DISCOVERING THE ART OF MATHEMATICS

Dance

MATHEMATICAL INQUIRY IN THE LIBERAL ARTS

Christine von Renesse,
with Volker Ecke, Julian F. Fleron, and Philip K. Hotchkiss

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Contents

Preface

This book is a very different type of mathematics textbook. Because of this, users new to it, and its companion books that form the Discovering the Art of Mathematics library[1], need context for the book's purpose and what it will ask of those that use it. This preface sets this context, addressing first the Explorers (students), then both Explorers and Guides (teachers) and finishing with important information for the Guides.

0.1 Notes to the Explorer

"Explorer?"

Yes, that's you - an Explorer. And these notes are for you.

We could have addressed you as "reader," but this is not a book intended to be read like a traditional book. This book is really a guide. It is a map. It is a route of trail markers along a path through part of the vast world of mathematics. This book provides you, our explorer, our heroine or hero, with a unique opportunity to explore - to take a surprising, exciting, and beautiful journey along a meandering path through a great mathematical continent.

"Surprising?" Yes, surprising. You will be surprised to be doing real mathematics. You will not be following rules or algorithms, nor will you be parroting what you have been dutifully shown in class or by the text. Unlike most mathematics textbooks, this book is not a transcribed lecture followed by exercises that mimic examples laid out for you to ape. Rather, the majority of each chapter is made up of Investigations. Each chapter has an introduction as well as brief surveys and narratives as accompaniment, but the Investigations form the heart of this book. They are landmarks for your expedition. In the form of a Socratic dialogue, the Investigations ask you to explore. They ask you to discover mathematics. This is not a sightseeing tour, you will be the active one here. You will see mathematics the only way it can be seen, with the eyes of the mind - your mind. You are the mathematician on this voyage.

"Exciting?" Yes, exciting. Mathematics is captivating, curious, and intellectually compelling if you are not forced to approach it in a mindless, stress-invoking and mechanical manner. In this journey you will find the mathematical world to be quite different from the static barren landscape most textbooks paint it to be. Mathematics is in the midst of a golden age - more mathematics is being discovered now than at any time in its long history. Each year there are 50,000 mathematical papers and books that are reviewed for *Mathematical Reviews*! Fundamental questions in mathematics - some hundreds of years old and others with $ 1 Million prizes - are

[1] All available freely online at `http://artofmathematics.org/books`.

being solved. In the time period between when these words were written and when you read them important new discoveries adjacent to the path laid out here have been made.

"Beautiful?" Yes, beautiful. Mathematics is beautiful. It is a shame, but most people finish high school after 10 - 12 years of mathematics *instruction* and have no idea that mathematics is beautiful. How can this happen? Well, they were busy learning arithmetical and quantitative skills, statistical reasoning, and applications of mathematics. These are important, to be sure. But there is more to mathematics than its usefulness and utility. There is its beauty. And the beauty of mathematics is perhaps its most powerful, driving force. As the famous **Henri Poincaré** (French mathematician; 1854 - 1912) said:

> The mathematician does not study pure mathematics because it is useful; [s]he studies it because [s]he delights in it and [s]he delights in it because it is beautiful.

Mathematics plays a dual role as a liberal art and as a science. As a powerful science, it shapes our technological society and serves as an indispensable tool and as a language in many fields. But it is not our purpose to explore these roles of mathematics here. This has been done in other fine, accessible books. Instead, our purpose is to journey down a path that values mathematics for its long tradition as a cornerstone of the liberal arts.

Mathematics was the organizing principle of the *Pythagorean society* (ca. 500 B.C.). It was a central concern of the great Greek philosophers like **Plato** (Greek philosopher; 427 - 347 B.C.). During the Dark Ages, classical knowledge was preserved in monasteries. The classical **liberal arts** organized knowledge in two components: the ***quadrivium*** (arithmetic, music, geometry, and astronomy) and the ***trivium*** (grammar, logic, and rhetoric) which were united by philosophy. Notice the central role of mathematics in both components. During the Renaissance and the Scientific Revolution the importance of mathematics as a science increased dramatically. Nonetheless, it also remained a central component of the liberal arts during these periods. Indeed, mathematics has never lost its place within the liberal arts except in contemporary classrooms and textbooks where the focus of attention has shifted solely to its utilitarian aspects. If you are a student of the liberal arts or if you want to study mathematics for its own sake, you should feel more at home on this expedition than in other mathematics classes.

"Surprise, excitement, and beauty? Liberal arts? In a mathematics textbook?" Yes. And more!

In your exploration here you will see that mathematics is a human endeavor with its own rich history of struggle and accomplishment. You will see many of the other arts in non-trivial roles: art, music, dance and literature. There is also philosophy and history. Students in the humanities and social sciences, you should feel at home here too. There are places in mathematics for anyone to explore, no matter their area of interest.

The great **Betrand Russell** (English mathematician and philosopher; 1872 - 1970) eloquently observed:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show.

We hope that your discoveries and explorations along this mathematical path will help you glimpse some of this beauty. And we hope they will help you appreciate Russell's claim:

> ...The true spirit of delight, the exultation, the sense of being more than [hu]man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

Finally, we hope that your discoveries and explorations enable you to make mathematics a part of your lifelong educational journey. For, in Russell's words once again:

> ...What is best in mathematics deserves not merely to be learned as a task but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement.

Bon voyage. May your journey be as fulfilling and enlightening as those that have beaconed people to explore the many continents of mathematics throughout humankind's history.

0.2 Navigating This Book

Intrepid Explorer, as you ready to begin your journey, it may be helpful for us to briefly describe basic customs used throughout this book.

As noted in the Preface, the central focus of this book is the **Investigations**. They are the sequences of problems that will help guide you on your active exploration of mathematics. In each chapter the Investigations are numbered sequentially in bold. Your role will be to work on these Investigation individually or cooperatively in groups, to consider them as part of homework assignments, to consider solutions to selected Investigations that are modeled by your fellow explorers - peers or your teacher - but always with you in an active role.

If you are stuck on an Investigation remember what **Frederick Douglass** (American slave, abolitionist, and writer; 1818 - 1895) told us:

> If there is no struggle, there is no progress.

Or what **Shelia Tobias** (American mathematics educator; 1935 -) tells us:

> There's a difference between not knowing and not knowing *yet*.

Keep thinking about the problem at hand, or let it ruminate a bit in your subconscious, think about it a different way, talk to peers, or ask your teacher for help. If you want you can temporarily put it aside and move on to the next section of the chapter. The sections are often somewhat independent.

Independent Investigations are so-called to point out that the task is more involved than the typical Investigations. They may require more significant mathematical epiphanies, additional research outside of class, or a significant writing component. They may also signify an opportunity for class discussion or group reporting once work has reached a certain stage of completion.

The **Connections** sections are meant to provide illustrations of the important connections between the mathematics you're exploring and other fields - especially in the liberal arts. Whether you complete a few of the Connections of your choice, all of the Connections in each section, or are asked to find your own Connections is up to your teacher. We hope that these Connections sections will help you see how rich mathematics' connections are to the liberal arts, the fine arts, culture, and the human experience.

Further Investigations, when included, are meant to continue the Investigations of the mathematical territory but with trails to significantly higher ground. Often the level of sophistication of these investigations will be higher. Additionally, our guidance will be more cursory - you are bushwhacking on less well-traveled trails.

In mathematics, proof plays an essential role. Proof is the arbiter for establishing truth and should be a central aspect of the sense-making at the heart of your exploration. Proof is reliant on logical deductions from agreed upon definitions and axioms. However, different contexts suggest different degrees of formality. In this book we use the following conventions regarding **definitions**:

- An *Undefined Term* is italicized the first time it is used. This signifies that the term is: a standard technical term which will not be defined and may be new to the reader; a term that will be defined a bit later; or an important non-technical term that may be new to the reader, suggesting a dictionary consultation may be helpful.

- An ***Informal Definition*** is italicized and bold- faced the first time it is used. This signifies that an implicit, non-technical, and/or intuitive definition should be clear from context. Often this means that a formal definition at this point would take the discussion too far afield or be overly pedantic.

- A **Formal Definition** is bolded the first time it is used. This is a formal definition that is suitably precise for logical, rigorous proofs to be developed from the definition.

In each chapter the first time a **Biographical Name** appears it is bolded and basic biographical information is included parenthetically to provide historical, cultural, and human connections.

In mapping out trails for your explorations of this fine mathematical continent we have tried to uphold the adage of **George Bernard Shaw** (Irish playwright and essayist; 1856 - 1950):

> I am not a teacher: only a fellow-traveler of whom you asked the way. I pointed ahead – ahead of myself as well as you.

We wish you wonderful explorations. May you make great discoveries, well beyond those we could imagine.

0.3 Directions for the Guides

Faithful Guide, you have already discovered great surprise, beauty and excitement in mathematics. This is why you are here. You are embarking on a wonderful journey with many explorers looking to you for bearings. You're being asked to lead, but in a way that seems new to many.

We believe telling is not teaching. Please don't tell them. Answer their questions with questions. They may protest, thinking that listening is learning. But we believe it is not.

This textbook is very different from typical mathematics textbooks in terms of structure (only questions, no explanations) and also of expectations it places on the students. They will likely protest, "We're supposed to figure this out? But you haven't explained anything yet!" It is important to communicate this shift in expectations to the students and explain some of the reasons. That's why we have written the earlier sections of this preface, which can help do the explaining for us (and for you).

You need support as well. A shift in pedagogy to a more inquiry-based approach may be subtle for some, but for many it is a great leap. Understanding this we have assembled an online resource to support teachers in the creation and nurturing of successful inquiry-based mathematics classrooms. Available online at `http://artofmathematics.org/classroom` it contains a wealth of information - in many different forms including text, data, videos, sample student work - on many critical topics:

- Why inquiry-based learning?
- How to get started using our books...
- A culture of curiosity
- Learning contracts
- Grouping students
- Choosing materials - Mixing It Up
- Asking good questions
- Creating inquiry-based activities
- Making mistakes
- Cool things
- Proof as sense-making
- Homework stories
- Exams
- Posters
- Assessment: Student Solution Sets
- Evaluating the effectiveness of inquiry-based learning
- ... and much more ...

We wrote the books that make up the Discovering the Art of Mathematics library because they have helped us have the most extraordinary experiences exploring mathematics with students who thought they hated mathematics and had been disenfranchised from the mathematical experience by their past experiences. We are encouraged that others have had similar experiences with these materials. We love to hear success stories and are also interested in hearing about things that might need to be changed or did not work so well. Please feel free to share your stories and suggestions with us: `http://artofmathematics.org/contact`.

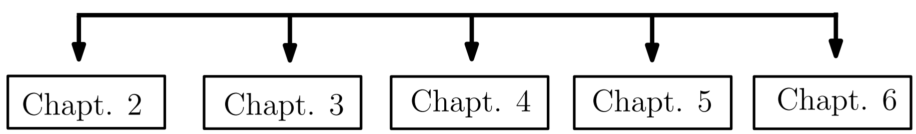
0.3.1 Chapter Dependencies

Guides are encouraged to pick and choose topics freely, from this book and others in the Discovering the Art of Mathematics series, depending on their interests and those of their students. The chapter dependencies in this book are as follows:

Discovering the Art of Mathematics: Dance

Chapter Dependencies

All chapters are independent.



Chapter 1

A little Introduction

1.1 Mathematics and Dancing

This is a not a regular textbook. This is a book which makes you move and think and write and discuss. I hope you read the “Notes to the explorer” preface to get a feeling for the spirit..

All chapters in this book are based on **movement**, as a motivation for the mathematical questions, as a representation of our strategies and to take a break from hard thinking and relax our brains. Don’t worry if you don’t like to dance, all the activities are really fun and it is not about “dancing perfectly”. You will be fine, trust me.

Chapter 2

Symmetry in Mathematics and Dance

The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful.

Aristotle (Greek Philosopher; 384 BC - 322 BC)

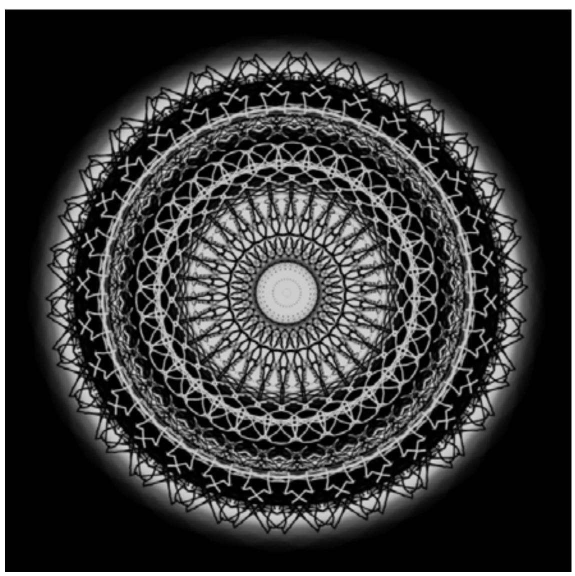


Figure 2.1: Picture Created with the App iornament

2.1 Moving in Symmetry in the Plane

Symmetry is beautiful. Most people find the balance of symmetry in nature, in architecture, in visual art, in clothing, etc. pleasing. The Dutch artist **M.C. Escher** (Dutch Graphic Artist; 1898 - 1972) uses symmetry in many of his beautiful pieces of art, see e.g. `https://en.wikipedia.org/wiki/M._C._Escher`. Although he was not a mathematician by training he was inspired by mathematics and worked with deep mathematical ideas in this artwork. He collaborated with mathematicians and later published his own mathematical ideas.

Figure 2.1 shows a picture with beautiful rotational symmetry. It was created with the app *iornament*, which allows the user to draw their own piece of art while playing with different symmetries. It is very easy to use (Figure 2.1 was drawn by my 6-year old daughter!) and can result in incredibly artistic pictures.

You can also look for symmetry in the realm of ideas, of patterns and reasoning – the more balance there is, the more pleasing a theory, a theorem or a proof is. Choreographers use symmetry (or the surprising lack thereof) as a stylistic feature in their dance creations. See Figure 2.2 for some beautiful examples of symmetry in dance compositions.

Figure 2.2: Examples of Symmetry in Dancing: Indian Dance, Estonian Dance, Morris Dancing and the Jabbawockeez

The goal of this chapter is to look at symmetry in dancing with the eye of a mathematician. But before we can do this we need some practice with symmetry.

2.2 Switching between Symmetries

2.2.1 The Mirror

Imagine you are standing in front of a mirror.

1. If you move your left arm, which arm is your mirror image going to move?

2. What happens if you move your left leg?

3. And how about turning to the right (clockwise, as viewed form above), away from the mirror, which way does your mirror image turn?

With a partner, explore this connection: One person is the active person while the other person is the mirror image who is permanently mirroring the moves. Tape the mirror line on the floor so you don't forget where the mirror is. Be creative as the active person, you can move in any way you want, except moving the mirror line itself.

We call this kind of symmetry ***reflectional symmetry*** or ***mirror symmetry***.

4. Which movements are easy for the mirroring person to follow? Give a few examples.

5. Which movements are hard for the mirroring person to follow? Give a few examples.

6. Why do you think some movements are harder to copy than others?

In the last questions you might have noticed that it is difficult to explain some of the positions in words. How about drawing a picture or a diagram?

7. How would you notate the position of the two dancers as viewed from the side? Give a few examples.

8. Which information are you missing in the side-view picture?

9. How would you notate the position of the two dancers from above? Give a few examples.

10. Which information are you missing in the top-view picture?

11. Can you draw a picture that shows all the information you need? Why or why not?

2.2.2 Same limbs...

Imagine the following situation: Both dancers face each other in the mirror and lift just their left arm.

12. Why is the above situation not a mirroring situation? Explain in detail.

We know that we can not use reflectional symmetry to describe the above position in which both dancers stand facing each other with just their left arms lifted. But clearly it looks and feels symmetric!

13. Think about the two dancers that face each other and both lift their left hand. Imagine you could pick up one person and move it around where ever you wanted. How would you move the person to match exactly with the other person? Act out the movement and describe or draw the process precisely. What would you call this movement?

We call this kind of symmetry ***rotational symmetry***.

14. Can you imagine why we call it rotational? What is being rotated?

15. And around which point do we rotate?

16. By how many degrees do we rotate?

17. Classroom Discussion: Compare the different representation we used to describe symmetry in dancing: actually moving, drawing movements, and describing movements in words. What are advantages and disadvantages of each representation? Do you have a preference?

A student invented the following notation, which will make it easier to show the difference between reflectional and rotational symmetry. She assumed that both people are facing each other and drew a circle for each foot and a rectangle for each hand. See Figure (2.3).

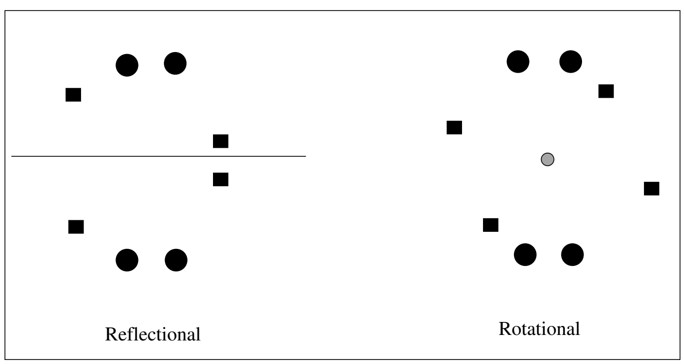


Figure 2.3: Reflectional and Rotational Symmetry Example

Practice with a partner again, this time one person (the follower) following the other (the leader) in rotational symmetry. Use tape on the floor to mark the point of rotation. Be creative!

18. Which movements are easy for the follower to follow? Give a few examples.

19. Which movements are difficult for the follower to follow? Give a few examples.

20. Why do you think some movements are harder to copy than others?

2.2.3 Switching between two kinds of Symmetry

Now that you know about two kinds of symmetry, we can practice using both. Start with reflectional symmetry, agreeing on a place for the mirror. After creating interesting movements for some time, the leader says "switch"[1]. Now the follower has to follow in rotational symmetry. But there is a problem: not in all positions can you switch smoothly between symmetries, meaning you don't have to quickly adjust your position.

21. Find a position in which you can not switch smoothly from reflectional to rotational symmetry. Explain why.

22. Find several positions in which you can switch from reflectional to rotational symmetry. Draw the corresponding pictures.

23. Describe **all** positions in which you can switch from reflectional to rotational symmetry. This is your *conjecture*.

If we want to be precise and prove a conjecture in mathematics it is helpful to have precise language for the definitions and terms we are using.

24. What do you think: where do definitions in mathematics come from? Who creates them and who decides which ones to use?

25. Is it ok for you to just invent something and call it a definition? Why or why not?

26. **Classroom Discussion:** In groups and as a whole class find precise definitions for reflectional and rotational symmetry. Compare your definitions and agree as a class on which one work best for our purpose.

Now you are ready for your first proof[2]:

27. Describe **all** positions in which you can switch from reflectional to rotational symmetry. Justify that you can actually use the positions you found to switch between symmetries. Explain how you can be sure that you found **all** of the positions.

2.2.4 Line Dancing

Have you ever seen or done *line dancing*? There is certainly symmetry involved but it doesn't seem to be reflectional or rotational symmetry. Watch a video on youtube when considering the following questions, e.g. `http://www.youtube.com/watch?v=rs5f8CYyLBo`[3].

28. Explain in detail why the relation between the line dancers in the video shows neither reflectional nor rotational symmetry.

29. Imagine again that you could pick up one of the line dancer and move them wherever. How would you move the dancer in order to match him or her up precisely with one of their neighboring dancers? Draw a picture and label clearly how you would move them, how far, etc.

[1] This exercise is taken from www.mathdance.org, [13].

[2] If you want to know more about proofs look at the guide Discovering the Art of Mathematics: Student Toolbox

[3] Try out the line dance by yourself or in your class!

30. Can you imagine line dances that have reflectional or rotational symmetry? Explain in detail.

The main symmetry you see in a line dance is called ***translational symmetry***. You can imagine sliding or "copying and pasting" a dancer to a different position in the room. The orientation of the dancer does not change, however.

Again, with a partner, practice following moves in translational symmetry. When you are comfortable with this, start switching between all three kinds of symmetry.

2.2.5 Switching between three kinds of Symmetry

Now that you know about three kinds of symmetry, we can dance using all of them. Start with reflectional symmetry, agreeing on a place for the mirror. After creating interesting movements for some time, the leader says "switch to . . . "[4]. The follower has then to follow in the symmetry called by the leader. But there is a problem: not in all positions can you switch smoothly between symmetries.

31. Is translational symmetry easier or harder to follow than the others? Explain why.

32. Find a position in which you can *not* switch from reflectional to translations symmetry. Explain why.

33. Is there a position in which you *can* switch from reflectional to translational symmetry. Explain.

34. Find a position in which you can *not* switch from rotational to translational symmetry. Explain why.

35. Is there a position in which you *can* switch from rotational to translational symmetry. Explain.

36. **Classroom Discussion:** What has to be true about positions where we can switch from one type of symmetry to another? How can we use this to find or describe *all* the different positions where such a switch may occur?

37. Independent Investigation: Find at dance clip that you like on `youtube.com` that exhibits different kinds of symmetries. Explain which symmetries are included and when they occur. Be prepared to share the clip with your class.

2.2.6 Glide Reflections

We are missing one very interesting kind of symmetry: the *glide reflection.* Imagine you are standing in front of a mirror but the mirror image is standing off to the side instead of in front of you: the mirror image is translated parallel to the mirror. See Figure 2.4 for an example.

[4]This exercise is taken from www.mathdance.org, [13]

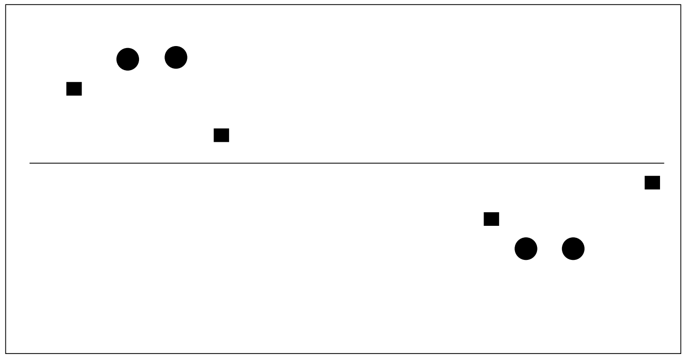


Figure 2.4: Dance Example of a Glide Reflection

38. Independent Investigation: With a partner, decide who is leading and who is following, and then move in glide reflections. How difficult is this compared to moving in the other kinds of symmetry?
Can you switch from glide reflections into any of the other symmetries or not? If yes, give examples of positions that allow you to switch. Describe all positions that allow such a switch and explain how you know that you found **all** such positions. If a switch is not possible explain why you can be sure that it is impossible.

2.3 Symmetry Choreograhy

So far we have been using the dance structure to ask interesting mathematical questions. But the aspect of choreography itself has similarities to mathematics. To explore those we will do a little dance performance.

We will use the 4 symmetries we discussed above (reflectional, rotational, translational, glide reflectional), but allow rotations of any degree. Get into groups of 4 dancers. Choose 3 of the above 4 symmetries. Now invent three different interesting dance poses that you all like. Be creative!

For each pose choose one of the symmetries. One person will get into the pose and the other group members will show a symmetric version of the pose. You could for instance stand in a circle and each show a 90 degree rotation of the original pose. Or you could all be in translational symmetry. You can also mix two symmetries and have two dancers in reflectional symmetry and the other two showing a rotational version of the first two dancers.

When you have composed the three poses in symmetry find interesting transitions to move between the poses. Make it easthetically pleasing to you. End your dance in an asymmetrical pose (why?). You can arrange your dance to music if you like. Now perform the dance sequences for each other.

39. What did you notice about the dance sequences? What did you enjoy? Why?

40. Describe the process of creating a dance, what did you do?

41. How is choreographing a dance similar to doing/discovering mathematics?

2.4 Further Investigation

2.4.1 Dance in Symmetry in a Line

Assume for the moment that your dancers all stand on one line.

F1. With a partner dance in translational symmetry (one leading, one following) while you are both standing on the same line. Does your definition of translational symmetry change if restricted to a line? In which direction can you translate? Explain.

F2. With a partner dance in reflectional symmetry (one leading, one following) while you are both standing on the same line. Does your definition of reflectional symmetry change if restricted to a line? Where can your mirrow be? Explain.

F3. With a partner dance in rotational symmetry (one leading, one following) while you are both standing on the same line. How would the definition of rotational symmetry change if restricted to a line? Where can the points of rotation be? How many degrees can you rotate? Explain.

F4. With a partner dance in glide reflectional symmetry (one leading, one following) while you are both standing on the same line. How would the definition of glide reflectional symmetry change if restricted to a line? Where can the mirror be and in which direction can you translate? Explain.

F5. Look at the position the dancers hold in Figure 2.4. They are in planar glide reflectional symmetry. Now move the dancers (either in your head, or on paper, or try it out!) until the dancers stand in a glide reflectional symmetry on the line as in Figure 2.5. Be careful: the dancers have to stay in planar glide reflectional symmetry while you move them.

Figure 2.5 shows an example of each of the four symmetries on the line. But what happens if we combine two symmetries? Do we get one of our four symmetries again or do we get a new, maybe asymmetric movement?

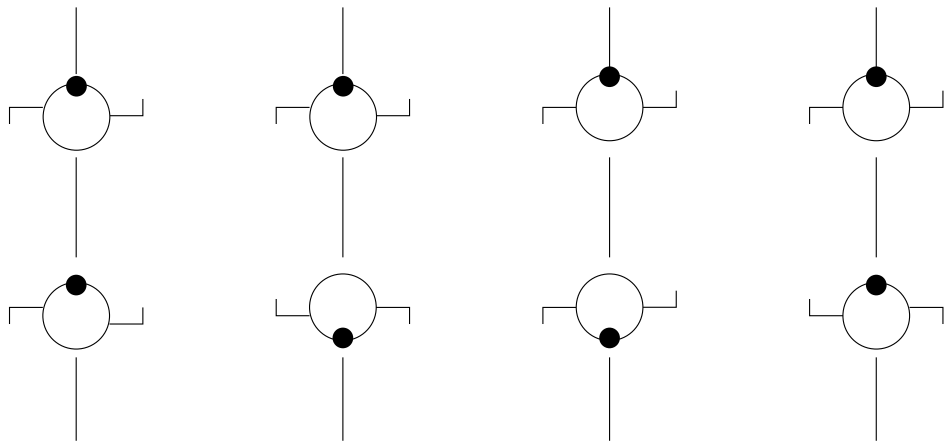


Figure 2.5: The 4 Symmetries on a Line

> **42.** Independent Investigation: Take the four symmetries in a line and combine two of them at a time. See if you can describe the result as one of our line symmetries. Use Figure 2.6 to record your answers. If you believe that you found the correct answers, prove them: how can you be sure that this will *always* be the result?

The pattern that you found is very special to mathematicians, they call any set of objects with this kind of combination table a ***Klein 4 group*** after **Fleix Klein** (German Mathematician; 1849 - 1925). The Klein 4 group can show up in many different contexts, its existance can for instance prove that a formula exists to find the x-values at which a polynomial of degree 4, e.g. $y = 5x^4 + 65x^3 - 16x^2 + 89x - 911$, is equal to zero.[5] Remember your quadratic equation from high school? This is the same idea but for much more complicated functions.

2.4.2 More Symmetries of the Plane

Let's see if we get a similar structur for our four symmetries of the plane (not the line). Recall that we allowed translations in any direction in the plane, reflections at any mirror in the plane, 180 degree rotations around any point, and glide reflections across any mirror with a translation parallel to the mirror.

> **43.** Independent Investigation: Take our four symmetries in the plane and combine two of them at a time. See if you can describe the result as one of our symmetries. Use

[5]The proof uses Galois groups and resolvents, see `http://en.wikipedia.org/wiki/Lagrange_resolvents`, `http://en.wikipedia.org/wiki/Quartic_function` or [4]

1st \ 2nd	T	M	R	G
T				
M				
R				
G				

Figure 2.6: Combinations of the 4 Symmetries on the Line

Figure 2.7 to record your answers. If you believe that you found the correct answers, prove them: how can you be sure that this will *always* be the result?

You probably noticed that this question is harder to answer. If I have, for instance, 2 mirrors, how do I know where in the plane they are? Look at the following investigations to see how complex the combinations of symmetries in the plane can be:

F6. Suppose you have two mirrors that are positioned at a radom angle. What happens when I combine the symmetries across these mirrors? Is the combination one of our plane symmetries? Why or why not?

Because of your result in Investigation **6** mathematicians do not call our set of symmetries in the plane a *group*. They say that "the set of symmetries is not closed under the combination". You will discover later what is needed for a mathematical *group*.

F7. Explain what the statement "the set of symmetries is not closed under the combination" means - in the context of our symmetries in the plane.

This might be dissapointing, we always hope to find more structure in the objects we are studying! But there is a neat trick to get some of the structure back.

First of all we noticed in Investigation **6** that we need to allow rotations of degrees other than 180 degrees. So from now on we will look at rotations around any point *using any degree*. Here is the trick: a translation can be considered a rotation around infinity. What? Infinity? Look at the following questions to make sense of this:

2nd / 1st	T	M	R	G
T				
M				
R				
G				

Figure 2.7: Combinations of the 4 Symmetries in the Plane

F8. Imagine a dancer is standing at point A in Figure 2.8, left scenario. If you choose your center of rotation at point P and rotate dancer A clockwise around P to a new point B, by how many degrees has the orientation of the dancer changed? Explain.

F9. Imagine a dancer is standing at point A in Figure 2.8, right scenario. If you choose your center of rotation at point P and rotate dancer A clockwise around P to a new point B, by how many degrees has the orientation of the dancer changed? Explain.

F10. Try a new scenario in which you move P even further to the left (A and B stay fixed). What happens to the orientation change? Explain.

F11. Imagine moving the point P further and further away, all the way to infinity. What do you think will happen to the orientation change? Explain.

F12. Using your reasoning from the above investigations, why can a rotation around infinity be considered a translation? Explain.

So this means we can combine our T and our R into one row or column in our table. But we can simplify even more: A glide reflection with no translation (or a translation of length 0) is really just a reflection, right? So in that sense we can combine the M and the rows and columns in our table, see Figure 2.9.

F13. Fill in table in Figure 2.9. How can you be sure that your answers will always be correct?

F14. Compare our new table with the tabe of the Klein 4 group. What is similar? What is different?

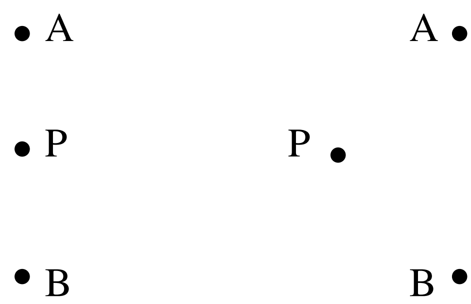


Figure 2.8: Rotations

2nd / 1st	T/R	M/G
T/R		
M/G		

Figure 2.9: Combinations of the 4 (adjusted) Symmetries in the Plane

2.4.3 Planar Symmetries and Groups

We want to look at one more way to define which symmetries we allow one the plane. We will look at the following 8 symmetries:

- translations,
- 90, 180 and 270 degree rotations around any point,
- glide reflections along vertical, horizontal, 45 and -45 degree mirrors. (Recall that the glide reflections of glide zero are just reflections.)

44. Independent Investigation: Draw and fill in the 8x8 table for the symmetries described above. Explain why you can be sure that you answers are correct for any possible combination.

F15. Is this set of 8 symmetries *closed*, i.e. is every combination of the symmetries again one of the 8 symmetries? Explain why or why not.

F16. Find one of the 8 symmetries that when combined with any other symmetry S will give just S? We call this symmetry the **identity element**.

F17. Given any symmetry S find a symmetry T that gives the identity element when combined. We call T the **inverse element** of S.

F18. Does the table have a symmetry line across the main diagonal (from top left to bottom right)? Look carefully.

Further Investigations **F15-F17** are asking about all the properties needed for a mathematical **group**. Further Investigation **F18** decides whether a group is **commutative** or not. This formal definition is extremely powerful, it provides us with a structure for mathematics that is used in many areas of higher mathematics, starting with ***abstract algebra***.

F19. Look back the Klein 4 group and our group of two elements. Are they commutative or not?

This process of looking at symmetries and how they influence the resulting structures is exactly what mathematicians did in the 18th century. Mathematicians all over the world were exploring geometric spaces that are different from our Euclidean geometry. The idea of different geometric spaces was (and is) mind blowing. If you are interested in learning more we suggest the investigations at `http://mathcs.slu.edu/escher/index.php/Math_and_the_Art_of_M._C._Escher`. And Felix Klein, the explorer of the Klein 4 group, was instrumental in describing geometry in a unified way encompassing Euclidean and non-Euclidean geometry. He published his results in the famous "Erlanger Program" in 1872.

The next section will introduce you to symmetry groups of our usual *Euclidean space*, and hint at the non-Euclidean symmetry groups.

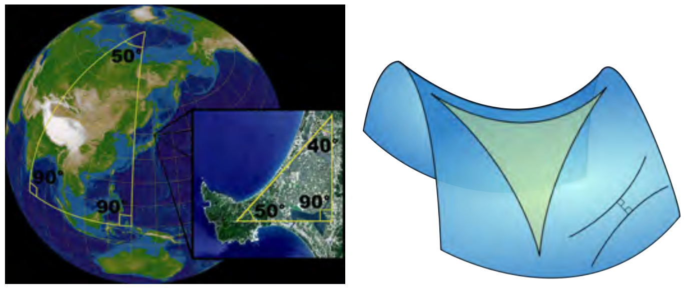


Figure 2.10: Triangles in Spherical and Hyperbolic Geometry

2.5 Frieze Patterns

> It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly and to comprehend at once both the ensemble and the details.
>
> **Henri Poincare** (French Mathematician; 1854 - 1912)

2.5.1 Frieze Patterns and Feet

Let's make a connection to symmetries in other areas of art, for instance in architecture. A ***frieze pattern*** is a pattern that has symmetry "in one direction". Friezes refer to the patterns right under the rim of a roof, a window etc. See Figure 2.11 for an example.

Figure 2.11: St. Louis Cathedral Basilica, Detail of Pulpit.

It is amazing to see that frieze patterns occur in so many different cultures all over the world.

45. Go online and find a few frieze patterns that you like from different cultures. Be prepared to share your patterns with the class. Can you say something about the story of the building, artwork, etc that the frieze pattern is connected to?

46. INDEPENDENT INVESTIGATION: Take large strips of paper and walk in a repeating pattern across the paper. The simplest pattern would be to just walk straight across the paper, so please be more creative. Draw your feet on the paper so you can see the pattern afterwards. Now analyze the symmetries of your pattern. Can you find translations, reflections, rotations and glide reflections?

The famous mathematician **John Horton Conway** (British Mathematician; 1937 -) used the idea of moving feet to describe seven different frieze patterns, see Figure 2.12, [5].

47. Find all symmetries (translational, rotational, reflectional, glide reflections) in the first pattern in Figure 2.12.

48. Find all symmetries (translational, rotational, reflectional, glide reflections) in the second pattern in Figure 2.12.

49. Find all symmetries (translational, rotational, reflectional, glide reflections) in the third pattern in Figure 2.12.

50. Find all symmetries (translational, rotational, reflectional, glide reflections) in the fourth pattern in Figure 2.12.

51. Find all symmetries (translational, rotational, reflectional, glide reflections) in the fifth pattern in Figure 2.12.

52. Find all symmetries (translational, rotational, reflectional, glide reflections) in the sixth pattern in Figure 2.12.

53. Find all symmetries (translational, rotational, reflectional, glide reflections) in the seventh pattern in Figure 2.12.

If you look at different cultures you can find **all** seven frieze patterns (see Figure 2.12) represented in each one of them, so the people must have "known" that there are not more than these seven, right? Would they not have used an 8th pattern if there was one? We would like to see if this is really true. So let's start with checking the claim using our own patterns. After all, if we find one example that is not in the list we should tell other people about it!

54. Analyze your own feet patterns and see if each feet pattern exhibits the exact symmetries of one of Conway's feet pattern in Figure 2.12. Be prepared to share your thinking with the class.

2.5.2 Are there exactly 7 Frieze Patterns?

So, are there just these seven frieze patterns (where patterns are considered the same if they have the same kind of symmetries? This question is investigated in a paper entitled "Classifying frieze Patterns Without Using Groups," by sarah-marie belcastro and Thomas C. Hull [10], available at `http://mars.wne.edu/~thull/papers/friezepaper.pdf`.

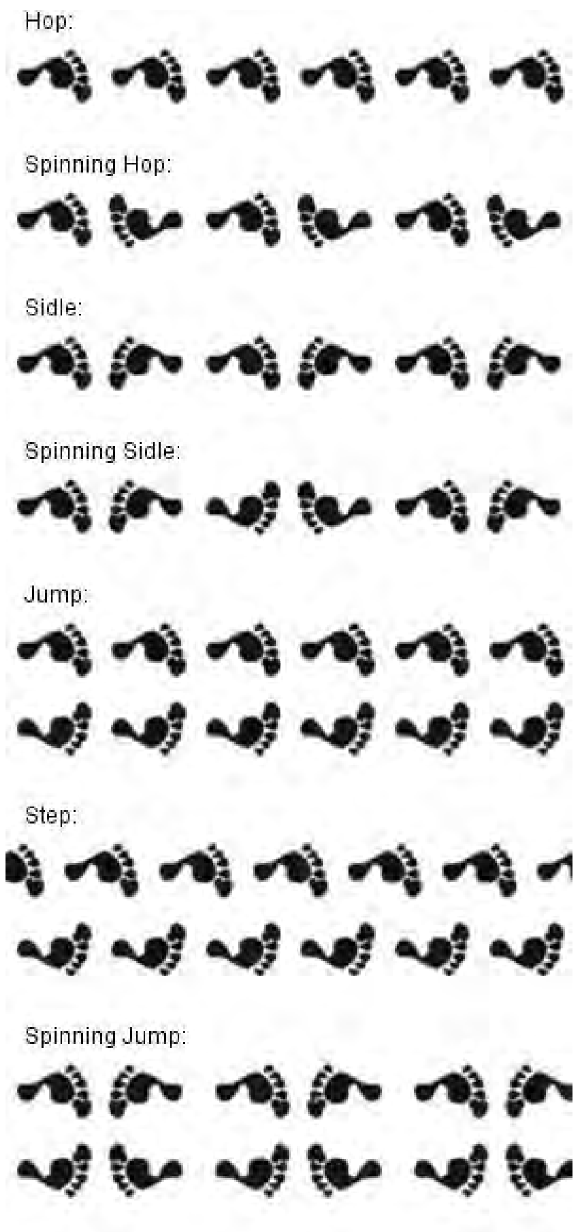


Figure 2.12: Frieze Patterns for Feet

55. Independent Investigation: Read the paper "Classifying frieze Patterns Without Using Groups" section by section, discussing the authors' approach, their pictures, ideas, and arguments. You will explore in the next investigations how these relate to and compare with your classroom experiences with symmetries. Take careful notes and use your group to make sure you fully understand the details.

56. The authors list five different *basic* symmetries they call t, r, h, v, and g. Among the frieze patterns in Figure 2.12, can you find examples for each of these five symmetries? Explain.

57. How do you think the authors came up with the list of sixteen combinations listed as part of Step 2? Explain.

58. Explain why the sixteen combinations listed as part of Step 2 are *all* the possible symmetries we need to consider? Why are there no others to consider?

59. How many of these are of length 1, length 2, length 3, length 4? (By "length" we mean the number of "basic" symmetries such as h or g. For example hvr has length 3, hv has length 2.)

60. Perhaps you have encountered expressions such as "4 choose 2" in the past? Imagine that I have four different kinds of candy. You get to choose two of them. How many different choices are possible?

61. Mathematicians usually write $\binom{4}{2}$ as a shorthand for "4 choose 2" and call them *binomials* or *binomial coefficients*. How many are $\binom{4}{2}$? How many are $\binom{4}{3}$? How many are $\binom{4}{1}$? How many are $\binom{4}{4}$? How many are $\binom{4}{0}$?

62. How are these numbers related to the list of sixteen combinations listed as part of Step 2? Explain. (Hint: Consider your answers to exploration Investigation **59**.)

63. The formula at the top of page 95 of the paper includes expressions such as $\binom{4}{3}$. What relationship do you think this equation expresses?

Next, we will take a look at the pictures the authors draw in their paper. Why do you think they choose the letter **P**?

64. The authors claim that the resulting pattern in Figure 3 does not have h, v, or g symmetry. Is that true or not? Explain.

65. The authors claim that the pattern in Figure 3 has *two* types of rotational symmetry. Is that true or not? Draw all the centers of rotation you see into the Figure and indicate the angle by which you would rotate.

66. Why is h "never lonely"? Explain in detail.

67. Why are v and g "happy being single?" Explain in detail.

68. Why does hr say "we are not alone?" Explain in detail.

69. What is the familiar pattern that vr produces? Explain in detail.

70. Which of the frieze feet patterns in Figure 2.12 corresponds to Figures 8–10 in the paper? Explain.

71. Explain the information in their "Final List," Table 1 on page 98 of the paper.

72. Independent Investigation: **In your own words, explain why there are only seven frieze patterns.**

Imagine that your audience is a fellow student who missed the last two classes, has therefore not read the paper, but who shared our explorations with dance and symmetry before that. The explanations and arguments in your writing should be clear to such a person.

Take your time to think carefully about the question; go back to the article we read for details; gather ideas about what you want to include in your essay; explore how to structure your writing; and how to express in a clear way your thinking. Use evidence from the article to support your arguments. You may use your previous explorations and your group as resources to help you clarify what is unclear, or to discuss what is controversial.

This was a long exploration of purely mathematical ideas. And **you** were the mathematician! Now we want to connect the mathematical idea back to the dance.

73. Independent Investigation: Create your own dance using the ideas of frieze patterns. Perform your dance in front of the class. Can your classmates find the patterns you used?

2.6 Further Investigations

2.6.1 Wallpaper Patterns

In the last section we understood all different symmetries of an infinite line, a frieze pattern. It is natural to wonder how this extends to the plane, to two dimensions. In how many ways can I have symmetric ***wallpaper patterns*** as in Figure 2.13? Are there just 7 again? Or more?

These patterns that fill the infinite plane are also called **tessellations**. With similar strategies as before we could now prove that there are exactly 17 wallpaper patterns that represent the symmetries.

This result was proven in the late 19th century simultaneously by **Evgraf Fedorov** (Russian Mathematician; 1853 - 1919), **Arthur Schoenflies** (German Mathematician; 1853 - 1928), and **William Barlow** (British Chrystallographer; 1845 - 1934). Check out the amazing website `http://clowder.net/hop/17walppr/17walppr.html#p2` to see the symmetries of the wallpaper patterns in action.

Figure 2.13: Examples of Egyptian Designs

Can we use the wallpaper symmetries to create beautiful movements? Karl Schaffer has played with arrangements of dancers that follow a lead-dancer according to a specific symmetry. The results are beautiful and complex, see Figures 2.14-2.15.

74. Independent Investigation: Research the 17 wallpaper symmetries and choose one. (You can choose one of the two example provoded here, see Figures 2.14-2.15, but you don't have to.) In your group or class invent a dance for one person that you then mimic using the "instructions" given by the symmetry. Do you like the result?

2.6.2 Non-Euclidean Symmetries

We saw in the last section how you can tessellate the plane. Now we want to tessellate the sphere and the Poincaré disk - a model of hyperbolic geomtery. See Figures 2.16-2.17, aren't they beautiful? There are about 31 different spherical symmetry patterns and 20 different hyperbolic symmetry patterns, depending on how you count them.

Again, we highly suggest the investigations at `http://mathcs.slu.edu/escher/index.php/Math_and_the_Art_of_M._C._Escher` to get involved with the mysteries of non-Euclidean geometries..

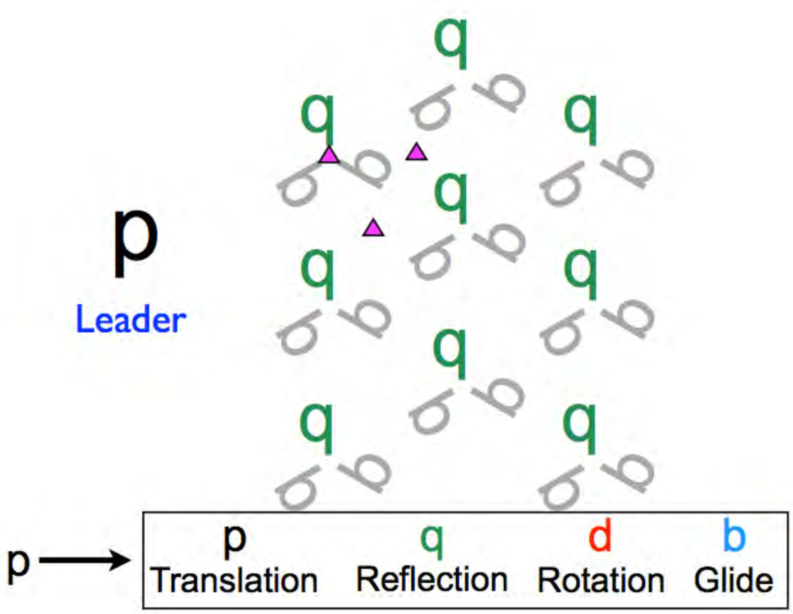


Figure 2.14: Dancing Symmetries of Wallpaper Patterns: Example 1

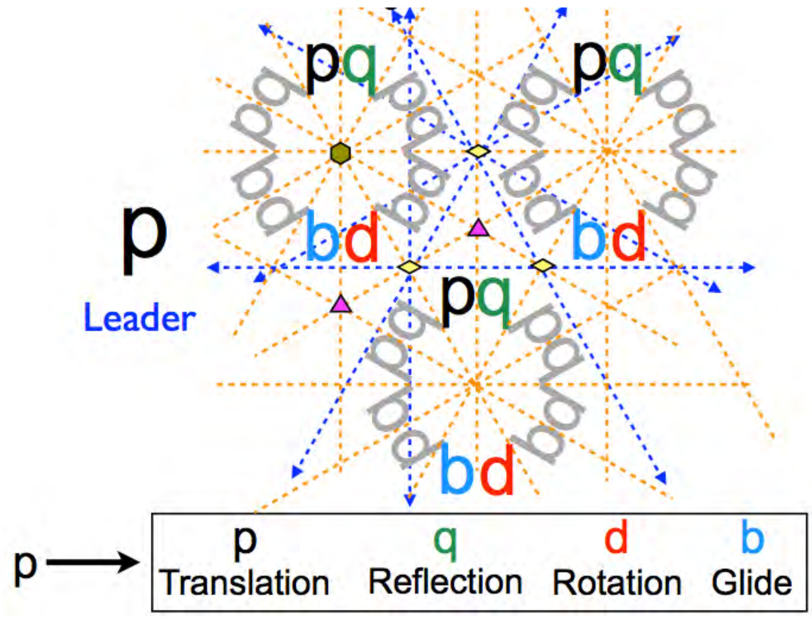


Figure 2.15: Dancing Symmetries of Wallpaper Patterns: Example 2

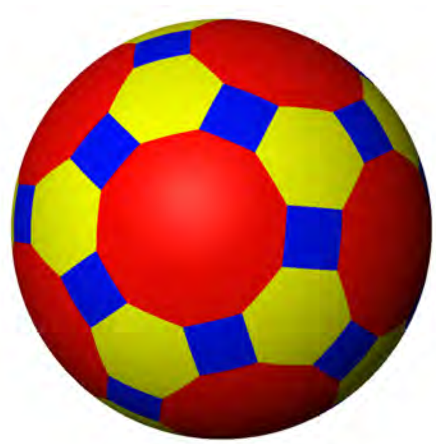


Figure 2.16: Spherical Tessellation

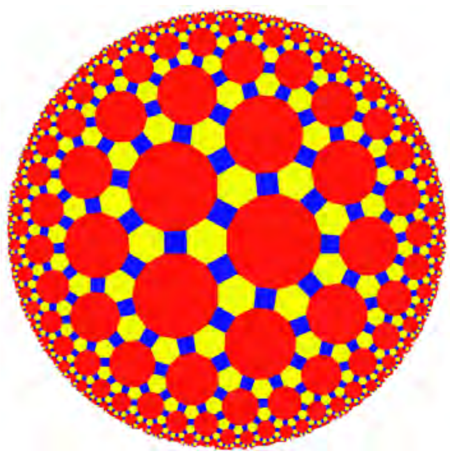


Figure 2.17: Hyperbolic Tessellation

Chapter 3

Salsa Rueda

Everything in the universe has rhythm. Everything dances.
Maya Angelou (American Author and Poet; 1928 -)

3.1 Learning the Basic Dance

Figure 3.1: Students practicing Salsa Rueda at Westfield State University

The underlying rhythm of a typical Salsa song is the 2-3 clave, also called Son [14]. You can think of it as 10010010 00101000 where 1's denote drum hits and 0's denote the ongoing beat. It is difficult to hear this underlying rhythm in the music, but for the dancing it in not necessary to do so. The steps are on different counts than the Salsa rhythm emphazises. For this chapter we will not focus on the 2-3 clave but practice the actual dance steps. Listen to some Salsa music, find the beat and count in eights. First step on every beat. Then step just on 1,2,3 and 5,6,7. It doesn't matter yet on which foot you start, but keep switching feet for every step. Practice this a lot in order to become more flexible. You want your feet to remember this rhythm by themselves, because later we need to add arm movements and directions to it. Try talking while you step, making turns, going backward, forwards and sideways.

Salsa Rueda was developed in Havana, Cuba in the 1950s. In Salsa Rueda couples are standing in a circle with the leader on the right side of the follower. Traditionally in partner dancing the men are leading and the women are following. One of the leaders is the "caller" telling the other leaders during every move which move is coming next.

Watch `http://www.youtube.com/watch?v=uisqUEMcH5U` and other video clips on `youtube.com` to get a feeling for salsa rueda.

You can use the rueda wiki `http://www.ruedawiki.org/rueda/index.php?title=Main_Page` for an introduction to the basic step *Guapea*. With your class, practice the guapea until you feel comfortable. The video shows the guapea step with the partners facing each other. When standing in the circle it is nice to face the other couples during the first 3 steps and only then turn towards your partner.

When your class tried to form a circle of couples for the first time, there might have been some shuffling around until everybody actually had a partner and all the couples were organized in a nice circle. This leads to our first set of mathematical questions: Imagine your whole class gets up to dance for the first time. Now everybody is standing and trying to find a partner to dance with. In how many different ways can we find dance partners for everybody?

1. 4 people are getting up to find a dance partner. How many ways are there to make couples out of these 4 people? (At this point we don't care who is leading or following, we just create pairs). We want all dancers to have a partner *at the same time*, since we cannot dance the salsa rueda otherwise. What makes you convinced that your answer makes sense?

2. How many ways are there to give everybody a partner if you have 6 people? (Hint: Acting this out with people, can you confirm your answer?) What makes you convinced that your answer makes sense?

3. How many ways are there to give everybody a partner if you have 8 people?

4. How many ways are there to give everybody a partner if you have 10 people?

5. Do you see a pattern? Describe.

6. Estimate how many ways there are to give everybody a partner if you have 20 people?

7. Now find out precisely how many ways are there to give everybody a partner if you have 20 people?

8. How does this result compare to your estimate? Does your answer surprise you? Explain.

9. Now you are ready to answer this question in general. How many ways are there to give everybody a partner if you have n people?

Now we want to also keep track of who is leading and who is following. We don't want to worry about gender for the moment, so everybody can lead or follow according to their liking or ability.

10. How many ways are there to give everybody a partner if you have 4 people and you want to distinguish between leaders and followers? As an example: if Jane is dancing with Sarah and Jane is leading then that is different from Jane dancing with Sarah and Sarah is leading. Again, we need all people to have a partner *at the same time* since we want to dance in a circle later.

11. How many ways are there to give everybody a partner (distinguishing leader and follower) if you have 6 people?

12. How many ways are there to give everybody a partner (distinguishing leader and follower) if you have 8 people?

13. How many ways are there to give everybody a partner (distinguishing leader and follower) if you have 10 people?

14. Do you see a pattern? Describe.

15. Estimate how many ways there are to give everybody a partner if you have 20 people?

16. Now find out precisely how many ways there are to give everybody a partner (distinguishing leader and follower) if you have 20 people?

17. Does your answer surprise you? Did you expect the answer to be larger or smaller than the one in the last exploration? Explain.

18. Now you are ready to answer this question in general. How many ways are there to give everybody a partner (distinguishing leader and follower) if you have n people?

To relax your math brain, dance some more Salsa Rueda and practice the basic step some more.

Since we dance Salsa Rueda we need to also arrange our couples in a circle. This next exploration comes with less help. See if you can find the necessary steps yourself. How can you check that you are correct?

19. If we have 20 dancers (10 couples), in how many different ways can we arrange them around the circle?

20. Does this answer surprise you? Explain.

3.2 Salsa Rueda – *Da Me* and *Da Me Dos*

Now we are ready to learn and practice *di le que no* and *da me* (using ruedawiki.org). It helps to first practice just the general motions of the dancers without bothering with the exact steps. To do this stand in a circle with the leaders on the right side of their followers. Now all the followers walk to the next "open space" to their left; now every follower has a new leader. In the next version let the leaders lead their next partner around them to their new position. Once these motions are comfortable, add in the music and the steps. The leader first get into standard dance position with their new partners (leader has the right hand on the left shoulder blade of the follower, leader's left hand holds follower's right hand), and then dance *di le que no* and *da me*.

21. While you are practicing Salsa Rueda, did you notice any kind of symmetry? If yes, which one? Is this symmetry helpful to you when you are learning the dance or not?

Da me dos is more difficult to dance because the leaders now want to dance with the *second* follower to their right. Get into the circle and let leaders make eye contact with the *second* follower to their right. On the first count, the leaders walk as fast as possible to their next partner (passing one follower) and dance a *di le que no* with their new partner. Watch the video clip on ruedawiki.org to watch the move.

The caller of the dance has to be creative in combining the different moves into a smooth and pleasant dance experience. But how many choices are there?

22. How many beats does it take to dance a *da me*?

23. How many beats does it take to dance a *da me dos*?

24. If you are only able to use *da me* and *da me dos* (not even the basic step!), how many different combinations of dance moves can you create in 64 beats? Explain.

We want to draw pictures of a whole salsa rueda dance position or dance move. One of the easiest dances would be to only dance *da me*'s. Given 4 couples, what would that look like? Leaders will be denoted by a black dot, followers by a grey dot. In *da me* only the follower change positions, hence the path looks like Figure 3.2.

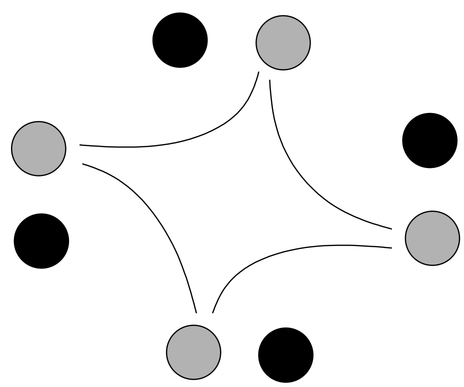


Figure 3.2: The path of a rueda dance using only *da me*.

If we dance only *da me dos*'s the path would look different. Now the leaders and the followers are moving! To make the path easier we will assume that only the follower moves towards the correct new spot. See Figure 3.3.

25. If there were 6 couples and we would dance only *da me*, would every leader dance with every follower? Why or why not?

26. If there were 6 couples and we would dance only *da me dos*, would every leader dance with every follower? Why or why not?

27. Using only *da me dos* and 12 couples, predict if every leader will dance with every follower. Explain.

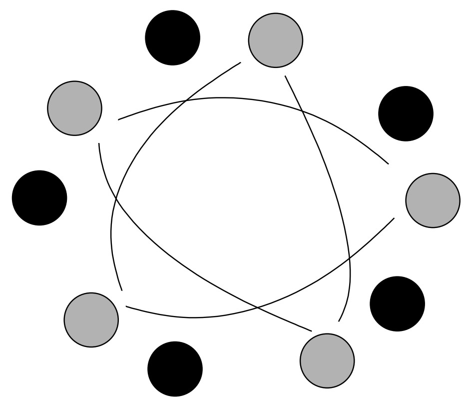


Figure 3.3: The path of a rueda dance using only *da me dos*.

28. Using only *da me dos* and 11 couples, predict if every leader will dance with every follower. Explain.

29. For any number of couples only dancing *da me*, predict if every leader will dance with every follower. Explain.

30. For any number of couples only dancing *da me dos*, predict if every leader will dance with every follower. Explain.

Now try dancing a *da me tres*, in which the leader goes to the **third** follower on his right (not the second as in *da me dos*). If you dare, try a *da me quatro*!

31. If there were 6 couples and we would dance only *da me tres* would every leader dance with every follower? Why or why not?

32. Using only *da me tres* and 12 couples, predict if every leader will dance with every follower. Explain.

33. Using only *da me tres* and 11 couples, predict if every leader will dance with every follower. Explain.

34. For any number of couples only dancing *da me tres*, predict of every leader will dance with every follower. Explain.

35. Independent Investigation: Imagine you could have any number of couples dancing in a circle and you could do any number of *da me*'s. Predict if every leader will dance with every follower or not.

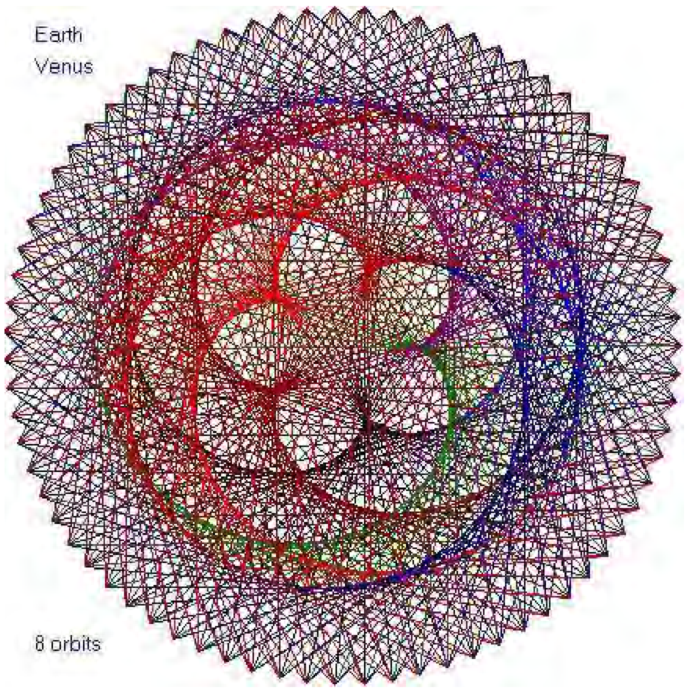


Figure 3.4: The dance of Earth and Venus.

Figure 3.4 should remind you of the patterns you drew to answer the above independent investigation. This picture was generated by drawing a line between the position of earth and the position of venus every few days. Isn't it an amazing dance pattern?

To explore more dance patterns of planets, see `http://ensign.editme.com/t43dances`.

3.3 Further Investigations and Connections

F1. How do we know that after dancing only dame dos every dancer will eventually dance again with their original partner? Find a proof.

F2. How do we know that after dancing only dame tres every dancer will eventually dance again with their original partner? Find a proof.

F3. How do we know that after dancing only dame k every dancer will eventually dance again with their original partner? Find a proof.

F4. Do you know Spirographs? Compare spirographs with the last independent investigation. You can find details in the book Discovering the Art of Mathematics: Patterns.

F5. Compare the last independent investigation with Star Polygons from the book Discovering the Art of Mathematics: Patterns.

Chapter 4

The Space of Partner Salsa Dancing

> I see dance being used as communication between body and soul, to express what is too deep to find words for.
>
> **Ruth St. Denis** (American Moder Dancer; 1879 - 1968)

Look at the video clips on `http://dmitri.tymoczko.com/ChordGeometries.html`. They show how Dmitri Tymoczko visualizes the space of all musical chords, using some cool geometric shapes like the Moebius strip and the torus [15]. Details can be found in the chapters on musical 2 and 3 chords in Discovering the Art of Mathematics: Music. When I saw and understood the space of musical chords for the first time I was intrigued: would it be possible to create a similar space for Salsa dancing? If yes, what would it look like, how big is it, and does it help me in leading/composing a dance?

These are very big questions to answer and at the time we started looking there was absolutely no research done in this area. Some people had invented notations for Salsa dance moves but there was no concept of measuring "how much there is". If a question in mathematics seems too big to understand, we begin small. What is a question we *can* answer? Well, how about counting positions with both hands held first?

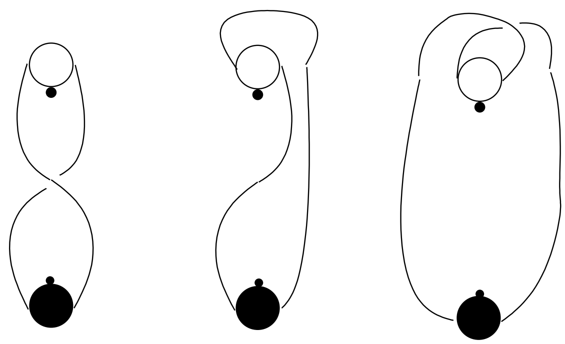
4.1 Counting Positions

[The following material is published as "Mathematics and Salsa Dancing," in the Journal of Mathematics and the Arts *[3].]*

In Partner Salsa dancing, the basic steps occur on beats 1, 2, 3 and 5, 6, 7 in an eight count rhythm. The steps alternate, with the leader starting on the left foot and the follower on the right foot. This basic step continues throughout the whole dance. The direction of the steps may change depending on the decisions of the dancers. Visit our web site at `http://www.westfield.ma.edu/renesse/salsa/` for a video clip showing the basic step.

Once one masters the basic step, it does not form a major part of the complexity of the Salsa dance any more. The real challenge lies—for the leader—in combining different moves in an interesting way and—for the follower—in styling. The most important aspect that makes moves "different" is the positioning and movement of the arms. To our knowledge, no other dance depends in such a major way on the arm positions. This is especially true in Cuban Salsa.

In this section, we will leave aside the details of the steps in Salsa dancing and focus on the stationary positions of the body and arms that you reach after four or eight counts of the music which is enough to lead a half turn or a full turn. We start by considering only moves where both partners' hands are held and will stay held throughout the moves. For now, we do not consider moves where we let go of one or both hands. Additionally, we will first consider only moves in which every position has one or fewer arm crossings: Crossings are counted as seen from above in diagrams such as Figure 4.1 with the lowest possible crossings count, i.e. we do not allow extra crossings which are easily undone by slight movements of the arms.



$(1, 1, RR, LL, C_L^R)$ $(1, 1, RR^*, LL, 0)$ $(1, 1, RR^*, LL^*, C_R^L)$

Figure 4.1: Examples of Salsa dance positions.

There are of course infinitely many ways how a couple could stand in any position: just move the hand an inch and you have a slightly different position. We only want to count positions that are significantly different, i.e. positions that need a defined "dance move" to change. Notice that the dancers always stand across from each other (as if they are standing on a line facing each other), never next to each other. These are the types of dance moves we allow:

- Turn the follower to the left (counter-clockwise) or right (clockwise) by multiples of 180 degrees (follower half turn).
- Turn the leader to the left or right by multiples of 180 degrees (leader half turn).
- Move one or both hands over the head of the follower.
- Move one or both hands over the head of the leader.
- The follower can duck under arms (difficult to lead).
- The leader can duck under arms.
- Move arms up or down.

Now grab a partner and try out how many different positions you can get yourself in keeping both hands held.

1. Write down the first 10 positions that you find. Draw a picture or a diagram similar to Figure 4.1.

2. Draw out or describe all stationary dance positions when both dancers face each other with *same* hands (right to right and left to left) held.

3. Draw out or describe all stationary dance positions when both dancers face each other with *opposite* hands (right to left) held.

4. What happens when one or both dancers face away from each other. Draw out or describe all stationary dance positions, when (a) same hands and (b) opposite hands are held.

5. How do these positions relate to your answers to Investigation **2** and Investigation **3**?

6. Use symmetry to find positions that you have left out. If, for instance, the leader has an arm behind the follower's back, could there be a position where the follower has an arm behind the leader's back?

7. Using symmetry, did you find positions that seemed to work on paper but not in practice? Show at least one.

8. **Classroom Discussion:** Share your solutions with your class and see if you can find a way to count all of the positons with both hands held. Use symmetry to argue why you found *all* the positions.

4.2 Salsa Dance Moves

You have a pretty good understanding now of how difficult some positions can get and probably how to get in and out of some of them. Now we would like to look at the *space of salsa dancing*. We will stay with the restriction of keeping both hands held for now. There are many options how one could create the space, and the following investigations will lead you through different approaches.

9. Draw two axes on a sheet of paper. Label the horizontal one with *follower half right turn*, the vertical one with *follower half left turn*. The origin is our basic position with leader and follower facing each other. Write down the positions you encounter when moving up or down the axes.

10. In investigation 9 which positions can you get to when you move to coordinates that do not lie on an axis? Name a few.

11. Are you missing positions from section4.1? If yes, which ones?

12. **Classroom Discussion:** What are the advantages and disadvantages of the approach in investigation 9?

The following investigation will show you another approach:

13. Draw two axes on a sheet of paper. Label the horizontal one with *follower half right turn* on the right and *follower half left turn* on the left. Label the vertical axis with *leader half left turn* on the bottom and *leader half right turn* on the top. The origin is our basic position with leader and follower facing each other. Write down the positions you encounter when moving up or down the axes.

14. In investigation 13 which positions can you get to when you move to coordinates that do not lie on an axis? Name a few.

15. Are you missing positions from section4.1? If yes, which ones?

16. Classroom Discussion: What are the advantages and disadvantages of the approach in investigation 13?

What you just did in understanding and evaluating the two approaches is called ***mathematical modeling***. In mathematical modeling you are trying to find a model for a given situation that is too complex to be fully understood as a whole. Often models only show special aspects of the whole picture and it is up to us to decide which one fits best. Mathematical models are used in all the sciences, in mechanics and engineering, and in the social sciences.

4.3 Further Investigations

F1. For the models in investigation 9 and 13 try the following. Pick a point other than the origin and pick two different "paths" through the space of salsa dancing. Following both paths, do you arrive at the same position?

F2. Now pick a different point other than the origin and pick two different "paths" through the space of salsa dancing to that point. Following both paths, do you arrive at the same position?

F3. Do you get the same result in investigation F1 and F2? Do you think your results will happen for *any* point and *any* paths you could pick? Explain.

F4. With your dance partner stand in basic position with both hands held, facing each other. Now try leading your partner in a full left turn with your right arm **down** and your left arm up. What happens?

F5. Can you think of other positions and moves where it matters if the arms are held high or low?

F6. Now you are ready! Read the paper *Mathematics and Salsa Dancing'* in the Journal of Mathematics and the Art [3] to understand how the complete space of salsa dance positions with both hands held is created.

Chapter 5

Contra Dancing

5.1 Contra Dancing and Permutations

Our nature consists in movement; absolute rest is death.

Pascal Blaise (French Mathematician; 1623 - 1662)

Contra dance is a partner dance in which couples dance in two facing lines of indefinite length. Originally coming from English and Scottish country dancing it has evolved into its own art-form. Many of the moves are similar to Square Dance moves. Watch the video `http://www.youtube.com/watch?v=-1cPyJWm-g4` on youtube to get an idea of a typical contra dance.

Now listen to the contra dance music in the video and see if you can detect any structure.

1. Can you hear how many beats form a measure or a smaller unit of the music?

2. Are there parts of the music (for instance the melody) that repeat after a while? If yes, after how many beats?

There are different ways how the couples can line up for a contra dance. We will only dance "improper" dances in this section, which means the couples are arranged as in figure 5.1. Couple 1 consists of leader 1 ($L1$) and follower 1 ($F1$) and couples two consists of leader 2 ($L2$) and follower 2 ($F2$). For simplicity we will use the male pronoun for the leader and the female pronoun for the follower. In contra dancing the leader is often called *gent* and the follower *lady*. The person next to you of the opposite gender that is not your **partner**, is called your **neighbor**. For instance $F2$ is the neighbor of $L1$ and $L2$ is the neighbor of $F1$.

Now we need to learn some basic moves of contra dancing. I suggest to make a field trip to a local contra dance and learn the moves while doing them. There are some introductory videos on you tube, for instance the CCD series `http://www.youtube.com/watch?v=qTtEOaruqr4`. See Figure (5.2) to get an idea what a contra dance event might look like. For the purpose of this book we will list a few important moves. Each step is on one beat.

- *long lines forward and back*
  Hold hands in long lines down the hall. Go forward 4 steps (the lines go toward each other) then go backwards 4 steps.

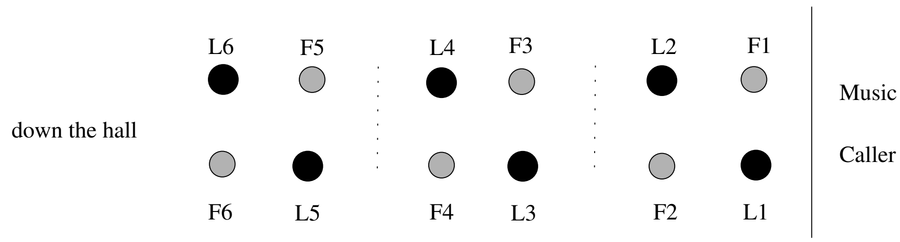


Figure 5.1: The typical setup for a Contra Dance.

- *circle left*
  Hold hands in your group of 4, then walk to your left around the circle. The caller will tell you how much, one quarter, half way round, three quarters or all the way around. (`http://www.youtube.com/watch?v=DBvhyVata9I`)

- *ladies chain*
  This move can only be done in the setup as in Figure (5.3). The two followers walk toward each other, pull by with the right hand and are being led around by the opposite leader. The leader can hold the follower in promenade position, with his right arm around her waist and his left hand holding her right hand. This move takes 8 beats. (`http://www.youtube.com/watch?v=DBvhyVata9I`)

- *swing*
  The caller announces who is swinging with whom. Get into regular dance position, with the leader's right hand behind the follower's back and the other hands held. Both dancer now walk around each other clockwise. Instead of just walking both dancers can bring their right feet into the middle and keep kicking off with the left feet. Through the momentum the couple moves fast and smooth in right turns until 8 beats are up. The follower ends at the right side of the leader. (`http://www.youtube.com/watch?v=N1o7tdtHZyE`).

- *right left through*
  We start in a setup as in Figure (5.4). Leaders and followers walk through the other couple to the other line and turn there as a couple, so that in the end the follower is again on the right side of the leader. (`http://www.youtube.com/watch?v=DBvhyVata9I`).

What makes contra dancing so interesting is that the dance progresses. After each round of music (64 beats) the first couple and the second couple exchange places and turn around, so that they dance with new neighbors next. The dance continues and in the end you danced with everybody in the whole line. See Figures 5.5-5.6 for the dance setup before and after the dance progressed once. Notice that couple 2 is out after the first progression and waiting to come in after the next progression to dance with couple 4.

Our main question for this chapter is the following: Imagine you are creating a contra dance yourself; how can you be sure that the dance progresses?

Figure 5.2: Contra Dance in Peterborough.

3. In the following simple dance, figure out if the dance progresses or not: neighbor swing, right left through, lady's chain, long lines forward and back, circle left three quarters. How did you do find your answer? Explain in detail.

Mathematicians like to think about setups using numbers, so let us draw numbers on the floor to label the positions of the dancers. See Figure (5.7).

Imagine that each dancer has a card that tells her for each move from which number she has to go to which number. Example of the 4 cards for ladies chain:

$$1 \to 3$$

$$2 \to 2$$

$$3 \to 1$$

$$4 \to 4$$

This notation is a bit cumbersome, so let's write $(1,3)(2)(4)$ instead. This means that positions 1 and 3 exchange places while positions 2 and 4 don't move. Think of $(1,3)$ as $1 \to 3$ **and** $3 \to 1$.

4. Write *right left through* in this new notation. Explain your solution.

5. Write *circle left three quarters* in this new notation. Explain your solution.

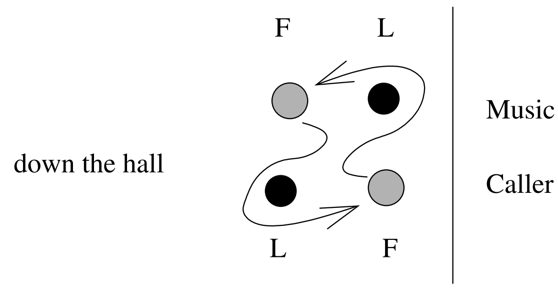


Figure 5.3: The setup for a ladies chain.

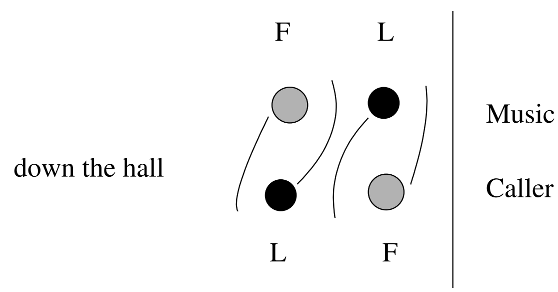


Figure 5.4: The setup for a right left through.

6. Write *long lines forward and back* in this new notation. Explain your solution.

Mathematicians call this new notation ***permutations***. Permutations describe in an easy way how to get a one-to-one correspondence between the two sets $\{1, \cdots, n\}$ and $\{1, \cdots, n\}$. n can be any number; in the above example n is equal to 4. One advantage of writing moves as permutations is that we can compute very easily if a dance progresses or not.[1]

7. Write the short dance from Investigation **3** in permutations.

8. Write all the permutations from the moves next to each other and see if you can figure out how to compute what happens in the whole dance? For instance, if I start in position 1 and follow all the different dance card instructions, can you predict where I will end up?

9. How can you tell if a dance progresses or not using permutations?

10. Does the short dance from investigation Investigation **3** progress or not?

[1] It is confusing if you think of the numbers as being numbers of the dancers. The numbers really stay on the floor! Remember that after every move, you are now at a new number and you have to see what happens with that new number on your next dance card!

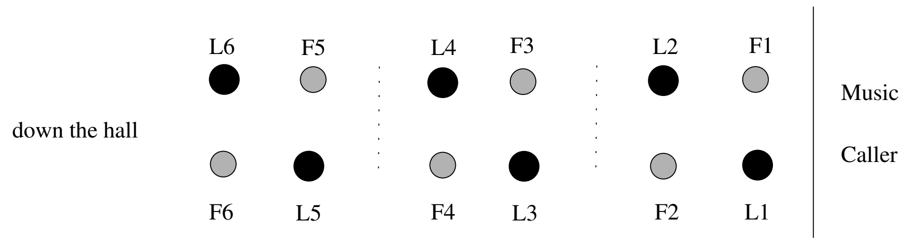


Figure 5.5: The setup for the dance.

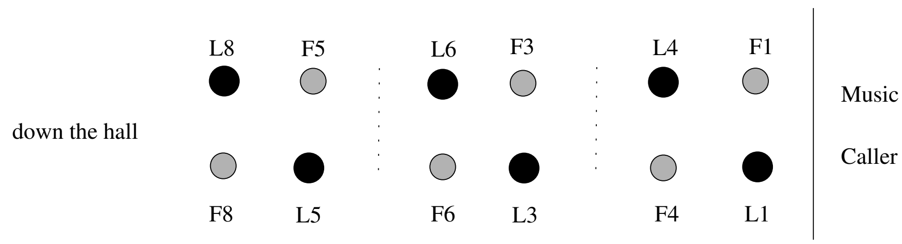


Figure 5.6: The first progression.

We will use our short dance from investigation Investigation **3** and add one more move: a neighbor-swing. Now you should be able to turn around after one round of the dance and start the dance over with the next set of neighbors. With some contra dance music try out the dance! You can find free contra dance music at `http://www.jefftk.com/news/2010-10-30`, for instance `http://ia600202.us.archive.org/22/items/Music_From_the_Contra_Dance_1/cast2.mp3`. If you don't have anyone comfortable enough to call the dance moves, just memorize the dance.

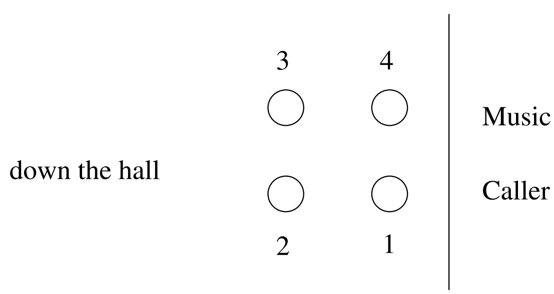


Figure 5.7: Numbering the positions on the floor.

11. Independent Investigation: In groups, invent your own contra dance using permutations and the moves you have learned so far. You can add more moves from the CCD videos if you like: `http://www.youtube.com/watch?v=N1o7tdtHZyE`, `http://www.youtube.com/watch?v=DBvhyVata9I` and `http://www.youtube.com/watch?v=oJs9MEhTP6Y`. Make sure that your dance really progresses. Then teach your dance to your whole class.

5.2 Further Investigations

F1. Discuss how to notate the moves of a contra dance so that they are easy to call. Some ideas are listed at `http://www.quiteapair.us/calling/cardformats.html`. Develop your own dance card and practice calling the dance from Investigation **3**. What is hard for you? What is easy for you? Explain.

F2. As a class, find and practice another (more advanced?) contra dance. You can use `http://www.quiteapair.us/calling/compositions.html` as a resource. Write the dance in permutation notation and show that it progresses.

5.3 Permutation Connections

In contra dancing you dance with different dancers in every progression. Similarly, in change ringing a different order of bells is being chosen for every "progression". Karl Schaffer wonders in [12] if there is a connection between the development of contra dance and change ringing in the 17th century because the mathematics they require (permutations) is so similar. See Figures 5.8-5.10 for images of change ringing. `http://www.bellringing.org/` has a wealth of information about change ringing as well.

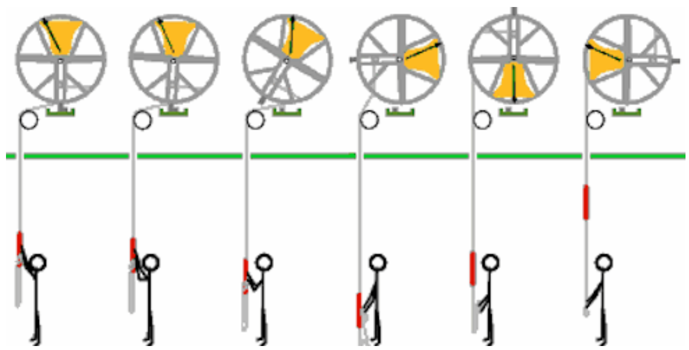


Figure 5.8: How one Bell is Roung in Change Ringing.

Figure 5.9: A Group of Bell Ringers at the Inveraray Bell Tower in Scotland. Image by Mr. William Dawson.

To physically understand how many permutations there are try the following activity:

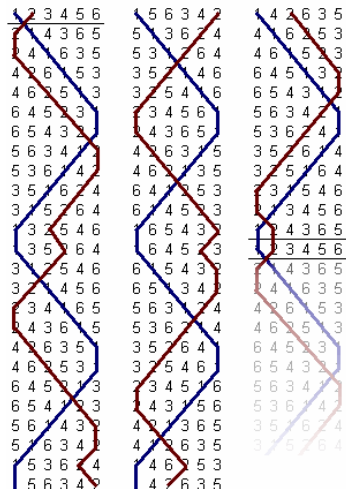


Figure 5.10: Recording the Order of 6 Bells in Change Ringing.

12. Independent Investigation: Take 3 people and try out all the different orders in which they can stand next to each other. How many orders did you find? Now try the same with 4 people. Create your own dance.
Now read Tim Verhoeff's paper [16] and realize some of his variations on dancing permutations.

In Michael Bush's and Gary Roodman's paper [1] you can find another beautiful connection of mathematics and folk dancing (four couple dances) in which the possiblilty of a specific dance configuration is discussed using permutations.

5.4 Contra Dancing and Groups

Take a piece of paper and draw a square on it. Label the vertices with 1,2,3 and 4 clockwise. What kind of symmetries does a square have? Any reflectional symmetry? If yes, across which axis? One way is to fold the square in half so that vertices 1 and 2 touch and vertices 3 and 4 touch. See Figure (5.11). To remember which vertices touch we write them as a permutation $(1,2)(3,4)$.

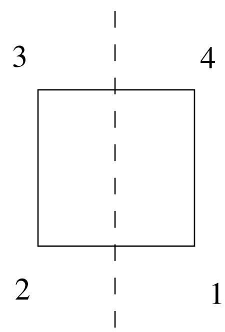


Figure 5.11: Reflectional symmetry of a square.

13. Find all the other symmetries of the square and draw the pictures.

14. Write all the other symmetries of the square as permutations.

15. If you combine two of the symmetries of the square will you get a new kind of symmetry? Check all the combinations.[2]

Now we know a lot about symmetries of the square, but how is this possibly connected to our contra dances? Well, try this:

16. Find for every symmetry of the square a contra dance move. You might have to invent your own move for some of them!

17. Do all the contra dance moves correspond to symmetries of the square? Explain.

One of the cool things in mathematics are structures that appear in or apply to several situations. You just found that symmetries of a square are in some way similar to a set of basic contra dance moves. The structure that you found mathematicians call a ***group***. A group consists of a set of elements (e.g. the symmetries or the dance moves) and an operation (e.g. execute two symmetries after one another, or, dance two contra dance moves after one another). In every group there has to be a ***identity element*** that you can combine with all the other elements without changing anything.

18. What is the identity element of the symmetries of the square?

[2]This is a lot of work, so think about how to organize your work. Maybe different groups of students can check different combinations? You can write your result as one big table, similar to the multiplication table.

19. What is the identity element of the contra dance moves?

Additionally every element in a group needs an ***inverse element***. If you combine an element with its inverse you get the neutral element from above.

20. What is the inverse element of a reflection of the square?

21. What is the inverse element of a rotation of the square?

22. Is there just one inverse element to each symmetry in the square?

23. What is the inverse element of *lady's chain*?

24. What is the inverse element of *circle to the left three quarters*?

25. Is there just one inverse element to each dance move? Explain why or why not.

26. For a mathematical group we want each element to have exactly one inverse element. Write down a list of contra dance moves that from a mathematical group.

You just convinced yourself that the symmetries of the square form a mathematical group and that the basic contra dance moves form a mathematical group[3]! See [6] and [2] for more in depth results about groups and contra dancing.

Groups by themselves might not seem that powerful, but they form the base on which many complicated and powerful theorems in *Abstract Algebra* can be proven. Groups are for example used in understanding *Special Relativity* and *Molecular Chemistry* but also in solving the *Rubik's Cube*.

Evariste Galois (French Mathematician; 1811 - 1832) was the first to use the mathematical term "group". Galois became interested in mathematics when he was 14 years old. He worked in the area now called *Group Theory*, inventing *Galois Groups* as a teenager while also being involved in the turbulent politics of his time. Galois died after a duel at the age of 20. See Figure 5.12.

Figure 5.12: Evariste Galois

[3] To be precise we would need to check closure and associativity as well, which is omitted here.

5.5 Further Investigation: Contra Dancing and Groups

F3. Read the following new article and understand how you can use matrices to create (all) contra dances:
`www.sciencenews.org/article/contra-dances-matrices-and-groups`

5.6 Other Connections

There are some interesting connections of Contra/Square Dancing to Japanese braiding. Watch the Kumihimo KumiLoom Braiding Instructions at `http://www.youtube.com/watch?v=0RNbFjvZycs`.

F4. Describe how a Japanese braid is similar to a square dance or a contra dance.

F5. Open question: Can you use a Japanese braid to see if the corresponding contra dance progresses?

Watch `http://www.youtube.com/watch?v=qSxMeQVkFZQ` for an amazing connection between contra dancing and visualizing DNA.

Chapter 6

Maypole Dancing

6.1 Which Ribbon Pattern is Created by the Dance?

> We don't accomplish anything in this world alone... Whatever happens is the result of the whole tapestry of one's life and all the weavings of individual threads from one to another that creates something.
>
> **Sandra Day O'Connor** (US Supreme Court justice; 1930 -)

Figure 6.1: Maypole Dancing in Westfield, Mass, 1939

In medieval village life, maypole dancing was a ritual to celebrate May Day. The pagan tradition was meant to increase vitality and fertility. May Day is still celebrated in this way in many places in Europe and also in the hill towns of Massachusetts; see Figure 6.2. The standard maypole dance has a certain number of dance couples arranged in a circle around a high wooden pole. Colored ribbons of fabric are strung from the top of the pole. Each dancer holds the end of one such ribbon. Figure 6.3 shows the starting position for a maypole dance with four couples.

Figure 6.2: Maypole Dancing in Ashfield, Mass, 2012

The large dashed circle indicates the outline of the dance circle, where pairs of leaders and followers (indicated by squares and circles, respectively) are arranged for the dance. Leaders and followers move in opposite directions around the maypole (leaders: counter-clockwise, followers: clockwise). For this initial pairing, leaders pass to the outside of the followers, as indicated by the arrows. When the dance starts, dancers will interweave with oncoming dancers by passing them on the inside and then outside in an alternating fashion. As the dance progresses, the colored ribbons wrap around the wooden pole, making patterns starting at the top of the pole and continuing lower as the dance progresses. You can watch the following video to see a maypole dance in action: `http://www.youtube.com/watch?v=FxcIqMmlVOs&feature=related` .

Building your own maypole: we find that a plastic pipe of about 2 inch diameter works well. A home building supply store such as Home Depot or Lowes' has sections that are 10 feet long in their plumbing supply area; they also have shorter sections of about 2 feet that we find ideal for each group. For ribbons we use satin ribbons of about 1 inch wide. Art supply stores such as Michael's has rolls of about 4 yards in various colors. Two colors, one dark, one bright, will be enough for the first few explorations. Later, a third color with good contrast will be needed (we use red in our figures).

1. Work in groups of at least 9 people. You will need one person holding the pole and 8 people for the dance. For this first dance, we will try to match the setup shown in Figure 6.3: All the leaders have light-colored bands and move in a counter-clockwise direction around the pole; all the followers have dark-colored bands and move in a clockwise direction around the pole. Decide on who of you will start in what position. The ribbons can be attached to the top of the maypole using wide rubber bands. To make it easy to see the ribbon patterns that emerge, it helps to attach each ribbon at a 45 degree angle to the top of the pole, hanging down in the direction that the dancer will take (different directions for leaders and followers).

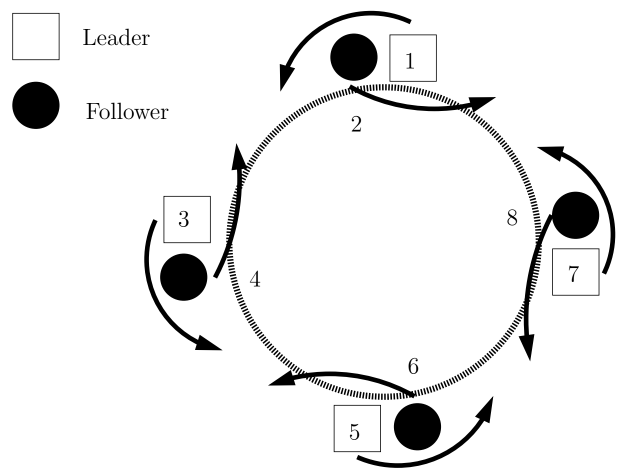


Figure 6.3: Starting position of a maypole dance with four pairs of dancers. Leaders (indicated by boxes) dance in a counter-clockwise directions; followers (indicated by small circles) dance in a clockwise direction.

> We find it easiest to attach all the leader bands first, using one wide rubber band. As you face each leader's ribbon where it is attached to the pole, you want it to move towards the right hand side. Attach all the leader ribbons. It helps to adjust the angle of the band so there is only very little extra space around the pole. Now attach all the follower ribbons, in such a way that the top of a follower's ribbon fits neatly on top of a leader's ribbon. In this case, the ribbons angle towards the left, neatly following the clockwise movement of the followers. Finish all the follower ribbons and adjust all the angles and spacings so it all looks neat.
> Now dance the above described maypole dance until you see a ribbon pattern emerge on the pole. Describe the pattern.

As a short-hand way, we describe the ribbon pattern of black and white ribbons shown for the eight dancers in Figure 6.3 as WBWBWBWB. Notice that this lists the ribbon colors for each dancer, starting with a particular leader, then their follower, on to the next dance pair in a counter-clockwise direction.

2. In your groups, describe how "neat" the maypole pattern is: do you have gaps between the ribbons or not? Are the ribbons tangled or squished in places? Do the angles of the ribbon stay the same or change?

We want to explore how to choose pole, ribbons and angles of ribbons so that we get a perfect beautifully arranged pattern.

3. Suppose we simply wanted to wrap one ribbon around the pole. Can you explain precisely how to determine the correct orientation so there are no gaps or overlaps?

4. Is the orientation you found in Investigation **3** unique, or are there additional orientations that will work? Explain precisely.

5. Suppose you wanted to two wrap two ribbons around the pole, in the same direction, so there was no overlap. Can you precisely determine the correct orientations for the ribbons? Explain.

6. Can you repeat Investigation **5** for three ribbons? Four? Explain, precisely, how to determine the maximum number of ribbons that can be wrapped around the pole.

Now that you have determined how to orient the ribbons, it will be easier to make tight, symmetric weavings. But lets start a bit smaller than the original eight.

7. Take two ribbons of the same size but different colors. Orient them in opposite directions at the appropriate orientations so your weaving will have no gaps.

8. What shapes do you see on the maypole pattern? Can you see how they are created by your dance moves?

9. Now take four ribbons, two of one color and two of another. Orient two appropriately, across from each other, and headed in the same direction. Orient the other two similarly, but headed the opposite direction.

10. What shapes is your weaving made up of? Can you see how they are created by your dance moves?

11. Suppose you continued adding couples. What would happen to the shapes?

12. Do you have to keep adding new colors as you continue to add couples? How should the colors be added?

13. Suppose we want to create a weaving from a Maypole dance with four couples so that the shapes the weaving is made up of are all squares. Determine precisely what the relationship between the width, w, of the ribbons and the diameter, d, of the pole so there will be no gaps or overlaps in the weaving.

Dancing a Maypole dance to get a nice, symmetric, gapless, non-overlapping weaving takes some time and appropriate materials, materials whose relative sizes change as more couples are added. It will not be feasible to physically experiment with all of the dances/weaving considered below. But the human brain is an amazing thing in its power to make abstractions. From the several examples that you have investigated physically, using body kinesthetics, you should be able to investigate most of the dances/weavings below in your mind.

Now that we have some experience creating Maypole dances to create certain patterns, we would like to understand how the choices of dance steps and colors impact the weavings. Conversely, given a Maypole dance weaving, we would like to be able to determine what dance steps and color combinations created this weaving.

14. Let's go back to the original dance with 8 dancers. As a single dancer, describe which colors you encounter in your dance, on the inside or outside. Where can you find the corresponding color pattern in the larger ribbon pattern on the pole.

15. Do the "encounters" repeat themselves, i.e. do you meet the same people in the same order after a while? If yes, after how many passings does that happen? If not, why not? Explain your thinking.

16. Did you notice a difference in passing a person on the outside or inside at this next encounter? Explain.

17. Is the number of "encounters" until the dance repeats the same for leaders and followers? Explain.

18. Using the dancers and their movements in your reasoning, explain why the whole ribbon pattern on the maypole looks the way it does. Can you be sure it will continue the same way?

We want to simplify the colors we have been using to get more interesting patterns. Let's say we use red, black and white: RBW.

19. Let two adjacent couples hold black ribbons, one couple red ribbons and one couple white ribbons, i.e. we are looking at BBBBRRWW. Which dancers are the leaders, which are the followers? Which ribbons would you put on the pole first? Set up the maypole with this ribbon pattern.

20. Once you have the maypole set up, dance the maypole dance until a clear ribbon pattern emerges. Describe the ribbon pattern on the pole, using drawings, tables, descriptions, photos, etc. Use your imagination in how to best record this data.

21. **Classroom Discussion:** Compare your representations (drawings, tables, descriptions, photos, etc.) of the ribbon patterns for BBBBRRWW with your classmates. What are the advantages and disadvantages of the different representations?

22. Consider John's representation of the ribbon pattern in Figure 6.5. Describe how this representation relates to the dancers and the ribbons. How does it compare to your representation of the last ribbon pattern? Label his picture so it is easier to understand for you.

To make it easier to analyze the pattern, John likes to turn it 45 degrees to the left, see Figure 6.5.

23. In John's ribbon pattern, Figure 6.5, describe in detail how a part of the pattern repeats.

We call the smallest part of a ribbon pattern that repeats itself the ***fundamental domain*** of the ribbon pattern. To understand why the whole ribbon patterns looks the way it does, it is enough to just look at the fundamental domain.

24. For example, you likely discovered a B/W checkerboard pattern for the WBWBWBWB ribbons. What size domain could you use to completely describe that pattern? What would be the smallest such domain that would fill the plane without gaps or overlaps?

Figure 6.4: John's ribbon pattern representation of BBBBRRWW

25. As one single dancer, describe which colors you meet in your BBBBRRWW dance and show where you can find that color pattern in the larger ribbon pattern on the pole and in John's representation of the ribbon pattern.

26. Now change the color pattern to BWBWBWBW. Describe the pattern that emerges *without actually dancing the maypole dance.*

27. As one single dancer, describe which colors you would meet in your BWBWBWBW dance and show where you would find that color pattern in the larger ribbon pattern on the pole and in John's representation of the ribbon pattern.

28. Explain in detail why the whole ribbon pattern for BWBWBWBW looks the way it does.

29. Now change the color pattern to BWWBBWWB. Describe the pattern that emerges *without actually dancing the maypole dance.*

30. As one single dancer, describe which colors you would meet in your BWWBBWWB dance and show where you could find that color pattern in the larger ribbon pattern on the pole and in John's representation of the ribbon pattern.

31. Explain in detail why the whole ribbon pattern for BWWBBWWB looks the way it does.

32. In you last investigations you had to predict ribbon patterns without trying out the dance. Dance those dances now and report if your thinking was correct or not. If not, explain how you would like to change your thinking.

We are going to change the number of dancers now. So let's first think about some basic questions:

33. Does the number of dancers have to be even or odd or does that not matter? Explain.

34. Does the number of pairs have to be even or odd or does that not matter? Explain.

We want to first consider what happens with 3 pairs of dancers.

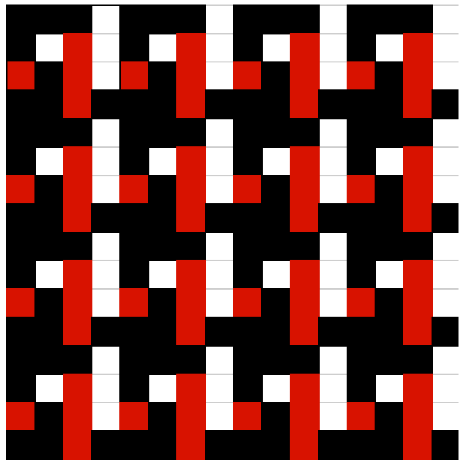


Figure 6.5: John's turned ribbon pattern representation of BBBBRRWW

35. Dance the BBWWRR maypole dance until you see a ribbon pattern emerge on the pole. Describe the pattern.

36. As one single dancer, describe which colors you meet in your dance and show where you can find that color pattern in the larger ribbon pattern on the pole.

37. Do the "encounters" repeat themselves, i.e. do you meet the same people in the same order after a while? If yes, after how many passings does that happen? If not, why not? Explain your thinking..

38. Did you notice a difference in passing this person one the outside or inside at this next encounter? Explain.

39. Find the fundamental domain for the BBWWRR dance.

40. Using the dancers and their movements in your reasoning, explain why the whole ribbon pattern on the maypole looks the way it does. Can you be sure it will continue the same way?

41. Classroom Discussion: Compare the size of the fundamental domain for 4 pairs with the size of the fundamental domain with 3 pairs. What do you notice? Explain your thinking. *(Hint: Consider your observations in Investigation* **16** *and Investigation* **38***.)*

6.2 Which Dance Arrangement Leads to this Ribbon Pattern?

If you stumble make it part of the dance.

Anonymous

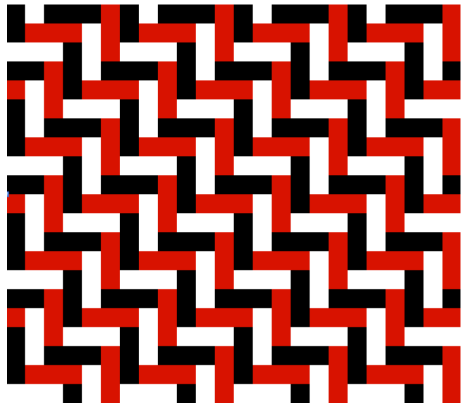


Figure 6.6: John's ribbon pattern representation of BBWWRR

As the maypole dance progresses and the ribbons are wound around the maypole, we may observe patterns like those shown in Figure 6.7.

42. Describe how this pattern is similar to, or different from, the patterns that you observed in your group when exploring maypole patterns.

In the Investigations of the previous chapter, we started with a particular arrangement of ribbon colors for a given number of dancers. The goal was to understand and predict the pattern that would result on the maypole. A problem like this is sometimes called a ***forward problem***: given some kind of physical process, we are asked to describe what kind of pattern will be created by this process. In our case, the dance is the process that creates a ribbon pattern: the forward problem attempts to predict the ribbon pattern resulting from a maypole dance with a certain number of people. As an example of a forward process in the sciences, consider a medical professional taking an X-ray of a part of your body. Forward problem: Given a particular body, predict what an X-ray in a particular direction will look like.

In this chapter, we will look at the ***inverse problem*** for the maypole dance process. Looking at a particular maypole pattern, such as Figure 6.8, we wonder: How many people participated in this dance, and what arrangement of ribbon colors did they start with?

As an example of a forward and inverse problem pair from mathematics consider the following. Forward problem: given two numbers, find their product. Compute, for example, 1234×4321. Even for large numbers, people, calculators, and computers can solve this problem fairly quickly. Now the inverse problem: take a ***composite*** number, that is a number resulting as a product of other numbers, and determine its factors. For large numbers this problem takes a lot of time even for computers, because many different possibly pairs have to be tried. Consider the following number with 13 digits: $9,449,772,114,007$, which is the product of two integers. How could we find these two numbers? A modern public key encryption system called RSA is based on the fact that multiplication is straightforward but factoring is computationally difficult (the numbers involved in RSA encryption are very large, e.g. a 1 followed by about 100 zeros).

But let us return now to Figure 6.8.

Figure 6.7: Maypole Pattern

43. How large is the fundamental domain of the pattern?

44. How many dancer will you need to dance the maypole dance that generates this pattern? Explain.

45. Which colors will you need to dance the maypole dance that generates this pattern? Explain.

46. Is there a different way how you could arrange dancers and get the **same** pattern or is the way of dancing a maypole pattern unique? Explain.

6.3 Further Investigations

Mathematicians often try to make a problem easier if they can't solve it. Unfortunately this doesn't seem to really work for maypole dancing!

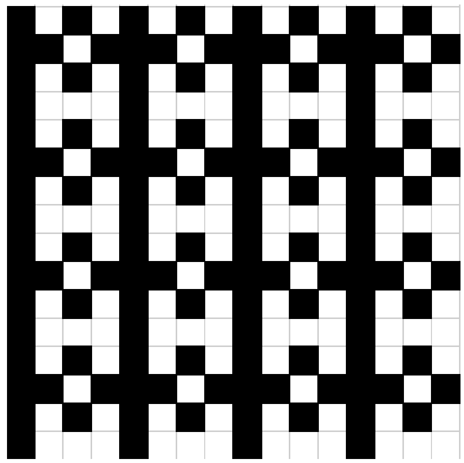


Figure 6.8: Backwards Problem

F1. Is it easier to understand a maypole patterns when there are more or less pairs? For both cases, explain which aspects of the pattern are harder or easier to understand.

F2. Is it easier to understand a maypole patterns when there are more or less colors? For both cases, explain which aspects of the pattern are harder or easier to understand.

Let's play a bit more with the possibilities of colors and patterns:

F3. Why does the ribbon pattern for BWBWBW look exactly the same as the ribbon pattern for BWBWBWBW? Doesn't that contradict our findings about fundamental domain sizes?

F4. Choose your own number of pairs and choose colors you like. Predict the maypole pattern. Then dance the pattern to check your work. Take a picture. Reflect on how well you were able to predict the pattern.

F5. Use paper strips to weave a dance pattern of your choice, similar to `http://mrhonner.com/2011/11/29/weavings-and-tilings/`. What did you learn from this activity that you did not know before about maypole dance patterns?

Let's solve a few more maypole dance puzzles: Given a pattern, can you find the dance?

F6. Find the number of dancers and colors to dance the pattern in Figure 6.10.

F7. Find the number of dancers and colors to dance the pattern in Figure 6.9.

F8. Open Problem: Can you find a maypole dance that will give you the pattern in Figure 6.11?

F9. How many dance pattern are there given a number of pairs and a number of colors? (This is a very big question to consider!)

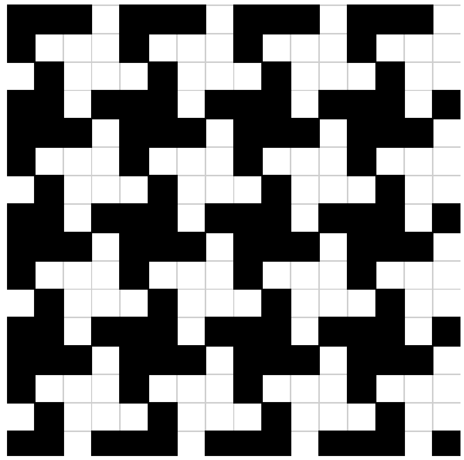


Figure 6.9: Pattern Puzzle 1

F10. Are there patterns (square-tilings) you can not create with a maypole dance? If not, explain why not, If yes, find a counter example.

F11. What happens if you change the dance? Can you create different patterns from before? Experiment with a few dances you invent.

6.4 Connections

47. Watch `http://www.youtube.com/watch?v=5fEnF6daTYQ`. Explain how maypole dancing is similar and different from african basket weaving.

48. Explain how maypole dancing is connected to salsa rueda dancing and to machines that make rope. Watch `http://www.youtube.com/watch?v=8ECeP5lHFr4`, and `http://www.youtube.com/watch?v=dER8DM3aYqk` to learn about rope making.

49. Watch `http://www.youtube.com/watch?v=G85f-C63CXg`. Explain how maypole dancing is connected to braid weaving.

50. Watch the Kumihimo KumiLoom Braiding Instructions at `http://www.youtube.com/watch?v=0RNbFjvZycs`. Explain how maypole dancing is connected to contra dancing and Japanese braiding.

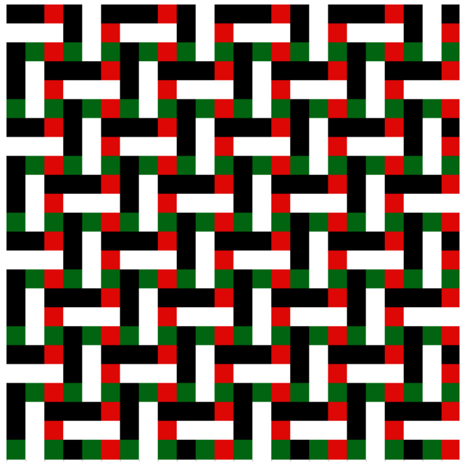


Figure 6.10: Pattern Puzzle 2

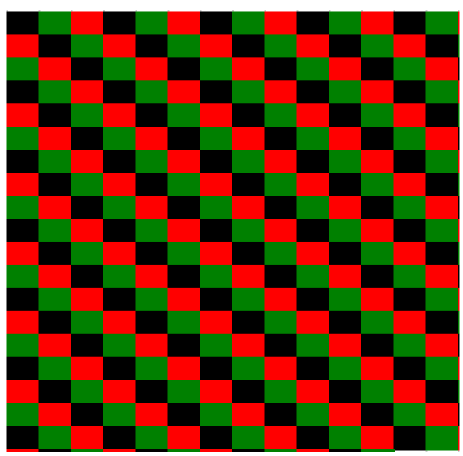


Figure 6.11: Find the dance?

Chapter 7

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