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XI.

THE RELATION BETWEEN METRIC AND AFFINITY.

(FROM THE DUBLIN INSTITUTE FOR ADVANCED STUDIES.)

BY ERWIN SCHRÖDINGER.

[Read 9 DECEMBER, 1946. Published 30 JUNE, 1947.]

§ 1. The Customary Relation is too Restricted.

As early as 1918 H. Weyl drew attention to the fact that in Einstein's relativisitc theory of gravitation of 1915, gravitation was based not directly on the metric g_{ik} but on the affine connection

$$\Gamma^i_{kl} = \begin{Bmatrix} i \\ k\,l \end{Bmatrix} \tag{1}$$

which is engendered by the metric, being the only symmetric connection which transfers the g_{ik} field into itself, in other words, makes its invariant derivative vanish:

$$g_{ik;l} \equiv g_{ik,l} - g_{\alpha k}\Gamma^\alpha_{il} - g_{i\alpha}\Gamma^\alpha_{kl} = 0\,. \tag{2}$$

This relationship between the g's and the Γ's is suggested because it makes the metric and the affinity "physically compatible" in the following sense.

The Γ's define a field of geodesics and distinguish on every geodesic a parameter s (up to a linear transformation $s' = as + b$ with arbitrary *constants* a and b), being the only one, for which the differential equation of the geodesic has the simple form:[1]

$$\frac{d^2 x_k}{d s^2} + \Gamma^k_{lm}\frac{d x_l}{d s}\frac{d x_m}{d s} = 0\,. \tag{3}$$

This parameter s constitutes sort of a metric along every geodesic. At least the ratio of two line-elements on the same geodesic can be defined as the ratio of their ds's—not for others, because the constant a is free

[1] L. P. Eisenhart, Riemannian Geometry (Princeton University Press, 1926), p. 50; Non-Riemannian Geometry (American Mathematical Society, New York, 1927), p. 57; E. Schrödinger, Proc. Roy. Irish Acad., 49, A, 285, 1944.

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on each of these curves. It is natural to demand that this "affine metric," as far as it goes, should be in accord with the metrical stipulation

$$ds^2 = g_{ik}\, dx_i\, dx_k\,, \tag{4}$$

and that is what I meant by the g's and the Γ's being physically compatible.

The relationship (2) secures compatibility, because it amounts to this: that for any vector A^k the invariant

$$g_{ik}\, A^i\, A^k \tag{4a}$$

does not change when A^k is parallel-transferred in *any* direction. It is easy to see that this is a *sufficient* condition for compatibility. It is also easy to see that it is *necessary* for compatibility that the invariant (4) should be conserved when the vector A^k is parallel-transferred *in its own direction.* But to decide whether the first, more exigent, demand is also necessary (which it is not) or whether the second relaxed one is also sufficient (which it is) needs a little further reflexion. The answer, which I have just indicated, is contained in a mémoire of L. P. Eisenhart[2]: The connection (1) is *not* the only symmetric connection, compatible with the metric g_{ik}. I beg permission to expose this here briefly. It does not seem to have found the attention it deserves. Indeed, while compatibility seems a very natural demand, to ask for more seems artificial.

§ 2. The General Relation.

Any symmetric connection Γ^i_{kl} can, without prejudice, be written in the form

$$\Gamma^i_{kl} = \left\{ {i \atop k\,l} \right\} + g^{is}\, T_{skl}\,, \tag{5}$$

where T is an arbitrary tensor, symmetric in its last two indices:

$$T_{slk} = T_{skl}\,. \tag{6}$$

Now, since the invariant (4a) does not change when A^k is transferred in any direction according to the curly bracket affinity, its change when A^k is displaced according to the connection (5) in the direction of A^k, is proportional to

$$-\,2 g_{ik}\, A^i\, g^{ks}\, T_{sml}\, A^m\, A^l = -\,2\, T_{sml}\, A^s\, A^m\, A^l\,. \tag{7}$$

This vanishes if and only if T is subjected to the further symmetry condition

$$T_{[ikl]} = 0\,, \tag{8}$$

where the [ ] indicate summation over the three cyclic permutations.

[2] L. P. Eisenhart, Transactions of the American Mathematical Society, 26, 378, 1924. Also idem, Non-Riemannian Geometry, p. 84. (See previous quotation.)

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This is a convenient formulation of our *necessary* condition for compatibility. The simplest way of seeing that it is also *sufficient* is the following. If you enhance (5) by an additional *skew* tensor

$$\Omega^{i}{}_{kl} \, (= - \, \Omega^{i}{}_{lk}) \, ,$$

then, according to (3), neither the geodesics nor the parameter s distinguished on each of them are changed. Now Ω can always be chosen so that the resulting non-symmetric affinity complies wtih the *sufficient* condition (2). In order that it should, we must have

$$T_{kil} + \Omega_{kil} + T_{ikl} + \Omega_{ikl} = 0 \, . \tag{9}$$

(The superscript of Ω is lowered with the help of g_{ik}.) If here you choose

$$\Omega_{kil} = \tfrac{1}{3} (T_{lik} - T_{ilk}) \tag{10}$$

(which is skew in i and l as it should be), you find that (9) is fulfilled, in virtue of (6) and (8).

Thus (5), with T subject to (6) and (8) is the most general affinity physically compatible with the metric g^{ik}.

§ 3. Einstein's Original Theory.

If in Einstein's 1915 theory the field-equations are based on the variational principle[3]

$$\delta \int \mathfrak{g}^{ik} R_{ik} \, dx^4 = 0 \tag{11}$$

then it makes no difference, whether we take R_{ik} to be the Einstein-tensor of the Christoffel-bracket-affinity and vary only the g_{ik}, or whether we take it to be formed of the more general affinity (5) and vary the g_{ik} and the T_{ikl}. Even more is true. In this case we need not even impose on T the additional symmetry condition (8). Even if we take the *general* symmetric connection $\Gamma^{i}{}_{kl}$ and vary the g_{ik} and the $\Gamma^{i}{}_{kl}$ independently, (11) yields, along with the field-equations $R_{ik} = 0$, the relation (2), which restricts Γ to the Christoffel-affinity (1).

I beg permission to recall the simple proof due to Palatini, since it is not all too well known. If we vary (11), use for δR_{ik} the precious Palatini-expression

$$\delta R_{ik} = - (\delta \Gamma^{\alpha}{}_{ik})_{;\,\alpha} + (\delta \Gamma^{\alpha}{}_{i\alpha})_{;\,k} \, , \tag{12}$$

and perform partial integration with respect to the semicolons, we get

$$\int (R_{ik} \, \delta \mathfrak{g}^{ik} + \mathfrak{g}^{ik}{}_{;\,\alpha} \delta \Gamma^{\alpha}{}_{ik} - \mathfrak{g}^{ik}{}_{;\,k} \delta \Gamma^{\alpha}{}_{i\alpha}) \, dx^4 = 0 \, . \tag{13}$$

[3] $\mathfrak{g}^{ik}$ is short for $g^{ik}\sqrt{-g}$.

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Hence, firstly,

$$R_{ik} = 0 \tag{14}$$

and secondly, (since $\delta\,\Gamma^{\alpha}{}_{ik} = \delta\,\Gamma^{\alpha}{}_{ki}$)

$$2\,\mathfrak{g}^{ik}{}_{;\,\alpha} - \delta^{k}{}_{\alpha}\,\mathfrak{g}^{i\beta}{}_{;\,\beta} - \delta^{i}{}_{\alpha}\,\mathfrak{g}^{k\beta}{}_{;\,\beta} = 0\,. \tag{15}$$

If this is contracted with respect to k and α, one gets $\mathfrak{g}^{i\beta}{}_{;\,\beta} = 0$, hence

$$\mathfrak{g}^{ik}{}_{;\,\alpha} = 0 \tag{16}$$

which is, of course, equivalent to (2) and thus to (1).

So the variational principle (11) is powerful enough, to select uniquely the Christoffel-affinity not only from those which I called "physically compatible," but indeed from *all* symmetric affinities.

§ 4. Outlook.

In any theory one is inclined, pending a more thorough investigation, to look upon the geodesics (3) as indicating the paths of particles. The T-tensor, if it does not vanish, yields additional "forces," which share with the gravitational "force" the characteristics of being proportional to the mass and independent of the charge, and which would thus seem well fit to depict classically the nuclear force.

The symmetry condition (8) seems a very natural, indeed the only natural, demand to impose at the outset—and then to adopt a variational principle, which must, of course, be different from (11) and yield the field-laws of the two fields, or if you like, of the one field that is proportional to the mass only.

The suggestion is akin to, but distinctly different from, the purely affine theory, which I have tried to develop in recent years,[4] and in which the eventually adopted metric and affinity are *not* in general in the relation (5) cum (8). Einstein, in a recent paper,[5] starts from complex g_{ik} and $\Gamma^{i}{}_{kl}$ both of them of hermitian symmetry. Their real parts *are* in the relation (5) cum (8).

Note added on proof, February 2nd, 1947: While the substance of this paper remains true, the outlook of § 4 has become dispensable. The purely affine point of view has in the meantime yielded the satisfactory extension of the General Theory of Relativity. It has been communicated to this Academy on the 27th of January. It is so simple that, before it is thoroughly investigated, no other attempt is called for.

E. S.

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[4] Proc. Roy. Irish Acad., **49**, A, 43, 1943, and 237, 275, 1944; **51**, A, 41, 1946.

[5] A. Einstein, Annals of Mathematics, **46**, 578, 1945.

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