1 < \tilde{M}.$$

This contradicts (4.116). The contradiction we have reached proves that

$$\|x_k - z\| \le 2\tilde{M},\ k = 0, \dots, T$$

and Theorem 4.6 itself.

In our second convergence result we assume that the feasibility point set is bounded.

Theorem 4.7 *Let* $r_0 \in (0, 1]$,

$$r_0 \ge 2\Delta,\ \alpha(L_0 + 1)M_1 \le r_0/2, \tag{4.124}$$

$$\{x \in X :\ f_i(x) \le r_0,\ i = 1, \dots, m\} \subset B(0, M_*), \tag{4.125}$$

$$\tilde{M} \ge 2M_* + 1, \tag{4.126}$$

T be a natural number satisfying

$$T \ge 128\tilde{M}^2\bar{\Delta}^{-1}\Delta^{-2}M_1^2, \tag{4.127}$$

$\{x_t\}_{t=0}^{T} \subset X$, $\{l_t\}_{t=0}^{T-1} \subset X$, $l_{t,i} \in X$, $t = 0, \dots, T-1$, $i = 1, \dots, m$,

$$\|x_0\| \le \tilde{M}, \tag{4.128}$$

$$w_t = (w_t(1), \dots, w_t(m)) \in R^m,\ t = 1, \dots, T, \tag{4.129}$$

$$w_t(i) \geq \bar{\Delta},\ i = 1, \ldots, m,\ t = 1, \ldots, T, \tag{4.130}$$

$$\sum_{i=1}^{m} w_t(i) = 1,\ t = 1, \ldots, T \tag{4.131}$$

and $y_{t,i} \in X$, $t = 1, \ldots, T$, $i = 1, \ldots, m$.

Assume that for all integers $t \in \{0, \ldots, T-1\}$ *and all integers* $i \in \{1, \ldots, m\}$,

$$B(l_t, \delta_f) \cap \partial f(x_t) \neq \emptyset, \tag{4.132}$$

$$\text{if } f_i(x_t - \alpha l_t) \leq \Delta,\ \text{then } y_{t+1,i} = x_t - \alpha l_t,\ l_{t,i} = 0, \tag{4.133}$$

if $f_i(x_t - \alpha l_t) > \Delta$, *then*

$$B(l_{t,i}, \delta_C) \cap \partial f_i(x_t - \alpha l_t) \neq \emptyset \tag{4.134}$$

(this implies that $l_{t,i} \neq 0$*)*,

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2} l_{t,i}, \bar{\delta}_C) \tag{4.135}$$

and that

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}\| \leq \bar{\delta}_C. \tag{4.136}$$

Then

$$\|x_t\| \leq 3\tilde{M},\ t = 0, \ldots, T,$$

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \delta_C(4\tilde{M} + 5),$$

$$\bar{\Delta} \sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$-\bar{\delta}_C(6\tilde{M} + 5) - 16\delta_C \Delta^{-1}(6\tilde{M} + 5)^3\} : t = 0, \ldots, T-1\} \leq 4M^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T-1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \delta_C(4\tilde{M} + 5),$$

$$\bar{\Delta}\sum_{i=1}^{m}\|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0+1)(6\tilde{M}+5)$$

$$-\bar{\delta}_C(6\tilde{M}+5) - 16\delta_C\Delta^{-2}(6\tilde{M}+5)^3\} \le 4M^2T^{-1}$$

then

$$f(x_t) \le \inf(f, C) + 2M^2(T\alpha)^{-1} + \alpha L_0^2 + \delta_f(6\tilde{M}+L_0+1)$$

$$+32(6\tilde{M}+2)\delta_C\Delta^{-2}(6\tilde{M}+1)^2\alpha^{-1} + \alpha^{-1}\bar{\delta}_C(4\tilde{M}+5),$$

$$f_i(x_t) \le \Delta + M_2\alpha(L_0+1),\ i = 1, \dots, m.$$

Proof By Proposition 4.5, in order to prove the theorem it is sufficient to show that for all $z \in B(0, M_*) \cap C$,

$$x_t \in B(z, 2\tilde{M}),\ t = 0, \dots, T.$$

Let

$$z \in B(0, M_*) \cap C. \tag{4.137}$$

By (4.128) and (4.137),

$$\|z - x_0\| \le M_* + \tilde{M} < 2\tilde{M}. \tag{4.138}$$

Assume that there exists an integer $k \in \{0, \dots, T\}$ such that

$$\|z - x_k\| > 2\tilde{M}. \tag{4.139}$$

By (4.138) and (4.139), $k > 0$. We may assume without loss of generality that

$$\|z - x_t\| \le 2\tilde{M},\ t = 0, \dots, k-1$$

and in particular,

$$\|z - x_{k-1}\| \le 2\tilde{M}. \tag{4.140}$$

By (4.79), (4.130)–(4.137), (4.140) and Lemma 4.4 applied with

$$\Delta_0 = \Delta \text{ see (condition (a))},\ x = x_{k-1},\ l = l_{k-1},$$

$$w(i) = w_{k-1}(i),\ l_i = l_{k-1,i},\ i = 1, \dots, m,\ y_i = y_{k,i},\ i = 1, \dots, m,\ y = x_k$$

we obtain that

$$\|x_{k-1} - z\|^2 - \|x_k - z\|^2$$

$$\geq \bar{\Delta} \sum_{i=1}^{m} \|x_{k-1} - y_{k,i}\| - 3\alpha(L_0 + 1)(6\tilde{M} + 5) - \bar{\delta}_C(6\tilde{M} + 5)$$

$$- 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3 \tag{4.141}$$

and that for every $j \in \{1, \ldots, m\}$, if $f_j(x_{k-1} - \alpha l_{k-1}) > \Delta$, then

$$\|x_{k-1} - y_{k,j}\| \geq 2^{-1}\Delta M_1^{-1}. \tag{4.142}$$

By (4.80), (4.81), (4.139)–(4.141), for every $i \in \{1, \ldots, m\}$,

$$\|x_{k-1} - y_{k,i}\|$$

$$\leq (\bar{\Delta}^{-1}(3\alpha(L_0 + 1)(6\tilde{M} + 5) + \bar{\delta}_C(6\tilde{M} + 5) + 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3))^{1/2}$$

$$< 2^{-1}\Delta M_1^{-1}. \tag{4.143}$$

Equations (4.142) and (4.143) imply that for every $i \in \{1, \ldots, m\}$,

$$f_i(x_{k-1} - \alpha l_{k-1}) \leq \Delta. \tag{4.144}$$

It follows from (4.137) and (4.140) that

$$\|x_{k-1}\| \leq 3\tilde{M}. \tag{4.145}$$

By (4.45), (4.124), (4.134) and (4.145),

$$\partial f(x_{k-1}) \subset B(0, L_0), \ \|l_{k-1}\| \leq L_0 + 1,$$

$$\alpha\|l_{k-1}\| \leq \alpha(L_0 + 1) \leq 1. \tag{4.146}$$

It follows from (4.44), (4.124), (4.125) and (4.144)–(4.146) that for $i = 1, \ldots, m$,

$$f_i(x_{k-1}) \leq f_i(x_{k-1} - \alpha l_{k-1}) + |f_i(x_{k-1}) - f_i(x_{k-1} - \alpha l_{k-1})|$$

$$\leq \Delta + M_1\alpha(L_0 + 1) \leq r_0. \tag{4.147}$$

By (4.125) and (4.147),

$$\|x_{k-1}\| \leq M_*. \tag{4.148}$$

In view of (4.80), (4.81), (4.141) and (4.148),

$$\|x_{k-1} - z\|^2 - \|x_k - z\|^2 \geq -1,$$

$$\|x_k - z\|^2 \leq \|x_{k-1} - z\|^2 + 1 \leq (\|x_{k-1} - z\| + 1)^2,$$

$$\|x_k - z\| \leq 2M_* + 1 < 2\tilde{M}.$$

This contradicts (4.139). The contradiction we have reached proves that

$$\|x_k - z\| \leq 2\tilde{M}, \ t = 0, \dots, T$$

and Theorem 4.7 itself.

4.4 The Third and Fourth Convergence Results

We use the notation and definitions introduced in Sections 4.1 and 4.2.

Let $\Delta \in (0, 1]$, $\delta_f, \delta_C, \bar{\delta}_C \in [0, 1]$, $\alpha \in (0, 1]$, $M_* > 0$, for each $i \in \{1, \dots, m\}$,

$$B(0, M_*) \cap \{v \in X : \ f_i(x) \leq -\Delta\} \neq \emptyset, \tag{4.149}$$

$$B(0, M_*) \cap C \neq \emptyset, \tag{4.150}$$

$\tilde{M} \geq M_*$, $M_0 \geq \max\{1, \tilde{M}\}$, $M_1 > 2$, $L_0 \geq 1$,

$$f_i(B(0, 3\tilde{M} + 4)) \subset [-M_0, M_0], \ i = 1, \dots, m, \tag{4.151}$$

$$|f_i(u) - f_i(v)| \leq (M_1 - 2)\|u - v\|$$

$$\text{for all } u, v \in B(0, 3\tilde{M} + 4) \text{ and all } i = 1, \dots, m, \tag{4.152}$$

$$|f(u) - f(v)| \leq L_0\|u - v\| \text{ for all } u, v \in (0, 3\tilde{M} + 4), \tag{4.153}$$

$$\begin{aligned} \epsilon_0 = {} & M_1(\alpha(L_0 + 1) + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2) \\ & + M_1(2\bar{\Delta}^{-1}(3\alpha(L_0 + 1)(6\tilde{M} + 5) + \bar{\delta}_C(6\tilde{M} + 5) \\ & + 16^{-1}\delta_C \Delta^{-2}(6\tilde{M} + 5)^3)^{1/2}, \end{aligned} \tag{4.154}$$

$$\alpha \leq 6^{-1}(L_0 + 1)^{-1}(6\tilde{M} + 5)^{-1}, \tag{4.155}$$

$$\delta_C \leq 32^{-1}\Delta^2(6\tilde{M} + 5)^{-2}(M_0 + 5)^{-1}. \tag{4.156}$$

We consider the following modification of our algorithm.

Cimmino Subgradient Projection Algorithm with Computational Errors

Initialization: fix $\epsilon \in [0, \epsilon_0]$ and select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_t \in X$ calculate

$$l_t \in \partial f(x_t) + B(0, \delta_f),$$

pick $w_{t+1} = (w_{t+1}(1), \dots, w_{t+1}(m)) \in R^m$ such that

$$w_{t+1}(i) \geq \bar{\Delta},\ i = 1, \dots, m,$$

$$\sum_{i=1}^{m} w_{t+1}(i) = 1,$$

for each $i \in \{1, \dots, m\}$,

$$\text{if } f_i(x_t - \alpha l_t) \leq \epsilon,\ \text{then } y_{t+1,i} = x_t - \alpha l_t,\ l_{t,i} = 0,$$

if $f_i(x_t - \alpha l_t) > \epsilon$, then we calculate

$$l_{t,i} \in \partial f_i(x_t - \alpha l_t) + B(0, \delta_C),$$

(this implies that $l_{t,i} \neq 0$),

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2} l_{t,i}, \bar{\delta}_C)$$

and the next iteration vector

$$x_{t+1} \in B(\sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}, \bar{\delta}_C).$$

Proposition 4.8 *Let* $\epsilon \in [0, \epsilon_0]$, T *be a natural number satisfying*

$$T^{-1} \leq (4\tilde{M}^2)^{-1}(3\alpha(L_0 + 1)(6\tilde{M} + 5)$$
$$+ \bar{\delta}_C(6\tilde{M} + 5) + 16^{-1}\Delta^{-2}\delta_C(6\tilde{M} + 5)^3), \tag{4.157}$$

$\{x_t\}_{t=0}^{T} \subset X$, $\{l_t\}_{t=0}^{T-1} \subset X$, $l_{t,i} \in X$, $t = 0, \dots, T-1$, $i = 1, \dots, m$,

$$x_t \in B(z, 2\tilde{M}) \textit{ for all } t = 0, \dots, T \textit{ and every } z \in B(0, M_*) \cap C, \tag{4.158}$$

$$w_t = (w_t(1), \dots, w_t(m)) \in R^m,\ t = 1, \dots, T,$$

$$w_t(i) \geq \bar{\Delta},\ i = 1, \dots, m,\ t = 1, \dots, T, \tag{4.159}$$

$$\sum_{i=1}^{m} w_t(i) = 1,\ t = 1, \ldots, T \tag{4.160}$$

and $y_{t,i} \in X$, $t = 1, \ldots, T$, $i = 1, \ldots, m$.

Assume that for all integers $t \in \{0, \ldots, T-1\}$ *and all integers* $i \in \{1, \ldots, m\}$,

$$B(l_t, \delta_f) \cap \partial f(x_t) \neq \emptyset, \tag{4.161}$$

$$\textit{if } f_i(x_t - \alpha l_t) \le \epsilon,\ \textit{then } y_{t+1,i} = x_t - \alpha l_t,\ l_{t,i} = 0, \tag{4.162}$$

if $f_j(x_t - \alpha l_t) > \epsilon$, *then*

$$B(l_{t,i}, \delta_C) \cap \partial f_i(x_t - \alpha l_t) \neq \emptyset \tag{4.163}$$

(this implies that $l_{t,i} \neq 0$*),*

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2} l_{t,i}, \bar{\delta}_C) \tag{4.164}$$

and that

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}\| \le \bar{\delta}_C. \tag{4.165}$$

Then

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \delta_C(4\tilde{M} + 5),$$

$$\bar{\Delta} \sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$-\bar{\delta}_C(6\tilde{M} + 5) - 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3\} : t = 0, \ldots, T-1\} \le 4M^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T-1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \delta_C(4\tilde{M} + 5),$$

$$\Delta \sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$-\bar{\delta}_C(6\tilde{M}+5) - 16\delta_C \Delta^{-2}(6\tilde{M}+5)^3\} \le 4M^2T^{-1}$$

then

$$f(x_t) \le \inf(f, C) + 2\tilde{M}^2(T\alpha)^{-1} + \alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1)$$

$$+32(6\tilde{M}+2)\alpha^{-1}\delta_C\Delta^{-2}(6\tilde{M}+1)^2 + \alpha^{-1}\bar{\delta}_C(2\tilde{M}+3),$$

$$f_i(x_t) \le \epsilon_0 + M_1\alpha(L_0+1),\ i = 1, \dots, m.$$

Proof In view of (4.10) and (4.150), there exists

$$z \in B(0, M_*) \cap C. \tag{4.166}$$

Let $t \in \{0, \dots, T-1\}$. By (4.149), (4.151)–(4.153), (4.155), (4.156), (4.158)–(4.165) and Lemma 4.4 (see condition (b)) applied with

$$\Delta_0 = \epsilon,\ x = x_t,\ l = l_t,\ w(i) = w_{t+1}(i),\ l_i = l_{t,i},\ i = 1, \dots, m,$$

$$y_i = y_{t+1,i},\ i = 1, \dots, m,\ y = x_{t+1}$$

we obtain that

$$\|x_t - z\|^2 - \|x_{t+1} - z\|^2$$

$$\ge 2\alpha(f(x_t) - f(z)) - 2\alpha^2L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M}+2)\delta_C\Delta^{-2}(6\tilde{M}+1)^2 - \bar{\delta}_C(4\tilde{M}+5), \tag{4.167}$$

$$\|x_t - z\|^2 - \|x_{t+1} - z\|^2$$

$$\ge \bar{\Delta}\sum_{i=1}^{m}\|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0+1)(6\tilde{M}+5) - \bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\Delta^{-2}(6\tilde{M}+5)^3, \tag{4.168}$$

and that for every $j \in \{1, \dots, m\}$, if $f_j(x_t - \alpha l_t) \le \epsilon$, then

$$\|x_t - y_{t+1,j}\| \le \alpha(L_0+1) \tag{4.169}$$

and if $f_j(x_t - \alpha l_t) > \epsilon$, then

$$\|x_t - y_{t+1,j}\|$$

$$\geq -\alpha(L_0+1) - \bar{\delta}_C - 16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2 + M_1^{-1} f_j(x_t - \alpha l_t). \tag{4.170}$$

By (4.158) and (4.166)–(4.168),

$$4\tilde{M}^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1} (\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\geq \sum_{t=0}^{T-1} (\max\{2\alpha(f(x_t) - f(z)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M}+2)\delta_C \Delta^{-2}(6\tilde{M}+1)^2 - \bar{\delta}_C(4\tilde{M}+5),$$

$$\bar{\Delta}\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0+1)(6\tilde{M}+5) - \bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C \Delta^{-2}(6\tilde{M}+5)^3\}. \tag{4.171}$$

Since z is an arbitrary element of $B(0, M_*) \cap C$ it follows from (4.11) and (4.171) that

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M}+2)\delta_C \Delta^{-2}(6\tilde{M}+1)^2 - \delta_C(4\tilde{M}+5),$$

$$\Delta\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0+1)(6\tilde{M}+5)$$

$$-\bar{\delta}_C(6\tilde{M}+5) - 16\delta_C \Delta^{-2}(6\tilde{M}+5)^3\} : t = 0, \ldots, T-1\} \leq 4\tilde{M}^2 T^{-1}.$$

Let $t \in \{0, \ldots, T-1\}$ and

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M}+2)\delta_C \Delta^{-2}(6\tilde{M}+1)^2 - \delta_C(4\tilde{M}+5),$$

$$\Delta\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0+1)(6\tilde{M}+5)$$

$$-\bar{\delta}_C(6\tilde{M}+5) - 16\delta_C \Delta^{-2}(6\tilde{M}+5)^3\} \leq 4\tilde{M}^2 T^{-1}. \tag{4.172}$$

In view of (4.172),

$$f(x_t) \le \inf(f, C) + 2\tilde{M}^2(T\alpha)^{-1} + \alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1)$$

$$+32(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2\alpha^{-1} + \alpha^{-1}\bar{\delta}_C(2\tilde{M} + 3).$$

By (4.172), for every $i \in \{1, \dots, m\}$,

$$\|x_t - y_{t+1,i}\|$$

$$\le (\bar{\Delta}^{-1}(3\alpha(L_0 + 1)(6\tilde{M} + 5) + \bar{\delta}_C(6\tilde{M} + 5)$$

$$+ 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3 + 4\tilde{M}^2T^{-1})^{1/2}. \tag{4.173}$$

Let $j \in \{1, \dots, m\}$. There are two cases:

$$f_j(x_t - \alpha l_t) \le \epsilon; \tag{4.174}$$

$$f_j(x_t - \alpha l_t) > \epsilon. \tag{4.175}$$

If (4.174) holds then

$$f_j(x_t - \alpha l_t) \le \epsilon_0. \tag{4.176}$$

Assume that (4.175) holds. Then (4.170) is true. Equations (4.157) and (4.173) imply that

$$\|x_t - y_{t+1,j}\|$$

$$\le (2\bar{\Delta}^{-1}(3\alpha(L_0 + 1)(6\tilde{M} + 5) + \bar{\delta}_C(6\tilde{M} + 5)$$

$$+ 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3))^{1/2}. \tag{4.177}$$

By (4.170) and (4.177),

$$M_1^{-1} f_j(x_t - \alpha l_t) - \alpha(L_0 + 1) - \bar{\delta}_C - 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2$$

$$\le \|x_t - y_{t+1,i}\|$$

$$\le (2\bar{\Delta}^{-1}(3\alpha(L_0+1)(6\tilde{M}+5)+\bar{\delta}_C(6\tilde{M}+5)+16\delta_C \Delta^{-2}(6\tilde{M}+5)^3))^{1/2}. \tag{4.178}$$

It follows from (4.154) and (4.178) that

$$f_j(x_t - \alpha l_t) \le M_1(\alpha(L_0 + 1) + \bar{\delta}_C + 16\delta_C \Delta^{-2}(6\tilde{M} + 1)^2)$$

$$+M_1(2\bar{\Delta}^{-1}(3\alpha(L_0+1)(6\tilde{M}+5)$$

$$+\bar{\delta}_C(6\tilde{M}+5)+16\delta_C\Delta^{-2}(6\tilde{M}+5)^3)^{1/2})=\epsilon_0.$$

Thus in both cases

$$f_j(x_t-\alpha l_t)\le\epsilon_0. \tag{4.179}$$

It follows from (4.158) and (4.166) that

$$\|x_t\|\le 3\tilde{M}. \tag{4.180}$$

By (4.135), (4.153), (4.161) and (4.180),

$$\|x_t-\alpha l_t\|\le 3\tilde{M}+1. \tag{4.181}$$

By (4.152), (4.153), (4.161), (4.180) and (4.181),

$$|f_j(x_t)-f_j(x_t-\alpha l_t)|\le M_1\alpha\|l_t\|\le M_1\alpha(L_0+1).$$

Together with (4.179) this implies that ,

$$f_j(x_t)\le\epsilon_0+M_1\alpha(L_0+1).$$

Proposition 4.8 is proved.

Analogously to Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations.

In the following result we assume that the objective function f satisfies some growth condition.

Theorem 4.9 *Let* $M_{*,0}>0$,

$$|f(u)|\le M_{*,0}\ \textit{for all}\ u\in B(0,M_*), \tag{4.182}$$

$$\tilde{M}\ge 2M_*+4, \tag{4.183}$$

$$f(u)>M_{*,0}+8\ \textit{for all}\ u\in X\setminus B(0,2^{-1}\tilde{M}), \tag{4.184}$$

$$\delta_f(6\tilde{M}+L_0+1)\le 1,\ \ \alpha L_0^2\le 1, \tag{4.185}$$

$$32(6\tilde{M}+2)\delta_C(6\tilde{M}+1)^2\le\alpha, \tag{4.186}$$

$$\epsilon\in[0,\epsilon_0], \tag{4.187}$$

T be a natural number satisfying

$$T^{-1} \le (4\tilde{M}^2)^{-1}(3\alpha(L_0+1)(6\tilde{M}+5) + \bar{\delta}_C(6\tilde{M}+5) + 16^{-1}\Delta^{-2}\delta_C(6\tilde{M}+5)^3), \tag{4.188}$$

$\{x_t\}_{t=0}^{T} \subset X$, $\{l_t\}_{t=0}^{T-1} \subset X$, $l_{t,i} \in X$, $t = 0, \dots, T-1$, $i = 1, \dots, m$,

$$\|x_0\| \le \tilde{M}, \tag{4.189}$$

$$w_t = (w_t(1), \dots, w_t(m)) \in R^m,\ t = 1, \dots, T,$$

$$w_t(i) \ge \bar{\Delta},\ i = 1, \dots, m,\ \sum_{i=1}^{m} w_t(i) = 1,\ t = 1, \dots, T \tag{4.190}$$

and $y_{t,i} \in X$, $t = 1, \dots, T$, $i = 1, \dots, m$.

Assume that for all integers $t \in \{0, \dots, T-1\}$ *and all integers* $i \in \{1, \dots, m\}$,

$$B(l_t, \delta_f) \cap \partial f(x_t) \ne \emptyset, \tag{4.191}$$

$$\text{if } f_i(x_t - \alpha l_t) \le \epsilon,\ \text{then } y_{t+1,i} = x_t - \alpha l_t,\ l_{t,i} = 0, \tag{4.192}$$

if $f_i(x_t - \alpha l_t) > \epsilon$, *then*

$$B(l_{t,i}, \delta_C) \cap \partial f_i(x_t - \alpha l_t) \ne \emptyset \tag{4.193}$$

(this implies that $l_{t,i} \ne 0$*),*

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2} l_{t,i}, \bar{\delta}_C) \tag{4.194}$$

and that

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}\| \le \bar{\delta}_C. \tag{4.195}$$

Then

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M}+2)\delta_C \Delta^{-2}(6\tilde{M}+1)^2 - \delta_C(4\tilde{M}+5),$$

$$\bar{\Delta}\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 2\alpha(L_0+1)(6\tilde{M}+5)$$

$$-\bar{\delta}_C(6\tilde{M}+5) - 16\delta_C \Delta^{-2}(6\tilde{M}+5)^3\} : t = 0, \dots, T-1\} \le 4\tilde{M}^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T-1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \delta_C(4\tilde{M} + 5),$$

$$\bar{\Delta} \sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$-\bar{\delta}_C(6\tilde{M} + 5) - 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3\} \le 4\tilde{M}^2 T^{-1}$$

then

$$f(x_t) \le \inf(f, C) + 2\tilde{M}^2(T\alpha)^{-1} + \alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1)$$

$$+32(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2\alpha^{-1} + \alpha^{-1}\bar{\delta}_C(2\tilde{M} + 3),$$

$$f_i(x_t) \le \epsilon_0 + M_1\alpha(L_0 + 1), \ i = 1, \ldots, m.$$

Proof By Proposition 4.8, in order to prove the theorem it is sufficient to show that for all $z \in B(0, M_*) \cap C$,

$$x_t \in B(z, 2\tilde{M}), \ t = 0, \ldots, T.$$

Let

$$z \in B(0, M_*) \cap C. \tag{4.196}$$

By (4.189) and (4.196),

$$\|z - x_0\| \le M_* + \tilde{M} < 2\tilde{M}. \tag{4.197}$$

Assume that there exists an integer $k \in \{0, \ldots, T\}$ such that

$$\|z - x_k\| > 2\tilde{M}. \tag{4.198}$$

By (4.197) and (4.198), $k > 0$. We may assume without loss of generality that

$$\|z - x_t\| \le 2\tilde{M}, \ t = 0, \ldots, k-1$$

and in particular,

$$\|z - x_{k-1}\| \le 2\tilde{M}. \tag{4.199}$$

By (4.149), (4.151)–(4.153), (4.156), (4.190), (4.191), (4.194), (4.196), (4.199) and Lemma 4.4 (see condition (b)) applied with

$$\Delta_0 = \epsilon,\ x = x_{k-1},\ l = l_{k-1},\ w(i) = w_k(i),\ l_i = l_{k-1,i},\ i = 1, \ldots, m,$$

$$y_i = y_{k,i},\ i = 1, \ldots, m,\ y = x_k$$

we obtain that

$$\|x_{k-1} - z\|^2 - \|x_k - z\|^2$$

$$\geq 2\alpha(f(x_{k-1}) - f(z)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$- 64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \bar{\delta}_C(4\tilde{M} + 5), \tag{4.200}$$

$$\|x_{k-1} - z\|^2 - \|x_k - z\|^2$$

$$-3\alpha(L_0 + 1)(6\tilde{M} + 5) - \bar{\delta}_C(6\tilde{M} + 5)$$

$$- 16\delta_C \Delta^{-2}(6\tilde{M} + 4)^3. \tag{4.201}$$

In view of (4.198) and (4.199),

$$\|x_{k-1} - z\| < \|x_k - z\|.$$

Combined with (4.182), (4.185)–(4.187), (4.196) and (4.200) this implies that

$$f(x_{k-1}) \leq f(z) + \alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1)$$

$$+32\alpha^{-1}(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 + \alpha^{-1}\bar{\delta}_C(4\tilde{M} + 5)$$

$$\leq M_{*,0} + 4.$$

By (4.184) and the relation above,

$$\|x_{k-1}\| \leq \tilde{M}/2. \tag{4.202}$$

It follows from (4.155), (4.156), (4.163), (4.196), (4.201) and (4.202) that

$$\|x_k - z\| \leq \|x_{k-1} - z\| + 2 \leq \tilde{M}/2 + M_* + 2 < \tilde{M}.$$

This contradicts (4.198). The contradiction we have reached proves that

$$\|x_k - z\| \leq 2\tilde{M},\ t = 0, \ldots, T$$

and Theorem 4.9 itself.

In the next result the feasibility point set is assumed to be bounded.

Theorem 4.10 *Let* $r_0 \in (0, 1]$,

$$r_0 > \epsilon_0, \tag{4.203}$$

$$\{x \in X : f_i(x) \le r_0, \ i = 1, \dots, m\} \subset B(0, M_*), \tag{4.204}$$

$$\tilde{M} \ge 2M_* + 2, \ \epsilon \in [0, \epsilon_0], \tag{4.205}$$

T be a natural number satisfying

$$T^{-1} \le (4\tilde{M}^2)^{-1}(3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$+ \bar{\delta}_C(6\tilde{M} + 5) + 16^{-1}\Delta^{-2}\delta_C(6\tilde{M} + 5)^3), \tag{4.206}$$

$\{x_t\}_{t=0}^T \subset X$, $\{l_t\}_{t=0}^{T-1} \subset X$, $l_{t,i} \in X$, $t = 0, \dots, T-1$, $i = 1, \dots, m$,

$$\|x_0\| \le \tilde{M}, \tag{4.207}$$

$$w_t = (w_t(1), \dots, w_t(m)) \in R^m, \ t = 1, \dots, T,$$

$$w_t(i) \ge \bar{\Delta}, \ i = 1, \dots, m, \ \sum_{i=1}^m w_t(i) = 1, \ t = 1, \dots, T \tag{4.208}$$

and $y_{t,i} \in X$, $t = 1, \dots, T$, $i = 1, \dots, m$.

Assume that for all integers $t \in \{0, \dots, T-1\}$ *and all integers* $i \in \{1, \dots, m\}$,

$$B(l_t, \delta_f) \cap \partial f(x_t) \ne \emptyset, \tag{4.209}$$

$$\text{if } f_i(x_t - \alpha l_t) \le \epsilon, \text{ then } y_{t+1,i} = x_t - \alpha l_t, \ l_{t,i} = 0, \tag{4.210}$$

if $f_i(x_t - \alpha l_t) > \epsilon$, *then*

$$B(l_{t,i}, \delta_C) \cap \partial f_i(x_t - \alpha l_t) \ne \emptyset \tag{4.211}$$

(this implies that $l_{t,i} \ne 0$*),*

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2} l_{t,i}, \bar{\delta}_C) \tag{4.212}$$

and that

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}\| \le \bar{\delta}_C. \tag{4.213}$$

Then

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \delta_C(4\tilde{M} + 5),$$

$$\bar{\Delta}\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$-\bar{\delta}_C(6\tilde{M} + 5) - 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3\} : t = 0, \ldots, T - 1\} \le 4\tilde{M}^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T - 1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1)$$

$$-64(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2 - \delta_C(4\tilde{M} + 5),$$

$$\bar{\Delta}\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5)$$

$$-\bar{\delta}_C(6\tilde{M} + 5) - 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3\} \le 4\tilde{M}^2 T^{-1}$$

then

$$f(x_t) \le \inf(f, C) + 2\tilde{M}^2(T\alpha)^{-1} + \alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1)$$

$$+32(6\tilde{M} + 2)\delta_C \Delta^{-2}(6\tilde{M} + 1)^2\alpha^{-1} + \alpha^{-1}\bar{\delta}_C(2\tilde{M} + 3),$$

$$f_i(x_t) \le \epsilon_0 + M_1\alpha(L_0 + 1),\ i = 1, \ldots, m.$$

Proof By Proposition 4.8, in order to prove the theorem it is sufficient to show that for all $z \in B(0, M_*) \cap C$,

$$x_t \in B(z, 2\tilde{M}),\ t = 0, \ldots, T.$$

Let

$$z \in B(0, M_*) \cap C. \tag{4.214}$$

By (4.205), (4.207) and (4.214),

$$\|z - x_0\| \le M_* + \tilde{M} < 2\tilde{M}.$$

Assume that there exists an integer $k \in \{0, \dots, T\}$ such that

$$\|z - x_k\| > 2\tilde{M}. \tag{4.215}$$

By (4.215), $k > 0$. We may assume without loss of generality that

$$\|z - x_t\| \le 2\tilde{M},\ t = 0, \dots, k-1$$

and in particular,

$$\|z - x_{k-1}\| \le 2\tilde{M}. \tag{4.216}$$

By (4.149), (4.151)–(4.153), (4.155), (4.156), (4.205), (4.208)–(4.210), (4.213), (4.214), (4.216) and Lemma 4.4 (see condition (b)) applied with

$$\Delta_0 = \epsilon,\ x = x_{k-1},\ l = l_{k-1},\ w(i) = w_k(i),\ l_i = l_{k-1,i},\ i = 1, \dots, m,$$

$$y_i = y_{k,i},\ i = 1, \dots, m,\ y = x_k$$

we obtain that

$$\begin{aligned} &\|x_{k-1} - z\|^2 - \|x_k - z\|^2 \\ &\ge \bar{\Delta} \sum_{i=1}^{m} \|x_{k-1} - y_{k,i}\|^2 - 3\alpha(L_0 + 1)(6\tilde{M} + 5) - \bar{\delta}_C(6\tilde{M} + 5) \\ &\quad - 16\delta_C \Delta^{-2}(6\tilde{M} + 4)^3, \end{aligned} \tag{4.217}$$

and that for every $j \in \{1, \dots, m\}$, if $f_j(x_{k-1} - \alpha l_{k-1}) > \epsilon$, then

$$\begin{aligned} \|x_{k-1} - y_{k,j}\| &\ge -\alpha(L_0 + 1) - \bar{\delta}_C - 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2 \\ &\quad + M_1^{-1} f_j(x_{k-1} - \alpha l_{k-1}). \end{aligned} \tag{4.218}$$

Equations (4.215) and (4.216) imply that

$$\|x_{k-1} - z\| \le \|x_k - z\|. \tag{4.219}$$

By (4.217) and (4.219), for every $i \in \{1, \dots, m\}$,

$$\|x_{k-1} - y_{k,i}\|$$
$$\leq (\bar{\Delta}^{-1}(3\alpha(L_0+1)(6\tilde{M}+5)+\bar{\delta}_C(6\tilde{M}+5)+16\delta_C\Delta^{-2}(6\tilde{M}+4)^3))^{1/2}. \quad (4.220)$$

Assume that $j \in \{1, \ldots, m\}$ and that

$$f_j(x_{k-1} - \alpha l_{k-1}) > \epsilon.$$

By (4.154), (4.218) and (4.220),

$$f_j(x_{k-1} - \alpha l_{k-1})$$
$$\leq M_1(\alpha(L_0 + 1) + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2 + \|x_k - y_{k,j}\|)$$
$$\leq M_1(\alpha(L_0 + 1) + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2)$$
$$+M_1(\bar{\Delta}^{-1}(3\alpha(L_0 + 1)(6\tilde{M} + 5) + \bar{\delta}_C(6\tilde{M} + 5)$$
$$+16\delta_C\Delta^{-2}(6\tilde{M} + 5)^3))^{1/2} \leq \epsilon_0.$$

Thus

$$f_j(x_{k-1} - \alpha l_{k-1}) \leq \epsilon_0, \;\; j = 1, \ldots, m.$$

Together with (4.203) and (4.204) this implies that

$$\|x_{k-1} - \alpha l_{k-1}\| \leq M_*. \quad (4.221)$$

By (4.153), (4.209), (4.214) and (4.216),

$$\|l_{k-1}\| \leq L_0 + 1. \quad (4.222)$$

Equations (4.221) and (4.222) imply that

$$\|x_{k-1}\| \leq \|x_{k-1} - \alpha l_{k-1}\| + \alpha\|l_{k-1}\|$$
$$\leq M_* + \alpha(L_0 + 1) \leq M_* + 1. \quad (4.223)$$

It follows from (4.155), (4.156), (4.205), (4.214), (4.217) and (4.223) that

$$\|x_{k-1} - z\|^2 - \|x_k - z\|^2 \geq -1,$$
$$\|x_k - z\|^2 \leq \|x_{k-1} - z\|^2 + 1 \leq (\|x_{k-1} - z\| + 1)^2,$$

$$\|x_k - z\| \le 2M_* + 2 < 2\tilde{M}.$$

This contradicts (4.215). The contradiction we have reached proves that

$$\|x_k - z\| \le 2\tilde{M},\ t = 0, \ldots, T$$

and Theorem 4.10 itself.

Chapter 5
Iterative Subgradient Projection Algorithm

In this chapter we consider a minimization of a convex function on a solution set of a convex feasibility problem in a general Hilbert space using the iterative subgradient projection algorithm. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We show that an algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a small constant. Moreover, if we known computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this.

5.1 Preliminaries

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

Let m be a natural number and $f_i : X \to R^1$, $i = 1, \dots, m$ be convex continuous functions.

For every integer $i = 1, \dots, m$ put

$$C_i = \{x \in X : \ f_i(x) \leq 0\}, \tag{5.1}$$

$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : \ f_i(x) \leq 0\}. \tag{5.2}$$

We suppose that

$$C \neq \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given positive number ϵ a point $x \in X$ is called an ϵ-approximate solution of the feasibility

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problem if

$$f_i(x) \le \epsilon \text{ for all } i = 1, \dots, m.$$

Let $M_* > 0$ and

$$C \cap B(0, M_*) \ne \emptyset. \tag{5.3}$$

Let $f : X \to R^1$ be a convex continuous function. We consider the minimization problem

$$f(x) \to \min, \ x \in C.$$

Assume that

$$\inf(f, C) = \inf(f, C \cap B(0, M_*)). \tag{5.4}$$

Fix an integer

$$\bar{n} \ge m.$$

Denote by $\mathcal{R}$ the set of all mappings $S : \{1, 2, \dots\} \to \{1, \dots, m\}$ such that for every integer $p \ge 0$,

$$\{1, \dots, m\} \subset \{S(i) : \ i \in \{p\bar{n} + 1, \dots, (p + 1)\bar{n}\}. \tag{5.5}$$

Let $\alpha > 0$ and $r \in \mathcal{R}$.
Let us describe our algorithm.

Iterative Subgradient Projection Algorithm
Initialization: select an arbitrary $x_0 \in X$.
Iterative step: given a current iteration vector $x_k \in X$ calculate

$$l_k \in \partial f(x_k),$$

$$y_{k+1,0} = x_k - \alpha l_k,$$

for each $i \in \{1, \dots, \bar{n}\}$,

$$\text{if } f_{r(k\bar{n}+i)}(y_{k+1,i-1}) \le 0 \text{ then } l_{k,i-1} = 0, \ y_{k+1,i} = y_{k+1,i-1},$$

and if $f_{r(k\bar{n}+i)}(y_{k+1,i-1}) > 0$ then

$$l_{k,i-1} \in \partial f_{r(k\bar{n}+i)}(y_{k+1,i-1}),$$

$$y_{k+1,i} = y_{k+1,i-1} - f_{r(k\bar{n}+i)}(y_{k+1,i-1})\|l_{k,i-1}\|^{-2}l_{k,i-1}$$

and define the next iteration vector

$$x_{k+1} = y_{k+1,\bar{n}}.$$

We analyze this algorithm under the presence of computational errors.

In this chapter we analyze this algorithm under the presence of computational errors. We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f, $\delta_C \in [0, 1]$ is a computational error produced by our computer system, when we calculate subgradients of the constraint functions f_i, $i = 1, \dots, m$ and $\bar{\delta}_C$ is a computational error produced by our computer system, when we calculate auxiliary projection operators. Let $\alpha > 0$ be a step size and $\Delta \in (0, 1]$.

Iterative Subgradient Projection Algorithm with Computational Errors

Initialization: select $S \in \mathcal{R}$ and an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_t \in X$ calculate

$$l_t \in \partial f(x_t) + B(0, \delta_f),$$

$$y_{t+1,0} = x_t - \alpha l_t,$$

for each $i \in \{1, \dots, \bar{n}\}$,

$$\text{if } f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \leq \Delta, \text{ then } y_{t+1,i} = y_{t+1,i-1},\ l_{t,i-1} = 0,$$

if

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1}) > \Delta,$$

then

$$l_{t,i-1} \in \partial f_{S(t\bar{n}+i)}(y_{t+1,i-1}) + B(0, \delta_C),$$

(this implies that $l_{t,i-1} \neq 0$),

$$y_{t+1,i} \in B(y_{t+1,i-1} - f_{S(t\bar{n}+i)}(y_{t+1,i-1})\|l_{t,i-1}\|^{-2}l_{t,i-1}, \bar{\delta}_C),$$

$$x_{t+1} = y_{t+1,\bar{n}}.$$

5.2 Auxiliary Results

In order to study the behavior of our algorithm we use the following auxiliary results.

Let $\delta_f, \delta_C, \bar{\delta}_C \in [0, 1]$, $\Delta \in (0, 1]$ and $\alpha \in (0, 1]$, $\bar{n}$ be a natural number, $\tilde{M} \geq M_* + 4$, $M_0 \geq 1$, $M_1 > 2$,

$$f_i(B(0, 3\tilde{M} + 4)) \subset [-M_0, M_0],\ i = 1, \ldots .m, \tag{5.6}$$

$$|f_i(u) - f_i(v)| \leq (M_1 - 2)\|u - v\| \text{ for all } u, v \in B(0, 3\tilde{M} + 4),\ i = 1, \ldots, m, \tag{5.7}$$

$$|f(u) - f(v)| \leq L_0\|u - v\| \text{ for all } u, v \in B(0, 3\tilde{M} + 4), \tag{5.8}$$

$$\alpha \leq (2L_0 + 2)^{-1}L_0^{-1},$$

$$(6\tilde{M} + 3)\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2) \leq \alpha, \tag{5.9}$$

$$\delta_f(6\tilde{M} + L_0) \leq 1 \tag{5.10}$$

and set

$$\gamma = \bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2. \tag{5.11}$$

Lemma 5.1 *Let $\Delta_0 \geq 0$ and at least one of the following conditions below holds:*

(a) $\Delta_0 = \Delta$;
(b) for each $i \in \{1, \ldots, m\}$,

$$B(0, M_*) \cap \{v \in X : \ f_i(x) \leq -\Delta\} \neq \emptyset.$$

Assume that

$$z \in B(0, M_*) \cap C, \tag{5.12}$$

$$x \in B(z, 2\tilde{M}), \tag{5.13}$$

$r : \{1, \ldots, \bar{n}\} \to \{1, \ldots, m\}$, $l \in X$ *satisfies*

$$B(l, \delta_f) \cap \partial f(x) \neq \emptyset, \tag{5.14}$$

$$y_0 = x - \alpha l, \tag{5.15}$$

for every $i \in \{1, \ldots, \bar{n}\}$, $l_{i-1} \in X$,

$$\textit{if } f_{r(i)}(y_{i-1}) \le \Delta_0, \textit{ then } l_{i-1}, \ y_i = y_{i-1}, \tag{5.16}$$

if $f_{r(i)}(y_{i-1}) > \Delta_0$, *then*

$$B(l_{i-1}, \delta_C) \cap \partial f_{r(i)}(y_{i-1}) \ne \emptyset \tag{5.17}$$

(this implies that $l_{i-1} \ne 0$*) and*

$$y_i \in B(y_{i-1} - f_{r(i)}(y_{i-1})\|l_{i-1}\|^{-2} l_{i-1}, \bar{\delta}_C). \tag{5.18}$$

Then

$$\|x - z\|^2 - \|y_{\bar{n}} - z\|^2$$

$$\ge 2\alpha(f(x) - f(z)) - 2\bar{n}\gamma(6\tilde{M} + 3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$

$$\|x - z\|^2 - \|y_{\bar{n}} - z\|^2$$

$$\ge \sum_{i=1}^{\bar{n}} \|y_{i-1} - y_i\| - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{n}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\bar{n}\delta_C \Delta^{-2}(6\tilde{M} + 5)^3$$

and for each $i \in \{1, \dots, \bar{n}\}$, *if* $f_{r(i)}(y_{i-1}) \le \Delta_0$, *then* $y_{i-1} = y_i$ *and if* $f_{r(i)}(y_{i-1}) > \Delta_0$, *then*

$$\|y_{i-1} - y_i\| \ge -\bar{\delta}_C - 16\delta_2 M_0 \Delta^{-2}(6\tilde{M} + 1)^2 + M_1^{-1} f_{r(i)}(y_{i-1}).$$

Proof In view of (5.8), (5.12) and (5.13),

$$\partial f(x) \subset B(0, L_0). \tag{5.19}$$

Equations (5.14) and (5.19) imply that

$$\|l\| \le L_0 + 1. \tag{5.20}$$

By (5.9), (5.13), (5.15) and (5.20),

$$\|z - y_0\| \le \|z - x\| + \|x - y_0\| \le 2\tilde{M} + \alpha(L_0 + 1) \le 2\tilde{M} + 1. \tag{5.21}$$

It follows from (5.12) and (5.13) that

$$\|x\| \le 2\tilde{M} + M_*. \tag{5.22}$$

By (5.9), (5.15), (5.20) and (5.22),

$$\|y_0\| \le \|x\| + \alpha(L_0 + 1) \le 2\tilde{M} + M_* + 1. \tag{5.23}$$

Assume that $i \in \{1, \dots, \bar{n}\}$ satisfies

$$\|z - y_{i-1}\| \le 2\tilde{M} + (i-1)\gamma + 1. \tag{5.24}$$

(In view of (5.21) and (5.24) is true for $i = 1$.) By (5.12) and (5.24),

$$\|y_{i-1}\| \le 2\tilde{M} + M_* + 1 + (i-1)\gamma. \tag{5.25}$$

If $f_{r(i)}(y_{i-1}) \le \Delta_0$, then by (5.16),

$$y_i = y_{i-1}, \ \|y_i\| \le 2\tilde{M} + M_* + 1 + i\gamma, \tag{5.26}$$

$$\|z - y_i\|^2 - \|z - y_{i-1}\|^2 = \|y_i - y_{i-1}\|^2 = 0 \tag{5.27}$$

and set

$$\xi_{i-1} = 0. \tag{5.28}$$

Assume that

$$f_{r(i)}(y_{i-1}) > \Delta_0. \tag{5.29}$$

In view of (5.17) and (5.29), there exists

$$\xi_{i-1} \in \partial f_{r(i)}(y_{i-1}) \tag{5.30}$$

such that

$$\|l_{i-1} - \xi_{i-1}\| \le \delta_C. \tag{5.31}$$

By (5.6), (5.7), (5.11), (5.12), (5.25) and (5.29)–(5.31), the inequality $\bar{n}\gamma \le 1$ and Lemma 4.2 applied with $\delta_1 = 0$, $\delta_2 = \delta_C$, $\bar{\delta}_2 = \bar{\delta}_C$, $M = 3\tilde{M}$, $x = x_0 = y_{i-1}$, $j = r(i)$, $\xi = \xi_{i-1}$, $l = l_{i-1}$, $u = y_i$, we have

$$\|\xi_{i-1}\| \ge \Delta(6\tilde{M} + 1)^{-1}, \tag{5.32}$$

$$\|l_{i-1}\| \ge 2^{-1}\Delta(6\tilde{M} + 1)^{-1}, \tag{5.33}$$

$$\|z - y_i\| \le \|z - y_{i-1}\| + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2, \tag{5.34}$$

$$\|y_i - (y_{i-1} - f_{r(i)}(y_{i-1})\|\xi_{i-1}\|^{-2}\xi_{i-1})\|$$

$$\leq \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2, \tag{5.35}$$

$$\|z - y_{i-1}\|^2 - \|z - y_i\|^2 - \|y_i - y_{i-1}\|^2$$

$$\geq -\bar{\delta}_C(6\tilde{M} + 5) - 16\delta_C \Delta^{-2}(6\tilde{M} + 5)^3, \tag{5.36}$$

$$\|y_{i-1} - y_i\| \geq -\bar{\delta}_C - 16\delta_C M_0 \Delta^2(6\tilde{M} + 1)^2 + M_1^{-1} f_{r(i)}(y_{i-1}). \tag{5.37}$$

In view of (5.11), (5.24) and (5.34),

$$\|z - y_i\|$$

$$\leq 2\tilde{M} + 1 + (i-1)\gamma + \bar{\delta}_C + 16M_0\delta_C \Delta^{-2}(6\tilde{M} + 1)^2$$

$$= 2\tilde{M} + 1 + \gamma i.$$

Therefore by induction we showed that the following property holds:

(a) for all $i = 1, \dots, \bar{n}$,

$$\|z - y_{i-1}\| \leq 2\tilde{M} + (i-1)\gamma + 1, \tag{5.38}$$

if $f_{r(i)}(y_{i-1}) \leq \Delta_0$, then (5.27) holds and if $f_{r(i)}(y_{i-1}) > \Delta_0$, then (5.30)–(5.37) hold.

By (5.12), (5.13), (5.15), (5.20) and (5.22) and Lemma 4.3 applied with $x_0 = y_0$, $\delta_1 = \alpha(L_0 + 1)$,

$$\|z - x\|^2 - \|z - y_0\|^2 - \|x - y_0\|^2 \geq -\alpha(L_0 + 1)(12\tilde{M} + 2). \tag{5.39}$$

Let $i \in \{1, \dots, \bar{n}\}$. Define a mapping $T_i : X \to X$ as follows. If $f_{r(i)}(y_{i-1}) \leq \Delta_0$, then $T_i = I$, the identity operator in X. If $f_{r(i)}(y_{i-1}) > \Delta_0$, then define

$$D_i = \{v \in X : \ f_{r(i)}(y_{i-1}) + \langle \xi_{i-1}, v - y_{i-1}\rangle \leq 0\} \tag{5.40}$$

and set

$$T_i = P_{D_i}. \tag{5.41}$$

We consider the product $\prod_{i=1}^{\bar{n}} T_i$ and estimate

$$\|\prod_{i=1}^{\bar{n}} T_i(x - \alpha l) - y_{\bar{n}}\|.$$

We show by induction that for all $k = 1, \dots, \bar{n}$,

$$\|\prod_{i=1}^{k} T_i(x - \alpha l) - y_k\| \le k\gamma. \tag{5.42}$$

Let $k = 1$. If $f_{r(1)}(y_0) \le \Delta_0$, then by (5.16) and the equality $T_1 = I$,

$$y_1 = y_0, \ \|T_1(y_0) - y_1\| = 0.$$

If $f_{r(1)}(y_0) > \Delta_0$, then Lemma 4.1, (5.11), (5.30), (5.35), (5.40) and (5.41) imply that

$$\|T_1(y_0) - y_1\| = \|y_1 - (y_0 - f_{r(1)}(y_0)\|\xi_0\|^{-2}\xi_0)\| \le \gamma.$$

In view of (5.15) and (5.42) is true for $k = 1$.

Assume that $k \in \{1, \dots, \bar{n}\} \setminus \{\bar{n}\}$ and that (5.42) holds. If

$$f_{r(k+1)}(y_k) \le \Delta_0,$$

then by definition and (5.16),

$$T_{k+1} = I, \ y_{k+1} = y_k,$$

$$\|\prod_{i=1}^{k+1} T_i(y_0) - y_{k+1}\| = \|\prod_{i=1}^{k} T_i(y_0) - y_k\| \le k\gamma.$$

Let

$$f_{r(k+1)}(y_k) > \Delta_0. \tag{5.43}$$

Lemma 4.1, (5.11), (5.30), (5.35) and (5.40)–(5.43) imply that

$$\|\prod_{i=1}^{k+1} T_i(y_0) - y_{k+1}\|$$

$$\le \|T_{k+1}(\prod_{i=1}^{k} T_i(y_0)) - T_{k+1}(y_k)\| + \|T_{k+1}(y_k) - y_{k+1}\|$$

$$\le \|\prod_{i=1}^{k} T_i(y_0) - y_{k+1}\| + \gamma \le (k+1)\gamma.$$

Thus (5.42) holds for all $1, \dots, \bar{n}$ and

$$\|\prod_{i=1}^{\bar{n}} T_i(x - \alpha l) - y_{\bar{n}}\| \leq \bar{n}\gamma. \tag{5.44}$$

By (5.8), (5.9), (5.12)–(5.14), (5.40), (5.41) and (5.44), relation $\bar{n}\gamma \leq 1$ and Lemma 2.7 applied with

$$F_0 = C,\ M_0 = 3\tilde{M},\ \delta_1 = \delta_f,\ \xi = l,$$

$$u = y_{\bar{n}},\ \delta_2 = \bar{n}\gamma = \bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2),\ Q = \prod_{i=1}^{\bar{n}} T_i,$$

we have

$$\alpha(f(x) - f(z)) \leq 2^{-1}\|x - z\|^2 - 2^{-1}\|y_{\bar{n}} - z\|^2$$

$$+\bar{n}\gamma(6\tilde{M}+3) + 2^{-1}\alpha^2 L_0^2 + \alpha\delta_f(6\tilde{M} + L_0 + 1)$$

and

$$\|x - z\|^2 - \|y_{\bar{n}} - z\|^2$$

$$\geq 2\alpha(f(x) - f(z)) - 2\bar{n}\gamma(6\tilde{M}+3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1).$$

By property (a), (5.12), (5.13), (5.15), (5.20), (5.27) and (5.36),

$$\|x - z\|^2 - \|y_{\bar{n}} - z\|^2$$

$$= \|x - z\|^2 - \|y_0 - z\|^2 + \|y_0 - z\|^2 - \|y_{\bar{n}} - z\|^2$$

$$\geq -\alpha(L_0 + 1)(4\tilde{M} + 2) + \sum_{i=1}^{\bar{n}}(\|y_{i-1} - z\|^2 - \|y_i - z\|^2)$$

$$\geq -\alpha(L_0 + 1)(4\tilde{M} + 2) + \sum_{i=1}^{\bar{n}}\|y_{i-1} - y_i\|^2$$

$$-\bar{n}\bar{\delta}_C(6\tilde{M}+5) - 16\bar{n}\delta_C(6\tilde{M}+5)^3.$$

This completes the proof of Lemma 5.1.

Set

$$\epsilon_1 = M_1(\bar{\delta}_C + 16M_0\Delta^2\delta_C(6\tilde{M}+1)^2)$$

$$+M_1[\alpha(L_0+1)(4\tilde{M}+2)+\bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$+16\bar{n}\delta_C\Delta^{-2}(6\tilde{M}+5)^3]^{1/2}. \tag{5.45}$$

Proposition 5.2 *Let T be a natural number, $S \in \mathcal{R}$, $\Delta_0 \geq 0$, and at least one of the following conditions holds:*

(a) $\Delta_0 = \Delta$;
(b) for each $i \in \{1, \dots, m\}$,

$$B(0, M_*) \cap \{x \in X : \ f_i(x) \leq -\Delta\} \neq \emptyset.$$

Let

$$\epsilon_T = M_1(\alpha(L_0+1)(4\tilde{M}+2)+\bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$+16\bar{n}\delta_C\Delta^{-2}(6\tilde{M}+5)^3+4\tilde{M}^2T^{-1})^{1/2}+M_1\bar{\delta}_C+16M_1\delta_C M_0\Delta^2(6\tilde{M}+1)^2, \tag{5.46}$$

$$\epsilon_T \leq 2^{-1}M_1\bar{n}^{-1}, \ \gamma \leq \bar{n}^{-1}, \tag{5.47}$$

$\{x_t\}_{t=0}^T \subset X$, $\{l_t\}_{t=0}^{T-1} \subset X$, $y_{t,i} \in X$, $t = 1, \dots, t$, $i = 0, \dots, \bar{n}$,

$$l_{t,i} \in X, \ t = 0, \dots, T-1, \ i = 1, \dots, \bar{n}.$$

Assume that for all integers $t \in \{0, \dots, T-1\}$ and all integers $i \in \{1, \dots, \bar{n}\}$,

$$B(l_t, \delta_f) \cap \partial f(x_t) \neq \emptyset, \ y_{t+1,0} = x_t - \alpha l_t, \tag{5.48}$$

$$\text{if } f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \leq \Delta_0, \ \text{then } y_{t+1,i} = y_{t+1,i-1}, \ l_{t,i-1} = 0, \tag{5.49}$$

if

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1}) > \Delta_0,$$

then

$$B(l_{t,i-1}, \delta_C) \cap \partial f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \neq \emptyset \tag{5.50}$$

(this implies that $l_{t,i-1} \neq 0$),

$$y_{t+1,i} \in B(y_{t+1,i-1} - f_{S(t\bar{n}+i)}(y_{t+1,i-1})\|l_{t,i-1}\|^{-2}l_{t,i-1}, \bar{\delta}_C),$$

$$x_{t+1} = y_{t+1,\bar{n}}. \tag{5.51}$$

Then the following assertions hold.

Assertion 1. Let $t \in \{0, \dots, T-1\}$ *and let*

$$x_t \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C. \tag{5.52}$$

Then for every $z \in B(0, M_*) \cap C$,

$$\|x_t - z\|^2 - \|x_{t+1} - z\|^2$$

$$\geq 2\alpha(f(x_t) - f(z)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2)(6\tilde{M}+3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$

$$\|x_t - z\|^2 - \|x_{t+1} - z\|^2$$

$$\geq \sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0+1)(4\tilde{M}+2) - \bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3.$$

Assertion 2. Assume that

$$\text{for every } z \in B(0, M_*) \cap C,\ \|x_t - z\| \leq 2\tilde{M},\ t = 0, \dots, T. \tag{5.53}$$

Then

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C))$$

$$-2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1))^2)(6\tilde{M}+3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$

$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0+1)(4\tilde{M}+2) - \bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3 :\ t = 0\dots, T-1\} \leq 4\tilde{M}^2 T^{-1}.$$

Moreover, if $t \in \{0, \dots, T-1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2)(6\tilde{M}+3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$

$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0+1)(4\tilde{M}+2) - \bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3\} \le 4\tilde{M}^2T^{-1},$$

then

$$f(x_t) \le \inf(f, C) + 2\tilde{M}^2(T\alpha)^{-1}$$

$$+\alpha^{-1}\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2)(6\tilde{M}+3)$$

$$+2^{-1}\alpha L_0^2 + \delta_f(6\tilde{M}+L_0+1),$$

$$f_i(x_t) \le \bar{n}\epsilon_T + M_1\alpha(L_0+1) + \max\{\Delta_0, \epsilon_T\},\ i = 1, \dots, m.$$

Assertion 3. Let $M_{*,0} > 0$, $\tilde{M} \ge 2M_* + 2$.

$$\alpha(L_0+1)(6\tilde{M}+2) \le (4\bar{n})^{-1},\ 4\bar{n}^2\bar{\delta}_C(6\tilde{M}+5) \le 1, \tag{5.54}$$

$$64\delta_C\bar{n}^2\Delta^{-2}(6\tilde{M}+5)^3) \le 1, \tag{5.55}$$

$$|f(u)| \le M_{*,0},\ u \in B(0, M_*), \tag{5.56}$$

$$f(u) > M_{*,0} + 8 \text{ for all } u \in X \setminus B(0, 2^{-1}\tilde{M}), \tag{5.57}$$

$t \in \{0, \dots, T-1\}$ *and that*

$$x_t \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C. \tag{5.58}$$

Then

$$x_{t+1} \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C.$$

Assertion 4. Let $r_0 \in (0, 1]$,

$$r_0 \ge \alpha(L_0+1)M_1 + \bar{n}\epsilon_1 + \max\{\epsilon_1, \Delta_0\},\ \bar{n}M_1^{-1}\epsilon_1 \le 1, \tag{5.59}$$

$$\{x \in X :\ f_i(x) \le r_0,\ i = 1, \dots, m\} \subset B(0, M_*), \tag{5.60}$$

$t \in \{0, \dots, T-1\}$ *and*

$$x_t \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C. \tag{5.61}$$

Then

$$x_{t+1} \in B(z, 2\tilde{M}) \textit{ for every } z \in B(0, M_*) \cap C.$$

Proof We prove Assertion 1. Let

$$z \in B(0, M) \cap C_*.$$

By (5.11), (5.48)–(5.52), (5.61) and Lemma 5.1 applied with

$$x = x_t,\ l = l_t,\ r(i) = S(t\bar{n} + i),\ i = 1, \dots, \bar{n},$$

$$l_i = l_{t,i},\ i = 1, \dots, \bar{n},\ y_i = y_{t+1,i},\ i = 0, \dots, \bar{n},\ x_{t+1} = y_{t+1,\bar{n}}$$

we obtain that

$$\|x_t - z\|^2 - \|x_{t+1} - z\|^2$$

$$\geq 2\alpha(f(x_t) - f(z))$$

$$-2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3)$$

$$- \alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1), \tag{5.62}$$

$$\|x - z\|^2 - \|x_{t+1} - z\|^2$$

$$\geq \sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{n}\bar{\delta}_C(6\tilde{M} + 5)$$

$$- 16\bar{n}\delta_C\Delta^{-2}(6\tilde{M} + 5)^3 \tag{5.63}$$

and for each $i \in \{1, \dots, \bar{n}\}$,

$$\text{if } f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \leq \Delta_0, \text{ then } y_{t+1,i} = y_{t+1,i-1} \tag{5.64}$$

and

$$\text{if } f_{S(t\bar{n}+i)}(y_{t+1,i-1}) > \Delta_0, \text{ then } \|y_{t+1,i-1} - y_{t+1,i}\|$$

$$\geq -\bar{\delta}_C - 16\delta_C M_0\Delta^2(6\tilde{M} + 1)^2 + M_1^{-1} f_{S(t\bar{n}+i)}(y_{t+1,i-1}). \tag{5.65}$$

Assertion 1 is proved.

We prove assertion 2. Let

$$z \in B(0, M) \cap C_*. \tag{5.66}$$

By (5.53), (5.66) and Assertion 1, for all $t = 0, \dots, T-1$ relations (5.62) and (5.63) hold and relation (5.64) and (5.65) are true for $i = 1, \dots, \bar{n}$. By (5.53), (5.64) and (5.65),

$$4\tilde{M}^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1} (\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\geq \sum_{t=0}^{T-1} \max\{2\alpha(f(x_t) - f(z)) - 2\bar{n}(\bar{\delta}_C$$

$$+16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2)(6\tilde{M}+3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M}+L_0+1),$$

$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0+1)(4\tilde{M}+2) - \bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$- 16\bar{n}\delta_C\Delta^{-2}(6\tilde{M}+5)^3\}, \tag{5.67}$$

Since z is an arbitrary element of $C \cap B(0, M_*)$ it follows from (4.11) and (5.67) that

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2)(6\tilde{M}+3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M}+L_0+1),$$

$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0+1)(4\tilde{M}+2) - \bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3 : t = 0\dots, T-1\} \leq 4\tilde{M}^2 T^{-1}.$$

Let $t \in \{0, \dots, T-1\}$ and

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2)(6\tilde{M}+3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M}+L_0+1),$$

$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0+1)(4\tilde{M}+2) - \bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3\}\le 4\tilde{M}^2T^{-1}. \tag{5.68}$$

In view of (5.68),

$$f(x_t)\le \inf(f,C)+2\tilde{M}^2(T\alpha)^{-1}$$
$$+\alpha^{-1}\bar{n}(\bar{\delta}_C+16M_0\Delta_C\Delta^{-2}(6\tilde{M}+1)^2)(6\tilde{M}+3)$$
$$+2^{-1}\alpha L_0^2+\delta_f(6\tilde{M}+L_0+1).$$

By (5.68), for $i=1,\dots,\bar{n}$,

$$\|y_{t+1,i-1}-y_{t+1,i}\|\le(\alpha(L_0+1)(4\tilde{M}+2)+\bar{n}\bar{\delta}_C(6\tilde{M}+5)$$
$$+16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3)+4\tilde{M}^2T^{-1})^{1/2}. \tag{5.69}$$

Let $i\in\{1,\dots,\bar{n}\}$. Assume that

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1})>\Delta_0. \tag{5.70}$$

It follows from (5.47), (5.65) and (5.70) that

$$M_1^{-1}f_{S(t\bar{n}+i)}(y_{t+1,i-1})-\bar{\delta}_C-16\delta_CM_0\Delta^2(6\tilde{M}+1)$$
$$\le\|y_{t+1,i-1}-y_{t+1,i}\|$$
$$\le(\alpha(L_0+1)(4\tilde{M}+2)+\bar{n}\bar{\delta}_C(6\tilde{M}+5)$$
$$+16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3+4\tilde{M}^2T^{-1})^{1/2}.$$

This implies that

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1})\le M_1(\alpha(L_0+1)(4\tilde{M}+2)+\bar{n}\bar{\delta}_C(6\tilde{M}+5)$$
$$+16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3+4\tilde{M}^2T^{-1})^{1/2}$$
$$+M_1\bar{\delta}_C+16M_1\delta_CM_0\Delta^2(6\tilde{M}+1)^2=\epsilon_T. \tag{5.71}$$

By (5.46), (5.70) and (5.71),

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1})\le\max\{\Delta_0,\epsilon_T\}. \tag{5.72}$$

In view (5.69),

$$\|y_{t+1,i-1} - y_{t+1,i}\| \le \epsilon_T M_1^{-1}. \tag{5.73}$$

Let $i \in \{1, \dots, \bar{n}\}$. It follows from (5.8), (5.9), (5.47)–(5.48) and (5.73) that

$$\|x_t - y_{t+1,i}\| \le \|x_t - y_{t+1,0}\| + i\epsilon_T M_1^{-1} \le \alpha(L_0 + 1) + \bar{n} M_1^{-1} \epsilon_T. \tag{5.74}$$

By (5.52), (5.66) and (5.74),

$$\|x_t\| \le M_* + 2\tilde{M}, \ \|y_{t+1,i}\| \le 3\tilde{M} + 2. \tag{5.75}$$

In view (5.7), (5.74) and (5.75),

$$|f(x_t) - f_{S(t\bar{n}+i)}(y_{t+1,i})| \le M_1\|x_t - y_{t+1,i}\| \le M_1\alpha(L_0 + 1) + \bar{n}\epsilon_T. \tag{5.76}$$

It follows from (5.72) and (5.76) that

$$f_{S(t\bar{n}+i)}(x_t) \le \bar{n}\epsilon_T + M_1\alpha(L_0 + 1) + \max\{\Delta_0, \epsilon_T\}.$$

Assertion 2 is proved.

We prove Assertion 3. Let

$$z \in B(0, M_*) \cap C. \tag{5.77}$$

In view of Assertion 1, (5.62) and (5.63) hold. We may assume without loss of generality that

$$\|x_{t+1} - z\| > \|x_t - z\|. \tag{5.78}$$

By (5.9), (5.10), (5.56), (5.62), (5.77) and (5.78),

$$\begin{aligned} f(x_t) &\le f(z) + \alpha^{-1}\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3) \\ &+ 2^{-1}\alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1) \\ &\le M_{*,0} + 3. \end{aligned} \tag{5.79}$$

In view of (5.53), (5.55) and (5.57),

$$\|x_t\| \le \tilde{M}/2. \tag{5.80}$$

It follows from (5.63) and (5.78) that for $i = 1, \dots, \bar{n}$,

$$\|y_{t+1,i-1} - y_{t+1,i}\|^2 \le \alpha(L_0 + 1)(4\tilde{M} + 2) + \bar{n}\bar{\delta}_C(6\tilde{M} + 5)$$

$$+16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3 \le \bar{n}^{-1}. \tag{5.81}$$

By the relation $x_{t+1} = y_{t+1,\bar{n}}$, (5.8), (5.48), (5.80) and (5.81),

$$\|x_t - x_{t+1}\| = \|x_t - y_{t+1,\bar{n}}\|$$

$$\le \|x_t - y_{t+1,0}\| + \|y_{t+1,0} - y_{t+1,\bar{n}}\| \le \alpha(L_0+1)+1 \le 2. \tag{5.82}$$

It follows from (5.77), (5.80) and (5.82) that

$$\|x_{t+1} - z\| \le \|x_{t+1} - x_t\| + \|x_t - z\| \le 2 + \tilde{M}/2 + M_* < 2\tilde{M}.$$

Assertion 3 is proved.

We prove Assertion 4.

$$z \in B(0, M) \cap C_*. \tag{5.83}$$

In view of Assertion 1, (5.63)–(5.65) hold. We may assume without loss of generality that

$$\|x_{t+1} - z\| > \|x_t - z\|. \tag{5.84}$$

Let $i \in \{1, \dots, \bar{n}\}$. By (5.63) and (5.84),

$$\|y_{t+1,i-1} - y_{t+1,i}\|$$

$$\le (\alpha(L_0+1)(4\tilde{M}+2) + \bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$+16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3)^{1/2}. \tag{5.85}$$

If

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1}) > \Delta_0,$$

then it follows from (5.45), (5.65) and (5.85) that

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \le M_1(\bar{\delta}_C + 16\delta_C M_0 \Delta^2(6\tilde{M}+1)^2)$$

$$+M_1\|y_{t+1,i-1} - y_{t+1,i}\|$$

$$\le M_1(\bar{\delta}_C + 16\delta_C M_0 \Delta^2(6\tilde{M}+1)^2)$$

$$+M_1(\alpha(L_0+1)(4\tilde{M}+2) + \bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$+16\delta_C \bar{n} \Delta^{-2}(6\tilde{M}+5)^3)^{1/2} = \epsilon_1.$$

Thus

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \le \max\{\epsilon_1, \Delta_0\}. \tag{5.86}$$

In view of (5.45) and (5.85),

$$\|y_{t+1,i-1} - y_{t+1,i}\| \le M_1^{-1}\epsilon_1. \tag{5.87}$$

By (5.8), (5.9), (5.47), (5.48), (5.61) and (5.87), for all $i = 1, \dots, \bar{n}$,

$$\|x_t\| \le 3\tilde{M}, \tag{5.88}$$

$$\|x_t - y_{t+1,i}\| \le \|x_t - y_{t+1,0}\| + \|y_{t+1,0} - y_{t+1,i}\|$$

$$\le \alpha(L_0 + 1) + \bar{n} M_1^{-1}\epsilon_1 \le 2, \tag{5.89}$$

$$\|y_{t+1,i}\| \le 3\tilde{M} + 2. \tag{5.90}$$

By (5.7) and (5.88)–(5.90), for all $i = 1, \dots, \bar{n}$,

$$|f_{S(t\bar{n}+i)}(x_t) - f_{S(t\bar{n}+i)}(y_{t+1,i})|$$

$$\le M_1\|x_t - y_{t+1,i}\| \le M_1\alpha(L_0+1) + \bar{n}\epsilon_1. \tag{5.91}$$

In view of (5.51), (5.59), (5.86) and (5.91), for $i = 1, \dots, m$,

$$f_i(x_t) \le M_1\alpha(L_0+1) + \bar{n}\epsilon_1 + \max\{\epsilon_1, \Delta_0\} \le r_0.$$

Equations (5.60) implies that

$$\|x_t\| \le M_*. \tag{5.92}$$

It follows from (5.89) and (5.92) that

$$\|x_{t+1}\| \le M_* + 2, \ \|z - x_{t+1}\| \le M_* + \tilde{M} + 2 < 2\tilde{M}.$$

Assertion 4 is proved. This completes the proof of Proposition 5.2.

Analogously to Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations in the case of Assertion 2.

5.3 The Main Results

We use the notation and definitions introduced in Section 5.1 and 5.2 and suppose that all the assumptions made there hold. The following result is easily deduced from Assertions 2 and 3 of Proposition 5.2. It is proved under assumption that the objective function f satisfies a growth condition.

Theorem 5.3 *Let $M_{*,0} > 0$, $\tilde{M} \geq 2M_* + 2$.*

$$|f(u)| \leq M_{*,0},\ u \in B(0, M_*),$$

$$f(u) > M_{*,0} + 8 \text{ for all } u \in X \setminus B(0, 2^{-1}\tilde{M}),$$

$S \in \mathcal{R}$, a natural number T satisfy

$$T \geq 4\tilde{M}^2 M_1^2 \epsilon_1^{-2},$$

$$4M_1^{-1}\bar{n}\epsilon_1 \leq 1,\ 4\bar{n}\bar{\delta}_C(6\tilde{M} + 5) \leq 1,$$

$$64\delta_C \bar{n}^2 \Delta^{-2}(6\tilde{M} + 5)^3 \leq 1,\ 2\bar{n}\alpha(L_0 + 1) < 1,$$

$\{x_t\}_{t=0}^T \subset X$,

$$\|x_0\| \leq \tilde{M},$$

$\{l_t\}_{t=0}^{T-1} \subset X$, $y_{t,i} \in X$, $t = 1, \dots, t$, $i = 0, \dots, \bar{n}$, $l_{t,i} \in X$, $t = 0, \dots, T-1$, $i = 0, \dots, \bar{n}$.

Assume that for all integers $t \in \{0, \dots, T-1\}$ and all integers $i \in \{1, \dots, \bar{n}\}$,

$$B(l_t, \delta_f) \cap \partial f(x_t) \neq \emptyset,\ y_{t+1,0} = x_t - \alpha l_t,$$

$$\text{if } f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \leq \Delta, \text{ then } y_{t+1,i} = y_{t+1,i-1},\ l_{t,i-1} = 0,$$

if

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1}) > \Delta,$$

then

$$B(l_{t,i-1}, \delta_C) \cap \partial f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \neq \emptyset$$

(this implies that $l_{t,i-1} \neq 0$),

$$y_{t+1,i} \in B(y_{t+1,i-1} - f_{S(t\bar{n}+i)}(y_{t+1,i-1})\|l_{t,i-1}\|^{-2} l_{t,i-1}, \bar{\delta}_C),$$

$$x_{t+1} = y_{t+1,\bar{n}}.$$

Then

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$

$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{n}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C\bar{n}\Delta^{-2}(6\tilde{M} + 5)^3\} : \ t = 0 \ldots, T - 1\} \le 4\tilde{M}^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T - 1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$

$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0 + 1)(6\tilde{M} + 1) - \bar{n}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C\bar{n}\Delta^{-2}(6\tilde{M} + 5)^3\} \le 4\tilde{M}^2 T^{-1},$$

then

$$f(x_t) \le \inf(f, C)$$

$$+2M^2(T\alpha)^{-1} + \alpha^{-1}\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3)$$

$$+2^{-1}\alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1),$$

$$f_i(x_t) \le \bar{n}\epsilon_T + M_1\alpha(L_0 + 1) + \max\{\Delta, \epsilon_T\}$$

$$\le M_1\alpha(L_0 + 1) + 4\bar{n}\epsilon_1 + \max\{\Delta, 4\epsilon_1\}, \ i = 1, \ldots, m.$$

The following result is easily deduced from (5.45), (5.47), Assertions 2 and 4 of Proposition 5.2. It is proved under assumption that the set

$$\{x \in X : \ f_i(x) \le r_0, \ i = 1, \ldots, m\}$$

is bounded for some $r_0 > 0$.

Theorem 5.4 *Let* $r_0 \in (0, 1]$, $\tilde{M} \geq 2M_*$, $S \in \mathcal{R}$,

$$\{x \in X : f_i(x) \leq r_0, \ i = 1, \dots, m\} \subset B(0, M_*),$$

$$r_0 \geq \alpha(L_0 + 1)M_1 + \bar{n}\epsilon_1 + \max\{\epsilon_1, \Delta\}, \ 8\bar{n}M_1^{-1}\epsilon_1 \leq 1,$$

a natural number T *satisfy*

$$T \geq 4\tilde{M}^2 M_1^2 \epsilon_1^{-2},$$

$\{x_t\}_{t=0}^{T} \subset X$,

$$\|x_0\| \leq \tilde{M},$$

$\{l_t\}_{t=0}^{T-1} \subset X$, $y_{t,i} \in X$, $t = 1, \dots, T$, $i = 0, \dots, \bar{n}$, $l_{t,i} \in X$, $t = 0, \dots, T-1$, $i = 0, \dots, \bar{n}$.

Assume that for all integers $t \in \{0, \dots, T-1\}$ *and all integers* $i \in \{1, \dots, \bar{n}\}$,

$$B(l_t, \delta_f) \cap \partial f(x_t) \neq \emptyset, \ y_{t+1,0} = x_t - \alpha l_t,$$

$$\text{if } f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \leq \Delta, \ \text{then } y_{t+1,i} = y_{t+1,i-1}, \ l_{t,i-1} = 0,$$

if

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1}) > \Delta,$$

then

$$B(l_{t,i-1}, \delta_C) \cap \partial f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \neq \emptyset$$

(this implies that $l_{t,i-1} \neq 0$*),*

$$y_{t+1,i} \in B(y_{t+1,i-1} - f_{S(t\bar{n}+i)}(y_{t+1,i-1})\|l_{t,i-1}\|^{-2} l_{t,i-1}, \bar{\delta}_C),$$

$$x_{t+1} = y_{t+1,\bar{n}}.$$

Then

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$

$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0+1)(4\tilde{M}+2) - \bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3\} : \ t = 0\ldots, T-1\} \le 4\tilde{M}^2T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T-1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2)(6\tilde{M}+3)$$

$$-\alpha^2L_0^2 - 2\alpha\delta_f(6\tilde{M}+L_0+1),$$

$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0+1)(4\tilde{M}+2) - \bar{n}\bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\bar{n}\Delta^{-2}(6\tilde{M}+5)^3\} \le 4\tilde{M}^2T^{-1},$$

then

$$f(x_t) \le \inf(f, C) + 2\tilde{M}^2(T\alpha)^{-1} + \alpha^{-1}\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2)(6\tilde{M}+3)$$

$$+2^{-1}\alpha L_0^2 + \delta_f(6\tilde{M}+L_0+1),$$

$$f_i(x_t) \le \bar{n}\epsilon_1 + M_1\alpha(L_0+1) + \max\{\Delta_0, \epsilon_T\}, \ i = 1, \ldots, m.$$

Let us consider a modification of our algorithm.

Iterative Subgradient Projection Algorithm with Computational Errors

Initialization: select $S \in \mathcal{R}$, $\epsilon \ge 0$ and an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_t \in X$ calculate

$$l_t \in \partial f(x_t) + B(0, \delta_f),$$

$$y_{t+1,0} = x_t - \alpha l_t,$$

for each $i \in \{1, \ldots, m\}$,

$$\text{if } f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \le \epsilon, \text{ then } y_{t+1,i} = y_{t+1,i-1}, \ l_{t,i-1} = 0,$$

if

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1}) > \epsilon,$$

then

$$l_{t,i-1} \in \partial f_{S(t\bar{n}+i)}(y_{t+1,i-1}) + B(0, \delta_C),$$

(this implies that $l_{t,i-1} \neq 0$),

$$y_{t+1,i} \in B(y_{t+1,i-1} - f_{S(t\bar{n}+i)}(y_{t+1,i-1})\|l_{t,i-1}\|^{-2}l_{t,i-1}, \bar{\delta}_C),$$

$$x_{t+1} = y_{t+1,\bar{n}}.$$

The following result is easily deduced from Assertions 2 and 3 of Proposition 5.2 with $\Delta_0 = \epsilon$. It is valid under a growth condition on the objective function f.

Theorem 5.5 *Let $M_{*,0} > 0$, $\tilde{M} \geq 2M_* + 2$, $\epsilon \geq 0$,*

$$B(0, M_*) \cap \{x \in X : f_i(x) \leq -\Delta\} \neq \emptyset, \ i = 1, \ldots, m,$$

$$|f(u)| \leq M_{*,0}, \ u \in B(0, M_*),$$

$$f(u) > M_{*,0} + 8 \textit{ for all } u \in X \setminus B(0, 2^{-1}M),$$

$S \in \mathcal{R}$, a natural number T satisfy

$$T \geq 4\tilde{M}^2 M_1^2 \epsilon_1^{-2},$$

$$4M_1^{-1}\bar{n}\epsilon_1 \leq 1, \ 4\bar{n}^2\bar{\delta}_C(6\tilde{M} + 5) \leq 1,$$

$$64\delta_C\bar{n}^2\Delta^{-2}(6\tilde{M} + 5)^3 \leq 1, \ 2\bar{n}\alpha(L_0 + 1) < (3\tilde{M} + 1)^{-1},$$

$\{x_t\}_{t=0}^T \subset X$,

$$\|x_0\| \leq \tilde{M},$$

$\{l_t\}_{t=0}^{T-1} \subset X$, $y_{t,i} \in X$, $t = 1, \ldots, t$, $i = 0, \ldots, \bar{n}$, $l_{t,i} \in X$, $t = 0, \ldots, T-1$, $i = 0, \ldots, \bar{n}$.

Assume that for all integers $t \in \{0, \ldots, T-1\}$ and all integers $i \in \{1, \ldots, \bar{n}\}$,

$$B(l_t, \delta_f) \cap \partial f(x_t) \neq \emptyset, \ y_{t+1,0} = x_t - \alpha l_t,$$

$$\textit{if } f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \leq \epsilon, \textit{ then } y_{t+1,i} = y_{t+1,i-1}, \ l_{t,i-1} = 0,$$

if

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1}) > \epsilon,$$

then

$$B(l_{t,i-1}, \delta_C) \cap \partial f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \neq \emptyset$$

(this implies that $l_{t,i-1} \neq 0$*),*

$$y_{t+1,i} \in B(y_{t+1,i-1} - f_{S(t\bar{n}+i)}(y_{t+1,i-1})\|l_{t,i-1}\|^{-2} l_{t,i-1}, \bar{\delta}_C),$$

$$x_{t+1} = y_{t+1,\bar{n}}.$$

Then

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3)$$
$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$
$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{n}\bar{\delta}_C(6\tilde{M} + 5)$$
$$-16\delta_C\bar{n}\Delta^{-2}(6\tilde{M} + 5)^3\} : \ t = 0\ldots, T - 1\} \leq 4\tilde{M}^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T - 1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3)$$
$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$
$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{n}\bar{\delta}_C(6\tilde{M} + 5)$$
$$-16\delta_C\bar{n}\Delta^{-2}(6\tilde{M} + 5)^3\} \leq 4\tilde{M}^2 T^{-1},$$

then

$$f(x_t) \leq \inf(f, C) + 2\tilde{M}^2(T\alpha)^{-1}$$
$$+\alpha^{-1}\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3)$$
$$+2^{-1}\alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1),$$

$$f_i(x_t) \leq 4\bar{n}\epsilon_1 + M_1\alpha(L_0 + 1) + \max\{\epsilon, 4\epsilon_1\}, \ i = 1, \ldots, m.$$

The following result is easily deduced from (5.45), (5.47), Assertions 2 and 4 of Proposition 5.2 with $\Delta_0 = \epsilon$. It holds under an assumption that the set

$$\{x \in X : f_i(x) \le r_0, \ i = 1, \dots, m\}$$

is bounded for some $r_0 > 0$.

Theorem 5.6 *Let $r_0 \in (0, 1]$, $\tilde{M} \ge 2M_*$, $S \in \mathcal{R}$, $\epsilon \ge 0$,*

$$B(0, M_*) \cap \{x \in X : f_i(x) \le -\Delta\} \ne \emptyset, \ i = 1, \dots, m,$$

$$\{x \in X : f_i(x) \le r_0, \ i = 1, \dots, m\} \subset B(0, M_*),$$

$$r_0 \ge \alpha(L_0 + 1)M_1 + \bar{n}\epsilon_1 + \max\{\epsilon_1, \epsilon\}, \ 4\bar{n}M_1^{-1}\epsilon_1 \le 1,$$

a natural number T satisfy

$$T \ge 4\tilde{M}^2 M_1^2 \epsilon_1^{-2},$$

$\{x_t\}_{t=0}^T \subset X$,

$$\|x_0\| \le \tilde{M},$$

$\{l_t\}_{t=0}^{T-1} \subset X$, $y_{t,i} \in X$, $t = 1, \dots, T$, $i = 0, \dots, \bar{n}$, $l_{t,i} \in X$, $t = 0, \dots, T-1$, $i = 0, \dots, \bar{n}$.

Assume that for all integers $t \in \{0, \dots, T-1\}$ and all integers $i \in \{1, \dots, \bar{n}\}$,

$$B(l_t, \delta_f) \cap \partial f(x_t) \ne \emptyset, \ y_{t+1,0} = x_t - \alpha l_t,$$

$$\text{if } f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \le \epsilon, \ \text{then } y_{t+1,i} = y_{t+1,i-1}, \ l_{t,i-1} = 0,$$

if

$$f_{S(t\bar{n}+i)}(y_{t+1,i-1}) > \epsilon,$$

then

$$B(l_{t,i-1}, \delta_C) \cap \partial f_{S(t\bar{n}+i)}(y_{t+1,i-1}) \ne \emptyset$$

(this implies that $l_{t,i-1} \ne 0$),

$$y_{t+1,i} \in B(y_{t+1,i-1} - f_{S(t\bar{n}+i)}(y_{t+1,i-1})\|l_{t,i-1}\|^{-2} l_{t,i-1}, \bar{\delta}_C),$$

$$x_{t+1} = y_{t+1,\bar{n}}.$$

Then

$$\min\{\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$

$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{n}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C \bar{n}\Delta^{-2}(6\tilde{M} + 5)^3\} : \ t = 0 \ldots, T - 1\} \le 4\tilde{M}^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T - 1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$

$$\sum_{i=1}^{\bar{n}} \|y_{t+1,i-1} - y_{t+1,i}\|^2 - \alpha(L_0 + 1)(6\tilde{M} + 2) - \bar{n}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C \bar{n}\Delta^{-2}(6\tilde{M} + 5)^3\} \le 4\tilde{M}^2 T^{-1},$$

then

$$f(x_t) \le \inf(f, C) + 2\tilde{M}^2(T\alpha)^{-1}$$

$$+\alpha^{-1}\bar{n}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)(6\tilde{M} + 3)$$

$$+2^{-1}\alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1),$$

$$f_i(x_t) \le 4\bar{n}\epsilon_1 + M_1\alpha(L_0 + 1) + \max\{\epsilon, 4\epsilon_1\}, \ i = 1, \ldots, m.$$

Chapter 6
Dynamic String-Averaging Subgradient Projection Algorithm

In this chapter we consider a minimization of a convex function on a solution set of a convex feasibility problem in a general Hilbert space using the dynamic string-averaging (DSA) subgradient projection algorithm. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We show that an algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a small constant. Moreover, if we known computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this.

6.1 Preliminaries

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

Let m be a natural number and $f_i : X \to R^1$, $i = 1, \dots, m$ be convex continuous functions.

For every integer $i = 1, \dots, m$ put

$$C_i = \{x \in X : \ f_i(x) \leq 0\}, \tag{6.1}$$

$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : \ f_i(x) \leq 0\}. \tag{6.2}$$

We suppose that

$$C \neq \emptyset.$$

A point $x \in C$ is called a solution of our feasibility problem. For a given positive number ϵ a point $x \in X$ is called an ϵ-approximate solution of the feasibility

A. J. Zaslavski, *Optimization on Solution Sets of Common Fixed Point Problems*,
Springer Optimization and Its Applications 178,
https://doi.org/10.1007/978-3-030-78849-0_6

problem if

$$f_i(x) \le \epsilon \text{ for all } i = 1, \dots, m.$$

Let $M_* > 0$ and

$$C \cap B(0, M_*) \ne \emptyset. \tag{6.3}$$

Let $f : X \to R^1$ be a convex continuous function. We consider the minimization problem

$$f(x) \to \min, \ x \in C.$$

Assume that

$$\inf(f, C) = \inf(f, C \cap B(0, M_*)) > -\infty. \tag{6.4}$$

By an index vector, we a mean a vector $t = (t_1, \dots, t_p)$ such that $t_i \in \{1, \dots, m\}$ for all $i = 1, \dots, p$.

For an index vector $t = (t_1, \dots, t_q)$ set

$$p(t) = q. \tag{6.5}$$

Fix a number

$$\bar{\Delta} \in (0, m^{-1}] \tag{6.6}$$

and an integer

$$\bar{q} \ge m. \tag{6.7}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of index vectors and

$$w : \Omega \to (0, \infty) \text{ satisfies } \sum_{t \in \Omega} w(t) = 1 \tag{6.8}$$

such that

$$p(t) \le \bar{q} \ \text{ for all } t \in \Omega, \tag{6.9}$$

$$w(t) \ge \bar{\Delta} \text{ for all } t \in \Omega, \tag{6.10}$$

$$\cup_{t \in \Omega} \{t_1, \dots, t_{p(t)}\} = \{1, \dots, m\}. \tag{6.11}$$

Let $x \in X$, $\delta, \bar{\delta}, \epsilon \geq 0$ and let $i \in \{1, \dots, m\}$. Set

$$A_i(x, \delta, \bar{\delta}, \epsilon) := \{x\} \text{ if } f_i(x) \leq \epsilon$$

and if $f_i(x) > \epsilon$, then set

$$A_i(x, \delta, \bar{\delta}, \epsilon)$$

$$= \{x - f_i(x)\|l\|^{-2}l : \ l \in \partial f_i(x) + B(0, \delta), \ l \neq 0\} + B(0, \bar{\delta}). \tag{6.12}$$

Let $x \in X$ and let $t = (t_1, \dots, t_{p(t)})$ be an index vector, $\delta \geq 0$, $\bar{\delta} \geq 0$, $\epsilon \geq 0$. Define

$$A_0(t, x, \delta, \bar{\delta}, \epsilon)$$

$$= \{(y, \lambda) \in X \times R^1 : \ \text{there is a sequence } \{y_i\}_{i=0}^{p(t)} \subset X \text{ such that}$$

$$y_0 = x,$$

for each $i = 1, \dots, p(t)$,

$$y_i \in A_{t_i}(y_{i-1}, \delta, \bar{\delta}, \epsilon),$$

$$y = y_{p(t)},$$

$$\lambda = \max\{\|y_i - y_{i-1}\| : \ i = 1, \dots, p(t)\}\}. \tag{6.13}$$

Let $x \in X$, $(\Omega, w) \in \mathcal{M}$, $\delta, \bar{\delta}, \epsilon \geq 0$. Define

$$A(x, (\Omega, w), \delta, \bar{\delta}, \epsilon) = \{(y, \lambda) \in X \times R^1 : \ \text{there exist}$$

$$(y_t, \lambda_t) \in A_0(t, x, \delta, \bar{\delta}, \epsilon), \ t \in \Omega$$

such that

$$\|y - \sum_{t \in \Omega} w(t) y_t\| \leq \bar{\delta},$$

$$\lambda = \max\{\lambda_t : \ t \in \Omega\}\}. \tag{6.14}$$

Let $\delta_f, \delta_C, \bar{\delta}_C, \epsilon \geq 0$ and $\alpha \in (0, 1]$. In this chapter we analyze the dynamic string-averaging (DSA) subgradient projection algorithm under the presence of computational errors. We suppose that δ_f is a computational error produced by our computer system, when we calculate a subgradient of the objective function f, δ_C is a computational error produced by our computer system, when we calculate subgradients of the constraint functions $f_i,\ i = 1, \dots, m$ and $\bar{\delta}_C$ is a computational error produced by our computer system, when we calculate auxiliary projection operators. Let $\alpha > 0$ be a step size.

The Dynamic String-Averaging (DSA) Subgradient Projection Algorithm Under the Presence of Computational Errors

Initialization: select an arbitrary $x_0 \in X$ and $\epsilon \geq 0$.

Iterative step: given a current iteration vector $x_k \in X$ calculate

$$l_k \in \partial f(x_k) + B(0, \delta_f),$$

pick a pair

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

and calculate the next iteration pair

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k - \alpha l_k, (\Omega_{k+1}, w_{k+1}), \delta_C, \bar{\delta}_C, \epsilon).$$

In order to proceed we make the following assumptions.

Assume that $\Delta \in (0, 1]$, $\tilde{M} \geq M_* + 4$, $M_0 \geq 1$, $L_0 \geq 1$, $M_1 > 2$,

$$f_i(B(0, 3\tilde{M} + 4)) \subset [-M_0, M_0],\ i = 1, \dots, m, \tag{6.15}$$

$$|f_i(u) - f_i(v)| \leq (M_1 - 2)\|u - v\| \text{ for all } u, v \in B(0, 3\tilde{M} + 4),\ i = 1, \dots, m, \tag{6.16}$$

$$|f(u) - f(v)| \leq L_0\|u - v\| \text{ for all } u, v \in B(0, 3\tilde{M} + 4), \tag{6.17}$$

$$\bar{q}(\bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 2)^2)(M_0 + 1) \leq \alpha, \tag{6.18}$$

$$\alpha \leq (L_0 + 1)^{-1}(6\tilde{M} + 2)^{-1},\ \alpha \leq 2^{-1}L_0^{-1}(L_0 + 1)^{-1} \tag{6.19}$$

and set

$$\gamma = \bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2, \tag{6.20}$$

$$\begin{aligned}\epsilon_1 = \max\{&(\bar{\Delta}^{-1} + \alpha(L_0 + 1)(4\tilde{M} + 2) + (\bar{q} + 1)\bar{\delta}_C(6\tilde{M} + 5)\\ &+ 16\bar{q}\delta_C\bar{\Delta}^{-2}(6\tilde{M} + 5)^3)^{1/2},\ \bar{\delta}_C + 16M_0\Delta^2\delta_C(6\tilde{M} + 1)^2),\ \Delta\}.\end{aligned} \tag{6.21}$$

We also assume that

$$2\bar{q}\epsilon_1 \leq 1, \tag{6.22}$$

$$\alpha \geq \bar{q}\gamma(6\tilde{M} + 3),\ \alpha \geq \bar{\delta}_C(4\tilde{M} + 5),\ \delta_f(6\tilde{M}_0 + 2) \leq 1, \tag{6.23}$$

$$\bar{\delta}_C(4\tilde{M}+5),\ \alpha(L_0+1)(6\tilde{M}+2),\ \bar{q}\bar{\delta}_C(6\tilde{M}+5),$$

$$16\bar{q}\delta_C\Delta^{-2}(6\tilde{M}+3)^3 \le 16^{-1}\bar{q}^{-2}. \tag{6.24}$$

The chapter contains four main results. The conclusions of the first two results hold for $\epsilon = \Delta$, the conclusion of the third result holds for any nonnegative ϵ while the conclusion of the final result is true for every nonnegative ϵ which does not exceed a certain positive constant. (see the description of the algorithm). This difference takes place because in the last two theorems we assume that for each $i \in \{1, \dots, m\}$,

$$B(0, M_*) \cap \{x \in X : \ f_i(x) \le -\Delta\} \ne \emptyset.$$

This condition is not assumed in the first two theorems.

6.2 The Basic Auxiliary Result

Proposition 6.1 *Let T be a natural number, $\Delta_0 \ge 0$, and at least one of the following conditions holds:*

(a) $\Delta_0 = \Delta$;
(b) for each $i \in \{1, \dots, m\}$,

$$B(0, M_*) \cap \{x \in X : \ f_i(x) \le -\Delta\} \ne \emptyset.$$

Assume that $\{x_k\}_{k=0}^T \subset X$, $(\Omega_k, w_k) \in \mathcal{M}_k$, $k = 1, \dots, T$, $\{\lambda_k\}_{k=1}^T \subset [0, \infty)$, $\{l_k\}_{k=0}^{T-1} \subset X$, *for all integers* $k \in \{0, \dots, T-1\}$,

$$B(l_k, \delta_f) \cap \partial f(x_k) \ne \emptyset, \tag{6.25}$$

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k - \alpha l_k, (\Omega_{k+1}, w_{k+1}), \delta_C, \bar{\delta}_C, \Delta_0). \tag{6.26}$$

Then the following assertions hold.

Assertion 1. Let $k \in \{0, \dots, T-1\}$ *and let*

$$x_k \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C.$$

Then for every $z \in B(0, M_*) \cap C$,

$$\|x_k - z\|^2 - \|x_{k+1} - z\|^2$$

$$\ge \max\{2\alpha(f(x_k) - f(z)) - 2\bar{q}\gamma(6\tilde{M}+3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M}+L_0+1) - \bar{\delta}_C(4\tilde{M}+5),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M}+5) - \alpha(L_0+1)(4\tilde{M}+2)$$

$$-\bar{q}\bar{\delta}_C(6\tilde{M}+5) - 16\delta_C\bar{q}\Delta^{-2}(6\tilde{M}+5)^3.$$

Assertion 2. Set

$$\epsilon_T = \max\{\bar{\Delta}^{-1}(\bar{\delta}_C(4\tilde{M}+5) + \alpha(L_0+1)(4\tilde{M}+2)$$

$$+\bar{q}\bar{\delta}_C(6\tilde{M}+5) + 16\delta_C\bar{q}\bar{\Delta}^{-2}(6\tilde{M}+5)^3 + 4\tilde{M}^2T^{-1})^{1/2},$$

$$\bar{\delta}_C + 16\delta_C M_0\Delta^2(6\tilde{M}+1)^2\}. \tag{6.27}$$

Assume that

$$\epsilon_T \le \bar{q}^{-1} \tag{6.28}$$

and that

$$\textit{for every } z \in B(0, M_*) \cap C,\ \|x_k - z\| \le 2\tilde{M},\ k = 0, \dots, T.$$

Then

$$\min\{\max\{2\alpha(f(x_k) - \inf(f, C)) - 2\bar{q}\gamma(6\tilde{M}+3)$$

$$-\alpha^2L_0^2 - 2\alpha\delta_f(6\tilde{M}+L_0+1) - \bar{\delta}_C(4\tilde{M}+5),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M}+5) - \alpha(L_0+1)(4\tilde{M}+2) - \bar{q}\bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\bar{q}\Delta^{-2}(6\tilde{M}+5)^3\} :\ k = 0 \dots, T-1\} \le 4\tilde{M}^2T^{-1}.$$

Moreover, if $k \in \{0, \dots, T-1\}$ *and*

$$\max\{2\alpha(f(x_k) - \inf(f, C)) - 2\bar{q}\gamma(6\tilde{M}+3)$$

$$-\alpha^2L_0^2 - 2\alpha\delta_f(6\tilde{M}+L_0+1) - \bar{\delta}_C(4\tilde{M}+5),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M}+5) - \alpha(L_0+1)(4\tilde{M}+2) - \bar{q}\bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\bar{q}\Delta^{-2}(6\tilde{M}+5)^3\} \le 4\tilde{M}^2T^{-1},$$

then

$$f(x_k) \le \inf(f, C) + 2\tilde{M}^2(T\alpha)^{-1} + \alpha^{-1}\bar{q}\gamma(6\tilde{M}+3)$$

$$+2^{-1}\alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1) + 2^{-1}\alpha\bar{\delta}_C(4\tilde{M} + 5),$$

$$f_i(x_k) \le \bar{q} M_1 \epsilon_T + M_1\alpha(L_0 + 1) + \max\{\Delta_0, 2M_1\epsilon_T\},\ i = 1, \dots, m.$$

Assertion 3. Let $M_{*,0} > 0$, $\tilde{M} \ge 2M_* + 4$.

$$|f(u)| \le M_{*,0},\ u \in B(0, M_*), \tag{6.29}$$

$$f(u) > M_{*,0} + 8 \text{ for all } u \in X \setminus B(0, 2^{-1}\tilde{M}), \tag{6.30}$$

$k \in \{0, \dots, T-1\}$ *and that*

$$x_k \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C. \tag{6.31}$$

Then

$$x_{k+1} \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C.$$

Assertion 4. Let $r_0 \in (0, 1]$,

$$\{x \in X :\ f_i(x) \le r_0,\ i = 1, \dots, m\} \subset B(0, M_*), \tag{6.32}$$

$$r_0 \ge \alpha(L_0 + 1)L_0 + L_0\bar{q}\epsilon_1 + 2M_1\epsilon_1, \tag{6.33}$$

$$\Delta_0 \le \min\{\Delta, r_0\}. \tag{6.34}$$

Assume that $k \in \{0, \dots, T-1\}$ *and that*

$$x_k \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C.$$

Then

$$x_{k+1} \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C.$$

Proof Let $k \in \{0, \dots, T-1\}$. By (6.14) and (6.26), there exist

$$(y_{k,t}, \lambda_{k,t}) \in A_0(t, x_k - \alpha l_k, \delta_C, \bar{\delta}_C, \Delta_0),\ t \in \Omega_{k+1} \tag{6.35}$$

such that

$$\|x_{k+1} - \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t}\| \le \bar{\delta}_C, \tag{6.36}$$

$$\lambda_{k+1} = \{\lambda_{k,t} :\ t \in \Omega_{k+1}\}. \tag{6.37}$$

By (6.13) and (6.35), for each $t \in \Omega_{k+1}$, there exists a sequence $\{y_i^{(k,t)}\}_{i=0}^{p(t)} \subset X$ such that

$$y_0^{(k,t)} = x_k - \alpha l_k, \tag{6.38}$$

for all $i = 1, \dots, p(t)$,

$$y_i^{(k,t)} \in A_{t_i}(y_{i-1}^{(k,t)}, \delta_C, \bar{\delta}_C, \Delta_0), \tag{6.39}$$

$$y_{k,t} = y_{p(t)}^{(k,t)}, \tag{6.40}$$

$$\lambda_{k,t} = \max\{\|y_i^{(k,t)} - y_{i-1}^{(k,t)}\| : \ i = 1, \dots, p(t)\}. \tag{6.41}$$

Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$. By (6.18) and (6.39), for every $i \in \{1, \dots, p(t)\}$,

$$\text{if } f_{t_i}(y_{i-1}^{(k,t)}) \le \Delta_0, \text{ then } y_i^{(k,t)} = y_{i-1}^{(k,t)}, \ l_{i-1}^{(k,t)} = 0,$$

$$\text{if } f_{t_i}(y_{i-1}^{(k,t)}) > \Delta_0 \text{ then}$$

$$l_{i-1}^{(k,t)} \in (\partial f_{t_i}(y_{i-1}^{(k,t)}) + B(0, \delta_C)) \setminus \{0\}, \tag{6.42}$$

$$y_i^{(k,t)} \in B(y_{i-1}^{(k,t)} - f_{t_i}(y_{i-1}^{(k,t)})\|l_{i-1}^{(k,t)}\|^{-2} l_{i-1}^{(k,t)}, \bar{\delta}_C). \tag{6.43}$$

We prove Assertion 1. Let $k \in \{0, \dots, T-1\}$ and

$$x_k \in B(z, 2\tilde{M}) \tag{6.44}$$

for every $z \in B(0, M_*) \cap C$. Fix

$$z \in B(0, M) \cap C_*. \tag{6.45}$$

Then (6.44) is true. Let $t = (t_1, \dots, t_{p(t)}) \in \Omega_{k+1}$. By (6.15)–(6.20), (6.25), (6.38), (6.44), (6.45) and Lemma 5.1 applied with $\bar{n} = p(t)$,

$$x = x_k, \ l = l_k, \ y_0 = x_k - \alpha l_k,$$

$$l_{i-1} = l_{i-1}^{(k,t)}, \ i = 1, \dots, p(t), \ y_i = y_i^{(k,t)}, \ i = 0, \dots, p(t)$$

we obtain that

$$\|x_k - z\|^2 - \|y_{p(t)}^{(k,t)} - z\|^2$$

$$\ge 2\alpha(f(x_k) - f(z))$$

$$- 2\bar{q}\gamma(6\tilde{M} + 3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1), \tag{6.46}$$

$$\|x_k - z\|^2 - \|y_{p(t)}^{(k,t)} - z\|^2$$

$$\geq \sum_{i=1}^{p(t)} \|y_{i-1}^{(k,t)} - y_i^{(k,t)}\|^2 - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{q}\bar{\delta}_C(6\tilde{M} + 5)$$

$$- 16\bar{q}\delta_C \Delta^{-2}(6\tilde{M} + 5)^3, \tag{6.47}$$

and for each $i \in \{1, \ldots, p(t)\}$,

$$\text{if } f_{t_i}(y_{i-1}^{(k,t)}) \leq \Delta_0, \text{ then } y_i^{(k,t)} = y_{i-1}^{(k,t)} \tag{6.48}$$

and if $f_{t_i}(y_{i-1}^{(k,t)}) > \Delta_0$, then

$$\|y_{i-1}^{(k,t)} - y_i^{(k,t)}\| \geq -\bar{\delta}_C - 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2 + M_1^{-1} f_{t_i}(y_{i-1}^{(k,t)}). \tag{6.49}$$

It follows from (6.8) and the convexity of the function $\|u - z\|^2$, $u \in X$ that

$$\|x_k - z\|^2 - \|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t} - z\|^2$$

$$\geq \|x_k - z\|^2 - \sum_{t \in \Omega_{k+1}} w_{k+1}(t)\|z - y_{k,t}\|^2$$

$$\geq \sum_{t \in \Omega_{k+1}} w_{k+1}(t)(\|x_k - z\|^2 - \|z - y_{k,t}\|^2) \tag{6.50}$$

By (6.8), (6.46) and (6.50),

$$\|x_k - z\|^2 - \|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t} - z\|^2$$

$$\geq 2\alpha(f(x_k) - f(z))$$

$$- 2\bar{q}\gamma(6\tilde{M} + 3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1), \tag{6.51}$$

In view of (6.8), (6.10), (6.40), (6.47) and (6.50),

$$\|x_k - z\|^2 - \|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t} - z\|^2$$

$$\geq \bar{\Delta} \sum_{t \in \Omega_{k+1}} \sum_{i=1}^{p(t)} \|y_{i-1}^{(k,t)} - y_i^{(k,t)}\|^2$$

$$- \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{q}\bar{\delta}_C(6\tilde{M} + 5) - 16\bar{q}\delta_C \Delta^{-2}(6\tilde{M} + 5)^3. \tag{6.52}$$

It follows from (6.18), (6.19) and (6.52) that

$$\|x_k - z\|^2 - \| \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t} - z\|^2 \geq -4. \tag{6.53}$$

By (6.53),

$$\| \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t} - z\| \leq \|x_k - z\| + 2 \leq 2\tilde{M} + 2. \tag{6.54}$$

Equations (6.36) and (6.54) imply that

$$\|x_{k+1} - z\| \leq 2\tilde{M} + 3. \tag{6.55}$$

It follows from (6.36), (6.54) and (6.55) that

$$\|x_k - z\|^2 - \|x_{k+1} - z\|^2$$

$$= \|x_k - z\|^2 - \| \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t} - z\|^2$$

$$+ \| \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t} - z\|^2 - \|x_{k+1} - z\|^2$$

$$\geq \|x_k - z\|^2 - \| \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t} - z\|^2$$

$$- \|x_{k+1} - \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t}\| (\| \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t} - z\|$$

$$+ \|x_{k+1} - z\|) \geq -\bar{\delta}_C(4\tilde{M} + 5)$$

$$+ \|x_k - z\|^2 - \| \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t} - z\|^2. \tag{6.56}$$

By (6.51) and (6.56),

$$\|x_k - z\|^2 - \|x_{k+1} - z\|^2$$

$$\geq -\bar{\delta}_C(4\tilde{M}+5) + 2\alpha(f(x_k) - f(z))$$

$$- 2\bar{q}\gamma(6\tilde{M}+3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1). \tag{6.57}$$

In view of (6.37), (6.41), (6.52) and (6.56),

$$\|x_k - z\|^2 - \|x_{k+1} - z\|^2$$

$$\geq -\bar{\delta}_C(4\tilde{M}+5) + \bar{\Delta}\lambda_{k+1}^2$$

$$- \alpha(L_0+1)(4\tilde{M}+2) - \bar{q}\bar{\delta}_C(6\tilde{M}+5) - 16\bar{q}\delta_C\Delta^{-2}(6\tilde{M}+5)^3. \tag{6.58}$$

Assertion 1 is proved.

We prove assertion 2. Let

$$z \in B(0, M) \cap C_*. \tag{6.59}$$

By Assertion 1, for all $k = 0, \dots, T-1$ relations (6.57) and (6.58) hold and for every $k \in \{0, \dots, T-1\}$, every $t = (t_1, \dots, t_{p(t)} \in \Omega_{k+1}$ and every $i \in \{1, \dots, p(t)\}$, (6.48) and (6.49) are true. By (6.57)–(6.59),

$$4\tilde{M}^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1} (\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\geq \sum_{t=0}^{T-1} \max\{2\alpha(f(x_k) - f(z))$$

$$-2\bar{q}\gamma(6\tilde{M}+3) - \alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1) - \bar{\delta}_C(4\tilde{M}+5),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M}+5)$$

$$- \alpha(L_0+1)(4\tilde{M}+2) - \bar{q}\bar{\delta}_C(6\tilde{M}+5) - 16\bar{q}\bar{\delta}_C\Delta^{-2}(6\tilde{M}+5)^3\}. \tag{6.60}$$

Since z is an arbitrary element of $C \cap B(0, M_*)$ it follows from (6.4) and (6.60) that

$$\min\{\max\{2\alpha(f(x_k) - \inf(f, C)) - 2\bar{q}\gamma(6\tilde{M}+3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1) - \bar{\delta}_C(4\tilde{M}+5),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M}+5) - \alpha(L_0+1)(4\tilde{M}+2) - \bar{q}\bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\bar{q}\Delta^{-2}(6\tilde{M}+5)^3\}:\ k=0\dots,T-1\}\le 4\tilde{M}^2T^{-1}.$$

Let $k\in\{0,\dots,T-1\}$ and

$$\max\{2\alpha(f(x_t)-\inf(f,C))-2\bar{q}\gamma(6\tilde{M}+3)$$

$$-\alpha^2L_0^2-2\alpha\delta_f(6\tilde{M}+L_0+1)-\bar{\delta}_C(4\tilde{M}+5),$$

$$\bar{\Delta}\lambda_{k+1}^2-\bar{\delta}_C(4\tilde{M}+5)-\alpha(L_0+1)(4\tilde{M}+2)$$

$$-\bar{q}\bar{\delta}_C(6\tilde{M}+5)-16\delta_C\bar{q}\Delta^{-2}(6\tilde{M}+5)^3\}\le 4\tilde{M}^2T^{-1}. \tag{6.61}$$

By (6.27) and (6.61),

$$f(x_t)\le\inf(f,C)+2\tilde{M}^2(T\alpha)^{-1}+\alpha^{-1}\bar{q}\gamma(6\tilde{M}+3)$$

$$+2^{-1}\alpha L_0^2+\delta_f(6\tilde{M}+L_0+1)+2^{-1}\alpha\bar{\delta}_C(4\tilde{M}+5),$$

$$\lambda_{k+1}^2\le\bar{\Delta}^{-1}(\delta_C(4\tilde{M}+5)+\alpha(L_0+1)(4\tilde{M}+2)+\bar{q}\bar{\delta}_C(6\tilde{M}+5)$$

$$+16\delta_C\bar{q}\bar{\Delta}^{-2}(6\tilde{M}+5)^3+4\tilde{M}^2T^{-1})\le\epsilon_T^2$$

and

$$\lambda_{k+1}\le\epsilon_T. \tag{6.62}$$

Let $t\in\Omega_{k+1}$ and $i\in\{1,\dots,p(t)\}$. In view of (6.37) and (6.41),

$$\|y_i^{(k,t)}-y_{i-1}^{(k,t)}\|\le\lambda_{k+1}. \tag{6.63}$$

Assume that

$$f_{t_i}(y_{i-1}^{(k,t)})>\Delta_0. \tag{6.64}$$

It follows from (6.27), (6.49) and (6.62)–(6.64) that

$$f_{t_i}(y_{i-1}^{(k,t)})\le M_1\bar{\delta}_C+16M_1\delta_CM_0\Delta^2(6\tilde{M}+1)^2+M_1\lambda_{k+1}$$

$$\le M_1\bar{\delta}_C+16M_1\delta_CM_0\Delta^2(6\tilde{M}+1)^2+M_1\epsilon_T\le 2M_1\epsilon_T. \tag{6.65}$$

By (6.64) and (6.65),

$$f_{t_i}(y_{i-1}^{(k,t)})\le\max\{\Delta_0,2M_1\epsilon_T\}. \tag{6.66}$$

Equations (6.17), (6.25), (6.38), (6.62) and (6.63) imply that

$$\|x_k - y_i^{(k,t)}\| \le \|x_k - y_0^{(k,t)}\| + \|y_0^{(k,t)} - y_i^{(k,t)}\|$$

$$\le \alpha(L_0 + 1) + \bar{q}\lambda_{k+1} \le \alpha(L_0 + 1) + \bar{q}\epsilon_T. \quad (6.67)$$

By (6.16), (6.19), (6.28), (6.59), (6.67) and the relation $x_k \in B(z, 2\tilde{M})$,

$$|f_{t_i}(x_k) - f_{t_i}(y_i^{(k,t)})| \le M_1\|x_k - y_i^{(k,t)}\|$$

$$\le M_1\alpha(L_0 + 1) + \bar{q}M_1\epsilon_T.$$

Together with (6.66) this implies that

$$f_{t_i}(x_k) \le M_1\bar{q}\epsilon_T + M_1\alpha(L_0 + 1) + \max\{\Delta_0, 2M_1\epsilon_T\}$$

for all $t \in \Omega_{k+1}$ and all $i \in \{1, \dots, p(t)\}$. Therefore

$$f_i(x_k) \le M_1\bar{q}\epsilon_T + M_1\alpha(L_0 + 1) + \max\{\Delta_0, 2M_1\epsilon_T\}$$

for all $i \in \{1, \dots, m\}$. Assertion 2 is proved.

We prove Assertion 3.

$$z \in B(0, M) \cap C_*. \quad (6.68)$$

In view of Assertion 1, (6.57) and (6.58) hold and (6.48) and (6.49) hold for all $t \in \Omega_{k+1}$ and all $i = 1, \dots, p(t)$. We may assume without loss of generality that

$$\|x_{k+1} - z\| > \|x_k - z\|. \quad (6.69)$$

By (6.19), (6.23), (6.29), (6.57), (6.68) and (6.69),

$$f(x_k) \le f(z) + \alpha^{-1}\bar{q}\gamma(6\tilde{M} + 3) + 2^{-1}\alpha^{-1}\bar{\delta}_C(4\tilde{M} + 5)$$

$$+2^{-1}\alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1)$$

$$\le M_{*,0} + 4.$$

Combined with (6.30) this implies that

$$\|x_k\| \le \tilde{M}/2. \quad (6.70)$$

It follows from (6.37), (6.41), (6.58) and (6.69) that for all $t \in \Omega_{k+1}$ and all $i = 1, \dots, p(t)$,

$$\bar{\Delta}\|y_{i-1}^{(k,t)} - y_i^{(k,t)}\|^2$$

$$\leq \bar{\delta}_C(4\tilde{M}+5) + \alpha(L_0+1)(4\tilde{M}+2) + \bar{q}\bar{\delta}_C(6\tilde{M}+5)$$

$$+ 16\delta_C\bar{q}\Delta^{-2}(6\tilde{M}+5)^3 \leq \bar{\Delta}(2\bar{q})^{-1}. \quad (6.71)$$

By (6.17), (6.19), (6.25) and (6.71), for all $t \in \Omega_{k+1}$,

$$\|x_k - y_{k,t}\| = \|x_k - y_0^{(k,t)}\| + \|y_0^{(k,t)} - y_{p(t)}^{(k,t)}\| \leq \alpha(L_0+1) + 2^{-1} \leq 2$$

and combined with (6.70) this implies that

$$\|y_{k,t}\| \leq \tilde{M}/2 + 2.$$

Together with (6.30) and (6.68) this implies that

$$\|x_{k+1}\| \leq \tilde{M}/2 + 3, \;\; \|x_{t+1} - z\| \leq 2 + \tilde{M}/2 + M_* < 2\tilde{M}.$$

Assertion 3 is proved.

We prove Assertion 4. Let

$$z \in B(0, M) \cap C_*. \quad (6.72)$$

Clearly,

$$\|z - x_k\| \leq 2\tilde{M}. \quad (6.73)$$

We need to show that

$$\|z - x_{k+1}\| \leq 2\tilde{M}.$$

We may assume without loss of generality that

$$\|x_{k+1} - z\| > \|x_k - z\|. \quad (6.74)$$

In view of Assertion 1, (6.57), (6.58) hold and (6.48) and (6.49) hold for all $t \in \Omega_{k+1}$ and all $i = 1, \ldots, p(t)$.

By (6.37), (6.41), (6.58) and (6.74), for all $t \in \Omega_{k+1}$ and all $i = 1, \ldots, p(t)$,

$$\|y_{i-1}^{(k,t)} - y_i^{(k,t)}\|^2 \leq \lambda_{k+1}^2$$

$$\leq \bar{\Delta}^{-1}(\alpha(L_0+1)(4\tilde{M}+2) + (\bar{q}+1)\bar{\delta}_C(6\tilde{M}+5)$$

$$+ 16\delta_C\bar{q}\Delta^{-2}(6\tilde{M}+5)^3). \quad (6.75)$$

Let $t = (t_1, \dots, t_{p(t)}\} \in \Omega_{k+1}$ and $i \in \{1, \dots, p(t)\}$. If

$$f_{t_i}(y_{i-1}^{(k,t)}) \le \Delta_0,$$

then $y_i^{(k,t)} = y_{i-1}^{(k,t)}$. If

$$f_{t_i}(y_{i-1}^{(k,t)}) > \Delta_0,$$

then it follows from (6.49) and (6.75) that

$$f_{t_i}(y_{i-1}^{(k,t)}) \le M_1(\bar{\delta}_C + 16\delta_C M_0 \Delta^2 (6\tilde{M} + 1)^2)$$

$$+M_1 \|y_{i-1}^{(k,t)} - y_i^{(k,t)}\|$$

$$\le M_1(\bar{\delta}_C + 16\delta_C M_0 \Delta^2 (6\tilde{M} + 1)^2)$$

$$+M_1 \Delta^{-1/2}(\alpha(L_0 + 1)(4\tilde{M} + 2) + (\bar{q} + 1)\bar{\delta}_C(6\tilde{M} + 5)$$

$$+16\delta_C \bar{q} \Delta^{-2}(6\tilde{M} + 5)^3)^{1/2}.$$

By the relation above and (6.21),

$$f_{t_i}(y_{i-1}^{(k,t)}) \le 2M_1\epsilon_1. \tag{6.76}$$

In view of (6.17), (6.21), (6.25), (6.72), (6.73) and (6.75),

$$\|x_k - y_{i-1}^{(k,t)}\| \le \|x_k - y_0^{(k,t)}\| + \|y_0^{(k,t)} - y_{i-1}^{(k,t)}\|$$

$$\le \alpha(L_0 + 1) + \bar{q}\epsilon_1. \tag{6.77}$$

By (6.19), (6.22), (6.72), (6.73) and (6.77),

$$\|x_t\| \le 3\tilde{M},\ \|y_{i-1}^{(k,t)}\| \le 3\tilde{M} + 2. \tag{6.78}$$

Equations (6.17) and (6.76)–(6.78) imply that for all $t \in \Omega_{k+1}$ and all $i = 1, \dots, p(t)$,

$$f_{t_i}(x_k) \le f_{t_i}(y_{i-1}^{(k,t)}) + L_0\|x_k - y_{i-1}^{(k,t)}\|$$

$$\le 2M_1\epsilon_1 + L_0\alpha(L_0 + 1) + L_0\bar{q}\epsilon_1.$$

Together with (6.33) this implies that for all $i = 1, \dots, m$,

$$f_i(x_k) \le r_0.$$

Combined with (6.32) implies that

$$\|x_k\| \le M_*. \tag{6.79}$$

It follows from (6.30) and (6.77) that

$$\|x_k - x_{k+1}\|$$

$$\le \|x_k - \sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t}\| + \|\sum_{t \in \Omega_{k+1}} w_{k+1}(t) y_{k,t} - x_{k+1}\|$$

$$\le \alpha(L_0 + 1) + \bar{q}\epsilon_1 + \bar{\delta}_C \le 4.$$

Combined with (6.72) and (6.79) this implies that

$$\|x_{k+1}\| \le M_* + 4$$

and

$$\|x_{k+1} - z\| \le 2\tilde{M}.$$

Assertion 4 is proved. This completes the proof of Proposition 6.1.

Analogously to Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations in the case of Assertion 2.

6.3 The Main Results

We use the notation and definitions introduced in Sections 6.1 and 6.2 and suppose that all the assumptions made there hold. The following result is easily deduced from Assertions 2 and 3 of Proposition 6.1 with $\Delta_0 = \Delta$. It holds under assumption that f satisfies a growth condition.

Theorem 6.2 *Let $M_{*,0} > 0$, $\tilde{M} \ge 2M_* + 4$.*

$$|f(u)| \le M_{*,0}, \ u \in B(0, M_*),$$

$$f(u) \ge M_{*,0} + 8 \text{ for all } u \in X \setminus B(0, 2^{-1}\tilde{M}),$$

T be a natural number, ϵ_T be defined by (6.27) and satisfy $\bar{q}\epsilon_T \le 1$.

Assume that $\{x_k\}_{k=0}^{T} \subset X$, $(\Omega_k, w_k) \in \mathcal{M}$, $k = 1, \dots, T$, $\{\lambda_k\}_{k=1}^{T} \subset [0, \infty)$, $\{l_k\}_{k=0}^{T-1} \subset X$, *for all integers* $k \in \{0, \dots, T-1\}$,

$$B(l_k, \delta_f) \cap \partial f(x_k) \neq \emptyset,$$

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k - \alpha l_k, (\Omega_{k+1}, w_{k+1}), \delta_C, \bar{\delta}_C, \Delta),$$

$$\|x_0\| \le \tilde{M}.$$

Then

$$\min\{\max\{2\alpha(f(x_k) - \inf(f, C)) - 2\bar{q}\gamma(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1) - \bar{\delta}_C(4\tilde{M} + 5),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M} + 5) - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{q}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C\bar{q}\bar{\Delta}^{-2}(6\tilde{M} + 5)^3\} : \ k = 0 \dots, T - 1\} \le 4\tilde{M}^2 T^{-1}.$$

Moreover, if $k \in \{0, \dots, T-1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{q}\gamma(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1) - \bar{\delta}_C(4\tilde{M} + 5),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M} + 5) - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{q}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C\bar{q}\Delta^{-2}(6\tilde{M} + 5)^3\} \le 4\tilde{M}^2 T^{-1},$$

then

$$f(x_k) \le \inf(f, C) + 2M^2(\tilde{T}\alpha)^{-1} + \alpha^{-1}\bar{q}\gamma(6\tilde{M} + 3)$$

$$+2^{-1}\alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1) + 2^{-1}\alpha\bar{\delta}_C(4\tilde{M} + 5),$$

$$f_i(x_k) \le 2\bar{q}M_1\epsilon_T + M_1\alpha(L_0 + 1) + \max\{\Delta, 2M_1\epsilon_T\}, \ i = 1, \dots, m.$$

The following result is easily deduced from Assertions 2 and 4 of Proposition 6.1 with $\Delta_0 = \Delta$. It holds under assumption that the set

$$\{x \in X : \ f_i(x) \le r_0, \ i = 1, \dots, m\}$$

is bounded for some $r_0 > 0$.

Theorem 6.3 *Let* $r_0 \in (0, 1]$,

$$\{x \in X : \ f_i(x) \le r_0, \ i = 1, \dots, m\} \subset B(0, M_*),$$

$$r_0 \ge \alpha(L_0 + 1)L_0 + L_0\bar{q}\epsilon_1 + 2M_1\epsilon_1, \ \Delta \le r_0,$$

T be a natural number, ϵ_T be defined by (6.27) and satisfy $\bar{q}\epsilon_T \le 1$.

Assume that $\{x_k\}_{k=0}^T \subset X$, $(\Omega_k, w_k) \in \mathcal{M}$, $k = 1, \dots, T$, $\{\lambda_k\}_{k=1}^T \subset [0, \infty)$, $\{l_k\}_{k=0}^{T-1} \subset X$, *for all integers* $k \in \{0, \dots, T-1\}$,

$$B(l_k, \delta_f) \cap \partial f(x_k) \ne \emptyset,$$

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k - \alpha l_k, (\Omega_{k+1}, w_{k+1}), \delta_C, \bar{\delta}_C, \Delta),$$

$$\|x_0\| \le \tilde{M}.$$

Then

$$\min\{\max\{2\alpha(f(x_k) - \inf(f, C)) - 2\bar{q}\gamma(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1) - \bar{\delta}_C(4\tilde{M} + 5),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M} + 5) - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{q}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C\bar{q}\bar{\Delta}^{-2}(6\tilde{M} + 5)^3\} : \ k = 0 \dots, T - 1\} \le 4\tilde{M}^2 T^{-1}.$$

Moreover, if $k \in \{0, \dots, T-1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{q}\gamma(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1) - \bar{\delta}_C(4\tilde{M} + 5),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M} + 5) - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{q}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C\bar{q}\bar{\Delta}^{-2}(6\tilde{M} + 5)^3\} \le 4\tilde{M}^2 T^{-1},$$

then

$$f(x_k) \le \inf(f, C) + 2M^2(\tilde{T}\alpha)^{-1} + \alpha^{-1}\bar{q}\gamma(6\tilde{M} + 3)$$

$$+2^{-1}\alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1) + 2^{-1}\alpha\bar{\delta}_C(4\tilde{M} + 5),$$

$$f_i(x_k) \le \bar{q}M_1\epsilon_T + M_1\alpha(L_0 + 1) + \max\{\Delta, 2M_1\epsilon_T\}, \ i = 1, \dots, m.$$

The following result is easily deduced from Assertions 2 and 3 of Proposition 6.1 with $\Delta_0 = \epsilon$. It holds under assumption that f satisfies a growth condition.

Theorem 6.4 *Let for each* $i \in \{1, \dots, m\}$,

$$B(0, M_*) \cap \{x \in X : \ f_i(x) \le -\Delta\} \neq \emptyset,$$

$M_{*,0} > 0$, $\tilde{M} \ge 2M_* + 4$.

$$|f(u)| \le M_{*,0}, \ u \in B(0, M_*),$$

$$f(u) \ge M_{*,0} + 8 \text{ for all } u \in X \setminus B(0, 2^{-1}\tilde{M}),$$

$\epsilon \ge 0$, *T be a natural number,* ϵ_T *be defined by (6.27) and satisfy* $\bar{q}\epsilon_T \le 1$.

Assume that $\{x_k\}_{k=0}^T \subset X$, $(\Omega_k, w_k) \in \mathcal{M}$, $k = 1, \dots, T$, $\{\lambda_k\}_{k=1}^T \subset [0, \infty)$, $\{l_k\}_{k=0}^{T-1} \subset X$, *for all integers* $k \in \{0, \dots, T-1\}$,

$$B(l_k, \delta_f) \cap \partial f(x_k) \neq \emptyset,$$

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k - \alpha l_k, (\Omega_{k+1}, w_{k+1}), \delta_C, \bar{\delta}_C, \epsilon),$$

$$\|x_0\| \le \tilde{M}.$$

Then

$$\min\{\max\{2\alpha(f(x_k) - \inf(f, C)) - 2\bar{q}\gamma(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1) - \bar{\delta}_C(4\tilde{M} + 5),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M} + 5) - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{q}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C\bar{q}\bar{\Delta}^{-2}(6\tilde{M} + 5)^3\} : \ k = 0 \dots, T - 1\} \le 4\tilde{M}^2 T^{-1}.$$

Moreover, if $k \in \{0, \dots, T-1\}$ *and*

$$\max\{2\alpha(f(x_t) - \inf(f, C)) - 2\bar{q}\gamma(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1) - \bar{\delta}_C(4\tilde{M} + 5),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M} + 5) - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{q}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C\bar{q}\bar{\Delta}^{-2}(6\tilde{M} + 5)^3\} \le 4\tilde{M}^2 T^{-1},$$

then

$$f(x_k) \le \inf(f, C) + 2\tilde{M}^2(T\alpha)^{-1} + \alpha^{-1}\bar{q}\gamma(6\tilde{M} + 3)$$

$$+2^{-1}\alpha L_0^2 + \delta_f(6\tilde{M} + L_0 + 1) + 2^{-1}\alpha\bar{\delta}_C(4\tilde{M} + 5),$$

$$f_i(x_k) \le \bar{q}M_1\epsilon_1 + M_1\alpha(L_0 + 1) + \max\{\epsilon, 2M_1\epsilon_T\},\ i = 1, \dots, m.$$

The following result is easily deduced from Assertions 2 and 4 of Proposition 6.1 with $\Delta_0 = \epsilon$ under condition (b). It holds under assumption that the set

$$\{x \in X :\ f_i(x) \le r_0,\ i = 1, \dots, m\}$$

is bounded for some $r_0 > 0$.

Theorem 6.5 *Let for each* $i \in \{1, \dots, m\}$,

$$B(0, M_*) \cap \{x \in X :\ f_i(x) \le -\Delta\} \ne \emptyset,$$

$r_0 \in (0, 1]$,

$$\{x \in X :\ f_i(x) \le r_0,\ i = 1, \dots, m\} \subset B(0, M_*),$$

$$r_0 \ge \alpha(L_0 + 1)L_0 + L_0\bar{q}\epsilon_1 + 2M_1\epsilon_1,$$

$$\epsilon \in [0, \min\{\Delta, r_0\}],$$

T *be a natural number satisfy*

$$T \ge 4\tilde{M}^2\epsilon_1^{-2}\Delta.$$

Assume that $\{x_k\}_{k=0}^T \subset X$, $(\Omega_k, w_k) \in \mathcal{M}$, $k = 1, \dots, T$, $\{\lambda_k\}_{k=1}^T \subset [0, \infty)$, $\{l_k\}_{k=0}^{T-1} \subset X$, *for all integers* $k \in \{0, \dots, T-1\}$,

$$B(l_k, \delta_f) \cap \partial f(x_k) \ne \emptyset,$$

$$(x_{k+1}, \lambda_{k+1}) \in A(x_k - \alpha l_k, (\Omega_{k+1}, w_{k+1}), \delta_C, \bar{\delta}_C, \epsilon),$$

$$\|x_0\| \le \tilde{M}.$$

Then

$$\min\{\max\{2\alpha(f(x_k) - \inf(f, C)) - 2\bar{q}\gamma(6\tilde{M} + 3)$$

$$-\alpha^2 L_0^2 - 2\alpha\delta_f(6\tilde{M} + L_0 + 1),$$

$$\bar{\Delta}\lambda_{k+1}^2 - \bar{\delta}_C(4\tilde{M} + 1) - \alpha(L_0 + 1)(4\tilde{M} + 2) - \bar{q}\bar{\delta}_C(6\tilde{M} + 5)$$

$$-16\delta_C\bar{q}\bar{\Delta}^{-2}(6\tilde{M}+5)^3\}: \ k=0\ldots,T-1\} \le 4\tilde{M}^2T^{-1}.$$

Moreover, if $k \in \{0,\ldots,T-1\}$ *and*

$$\max\{2\alpha(f(x_t)-\inf(f,C))-2\bar{q}\gamma(6\tilde{M}+3)$$

$$-\alpha^2L_0^2-2\alpha\delta_f(6\tilde{M}+L_0+1),$$

$$\bar{\Delta}\lambda_{k+1}^2-\bar{\delta}_C(4\tilde{M}+5)-\alpha(L_0+1)(4\tilde{M}+2)-\bar{q}\bar{\delta}_C(6\tilde{M}+5)$$

$$-16\delta_C\bar{q}\bar{\Delta}^{-2}(6\tilde{M}+5)^3\} \le 4\tilde{M}^2T^{-1},$$

then

$$f(x_k) \le \inf(f,C)+2\tilde{M}^2(T\alpha)^{-1}+\alpha^{-1}\bar{q}\gamma(6\tilde{M}+3)$$

$$+2^{-1}\alpha L_0^2+\delta_f(6\tilde{M}+L_0+1)+2^{-1}\alpha\bar{\delta}_C(4\tilde{M}+5),$$

$$f_i(x_k) \le \bar{q}M_1\epsilon_T+M_1\alpha(L_0+1)+\max\{\epsilon,2M_1\epsilon_T\}, \ i=1,\ldots,m.$$

Chapter 7
Fixed Point Gradient Projection Algorithm

In this chapter we consider a minimization of a convex smooth function on a solution set of a convex feasibility problem in a general Hilbert space using the fixed point gradient projection algorithm. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We show that an algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a small constant. Moreover, if we known computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this.

7.1 Preliminaries

Let X be a Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ which induces a complete norm $\|\cdot\|$. Let C be a nonempty closed convex subset of X, U be an open convex subset of X such that $C \subset U$ and let $f : U \to R^1$ be a convex continuous function.

We suppose that the function f is Frechet differentiable at every point $x \in U$ and for every $x \in U$ we denote by $f'(x) \in X$ the Frechet derivative of f at x. It is clear that for any $x \in U$ and any $h \in X$

$$\langle f'(x), h\rangle = \lim_{t\to 0} t^{-1}(f(x+th) - f(x)).$$

Recall that for each nonempty set D and each function $g : D \to R^1$,

$$\inf(g, D) = \inf\{g(y) : \ y \in D\},$$

$$\text{argmin}(g, D) = \text{argmin}\{g(z) : \ z \in D\}$$

A. J. Zaslavski, *Optimization on Solution Sets of Common Fixed Point Problems*, Springer Optimization and Its Applications 178,
https://doi.org/10.1007/978-3-030-78849-0_7

$$= \{z \in D : \; g(z) = \inf(g, D)\}.$$

For every $x \in X$ and every $D \subset X$,

$$d(x, A) = \inf\{\|x - y\| : \; y \in D\}.$$

Proposition 7.1 *Let D be a nonempty closed convex subset of X, $x \in X$ and $y \in D$. Assume that for each $z \in D$,*

$$\langle z - y, x - y \rangle \le 0. \tag{7.1}$$

Then $y = P_D(x)$.

Proof Let $z \in D$. By (7.1),

$$\langle z - x, z - x \rangle = \langle z - y + (y - x), z - y + (y - x) \rangle$$

$$= \langle y - x, y - x \rangle + 2\langle z - y, y - x \rangle + \langle z - y, z - y \rangle$$

$$\ge \langle y - x, y - x \rangle + \langle z - y, z - y \rangle$$

$$= \|y - x\|^2 + \|z - y\|^2.$$

Thus $y = P_D(x)$. Proposition 7.1 is proved. □

Proposition 7.2 *Assume that $x, u \in U$, $L > 0$ and that for each $v_1, v_2 \in \{tx + (1 - t)u : \; t \in [0, 1]\}$,*

$$\|f'(v_1) - f'(v_2)\| \le L\|v_1 - v_2\|.$$

Then

$$f(u) \le f(x) + \langle f'(x), u - x \rangle + 2^{-1}L\|u - x\|^2.$$

Proof For each $t \in [0, 1]$ set

$$\phi(t) = f(x + t(u - x)). \tag{7.2}$$

Clearly, ϕ is a differentiable function and for each $t \in [0, 1]$,

$$\phi'(t) = \langle f'(x + t(u - x)), u - x \rangle. \tag{7.3}$$

By (7.2) and (7.3) and the proposition assumptions,

$$f(u) - f(x) = \phi(1) - \phi(0)$$

$$= \int_0^1 \phi'(t)dt = \int_0^1 \langle f'(x + t(u - x)), u - x \rangle dt$$

$$= \int_0^1 \langle f'(x), u - x \rangle dt + \int_0^1 \langle f'(x + t(u - x)) - f'(x), u - x \rangle dt$$

$$\leq \langle f'(x), u - x \rangle + \int_0^1 Lt \|u - x\|^2 dt$$

$$= \langle f'(x), u - x \rangle + L\|u - x\|^2 \int_0^1 t dt$$

$$= \langle f'(x), u - x \rangle + L\|u - x\|^2/2.$$

Proposition 7.1 is proved.

7.2 The Basic Lemma

Lemma 7.3 *Let Let D be a nonempty closed convex subset of X, U be an open convex subset of X such that $D \subset U$ and let $f : U \to R^1$ be a convex Frechet differentiable function, $M_0, L > 0$,*

$$|f(v_1) - f(v_2)| \leq L\|v_1 - v_2\| \text{ for all } v_1, v_2 \in U \cap B(0, 3M_0 + 6), \tag{7.4}$$

$$\|f'(v_1) - f'(v_2)\| \leq L\|v_1 - v_2\| \text{ for all } v_1, v_2 \in U \cap B(0, 3M_0 + 6), \tag{7.5}$$

$$\delta_1, \delta_2 \in [0, 1], \ \alpha \in (0, \min\{1, L^{-1}\}], \tag{7.6}$$

$$x \in B(0, M_0) \cap D, \tag{7.7}$$

$$u \in B(0, M_0 + 1) \cap U, \tag{7.8}$$

$\xi \in X$ satisfy

$$\|\xi - f'(u)\| \leq \delta_1 \tag{7.9}$$

and let $v \in U$ satisfy

$$\|v - P_D(u - \alpha\xi)\| \leq \delta_2. \tag{7.10}$$

Then

$$\|x-u\|^2-\|x-v\|^2$$

$$\geq 2\alpha(f(v)-f(x))-(\delta_2+\alpha\delta_1)(4M_0+6)-2\alpha(L\delta_2+\delta_1),$$

$$\|x-u\|^2-\|x-v\|^2$$

$$\geq \|u-\alpha\xi-P_D(u-\alpha\xi)\|^2-(\delta_2+\alpha\delta_1)(4M_0+6)-\alpha^2L^2$$

$$-2\alpha\delta_1(2\alpha\delta_1+4M_0+4)-2\alpha L(2M_0+1),$$

$$\|x-u\|^2-\|x-v\|^2$$

$$\geq \|u-\alpha\xi-v\|^2-\delta_2(2M_0+2)-2\alpha\delta_1(2\alpha\delta_1+4M_0+4)$$

$$-(\delta_2+\alpha\delta_1)(4M_0+6)-\alpha^2L^2-2\alpha L(2M_0+1).$$

Proof For each $y \in X$ define

$$g(y)=f(u)+\langle f'(u),y-u\rangle+2^{-1}\alpha^{-1}\|y-u\|^2. \tag{7.11}$$

Clearly, $g: X \to R^1$ is a convex Frechet differentiable function, for each $y \in U$,

$$g'(y)=f'(u)+\alpha^{-1}(y-u), \tag{7.12}$$

$$\lim_{\|y\|\to\infty} g(y)=\infty \tag{7.13}$$

and there exists

$$x_0 \in D \tag{7.14}$$

such that

$$g(x_0) \leq g(z) \text{ for all } z \in D. \tag{7.15}$$

By (7.14) and (7.15), for all $z \in D$,

$$\langle g'(x_0), z-x_0\rangle \geq 0. \tag{7.16}$$

In view of (7.12) and (7.16),

$$\langle \alpha f'(u)+x_0-u, z-x_0\rangle \geq 0 \text{ for all } z \in D. \tag{7.17}$$

Proposition 7.1, (7.14), and (7.17) imply that

$$x_0 = P_D(u - \alpha f'(u)). \tag{7.18}$$

It follows from (7.4) and (7.8) that

$$\|f'(u)\| \le L. \tag{7.19}$$

Since the mapping P_D is nonexpansive equations (7.9), (7.10), and (7.18) imply that

$$\|v - x_0\|$$
$$\le \|v - P_D(u - \alpha\xi)\|$$
$$+\|P_D(u - \alpha\xi) - P_D(u - \alpha f'(u))\|$$
$$\le \delta_2 + \alpha\|\xi - f'(u)\| \le \delta_2 + \alpha\delta_1. \tag{7.20}$$

Since the mapping P_D is nonexpansive it follows from (7.6)–(7.8), (7.18), and (7.19) that

$$\|x - x_0\| = \|x - P_D(u - \alpha f'(u))\|$$
$$\|x - u + \alpha f'(u)\| \le \|x - u\| + \alpha\|f'(u)\|$$
$$\le 2M_0 + 1 + \alpha L \le 2M_0 + 2. \tag{7.21}$$

In view of (7.7) and (7.21),

$$\|x_0\| \le 3M_0 + 2. \tag{7.22}$$

Relations (7.6), (7.20), and (7.22) imply that

$$\|v\| \ \le 3M_0 + 4. \tag{7.23}$$

By (7.4)–(7.6), (7.8), and Proposition 7.2, for all $y \in U \cap B(0, 3M_0 + 4)$,

$$f(y) \le f(u) + \langle f'(u), y - u\rangle + 2^{-1}L\|u - y\|^2 \le g(y). \tag{7.24}$$

It follows from (7.4), (7.6), (7.14), (7.20), (7.22), and (7.23) that

$$|f(x_0) - f(v)| \le L\|v - x_0\| \le \delta_2 L + L\alpha\delta_1 \le L\delta_2 + \delta_1. \tag{7.25}$$

In view of (7.14), (7.22), and (7.24),

$$g(x_0) \ge f(x_0). \tag{7.26}$$

By (7.11) and (7.26) and convexity of f,

$$f(x) - f(x_0) \geq f(x) - g(x_0)$$

$$= f(x) - f(u) - \langle f'(u), x_0 - u \rangle - 2^{-1}\alpha^{-1}\|u - x_0\|^2$$

$$\geq f(u) + \langle f'(u), x - u \rangle$$

$$-f(u) - \langle f'(u), x_0 - u \rangle - 2^{-1}\alpha^{-1}\|u - x_0\|^2$$

$$= \langle f'(u), x - x_0 \rangle - 2^{-1}\alpha^{-1}\|u - x_0\|^2. \tag{7.27}$$

Relation (7.16) (with $z = x$) imply that

$$0 \leq \langle g'(x_0), x - x_0 \rangle. \tag{7.28}$$

By (7.12) and (7.14),

$$\langle g'(x_0), x - x_0 \rangle = \langle f'(u) + \alpha^{-1}(x_0 - u), x - x_0 \rangle. \tag{7.29}$$

It follows from (7.28) and (7.29) that

$$\langle f'(u), x - x_0 \rangle$$

$$= \langle g'(x_0), x - x_0 \rangle$$

$$- \alpha^{-1}\langle x_0 - u, x - x_0 \rangle \geq -\alpha^{-1}\langle x_0 - u, x - x_0 \rangle. \tag{7.30}$$

In view of (7.27) and (7.30),

$$f(x) - f(x_0) \geq \langle f'(u), x - x_0 \rangle - (2\alpha)^{-1}\|u - x_0\|^2$$

$$\geq -\alpha^{-1}\langle x_0 - u, x - x_0 \rangle - (2\alpha)^{-1}\|u - x_0\|^2. \tag{7.31}$$

By (7.31),

$$f(x) - f(x_0)$$

$$\geq -2^{-1}\alpha^{-1}\|x_0 - u\|^2 - 2^{-1}\alpha^{-1}[\|x - u\|^2 - \|x - x_0\|^2 - \|u - x_0\|^2]$$

$$= 2^{-1}\alpha^{-1}\|x - x_0\|^2 - 2^{-1}\alpha^{-1}\|x - u\|^2. \tag{7.32}$$

By (7.25),

$$f(x) - f(v) \geq f(x) - f(x_0) - \delta_1 - L\delta_2. \tag{7.33}$$

It follows from (7.6), (7.8), and (7.20)–(7.23) that

$$\begin{aligned}
&|\|x-x_0\|^2-\|x-v\|^2| \\
&= |\|x-x_0\|-\|x-v\||(\|x-x_0\|+\|x-v\|) \\
&\le (\delta_2+\alpha\delta_1)(4M_0+6) \qquad (7.34)
\end{aligned}$$

and

$$\begin{aligned}
&|\|u-x_0\|^2-\|u-v\|^2| \\
&= |\|u-x_0\|-\|u-v\||(\|u-x_0\|+\|u-v\|) \\
&\le (\delta_2+\alpha\delta_1)(8M_0+8). \qquad (7.35)
\end{aligned}$$

In view of (7.25), (7.32), and (7.34),

$$\begin{aligned}
&f(x)-f(v) \\
&\ge f(x)-f(x_0)-\delta_2 L-\delta_1 \\
&\ge 2^{-1}\alpha^{-1}\|x-x_0\|^2-2^{-1}\alpha^{-1}\|x-u\|^2-L\delta_2-\delta_1 \\
&\ge 2^{-1}\alpha^{-1}\|x-v\|^2-2^{-1}\alpha^{-1}\|x-u\|^2-(2\alpha)^{-1}(\delta_2+\alpha\delta_1)(4M_0+6) \\
&\quad -L\delta_2-\delta_1. \qquad (7.36)
\end{aligned}$$

Equations (7.4), (7.7), (7.8), (7.18), (7.19), and (7.34) imply that

$$\begin{aligned}
&\|x-u\|^2-\|x-v\|^2 \\
&\ge -(\delta_2+\alpha\delta_1)(4M_0+6)+\|x-u\|^2-\|x-x_0\|^2 \\
&\ge -(\delta_2+\alpha\delta_1)(4M_0+6)+\|x-u\|^2-\|x-P_D(u-\alpha f'(u))\|^2 \\
&\ge -(\delta_2+\alpha\delta_1)(4M_0+6)+\|x-u\|^2-\|x-(u-\alpha f'(u))\|^2 \\
&\quad +\|u-\alpha f'(u)-P_D(u-\alpha f'(u))\|^2 \\
&\ge \|u-\alpha f'(u)-P_D(u-\alpha f'(u))\|^2 \\
&\quad -(\delta_2+\alpha\delta_1)(4M_0+6)-\alpha^2L^2-2\alpha L(2M_0+1). \qquad (7.37)
\end{aligned}$$

In view of (7.9) and the properties of the mapping P_D,

$$|\|u-\alpha\xi - P_D(u-\alpha\xi)\|^2 - \|u-\alpha f'(u) - P_D(u-\alpha f'(u))\|^2|$$

$$\leq |\|u-\alpha\xi - P_D(u-\alpha\xi)\| - \|u-\alpha f'(u) - P_D(u-\alpha f'(u))\||$$

$$\times(\|u-\alpha\xi - P_D(u-\alpha\xi)\| + \|u-\alpha f'(u) - P_D(u-\alpha f'(u))\|)$$

$$\leq 2\alpha\|\xi - f'(u)\|(\|u-\alpha\xi - P_D(u-\alpha\xi)\|$$

$$+\|u-\alpha f'(u) - P_D(u-\alpha f'(u))\|)$$

$$\leq 2\alpha\|\xi - f'(u)\|(2\|u-\alpha f'(u) - P_D(u-\alpha f'(u))\|$$

$$+2\alpha\|\xi - f'(u)\|)$$

$$\leq 2\alpha\delta_1(2\alpha\delta_1 + 2\|u-\alpha f'(u) - P_D(u-\alpha f'(u))\|). \tag{7.38}$$

By (4.4) and (7.6)–(7.8),

$$\|u-\alpha f'(u) - P_D(u-\alpha f'(u))\| \leq \|x-(u-\alpha f'(u))\|$$

$$\leq 2M_0 + 1 + \alpha L \leq 2M_0 + 2. \tag{7.39}$$

Equations (7.38) and (7.39) imply that

$$|\|u-\alpha\xi - P_D(u-\alpha\xi)\|^2 - \|u-\alpha f'(u) - P_D(u-\alpha f'(u))\|^2|$$

$$\leq 2\alpha\delta_1(2\alpha\delta_1 + 4M_0 + 4). \tag{7.40}$$

By (7.37) and (7.40),

$$\|x-u\|^2 - \|x-v\|^2$$

$$\geq \|u-\alpha\xi - P_D(u-\alpha\xi)\|^2$$

$$-2\alpha\delta_1(2\alpha\delta_1 + 4M_0 + 4) - (\delta_2 + \alpha\delta_1)(4M_0 + 6)$$

$$-\alpha^2 L^2 - 2\alpha L(2M_0 + 1).$$

In view of (4.4), (7.7), (7.8), and (7.10),

$$|\|u-\alpha\xi - v\|^2 - \|u-\alpha\xi - P_D(u-\alpha\xi)\|^2|$$

$$\leq \|v - P_D(u-\alpha\xi)\|(\|u-\alpha\xi - v\| + \|u - P_D(u-\alpha\xi)\|)$$

$$\leq \delta_2(2\|u - P_D(u - \alpha\xi)\| + \delta_2) \leq \delta_2(2\|x - u\| + 1) \leq \delta_2(2M_0 + 2).$$

It follows from the inequalities above that

$$\|x - u\|^2 - \|x - v\|^2$$

$$\geq \|u - \alpha\xi - v\|^2 - \delta_2(2M_0 + 2) - 2\alpha\delta_1(2\alpha\delta_1 + 4M_0 + 4)$$

$$-(\delta_2 + \alpha\delta_1)(4M_0 + 6) - \alpha^2 L^2 - 2\alpha L(2M_0 + 1).$$

Combined with (7.36) this completes the proof of Lemma 7.3.

7.3 An optimization problem

Let $U \subset X$ be a nonempty convex open set, $C_i \subset U$, $i = 1, \dots, m$ be nonempty convex closed sets in X,

$$C = \cap_{i=1}^m C_i,$$

$f : U \to R^1$ be a convex Frechet differentiable function, $M_* > 0$, $\tilde{M} > M_* + 4$, $L \geq 1$,

$$C \cap B(0, M_*) \neq \emptyset, \tag{7.41}$$

$$|f(v_1) - f(v_2)| \leq L\|v_1 - v_2\| \text{ for all } v_1, v_2 \in U \cap B(0, 9\tilde{M}_0 + 6), \tag{7.42}$$

$$\|f'(v_1) - f'(v_2)\| \leq L\|v_1 - v_2\| \text{ for all } v_1, v_2 \in \cap B(0, 9\tilde{M}_0 + 6), \tag{7.43}$$

$$\delta_f, \delta_C \in [0, 1], \ \alpha \in (0, , L^{-1}], \Delta \in (0, m^{-1}],$$

$$P_i = P_{C_i}, \ i = 1, \dots, m, \tag{7.44}$$

$$\inf(f, C) = \inf(f, C \cap B(0, M_*)). \tag{7.45}$$

We consider the problem

$$f(x) \to \min, \ x \in C.$$

Let us describe our algorithm.

Cimmino Gradient Algorithm
Initialization: select an arbitrary $x_0 \in U$.
Iterative step: given a current iteration vector $x_k \in U$ calculate

$$f'(x_k),$$

pick $w_{k+1} = (w_{k+1}(1), \ldots, w_{k+1}(m)) \in R^m$ such that

$$w_{k+1}(i) \geq \Delta, \ i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} w_{k+1}(i) = 1$$

and define the next iteration vector

$$x_{k+1} = \sum_{i=1}^{m} w_{k+1}(i) P_i(x_k - \alpha_k f'(x_k).$$

This algorithm is studied under the presence of computational errors and two convergence results are obtained. We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate the gradient of the objective function f while $\delta_C \in [0, 1]$ is a computational error produced by our computer system, when we calculate the operators P_i, $i = 1, \ldots, m$.

Cimmino Gradient Algorithm with Computational Errors
Initialization: select an arbitrary $x_0 \in U$.

Iterative step: given a current iteration vector $x_k \in U$ calculate

$$\xi_k \in B(f'(x_k), \delta_f),$$

pick $w_{k+1} = (w_{k+1}(1), \ldots, w_{k+1}(m)) \in R^m$ such that

$$w_{k+1}(i) \geq \Delta, \ i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} w_{k+1}(i) = 1,$$

calculate

$$y_{k,i} \in B(P_i(x_k - \alpha\xi_k), \delta_C), \ i = 1, \ldots, m$$

and the next iteration vector and $x_{k+1} \in U$ such that

$$\|x_{k+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t,i}\| \leq \delta_C.$$

Our main results are deduced from the following result.

Proposition 7.4 *T be a natural number and let*

$$\gamma_T = (\Delta^{-1}(4\tilde{M}^2 T^{-1} + 2\alpha\delta_f(18\tilde{M} + 8)$$

$$+ \delta_C(16\tilde{M} + 16) + 2\alpha L(6\tilde{M} + 2))^{1/2} + \delta_C. \tag{7.46}$$

Assume that $\{x_t\}_{t=0}^{T} \subset U$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$w_t = (w_t(1), \ldots, w_t(m)) \in R^m,\ t = 1, \ldots, T,$$

$$w_t(i) \geq \Delta,\ i = 1, \ldots, m,\ t = 1, \ldots, T, \tag{7.47}$$

$$\sum_{i=1}^{m} w_t(i) = 1,\ t = 1, \ldots, T, \tag{7.48}$$

and that for all integers $t \in \{0, \ldots, T-1\}$,

$$\|\xi_t - f'(x_t)\| \leq \delta_f, \tag{7.49}$$

$$y_{t,i} \in B(P_i(x_t - \alpha\xi_t), \delta_C) \cap U,\ i = 1, \ldots, m, \tag{7.50}$$

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t,i}\| \leq \delta_C. \tag{7.51}$$

Then the following assertions hold.

1. Let

$$z \in B(0, M_*) \cap C,$$

$t \in \{0, \ldots, T-1\}$,

$$\|x_t - z\| \leq 2\tilde{M}. \tag{7.52}$$

Then

$$\|x_t - z\|^2 - \|x_{t+1} - z\|^2$$

$$\geq 2\alpha(f(x_{t+1}) - f(z)) - 2\alpha L\delta_C - (\delta_C + \alpha\delta_f)(12\tilde{M} + 6)$$

$$- 2\alpha(L\delta_C + \delta_f) - \delta_C(4\tilde{M} + 5), \tag{7.53}$$

$$\|x_t - z\|^2 - \|x_{t+1} - z\|^2$$

$$\geq \Delta \sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-2\alpha\delta_f(12\tilde{M} + 5) - (\delta_C + \alpha\delta_f)(12\tilde{M} + 6)$$

$$- \alpha^2 L^2 - 2\alpha L(6\tilde{M} + 1) - \delta_C(4\tilde{M} + 7). \tag{7.54}$$

Assertion 2. Assume that

$$\text{for every } z \in B(0, M_*) \cap C,\ \|x_t - z\| \leq 2\tilde{M},\ t = 0, \ldots, T - 1.$$

Then

$$\min\{\max\{2\alpha(f(x_{t+1}) - \inf(f, C))$$

$$-\delta_C(16\tilde{M} + 18) - 4\alpha L\delta_C - \alpha\delta_f(12\tilde{M} + 8),$$

$$\Delta \sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-2\alpha\delta_f(18\tilde{M} + 18) - \delta_C(16\tilde{M} + 16)$$

$$-2\alpha L(6\tilde{M} + 2) : t = 0 \ldots, T - 1\} \leq 4\tilde{M}^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T - 1\}$ *and*

$$\max\{2\alpha(f(x_{t+1}) - \inf(f, C))$$

$$-\delta_C(16\tilde{M} + 18) - 4\alpha L\delta_C - \alpha\delta_f(12\tilde{M} + 8),$$

$$\Delta \sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-2\alpha\delta_f(18\tilde{M} + 8) - \delta_C(16\tilde{M} + 16)$$

$$-2\alpha L(6\tilde{M} + 2)\} \leq 4\tilde{M}^2 T^{-1},$$

then

$$f(x_{t+1}) \leq \inf(f, C)$$

$$+2^{-1}\alpha^{-1}\delta_C(16\tilde{M} + 18) + 2^{-1}\delta_f(12\tilde{M} + 8)$$

$$+2L\delta_C + 2\tilde{M}^2(T\alpha)^{-1},$$

$$d(x_{t+1}, C_i) \le 2\gamma_T,\ i = 1, \dots, m.$$

Assertion 3. Let $M_{*,0} > 0$, $\tilde{M} > 2M_*$, $\delta_C \le L^{-1}$, $\delta_f \le (12\tilde{M} + 6)^{-1}$,

$$\alpha \ge \delta_C(14\tilde{M} + 9),$$

$$|f(u)| \le M_{*,0},\ u \in B(0, M_*) \cap U,$$

$$f(u) > M_{*,0} + 8 \text{ for all } u \in U \setminus B(0, 2^{-1}\tilde{M}),$$

$t \in \{0, \dots, T-1\}$ *and that*

$$x_t \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C.$$

Then

$$x_{t+1} \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C.$$

Assertion 4. Let $r_0 \in (0, 1]$,

$$\{u \in X :\ d(u, C_i) \le r_0,\ i = 1, \dots, m\} \subset B(0, M_*),$$

$$2\delta_C + 2[\Delta^{-1}(2\alpha\delta_f(24\tilde{M} + 12) + \delta_C(16\tilde{M} + 18) + 2\alpha L(6\tilde{M} + 2))]^{1/2} \le r_0.$$

Assume that $t \in \{0, \dots, T-1\}$ *and that*

$$x_t \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C.$$

Then

$$x_{t+1} \in B(z, 2\tilde{M}) \text{ for every } z \in B(0, M_*) \cap C.$$

Proof We prove Assertion 1. Let $i \in \{1, \dots, m\}$. By (7.42)–(7.44), (7.49), (7.50), and Lemma 7.3 applied with $M_0 = 3\tilde{M}$, $\delta_1 = \delta_f$, $\delta_2 = \delta_C$

$$x = z,\ u = x_t,\ \xi = \xi_t,\ v = y_{t,i}$$

we obtain that

$$\|z - x_t\|^2 - \|z - y_{t,i}\|^2$$

$$\ge 2\alpha(f(y_{t,i}) - f(z))$$

$$- (\delta_C + \alpha\delta_f)(12\tilde{M} + 6) - 2\alpha(L\delta_C + \delta_f), \tag{7.55}$$

$$\|z - x_t\|^2 - \|z - y_{t,i}\|^2$$

$$\geq \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$- 2\alpha\delta_f(12\tilde{M} + 6) - (\delta_C + \alpha\delta_f)(12\tilde{M} + 6) - \alpha^2L^2 - 2\alpha L(6\tilde{M} + 1). \tag{7.56}$$

It follows from (7.40), (7.47), and (7.55) and the convexity of the functions f and $\|z - u\|^2$, $u \in X$ that

$$\|z - x_t\|^2 - \|z - \sum_{i=1}^m w_t(i)y_{t,i}\|^2$$

$$\geq \|z - x_t\|^2 - \sum_{i=1}^m w_t(i)\|z - y_{t,i}\|^2$$

$$= \sum_{i=1}^m w_t(i)(\|z - x_t\|^2 - \|z - y_{t,i}\|^2)$$

$$\geq \sum_{i=1}^m w_t(i)[2\alpha(f(y_{t,i}) - f(z))$$

$$-(\delta_C + \alpha\delta_f)(12\tilde{M} + 6) - 2\alpha(L\delta_C + \delta_f)]$$

$$\geq 2\alpha(f(\sum_{i=1}^m w_t(i)y_{t,i}) - f(z))$$

$$- (\delta_C + \alpha\delta_f)(12\tilde{M} + 6) - 2\alpha(L\delta_C + \delta_f). \tag{7.57}$$

By (7.47), (7.48), and (7.56) and the convexity of the function $\|\cdot\|^2$,

$$\|z - x_t\|^2 - \|z - \sum_{i=1}^m w_t(i)y_{t,i}\|^2$$

$$\geq \sum_{i=1}^m w_t(i)(\|z - x_t\|^2 - \|z - y_{t,i}\|^2)$$

$$\geq \sum_{i=1}^{m} w_t(i)\|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-2\alpha\delta_f(12\tilde{M}+6) - (\delta_C + \alpha\delta_f)(12\tilde{M}+6) - \alpha^2 L^2 - 2\alpha L(6\tilde{M}+1). \quad (7.58)$$

In view of (7.51) and (7.52),

$$\|x_t\| \leq 3\tilde{M}. \quad (7.59)$$

Equations (7.42), (7.49), and (7.59) imply that

$$\|f'(x_t)\| \leq L, \ \|\xi_t\| \leq L+1. \quad (7.60)$$

By (4.4), (7.43), (7.47), (7.48), (7.50), (7.51), and (7.60) and the convexity of the norm

$$\|z - \sum_{i=1}^{m} w_t(i) y_{t,i}\| \leq \|z - \sum_{i=1}^{m} w_t(i) P_{C_i}(x_t - \alpha\xi_t)\| + \delta_C$$

$$\leq \sum_{i=1}^{m} w_t(i)\|z - P_{C_i}(x_t - \alpha\xi_t)\| + \delta_C$$

$$\leq \|z - x_t\| + \alpha\|\xi_t\| + \delta_C$$

$$\leq 2\tilde{M} + \alpha(L+1) + 1 \leq 2\tilde{M} + 3. \quad (7.61)$$

It follows from (7.51) and (7.61) that

$$\|z - x_{t+1}\|$$

$$\leq \|z - \sum_{i=1}^{m} w_t(i) y_{t,i}\| + \|\sum_{i=1}^{m} w_t(i) y_{t,i} - x_{t+1}\|$$

$$\leq 2\tilde{M} + 3 + \delta_C \leq 2\tilde{M} + 4. \quad (7.62)$$

In view of (7.51) and (7.61),

$$\|\sum_{i=1}^{m} w_t(i) y_{t,i}\| \leq 2\tilde{M} + 3 + M_* < 3\tilde{M}. \quad (7.63)$$

By (7.42), (7.51), (7.59), and (7.63),

$$|f(x_{t+1}) - f(\sum_{i=1}^{m} w_t(i) y_{t,i})|$$

$$\leq L\|x_{t+1} - \sum_{i=1}^{m} w_t(i) y_{t,i}\| \leq L\delta_C. \tag{7.64}$$

Equations (7.51), (7.61), and (7.62) imply that

$$|\|z - x_{t+1}\|^2 - \|z - \sum_{i=1}^{m} w_t(i) y_{t,i}\|^2|$$

$$\leq \|x_{t+1} - \sum_{i=1}^{m} w_t(i) y_{t,i}\|$$

$$\times (\|z - x_{t+1}\| + \|z - \sum_{i=1}^{m} w_t(i) y_{t,i}\|) \leq \delta_C(4\tilde{M} + 7). \tag{7.65}$$

By (7.57), (7.64), and (7.65),

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2$$

$$\|z - x_t\|^2 - \|z - \sum_{i=1}^{m} w_t(i) y_{t,i}\|^2 - \delta_C(4\tilde{M} + 7)$$

$$\geq 2\alpha(f(\sum_{i=1}^{m} w_t(i) y_{t,i}) - f(z))$$

$$-(\delta_C + \alpha\delta_f)(12\tilde{M} + 6) - 2\alpha(L\delta_C + \delta_f) - \delta_C(4\tilde{M} + 7)$$

$$\geq 2\alpha(f(x_{t+1}) - f(z)) - 2\alpha L\delta_C$$

$$-(\delta_C + \alpha\delta_f)(12\tilde{M} + 6) - 2\alpha(L\delta_C + \delta_f) - \delta_C(4\tilde{M} + 7).$$

In view of (7.58) and (7.65),

$$\|z - x_t\|^2 - \|z - x_{t+1}\|^2$$

$$\|z - x_t\|^2 - \|z - \sum_{i=1}^{m} w_t(i) y_{t,i}\|^2 - \delta_C(4\tilde{M} + 7)$$

$$\geq \Delta \sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-2\alpha\delta_f(12\tilde{M} + 5) - (\delta_C + \alpha\delta_f)(12\tilde{M} + 6)$$

$$-\alpha^2 L^2 - 2\alpha L(6\tilde{M} + 1) - \delta_C(4\tilde{M} + 7).$$

Assertion 1 is proved.

We prove Assertion 2. By Assertion 1, for $z \in B(0, M_*) \cap C$ and every $t \in \{0, \dots, T-1\}$, (7.53) and (7.54) hold.

Let

$$z \in B(0, M_*) \cap C. \tag{7.66}$$

By (7.43), (7.53), (7.54), and (7.66),

$$4\tilde{M}^2 \geq \|z - x_0\|^2 \geq \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1} (\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\geq \sum_{t=0}^{T-1} \max\{2\alpha(f(x_{t+1}) - f(z)) - 2\alpha L\delta_C - (\delta_C + \alpha\delta_f)(12\tilde{M} + 6)$$

$$-2\alpha(L\delta_C + \delta_f) - \delta_C(4\tilde{M} + 7),$$

$$\Delta \sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-2\alpha\delta_f(12\tilde{M} + 5) - (\delta_C + \alpha\delta_f)(12\tilde{M} + 6)$$

$$-\alpha^2 L^2 - 2\alpha L(6\tilde{M} + 1) - \delta_C(4\tilde{M} + 7)\}. \tag{7.67}$$

Since z is any element of $B(0, M_*) \cap C$ it follows from (7.45) and (7.67) that

$$\min\{\max\{2\alpha(f(x_{t+1}) - \inf(f, C))$$

$$-4\alpha L\delta_C - \delta_C(16\tilde{M} + 8) - \alpha\delta_f(12\tilde{M} + 8),$$

$$\Delta \sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-2\alpha\delta_f(18\tilde{M}+8)-\delta_C(16\tilde{M}+16)$$

$$-2\alpha L(6\tilde{M}+2)\}:\ t\in\{0,\dots,T-1\}\le 4\tilde{M}^2T^{-1}.$$

Let $t\in\{0,\dots,T-1\}$,

$$\max\{2\alpha(f(x_{t+1})-\inf(f,C))$$

$$-4\alpha L\delta_C-\delta_C(16\tilde{M}+8)-\alpha\delta_f(12\tilde{M}+8),$$

$$\Delta\sum_{i=1}^{m}\|x_t-\alpha\xi_t-y_{t,i}\|^2$$

$$-2\alpha\delta_f(18\tilde{M}+8)-\delta_C(16\tilde{M}+16)$$

$$-2\alpha L(6\tilde{M}+2)\}\le 4\tilde{M}^2T^{-1}. \tag{7.68}$$

By (7.68),

$$f(x_{t+1})\le\inf(f,C)+2^{-1}\alpha^{-1}\delta_C(16\tilde{M}+18)$$

$$+2^{-1}\delta_f(12\tilde{M}+8)+2\tilde{M}^2(T\alpha)^{-1}+2L\delta_C.$$

In view (7.46), (7.50), and (7.68), for all $i=1,\dots,m$,

$$\|x_t-\alpha\xi_t-y_{t,i}\|$$

$$\le(\Delta^{-1}(4\tilde{M}^2T^{-1}+2\alpha\delta_f(18\tilde{M}+8)$$

$$+\delta_C(16\tilde{M}+16)+2\alpha L(6\tilde{M}+2))^{1/2}\le\gamma_T, \tag{7.69}$$

$$\|x_t-\alpha\xi_t-P_{C_i}(x_t-\alpha\xi_t)\|$$

$$\le(\Delta^{-1}(4\tilde{M}^2T^{-1}+2\alpha\delta_f(18\tilde{M}+8)$$

$$+\delta_C(16\tilde{M}+16)+2\alpha L(6\tilde{M}+2))^{1/2}+\delta_C\le\gamma_T. \tag{7.70}$$

By (7.46)–(7.48), and (7.51),

$$\|x_t-\alpha\xi_t-x_{t+1}\|$$

$$\le\|x_t-\alpha\xi_t\sum_{i=1}^{m}w_t(i)y_{t,i}\|+\|\sum_{i=1}^{m}w_t(i)y_{t,i}-x_{t+1}\|$$

$$\leq \sum_{i=1}^{m} w_t(i)\|x_t - \alpha\xi_t - y_{t,i}\| + \delta_C \leq \gamma_T. \tag{7.71}$$

Equations (7.70) and (7.71) imply that for $i = 1, \ldots, m$,

$$\|x_{t+1} - P_{C_i}(x_t - \alpha\xi_t)\|$$

$$\|x_{t+1} - (x_t - \alpha\xi_t)\| + \|x_t - \alpha\xi_t - P_{C_i}(x_t - \alpha\xi_t)\| \leq 2\gamma_T$$

and

$$d(x_{t+1}, C_i) \leq 2\gamma_T.$$

Assertion 2 is proved.

We prove Assertion 3.

$$z \in B(0, M) \cap C. \tag{7.72}$$

In view of Assertion 1, (7.53), and (7.54) hold. We may assume without loss of generality that

$$\|x_{t+1} - z\| > \|x_t - z\|. \tag{7.73}$$

By (7.33), (7.72), (7.73) and the assumptions of Assertion 3,

$$f(x_{t+1}) \leq f(z) + 2L\delta_C + \alpha^{-1}(\delta_C + \alpha\delta_f)(6\tilde{M} + 3)$$

$$+\delta_f + \alpha^{-1}\delta_C(2\tilde{M} + 4) \leq M_{*,0} + 6$$

and

$$\|x_{t-1}\| \leq \tilde{M}/2.$$

Together with (7.72) this implies that

$$\|x_{t+1} - z\| \leq 2\tilde{M}.$$

Assertion 3 is proved.

We prove Assertion 4. Let

$$z \in B(0, M_*) \cap C. \tag{7.74}$$

In view of Assertion 1, (7.54) holds.

We may assume without loss of generality that

$$\|x_{t+1} - z\| > \|x_t - z\|. \tag{7.75}$$

Equations (7.54) and (7.75) imply that for all $i = 1, \dots, m$,

$$\|x_t - \alpha\xi_t - y_{t,i}\|$$
$$\leq (\Delta^{-1}(2\alpha\delta_f(24\tilde{M} + 12)$$
$$+ \delta_C(16\tilde{M} + 16) + 2\alpha L(6\tilde{M} + 2))^{1/2}. \tag{7.76}$$

By (7.47), (7.48), (7.51), and (7.76),

$$\|x_t - \alpha\xi_t - x_{t+1}\|$$
$$\leq \|x_t - \alpha\xi_t - \sum_{i=1}^{m} w_t(i)y_{t,i}\| + \|\sum_{i=1}^{m} w_t(i)y_{t,i} - x_{t+1}\|$$
$$\leq \sum_{i=1}^{m} w_t(i)\|x_t - \alpha\xi_t - y_{t,i}\| + \delta_C$$
$$\leq \delta_C + [\Delta^{-1}(2\alpha\delta_f(24\tilde{M} + 12)$$
$$+ \delta_C(16\tilde{M} + 16) + 2\alpha L(6\tilde{M} + 2))]^{1/2}. \tag{7.77}$$

It follows from (4.4), (7.50), (7.70), (7.74), and (7.77) and the assumptions of Assertion 4 that

$$d(x_{t+1}, C_i) \leq \|x_{t+1} - P_{C_i}(x_t - \alpha\xi_t)\|$$
$$\leq \|x_{t+1} - (x_t - \alpha\xi_t)\| + \|x_t - \alpha\xi_t - P_{C_i}(x_t - \alpha\xi_t)\|$$
$$\leq 2(\delta_C + 2\Delta^{-1}(2\alpha\delta_f(24\tilde{M} + 12)$$
$$+\delta_C(16\tilde{M} + 16) + 2\alpha L(6\tilde{M} + 2))^{1/2}) \leq r_0$$

and

$$\|x_{t+1}\| \leq M_*.$$

Together with (7.74) implies that

$$\|z - x_{t+1}\| \le 2\tilde{M}.$$

Assertion 4 is proved. This completes the proof of Proposition 7.4.

Analogously to Theorem 2.9 we choose α, T and an approximate solution of our problem after T iterations in the case of Assertion 2.

The following result is easily deduced from Assertions 2 and 3 of Proposition 7.4. It holds under some growth condition on the objective function f.

Theorem 7.5 *Let* $M_{*,0} > 0$*,* $\tilde{M} > 2M_*$*.*

$$|f(u)| \le M_{*,0},\ u \in B(0, M_*) \cap U,$$

$$f(u) \ge M_{*,0} + 8 \textit{ for all } u \in U \setminus B(0, 2^{-1}\tilde{M}),$$

$$\delta_C \le L^{-1},\ \delta_f \le (12\tilde{M} + 6)^{-1},\ \alpha \ge \delta_C(14\tilde{M} + 9))^{-1},$$

T be a natural number,

$$\gamma_T = (\Delta^{-1}(4\tilde{M}^2 T^{-1} + 2\alpha\delta_f(18\tilde{M} + 8)$$

$$+\delta_C(16\tilde{M} + 16) + 2\alpha L_0(6\tilde{M} + 2))^{1/2} + \delta_C.$$

Assume that $\{x_t\}_{t=0}^{T} \subset U$*,* $\{\xi_t\}_{t=0}^{T-1} \subset X$*,*

$$w_t = (w_t(1), \dots, w_t(m)) \in R^m,\ t = 1, \dots, T,$$

$$w_t(i) \ge \Delta,\ i = 1, \dots, m,\ t = 1, \dots, T,$$

$$\sum_{i=1}^{m} w_t(i) = 1,\ t = 1, \dots, T,$$

and that for all integers $t \in \{0, \dots, T-1\}$*,*

$$\|\xi_t - f'(x_t)\| \le \delta_f,$$

$$y_{t,i} \in B(P_{C_i}(x_t - \alpha\xi_t), \delta_C) \cap U,\ i = 1, \dots, m,$$

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t,i}\| \le \delta_C$$

and

$$\|x_0\| \le \tilde{M}.$$

Then

$$\min\{\max\{2\alpha(f(x_{t+1}) - \inf(f, C))$$

$$-\delta_C(16\tilde{M} + 18) - 4\alpha L\delta_C - \alpha\delta_f(12\tilde{M} + 8),$$

$$\Delta\sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-2\alpha\delta_f(18\tilde{M} + 18) - \delta_C(16\tilde{M} + 16)$$

$$-2\alpha L(6\tilde{M} + 2)\} : t = 0\ldots, T - 1\} \le 4\tilde{M}^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T - 1\}$ *and*

$$\max\{2\alpha(f(x_{t+1}) - \inf(f, C))$$

$$-\delta_C(16\tilde{M} + 18) - 4\alpha L\delta_C - \alpha\delta_f(12\tilde{M} + 8),$$

$$\Delta\sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-2\alpha\delta_f(18\tilde{M} + 8) - \delta_C(16\tilde{M} + 16)$$

$$-2\alpha L(6\tilde{M} + 2)\} \le 4\tilde{M}^2 T^{-1},$$

then

$$f(x_{t+1}) \le \inf(f, C)+$$

$$2^{-1}\alpha^{-1}\delta_C(16\tilde{M} + 18) + 2^{-1}\delta_f(12\tilde{M} + 8)$$

$$+2L\delta_C + 2\tilde{M}^2(T\alpha)^{-1},$$

$$d(x_{t+1}, C_i) \le 2\gamma_T,\ i = 1, \ldots, m.$$

The following result is easily deduced from Assertions 2 and 4 of Proposition 7.4. It holds under the assumption that the set

$$\{u \in X:\ d(u, C_i) \le r_0,\ i = 1, \ldots, m\}$$

is bounded for some positive number r_0.

Theorem 7.6 *Let* $r_0 \in (0, 1]$,

$$\{u \in X : \ d(u, C_i) \le r_0, \ i = 1, \dots, m\} \subset B(0, M_*),$$

$$2\delta_C + 2[\Delta^{-1}(2\alpha\delta_f(24\tilde{M} + 12) + \delta_C(16\tilde{M} + 18) + 2\alpha L(6\tilde{M} + 2)]^{1/2} \le r_0.$$

T *be a natural number,*

$$\gamma_T = (\Delta^{-1}(4\tilde{M}^2 T^{-1} + 2\alpha\delta_f(18\tilde{M} + 8)$$

$$+\delta_C(16\tilde{M} + 16) + 2\alpha L(6\tilde{M} + 2))^{1/2} + \delta_C.$$

Assume that $\{x_t\}_{t=0}^{T} \subset U$, $\{\xi_t\}_{t=0}^{T-1} \subset X$,

$$(w_t(1), \dots, w_t(m)) \in R^m, \ t = 1, \dots, T,$$

$$w_t(i) \ge \Delta, \ i = 1, \dots, m, \ t = 1, \dots, T,$$

$$\sum_{i=1}^{m} w_t(i) = 1, \ t = 1, \dots, T,$$

and that for all integers $t \in \{0, \dots, T - 1\}$,

$$\|\xi_t - f'(x_t)\| \le \delta_f,$$

$$y_{t,i} \in B(P_{C_i}(x_t - \alpha\xi_t), \delta_C) \cap U, \ i = 1, \dots, m,$$

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t,i}\| \le \delta_C$$

and

$$\|x_0\| \le \tilde{M}.$$

Then

$$\min\{\max\{2\alpha(f(x_{t+1}) - \inf(f, C))$$

$$-\delta_C(16\tilde{M} + 18) - 4\alpha L\delta_C - \alpha\delta_f(12\tilde{M} + 8),$$

$$\Delta \sum_{i=1}^{m} \|x_t - \alpha\xi_t - y_{t,i}\|^2$$

$$-2\alpha\delta_f(18\tilde{M}+18)-\delta_C(16\tilde{M}+16)$$

$$-2\alpha L(6\tilde{M}+2)\}: t=0\dots,T-1\}\le 4\tilde{M}^2T^{-1}.$$

Moreover, if $t\in\{0,\dots,T-1\}$ *and*

$$\max\{2\alpha(f(x_{t+1})-\inf(f,C))$$

$$-\delta_C(16\tilde{M}+18)-4\alpha L\delta_C-\alpha\delta_f(12\tilde{M}+8),$$

$$\Delta\sum_{i=1}^{m}\|x_t-\alpha\xi_t-y_{t,i}\|^2$$

$$-2\alpha\delta_f(18\tilde{M}+8)-\delta_C(16\tilde{M}+16)$$

$$-2\alpha L(6\tilde{M}+2)\}\le 4\tilde{M}^2T^{-1},$$

then

$$f(x_{t+1})\le\inf(f,C)$$

$$+2^{-1}\alpha^{-1}\delta_C(16\tilde{M}+18)+2^{-1}\delta_f(12\tilde{M}+8)$$

$$+2L\delta_C+2\tilde{M}^2(T\alpha)^{-1},$$

$$d(x_{t+1},C_i)\le 2\gamma_T,\ i=1,\dots,m.$$

Chapter 8
Cimmino Gradient Projection Algorithm

In this chapter we consider a minimization of a convex smooth function on a solution set of a convex feasibility problem in a general Hilbert space using the Cimmino gradient projection algorithm. Our goal is to obtain a good approximate solution of the problem in the presence of computational errors. We show that an algorithm generates a good approximate solution, if the sequence of computational errors is bounded from above by a small constant. Moreover, if we known computational errors for our algorithm, we find out what an approximate solution can be obtained and how many iterates one needs for this.

8.1 Preliminaries

Let X be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$. Let m be a natural number and $f_i : X \to R^1$, $i = 1, \dots, m$ be convex continuous functions.

For every integer $i = 1, \dots, m$ put

$$C_i = \{x \in X : \ f_i(x) \le 0\},$$

$$C = \cap_{i=1}^m C_i = \cap_{i=1}^m \{x \in X : \ f_i(x) \le 0\}. \tag{8.1}$$

We suppose that

$$C \ne \emptyset.$$

Let $M_* > 0$ and

$$C \cap B(0, M_*) \ne \emptyset. \tag{8.2}$$

A. J. Zaslavski, *Optimization on Solution Sets of Common Fixed Point Problems*,
Springer Optimization and Its Applications 178,
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Let $f : X \to R^1$ be a convex Frechet differentiable function. We consider the minimization problem

$$f(x) \to \min, \ x \in C.$$

Assume that

$$\inf(f, C) = \inf(f, C \cap B(0, M_*)). \tag{8.3}$$

8.2 Cimmino Type Gradient Algorithm

Let

$$\bar{\Delta} \in (0, m^{-1}], \ \tilde{M} > M_* + 4, \ L \geq 1, \tag{8.4}$$

$$|f(v_1) - f(v_2)| \leq L\|v_1 - v_2\| \text{ for all } v_1, v_2 \in B(0, 9\tilde{M} + 6), \tag{8.5}$$

$$\|f'(v_1) - f'(v_2)\| \leq L\|v_1 - v_2\| \text{ for all } v_1, v_2 \in \cap B(0, 9\tilde{M} + 6), \tag{8.6}$$

$$\delta_f, \delta_C, \bar{\delta}_C \in [0, 1], \ \alpha \in (0, (L + 1)^{-1}]. \tag{8.7}$$

Let us describe our algorithm. Fix $\alpha > 0$.

Cimmino Gradient Algorithm
Initialization: select an arbitrary $x_0 \in X$.

Iterative step: given a current iteration vector $x_k \in X$ calculate $f'(x_k)$, pick $w_{k+1} = (w_{k+1}(1), \ldots, w_{k+1}(m)) \in R^m$ such that

$$w_{k+1}(i) \geq \bar{\Delta}, \ i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} w_{k+1}(i) = 1,$$

for each $i \in \{1, \ldots, m\}$,

$$\text{if } f_i(x_k - \alpha f'(x_k)) \leq 0 \text{ then } x_{k,i} = x_k - \alpha f'(x_k),$$

and if $f_i(x_k - \alpha f'(x_k)) > 0$ then

$$l_{k,i} \in \partial f_i(x_k - \alpha f'(x_k)),$$

$$x_{k,i} = x_k - \alpha f'(x_k) - f_i(x_k - \alpha f'(x_k))\|l_{k,i}\|^{-2} l_{k,i}$$

and define the next iteration vector

$$x_{k+1} = \sum_{i=1}^{m} w_{k+1}(i) x_{k,i}.$$

In this chapter this algorithm is studied under the presence of computational errors.

Cimmino Gradient Projection Algorithm with Computational Errors

We suppose that $\delta_f \in (0, 1]$ is a computational error produced by our computer system, when we calculate a subgradient of the objective function f, $\delta_C \in [0, 1]$ is a computational error produced by our computer system, when we calculate subgradients of the constraint functions f_i, $i = 1, \dots, m$ and $\bar{\delta}_C$ is a computational error produced by our computer system, when we calculate auxiliary projection operators. Let $\alpha > 0$ be a step size and $\Delta \in (0, 1]$.

Initialization: select an arbitrary $x_0 \in X$.
Iterative step: given a current iteration vector $x_t \in X$ calculate

$$l_t \in B(f'(x_t), \delta_f),$$

pick $w_{t+1} = (w_{t+1}(1), \dots, w_{t+1}(m)) \in R^m$ such that

$$w_{t+1}(i) \geq \bar{\Delta}, \ i = 1, \dots, m,$$

$$\sum_{i=1}^{m} w_{t+1}(i) = 1,$$

for each $i \in \{1, \dots, m\}$,

$$\text{if } f_i(x_t - \alpha l_t) \leq \Delta, \text{ then } y_{t+1,i} = x_t - \alpha l_t, \ l_{t,i} = 0,$$

if $f_i(x_t - \alpha l_t) > \Delta$, then we calculate

$$l_{t,i} \in \partial f_i(x_t - \alpha l_t) + B(0, \delta_C),$$

(this implies that $l_{t,i} \neq 0$),

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2} l_{t,i}, \bar{\delta}_C)$$

and the next iteration vector

$$x_{t+1} \in B(\sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}, \bar{\delta}_C).$$

8.3 The Basic Lemma

We use all the notation and definitions introduced in Section 8.1 and assume that all the assumption made there hold.

Let $M_0 \geq 1,\ M_1 > 2,$

$$f_i(B(0, \tilde{M} + 1)) \subset [-M_0, M_0],\ i = 1, \dots, m, \tag{8.8}$$

$$|f_i(u) - f_i(v)| \leq M_1 \|u - v\| \text{ for all } u, v \in (0, 3\tilde{M} + 2) \text{ and all } i = 1, \dots, m, \tag{8.9}$$

$$0 \leq \delta_C \leq 2^{-1} \Delta (6\tilde{M} + 1)^{-1},\ \alpha(L + 1)(6\tilde{M} + 6) \leq 1, \tag{8.10}$$

$$16 L M_0 \delta_C \Delta^{-2} (6\tilde{M} + 5)^3 \leq 1,\ \bar{\delta}_C (6\tilde{M} + 16) \leq 1. \tag{8.11}$$

Lemma 8.1 *Let $\Delta_0 \geq 0$ and let at least one of the following conditions hold:*

(a) $\Delta_0 = \Delta$;
(b) for each $i \in \{1, \dots, m\}$,

$$B(0, M_*) \cap \{x \in X :\ f_i(x) \leq -\Delta\} \neq \emptyset.$$

Assume that

$$z \in B(0, M_*) \cap C, \tag{8.12}$$

$$x \in B(z, 2\tilde{M}), \tag{8.13}$$

$l \in X$ *satisfies*

$$\|l - f'(x)\| \leq \delta_f, \tag{8.14}$$

$$w(i) \geq \bar{\Delta},\ i = 1, \dots, m, \tag{8.15}$$

$$\sum_{i=1}^{m} w(i) = 1, \tag{8.16}$$

$l_i, y_i \in X,\ i = 1, \dots, m,$ *for every* $j \in \{1, \dots, m\}$,

$$\text{if } f_j(x - \alpha l) \leq \Delta_0, \text{ then } y_j = x - \alpha l,\ l_j = 0, \tag{8.17}$$

if $f_j(x - \alpha l) > \Delta_0$, *then*

$$B(l_j, \delta_C) \cap \partial f_j(x - \alpha l) \neq \emptyset \tag{8.18}$$

(this implies that $l_j \neq 0$*),*

$$y_j \in B(x - \alpha l - f_j(x - \alpha l)\|l_j\|^{-2} l_j, \bar{\delta}_C) \tag{8.19}$$

and that

$$\|y - \sum_{i=1}^{m} w(i) y_i\| \leq \bar{\delta}_C. \tag{8.20}$$

Then

$$\|x - z\|^2 - \|y - z\|^2$$

$$\geq 2\alpha(f(y) - f(z)) - L\bar{\delta}_C - \bar{\delta}_C(4\tilde{M} + 8) -$$

$$-(12\tilde{M} + 6)(\alpha\delta_f + \bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2)$$

$$-2\alpha(\delta_f + L\bar{\delta}_C + 16M_0 L\delta_C\Delta^{-2}(6\tilde{M} + 1)^2),$$

$$\|x - z\|^2 - \|y - z\|^2$$

$$\geq \bar{\Delta}\sum_{j=1}^{m} \|x - y_j\|^2 - \delta_C(4\tilde{M} + 8) - \alpha(L_0 + 1)(6\tilde{M} + 1)$$

$$-16\delta_C\Delta^{-2}(6\tilde{M} + 4)^3 - 2(\alpha(L + 1) + \bar{\delta}_C)(6\tilde{M} + 5)$$

and for each $j \in \{1, \ldots, m\}$*, if* $f_j(x - \alpha l) \leq \Delta_0$*, then*

$$\|x - y_j\| \leq \alpha(L_0 + 1)$$

and if $f_j(x - \alpha l) > \Delta_0$*, then*

$$f_j(x - \alpha l) \leq M_1\|x - y_j\| + M_1[\alpha(L_0 + 1) + \bar{\delta}_C + 16\delta_C M_0\Delta^{-2}(6\tilde{M} + 1)^2].$$

Proof Let $j \in \{1, \ldots, m\}$. By (8.5) and (8.12)–(8.14),

$$\|f'(x)\| \leq L, \ \|l\| \leq L + 1. \tag{8.21}$$

Assume that

$$f_j(x - \alpha l) \leq \Delta_0. \tag{8.22}$$

In view of (8.17) and (8.22),

$$y_j = x - \alpha l. \tag{8.23}$$

Equations (8.21) and (8.23) imply that

$$\|x - y_j\| \le \alpha(L_0 + 1). \tag{8.24}$$

By (8.5), (8.6), (8.10), (8.12)–(8.14), (8.23), (8.24), and Lemma 7.3 applied with $D = U = X$, $M_0 = 3\tilde{M}$, $\delta_1 = \delta_f$, $\delta_2 = \delta_C$, $P_D = I$ (the identity mapping in X), $u = x$, $x = z$, $\xi = l$, $v = y_j$ that

$$\|x - z\|^2 - \|y_j - z\|^2$$

$$\ge 2\alpha(f(y_j) - f(z)) - (\delta_C + \alpha\delta_f)(12\tilde{M}_0 + 6) - 2\alpha(L\delta_C + \delta_f),$$

$$\|x - z\|^2 - \|y_j - z\|^2 - \|x - y_j\|^2$$

$$\ge -(\delta_C + \alpha\delta_f)(12\tilde{M}_0 + 6) - 2\alpha^2 L^2$$

$$- 2\alpha L(6\tilde{M} + 1) - 2\alpha\delta_f(12\tilde{M} + 6). \tag{8.25}$$

Assume that

$$f_j(x - \alpha l) > \Delta_0. \tag{8.26}$$

In view of (8.18) and (8.26), there exists

$$\xi_j \in \partial f_j(x - \alpha l) \tag{8.27}$$

such that

$$\|\xi_j - l_j\| \le \delta_C. \tag{8.28}$$

By (8.8)–(8.13), (8.19), (8.21), (8.26)–(8.28), and Lemma 4.2 applied with $\delta_1 = \alpha(L_0 + 1)$, $\delta_2 = \delta_C$, $\bar{\delta}_2 = \bar{\delta}_C$, $M = 3\tilde{M}$, $x_0 = x - \alpha l$, $\xi = \xi_j$, $l = l_j$, $u = y_j$, we have

$$\xi_j \ne 0, \ l_j \ne 0,$$

$$\|\xi_j\| \ge \Delta(6\tilde{M} + 1)^{-1}, \ \|l_j\| \ge 2^{-1}\Delta(6\tilde{M} + 1)^{-1} \tag{8.29}$$

$$\|z - y_j\| \le \|z - x\| + \alpha(L_0 + 1) + \bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M} + 1)^2, \tag{8.30}$$

$$\|y_j - (x - \alpha l - f_j(x - \alpha l)\|\xi_j\|^{-2}\xi_j)\|$$

$$\leq \bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2, \tag{8.31}$$

$$\|z-x\|^2 - \|z-y_j\|^2 - \|x-y_j\|^2$$

$$\geq -\alpha(L+1)(6\tilde{M}+1) - 16\delta_C\Delta^{-2}(6\tilde{M}+4)^3$$

$$-(\alpha(L_0+1)+\bar{\delta}_C)(6\tilde{M}+5), \tag{8.32}$$

$$\|x-y_j\| \geq -\alpha(L_0+1) - \bar{\delta}_C$$

$$-16\delta_C M_0\Delta^{-2}(6\tilde{M}+1)^2 + M_1^{-1} f_j(x-\alpha l). \tag{8.33}$$

By (8.5), (8.8), (8.10)–(8.13), and Lemma 4.1 applied with

$$D = U = X,\ M_0 = 3\tilde{M},\ \delta_1 = \delta_f,\ \xi = l,$$

$$x = z,\ u = x,\ v = y_j,\ \delta_2 = \bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2,$$

$$D = \{v \in X :\ f_j(x-\alpha l) + \langle \xi_j, v-(x-\alpha\xi_j)\rangle \leq 0\},$$

we have

$$\|x-z\|^2 - \|y_j-z\|^2$$

$$\geq 2\alpha(f(y_j)-f(z))$$

$$-(12\tilde{M}+6)(\alpha\delta_f + \bar{\delta}_C + 16M_0\delta_C\Delta^{-2}(6\tilde{M}+1)^2)$$

$$-2\alpha(\delta_f + L\bar{\delta}_C + 16M_0L\delta_C\Delta^{-2}(6\tilde{M}+1)^2). \tag{8.34}$$

In both cases, in view of (8.25), Equation (8.34) holds. In both cases, by (8.25) and (8.32),

$$\|z-x\|^2 - \|z-y_j\|^2$$

$$\geq \|x-y_j\|^2 - \alpha(L_0+1)(6\tilde{M}+1)$$

$$-16\delta_C\Delta^{-2}(6\tilde{M}+4)^3 - 2(\alpha(L+1)+\bar{\delta}_C)(6\tilde{M}+6). \tag{8.35}$$

It follows from (8.22) and (8.24) that if $f_j(x-\alpha l) \leq \Delta_0$, then

$$\|x-y_j\| \leq \alpha(L_0+1). \tag{8.36}$$

By (8.26) and (8.33), if $f_j(x-\alpha l) > \Delta_0$, then

$$f_j(x-\alpha l)$$

$$\le M_1\|x-y_j\| + M_1[\alpha(L_0+1)+\bar{\delta}_C+16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2]. \tag{8.37}$$

Since the function $v \to \|v-z\|^2$, $v \in X$ is convex it follows from (8.15), (8.16), and (8.34) that

$$\|x-z\|^2 - \|\sum_{j=1}^m w_j y_j - z\|^2$$

$$\ge \|x-z\|^2 - \sum_{j=1}^m w_j \|y_j - z\|^2$$

$$\ge \sum_{j=1}^m (w_j(\|x-z\|^2 - \|y_j-z\|^2)$$

$$\ge 2\alpha \sum_{j=1}^m w_j(f(y_j)-f(z))$$

$$-(12\tilde{M}+6)(\alpha\delta_f+\bar{\delta}_C+16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2)$$

$$-2\alpha(\delta_f+L\bar{\delta}_C+16\delta_C M_0 L\Delta^{-2}(6\tilde{M}+1)^2)$$

$$\ge 2\alpha(f(\sum_{j=1}^m w_j y_j)-f(z))$$

$$-(12\tilde{M}+6)(\alpha\delta_f+\bar{\delta}_C+16\delta_C L M_0 \Delta^{-2}(6\tilde{M}+1)^2)$$

$$-2\alpha(\delta_f+L\bar{\delta}_C+16\delta_C M_0 L\Delta^{-2}(6\tilde{M}+1)^2). \tag{8.38}$$

By (8.15), (8.16), and (8.35) and the convexity of the function $\|\cdot\|^2$,

$$\|z-x\|^2 - \|\sum_{j=1}^m w_j y_j - z\|^2$$

$$\ge \sum_{j=1}^m (w_j(\|z-x\|^2 - \|y_j-z\|^2)$$

$$\geq \sum_{j=1}^{m} w_j \|x - y_j\|^2$$

$$-\alpha(L_0 + 1)(6\tilde{M} + 1) - 16\delta_C L \Delta^{-2}(6\tilde{M} + 4)^3$$

$$-2(\alpha(L + 1) + \bar{\delta}_C)(6\tilde{M} + 6)$$

$$\geq \bar{\Delta} \sum_{j=1}^{m} \|x - y_j\|^2$$

$$-\alpha(L_0 + 1)(6\tilde{M} + 1) - 16\delta_C \Delta^{-2}(6\tilde{M} + 4)^3$$

$$- 2(\alpha(L + 1) + \bar{\delta}_C)(6\tilde{M} + 6). \tag{8.39}$$

In view of (8.10)–(8.13), and (8.35), for every $j \in \{1, \ldots, m\}$,

$$\|z - y_j\|^2 \leq \|z - x\|^2 + 5 \leq (\|z - x\| + 3)^2 \leq (2\tilde{M} + 3)^2,$$

$$\|z - y_j\| \leq 2\tilde{M} + 3, \ \|y_i\| \leq 3\tilde{M} + 2. \tag{8.40}$$

It follows from (8.15), (8.16), and (8.40) that

$$\|\sum_{j=1}^{m} w_j y_j\| \leq 3\tilde{M}. \tag{8.41}$$

By (8.20) and (8.41),

$$\|y\| \leq 3\tilde{M} + 1. \tag{8.42}$$

In view of (8.5), (8.20), (8.41), and (8.42),

$$|f(y) - f(\sum_{j=1}^{m} w_j y_j)| \leq L\|y - \sum_{j=1}^{m} w_j y_j\| \leq L\bar{\delta}_C. \tag{8.43}$$

By (8.15), (8.16), (8.20), and (8.40),

$$\|z - \sum_{j=1}^{m} w_j y_j\| \leq 2\tilde{M} + 3, \ \|z - y\| \leq 2\tilde{M} + 4,$$

$$|\|z-y\|^2 - \|\sum_{j=1}^{m} w_j y_j - z\|^2|$$

$$\leq \|\sum_{j=1}^{m} w_j y_j - y\|(\|\sum_{j=1}^{m} w_j y_j - z\| + \|y-z\|) \leq \bar{\delta}_C(4\tilde{M}+8). \tag{8.44}$$

It follows from (8.38), (8.43), and (8.44) that

$$\|x-z\| - \|z-y\|^2$$

$$\geq \|x-z\|^2 - \|\sum_{j=1}^{m} w_j y_j - z\|^2 - \bar{\delta}_C(4\tilde{M}+8)$$

$$\geq 2\alpha(f(\sum_{j=1}^{m} w_j y_j) - f(z)) - \bar{\delta}_C(4\tilde{M}+8)$$

$$-(12\tilde{M}+6)(\alpha\delta_f + \bar{\delta}_C + 16\delta_C L M_0 \Delta^{-2}(6\tilde{M}+1)^2)$$

$$-2\alpha(\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L \Delta^{-2}(6\tilde{M}+1)^2)$$

$$\geq 2\alpha(f(y) - f(z)) - L\bar{\delta}_C - \bar{\delta}_C(4\tilde{M}+8)$$

$$-(12\tilde{M}+6)(\alpha\delta_f + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2)$$

$$-2\alpha(\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L \Delta^{-2}(6\tilde{M}+1)^2).$$

By (8.39) and (8.44),

$$\|x-z\|^2 - \|z-y\|^2$$

$$\geq \|x-z\|^2 - \|\sum_{j=1}^{m} w_j y_j - z\|^2 - \bar{\delta}_C(4\tilde{M}+8)$$

$$\geq -\bar{\delta}_C(4\tilde{M}+8) + \bar{\Delta}\sum_{j=1}^{m} \|x-y_j\|^2$$

$$-\alpha(L_0+1)(6\tilde{M}+1) - 16\delta_C \Delta^{-2}(6\tilde{M}+4)^3$$

$$-2(\alpha(L+1) + \bar{\delta}_C)(6\tilde{M}+5).$$

Lemma 8.1 is proved.

8.4 Main Results

We use the notation and definitions introduced in Sections 8.1 and 8.2 and suppose that all the assumptions made there hold.

Proposition 8.2 *Let T be a natural number, $\Delta_0 \geq 0$,*

$$\gamma_T = M_1[\alpha(L+1) + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2]$$
$$+M_1[\Delta^{-1}(\delta_C(4\tilde{M}+8) + \alpha(L+1)(6\tilde{M}+1) + 16\delta_C\Delta^{-2}(6\tilde{M}+5)^3$$
$$+ 2(\alpha(L+1) + \bar{\delta}_C)(6\tilde{M}+8) + 4\tilde{M}^2T^{-1})]^{1/2}, \tag{8.45}$$

and at least one of the following conditions holds:

(a) $\Delta_0 = \Delta$;
(b) for each $i \in \{1, \dots, m\}$,

$$B(0, M_*) \cap \{x \in X : \ f_i(x) \leq -\Delta\} \neq \emptyset.$$

Assume that $\{x_t\}_{t=0}^{T} \subset X$, $\{l_t\}_{t=0}^{T-1} \subset X$, $l_{t,i} \in X$, $t = 0, \dots, T-1$, $i = 1, \dots, m$,

$$w_t = (w_t(1), \dots, w_t(m)) \in R^m, \ t = 1, \dots, T,$$

$$\sum_{i=1}^{m} w_t(i) = 1, \ t = 1, \dots, T, \tag{8.46}$$

$$w_t(i) \geq \bar{\Delta}, \ i = 1, \dots, m, \ t = 1, \dots, T, \tag{8.47}$$

and $y_{t,i} \in X$, $t = 1, \dots, T$, $i = 1, \dots, m$.

Assume that for all integers $t \in \{0, \dots, T-1\}$ *and all integers* $i \in \{1, \dots, m\}$,

$$\|l_t - f'(x_t)\| \leq \delta_f, \tag{8.48}$$

$$\text{if } f_i(x_t - \alpha l_t) \leq \Delta_0, \text{ then } y_{t+1,i} = x_t - \alpha l_t, \ l_{t,i} = 0, \tag{8.49}$$

if $f_j(x_t - \alpha l_t) > \Delta_0$, *then*

$$B(l_{t,i}, \delta_C) \cap \partial f_i(x_t - \alpha l_t) \neq \emptyset \tag{8.50}$$

(this implies that $l_{t,i} \neq 0$*),*

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2}l_{t,i}, \bar{\delta}_C) \tag{8.51}$$

and that

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}\| \leq \bar{\delta}_C. \tag{8.52}$$

Then the following assertions hold.

Assertion 1. Assume that $t \in \{0, \ldots, T-1\}$,

$$z \in B(0, M_*) \cap C \tag{8.53}$$

and that

$$x_t \in B(z, 2\tilde{M}). \tag{8.54}$$

Then

$$\|x_t - z\|^2 - \|x_{t+1} - z\|^2$$

$$\geq 2\alpha(f(x_{t+1}) - f(z)) - L\bar{\delta}_C - \bar{\delta}_C(4\tilde{M} + 8)$$

$$-(12\tilde{M} + 6)(\alpha\delta_f + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2)$$

$$- 2\alpha(\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L \Delta^{-2}(6\tilde{M} + 1)^2), \tag{8.55}$$

$$\|x_t - z\|^2 - \|x_{t+1} - z\|^2$$

$$\geq \bar{\Delta} \sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - \bar{\delta}_C(4\tilde{M} + 8)$$

$$-\alpha(L_0 + 1)(6\tilde{M} + 1) - 16\delta_C \Delta^{-2}(6\tilde{M} + 4)^3$$

$$- 2(\alpha(L + 1) + \bar{\delta}_C)(6\tilde{M} + 6), \tag{8.56}$$

and that for every $j \in \{1, \ldots, m\}$, *if* $f_j(x_t - \alpha l_t) \leq \Delta_0$, *then*

$$\|x_t - y_{t+1,j}\| \leq \alpha(L_0 + 1)$$

and if $f_j(x_t - \alpha l_t) > \Delta_0$, *then ,*

$$f_j(x_t - \alpha l_t) \leq M_1 \|x_t - y_{t+1,j}\|$$

$$+M_1[\alpha(L_0 + 1) + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2].$$

Assertion 2. Assume that

$$\textit{for every } z \in B(0, M_*) \cap C,\ \|x_t - z\| \le 2\tilde{M},\ t = 0, \dots, T.$$

Then

$$\min\{\max\{2\alpha(f(x_{t+1}) - \inf(f, C)) - L\bar{\delta}_C$$

$$-\bar{\delta}_C(4\tilde{M} + 8) - (12\tilde{M} + 6)(\alpha\delta_f + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2)$$

$$-2\alpha(\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L \Delta^{-2}(6\tilde{M} + 1)^2,$$

$$\bar{\Delta}\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - \bar{\delta}_C(4\tilde{M} + 8)$$

$$-\alpha(L_0 + 1)(6\tilde{M} + 1) - 16\delta_C \Delta^{-2}(6\tilde{M} + 4)^3$$

$$-2(\alpha(L + 1) + \bar{\delta}_C)(6\tilde{M} + 4)\} :$$

$$t = 0 \dots, T - 1\} \le 4\tilde{M}^2 T^{-1}.$$

Moreover, if $t \in \{0, \dots, T - 1\}$ *and*

$$\max\{2\alpha(f(x_{t+1}) - \inf(f, C)) - L\bar{\delta}_C$$

$$-\bar{\delta}_C(4\tilde{M} + 8) - (12\tilde{M} + 6)(\alpha\delta_f + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2)$$

$$-2\alpha(\delta_f + L\bar{\delta}_C) + 16\delta_C M_0 L \Delta^{-2}(6\tilde{M} + 1)^2,$$

$$\bar{\Delta}\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - \bar{\delta}_C(4\tilde{M} + 8)$$

$$-\alpha(L_0 + 1)(6\tilde{M} + 1) - 16\delta_C \Delta^{-2}(6\tilde{M} + 4)^3$$

$$-2(\alpha(L + 1) + \bar{\delta}_C)(6\tilde{M} + 6)\} \le 4\tilde{M}^2 T^{-1}$$

then

$$f(x_{t+1}) \le \inf(f, C) + (2\alpha)^{-1} L\bar{\delta}_C$$

$$+(2\alpha)^{-1}\bar{\delta}_C(4\tilde{M} + 8) + (6\tilde{M} + 3)\delta_f$$

$$+(6\tilde{M} + 3)\alpha^{-1}\bar{\delta}_C + 8\alpha^{-1}\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2)$$

$$+\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L \Delta^{-2}(6\tilde{M} + 1)^2) + 2\tilde{M}^2(T\alpha)^{-1},$$

$$f_i(x_t) \leq M_1\alpha(L+1) + \max\{\Delta_0, \gamma_T\},\ i = 1, \dots, m.$$

Assertion 3. Let $M_{*,0} > 0$, .

$$|f(u)| \leq M_{*,0},\ u \in B(0, M_*), \tag{8.57}$$

$$f(u) > M_{*,0} + 8 \textit{ for all } u \in X \setminus B(0, 2^{-1}\tilde{M}), \tag{8.58}$$

$$\delta_f(6\tilde{M} + 4) \leq 1,\ L\bar{\delta}_C \leq 1, \tag{8.59}$$

$$\alpha \geq \bar{\delta}_C(6\tilde{M} + 8),\ \alpha \geq L\bar{\delta}_C,$$

$$8(12\tilde{M} + 6)\alpha^{-1}M_0(L+1)\delta_C\Delta^{-2}(6\tilde{M} + 1)^2 \leq 1, \tag{8.60}$$

$t \in \{0, \dots, T-1\}$ *and that*

$$x_t \in B(z, 2\tilde{M}) \textit{ for every } z \in B(0, M_*) \cap C.$$

Then

$$x_{t+1} \in B(z, 2\tilde{M}) \textit{ for every } z \in B(0, M_*) \cap C.$$

Assertion 4. Let $r_0 \in (0, 1]$,

$$\{x \in X :\ f_i(x) \leq r_0,\ i = 1, \dots, m\} \subset B(0, M_*), \tag{8.61}$$

$$\Delta_0 \leq r_0, \tag{8.62}$$

$$M_1[\alpha(L+1) + \bar{\delta}_C + 16\delta_C M_0\Delta^{-2}(6\tilde{M} + 1)^2]$$

$$+M_1[\Delta^{-1}(\delta_C(4\tilde{M} + 8) + \alpha(L+1)(6\tilde{M} + 1) + 16\delta_C\Delta^{-2}(6\tilde{M} + 4)^3$$

$$+ 2(\alpha(L+1) + \bar{\delta}_C)(6\tilde{M} + 6))^{1/2} \leq r_0. \tag{8.63}$$

Assume that $t \in \{0, \dots, T-1\}$ *and that*

$$x_t \in B(z, 2\tilde{M}) \textit{ for every } z \in B(0, M_*) \cap C.$$

Then

$$x_{t+1} \in B(z, 2\tilde{M}) \textit{ for every } z \in B(0, M_*) \cap C.$$

Proof Assertion 1 follows from Lemma 6.1. We prove Assertion 2. By Assertion 1, for every $t \in \{0, \dots, T-1\}$ and every $z \in B(0, M_*) \cap C$, (8.55), (8.56) hold, and if $j \in \{1, \dots, m\}$ and $f_j(x_t - \alpha l_t) \le \Delta_0$, then

$$\|x_t - y_{t+1,j}\| \le \alpha(L_0 + 1) \tag{8.64}$$

and if $f_j(x_t - \alpha l_t) > \Delta_0$, then

$$f_j(x_t - \alpha l_t) \le M_1\|x_t - y_{t+1,j}\|$$

$$+ M_1[\alpha(L_0 + 1) + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2]. \tag{8.65}$$

Let

$$z \in B(0, M_*) \cap C. \tag{8.66}$$

By (8.55), (8.56), and (8.66) and the assumptions of Assertion 2,

$$4\tilde{M}^2 \ge \|z - x_0\|^2 \ge \|z - x_0\|^2 - \|z - x_T\|^2$$

$$= \sum_{t=0}^{T-1} (\|z - x_t\|^2 - \|z - x_{t+1}\|^2)$$

$$\ge \sum_{t=0}^{T-1} (\max\{2\alpha(f(x_{t+1}) - f(z)) - L\bar{\delta}_C - \bar{\delta}_C(4\tilde{M} + 8)$$

$$-(12\tilde{M} + 6)(\alpha\delta_f + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2)$$

$$-2\alpha(\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L \Delta^{-2}(6\tilde{M} + 1)^2),$$

$$\bar{\Delta} \sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - \bar{\delta}_C(4\tilde{M} + 8)$$

$$-\alpha(L_0 + 1)(6\tilde{M} + 1) - 16\delta_C \Delta^{-2}(6\tilde{M} + 4)^3 - 2(\alpha(L + 1) + \bar{\delta}_C)(6\tilde{M} + 6)\}).$$

Since z is an arbitrary element of $C \cap B(0, M_*) \cap C$ it follows from (8.3) and the relation above that

$$\min\{\max\{2\alpha(f(x_{t+1}) - \inf(f, C)) - L\bar{\delta}_C - \bar{\delta}_C(4\tilde{M} + 8)$$

$$-(12\tilde{M} + 6)(\alpha\delta_f + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2)$$

$$-2\alpha(\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L \Delta^{-2}(6\tilde{M} + 1)^2),$$

$$\bar{\Delta}\sum_{i=1}^{m}\|x_t - y_{t+1,i}\|^2 - \bar{\delta}_C(4\tilde{M}+8)$$

$$-\alpha(L_0+1)(6\tilde{M}+1) - 16\delta_C\Delta^{-2}(6\tilde{M}+4)^3$$

$$-2(\alpha(L+1)+\bar{\delta}_C)(6\tilde{M}+6)\}: \ t = 0,\dots,T-1\} \le 4\tilde{M}^2T^{-1}. \tag{8.67}$$

Let $\{t \in \{0,\dots,T-1\}$ and

$$\min\{\max\{2\alpha(f(x_{t+1}) - \inf(f,C)) - L\bar{\delta}_C - \bar{\delta}_C(4\tilde{M}+8)$$

$$-(12\tilde{M}+6)(\alpha\delta_f + \bar{\delta}_C + 16\delta_C M_0\Delta^{-2}(6\tilde{M}+1)^2)$$

$$-2\alpha(\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L\Delta^{-2}(6\tilde{M}+1)^2),$$

$$\bar{\Delta}\sum_{i=1}^{m}\|x_t - y_{t+1,i}\|^2 - \bar{\delta}_C(4\tilde{M}+8)$$

$$-\alpha(L_0+1)(6\tilde{M}+1) - 16\delta_C\Delta^{-2}(6\tilde{M}+4)^3$$

$$-2(\alpha(L+1)+\bar{\delta}_C)(6\tilde{M}+6)\} \le 4\tilde{M}^2T^{-1}. \tag{8.68}$$

By (8.68),

$$f(x_{t+1}) \le \inf(f,C) + (2\alpha)^{-1}L\bar{\delta}_C + (2\alpha)^{-1}\bar{\delta}_C(4\tilde{M}+8)$$

$$+(6\tilde{M}+3)\delta_f + (6\tilde{M}+3)\alpha^{-1}\bar{\delta}_C + 8\alpha^{-1}\delta_C M_0\Delta^{-2}(6\tilde{M}+1)^2$$

$$+\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L\Delta^{-2}(6\tilde{M}+1)^2 + 2\tilde{M}^2(T\alpha)^{-1}. \tag{8.69}$$

In view of (8.68), for $j = 1,\dots,m$,

$$\|x_t - y_{t+1,i}\| \le [\Delta^{-1}(\bar{\delta}_C(4\tilde{M}+8)$$

$$+\alpha(L_0+1)(6\tilde{M}+1) + 16\delta_C\Delta^{-2}(6\tilde{M}+5)^3$$

$$+2(\alpha(L+1)+\bar{\delta}_C)(6\tilde{M}+8) + 4\tilde{M}^2T^{-1}]^{1/2}. \tag{8.70}$$

Let $j \in \{1,\dots,m\}$. Assume that

$$f_j(x_t - \alpha l_t) > \Delta_0. \tag{8.71}$$

In view of (8.45), (8.65), and (8.71),

$$f_j(x_t - \alpha l_t) \le M_1\|x_t - y_{t+1,j}\|$$

$$+M_1[\alpha(L_0+1) + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2]$$

$$\le M_1[\alpha(L_0+1) + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2]$$

$$+M_1[\Delta^{-1}(\bar{\delta}_C(4\tilde{M}+8)$$

$$+\alpha(L_0+1)(6\tilde{M}+1) + 16\delta_C \Delta^{-2}(6\tilde{M}+5)^3$$

$$+2(\alpha(L+1)+\bar{\delta}_C)(6\tilde{M}+8) + 4\tilde{M}^2 T^{-1}]^{1/2} = \gamma_T.$$

Thus

$$f_j(x_t - \alpha l_t) \le \max\{\gamma_T, \Delta_0\},\ \ j = 1, \dots, m. \tag{8.72}$$

Clearly,

$$\|x_t\| \le 3\tilde{M}. \tag{8.73}$$

It follows from (8.5), (8.10), (8.14), and (8.73),

$$\|l_t\| \le L + 1, \tag{8.74}$$

$$\|x_t - \alpha l_t\| \le \|x_t\| + \alpha(L+1) \le 3\tilde{M} + 1. \tag{8.75}$$

By (8.9) and (8.73)–(8.75), for all $j = 1, \dots, m$,

$$|f_j(x_t - \alpha l_t) - f_j(x_t)| \le M_1\alpha\|l_t\| \le M_1\alpha(L+1),$$

$$f_j(x_t) \le M_1\alpha(L+1) + \max\{\gamma_T, \Delta_0\}.$$

Assertion 2 is proved.

We prove Assertion 3. Let

$$z \in B(0, M_*) \cap C.$$

We may assume without loss of generality that

$$\|x_{t+1} - z\| > \|x_t - z\|.$$

By (8.11), (8.55), (8.57), (8.59), (8.60), and Assertion 1,

$$f(x_{t+1}) \le f(z) + (2\alpha)^{-1} L\bar{\delta}_C + (2\alpha)^{-1}\bar{\delta}_C(4\tilde{M}+8)$$

$$+(6\tilde{M}+3)\delta_f+(6\tilde{M}+3)\alpha^{-1}\bar{\delta}_C$$

$$+8\alpha^{-1}\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2)(12\tilde{M}+6)$$

$$+\delta_f+L\bar{\delta}_C+16\delta_C M_0 L\Delta^{-2}(6\tilde{M}+1)^2 \le M_{*,0}+8.$$

Together with (8.58) this implies that

$$\|x_{t+1}\| \le \tilde{M}/2.$$

This implies that

$$\|z-x_{t+1}\| \le 2\tilde{M}.$$

Assertion 3 is proved.

We prove Assertion 4. Let

$$z \in B(0, M_*) \cap C.$$

Then

$$\|z-x_t\| \le 2\tilde{M}. \tag{8.76}$$

We may assume without loss of generality that

$$\|x_{t+1}-z\| > \|x_t-z\|. \tag{8.77}$$

Let $j=1,\dots,m$. By (8.56), (8.67), (8.77), and Assertion 1,

$$\|x_t-y_{t+1,j}\| \le [\Delta^{-1}(\bar{\delta}_C(4\tilde{M}+8)$$

$$+\alpha(L_0+1)(6\tilde{M}+1)+16\delta_C\Delta^{-2}(6\tilde{M}+4)^3$$

$$+2(\alpha(L+1)+\bar{\delta}_C)(6\tilde{M}+6)]^{1/2}. \tag{8.78}$$

Assertion 1, (8.63) and (8.78) imply that if

$$f_j(x_t-\alpha l_t) > \Delta_0,$$

then

$$f_j(x_t-\alpha l_t) \le M_1[\alpha(L_0+1)+\bar{\delta}_C+16\delta_C M_0\Delta^{-2}(6\tilde{M}+1)^2]$$

$$+M_1[\Delta^{-1}(\bar{\delta}_C(4\tilde{M}+8)$$

$$+\alpha(L_0+1)(6\tilde{M}+1)+16\delta_C\Delta^{-2}(6\tilde{M}+4)^3$$

$$+2(\alpha(L+1)+\bar{\delta}_C)(6\tilde{M}+6)]^{1/2} \le r_0. \tag{8.79}$$

By (8.68) and (8.79), for $j=1,\ldots,m$,

$$f_j(x_t-\alpha l_t) \le \max\{\Delta_0, r_0\} = r_0.$$

Combined with (8.61) this implies that

$$\|x_t-\alpha l_t\| \le M_*. \tag{8.80}$$

In view of (8.5), (8.48), (8.75), and (8.76),

$$\|l_t\| \le L+1. \tag{8.81}$$

It follows from (8.7), (8.80), and (8.81) that

$$\|x_t\| \le M_*+\alpha(L+1) \le M_*+1. \tag{8.82}$$

By (8.78), for all $j=1,\ldots,m$,

$$\|x_t-y_{t+1,j}\| \le r_0 \le 1. \tag{8.83}$$

In view of (8.52) and (8.83),

$$\|x_t-x_{t+1}\| \le \|x_t-\sum_{i=1}^m w_{t+1}(i)y_{t+1,i}\|$$

$$+\|\sum_{i=1}^m w_{t+1}(i)y_{t+1,i}-x_{t+1}\| \le r_0+1 \le 2.$$

Combined with (8.75) and (8.82) this implies that

$$\|x_{t+1}\| \le \|x_t\|+2 \le M_*+3,$$

$$\|z-x_{t+1}\| \le 2M_*+3 < 2\tilde{M}.$$

Assertion 4 is proved. This completes the proof of Proposition 8.2.

For Assertion 2 the right-hand side of the estimation of $f(x_{t+1})$ is at the same order as $\alpha^{-1}(\delta_C+\bar{\delta}_C)$. If δ_f, δ_C, $\bar{\delta}_C$ are sufficiently small, then γ_T is the same order as $\alpha^{1/2}$. Since T can be arbitrary large α should be at the same order as $(\delta_C+\bar{\delta}_C)^{2/3}$. The T should be at the same order as $\max\{\delta_C^{-1}, \bar{\delta}_C^{-1}\}$.

Proposition 6.2 easily implies the following result.

Theorem 8.3 *Let* $\Delta_0 \geq 0$, T *be a natural number,*

$$\gamma_T = M_1[\alpha(L+1) + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2]$$

$$+M_1[\Delta^{-1}(\delta_C(4\tilde{M}+8) + \alpha(L+1)(6\tilde{M}+1) + 16\delta_C\Delta^{-2}(6\tilde{M}+5)^3$$

$$+2(\alpha(L+1)+\bar{\delta}_C)(6\tilde{M}+5) + 4\tilde{M}^2T^{-1})]^{1/2}$$

and let at least one of the following conditions hold:

(a) $\Delta_0 = \Delta$;
(b) for each $i \in \{1, \ldots, m\}$,

$$B(0, M_*) \cap \{x \in X : \ f_i(x) \leq -\Delta\} \neq \emptyset.$$

Assume that at least of the following conditions holds:

(a) there exists $M_{*,0} > 0$ *such that*

$$|f(u)| \leq M_{*,0}, \ u \in B(0, M_*),$$

$$f(u) > M_{*,0} + 8 \text{ for all } u \in X \setminus B(0, 2^{-1}\tilde{M}/2),$$

$$\delta_f(6\tilde{M}+4) \leq 1, \ L\bar{\delta}_C \leq 1,$$

$$\alpha \geq \bar{\delta}_C(6\tilde{M}+8), \ \alpha \geq L\bar{\delta}_C,$$

$$8(12\tilde{M}+6)\alpha^{-1}M_0(L+1)\delta_C\Delta^{-2}(6\tilde{M}+1)^2 \leq 1;$$

(b) there exists $r_0 \in (0, 1]$ *such that*

$$\{x \in X : \ f_i(x) \leq r_0, \ i = 1, \ldots, m\} \subset B(0, M_*),$$

$$\Delta_0 \leq r_0,$$

$$M_1[\alpha(L+1) + \bar{\delta}_C + 16\delta_C M_0\Delta^{-2}(6\tilde{M}+1)^2]$$

$$+M_1(\Delta^{-1}(\delta_C(4\tilde{M}+8) + \alpha(L+1)(6\tilde{M}+1) + 16\delta_C\Delta^{-2}(6\tilde{M}+5)^3$$

$$+\alpha(L+1) + \bar{\delta}_C)(6\tilde{M}+6))^{1/2} \leq r_0.$$

Let $\{x_t\}_{t=0}^T \subset X$, $\{l_t\}_{t=0}^{T-1} \subset X$, $l_{t,i} \in X$, $t = 0, \ldots, T-1$, $i = 1, \ldots, m$,

$$w_t = (w_t(1), \dots, w_t(m)) \in R^m,\ t = 1, \dots, T,$$

$$\sum_{i=1}^{m} w_t(i) = 1,\ t = 1, \dots, T,$$

$$w_t(i) \geq \bar{\Delta},\ i = 1, \dots, m,\ t = 1, \dots, T,$$

and $y_{t,i} \in X$, $t = 1, \dots, T$, $i = 1, \dots, m$.

$$\|x_0\| \leq \tilde{M}.$$

Assume that for all integers $t \in \{0, \dots, T-1\}$ *and all integers* $i \in \{1, \dots, m\}$,

$$\|l_t - f'(x_t)\| \leq \delta_f,$$

$$\text{if } f_i(x_t - \alpha l_t) \leq \Delta_0,\ \text{then } y_{t+1,i} = x_t - \alpha l_t,\ l_{t,i} = 0,$$

if $f_i(x_t - \alpha l_t) > \Delta_0$, *then*

$$B(l_{t,i}, \delta_C) \cap \partial f_i(x_t - \alpha l_t) \neq \emptyset$$

(this implies that $l_{t,i} \neq 0$*),*

$$y_{t+1,i} \in B(x_t - \alpha l_t - f_i(x_t - \alpha l_t)\|l_{t,i}\|^{-2} l_{t,i}, \bar{\delta}_C)$$

and that

$$\|x_{t+1} - \sum_{i=1}^{m} w_{t+1}(i) y_{t+1,i}\| \leq \bar{\delta}_C.$$

Then

$$\|x_t\| \leq 3\tilde{M},\ t = 0, \dots, T,$$

$$\min\{\max\{2\alpha(f(x_{t+1}) - \inf(f, C)) - L\bar{\delta}_C$$

$$-\bar{\delta}_C(4\tilde{M} + 8) - (12\tilde{M} + 6)(\alpha\delta_f + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M} + 1)^2)$$

$$-2\alpha(\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L \Delta^{-2}(6\tilde{M} + 1)^2,$$

$$\bar{\Delta} \sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - \bar{\delta}_C(4\tilde{M} + 8)$$

$$-\alpha(L_0+1)(6\tilde{M}+1) - 16\delta_C \Delta^{-2}(6\tilde{M}+5)^3(M_0+1)$$

$$-2(\alpha(L+1)+\bar{\delta}_C)(6\tilde{M}+4) : t = 0\ldots, T-1\} \le 4\tilde{M}^2 T^{-1}.$$

Moreover, if $t \in \{0, \ldots, T-1\}$ *and*

$$\max\{2\alpha(f(x_{t+1}) - \inf(f, C)) - L\bar{\delta}_C$$

$$-\bar{\delta}_C(4\tilde{M}+8) - (12\tilde{M}+6)(\alpha\delta_f + \bar{\delta}_C + 16\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2)$$

$$-2\alpha(\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L \Delta^{-2}(6\tilde{M}+1)^2),$$

$$\bar{\Delta}\sum_{i=1}^{m} \|x_t - y_{t+1,i}\|^2 - \bar{\delta}_C(4\tilde{M}+8)$$

$$-\alpha(L_0+1)(6\tilde{M}+1) - 16\delta_C \Delta^{-2}(6\tilde{M}+5)^3$$

$$-2(\alpha(L+1)+\bar{\delta}_C)(6\tilde{M}+4)\} \le 4\tilde{M}^2 T^{-1}.$$

then

$$f(x_{t+1}) \le \inf(f, C) + (2\alpha)^{-1} L\bar{\delta}_C$$

$$+(2\alpha)^{-1}\bar{\delta}_C(4\tilde{M}+8) + (6\tilde{M}+3)\delta_f + (6\tilde{M}+3)\alpha^{-1}\bar{\delta}_C$$

$$+8\alpha^{-1}\delta_C M_0 \Delta^{-2}(6\tilde{M}+1)^2$$

$$+\delta_f + L\bar{\delta}_C + 16\delta_C M_0 L \Delta^{-2}(6\tilde{M}+1)^2) + 2\tilde{M}^2(T\alpha)^{-1},$$

$$f_i(x_t) \le M_1\alpha(L+1) + \max\{\Delta_0, \gamma_T\},\ i = 1, \ldots, m.$$

Chapter 9
A Class of Nonsmooth Convex Optimization Problems

In this chapter we study the convergence of the projected subgradient method for a class of constrained optimization problems in a Hilbert space. For this class of problems, an objective function is assumed to be convex but a set of admissible points is not necessarily convex. Our goal is to obtain an ϵ-approximate solution in the presence of computational errors, where ϵ is a given positive number.

9.1 Preliminaries

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with a inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$.

For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) = \inf\{\|x - y\| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$,

$$B_X(x, r) = \{y \in X : \ \|y - x\| \le r\}$$

and use the notation $B(x, r) = B_X(x, r)$ is the space X is understood.

Let C be a nonempty closed subset of X, U be an open convex subset of X and that

$$C \subset U. \tag{9.1}$$

Assume that $f : U \to R^1$ is a convex continuous function which is Lipschitz on all bounded subsets of U.

For each point $x \in U$ and each number $\epsilon \ge 0$ let

A. J. Zaslavski, *Optimization on Solution Sets of Common Fixed Point Problems*, Springer Optimization and Its Applications 178,
https://doi.org/10.1007/978-3-030-78849-0_9

$$\partial f(x) = \{l \in X : \ f(y) - f(x) \ge \langle l, y - x\rangle \text{ for all } y \in U\} \tag{9.2}$$

be the subdifferential of f at x and let

$$\partial_\epsilon f(x) = \{l \in X : \ f(y) - f(x) \ge \langle l, y - x\rangle - \epsilon \text{ for all } y \in U\} \tag{9.3}$$

be the ϵ-subdifferential of f at x.

Assume that

$$\lim_{\|x\| \to \infty, x \in U} f(x) = \infty. \tag{9.4}$$

It means that for each $M_0 > 0$ there exists $M_1 > 0$ such that if a point $x \in U$ satisfies the inequality $\|x\| \ge M_1$, then $f(x) > M_0$.

We consider the problem

$$f(x) \to \min, \ \ x \in C.$$

Define

$$\inf(f, C) = \inf\{f(z) : \ z \in C\}.$$

Since the function f is Lipschitz on all bounded subsets of the set U it follows from (9.4) that $\inf(f, C)$ is finite.

Set

$$C_{min} = \{x \in C : \ f(x) = \inf(f, C)\}. \tag{9.5}$$

It is well-known that if the set C is convex, then the set C_{min} is nonempty. Clearly, the set $C_{min} \neq \emptyset$ if the space X is finite-dimensional.

In this chapter we assume that

$$C_{min} \neq \emptyset. \tag{9.6}$$

It is clear that C_{min} is a closed subset of X.

As in particular cases we are interested in the case where $U = X$ and where U is bounded.

Denote by $\mathcal{M}_U$ (or $\mathcal{M}$ if the set U is understood) the set of all mappings $P : X \to U$ such that

$$P(z) = z \text{ for all } x \in C. \tag{9.7}$$

$$\|x - P(z)\| \le \|z - x\| \text{ for all } x \in C \text{ and all } z \in X. \tag{9.8}$$

For each $P \in \mathcal{M}$, set $P^0(x) = x$, $x \in X$.

Fix

$$\theta_0 \in C. \tag{9.9}$$

Define

$$U_0 = \{x \in U : \ f(x) \le f(\theta_0) + 4\}. \tag{9.10}$$

In view of (9.4), there exists a number $\bar{K} > 1$ such that

$$U_0 \subset B_X(0, \bar{K}). \tag{9.11}$$

Since the function f is Lipschtiz on all bounded subsets of the space U there exists a number $\bar{L} > 1$ such that

$$|f(z_1) - f(z_2)| \le \bar{L}\|z_1 - z_2\| \text{ for all } z_1, z_2 \in U \cap B_X(0, \bar{K} + 4). \tag{9.12}$$

Let $\mathcal{A} \subset \mathcal{M}$ be nonempty. We say that the family $\mathcal{A}$ is (C)-quasi-contractive if the following assumption holds:

(A1) For each $M > 0$ and each $r > 0$ there exists $\delta > 0$ such that for each $x \in B_X(0, M)$ satisfying $d(x, C) \ge r$ and each $z \in B_X(0, M) \cap C$ and each $P \in \mathcal{A}$, we have

$$\|P(x) - z\| \le \|x - z\| - \delta.$$

Let $P \in \mathcal{M}$ and $\mathcal{A} = \{P\}$. If $\{P\}$ is (C)-quasi-contractive in Section 3.13, Chapter 3 of [94] it is called a (C)-quasi-contractive mapping. It was shown there that in appropriate subspaces of $\mathcal{M}$ most of mappings are (C)-quasi-contractive. More precisely, a subspace of $\mathcal{M}$ is equipped with an appropriate complete metric and it is shown that the subspace contains a G_δ everywhere dense set such that all its elements are (C)-quasi-contractive. In this case we say that a generic (typical) mapping is (C)-quasi-contractive. Many generic results in nonlinear analysis are collected in [69].

Let $\mathcal{B}$ be a nonempty set of sequences $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}$.

In this chapter we also use the following assumption.

(A2) For each $\epsilon > 0$ and each $K > 0$ there exist $\delta > 0$ and a natural number m_0 such that:

(i) for each $x, y \in B_X(0, K)$ satisfying $\|x - y\| \le \delta$, each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $t \ge 0$, $\|P_t(x) - P_t(y)\| \le \epsilon$ holds;

(ii) for each $x \in B_X(0, K)$, each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $q \ge 0$,

$$d(P_{q+m_0} \dots P_{q+1} P_q(x), C) \le \epsilon.$$

In order to solve the optimization problem

$$f(x) \to \min, \ x \in C$$

we use an extended subgradient algorithm with computational errors.

Subgradient Projection Algorithm Fix $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}$ and a sequence $\{\alpha_i\}_{i=0}^{\infty} \subset (0, 1]$ satisfying

$$\lim_{i\to\infty} \alpha_i = 0, \ \sum_{i=0}^{\infty} \alpha_i = \infty.$$

We assume that $\delta > 0$ is a computational error produced by our computer system.

Initialization: select an arbitrary $x_0 \in U$.

Iterative step: given a current iteration vector $x_k \in U$ calculate

$$\xi_k \in \partial f(x_k) + B_X(0, \delta)$$

and the next iteration vector

$$x_{k+1} \in B_X(P_k(x_k - \alpha_k \xi_k), \alpha_t \delta).$$

It should be mentioned that in [96] this algorithm is considered in the case when all operators P satisfy $P(X) = C$.

9.2 Approximate Solutions of Problems on Bounded Sets

We use the notation and definitions introduced in Section 9.1. Assume that the set C is bounded. Then f is bounded on C. We assume without loss of generality (see (9.9) that

$$f(\theta_0) > \sup\{f(z) : \ z \in C\} - 1. \tag{9.13}$$

In view of (9.10)–(9.12),

$$C \subset U_0 \subset B_X(0, \bar{K}). \tag{9.14}$$

For each $r \geq 0$ set

$$U_r = \{x \in U : \ B(x, r) \subset U\}, \tag{9.15}$$

$$\mathcal{M}_r = \{T \in \mathcal{M} : \ T(X) \subset U_r\}. \tag{9.16}$$

In this chapter we prove the following results.

Theorem 9.1 *Let $\bar{r} > 0$, $\mathcal{B}$ be a nonempty set of sequences $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{\bar{r}}$ such that*

$$P_t(X) \subset B_X(0, \bar{K}),\ t = 0, 1, \ldots \tag{9.17}$$

and at least one of the following conditions holds:

there exists a nonempty (C)-quasi-contractive set $\mathcal{A} \subset \mathcal{M}$ such that

$$\mathcal{B} \subset \{\{P_t\}_{t=0}^{\infty} :\ P_t \in \mathcal{A},\ t = 0, 1, \ldots\};$$

$\mathcal{B}$ satisfies (A2).

Let $\{\alpha_i\}_{i=0}^{\infty} \subset (0, 1]$ satisfy

$$\lim_{i\to\infty} \alpha_i = 0, \tag{9.18}$$

$$\sum_{i=0}^{\infty} \alpha_i = \infty \tag{9.19}$$

and let $\epsilon > 0$. Then there exist $\delta \in (0, 2^{-1}\bar{r})$ and a pair of natural numbers n_0, n_1 such that the following assertion holds.

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, $\{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}}$, $\{\xi_t\}_{t=0}^{\infty} \subset X$,

$$\|x_0\| \leq \bar{K} + 1,$$

for all integers $t \geq 0$,

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \neq \emptyset,$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \leq \alpha\delta.$$

Then

1. *for all integers $t \geq n_0$, $d(x_t, C) \leq \epsilon \bar{L}^{-1}$;*
2. $\liminf_{t\to\infty} f(x_t) \leq \inf(f, C) + \epsilon$*;*
3. *for all integers $n \geq n_0$,*

$$f((\sum_{i=0}^{n-1} \alpha_i)^{-1} \sum_{t=0}^{n-1} \alpha_t x_t) - \inf(f, C) < \epsilon;$$

4. *for all integers $n \geq n_0$,*

$$d((\sum_{i=0}^{n-1} \alpha_i)^{-1} \sum_{t=0}^{n-1} \alpha_t x_t, C) < \epsilon.$$

Theorem 9.2 *Let $\bar{r} > 0$, $\mathcal{B}$ be a nonempty set of sequences $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{\bar{r}}$ such that*

$$P_t(X) \subset B(0, \bar{K}),\ t = 0, 1, \dots \tag{9.20}$$

and at least one of the following conditions holds:

there exists a nonempty (C)-quasi-contractive set $\mathcal{A} \subset \mathcal{M}$ such that

$$\mathcal{B} \subset \{\{P_t\}_{t=0}^{\infty} :\ P_t \in \mathcal{A},\ t = 0, 1, \dots\};$$

$\mathcal{B}$ satisfies (A2).

Let $\epsilon > 0$. Then there exist $\beta_0 > 0$ such that for each $\beta_1 \in (0, \beta_0)$ there exist $\delta \in (0, 2^{-1}\bar{r}]$ and a pair of natural numbers n_0, n_1 such that the following assertion holds.

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, $\{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}}$, $\{\xi_t\}_{t=0}^{\infty} \subset X$,

$$\|x_0\| \le \bar{K} + 1,$$

for all integers $t \ge 0$,

$$\alpha_t \in [\beta_1, \beta_0],$$

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \ne \emptyset,$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \le \delta.$$

Then

1. *for all integers $t \ge n_0$, $d(x_t, C) \le \epsilon \bar{L}^{-1}$;*
2. *for all integers $m > n \ge 0$ satisfying $m - n \ge n_0$,*

$$\min\{f(x_t) :\ t = n, \dots, m-1\},\ f((\sum_{i=n}^{m-1} \alpha_i)^{-1} \sum_{t=n}^{m-1} \alpha_t x_t)$$

$$\le (\sum_{i=n}^{m-1} \alpha_i)^{-1} \sum_{t=n}^{m-1} \alpha_t f(x_t) < \inf(f, C) + \epsilon;$$

3. *if C is convex, then for all integers $n \ge n_1$,*

$$d((\sum_{i=0}^{n-1} \alpha_i)^{-1} \sum_{t=0}^{n-1} \alpha_t x_t, C) < \epsilon.$$

9.3 Approximate Solutions of Problems on Unbounded Sets

We use the notation and definitions introduced in Section 9.1. Recall that for each $r \ge 0$,

$$U_r = \{x \in U : B(x, r) \subset U\},$$

$$\mathcal{M}_r = \{T \in \mathcal{M} : T(X) \subset U_r\}.$$

In this chapter we prove the following results.

Theorem 9.3 *Let $\bar{r} \in (0, 1]$, $\mathcal{B}$ be a nonempty set of sequences $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{\bar{r}}$ such that and at least one of the following conditions holds:*

there exists a nonempty (C)-quasi-contractive set $\mathcal{A} \subset \mathcal{M}$ such that

$$\mathcal{B} \subset \{\{P_t\}_{t=0}^{\infty} : P_t \in \mathcal{A}, \ t = 0, 1, \ldots\};$$

$\mathcal{B}$ satisfies (A2).

Let $M, \epsilon > 0$,

$$M > 4\bar{K} + 8, \ L_0 > \bar{L}, \tag{9.21}$$

$$\{x \in U : f(x) < \inf(f, C) + 8\bar{L}\} \subset B_X(0, M/2 - 1), \tag{9.22}$$

$$|f(z_1) - f(z_2)| \le L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in U \cap B_X(0, 3M + 4), \tag{9.23}$$

$\{\alpha_i\}_{i=0}^{\infty} \subset (0, 1]$ satisfy

$$\lim_{i\to\infty} \alpha_i = 0, \tag{9.24}$$

$$\sum_{i=0}^{\infty} \alpha_i = \infty \tag{9.25}$$

and for all integers $i \ge 0$,

$$\alpha_i \le (L_0 + 2)^{-2}. \tag{9.26}$$

Then there exist $\delta \in (0, 2^{-1}\bar{r}]$ and a pair of natural numbers $n_0 < n_1$ such that the following assertion holds.

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, $\{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}}$, $\{\xi_t\}_{t=0}^{\infty} \subset X$,

$$\|x_0\| \le M,$$

for all integers $t \ge 0$,

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \neq \emptyset,$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \leq \alpha_t \delta.$$

Then

1. $\|x_t\| \leq 3M$, $t = 0, 1, \dots$ *and for all integers* $t \geq n_0$, $d(x_t, C) \leq \epsilon L_0$;
2. $\liminf_{t\to\infty} f(x_t) \leq \inf(f, C) + \epsilon$;
3. *for all integers* $m > n \geq 0$ *satisfying* $m - n \geq n_0$,

$$f((\sum_{i=n}^{m-1} \alpha_i)^{-1} \sum_{t=n}^{m-1} \alpha_t x_t) < \inf(f, C) + \epsilon;$$

3. *if C is convex, then for all integers* $n \geq n_1$,

$$d((\sum_{i=0}^{n-1} \alpha_i)^{-1} \sum_{t=0}^{n-1} \alpha_t x_t, C) < \epsilon.$$

Theorem 9.4 *Let* $\bar{r} \in (0, 1]$, $\mathcal{B}$ *be a nonempty set of sequences* $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{\bar{r}}$ *such that and at least one of the following conditions holds:*

there exists a nonempty (C)*-quasi-contractive set* $\mathcal{A} \subset \mathcal{M}$ *such that*

$$\mathcal{B} \subset \{\{P_t\}_{t=0}^{\infty} : \ P_t \in \mathcal{A}, \ t = 0, 1, \dots\};$$

$\mathcal{B}$ *satisfies (A2).*

Let $M, \epsilon > 0$,

$$M > 4\bar{K} + 8, \ L_0 > \bar{L},$$

$$\{x \in U : \ f(x) < \inf(f, C) + 8\bar{L}\} \subset B_X(0, M/2 - 1), \tag{9.27}$$

$$|f(z_1) - f(z_2)| \leq \bar{L}_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in U \cap B_X(0, 3M + 4). \tag{9.28}$$

Then there exists $\beta_0 > 0$ *such that for each* $\beta_1 \in (0, \beta_0)$ *there exist* $\delta \in (0, 2^{-1}\bar{r}]$ *and a pair of natural numbers* $n_0 \leq n_1$ *such that the following assertion holds.*

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, $\{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}}$, $\{\xi_t\}_{t=0}^{\infty} \subset X$,

$$\|x_0\| \leq M,$$

for all integers $t \geq 0$,

$$\alpha_t \in [\beta_1, \beta_0],$$

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \neq \emptyset,$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \le \delta.$$

Then

1. $\|x_t\| \le 3M$, $t = 0, 1, \ldots$ *and for all integers* $t \ge n_0$, $d(x_t, C) \le \epsilon L_0^{-1}$;
2. $\liminf_{t\to\infty} f(x_t) \le \inf(f, C) + \epsilon$;
3. *for all integers* $m > n \ge 0$ *satisfying* $m - n \ge n_0$,

$$f((\sum_{i=n}^{m-1} \alpha_i)^{-1} \sum_{t=n}^{m-1} \alpha_t x_t) < \inf(f, C) + \epsilon;$$

3. *if* C *is convex, then for all integers* $n \ge n_1$,

$$d((\sum_{i=0}^{n-1} \alpha_i)^{-1} \sum_{t=0}^{n-1} \alpha_t x_t, C) < \epsilon.$$

9.4 Convergence to the Set of Minimizers

We use the notation and definitions introduced in Section 9.1 and the following assumption.

(A3) For every positive number ϵ there exists $\delta > 0$ such that if a point $x \in C$ satisfies the inequality $f(x) \le \inf(f, C) + \delta$, then $d(x, C_{min}) \le \epsilon$.

It is clear that (A3) holds if the space X is finite-dimensional.

If (A3) holds, then for every number $\epsilon \in (0, \infty)$ set

$$\phi(\epsilon) = \sup\{\delta \in (0, 1] : \text{ if } x \in C \text{ satisfies } f(x) \le \inf(f, C) + \delta,$$

$$\text{then } d(x, C_{min}) \le \min\{1, \epsilon\}\}. \tag{9.29}$$

In view of (A3), $\phi(\epsilon)$ is well-defined for every positive number $\epsilon > 0$. Recall that for each $r > 0$

$$U_r = \{x \in U : B(x, r) \subset U\}, \tag{9.30}$$

$$\mathcal{M}_r = \{T \in \mathcal{M} : T(X) \subset U_r\}. \tag{9.31}$$

In this chapter we prove the following results.

Theorem 9.5 *Let* $\bar{r} \in (0, 1]$, *(A3) hold,* $\mathcal{B}$ *be a nonempty set of sequences* $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{\bar{r}}$ *such that and at least one of the following conditions holds:*

there exists a nonempty (C)*-quasi-contractive set* $\mathcal{A} \subset \mathcal{M}$ *such that*

$$\mathcal{B} \subset \{\{P_t\}_{t=0}^{\infty} : \ P_t \in \mathcal{A}, \ t = 0, 1, \dots\};$$

$\mathcal{B}$ satisfies (A2).

Let $\{\alpha_i\}_{i=0}^{\infty} \subset (0, 1]$ *satisfy for all integers* $i \geq 0$,

$$\alpha_i \leq \bar{r}/8, \tag{9.32}$$

$$\lim_{i\to\infty} \alpha_i = 0, \ \sum_{i=0}^{\infty} \alpha_i = \infty \tag{9.33}$$

and $M, \epsilon > 0$*. Then there exist* $\delta \in (0, 2^{-1}\bar{r})$ *and a natural number* n_0 *such that the following assertion holds.*

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$*, an integer* $n \geq n_0$*,* $\{x_t\}_{t=0}^{n} \subset U_{\bar{r}}$,

$$\|x_0\| \leq M,$$

$$v_k \in \partial_\delta f(x_k) \setminus \{0\}, \ k = 0, 1, \dots, n-1,$$

$$\{\eta_k\}_{k=0}^{n-1}, \ \{\xi_k\}_{k=0}^{n-1} \subset B_X(0, \delta),$$

and that for $k = 0, \dots, n-1$ *,*

$$x_{k+1} = P_k(x_k - \alpha_k \|v_k\|^{-1} v_k - \alpha_k \xi_k) - \alpha_k \eta_k.$$

Then the inequality $d(x_k, C_{min}) \leq \epsilon$ *hods for all integers* k *satisfying* $n_0 \leq k \leq n$*.*

Theorem 9.6 *Let* $\bar{r} \in (0, 1]$*, (A3) hold,* $\mathcal{B}$ *be a nonempty set of sequences* $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{\bar{r}}$ *such that and at least one of the following conditions holds:*

there exists a nonempty (C)*-quasi-contractive set* $\mathcal{A} \subset \mathcal{M}$ *such that*

$$\mathcal{B} \subset \{\{P_t\}_{t=0}^{\infty} : \ \ P_t \in \mathcal{A}, \ t = 0, 1, \dots\};$$

$\mathcal{B}$ satisfies (A2).

Let $M, \epsilon > 0$*. Then there exists* $\beta_0 > 0$ *such that for each* $\beta_1 \in (0, \beta_0)$ *there exist* $\delta \in (0, 2^{-1}\bar{r})$ *and a natural number* n_0 *such that the following assertion holds.*

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$*, an integer* $n \geq n_0$*,* $\{x_t\}_{t=0}^{n} \subset U_{\bar{r}}$,

$$\|x_0\| \leq M,$$

$$v_k \in \partial_\delta f(x_k) \setminus \{0\}, \ k = 0, 1, \dots, n-1,$$

$\{\alpha_i\}_{i=0}^{n-1} \subset [\beta_1, \beta_0]$,

$$\{\eta_k\}_{k=0}^{n-1}, \ \{\xi_k\}_{k=0}^{n-1} \subset B_X(0, \delta),$$

and that for $k = 0, \dots, n-1$,

$$x_{k+1} = P_k(x_k - \alpha_k \|v_k\|^{-1} v_k - \alpha_k \xi_k) - \eta_k.$$

Then the inequality $d(x_k, C_{min}) \le \epsilon$ *hods for all integers* k *satisfying* $n_0 \le k \le n$.

In the next theorem with the bounded set C we use a non-normalized subgradient method.

Theorem 9.7 *Let* $\bar{r} \in (0, 1]$, *(A3) hold,* $\mathcal{B}$ *be a nonempty set of sequences* $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{\bar{r}}$ *such that and at least one of the following conditions holds:*

there exists a nonempty (C)*-quasi-contractive set* $\mathcal{A} \subset \mathcal{M}$ *such that*

$$\mathcal{B} \subset \{\{P_t\}_{t=0}^{\infty} : \ P_t \in \mathcal{A}, \ t = 0, 1, \dots\};$$

$\mathcal{B}$ *satisfies (A2).*

Let $M > \bar{K} + 1, \epsilon > 0$,

$$C \subset B_X(0, M), \tag{9.34}$$

$$P_t(X) \subset B_X(0, M), \ t = 0, 1 \dots, \tag{9.35}$$

a sequence $\{\alpha_i\}_{i=0}^{\infty}$ *satisfy*

$$\lim_{i \to \infty} \alpha_i = 0, \ \sum_{i=0}^{\infty} \alpha_i = \infty. \tag{9.36}$$

Then there exist $\delta \in (0, 2^{-1}\bar{r}]$ *and a natural number* n_0 *such that the following assertion holds.*

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, *an integer* $n \ge n_0$, $\{x_t\}_{t=0}^{n} \subset U_{\bar{r}}$,

$$\|x_0\| \le M,$$

$\{\xi_t\}_{t=0}^{n-1} \subset X$, *for all integers* $t = 0, \dots, n-1$,

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \ne \emptyset,$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \le \alpha_t \delta.$$

Then the inequality $d(x_t, C_{min}) \le \epsilon$ *hods for all integers* t *satisfying* $n_0 \le t \le n$.

In the following two theorems the set C is not assumed to be bounded.

Theorem 9.8 *Let* $\bar{r} \in (0, 1]$, *(A3) hold,* $\mathcal{B}$ *be a nonempty set of sequences* $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{\bar{r}}$ *such that and at least one of the following conditions holds:*

there exists a nonempty (C)*-quasi-contractive set* $\mathcal{A} \subset \mathcal{M}$ *such that*

$$\mathcal{B} \subset \{\{P_t\}_{t=0}^{\infty} : \ P_t \in \mathcal{A}, \ t = 0, 1, \dots\};$$

$\mathcal{B}$ satisfies (A2).

Let $M > \bar{K}, \epsilon > 0$, $L_0 > \bar{L}$,

$$|f(z_1) - f(z_2)| \le L_0 \|z_1 - z_2\| \text{ for all } z_1, z_2 \in B_X(0, 3M + 4) \cap U, \tag{9.37}$$

$\{\alpha_i\}_{i=0}^{\infty} \subset (0, 1]$, *for all integers $t \ge 0$,*

$$\alpha_i \le (2L_0 + 2)^{-2}, \tag{9.38}$$

$$\lim_{i \to \infty} \alpha_i = 0, \ \sum_{i=0}^{\infty} \alpha_i = \infty. \tag{9.39}$$

Then there exist $\delta \in (0, 2^{-1}\bar{r}]$ and a natural number n_0 such that the following assertion holds.

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, an integer $n \ge n_0$, $\{x_t\}_{t=0}^{n} \subset U_{\bar{r}}$,

$$\|x_0\| \le M,$$

$\{\xi_t\}_{t=0}^{n-1} \subset X$, *for all integers $t = 0, \dots, n - 1$,*

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \ne \emptyset,$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \le \alpha_t \delta.$$

Then the inequality $d(x_t, C_{min}) \le \epsilon$ hods for all integers t satisfying $n_0 \le t \le n$.

Theorem 9.9 *Let $\bar{r} \in (0, 1]$, (A3) hold, $\mathcal{B}$ be a nonempty set of sequences $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{\bar{r}}$ such that and at least one of the following conditions holds:*

there exists a nonempty (C)-quasi-contractive set $\mathcal{A} \subset \mathcal{M}$ such that

$$\mathcal{B} \subset \{\{P_t\}_{t=0}^{\infty} : \ P_t \in \mathcal{A}, \ t = 0, 1, \dots\};$$

$\mathcal{B}$ satisfies (A2).

Let $M, \epsilon > 0$. Then there exists $\beta_0 \in (0, 1)$ such that for each $\beta_1 \in (0, \beta_0)$ there exist $\delta \in (0, 2^{-1}\bar{r}]$ and a natural number n_0 such that the following assertion holds.

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, an integer $n \ge n_0$, $\{x_t\}_{t=0}^{n} \subset U_{\bar{r}}$,

$$\|x_0\| \le M,$$

$\{\xi_t\}_{t=0}^{n-1} \subset X$, $\{\alpha_t\}_{t=0}^{n-1} \subset [\beta_1, \beta_0]$ *and that for all integers $t = 0, \dots, n - 1$,*

$$B(\xi_t, \delta) \cap \partial_\delta f(x_t) \ne \emptyset,$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \le \delta.$$

Then the inequality $d(x_t, C_{min}) \le \epsilon$ hods for all integers t satisfying $n_0 \le t \le n$.

9.5 Auxiliary Results on the Convergence of Infinite Products

Lemma 9.10 *Assume that a nonempty set $\mathcal{A} \subset \mathcal{M}$ is (C)-quasi-contractive, $K_0, \epsilon > 0$. Then there exist a natural number m_0 and $\delta_0 \in (0, 1)$ such that for each integer $n \ge m_0$, each $\{P_i\}_{i=0}^{n-1} \subset \mathcal{A}$ and each finite sequence $\{x_i\}_{i=0}^{n} \subset X$ satisfying*

$$\|x_i\| \le K_0, \ i = 0, \dots, n \tag{9.40}$$

and

$$B_X(x_{i+1}, \delta_0) \cap P_i(B_X(x_i, \delta_0)) \ne \emptyset, \ i = 0, \dots, n-1 \tag{9.41}$$

the inequality

$$d(x_i, C) \le \epsilon$$

holds for all integers $i \in [m_0, n]$.

Proof Let

$$\bar{x} \in C_{min}. \tag{9.42}$$

Since $\mathcal{A} \subset \mathcal{M}$ is (C)-quasi-contractive that there exists $\gamma_0 \in (0, 1)$ such that the following property holds:

(i) for each $x \in B_X(0, K_0 + 1)$ satisfying

$$d(x, C) \ge \epsilon/8$$

and each $z \in B_X(0, \bar{K} + 2K_0 + 5) \cap C$ and each $P \in \mathcal{A}$, we have

$$\|P(x) - z\| \le \|x - z\| - \gamma_0.$$

Choose a natural number

$$m_0 > 2(K_0 + \bar{K} + 1)\gamma_0^{-1} + 2 \tag{9.43}$$

and a positive number

$$\delta_0 < \min\{\epsilon/8, 1, \gamma_0/4\}. \tag{9.44}$$

Assume that an integer $n \geq m_0$, $\{P_i\}_{i=0}^{n-1} \subset \mathcal{A}$ and that a finite sequence $\{x_i\}_{i=0}^{n} \subset X$ satisfies (9.40) and (9.41). We show that there exists $j \in \{0, \ldots, m_0\}$ such that

$$d(x_j, C) \leq \epsilon/4.$$

We may assume without loss of generality that

$$d(x_0, C) > \epsilon/4. \tag{9.45}$$

By (9.41), there exists

$$y_0 \in B_X(x_0, \delta_0) \tag{9.46}$$

such that

$$\|x_1 - P_0(y_0)\| \leq \delta_0. \tag{9.47}$$

By (9.44)–(9.46),

$$d(y_0, C) \geq d(x_0, C) - \delta_0 > \epsilon/4 - \delta_0 > \epsilon/8. \tag{9.48}$$

In view of (9.40), (9.44), and (9.46),

$$\|y_0\| \leq \|x_0\| + \delta_0 \leq K_0 + 1. \tag{9.49}$$

Property (i), (9.5), (9.9)–(9.11), (9.46), (9.48), and (9.49) imply that

$$\begin{aligned}\|P_0(y_0) - \bar{x}\| &\leq \|y_0 - \bar{x}\| - \gamma_0 \\ &\leq \|x_0 - \bar{x}\| + \|x_0 - y_0\| - \gamma_0 \\ &\leq \|x_0 - \bar{x}\| + \delta_0 - \gamma_0. \end{aligned} \tag{9.50}$$

By (9.44), (9.47), and (9.50),

$$\begin{aligned}\|x_1 - \bar{x}\| &\leq \|x_1 - P_0(y_0)\| + \|P_0(y_0) - \bar{x}\| \\ &\leq \delta_0 + \|x_0 - \bar{x}\| - \gamma_0 + \delta_0 \\ &= \|x_0 - \bar{x}\| - \gamma_0 + 2\delta_0 \leq \|x_0 - \bar{x}\| - \gamma_0/2. \end{aligned} \tag{9.51}$$

We may assume without loss of generality that

$$d(x_1, C) > \epsilon/4. \tag{9.52}$$

Assume that p is a natural number such that for all $i = 0, \ldots, p-1$,

$$\|x_{i+1} - \bar{x}\| \le \|x_i - \bar{x}\| - \gamma_0/2. \tag{9.53}$$

(In view of (9.51), this assumption holds for $p = 1$.) It follows from (9.9), (9.11), (9.40), (9.42), and (9.53) that

$$K_0 + \bar{K} \ge \|x_0 - \bar{x}\|$$

$$\ge \|x_0 - \bar{x}\| - \|x_p - \bar{x}\|$$

$$= \sum_{i=0}^{p-1} (\|x_i - \bar{x}\| - \|x_{i+1} - \bar{x}\|) \ge (\gamma_0/2)p$$

and

$$p \le 2(K_0 + \bar{K})\gamma_0^{-1}.$$

Together with (9.43) this implies that

$$p < m_0. \tag{9.54}$$

In view of (9.54), we mat assume without loss of generality that p is the largest natural number such that (9.53) holds for $i = 0, \ldots, p-1$. Thus

$$\|x_{p+1} - \bar{x}\| > \|x_p - \bar{x}\| - \gamma_0/2. \tag{9.55}$$

Assume that

$$d(x_p, C) > \epsilon/4. \tag{9.56}$$

By (9.41), there exists

$$y_p \in B_X(x_p, \delta_0) \tag{9.57}$$

such that

$$\|x_{p+1} - P_p(y_p)\| \le \delta_0. \tag{9.58}$$

Relations (9.40), (9.44), and (9.57) imply that

$$\|y_p\| \le K_0 + 1. \tag{9.59}$$

By (9.44), (9.56), and (9.57),

$$d(y_p, C) \geq d(x_p, C) - \|x_p - y_p\| > \epsilon/4 - \delta_0 \geq \epsilon/8. \tag{9.60}$$

It follows from property (i), (9.9)–(9.11), (9.48), (9.57), (9.59), and (9.60) that

$$\begin{aligned}\|P_p(y_p) - \bar{x}\| &\leq \|y_p - \bar{x}\| - \gamma_0 \\ &\leq \|x_p - \bar{x}\| + \|y_p - x_p\| - \gamma_0 \\ &\leq \|x_p - \bar{x}\| - \gamma_0 + \delta_0.\end{aligned} \tag{9.61}$$

By (9.44), (9.58), and (9.61),

$$\begin{aligned}\|x_{p+1} - \bar{x}\| &\leq \|x_{p+1} - P_p(y_p)\| + \|P_p(y_p) - \bar{x}\| \\ &\leq \|x_p - \bar{x}\| - \gamma_0 + 2\delta_0 \\ &\leq \|x_p - \bar{x}\| - \gamma_0/2.\end{aligned}$$

This contradicts (9.55). The contradiction we have reached proves that

$$d(x_p, C) \leq \epsilon/4.$$

Thus we proved the existence of an integer $p \geq 0$ such that

$$p < m_0, \tag{9.62}$$

$$d(x_p, C) \leq \epsilon/4. \tag{9.63}$$

We show that

$$d(x_j, C) \leq \epsilon \text{ for all integers } j \in [p, n].$$

Assume that $j \geq p$ is an integer, $j < n$ and that

$$d(x_j, C) \leq \epsilon. \tag{9.64}$$

There are two cases:

$$d(x_j, C) \leq \epsilon/4; \tag{9.65}$$

$$d(x_j, C) > \epsilon/4. \tag{9.66}$$

By (9.41), there exists

$$y \in B_X(x_j, \delta_0) \tag{9.67}$$

such that

$$\|x_{j+1} - P_j(y)\| \le \delta_0. \tag{9.68}$$

Assume that (9.65) holds. It follows from (9.8), (9.44), (9.65), and (9.67) that

$$\begin{aligned} d(x_{j+1}, C) &\le \|x_{j+1} - P_j(y)\| + d(P_j(y), C) \\ &\le \delta_0 + d(y, C) \\ &\le \delta_0 + d(x_j, C) + \|y - x_j\| \\ &\le 2\delta_0 + d(x_j, C) \le \epsilon/4 + 2\delta_0 \le \epsilon. \end{aligned} \tag{9.69}$$

Assume that (2.66) holds. By (9.9)–(9.11), (9.40), (9.42), (9.44), and (9.67),

$$d(y, C) \le \|y - \bar{x}\| \le K_0 + \bar{K} + 1. \tag{9.70}$$

In view of (9.44) and (9.70), there exists $z \in C$ such that

$$\|z - y\| \le d(y, C) + \delta_0 \le K_0 + \bar{K} + 2. \tag{9.71}$$

By (9.40), (9.44), and (9.67),

$$\begin{aligned} \|z\| &\le K_0 + \bar{K} + 2 + \|y\| \le K_0 + \bar{K} + 2 \\ &+ \|x_j\| + \|y - x_j\| \le 2K_0 + \bar{K} + 3. \end{aligned} \tag{9.72}$$

It follows from (9.40), (9.44), and (9.67) that

$$\|y\| \le \|x_j\| + 1 \le K_0 + 1. \tag{9.73}$$

By (9.44), (9.66), and (9.67),

$$d(y, C) \ge d(x_j, C) - \|x_j - y\| \ge \epsilon/4 - \delta_0 > \epsilon/8. \tag{9.74}$$

Property (i), the inclusion $z \in C$, (9.71), (9.73), and (9.74) imply that

$$\|P_j(y) - z\| \le \|y - z\| - \gamma_0. \tag{9.75}$$

The inclusion $z \in C$, (9.44), (9.64), (9.67), (9.68), (9.71), and (9.75) imply that

$$d(x_{j+1}, C) \le \|x_{j+1} - z\|$$

$$\le \|P_j(y) - z\| + \|x_{j+1} - P_j(y)\|$$

$$\le \|y - z\| - \gamma_0 + \delta_0$$

$$\le d(y, C) + \delta_0 - \gamma_0 + \delta_0$$

$$\le d(x_j, C) + \|y - x_j\| + \delta_0 - \gamma_0 + \delta_0$$

$$\le d(x_j, C) + 2\delta_0 - \gamma_0 + \delta_0 \le d(x_j, C) - \gamma_0/4 \le \epsilon.$$

Thus in both cases

$$d(x_{j+1}, C) \le \epsilon.$$

Therefore $d(x_i, C) \le \epsilon$ for all $i = j, \dots, n$. Lemma 9.10 is proved.

Lemma 9.11 *Let $\mathcal{B}$ be a nonempty set of sequences $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}$ such that the following properties hold:*

(i) for each $\epsilon > 0$ and each $K > 0$ there exists $\delta > 0$ such that for each $x, y \in B_X(0, K)$ satisfying $\|x - y\| \le \delta$, each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $t \ge 0$, $\|P_t(x) - P_t(y)\| \le \epsilon$ holds;
(ii) for each $\epsilon, K > 0$ there exists a natural number m_0 such that for each $x \in B_X(0, K)$, each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $q \ge 0$,

$$d(P_{q+m_0} \dots P_{q+1} P_q(x), C) \le \epsilon.$$

Let $K_0 > 0$, $\epsilon \in (0, 1)$. There exist a natural number m_0 and $\delta_0 \in (0, 1)$ such that for each integer $n \ge m_0$, each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each finite sequence $\{x_i\}_{i=0}^{n} \subset X$ satisfying

$$\|x_i\| \le K_0, \ i = 0, \dots, n \tag{9.76}$$

and

$$B_X(x_{i+1}, \delta_0) \cap P_i(B_X(x_i, \delta_0)) \ne \emptyset, \ i = 0, \dots, n-1 \tag{9.77}$$

the inequality

$$d(x_i, C) \le \epsilon$$

holds for all integers $i \in [m_0, n]$.

Proof Property (iii) implies that there exists a natural number m_0 such that the following property holds:

(iv) for each $x \in B_X(0, K_0 + \bar{K} + 4)$, each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $q \geq 0$,

$$d(P_{q+m_0-1} \dots P_{q+1}P_q(x), C) < \epsilon/8. \tag{9.78}$$

Set

$$\delta_{m_0} = \epsilon/4. \tag{9.79}$$

In view (ii), there exists

$$\delta_{m_0-1} \in (0, \delta_{m_0}/4)$$

such that the following property holds:

(v)

$$\|P_t(z_1) - P_t(z_2)\| \leq \delta_{m_0}/4$$

for every $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, each integer $t \geq 0$, and all $z_1, z_2 \in B_X(0, \bar{K} + K_0 + 4)$ satisfying

$$\|z_1 - z_2\| \leq 4\delta_{m_0-1}.$$

By induction, using property (ii), we construct a finite sequence $\{\delta_i\}_{i=0}^{m_0} \subset (0, \infty)$ such that for each integer $i \in [0, m_0 - 1]$ we have

$$\delta_i < \delta_{i+1}/4 \tag{9.80}$$

and for every $z_1, z_2 \in B_X(0, K_0 + \bar{K} + 4)$ satisfying

$$\|z_1 - z_2\| \leq 4\delta_i, \tag{9.81}$$

every $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and every integer $t \geq 0$, we have

$$\|P_t(z_1) - P_t(z_2)\| \leq \delta_{i+1}/4. \tag{9.82}$$

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, $n \geq m_0$ is an integer,

$$\{x_i\}_{i=0}^{n} \subset B_X(0, K_0)$$

and that (9.77) holds.

Let

$$k \in [m_0, n] \tag{9.83}$$

be an integer. In order to complete the proof it is sufficient to show that

$$d(x_k, C) \le \epsilon.$$

By (9.76), (9.83), and property (iv),

$$d(P_{k-1} \dots P_{k-m_0}(x_{k-m_0}), C) < \epsilon/8. \tag{9.84}$$

In view of (9.77), there exists

$$y_{k-m_0} \in B_X(x_{k-m_0}, \delta_0) \tag{9.85}$$

such that

$$\|P_{k-m_0}(y_{k-m_0}) - x_{k-m_0+1}\| \le \delta_0. \tag{9.86}$$

By (9.76), (9.80)–(9.82), and (9.85),

$$\|P_{k-m_0}(y_{k-m_0}) - P_{k-m_0}(x_{k-m_0})\| \le \delta_1/4. \tag{9.87}$$

It follows from (9.80), (9.86), and (9.87) that

$$\|x_{k-m_0+1} - P_{k-m_0}(x_{k-m_0})\| \le \delta_1/2. \tag{9.88}$$

We show that for all $i = 1, \dots, m_0$,

$$\|x_{i+k-m_0} - \prod_{t=k-m_0}^{k-m_0-1+i} P_t(x_{k-m_0})\| \le \delta_i/2. \tag{9.89}$$

(Note that in view of (9.88), inequality (9.89) holds for $i = 1$.) Assume that $i \in \{1, \dots, m_0\} \setminus \{m_0\}$ and that (9.89) holds. By (9.77), there exists

$$y_{i+k-m_0} \in B_X(x_{i+k-m_0}, \delta_0) \tag{9.90}$$

such that

$$\|P_{i+k-m_0}(y_{i+k-m_0}) - x_{i+k-m_0+1}\| \le \delta_0. \tag{9.91}$$

In view of (9.89) and (9.90),

$$\|y_{i+k-m_0} - \prod_{t=k-m_0}^{k-m_0-1+i} P_t(x_{k-m_0})\| \le \delta_i. \tag{9.92}$$

It follows from (9.76), (9.80)–(9.82), (9.89), (9.90), and (9.92) that

$$\|P_{i+k-m_0}(y_{i+k-m_0}) - \prod_{t=k-m_0}^{k-m_0+i} P_t(x_{k-m_0})\| \le \delta_{i+1}/4. \tag{9.93}$$

It follows from (9.80), (9.91), and (9.93) that

$$\|x_{i+k-m_0+1} - \prod_{t=k-m_0}^{k-m_0-1+i} P_t(x_{k-m_0})\| \le \delta_{i+1}/2.$$

Thus our assumption holds for $i+1$ too. Therefore we showed by induction that

$$\|x_{i+k-m_0} - \prod_{t=k-m_0}^{k-m_0-1+i} P_t(x_{k-m_0})\| \le \delta_i$$

for all $i = 1, \dots, m_0$ and in particular (see (9.79))

$$\|x_k - \prod_{t=k-m_0}^{k-1} P_t(x_{k-m_0})\| \le \delta_{m_0} = \epsilon/4.$$

Together with (9.84) this implies that

$$d(x_k, C) \le \epsilon.$$

Lemma 9.11 is proved.

9.6 Auxiliary Results

Lemma 9.12 *Let* K_0, L_0,

$$|f(z_1) - f(z_2)| \le L_0\|z_1 - z_2\| \textit{ for all } z_1, z_2 \in B_X(0, K_0 + 1) \cap U, \tag{9.94}$$

$$x \in B_X(0, K_0) \cap U, \ \ v \in \partial f(x). \tag{9.95}$$

Then $\|v\| \le L_0$.

Proof By (9.95), for all $u \in U$,

$$f(u) - f(x) \ge \langle v, u - x\rangle. \tag{9.96}$$

There exists $r \in (0, 1)$ such that

$$B_X(x, r) \subset U. \tag{9.97}$$

By (9.95) and (9.97),

$$B_X(x, r) \subset B_X(0, K_0 + 1). \tag{9.98}$$

It follows from (9.94)–(9.98) that for all $h \in B_X(0, 1)$,

$$x + rh \in U \cap B_X(0, K_0 + 1),$$

$$\langle v, rh \rangle \le +f(x + rh) - f(x) \le L_0 r \|h\| \le L_0 r$$

and

$$\langle v, h \rangle \le L_0.$$

This implies that $\|v\| \le L_0$. Lemma 9.12 is proved.

Lemma 9.13 $K_0, L_0 > 0$, $r \in (0, 1]$, $\Delta > 0$,

$$|f(z_1) - f(z_2)| \le L_0 \|z_1 - z_2\| \text{ for all } z_1, z_2 \in B_X(0, K_0 + 1) \cap U, \tag{9.99}$$

$$x \in B_X(0, K_0) \cap U, \tag{9.100}$$

$$B_X(x, r) \subset U. \tag{9.101}$$

Then

$$\partial_\Delta f(x) \subset B_X(0, L_0 + \Delta r^{-1}).$$

Proof Let

$$\xi \in \partial_\Delta f(x). \tag{9.102}$$

In view of (9.100) and (9.101),

$$B_X(x, r) \subset U \cap B_X(0, K_0 + 1). \tag{9.103}$$

It follows from (9.103) that for all $h \in B_X(0, 1)$,

$$x + rh \in U \cap B_X(0, K_0 + 1).$$

Combined with (9.99), (9.100), and (9.102) this implies that

$$\langle \xi, rh \rangle - \Delta \le f(x + rh) - f(x) \le L_0 r \|h\| \le L_0 r$$

and

$$\langle \xi, h \rangle \leq L_0 + \Delta/r.$$

This implies that $\|\xi\| \leq L_0 + \Delta/r$. Lemma 9.13 is proved.

Lemma 9.14 *Let $P \in \mathcal{M}$, $K_0 \geq \bar{K}$, $L_0 > 0$*

$$|f(z_1) - f(z_2)| \leq L_0 \|z_1 - z_2\| \text{ for all } z_1, z_2 \in B_X(0, K_0 + 2) \cap U, \tag{9.104}$$

$r, \alpha, \delta_f, \delta_C \in (0, 1]$, $\Delta > 0$, let a point

$$x \in U \cap B_X(0, K_0 + 1) \tag{9.105}$$

satisfy

$$B_X(x, r) \subset U, \tag{9.106}$$

$\xi \in X$ satisfy

$$B(\xi, \delta_f) \cap \partial_\Delta f(x) \neq \emptyset \tag{9.107}$$

and let

$$y \in B_X(P(x - \alpha\xi), \delta_C) \cap U. \tag{9.108}$$

Then for all $z \in C$ satisfying $f(z) \leq f(\theta_0) + 4$,

$$2\alpha(f(x) - f(z))$$

$$\leq \|x - z\|^2 - \|y - z\|^2 + 2\delta_C(K_0 + \bar{K} + L_0 + 3 + \Delta r^{-1})$$

$$+2\alpha\Delta + \alpha^2(L_0 + \Delta r^{-1})^2 + 2\alpha\delta_f(K_0 + \bar{K} + L_0 + 2 + \Delta r^{-1}).$$

Proof Let

$$z \in C \tag{9.109}$$

satisfy

$$f(z) \leq f(\theta_0) + 4. \tag{9.110}$$

By (9.10), (9.11), (9.109), and (9.110),

$$\|z\| \leq \bar{K}. \tag{9.111}$$

In view of (9.107), there exists

$$v \in \partial f_{\Delta}(x) \tag{9.112}$$

such that

$$\|\xi - v\| \le \delta_f. \tag{9.113}$$

Lemma 9.13 and (9.104)–(9.106) imply that

$$\partial_{\Delta} f(x) \subset B_X(0, L_0 + \Delta r^{-1}). \tag{9.114}$$

Equations (9.112) and (9.114) imply that

$$\|v\| \le L_0 + \Delta r^{-1}. \tag{9.115}$$

By (9.113) and (9.115),

$$\|z\| \le L_0 + \Delta r^{-1} + 1. \tag{9.116}$$

It follows from (9.105), (9.111), (9.113), and (9.115) that

$$\|x - \alpha\xi - z\|^2 = \|x - \alpha v + (\alpha v - \alpha\xi) - z\|^2$$

$$\le \|x - \alpha v - z\|^2 + \alpha^2\|v - \xi\|^2 + 2\alpha\langle v - \xi, x - \alpha v - z\rangle$$

$$\le \|x - \alpha v - z\|^2 + \alpha^2\delta_f^2 + 2\alpha\delta_f\|x - \alpha v - z\|$$

$$\le \|x - \alpha v - z\|^2 + \alpha^2\delta_f^2 + 2\alpha\delta_f(K_0 + \bar{K} + 1 + L_0 + \Delta r^{-1})$$

$$\le \|x - z\|^2 - 2\alpha\langle x - z, v\rangle + \alpha^2\|v\|^2$$

$$+\alpha^2\delta_f^2 + 2\alpha\delta_f(K_0 + \bar{K} + 1 + L_0 + \Delta r^{-1})$$

$$\le \|x - z\|^2 - 2\alpha\langle x - z, v\rangle + \alpha^2(L_0 + \Delta r^{-1})^2$$

$$+ \alpha^2\delta_f^2 + 2\alpha\delta_f(K_0 + \bar{K} + 1 + L_0 + \Delta r^{-1}). \tag{9.117}$$

In view of (9.112),

$$\langle v, z - x\rangle \le f(z) - f(x) + \Delta. \tag{9.118}$$

By (9.105), (9.111), and (9.116),

$$\|x - \alpha\xi - z\| \le K_0 + \bar{K} + 2 + L_0 + \Delta r^{-1}. \tag{9.119}$$

By (9.117) and (9.118),

$$\|x - \alpha\xi - z\|^2 \le \|x - z\|^2 + 2\alpha(f(z) - f(x)) + \alpha^2(L_0 + \Delta r^{-1})^2$$

$$+ 2\alpha\Delta + 2\alpha\delta_f(K_0 + \bar{K} + 2 + L_0 + \Delta r^{-1}). \tag{9.120}$$

It follows from (9.8), (9.108), (9.109), (9.119), and (9.120) that

$$\|y - z\|^2 = \|y - P(x - \alpha\xi) + P(x - \alpha\xi) - z\|^2$$

$$\le \|y - P(x - \alpha\xi)\|^2 + 2\|y - P(x - \alpha\xi)\|\|P(x - \alpha\xi) - z\|$$

$$+\|P(x - \alpha\xi) - z\|^2$$

$$\le \delta_C^2 + 2\delta_C(K_0 + \bar{K} + 2 + L_0 + \Delta r^{-1}) + \|P(x - \alpha\xi) - z\|^2$$

$$\le 2\delta_C(K_0 + \bar{K} + 3 + L_0 + \Delta r^{-1}) + \|x - z\|^2 + 2\alpha(f(z) - f(x))$$

$$+\alpha^2(L_0 + \Delta r^{-1})^2 + 2\alpha\Delta + 2\alpha\delta_f(K_0 + \bar{K} + 2 + L_0 + \Delta r^{-1}).$$

This implies that

$$2\alpha(f(x) - f(z)) \le \|x - z\|^2 - \|y - z\|^2$$

$$+2\delta_C(K_0 + \bar{K} + 3 + L_0 + \Delta r^{-1}) + 2\alpha\Delta + \alpha^2(L_0 + \Delta r^{-1})^2$$

$$+2\alpha\delta_f(K_0 + \bar{K} + 2 + L_0 + \Delta r^{-1}).$$

Lemma 9.14 is proved.

Corollary 9.15 *Let* $P \in \mathcal{M}$, $K_0 \ge \bar{K}$, $L_0 > 0$, *(9.104) hold,* $r, \alpha, \delta_f, \delta_C \in (0, 1]$, $\Delta, \epsilon > 0$, *let a point*

$$x \in U \cap B_X(0, K_0 + 1)$$

satisfy

$$B_X(x, r) \subset U,$$

$$f(x) > \inf(f, C) + \epsilon,$$

$\xi \in X$ *satisfy (9.107) and let* $y \in X$ *satisfy (9.108). Then for every* $\bar{x} \in C_{min}$,

$$\|y - \bar{x}\|^2 \le \|x - \bar{x}\|^2 - 2\alpha\epsilon$$

$$+2\delta_C(K_0 + \bar{K} + L_0 + 3 + \Delta r^{-1}) + 2\alpha\Delta$$

$$+\alpha^2(L_0 + \Delta r^{-1})^2 + 2\alpha\delta_f(K_0 + \bar{K} + L_0 + 2 + \Delta r^{-1}),$$

$$d(y, C_{min})^2 \le d(x, C_{min})^2 - 2\alpha\epsilon$$

$$+2\delta_C(K_0 + \bar{K} + L_0 + 3 + \Delta r^{-1}) + 2\alpha\Delta$$

$$+\alpha^2(L_0 + \Delta r^{-1})^2 + 2\alpha\delta_f(K_0 + \bar{K} + L_0 + 2 + \Delta r^{-1}).$$

Lemma 9.16 *Let $P \in \mathcal{M}$, $r \in (0, 1]$,*

$$C_{min} \subset U_r, \tag{9.121}$$

$$\bar{x} \in C_{min}, \tag{9.122}$$

$K_0 \ge \bar{K}$, $L_0 > 0$,

$$|f(z_1) - f(z_2)| \le L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B_X(0, K_0 + 2) \cap U, \tag{9.123}$$

$r, \alpha, \delta_f, \delta_C \in (0, 1]$, $\Delta > 0$,

$$\epsilon \in (0, \min\{16\bar{L},\ 4\bar{L}r\}), \tag{9.124}$$

$$\Delta \le \epsilon/2, \tag{9.125}$$

$$\delta_f(K_0 + \bar{K} + 2) \le (8\bar{L})^{-1}\epsilon, \tag{9.126}$$

let a point $x \in X$ satisfy

$$x \in U \cap B_X(0, K_0 + 1), \tag{9.127}$$

$$B_X(x, r) \subset U, \tag{9.128}$$

$$f(x) > \inf(f, C) + \epsilon, \tag{9.129}$$

$$v \in \partial_\Delta f(x). \tag{9.130}$$

Then

$$\|v\| \ge 2^{-1}\epsilon(K_0 + 1 + \bar{K})^{-1} \tag{9.131}$$

and for each

$$\xi \in B_X(\|v\|^{-1}v, \delta_f) \tag{9.132}$$

and each

$$y \in B_X(P(x - \alpha\xi), \delta_C) \tag{9.133}$$

the following inequalities hold:

$$\|y - \bar{x}\|^2 \le \|x - \bar{x}\|^2 - (4\bar{L})^{-1}\alpha\epsilon + \delta_C^2 + 2\delta_C(K_0 + \bar{K} + 3) + 2\alpha^2,$$

$$d(y, C_{min})^2 \le d(x, C_{min})^2 - (4\bar{L})^{-1}\alpha\epsilon + \delta_C^2 + 2\delta_C(K_0 + \bar{K} + 3) + 2\alpha^2.$$

Proof By (9.124),

$$B_X(\bar{x}, (4\bar{L})^{-1}\epsilon) \subset B_X(\bar{x}, r). \tag{9.134}$$

In view of (9.9)–(9.12), and (9.134) for every point

$$z \in B_X(\bar{x}, 4^{-1}\epsilon\bar{L}^{-1}),$$

we have

$$f(z) \le f(\bar{x}) + \bar{L}\|z - \bar{x}\| \le f(\bar{x}) + \epsilon/4 = \inf(f, C) + \epsilon/4. \tag{9.135}$$

Equations (9.122) and (9.129) imply that

$$-\epsilon > f(\bar{x}) - f(x) \ge \langle v, \bar{x} - x\rangle - \Delta. \tag{9.136}$$

By (9.9)–(9.11), (9.122), (9.125), (9.127), and (9.136),

$$2^{-1}\epsilon \le \epsilon - \Delta < \langle v, \bar{x} - x\rangle$$

$$\le \|v\|\|x - \bar{x}\| \le \|v\|(K_0 + 1 + \bar{K})$$

and

$$\|v\| \ge 2^{-1}\epsilon(K_0 + 1 + \bar{K})^{-1}.$$

Therefore (9.131) is true.

Let $\xi, y \in X$ satisfy (9.132) and (9.133). It follows from (9.125), (9.128), (9.129), (9.134), and (9.135) that for every point

$$z \in B_X(\bar{x}, 4^{-1}\epsilon\bar{L}^{-1}),$$

we have

$$\langle v, z - x\rangle \le f(z) - f(x) + \Delta$$

$$\le \inf(f, C) + \epsilon/4 - \inf(f, C) - \epsilon + \Delta$$

$$= -(3/4)\epsilon + \Delta < 0.$$

Thus

$$\langle \|v\|^{-1} v, z - x\rangle < 0 \text{ for all } z \in B_X(\bar{x}, (4\bar{L})^{-1}\epsilon). \tag{9.137}$$

Set

$$\tilde{z} = \bar{x} + 4^{-1}\bar{L}^{-1}\epsilon\|v\|^{-1}v. \tag{9.138}$$

In view of (9.128), (9.134) and (9.138),

$$\tilde{z} \in U.$$

Equations (9.137) and (9.138) imply that

$$0 > \langle \|v\|^{-1}v, \tilde{z} - x\rangle = \langle \|v\|^{-1}v, \bar{x} + 4^{-1}\bar{L}^{-1}\epsilon\|v\|^{-1}v - x\rangle$$

$$= \langle \|v\|^{-1}v, \bar{x} - x\rangle + 4^{-1}\bar{L}^{-1}\epsilon. \tag{9.139}$$

Set

$$y_0 = x - \alpha\xi. \tag{9.140}$$

It follows from (9.9)–(9.11), (9.122), (9.126), (9.127), (2.132), (9.139), and (9.140) that

$$\|y_0 - \bar{x}\|^2 = \|x - \alpha\xi - \bar{x}\|^2$$

$$= \|x - \alpha\|v\|^{-1}v + \alpha(\|v\|^{-1}v - \xi) - \bar{x}\|^2$$

$$\|x - \alpha\|v\|^{-1}v - \bar{x}\|^2 + \alpha^2\|\|v\|^{-1}v - \xi\|^2$$

$$+2\alpha\langle \|v\|^{-1}v - \xi, x - \alpha\|v\|^{-1}v - \bar{x}\rangle$$

$$\le \|x - \alpha\|v\|^{-1}v - \bar{x}\|^2$$

$$+\alpha^2\delta_f^2 + 2\alpha\delta_f K_0 + \bar{K} + 2)$$

$$\leq \|x - \bar{x}\|^2 - 2\langle x - \bar{x}, \alpha\|v\|^{-1}v\rangle$$

$$+\alpha^2(1 + \delta_f)^2 + 2\alpha\delta_f(K_0 + \bar{K} + 2)$$

$$< \|x - \bar{x}\|^2 - 2\alpha(4^{-1}\bar{L}^{-1}\epsilon)$$

$$+\alpha^2(1 + \delta_f^2) + 2\alpha\delta_f(K_0 + \bar{K} + 2)$$

$$\leq \|x - \bar{x}\|^2 - \alpha(4\bar{L})^{-1}\epsilon + 2\alpha^2. \tag{9.141}$$

In view of (9.122), (9.127), and (9.141),

$$\|y_0 - \bar{x}\|^2 \leq \|x - \bar{x}\|^2 + 2 \leq (K_0 + \bar{K} + 1)^2 + 2$$

and

$$\|y_0 - \bar{x}\| \leq K_0 + \bar{K} + 3. \tag{9.142}$$

By (9.8), (9.122), (9.133), (9.141), and (9.142),

$$\|y - \bar{x}\|^2 = \|y - P(x - \alpha\xi) + P(x - \alpha\xi) - \bar{x}\|^2$$

$$\|y - P(x - \alpha\xi)\|^2 + \|P(x - \alpha\xi) - \bar{x}\|^2 + 2\|y - P(x - \alpha\xi)\|\|P(x - \alpha\xi) - \bar{x}\|$$

$$\leq \|y_0 - \bar{x}\|^2 + \delta_C^2 + 2\delta_C\|y_0 - \bar{x}\|$$

$$\leq \|x - \bar{x}\|^2 - \alpha(4\bar{L})^{-1}\epsilon$$

$$+2\alpha^2 + \delta_C^2 + 2\delta_C\|y_0 - \bar{x}\|$$

$$\leq \|x - \bar{x}\|^2 - \alpha(4\bar{L})^{-1}\epsilon$$

$$+2\alpha^2 + \delta_C^2 + 2\delta_C(K_0 + \bar{K} + 3).$$

This completes the proof of Lemma 9.16.

Proposition 9.17 *Let (A3) hold, the function ϕ be defined by (9.29) and $\epsilon \in (0, 1]$. Then for each $x \in X$ satisfying*

$$d(x, C) < \min\{\bar{L}^{-1}2^{-1}\phi(\epsilon/2),\ \epsilon/2\}, \tag{9.143}$$

$$f(x) \leq \inf(f, C) + \min\{2^{-1}\phi(\epsilon/2),\ \epsilon/2\}, \tag{9.144}$$

the inequality $d(x, C_{min}) \leq \epsilon$ holds.

Proof In view of the definition of ϕ, $\phi(\epsilon/2) \in (0, 1]$ and the following property holds:

(i)

$$\text{if } x \in C \text{ satisfies } f(x) < \inf(f, C) + \phi(\epsilon/2),$$

$$\text{then } d(x, C_{min}) \le \min\{1, \epsilon/2\}. \tag{9.145}$$

Assume that a point $x \in U$ satisfies (9.143) and (9.144). In view (9.143), there exists a point $y \in C$ which satisfies

$$\|x - y\| < 2^{-1}\bar{L}^{-1}\phi(\epsilon/2) \text{ and } \|x - y\| < \epsilon/2. \tag{9.146}$$

Relations (9.9), (9.111), (9.144), and (9.146) imply that

$$x \in B_X(0, \bar{K}),\ \ y \in B_X(0, \bar{K} + 1). \tag{9.147}$$

By (9.11), (9.146), and (9.147) and the definition of $\bar{L}$,

$$|f(x) - f(y)| \le \bar{L}\|x - y\| < \phi(\epsilon/2)2^{-1}. \tag{9.148}$$

It follows from (9.144) and (9.148) that

$$f(y) < f(x) + \phi(\epsilon/2)2^{-1} \le \inf(f, C) + \phi(\epsilon/2).$$

Combined with property (ii) and the inclusion $y \in C$ this implies that

$$d(y, C_{min}) \le \epsilon/2. \tag{9.149}$$

By (9.146) and (9.149),

$$d(x, C_{min}) \le \|x - y\| + d(y, C_{min}) \le \epsilon.$$

This completes the proof of Proposition 9.17.

The next result easily follows from Lemma 9.14.

Lemma 9.18 *Let* $P \in \mathcal{M}$, $r, \alpha, \delta_f, \delta_C \in (0, 1)$*,* $\Delta > 0$,

$$C \subset U,$$

let a point

$$x \in U \cap B_X(0, \bar{K} + 1)$$

satisfy

$$B_X(x,r) \subset U,$$

$\xi \in X$ satisfy

$$B_X(\xi, \delta_f) \cap \partial_\Delta f(x) \neq \emptyset$$

and let

$$y \in B_X(P(x - \alpha\xi), \delta_C) \cap U.$$

Then for all $z \in C$,

$$2\alpha(f(x) - f(z))$$

$$\leq \|x - z\|^2 - \|y - z\|^2 + 2\delta_C(2\bar{K} + \bar{L} + 3 + \Delta r^{-1})$$

$$2\alpha\Delta + \alpha^2(\bar{L} + \Delta r^{-1})^2 + 2\alpha\delta_f(2\bar{K} + \bar{L} + 2 + \Delta r^{-1}).$$

9.7 Proof of Theorem 9.1

We may assume without loss of generality that

$$\epsilon < \min\{1, \bar{r}\}. \tag{9.150}$$

We may assume that if (A2) holds, then the following property holds:

(i) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $s \geq 0$, $\{P_{t+s}\}_{t=0}^{\infty} \in \mathcal{B}$.
 Fix

$$\epsilon_1 \in (0, \epsilon(64\bar{L})^{-1}). \tag{9.151}$$

Lemmas 9.10 and 9.11 imply that there exist $\delta_1 \in (0,1)$ and a natural number m_0 such that the following property holds:

(ii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, each integer $n \geq m_0$, each integer $s \geq 0$ and each finite sequence $\{y_i\}_{i=0}^{n} \subset B_X(0, \bar{K} + 4)$ satisfying

$$B_X(y_{t+1}, \delta_1) \cap P_{t+s}(B_X(y_t, \delta_1)) \neq \emptyset,\ t = 0, \ldots, n-1$$

the inequality

$$d(y_t, C) \leq \epsilon_1$$

holds for all integers $i \in [m_0, n]$.

Since $\lim_{i\to\infty}\alpha_i = 0$ (see (9.18)) there is an integer $p_0 > 0$ such that for all integers $i \geq p_0$, we have

$$\alpha_i \leq \min\{\delta_1/2(\bar{L}+2)^{-1},\ 16^{-1}\epsilon_1(64(\bar{L}+4))^{-2}\}. \tag{9.152}$$

Since $\sum_{i=0}^{\infty}\alpha_i = \infty$ (see (9.19)) there exists a natural number

$$n_0 > p_0 + 4 + m_0 \tag{9.153}$$

such that

$$\sum_{i=p_0+m_0}^{n_0-1}\alpha_i > 128(4\bar{K}+4)^2\bar{\epsilon}^{-1}(\bar{L}+1)^2 p_0. \tag{9.154}$$

Fix a positive number δ such that

$$\delta(4\bar{K}+4\bar{L}+8) < 8^{-1}(64\bar{L})^{-1}\epsilon_1. \tag{9.155}$$

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, $\{\xi_t\}_{t=0}^{\infty} \subset X$,

$$\|x_0\| \leq \bar{K}+1, \tag{9.156}$$

$$\{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}}, \tag{9.157}$$

for all integers $t \geq 0$,

$$B_X(\xi_t,\delta) \cap \partial_\delta f(x_t) \neq \emptyset, \tag{9.158}$$

$$\|x_{t+1} - P_t(x_t - \alpha_t\xi_t)\| \leq \alpha_t\delta. \tag{9.159}$$

In view of (9.17) and (9.158),

$$\|x_t\| \leq \bar{K}+3,\ t = 0, 1, \ldots. \tag{9.160}$$

Set

$$y_t = x_{t+p_0},\ t = 0, 1, \ldots. \tag{9.161}$$

In view of (9.159) and (9.160),

$$\{y_t\}_{t=0}^{\infty} \subset B_X(0, \bar{K}+3). \tag{9.162}$$

Lemma 9.13, (9.12), (9.155), and (9.157) imply that for all integers $t \geq 0$,

$$\partial_\delta f(x_t) \subset B_X(0, \bar{L} + \delta \bar{r}^{-1}) \subset B_X(0, \bar{L} + 1). \tag{9.163}$$

It follows from (9.60), (9.152), (9.158), (9.160), and (9.163) that for all integers $t \geq 0$,

$$\|y_t - (y_t - \alpha_{t+p_0}\xi_{t+p_0})\| \leq \alpha_{t+p_0}\|\xi_{t+p_0}\| \leq \alpha_{t+p_0}(\bar{L} + 2) < \delta_1,$$

$$\|y_{t+1} - P_{t+p_0}(y_t - \alpha_{t+p_0}\xi_{t+p_0})\|$$

$$= \|x_{t+1+p_0} - P_{t+p_0}(y_{t+p_0} - \alpha_{t+p_0}\xi_{t+p_0})\| \leq \alpha_{t+p_0} < \delta_1,$$

$$B_X(y_{t+1}, \delta_1) \cap P_{t+p_0}(B_X(y_t, \delta_1)) \neq \emptyset. \tag{9.164}$$

Properties (i) and (iv), (9.162) and (9.163) imply that

$$d(y_t, C) \leq \epsilon_1 \text{ for all integers } t \geq m_0. \tag{9.165}$$

In view of (9.161) and (9.165),

$$d(x_t, C) \leq \epsilon_1 \text{ for all integers } t \geq m_0 + p_0. \tag{9.166}$$

Assertion 1 follows from (9.151), (9.153), and (9.166) while Assertion 4 and the existence of n_1 follows from Assertion 1, (9.19), and (9.160).

Let $t \geq 0$ be an integer. By (9.17), (9.158), and (9.159) and Lemma 9.18 applied with $P = P_t, \alpha = \alpha_t, \Delta = \delta, \delta_f = \delta, \delta_C = \alpha_t\delta, x = x_t, \ y = x_{t+1}, \xi = \xi_t$, for all $z \in C$ we have

$$\alpha(f(x_t) - f(z))$$

$$\leq 2^{-1}\|x_t - z\|^2 - 2^{-1}\|x_{t+1} - z\|^2 + \alpha_t\delta(2\bar{K} + \bar{L} + 4)$$

$$+ \alpha_t\delta + \alpha^2(\bar{L} + 1)^2 + \alpha_t\delta(2\bar{K} + \bar{L} + 3) \tag{9.167}$$

In view of (9.167), for each pair of integers $m > n \geq 0$ and each $z \in C$,

$$\sum_{t=n}^{m-1} \alpha_t(f(x_t) - f(z))$$

$$\leq 2^{-1} \sum_{t=n}^{m-1} (\|x_t - z\|^2 - \|x_{t+1} - z\|^2)$$

$$+\sum_{t=n}^{m-1}\alpha_t\delta(4\bar{K}+2\bar{L}+8)+\sum_{t=n}^{m-1}\alpha_t^2(\bar{L}+1). \tag{9.168}$$

By (9.14), (9.160), and (9.168), for every $z \in C$,

$$(\sum_{t=n}^{m-1}\alpha_t)^{-1}\sum_{t=n}^{m-1}\alpha_t f(x_t) - f(z)$$

$$\leq \delta(4\bar{K}+2\bar{L}+8)+(\sum_{t=n}^{m-1}\alpha_t)^{-1}\sum_{t=n}^{m-1}\alpha_t^2(\bar{L}+1)+2^{-1}(\sum_{t=n}^{m-1}\alpha_t)^{-1}\|x_n - z\|^2$$

$$\leq (\sum_{t=n}^{m-1}\alpha_t)^{-1}(2\bar{K}+3)^2+\delta(4\bar{K}+2\bar{L}+8)+(\sum_{t=n}^{m-1}\alpha_t)^{-1}\sum_{t=n}^{m-1}\alpha_t^2(\bar{L}+1). \tag{9.169}$$

Let $m > n \geq p_0$ be integers. By (9.19), (9.152), (9.155), and (9.169),

$$\min\{f(x_t) :\ t = n, \dots, m-1\} - \inf(f, C)$$

$$\leq (\sum_{i=n}^{m-1}\alpha_i)^{-1}\sum_{t=n}^{m-1}\alpha_t f(x_t) - \inf(f, C)$$

$$\leq (\sum_{t=n}^{m-1}\alpha_t)^{-1}(2\bar{K}+3)^2+\delta(4\bar{K}++2\bar{L}+8)$$

$$+(\bar{L}+1)^2\max\{\alpha_t :\ t = n, \dots, m-1\}$$

$$\leq (\sum_{i=n}^{m-1}\alpha_i)^{-1}(2\bar{K}+3)^2+\epsilon/8+16^{-1}\epsilon < \epsilon$$

where $m \to \infty$ is sufficiently large. This implies Assertion 2.

We prove Assertion 3. Let an integer $n \geq n_0$. By (9.154), (9.155), and (9.169),

$$f((\sum_{i=0}^{n-1}\alpha_i)^{-1}\sum_{t=0}^{n-1}\alpha_t x_t) - \inf(f, C)$$

$$\leq (\sum_{i=0}^{n-1}\alpha_i)^{-1}\sum_{t=0}^{n-1}\alpha_t f(x_t) - \inf(f, C)$$

$$\leq (\sum_{i=0}^{n-1} \alpha_i)^{-1}(2\bar{K}+3)^2 + \delta(4\bar{K}+2\bar{L}+8) + (\sum_{i=0}^{n-1} \alpha_i)^{-1} \sum_{i=0}^{n-1} \alpha_i^2(\bar{L}+1)$$

$$< 64^{-1}\epsilon + 8^{-1}\epsilon + (\bar{L}+1)^2(\sum_{i=0}^{n-1} \alpha_i)^{-1} \sum_{i=p_0}^{n-1} \alpha_i^2$$

$$+(\sum_{i=0}^{n-1} \alpha_i)^{-1}(\bar{L}+1)^2 p_0 \leq 64^{-1}\epsilon + 8^{-1}\epsilon + 8^{-1}\epsilon + 64^{-1}\epsilon < \epsilon.$$

Assertion 3 is proved. This completes the proof of Theorem 2.1.

9.8 Proof of Theorem 9.2

We may assume without loss of generality that

$$\epsilon < \min\{1, \bar{r}\}. \tag{9.170}$$

We may assume that if (A2) holds, then the following property is true:

(i) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $s \geq 0$, $\{P_{t+s}\}_{t=0}^{\infty} \in \mathcal{B}$.
Fix

$$\epsilon_1 \in (0, \epsilon(64\bar{L})^{-1}). \tag{9.171}$$

Lemmas 9.10 and 9.11 imply that there exist $\delta_1 \in (0,1)$ and a natural number m_0 such that the following property holds:

(ii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, each integer $n \geq m_0$, each integer $s \geq 0$ and each finite sequence $\{y_i\}_{i=0}^{n} \subset B_X(0, \bar{K}+4)$ satisfying

$$B_X(y_{t+1}, \delta_1) \cap P_{t+s}(B_X(y_t, \delta_1)) \neq \emptyset,\ t = 0, \ldots, n-1$$

the inequality

$$d(y_t, C) \leq \epsilon_1$$

holds for all integers $i \in [m_0, n]$.

Choose a positive number β_0 such that

$$\beta_0 \leq (8(\bar{L}+2))^{-1} \min\{\delta_1, \epsilon_1\}. \tag{9.172}$$

Let

$$\beta_1 \in (0, \beta_0). \tag{9.173}$$

Fix a natural number

$$n_0 > m_0 + 16\epsilon_1^{-1}\beta_1^{-1}(2\bar{K} + 3)^2 \tag{9.174}$$

and a positive number δ such that

$$\delta(4\bar{K} + 4\bar{L} + 4) < 8^{-1}\bar{L}^{-1}\epsilon_1\beta_1. \tag{9.175}$$

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, $\{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}}$, $\{\xi_t\}_{t=0}^{\infty} \subset X$,

$$\|x_0\| \le \bar{K} + 1, \tag{9.176}$$

for all integers $t \ge 0$,

$$\alpha_t \in [\beta_1, \beta_0], \tag{9.177}$$

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \ne \emptyset, \tag{9.178}$$

$$\|x_{t+1} - P_t(x_t - \alpha_t\xi_t)\| \le \delta. \tag{9.179}$$

In view of (9.20), (9.176), and (9.179),

$$\|x_t\| \le \bar{K} + 1,\ \ t = 0, 1, \ldots. \tag{9.180}$$

Lemma 9.13 and (9.175) imply that for all integers $t \ge 0$,

$$\partial_\delta f(x_t) \subset B_X(0, \bar{L} + \delta \min\{1, \bar{r}\}^{-1}) \subset B_X(0, \bar{L} + 1). \tag{9.181}$$

It follows from (9.172), (9.177), (9.178), and (9.181) that for all integers $t \ge 0$,

$$\|x_t - (x_t - \alpha_t\xi_t)\| \le \alpha_t\|\xi_t\| \le \alpha_t(\bar{L} + 2) < \delta_1,$$

$$B_X(x_{t+1}, \delta_1) \cap P_t(B_X(x_t, \delta_1)) \ne \emptyset. \tag{9.182}$$

Property (ii), (9.180), and (9.182) imply that

$$d(x_t, C) \le \epsilon_1 \text{ for all integers } t \ge m_0. \tag{9.183}$$

Thus Assertion 1 holds. Assertion 3 and the existence of n_1 follows from (9.183).

Let $t \geq 0$ be an integer. By (9.178)–(9.180) and Lemma 9.18 applied with $P = P_t$, $\Delta = \delta$, $\delta_f = \delta$, $\delta_C = \delta$, $x = x_t$, $y = x_{t+1}$, $\xi = \xi_t$, for all $z \in C$ we have

$$\alpha(f(x_t) - f(z))$$
$$\leq 2^{-1}\|x_t - z\|^2 - 2^{-1}\|x_{t+1} - z\|^2 + 2\delta(2\bar{K} + \bar{L} + 4)$$
$$+ \alpha_t^2(\bar{L} + 1)^2. \tag{9.184}$$

Let $m > n \geq n_0$ be integers such that

$$m - n \geq n_0. \tag{9.185}$$

By (9.14), (9.180), and (9.190), for each $z \in C$,

$$\sum_{t=n}^{m-1} \alpha_t(f(x_t) - f(z))$$
$$\leq 2^{-1}\sum_{t=n}^{m-1}(\|x_t - z\|^2 - \|x_{t+1} - z\|^2)$$
$$+2\delta(m - n)(2\bar{K} + \bar{L} + 4) + \sum_{t=n}^{m-1} \alpha_t^2(\bar{L} + 1)^2$$
$$\leq 2^{-1}(2\bar{K} + 3) + 2\delta(m - n)(2\bar{K} + \bar{L} + 4) + \sum_{t=n}^{m-1} \alpha_t^2(\bar{L} + 1)^2. \tag{9.186}$$

By (9.174), (9.175), (9.177), (9.185), and (9.186),

$$\min\{f(x_t) : \ t = n, \dots, m - 1\} - \inf(f, C),$$
$$f((\sum_{i=n}^{m-1} \alpha_i)^{-1} \sum_{t=n}^{m-1} \alpha_t x_t) - \inf(f, C)$$
$$\leq (\sum_{i=n}^{m-1} \alpha_i)^{-1} \sum_{t=n}^{m-1} \alpha_t f(x_t) - \inf(f, C)$$
$$\leq 2^{-1}(\sum_{t=n}^{m-1} \alpha_t)^{-1}(2\bar{K} + 3)^2 + (\sum_{i=n}^{m-1} \alpha_i)^{-1}\delta(m - n)(2\bar{K} + \bar{L} + 4)$$

$$+(\sum_{i=n}^{m-1}\alpha_i)^{-1}\sum_{t=n}^{m-1}\alpha_t^2(\bar{L}+1)^2$$

$$\le 2^{-1}(2\bar{K}+3)^2 n_0^{-1}\beta_1^{-1}+\beta_1^{-1}\delta(2\bar{K}+\bar{L}+4)$$

$$+\beta_0(\bar{L}+1)^2<\epsilon_1/4+8^{-1}\epsilon_1+\epsilon_1/8.$$

Assertion 2 is proved. This completes the proof of Theorem 9.2.

9.9 Proof of Theorem 9.3

We may assume without loss of generality that

$$\epsilon < \min\{1, \bar{r}\}. \tag{9.187}$$

We may assume that if (A2) holds, then the following property holds:

(i) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $s \ge 0$, $\{P_{t+s}\}_{t=0}^{\infty} \in \mathcal{B}$.
Fix

$$\epsilon_1 \in (0, \epsilon(64L_0)^{-1}). \tag{9.188}$$

Lemmas 9.10 and 9.11 imply that there exist $\delta_1 \in (0, 1)$ and a natural number m_0 such that the following property holds:

(ii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, each integer $n \ge m_0$, each integer $s \ge 0$ and each finite sequence $\{y_i\}_{i=0}^{\infty} \subset B_X(0, 3M)$ satisfying

$$B_X(y_{t+1}, \delta_1) \cap P_{t+s}(B_X(y_t, \delta_1)) \ne \emptyset,\ t = 0, 1, \dots,$$

the inequality

$$d(y_t, C) \le \epsilon_1$$

holds for all integers $t \ge m_0$.

Since $\lim_{i\to\infty}\alpha_i = 0$ (see (9.28)) there is an integer $p_0 > 0$ such that for all integers $i \ge p_0$, we have

$$\alpha_i \le \min\{(\delta_1/2)(L_0+2)^{-1},\ 16^{-1}\epsilon_1(64(L_0)^{-2}\}. \tag{9.189}$$

Since $\sum_{i=0}^{\infty}\alpha_i = \infty$ (see (9.25)) there exists a natural number

$$n_0 > p_0 + 4 + m_0$$

such that

$$\sum_{i=p_0+m_0}^{n_0-1} \alpha_i > 128M^2\epsilon^{-1}(L_0+1)^2 p_0. \tag{9.190}$$

Fix a positive number δ such that

$$\delta(4M+4L_0+8) < 8^{-1}(64L_0)^{-1}\epsilon_1. \tag{9.191}$$

Fix

$$\bar{x} \in C_{min}. \tag{9.192}$$

Assume that

$$\{P_t\}_{t=0}^{\infty} \in \mathcal{B},\ \{\xi_t\}_{t=0}^{\infty} \subset X,\ \{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}}, \tag{9.193}$$

$$\|x_0\| \le M, \tag{9.194}$$

for all integers $t \ge 0$,

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \ne \emptyset, \tag{9.195}$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \le \alpha_t \delta. \tag{9.196}$$

In view of (9.9)–(9.11), and (9.192),

$$\|\bar{x}\| \le \bar{K}. \tag{9.197}$$

We show that

$$\|x_t - \bar{x}\| \le 2M \tag{9.198}$$

for all integers $t \ge 0$. Equations (9.194) and (9.197) imply that (9.198) holds for $t = 0$.

Assume that $t \ge 0$ is an integer and that (2.197) holds. In view of (9.197) and (9.198),

$$\|x_t\| \le 3M. \tag{9.199}$$

Lemma 9.13 implies that

$$\partial_\delta f(x_t) \subset B_X(0, L_0 + \delta \min\{1, \bar{r}\}^{-1}) \subset B_X(0, L_0+1). \tag{9.200}$$

By (9.195) and (9.200),

$$\|\xi_t\| \leq L_0 + 2. \tag{9.201}$$

There are two cases:

$$f(x_t) \leq \inf(f, C) + 8\bar{L}; \tag{9.202}$$

$$f(x_t) > \inf(f, C) + 8\bar{L}. \tag{9.203}$$

Assume that (9.202) holds. In view of (9.22) and (9.202),

$$\|x_t\| \leq M/2 - 1. \tag{9.204}$$

Equations (9.197) and (9.204) imply that

$$\|x_t - \bar{x}\| \leq \bar{K} + M/2. \tag{9.205}$$

It follows from (9.8), (9.21), (9.26), (9.193), (9.196), (9.201), and (9.205) that

$$\|x_{t+1} - \bar{x}\| \leq \alpha_t \delta + \|P_t(x_t - \alpha_t \xi_t) - \bar{x}\|$$

$$\leq \alpha_t \delta + \|x_t - \bar{x}\| + \alpha_t \|\xi_t\|$$

$$\leq 1 + \bar{K} + M/2 + \alpha_t(L_0 + 2) \leq 2 + \bar{K} + M/2 < M,$$

$$\|x_{t+1} - \bar{x}\| < M.$$

Assume that (9.203) holds. In view of (9.23), (9.192), (9.193), (9.195), (9.196), and (9.199), we apply Lemma 9.14 with

$$P = P_t,\ K_0 = 3M,\ \delta_f, \Delta = \delta,\ \delta_C = \delta\alpha_t,\ r = \bar{r},$$

$$\xi = \xi_t,\ x = x_t,\ y = x_{t+1},\ z = \bar{x}$$

and obtain that

$$8\alpha_t \bar{L} \leq \alpha_t (f(x_t) - f(\bar{x}))$$

$$\leq 2^{-1}\|x_t - \bar{x}\|^2 - 2^{-1}\|x_{t+1} - \bar{x}\|^2$$

$$+\delta\alpha_t(3M + \bar{K} + L_0 + 4) + \alpha_t \delta + (L_0 + 1)^2 \alpha_t^2$$

$$+ \alpha_t \delta(3M + \bar{K} + L_0 + 3). \tag{9.206}$$

By (9.26), (9.191), (9.198), and (9.206),

$$\|x_{t+1}-\bar{x}\|^2 \le \|x_t-\bar{x}\|^2 - 16\alpha_t\bar{L}$$

$$+4\delta\alpha_t(3M+\bar{K}+L_0+5)+2(L_0+1)^2\alpha_t^2$$

$$\le \|x_t-\bar{x}\|^2 - 12\alpha_t\bar{L} + 2\alpha_t^2(L_0+1)^2$$

$$\le \|x_t-\bar{x}\|^2 - 12\alpha_t\bar{L} + \alpha_t \le \|x_t-\bar{x}\|^2,$$

$$\|x_{t+1}-\bar{x}\| \le \|x_t-\bar{x}\| \le 2M.$$

Thus in the both cases

$$\|x_{t+1}-\bar{x}\| \le 2M.$$

By induction we showed that

$$\|x_t-\bar{x}\| \le 2M,\ t=0,1,\dots. \tag{9.207}$$

Equations (9.197) and (9.207) imply that

$$\|x_t\| \le 3M,\ t=0,1,\dots.$$

By (9.12), (9.193), (9.195), and (9.208), the relation above and Lemma 9.13,

$$\partial f_\delta(x_t) \subset B_X(0, L_0+1),\ \|\xi_t\| \le L_0+2. \tag{9.208}$$

Set

$$y_t = x_{t+p_0},\ t=0,1,\dots. \tag{9.209}$$

In view of (9.207) and (9.209),

$$\{y_t\}_{t=0}^{\infty} \subset B_X(0,3M). \tag{9.210}$$

It follows from (9.189), (9.196), (9.208), and (9.209) that for all integers $t \ge 0$,

$$\|y_t-(y_t-\alpha_{t+p_0}\xi_{t+p_0})\| \le \|x_{t+p_0}-(x_{t+p_0}-\alpha_{t+p_0}\xi_{t+p_0})\|$$

$$\le \alpha_{t+p_0}\|\xi_{t+p_0}\| \le \alpha_{t+p_0}(L_0+2) \le \delta_1,$$

$$\|y_{t+1}-P_{t+p_0}(y_t-\alpha_{t+p_0}\xi_{t+p_0})\|$$

$$= \|x_{t+1+p_0} - P_{t+p_0}(x_{t+p_0} - \alpha_{t+p_0}\xi_{t+p_0})\| \le \alpha_{t+p_0} \le \delta_1,$$

$$B_X(y_{t+1}, \delta_1) \cap P_{t+p_0}(B_X(y_t, \delta_1)) \ne \emptyset. \tag{9.211}$$

Property (ii), (9.293), (9.210), and (9.211) imply that

$$d(y_t, C) \le \epsilon_1 \text{ for all integers } t \ge m_0.$$

Together with (9.209) this implies that ,

$$d(x_t, C) \le \epsilon_1 \text{ for all integers } t \ge m_0 + p_0. \tag{9.212}$$

Assertion 1 is proved while Assertion 4 and the existence of n_1 follows from (9.25) and (9.212).

Let $t \ge 0$ be an integer. By (9.23), (9.193), (9.195), (9.196), and (9.199) and Lemma 9.14 applied with $P = P_t$, $K_0 = 3M$, $\alpha = \alpha_t$, $\Delta = \delta$, $\delta_f = \delta$, $\delta_C = \alpha_t\delta$, $r = \bar{r}$, $x = x_t$, $y = x_{t+1}$, $\xi = \xi_t$, $z = \bar{x}$, we have

$$\alpha_t(f(x_t) - f(\bar{x}))$$

$$\le 2^{-1}\|x_t - \bar{x}\|^2 - 2^{-1}\|x_{t+1} - \bar{x}\|^2 + \alpha_t\delta(3M + \bar{K} + L_0 + 4)$$

$$+ \alpha_t\delta + \alpha^2(L_0 + 1)^2 + \alpha_t\delta(3M + \bar{K} + L_0 + 3). \tag{9.213}$$

Let $m > n \ge 0$ be integers. By (9.197), (9.208), and (9.213),

$$\sum_{t=n}^{m-1} \alpha_t(f(x_t) - f(\bar{x}))$$

$$\le 2^{-1}\sum_{t=n}^{m-1}(\|x_t - \bar{x}\|^2 - \|x_{t+1} - \bar{x}\|^2)$$

$$+ \sum_{t=n}^{m-1} \alpha_t\delta(3M + \bar{K} + L_0 + 4) + \sum_{t=n}^{m-1} \alpha_t\delta$$

$$+ \sum_{t=n}^{m-1} \alpha_t^2(L_0 + 1)^2 + (\sum_{t=n}^{m-1} \alpha_t)\delta(3M + \bar{K} + L_0 + 3)$$

$$\le 2^{-1}(3M + \bar{K})^2 + 2\delta(3M + \bar{K} + L_0 + 6)\sum_{t=n}^{m-1} \alpha_t + (L_0 + 1)^2\sum_{t=n}^{m-1} \alpha_t^2. \tag{9.214}$$

In view of (9.192) and (9.214),

$$\min\{f(x_t):\ t=n,\dots,m-1\}-\inf(f,C),$$

$$f((\sum_{i=n}^{m-1}\alpha_i)^{-1}\sum_{t=n}^{m-1}\alpha_t x_t)-\inf(f,C)$$

$$\le(\sum_{i=n}^{m-1}\alpha_i)^{-1}\sum_{t=n}^{m-1}\alpha_t f(x_t)-\inf(f,C)$$

$$\le 2^{-1}(\sum_{t=n}^{m-1}\alpha_t)^{-1}(3M+\bar{K})^2+2\delta(3M+\bar{K}+L_0+6)$$

$$+(L_0+1)^2(\sum_{i=n}^{m-1}\alpha_i)^2)(\sum_{t=n}^{m-1}\alpha_t)^{-1}. \tag{9.215}$$

Let

$$m-n\ge p_0. \tag{9.216}$$

It follows from (9.189), (9.191), (9.215), and (9.216) that

$$\min\{f(x_t):\ t=n,\dots,m-1\}-\inf(f,C),$$

$$f((\sum_{i=n}^{m-1}\alpha_i)^{-1}\sum_{t=n}^{m-1}\alpha_t x_t)-\inf(f,C)$$

$$\le(\sum_{i=n}^{m-1}\alpha_i)^{-1}\sum_{t=n}^{m-1}\alpha_t f(x_t)-\inf(f,C)$$

$$\le(\sum_{t=n}^{m-1}\alpha_t)^{-1}(3M+\bar{K})^2+\epsilon/8$$

$$+(L_0+1)^2\max\{\alpha_t:\ t=n,\dots,m-1\}$$

$$\le(\sum_{t=n}^{m-1}\alpha_t)^{-1}(3M+\bar{K})^2+\epsilon/8+\epsilon/8<\epsilon$$

when m is sufficiently large. Thus Assertion 2 holds.

Let an integer $m > n_0$. By (9.189)–(9.191), and (9.215),

$$\min\{f(x_t) : \ t = 0, \dots, m-1\} - \inf(f, C),$$

$$f((\sum_{i=0}^{m-1} \alpha_i)^{-1} \sum_{t=0}^{m-1} \alpha_t x_t) - \inf(f, C)$$

$$\leq (\sum_{i=0}^{m-1} \alpha_i)^{-1} \sum_{t=0}^{m-1} \alpha_t f(x_t) - \inf(f, C)$$

$$< 8^{-1}\epsilon + 8^{-1}\epsilon + (\sum_{i=0}^{m-1} \alpha_i)^{-1} \sum_{t=p_0}^{m-1} \alpha_t^2 (L_0+1)^2 + (L_0+1)^2 p_0 (\sum_{i=0}^{m-1} \alpha_i)^{-1}$$

$$< 4^{-1}\epsilon + 8^{-1}\epsilon + 4^{-1}\epsilon < \epsilon.$$

Assertion 3 is proved. This completes the proof of Theorem 9.3.

9.10 Proof of Theorem 9.4

We may assume without loss of generality that

$$\epsilon < \min\{1, \bar{r}\}. \tag{9.217}$$

We may assume that if (A2) is true, then the following property holds:

(i) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $s \geq 0$, $\{P_{t+s}\}_{t=0}^{\infty} \in \mathcal{B}$.
Fix

$$\epsilon_1 \in (0, \epsilon(64L_0)^{-1}). \tag{9.218}$$

Lemmas 9.10 and 9.11 imply that there exist $\delta_1 \in (0, 1)$ and a natural number m_0 such that the following property holds:

(ii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, each integer $n \geq m_0$, each integer $s \geq 0$ and each finite sequence $\{y_i\}_{i=0}^{\infty} \subset B_X(0, 3M)$ satisfying

$$B_X(y_{t+1}, \delta_1) \cap P_{t+s}(B_X(y_t, \delta_1)) \neq \emptyset, \ t = 0, 1, \dots,$$

the inequality

$$d(y_t, C) \leq \epsilon_1$$

holds for all integers $t \geq m_0$.

Choose a positive number β_0 such that

$$\beta_0 \le \min\{\delta_1, 76^{-1}\epsilon_1(L_0+2)^{-2}\}. \tag{9.219}$$

Let

$$\beta_1 \in (0, \beta_0). \tag{9.220}$$

Choose a natural number

$$n_0 > m_0 + 16\epsilon_1^{-1}\beta_1^{-1}(2M+3)^2 \tag{9.221}$$

a positive number δ such that

$$\delta(4M + 4L_0 + 8) < 16^{-1}L_0^{-1}\epsilon_1\beta_1. \tag{9.222}$$

Fix

$$\bar{x} \in C_{min}. \tag{9.223}$$

Assume that

$$\{P_t\}_{t=0}^{\infty} \in \mathcal{B},\ \{\xi_t\}_{t=0}^{\infty} \subset X,\ \{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}}, \tag{9.224}$$

$$\|x_0\| \le M, \tag{9.225}$$

$$\alpha_t \in [\beta_1, \beta_0],\ t = 0, 1, \dots, \tag{9.226}$$

for all integers $t \ge 0$,

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \ne \emptyset, \tag{9.227}$$

$$\|x_{t+1} - P_t(x_t - \alpha_t\xi_t)\| \le \delta. \tag{9.228}$$

We show that

$$\|x_t - \bar{x}\| \le 2M \tag{9.229}$$

for all integers $t \ge 0$. Equations (9.25) and (9.223) imply that (9.229) holds for $t = 0$.

Assume that $t \ge 0$ is an integer and that (9.229) holds. In view of (9.223) and (9.229),

$$\|x_t\| \le 3M. \tag{9.230}$$

Lemma 9.13, (9.225), and (9.230) imply that

$$\partial_\delta f(x_t) \subset B_X(0, L_0 + \delta \bar{r}^{-1}) \subset B_X(0, L_0 + 1). \tag{9.231}$$

By (9.227) and (9.231),

$$\|\xi_t\| \le L_0 + 2. \tag{9.232}$$

There are two cases:

$$f(x_t) \le \inf(f, C) + 8\bar{L}; \tag{9.233}$$

$$f(x_t) > \inf(f, C) + 8\bar{L}. \tag{9.234}$$

Assume that (9.233) holds. In view of (9.223), (9.227) and (9.233),

$$\|x_t\| \le M/2 - 1, \ \|x_t - \bar{x}\| \le \bar{K} + M/2. \tag{9.235}$$

It follows from (9.8), (9.128), (9.222), (9.223), (9.226), (9.228), (9.232), and (9.235) that

$$\|x_{t+1} - \bar{x}\| \le \delta + \|P_t(x_t - \alpha_t \xi_t) - \bar{x}\|$$

$$\le \delta + \|x_t - \alpha_t \xi_t - \bar{x}_t\|$$

$$\le \delta + \|x_t - \bar{x}\| + \beta_0(L_0 + 2) \le 2 + \bar{K} + M/2 < M.$$

Assume that (9.234) holds. In view of (9.104), (9.224), (9.227), (9.228), and (9.230), we apply Lemma 9.14 with

$$P = P_t, \ r = \bar{r}, \ K_0 = 3M, \ \delta_f, \Delta, \delta_C = \delta,$$

$$\xi = \xi_t, \ x = x_t, \ y = x_{t+1}, \ z = \bar{x}$$

and obtain that

$$2\alpha_t(f(x_t) - f(\bar{x}))$$

$$\le \|x_t - \bar{x}\|^2 - \|x_{t+1} - \bar{x}\|^2$$

$$+2\delta(3M + \bar{K} + L_0 + 4) + 2\alpha_t\delta + (L_0 + 1)^2\alpha_t^2 + 2\alpha_t\delta(3M + \bar{K} + L_0 + 3)$$

$$\le \|x_t - \bar{x}\|^2 - \|x_{t+1} - \bar{x}\|^2$$

$$+4\delta(3M + \bar{K} + L_0 + 6) + (L_0 + 1)^2\alpha_t^2.$$

Combined with (9.128), (9.222), (9.223), (9.226), (9.229), and (9.234) this implies that

$$\|x_{t+1} - \bar{x}\|^2 \le \|x_t - \bar{x}\|^2$$

$$+4\delta(3M + \bar{K} + L_0 + 6) - 16\alpha_t \bar{L} + (L_0 + 1)^2 \alpha_t^2$$

$$\le \|x_t - \bar{x}\|^2 + \beta_1 - 16\alpha_t \bar{L} + \alpha_t \beta_0 (L_0 + 1)^2$$

$$\le \|x_t - \bar{x}\|^2 + \beta_1 - 16\alpha_t \bar{L} + \alpha_t$$

$$\le \|x_t - \bar{x}\|^2 + \beta_1 - 8\bar{L}\beta_1 < \|x_t - \bar{x}\|^2,$$

$$\|x_{t+1} - \bar{x}\| \le \|x_t - \bar{x}\| \le 2M.$$

Thus in the both cases

$$\|x_{t+1} - \bar{x}\| \le 2M.$$

By induction we showed that

$$\|x_t - \bar{x}\| \le 2M, \ t = 0, 1, \dots. \tag{9.236}$$

Equations (9.223) and (9.236) imply that

$$\|x_t\| \le 3M, \ t = 0, 1, \dots. \tag{9.237}$$

By (9.227), (9.237) and Lemma 9.13, for all integers $t \ge 0$,

$$\partial f_\delta(x_t) \subset B_X(0, L_0 + 1), \ \|\xi_t\| \le L_0 + 2.. \tag{9.238}$$

It follows from (9.128), (9.220), (9.222), (9.226), (9.228), and (9.238) that for all integers $t \ge 0$,

$$\|x_t - (x_t - \alpha_t \xi_t)\| \le \alpha_t \|\xi_t)\| \le \beta_0(L_0 + 2) \le \delta_1,$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \le \delta \le \delta_1,$$

$$B_X(x_{t+1}, \delta_1) \cap P_t(B_X(x_t, \delta_1)) \ne \emptyset. \tag{9.239}$$

In view of (9.239),

$$d(x_t, C) \le \epsilon_1 \text{ for all integers } t \ge m_0. \tag{9.240}$$

By (9.237) and (9.240), Assertion 1 is proved while Assertion 4 and the existence of n_1 follows from (9.226) and (9.240).

Let $t \geq 0$ be an integer. By (9.224), (9.227), (9.228), (9.237) and Lemma 9.14 applied with $P = P_t$, $K_0 = 3M$, $\alpha = \alpha_t$, $\Delta = \delta$, $\delta_f = \delta$, $\delta_C = \delta$, $r = \bar{r}$, $x = x_t$, $y = x_{t+1}, \xi = \xi_t$, we have

$$\alpha_t(f(x_t) - f(\bar{x}))$$

$$\leq 2^{-1}\|x_t - \bar{x}\|^2 - 2^{-1}\|x_{t+1} - \bar{x}\|^2 + \delta(3M + \bar{K} + L_0 + 4)$$

$$+\alpha_t\delta + \alpha^2(L_0 + 1)^2 + \alpha_t\delta(3M + \bar{K} + L_0 + 3)$$

$$\leq 2^{-1}\|x_t - \bar{x}\|^2 - 2^{-1}\|x_{t+1} - \bar{x}\|^2$$

$$+ 2\delta(3M + \bar{K} + L_0 + 6) + \alpha^2(L_0 + 1)^2. \tag{9.241}$$

Let $m > n \geq 0$ be integers. By (9.226) and (9.241),

$$\sum_{t=n}^{m-1} \alpha_t(f(x_t) - f(\bar{x}))$$

$$\leq 2^{-1}\sum_{t=n}^{m-1}(\|x_t - \bar{x}\|^2 - \|x_{t+1} - \bar{x}\|^2)$$

$$+2(m-n)\delta(3M + \bar{K} + L_0 + 6) + \sum_{t=n}^{m-1} \alpha_t^2(L_0 + 1)^2$$

$$\leq 2^{-1}\|x_n - \bar{x}\|^2 + 2(m-n)\delta(3M + \bar{K} + L_0 + 6) + \sum_{t=n}^{m-1} \alpha_t\beta_0(L_0 + 1). \tag{9.242}$$

By (9.128), (9.222), (9.223), (9.226), (9.236), and (9.242),

$$(\sum_{i=n}^{m-1} \alpha_i)^{-1} \sum_{t=n}^{m-1} \alpha_t f(x_t) - \inf(f, C)$$

$$\leq 2M^2(\sum_{i=n}^{m-1} \alpha_i)^{-1} + 2(m-n)\delta(3M + \bar{K} + L_0 + 6)((m-n)\beta_1)^{-1} + \beta_0(L_0 + 1)$$

$$\leq 2M^2\beta_1^{-1}(m-n)^{-1} + 2\delta\beta_1^{-1}(3M + \bar{K} + L_0 + 6) + \beta_0(L_0 + 1)$$

$$\leq 2M^2\beta_1^{-1}(m-n)^{-1} + \epsilon/4. \tag{9.243}$$

Let

$$m - n \geq n_0. \tag{9.244}$$

It follows from (9.221), (9.243), and (9.244) that

$$\min\{f(x_t) : \ t = n, \dots, m-1\} - \inf(f, C),$$

$$f((\sum_{i=n}^{m-1} \alpha_i)^{-1} \sum_{t=n}^{m-1} \alpha_t x_t) - \inf(f, C)$$

$$\leq (\sum_{i=n}^{m-1} \alpha_i)^{-1} \sum_{t=n}^{m-1} \alpha_t f(x_t) - \inf(f, C)$$

$$\leq \epsilon/4 + 2M^2\beta_1^{-1}n_0^{-1} < \epsilon/2 + \epsilon_1 < \epsilon.$$

Thus Assertions 2 and 4 hold. Theorem 9.4 is proved.

9.11 Proof of Theorem 9.5

We may assume without loss of generality that

$$\epsilon < 1, \ \bar{r} \leq 1. \tag{9.245}$$

In view of Proposition 9.17, there exists a number

$$\bar{\epsilon} \in (0, \min\{\epsilon/8, \bar{r}/8\}) \tag{9.246}$$

such that the following property holds:

(i)

$$\text{if } x \in U, \ d(x, C) \leq 2\bar{\epsilon} \text{ and } f(x) \leq \inf(f, C) + 2\bar{\epsilon},$$

$$\text{then } d(x, C_{min}) \leq \epsilon.$$

We may assume that if (A2) holds, the following property holds:

(ii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $s \geq 0$, $\{P_{t+s}\}_{t=0}^{\infty} \in \mathcal{B}$.

In view of (9.4), we may assume without loss of generality that

$$M > 4\bar{K} + 8, \tag{9.247}$$

$$\{x \in U : f(x) \leq \inf(f, C) + 16\bar{L}\} \subset B_X(0, 2^{-1}M - 1). \tag{9.248}$$

Fix

$$\bar{\epsilon}_1 \in (0, \bar{\epsilon}(64\bar{L})^{-1}). \tag{9.249}$$

Lemmas 9.10 and 9.11 imply that there exist $\delta_1 \in (0, 1)$ and a natural number m_0 such that the following property holds:

(iii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, each integer $n \geq m_0$, each integer $s \geq 0$ and each finite sequence $\{y_i\}_{i=0}^{n} \subset B_X(0, 3M)$ satisfying

$$B_X(y_{t+1}, \delta_1) \cap P_{t+s}(B_X(y_t, \delta_1)) \neq \emptyset,\ t = 0, \ldots, n-1$$

the inequality

$$d(y_t, C) \leq \bar{\epsilon}_1$$

holds for all integers $t \in [m_0, n]$.

Since $\lim_{i\to\infty} \alpha_i = 0$ (see (9.33)) there is an integer $p_0 > 0$ such that for all integers $i \geq p_0$, we have

$$\alpha_i \leq \delta_1/2. \tag{9.250}$$

Fix

$$\bar{x} \in C_{min}. \tag{9.251}$$

Since $\lim_{i\to\infty} \alpha_i = 0$ (see (9.33)) there is an integer

$$p_1 > p_0 \tag{9.252}$$

such that for every integer $i \geq p_1$,

$$\alpha_i < 16^{-1}(32\bar{L})^{-1}\bar{\epsilon}_1. \tag{9.253}$$

Since $\sum_{i=0}^{\infty} \alpha_i = \infty$ (see (9.33)) there exists a natural number

$$n_0 > p_0 + p_1 + 4 + m_0 \tag{9.254}$$

such that

$$\sum_{i=p_0+p_1+m_0}^{n_0-1} \alpha_i > 128(3M+\bar{K})^2\epsilon^{-1}\bar{L}+1. \quad (9.255)$$

Fix a positive number δ such that

$$\delta(3M+\bar{K}+4) < 8^{-1}(64\bar{L})^{-1}\bar{\epsilon}_1 \min\{1,\bar{r}\}. \quad (9.256)$$

Assume that $n \geq n_0$ is an integer,

$$\{P_t\}_{t=0}^{\infty} \in \mathcal{B}, \quad (9.257)$$

$$\{x_t\}_{t=0}^{n} \subset U_{\bar{r}}, \ \|x_0\| \leq M, \quad (9.258)$$

$$v_t \in \partial_\delta f(x_t) \setminus \{0\}, \ t=0,1,\dots,n-1 \quad (9.259)$$

$$\{\eta_t\}_{t=0}^{n-1}, \ \{\xi_t\}_{t=0}^{n-1} \subset B_X(0,\delta), \quad (9.260)$$

and that for all integers $t=0,\dots,n-1$, we have

$$x_{t+1} = P_t(x_t - \alpha_t\|v_t\|^{-1}v_t - \alpha_t\xi_t) - \alpha_t\eta_t. \quad (9.261)$$

In order to prove the theorem it is sufficient to show that

$$d(x_t, C_{min}) \leq \epsilon \text{ for all integers } t \text{ satisfying } n_0 \leq t \leq n.$$

First we show that for all integers $t=0,\dots,n$,

$$d(x_t, C_{min}) \leq 2M. \quad (9.262)$$

In view of (9.9)–(9.11), (9.247), and (9.258), inequality (9.262) holds for $t=0$. Assume that

$$t \in \{0,\dots,n\} \setminus \{n\}$$

and that (9.262) is true. There are two cases:

$$f(x_t) \leq \inf(f,C) + 4\bar{L}\bar{r}; \quad (9.263)$$

$$f(x_t) > \inf(f,C) + 4\bar{L}\bar{r}. \quad (9.264)$$

Assume that (9.263) holds. In view of (9.248) and (9.263),

$$\|x_t\| \le M/2 - 1. \tag{9.265}$$

By (9.9)–(9.11), and (9.251),

$$\|\bar{x}\| \le \bar{K}. \tag{9.266}$$

It follows from (9.265) and (9.266) that

$$\|x_t - \bar{x}\| \le \bar{K} + M/2. \tag{9.267}$$

By (9.8), (9.247), (9.260), (9.261), and (9.267),

$$\|x_{t+1} - \bar{x}\|$$

$$\le \alpha_t \|\eta_t\| + \|\bar{x} - P_t(x_t - \alpha_t \|v_t\|^{-1} v_t - \alpha_t \xi_t)\|$$

$$\le \alpha_t \delta + \|\bar{x} - (x_t - \alpha_t \|v_t\|^{-1} v_t - \alpha_t \xi_t)\|$$

$$\le \alpha_t \delta + \|\bar{x} - x_t\| + \alpha_t + \alpha_t \delta$$

$$\le \|\bar{x} - x_t\| + 3$$

$$\le \bar{K} + M/2 + 3 < 2M$$

and

$$d(x_{t+1}, C_{min}) \le 2M. \tag{9.268}$$

Assume that (9.264) holds. It follows from ((9.32), (9.36), (9.247), (9.256), (9.258), (9.259), (9.262), (9.264) and Lemma 9.16 applied with

$$P = P_t,\ \epsilon = 4\bar{L}\bar{r},\ r = \bar{r},\ \Delta = \delta,\ K_0 = 3M,$$

$$\alpha = \alpha_t,\ \delta_f = \delta,\ \delta_C = \delta\alpha_t,$$

$$x = x_t,\ \xi = \xi_t + \|v_t\|^{-1} v_t,\ v = v_t,\ y = x_{t+1},\ \eta = \alpha_t \eta_t$$

that

$$d(x_{t+1}, C_{min})^2 \le d(x_t, C_{min})^2 - \alpha_t \bar{r} + 4\alpha_t^2 + 2\alpha_t \delta(3M + \bar{K} + 3)$$

$$\le d(x_t, C_{min})^2 - \alpha_t \bar{r}/2 + 2\alpha_t \delta(3M + \bar{K} + 3) \le d(x_t, C_{min})^2.$$

Together with (9.262) this implies that

$$d(x_{t+1}, C_{min}) \le 2M.$$

Thus in both cases

$$d(x_{i+1}, C_{min}) \le 2M.$$

Thus we have shown by induction that for all integers $i = 0, \dots, n$,

$$d(x_i, C_{min}) \le 2M.$$

Together with (9.9)–(9.11), and (9.247) this implies that

$$\|x_t\| \le 3M, \ t = 0, \dots, n. \tag{9.269}$$

Set

$$y_t = x_{t+p_0}, \ t = 0, \dots, n - p_0. \tag{9.270}$$

By (9.269) and (9.270),

$$\{y_t\}_{t=0}^{n-p_0} \subset B_X(0, 3M). \tag{9.271}$$

In view of (9.254),

$$n - p_0 > m_0. \tag{9.272}$$

It follows from (9.250), (9.260), (9.261), and (9.270) that for all integers $t = 0, \dots, n - p_0 - 1$,

$$\|y_t - (y_t - \alpha_{t+p_0}\|v_{t+p_0}\|^{-1}v_{t+p_0} - \alpha_{t+p_0}\xi_{t+p_0})\|$$

$$\le \|x_{t+p_0} - (x_{t+p_0} - \alpha_{t+p_0}\|v_{t+p_0}\|^{-1}v_{t+p_0} - \alpha_{t+p_0}\xi_{t+p_0})\|$$

$$\le 2\alpha_{t+p_0} \le \delta_1,$$

$$\|y_{t+1} - P_{t+p_0}(y_t - \alpha_{t+p_0}\|v_{t+p_0}\|^{-1}v_{t+p_0} - \alpha_{t+p_0}\xi_{t+p_0})\|$$

$$\le \alpha_{t+p_0} \le \delta_1,$$

$$B_X(y_{t+1}, \delta_1) \cap P_{t+p_0}(B_X(y_t, \delta_1)) \ne \emptyset. \tag{9.273}$$

Property (iii) and (9.271)–(9.273) imply that

$$d(y_t, C) \le \bar{\epsilon}_1, \ t = m_0, \dots, n - p_0.$$

Together with (9.270) this implies that ,

$$d(x_t, C) \le \epsilon_1 < \bar{\epsilon},\ t = m_0 + p_0, \ldots, n. \tag{9.274}$$

. Assume that an integer

$$t \in [p_0 + p_1 + m_0, n - 1], \tag{9.275}$$

$$f(x_t) > \inf(f, C) + \bar{\epsilon}/8. \tag{9.276}$$

It follows from (9.30), (9.31), (9.209), (9.246), (9.251), (9.253), (9.256), (9.258), (9.261), (9.269), and (9.276) and Lemma 9.16 applied with

$$P = P_t,\ r = \bar{r},\ K_0 = 3M,\ \epsilon = \bar{\epsilon}/8,\ \alpha = \alpha_t,$$

$$\delta_f = \delta,\ \delta_C = \delta\alpha_t,\ \Delta = \delta,\ x = x_t,\ \xi = \xi_t + \|v_t\|^{-1}v_t,\ y = x_{t+1},$$

that

$$\|x_{t+1} - \bar{x}\|^2 \le \|x_t - \bar{x}\|^2 - \alpha_t(4\bar{L})^{-1}\bar{\epsilon}/8$$

$$+2\alpha_t^2 + \alpha_t^2\delta^2 + 2\alpha_t\delta(3M + \bar{K} + 3)$$

$$\le \|x_t - \bar{x}\|^2 - \alpha_t(64\bar{L})^{-1}\bar{\epsilon}$$

$$+2\alpha_t\delta(3M + \bar{K} + 4)$$

$$\le \|x_t - \bar{x}\|^2 - \alpha_t(128\bar{L})^{-1}\bar{\epsilon}.$$

Thus we have shown that the following property holds:

(iv) if an integer t satisfies (9.275) and (9.276), then we have

$$\|x_{t+1} - \bar{x}\|^2 \le \|x_t - \bar{x}\|^2 - (128\bar{L})^{-1}\alpha_t\bar{\epsilon}.$$

We claim that there exists an integer $j \in \{p_0 + p_1 + m_0, \ldots, n_0\}$ such that

$$f(x_j) \le \inf(f, C) + \bar{\epsilon}/8.$$

Assume the contrary. Then

$$f(x_j) > \inf(f, C) + \bar{\epsilon}/8,\ j = p_0 + p_1 + m_0, \ldots, n_0. \tag{9.277}$$

Property (v) and (9.277) imply that for all integers $t = p_0 + p_1 + m_0, \ldots, n_0 - 1$,

$$\|x_{t+1} - \bar{x}\|^2 \leq \|x_t - \bar{x}\|^2 - (128\bar{L})^{-1}\alpha_t\bar{\epsilon}. \tag{9.278}$$

Relations (9.9)–(9.11), (9.251), (9.269), and (9.278) imply that

$$(3M + \bar{K})^2 \geq \|x_{p_0+p_1+m_0} - \bar{x}\|^2$$

$$\geq \|x_{p_0+p_1+m_0} - \bar{x}\|^2 - \|x_{n_0} - \bar{x}\|^2$$

$$= \sum_{t=p_0+p_1+m_0}^{n_0-1} [\|x_t - \bar{x}\|^2 - \|x_{t+1} - \bar{x}\|^2]$$

$$\geq (128\bar{L})^{-1}\bar{\epsilon} \sum_{t=p_0+p_1+m_0}^{n_0-1} \alpha_t. \tag{9.279}$$

In view of (9.279),

$$\sum_{t=p_0+p_1+m_0}^{n_0-1} \alpha_t \leq 128(3M + \bar{K})^2\bar{L}\bar{\epsilon}^{-1}.$$

This contradicts (9.255). The contradiction we have reached proves that there exists an integer

$$j \in \{p_0 + p_1 + m_0, \ldots, n_0\} \tag{9.280}$$

such that

$$f(x_j) \leq \inf(f, C) + \bar{\epsilon}/8. \tag{9.281}$$

By property (i), (9.274), (9.280), and (9.281), we have

$$d(x_j, C_{min}) \leq \epsilon. \tag{9.282}$$

We claim that for all integers t satisfying $j \leq t \leq n$,

$$d(x_t, C_{min}) \leq \epsilon.$$

Assume the contrary. Then there exists an integer $k \in [j, n]$ for which

$$d(x_k, C_{min}) > \epsilon. \tag{9.283}$$

By (9.280), (9.282), and (9.283), we have

$$k > j \geq p_0 + p_1 + m_0. \tag{9.284}$$

We may assume without loss of generality that

$$d(x_t, C_{min}) \leq \epsilon \text{ for all integers } t \text{ satisfying } j \leq t < k. \tag{9.285}$$

Thus

$$d(x_{k-1}, C_{min}) \leq \epsilon. \tag{9.286}$$

There are two cases:

$$f(x_{k-1}) \leq \inf(f, C) + \bar{\epsilon}/8; \tag{9.287}$$

$$f(x_{k-1}) > \inf(f, C) + \bar{\epsilon}/8. \tag{9.288}$$

Assume that (9.287) is valid. It follows from (9.274) and (9.284) that

$$d(x_{k-1}, C) \leq \bar{\epsilon}_1. \tag{9.289}$$

By (9.289), there exists a point

$$z \in C \tag{9.290}$$

such that

$$\|x_{k-1} - z\| < 2\bar{\epsilon}_1. \tag{9.291}$$

By (9.8), (9.253), (9.256), (9.260), (9.261), (9.290), and (9.291),

$$\|x_k - z\| \leq \alpha_{k-1}\delta$$

$$+\|z - P_{k-1}(x_{k-1} - \alpha_{k-1}\|v_{k-1}\|^{-1}v_{k-1} - \alpha_{k-1}\xi_{k-1})\|$$

$$\leq \delta + \|z - x_{k-1}\| + \alpha_{k-1} + \delta$$

$$\leq 2\bar{\epsilon}_1 + 2\delta + \alpha_{k-1} < 3\bar{\epsilon}_1. \tag{9.292}$$

In view of (9.291) and (9.292),

$$\|x_k - x_{k-1}\| \leq \|x_k - z\| + \|z - x_{k-1}\| < 5\bar{\epsilon}_1. \tag{9.293}$$

It follows from (9.7)–(9.9), (9.249), (9.286), and (9.293) that

$$\|x_{k-1}\| \leq \bar{K} + 2\epsilon,$$

$$\|x_k\| \le \|x_{k-1}\| + 5\bar{\epsilon}_1 \le \bar{K} + 3\epsilon$$

and

$$\|x_{k-1}\|, \ \|x_k\| \le \bar{K} + 4.$$

Combined with (9.12) and (9.293) the relation above implies that

$$|f(x_{k-1}) - f(x_k)| \le \bar{L}\|x_{k-1} - x_k\| \le 5\bar{L}\bar{\epsilon}_1.$$

Together with (9.128) and (9.287) this implies that

$$f(x_k) \le f(x_{k-1}) + 5\bar{L}\bar{\epsilon}_1$$

$$\le \inf(f, C) + \bar{\epsilon}/8 + 5\bar{L}\bar{\epsilon}_1 \le \inf(f, C) + \bar{\epsilon}/4. \tag{9.294}$$

Property (i), (9.274), (9.284), and (9.294) imply that

$$d(x_k, C_{min}) \le \epsilon.$$

This inequality contradicts (9.283). The contradiction we have reached proves (9.288). It follows from (9.30), (9.31), (9.255), (9.258), (9.260), (9.261), (9.269), and (9.288) that Lemma 9.16 holds with

$$P = P_t, \ r = \bar{r}, \ K_0 = 3M, \ \epsilon = \bar{\epsilon}/8, \ \alpha = \alpha_t,$$

$$\delta_f = \delta, \ \delta_C = \delta\alpha_{k-1}, \ \Delta = \delta, \ x = x_{k-1}, \ \xi = \xi_{k-1} + \|v_{k-1}\|^{-1}v_{k-1}, \ y = x_k$$

and combined with (9.253), (9.256), (9.284), and (9.286) this implies that

$$d(x_k, C_{min})^2 \le d(x_{k-1}, C_{min})^2 - \alpha_{k-1}(4\bar{L})^{-1}\bar{\epsilon}/8$$

$$+2\alpha_{k-1}^2 + \alpha_{k-1}^2\delta^2 + 2\alpha_{k-1}\delta(3M + \bar{K} + 3)$$

$$\le d(x_{k-1}, C_{min})^2 - \alpha_{k-1}(32\bar{L})^{-1}\bar{\epsilon}$$

$$+3\alpha_{k-1}^2 + 2\alpha_{k-1}\delta(3M + \bar{K} + 3)$$

$$\le d(x_{k-1}, C_{min})^2 - \alpha_{k-1}(64\bar{L})^{-1}\bar{\epsilon} + 2\alpha_{k-1}\delta(3M + \bar{K} + 3)$$

$$\le d(x_{k-1}, C_{min})^2 - \alpha_{k-1}(128\bar{L})^{-1}\bar{\epsilon}$$

and

$$d(x_k, C_{min}) \le d(x_{k-1}, C_{min}) \le \epsilon.$$

This contradicts (9.283). The contradiction we have reached proves that $d(x_t, C_{min}) \le \epsilon$ for all integers t satisfying $j \le t \le n$. This completes the proof of Theorem 9.5.

9.12 Proof of Theorem 9.6

In view of (9.4), we may assume without loss of generality that

$$\epsilon < 1, \ M > 8\bar{K} + 8 \tag{9.295}$$

and that

$$\{x \in U : f(x) \le \inf(f, C) + 16\bar{L}\} \subset B_X(0, 2^{-1}M - 1). \tag{9.296}$$

Proposition 9.17 implies that there exists a number

$$\bar{\epsilon} \in (0, \min\{\epsilon/8, \bar{r}/8\}) \tag{9.297}$$

such that the following property holds:

(i)

$$\text{if } x \in X, \ d(x, C) \le 2\bar{\epsilon} \text{ and } f(x) \le \inf(f, C) + 2\bar{\epsilon},$$

$$\text{then } d(x, C_{min}) \le \epsilon/4.$$

Fix

$$\bar{x} \in C_{min} \tag{9.298}$$

and

$$\bar{\epsilon}_1 \in (0, \bar{\epsilon}(64\bar{L})^{-1}). \tag{9.299}$$

We may assume that if (A2) holds, the following property holds:

(ii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $s \ge 0$, $\{P_{t+s}\}_{t=0}^{\infty} \in \mathcal{B}$.

Lemmas 9.10 and 9.11 imply that there exist $\delta_1 \in (0, 1)$ and a natural number m_0 such that the following property holds:

(iii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, each integer $n \ge m_0$, each integer $s \ge 0$ and each finite sequence $\{y_i\}_{i=0}^{n} \subset B_X(0, 3M)$ satisfying

$$B_X(y_{t+1}, \delta_1) \cap P_{t+s}(B_X(y_t, \delta_1)) \neq \emptyset, \ t = 0, \dots, n-1$$

the inequality

$$d(y_t, C) \leq \epsilon_1$$

holds for all integers $t \in [m_0, n]$.

Choose a positive number β_0 such that

$$\beta_0 \leq \delta_1/2, \ 2\beta_0 < (64^2 \bar{L})^{-1} \bar{\epsilon}_1. \tag{9.300}$$

Let

$$\beta_1 \in (0, \beta_0). \tag{9.301}$$

Fix a natural number n_0 such that

$$n_0 > m_0 + 4 \cdot 32^2 M^2 \bar{L} \bar{\epsilon}^{-1} \beta_1^{-1} \tag{9.302}$$

and a positive number δ such that

$$\delta(8M + \bar{K} + 4) \leq (128)^{-1} \bar{\epsilon}_1 \beta_1 \min\{1, \bar{r}\}. \tag{9.303}$$

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, $n \geq n_0$ is an integer,

$$\{x_t\}_{t=0}^{n} \subset U_{\bar{r}}, \ \|x_0\| \leq M,$$

$$v_t \in \partial_\delta f(x_t) \setminus \{0\}, \ t = 0, 1, \dots, n-1 \tag{9.304}$$

$$\{\alpha_t\}_{t=0}^{n-1} \subset [\beta_1, \beta_0], \tag{9.305}$$

$$\{\eta_t\}_{t=0}^{n-1}, \ \{\xi_t\}_{t=0}^{n-1} \subset B_X(0, \delta), \tag{9.306}$$

and that for all integers $t = 0, \dots, n-1$, we have

$$x_{t+1} = P_t(x_t - \alpha_t \|v_t\|^{-1} v_t - \alpha_t \xi_t) - \eta_t. \tag{9.307}$$

In order to prove the theorem it is sufficient to show that

$$d(x_t, C_{min}) \leq \epsilon \text{ for all integers } t \text{ satisfying } n_0 \leq t \leq n.$$

First we show that for all integers $t = 0, \dots, n$,

$$d(x_t, C_{min}) \leq 2M. \tag{9.308}$$

In view of (9.296), (9.304), inequality (9.308) holds for $t = 0$. Assume that

$$t \in \{0, \dots, n\} \setminus \{n\}$$

and that (9.308) is true. There are two cases:

$$f(x_t) \le \inf(f, C) + 4\bar{L}\min\{1, \bar{r}\}; \tag{9.309}$$

$$f(x_t) > \inf(f, C) + 4\bar{L}\min\{1, \bar{r}\}. \tag{9.310}$$

Assume that (9.309) holds. In view of (9.296) and (9.309),

$$\|x_t\| \le M/2 - 1. \tag{9.311}$$

By (9.9)–(9.11), and (9.298),

$$\|\bar{x}\| \le \bar{K}. \tag{9.312}$$

It follows from (9.311) and (9.312) that

$$\|x_t - \bar{x}\| \le \bar{K} + M/2. \tag{9.313}$$

By (9.8), (9.195), (9.298), (9.300), (9.303), (9.305)–(9.307), and (9.313),

$$d(x_{t+1}, C_{min}) \le \|x_{t+1} - \bar{x}\|$$

$$\le \|\eta_t\| + \|\bar{x} - P_t(x_t - \alpha_t\|v_t\|^{-1}v_t - \alpha_t\xi_t)\|$$

$$\le \delta + \|\bar{x} - x_t\| + \alpha_t + \alpha_t\delta$$

$$\le 2\beta_0 + \delta + \bar{K} + M/2 \le \bar{K} + M/2 + 3 \le M.$$

Assume that (9.310) holds. In view of (9.9)–(9.11), and (9.308),

$$\|x_t\| \le 3M. \tag{9.314}$$

It follows from (9.300), (9.303)–(9.307), (9.309), (9.310), and Lemma 9.16 applied with

$$P = P_t,\ \epsilon = 4\bar{L}\min\{1, \bar{r}\},\ r = \bar{r},\ \Delta = \delta,\ K_0 = 3M,$$

$$\alpha = \alpha_t,\ \delta_f = \delta,\ \delta_C = \delta,\ x = x_t,\ \xi = \xi_t + \|v_t\|^{-1}v_t,\ v = v_t,\ y = x_{t+1},$$

that

$$d(x_{t+1}, C_{min})^2 \le d(x_t, C_{min})^2 - \alpha_t \min\{1, \bar{r}\} + 2\alpha_t^2 + \delta^2 + 2\delta(3M + \bar{K} + 3)$$

$$\le d(x_t, C_{min})^2 - 2^{-1}\beta_1 \min\{1, \bar{r}\} + 2\delta(3M + \bar{K} + 4)$$

$$\le d(x_t, C_{min})^2 - 2^{-1}\beta_1 \bar{r} + 8\delta M \le d(x_t, C_{min})^2 - 4^{-1}\beta_1$$

and in view of (9.308),

$$d(x_{t+1}, C_{min}) \le d(x_t, C_{min}) \le 2M.$$

Thus in both cases

$$d(x_{t+1}, C_{min}) \le 2M.$$

Thus we have shown by induction that for all integers $t = 0, \dots, n$,

$$d(x_t, C_{min}) \le 2M. \tag{9.315}$$

By (9.9)–(9.11), (9.295), and (9.315),

$$\|x_t\| \le 3M, \; t = 0, \dots, n.$$

It follows from (9.300), (9.301), (9.303), (9.305)–(9.307) that for all integers $t = 0, \dots, n-1$,

$$\|x_t - (x_t - \alpha_t \|v_t\|^{-1} v_t - \alpha_t \xi_t)\|$$

$$\le \alpha_t + \alpha_t \delta \le 2\alpha_t \le 2\beta_0 \le \delta_1,$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \|v_t\|^{-1} v_t - \alpha_t \xi_t)\|$$

$$\le \|\eta_t\| \le \delta \le \beta_1 < \delta$$

and

$$B_X(x_{t+1}, \delta_1) \cap P_t(B_C(x_t, \delta_1)) \ne \emptyset. \tag{9.316}$$

Property (iii) and (9.302), and (9.316) imply that

$$d(x_t, C) \le \bar{\epsilon}_1, \; t = m_0, \dots, n. \tag{9.317}$$

Assume that an integer

$$t \in [m_0, n-1], \tag{9.318}$$

$$f(x_t) > \inf(f, C) + \bar{\epsilon}/8. \tag{9.319}$$

It follows from (9.198), (9.300), (9.303)–(9.307), (9.316), (9.319), and Lemma 9.16 applied with

$$P = P_t,\ r = \bar{r},\ K_0 = 3M,\ \epsilon = \bar{\epsilon}/8,\ \alpha = \alpha_t,$$

$$\delta_f = \delta,\ \delta_C = \delta,\ x = x_t,\ \xi = \xi_t + \|v_t\|^{-1} v_t,\ y = x_{t+1},$$

that

$$\|x_{t+1} - \bar{x}\|^2 \le \|x_t - \bar{x}\|^2 - \alpha_t (4\bar{L})^{-1} \bar{\epsilon}/8$$

$$+2\alpha_t^2 + \delta^2 + 2\delta(3M + \bar{K} + 3)$$

$$\le \|x_t - \bar{x}\|^2 - \alpha_t (64\bar{L})^{-1} \bar{\epsilon} + 2\delta(3M + \bar{K} + 4)$$

$$\le \|x_t - \bar{x}\|^2 - \beta_1 (64\bar{L})^{-1} \bar{\epsilon} + 2\delta(3M + \bar{K} + 4)$$

$$\le \|x_t - \bar{x}\|^2 - \beta_1 (128\bar{L})^{-1} \bar{\epsilon}.$$

Thus we have shown that the following property holds:

(iv) if an integer $k \in [m_0, n-1]$ satisfies

$$f(x_k) > \inf(f, C) + \bar{\epsilon}/8,$$

then we have

$$\|x_{k+1} - \bar{x}\|^2 \le \|x_k - \bar{x}\|^2 - (128\bar{L})^{-1} \beta_1 \bar{\epsilon}.$$

We claim that there exists an integer $j \in \{m_0, \dots, n_0\}$ such that

$$f(x_j) \le \inf(f, C) + \bar{\epsilon}/8.$$

Assume the contrary. Then

$$f(x_j) > \inf(f, C) + \bar{\epsilon}/8,\ i = m_0, \dots, n_0. \tag{9.320}$$

Property (iv) and (9.320) imply that for all integers $t = m_0, \dots, n_0 - 1$,

$$\|x_{t+1} - \bar{x}\|^2 \le \|x_t - \bar{x}\|^2 - (128\bar{L})^{-1} \beta_1 \bar{\epsilon}. \tag{9.321}$$

Relations (9.316) and (9.321) imply that

$$(4M)^2 \geq \|x_{m_0} - \bar{x}\|^2$$

$$\geq \|x_{m_0} - \bar{x}\|^2 - \|x_{n_0} - \bar{x}\|^2$$

$$= \sum_{t=m_0}^{n_0-1} [\|x_t - \bar{x}\|^2 - \|x_{t+1} - \bar{x}\|^2]$$

$$\geq (n_0 - m_0)(128\bar{L})^{-1}\bar{\epsilon}\beta_1$$

and

$$n_0 - m_0 \leq 128 \cdot 16M^2\bar{L}\bar{\epsilon}^{-1}\beta_1^{-1}.$$

This contradicts (9.302). The contradiction we have reached proves that there exists an integer

$$j \in \{m_0, \dots, n_0\} \tag{9.322}$$

such that

$$f(x_j) \leq \inf(f, C) + \bar{\epsilon}/8. \tag{9.323}$$

By property (i), (9.317), (9.322), and (9.323), we have

$$d(x_j, C_{min}) \leq \epsilon/4. \tag{9.324}$$

We claim that for all integers k satisfying $j \leq k \leq n$,

$$d(x_k, C_{min}) \leq \epsilon.$$

Assume the contrary. Then there exists an integer $k \in [j, n]$ for which

$$d(x_k, C_{min}) > \epsilon. \tag{9.325}$$

By (9.324) and (9.325), we have

$$k > j.$$

We may assume without loss of generality that

$$d(x_i, C_{min}) \leq \epsilon \text{ for all integers } i \text{ satisfying } j \leq i < k.$$

Thus

$$d(x_{k-1}, C_{min}) \leq \epsilon. \tag{9.326}$$

There are two cases:

$$f(x_{k-1}) \leq \inf(f, C) + \bar{\epsilon}/8; \tag{9.327}$$

$$f(x_{k-1}) > \inf(f, C) + \bar{\epsilon}/8. \tag{9.328}$$

Assume that (9.327) is valid. It follows from (9.317) and (9.322) that

$$d(x_{k-1}, C) \leq \bar{\epsilon}_1. \tag{9.329}$$

By (9.329), there exists a point

$$z \in C \tag{9.330}$$

such that

$$\|x_{k-1} - z\| < 2\bar{\epsilon}_1. \tag{9.331}$$

By (9.8), (9.230), (9.303), (9.305)–(9.307), and (9.331),

$$\|x_k - z\| \leq \delta$$

$$+\|z - P_{k-1}(x_{k-1} - \alpha_{k-1}\|v_{k-1}\|^{-1}v_{k-1} - \alpha_{k-1}\xi_{k-1})\|$$

$$\leq \delta + \|z - x_{k-1}\| + 2\alpha_{k-1}$$

$$\leq 2\bar{\epsilon}_1 + \delta + 2\beta_0 \leq 3\bar{\epsilon}_1. \tag{9.332}$$

In view of (9.331) and (9.332),

$$\|x_k - x_{k-1}\| \leq \|x_k - z\| + \|z - x_{k-1}\| \leq 5\bar{\epsilon}_1. \tag{9.333}$$

It follows from (9.9)–(9.11), and (9.326) that

$$\|x_{k-1}\| \leq \bar{K} + 1. \tag{9.334}$$

By (9.299), (9.333), and (9.334),

$$\|x_k\| \leq \|x_{k-1}\| + 5\bar{\epsilon}_1 \leq \bar{K} + 4. \tag{9.335}$$

Equations (9.213) and (9.333)–(9.335) imply that

$$|f(x_{k-1}) - f(x_k)| \leq \bar{L}\|x_{k-1} - x_k\| \leq 5\bar{L}\bar{\epsilon}_1.$$

Together with (9.249) and (9.327) this implies that

$$f(x_k) \leq f(x_{k-1}) + 5\bar{L}\bar{\epsilon}_1$$
$$\leq \inf(f, C) + \bar{\epsilon}/8 + 5\bar{L}\bar{\epsilon}_1 \leq \inf(f, C) + \bar{\epsilon}/4. \quad (9.336)$$

By (9.299), (9.329), and (9.333),

$$d(x_k, C) \leq 6\bar{\epsilon}_1 < \bar{\epsilon}. \quad (9.337)$$

Property (i), (9.336), and (9.337) imply that

$$d(x_k, C_{min}) \leq \epsilon.$$

This inequality contradicts (9.325). The contradiction we have reached proves (9.328). It follows from (9.322) and (9.328), the inequality $k > j$ and property (iv) that

$$\|x_k - \bar{x}\| \leq \|x_{k-1} - \bar{x}\|.$$

Since $\bar{x}$ is an arbitrary element of C_{min} in view of (9.326),

$$d(x_k, C_{min}) \leq d(x_{k-1}, C_{min}) \leq \epsilon.$$

This contradicts (9.325). The contradiction we have reached proves that $d(x_t, C_{min}) \leq \epsilon$ for all integers t satisfying $j \leq t \leq n$. This completes the proof of Theorem 9.6.

9.13 Proof of Theorem 9.7

We may assume without loss of generality that

$$\epsilon < 1.$$

In view of Proposition 9.17, there exists a number

$$\bar{\epsilon} \in (0, (\epsilon/8)\min\{1, \bar{r}\}) \quad (9.338)$$

such that the following property holds:

(i)

$$\text{if } x \in U,\ d(x, C) \leq 2\bar{\epsilon} \text{ and } f(x) \leq \inf(f, C) + 2\bar{\epsilon},$$

then $d(x, C_{min}) \leq \epsilon$.

We may assume that if (A2) holds, then the following property holds: for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $s \geq 0$, $\{P_{t+s}\}_{t=0}^{\infty} \in \mathcal{B}$.

We may assume without loss of generality that

$$M > 4\bar{K} + 8. \tag{9.339}$$

There exists $L_0 > \bar{L}$ such that

$$|f(z_1) - f(z_2)| \leq L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B_X(0, M+2) \cap U. \tag{9.340}$$

Fix

$$\bar{\epsilon}_1 \in (0, \bar{\epsilon}(64L_0)^{-1}). \tag{9.341}$$

Lemmas 9.10 and 9.11 imply that there exist $\delta_1 \in (0, 1)$ and a natural number m_0 such that the following property holds:

(ii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, each integer $n \geq m_0$, each integer $s \geq 0$ and each finite sequence $\{y_i\}_{i=0}^{n} \subset B_X(0, M+2)$ satisfying

$$B_X(y_{t+1}, \delta_1) \cap P_{t+s}(B_X(y_t, \delta_1)) \neq \emptyset,\ \ t = 0, \ldots, n-1$$

the inequality

$$d(y_t, C) \leq \bar{\epsilon}_1$$

holds for all integers $t \in [m_0, n]$.

Since $\lim_{i\to\infty} \alpha_i = 0$ (see (9.36)) there is an integer $p_0 > 0$ such that for all integers $i \geq p_0$, we have

$$\alpha_i \leq \min\{(\delta_1/2)(L_0+2)^{-1},\ 64^{-1}\bar{\epsilon}_1(L_0+\bar{L}+2)^{-2}\}. \tag{9.342}$$

Fix

$$\bar{x} \in C_{min}. \tag{9.343}$$

Since $\lim_{i\to\infty} \alpha_i = \infty$ (see (9.36)) there is an integer

$$n_0 > p_0 + m_0 + 4 \tag{9.344}$$

such that

$$\sum_{i=2p_0+m_0}^{n_0-1} \alpha_i > 128(4M)^2\bar{\epsilon}^{-1}L_0 + 1. \tag{9.345}$$

Fix a positive number δ such that

$$2\delta(4M + 3 + 10L_0) < 8^{-1}(64L_0)^{-1}\bar{\epsilon}_1. \tag{9.346}$$

Assume that

$$\{P_t\}_{t=0}^{\infty} \in \mathcal{B},$$

$n \geq n_0$ is an integer,

$$\{x_t\}_{t=0}^{n} \subset U_{\bar{r}},\ \ \|\xi_t\}_{t=0}^{n-1} \subset X, \tag{9.347}$$

$$\|x_0\| \leq M, \tag{9.348}$$

and that for all integers $t = 0, \ldots, n-1$, we have

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \neq \emptyset, \tag{9.349}$$

$$\|x_{t+1} - P_t(x_t - \alpha_t\xi_t)\| \leq \alpha_t\delta. \tag{9.350}$$

In order to prove the theorem it is sufficient to show that

$$d(x_t, C_{min}) \leq \epsilon \text{ for all integers } t \text{ satisfying } n_0 \leq t \leq n.$$

By (9.34), (9.35), (9.343), and (9.350), for all integers $t = 0, \ldots, n$,

$$\|x_t\| \leq M + 1, \tag{9.351}$$

$$d(x_t, C_{min}) \leq \|x_t - \bar{x}\| \leq 2M. \tag{9.352}$$

Set

$$y_t = x_{t+p_0},\ \ t = 0, \ldots, n - p_0. \tag{9.353}$$

By (9.351) and (9.353),

$$\{y_t\}_{t=0}^{n-p_0} \subset B_X(0, M+1). \tag{9.354}$$

In view of (9.344),

$$n - p_0 > m_0.$$

Lemma 9.13, (9.340), (9.345)–(9.347), (9.349), and (9.351) imply that for all integers $t = 0, \ldots, n$,

$$\partial_\delta f(x_t) \subset B_X(0, L_0 + \delta \bar{r}^{-1}) \subset B_X(0, L_0 + 1). \tag{9.355}$$

It follows from (9.342), (9.346), (9.349), (9.350), (9.353), and (9.355) that for all integers $t = 0, \ldots, n - p_0 - 1$,

$$\|y_t - (y_t - \alpha_{t+p_0}\xi_{t+p_0})\| \le \alpha_{t+p_0}\|\xi_{t+p_0}\| \le \alpha_{t+p_0}(L_0 + 2) \le \delta_1,$$

$$\|y_{t+1} - P_{t+p_0}(y_t - \alpha_{t+p_0}\xi_{t+p_0})\|$$

$$= \|x_{t+1+p_0} - P_{t+p_0}(x_{t+p_0} - \alpha_{t+p_0}\xi_{t+p_0})\|$$

$$\le \alpha_{t+p_0} \le \delta_1,$$

$$B_X(y_{t+1}, \delta_1) \cap P_{t+p_0}(B_X(y_t, \delta_1)) \neq \emptyset. \tag{9.356}$$

Property (ii), (9.344), (9.354), and (9.356) imply that

$$d(y_t, C) \le \bar{\epsilon}_1, \; t = m_0, \ldots, n - p_0.$$

Together with (9.353) this implies that

$$d(x_t, C) \le \bar{\epsilon}_1, \; t = m_0 + p_0, \ldots, n. \tag{9.357}$$

Assume that an integer

$$t \in [p_0 + m_0, n - 1], \tag{9.358}$$

$$f(x_t) > \inf(f, C) + \bar{\epsilon}/8. \tag{9.359}$$

It follows from equations (9.40), (9.342), (9.346), (9.347), (9.349)–(9.351), (9.358), and (9.359) and Corollary 9.15 applied with

$$P = P_t, \; K_0 = M + 1, \; \epsilon = \bar{\epsilon}/8, \; \alpha = \alpha_t,$$

$$\delta_f = \delta, \; \delta_C = \delta\alpha_t, \; \Delta = \delta, \; x = x_t, \; \xi = \xi_t, \; y = x_{t+1},$$

that

$$\|x_{t+1} - \bar{x}\|^2 \le \|x_t - \bar{x}\|^2 - \alpha_t 4^{-1}\bar{\epsilon}$$

$$+\alpha_t^2(L_0 + 2)^2 + 2\alpha_t\delta(4M + 4L_0 + 8)$$

$$\leq \|x_t - \bar{x}\|^2 - 8^{-1}\alpha_t\bar{\epsilon}$$

$$+2\alpha_t\delta(4M + 4L_0 + 8)$$

$$\leq \|x_t - \bar{x}\|^2 - 16^{-1}\alpha_t\bar{\epsilon}.$$

Thus we have shown that the following property holds:

(iii) if an integer t satisfies (9.358) and (9.359), then we have

$$\|x_{t+1} - \bar{x}\|^2 \leq \|x_t - \bar{x}\|^2 - 16^{-1}\alpha_t\bar{\epsilon}.$$

We claim that there exists an integer $j \in \{p_0 + m_0, \ldots, n_0\}$ such that

$$f(x_j) \leq \inf(f, C) + \bar{\epsilon}/8.$$

Assume the contrary. Then

$$f(x_j) > \inf(f, C) + \bar{\epsilon}/8, \ i = p_0 + m_0, \ldots, n_0. \tag{9.360}$$

Property (iii) implies that for all integers $t = p_0 + m_0, \ldots, n_0 - 1$,

$$\|x_{t+1} - \bar{x}\|^2 \leq \|x_t - \bar{x}\|^2 - 16^{-1}\alpha_t\bar{\epsilon}. \tag{9.361}$$

Relations (9.9)–(9.11), (9.343), (9.351), and (9.361) imply that

$$(4M)^2 \geq \|x_{p_0+m_0} - \bar{x}\|^2$$

$$\geq \|x_{p_0+m_0} - \bar{x}\|^2 - \|x_{n_0} - \bar{x}\|^2$$

$$= \sum_{t=p_0+m_0}^{n_0-1} [\|x_t - \bar{x}\|^2 - \|x_{t+1} - \bar{x}\|^2]$$

$$\geq 16^{-1}\bar{\epsilon} \sum_{t=p_0+m_0}^{n_0-1} \alpha_t. \tag{9.362}$$

In view of (9.362),

$$\sum_{i=p_0+m_0}^{n_0-1} \alpha_t \leq 16^2\bar{\epsilon}^{-1}M^2.$$

This contradicts (9.345). The contradiction we have reached proves that there exists an integer

$$j \in \{p_0 + m_0, \dots, n_0\} \tag{9.363}$$

such that

$$f(x_j) \le \inf(f, C) + \bar{\epsilon}/8. \tag{9.364}$$

By property (i), (9.357), (9.363), and (9.364), we have

$$d(x_j, C_{min}) \le \epsilon. \tag{9.365}$$

We claim that for all integers t satisfying $j \le t \le n$,

$$d(x_t, C_{min}) \le \epsilon.$$

Assume the contrary. Then there exists an integer $k \in [j, n]$ for which

$$d(x_k, C_{min}) > \epsilon. \tag{9.366}$$

By (9.363), (9.365), and (9.366), we have

$$k > j \ge p_0 + m_0. \tag{9.367}$$

We may assume without loss of generality that

$$d(x_t, C_{min}) \le \epsilon \text{ for all integers } t \text{ satisfying } j \le t < k.$$

Thus

$$d(x_{k-1}, C_{min}) \le \epsilon. \tag{9.368}$$

There are two cases:

$$f(x_{k-1}) \le \inf(f, C) + \bar{\epsilon}/8; \tag{9.369}$$

$$f(x_{k-1}) > \inf(f, C) + \bar{\epsilon}/8. \tag{9.370}$$

Assume that (9.369) is valid. It follows from (9.357) and (9.367) that

$$d(x_{k-1}, C) \le \bar{\epsilon}_1. \tag{9.371}$$

By (9.371), there exists a point

$$z \in C \tag{9.372}$$

such that

$$\|x_{k-1} - z\| < 2\bar{\epsilon}_1. \tag{9.373}$$

By (9.8), (9.342), (9.346), (9.349), (9.350), (9.372), and (9.373),

$$\|x_k - z\| \le \alpha_{k-1}\delta$$

$$+\|z - P_{k-1}(x_{k-1} - \alpha_{k-1}\xi_{k-1}\|$$

$$\le \delta + \|z - x_{k-1}\| + \alpha_{k-1}\|\xi_{k-1}\|$$

$$\le \delta + 2\bar{\epsilon}_1 + \alpha_{k-1}(L_0 + 2) < 3\bar{\epsilon}_1. \tag{9.374}$$

In view of (9.373) and (9.374),

$$\|x_k - x_{k-1}\| \le \|x_k - z\| + \|z - x_{k-1}\| < 5\bar{\epsilon}_1. \tag{9.375}$$

It follows from (9.9)–(9.11), and (9.368) that

$$\|x_{k-1}\| \le \bar{K} + 1,$$

$$\|x_k\| \le \|x_{k-1}\| + 5\bar{\epsilon}_1 \le \bar{K} + 4$$

and

$$\|x_{k-1}\|,\ \|x_k\| \le \bar{K} + 4.$$

Combined with (9.12) and (9.375) the relation above implies that

$$|f(x_{k-1}) - f(x_k)| \le 5\bar{K}\bar{L}\bar{\epsilon}_1. \tag{9.376}$$

Together with (9.341) and (9.369) this implies that

$$f(x_k) \le f(x_{k-1}) + 5\bar{L}\bar{\epsilon}_1$$

$$\le \inf(f, C) + \bar{\epsilon}/8 + 5\bar{L}\bar{\epsilon}_1 \le \inf(f, C) + \bar{\epsilon}/4. \tag{9.377}$$

Property (i), (9.357), (9.367), and (9.377) imply that

$$d(x_k, C_{min}) \le \epsilon.$$

This inequality contradicts (9.366). The contradiction we have reached proves (9.370). It follows from (9.367), (9.370), and property (iii) that

$$\|x_k - \bar{x}\| \le \|x_{k-1} - \bar{x}\|.$$

Since $\bar{x}$ is an arbitrary element of C_{min} in view of (9.374),

$$d(x_k, C_{min}) \le d(x_{k-1}, C_{min}) \le \epsilon.$$

This contradicts (9.366). The contradiction we have reached proves that $d(x_t, C_{min}) \le \epsilon$ for all integers t satisfying $j \le t \le n$. This completes the proof of Theorem 9.7.

9.14 Proof of Theorem 9.8

We may assume without loss of generality that

$$\epsilon < 1.$$

Proposition 9.17 implies that there exists a number

$$\bar{\epsilon} \in (0, \min\{\epsilon/8, \bar{r}\}) \tag{9.378}$$

such that the following property holds:

(i)

$$\text{if } x \in U,\ d(x, C) \le 2\bar{\epsilon} \text{ and } f(x) \le \inf(f, C) + 2\bar{\epsilon},$$

$$\text{then } d(x, C_{min}) \le \epsilon/4.$$

We may assume that if (A2) holds, then the following property holds:

(ii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $s \ge 0$, $\{P_{t+s}\}_{t=0}^{\infty} \in \mathcal{B}$.

In view of (9.4), we may assume without loss of generality that

$$M > 4\bar{K} + 8, \tag{9.379}$$

$$\{x \in X : f(x) \le \inf(f, C) + 16\bar{L}\} \subset B(0, 2^{-1}M - 1). \tag{9.380}$$

Fix

$$\bar{\epsilon}_1 \in (0, \bar{\epsilon}(64\bar{L})^{-1}). \tag{9.381}$$

Lemmas 9.10 and 9.11 imply that there exist $\delta_1 \in (0, 1)$ and a natural number m_0 such that the following property holds:

(iii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, each integer $n \ge m_0$, each integer $s \ge 0$ and each finite sequence $\{y_i\}_{i=0}^{n} \subset B_X(0, 3M)$ satisfying

$$B_X(y_{t+1}, \delta_1) \cap P_{t+s}(B_X(y_t, \delta_1)) \neq \emptyset,\ t = 0, \dots, n-1$$

the inequality

$$d(y_t, C) \le \epsilon_1$$

holds for all integers $t \in [m_0, n]$.

Fix

$$\bar{x} \in C_{min}. \tag{9.382}$$

Since $\lim_{i\to\infty} \alpha_i = 0$ (see (9.39)) there is an integer $p_0 > 0$ such that for all integers $i \ge p_0$, we have

$$\alpha_i \le \min\{(\delta_1/2)(L_0+2)^{-1},\ 2^{-9}\bar{\epsilon}_1 L_0^{-2}\}. \tag{9.383}$$

Since $\lim_{i\to\infty} \alpha_i = \infty$ (see (9.39)) there is an integer

$$n_0 > p_0 + m_0 + 4 \tag{9.384}$$

such that

$$\sum_{i=p_0+m_0}^{n_0-1} \alpha_i > 128(4M)^2 \bar{\epsilon}^{-1} L_0 + 1. \tag{9.385}$$

Fix a positive number δ such that

$$\delta(4M + 6 + 10L_0) < 8^{-1}(64L_0)^{-1}\bar{\epsilon}_1. \tag{9.386}$$

Assume that

$$\{P_t\}_{t=0}^{\infty} \in \mathcal{B},$$

$n \ge n_0$ is an integer,

$$\{x_t\}_{t=0}^{n} \subset U_{\bar{r}},\ \{\xi_t\}_{t=0}^{n-1} \subset X, \tag{9.387}$$

$$\|x_0\| \le M,\ , \tag{9.388}$$

and that for all integers $t = 0, \dots, n-1$, we have

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \neq \emptyset, \tag{9.389}$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \le \alpha_t \delta. \tag{9.390}$$

In order to prove the theorem it is sufficient to show that

$$d(x_t, C_{min}) \leq \epsilon \text{ for all integers } t \text{ satisfying } n_0 \leq t \leq n.$$

First we show that for all integers $t = 0, \ldots, n$,

$$d(x_t, C_{min}) \leq 2M. \tag{9.391}$$

In view of (9.9)–(9.11), and (9.388), inequality (9.391) holds for $t = 0$. Assume that

$$t \in \{0, \ldots, n\} \setminus \{n\}$$

and that (9.391) is true. In view of (9.9)–(9.11), (9.387), (9.389), and (9.391),

$$\|x_t\| \leq 3M, \; \partial_\delta f(x_t) \subset B_X(L_0 + 1), \; \|\xi_t\| \leq L_0 + 2. \tag{9.392}$$

There are two cases:

$$f(x_t) \leq \inf(f, C) + 16\bar{L}; \tag{9.393}$$

$$f(x_t) > \inf(f, C) + 16\bar{L}. \tag{9.394}$$

Assume that (9.393) holds. In view of (9.380) and (9.393),

$$\|x_t\| \leq M/2 - 1.$$

By (9.9)–(9.11), and (9.382),

$$\|\bar{x}\| \leq \bar{K}. \tag{9.395}$$

It follows from (9.394) and (9.395) that

$$\|x_t - \bar{x}\| \leq \bar{K} + M/2.$$

By (9.38), (9.382), (9.390), and (9.392) and the relation above,

$$\|x_{t+1} - \bar{x}\| \leq \alpha_t \delta + \|\bar{x} - P_t(x_t - \alpha_t \xi_t)\|$$

$$\leq \alpha_t \delta + \|\bar{x} - (x_t - \alpha_t \xi_t)\|$$

$$\leq \alpha_t \delta + \|\bar{x} - x_t\| + \alpha_t \|\xi_t)\|$$

$$\leq 2 + \bar{K} + M/2 < 2M$$

and

$$d(x_{t+1}, C_{min}) \leq 2M.$$

Assume that (9.394) holds. It follows from (9.38), (9.386), (9.387), (9.389), (9.390), (9.394), and Corollary 9.15 applied with

$$P = P_t, \ \epsilon = 16\bar{L}, \ r = \bar{r}, \ \Delta = \delta, \ K_0 = 3M,$$

$$\alpha = \alpha_t, \ \delta_f = \delta, \ \delta_C = \delta\alpha_t, \ x = x_t, \ \xi = \xi_t, \ y = x_{t+1}$$

that

$$d(x_{t+1}, C_{min})^2 \leq d(x_t, C_{min})^2 - 32\alpha_t \bar{L} + \alpha_t^2 (L_0 + 1)^2 + 4\delta(4M + 2L_0 + 4)\alpha_t$$

$$\leq d(x_t, C_{min})^2 - 8\alpha_t \bar{L} + 4\delta(4M + 2L_0 + 4)\alpha_t$$

$$\leq d(x_t, C_{min})^2 - \alpha_t$$

and

$$d(x_{t+1}, C_{min}) \leq d(x_t, C_{min}) \leq 2M.$$

Thus in both cases

$$d(x_{t+1}, C_{min}) \leq 2M.$$

Thus we have shown by induction that for all integers $t = 0, \ldots, n$, (9.391) is true. Together with (9.9)–(9.11), and (9.379),

$$\|x_t\| \leq 3M, \ t = 0, \ldots, n.$$

Set

$$y_t = x_{t+p_0}, \ t = 0, \ldots, n - p_0.$$

By the relations above,

$$\{y_t\}_{t=0}^{n-p_0} \subset B_X(0, 3M).$$

In view of (9.384),

$$n - p_0 > m_0. \tag{9.396}$$

It follows from the relations above that for all integers $t = 0, \ldots, n - p_0 - 1$,

$$\|y_t - (y_t - \alpha_{t+p_0}\xi_{t+p_0})\|$$

$$= \|x_{t+p_0} - (x_{t+p_0} - \alpha_{t+p_0}\xi_{t+p_0})\|$$

$$\leq \alpha_{t+p_0}\|\xi_{t+p_0}\| \leq \alpha_{t+p_0}(L_0 + 2) \leq \delta_1,$$

$$\|y_{t+1} - P_{t+p_0}(y_t - \alpha_{t+p_0}\xi_{t+p_0})\|$$

$$= \|x_{t+1+p_0} - P_{t+p_0}(x_{t+p_0} - \alpha_{t+p_0}\xi_{t+p_0})\|$$

$$\leq \alpha_{t+p_0} \leq \delta_1,$$

$$B_X(y_{t+1}, \delta_1) \cap P_{t+p_0}(B_X(t_t, \delta_1)) \neq \emptyset. \tag{9.397}$$

Equations (9.396) and (9.397) and property (iii) imply that

$$d(y_t, C) \leq \bar{\epsilon}_1,\ t = m_0, \dots, n - p_0.$$

Together with (9.394) this implies that

$$d(x_t, C) \leq \bar{\epsilon}_1,\ t = m_0 + p_0, \dots, n. \tag{9.398}$$

Assume that an integer

$$t \in [m_0 + p_0, n - 1], \tag{9.399}$$

$$f(x_t) > \inf(f, C) + \bar{\epsilon}/8. \tag{9.400}$$

It follows from (9.31), (9.387), (9.389), (9.390), and (9.400) and Corollary 9.15 applied with

$$P = P_t,\ r = \bar{r},\ K_0 = 3M,\ \epsilon = \bar{\epsilon}/8,\ \alpha = \alpha_t\ \Delta = \delta,$$

$$\delta_f = \delta,\ \delta_C = \delta\alpha_t,\ x = x_t,\ \xi = \xi_t,\ y = x_{t+1},$$

that

$$\|x_{t+1} - \bar{x}\|^2 \leq \|x_t - \bar{x}\|^2 - 4^{-1}\alpha_t\bar{\epsilon}$$

$$+\alpha_t^2(L_0 + 1)^2 + 2\delta^2\alpha_t^2 + 2\delta(4M + 4L_0)$$

$$\leq \|x_t - \bar{x}\|^2 - 8^{-1}\alpha_t\bar{\epsilon} + 2\delta\alpha_t(4M + 4L_0 + 4)$$

$$\leq \|x_t - \bar{x}\|^2 - 16^{-1}\alpha_t\bar{\epsilon}.$$

Thus we have shown that the following property holds:

(iv) if an integer t satisfies (9.399) and (9.400), then we have

$$\|x_{t+1} - \bar{x}\|^2 \le \|x_t - \bar{x}\|^2 - 16^{-1}\alpha_t\bar{\epsilon}.$$

We claim that there exists an integer $j \in \{m_0 + p_0, \dots, n_0\}$ such that

$$f(x_j) \le \inf(f, C) + \bar{\epsilon}/8.$$

Assume the contrary. Then

$$f(x_j) > \inf(f, C) + \bar{\epsilon}/8, \;\; i = m_0 + p_0, \dots, n_0. \tag{9.401}$$

Property (iv) and (9.401) imply that for all integers $t = m_0 + p_0, \dots, n_0 - 1$,

$$\|x_{t+1} - \bar{x}\|^2 \le \|x_t - \bar{x}\|^2 - 16^{-1}\alpha_t\bar{\epsilon}. \tag{9.402}$$

Relations (9.379), (9.386), (9.393), and (9.402) imply that

$$(4M)^2 \ge \|x_{p_0+m_0} - \bar{x}\|^2$$

$$\ge \|x_{p_0+m_0} - \bar{x}\|^2 - \|x_{n_0} - \bar{x}\|^2$$

$$= \sum_{t=m_0+p_0}^{n_0-1} [\|x_t - \bar{x}\|^2 - \|x_{t+1} - \bar{x}\|^2]$$

$$\ge 16^{-1}\bar{\epsilon} \sum_{t=p_0+m_0}^{n_0-1} \alpha_t. \tag{9.403}$$

In view of (9.403),

$$\sum_{t=p_0+m_0}^{n_0-1} \alpha_t \le 16^2\bar{\epsilon}^{-1}M^2.$$

This contradicts (9.385). The contradiction we have reached proves that there exists an integer

$$j \in \{m_0 + p_0, \dots, n_0\} \tag{9.404}$$

such that

$$f(x_j) \le \inf(f, C) + \bar{\epsilon}/8. \tag{9.405}$$

By property (i), (9.398), (9.404) and (9.405), we have

$$d(x_j, C_{min}) \le \epsilon. \tag{9.406}$$

We claim that for all integers t satisfying $j \le t \le n$,

$$d(x_t, C_{min}) \le \epsilon.$$

Assume the contrary. Then there exists an integer $k \in [j, n]$ for which

$$d(x_k, C_{min}) > \epsilon. \tag{9.407}$$

By (9.404), (9.406), and (9.407), we have

$$k > j \ge p_0 + m_0. \tag{9.408}$$

We may assume without loss of generality that

$$d(x_t, C_{min}) \le \epsilon \text{ for all integers } t \text{ satisfying } j \le t < k.$$

Thus

$$d(x_{k-1}, C_{min}) \le \epsilon. \tag{9.409}$$

There are two cases:

$$f(x_{k-1}) \le \inf(f, C) + \bar{\epsilon}/8; \tag{9.410}$$

$$f(x_{k-1}) > \inf(f, C) + \bar{\epsilon}/8. \tag{9.411}$$

Assume that (9.410) is valid. It follows from (9.398) and (9.408) that

$$d(x_{k-1}, C) \le \bar{\epsilon}_1. \tag{9.412}$$

By (9.412), there exists a point

$$z \in C \tag{9.413}$$

such that

$$\|x_{k-1} - z\| < 2\bar{\epsilon}_1. \tag{9.414}$$

By (9.383), (9.386), (9.390)–(9.397), (9.408), and (9.414),

$$\|x_k - z\| \le \alpha_t \delta$$

$$+\|z - P_{k-1}(x_{k-1} - \alpha_{k-1}\xi_{k-1})\|$$

$$\leq \delta + \|z - x_{k-1}\| + \alpha_{k-1}\|\xi_{k-1}\|$$

$$\leq \delta + 2\bar{\epsilon}_1 + \alpha_{k-1}(L_0 + 2) < 3\bar{\epsilon}_1. \tag{9.415}$$

In view of (9.414) and (9.415),

$$\|x_k - x_{k-1}\| \leq \|x_k - z\| + \|z - x_{k-1}\| \leq 5\bar{\epsilon}_1. \tag{9.416}$$

It follows from (9.9)–(9.11), (9.409), and (9.416) that

$$\|x_{k-1}\| \leq \bar{K} + 2\epsilon,$$

$$\|x_k\| \leq \|x_{k-1}\| + 5\bar{\epsilon}_1 \leq \bar{K} + 3\epsilon,$$

$$\|x_{k-1}\|,\ \|x_k\| \leq \bar{K} + 4.$$

Together with (9.12) and (9.416) this implies that

$$|f(x_k) - f(x_{k-1})| \leq \bar{L}\|x_{k-1} - x_k\| \leq 5\bar{L}\bar{\epsilon}_1.$$

Combined with (9.381) and (9.409) this implies that

$$f(x_k) \leq f(x_{k-1}) + 5\bar{L}\bar{\epsilon}_1$$

$$\leq \inf(f, C) + \bar{\epsilon}/8 + 5\bar{L}\bar{\epsilon}_1 \leq \inf(f, C) + \bar{\epsilon}/4.$$

By (9.398), (9.408), and property (i) and the relation above,

$$d(x_k, C_{min}) \leq \epsilon.$$

This inequality contradicts (9.407). The contradiction we have reached proves (9.411). It follows from (9.408), (9.411), and property (iv) that

$$\|x_k - \bar{x}\| \leq \|x_{k-1} - \bar{x}\|.$$

Since $\bar{x}$ is an arbitrary element of C_{min} in view of (9.406),

$$d(x_k, C_{min}) \leq d(x_{k-1}, C_{min}) \leq \epsilon.$$

This contradicts (9.407). The contradiction we have reached proves that $d(x_t, C_{min}) \leq \epsilon$ for all integers t satisfying $j \leq t \leq n$. This completes the proof of Theorem 9.8.

9.15 Proof of Theorem 9.9

In view of (9.4), we may assume without loss of generality that

$$\epsilon < \min\{1, \bar{r}\}, \ \ M > 8\bar{K} + 8 \tag{9.417}$$

and that

$$\{x \in U : f(x) \le \inf(f, C) + 16\bar{L}\} \subset B_X(0, 2^{-1}M - 1). \tag{9.418}$$

There exists $L_0 > \bar{L}$ such that

$$|f(z_1) - f(z_2)| \le L_0\|z_1 - z_2\| \text{ for all } z_1, z_2 \in B_X(0, 3M + 4) \cap U. \tag{9.419}$$

Proposition 9.17 implies that there exists a number

$$\bar{\epsilon} \in (0, \epsilon/8) \tag{9.420}$$

such that the following property holds:

(i)

$$\text{if } x \in U, \ d(x, C) \le 2\bar{\epsilon} \text{ and } f(x) \le \inf(f, C) + 2\bar{\epsilon},$$

$$\text{then } d(x, C_{min}) \le \epsilon/4.$$

We may assume that if (A2) holds, then the following property holds:

(ii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $s \ge 0$, $\{P_{t+s}\}_{t=0}^{\infty} \in \mathcal{B}$.

Fix

$$\bar{x} \in C_{min}, \tag{9.421}$$

$$\bar{\epsilon}_1 \in (0, \bar{\epsilon}(64\bar{L})^{-1}). \tag{9.422}$$

Lemmas 9.10 and 9.11 imply that there exist $\delta_1 \in (0, 1)$ and a natural number m_0 such that the following property holds:

(iii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$, each integer $n \ge m_0$, each integer $s \ge 0$ and each finite sequence $\{y_i\}_{i=0}^{n} \subset B_X(0, 3M)$ satisfying

$$B_X(y_{t+1}, \delta_1) \cap P_{t+s}(B_X(y_t, \delta_1)) \ne \emptyset, \ t = 0, \dots, n-1$$

the inequality

$$d(y_t, C) \le \bar{\epsilon}_1$$

holds for all integers $t \in [m_0, n]$.

Choose a positive number β_0 such that

$$\beta_0 \le 800^{-1}(2L_0 + 2)^{-2} \min\{\delta_1, \bar{\epsilon}_1\}. \tag{9.423}$$

Let

$$\beta_1 \in (0, \beta_0).$$

Let a natural number n_0 be such that

$$n_0 > m_0 + 32^3 M^2 \bar{\epsilon}^{-1} \beta_1^{-1} \tag{9.424}$$

and a positive number δ satisfy

$$\delta(4M + 16L_0 + 8) \le 2^{-9} L_0^{-1} \beta_1 \bar{\epsilon}_1. \tag{9.425}$$

Assume that

$$\{P_t\}_{t=0}^{\infty} \in \mathcal{B},$$

$n \ge n_0$ is an integer,

$$\{x_t\}_{t=0}^{n} \subset U_{\bar{r}}, \ \|\xi_t\}_{t=0}^{n-1} \subset X, \tag{9.426}$$

$$\alpha_t \in [\beta_1, \beta_0], \ t = 0, \dots, n-1, \tag{9.427}$$

$$\|x_0\| \le M, \tag{9.428}$$

and that for all integers $t = 0, \dots, n-1$, we have

$$B_X(\xi_t, \delta) \cap \partial_\delta f(x_t) \ne \emptyset, \tag{9.429}$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \le \delta. \tag{9.430}$$

In order to prove the theorem it is sufficient to show that

$$d(x_t, C_{min}) \le \epsilon \text{ for all integers } t \text{ satisfying } n_0 \le t \le n.$$

First we show that for all integers $t = 0, \dots, n$,

$$d(x_t, C_{min}) \le 2M. \tag{9.431}$$

In view of (9.9)–(9.11), (9.417), and (9.427), inequality (9.431) holds for $t = 0$. Assume that

$$t \in \{0, \dots, n\} \setminus \{n\}$$

and that (9.431) is true. There are two cases:

$$f(x_t) \le \inf(f, C) + 16\bar{L}; \tag{9.432}$$

$$f(x_t) > \inf(f, C) + 16\bar{L}. \tag{9.433}$$

It follows from (9.9)–(9.11), (9.214), (9.419), (9.429), and (9.431) that

$$\|x_t\| \le 3M, \tag{9.434}$$

$$\partial_\delta f(x_t) \subset B_X(0, L_0 + 1), \tag{9.435}$$

$$\|\xi_t\| \le L_0 + 2. \tag{9.436}$$

Assume that (9.432) holds. In view of (9.267) and (9.418),

$$\|x_t\| \le M/2 - 1. \tag{9.437}$$

By (9.9)–(9.11), and (9.421),

$$\|\bar{x}\| \le \bar{K}. \tag{9.438}$$

It follows from (9.437) and (9.438) that

$$\|x_t - \bar{x}\| \le \bar{K} + M/2. \tag{9.439}$$

By (9.8), (9.417), (9.421), (9.423), (9.427), (9.430), (9.436), and (9.439),

$$d(x_{t+1}, C_{min}) \le \|x_{t+1} - \bar{x}\|$$

$$\le \delta + \|\bar{x} - P_t(x_t - \alpha_t \xi_t)\|$$

$$\le \delta + \|\bar{x} - (x_t - \alpha_t \xi_t)\|$$

$$\le \delta + \|\bar{x} - x_t\| + \alpha_t \|\xi_t\|$$

$$\le 1 + \bar{K} + M/2 + \beta_0(L_0 + 2) \le \bar{K} + M/2 + 2 \le M.$$

Assume that (9.433) holds. In view of (9.9)–(9.11), and (9.431),

$$\|x_t\| \le 3M. \tag{9.440}$$

It follows from (9.419), (9.425)–(9.427), (9.429), (9.431), (9.433), (9.440), and Corollary 9.15 applied with

$$P = P_t,\ \epsilon = 16\bar{L},\ r = \bar{r},\ \Delta = \delta,\ K_0 = 3M,$$

$$\delta_f = \delta,\ \delta_C = \delta,\ x = x_t,\ \xi = \xi_t,\ y = x_{t+1}$$

that

$$d(x_{t+1}, C_{min})^2 \le d(x_t, C_{min})^2 - 16\alpha_t + \alpha_t^2(L_0+1)^2 + 4\delta(4M + 10L_0 + 4)$$

$$\le d(x_t, C_{min})^2 - \alpha_t + 4\delta(4M + 10L_0 + 4)$$

$$\le d(x_t, C_{min})^2 - \beta_1 + 4\delta(4M + 10L_0 + 4)$$

$$\le d(x_t, C_{min})^2 - 2^{-1}\beta_1.$$

We have

$$d(x_{t+1}, C_{min}) \le d(x_t, C_{min}) \le 2M.$$

Thus in both cases

$$d(x_{t+1}, C_{min}) \le 2M.$$

Thus we have shown by induction that for all integers $t = 0, \dots, n$,

$$d(x_t, C_{min}) \le 2M. \tag{9.441}$$

By (9.9)–(9.11), (9.417), and (9.441),

$$\|x_t\| \le 3M,\ t = 0, \dots, n. \tag{9.442}$$

It follows from (9.405), (9.423), (9.427), (9.430), and (9.436) that for all integers $t = 0, \dots, n-1$,

$$\|x_t - (x_t - \alpha_t\xi_t)\| \le \alpha_t(L_0+2) \le \beta_0(L_0+2) \le \delta_1,$$

$$\|x_{t+1} - P_t(x_t - \alpha_t\xi_t)\| \le \delta \le \delta_1$$

and

$$B_X(x_{t+1}, \delta_1) \cap P_t(B_X(x_t, \delta_1)) \ne \emptyset. \tag{9.443}$$

Property (iii), (9.442), and (9.443) imply that

$$d(x_t, C) \le \bar{\epsilon}_1, \ t = m_0, \dots, n. \tag{9.444}$$

Assume that an integer

$$t \in [m_0, n-1], \tag{9.445}$$

$$f(x_t) > \inf(f, C) + \bar{\epsilon}/8. \tag{9.446}$$

It follows from (9.419), (9.423), (9.426), (9.427), (9.429), (9.430), (9.442), (9.446), and Corollary 9.15 applied with

$$P = P_t, \ r = \bar{r}, \ K_0 = 3M, \ \epsilon = \bar{\epsilon}/8, \ \alpha = \alpha_t,$$

$$\Delta, \delta_f = \delta, \ \delta_C = \delta, \ x = x_t, \ \xi = \xi_t, \ y = x_{t+1},$$

that

$$\|x_{t+1} - \bar{x}\|^2 \le \|x_t - \bar{x}\|^2 - 4^{-1}\alpha_t\bar{\epsilon}$$

$$+\alpha_t^2(L_0+1)^2 + 4\delta(\bar{K} + 3M + 2L_0 + 8)$$

$$\le \|x_t - \bar{x}\|^2 - 8^{-1}\alpha_t\bar{\epsilon} + 4\delta(4M + 2L_0 + 8)$$

$$\le \|x_t - \bar{x}\|^2 - 8^{-1}\beta_1\bar{\epsilon} + 2\delta(4M + 2L_0 + 8)$$

$$\le \|x_t - \bar{x}\|^2 - 16^{-1}\beta_1\bar{\epsilon}.$$

Thus we have shown that the following property holds:

(iv) if an integer $t \in [m_0, n-1]$ satisfies

$$f(x_t) > \inf(f, C) + \bar{\epsilon}/8$$

then we have

$$\|x_{t+1} - \bar{x}\|^2 \le \|x_t - \bar{x}\|^2 - 16^{-1}\beta_1\bar{\epsilon}.$$

We claim that there exists an integer $j \in \{m_0, \dots, n_0\}$ such that

$$f(x_j) \le \inf(f, C) + \bar{\epsilon}/8.$$

Assume the contrary. Then

$$f(x_j) > \inf(f, C) + \bar{\epsilon}/8, \;\; i = m_0, \ldots, n_0. \tag{9.447}$$

Property (iv) and (9.447) imply that for all integers $k = m_0, \ldots, n_0 - 1$,

$$\|x_{k+1} - \bar{x}\|^2 \le \|x_k - \bar{x}\|^2 - 16^{-1}\beta_1\bar{\epsilon}. \tag{9.448}$$

Relations (9.438), (9.442), and (9.448) imply that

$$(4M)^2 \ge \|x_{m_0} - \bar{x}\|^2$$

$$\ge \|x_{m_0} - \bar{x}\|^2 - \|x_{n_0} - \bar{x}\|^2$$

$$= \sum_{t=m_0+p_0}^{n_0-1} [\|x_t - \bar{x}\|^2 - \|x_{t+1} - \bar{x}\|^2]$$

$$\ge 16^{-1}\bar{\epsilon}\beta_1(n_0 - m_0)$$

and

$$n_0 - m_0 \le 16^2 M^2 \beta_1^{-1}\bar{\epsilon}^{-1}.$$

This contradicts (9.424). The contradiction we have reached proves that there exists an integer

$$j \in \{m_0, \ldots, n_0\} \tag{9.449}$$

such that

$$f(x_j) \le \inf(f, C) + \bar{\epsilon}/8. \tag{9.450}$$

By property (i), (9.444), (9.449), and (9.450), we have

$$d(x_j, C_{min}) \le \epsilon. \tag{9.451}$$

We claim that for all integers k satisfying $j \le k \le n$,

$$d(x_k, C_{min}) \le \epsilon.$$

Assume the contrary. Then there exists an integer $k \in [j, n]$ for which

$$d(x_k, C_{min}) > \epsilon. \tag{9.452}$$

By (9.451) and (9.452), we have

$$k > j. \tag{9.453}$$

We may assume without loss of generality that

$$d(x_t, C_{min}) \leq \epsilon \text{ for all integers } t \text{ satisfying } j \leq t < k.$$

Thus

$$d(x_{k-1}, C_{min}) \leq \epsilon. \tag{9.454}$$

There are two cases:

$$f(x_{k-1}) \leq \inf(f, C) + \bar{\epsilon}/8; \tag{9.455}$$

$$f(x_{k-1}) > \inf(f, C) + \bar{\epsilon}/8. \tag{9.456}$$

Assume that (9.455) is valid. It follows from (9.444), (9.449), and (9.453) that

$$d(x_{k-1}, C) \leq \bar{\epsilon}_1. \tag{9.457}$$

By (9.457), there exists a point

$$z \in C \tag{9.458}$$

such that

$$\|x_{k-1} - z\| < 2\bar{\epsilon}_1. \tag{9.459}$$

By (9.8), (9.423), (9.425), (9.427), (9.430), (9.436), and (9.459),

$$\begin{aligned} \|x_k - z\| &\leq \delta \\ &+ \|z - P_{k-1}(x_{k-1} - \alpha_{k-1}\xi_{k-1})\| \\ &\leq \delta + \|z - x_{k-1}\| + \alpha_{k-1}\|\xi_{k-1}\| \\ &\leq \delta + 2\bar{\epsilon}_1 + \beta_0(L_0 + 2) < 3\bar{\epsilon}_1. \end{aligned} \tag{9.460}$$

In view of (9.459) and (9.460),

$$\|x_k - x_{k-1}\| \leq \|x_k - z\| + \|z - x_{k-1}\| \leq 5\bar{\epsilon}_1. \tag{9.461}$$

It follows from (9.9)–(9.11), and (9.454) that

$$\|x_{k-1}\| \leq \bar{K} + 1. \tag{9.462}$$

In view of (9.461) and (9.462),

$$\|x_k\| \le \|x_{k-1}\| + 5\bar{\epsilon}_1 \le \bar{K} + 4. \tag{9.463}$$

Equations (9.12), (9.462), and (9.463) imply that

$$|f(x_k) - f(x_{k-1})| \le \bar{L}\|x_{k-1} - x_k\| \le 5\bar{L}\bar{\epsilon}_1.$$

Combined with (9.250) and (9.422) this implies that

$$f(x_k) \le f(x_{k-1}) + 5\bar{L}\bar{\epsilon}_1$$

$$\le \inf(f, C) + \bar{\epsilon}/8 + 5\bar{L}\bar{\epsilon}_1 \le \inf(f, C) + \bar{\epsilon}/4. \tag{9.464}$$

By (9.444), (9.449), and (9.453),

$$d(x_k, C) \le \bar{\epsilon}_1 < \bar{\epsilon}. \tag{9.465}$$

Property (i), (9.464), and (9.465) imply that

$$d(x_k, C_{min}) \le \epsilon.$$

This inequality contradicts (9.452). The contradiction we have reached proves (9.456). It follows from (9.449), (9.453), (9.456), and property (iv) that

$$\|x_k - \bar{x}\| \le \|x_{k-1} - \bar{x}\|.$$

Since $\bar{x}$ is an arbitrary element of C_{min} in view of (9.424),

$$d(x_k, C_{min}) \le d(x_{k-1}, C_{min}) \le \epsilon.$$

This contradicts (9.452). The contradiction we have reached proves that $d(x_t, C_{min}) \le \epsilon$ for all integers t satisfying $j \le t \le n$. This completes the proof of Theorem 9.9.

Chapter 10
Zero-Sum Games with Two Players

In this chapter we study an extension of the projected subgradient method for zero-sum games with two players under the presence of computational errors. We show that our algorithm generate a good approximate solution, if all the computational errors are bounded from above by a small positive constant.

10.1 Preliminaries and an Auxiliary Result

Let $(X, \langle \cdot, \cdot \rangle)$, $(Y, \langle \cdot, \cdot \rangle)$ be Hilbert spaces equipped with the complete norms $\| \cdot \|$ which are induced by their inner products. Let C be a nonempty closed convex subset of X, D be a nonempty closed convex subset of Y, U be an open convex subset of X and V be an open convex subset of Y such that

$$C \subset U, \ D \subset V \tag{10.1}$$

and let a function $f : U \times V \to R^1$ possess the following properties:

(i) for each $v \in V$, the function $f(\cdot, v) : U \to R^1$ is convex;
(ii) for each $u \in U$, the function $f(u, \cdot) : V \to R^1$ is concave.

Assume that positive numbers M_1, M_2, L_1, L_2 satisfy

$$C \subset B_X(0, M_1), \ D \subset B_Y(0, M_2), \tag{10.2}$$

$$|f(u_1, v) - f(u_2, v)| \le L_1 \|u_1 - u_2\|$$

$$\text{for all } v \in V \text{ and all } u_1, u_2 \in U, \tag{10.3}$$

$$|f(u, v_1) - f(u, v_2)| \le L_2 \|v_1 - v_2\|$$

A. J. Zaslavski, *Optimization on Solution Sets of Common Fixed Point Problems*,
Springer Optimization and Its Applications 178,
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$$\text{for all } u \in U \text{ and all } v_1, v_2 \in V. \tag{10.4}$$

Let

$$x_* \in C \text{ and } y_* \in D \tag{10.5}$$

satisfy

$$f(x_*, y) \le f(x_*, y_*) \le f(x, y_*) \tag{10.6}$$

for each $x \in C$ and each $y \in D$.

For each $x \in X$ and each nonempty set $A \subset X$ put

$$d(x, A) = \inf\{\|x - y\| : \ y \in A\}.$$

For each $x \in X$ and each $r > 0$,

$$B_X(x, r) = \{y \in X : \ \|y - x\| \le r\}.$$

Assume that a function $\phi : R^1 \to [0, \infty)$ is bounded on all bounded sets

The following result was obtained in [95].

Proposition 10.1 *Let T be a natural number, $\delta_C, \ \delta_D \in (0, 1]$, $\{a_t\}_{t=0}^T \subset (0, \infty)$ and let $\{b_{t,1}\}_{t=0}^T, \ \{b_{t,2}\}_{t=0}^T \subset (0, \infty)$. Assume that $\{x_t\}_{t=0}^{T+1} \subset U$, $\{y_t\}_{t=0}^{T+1} \subset V$, for each $t \in \{0, \ldots, T+1\}$,*

$$B_X(x_t, \delta_C) \cap C \neq \emptyset,$$

$$B_Y(y_t, \delta_D) \cap D \neq \emptyset,$$

for each $z \in C$ and each $t \in \{0, \ldots, T\}$,

$$a_t(f(x_t, y_t) - f(z, y_t))$$

$$\le \phi(\|z - x_t\|) - \phi(\|z - x_{t+1}\|) + b_{t,1}$$

and that for each $v \in D$ and each $t \in \{0, \ldots, T\}$,

$$a_t(f(x_t, v) - f(x_t, y_t))$$

$$\le \phi(\|v - y_t\|) - \phi(\|v - y_{t+1}\|) + b_{t,2}.$$

Let

$$\widehat{x}_T = (\sum_{i=0}^{T} a_i)^{-1} \sum_{t=0}^{T} a_t x_t,$$

$$\widehat{y}_T = (\sum_{i=0}^{T} a_i)^{-1} \sum_{t=0}^{T} a_t y_t.$$

Then

$$B_X(\widehat{x}_T, \delta_C) \cap C \neq \emptyset, \;\; B_Y(\widehat{y}_T, \delta_D) \cap D \neq \emptyset,$$

$$|(\sum_{t=0}^{T} a_t)^{-1} \sum_{t=0}^{T} a_t f(x_t, y_t) - f(x_*, y_*)|$$

$$\leq (\sum_{t=0}^{T} a_t)^{-1} \max\{\sum_{t=0}^{T} b_{t,1}, \;\; \sum_{t=0}^{T} b_{t,2}\}$$

$$+ \max\{L_1 \delta_C, \;\; L_2 \delta_D\}$$

$$+(\sum_{t=0}^{T} a_t)^{-1} \sup\{\phi(s) : \;\; s \in [0, \max\{2M_1, \;\; 2M_2\} + 1]\},$$

$$|f(\widehat{x}_T, \widehat{y}_T) - (\sum_{t=0}^{T} a_t)^{-1} \sum_{t=0}^{T} a_t f(x_t, y_t)|$$

$$\leq (\sum_{t=0}^{T} a_t)^{-1} \sup\{\phi(s) : \;\; s \in [0, \max\{2M_1, \;\; 2M_2\} + 1]\}$$

$$+(\sum_{t=0}^{T} a_t)^{-1} \max\{\sum_{t=0}^{T} b_{t,1}, \;\; \sum_{t=0}^{T} b_{t,2}\}$$

$$+ \max\{L_1 \delta_C, \;\; L_2 \delta_D\}$$

and for each $z \in C$ and each $v \in D$,

$$f(z, \widehat{y}_T) \geq f(\widehat{x}_T, \widehat{y}_T)$$

$$-2(\sum_{t=0}^{T} a_t)^{-1} \sup\{\phi(s) : \;\; s \in [0, \max\{2M_1, \;\; 2M_2\} + 1]\}$$

$$-2(\sum_{t=0}^{T} a_t)^{-1} \max\{\sum_{t=0}^{T} b_{t,1},\ \sum_{t=0}^{T} b_{t,2}\}$$

$$-\max\{L_1\delta_C,\ L_2\delta_D\},$$

$$f(\widehat{x}_T, v) \le f(\widehat{x}_T, \widehat{y}_T)$$

$$+2(\sum_{t=0}^{T} a_t)^{-1} \sup\{\phi(s):\ s \in [0, \max\{2M_1,\ 2M_2\}+1]\}$$

$$+2(\sum_{t=0}^{T} a_t)^{-1} \max\{\sum_{t=0}^{T} b_{t,1},\ \sum_{t=0}^{T} b_{t,2}\}$$

$$+\max\{L_1\delta_C,\ L_2\delta_D\}.$$

The following corollary was obtained in [95].

Corollary 10.2 *Suppose that all the assumptions of Proposition 10.1 hold and that*

$$\tilde{x} \in C,\ \tilde{y} \in D$$

satisfy

$$\|\widehat{x}_T - \tilde{x}\| \le \delta_C,\ \|\widehat{y}_T - \tilde{y}\| \le \delta_D.$$

Then

$$|f(\tilde{x}, \tilde{y}) - f(\widehat{x}_T, \widehat{y}_T)| \le L_1\delta_C + L_2\delta_D$$

and for each $z \in C$ and each $v \in D$,

$$f(z, \tilde{y}) \ge f(\tilde{x}, \tilde{y})$$

$$-2(\sum_{t=0}^{T} a_t)^{-1} \sup\{\phi(s):\ s \in [0, \max\{2M_1,\ 2M_2\}+1]\}$$

$$-2(\sum_{t=0}^{T} a_t)^{-1} \max\{\sum_{t=0}^{T} b_{t,1},\ \sum_{t=0}^{T} b_{t,2}\} - 4\max\{L_1\delta_C,\ L_2\delta_D\}$$

and

$$f(\tilde{x}, v) \le f(\tilde{x}, \tilde{y})$$

$$+2(\sum_{t=0}^{T} a_t)^{-1} \sup\{\phi(s) :\ s \in [0, \max\{2M_1,\ 2M_2\} + 1]\}$$

$$+2(\sum_{t=0}^{T} a_t)^{-1} \max\{\sum_{t=0}^{T} b_{t,1},\ \sum_{t=0}^{T} b_{t,2}\} + 4\max\{L_1\delta_C,\ L_2\delta_D\}.$$

Lemma 10.3 *Let $\Delta > 0$, $\alpha_i \in [0, 1]$, $i = 0, 1, \dots, n_2 > n_1$ be natural numbers, $\sum_{i=n_1}^{n_2} \alpha_i > 0$, $z_i \in B_X(0, \Delta),\ i = 0, \dots, n_2$. Then*

$$\|(\sum_{i=0}^{n_2} \alpha_i)^{-1} \sum_{i=0}^{n_2} \alpha_i z_i - (\sum_{i=n_1}^{n_2} \alpha_i)^{-1} \sum_{i=n_1}^{n_2} \alpha_i z_i\| \le 2n_1\Delta(\sum_{i=0}^{n_2} \alpha_i)^{-1}.$$

Proof It is not difficult to see that

$$\|(\sum_{i=0}^{n_2} \alpha_i)^{-1} \sum_{i=0}^{n_2} \alpha_i z_i - (\sum_{i=n_1}^{n_2} \alpha_i)^{-1} \sum_{i=n_1}^{n_2} \alpha_i z_i\|$$

$$\le \|(\sum_{i=0}^{n_2} \alpha_i)^{-1} \sum_{i=0}^{n_1-1} \alpha_i z_i\|$$

$$+\|(\sum_{i=0}^{n_2} \alpha_i)^{-1} \sum_{i=n_1}^{n_2} \alpha_i z_i - (\sum_{i=n_1}^{n_2} \alpha_i)^{-1} \sum_{i=n_1}^{n_2} \alpha_i z_i\|$$

$$\le (\sum_{i=0}^{n_2} \alpha_i)^{-1} n_1\Delta + \|\sum_{i=n_1}^{n_2} \alpha_i z_i\| |(\sum_{i=0}^{n_2} \alpha_i)^{-1} - (\sum_{i=n_1}^{n_2} \alpha_i)^{-1}|$$

$$\le n_1\Delta(\sum_{i=0}^{n_2} \alpha_i)^{-1} + (\sum_{i=0}^{n_2} \alpha_i)^{-1}(\sum_{i=n_1}^{n_2} \alpha_i)^{-1} \sum_{i=0}^{n_1-1} \alpha_i \|\sum_{i=n_1}^{n_2} \alpha_i z_i\|$$

$$\le n_1\Delta(\sum_{i=0}^{n_2} \alpha_i)^{-1} + n_1(\sum_{i=0}^{n_2} \alpha_i)^{-1}(\sum_{i=n_1}^{n_2} \alpha_i)^{-1} \|\sum_{i=n_1}^{n_2} \alpha_i z_i\|$$

$$\le 2n_1\Delta(\sum_{i=0}^{n_2} \alpha_i)^{-1}.$$

Lemma 10.3 is proved.

10.2 Zero-Sum Games on Bounded Sets

Let $(X, \langle\cdot,\cdot\rangle)$, $(Y, \langle\cdot,\cdot\rangle)$ be Hilbert spaces equipped with the complete norms $\|\cdot\|$ which are induced by their inner products. Let C be a nonempty closed convex subset of X, D be a nonempty closed convex subset of Y, U be an open convex subset of X and V be an open convex subset of Y such that

$$C \subset U, \;\; D \subset V. \tag{10.7}$$

For each concave function $g : V \to R^1$, each $x \in V$ and each $\epsilon > 0$, set

$$\partial g(x) = \{l \in Y : \; \langle l, y - x\rangle \geq g(y) - g(x) \text{ for all } y \in V\}, \tag{10.8}$$

$$\partial_\epsilon g(x) = \{l \in Y : \; \langle l, y - x\rangle + \epsilon \geq g(y) - g(x) \text{ for all } y \in V\}. \tag{10.9}$$

Clearly, for each $x \in V$ and each $\epsilon > 0$,

$$\partial g(x) = -(\partial(-g)(x)), \;\; \partial_\epsilon g(x) = -(\partial_\epsilon(-g)(x)). \tag{10.10}$$

Suppose that there exist $L_1, L_2, M_1, M_2 > 1$ such that

$$U \subset B_X(0, M_1), \;\; V \subset B_Y(0, M_2),$$

a function $f : U \times V \to R^1$ possesses the following properties:

(i) for each $v \in V$, the function $f(\cdot, v) : U \to R^1$ is convex;
(ii) for each $u \in U$, the function $f(u, \cdot) : V \to R^1$ is concave,

for each $v \in V$,

$$|f(u_1, v) - f(u_2, v)| \leq L_1\|u_1 - u_2\|$$
$$\text{for all } u_1, u_2 \in U \tag{10.11}$$

and that for each $u \in U$,

$$|f(u, v_1) - f(u, v_2)| \leq L_2\|v_1 - v_2\|$$
$$\text{for all } v_1, v_2 \in V. \tag{10.12}$$

Set

$$\|f\| = \sup\{|f(x, y)| : \; x \in U, \;\; y \in V\}.$$

For each $(\xi, \eta) \in U \times V$ and each $\epsilon > 0$, set

$$\partial_x f(\xi, \eta) = \{l \in X :$$

$$f(y, \eta) - f(\xi, \eta) \geq \langle l, y - \xi \rangle \text{ for all } y \in U\},$$

$$\partial_y f(\xi, \eta) = \{l \in Y :$$

$$\langle l, y - \eta \rangle \geq f(\xi, y) - f(\xi, \eta) \text{ for all } y \in V\},$$

$$\partial_{x,\epsilon} f(\xi, \eta) = \{l \in X :$$

$$f(y, \eta) - f(\xi, \eta) + \epsilon \geq \langle l, y - \xi \rangle \text{ for all } y \in U\},$$

$$\partial_{y,\epsilon} f(\xi, \eta) = \{l \in Y :$$

$$\langle l, y - \eta \rangle + \epsilon \geq f(\xi, y) - f(\xi, \eta) \text{ for all } y \in V\}.$$

In view of properties (i) and (ii), (10.11) and (10.12), for each $\xi \in U$ and each $\eta \in V$,

$$\emptyset \neq \partial_x f(\xi, \eta) \subset B_X(0, L_1),$$

$$\emptyset \neq \partial_y f(\xi, \eta) \subset B_Y(0, L_2).$$

Let

$$x_* \in C \text{ and } y_* \in D$$

satisfy

$$f(x_*, y) \leq f(x_*, y_*) \leq f(x, y_*) \tag{10.13}$$

for each $x \in C$ and each $y \in D$.

Denote by $\mathcal{M}_U$ the set of all mappings $P : X \to U$ such that

$$Px = x, \ \ x \in C, \tag{10.14}$$

$$\|Pz - x\| \leq \|x - z\| \text{ for all } x \in C \text{ and all } z \in X \tag{10.15}$$

and by $\mathcal{M}_V$ the set of all mappings $Q : Y \to V$ such that

$$Qy = y, \ \ y \in D, \tag{10.16}$$

$$\|Qz - y\| \leq \|y - z\| \text{ for all } y \in D \text{ and all } z \in Y. \tag{10.17}$$

Let $\mathcal{A} \subset \mathcal{M}_U$ be nonempty. We say that the family $\mathcal{A}$ is (C)-quasi-contractive if the following assumption holds:

(a) For each $M > 0$ and each $r > 0$ there exists $\delta > 0$ such that for each $x \in B_X(0, M)$ satisfying $d(x, C) \geq r$, each $z \in B_X(0, M) \cap C$ and each $P \in \mathcal{A}$, we have

$$\|P(x) - z\| \leq \|x - z\| - \delta.$$

Let $\mathcal{A} \subset \mathcal{M}_V$ be nonempty. We say that the family $\mathcal{A}$ is (D)-quasi-contractive if the following assumption holds:

(b) For each $M > 0$ and each $r > 0$ there exists $\delta > 0$ such that for each $x \in B_Y(0, M)$ satisfying $d(x, C) \geq r$, each $z \in B_Y(0, M) \cap D$ and each $P \in \mathcal{A}$, we have

$$\|P(x) - z\| \leq \|x - z\| - \delta.$$

Let $\mathcal{B}$ be a nonempty set of sequences $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_U$ ($\mathcal{M}_V$ respectively). In the chapter we also use the following assumption.

(C1) for each $\epsilon > 0$ and each $K > 0$ there exist $\delta > 0$ and a natural number m_0 such that:

(i) for each $x, y \in B_X(0, K)$ ($B_Y(0, K)$ respectively) satisfying $\|x - y\| \leq \delta$, each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $t \geq 0$, $\|P_t(x) - P_t(y)\| \leq \epsilon$ holds;

(ii) for each $x \in B_X(0, K)$ ($B_Y(0, K)$ respectively), each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}$ and each integer $q \geq 0$,

$$d(P_{q+m_0} \dots P_{q+1} P_q(x), C) \leq \epsilon$$

$$(d(P_{q+m_0} \dots P_{q+1} P_q(x), D) \leq \epsilon, \text{ respectively}).$$

For each $r > 0$ set

$$U_r = \{x \in U : B_X(x, , r) \subset U\},$$

$$V_r = \{x \in V : B_Y(x, , r) \subset V\},$$

$$\mathcal{M}_{U,r} = \{T \in \mathcal{M}_U : T(X) \subset U_r\},$$

$$\mathcal{M}_{V,r} = \{T \in \mathcal{M}_V : T(X) \subset V_r\}.$$

In order to solve our saddle-point problem we use an extended subgradient algorithm with computational errors.

Subgradient Projection Algorithm Fix $\bar{r} \in (0, 1]$, $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{U,\bar{r}}$, $\{Q_t\}_{t=0}^{\infty} \subset \mathcal{M}_{V,\bar{r}}$ and a sequence $\{\alpha_i\}_{i=0}^{\infty} \subset (0, 1]$ satisfying

$$\lim_{i\to\infty}\alpha_i = 0,\ \sum_{i=0}^{\infty}\alpha_i = \infty.$$

We assume that $\delta > 0$ is a computational error produced by our computer system.

Initialization: select an arbitrary $x_0 \in U$, $y_0 \in V$.

Iterative step: given a current iteration pair of vectors $x_k \in U$ and $y_k \in V$ calculate

$$\xi_t \in \partial_{x,\delta} f(x_t, y_t) + B_X(0, \delta),$$

$$\eta_t \in \partial_{y,\delta} f(x_t, y_t) + B_Y(0, \delta)$$

and the next iteration pair of vectors

$$x_{k+1} \in B_X(P_k(x_k - \alpha_k \xi_k), \alpha_t \delta)$$

and

$$y_{k+1} \in B_Y(Q_k(y_k + \alpha_k \eta_k), \alpha_t \delta).$$

It should be mentioned that in [96] this algorithm is considered in the case when when the images of all operators involved in the algorithms are C or D respectively.

10.3 The First Main Result

Theorem 10.4 *Let $\bar{r} \in (0, 1]$, $\mathcal{B}_1$ be a nonempty set of sequences $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{U,\bar{r}}$ such that at least one of the following conditions holds:*

there exists a nonempty (C)-quasi-contractive set $\mathcal{A}_1 \subset \mathcal{M}_U$ such that

$$\mathcal{B}_1 \subset \{\{P_t\}_{t=0}^{\infty} :\ P_t \in \mathcal{A}_1,\ t = 0, 1, \dots\};$$

$\mathcal{B}_1$ satisfies (C1).

Let $\mathcal{B}_2$ be a nonempty set of sequences $\{Q_t\}_{t=0}^{\infty} \subset \mathcal{M}_{V,\bar{r}}$ such that at least one of the following conditions holds:

there exists a nonempty (D)-quasi-contractive set $\mathcal{A}_2 \subset \mathcal{M}_V$ such that

$$\mathcal{B}_2 \subset \{\{Q_t\}_{t=0}^{\infty} :\ Q_t \in \mathcal{A}_2,\ t = 0, 1, \dots\};$$

$\mathcal{B}_2$ satisfies (C2).

Let $\{\alpha_i\}_{i=0}^{\infty} \subset (0, 1]$ satisfy

$$\lim_{i\to\infty} \alpha_i = 0,\ \sum_{i=0}^{\infty} \alpha_i = \infty \tag{10.18}$$

and let $\epsilon > 0$. *Then there exist* $\delta \in (0, 2^{-1}\bar{r})$ *and a natural number* n_0 *such that the following assertion holds.*

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}_1$, $\{Q_t\}_{t=0}^{\infty} \in \mathcal{B}_2$, $\{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}}$, $\{y_t\}_{t=0}^{\infty} \subset V_{\bar{r}}$, $\{\xi_t\}_{t=0}^{\infty} \subset X$, $\{\eta_t\}_{t=0}^{\infty} \subset Y$ *and that for each integer* $t \geq 0$,

$$\xi_t \in \partial_{x,\delta} f(x_t, y_t) + B_X(0, \delta), \tag{10.19}$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \leq \delta \alpha_t, \tag{10.20}$$

and

$$\eta_t \in \partial_{y,\delta} f(x_t, y_t) + B_Y(0, \delta), \tag{10.21}$$

$$\|y_{t+1} - Q_t(y_t + \alpha_t \eta_t)\| \leq \delta \alpha_t. \tag{10.22}$$

Let for each natural number T,

$$\widehat{x}_T = (\sum_{i=0}^{T} \alpha_t)^{-1} \sum_{t=0}^{T} \alpha_t x_t,\ \widehat{y}_T = (\sum_{i=0}^{T} \alpha_t)^{-1} \sum_{t=0}^{T} \alpha_t y_t. \tag{10.23}$$

Then the following assertion holds.

1. *For each integer* $t \geq n_0$,

$$d(x_t, C) \leq \epsilon,\ d(y_t, D) \leq \epsilon$$

and for each integer $T \geq n_0$,

$$d(\widehat{x}_T, C) \leq \epsilon,\ d(\widehat{y}_T, D) \leq \epsilon.$$

2. *For each natural number* $T \geq n_0$,

$$|(\sum_{t=0}^{T} \alpha_t)^{-1} \sum_{t=0}^{T} \alpha_t f(x_t, y_t) - f(x_*, y_*)| \leq \epsilon,$$

$$|f(\widehat{x}_T, \widehat{y}_T) - (\sum_{t=0}^{T} \alpha_t)^{-1} \sum_{t=0}^{T} \alpha_t f(x_t, y_t)| \leq \epsilon$$

and for each $z \in C$ *and each* $v \in D$,

$$f(z, \widehat{y}_T) \geq f(\widehat{x}_T, \widehat{y}_T) - \epsilon,$$

$$f(\widehat{x}_T, v) \leq f(\widehat{x}_T, \widehat{y}_T) + \epsilon.$$

Proof We may assume without loss of generality that

$$\epsilon < \min\{1, \bar{r}\}. \tag{10.24}$$

We may assume for $i = 1, 2$ that the following property holds:

if (C1) holds for $\mathcal{B}_i$, then for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}_i$ and each integer $s \geq 0$, $\{P_{t+s}\}_{t=0}^{\infty} \in \mathcal{B}_i$.

Fix

$$\epsilon_1 \in (0, \epsilon(64(L_1 + L_2 + 1))^{-1}). \tag{10.25}$$

Lemmas 9.10 and 9.11 imply that there exist $\delta_1 \in (0, \epsilon_1)$ and a natural number m_0 such that the following properties hold:

(iii) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}_1$, each integer $n \geq m_0$, each integer $s \geq 0$ and each finite sequence $\{y_i\}_{i=0}^{n} \subset B_X(0, M_1 + M_1 + L_1)$ satisfying

$$B_X(y_{t+1}, \delta_1) \cap P_{t+s}(B_X(y_t, \delta_1)) \neq \emptyset, \ t = 0, \ldots, n-1$$

the inequality

$$d(y_t, C) \leq \epsilon_1$$

holds for all integers $i \in [m_0, n]$.

(iv) for each $\{Q_t\}_{t=0}^{\infty} \in \mathcal{B}_2$, each integer $n \geq m_0$, each integer $s \geq 0$ and each finite sequence $\{y_i\}_{i=0}^{n} \subset B_Y(0, M_1 + M_2 + 4)$ satisfying

$$B_Y(y_{t+1}, \delta_1) \cap Q_{t+s}(B_Y(y_t, \delta_1)) \neq \emptyset, \ t = 0, \ldots, n-1$$

the inequality

$$d(y_t, C) \leq \epsilon_1$$

holds for all integers $t \in [m_0, n]$.

Since $\lim_{i\to\infty} \alpha_i = 0$ (see (10.18)) there is an integer $p_0 > 0$ such that for all integers $i > p_0$, we have

$$\alpha_i < 16^{-1}\delta_1(64(L_1 + L_2 + 4))^{-2}. \tag{10.26}$$

Since $\sum_{i=0}^{\infty} \alpha_i = \infty$ (see (10.18)) there exists a natural number

$$n_0 > p_0 + 4 + m_0 \tag{10.27}$$

such that

$$\sum_{i=p_0+m_0}^{n_0-1} \alpha_i$$

$$> 128(4M_1 + 4M_2 + 4)^2 \epsilon_1^{-1}(L_1 + L_2 + 1)(p_0 + m_0 + 1)(\|f\| + 1). \tag{10.28}$$

Fix a positive number δ such that

$$4\delta(M_1 + M_2 + L_1 + L_2 + 10) < 8^{-1}(64(L_1 + L_2))^{-1}\epsilon_1. \tag{10.29}$$

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}_1$, $\{Q_t\}_{t=0}^{\infty} \in \mathcal{B}_2$,

$$\{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}},\ \{y_t\}_{t=0}^{\infty} \subset V_{\bar{r}}, \tag{10.30}$$

$\{\xi_t\}_{t=0}^{\infty} \subset X$, $\{\eta_t\}_{t=0}^{\infty} \subset Y$, and that for each integer $t \geq 0$,

$$\xi_t \in \partial_{x,\delta} f(x_t, y_t) + B_X(0, \delta), \tag{10.31}$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \leq \delta\alpha_t, \tag{10.32}$$

$$\eta_t \in \partial_{y,\delta} f(x_t, y_t) + B_Y(0, \delta), \tag{10.33}$$

$$\|y_{t+1} - Q_t(y_t + \alpha_t \eta_t)\| \leq \delta\alpha_t. \tag{10.34}$$

Let for each natural number T,

$$\widehat{x}_T = (\sum_{i=0}^{T} \alpha_t)^{-1} \sum_{t=0}^{T} \alpha_t x_t,\ \widehat{y}_T = (\sum_{i=0}^{T} \alpha_t)^{-1} \sum_{t=0}^{T} \alpha_t y_t. \tag{10.35}$$

In view of (10.30), for all integers $t \geq 0$,

$$\|x_t\| \leq M_1,\ \|y_t\| \leq M_2. \tag{10.36}$$

Set

$$\tilde{x}_t = x_{t+p_0},\ \tilde{y}_t = y_{t+p_0},\ t = 0, 1, \ldots. \tag{10.37}$$

Lemma 9.13, (10.11), (10.12) and (10.30) imply that for all integers $t \geq 0$,

$$\partial_{x,\delta} f(x_t, y_t) \subset B_X(0, L_1 + \delta\bar{r}^{-1}) \subset B_X(0, L_1 + 1). \tag{10.38}$$

$$\partial_{y,\delta} f(x_t, y_t) \subset B_Y(0, L_2 + \delta\bar{r}^{-1}) \subset B_Y(0, L_2 + 1). \tag{10.39}$$

It follows from (10.26), (10.31), (10.33), (10.38) and (10.39) that for all integers $t \geq 0$,

$$\|\tilde{x}_t - (\tilde{x}_t - \alpha_{t+p_0}\xi_{t+p_0})\| \leq \alpha_{t+p_0}\|\xi_{t+p_0}\| \leq \alpha_{t+p_0}(L_1 + 1) < \delta_1, \tag{10.40}$$

$$\|\tilde{y}_t - (\tilde{y}_t - \alpha_{t+p_0}\eta_{t+p_0})\| \leq \alpha_{t+p_0}\|\eta_{t+p_0}\| \leq \alpha_{t+p_0}(L_2 + 1) < \delta_1. \tag{10.41}$$

By (10.32), (10.37) and (10.34),

$$\|\tilde{x}_{t+1} - P_{t+p_0}(\tilde{x}_t - \alpha_{t+p_0}\xi_{t+p_0})\|$$

$$= \|x_{t+1+p_0} - P_{t+p_0}(x_{t+p_0} - \alpha_{t+p_0}\xi_{t+p_0})\| \leq \delta_1, \tag{10.42}$$

$$\|\tilde{y}_{t+1} - Q_{t+p_0}(\tilde{y}_t + \alpha_{t+p_0}\eta_{t+p_0})\| \leq \delta_1. \tag{10.43}$$

It follows from (10.40)–(10.43) that

$$B_X(\tilde{x}_{t+1}, \delta_1) \cap P_{t+p_0}(B_X(\tilde{x}_t, \delta_1)) \neq \emptyset, \tag{10.44}$$

$$B_Y(\tilde{y}_{t+1}, \delta_1) \cap Q_{t+p_0}(B_Y(\tilde{y}_t, \delta_1)) \neq \emptyset. \tag{10.45}$$

Properties (iii) and (iv), (1035), (10.37), (10.44) and (10.45) imply that

$$d(\tilde{x}_t, C) \leq \epsilon_1 \text{ for all integers } t \geq m_0. \tag{10.46}$$

$$d(\tilde{y}_t, D) \leq \epsilon_1 \text{ for all integers } t \geq m_0. \tag{10.47}$$

In view of (10.37), (10.46) and (10.47),

$$d(x_t, C) \leq \epsilon_1 \text{ for all integers } t \geq m_0 + p_0, \tag{10.48}$$

$$d(y_t, D) \leq \epsilon_1 \text{ for all integers } t \geq m_0 + p_0. \tag{10.49}$$

Let $t \geq 0$ be an integer. Applying Lemma 9.14 with

$$P = P_t,\ r = \bar{r},\ K_0 = \bar{K} = M_1,\ L_0 = L_1,\ \delta_f = \delta,\ \alpha = \alpha_t,\ \delta_C = \alpha_t\delta,\ \Delta = \delta,$$

$$x = x_t,\ \ f = f(\cdot, y_t),\ \ \xi = \xi_t,\ \ y = x_{t+1}$$

and $\theta_0 \in C$ satisfying

$$f(\theta_0, y_t) > \sup\{f(z, y_t) :\ z \in C\} - 1$$

we obtain that for each $z \in C$,

$$\alpha_t(f(x_t, y_t) - f(z, y_t)) \le 2^{-1}\|z - x_t\|^2 - 2^{-1}\|z - x_{t+1}\|^2$$

$$+2\alpha_t\delta(2M_1 + L_1 + 4) + \alpha_t^2(L_1 + 1)^2 + \alpha_t\delta$$

$$\le 2^{-1}\|z-x_t\|^2-2^{-1}\|z-x_{t+1}\|^2+2\alpha_t\delta(2M_1+L_1+5)+\alpha_t^2(L_1+1)^2. \quad (10.50)$$

Applying Lemma 9.14 with

$$P = Q_t,\ r = \bar{r},\ K_0 = \bar{K} = M_2,\ L_0 = L_2,\ \delta_f = \delta,\ \alpha = \alpha_t,\ \delta_C = \alpha_t\delta,\ \Delta = \delta,$$

$$x = y_t,\ f = -f(x_t, \cdot),\ \xi = -\eta_t,\ y = y_{t+1}$$

and $\theta_0 \in D$ satisfying

$$-f(x_t, \theta_0) > \sup\{-f(x_t, v) :\ v \in D\} - 1$$

we obtain that for each $v \in D$,

$$\alpha_t(f(x_t, v) - f(x_t, y_t)) \le 2^{-1}\|v - y_t\|^2 - 2^{-1}\|v - y_{t+1}\|^2$$

$$+ 2\alpha_t\delta(2M_2 + L_2 + 5) + \alpha_t^2(L_2 + 1)^2. \quad (10.51)$$

Set

$$b_{t,1} = 2\alpha_t\delta(2M_1 + L_1 + 5) + \alpha_t^2(L_1 + 1)^2, \quad (10.52)$$

$$b_{t,2} = 2\alpha_t\delta(2M_2 + L_2 + 5) + \alpha_t^2(L_2 + 1)^2. \quad (10.53)$$

By (10.50)–(10.53), for each $z \in C$,

$$\alpha_t(f(x_t, y_t) - f(z, y_t)) \le 2^{-1}\|z - x_t\|^2 - 2^{-1}\|z - x_{t+1}\|^2 + b_{t,1} \quad (10.54)$$

and for each $v \in D$,

$$\alpha_t(f(x_t, v) - f(x_t, y_t)) \le 2^{-1}\|v - y_t\|^2 - 2^{-1}\|v - y_{t+1}\|^2 + b_{t,2}. \quad (10.55)$$

Define

$$\phi(s) = 2^{-1}s^2,\ s \in R^1.$$

For all integers $t \ge 0$ set

$$\tilde{x}_t = x_{t+m_0+p_0},\ \tilde{y}_t = y_{t+m_0+p_0},\ \tilde{\alpha}_t = \alpha_{t+m_0+p_0},\ \tilde{b}_{t,i} = b_{t+m_0+p_0},\ i = 1, 2. \tag{10.56}$$

For each natural number T set

$$\bar{x}_T = (\sum_{i=0}^{T} \tilde{\alpha}_i)^{-1} \sum_{t=0}^{T} \tilde{\alpha}_t \tilde{x}_t = (\sum_{i=p_0+m_0}^{T+p_0+m_0} \alpha_i)^{-1} \sum_{t=p_0+m_0}^{T+p_0+m_0} \alpha_t x_t, \tag{10.57}$$

$$\bar{y}_T = (\sum_{i=0}^{T} \tilde{\alpha}_i)^{-1} \sum_{t=0}^{T} \tilde{\alpha}_t \tilde{y}_t = (\sum_{i=p_0+m_0}^{T+p_0+m_0} \alpha_i)^{-1} \sum_{t=p_0+m_0}^{T+p_0+m_0} \alpha_t y_t. \tag{10.58}$$

Let $T \geq 1$ be an integer. By Proposition 10.1 applied to $\tilde{x}_t, \tilde{y}_t, t = 0, \dots, T+1$ with $\delta_C = \delta_C = \epsilon_1$ and (10.52)–(10.56),

$$B_X(\widehat{x}_T, \epsilon_1) \cap C \neq \emptyset,\ B_Y(\widehat{y}_T, \epsilon_1) \cap D \neq \emptyset, \tag{10.59}$$

$$|(\sum_{t=0}^{T} \tilde{\alpha}_t)^{-1} \sum_{t=0}^{T} \tilde{\alpha}_t f(\tilde{x}_t, \tilde{y}_t) - f(x_*, y_*)|$$

$$\leq (\sum_{t=0}^{T} \tilde{\alpha}_t)^{-1} \max\{\sum_{t=0}^{T} \tilde{b}_{t,1},\ \sum_{t=0}^{T} \tilde{b}_{t,2}\}$$

$$+ (L_1 + L_2)\epsilon_1 + 2^{-1}(\sum_{t=0}^{T} \tilde{\alpha}_t)^{-1}(2M_1 + 2M_2 + 1)^2, \tag{10.60}$$

$$|f(\bar{x}_T, \bar{y}_T) - (\sum_{t=0}^{T} \tilde{\alpha}_t)^{-1} \sum_{t=0}^{T} \tilde{\alpha}_t f(x_t, y_t)|$$

$$\leq 2^{-1}(\sum_{t=0}^{T} \tilde{\alpha}_t)^{-1}(2M_1 + 2M_2 + 1)^2$$

$$+ (\sum_{t=0}^{T} \tilde{\alpha}_t)^{-1} \max\{\sum_{t=0}^{T} \tilde{b}_{t,1},\ \sum_{t=0}^{T} \tilde{b}_{t,2}\} + (L_1 + L_2)\epsilon_1 \tag{10.61}$$

and for each $z \in C$ and each $v \in D$,

$$f(z, \bar{y}_T) \geq f(\bar{x}_T, \bar{y}_T)$$

$$-(\sum_{t=0}^{T}\tilde{\alpha}_t)^{-1}(2M_1+2M_2+1)^2$$

$$-2(\sum_{t=0}^{T}\tilde{\alpha}_t)^{-1}\max\{\sum_{t=0}^{T}\tilde{b}_{t,1},\ \sum_{t=0}^{T}\tilde{b}_{t,2}\}-(L_1+L_2)\epsilon_1, \tag{10.62}$$

$$f(\bar{x}_T, v) \le f(\bar{x}_T, \bar{y}_T)$$

$$+(\sum_{t=0}^{T}\tilde{\alpha}_t)^{-1}(2M_1+2M_2+1)^2$$

$$+2(\sum_{t=0}^{T}\tilde{\alpha}_t)^{-1}\max\{\sum_{t=0}^{T}\tilde{b}_{t,1},\ \sum_{t=0}^{T}\tilde{b}_{t,2}\}+(L_1+L_2)\epsilon_1. \tag{10.63}$$

Assume that

$$T \ge n_0 - p_0 - m_0 \tag{10.64}$$

is an integer. By (10.26), (10.28), (10.29), (10.52), (10.53), (10.56) and (10.64),

$$(\sum_{t=0}^{T}\tilde{\alpha}_t)^{-1}(2M_1+2M_2+1)^2 = (\sum_{t=p_0+m_0}^{T+p_0+m_0}\alpha_t)^{-1}(2M_1+2M_2+1)^2$$

$$\le (\sum_{t=p_0+m_0}^{n_0}\alpha_t)^{-1}(2M_1+2M_2+1)^2 < 128^{-1}\epsilon_1, \tag{10.65}$$

$$(\sum_{t=0}^{T}\tilde{\alpha}_t)^{-1}\max\{\sum_{t=0}^{T}\tilde{b}_{t,1},\ \sum_{t=0}^{T}\tilde{b}_{t,2}\}$$

$$\le (\sum_{t=p_0+m_0}^{T+p_0+m_0}\alpha_t)^{-1}((L_1+1)^2+(L_2+1)^2)\sum_{t=p_0+m_0}^{T+p_0+m_0}\alpha_t^2$$

$$+2\delta(2M_1+2M_2+L_1+L_2+10)$$

$$\le 2\delta(2M_1+2M_2+L_1+L_2+10)$$

$$+(L_1+L_2+2)^2\sup\{\alpha_t :\ t \ge m_0+p_0 \text{ is an integer}\}$$

$$< 128^{-1}\epsilon_1 + 64^{-1}\epsilon_1. \tag{10.66}$$

It follows from (10.22), (10.60)–(10.62), (10.65) and (10.66) that

$$|(\sum_{t=0}^{T}\tilde{\alpha}_t)^{-1}\sum_{t=0}^{T}\tilde{\alpha}_t f(\tilde{x}_t,\tilde{y}_t) - f(x_*,y_*)|$$

$$\leq 3\cdot 128^{-1}\epsilon_1 + 64^{-1}\epsilon_1 + (L_1+L_2)\epsilon_1 \leq (L_1+L_2+1)\epsilon_1, \tag{10.67}$$

$$|f(\widehat{x}_T,\widehat{y}_T) - (\sum_{t=0}^{T}\tilde{\alpha}_t)^{-1}\sum_{t=0}^{T}\tilde{\alpha}_t f(\tilde{x}_t,\tilde{y}_t)|$$

$$\leq 3\cdot 128^{-1}\epsilon_1 + 128^{-1}\epsilon_1 + (L_1+L_2)\epsilon_1 \tag{10.68}$$

and for each $z \in C$ and each $v \in D$,

$$f(z,\widehat{y}_T) \geq f(\widehat{x}_T,\widehat{y}_T) - 3\cdot 64^{-1}\epsilon_1 - 128^{-1}\epsilon_1 - (L_1+L_2)\epsilon_1,$$

$$f(\widehat{x}_T,v) \leq f(\widehat{x}_T,\widehat{y}_T) + 3\cdot 32^{-1}\epsilon_1 + (L_1+L_2)\epsilon_1.$$

Lemma 10.3, (10.28), (10.56), (10.64) and (10.68) imply that

$$|(\sum_{t=0}^{T}\tilde{\alpha}_t)^{-1}\sum_{t=0}^{T}\tilde{\alpha}_t f(\tilde{x}_t,\tilde{y}_t) - (\sum_{t=0}^{T+p_0+m_0}\alpha_t)^{-1}\sum_{t=0}^{T+p_0+m_0}\alpha_t f(x_t,y_t)|$$

$$= |(\sum_{t=p_0+m_0}^{T+p_0+m_0}\alpha_t)^{-1}\sum_{t=p_0+m_0}^{T+p_0+m_0}\alpha_t f(x_t,y_t) - (\sum_{t=0}^{T+p_0+m_0}\alpha_t)^{-1}\sum_{t=0}^{T+p_0+m_0}\alpha_t f(x_t,y_t)|$$

$$\leq 2\|f\|(p_0+m_0+1)(\sum_{i=0}^{n_0}\alpha_i)^{-1} < 64^{-1}\epsilon_1. \tag{10.69}$$

In view of (10.15), (10.67) and (10.69),

$$|(\sum_{t=0}^{T+p_0+m_0}\alpha_t)^{-1}\sum_{t=0}^{T+p_0+m_0}\alpha_t f(x_t,y_t) - f(x_*,y_*)| < 2^{-1}\epsilon. \tag{10.70}$$

Lemma 10.3, (10.28), (10.35), (10.56)–(10.58) and (10.64) imply that

$$\|\bar{x}_T - \widehat{x}_{T+p_0+m_0}\|$$

$$= \|(\sum_{t=p_0+m_0}^{T+p_0+m_0} \alpha_t)^{-1} \sum_{t=p_0+m0}^{T+p_0+m_0} \alpha_t x_t$$

$$-(\sum_{t=0}^{T+p_0+m_0} \alpha_t)^{-1} \sum_{t=0}^{T+p_0+m_0} \alpha_t x_t\|$$

$$\leq 2(p_0 + m_0 + 1)M_1(\sum_{i=0}^{n_0} \alpha_i)^{-1}$$

$$< 128^{-1}\epsilon_1(L_1 + L_2 + 1)^{-1}(\|f\| + 1)^{-1}, \tag{10.71}$$

$$\|\bar{y}_T - \widehat{y}_{T+p_0+m_0}\|$$

$$= \|(\sum_{t=p_0+m_0}^{T+p_0+m_0} \alpha_t)^{-1} \sum_{t=p_0+m0}^{T+p_0+m_0} \alpha_t y_t$$

$$-(\sum_{t=0}^{T+p_0+m_0} \alpha_t)^{-1} \sum_{t=0}^{T+p_0+m_0} \alpha_t y_t\|$$

$$\leq 2(p_0 + m_0 + 1)M_2(\sum_{i=0}^{n_0} \alpha_i)^{-1}$$

$$< 128^{-1}\epsilon_1(L_1 + L_2 + 1)^{-1}(\|f\| + 1)^{-1}, \tag{10.72}$$

Equations (10.59), (10.71) and (10.72) imply that

$$d(\widehat{x}_{T+p_0+m_0}, C) \leq 128^{-1}\epsilon_1 + d(\widehat{x}_T, C) < 2\epsilon_1 < \epsilon,$$

$$d(\widehat{y}_{T+p_0+m_0}, C) \leq 128^{-1}\epsilon_1 + d(\widehat{y}_T, D) < 2\epsilon_1 < \epsilon.$$

Together with (10.48) and (10.49) this implies the validity of Assertion 1. By (10.11), (10.12), (10.71) and (10.72), for each $z \in U$ and each $v \in V$,

$$|f(\bar{x}_T, v) - f(\widehat{x}_{T+p_0+m_0}, v) \leq L_1\|\bar{x}_T - \widehat{x}_{T+p_0+m_0}\| \leq 128^{-1}\epsilon_1(\|f\| + 1)^{-1},$$

$$|f(z, \bar{y}_T) - f(z, \widehat{y}_{T+p_0+m_0}) \leq L_2\|\bar{y}_T - \widehat{y}_{T+p_0+m_0}\| \leq 128^{-1}\epsilon_1(\|f\| + 1)^{-1}, \tag{10.73}$$

$$|f(\bar{x}_T, \bar{y}_T) - f(\widehat{x}_{T+p_0+m_0}, \widehat{y}_{T+p_0+m_0})|$$

$$\leq |f(\bar{x}_T, \bar{y}_T) - f(\widehat{x}_{T+p_0+m_0}, \bar{y}_T)|$$

$$+ |f(\widehat{x}_{T+p_0+m_0}, \bar{y}_T) - f(\widehat{x}_{T+p_0+m_0}, \widehat{y}_{T+p_0+m_0})| < 64^{-1}\epsilon_1. \tag{10.74}$$

It follows from (10.67), (10.69) and (10.74) that

$$(\sum_{t=0}^{T+p_0+m_0} \alpha_t)^{-1} \sum_{t=0}^{T+p_0+m_0} \alpha_t f(x_t, y_t) - f(\widehat{x}_{T+p_0+m_0}, \widehat{y}_{T+p_0+m_0})|$$

$$\leq |(\sum_{t=0}^{T+p_0+m_0} \alpha_t)^{-1} \sum_{t=0}^{T+p_0+m_0} \alpha_t f(x_t, y_t) - (\sum_{t=p_0+m_0}^{T+p_0+m_0} \alpha_t)^{-1} \sum_{t=p_0+m_0}^{T+p_0+m_0} \alpha_t f(x_t, y_t)|$$

$$+|(\sum_{t=p_0+m_0}^{T+p_0+m_0} \alpha_t)^{-1} \sum_{t=p_0+m_0}^{T+p_0+m_0} \alpha_t f(x_t, y_t) - f(\bar{x}_T, \bar{y}_T)|$$

$$+|f(\bar{x}_T, \bar{y}_T) - f(\widehat{x}_{T+p_0+m_0}, \widehat{y}_{T+p_0+m_0})|$$

$$\leq 64^{-1}\epsilon_1 + 64^{-1}\epsilon_1 + 32^{-1}\epsilon_1 + (L_1 + L_2)\epsilon_1 + 64^{-1}\epsilon_1.$$

Let $z \in C$ and $v \in D$. By (10.25), (10.68), (10.73) and (10.74),

$$f(z, \widehat{y}_{T+p_0+m_0}) \geq f(z, \bar{y}_T) - 128^{-1}\epsilon_1$$

$$\geq f(\bar{x}_T, \bar{y}_T) - 16^{-1}\epsilon_1 - (L_1 + L_2)\epsilon_1$$

$$\geq f(\widehat{x}_{T+p_0+m_0}, \widehat{y}_{T+p_0+m_0}) - 64^{-1}\epsilon_1 - 16^{-1}\epsilon_1 - (L_1 + L_2)\epsilon_1$$

$$\geq f(\widehat{x}_{T+p_0+m_0}, \widehat{y}_{T+p_0+m_0}) - \epsilon,$$

$$f(\widehat{x}_{T+p_0+m_0}, v) \leq f(\bar{x}_T, v) + 128^{-1}\epsilon_1$$

$$\leq f(\bar{x}_T, \bar{y}_T) + 8^{-1}\epsilon_1 + (L_1 + L_2)\epsilon_1$$

$$\leq f(\widehat{x}_{T+p_0+m_0}, \widehat{y}_{T+p_0+m_0}) + 64^{-1}\epsilon_1 + 8^{-1}\epsilon_1 + (L_1 + L_2)\epsilon_1$$

$$\leq f(\widehat{x}_{T+p_0+m_0}, \widehat{y}_{T+p_0+m_0}) + \epsilon.$$

Assertion 2 is proved. This completes the proof of Theorem 10.4.

10.4 The Second Main Result

Theorem 10.5 *Let $\bar{r} \in (0, 1]$ and $\mathcal{B}_1$ be a nonempty set of sequences $\{P_t\}_{t=0}^{\infty} \subset \mathcal{M}_{U,\bar{r}}$ such that at least one of the following conditions holds:*

there exists a nonempty (C)-quasi-contractive set $\mathcal{A}_1 \subset \mathcal{M}_U$ such that

$$\mathcal{B}_1 \subset \{\{P_t\}_{t=0}^{\infty} : \ P_t \in \mathcal{A}_1, \ t = 0, 1, \dots\};$$

$\mathcal{B}_1$ satisfies (C1).

Let $\mathcal{B}_2$ be a nonempty set of sequences $\{Q_t\}_{t=0}^{\infty} \subset \mathcal{M}_{V,\bar{r}}$ such that at least one of the following conditions holds:

there exists a nonempty (D)-quasi-contractive set $\mathcal{A}_2 \subset \mathcal{M}_V$ such that

$$\mathcal{B}_2 \subset \{\{Q_t\}_{t=0}^{\infty} : \ Q_t \in \mathcal{A}_2, \ t = 0, 1, \dots\};$$

$\mathcal{B}_2$ satisfies (C1).

Let $\epsilon > 0$. Then there exists $\beta_0 \in (0, 1)$ such that for each $\beta_1 \in (0, \beta_0)$ there exist $\delta \in (0, 2^{-1}\bar{r}]$ and a natural number n_0 such that the following assertion holds.

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}_1$, $\{Q_t\}_{t=0}^{\infty} \in \mathcal{B}_2$, $\{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}}$, $\{y_t\}_{t=0}^{\infty} \subset V_{\bar{r}}$, $\{\xi_t\}_{t=0}^{\infty} \subset X$, $\{\eta_t\}_{t=0}^{\infty} \subset Y$, $\{\alpha_t\}_{t=0}^{\infty} \subset [\beta_1, \beta_0]$, and that for each integer $t \geq 0$,

$$\xi_t \in \partial_{x,\delta} f(x_t, y_t) + B_X(0, \delta),$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \leq \delta,$$

and

$$\eta_t \in \partial_{y,\delta} f(x_t, y_t) + B_Y(0, \delta),$$

$$\|y_{t+1} - Q_t(y_t + \alpha_t \eta_t)\| \leq \delta.$$

Let for each pair of natural numbers $m > n$,

$$\widehat{x}_{n,m} = (\sum_{i=n}^{m} \alpha_i)^{-1} \sum_{t=n}^{m} \alpha_t x_t, \ \widehat{y}_{n,m} = (\sum_{i=n}^{m} \alpha_t)^{-1} \sum_{t=n}^{m} \alpha_t y_t.$$

Then for each integer $t \geq n_0$,

$$d(x_t, C) \leq \epsilon, \ d(y_t, D) \leq \epsilon,$$

for each pair of natural numbers $m > n$ satisfying $m - n \geq n_0$,

$$d(\widehat{x}_{n,m}, C) \leq \epsilon, \ d(\widehat{y}_{n,m}, D) \leq \epsilon$$

and for each pair of natural numbers $m > n \geq n_0$ satisfying $m - n \geq n_0$,

$$|(\sum_{t=n}^{m} \alpha_t)^{-1} \sum_{t=n}^{m} \alpha_t f(x_t, y_t) - f(x_*, y_*)| \leq \epsilon,$$

$$|f(\widehat{x}_{n,m}, \widehat{y}_{n,m}) - (\sum_{t=n}^{m} \alpha_t)^{-1} \sum_{t=n}^{m} \alpha_t f(x_t, y_t)| \leq \epsilon$$

and for each $z \in C$ and each $v \in D$,

$$f(z, \widehat{y}_{n,m}) \geq f(\widehat{x}_{n,m}, \widehat{y}_{n,m}) - \epsilon,$$

$$f(\widehat{x}_{n,m}, v) \leq f(\widehat{x}_{n,m}, \widehat{y}_{n,m}) + \epsilon.$$

Proof We may assume without loss of generality that

$$\epsilon < \min\{1, \bar{r}\}.$$

We may assume that for $i = 1, 2$ the following property holds:

if (C1) holds for $\mathcal{B}_i$, then for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}_i$ and each integer $s \geq 0$, $\{P_{t+s}\}_{t=0}^{\infty} \in \mathcal{B}_i$.

Fix

$$\epsilon_1 \in (0, \epsilon(64L_1 + L_2 + 10)^{-1}). \tag{10.75}$$

Lemmas 9.10 and 9.11 imply that there exist $\delta_1 \in (0, \epsilon_1)$ and a natural number m_0 such that the following properties hold:

(v) for each $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}_1$, each integer $n \geq m_0$, each integer $s \geq 0$ and each finite sequence $\{y_i\}_{i=0}^{n} \subset B_X(0, M_1 + M_1 + 4)$ satisfying

$$B_X(y_{t+1}, \delta_1) \cap P_{t+s}(B_X(y_t, \delta_1)) \neq \emptyset, \ t = 0, \dots, n-1$$

the inequality

$$d(y_t, C) \leq \epsilon_1$$

holds for all integers $t \in [m_0, n]$.

(vi) for each $\{Q_t\}_{t=0}^{\infty} \in \mathcal{B}_2$, each integer $n \geq m_0$, each integer $s \geq 0$ and each finite sequence $\{y_i\}_{i=0}^{n} \subset B_Y(0, M_1 + M_2 + 4)$ satisfying

$$B_Y(y_{t+1}, \delta_1) \cap Q_{t+s}(B_Y(y_t, \delta_1)) \neq \emptyset, \ t = 0, \dots, n-1$$

the inequality

$$d(y_t, D) \le \epsilon_1$$

holds for all integers $t \in [m_0, n]$.

Choose a positive number β_0 such that

$$\beta_0 \le 16^{-1}\delta_1(64(L_1 + L_2 + 4))^{-2}. \tag{10.76}$$

Let

$$\beta_1 \in (0, \beta_0). \tag{10.77}$$

Fix a natural number

$$n_0 > m_0$$
$$+ 128\epsilon_1^{-1}\beta_1^{-1}(4M_1 + 4M_2 + 4)^2(L_1 + L_2 + 1)(m_0 + 1)(\|f\| + 1) \tag{10.78}$$

and a positive number δ such that

$$4\delta(M_1 + M_2 + L_1 + L_2 + 5) < 8^{-1}(64(L_1 + L_2 + 10))^{-1}\epsilon_1\beta_1. \tag{10.79}$$

Assume that $\{P_t\}_{t=0}^{\infty} \in \mathcal{B}_1$, $\{Q_t\}_{t=0}^{\infty} \in \mathcal{B}_2$,

$$\{x_t\}_{t=0}^{\infty} \subset U_{\bar{r}},\ \{y_t\}_{t=0}^{\infty} \subset V_{\bar{r}}, \tag{10.80}$$

$\{\xi_t\}_{t=0}^{\infty} \subset X$, $\{\eta_t\}_{t=0}^{\infty} \subset Y$,

$$\{\alpha_t\}_{t=0}^{\infty} \subset [\beta_1, \beta_0], \tag{10.81}$$

and that for each integer $t \ge 0$,

$$\xi_t \in \partial_{x,\delta} f(x_t, y_t) + B_X(0, \delta), \tag{10.82}$$

$$\|x_{t+1} - P_t(x_t - \alpha_t\xi_t)\| \le \delta, \tag{10.83}$$

$$\eta_t \in \partial_{y,\delta} f(x_t, y_t) + B_Y(0, \delta), \tag{10.84}$$

$$\|y_{t+1} - Q_t(y_t + \alpha_t\eta_t)\| \le \delta. \tag{10.85}$$

Let $m.n$ be natural numbers. Recall that

$$\widehat{x}_{n,m} = (\sum_{i=n}^{m} \alpha_t)^{-1} \sum_{t=n}^{m} \alpha_t x_t,\ \widehat{y}_{n,m} = (\sum_{i=n}^{m} \alpha_t)^{-1} \sum_{t=n}^{m} \alpha_t y_t. \tag{10.86}$$

In view of (10.80), for all integers $t \geq 0$,

$$\|x_t\| \leq M_1, \ \|y_t\| \leq M_2. \tag{10.87}$$

Lemma 9.13, (10.80) and (10.87) imply that for all integers $t \geq 0$,

$$\partial_{x,\delta} f(x_t, y_t) \subset B_X(0, L_1 + \delta\bar{r}^{-1}) \subset B_X(0, L_1 + 1). \tag{10.88}$$

$$\partial_{y,\delta} f(x_t, y_t) \subset B_Y(0, L_2 + \delta\bar{r}^{-1}) \subset B_Y(0, L_2 + 1). \tag{10.89}$$

It follows from (10.76), (10.79), (10.81)–(10.85), (10.88) and (10.89) that for all integers $t \geq 0$,

$$\|x_t - (x_t - \alpha_t \xi_t)\| \leq \alpha_t \|\xi_t\| \leq \beta_0(L_1 + 2) < \delta_1,$$

$$\|y_t - (y_t + \alpha_t \eta_t)\| \leq \alpha_t \|\eta_t\| \leq \beta_0(L_2 + 2) < \delta_1,$$

$$\|x_{t+1} - P_t(x_t - \alpha_t \xi_t)\| \leq \delta_1,$$

$$\|y_{t+1} - Q_t(y_t + \alpha_t \eta_t)\| \leq \delta_1,$$

$$B_X(x_{t+1}, \delta_1) \cap P_t(B_X(x_t, \delta_1)) \neq \emptyset, \tag{10.90}$$

$$B_Y(y_{t+1}, \delta_1) \cap Q_t(B_Y(y_t, \delta_1)) \neq \emptyset. \tag{10.91}$$

Properties (v) and (vi), (10.87), (10.90) and (10.91) imply that

$$d(x_t, C) \leq \epsilon_1 \text{ for all integers } t \geq m_0. \tag{10.92}$$

$$d(y_t, D) \leq \epsilon_1 \text{ for all integers } t \geq m_0. \tag{10.93}$$

Let $t \geq 0$ be an integer. In view of (10.80) and (10.82)–(10.85), applying Lemma 9.14 with

$$P = P_t, \ r = \bar{r}, \ K_0 = \bar{K} = M_1, \ L_0 = L_1, \ \delta_f = \delta, \ \alpha = \alpha_t, \ \delta_C = \delta, \ \Delta = \delta,$$

$$x = x_t, \ f = f(\cdot, y_t), \ \xi = \xi_t, \ y = x_{t+1}$$

and $\theta_0 \in C$ satisfying

$$f(\theta_0, y_t) > \sup\{f(z, y_t) : \ z \in C\} - 1$$

we obtain that for each $z \in C$,

$$\alpha_t(f(x_t, y_t) - f(z, y_t)) \leq 2^{-1}\|z - x_t\|^2 - 2^{-1}\|z - x_{t+1}\|^2$$

$$+2\delta(2M_1 + L_1 + 4) + \alpha_t^2(L_1 + 1)^2 + \alpha_t\delta$$

$$\leq 2^{-1}\|z - x_t\|^2 - 2^{-1}\|z - x_{t+1}\|^2 + 2\delta(2M_1 + L_1 + 5) + \alpha_t^2(L_1 + 1)^2. \quad (10.94)$$

In view of (10.80) and (10.82)–(10.85), applying Lemma 9.14 with

$$P = Q_t,\ r = \bar{r},\ K_0 = \bar{K} = M_2,\ L_0 = L_2,\ \delta_f = \delta,\ \alpha = \alpha_t,\ \delta_C = \delta,\ \Delta = \delta,$$

$$x = y_t,\ f = -f(x_t, \cdot),\ \xi = -\eta_t,\ y = y_{t+1}$$

we obtain that for each $v \in D$,

$$\alpha_t(f(x_t, v) - f(x_t, y_t)) \leq 2^{-1}\|v - y_t\|^2 - 2^{-1}\|v - y_{t+1}\|^2$$

$$+ 2\delta(2M_2 + L_2 + 5) + \alpha_t^2(L_2 + 1)^2. \quad (10.95)$$

Set

$$b_{t,1} = 2\delta(2M_1 + L_1 + 5) + \alpha_t^2(L_1 + 1)^2, \quad (10.96)$$

$$b_{t,2} = 2\delta(2M_2 + L_2 + 5) + \alpha_t^2(L_2 + 1)^2. \quad (10.97)$$

By (10.94)–(10.97), for each $z \in C$ and for each $v \in D$,

$$\alpha_t(f(x_t, y_t) - f(z, y_t)) \leq 2^{-1}\|z - x_t\|^2 - 2^{-1}\|z - x_{t+1}\|^2 + b_{t,1} \quad (10.98)$$

$$\alpha_t(f(x_t, v) - f(x_t, y_t)) \leq 2^{-1}\|v - y_t\|^2 - 2^{-1}\|v - y_{t+1}\|^2 + b_{t,2}. \quad (10.99)$$

Define

$$\phi(s) = 2^{-1}s^2,\ s \in R^1.$$

Let m, n be a natural numbers such that

$$m \geq n + m_0. \quad (10.100)$$

By Proposition 10.1, (10.92), (10.93) and (10.96)–(10.99),

$$B_X(\widehat{x}_{n+m_0,m}, \epsilon_1) \cap C \neq \emptyset,\ B_Y(\widehat{y}_{n+m_0,m}, \epsilon_1) \cap D \neq \emptyset, \quad (10.101)$$

$$|(\sum_{t=n+m_0}^{m} \alpha_t)^{-1} \sum_{t=n+m_0}^{m} \alpha_t f(x_t, y_t) - f(x_*, y_*)|$$

$$\le (\sum_{t=n+m_0}^{m} \alpha_t)^{-1} \max\{\sum_{t=n+m_0}^{m} b_{t,1}, \sum_{t=n+m_0}^{m} b_{t,2}\}$$

$$+ (L_1 + L_2)\epsilon_1 + 2^{-1}(\sum_{t=n+m_0}^{m} \alpha_t)^{-1}(2M_1 + 2M_2 + 1)^2, \tag{10.102}$$

$$|f(\widehat{x}_{n+m_0,m}, \widehat{y}_{n+m_0,m}) - (\sum_{t=n+m_0}^{T} \alpha_t)^{-1} \sum_{t=n+m_0}^{m} \alpha_t f(x_t, y_t)|$$

$$\le 2^{-1}(\sum_{t=n+m_0}^{m} \alpha_t)^{-1}(2M_1 + 2M_2 + 1)^2$$

$$+ (\sum_{t=n+m_0}^{m} \alpha_t)^{-1} \max\{\sum_{t=n+m_0}^{m} b_{t,1}, \sum_{t=n+m_0}^{m} b_{t,2}\} + (L_1 + L_2)\epsilon_1 \tag{10.103}$$

and for each $z \in C$ and each $v \in D$,

$$f(z, \widehat{y}_{n+m_0,m}) \ge f(\widehat{x}_{n+m_0,m}, \widehat{y}_{n+m_0,m})$$

$$-2(\sum_{t=n+m_0}^{m} \alpha_t)^{-1}(2M_1 + 2M_2 + 1)^2\}$$

$$- 2(\sum_{t=n+m_0}^{m} \alpha_t)^{-1} \max\{\sum_{t=n+m_0}^{m} b_{t,1}, \sum_{t=n+m_0}^{m} b_{t,2}\} - (L_1 + L_2)\epsilon_1, \tag{10.104}$$

$$f(\widehat{x}_{n+m_0,m}, v) \le f(\widehat{x}_{n+m_0,m}, \widehat{y}_{n+m_0,m})$$

$$+(\sum_{t=n+m_0}^{m} \alpha_t)^{-1}(2M_1 + 2M_2 + 1)^2$$

$$+ 2(\sum_{t=n+m_0}^{m} \alpha_t)^{-1} \max\{\sum_{t=n+m_0}^{m} b_{t,1}, \sum_{t=n+m_0}^{m} b_{t,2}\} + (L_1 + L_2)\epsilon_1. \tag{10.105}$$

By (10.76), (10.78), (10.79), (10.81) and (10.100),

$$(\sum_{t=n+m_0}^{m} \alpha_t)^{-1}(2M_1+2M_2+1)^2 \le \beta_1^{-1}(n_0-m_0)^{-1}(2M_1+2M_2+1)^2 < 128^{-1}\epsilon_1, \tag{10.106}$$

$$(\sum_{t=n+m_0}^{m} \alpha_t)^{-1} \max\{\sum_{t=n+m_0}^{m} (2\delta(2M_1 + L_1 + 5) + \alpha_t^2(L_1 + 1)^2),$$

$$\sum_{t=n+m_0}^{m} (2\delta(2M_2 + L_2 + 5) + \alpha_t^2(L_2 + 1)^2)\}$$

$$\leq \beta_0(L_1 + L_2 + 1)^2 + 2(2M_1 + 2M_2 + L_1 + L_2 + 5)\delta\beta_1^{-1} \leq 16^{-1}\epsilon_1 + 64^{-1}\epsilon_1. \tag{10.107}$$

It follows from (10.91), (10.96), (10.101), (10.106) and (10.107) that

$$|(\sum_{t=n+m_0}^{m} \alpha_t)^{-1} \sum_{t=n+m_0}^{m} \alpha_t f(x_t, y_t) - f(x_*, y_*)| \leq (L_1 + L_2 + 1)\epsilon_1. \tag{10.108}$$

By (10.96), (10.97), (10.103), (10.106) and (10.107),

$$|f(\widehat{x}_{n+m_0,m}, \widehat{y}_{n+m_0,m}) - (\sum_{t=n+m_0}^{m} \alpha_t)^{-1} \sum_{t=n+m_0}^{m} \alpha_t f(x_t, y_t)|$$

$$\leq (L_1 + L_2 + 1)\epsilon_1. \tag{10.109}$$

In view of (10.96), (10.97) and (10.104)–(10.107), for each $z \in C$ and each $v \in D$,

$$f(z, \widehat{y}_{n+m_0,m}) \geq f(\widehat{x}_{n+m_0,m}, \widehat{y}_{n+m_0,m}) - (L_1 + L_2 + 1)\epsilon_1, \tag{10.110}$$

$$f(\widehat{x}_{n+m_0,m}, v) \leq f(\widehat{x}_{n+m_0,m}, \widehat{y}_{n+m_0,m}) + (L_1 + L_2 + 1)\epsilon_1. \tag{10.111}$$

Lemma 10.3, (10.81), (10.86), (10.87) and (10.110) imply that

$$\|\widehat{x}_{n,m} - \widehat{x}_{n+m_0,m}\| \leq 2M_1(m_0 + 1)(\sum_{i=n}^{m} \alpha_i)^{-1}$$

$$\leq 2M_1(m_0 + 1)\beta_1^{-1} n_0^{-1}, \tag{10.112}$$

$$\|\widehat{y}_{n,m} - \widehat{y}_{n+m_0,m}\| \leq 2M_2(m_0 + 1)(\sum_{i=n}^{m} \alpha_i)^{-1}$$

$$\leq 2M_2(m_0 + 1)\beta_1^{-1} n_0^{-1}. \tag{10.113}$$

By (10.3), (10.4), (10.12), (10.78) and (10.112), for each $z \in U$ and each $v \in V$,

$$|f(\widehat{x}_{n+m}, v) - f(\widehat{x}_{n+m_0,m}, v)| \leq 2M_1 L_1(m_0 + 1)\beta_1^{-1} n_0^{-1} < \epsilon_1,$$

$$|f(z, \widehat{y}_{n,m}) - f(z, \widehat{y}_{n+m_0,m})| \le 2M_2L_2(m_0+1)\beta^{-1}n_0^{-1} < \epsilon_1,$$

$$|f(\widehat{x}_{n,m}, \widehat{y}_{n,m}) - f(\widehat{x}_{n+m_0,m}, \widehat{y}_{n+m_0,m})|$$

$$\le f(\widehat{x}_{n,m}, \widehat{y}_{n,m}) - f(\widehat{x}_{n,m}, \widehat{y}_{n+m_0,m})|$$

$$+|f(\widehat{x}_{n,m}, \widehat{y}_{n+m_0,m}) - f(\widehat{x}_{n+m_0,m}, \widehat{y}_{n+m_0,m})|$$

$$\le 2(m_0+1)\beta_1^{-1}n_0^{-1}(M_1+M_2)(L_1+L_2) < \epsilon_1. \tag{10.114}$$

Lemma 10.3, (10.78), (10.81) and (10.100) imply that

$$(\sum_{t=n}^{m}\alpha_t)^{-1}\sum_{t=n}^{m}\alpha_t f(x_t, y_t) - (\sum_{t=n+m_0}^{m}\alpha_t)^{-1}\sum_{t=n+m_0}^{m}\alpha_t f(x_t, y_t)|$$

$$\le 2\|f\|(m_0+1)\sum_{i=n}^{m-1}\alpha_i \le 2\|f\|(m_0+1)\beta_1^{-1}n_0^{-1} < \epsilon_1. \tag{10.115}$$

Equations (10.78), (10.101), (10.112) and (10.113) imply that

$$d(\widehat{x}_{n,m}, C) \le 2M_1(m_0+1)\beta_1^{-1}n_0^{-1} + d(\widehat{x}_{n+m_0,m}, C) \le 2\epsilon_1 < \epsilon,$$

$$d(\widehat{y}_{n,m}, D) \le 2M_2(m_0+1)\beta_1^{-1}n_0^{-1} + d(\widehat{y}_{n+m_0,m}, D) \le 2\epsilon_1 < \epsilon.$$

By (10.75), (10.108) and (10.115),

$$|(\sum_{t=n}^{m}\alpha_t)^{-1}\sum_{t=n}^{m}\alpha_t f(x_t, y_t) - f(x_*, y_*)| \le \epsilon_1(L_1+L_2+2) < \epsilon.$$

In view of (10.75), (10.109) and (10.114),

$$|f(\widehat{x}_{n,m}, \widehat{y}_{n,m}) - (\sum_{t=n}^{m}\alpha_t)^{-1}\sum_{t=n}^{m}\alpha_t f(x_t, y_t)| \le \epsilon_1(L_1+L_2+2) < \epsilon.$$

It follows from (10.75), (10.104), (10.110), (10.111) and (10.114), for each $z \in C$ and each $v \in D$,

$$f(z, \widehat{y}_{n,m}) \ge f(\widehat{x}_{n,m}, \widehat{y}_{n,m}) - \epsilon_1(L_1+L_2+4) \ge f(\widehat{x}_{n,m}, \widehat{y}_{n,m}) - \epsilon,$$

$$f(\widehat{x}_{n,m}, v) \le f(\widehat{x}_{n,m}, \widehat{y}_{n,m}) + \epsilon(L_1+L_2+4) < \epsilon.$$

Theorem 10.5 is proved.

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