An Introduction to Convexity, Optimization, and Algorithms

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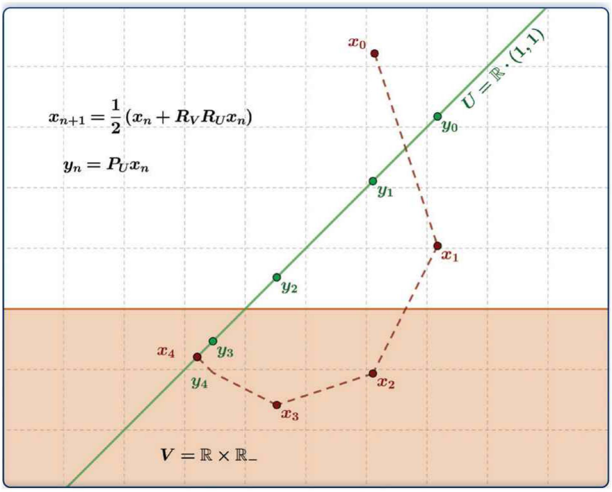

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Library of Congress Control Number: 2023043614

To our students

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Preface

Convex analysis, convex optimization, and algorithms are important topics in modern applied mathematics. In this text, we provide an introduction to a selection of these topics accessible at the advanced undergraduate or beginning graduate level. The only background required is some core knowledge of calculus, linear algebra, and analysis.

This book is the core of a one-semester course (12 weeks with 3 lecture hours/week) that the authors taught at University of British Columbia Okanagan and at University of Waterloo. The material is complementary to existing books aimed at this level — we single out in particular the books by Beck [4] and by Mordukhovich and Nam [34]. This book is also immensely suitable for self-study.

In a lifetime, a mathematician typically reads only a few books cover-to-cover. This book aims to be an exception: you should be able to completely read this in a relatively short — and also enjoyable — period of time. The material is largely self-contained; however, when we rely on external results that we do not fully prove, we flag them as **Facts**. After you've read this book, you are ready to tackle more advanced texts we comment upon in the last section of the book; in particular, you will be able to dive into [3], Beck's [5], and Rockafellar's seminal book [39], which served as main motivations for the selection of the material presented.

We thank the following colleagues for their encouragement and support: Fran Aragón Artacho, Sedi Bartz, Amir Beck, Radu Boţ, Yunier Bello-Cruz, Minh Dao, Warren Hare, Mau Nam Nguyen, Hung Phan, Nghia Tran, Levent Tunçel, Jon Vanderwerff, Stephen Vavasis, Shawn Wang, Henry Wolkowicz, Yaoliang Yu, and Jim Zhu. Specials thanks go to Paula Callaghan, Cheryl Hufnagle, Rose Kolassiba, and Louis Primus for all their kind and encouraging help shepherding this book to production! Last but not least, we thank our students and teaching assistants for valuable comments and feedback, in particular Alvaro Carbonero, Woosuk Jung, Tanmaya Karmarkar, Hongda Li, Joao Paulo Pinto Galdino Marques, Heejun Song, and Samuel Street. We welcome you contacting us with suggestions and corrections!

August 2023 Heinz Bauschke and Walaa Moursi

Symbols and Notation

Real Line

$\mathbb{R}$	The set of real numbers
$\mathbb{R}_+$	The set of nonnegative real numbers $[0, +\infty[$
$\mathbb{R}_{++}$	The set of positive real numbers $]0, +\infty[$
$\mathbb{R}_-$	The set of nonpositive real numbers $]-\infty, 0]$
$\mathbb{R}_{--}$	The set of negative real numbers $]-\infty, 0[$
$\mathbb{Q}$	The set of rational numbers
$\mathbb{Z}$	The set of integers
$\mathbb{N}$	The set of nonnegative integers $\{0, 1, \dots\}$
$\inf, \min$	Infimum and minimum
$\sup, \max$	Supremum and maximum
$\varliminf$	Limit inferior
$\varlimsup$	Limit superior

Euclidean Spaces

$\langle \cdot, \cdot \rangle$		Inner product
$\| \cdot \|$		Norm
$\| \cdot \|_*$	p. 5	Dual norm
Id		Identity operator
$x_k \to x$		The sequence $(x_k)_{k \in \mathbb{N}}$ converges to x
$\mathbb{S}^n$		Space of real symmetric $n \times n$ matrices
$\mathbb{S}^n_+$		$\{A \in \mathbb{S}^n \mid A \text{ is positive semidefinite}\}$

Sets

$C \times D$		Cartesian product of the sets C and D
$C + D$	p. 14	$\{c + d \mid (c, d) \in C \times D\}$
λC		$\{\lambda c \mid c \in C\}$
ΛC		$\{\lambda c \mid c \in C, \lambda \in \Lambda\}$.
$z + C$	p. 9	$\{z + c \mid c \in C\}$
$\operatorname{span} C$		Linear span
$\operatorname{aff} C$	p. 9	Affine hull
$A(C)$		$\{A(c) \bigm\vert c \in C\}$
$B[y; \rho]$		Closed ball $\{x \in X \mid \\|x - y\\| \leqslant \rho\}$
$B(y; \rho)$		Open ball $\{x \in X \mid \\|x - y\\| < \rho\}$
$\overline{C}$		Closure
$\operatorname{int} C$	p. 13	Interior
$\operatorname{bdry} C$		Boundary $\overline{C} \smallsetminus \operatorname{int} C$
$\operatorname{ri} C$	p. 13	Relative interior
$\operatorname{conv} C$	p. 9	Convex hull
$\overline{\operatorname{conv}}\, C$		$\overline{\operatorname{conv} C}$
P_C	p. 39	Projection onto the set C
$\operatorname{cone} C$		Conical hull $\{0\} \cup \mathbb{R}_+ C$
$\overline{\operatorname{cone}}\, C$		Closed conical hull $\overline{\operatorname{cone} C}$
$C^\perp$		Orthogonal complement
$C^\ominus$	p. 36	Polar cone of a set C
N_C	p. 49	Normal cone
T_C	p. 49	Tangent cone operator of a set C
σ_C	p. 35	Support function
ι_C	p. 17	Indicator function
d_C	p. 1	Distance function
$[a, b]$	p. 10	Line segment

Functions and Operators

$f_1 \oplus \cdots \oplus f_m$	p. 23	Separable sum
$f^\vee$	p. 157	$f \circ (-\mathrm{Id})$
$f\vert_C$		f restricted to C
$\operatorname{dom} f$	p. 17	Domain of a function
$\overline{\operatorname{dom}}\, f$		$\overline{\operatorname{dom} f}$
$\operatorname{dom} \partial f$	p. 45	Domain of a subdifferential operator
$\operatorname{epi} f$	p. 17	Epigraph of a function
$\overline{\operatorname{epi}}\, f$		$\overline{\operatorname{dom} f}$
$\operatorname{lev}_{\leqslant \alpha} f$	p. 20	Lower level set
$\operatorname{Argmin} f$	p. 28	Set of global minimizers of a function f
$\operatorname{Argmin}_C f$		$\operatorname{Argmin}(f + \iota_C)$
$f \mathbin{\square} g$	p. 32	Infimal convolution of the functions f and g
$\operatorname{env}_\mu f$	p. 123	Envelope
P_f	p. 115	Proximal mapping
R_f	p. 152	Reflectant $2\mathrm{P}_f - \mathrm{Id}$
f^*	p. 85	Fenchel conjugate
f^{**}		Biconjugate $(f^*)^*$
∂f	p. 45	Subdifferential

∇f		Gradient
$f'(x;d)$	p. 53	Directional derivative
f'_+		Right derivative
f'_-		Left derivative
ι_C	p. 17	Indicator function
d_C	p. 1	Distance function
σ_C	p. 35	Support function
$\lambda_{\max}$		Largest eigenvalue map
$\lambda_{\min}$		Smallest eigenvalue map

Single-Valued Operators

$\operatorname{Fix} T$	p. 101	Fixed points $\{x \in X \mid Tx = x\}$
$\ker A$		Kernel $\{x \in X \mid Ax = 0\}$
$\|A\|_2$	p. 4	Operator/spectral norm
$\|A\| = \|A\|_{\mathsf{F}}$		Frobenius norm of a matrix A
A^*		Adjoint of a linear operator
A^{T}		Transpose of a matrix
$A^{\dagger}$		Moore–Penrose (pseudo) inverse
$\operatorname{tra}(A)$		Trace of a square matrix A
$A \succ B$		The matrix $A - B$ is positive definite
$A \succeq B$		The matrix $A - B$ is positive semidefinite

Set-Valued Operators

$\operatorname{zer} A$		zeros of A, i.e., $A^{-1}(0) = \{x \in X \mid 0 \in Ax\}$
A^{-1}		Set-valued inverse: $x \in A^{-1}(y) \Leftrightarrow y \in Ax$

Chapter 1

Setting the Stage

1.1 ▪ Reminders from Analysis and Linear Algebra

Recall that if S is a subset of $\mathbb{R}$, then $\inf S$ and $\sup S$ denote the *infimum* and *supremum* of S. These numbers may lie in $[-\infty, +\infty]$. If $\inf S \in \mathbb{R}$ and the infimum is attained, i.e., a *minimum*, then we will also write $\min S$. A similar comment applies to the supremum and the *maximum*, written $\max S$.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$, then $\underline{\lim}_{n\in\mathbb{N}} x_n$ denotes its *limit inferior* while $\overline{\lim}_{n\in\mathbb{N}} x_n$ denotes its *limit superior*. If $(y_n)_{n\in\mathbb{N}}$ is another sequence in $\mathbb{R}$, then

$$\varliminf_{n\in\mathbb{N}} x_n + \varliminf_{n\in\mathbb{N}} y_n \leqslant \varliminf_{n\in\mathbb{N}} (x_n + y_n). \tag{1.1}$$

The famous *Bolzano–Weierstrass Theorem* states that every bounded sequence (residing in $\mathbb{R}^n$) has a convergent subsequence.

Recall that a subset S of $\mathbb{R}^n$ is *compact* if it is both bounded and closed.

The famous *Weierstrass Theorem* states that if $f\colon S \to \mathbb{R}$ is continuous, where S is compact, then f attains its minimum and maximum on S: There exist s_1 and s_2 in S such that

$$f(s_1) = \min f(S) \quad \text{and} \quad f(s_2) = \max f(S). \tag{1.2}$$

If a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is *positive semidefinite*, i.e., $(\forall x \in \mathbb{R}^n)\ x^\mathsf{T} A x \geqslant 0$, then we write $A \succeq 0$. If $(\forall x \in \mathbb{R}^n \smallsetminus \{0\})\ x^\mathsf{T} A x > 0$, then A is *positive definite*, written $A \succ 0$.

Given a subset S of X, its *distance function* d_S is defined by

$$d_S(x) := \inf \|x - S\| = \inf_{s\in S} \|x - s\|, \tag{1.3}$$

where $\|\cdot\|$ denotes a norm (usually the Euclidean norm) on X.

Proposition 1.1 *Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric and positive semidefinite. Suppose that there exists $\beta \geqslant 0$ such that $A \preceq \beta B$. Denote the nonnegative eigenvectors of B by $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_d > \beta_{d+1} = \cdots = \beta_n = 0$. If $d = 0$, then set $\gamma = 0$. If $d \geqslant 1$, then proceed as follows: Obtain orthonormal eigenvectors $u_1, \ldots, u_d$ of B with corresponding eigenvalues $\beta_1, \ldots, \beta_d$. Build the matrix $U := [u_1|u_2|\ldots|u_d] \in \mathbb{R}^{n\times d}$, the diagonal matrix $D \in \mathbb{R}^{d\times d}$ with diagonal entries $\sqrt{\beta_1}, \ldots, \sqrt{\beta_d}$, and finally the matrix $C := D^{-1}U^\mathsf{T} A U D^{-1}$. Denote its largest eigenvalue by γ. No matter which case we are in ($d = 0$ or $d \geqslant 1$), the γ constructed is the smallest nonnegative real number such that $A \preceq \gamma B$.*

1.2 ▪ Euclidean Spaces

Throughout this book,

$$X \text{ is a Euclidean space,}$$

with inner product $\langle x, y\rangle$ and induced Euclidean norm

$$\|x\| := \sqrt{\langle x, x\rangle}.$$

Sometimes, a second Euclidean space is present, usually denoted by Y.

We will occasionally employ other norms; however, if we don't mention anything, it will be the Euclidean norm.

We recall here that an inner product is a mapping

$$\langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}$$

satisfying

(i) $\langle x, y\rangle = \langle y, x\rangle$;

(ii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;

(iii) $\langle x, x\rangle \geqslant 0$, with equality if and only if $x = 0$.

Proposition 1.2 (classical Cauchy–Schwarz) *For vectors x, y in X, we have*

$$\langle x, y\rangle \leqslant \|x\|\|y\|, \quad \textit{with equality if and only if } \|y\|x = \|x\|y.$$

Proof. This is clearly true if $x = 0$ or $y = 0$. So assume that $x \neq 0$ and $y \neq 0$. Then

$$\begin{aligned} 0 &\leqslant \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = \left\| \frac{x}{\|x\|} \right\|^2 + \left\| \frac{y}{\|y\|} \right\|^2 - 2\left\langle \frac{x}{\|x\|} \,\middle|\, \frac{y}{\|y\|} \right\rangle \\ &= 1 + 1 - 2\frac{1}{\|x\|\|y\|}\langle x, y\rangle = 2\left(1 - \frac{\langle x, y\rangle}{\|x\|\|y\|}\right). \end{aligned}$$

Now rearrange and we are done. $\square$

Example 1.3 (standard Euclidean space) Suppose $X = \mathbb{R}^n$. Then X is a Euclidean space with the inner product

$$\langle x, y\rangle := x^{\mathsf{T}} y$$

being equal to the dot product; thus,

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Example 1.4 If $X = \mathbb{R}^n$ and $Q \in \mathbb{R}^{n\times n}$ is *positive definite*, i.e., $Q = Q^{\mathsf{T}}$ and $\langle x, Qx\rangle > 0$ if $x \neq 0$, then

$$\langle x, y\rangle_Q := x^{\mathsf{T}} Q y$$

is the Q-inner product and

$$\|x\|_Q := \sqrt{\langle x, x\rangle_Q}$$

is the Q-norm.

Example 1.5 (matrix spaces) Suppose $X = \mathbb{R}^{m\times n}$. Then X is a Euclidean space with the inner product

$$\langle A, B\rangle := \operatorname{tra} A^\mathsf{T} B = \sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} A_{i,j}B_{i,j},$$

where tra denotes the trace function. Thus

$$\|A\| = \sqrt{\sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}} A_{i,j}^2},$$

which is also known as the *Frobenius norm* and also written as $\|A\|_\mathsf{F}$. If $m = n$, then an important subspace is

$$\boxed{\mathbb{S}^n := \big\{A \in \mathbb{R}^{n\times n} \;\big|\; A^\mathsf{T} = A\big\},}$$

the space of symmetric matrices of size $n \times n$.

Example 1.6 (product space) Let X and Y be Euclidean spaces, with inner products $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$, respectively. Then the *Cartesian product*

$$X \times Y = \big\{(x, y) \;\big|\; x \in X,\ y \in Y\big\}$$

is a real vector space with

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &:= (x_1 + x_2, y_1 + y_2),\\ \alpha(x, y) &:= (\alpha x, \alpha y).\end{aligned}$$

$X \times Y$ is also an inner product space with

$$\langle (x_1, y_1), (x_2, y_2)\rangle := \langle x_1, x_2\rangle_X + \langle y_1, y_2\rangle_Y\,,$$

which gives rise to the corresponding Euclidean norm

$$\|(x, y)\| := \sqrt{\langle (x, y), (x, y)\rangle} = \sqrt{\langle x, x\rangle_X + \langle y, y\rangle_Y} = \sqrt{\|x\|^2 + \|y\|^2}.$$

We can also deal with 3 or more Euclidean spaces in a similar fashion.

Next are some identities and inequalities that will be useful later.

Lemma 1.7 (Euclidean norm identity) *Let u, v be in X and let $\lambda \in \mathbb{R}$. Then*

$$\|(1-\lambda)u + \lambda v\|^2 + \lambda(1-\lambda)\|u - v\|^2 = (1-\lambda)\|u\|^2 + \lambda\|v\|^2. \tag{1.4}$$

Proof. Exercise 1.2. ☐

Lemma 1.8 *Let u, v be in X, and let $\alpha \in \,]0, 1[$. Then*

$$\alpha\big(\|u\|^2 - \|(1-\alpha^{-1})u + \alpha^{-1}v\|^2\big) = \|u\|^2 - (1-\alpha)\alpha^{-1}\|u - v\|^2 - \|v\|^2. \tag{1.5}$$

Proof. Exercise 1.3. □

Proposition 1.9 *Let u, v be in X. Then*

$$\langle u, v\rangle \leqslant 0 \quad \Leftrightarrow \quad (\forall \lambda \in [0,1]) \;\; \|u\| \leqslant \|u - \lambda v\|. \tag{1.6}$$

Proof. Let $\lambda \in \,]0,1]$. Then

$$\|u\| \leqslant \|u - \lambda v\| \Leftrightarrow \|u\|^2 \leqslant \|u - \lambda v\|^2 = \|u\|^2 - 2\lambda \langle u, v\rangle + \lambda^2 \|v\|^2 \tag{1.7a}$$

$$\Leftrightarrow 2\langle u, v\rangle \leqslant \lambda \|v\|^2. \tag{1.7b}$$

We now prove the equivalence (1.6). "$\Rightarrow$": If $\langle u, v\rangle \leqslant 0$, then $2\langle u, v\rangle \leqslant 0 \leqslant \lambda\|v\|^2$ and so (1.7) yields $\|u\| \leqslant \|u - \lambda v\|$. "$\Leftarrow$": If $(\forall \lambda \in \,]0,1])$ $\|u\| \leqslant \|u - \lambda v\|$, then (1.7) yields $(\forall \lambda \in \,]0,1])$ $2\langle u, v\rangle \leqslant \lambda\|v\|^2$ and taking $\lambda \to 0^+$ yields $2\langle u, v\rangle \leqslant 0$. □

Proposition 1.10 *Let u, v be in X. Then*

$$u = v \quad \Leftrightarrow \quad \langle \cdot, u\rangle \leqslant \langle \cdot, v\rangle. \tag{1.8}$$

Proof. Exercise 1.4. □

1.3 ▪ General Norms

Recall that $\|\cdot\|$ is a *norm* on X if the following hold:

(i) $\|x\| \geqslant 0$, and $\|x\| = 0 \Leftrightarrow x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$;

(iii) **(triangle inequality)** $\|x + y\| \leqslant \|x\| + \|y\|$.

In this course, if we don't say anything, the norm will be the Euclidean norm; however, sometimes results hold true for general norms, and these are worth pointing out.

Associated with any norm is the *closed ball centered at c of radius $\rho \geqslant 0$*,

$$B[c;\rho] := \big\{x \in X \;\big|\; \|x - c\| \leqslant \rho\big\},$$

and the *open ball* $B(c;\rho) := \big\{x \in X \;\big|\; \|x - c\| < \rho\big\}$.

Example 1.11 [25, Example 1.2-3] If $X = \mathbb{R}^n$ and $1 \leqslant p < \infty$, then

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

is the p-norm, and

$$\|x\|_\infty := \max\big\{|x_1|, \ldots, |x_n|\big\}$$

is the max or infinity norm.

Example 1.12 (spectral norm) If $A \in \mathbb{R}^{m\times n}$, then the *operator/spectral norm* of A is given by

$$\|A\|_2 := \max_{\|x\|\leqslant 1} \|Ax\| = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^\mathsf{T} A)},$$

where $\sigma_{\max}(A)$ is the largest singular value of A, which is also equal to the square root of $\lambda_{\max}(A^\mathsf{T} A)$, the largest eigenvalue of $A^\mathsf{T} A$. Note that $\|A\|_2 \leqslant \|A\| = \sqrt{\operatorname{tra}(A^\mathsf{T} A)} = \sqrt{\sum_{i=1}^{\min\{m,n\}} \lambda_i(A^\mathsf{T} A)}$, the Frobenius norm of A.

Example 1.13 (dual norm) Let $\|\cdot\|$ denote any norm on X. Then the *dual norm* on X is defined by

$$\|y\|_* := \max_{\|x\|\leqslant 1} \langle x, y\rangle ;$$

it is also equal to $\max_{\|x\|=1} \langle x, y\rangle$, provided that $X \neq \{0\}$.

Example 1.14 (Euclidean norm is self-dual) Suppose that $\|\cdot\|$ is the Euclidean norm. Then its dual norm is the same as the original norm.

Proof. Exercise 1.5. □

Proposition 1.15 (generalized Cauchy–Schwarz) *If* $\|\cdot\|$ *is any norm on* X*, with dual norm* $\|\cdot\|_*$*, and* x, y *are in* X*, then*

$$|\langle x, y\rangle| \leqslant \|x\|\|y\|_*.$$

Proof. Assume without loss of generality that $x \neq 0$ and set $z := x/\|x\|$. Then $\|z\| = 1$, $\|-z\| = 1$, $\|y\|_* \geqslant \langle z, y\rangle = \langle x, y\rangle/\|x\|$, and $\|y\|_* \geqslant \langle -z, y\rangle = -\langle x, y\rangle/\|x\|$. Hence $\|y\|_* \geqslant \max\{\langle x, y\rangle/\|x\|, -\langle x, y\rangle/\|x\|\} = |\langle x, y\rangle|/\|x\|$ and we are done. □

Example 1.16 Suppose that $X = \mathbb{R}^n$ with the standard Euclidean norm. Then the dual norm of the Q-norm $\sqrt{\langle x, Qx\rangle}$ (see also Example 1.4) is $\sqrt{\langle y, Q^{-1}y\rangle}$, where Q is positive definite.

Example 1.17 [25, Example 2.10-7] The dual norm of the $\|\cdot\|_p$ norm is $\|\cdot\|_q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Example 1.18 The dual norm of

$$\sqrt{\sum_{i=1}^{m} \omega_i \|x_i\|^2},$$

where $\omega_1 > 0, \ldots, \omega_m > 0$ and where $(x_1, \ldots, x_m) \in X_1 \times \cdots \times X_m$ is a Cartesian product of Euclidean spaces, is

$$\sqrt{\sum_{i=1}^{m} \frac{1}{\omega_i} \|y_i\|^2}.$$

Remark 1.19 The three previous examples all show that the Euclidean norm is self-dual: $\|\cdot\|_* = \|\cdot\|$. (Use $Q = \mathrm{Id}$; $p = 2$; or $\omega_i \equiv 1$.)

We conclude this chapter with two well-known facts which we state for future reference.

Fact 1.20 (automatic continuity) [25, Theorems 2.7-8 and 2.7-9] *Let* $A\colon X \to Y$ *be a linear map. Then* A *is continuous.*

Fact 1.21 (all norms are equivalent) [25, Theorem 2.4-5] *Suppose that* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are two norms*[1] *on* X*. Then there exist positive constants* α, β *such that*

$$(\forall x \in X) \quad \alpha\|x\|_1 \leqslant \|x\|_2 \leqslant \beta\|x\|_1.$$

[1] These norms are not necessarily the p-norms from Example 1.17.

Exercises

Exercise 1.1 Provide the details for Proposition 1.1.

Exercise 1.2 Prove (1.4).

Exercise 1.3 Prove (1.5).

Exercise 1.4 Prove Proposition 1.10.

Exercise 1.5 Prove Example 1.14.

Exercise 1.6 (parallelogram law) Let x, y be in X. Show that $\|x+y\|^2+\|x-y\|^2 = 2(\|x\|^2+\|y\|^2)$.

Exercise 1.7 Consider $X = \mathbb{R}^2$ with the norms $\|\cdot\|_1 \colon X \to \mathbb{R}_+ \colon (\xi_1,\xi_2) \mapsto |\xi_1|+|\xi_2|$ and $\|\cdot\|_\infty \colon X \to \mathbb{R}_+ \colon (\xi_1,\xi_2) \mapsto \max\{|\xi_1|,|\xi_2|\}$. Show that neither norm satisfies the parallelogram law (see Exercise 1.6).

Exercise 1.8 Let x, y be in X. Show that the following are equivalent:

(i) $\|y\|^2 + \|x-y\|^2 = \|x\|^2$.

(ii) $\|y\|^2 = \langle x, y\rangle$.

(iii) $\langle y, x-y\rangle = 0$.

(iv) $\|2y - x\| = \|x\|$.

(v) $(\forall \alpha \in [-1,1])\ \|y\| \leqslant \|\alpha x + (1-\alpha)y\|$.

(vi) $(\forall \alpha \in \mathbb{R})\ \|y\| \leqslant \|\alpha x + (1-\alpha)y\|$.

Exercise 1.9 Let x and y be nonzero vectors. Define the angle α between x and y, where 0 is thought of as the vertex, by

$$\cos(\alpha) := \frac{\langle x, y\rangle}{\|x\|\|y\|} \tag{1.9}$$

and also set

$$\sin\alpha := \begin{cases} \frac{\sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2}}{\|x\|\|y\|} & \text{if } \cos(\alpha) \geqslant 0; \\ \frac{-\sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2}}{\|x\|\|y\|} & \text{if } \cos(\alpha) \leqslant 0. \end{cases} \tag{1.10}$$

Show that $\cos^2(\alpha) + \sin^2(\alpha) = 1$.

Exercise 1.10 Let x and y be linearly independent. Then $\delta := \|x\|^2\|y\|^2 - \langle x, y\rangle^2 > 0$ by Cauchy–Schwarz; thus, the point

$$z := \frac{\|y\|^2 \langle x-y, x\rangle x + \|x\|^2 \langle y, y-x\rangle y}{2\big(\|x\|^2\|y\|^2 - \langle x,y\rangle^2\big)} \tag{1.11}$$

is well defined. Note that $\delta = \|x\|^2\|y-x\|^2 - \langle x, y-x\rangle^2 = \|y\|^2\|x-y\|^2 - \langle y, x-y\rangle^2$. Show that

$$\|0 - z\| = \|x - z\| = \|y - z\| = \frac{\|x\|\|y\|\|x-y\|}{2\sqrt{\delta}} =: R. \tag{1.12}$$

The point z is called the *circumcenter* of the triangle $\operatorname{conv}\{0, x, y\}$, and R is the *circumradius*. Note that in the acute case,

$$2R = \frac{\|x\|\|y\|\|x-y\|}{\sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}} = \frac{\|x-y\|}{\sin(\alpha)}, \tag{1.13}$$

where α is the angle between x and y, with 0 as the vertex. This yields the *Law of Sines*. The area of a triangle is

$$\tfrac{1}{2}\|x\|\|y\|\sin(\alpha) = \tfrac{1}{2}\sqrt{\delta} = \frac{\sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}}{2}. \tag{1.14}$$

Chapter 2

Affine and Convex Sets

2.1 ▪ Convex Sets

Definition 2.1 (affine subspace) Let S be a nonempty subset of X. Then S is an *affine subspace* if

$$\boxed{(\forall x \in S)(\forall y \in S)(\forall \lambda \in \mathbb{R}) \quad (1-\lambda)x + \lambda y \in S.}$$

The smallest affine subspace containing S is the *affine hull* of S, written $\operatorname{aff} S$.

Example 2.2 Let Y be a linear subspace of X. Then clearly Y is also an affine subspace of X. More generally, if $s \in X$, then

$$s + Y := \{s + y \mid y \in Y\}$$

is affine.

Example 2.3 Suppose $A\colon X \to Y$ is linear and $b \in Y$. If $A^{-1}(b) := \{x \in X \mid Ax = b\} \neq \varnothing$, then $A^{-1}(b)$ is an affine subspace.

Definition 2.4 (convex set) Let C be a (possibly empty) subset of X. Then C is *convex* if

$$\boxed{(\forall x \in C)(\forall y \in C)(\forall \lambda \in [0,1]) \quad (1-\lambda)x + \lambda y \in C.}$$

Note that this is equivalent to requiring that $(\forall x \in C)(\forall y \in C)(\forall \lambda \in \,]0,1[)\ (1-\lambda)x + \lambda y \in C$. The *convex hull* of C, written $\operatorname{conv} C$, is the smallest convex superset containing C (this is well defined in view of Theorem 2.13 below!).

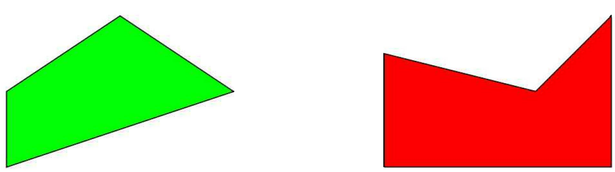


Figure 2.1. *The green set on the left is convex, while the red set on the right is not convex.*

Example 2.5 (line segment) If a and b are in X, then the *line segment*

$$[a,b] := \big\{(1-\lambda)a + \lambda b \;\big|\; 0 \leqslant \lambda \leqslant 1\big\}$$

is convex.

Example 2.6 $\varnothing$ and X are convex. If S is an affine subspace, then S is convex.

Example 2.7 (closed ball) If $c \in X$, $\rho \geqslant 0$, and $\|\cdot\|$ denotes *any* norm, then the closed ball $B[c;\rho] = \big\{x \in X \;\big|\; \|x-c\| \leqslant \rho\big\}$ is convex.

Example 2.8 (closed halfspace) Let $a \in X \smallsetminus \{0\}$ and let $\beta \in \mathbb{R}$. Then the *halfspace*

$$\big\{x \in X \;\big|\; \langle x, a\rangle \leqslant \beta\big\}$$

is convex.

Example 2.9 (nonnegative orthant) The *nonnegative orthant*

$$\mathbb{R}^n_+ = \big\{x \in \mathbb{R}^n \;\big|\; \text{each } x_i \geqslant 0\big\}$$

is convex but not affine. The *positive orthant* $\mathbb{R}^n_{++} := \big\{x \in \mathbb{R}^n \;\big|\; \text{each } x_i > 0\big\}$ is convex but not affine.

Example 2.10 (probability simplex) The *probability simplex*

$$\boxed{\mathbb{P}_n := \big\{x \in \mathbb{R}^n \;\big|\; \textstyle\sum_{i=1}^n x_i = 1 \text{ and each } x_i \geqslant 0\big\}}$$

is convex but not affine (unless $n = 1$).

Proof. Take x, y in $\mathbb{P}_n$, and let $\lambda \in [0,1]$. Then each $x_i \geqslant 0$, each $y_i \geqslant 0$, and $\sum_{k=1}^n x_k = 1 = \sum_{k=1}^n y_k$.

Now set $z := (1-\lambda)x + \lambda y$. Note that $\lambda \geqslant 0$ and $1 - \lambda \geqslant 0$; thus,

$$z_i = \underbrace{(1-\lambda)}_{\geqslant 0}\underbrace{x_i}_{\geqslant 0} + \underbrace{\lambda}_{\geqslant 0}\underbrace{y_i}_{\geqslant 0} \geqslant 0.$$

Moreover,

$$\begin{aligned}\sum_{k=1}^n z_k &= \sum_{k=1}^n \big((1-\lambda)x_k + \lambda y_k\big) = (1-\lambda)\sum_{k=1}^n x_k + \lambda \sum_{k=1}^n y_k \\ &= (1-\lambda)\cdot 1 + \lambda \cdot 1 = 1.\end{aligned}$$

Hence $z \in \mathbb{P}_n$ and we are done. $\square$

Example 2.11 (box) Suppose that $l \in [-\infty, +\infty]^n$ and $r \in [-\infty, +\infty]^n$. Then the (possibly empty or unbounded) *box*

$$\big\{x \in \mathbb{R}^n \;\big|\; \text{each } l_i \leqslant x_i \leqslant r_i\big\}$$

is convex.

Example 2.12 (positive (semi)definite matrices) In $\mathbb{S}^n$, the set of real symmetric $n \times n$ matrices, the sets

$$\boxed{\mathbb{S}^n_+ := \{A \in \mathbb{S}^n \mid A \succeq 0\} \text{ and } \mathbb{S}^n_{++} := \{A \in \mathbb{S}^n \mid A \succ 0\}}$$

are convex. Here we recall that $A \in \mathbb{S}^n$ is *positive definite* if $(\forall x \in \mathbb{R}^n \smallsetminus \{0\})$ $\langle x, Ax\rangle > 0$, while $A \in \mathbb{S}^n$ is *positive semidefinite* if $(\forall x \in \mathbb{R}^n)$ $\langle x, Ax\rangle \geqslant 0$.

Theorem 2.13 *The intersection of any collection of convex sets is convex.*

Proof. Suppose $(\forall i \in I)$ C_i is convex, and set

$$C := \bigcap_{i\in I} C_i.$$

If $C = \varnothing$, then C is trivially convex. So suppose that $C \neq \varnothing$. Let $(x, y) \in C \times C$ and $\lambda \in [0, 1]$. Because each C_i is convex, we learn that

$$(\forall i \in I) \quad (1-\lambda)x + \lambda y \in C_i,$$

which implies that $(1-\lambda)x + \lambda y \in C$ and we are done. ☐

Corollary 2.14 *Let I be an index set (finite or infinite), let $(a_i)_{i\in I}$ be a family of vectors in X, and let $(\beta_i)_{i\in I}$ be a family of real numbers. Then*

$$\{x \in X \mid (\forall i \in I)\ \langle x, a_i\rangle \leqslant \beta_i\}$$

is convex.

Proof. Combine Example 2.8 with Theorem 2.13. ☐

2.2 ▪ Convex Combinations

Definition 2.15 (convex combination) The linear combination of vectors

$$\lambda_1 x_1 + \cdots + \lambda_n x_n$$

is called a *convex combination* if $n \in \{1, 2, 3, \ldots\}$, each x_i belongs to X, each $\lambda_i \in [0, 1]$, and $\lambda_1 + \cdots + \lambda_n = 1$.

Theorem 2.16 *Let C be a nonempty subset of X. Then C is convex $\Leftrightarrow$ C contains all of its convex combinations of vectors in C.*

Proof. "$\Leftarrow$": Let $x \in C$, $y \in C$, and $\lambda \in [0, 1]$. Then $(1-\lambda)x + \lambda y$ is a convex combination of two vectors in C; hence, it lies in C.

"$\Rightarrow$": Suppose that C is convex. We show that each convex combination of n vectors in C lies in C, where $n \geqslant 1$. This is trivial for $n = 1$, and clear for $n = 2$ by convexity. Now assume we have all convex combinations of n vectors lying in C, where $n \geqslant 2$. Consider $n+1$ points $x_1, x_2, \ldots, x_n, x_{n+1}$ drawn from C, and let $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0, \ldots, \lambda_{n+1} \geqslant 0$ with $\lambda_1 + \cdots + \lambda_{n+1} = 1$. Our goal is to show that

$$z := \sum_{i=1}^{n+1} \lambda_i x_i \overset{?}{\in} C.$$

We may and do assume without loss of generality that all $\lambda_i > 0$. Because $n+1 \geqslant 3$, it follows that all $\lambda_i < 1$. Then

$$z = \Big(\sum_{i=1}^{n} \lambda_i x_i\Big) + \lambda_{n+1} x_{n+1} = (1-\lambda_{n+1})\Big(\sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_{n+1}} x_i\Big) + \lambda_{n+1} x_{n+1}.$$

Now observe that $\sum_{i=1}^{n} \lambda_i = (\sum_{i=1}^{n+1} \lambda_i) - \lambda_{n+1} = 1 - \lambda_{n+1}$. Hence

$$\sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_{n+1}} = 1;$$

in turn this and the inductive hypothesis imply that

$$y := \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_{n+1}} x_i \in C.$$

It follows from the convexity of C and the definition of z that

$$z = (1-\lambda_{n+1})y + \lambda_{n+1} x_{n+1} \in C,$$

which completes the proof. ☐

Theorem 2.17 *Let S be a nonempty subset of X. Then* conv S *is equal to the set of all convex combinations of elements of S.*

Proof. Let D be the set containing all convex combinations of the form

$$\sum_{i \in I} \lambda_i x_i,$$

where I is a nonempty finite index set, each $x_i \in S$, each $\lambda_i \geqslant 0$, and $\sum_{i \in I} \lambda_i = 1$.

"conv $S \subseteq D$": Clearly, $S \subseteq D$. We now show that D is convex — this implies that conv $S \subseteq D$. To this end, let d and e be in D, and let $\gamma \in [0,1]$. Then there exist $x_1, \ldots, x_m$ in S and $\lambda_1 \geqslant 0, \ldots, \lambda_m \geqslant 0$ such that $\sum_{i=1}^{m} \lambda_i = 1$ and $d = \sum_{i=1}^{m} \lambda_i x_i$. Similarly, there exist $y_1, \ldots, y_n$ in S and $\mu_1 \geqslant 0, \ldots, \mu_n \geqslant 0$ such that $\sum_{j=1}^{n} \mu_j = 1$ and $e = \sum_{j=1}^{n} \mu_j y_j$. Then

$$(1-\gamma)d + \gamma e = (1-\gamma)\sum_{i=1}^{m} \lambda_i x_i + \gamma \sum_{j=1}^{n} \mu_j y_j = \sum_{i=1}^{m} \big((1-\gamma)\lambda_i\big) x_i + \sum_{j=1}^{n} (\gamma \mu_j) y_j;$$

moreover all coefficients in the last linear combination are nonnegative, and the coefficients sum up to 1. Put differently, we have revealed $(1-\gamma)d + \gamma e$ to be a convex combination of $x_1, \ldots, x_m, y_1, \ldots, y_n$. By hypothesis, $(1-\gamma)d + \gamma e \in D$. We have shown that conv $S \subseteq D$.

"conv $S \supseteq D$": It is clear that $S \subseteq$ conv S. By Theorem 2.16, $D \subseteq$ conv S. ☐

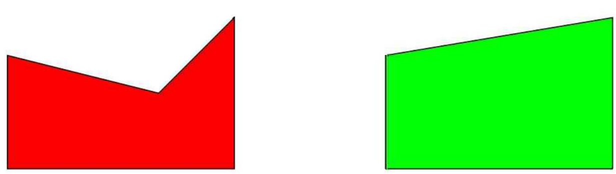


Figure 2.2. *The red set on the left is not convex — its convex hull is the green set on the right.*

2.3 ▪ Relative Interior and the Accessibility Lemma

Recall that the *interior* of C is given by

$$\operatorname{int} C = \big\{x \in X \,\big|\, (\exists\, \rho > 0) \;\; B[x;\rho] \subseteq C\big\}. \tag{2.1}$$

We have the following refinement, the so-called *relative interior*

$$\operatorname{ri} C := \big\{x \in C \,\big|\, (\exists\, \rho > 0) \;\; \operatorname{aff}(C) \cap B[x;\rho] \subseteq C\big\}. \tag{2.2}$$

Clearly, $\operatorname{int} C \subseteq \operatorname{ri} C$; however, the converse is false.

Example 2.18 Consider $C = \mathbb{R} \times \{0\}$ in $\mathbb{R}^2$. Then $\operatorname{int} C = \varnothing$ but $\operatorname{ri} C = C$.

Fact 2.19 [39, Theorem 6.2] *Let C be a nonempty convex subset of X. Then* $\operatorname{ri} C \neq \varnothing$.

Proposition 2.20 *Let C be a convex subset of X such that* $\operatorname{int} C \neq \varnothing$. *Then* $\operatorname{aff}(C) = X$.

Proof. Let $x \in X$. It suffices to show that $x \in \operatorname{aff}(C)$. If $x \in C$, then $x \in \operatorname{aff}(C)$, as claimed. So assume that $x \notin C$. Because $\operatorname{int} C \neq \varnothing$, there exists $x_0 \in \operatorname{int} C$, say $B[x_0;\delta] \subseteq C$, where $\delta > 0$. Then

$$c_0 := x_0 + \delta \frac{x - x_0}{\|x - x_0\|} \in C.$$

But then

$$\begin{aligned} x = x_0 + (x - x_0) = x_0 + \frac{\|x - x_0\|}{\delta}(c_0 - x_0) &= \Big(1 - \frac{\|x - x_0\|}{\delta}\Big)x_0 + \frac{\|x - x_0\|}{\delta} c_0 \\ &\in \operatorname{aff}\{x_0, c_0\} \subseteq \operatorname{aff}(C) \end{aligned}$$

and we are done. □

Corollary 2.21 *Suppose C is a convex subset of X such that* $\operatorname{int} C \neq \varnothing$. *Then* $\operatorname{ri} C = \operatorname{int} C$.

Lemma 2.22 (Accessibility Lemma) *Let C be a convex subset of X. Suppose that* $x_0 \in \operatorname{int} C$ *and $x_1 \in \overline{C}$. Then*

$$(\forall \lambda \in [0,1[) \quad x_\lambda := (1-\lambda)x_0 + \lambda x_1 \in \operatorname{int} C.$$

Proof. It will be convenient to work with $B := B[0;1]$. Let $\lambda \in [0,1[$. Because $x_0 \in \operatorname{int} C$, there exists $\varepsilon > 0$ such that

$$x_0 + \tfrac{1+\lambda}{1-\lambda}\varepsilon B \subseteq C.$$

Because $x_1 \in \overline{C}$, we have $x_1 \in C + \varepsilon B$. Therefore,

$$\begin{aligned} x_\lambda + \varepsilon B &= (1-\lambda)x_0 + \lambda x_1 + \varepsilon B \\ &\subseteq (1-\lambda)x_0 + \lambda(C + \varepsilon B) + \varepsilon B \\ &= (1-\lambda)x_0 + (1+\lambda)\varepsilon B + \lambda C \\ &= (1-\lambda)\big(x_0 + \tfrac{1+\lambda}{1-\lambda}\varepsilon B\big) + \lambda C \\ &\subseteq (1-\lambda)C + \lambda C \\ &\subseteq C; \end{aligned}$$

hence, $x_\lambda \in \operatorname{int} C$ and we are done. □

Remark 2.23 Similarly to Lemma 2.22, one can prove that if $x_0 \in \operatorname{ri} C$ and $x_1 \in \overline{C}$, then $(1-\lambda)x_0 + \lambda x_1 \in \operatorname{ri} C$ for $0 \leqslant \lambda < 1$.

Exercises

Exercise 2.1 Provide the details for Example 2.2.

Exercise 2.2 Suppose that Y is an affine subspace of X and that $s \in Y$. Show that $L := Y - s$ is a linear subspace of X.

Exercise 2.3 Suppose that Y is an affine subspace of X and let y_1, y_2, y_3 be in Y, and let $\lambda_1, \lambda_2, \lambda_3$ be in $\mathbb{R}$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Show that $\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 \in Y$ and further deduce that $Y + Y - Y = Y$.

Exercise 2.4 Suppose that Y is an affine subspace of X. We saw in Exercise 2.2 that $Y - s$ is a linear subspace whenever $s \in Y$. Show that this linear subspace does not depend on the choice of s, and deduce that

$$(\forall s_1 \in Y)(\forall s_2 \in Y) \quad Y - s_1 = Y - s_2 = Y - Y.$$

The set $Y - Y$ is also called the *parallel space* of Y.

Exercise 2.5 Verify Example 2.3.

Exercise 2.6 Verify Example 2.5 and also characterize when $[a, b]$ is affine.

Exercise 2.7 Verify Example 2.7.

Exercise 2.8 (open ball) If $c \in X$, $\rho \geqslant 0$, and $\|\cdot\|$ denotes *any* norm, then the open ball $B(c;\rho) = \big\{x \in X \mid \|x - c\| < \rho\big\}$ is convex.

Exercise 2.9 Provide the details for Example 2.8.

Exercise 2.10 (open halfspace) Let $a \in X \smallsetminus \{0\}$ and let $\beta \in \mathbb{R}$. Show that the *open halfspace* $\big\{x \in X \mid \langle x, a\rangle < \beta\big\}$ is convex.

Exercise 2.11 (complement of a hyperplane) Suppose that $a \in X \smallsetminus \{0\}$ and let $\beta \in \mathbb{R}$. When is $\big\{x \in X \mid \langle x, a\rangle \neq \beta\big\}$ convex?

Exercise 2.12 Verify Example 2.9.

Exercise 2.13 Verify Example 2.11.

Exercise 2.14 Verify Example 2.12.

Exercise 2.15 Show that when $X \neq \{0\}$, then the union of two convex sets may fail to be convex.

Exercise 2.16 What is the relative interior of the subset $\{0, 1\}$ of $\mathbb{R}$? Comment on Fact 2.19.

Exercise 2.17 Suppose that C and D are convex subsets of X. Show that their sum $C + D = \big\{c + d \bigm| c \in C,\ d \in D\big\}$ is also convex.

Exercise 2.18 Suppose that C and D are closed subsets of X. Show that if D is compact, then their sum $C + D = \big\{c + d \bigm| c \in C,\ d \in D\big\}$ is closed.

Exercise 2.19 Find two closed convex subsets C, D of X such that $C + D$ is not closed.

Exercise 2.20 Let C be nonempty and convex and let $\alpha \geq 0$. Suppose that $(\forall x \in C)\ \|x\| = \alpha$. Prove that C is singleton.

Exercise 2.21 Suppose that C and D are convex subsets of X. Let α, β be in $\mathbb{R}$. Show that the sum $\alpha C + \beta D = \big\{\alpha c + \beta d \;\big|\; c \in C,\ d \in D\big\}$ is also convex.

Exercise 2.22 Suppose that C and D are subsets of X and that $\varnothing \neq C \subseteq D$. Prove that $\operatorname{aff} C \subseteq \operatorname{aff} D$.

Exercise 2.23 Suppose that C and D are subsets of X and that $C \subseteq D$. Prove that $\operatorname{conv} C \subseteq \operatorname{conv} D$.

Exercise 2.24 Suppose that C and D are subsets of X and that $C \subseteq D$. Is it true that $\operatorname{ri} C \subseteq \operatorname{ri} D$? Justify your answer.

Chapter 3

Convex and Lower Semicontinuous Functions

3.1 ▪ Convex Functions

Definition 3.1 (epigraph) Let $f\colon X \to [-\infty,+\infty]$. Then the *epigraph* is defined by

$$\operatorname{epi} f := \big\{(x,\rho) \in X \times \mathbb{R} \,\big|\, f(x) \leqslant \rho\big\}.$$

The *(effective) domain* of f is

$$\operatorname{dom} f := \big\{x \in X \,\big|\, f(x) < +\infty\big\};$$

f is called *proper* if $\operatorname{dom} f \neq \varnothing$ and every $x \in X$ satisfies $-\infty < f(x)$.

Example 3.2 Any function $f\colon X \to \mathbb{R}$ is proper.

Remark 3.3 If a function f is defined only on a subset D of X, then we can and often do identify f with its extension

$$x \mapsto \begin{cases} f(x) & \text{if } x \in D; \\ +\infty & \text{if } x \notin D. \end{cases}$$

Example 3.4 (indicator function) Let C be a subset of X. The *indicator function* of C is defined by

$$\iota_C\colon X \to \left]-\infty,+\infty\right] : x \mapsto \begin{cases} 0 & \text{if } x \in C; \\ +\infty & \text{if } x \notin C. \end{cases}$$

Note that $\operatorname{dom} \iota_C = C$, and that ι_C is proper $\Leftrightarrow C \neq \varnothing$.

Definition 3.5 (convex function) Let $f\colon X \to [-\infty,+\infty]$. Then f is *convex* if $\operatorname{epi} f$ is a convex set (in the Cartesian product space $X \times \mathbb{R}$).

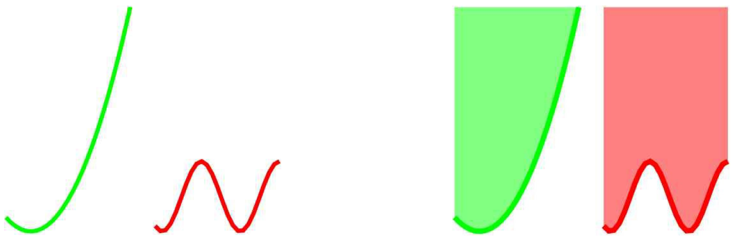


Figure 3.1. *The green function on the left is convex, while the red one is not. On the right, parts of their respective epigraphs are shown.*

While the modern epigraph-based definition of convexity is very elegant, one often works with the following classical description:

Theorem 3.6 (Jensen's inequality) *Let $f\colon X \to [-\infty, +\infty]$. Then f is convex $\Leftrightarrow$*

$$\begin{aligned}&(\forall x \in \operatorname{dom} f)(\forall y \in \operatorname{dom} f)(\forall \lambda \in]0,1[)\\ &\qquad f\big((1-\lambda)x + \lambda y\big) \leqslant (1-\lambda)f(x) + \lambda f(y).\end{aligned} \tag{3.1}$$

Proof. If $\operatorname{dom} f = \varnothing$, then $f \equiv +\infty$, $\operatorname{epi} f = \varnothing$, and (3.1) holds.

We now assume that $\operatorname{dom} f \neq \varnothing$.

"$\Rightarrow$": Let x, y be in $\operatorname{dom} f$, let $\lambda \in]0,1[$, and take $(x,\xi), (y,\eta) \in \operatorname{epi} f$. By convexity of $\operatorname{epi} f$, we have

$$\big((1-\lambda)x + \lambda y, (1-\lambda)\xi + \lambda\eta\big) = (1-\lambda)\big(x,\xi\big) + \lambda\big(y,\eta\big) \in \operatorname{epi} f;$$

thus,

$$f\big((1-\lambda)x + \lambda y\big) \leqslant (1-\lambda)\xi + \lambda\eta.$$

Letting $\xi \to f(x)$ and $\eta \to f(y)$, we obtain (3.1).

"$\Leftarrow$": Let $(x,\alpha) \in \operatorname{epi} f$, $(y,\beta) \in \operatorname{epi} f$, and $\lambda \in]0,1[$. Then $f(x) \leqslant \alpha$, $f(y) \leqslant \beta$, and $1-\lambda > 0$. Hence

$$\begin{aligned} f\big((1-\lambda)x + \lambda y\big) &\overset{(3.1)}{\leqslant} (1-\lambda)f(x) + \lambda f(y) &\quad (3.2\text{a})\\ &\leqslant (1-\lambda)\alpha + \lambda\beta &\quad (3.2\text{b})\end{aligned}$$

and thus

$$(1-\lambda)(x,\alpha) + \lambda(y,\beta) = \big((1-\lambda)x + \lambda y, (1-\lambda)\alpha + \lambda\beta\big) \overset{(3.2)}{\in} \operatorname{epi} f.$$

Therefore, $\operatorname{epi} f$ is convex. $\square$

Corollary 3.7 *Let $f\colon X \to [-\infty, +\infty]$ be convex, and let $\alpha \in \mathbb{R}$. Then $\operatorname{dom} f$ and $\big\{x \in X \mid f(x) \leqslant \alpha\big\}$ are convex.*

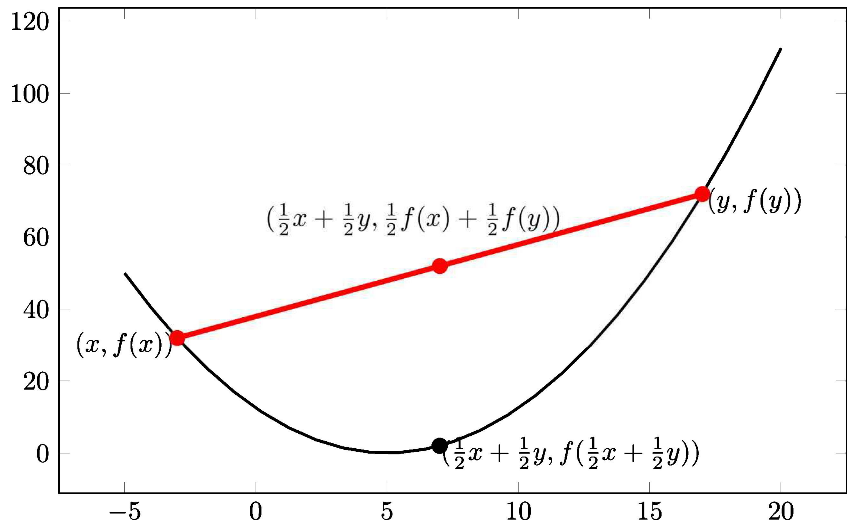


Figure 3.2. *Jensen's inequality says that the line segment connecting any two points on the graph is always on or above the graph. Illustrated here is the case when* $\lambda = \frac{1}{2}$.

Proof. We may and do assume without loss of generality that $\operatorname{dom} f \neq \varnothing$. Take x, y in $\operatorname{dom} f$, and let $\lambda \in \,]0,1[$. By Theorem 3.6,

$$f\big((1-\lambda)x + \lambda y\big) \leqslant (1-\lambda)f(x) + \lambda f(y) < +\infty;$$

hence, $(1-\lambda)x + \lambda y \in \operatorname{dom} f$.

Now set $C := \big\{x \in X \;\big|\; f(x) \leqslant \alpha\big\}$ and note that $C \subseteq \operatorname{dom} f$. Arguing as before for x, y in C and $\lambda \in \,]0,1[$, $f((1-\lambda)x + \lambda y) \leqslant (1-\lambda)f(x) + \lambda f(y) \leqslant (1-\lambda)\alpha + \lambda\alpha = \alpha$, and thus $(1-\lambda)x + \lambda y \in C$. ☐

Example 3.8 The functions $\|\cdot\|$ and $\|\cdot\|^2$ are convex. Indeed, the former function is a norm, and the latter is convex by (1.4).

Remark 3.9 We know from calculus that a twice differentiable function f is convex if and only if $f'' \geqslant 0$; this quickly leads to additional examples. For more details, see the upcoming Theorem 11.10.

3.2 ▪ Strictly Convex Functions

Definition 3.10 Let $f\colon X \to \,]-\infty, +\infty]$ be proper. Then f is *strictly convex* if

$$\boxed{\left.\begin{array}{c} x \in \operatorname{dom} f \\ y \in \operatorname{dom} f \\ x \neq y \\ \lambda \in \,]0,1[\end{array}\right\} \;\Rightarrow\; f\big((1-\lambda)x + \lambda y\big) < (1-\lambda)f(x) + \lambda f(y).}$$

Example 3.11 The Euclidean norm $\|\cdot\|^2$ is strictly convex by (1.4).

Example 3.12 If $X \neq \{0\}$, then no norm on X is strictly convex.

Proposition 3.13 *Let $f\colon X \to\,]-\infty,+\infty]$ be strictly convex and proper. Then f has either no or a unique minimizer.*

Proof. Suppose that x_0 and x_1 are two distinct minimizers: $x_0 \neq x_1$. Then $f(x_0) = f(x_1) = \min f(X) =: \mu$. However, by strict convexity, we obtain $\mu \leqslant f(\frac{1}{2}x_0 + \frac{1}{2}x_1) < \frac{1}{2}f(x_0) + \frac{1}{2}f(x_1) = \mu$, which is absurd! Hence f has at most one minimizer. ☐

3.3 ▪ Lower Semicontinuous Functions

Definition 3.14 Let $f\colon X \to [-\infty,+\infty]$. Then f is *lower semicontinuous (lsc)* at $x \in X$ if, for every sequence $(x_n)_{n\in\mathbb{N}}$ in X, we have

$$\boxed{\lim_{n\in\mathbb{N}} x_n = x \quad\Rightarrow\quad f(x) \leqslant \varliminf_{n\in\mathbb{N}} f(x_n).}$$

Moreover, f is simply lsc if f is lsc at every point $x \in X$.

Remark 3.15 Clearly, if f is continuous at x, then f is lsc at x. The converse is false in general.

Recall that a subset S of X is *closed* if

$$\left.\begin{array}{c}(s_n)_{n\in\mathbb{N}} \text{ lies in } S\\ s_n \to x \in X\end{array}\right\} \quad\Rightarrow\quad x \in S.$$

Example 3.16 Consider $f\colon X \to [-\infty,+\infty]$ defined by

$$f(x) = \begin{cases} +\infty & \text{if } x \leqslant 0;\\ \dfrac{1}{x} & \text{if } x > 0.\end{cases} \tag{3.3}$$

Then f is lower semicontinuous yet $\operatorname{dom} f = \,]0,+\infty[$ is not closed.

Definition 3.17 (lower level set) Let $f\colon X \to [-\infty,+\infty]$, and let $\alpha \in \mathbb{R}$. The *lower level set* of f at height α is

$$\boxed{\operatorname{lev}_\alpha f := \big\{x \in X \;\big|\; f(x) \leqslant \alpha\big\}.}$$

Theorem 3.18 *Let $f\colon X \to [-\infty,+\infty]$. Then the following are equivalent:*

(i) *f is lower semicontinuous.*

(ii) *f is* closed, *i.e., epi f is closed.*

(iii) $(\forall\alpha \in \mathbb{R})$ $\operatorname{lev}_\alpha f$ *is closed.*

Proof. "(i)⇒(ii)": Suppose that f is lsc, and suppose that $(x_n,\alpha_n)_{n\in\mathbb{N}}$ is a sequence in epi f such that $(x_n,\alpha_n) \to (x,\alpha) \in X \times \mathbb{R}$. Because $x_n \to x$, $\alpha_n \to \alpha$, and f is lsc at x, we have

$$f(x) \leqslant \varliminf_{n\in\mathbb{N}} f(x_n) \leqslant \varliminf_{n\in\mathbb{N}} \alpha_n = \alpha. \tag{3.4}$$

Hence $(x,\alpha) \in$ epi f and thus epi f is closed.

"(ii)⇒(iii)": Suppose epi f is closed, and let $\alpha \in \mathbb{R}$. Suppose that $(x_n)_{n\in\mathbb{N}}$ lies in $\operatorname{lev}_\alpha f$ such that $x_n \to x \in X$. Then $(\forall n \in \mathbb{N})$ $f(x_n) \leqslant \alpha$, i.e., $(x_n, \alpha) \in \operatorname{epi} f$. Clearly $(x_n, \alpha) \to (x, \alpha)$. Because epi f is closed, we deduce that $(x, \alpha) \in \operatorname{epi} f$, i.e., $f(x) \leqslant \alpha$, i.e., $x \in \operatorname{lev}_\alpha f$. Hence $\operatorname{lev}_\alpha f$ is closed.

"(iii)⇒(i)": Suppose that (iii) holds. We argue by contradiction and assume that (i) fails. Then there exist $x \in X$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$x_n \to x \quad \text{yet} \quad f(x) > \varliminf_{n\in\mathbb{N}} f(x_n). \tag{3.5}$$

Pick $\alpha \in \mathbb{R}$ such that

$$f(x) > \alpha > \varliminf_{n\in\mathbb{N}} f(x_n). \tag{3.6}$$

Hence there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$(\forall n \in \mathbb{N}) \quad \alpha > f(x_{k_n}). \tag{3.7}$$

Hence $(x_{k_n})_{n\in\mathbb{N}}$ lies in $\operatorname{lev}_\alpha f$, which is a closed set by assumption. Therefore $x_{k_n} \to x \in \operatorname{lev}_\alpha f$, i.e., $f(x) \leqslant \alpha$ and this contradicts (3.6). Thus f is indeed lsc. ☐

Proposition 3.19 *Let $f\colon X \to \left]-\infty, +\infty\right]$, suppose that $C := \operatorname{dom} f$ is closed, and that $f|_C$ is continuous. Then f is lower semicontinuous.*

Proof. Suppose that $(x_n, \rho_n)_{n\in\mathbb{N}}$ is a sequence in epi f such that $(x_n, \rho_n) \to (x, \rho) \in X \times \mathbb{R}$. Then $(\forall n \in \mathbb{N})$ $f(x_n) \leqslant \rho_n$. Note that $(x_n)_{n\in\mathbb{N}}$ lies in C. Because C is closed, $x \in C$. And because $f|_C$ is continuous, we have $f(x_n) \to f(x)$. Therefore, $f(x) \leftarrow f(x_n) \leqslant \rho_n \to \rho$ and so $f(x) \leqslant \rho$, i.e., $(x, \rho) \in \operatorname{epi} f$. Thus epi f is closed and we are done by Theorem 3.18. ☐

Example 3.20 Let C be a subset of X. Then the following hold:

(i) ι_C is proper ⇔ $C \neq \varnothing$.

(ii) ι_C is convex ⇔ C is convex.

(iii) ι_C is lower semicontinuous ⇔ C is closed.

Proof. Note that

$$\operatorname{epi} \iota_C = C \times \mathbb{R}_+. \tag{3.8}$$

(i)&(ii): Exercise 3.5.

(iii): In view of (3.8) and Theorem 3.18, it suffices to show that $C \times \mathbb{R}_+$ is closed ⇔ C is closed. But this equivalence follows easily because $\mathbb{R}_+$ is already closed: indeed, suppose first that $C \times \mathbb{R}_+$ is closed. Take a sequence $(c_n)_{n\in\mathbb{N}}$ in C such that $c_n \to x \in X$. Then $(c_n, 0)_{n\in\mathbb{N}}$ lies in $C \times \mathbb{R}_+$ and $(c_n, 0) \to (x, 0)$. Hence $(x, 0) \in C \times \mathbb{R}_+$ and thus $x \in C$. The converse is proved similarly. ☐

Remark 3.21 The indicator function is of fundamental importance because it allows us to model constraints with ease. To illustrate, suppose our problem is to

$$\text{minimize } f(x) \text{ subject to } x \in C. \tag{3.9}$$

But this is the same as

$$\text{minimize } (f + \iota_C)(x) \text{ subject to } x \in X. \tag{3.10}$$

Note that (3.9) is a constrained minimization problem, while (3.10) is unconstrained; however, the constraint is absorbed in the objective function $f + \iota_C$, which is nonsmooth even if the original function f is smooth.

Exercises

Exercise 3.1 Consider Corollary 3.7. Is the converse also true, i.e., if $\operatorname{dom} f$ is convex, must f be convex?

Exercise 3.2 Verify Example 3.12.

Exercise 3.3 This problem is related to Proposition 3.13. Provide (i) a strictly convex function with no minimizer, and (ii) a function that is not strictly convex with a unique minimizer.

Exercise 3.4 Consider Remark 3.15. Provide a function $f\colon \mathbb{R} \to \mathbb{R}$ that is lsc everywhere, but not continuous everywhere.

Exercise 3.5 Provide the details for the proof of Example 3.20(i)&(ii).

Chapter 4

More on Convex and Lower Semicontinuous Functions

4.1 ▪ Preservation

Proposition 4.1 (sum) *Let I be a nonempty finite index set, and let f_i be proper functions from X to $]-\infty, +\infty]$ for each $i \in I$. Then the following hold:*

(i) *If each f_i is convex, then so is $\sum_{i\in I} f_i$.*

(ii) *If each f_i is lower semicontinuous, then so is $\sum_{i\in I} f_i$.*

Proof. The proof is relatively straightforward. For (i), work with Theorem 3.6 (Jensen). For (ii), work with the definition of lower semicontinuity. □

Remark 4.2 Taking the difference generally preserves neither convexity nor lower semicontinuity: (i) 0 and $|x|$ are convex functions, but $0 - |x| = -|x|$ is not; (ii) 0 and $\iota_{\{0\}}$ are lower semicontinuous, but $0 - \iota_{\{0\}} = -\iota_{\{0\}}$ is not.

Proposition 4.3 (separable sum) *Let $X_1, \ldots, X_m$ be finite-dimensional Euclidean spaces and let $f_1, \ldots, f_m$ be lower semicontinuous on $X_1, \ldots, X_m$, respectively. Then*

$$f_1 \oplus \cdots \oplus f_m \colon (x_1, \ldots, x_m) \mapsto \sum_{i=1}^m f_i(x_i)$$

is also lower semicontinuous.

Proof. Suppose that $(x_{1,n}, \ldots, x_{m,n})_{n\in\mathbb{N}}$ converges to $(x_1, \ldots, x_m)$. Then

$$\begin{aligned}
\big(f_1 \oplus \cdots \oplus f_m\big)(x_1, \ldots, x_m) &= \sum_{i=1}^m f_i(x_i) \\
&\leqslant \sum_{i=1}^m \varliminf_{n\in\mathbb{N}} f_i(x_{i,n}) \\
&\leqslant \varliminf_{n\in\mathbb{N}} \sum_{i=1}^m f_i(x_{i,n}) \\
&= \varliminf_{n\in\mathbb{N}} \big(f_1 \oplus \cdots \oplus f_m\big)(x_{1,n}, \ldots, x_{m,n}),
\end{aligned}$$

and we are done. □

Example 4.4 On $\mathbb{R}^m$, set $f(x) := \operatorname{card}\big\{i \mid x_i \neq 0\big\}$, where card denotes the cardinality of a set. Then $f = f_1 \oplus \cdots \oplus f_m$, where each f_i defined by $f_i(\xi) = 0$ if $\xi = 0$; $f_i(\xi) = 1$ if $\xi \neq 0$ is lower semicontinuous. Then f is lower semicontinuous by Proposition 4.3.

Proposition 4.5 (positive multiple) *Let $f\colon X \to [-\infty, +\infty]$, and let $\alpha > 0$. Then the following hold:*

(i) *If f is convex, then so is αf.*

(ii) *If f is lower semicontinuous, then so is αf.*

Proof. The proof is relatively straightforward. For the convexity part, work with Theorem 3.6 (Jensen). ☐

Proposition 4.6 *Let $A\colon X \to Y$ be linear, and let $g\colon Y \to \,]-\infty, +\infty]$. Set $f := g \circ A$. Then the following hold:*

(i) *If g is convex, then so is f.*

(ii) *If g is lower semicontinuous, then so is f.*

Proof. (i): Suppose x, y belong to $\operatorname{dom} f$ and let $\lambda \in \,]0,1[$. Then Ax, Ay belong to $\operatorname{dom} g$. By Jensen's inequality (see Theorem 3.6), we get

$$\begin{aligned} f\big((1-\lambda)x + \lambda y\big) &= g\Big(A\big((1-\lambda)x + \lambda y\big)\Big) = g\big((1-\lambda)Ax + \lambda Ay\big) \\ &\leqslant (1-\lambda)g(Ax) + \lambda g(Ay) = (1-\lambda)f(x) + \lambda f(y). \end{aligned}$$

(ii): Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $x_n \to x$. Because X and Y are finite-dimensional, the linear operator A is continuous (see Fact 1.20). Hence $Ax_n \to Ax$. Using the lower semicontinuity of g at Ax, we obtain

$$f(x) = g(Ax) \leqslant \varliminf_{n\in\mathbb{N}} g(Ax_n) = \varliminf_{n\in\mathbb{N}} f(x_n)$$

and we are done. ☐

Lemma 4.7 *Let I be a nonempty index set, and let $(f_i)_{i\in I}$ be a family of functions from X to $[-\infty, +\infty]$. Then*

$$\operatorname{epi}\Big(\sup_{i\in I} f_i\Big) = \bigcap_{i\in I} \operatorname{epi} f_i.$$

Proof. Let $(x, \rho) \in X \times \mathbb{R}$. Then

$$\begin{aligned} (x,\rho) \in \operatorname{epi}\Big(\sup_{i\in I} f_i\Big) &\Leftrightarrow \sup_{i\in I} f_i(x) \leqslant \rho \Leftrightarrow (\forall i \in I)\ \ f_i(x) \leqslant \rho \\ &\Leftrightarrow (\forall i \in I)\ \ (x,\rho) \in \operatorname{epi} f_i \Leftrightarrow (x,\rho) \in \bigcap_{i\in I} \operatorname{epi} f_i \end{aligned}$$

and we are done. ☐

Corollary 4.8 *Let I be a nonempty index set, and let $(f_i)_{i\in I}$ be a family of functions from X to $[-\infty,+\infty]$. Then the following hold:*

(i) *If each f_i is convex, then so is $\sup_{i\in I} f_i$.*

(ii) *If each f_i is lower semicontinuous, then so is $\sup_{i\in I} f_i$.*

(iii) *If each f_i is convex and lower semicontinuous, then so is $\sup_{i\in I} f_i$.*

Proof. Recall from Lemma 4.7 that

$$\operatorname{epi}\Big(\sup_{i\in I} f_i\Big) = \bigcap_{i\in I} \operatorname{epi} f_i. \tag{4.1}$$

(i): Combine (4.1) with Theorem 2.13.
(ii): Combine (4.1) with Theorem 3.18.
(iii): Combine (i) and (ii). □

Proposition 4.9 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of convex function such that $f_n \to f$ pointwise, i.e., $(\forall x\in X)$ $f_n(x)\to f(x)$. Then f is convex.*

Proof. This follows using Theorem 3.6 (Jensen). □

4.2 ▪ Examples

The following is a great example to show the power of Corollary 4.8:

Example 4.10 (Asplund function) Let S be a nonempty subset of X. Then the function f defined by

$$f(x) := \tfrac{1}{2}\|x\|^2 - \tfrac{1}{2}d_S^2(x)$$

is convex and lower semicontinuous with $\operatorname{dom} f = X$. Indeed, we note that for every $s\in S$, the function $x\mapsto 2\langle x,s\rangle - \|s\|^2$ is (linear $\Rightarrow$) convex and (continuous $\Rightarrow$) lower semicontinuous. Using Corollary 4.8(iii), the function

$$\begin{aligned}
\sup_{s\in S}\big(2\langle x,s\rangle - \|s\|^2\big) &= \|x\|^2 + \sup_{s\in S}\big(-\|x\|^2 + 2\langle x,s\rangle - \|s\|^2\big)\\
&= \|x\|^2 - \inf_{s\in S}\big(\|x\|^2 - 2\langle x,s\rangle + \|s\|^2\big)\\
&= \|x\|^2 - \inf_{s\in S}\|x-s\|^2\\
&= \|x\|^2 - d_S^2(x)
\end{aligned}$$

is convex and lsc. It follows that $\frac{1}{2}\|x\|^2 - \frac{1}{2}d_S^2(x)$ is convex and lsc by Proposition 4.5. The domain statement is clear because S is nonempty and so $\operatorname{dom} d_S = X$.

Even the minimum of just two convex functions may fail to be convex.

Example 4.11 The functions x and $-x$ are linear and hence convex on $\mathbb{R}$; however, $\min\{x,-x\} = -|x|$ is not convex.

On the positive side, we have the following:

Theorem 4.12 (marginal function) *Suppose $F\colon X\times Y\to\,]-\infty,+\infty]$ is convex. Then the* marginal function

$$f\colon X\to[-\infty,+\infty] : x\mapsto \inf F(x,Y) = \inf_{y\in Y} F(x,y)$$

is convex.

Proof. Take x_0 and x_1 in $\operatorname{dom} f$, and let $\lambda \in]0,1[$. Pick ρ_0, ρ_1 in $\mathbb{R}$ such that each $\rho_i > f(x_i)$. Then obtain y_0, y_1 in Y such that each $\rho_i > F(x_i, y_i)$. The convexity of F and Theorem 3.6 yield

$$\begin{aligned} f\big((1-\lambda)x_0 + \lambda x_1\big) &\leqslant F\big((1-\lambda)x_0 + \lambda x_1, (1-\lambda)y_0 + \lambda y_1\big) \\ &= F\big((1-\lambda)(x_0, y_0) + \lambda(x_1, y_1)\big) \\ &\leqslant (1-\lambda)F(x_0, y_0) + \lambda F(x_1, y_1) \\ &< (1-\lambda)\rho_0 + \lambda\rho_1. \end{aligned}$$

Letting ρ_i tend to $f(x_i)$ from the right, we now deduce that $f((1-\lambda)x_0+\lambda x_1) \leqslant (1-\lambda)f(x_0)+\lambda f(x_1)$, which completes the proof. ☐

Exercises

Exercise 4.1 Provide the details of the proof for Proposition 4.1.

Exercise 4.2 Provide the details of the proof for Proposition 4.5.

Exercise 4.3 Verify Proposition 4.9.

Exercise 4.4 Show that the minimum of two lsc functions is again lsc.

Exercise 4.5 Provide an example of a family $(f_i)_{i\in I}$ of lsc functions on $\mathbb{R}$ such that $\inf_{i\in I} f_i$ is not lsc.

Exercise 4.6 Provide a sequence of continuous convex functions $(f_n)_{n\in\mathbb{N}}$ from $\mathbb{R}$ to $\mathbb{R}$ which converge pointwise to some function f that is not lsc.

Chapter 5

Global and Local Minimizers

5.1 ▪ Coercivity and the Existence of Minimizers

Definition 5.1 ((super) coercivity) Let $f \colon X \to [-\infty, +\infty]$. We say that f is *coercive* if $\lim_{\|x\|\to+\infty} f(x) = +\infty$; equivalently,

$$\left.\begin{array}{c}(x_n)_{n\in\mathbb{N}} \text{ lies in } X,\\ \|x_n\| \to +\infty\end{array}\right\} \quad\Rightarrow\quad f(x_n) \to +\infty.$$

And f is *supercoercive* if $\lim_{\|x\|\to+\infty} f(x)/\|x\| = +\infty$; equivalently,

$$\left.\begin{array}{c}(x_n)_{n\in\mathbb{N}} \text{ lies in } X \smallsetminus \{0\},\\ \|x_n\| \to +\infty\end{array}\right\} \quad\Rightarrow\quad \frac{f(x_n)}{\|x_n\|} \to +\infty.$$

Remark 5.2 It is clear that every supercoercive function is coercive. We also note that the definition of coercivity and supercoercivity remains unchanged if we replace the Euclidean norm by another other norm because all norms on X are equivalent (Fact 1.21).

Example 5.3 Suppose that $X = \mathbb{R}$. Then

(i) x^2 is supercoercive;

(ii) $|x|$ is coercive but not supercoercive;

(iii) $\exp(x)$ is not coercive.

Theorem 5.4 (Key Existence Theorem) *Let $f \colon X \to\,]-\infty, +\infty]$ be coercive, lower semicontinuous, and proper. Then f has a (global) minimizer, i.e., there exists $\bar{x} \in X$ such that*

$$f(\bar{x}) = \min f(X).$$

Proof. Because f is proper, we note that

$$\inf f(X) < +\infty. \tag{5.1}$$

Now take a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$f(x_n) \to \inf f(X). \tag{5.2}$$

We claim that $\underline{\lim}_{n\in\mathbb{N}} \|x_n\| < +\infty$, i.e.,

$$(x_n)_{n\in\mathbb{N}} \text{ has a bounded subsequence.} \tag{5.3}$$

Suppose to the contrary that (5.3) fails. Then $\|x_n\| \to +\infty$. By coercivity of f, we have $f(x_n) \to +\infty$. By (5.2), $\inf f(X) = +\infty$; however, this contradicts (5.1). We have thus verified (5.3).

After passing to subsequences if necessary, we may and do assume that $(x_n)_{n\in\mathbb{N}}$ itself is bounded and also (Bolzano–Weierstrass!) convergent, say

$$x_n \to \bar{x}.$$

Using the assumption that f is lsc at $\bar{x}$ and (5.2), we deduce that

$$\inf f(X) \leqslant f(\bar{x}) \leqslant \underline{\lim}_{n\in\mathbb{N}} f(x_n) = \inf f(X).$$

Hence all inequalities are actually equalities, and we see that $f(\bar{x}) = \inf f(X)$, which completes the proof. ☐

Corollary 5.5 *Let $f\colon X \to \left]-\infty, +\infty\right]$ be coercive and lower semicontinuous, let C be a closed subset of X, and assume that $C \cap \operatorname{dom} f \neq \varnothing$. Then $f|_C$ has a minimizer.*

Proof. First, observe that $f + \iota_C$ is still coercive because

$$f(x) + \iota_C(x) \geqslant f(x) \to +\infty \quad \text{as } \|x\| \to +\infty.$$

Second, ι_C is lsc by Example 3.20(iii). Proposition 4.1(ii) yields the lower semicontinuity of $f + \iota_C$. Third, $f + \iota_C$ is proper because $C \cap \operatorname{dom} f \neq \varnothing$. Therefore, the conclusion follows by applying Theorem 5.4 to $f + \iota_C$. ☐

Corollary 5.6 ((one-sided!) Weierstrass) *Let $f\colon X \to \left]-\infty, +\infty\right]$ be lower semicontinuous, let C be a bounded closed (i.e., compact) subset of X, and assume that $C \cap \operatorname{dom} f \neq \varnothing$. Then $f|_C$ has a minimizer.*

Proof. If $(x_n)_{n\in\mathbb{N}}$ is a sequence such that $\|x_n\| \to +\infty$, then eventually $x_n \notin C$ (because C is bounded!) and thus $f(x_n) + \iota_C(x_n) = +\infty$. Hence $f + \iota_C$ is coercive. $f + \iota_C$ is lower semicontinuous because of Example 3.20(iii) and Proposition 4.1(ii). And $f + \iota_C$ is proper since $C \cap \operatorname{dom} f \neq \varnothing$. The conclusion thus follows from Theorem 5.4 applied to $f + \iota_C$. ☐

5.2 ▪ Global and Local Minimizers, and Role of Convexity

Definition 5.7 Let f be proper on X, and let $\bar{x} \in X$. Then

(i) $\bar{x}$ is a *(global) minimizer of f* if $(\forall x \in X)\ f(\bar{x}) \leqslant f(x)$;

(ii) $\bar{x}$ is a *local minimizer of f* if $(\exists\, \varepsilon > 0)(\forall x \in B[\bar{x}; \varepsilon])\ f(\bar{x}) \leqslant f(x)$.

The (possibly empty) set of global minimizers is $\operatorname{Argmin} f$. If $\operatorname{Argmin} f$ is a singleton, we may identify $\operatorname{Argmin} f$ with its unique element. Global and local maximizers and $\operatorname{Argmax} f$ are defined similarly.

It is obvious that every global minimizer is a local minimizer, but the converse may fail, as the next example shows.

Example 5.8 Set $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - x^2 - 1$. Then $f'(x) = x^3 + x^2 - 2x = x(x-1)(x+2)$. Then f has two local minimizers, at 1 and at -2; the global minimizer is $x = -2$. And f has a local maximizer at $x = 0$ but no global maximizer (f is coercive!).

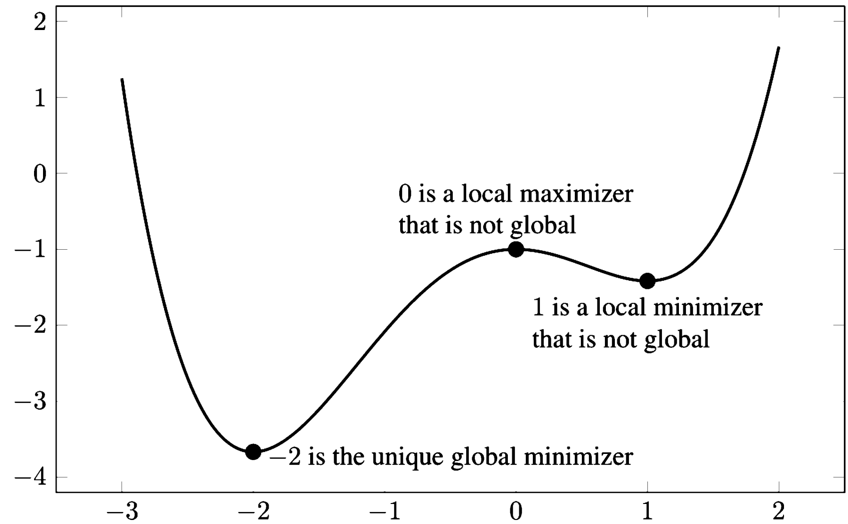


Figure 5.1. *Minimizers and maximizers for the function $f(x)$ from Example* 5.8.

Magic happens in the presence of convexity.

Proposition 5.9 *Let $f\colon X \to \left]-\infty,+\infty\right]$ be convex and proper. Then every local minimizer is also a global minimizer.*

Proof. Suppose that x is a local minimizer of f. Then there exists $\rho > 0$ such that

$$f(x) = \min f\big(B[x;\rho]\big). \tag{5.4}$$

We will show that

$$(\forall y \in X) \quad f(x) \leqslant f(y). \tag{5.5}$$

Let $y \in X$. Then (5.5) clearly holds if $y \notin \operatorname{dom} f$ and if $y \in B[x;\rho]$. So assume $y \in (\operatorname{dom} f) \smallsetminus B[x;\rho]$. Then $\|x-y\| > \rho$. Set

$$\lambda := \frac{\rho}{\|x-y\|} \in \left]0,1\right[.$$

In view of the convexity of $\operatorname{dom} f$ (see Corollary 3.7), we obtain that

$$z := (1-\lambda)x + \lambda y \in \operatorname{dom} f.$$

Moreover,

$$z - x = (1-\lambda)x + \lambda y - x = \lambda(y-x)$$

and hence

$$\|z-x\| = \|\lambda(y-x)\| = \lambda\|y-x\| = \frac{\rho}{\|x-y\|}\|y-x\| = \rho.$$

Thus, $z \in B[x;\rho]$ and so

$$f(x) \leqslant f(z)$$

by (5.4). Combining this with the definition of z, the convexity of f and Jensen's inequality (Theorem 3.6) yields

$$f(x) \leqslant f(z) = f\big((1-\lambda)x + \lambda y\big) \leqslant (1-\lambda)f(x) + \lambda f(y).$$

Therefore, $f(x) \leqslant f(y)$ and we are done. ☐

Corollary 5.10 *Let $f\colon X \to \,]-\infty, +\infty]$ be convex and proper, and let C be a subset of X. Suppose that $x \in \operatorname{int} C$ is a minimizer of $f|_C$. Then x is a global minimizer of f.*

Proof. Because $x \in \operatorname{int} C$, there exists $\delta > 0$ such that $B[x;\delta] \subseteq C$. The assumption implies that $f(x) = \min f(B[x;\delta]) = \min f(C)$. Now apply Proposition 5.9. ☐

Exercises

Exercise 5.1 Consider Theorem 5.4. Show that neither coercivity nor lower semicontinuity can be dropped as an assumption: (i) Find $f\colon \mathbb{R} \to \mathbb{R}$ such that f is coercive, but without minimizers. (ii) Find $f\colon \mathbb{R} \to \mathbb{R}$ such that f is lower semicontinuous, but without minimizers.

Exercise 5.2 Construct f and C as in Corollary 5.6 such that $f|_C$ has no maximizer.

Exercise 5.3 Let f be convex and proper on X. Show that $\operatorname{Argmin} f$, the set of minimizers of f, is convex.

Exercise 5.4 Provide details for Example 5.8.

Exercise 5.5 Let f be convex and proper on X, and let C be a convex subset of $\operatorname{dom} f$. Show that if $y \in \operatorname{int} C$ is a maximizer of $f|_C$, then $f|_C$ is constant. (Consequently, if f is not constant and $f|_C$ has maximizers, then there is a maximizer in $\operatorname{bdry} C$.)

Chapter 6

Even More on Convex Functions

6.1 ▪ General Jensen Inequality

Proposition 6.1 *Let $f\colon X \to [-\infty, +\infty]$. Then the following are equivalent:*

(i) *f is convex.*

(ii) $(\forall x \in \operatorname{dom} f)(\forall y \in \operatorname{dom} f)(\forall \lambda \in]0,1[)$
$f((1-\lambda)x + \lambda y) \leqslant (1-\lambda)f(x) + \lambda f(y)$.

(iii) $(\forall x_1 \in \operatorname{dom} f, \ldots, x_n \in \operatorname{dom} f)(\forall \lambda_1 > 0, \ldots, \lambda_n > 0 : \sum_{i=1}^n \lambda_i = 1)$
$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leqslant \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$.

Proof. "(i)⇔(ii)": Theorem 3.6. "(ii)⇐(iii)": This follows when $n = 2$. "(ii)⇒(iii)": We show this by induction on n. The case $n = 1$ is trivial, and $n = 2$ is (ii). Now assume that $n \geqslant 2$ and the result is true for n. Let $x_1 \in \operatorname{dom} f, \ldots x_{n+1} \in \operatorname{dom} f$, and $\lambda_1 > 0, \ldots, \lambda_{n+1} > 0$ such that $\sum_{i=1}^{n+1} \lambda_i = 1$. Using the base case and the inductive hypothesis, we estimate

$$\begin{aligned}
&f(\lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_{n+1} x_{n+1})\\
&= f\Big((1-\lambda_{n+1})\big(\tfrac{\lambda_1}{1-\lambda_{n+1}} x_1 + \cdots + \tfrac{\lambda_n}{1-\lambda_{n+1}} x_n\big) + \lambda_{n+1} x_{n+1}\Big)\\
&\leqslant (1-\lambda_{n+1}) f\big(\tfrac{\lambda_1}{1-\lambda_{n+1}} x_1 + \cdots + \tfrac{\lambda_n}{1-\lambda_{n+1}} x_n\big) + \lambda_{n+1} f(x_{n+1})\\
&\leqslant (1-\lambda_{n+1})\big(\tfrac{\lambda_1}{1-\lambda_{n+1}} f(x_1) + \cdots + \tfrac{\lambda_n}{1-\lambda_{n+1}} f(x_n)\big) + \lambda_{n+1} f(x_{n+1})\\
&= \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) + \lambda_{n+1} f(x_{n+1}),
\end{aligned}$$

which completes the proof. ☐

6.2 ▪ Recognizing Convexity via Calculus

The following result, which will be proven in Theorem 11.10 below, is very useful:

Fact 6.2 *Let C be an open convex subset of X, and let $f\colon C \to \mathbb{R}$ be differentiable. Then the following are equivalent:*

(i) *f is convex.*

(ii) $(\forall x \in C)(\forall y \in C)\ \langle x - y, \nabla f(y)\rangle + f(y) \leqslant f(x)$.

(iii) $(\forall x \in C)(\forall y \in C)\ \langle x - y, \nabla f(x) - \nabla f(y)\rangle \geqslant 0$.

If f is twice differentiable, then we can add to this list

(iv) $(\forall x \in C)\ \nabla^2 f(x) \succeq 0$.

Remark 6.3 Fact 6.2(ii) reveals the nonnegativity of the *Bregman distance*, i.e.,

$$D_f(x,y) := f(x) - f(y) - \langle x-y, \nabla f(y)\rangle \geqslant 0.$$

Fact 6.2(iii) means that the operator ∇f is *monotone*. When $X = \mathbb{R}$, the condition Fact 6.2(iv) turns into $f'' \geqslant 0$, which is known from Calculus I as a way to check that a function is "concave up" (as convexity is referred to in those texts).

Example 6.4 Let $A \in \mathbb{R}^{n\times n}$ be symmetric, and set $f(x) := \frac{1}{2}\langle x, Ax\rangle$. Then $\nabla^2 f(x) = A$; consequently, f is convex if and only if A is positive semidefinite.

Example 6.5 The function f defined by

$$f(x) = \begin{cases} -\ln(x) & \text{if } x > 0; \\ +\infty & \text{otherwise} \end{cases}$$

is convex because $f''(x) = 1/x^2 > 0$ when $x > 0$.

Regarding Fact 6.2, if we can extend f continuously to $\overline{C}$, then the extension is convex. This allows us to deal with the next example.

Example 6.6 The function $-\sqrt{x}$ is convex by Fact 6.2 on $\mathbb{R}_{++}$; hence, so is its extension to the closure of its domain, i.e., to $\mathbb{R}_+$.

Example 6.7 If $a \in X$ and $\beta \in \mathbb{R}$, then $f(x) = \langle x, a\rangle + \beta$ is convex. Indeed, this can be seen by noting that the Hessian $\nabla^2 f(x) = 0$ is positive semidefinite. In fact, f is an *affine* function, i.e., both f and $-f$ are convex.

6.3 ▪ Infimal Convolution

Definition 6.8 (infimal convolution) Let g and h be functions from X to $]-\infty, +\infty]$. Then their *infimal convolution* is defined by

$$\boxed{(g \,\Box\, h)(x) = \inf_{y\in X} \big(g(y) + h(x-y)\big).}$$

Proposition 6.9 *Let g, h be proper convex functions from X to $]-\infty, +\infty]$. Then $g \,\Box\, h = h \,\Box\, g$ is convex, with* $\operatorname{dom}(g \,\Box\, h) = \operatorname{dom} g + \operatorname{dom} h$.

Proof. The symmetry follows readily by a change of variables. It is not hard to see that

$$F\colon X \times X \to\ ]-\infty, +\infty] : (x,y) \mapsto g(y) + h(x-y)$$

is convex and proper. By Theorem 4.12, the function

$$x \mapsto \inf F(x, X) = \inf_{y\in X}\big(g(y) + h(x-y)\big) = (g \,\Box\, h)(x)$$

is convex. The domain statement is clear from the definition of the infimal convolution. $\square$

Example 6.10 Let C be a nonempty convex subset of X. Then d_C and $\frac{1}{2}d_C^2$ are convex and proper.

Proof. Because C is convex (as a set), the function ι_C is convex (as a function) by Example 3.20(ii). Now $\|\cdot\|$ is a norm, hence convex, while $\frac{1}{2}\|\cdot\|^2$ is convex because $\nabla\|\cdot\|^2 = \mathrm{Id} \succ 0$ or by using (1.4). In view of Proposition 6.9, we see that both

$$d_C = \iota_C \,\square\, \|\cdot\| \quad \text{and} \quad \tfrac{1}{2}d_C^2 = \iota_C \,\square\, \tfrac{1}{2}\|\cdot\|^2$$

are convex. □

6.4 ▪ Convexity and Continuity

Proposition 6.11 *Let $f\colon X \to \left]-\infty, +\infty\right]$ be convex, lower semicontinuous, and proper. Let $x_0 \in X$, $x_1 \in \operatorname{dom} f$, and set $(\forall \lambda \in \left]0,1\right[)$ $x_\lambda := (1-\lambda)x_0 + \lambda x_1$. Then $\lim_{\lambda \to 0^+} f(x_\lambda) = f(x_0)$.*

Proof. Using the lower semicontinuity of f at x_0 and then the convexity in the form of Theorem 3.6 (Jensen), we have

$$\begin{aligned} f(x_0) &\leqslant \varliminf_{\lambda \to 0^+} f(x_\lambda) \leqslant \varlimsup_{\lambda \to 0^+} f(x_\lambda) \\ &\leqslant \varlimsup_{\lambda \to 0^+} (1-\lambda) f(x_0) + \lambda f(x_1) \\ &= f(x_0) \end{aligned}$$

and we are done. □

Corollary 6.12 *Let $f\colon \mathbb{R} \to \left]-\infty, +\infty\right]$ be convex, lower semicontinuous, and proper. Then $f|_{\operatorname{dom} f}$ is continuous.*

We also have the following strong continuity property of convex functions.

Fact 6.13 [39, Theorem 10.4] *Let $f\colon X \to \left]-\infty, +\infty\right]$ be convex, and assume that $x_0 \in \operatorname{ri} \operatorname{dom} f$. Then $f|_{\operatorname{dom} f}$ is* locally Lipschitz around x_0*: There exists $L \geqslant 0$ and $\varepsilon > 0$ such that for all x, y in $B[x_0; \varepsilon] \cap \operatorname{dom} f$, we have $|f(x) - f(y)| \leqslant L\|x - y\|$.*

Exercises

Exercise 6.1 Provide the details for Example 6.6.

Exercise 6.2 Consider the function $f(x) = x\ln(x) - x$ with domain $\mathbb{R}_{++}$. Determine whether or not f is convex.

Exercise 6.3 Consider the function $f(x, y) := x/y$ with domain $\mathbb{R}_{++} \times \mathbb{R}_{++}$. Determine whether or not f is convex.

Exercise 6.4 Consider the function $f(x, y) := x^2/y$ with domain $\mathbb{R} \times \mathbb{R}_{++}$. Determine whether or not f is convex.

Exercise 6.5 Consider the function $f(x, y) := \ln(e^x + e^y)$ with domain $\mathbb{R} \times \mathbb{R}$. Show that f is convex.

Exercise 6.6 (log-sum-exp) Consider the full-domain function $f(x) := \ln(\sum_{i=1}^n e^{x_i})$ on $\mathbb{R}^n$. Show that f is convex. *Aside:* ∇f is a famous function in machine learning and known there as the *softmax* or *softargmax* function.

Exercise 6.7 Consider the function $f(x, y) := \exp(\ln(x) - \ln(y))$ with domain $\mathbb{R}_{++} \times \mathbb{R}_{++}$. Determine whether or not f is convex.

Exercise 6.8 Consider the function $f(x, y) := xy$ with domain $\mathbb{R}_{++} \times \mathbb{R}_{++}$. Determine whether or not f is convex.

Exercise 6.9 Consider the function $f(x, y) := -\sqrt{xy}$ with domain $\mathbb{R}_{+} \times \mathbb{R}_{+}$. Determine whether or not f is convex.

Exercise 6.10 Consider the function $f(x, y) := x \ln(x/y) - x + y$ with domain $\mathbb{R}_{++} \times \mathbb{R}_{++}$. Determine whether or not f is convex.

Exercise 6.11 In Example 6.10 we saw that d_C is convex, provided that C is convex and nonempty. Does the conclusion remain true when we drop the assumption on convexity of C?

Exercise 6.12 By providing an example, show that Fact 6.13 fails when x_0 is merely assumed to lie in $\operatorname{dom} f$.

Chapter 7

Support Functions and Polar Cones

7.1 ▪ Support Functions

Definition 7.1 (support function) Let C be a subset of X. The corresponding *support function* σ_C is defined by

$$\boxed{\sigma_C : X \to [-\infty, +\infty] : x \mapsto \sup \langle C, x \rangle = \sup_{c \in C} \langle c, x \rangle .}$$

Proposition 7.2 *Let C be a nonempty subset of X. Then σ_C is convex, lower semicontinuous, and proper.*

Proof. Set $(\forall c \in C)$ $f_c : X \to \mathbb{R} : x \mapsto \langle c, x \rangle$, which is linear ($\Rightarrow$ convex) and continuous ($\Rightarrow$ lsc). By Corollary 4.8(iii),

$$\sigma_C = \sup_{c \in C} f_c$$

is convex and lower semicontinuous. Fix $c_0 \in C$. On the one hand, $(\forall x \in X)$ $-\infty < \langle c_0, x \rangle \leqslant \sigma_C(x)$. On the other hand, $\sigma_C(0) = \sup \langle C, 0 \rangle = \sup 0 = 0 < +\infty$. Altogether, σ_C is proper. □

Example 7.3 If $\|\cdot\|$ is any norm on X and we set $C := B[0;1]$, then we recover the dual norm via

$$\sigma_C(y) = \sup_{\|x\| \leqslant 1} \langle x, y \rangle = \|y\|_*.$$

Example 7.4 Suppose that $0 \leqslant a \leqslant b$ and set $C := [a, b]$. Then

$$\sigma_C(x) = \begin{cases} bx & \text{if } x \geqslant 0; \\ ax & \text{if } x < 0. \end{cases}$$

Example 7.5 Set $C := \mathbb{R}_+ \subseteq \mathbb{R}$, and let $x \in \mathbb{R}$. If $x \leqslant 0$, then $\sigma_C(x) = \sup \mathbb{R}_+ \cdot x = 0$, whereas if $x > 0$, then $\sigma_C(x) = \sup \mathbb{R}_+ \cdot x = +\infty$. Altogether,

$$\sigma_{\mathbb{R}_+} = \iota_{\mathbb{R}_-}.$$

Example 7.6 In $X = \mathbb{R}^n$, set $C := \{\pm e_1, \ldots, \pm e_n\}$, where e_i denotes the standard ith unit vector. Then

$$\sigma_C(x) = \sup\{\pm x_1, \ldots, \pm x_n\} = \max\{|x_1|, \ldots, |x_n|\} = \|x\|_\infty$$

and we recover the max-norm as a support function!

Recall that for subsets A, B of X, and $\gamma \in \mathbb{R}$, we write

$$A + B := \{a + b \mid a \in A,\ b \in B\} \quad \text{and} \quad \gamma A := \{\gamma a \mid a \in A\}.$$

Known properties of the supremum thus yield to the following calculus rules for the support function:

Proposition 7.7 *Let A, B, C be nonempty subsets of X, let $\rho > 0$, and let x, y be in X. Then the following hold:*

(i) $\sigma_C(\rho x) = \rho\sigma_C(x) = \sigma_{\rho C}(x)$.

(ii) $\sigma_C(x + y) \leqslant \sigma_C(x) + \sigma_C(y)$.

(iii) $\sigma_C(x) = \sigma_{\overline{C}}(x)$.

(iv) $\sigma_C(x) = \sigma_{\operatorname{conv} C}(x)$.

(v) $\sigma_C(x) = \sigma_{\overline{\operatorname{conv}}\, C}(x)$.

(vi) $\sigma_{A+B}(x) = \sigma_A(x) + \sigma_B(x)$.

(vii) $\sigma_{A \cup B}(x) = \max\{\sigma_A(x), \sigma_B(x)\}$.

(viii) $\sigma_{A \cap B}(x) \leqslant \min\{\sigma_A(x), \sigma_B(x)\}$.

Example 7.8 It is clear that the probability simplex $\mathbb{P}_n$ in $\mathbb{R}^n$ is the convex hull of the set of unit vectors $C := \{e_1, \ldots, e_n\}$. Hence for every $x = (x_1, \ldots, x_n)$, we have $\sigma_{\mathbb{P}_n}(x) = \sigma_C(x) = \max\{x_1, \ldots, x_n\}$.

7.2 ▪ Cones and Polar Cones

A *cone* K is a nonempty subset of X such that $\mathbb{R}_+ K \subseteq K$. The *polar cone* of K is

$$K^\ominus := \{y \in X \mid (\forall x \in K)\ \ \langle x, y\rangle \leqslant 0\} = \operatorname{lev}_0 \sigma_K. \tag{7.1}$$

The *conical hull* of K, written cone K, is the smallest cone containing K.

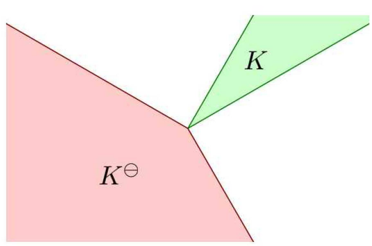


Figure 7.1. *The green closed convex cone K and its red polar cone $K^\ominus$. A rectangular clip is shown because the cones are unbounded.*

Proposition 7.9 *Let K be a cone in X. Then $K^{\ominus}$ is a closed convex cone and*

$$\sigma_K = \iota_{K^{\ominus}}. \tag{7.2}$$

Proof. We omit the proof that $K^{\ominus}$ is a closed convex cone — see Exercise 7.3 for a more general result. To prove (7.2), let's fix $y \in X$.

Case 1: $y \in K^{\ominus}$. Then

$$\sigma_K(y) = \sup_{x \in K} \langle x, y\rangle \leqslant 0 = \langle 0, y\rangle \leqslant \sup \langle K, y\rangle = \sigma_K(y).$$

Hence equalities hold throughout and we deduce that $\sigma_K(y) = 0 = \iota_{K^{\ominus}}(y)$.

Case 2: $y \notin K^{\ominus}$. Then there exists $x \in K$ such that $\langle x, y\rangle > 0$. Because $\alpha x \in K$ for every $\alpha > 0$, we deduce that

$$+\infty \geqslant \sigma_K(y) = \sup \langle K, y\rangle \geqslant \sup_{\alpha>0} \langle \alpha x, y\rangle = \langle x, y\rangle \sup_{\alpha>0} \alpha = +\infty.$$

Hence equalities hold throughout; thus, $\sigma_K(y) = +\infty = \iota_{K^{\ominus}}(y)$. □

Example 7.5 quickly leads to the following.

Example 7.10 $\sigma_{\mathbb{R}^n_+} = \iota_{\mathbb{R}^n_-}$.

The idea of the polar cone is a generalization of the orthogonal complement due to the following.

Example 7.11 Let Y be a linear subspace of X. Then $Y^{\ominus} = Y^{\perp}$.

The following examples are more complicated but recorded for future use.

Fact 7.12 [5, Example 2.29] *Suppose $A \in \mathbb{R}^{m\times n}$ and set $K = \{x \in \mathbb{R}^n \mid Ax \leqslant 0\}$. Then K is a closed convex cone and*

$$K^{\ominus} = A^{\mathsf{T}}(\mathbb{R}^m_+);$$

consequently, $\sigma_K = \iota_{A^{\mathsf{T}}(\mathbb{R}^m_+)}$.

Fact 7.13 [5, Example 2.30] *Suppose $A \in \mathbb{R}^{m\times n}$, let $b \in \mathbb{R}^m$, and set $C := \{x \in \mathbb{R}^n \mid Ax = b\}$. Assume that $C \neq \varnothing$, say $c \in C$. Then*

$$\sigma_C(x) = \langle x, c\rangle + \iota_{\operatorname{ran} A^{\mathsf{T}}}(x).$$

Fact 7.14 [39, page 115] *In $\mathbb{R}^2$, set*

$$C := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + \tfrac{1}{2}x_2^2 \leqslant 0\}.$$

Then C is convex, closed, and nonempty, and

$$\sigma_C(y_1, y_2) = \begin{cases} \frac{y_2^2}{2y_1} & \text{if } y_1 > 0; \\ 0 & \text{if } y_1 = y_2 = 0; \\ +\infty & \text{otherwise} \end{cases}$$

is — being a support function — convex and lower semicontinuous. Note that $\sigma_C(0,0) = 0$. On the other hand, if $\alpha > 0$ and $t > 0$, then $\sigma_C(t^2/(2\alpha), t) = t^2/(2t^2/(2\alpha)) = \alpha$, yet $(t^2/(2\alpha), t) \to (0,0)$ as $t \to 0^+$. Hence σ_C is not *continuous at $(0,0)$. (Contrast this with continuity along line segments in Proposition* 6.11.*)*

Exercises

Exercise 7.1 Provide the details for Example 7.4.

Exercise 7.2 Verify Proposition 7.7.

Exercise 7.3 Let C be a nonempty subset of X. Show that $C^{\ominus}$ is a closed convex cone.

Exercise 7.4 Provide the details for Example 7.10.

Exercise 7.5 Provide the details for Example 7.11.

Exercise 7.6 Show that the set C of Fact 7.14 is convex, closed, and nonempty. Moreover, show that $C = \operatorname{conv}\big\{(x_1, x_2) \in \mathbb{R}^2 \;\big|\; x_1 + \frac{1}{2}x_2^2 = 0\big\}$.

Exercise 7.7 Verify the formula for σ_C from Fact 7.14.

Chapter 8

Projection and Separation

8.1 ▪ Projections

Definition 8.1 (projection) Let C be a nonempty closed subset of X, and let $z \in X$. Suppose $x \in C$ satisfies

$$\|x - z\| = d_C(z) = \inf_{c \in C} \|c - z\|.$$

Then the point x is called a *projection* of z onto C, written $x \in \mathrm{P}_C(z)$. If we know that $\mathrm{P}_C(z)$ is a singleton, we will also write (abusing notation slightly) $x = \mathrm{P}_C(z)$.

Remark 8.2 (projections exist) Let C be a nonempty closed subset of X, and let $z \in X$. The function

$$f \colon x \mapsto \|x - z\|$$

is continuous, hence lower semicontinuous. Moreover, since

$$\|x - z\| \geqslant \|x\| - \|z\| \to +\infty$$

as $\|x\| \to +\infty$, we see that f is coercive. Finally $C \cap \operatorname{dom} f = C \cap X = C$ is nonempty. By Corollary 5.5, $f|_C$ has a minimizer, i.e., $\mathrm{P}_C(z) \neq \varnothing$. Note that we didn't employ any special properties of the Euclidean norm — nearest points exist for *any* norm!

Proposition 8.3 *Let C be a nonempty closed convex subset of X and let $z \in X$. Then $\mathrm{P}_C(z)$ is a singleton.*

Proof. Take x_0, x_1 in $\mathrm{P}_C(z)$, and set

$$x_\lambda := (1 - \lambda)x_0 + \lambda x_1,$$

where $0 < \lambda < 1$. Using that $x_\lambda \in C$ and (1.4), we have

$$\begin{aligned}
d_C^2(z) &\leqslant \|x_\lambda - z\|^2 = \|(1 - \lambda)(x_0 - z) + \lambda(x_1 - z)\|^2 \\
&= (1 - \lambda)\|x_0 - z\|^2 + \lambda\|x_1 - z\|^2 - \lambda(1 - \lambda)\|x_0 - x_1\|^2 \\
&= (1 - \lambda)d_C^2(z) + \lambda d_C^2(z) - \lambda(1 - \lambda)\|x_0 - x_1\|^2 \\
&= d_C^2(z) - \lambda(1 - \lambda)\|x_0 - x_1\|^2 \\
&\leqslant d_C^2(z).
\end{aligned}$$

So all inequalities are actually equalities and we deduce that $\lambda(1 - \lambda)\|x_0 - x_1\|^2 = 0$, which in turn yields $x_0 = x_1$. ☐

Theorem 8.4 (Projection Theorem) *Let C be a nonempty closed convex subset of X, and let $z \in X$. Then there exists a unique point $\mathrm{P}_C(z) \in C$ such that $d_C(z) = \|\mathrm{P}_C(z) - z\|$. Moreover, let $p \in X$. Then $\mathrm{P}_C(z)$ is characterized by the* obtuse angle condition

$$\boxed{p = \mathrm{P}_C(z) \quad \Leftrightarrow \quad p \in C \textit{ and } (\forall c \in C) \quad \langle c - p, z - p \rangle \leqslant 0.} \tag{8.1}$$

Proof. We already observed the uniqueness in Proposition 8.3. Now (1.6) yields

$$\begin{aligned} p = \mathrm{P}_C(z) &\Leftrightarrow p \in C \wedge (\forall c \in C) \quad \|p - z\| \leqslant \|c - z\| \\ &\Leftrightarrow p \in C \wedge (\forall c \in C)(\forall \lambda \in [0,1]) \quad \|p - z\| \leqslant \|(1-\lambda)p + \lambda c - z\| \\ &\Leftrightarrow p \in C \wedge (\forall c \in C)(\forall \lambda \in [0,1]) \quad \|p - z\| \leqslant \|(p - z) - \lambda(p - c)\| \\ &\Leftrightarrow p \in C \wedge (\forall c \in C) \quad \langle p - z, p - c \rangle \leqslant 0, \end{aligned}$$

and this proves (8.1). ☐

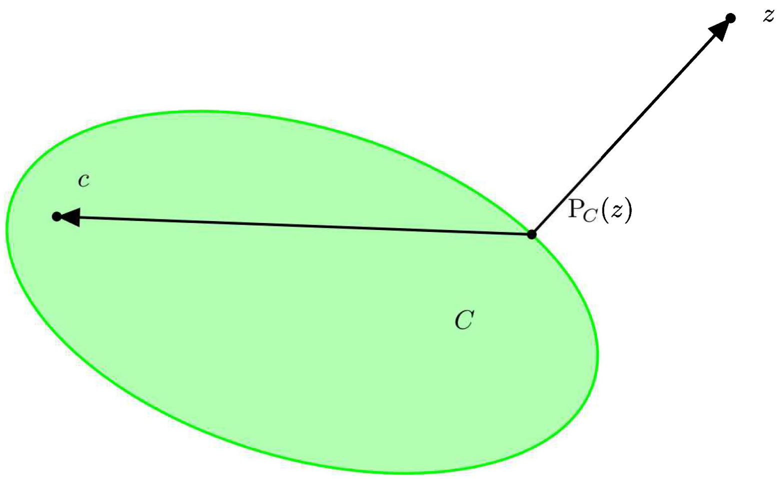


Figure 8.1. *For every $c \in C$, the angle between $c - \mathrm{P}_C(z)$ and $z - \mathrm{P}_C(z)$ is obtuse.*

Corollary 8.5 *Let C be a nonempty closed convex subset of X. Then the operator P_C satisfies*

$$(\forall x \in X)(\forall y \in X) \quad \|\mathrm{P}_C(x) - \mathrm{P}_C(y)\|^2 \leqslant \langle \mathrm{P}_C(x) - \mathrm{P}_C(y), x - y \rangle\,; \tag{8.2}$$

consequently,

$$(\forall x \in X)(\forall y \in X) \quad \|\mathrm{P}_C(x) - \mathrm{P}_C(y)\| \leqslant \|x - y\|. \tag{8.3}$$

Proof. Let x, y be in X. The obtuse angle characterization (see (8.1)) gives $\langle \mathrm{P}_C(y) - \mathrm{P}_C(x), x - \mathrm{P}_C(x) \rangle \leqslant 0$ and $\langle \mathrm{P}_C(x) - \mathrm{P}_C(y), y - \mathrm{P}_C(y) \rangle \leqslant 0$. Adding these yields $\langle \mathrm{P}_C(x) - \mathrm{P}_C(y), (\mathrm{P}_C(x) - \mathrm{P}_C(y)) - (x - y) \rangle \leqslant 0$, i.e., $\|\mathrm{P}_C(x) - \mathrm{P}_C(y)\|^2 - \langle \mathrm{P}_C(x) - \mathrm{P}_C(y), x - y \rangle \leqslant 0$, from which (8.2) follows. In turn, (8.2) and Cauchy–Schwarz yield (8.3). ☐

8.2 ▪ Examples

Example 8.6 Let $-\infty \leqslant \alpha \leqslant \beta \leqslant +\infty$, set $C := [\alpha, \beta] \cap \mathbb{R} = \{x \in \mathbb{R} \mid \alpha \leqslant x \leqslant \beta\}$, and suppose that $C \neq \varnothing$. Then C is convex and closed, and

$$(\forall x \in \mathbb{R}) \quad \mathrm{P}_C(x) = \begin{cases} \alpha & \text{if } x < \alpha; \\ x & \text{if } \alpha \leqslant x \leqslant \beta; \\ \beta & \text{if } \beta < x \end{cases} = \min\big\{\max\{x, \alpha\}, \beta\big\}. \tag{8.4}$$

Example 8.7 $\mathrm{P}_{\mathbb{R}_+}(x) = \max\{x, 0\}$.

Proposition 8.8 *Let $C_1, \ldots, C_m$ be nonempty closed convex subsets of $X_1, \ldots, X_m$, respectively. Then*

$$\mathrm{P}_{C_1 \times \cdots \times C_m}(x_1, \ldots, x_m) = \big(\mathrm{P}_{C_1}(x_1), \ldots, \mathrm{P}_{C_m}(x_m)\big).$$

Example 8.9 $\mathrm{P}_{\mathbb{R}^n_+}(x_1, \ldots, x_n) = (x_1^+, \ldots, x_n^+)$, where $\xi^+ = \max\{\xi, 0\}$ is the positive part of ξ.

Proposition 8.10 *Let $A \in \mathbb{R}^{m \times n}$ be such that AA^T is invertible, let $b \in \mathbb{R}^m$, and suppose that the preimage $C := A^{-1}(b)$ is nonempty. Then*

$$\mathrm{P}_C(x) = x - A^\mathsf{T}(AA^\mathsf{T})^{-1}(Ax - b).$$

Proof. Let $z \in X$, and set $p := z - A^\mathsf{T}(AA^\mathsf{T})^{-1}(Az - b)$. Then $Ap = Az - (AA^\mathsf{T})(AA^\mathsf{T})^{-1}(Az - b) = Az - (Az - b) = b$; thus, $p \in C$.

Now let $c \in C$, i.e., $Ac = b$. Then

$$\begin{aligned} \langle c - p, z - p \rangle &= \big\langle c - p, z - (z - A^\mathsf{T}(AA^\mathsf{T})^{-1}(Az - b))\big\rangle \\ &= \big\langle c - p, A^\mathsf{T}(AA^\mathsf{T})^{-1}(Az - b)\big\rangle \\ &= \big\langle Ac - Ap, (AA^\mathsf{T})^{-1}(Az - b)\big\rangle \\ &= \big\langle b - b, (AA^\mathsf{T})^{-1}(Az - b)\big\rangle \\ &= 0 \end{aligned}$$

and we are done by the Projection Theorem (Theorem 8.4). □

Example 8.11 Suppose that $X = \mathbb{R}^n$, let $a \in \mathbb{R}^n \smallsetminus \{0\}$, and let $\beta \in \mathbb{R}$. Then $C = \{x \in \mathbb{R}^n \mid \langle x, a \rangle = \beta\} \neq \varnothing$, and

$$\mathrm{P}_C(x) = x - \frac{\langle x, a \rangle - \beta}{\|a\|^2} a.$$

Proof. Set $A = a^\mathsf{T} \in \mathbb{R}^{1 \times n}$. Then $AA^\mathsf{T} = \|a\|^2 > 0$ so AA^T is invertible. Now apply Proposition 8.10. □

The following extension of Proposition 8.10 holds.

Fact 8.12 [3, Example 29.17(ii)] *Let $A \in \mathbb{R}^{m \times n}$, let $b \in \mathbb{R}^m$, and suppose that $C := A^{-1}(b) \neq \varnothing$. Then*

$$\mathrm{P}_C(x) = x - A^\dagger(Ax - b),$$

where $A^\dagger$ denotes the Moore–Penrose (a.k.a. pseudo) inverse *of A.*

Example 8.13 (unit ball projection) Set $C := B[0;1]$. Then

$$\mathrm{P}_C(x) = \begin{cases} x & \text{if } \|x\| \leqslant 1; \\ \dfrac{x}{\|x\|} & \text{if } \|x\| > 1 \end{cases} = \frac{x}{\max\{\|x\|, 1\}}.$$

Proof. Let $z \in X$. The formula is clear if $z \in C$, i.e., $\|z\| \leqslant 1$. So assume that $\|z\| > 1$ and set $p := z/\|z\|$. Then $\|p\| = 1$ and so $p \in C$. Now let $c \in C$, i.e., $\|c\| \leqslant 1$. Then, using also Cauchy–Schwarz,

$$\begin{aligned} \langle c - p, z - p \rangle &= \langle c - p, z - z/\|z\| \rangle \\ &= \big(1 - \tfrac{1}{\|z\|}\big) \langle c - p, z \rangle \\ &= \big(1 - \tfrac{1}{\|z\|}\big)\big(\langle c, z \rangle - \|z\|\big) \\ &\leqslant \big(1 - \tfrac{1}{\|z\|}\big)\big(\|c\|\|z\| - \|z\|\big) \\ &= \big(\|z\| - 1\big)\big(\|c\| - 1\big) \\ &\leqslant 0 \end{aligned}$$

and we are done by the Projection Theorem (Theorem 8.4). □

8.3 ▪ Separation

Corollary 8.14 (Separation Theorem) *Let C be a nonempty closed convex subset of X, and let $z \in X \smallsetminus C$. Then there exists $u \in X \smallsetminus \{0\}$ and $\alpha \in \mathbb{R}$ such that*

$$\langle z, u \rangle > \alpha > \sup \langle C, u \rangle ;$$

in other words, z and C are separated by the hyperplane $\{x \in X \mid \langle x, u \rangle = \alpha\}$.

Proof. Set

$$u := z - \mathrm{P}_C(z) \neq 0 \ \text{ and } \ \beta := \langle \mathrm{P}_C(z), z - \mathrm{P}_C(z) \rangle .$$

Let $c \in C$. Then (8.1) yields $\langle c - \mathrm{P}_C(z), z - \mathrm{P}_C(z) \rangle \leqslant 0 \Leftrightarrow \langle c, z - \mathrm{P}_C(z) \rangle \leqslant \langle \mathrm{P}_C(z), z - \mathrm{P}_C(z) \rangle$ $\Leftrightarrow \langle c, u \rangle \leqslant \beta$. Hence $\sup \langle C, u \rangle \leqslant \beta$. On the other hand,

$$\begin{aligned} \langle z, u \rangle &= \langle z, z - \mathrm{P}_C(z) \rangle \\ &= \langle z - \mathrm{P}_C(z), z - \mathrm{P}_C(z) \rangle + \langle \mathrm{P}_C(z), z - \mathrm{P}_C(z) \rangle \\ &= d_C^2(z) + \beta \\ &> \beta \end{aligned}$$

because $z \notin C$. Altogether, $\langle z, u \rangle > \beta \geqslant \sup \langle C, u \rangle$. Finally, any α in $]\beta, \langle z, u \rangle[$ does the job. □

Lemma 8.15 *Let A, B be nonempty closed convex subsets of X. Then*

$$A = B \ \Leftrightarrow \ \sigma_A = \sigma_B.$$

Proof. "$\Rightarrow$": This is clear. "$\Leftarrow$": By symmetry, it suffices to show that $A \subseteq B$. Suppose to the contrary that $A \not\subseteq B$. Then there exists $z \in A \smallsetminus B$. By Corollary 8.14, there exists $u \in X$ such that $\langle z, u \rangle > \sup \langle B, u \rangle$. But then $\sigma_A(u) \geqslant \langle z, u \rangle > \sup \langle B, u \rangle = \sigma_B(u) = \sigma_A(u)$, which is absurd. □

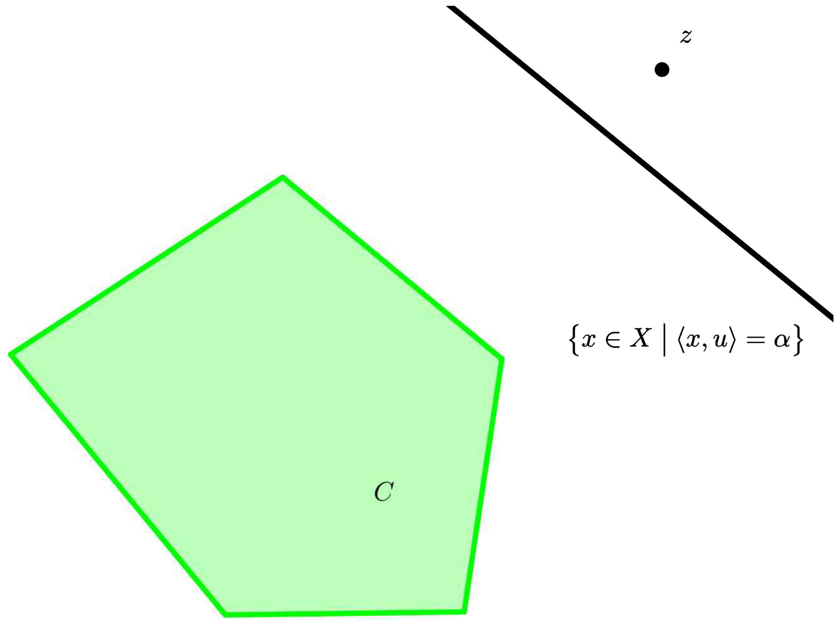


Figure 8.2. *An illustration of Corollary* 8.14.

Exercises

Exercise 8.1 Provide an example of a nonempty set C that is not closed and for which $\mathrm{P}_C(x) = \varnothing$ for some $x \in X$.

Exercise 8.2 Show that the conclusion of Proposition 8.3 fails if the set C is not convex.

Exercise 8.3 Show that the conclusion of Proposition 8.3 fails if the implicit Euclidean norm is replaced by an arbitrary norm.

Exercise 8.4 Consider (8.3) from Corollary 8.5. Is it possible that there exists a nonempty closed convex subset C of X such that $\|\mathrm{P}_C(x) - \mathrm{P}_C(y)\| < \|x - y\|$ whenever $x \neq y$?

Exercise 8.5 Let $a \in X \smallsetminus \{0\}$ and $\beta \in \mathbb{R}$, and set $C := \{x \in X \mid \langle x, a\rangle = \beta\}$. Show that $\mathrm{P}_C(z) = z - (\langle z, a\rangle - \beta)/\|a\|^2 a$ by using only Theorem 8.4.

Exercise 8.6 Prove Proposition 8.8.

Exercise 8.7 (projection onto a translated set) Let C be a nonempty closed convex subset of X, let $a \in X$, and let $z \in X$. Show that $\mathrm{P}_{a+C}(z) = a + \mathrm{P}_C(z - a)$.

Exercise 8.8 (projection onto a scaled set) Let C be a nonempty closed convex subset of X, let $\alpha \in \mathbb{R} \smallsetminus \{0\}$, and let $z \in X$. Show that $\mathrm{P}_{\alpha C}(z) = \alpha \mathrm{P}_C(z/\alpha)$.

Exercise 8.9 (projection onto a general ball) Let $c \in X$, let $\rho > 0$, and consider the closed ball centered at c of radius ρ, i.e., $C := B[c; \rho] = \{x \in X \mid \|x - c\| \leqslant \rho\}$. Let $z \in X$. Show that

$$\mathrm{P}_C(z) = c + \frac{\rho(z - c)}{\max\{\|z - c\|, \rho\}}. \tag{8.5}$$

Exercise 8.10 Show that the assumption on convexity in Corollary 8.14 is critical, i.e., find a nonempty closed set C that is not convex and a point $z \in X \smallsetminus C$ such that the conclusion of Corollary 8.14 fails.

Chapter 9

Subgradients

9.1 ▪ Optimization and Examples

Definition 9.1 (subgradients and subdifferential) Let $f\colon X \to \left]-\infty,+\infty\right]$ be proper. Then the *subdifferential* of f at $x \in X$ is

$$\boxed{\partial f(x) := \big\{u \in X \;\big|\; (\forall y \in X)\ \ f(x) + \langle y - x, u\rangle \leqslant f(y)\big\}.}$$

The elements of $\partial f(x)$ are called *subgradients* of f at x. Moreover,

$$\boxed{\operatorname{dom}\partial f := \big\{x \in X \;\big|\; \partial f(x) \neq \varnothing\big\} \subseteq \operatorname{dom} f.}$$

Easy but powerful is the following result, which makes the importance of the subdifferential in optimization clear:

Lemma 9.2 (Fermat's rule) *Let $f\colon X \to \left]-\infty,+\infty\right]$ be proper, and let $x \in X$. Then*

$$\boxed{0 \in \partial f(x) \Leftrightarrow x \textit{ is a global minimizer of } f.}$$

Proof. Indeed,

$$\begin{aligned}
0 \in \partial f(x) &\Leftrightarrow (\forall y \in X)\ \ f(x) + \langle y - x, 0\rangle \leqslant f(y)\\
&\Leftrightarrow (\forall y \in X)\ \ f(x) \leqslant f(y)\\
&\Leftrightarrow x \text{ is a global minimizer of } f,
\end{aligned}$$

as claimed. ☐

Note that $u \in \partial f(x)$ is the slope of the affine function $h\colon y \mapsto f(x) + \langle y - x, u\rangle$, also called an *affine minorant* of f: the graph of this affine function h never cuts above the graph of f and its value at x is $h(x) = f(x)$.

Example 9.3 Let $f\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto |x|$. Then

$$\partial f(x) = \begin{cases} \{-1\} & \text{if } x < 0; \\ [-1, 1] & \text{if } x = 0; \\ \{1\} & \text{if } x > 0. \end{cases}$$

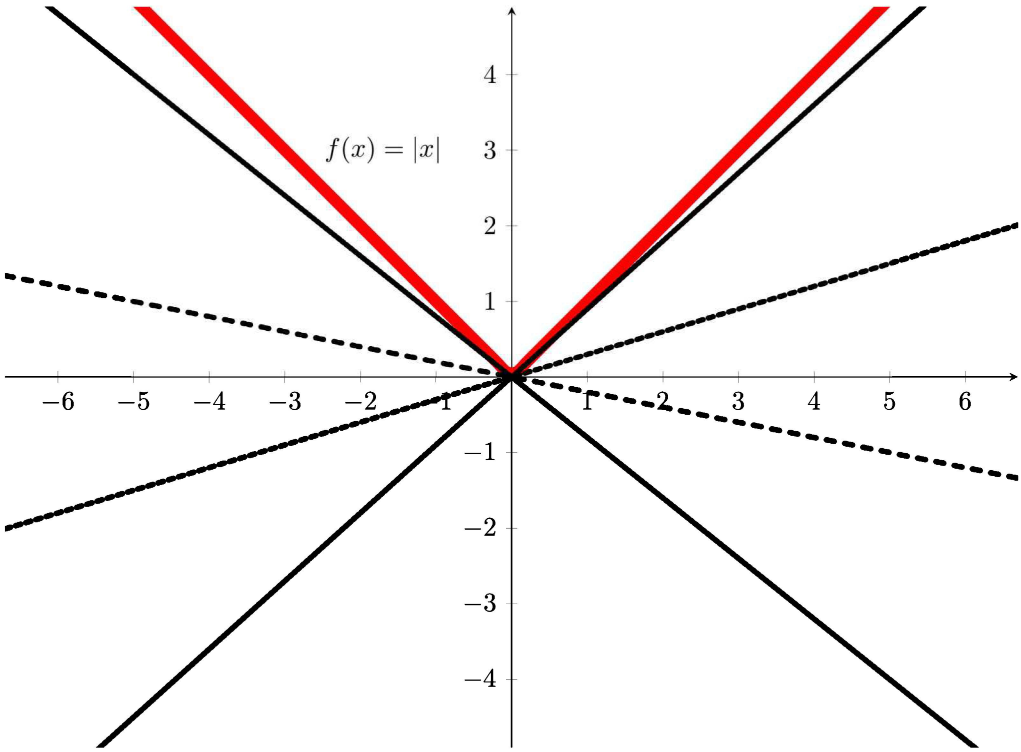


Figure 9.1. *For the red function $f(x) = |x|$, we have drawn four affine minorants whose slopes correspond to subgradients taken from $\partial f(0) = [-1, 1]$.*

Proposition 9.4 *Let $f = \|\cdot\|$ be any norm on X. Then*

$$\partial f(0) = \big\{u \in X \;\big|\; \|u\|_* \leqslant 1\big\}$$

is the dual unit ball.

Proof. We have $u \in \partial f(0) \Leftrightarrow (\forall y \in X)\ \langle y, u\rangle = f(0) + \langle y - 0, u\rangle \leqslant f(y) = \|y\| \Leftrightarrow \|u\|_* \leqslant 1$ for $u \in X$. ☐

Example 9.5 $\partial\|\cdot\|_1(0) = [-1, 1]^n$ on $\mathbb{R}^n$.

Proof. Indeed, because the dual norm of the 1-norm is the max-norm, this follows from Proposition 9.4. ☐

Example 9.6 Suppose that $X = \mathbb{S}^n$ and that f maps $A \in X$ to its largest eigenvalue. Given $A \in X$, let v be a normalized eigenvector of A corresponding to the eigenvalue $f(A)$. Then $vv^{\mathsf{T}} \in \partial f(A)$.

9.2 ▪ Further Properties

Proposition 9.7 *Let $f\colon X \to \left]-\infty,+\infty\right]$ be proper, and let $x \in X$. Then $\partial f(x)$ is convex and closed.*

Proof. The conclusion is clear if $\partial f(x) = \varnothing$, so let us assume that $\partial f(x) \neq \varnothing$. Take u_0, u_1 in $\partial f(x)$, and let $0 < \lambda < 1$. Let $y \in X$. Then

$$f(x) + \langle y - x, u_0\rangle \leqslant f(y) \quad\text{and}\quad f(x) + \langle y - x, u_1\rangle \leqslant f(y).$$

Multiplying the first and second inequality by $(1-\lambda)$ and λ, respectively, and then adding and simplifying yields

$$f(x) + \langle y - x, (1-\lambda)u_0 + \lambda u_1\rangle \leqslant f(y).$$

Hence $(1-\lambda)u_0 + \lambda u_1 \in \partial f(x)$ and thus $\partial f(x)$ is convex.

Now let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\partial f(x)$ such that $u_n \to u \in X$. Let $y \in X$. Then $(\forall n \in \mathbb{N})$ $f(x) + \langle y - x, u_n\rangle \leqslant f(y)$. Taking the limit as $n \to +\infty$ yields $f(x) + \langle y - x, u\rangle \leqslant f(y)$. Therefore $u \in \partial f(x)$ and hence $\partial f(x)$ is closed. ☐

Proposition 9.8 (monotonicity) *Let $f\colon X \to \left]-\infty,+\infty\right]$ be proper. Then ∂f is a* monotone operator*: $(\forall u \in \partial f(x))(\forall v \in \partial f(y))$ $\langle x - y, u - v\rangle \geqslant 0$.*

Proposition 9.9 *Let $f\colon X \to \left]-\infty,+\infty\right]$ be proper, and let $x \in \operatorname{dom}\partial f$. Then f is lower semicontinuous at x.*

Proposition 9.10 *Let $f\colon X \to \left]-\infty,+\infty\right]$ be proper. Suppose that $\operatorname{dom} f$ is convex and that $\operatorname{dom} f = \operatorname{dom}\partial f$. Then f is convex.*

Proof. Take x_0, x_1 in $\operatorname{dom} f$, and let $\lambda \in \left]0,1\right[$. Set $x = (1-\lambda)x_0 + \lambda x_1$. Then $x \in \operatorname{dom} f = \operatorname{dom}\partial f$, so let $u \in \partial f(x)$. Then

$$(\forall y \in X) \quad f(x) + \langle y - x, u\rangle \leqslant f(y).$$

Specializing to $y = x_0$ and x_1, and recalling the definition of x, we have

$$\begin{aligned} f(x) + \langle x_0 - ((1-\lambda)x_0 + \lambda x_1), u\rangle &\leqslant f(x_0),\\ f(x) + \langle x_1 - ((1-\lambda)x_0 + \lambda x_1), u\rangle &\leqslant f(x_1), \end{aligned}$$

which simplifies to

$$f(x) + \lambda\,\langle x_0 - x_1, u\rangle \leqslant f(x_0), \tag{9.1a}$$

$$f(x) - (1-\lambda)\,\langle x_0 - x_1, u\rangle \leqslant f(x_1). \tag{9.1b}$$

Finally, $(1-\lambda)$(9.1a) $+\, \lambda$(9.1b) simplifies to

$$f((1-\lambda)x_0 + \lambda x_1) = f(x) \leqslant (1-\lambda)f(x_0) + \lambda f(x_1)$$

and we are done. ☐

The domain of the subdifferential operator ∂f may be a proper subset of $\operatorname{dom} f$, as the next example illustrates:

Example 9.11 Consider the proper lower semicontinuous convex function

$$f\colon \mathbb{R} \to \left]-\infty,+\infty\right] : x \mapsto \begin{cases} +\infty & \text{if } x < 0;\\ -\sqrt{x} & \text{if } x \geqslant 0. \end{cases}$$

Then $\partial f(0) = \varnothing$.

Proof. Suppose to the contrary that $\partial f(0) \neq \varnothing$, say $u \in \partial f(0)$. Then $(\forall y \geqslant 0)\ uy = f(0) + (y-0)u \leqslant f(y) = -\sqrt{y}$ and therefore

$$-\infty < u \leqslant -\frac{\sqrt{y}}{y} = -\frac{1}{\sqrt{y}} \to -\infty \quad \text{as } y \to 0^+,$$

which is absurd! ☐

Fact 9.12 [39, Theorem 23.4] *Let $f\colon X \to \left]-\infty,+\infty\right]$ be convex and proper. Then* $\operatorname{ri}\operatorname{dom} f \subseteq \operatorname{dom}\partial f \subseteq \operatorname{dom} f$. *Moreover, if $x \in \operatorname{ri}\operatorname{dom} f$, then $\partial f(x)$ is bounded $\Leftrightarrow x \in \operatorname{int}\operatorname{dom} f$.*

Corollary 9.13 *Let $f\colon X \to \mathbb{R}$ be convex. Then f is subdifferentiable everywhere, i.e., $(\forall x \in X)\ \partial f(x) \neq \varnothing$.*

Fact 9.14 [39, Theorem 24.7] *Let $f\colon X \to \left]-\infty,+\infty\right]$ be convex, lower semicontinuous, and proper. Suppose that C is a nonempty compact subset of* $\operatorname{int}\operatorname{dom} f$. *Then $\partial f(C)$ is compact and nonempty.*

Exercises

Exercise 9.1 Provide the details for Example 9.3.

Exercise 9.2 Verify Example 9.6.

Exercise 9.3 Verify Proposition 9.8.

Exercise 9.4 Verify Proposition 9.9.

Exercise 9.5 Let $f\colon X \to \left]-\infty,+\infty\right]$ be convex and proper, and let $x \in \operatorname{dom}\partial f$. Set $V := (\operatorname{aff}\operatorname{dom} f) - x = \operatorname{span}((\operatorname{dom} f) - x)$. Show that $V^{\perp} + \partial f(x) \subseteq \partial f(x)$.

Exercise 9.6 Let $f\colon X \to \left]-\infty,+\infty\right]$ be convex and proper, set $\mu := \inf f(X)$ and $S := \operatorname{Argmin} f$, and let $x \in X$. Show that if $S \neq \varnothing$, then

$$f(x) - \mu \leqslant d_{\partial f(x)}(0) \cdot d_S(x).$$

Chapter 10
Normal Cones

10.1 ▪ Normal Cone

Definition 10.1 (normal cone) The *normal cone* of a nonempty subset C of X at $x \in X$ is

$$N_C(x) := \begin{cases} \{u \in X \mid (\forall c \in C) \quad \langle c - x, u \rangle \leqslant 0\} & \text{if } x \in C; \\ \varnothing & \text{if } x \notin C. \end{cases}$$

Proposition 10.2 *Let C be a nonempty subset of X. Then $N_C = \partial\iota_C$. Moreover, if $x \in C$, then $N_C(x)$ is a closed convex cone.*

Proof. Let $x \in X$. If $x \notin C$, then $N_C(x) = \varnothing = \partial\iota_C(x)$. Now assume that $x \in C$, and let $u \in X$. Then

$$\begin{aligned} u \in N_C(x) &\Leftrightarrow (\forall c \in C) \quad \langle c - x, u \rangle \leqslant 0 \\ &\Leftrightarrow (\forall y \in X) \quad \iota_C(x) + \langle y - x, u \rangle \leqslant \iota_C(y) \\ &\Leftrightarrow u \in \partial\iota_C(x). \end{aligned}$$

This and Proposition 9.7 imply that $N_C(x)$ is closed and convex. Clearly $0 \in N_C(x)$. Now let $u \in N_C(x)$ and suppose that $\alpha > 0$. Then $(\forall c \in C)$ $\langle c - x, u \rangle \leqslant 0$ and hence $(\forall c \in C)$ $\langle c - x, \alpha u \rangle \leqslant 0$, i.e., $\alpha u \in N_C(x)$. Hence $N_C(x)$ is also a cone. ☐

Remark 10.3 Suppose C is a nonempty closed convex subset of X, and let $x \in C$. Then Definition 10.1 yields $N_C(x) = (C - x)^\ominus$.

Remark 10.4 Let C be a nonempty closed convex subset of X, and let $x \in C$. It can be shown (see [3, Proposition 6.44(i)]) that

$$T_C(x) := \overline{\text{cone}}\,(C - x) = N_C(x)^\ominus \tag{10.1}$$

is the *tangent cone* of C at x.

10.2 ▪ Examples

Example 10.5 Let $-\infty \leqslant \alpha \leqslant \beta \leqslant +\infty$, and set $C := \{x \in \mathbb{R} \mid \alpha \leqslant x \leqslant \beta\}$. Assume that $C \neq \varnothing$, and let $x \in C$. Then

$$N_C(x) = \begin{cases} \mathbb{R} & \text{if } \alpha = x = \beta; \\ \{0\} & \text{if } \alpha < x < \beta; \\ \mathbb{R}_- & \text{if } \alpha = x < \beta; \\ \mathbb{R}_+ & \text{if } \alpha < x = \beta. \end{cases}$$

Example 10.6 Let C be the closed unit ball $B[0;1]$ in X, and let $x \in C$. Then

$$N_C(x) = \begin{cases} \{0\} & \text{if } \|x\| < 1; \\ \mathbb{R}_+ x & \text{if } \|x\| = 1. \end{cases}$$

Moreover, $T_C(x) = X$ if $\|x\| < 1$, while $T_C(x) = \{y \in X \mid \langle y, x\rangle \leqslant 0\}$.

Proposition 10.7 *Let C be a convex subset of X, and let $x \in \operatorname{int} C$. Then $N_C(x) = \{0\}$.*

Proof. Get $\delta > 0$ such that $B[x;\delta] \subseteq C$. Now let $u \in N_C(x)$. Then $0 \geqslant \sup \langle B[x;\delta] - x, u\rangle = \sup \langle B[0;\delta], u\rangle = \delta \sup \langle B[0;1], u\rangle = \delta\|u\| \geqslant 0$. Hence $\|u\| = 0$ and so $u = 0$. ☐

Example 10.8 Suppose that $X = \mathbb{R}^2$, and set $C := \mathbb{R}^2_+$. Then the following hold:

(i) $N_C(1,2) = \{(0,0)\}$.

(ii) $N_C(1,0) = \{0\} \times \mathbb{R}_-$.

(iii) $N_C(0,2) = \mathbb{R}_- \times \{0\}$.

(iv) $N_C(0,0) = \mathbb{R}^2_-$.

Example 10.9 Set $C := \mathbb{R} \times \{1\}$ in $X = \mathbb{R}^2$. Then $\operatorname{int} C = \varnothing$ while $\operatorname{ri} C = \overline{C} = C$. Moreover, if $x \in C$, then $N_C(x) = \{0\} \times \mathbb{R}$.

Example 10.10 Let C be the probability simplex in $X = \mathbb{R}^n$:

$$C := \mathbb{P}_n = \{x \in \mathbb{R}^n_+ \mid x_1 + \cdots + x_n = 1\}.$$

Let $x \in C$ and $u \in X$. Then

$$u \in N_C(x) \Leftrightarrow (\exists\, \mu \in \mathbb{R})(\forall i \in \{1,\ldots,n\}) \tag{10.2a}$$
$$\text{either } (x_i > 0 \text{ and } u_i = \mu) \tag{10.2b}$$
$$\text{or } (x_i = 0 \text{ and } u_i \leqslant \mu). \tag{10.2c}$$

Proof. Set $I := \{1,\ldots,n\}$, $I_+ := \{i \in I \mid x_i > 0\}$, and $I_0 := \{i \in I \mid x_i = 0\}$. Note that $I_+ \neq \varnothing$ because $x \in C$.

"$\Leftarrow$": Suppose the right side of (10.2) holds, and let $c \in C$. Then

$$\begin{aligned} \langle c - x, u\rangle &= \sum_{i\in I}(c_i - x_i)u_i = \mu \sum_{i\in I_+}(c_i - x_i) + \sum_{i\in I_0} c_i u_i \\ &\leqslant \mu \sum_{i\in I_+}(c_i - x_i) + \mu \sum_{i\in I_0} c_i = \mu \sum_{i\in I} c_i - \mu \sum_{i\in I_+} x_i \\ &= \mu \sum_{i\in I} c_i - \mu \sum_{i\in I} x_i \\ &= \mu - \mu = 0; \end{aligned}$$

thus, $u \in N_C(x)$, as claimed.

"⇒": Suppose the left side of (10.2) holds. Suppose that $i \in I_+$ and that $j \in I \smallsetminus \{i\}$. Cleverly set

$$c_k := \begin{cases} x_k & \text{if } k \notin \{i, j\}; \\ x_i - \frac{1}{2}x_i & \text{if } k = i; \\ x_j + \frac{1}{2}x_i & \text{if } k = j. \end{cases}$$

Then $\sum_{k \in I} c_k = 1$ and $c \in \mathbb{R}^n_+$ because $x \in C$. Hence $c \in C$. Because $u \in N_C(x)$, we obtain

$$\begin{aligned} 0 \geqslant \langle c - x, u \rangle &= \sum_{k \in I} (c_k - x_k) u_k \\ &= \big(x_i - \tfrac{1}{2}x_i - x_i\big)u_i + \big(x_j + \tfrac{1}{2}x_i - x_j\big)u_j \\ &= \tfrac{1}{2}x_i(u_j - u_i). \end{aligned}$$

We have thus shown the implication

$$\left.\begin{array}{c} x_i > 0 \\ j \neq i \end{array}\right\} \quad \Rightarrow \quad u_i \geqslant u_j. \tag{10.3}$$

(Note that this implication is trivially true if $n = 1$.) Consequently,

$$\left.\begin{array}{c} x_i > 0 \\ j \neq i \\ x_j > 0 \end{array}\right\} \quad \Rightarrow \quad u_i = u_j \tag{10.4}$$

and we now well define

$$\mu := u_i, \quad \text{where } i \in I_+.$$

In view of (10.3) and (10.4), we see that the right side of (10.2) holds. ☐

10.3 ▪ Further Properties

Proposition 10.11 *Let C be a nonempty closed convex subset of X, let $x \in C$, and let $u \in X$. Then $u \in N_C(x) \Leftrightarrow \mathrm{P}_C(x + u) = x$.*

Proof. Using Theorem 8.4, we have $u \in N_C(x) \Leftrightarrow (\forall c \in C)\ \langle c - x, u \rangle \leqslant 0 \Leftrightarrow (\forall c \in C)$ $\langle c - x, (x + u) - x \rangle \leqslant 0 \Leftrightarrow x = \mathrm{P}_C(x + u)$. ☐

Proposition 10.12 *Let f be convex and proper on X, and let x, u be in X. Then*

$$u \in \partial f(x) \quad \Leftrightarrow \quad (u, -1) \in N_{\operatorname{epi} f}(x, f(x)). \tag{10.5}$$

Remark 10.13 (relevance in optimization) Suppose we wish to

$$\text{minimize } f(x) \text{ subject to } x \in C.$$

This is the same as to

$$\text{minimize } f + \iota_C.$$

In view of Fermat's rule (Lemma 9.2), every solution x to this problem must satisfy $0 \in \partial(f + \iota_C)(x)$. In many cases, we actually have $\partial(f + \iota_C)(x) = \partial f(x) + \partial \iota_C(x) = \partial f(x) + N_C(x)$ so that the minimizer x satisfies

$$(\exists\, u \in \partial f(x)) \quad -u \in N_C(x).$$

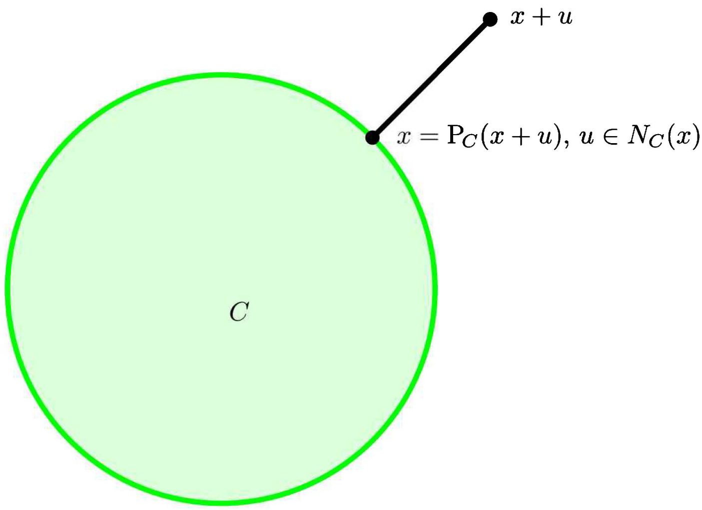


Figure 10.1. *An illustration of Proposition* 10.11.

Moreover, if f is differentiable at x, then this turns into

$$-\nabla f(x) \in N_C(x).$$

We will study this in more detail in Section 15.1 below.

Exercises

Exercise 10.1 Let C be a nonempty closed convex subset of X, let $x \in X$, and let $a \in X$. Show that $N_C(x) = N_{a+C}(a + x)$.

Exercise 10.2 Verify the formula for $N_C(x)$ in Example 10.5.

Exercise 10.3 Verify the formula for $N_C(x)$ in Example 10.6.

Exercise 10.4 Let $C_1, \ldots, C_m$ be nonempty closed convex subsets of Euclidean spaces $X_1, \ldots,$ X_m, respectively. Show that $N_{C_1 \times \cdots \times C_m} = N_{C_1} \times \cdots \times N_{C_m}$.

Exercise 10.5 Verify Example 10.8.

Exercise 10.6 Suppose that $C = c_0 + Y$, where $c_0 \in X$ and Y is a linear subspace of X. Show that $N_C(x) = Y^\perp$ for every $x \in C$.

Exercise 10.7 Verify Proposition 10.12.

Chapter 11

Directional and Classical Derivatives

11.1 ▪ Directional Derivative

Definition 11.1 (directional derivative) Let f be proper on X, suppose that $x \in \operatorname{dom} f$, and let $d \in X$. Then the *directional derivative* of f at x in direction d is

$$\boxed{f'(x;d) := \lim_{\alpha \to 0^+} \frac{f(x+\alpha d) - f(x)}{\alpha},}$$

provided this limit exists in $[-\infty, +\infty]$.

Remark 11.2 Let f be proper on X, suppose that $x \in \operatorname{int} \operatorname{dom} f$, let $d \in X$, and suppose that $f'(x;d) \in \mathbb{R}$. Then

$$\lim_{\alpha \to 0^+} f(x+\alpha d) = f(x)$$

because $f(x+\alpha d) = \alpha(f(x+\alpha d) - f(x))/\alpha + f(x) \to 0 \cdot f'(x;d) + f(x) = f(x)$ as $\alpha \to 0^+$.

Theorem 11.3 *Let f be convex and proper on X, suppose that $x \in \operatorname{int} \operatorname{dom} f$, and let $d \in X$. Then*

$$\Phi_d(\alpha) := \frac{f(x+\alpha d) - f(x)}{\alpha}$$

is increasing on $\mathbb{R}_{++}$; consequently, $f'(x;d)$ exists and

$$f'(x;d) = \inf_{\alpha > 0} \Phi_d(\alpha) = \inf_{\alpha > 0} \frac{f(x+\alpha d) - f(x)}{\alpha}.$$

Moreover, $-\infty < -f'(x;-d) \leqslant f'(x;d) < +\infty$.

Proof. Let $0 < \alpha < \beta$, and set $\lambda := \alpha/\beta$. Then $0 < \lambda < 1$ and the convexity of f gives

$$\begin{aligned} f(x+\alpha d) &= f\big((1-\lambda)x + \lambda(x+\beta d)\big) \\ &\leqslant (1-\lambda)f(x) + \lambda f(x+\beta d) \\ &= f(x) + \lambda\big(f(x+\beta d) - f(x)\big) \end{aligned}$$

and thus

$$f(x + \alpha d) - f(x) \leqslant \frac{\alpha}{\beta}\big(f(x + \beta d) - f(x)\big).$$

Now divide by $\alpha > 0$ to learn that $\Phi_d(\alpha) \leqslant \Phi_d(\beta)$.

Now let us assume additionally that $\beta > 0$ is so small that $[x - \beta d, x + \beta d] \subseteq \operatorname{dom} f$ (which is possible because $x \in \operatorname{int}\operatorname{dom} f$). The convexity of f yields

$$f(x) \leqslant \tfrac{1}{2} f(x - \alpha d) + \tfrac{1}{2} f(x + \alpha d). \tag{11.1}$$

Then we obtain using (11.1) that

$$\begin{aligned} -\Phi_{-d}(\beta) \leqslant -\Phi_{-d}(\alpha) &= \frac{f(x) - f(x - \alpha d)}{\alpha} \leqslant \frac{f(x + \alpha d) - f(x)}{\alpha} \\ &= \Phi_d(\alpha) \\ &\leqslant \Phi_d(\beta). \end{aligned}$$

Hence $-\infty < -\Phi_{-d}(\beta) \leqslant -f'(x; -d) \leqslant f'(x; d) \leqslant \Phi_d(\beta) < \infty$ and we are done. ☐

Corollary 11.4 *Let f_1, f_2 be convex and proper on X, and suppose that $x \in \operatorname{int}\operatorname{dom} f_1 \cap \operatorname{int}\operatorname{dom} f_2$. Then*

$$(f_1 + f_2)'(x; \cdot) = f_1'(x; \cdot) + f_2'(x; \cdot).$$

Corollary 11.5 *Let f be convex and proper on X, suppose that $x \in \operatorname{int}\operatorname{dom} f$, let $d \in X$, let $\gamma \geqslant 0$, and let $y \in X$. Then the following hold:*

(i) $f'(x; d) \leqslant f(x + d) - f(x)$.

(ii) $f'(x; \cdot)$ *is convex.*

(iii) $f'(x; \gamma d) = \gamma f'(x; d)$

(iv) $f'(x; y - x) \leqslant f(y) - f(x)$.

Proof. Consider Theorem 11.3 and its notation.

(i): This is a restatement of $f'(x; d) \leqslant \Phi_d(1)$, which we know to be true.

(ii): For each $\alpha > 0$, the function $d \mapsto (f(x + \alpha d) - f(x))/\alpha$ is convex, and hence so is the limit (see Proposition 4.9) as $\alpha \to 0^+$, which is $f'(x; d)$.

(iii): This is clear from the definition when $\gamma = 0$, so assume $\gamma > 0$. We have $f'(x; \gamma d) \leftarrow \Phi_{\gamma d}(\alpha) = \gamma \Phi_d(\alpha\gamma) \to \gamma f'(x; d)$ as $\alpha \to 0^+$.

(iv): Clear by setting $d = y - x$ and applying (i). ☐

Fact 11.6 [5, Theorem 3.24] *Let $f_1, \ldots, f_m$ be proper convex functions on X. Set $I := \{1, \ldots, m\}$ and $f := \max\{f_1, \ldots, f_m\}$, suppose that $x \in \bigcap_{i \in I} \operatorname{int}\operatorname{dom} f_i$, and set $I(x) := \big\{i \in I \mid f_i(x) = f(x)\big\}$. Then*

$$f'(x; d) = \max_{i \in I(x)} f_i'(x; d).$$

Theorem 11.7 (Max Formula) *Let f be convex and proper on X, suppose that $x \in \operatorname{int}\operatorname{dom} f$, and let $d \in X$. Then*

$$f'(x; d) = \max \langle d, \partial f(x) \rangle = \sigma_{\partial f(x)}(d). \tag{11.2}$$

Proof. Let $u \in \partial f(x)$. Then $(\forall \alpha > 0)$ $f(x) + \langle \alpha d, u\rangle \leqslant f(x+\alpha d)$. It follows that

$$\langle d, u\rangle \leq \frac{f(x+\alpha d) - f(x)}{\alpha} \to f'(x;d) \quad \text{as } \alpha \to 0^+.$$

Thus,

$$f'(x;d) \geqslant \sup \langle d, \partial f(x)\rangle . \tag{11.3}$$

We now tackle the converse inequality as well as attainment. To this end, set $h := f'(x;\cdot)$. By Corollary 11.5(ii) and Theorem 11.3, the function h is convex and full domain. Let $v \in X$. By Corollary 9.13, $\partial h(d) \neq \varnothing$, say $g \in \partial h(d)$. This implies the inequality in

$$\begin{aligned}(\forall \alpha \geqslant 0) \quad f'(x;d) + \langle \alpha v - d, g\rangle &= h(d) + \langle \alpha v - d, g\rangle \\ &\leqslant h(\alpha v) = f'(x;\alpha v) = \alpha f'(x;v),\end{aligned}$$

while the last equality stems from Corollary 11.5(iii). Rearranging gives

$$(\forall \alpha \geqslant 0) \quad f'(x;d) - \langle d, g\rangle \leqslant \alpha\big(f'(x;v) - \langle v, g\rangle\big). \tag{11.4}$$

This implies

$$\langle v, g\rangle \leqslant f'(x;v) \tag{11.5}$$

because if $\langle v, g\rangle > f'(x;v)$, then letting $\alpha \to +\infty$ in (11.4) gives a contradiction. In turn, (11.5), setting $v = y - x$, and Corollary 11.5(iv) yield

$$(\forall y \in X) \quad f(x) + \langle y - x, g\rangle \leqslant f(x) + f'(x; y-x) \leqslant f(y).$$

Thus, $g \in \partial f(x)$. This, combined with setting $\alpha = 0$ in (11.4), gives

$$f'(x;d) \leqslant \langle d, g\rangle \leqslant \sup \langle d, \partial f(x)\rangle . \tag{11.6}$$

Combining (11.3) and (11.6) results in $f'(x;d) = \langle d, g\rangle = \sup \langle d, \partial f(x)\rangle = \sigma_{\partial f(x)}(d)$. Because $g \in \partial f(x)$, the supremum is actually a maximum. ☐

11.2 ▪ Classical Derivative

Recall that f is differentiable at $x \in \operatorname{int}\operatorname{dom} f$ if there exists $g \in X$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - \langle h, g\rangle}{\|h\|} = 0;$$

if this is the case, then g is unique and one writes $g = \nabla f(x)$.

Proposition 11.8 *Let f be proper on X, and suppose that f is differentiable at $x \in \operatorname{int}\operatorname{dom} f$. Then $f'(x;\cdot) = \langle \cdot, \nabla f(x)\rangle$.*

Proof. The identity holds at 0. Now pick $d \in X \smallsetminus \{0\}$, $\alpha > 0$ and set $h = \alpha d$. Then the definition yields $0 = \lim_{\alpha \to 0^+} (f(x+\alpha d) - f(x))/(\alpha\|d\|) - \langle d, \nabla f(x)\rangle / \|d\|$ and the conclusion follows. ☐

Fact 11.9 [3, Proposition 17.31] *Let f be convex and proper on X, and suppose that $x \in \operatorname{int}\operatorname{dom} f$. Then*

$$f \text{ is differentiable at } x \Leftrightarrow \partial f(x) \text{ is a singleton,}$$

in which case $\partial f(x) = \{\nabla f(x)\}$.

Proof. Because f is convex and proper, and $x \in \operatorname{int} \operatorname{dom} f$, by Fact 9.12, $\partial f(x) \neq \varnothing$. Let us show at least "$\Rightarrow$": Let $u \in \partial f(x)$. Fix $h \in X$ and $t > 0$. Then

$$\begin{aligned} f(x) + t\langle h, u\rangle &= f(x) + \langle (x+th) - x, u\rangle \\ &\leqslant f(x+th). \end{aligned}$$

Rearranging and recalling Proposition 11.8 gives $\langle h, u\rangle \leqslant (f(x+th) - f(x))/t \to \langle h, \nabla f(x)\rangle$. By Proposition 1.10, $u = \nabla f(x)$. ☐

Theorem 11.10 *Let f be proper on X, suppose that $D := \operatorname{dom} f$ is convex and open, and that f is differentiable on D. Then the following are equivalent:*

(i) *f is convex.*

(ii) $(\forall x \in D)(\forall y \in D)\ f(x) + \langle y - x, \nabla f(x)\rangle \leqslant f(y)$.

(iii) $(\forall x \in D)(\forall y \in D)\ \langle x - y, \nabla f(x) - \nabla f(y)\rangle \geqslant 0$.

If f is twice differentiable on D, then we may add another item to this list, namely:

(iv) $(\forall x \in D)\ \nabla^2 f(x) \succeq 0$.

Proof. Let x, y be in D.

"(i)$\Rightarrow$(ii)": Corollary 11.5(iv) yields $f'(x; y-x) \leqslant f(y) - f(x)$. On the other hand, $f'(x; y-x) = \langle y - x, \nabla f(x)\rangle$ by Proposition 11.8.

"(ii)$\Rightarrow$(iii)": We have $f(x) + \langle y - x, \nabla f(x)\rangle \leqslant f(y)$ and also (switching x and y) $f(y) + \langle x - y, \nabla f(y)\rangle \leqslant f(x)$. Now add and simplify.

"(iii)$\Rightarrow$(i)": Because D is open, there exists $\delta > 0$ such that $\{x + \delta(x-y), y + \delta(y-x)\} \subseteq D$. Set $I := \,]-\delta, 1+\delta[$ and

$$\phi(\alpha) := \iota_I(\alpha) + f(y + \alpha(x-y))$$

for which

$$(\forall \alpha \in I) \quad \phi'(\alpha) = \langle x - y, \nabla f(y + \alpha(x-y))\rangle\,. \tag{11.7}$$

Now let $\alpha < \beta$ in C and set $z_\alpha := y + \alpha(x-y)$ and $z_\beta := y + \beta(x-y)$. Then

$$\begin{aligned} \phi'(\beta) - \phi'(\alpha) &= \langle x - y, \nabla f(z_\beta) - \nabla f(z_\alpha)\rangle \\ &= \frac{1}{\beta - \alpha}\langle z_\beta - z_\alpha, \nabla f(z_\beta) - \nabla f(z_\alpha)\rangle \\ &\geqslant 0. \end{aligned}$$

Hence ϕ' is increasing on I. It is known from calculus that ϕ is therefore convex. Hence,

$$\begin{aligned} f\big(\alpha x + (1-\alpha)y\big) &= \phi(\alpha) \\ &\leqslant \alpha\phi(1) + (1-\alpha)\phi(0) = \alpha f(x) + (1-\alpha) f(y) \end{aligned}$$

and we are done.

Now assume that f is twice differentiable.

"(iii)$\Rightarrow$(iv)": Let $d \in X$. Then for all $\alpha > 0$ sufficiently small, we have $x + \alpha d \in D$ and

$$\begin{aligned} \langle d, \nabla f(x + \alpha d) - \nabla f(x)\rangle &= \frac{\langle (x + \alpha d) - x, \nabla f(x + \alpha d) - \nabla f(x)\rangle}{\alpha} \\ &\geqslant 0. \end{aligned}$$

Dividing by α and taking the limit as $\alpha \to 0^+$ yields $\big\langle d, \nabla^2 f(x) d\big\rangle \geqslant 0$.

"(iv)⇒(i)": Taking the derivative of equation (11.7) another time yields $\phi''(\alpha) = \langle x - y, (\nabla^2 f(y + \alpha(x-y)))(x-y)\rangle \geqslant 0$. Hence ϕ' is increasing and we complete the proof as in "(iii)⇒(i)". □

Corollary 11.11 *Let $f_1, \ldots, f_m$ be convex and proper on X. Suppose that each f_i is differentiable at $x \in \operatorname{int}\operatorname{dom} f_1 \cap \cdots \cap \operatorname{int}\operatorname{dom} f_m$. Set $f := \max\{f_1, \ldots, f_m\}$, and let $d \in X$. Then*

$$f'(x; d) = \max\big\{\langle d, \nabla f_i(x)\rangle \;\big|\; f_i(x) = f(x)\big\}.$$

Fact 11.12 [3, Corollary 17.42(ii)] *Let f be convex and proper on X, and suppose that f is differentiable on $C := \operatorname{int}\operatorname{dom} f \neq \varnothing$. Then f is continuously differentiable, i.e., ∇f is continuous on C.*

Exercises

Exercise 11.1 Consider the function $f(x) = x\ln(x) - x$ if $x > 0$; $f(0) := 0$; $f(x) = +\infty$ if $x < 0$. Compute $f'(0; \cdot)$.

Exercise 11.2 Prove Corollary 11.4.

Exercise 11.3 Let f be convex and proper on X, suppose that $x \in \operatorname{int}\operatorname{dom} f$, and let $\gamma > 0$. Show that $(\gamma f)'(x; \cdot) = \gamma f'(x; \cdot)$.

Exercise 11.4 Verify Corollary 11.11. *Hint:* See Proposition 11.8 and Fact 11.6.

Exercise 11.5 Consider the ReLU function $f(x) := \max\{x, 0\}$ defined on $\mathbb{R}$. Show that for every $d \in \mathbb{R}$,

$$f'(x; d) = \begin{cases} 0 & \text{if } x < 0; \\ \max\{d, 0\} & \text{if } x = 0; \\ d & \text{if } x > 0. \end{cases}$$

Exercise 11.6 Consider the function $f(x) := |x|$, defined on $\mathbb{R}$. Show that for every $d \in \mathbb{R}$,

$$f'(x; d) = \begin{cases} -d & \text{if } x < 0; \\ |d| & \text{if } x = 0; \\ d & \text{if } x > 0. \end{cases}$$

Chapter 12

Subgradients, Derivatives, and the Bregman Distance

12.1 ▪ Computing Subgradients and Derivatives

Fact 12.1 [3, Proposition 17.16] *Suppose that $X = \mathbb{R}$ and f is convex and proper on $\mathbb{R}$, and let $x \in \operatorname{dom} f$. Then $f'_-(x) \leqslant f'_+(x)$ and*

$$\boxed{\partial f(x) = [f'_-(x), f'_+(x)] \cap \mathbb{R}, \qquad (12.1)}$$

where

$$f'_-(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = -f'(x; -1),$$
$$f'_+(x) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = f'(x; 1).$$

Moreover, f'_-, f'_+ are increasing on $\operatorname{dom} f$ *and if $x < y \in \operatorname{dom} f$, then $f'_+(x) \leqslant f_-(y)$.*

The above facts allow an often painless computation of subdifferentials:

Example 12.2 Let $f(x) = |x|$ on $X = \mathbb{R}$. If $x > 0$, then clearly $f'(x) = 1$ and so $\partial f(x) = \{1\}$. Similarly, if $x < 0$, then $\partial f(x) = \{-1\}$. If $x = 0$, then $f'_+(0) = 1$ and $f'_-(0) = -1$ and so $\partial f(0) = [-1, 1] \cap \mathbb{R} = [-1, 1]$, as seen earlier in Example 9.3.

Example 12.3 (negative entropy) The subdifferential operator of the negative entropy function

$$f(x) := \begin{cases} +\infty & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ x \ln(x) - x & \text{if } x > 0 \end{cases}$$

is

$$\partial f(x) = \begin{cases} \varnothing & \text{if } x \leqslant 0; \\ \{\ln(x)\} & \text{if } x > 0. \end{cases}$$

Example 12.4 Consider $f(x) = -\sqrt{x}$ with $\operatorname{dom} f = \mathbb{R}_+$, which is convex, lower semicontinuous, and proper. If $x > 0$, then $f'(x) = -1/(2\sqrt{x})$. If $x < 0$, then $x \notin \operatorname{dom} f$ and so

$\partial f(x) = \varnothing$. Now let's focus on $x = 0$. Then

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{-\sqrt{h}}{h} = -\infty.$$

But $f'_-(0) = -\infty$ as well and so $\partial f(0) = [-\infty, -\infty] \cap \mathbb{R} = \varnothing$. To sum up,

$$\partial f(x) = \begin{cases} \varnothing & \text{if } x \leqslant 0; \\ \left\{ \dfrac{-1}{2\sqrt{x}} \right\} & \text{if } x > 0. \end{cases}$$

Example 12.5 Suppose that $f(x) = \|x\|$, the Euclidean norm. Then

$$\partial f(x) = \begin{cases} B[0;1] & \text{if } x = 0; \\ \left\{ \dfrac{x}{\|x\|} \right\} & \text{if } x \neq 0. \end{cases}$$

Proof. We have seen the case when $x = 0$ in Proposition 9.4. So assume that $x \neq 0$. Now $\nabla \|x\|^2 = 2x$. Hence the change rule gives $\nabla \|x\| = \nabla \sqrt{\|x\|^2} = \frac{1}{2\sqrt{\|x\|^2}}(2x) = x/\|x\|$ and we are done. ☐

Example 12.6 Let C be a nonempty closed convex subset of X. Then

$$\boxed{\nabla \tfrac{1}{2} d_C^2 = \mathrm{Id} - \mathrm{P}_C.}$$

Proof. Let $z \in X$. Define

$$f(x) := \tfrac{1}{2} d_C^2(z + x) - \tfrac{1}{2} d_C^2(z) - \langle x, z - \mathrm{P}_C z \rangle . \tag{12.2}$$

We have seen in Example 6.10 that f is convex. To complete the proof, we must show that

$$\lim_{x \to 0} \frac{f(x)}{\|x\|} = 0. \tag{12.3}$$

Clearly, we have

$$\|z + x - \mathrm{P}_C(z + x)\| = d_C(z + x) \leqslant \|z + x - \mathrm{P}_C(z)\|. \tag{12.4}$$

Also note that $(\forall y \in X)$

$$\tfrac{1}{2}\|y + x\|^2 - \tfrac{1}{2}\|y\|^2 - \langle x, y \rangle = \tfrac{1}{2}\|x\|^2. \tag{12.5}$$

Then (12.2), (12.4) (with $y = z - \mathrm{P}_C(z)$), and (12.5) yield

$$f(x) \leqslant \tfrac{1}{2}\|z + x - \mathrm{P}_C(z)\|^2 - \tfrac{1}{2}\|z - \mathrm{P}_C(z)\|^2 - \langle x, z - \mathrm{P}_C(z) \rangle = \tfrac{1}{2}\|x\|^2.$$

Hence

$$f(x) \leqslant \tfrac{1}{2}\|x\|^2 \quad \text{and} \quad f(-x) \leqslant \tfrac{1}{2}\|-x\|^2 = \tfrac{1}{2}\|x\|^2.$$

The definition of f, (12.2), gives $f(0) = 0$. Because f is convex, Jensen (Theorem 3.6) now gives

$$0 = f(0) = f\big(\tfrac{1}{2}x + \tfrac{1}{2}(-x)\big) \leqslant \tfrac{1}{2} f(x) + \tfrac{1}{2} f(-x);$$

thus, $-f(-x) \leqslant f(x)$. Combining all of the above, we deduce

$$-\tfrac{1}{2}\|x\|^2 \leqslant -f(-x) \leqslant f(x) \leqslant \tfrac{1}{2}\|x\|^2,$$

which, upon dividing by $\|x\|$, yields

$$-\tfrac{1}{2}\|x\| \leqslant \frac{f(x)}{\|x\|} \leqslant \tfrac{1}{2}\|x\|.$$

Applying the squeeze theorem as $x \to 0$, we obtain (12.3). ☐

12.2 ▪ The Bregman Distance

Definition 12.7 (Bregman distance) Let f be proper on X, and assume that f is differentiable on $\operatorname{int}\operatorname{dom} f$. Then

$$\boxed{D_f(x,y) := \begin{cases} f(x) - f(y) - \langle x - y, \nabla f(y)\rangle & \text{if } y \in \operatorname{int}\operatorname{dom} f; \\ +\infty & \text{if } y \notin \operatorname{int}\operatorname{dom} f \end{cases}}$$

is called the *Bregman distance* between x and y.

Definition 12.7 and Theorem 11.10 imply the following:

Proposition 12.8 *Let $f\colon X \to\,]-\infty, +\infty]$ be convex and proper, and assume that f is differentiable on* $\operatorname{int}\operatorname{dom} f$. *Let $x \in X$ and let $y \in$* $\operatorname{int}\operatorname{dom} f$. *Then*

$$D_f(y,y) = 0 \quad \textit{and} \quad D_f(x,y) \geqslant 0;$$

hence, y is a minimizer of the convex function $D_f(\cdot, y)$. If $x \in$ $\operatorname{int}\operatorname{dom} f$, *then $\nabla D_f(\cdot,y)(x) = \nabla f(x) - \nabla f(y)$.*

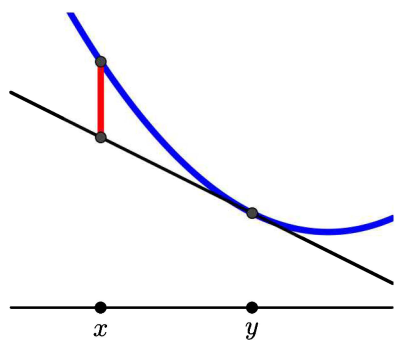


Figure 12.1. *The Bregman distance $D_f(x,y)$ is the length of the red line segment.*

Example 12.9 Suppose that $f = \frac{1}{2}\|\cdot\|^2$. Then $D_f(x,y) = \frac{1}{2}\|x-y\|^2$.

Proposition 12.10 *Let f, g be proper on X, and let $\beta > 0$. Assume that f, g are differentiable on* $\operatorname{int}\operatorname{dom} f \cap \operatorname{int}\operatorname{dom} g$. *Then $D_{f+\beta g}(x,y) = D_f(x,y) + \beta D_g(x,y)$ when $x \in X$ and $y \in$* $\operatorname{int}\operatorname{dom} f \cap \operatorname{int}\operatorname{dom} g$.

Example 12.11 Let f be convex, proper, and differentiable on $\operatorname{int}\operatorname{dom} f$ on X, let $v \in X$, and let $\beta \in \mathbb{R}$. Then $D_{f+\langle\cdot,v\rangle+\beta} = D_f$.

Proof. Set $g := \langle\cdot, v\rangle + \beta$. Then $\nabla g \equiv v$, which implies $D_g(x,y) = \langle x,v\rangle + \beta - \langle y,v\rangle - \beta - \langle x-y,v\rangle = 0$. Now assume that $y \in \operatorname{int}\operatorname{dom} f = \operatorname{int}\operatorname{dom}(f+g)$. By Proposition 12.10, $D_{f+\langle\cdot,v\rangle+\beta} = D_{f+g} = D_f + D_g = D_f$. ☐

Example 12.12 (Kullback–Leibler) Set $f(x) := x\ln(x) - x$ if $x > 0$; $f(0) := 0$; $f(x) := +\infty$ if $x < 0$. Then the associated Bregman distance of f is

$$D_f(x,y) = \begin{cases} x\ln(x/y) - x + y & \text{if } x > 0 \text{ and } y > 0; \\ y & \text{if } x = 0 \text{ and } y > 0; \\ +\infty & \text{otherwise,} \end{cases} \tag{12.6}$$

which is also known as the *Kullback–Leibler Information Divergence*.

Proposition 12.13 (4-point and 3-point identities) *Let f be convex and proper on X, and assume that f is differentiable on* $\operatorname{int}\operatorname{dom} f$. *Let a, b be in* $\operatorname{dom} f$ *and let x, y be in* $\operatorname{int}\operatorname{dom} f$. *Then the Bregman distance D_f satisfies the* 4-point identity

$$\langle a - b, \nabla f(x) - \nabla f(y)\rangle = D_f(b,x) + D_f(a,y) - D_f(a,x) - D_f(b,y) \tag{12.7}$$

and consequently the 3-point identity

$$\langle x - b, \nabla f(x) - \nabla f(y)\rangle = D_f(b,x) + D_f(x,y) - D_f(b,y). \tag{12.8}$$

Proof. Indeed, we have

$$\begin{aligned} & D_f(b,x) + D_f(a,y) - D_f(a,x) - D_f(b,y) \\ &\quad = f(b) - f(x) - \langle b - x, \nabla f(x)\rangle + f(a) - f(y) - \langle a - y, \nabla f(y)\rangle \\ &\qquad - f(a) + f(x) + \langle a - x, \nabla f(x)\rangle - f(b) + f(y) + \langle b - y, \nabla f(y)\rangle \\ &\quad = \langle -b + x + a - x, \nabla f(x)\rangle + \langle -a + y + b - y, \nabla f(y)\rangle \\ &\quad = \langle a - b, \nabla f(x) - \nabla f(y)\rangle, \end{aligned}$$

which is (12.7). Note that (12.8) follows from (12.7) by setting $a = x$. ☐

Exercises

Exercise 12.1 Provide the details for Example 12.3 using (12.1).

Exercise 12.2 (ball pen function) Set $f(x) := -\sqrt{1 - x^2}$ if $|x| \leqslant 1$; $f(x) := +\infty$ if $|x| > 1$. Determine $\partial f(x)$ using (12.1).

Exercise 12.3 Provide the details for Example 12.9

Exercise 12.4 Provide the details for Proposition 12.10.

Exercise 12.5 Verify (12.6).

Exercise 12.6 Compute the Bregman distance of the exponential function and determine whether or not it is convex on $\mathbb{R}^2$.

Exercise 12.7 (Iusem's characterization) Let f be a twice differentiable convex function on X with full domain. Show that f is a quadratic function $\Leftrightarrow$ D_f is symmetric.

Exercise 12.8 Let f, g be convex, lower semicontinuous, and proper on $\mathbb{R}$. Suppose that $\operatorname{dom} f = \operatorname{dom} g = \mathbb{R}_+$, f, g are differentiable on $\mathbb{R}_{++}$, $f(0) = g(0) = 0$, and $f'_+(0) = -\infty$. Let $x > 0$. Show that $f(h) + g(x - h) < f(0) + g(x)$ for all $h > 0$ sufficiently small.

Chapter 13

Subgradient Calculus

13.1 ▪ Positive Multiples and Sums

Proposition 13.1 *Let f be proper on X, and let $\alpha > 0$. Then*

$$\partial(\alpha f)(x) = \alpha \partial f(x). \tag{13.1}$$

Proof. Let u, y be in X. Then we have

$$\begin{aligned} u \in \partial(\alpha f)(x) &\Leftrightarrow \alpha f(x) + \langle y - x, u \rangle \leqslant \alpha f(y) \\ &\Leftrightarrow f(x) + \langle y - x, u/\alpha \rangle \leqslant f(y) \\ &\Leftrightarrow u/\alpha \in \partial f(x) \\ &\Leftrightarrow u \in \alpha \partial f(x), \end{aligned}$$

and we are done. □

Proposition 13.2 *Let f_1, f_2 be proper on X. Then*

$$\partial f_1(x) + \partial f_2(x) \subseteq \partial(f_1 + f_2)(x). \tag{13.2}$$

Proof. Suppose that $u_i \in \partial f_i(x)$. Let $y \in X$. Then

$$\begin{aligned} f_1(x) + \langle y - x, u_1 \rangle &\leqslant f_1(y), \\ f_2(x) + \langle y - x, u_2 \rangle &\leqslant f_2(y). \end{aligned}$$

Adding these inequalities yields $(f_1 + f_2)(x) + \langle y - x, u_1 + u_2 \rangle \leqslant (f_1 + f_2)(y)$. Thus $u_1 + u_2 \in \partial(f_1 + f_2)(x)$. □

Equality in (13.2) may fail:

Example 13.3 (failure of the sum rule) In $X = \mathbb{R}^2$, consider the balls $C_1 := B[(-1, 0); 1]$ and $C_2 := B[(1, 0); 1]$. $(-1, 0)$ and $(1, 0)$, respectively. Set $f_1 := \iota_{C_1}$ and $f_2 := \iota_{C_2}$. Then the following hold:

(i) $f_1 + f_2 = \iota_{\{(0,0)\}}$.

(ii) $\partial(f_1 + f_2)(0, 0) = \mathbb{R}^2$.

(iii) $\partial f_1(0, 0) = \mathbb{R}_+ \times \{0\}$ and $\partial f_2(0, 0) = \mathbb{R}_- \times \{0\}$.

(iv) $\partial f_1(0, 0) + \partial f_2(0, 0) = \mathbb{R} \times \{0\}$.

If the domains "overlap sufficiently," then (13.2) turns into an equality.

Theorem 13.4 *Let f_1, f_2 be convex and proper on X. Suppose that $x \in \operatorname{int}(\operatorname{dom} f_1 \cap \operatorname{dom} f_2) = \operatorname{int}\operatorname{dom} f_1 \cap \operatorname{int}\operatorname{dom} f_2$. Then*

$$\partial f_1(x) + \partial f_2(x) = \partial(f_1 + f_2)(x). \tag{13.3}$$

Proof. Set $f := f_1 + f_2$, and let $d \in X$. On the one hand, by Theorem 11.7,

$$f'(x; d) = \sigma_{\partial f(x)}(d).$$

On the other hand, by Corollary 11.4, again Theorem 11.7, and Proposition 7.7(vi),

$$f'(x; d) = f_1'(x; d) + f_2'(x; d) = \sigma_{\partial f_1(x)}(d) + \sigma_{\partial f_2(x)}(d) = \sigma_{\partial f_1(x) + \partial f_2(x)}(d).$$

Altogether,

$$\sigma_{\partial f(x)}(d) = \sigma_{\partial f_1(x) + \partial f_2(x)}(d). \tag{13.4}$$

Next, $\partial f(x)$ is compact and convex. Each $\partial f_i(x)$ is compact and convex, hence so is $\partial f_1(x) + \partial f_2(x)$ by Exercise 2.18. With the help of Lemma 8.15, we deduce from (13.4) that $\partial f(x) = \partial f_1(x) + \partial f_2(x)$. $\square$

Note that by induction, one may easily obtain a formula for more than 2 summands.

Example 13.5 (the median) Let $a_1 < a_2 < \cdots < a_n$ be real numbers where $n = 2m - 1$ is an odd integer. Then the middle index is $m := (n+1)/2$ and a_m, the median, is the unique minimizer of the function

$$f(x) := \sum_{i=1}^{n} |x - a_i|.$$

Proof. Set $f_i(x) := |x - a_i|$. Then f_i is continuous, convex, with full domain, and

$$\partial f_i(x) = \begin{cases} \{-1\} & \text{if } x < a_i; \\ [-1, 1] & \text{if } x = a_i; \\ \{1\} & \text{if } a_i < x. \end{cases}$$

Set $A := \{a_1, \ldots, a_n\}$, and denote the cardinality of a set S by $\operatorname{card}(S)$. Because $f = f_1 + \cdots + f_n$, the sum rule gives

$$\begin{aligned}
\partial f(x) &= \sum_{i=1}^{n} \partial f_i(x) \\
&= \sum_{i=1}^{n} \begin{cases} \{-1\} & \text{if } x < a_i; \\ [-1, 1] & \text{if } x = a_i; \\ \{1\} & \text{if } a_i < x \end{cases} \\
&= -\operatorname{card}\big(A \cap {]x, +\infty[}\big) + \operatorname{card}\big(A \cap {]-\infty, x[}\big) + \begin{cases} \{0\} & \text{if } x \notin A; \\ [-1, 1] & \text{if } x \in A. \end{cases}
\end{aligned}$$

Because n is odd, we see that $0 \in \partial f(x) \Leftrightarrow x = a_m$. Now recall Fermat's rule (Lemma 9.2). $\square$

Remark 13.6 (median vs. mean) Example 13.5 is very nice: we have recognized the median as the minimizer of the nonsmooth function

$$\sum_{i=1}^{n} |x - a_i|.$$

In contrast, the mean is the minimizer of the smooth function

$$\sum_{i=1}^{n} |x - a_i|^2.$$

Sum rules like the one we just proved are hard. Better ones are available. We list a popular one next.

Fact 13.7 (a strong sum rule) [3, Corollary 16.48(ii)] [39, Theorem 23.9] *Let f_1, f_2 be convex and proper on X. Suppose that* $\operatorname{ri}\operatorname{dom} f_1 \cap \operatorname{ri}\operatorname{dom} f_2 \neq \varnothing$, *that* $\operatorname{dom} f_1 \cap \operatorname{int}\operatorname{dom} f_2 \neq \varnothing$, *or that* ($\operatorname{ri}\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \neq \varnothing$ *and f_2 is polyhedral*[2]*). Then*

$$\partial(f_1 + f_2) = \partial f_1 + \partial f_2$$

everywhere.

As is often the case, *separable* sums are much easier:

Proposition 13.8 (separable sum) *Let f_i be convex and proper on X_i for $i \in \{1, \ldots, m\}$, and set*

$$f(x_1, \ldots, x_m) := f_1(x_1) + \cdots + f_m(x_m).$$

Then

$$\partial f(x_1, \ldots, x_m) = \partial f_1(x_1) \times \cdots \times \partial f_m(x_m).$$

Example 13.9 Suppose that $f(x) = \|x\|_1 = \sum_{i=1}^{m} |x_i|$ on $\mathbb{R}^m$. Then

$$\partial f(x) = \operatorname{Sign}(x_1) \times \cdots \times \operatorname{Sign}(x_m),$$

where $\operatorname{Sign}(\xi) := \partial|\cdot|(\xi)$. In particular $(\operatorname{sign}(x_1), \ldots, \operatorname{sign}(x_m)) \in \partial f(x)$.

13.2 ▪ The Sum Rule and Optimization

Proposition 13.10 *Let f be convex, lower semicontinuous, and proper on X, and let C be a nonempty closed convex subset of X. Suppose that* $\operatorname{ri} C \cap \operatorname{ri}\operatorname{dom} f \neq \varnothing$, *that* $\operatorname{int} C \cap \operatorname{dom} f \neq \varnothing$, *that* $C \cap \operatorname{int}\operatorname{dom} f \neq \varnothing$, *or that ($C$ is polyhedral and* $C \cap \operatorname{ri}\operatorname{dom} f \neq \varnothing$*). Consider the problem (P), which asks to*

$$\textit{minimize } f(x) \textit{ subject to } x \in C.$$

Let $x \in C$. Then the following are equivalent:

(i) *x solves (P).*

(ii) $0 \in \partial(f + \iota_C)(x)$.

(iii) $0 \in \partial f(x) + N_C(x)$.

(iv) $-N_C(x) \cap \partial f(x) \neq \varnothing$.

(v) $(\exists\, g \in \partial f(x))(\forall c \in C)\ \langle c - x, g\rangle \geqslant 0$.

Proof. "(i)⇔(ii)": Clear from Fermat's rule (Lemma 9.2). "(ii)⇔(iii)": Clear from the sum rule (Fact 13.7). "(iii)⇔(iv)": Obvious. "(iv)⇔(v)": Clear from the definition of the normal cone. □

[2] A function is polyhedral if it is the sum of two functions: one is the finite maximum of affine (i.e., linear + constant) functions and the other is the indicator function of a set that is the nonempty intersection of finitely many halfspaces. Note that this allows for the intersection to be X by considering the intersection over an empty index set.

Corollary 13.11 *Suppose that f is convex and proper on $X = \mathbb{R}^n$, and that $C = \mathbb{P}_n$ is the probability simplex in $\mathbb{R}^n$. Suppose that $C \cap \operatorname{ri} \operatorname{dom} f \neq \varnothing$, and let $x \in C$. Then x minimizes f over $C \Leftrightarrow$*

$$(\exists\, g \in \partial f(x))(\exists\, \mu \in \mathbb{R})\ \ g_i \begin{cases} = \mu & \text{if } x_i > 0; \\ \geqslant \mu & \text{if } x_i = 0. \end{cases}$$

Proof. Note that C is polyhedral. Hence the result follows by combining Proposition 13.10 with Example 10.10. □

Example 13.12 Let $y \in X := \mathbb{R}^n$ be fixed, set $I := \{1, \ldots, n\}$, and consider the problem (P) of minimizing

$$f(x) := \sum_{i \in I} x_i \ln(x_i) - x_i y_i$$

over the probability simplex $C := \mathbb{P}_n$ in X, and where $0 \ln(0) = 0$. Then (P) has a unique solution x which is given by

$$(\forall i \in I) \quad x_i = \frac{\exp(y_i)}{\sum_{j \in I} \exp(y_j)},$$

i.e., x is the *softmax* of y.

Proof. Note that f is convex, lower semicontinuous, and proper on X (see also Example 18.12 below). Moreover, C is compact and convex. By Corollary 5.6, $f|_C$ has a minimizer. It follows from Exercise 12.8 that $x \in \mathbb{R}^n_{++}$ and so $x \in \operatorname{ri} C$. Note that then

$$\nabla f(x) = (\ln(x_i) + 1 - y_i)_{i \in I}.$$

By Corollary 13.11, there exists $\mu \in \mathbb{R}$ such that

$$(\forall i \in I) \ \ \mu = \ln(x_i) + 1 - y_i.$$

Hence $(\forall i \in I)\ \ln(x_i) = \mu - 1 + y_i$, i.e.,

$$x_i = \exp(\mu - 1 + y_i) = \alpha \exp(y_i), \quad \text{where } \alpha := \exp(\mu - 1). \tag{13.5}$$

On the other hand, $x \in C$ and so $\sum_{i \in I} x_i = 1$, which means $\alpha \sum_{i \in I} \exp(y_i) = 1$, i.e.,

$$\alpha = \frac{1}{\sum_{j \in I} \exp(y_j)}. \tag{13.6}$$

Combining (13.5) and (13.6), we obtain the result. □

13.3 ▪ Minimizers of the Sum vs. Zeros of the Sum of the Subdifferentials

We now discuss the subtle interplay between minimizers of the sum of two functions and zeros of the sum of the corresponding subdifferential operators.

Theorem 13.13 *Let f, g be convex and lower semicontinuous on X such that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \varnothing$. *Set*

$$S := \operatorname{Argmin}(f + g) \ \ \text{and} \ \ Z := \operatorname{zer}(\partial f + \partial g).$$

Then the following hold:

(i) $Z \subseteq S$.

(ii) *If $Z \neq \varnothing$, then $Z = S$.*

Proof. (i): Suppose that $z \in Z$. Then $0 \in \partial f(z) + \partial g(z)$. By (13.2), $0 \in \partial(f+g)(z)$. Hence, Fermat's rule (Lemma 9.2) yields $z \in S$.

(ii): Suppose that $Z \neq \varnothing$, say $z \in Z$. By (i), $z \in S \neq \varnothing$. Let $s \in S$. Now s and z are both minimizers of $f+g$; thus,

$$f(s) - f(z) = g(z) - g(s). \tag{13.7}$$

Because $z \in Z$, there exists $w \in \partial f(z)$ such that $-w \in \partial g(z)$. Thus,

$$(\forall x \in X) \quad f(x) \geqslant f(z) + \langle x - z, w\rangle \quad \text{and} \quad g(x) \geqslant g(z) + \langle x - z, -w\rangle. \tag{13.8}$$

In particular, $f(s) \geqslant f(z) + \langle s - z, w\rangle$ and $g(s) \geqslant g(z) + \langle s - z, -w\rangle$; in turn, this implies

$$f(s) - f(z) \geqslant \langle s - z, w\rangle \geqslant g(z) - g(s). \tag{13.9}$$

Combining (13.7) and (13.9) yields

$$f(s) - f(z) = \langle s - z, w\rangle = g(z) - g(s). \tag{13.10}$$

Finally, let $x \in X$. Then

$$\begin{aligned} f(x) &\geqslant f(z) + \langle x - z, w\rangle && \text{(by (13.8))} \\ &= \big(f(z) + \langle s - z, w\rangle\big) + \langle x - s, w\rangle \\ &= f(s) + \langle x - s, w\rangle, && \text{(by (13.10))} \end{aligned}$$

which yields $w \in \partial f(s)$. Similarly, $-w \in \partial g(s)$. Therefore, $0 = w + (-w) \in \partial f(s) + \partial g(s)$ and thus $s \in Z$. ☐

Remark 13.14 Consider Theorem 13.13 and its notation.

A way to guarantee that $Z = S$ is that a constraint qualification such as $\operatorname{ri} \operatorname{dom} f \cap \operatorname{ri} \operatorname{dom} g \neq \varnothing$ holds (see Fact 13.7).

Theorem 13.13 yields the following *trichotomy*, i.e., exactly one of the following holds:

(i) $Z = S = \varnothing$.

(ii) $Z = \varnothing$ and $S \neq \varnothing$.

(iii) $Z = S \neq \varnothing$.

Exercises

Exercise 13.1 Provide the details for Example 13.3.

Exercise 13.2 Prove the statement regarding the mean in Remark 13.6

Exercise 13.3 Provide the details for Proposition 13.8.

Exercise 13.4 Consider the proof of Example 13.12. Provide more details for the step that shows that every solution of (P) must lie in $\mathbb{R}^n_{++}$.

Exercise 13.5 Provide examples for each of the three cases considered in the trichotomy of Remark 13.14.

Chapter 14

Composition and Maximum

14.1 ▪ Composition

Fact 14.1 [3, Corollary 16.72] *Let f be convex on X, and let g be convex and increasing on $\mathbb{R}$. Suppose that g is differentiable at $f(x)$. Then*

$$\partial(g \circ f)(x) = g'(f(x))\partial f(x).$$

Example 14.2 Let C be a nonempty closed convex subset of X. Then

$$(\forall x \in X) \quad \partial d_C(x) = \begin{cases} B[0;1] \cap N_C(x) & \text{if } x \in C; \\ \left\{ \dfrac{x - \mathrm{P}_C(x)}{d_C(x)} \right\} & \text{if } x \notin C. \end{cases} \tag{14.1}$$

Proof. Because C is convex, the functions d_C and $f := \frac{1}{2}d_C^2$ are convex by Example 6.10; moreover, $\nabla f = \mathrm{Id} - \mathrm{P}_C$ (see Example 12.6). Furthermore, set $g(x) := \frac{1}{2}\max^2\{0, x\}$, which is convex, increasing, with $g'(x) = \max\{0, x\}$. Then the chain rule (Fact 14.1) yields

$$\begin{aligned} x - \mathrm{P}_C(x) &= \nabla f(x) = \nabla(g \circ d_C)(x) \\ &= \partial(g \circ d_C)(x) = g'(d_C(x))\partial d_C(x) \\ &= \max\{0, d_C(x)\}\partial d_C(x) \\ &= d_C(x)\partial d_C(x). \end{aligned}$$

If $x \notin C$, then $d_C(x) > 0$ and (14.1) follows. We thus assume that $x \in C$. Let $u \in X$.

Assume first that $u \in \partial d_C(x)$. Then

$$(\forall y \in X) \quad \langle y - x, u\rangle = d_C(x) + \langle y - x, u\rangle \leqslant d_C(y); \tag{14.2}$$

in particular, $(\forall c \in C)$ $\langle c - x, u\rangle \leqslant 0$ and so $u \in N_C(x)$. Setting $y = x + u$ in (14.2) yields $\|u\|^2 = \langle u, (x+u) - x\rangle \leqslant d_C(x+u) \leqslant \|(x+u) - x\| = \|u\|$ and hence $\|u\| \leqslant 1$, i.e., $u \in B[0;1]$.

Now assume that $u \in N_C(x) \cap B[0;1]$. Then for every $y \in X$, we have

$$\begin{aligned} \langle y - x, u\rangle &= \langle y - \mathrm{P}_C(y), u\rangle + \langle \mathrm{P}_C(y) - x, u\rangle \\ &\leqslant \langle y - \mathrm{P}_C(y), u\rangle \leqslant \|y - \mathrm{P}_C(y)\|\|u\| \\ &\leqslant \|y - \mathrm{P}_C(y)\| = d_C(y) \\ &= d_C(y) - d_C(x); \end{aligned}$$

thus, $u \in \partial d_C(x)$ and we are done. $\square$

Remark 14.3 Specializing Example 14.2 to $X = \mathbb{R}$ and $C = \{0\}$ yields another way to obtain $\partial|\cdot|$.

Proposition 14.4 *Let $A\colon X \to Y$ be linear, and let g be proper on Y. Then*

$$A^*\partial g(Ax) \subseteq \partial(g \circ A)(x).$$

Fact 14.5 [39, Theorem 23.9] *Let $A\colon X \to Y$ be linear, and let g be convex and proper on Y. Suppose that* $\operatorname{ran} A \cap \operatorname{ri}\operatorname{dom} g \neq \varnothing$. *Then*

$$A^*\partial g(Ax) = \partial(g \circ A)(x).$$

Corollary 14.6 *Let $A\colon X \to Y$ be linear, and let g be convex and proper on Y. Suppose that A is surjective. Then $A^*\partial g(Ax) = \partial(g \circ A)(x)$.*

Example 14.7 Suppose that $X = \mathbb{R}^2$. Set $C := B[(-1,0);1]$, $V := \{0\} \times \mathbb{R}$, and $A := \mathrm{P}_V$. Then $A^*\partial\iota_C(A(0,0)) \subsetneqq \partial(\iota_C \circ A)(0,0)$.

14.2 ▪ Maximum

Fact 14.8 [3, Theorem 18.5] *Let $f_1,\ldots,f_m$ be convex and proper on X, and set $I := \{1,\ldots,m\}$ and $f := \max_{i\in I} f_i$. Suppose that $x \in \bigcap_{i\in I} \operatorname{int}\operatorname{dom} f_i$ and set $I(x) := \big\{i \in I \;\big|\; f_i(x) = f(x)\big\}$. Then*

$$\partial f(x) = \operatorname{conv} \bigcup_{i\in I(x)} \partial f_i(x). \tag{14.3}$$

Remark 14.9 Because $|x| = \max\{x,-x\}$, we can use Fact 14.8 to obtain once again $\partial|\cdot|$.

Exercises

Exercise 14.1 Provide the details for Remark 14.3.

Exercise 14.2 Verify Proposition 14.4.

Exercise 14.3 Verify Corollary 14.6.

Exercise 14.4 Provide the details for Example 14.7.

Exercise 14.5 Verify the inclusion "$\supseteq$" from (14.3).

Exercise 14.6 Provide the details for Remark 14.9.

Chapter 15

Minimizing a Sum and the Fritz John Necessary Conditions

15.1 ▪ Minimizing a Sum

Theorem 15.1 *Let f be proper on X, and let g be convex, lower semicontinuous, and proper. Assume that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \varnothing$. *Consider the problem (P) of minimizing $f + g$.*

(i) *If x^* is local minimizer of (P) and f is differentiable at x^*, then $-\nabla f(x^*) \in \partial g(x^*)$.*

(ii) *If f is convex and differentiable at $x^* \in \operatorname{dom} g$, then x^* is a global minimizer of (P) $\Leftrightarrow$ $-\nabla f(x^*) \in \partial g(x^*)$.*

Proof. (i): Let $y \in \operatorname{dom} g$. Because g is convex, so is $\operatorname{dom} g$. Set $(\forall \lambda \in \,]0,1[)$ $x_\lambda := (1-\lambda)x^* + \lambda y \in \operatorname{dom} g$. Because x^* is a local minimizer of (P), we have for λ sufficiently small

$$\begin{aligned} f(x^*) + g(x^*) &\leqslant f(x_\lambda) + g(x_\lambda) \\ &= f(x^* + \lambda(y - x^*)) + g((1-\lambda)x^* + \lambda y) \\ &\leqslant f(x^* + \lambda(y - x^*)) + (1-\lambda)g(x^*) + \lambda g(y), \end{aligned}$$

which, after rearranging, yields

$$g(x^*) - g(y) \leqslant \frac{f(x^* + \lambda(y - x^*)) - f(x^*)}{\lambda} \to f'(x^*; y - x^*) = \langle y - x^*, \nabla f(x^*)\rangle$$

as $\lambda \to 0^+$. Thus, $g(x^*) + \langle y - x^*, -\nabla f(x^*)\rangle \leqslant g(y)$ and hence $-\nabla f(x^*) \in \partial g(x^*)$.

(ii): The direction "$\Rightarrow$" is clear from (i). To prove "$\Leftarrow$", assume that $-\nabla f(x^*) \in \partial g(x^*)$. Then, using (13.2), $0 \in \nabla f(x^*) + \partial g(x^*) = \partial f(x^*) + \partial g(x^*) \subseteq \partial(f+g)(x^*)$; hence, by Fermat's rule (Lemma 9.2), x^* is a (global) minimizer of $f + g$. $\square$

Example 15.2 Consider the setting of Theorem 15.1. Let C be a nonempty closed convex subset of X such that $C \cap \operatorname{dom} f \neq \varnothing$. Consider the problem (P) of minimizing f over C. If x is a local minimizer of (P) and x is differentiable at x, then $-\nabla f(x) \in N_C(x)$.

Example 15.3 Consider the setting of Theorem 15.1 with $g = \lambda\|\cdot\|_1$. If x minimizes $f + \lambda\|\cdot\|_1$ and f is differentiable at x, then

$$\frac{\partial f(x)}{\partial x_i} \begin{cases} = \lambda & \text{if } x_i < 0; \\ \in [-\lambda, \lambda] & \text{if } x_i = 0; \\ -\lambda & \text{if } x_i > 0. \end{cases}$$

15.2 ▪ Fritz John Necessary Conditions

For the remainder of this chapter, we assume that

$$f, g_1, \dots, g_m \quad \text{are functions from } X \text{ to } \mathbb{R},$$

that $I := \{1, \dots, m\}$, and we consider the problem (P), which asks to

$$\text{minimize } f(x) \text{ subject to } g_i(x) \leqslant 0 \text{ for every } i \in I.$$

We refer to f as the *objective function*, while the functions g_i give rise to the *constraints*. The *optimal value* of (P) is defined by

$$\mu := \inf \big\{ f(x) \bigm| (\forall i \in I)\ g_i(x) \leqslant 0 \big\}.$$

If $x \in X$ and $g_i(x) \leqslant 0$ for all $i \in I$, then x is a *feasible point*. If $x \in X$ is a feasible point and $f(x) = \mu$, then x is a *solution* of the constrained problem (P). Now define the function

$$F(x) := \max \big\{ f(x) - \mu, g_1(x), \dots, g_m(x) \big\}. \tag{15.1}$$

Proposition 15.4 *If* (P) *has solutions, then* $\min F(X) = 0$ *and the minimizers of* F *are exactly the solutions of* (P).

Theorem 15.5 (Fritz John) *Suppose that* $f, g_1, \dots, g_m$ *are all convex and that* x *solves* (P). *Then there exist* $\alpha_0 \geqslant 0, \dots, \alpha_m \geqslant 0$, *not all* α_i *are equal to* 0, *such that*

$$0 \in \alpha_0 \partial f(x) + \sum_{i \in I} \alpha_i \partial g_i(x),$$

and we have complementary slackness, *i.e.,*

$$(\forall i \in I) \quad \alpha_i g_i(x) = 0.$$

Proof. It is convenient to set $g_0(\cdot) := f(\cdot) - \mu$. By Proposition 15.4, $F(x) = 0 = \min F(X)$. Hence, by Fermat's rule (Lemma 9.2) and the max rule (Fact 14.8),

$$0 \in \partial F(x) = \operatorname{conv} \bigcup_{i \in I_0(x)} \partial g_i(x),$$

where $I_0(x) := \{0\} \cup \big\{ i \in I \bigm| g_i(x) = 0 \big\}$. Hence $(\forall i \in I_0(x))$ $(\exists\, \alpha_i \geqslant 0)$ $\sum_{j \in I_0(x)} \alpha_j = 1$, which implies that some $\alpha_j > 0$, and

$$0 \in \sum_{i \in I_0(x)} \alpha_i \partial g_i(x) = \alpha_0 \partial f(x) + \sum_{i \in I_0(x) \smallsetminus \{0\}} \alpha_i \partial g_i(x).$$

Finally, we simply set $\alpha_i = 0$ for $i \in I \smallsetminus I_0(x)$. With that choice, the conclusion holds. □

Exercises

Exercise 15.1 Provide an example that shows that Theorem 15.1(ii) fails if f is not convex.

Exercise 15.2 Illustrate that F in (15.1) may fail to be differentiable even when f and all g_i are.

Exercise 15.3 Verify Proposition 15.4.

Exercise 15.4 Suppose $X = \mathbb{R}$, $m = 1$, $f(x) = x$, and $g_1(x) = \frac{1}{2}x^2$. Find all solutions of (P) and also find α_0 and α_1 as in Theorem 15.5.

Exercise 15.5 Suppose $X = \mathbb{R}^2$, $m = 1$, $f(x) = f(x_1, x_2) = \frac{1}{2}x_1^2 - x_2$, and $g_1(x) = |x_2|$. Discuss this in the context of Theorem 15.5.

Exercise 15.6 Suppose $X = \mathbb{R}^2$, $m = 1$, $f(x) = f(x_1, x_2) = \frac{1}{2}x_1^2 - x_2$, and $g_1(x) = \frac{1}{2}x_2^2$. Discuss this in the context of Theorem 15.5.

Chapter 16

Karush–Kuhn–Tucker Conditions

In this chapter, we assume that

$$f, g_1, \dots, g_m \quad \text{are functions from } X \text{ to } \mathbb{R},$$

that $I := \{1, \dots, m\}$, and we consider the problem (P), which asks to

$$\text{minimize } f(x) \text{ subject to } g_i(x) \leqslant 0 \text{ for every } i \in I.$$

The *optimal value* of (P) is defined by

$$\mu := \inf \big\{ f(x) \;\big|\; (\forall i \in I)\ g_i(x) \leqslant 0 \big\}.$$

16.1 ▪ KKT Necessary Conditions

Theorem 16.1 *Suppose that all functions $f, g_1, \dots, g_m$ are convex, and that x solves (P). Assume that* Slater's condition *holds, i.e.,*

$$(\exists\, s \in X)(\forall i \in I) \quad g_i(s) < 0.$$

Then there exist Lagrange multipliers $\lambda_1 \geqslant 0, \dots, \lambda_m \geqslant 0$ *such that the* KKT conditions

$$0 \in \partial f(x) + \sum_{i \in I} \lambda_i \partial g_i(x), \qquad \text{(stationarity)}$$

$$(\forall i \in I)\ \ \lambda_i g_i(x) = 0 \qquad \text{(complementary slackness)}$$

hold.

Proof. By Theorem 15.5,

$$(\exists\, \alpha_0 \geqslant 0)(\exists\, \alpha_1 \geqslant 0) \cdots (\exists\, \alpha_m \geqslant 0) \ \text{ not all } \alpha_i = 0, \tag{16.1}$$

such that

$$0 \in \alpha_0 \partial f(x) + \sum_{i \in I} \alpha_i \partial g_i(x) \tag{16.2}$$

and

$$(\forall i \in I) \quad \alpha_i g_i(x) = 0. \tag{16.3}$$

Claim: $\alpha_0 > 0$.

Assume to the contrary that $\alpha_0 = 0$. By (16.1) and (16.2), we obtain $(\forall i \in I)(\exists\, v_i \in \partial g_i(x))$ such that

$$\sum_{i \in I} \alpha_i v_i = 0 \tag{16.4}$$

as well as

$$(\forall i \in I) \quad g_i(x) + \langle s - x, v_i \rangle \leqslant g_i(s). \tag{16.5}$$

Recalling complementary slackness (16.3) and multiplying (16.5) by $\alpha_i \geqslant 0$, we learn that

$$(\forall i \in I) \quad \langle s - x, \alpha_i v_i \rangle = \alpha_i g_i(x) + \langle s - x, \alpha_i v_i \rangle \leqslant \alpha_i g_i(s). \tag{16.6}$$

Recalling (16.4) and summing (16.6) yields

$$0 = \langle s - x, 0 \rangle = \sum_{i \in I} \langle s - x, \alpha_i v_i \rangle \leqslant \sum_{i \in I} \alpha_i g_i(s),$$

which is absurd because $\sum_{i \in I} \alpha_i g_i(s) < 0$ by (16.1) and the assumption that s is a Slater point. We have thus verified the claim.

Therefore, we can and do divide (16.2) by $\alpha_0 > 0$, and the conclusion now follows after setting each $\lambda_i := \alpha_i / \alpha_0$. ☐

16.2 ▪ KKT Sufficient Conditions

Theorem 16.2 *Suppose that all functions $f, g_1, \ldots, g_m$ are convex and that $x \in X$ and $\lambda \in \mathbb{R}^m$ satisfy the following:*

$$(\forall i \in I) \quad g_i(x) \leqslant 0, \qquad \text{(primal feasibility)}$$

$$(\forall i \in I) \quad \lambda_i \geqslant 0, \qquad \text{(dual feasibility)}$$

$$0 \in \partial f(x) + \sum_{i \in I} \lambda_i \partial g_i(x), \qquad \text{(stationarity)}$$

$$(\forall i \in I) \quad \lambda_i g_i(x) = 0. \qquad \text{(complementary slackness)}$$

Then x solves (P).

Proof. (primal feasibility) guarantees that x is feasible for (P). Now define the function

$$h := f + \sum_{i \in I} \lambda_i g_i.$$

Because $f, g_1, \ldots, g_m$ are convex, so is h by (dual feasibility) and Propositions 4.1 and 4.5. (stationarity), the weak sum rule (13.2), the scalar multiple rule (13.1), and again (dual feasibility) yield

$$0 \in \partial h(x).$$

By Fermat's rule (Lemma 9.2), we deduce that

$$x \text{ is a (global) minimizer of } h.$$

Now let $y \in X$ be feasible for (P), i.e.,

$$(\forall i \in I) \quad g_i(y) \leqslant 0.$$

Then

$$\begin{aligned}
f(x) &= f(x) + \sum_{i \in I} \lambda_i g_i(x) && \text{(using (complementary slackness))} \\
&= h(x) && \text{(using the definition of } h\text{)} \\
&\leqslant h(y) && (x \text{ minimizes } h) \\
&= f(y) + \sum_{i \in I} \lambda_i g_i(y) && \text{(using the definition of } h\text{)} \\
&\leqslant f(y), && \text{(using } \lambda_i \geqslant 0 \text{ and } g_i(y) \leqslant 0\text{)}
\end{aligned}$$

which shows that x solves (P). ☐

Corollary 16.3 *Suppose that all functions $f, g_1, \dots, g_m$ are convex, and Slater's condition holds:*

$$(\exists\, s \in X)(\forall i \in I) \quad g_i(s) < 0.$$

Let $x \in X$. Then x solves (P) if and only if there exists $\lambda \in \mathbb{R}^m$ such that the following hold:

$$\begin{aligned}
&(\forall i \in I) \quad g_i(x) \leqslant 0, && \text{(primal feasibility)} \\
&(\forall i \in I) \quad \lambda_i \geqslant 0, && \text{(dual feasibility)} \\
&0 \in \partial f(x) + \sum_{i \in I} \lambda_i \partial g_i(x), && \text{(stationarity)} \\
&(\forall i \in I) \quad \lambda_i g_i(x) = 0. && \text{(complementary slackness)}
\end{aligned}$$

Proof. "$\Rightarrow$": Theorem 16.1. "$\Leftarrow$": Theorem 16.2. ☐

Exercises

Exercise 16.1 (no Lagrange multiplier) Suppose that $X = \mathbb{R}$, $m = 1$, $f(x) = x$, and $g_1(x) = \frac{1}{2}x^2$. Discuss this situation in the context of Theorems 16.1 and 16.2.

Exercise 16.2 Suppose that $X = \mathbb{R}^2$, $m = 1$, $f(x) = f(x_1, x_2) = \frac{1}{2}x_1^2 - x_2$, and $g_1(x) = |x_2|$. Discuss this situation in the context of Theorems 16.1 and 16.2.

Exercise 16.3 Suppose that $X = \mathbb{R}^2$, $m = 1$, $f(x) = f(x_1, x_2) = \frac{1}{2}x_1^2 - x_2$, and $g_1(x) = \frac{1}{2}x_2^2$. Discuss this situation in the context of Theorems 16.1 and 16.2.

Chapter 17

A Worked-Out KKT Example

We specialize the setting in the last chapter to $X = \mathbb{R}^2$ and $m = 2$ so that $I := \{1, 2\}$. Instead of writing $(x_1, x_2) \in \mathbb{R}^2$, we shall write $(x, y) \in \mathbb{R}^2$.

We start by introducing and analyzing the objective function: Set

$$\boxed{f\colon \mathbb{R}^2 \to \mathbb{R}\colon (x, y) \mapsto x^2 - 4x + 4y^2 - 2y.}$$

Proposition 17.1 *The objective function f is convex, and*

$$\nabla f(x, y) = (2x - 4, 8y - 2).$$

Proof. The formula for ∇f is clear. The Hessian turns into

$$\nabla^2 f(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \succ 0$$

because the diagonal elements as well as the determinant are positive. Hence f is convex, even strictly. $\square$

Next, we introduce the two constraint functions

$$\boxed{g_1(x, y) := x + y - 1 \quad \text{and} \quad g_2(x, y) := x + 5y - 3.}$$

Proposition 17.2 *The constraint functions g_1, g_2 are convex, with*

$$\nabla g_1(x, y) = (1, 1) \quad \textit{and} \quad \nabla g_2(x, y) = (1, 5).$$

Proof. The gradient formulas are clear; convexity follows because both functions are even affine. $\square$

Proposition 17.3 *The objective function f is continuous and coercive.*

Proof. Clearly, f is continuous (even continuously differentiable). As to coercivity, observe that

$$\begin{aligned} f(x, y) &= x^2 - 4x + 4y^2 - 2y \\ &= (x - 2)^2 - 4 + (2y - \tfrac{1}{2})^2 - \tfrac{1}{4} \\ &\to +\infty \end{aligned}$$

as $\|(x, y)\| \to +\infty$. $\square$

We investigate the problem (P), which asks to

$$\text{minimize } f(x,y) \text{ subject to } g_1(x,y) \leqslant 0 \text{ and } g_2(x,y) \leqslant 0. \tag{17.1}$$

The *optimal value* of (P) is defined by

$$\mu := \inf\big\{f(x,y) \mid g_1(x,y) \leqslant 0,\ g_2(x,y) \leqslant 0\big\}. \tag{17.2}$$

Proposition 17.4 *The problem (P) has a Slater point, i.e., there exists $(s,t) \in \mathbb{R}^2$ such that $g_1(s,t) < 0$ and $g_2(s,t) < 0$.*

Proof. Many solutions are possible; perhaps the easiest is $(s,t) = (0,0)$. □

Given $(x,y) \in \mathbb{R}^2$ and $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, the KKT conditions for (P) are — in their abstract form — the following:

$$\begin{aligned}
(\forall i \in \{1,2\}) \quad g_i(x,y) &\leqslant 0, && \text{(primal feasibility)}\\
(\forall i \in \{1,2\}) \quad \lambda_i &\geqslant 0, && \text{(dual feasibility)}\\
(0,0) \in \partial f(x,y) + \sum_{i\in\{1,2\}} \lambda_i \partial g_i(x,y)&, && \text{(stationarity)}\\
(\forall i \in \{1,2\}) \quad \lambda_i g_i(x,y) &= 0. && \text{(complementary slackness)}
\end{aligned}$$

In view of our work above, we obtain the following concrete version:

Proposition 17.5 *The KKT conditions for (P) are*

$$\begin{aligned}
x + y - 1 \leqslant 0 \quad &\textit{and} \quad x + 5y - 3 \leqslant 0, && \text{(pf)}\\
\lambda_1 \geqslant 0 \quad &\textit{and} \quad \lambda_2 \geqslant 0, && \text{(df)}\\
0 = 2x - 4 + \lambda_1 + \lambda_2 \quad &\textit{and} \quad 0 = 8y - 2 + \lambda_1 + 5\lambda_2, && \text{(st)}\\
\lambda_1(x + y - 1) = 0 \quad &\textit{and} \quad \lambda_2(x + 5y - 3) = 0. && \text{(cs)}
\end{aligned}$$

Proof. (pf), (df), and (cs) are clear from the definition of f, g_1, g_2. (stationarity) turns into

$$(0,0) = (2x - 4, 8y - 2) + \lambda_1(1,1) + \lambda_2(1,5),$$

which yields (st). □

The KKT conditions are powerful — they transform the optimization problem (P) into a feasibility problem. The latter is easier but still not easy to solve analytically. Especially (cs) is nasty and requires us to discuss cases, which we will do in the following.

Before we do so, we point out the following:

Proposition 17.6 *The problem (P) has a solution.*

Proof. Indeed, f is coercive so Corollary 5.5 applies. □

Proposition 17.7 *When $\lambda_1 = 0$ and $\lambda_2 = 0$, then the KKT conditions have no solution.*

Proof. Suppose that $\lambda_1 = 0$ and $\lambda_2 = 0$. Then (pf) is unchanged, (df) and (cs) are trivially true, and (st) simplifies to

$$(0,0) = (2x - 4, 8y - 2), \quad \text{i.e.,} \quad (x,y) = (2, \tfrac{1}{4}).$$

Unfortunately, the vector $(2, \frac{1}{4})$ fails both conditions in (pf). Hence this KKT system has no solution. □

Remark 17.8 The vector $(2, \frac{1}{4})$ found in the proof of Proposition 17.7 is the unique solution to the problem of minimizing f without any constraints.

Proposition 17.9 *When $\lambda_1 > 0$ and $\lambda_2 > 0$, then the KKT conditions have no solution.*

Proof. Suppose that $\lambda_1 > 0$ and $\lambda_2 > 0$. Then (pf) is unchanged, (df) is true by assumption, (st) stays

$$0 = 2x - 4 + \lambda_1 + \lambda_2 \quad \text{and} \quad 0 = 8y - 2 + \lambda_1 + 5\lambda_2, \tag{st}$$

while (cs) simplifies to

$$x + y - 1 = 0 \quad \text{and} \quad x + 5y - 3 = 0. \tag{cs}$$

We solve (cs) for x and get $x = -y + 1$ and $x = -5y + 3$. Equating gives $-y + 1 = -5y + 3$, i.e., $4y = 2$, i.e., $y = \frac{1}{2}$ and thus $x = \frac{1}{2}$. Plugging $(x, y) = (\frac{1}{2}, \frac{1}{2})$ into (st) gives

$$0 = -3 + \lambda_1 + \lambda_2 \quad \text{and} \quad 0 = 2 + \lambda_1 + 5\lambda_2. \tag{st}$$

Solving this for λ_1 gives $3 - \lambda_2 = \lambda_1 = -2 - 5\lambda_2$; hence, $4\lambda_2 = -5$, i.e., $\lambda_2 = -5/4 < 0$. But this contradicts (df)! □

Proposition 17.10 *When $\lambda_1 = 0$ and $\lambda_2 > 0$, then the KKT conditions have no solution.*

Proof. Suppose that $\lambda_1 = 0$ and $\lambda_2 > 0$. Of course, (pf) is unchanged, (df) is trivially true, and (st) turns into

$$0 = 2x - 4 + \lambda_2 \quad \text{and} \quad 0 = 8y - 2 + 5\lambda_2, \tag{st}$$

while we learn from (cs) that

$$x + 5y - 3 = 0. \tag{cs}$$

Solving (cs) for x and also the left side of (st) for x yields $-5y + 3 = x = 2 - \frac{1}{2}\lambda_2$. So $5y = 1 + \frac{1}{2}\lambda_2$, i.e.,

$$y = \tfrac{1}{5} + \tfrac{1}{10}\lambda_2.$$

Plugging this into the right side of (st) gives $0 = \frac{8}{5} + \frac{4}{5}\lambda_2 - 2 + 5\lambda_2 \Leftrightarrow \frac{2}{5} = \frac{10-8}{5} = 2 - \frac{8}{5} = (\frac{4}{5} + 5)\lambda_2 = \frac{4+25}{5}\lambda_2 = \frac{29}{5}\lambda_2 \Leftrightarrow$

$$\lambda_2 = \tfrac{2}{29}.$$

This is consistent with (df). Thus

$$x = 2 - \tfrac{1}{2}\lambda_2 = 2 - \tfrac{1}{29} = \tfrac{57}{29}$$

and also

$$y = \tfrac{1}{5} + \tfrac{1}{10}\lambda_2 = \tfrac{1}{5} + \tfrac{1}{5 \cdot 29} = \tfrac{30}{5 \cdot 29} = \tfrac{6}{29}.$$

Finally, (pf) does not hold because $x + y - 1 = \frac{1}{29}(57 + 6 - 29) = \frac{34}{29} > 0$. □

Proposition 17.11 *When $\lambda_1 > 0$ and $\lambda_2 = 0$, then the KKT conditions have a unique solution, namely, $(x, y) = (1, 0)$.*

Proof. Suppose that $\lambda_1 > 0$ and $\lambda_2 = 0$. Again, (pf) stays unchanged, while (df) trivially holds. Next, (st) turns into

$$0 = 2x - 4 + \lambda_1 \quad \text{and} \quad 0 = 8y - 2 + \lambda_1, \tag{st}$$

while (cs) yields

$$x + y - 1 = 0. \tag{cs}$$

Solving (cs) for x and the left side of (st) for x gives $-y+1 = x = 2-\frac{1}{2}\lambda_1$. Hence $y = -1+\frac{1}{2}\lambda_1$. Plugging this into the right side of (st) yields $0 = -8+4\lambda_1 - 2 + \lambda_1 = -10+5\lambda_1$ and so

$$\lambda_1 = \frac{10}{5} = 2.$$

This is consistent with (df)! Also, it yields $y = -1 + \frac{1}{2}\lambda_1 = 0$ and $x = -y+1 = 1$, i.e.,

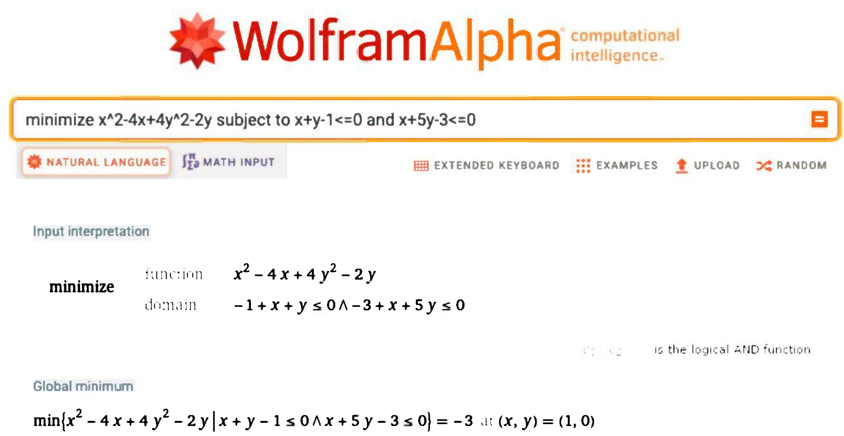
$$(x, y) = (1, 0).$$

Thankfully, (pf) holds! $\square$

Corollary 17.12 *The unique solution of (P) is $(x, y) = (1, 0)$, and the optimal value is $\mu = -3$.*

Proof. The above and Corollary 16.3 show that $(x, y) = (1, 0)$ is the unique solution of (P). The optimal value is $\mu = f(1, 0) = 1^2 - 4(1) + 4(0)^2 - 2(0) = -3$. $\square$

Remark 17.13 (WolframAlpha) We can do this computation with WolframAlpha as follows:



Remark 17.14 (Julia) Here is a numerical optimizer in `Julia`:

```
In [1]: using JuMP; using Ipopt;

In [2]: m = Model(optimizer_with_attributes(Ipopt.Optimizer,"print_level" => 0));
        @variable(m,x);
        @variable(m,y);
        @NLobjective(m,Min,x^2-4*x+4*y^2-2*y);
        @constraint(m,x+y-1<=0);
        @constraint(m,x+5*y-3<=0);

In [3]: optimize!(m);
        optimalvalue = objective_value(m);
        minimizer   value.([x,y]);
Out[3]:
        ******************************************************************************
        This program contains Ipopt, a library for large-scale nonlinear optimization.
         Ipopt is released as open source code under the Eclipse Public License (EPL).
                 For more information visit https://github.com/coin-or/Ipopt
        ******************************************************************************


In [4]: println("The minimizer is: ",minimizer);
        println("The optimal value is: ",optimalvalue);

Out[4]: The minimizer is: [1.0000000074994715, 1.2476711517184339e-9]
        The optimal value is: -3.000000017494285
```

Exercises

Exercise 17.1 For the problem considered in this chapter, find a point that is feasible, though not a Slater point, as well as a point that is not feasible.

Exercise 17.2 Do the steps of the analysis in this chapter when $f(x, y) = x^2 - 14x + y^2 - 6y - 7$, $g_1(x, y) = x + y - 2$, and $g_2(x, y) = x + 2y - 3$:

(i) Show that f is convex and compute ∇f.

(ii) Show that g_1 and g_2 are convex, and compute their gradients.

(iii) Provide a Slater point for (P).

(iv) Write down and simplify the KKT conditions for (P).

(v) Discuss the KKT conditions when $\lambda_1 = 0$ and $\lambda_2 = 0$.

(vi) Discuss the KKT conditions when $\lambda_1 > 0$ and $\lambda_2 = 0$.

(vii) Discuss the KKT conditions when $\lambda_1 = 0$ and $\lambda_2 > 0$.

(viii) Discuss the KKT conditions when $\lambda_1 > 0$ and $\lambda_2 > 0$.

(ix) Provide the minimizers and the optimal value for (P).

(x) Solve (P) using two different solvers.

Chapter 18

Fenchel Conjugates

18.1 ▪ Definition and Examples

Definition 18.1 Let $f\colon X \to [-\infty, +\infty]$. Then

$$f^*\colon X \to [-\infty, +\infty] : y \mapsto \sup_{x\in X} \big(\langle x, y\rangle - f(x)\big)$$

is the *Fenchel conjugate* of f. The Fenchel biconjugate is $f^{**} := (f^*)^*$.

Proposition 18.2 *Let $f\colon X \to [-\infty, +\infty]$. Then f^* is convex and lower semicontinuous.*

Proof. Indeed, f^* is the supremum of the family $(\langle x, \cdot\rangle - f(x))_{x\in X}$, whose members are all affine and continuous. The result thus follows from Corollary 4.8(iii). ☐

Example 18.3 Let C be a subset of X. Then

$$\iota_C^* = \sigma_C$$

because $\iota_C^*(y) = \sup_{x\in X}(\langle x, y\rangle - \iota_C(x)) = \sup_{x\in C} \langle x, y\rangle = \sigma_C(y)$.

Example 18.4 Let $\alpha > 0$. Then

$$(\alpha|\cdot|)^* = \iota_{[-\alpha,\alpha]}$$

because

$$\begin{aligned}(\alpha|\cdot|)^*(y) &= \sup_{x\in\mathbb{R}} \big(xy - \alpha|x|\big) = \max\Big\{ \sup_{x\geqslant 0} \big(x(y-\alpha)\big), \sup_{x\leqslant 0} \big(x(y+\alpha)\big)\Big\}\\ &= \max\big\{\iota_{]-\infty,\alpha]}(y), \iota_{[-\alpha,+\infty\}}(y)\big\} = \iota_{[-\alpha,\alpha]}.\end{aligned}$$

Example 18.5 Set $K := \mathbb{R}^n_+$. By Examples 18.3 and 7.10, $\iota_K^* = \sigma_K = \iota_{K^\ominus}$ and $K^\ominus = -K$; thus,

$$\iota^*_{\mathbb{R}^n_+} = \iota_{\mathbb{R}^n_-}.$$

Example 18.6 Suppose that $f(x) = \langle x, a\rangle + \beta$, where $a \in X$ and $\beta \in \mathbb{R}$. Then

$$f^*(y) = \sup_{x\in X} \big(\langle x, y\rangle - \langle x, a\rangle - \beta\big) = -\beta + \sup_{x\in X} \langle x, y - a\rangle = \iota_{\{a\}}(y) - \beta.$$

Example 18.7 Suppose that $g(x) = \iota_{\{a\}}(x) - \beta$, where $a \in X$ and $\beta \in \mathbb{R}$. Then

$$g^*(y) = \sup_{x \in X} \big(\langle x, y \rangle - \iota_{\{a\}}(x) + \beta \big) = \langle a, y \rangle + \beta.$$

Combining Example 18.6 with Example 18.7, we see that $f^{**} = f$ and $g^{**} = g$; indeed, this is not a coincidence (peek at Theorem 19.2 if you are curious).

Remark 18.8 We saw earlier (see Example 4.10) that the Asplund function $A_C(y) = \frac{1}{2}\|y\|^2 - d_C^2(y)$ is convex for any nonempty subset C of X. An inspection of the proof reveals that we actually proved

$$A_C = \big(\tfrac{1}{2}\|\cdot\|^2 + \iota_C\big)^*.$$

Remark 18.9 (calculus approach) Suppose f is convex and differentiable, and our job is to determine f^*. Let $y \in X$ and observe that

$$f^*(y) = -\inf_{x \in X} \big(f(x) - \langle x, y \rangle \big) \tag{18.1}$$

is the negative minimum value of an optimization problem featuring the sum of two differentiable convex functions, one of which even has full domain. We tackle this minimization problem via Fermat's rule (Lemma 9.2) and the sum rule: x is a minimizer in (18.1) $\Leftrightarrow 0 = \nabla f(x) - y \Leftrightarrow \nabla f(x) = y$. Now suppose that $y \in \operatorname{ran} \nabla f$ and ∇f is so nice that we can invert: we obtain the minimizer $x = x_y$ such that

$$\nabla f(x_y) = y.$$

Then we can substitute this minimizer x_y back into the definition of f^* to determine $f^*(y)$:

$$f^*(y) = \langle x_y, y \rangle - f(x_y). \tag{18.2}$$

In particular, if we *start* with any x such that $\nabla f(x)$ exists, then we can do this with $y = \nabla f(x)$ and obtain $x_y = x$. Put differently,

$$f^*(\nabla f(x)) = \langle x, \nabla f(x) \rangle - f(x) \quad \text{when } \nabla f(x) \text{ exists.} \tag{18.3}$$

For this reason, the Fenchel conjugate is sometimes known as the Legendre transform.

Let us see the calculus approach in action:

Example 18.10 Suppose that $f = \frac{1}{2}\|\cdot\|^2$. Then $f^* = f$.

Proof. We have $\nabla f = \mathrm{Id}$, so the equation $\nabla f(x) = y$ turns into $x = y$. Hence $f^*(y) = \langle y, y \rangle - \frac{1}{2}\|y\|^2 = \frac{1}{2}\|y\|^2 = f(y)$. □

Example 18.11 Let $p > 1$ and set $q := p/(p-1)$ and hence $\frac{1}{p} + \frac{1}{q} = 1$. Let $f_p(x) := f(x) := \frac{1}{p}|x|^p$. Then $f^*(y) = \frac{1}{q}|y|^q$ and so $f_p^* = f_q$.

Proof. Note that f is differentiable with $f'(x) = \operatorname{sign}(x)|x|^{p-1}$. Let $y \in \mathbb{R}$. Luckily, we can invert $f'(x) = y$ and find

$$x = \operatorname{sign}(y)|y|^{1/(p-1)}.$$

It follows that

$$\begin{aligned} f^*(y) &= \operatorname{sign}(y)|y|^{1/(p-1)}y - \tfrac{1}{p}\big|\operatorname{sign}(y)|y|^{1/(p-1)}\big|^p \\ &= |y|^{1+1/(p-1)} - \tfrac{1}{p}|y|^{p/(p-1)} = \big(1 - \tfrac{1}{p}\big)|y|^{p/(p-1)} \\ &= \tfrac{1}{q}|y|^q, \end{aligned}$$

as announced. □

Example 18.12 We have

$$\exp^*(y) = \begin{cases} +\infty & \text{if } y < 0; \\ 0 & \text{if } y = 0; \\ y\ln(y) - y & \text{if } y > 0. \end{cases}$$

Proof. Since $\exp' = \exp$, it is possible to invert $\exp(x) = y$ whenever $y > 0$ and in which case $x = \ln(y)$ and so $\exp^*(y) = \ln(y)y - \exp(\ln(y)) = y\ln(y) - y$. When $y \leqslant 0$, we work directly with the definition of the Fenchel conjugate: $\exp^*(0) = \sup_{x\in\mathbb{R}}(x \cdot 0 - \exp(x)) = -\inf_{x\in\mathbb{R}} \exp(x) = -0 = 0$. Finally, suppose that $y < 0$. Then $f^*(y) \geqslant \lim_{x\to-\infty}(xy - \exp(x)) = +\infty$. $\square$

Example 18.13 If $f(x) := \begin{cases} +\infty & \text{if } x \leqslant 0; \\ -\ln(x) & \text{if } x > 0 \end{cases}$, then $f^*(y) = -1 + f(-y)$.

Example 18.14 If $f(x) := \begin{cases} +\infty & \text{if } x < 0; \\ -2\sqrt{x} & \text{if } x \geqslant 0 \end{cases}$, then $f^*(y) = \begin{cases} \dfrac{-1}{y} & \text{if } y < 0; \\ +\infty & \text{if } y \geqslant 0. \end{cases}$

18.2 ▪ Properness of f^* and Fenchel–Young Inequality

Theorem 18.15 *Let $f\colon X \to\ ]-\infty, +\infty]$ be convex and proper. Then f^* is convex, lower semicontinuous, and proper; in fact,* $\operatorname{ran}\partial f \subseteq \operatorname{dom} f^*$.

Proof. We know already (see Proposition 18.2) that f^* is convex and lower semicontinuous. Because f is convex and proper, we have $\operatorname{dom}\partial f \neq \varnothing$ (see Fact 9.12), say $z \in \operatorname{dom}\partial f \subseteq \operatorname{dom} f$ and then take $w \in \partial f(z)$. Then

$$(\forall x \in X) \quad f(z) + \langle x - z, w\rangle \leqslant f(x). \tag{18.4}$$

Then $(\forall y \in X)\ f^*(y) \geqslant \langle z, y\rangle - f(z) > -\infty$. Finally, recalling (18.4), we get $\langle x, w\rangle - f(x) \leqslant \langle z, w\rangle - f(z)$ and so $f^*(w) \leqslant \langle z, w\rangle - f(z) < +\infty$. $\square$

Remark 18.16 With more work one may show that $\overline{\operatorname{ran}}\,\partial f = \overline{\operatorname{dom}}\, f^*$ (see, e.g., [3, Theorem 16.58]). More precisely, f^* is the lower semicontinuous hull/extension of $f^*|_{\operatorname{ran}\partial f}$.

Theorem 18.17 (Fenchel–Young inequality) *Let $f\colon X \to\ ]-\infty, +\infty]$ be proper. Then for all x, y in X, we have*

$$\boxed{f(x) + f^*(y) \geqslant \langle x, y\rangle\,.} \tag{18.5}$$

Proof. Because f is proper, $f \not\equiv +\infty$ and so $f^* > -\infty$. Clearly, $f^*(y) \geqslant \langle x, y\rangle - f(x)$ and now the result follows. $\square$

Remark 18.18 (Cauchy–Schwarz revisited) Suppose that $f = \frac{1}{2}\|\cdot\|^2$. By Example 18.10, $f^* = f$. Hence Fenchel–Young (Theorem 18.17) yields

$$\tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2 \geqslant \langle x, y\rangle\,, \tag{18.6}$$

which is also equivalent to $\frac{1}{2}\|x - y\|^2 \geqslant 0$. Now suppose that u, v are nonzero vectors in X and that $x = u/\|u\|$ and $y = v/\|v\|$. Then $\|x\| = \|y\| = 1$ and (18.6) yields $1 \geqslant \langle x, y\rangle = \langle u/\|u\|, v/\|v\|\rangle$ and we recover Cauchy–Schwarz.

Similarly, one may recover the Hölder inequality (which in turn gives rise to Minkowski's inequality) from Example 18.11.

Exercises

Exercise 18.1 Compute f^* when $f(x) = -|x|$.

Exercise 18.2 Compute f^* when f is not proper.

Exercise 18.3 Compute $\iota^*_{\{-1,1\}}$ and $\iota^*_{[-1,1]}$.

Exercise 18.4 Provide the details for Example 18.13.

Exercise 18.5 Provide the details for Example 18.14.

Exercise 18.6 Let $f(x) := \alpha\frac{1}{2}\|x\|^2$, where $\alpha > 0$. Show that $f^*(y) = \alpha^{-1}\frac{1}{2}\|y\|^2$.

Exercise 18.7 (hinge loss) Let $f(x) := \max\{1 - x, 0\}$ be the *hinge loss* function. Show that $f^*(y) = y + \iota_{[-1,0]}(y)$.

Exercise 18.8 Let $A \succ 0$ and set $f(x) := \frac{1}{2}\langle x, Ax\rangle$. Show that $f^*(y) = \frac{1}{2}\langle y, A^{-1}(y)\rangle$.

Exercise 18.9 Suppose that $f = \|\cdot\|$ is any norm on X. Show that $f^* = \iota_C$, where C is the unit ball of the dual norm.

Exercise 18.10 (Hölder's inequality) Suppose that $1 < p, q < +\infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let x, y be in $\mathbb{R}^n$. Show that $\langle x, y\rangle \leqslant \|x\|_p\|y\|_q$ by following the steps in Remark 18.18.

Exercise 18.11 (log-sum-exp) For $x \in \mathbb{R}^n$, set $f(x) := \ln(\sum_{i=1}^n \exp(x_i))$. Show that $f^*(y) = \sum_{i=1}^n y_i \ln(y_i)$ when $y \in \mathbb{R}^n_{++}$ and $y \in \mathbb{P}_n$. In fact, more work with Remark 18.16 shows that

$$f^*(y) = \begin{cases} \sum_{i=1}^n y_i \ln(y_i) & \text{if } y \text{ is in the probability simplex } \mathbb{P}_n; \\ +\infty & \text{otherwise,} \end{cases}$$

where as usual $0\ln(0) = 0$.

Chapter 19

Biconjugates and Fenchel Calculus

19.1 ▪ The Biconjugate Theorem

Lemma 19.1 *Let $f\colon X \to \left]-\infty,+\infty\right]$ be proper. Then $f^{**} \leqslant f$.*

Proof. By Fenchel–Young (Theorem 18.17), we have $f(x) \geqslant \langle x, y\rangle - f^*(y)$. Hence $f(x) \geqslant \sup_{y\in X}(\langle x,y\rangle - f^*(y)) = (f^*)^*(x)$. □

Theorem 19.2 (Biconjugate Theorem) *Let $f\colon X \to \left]-\infty,+\infty\right]$ be proper. Then*

$$f^{**} = f \quad \Leftrightarrow \quad f \textit{ is convex and lower semicontinuous.}$$

Proof. "$\Rightarrow$": This is clear from Proposition 18.2. "$\Leftarrow$": Assume that f is convex and lower semicontinuous. Then $\operatorname{epi}(f)$ is convex, closed, and nonempty by Theorem 3.18. Applying Theorem 18.15 twice, we see that f^* is proper and then so is f^{**}. We show the desired conclusion by contradiction. In view of Lemma 19.1, we thus assume that there exists $w \in X$ such that $f^{**}(w) < f(w)$. Then $f(w) \not\leqslant f^{**}(w)$ and so $(w, f^{**}(w)) \notin \operatorname{epi}(f)$. Applying the separation theorem (Corollary 8.14), we obtain $a \in X$, $\beta \in \mathbb{R}$, $\gamma_1 \in \mathbb{R}$, and $\gamma_2 \in \mathbb{R}$ such that

$$(\forall (x,r) \in \operatorname{epi} f) \quad \langle x, a\rangle + \beta r \leqslant \gamma_1 < \gamma_2 \leqslant \langle w, a\rangle + \beta f^{**}(w).$$

Rearranging gives

$$(\forall (x,r) \in \operatorname{epi} f) \quad \langle x - w, a\rangle + \beta(r - f^{**}(w)) \leqslant \gamma_1 - \gamma_2 =: \gamma < 0. \tag{19.1}$$

If $\beta > 0$, then letting $r \to +\infty$ in (19.1) gives a contradiction. Thus, $\beta \leqslant 0$. We discuss two cases.

Case 1: $\beta = -|\beta| < 0$. Dividing (19.1) by $|\beta| = -\beta > 0$ and using $r = f(x)$ gives

$$(\forall x \in X) \quad \langle x, a/|\beta|\rangle - f(x) - \langle w, a/|\beta|\rangle + f^{**}(w) \leqslant \gamma/|\beta| < 0.$$

Taking now the supremum over $x \in X$ results in $f^*(a/|\beta|) + f^{**}(w) - \langle w, a/|\beta|\rangle < 0$, which contradicts the Fenchel–Young inequality!

Case 2: $\beta = 0$. Let $y \in \operatorname{dom} f^*$. Let $\varepsilon > 0$, set $a_\varepsilon := a + \varepsilon y$, and let $z \in \operatorname{dom} f$. Then, for ε sufficiently small and using (19.1) with $(x,r) = (z, f(z))$ as well as the definition of $f^*(y)$ in the inequality below, we obtain

$$\begin{aligned}
&\langle z - w, a_\varepsilon\rangle - \varepsilon\big(f(z) - f^{**}(w)\big)\\
&\quad = \langle z - w, a\rangle + \varepsilon\big(\langle z, y\rangle - \langle w, y\rangle - f(z) + f^{**}(w)\big)\\
&\quad \leqslant \gamma + \varepsilon\big(f^*(y) - \langle w, y\rangle + f^{**}(w)\big) =: \gamma_\epsilon < 0.
\end{aligned}$$

Dividing by $\varepsilon > 0$ gives

$$\langle z, a_\varepsilon/\varepsilon\rangle - f(z) - \langle w, a_\varepsilon/\varepsilon\rangle + f^{**}(w) < \gamma_\varepsilon/\varepsilon < 0.$$

Now taking the supremum over $z \in \operatorname{dom} f$ yields

$$f^*(a_\varepsilon/\varepsilon) + f^{**}(w) - \langle w, a_\varepsilon/\varepsilon\rangle < 0.$$

Once again, this contradicts the Fenchel–Young inequality. $\square$

Example 19.3 Let C be a nonempty subset of X. We saw earlier that $\iota_C^* = \sigma_C$ (Example 18.3) and also that $\sigma_C = \sigma_{\overline{C}} = \sigma_{\operatorname{conv} C} = \sigma_{\overline{\operatorname{conv}}\, C}$ (Proposition 7.7). By Example 3.20, $\iota_{\overline{\operatorname{conv}}\, C}$ is convex, lower semicontinuous, and proper. Hence Theorem 19.2 yields

$$\iota_{\overline{\operatorname{conv}}\, C} = \iota_{\overline{\operatorname{conv}}\, C}^{**} = \sigma_{\overline{\operatorname{conv}}\, C}^{*} = \sigma_C^* = \iota_C^{**}.$$

We can further specialize this to the following:

Example 19.4 Set $f(x) := \max\{x_1, \ldots, x_n\}$ on $\mathbb{R}^n$. Then $f = \sigma_{\{e_1,\ldots,e_n\}}$, where e_i denotes the ith standard unit vector. Then $f^* = \sigma^*_{\{e_1,\ldots,e_n\}} = \iota_{\overline{\operatorname{conv}}\,\{e_1,\ldots,e_n\}}$ is the indicator function of the probability simplex $\mathbb{P}_n$, and its conjugate is again f.

Theorem 19.5 *Let $f\colon X \to\,]-\infty, +\infty]$ be convex and proper. Let x, y be in X. Then the following are equivalent:*

(i) $f(x) + f^*(y) = \langle x, y\rangle$.

(ii) $y \in \partial f(x)$.

If f is also lower semicontinuous, then we can add another item to this list:

(iii) $x \in \partial f^*(y)$.

Proof. Indeed, using Fermat's rule (Lemma 9.2) and the sum rule (Fact 13.7), we get

$$\begin{aligned}
\text{(i)} &\Leftrightarrow f^*(y) = \langle x, y\rangle - f(x)\\
&\Leftrightarrow \sup_{z\in X}\big(\langle z, y\rangle - f(z)\big) = \langle x, y\rangle - f(x)\\
&\Leftrightarrow x \text{ maximizes } z \mapsto \langle z, y\rangle - f(z)\\
&\Leftrightarrow x \text{ minimizes } z \mapsto \langle z, -y\rangle + f(z)\\
&\Leftrightarrow 0 \in \partial(\langle\cdot, -y\rangle + f)(x)\\
&\Leftrightarrow 0 \in -y + \partial f(x)\\
&\Leftrightarrow \text{(ii)}.
\end{aligned}$$

Now assume that f is also lower semicontinuous. Then the Biconjugate Theorem (Theorem 19.2) yields $f^{**} = f$. Now applying the just-proved equivalence to f^* instead of f (with x and y interchanged) yields (i) $\Leftrightarrow f^*(y) + (f^*)^*(x) = \langle y, x\rangle \Leftrightarrow x \in \partial f^*(y) \Leftrightarrow$ (iii). $\square$

Corollary 19.6 *Let $f\colon X \to\,]-\infty, +\infty]$ be convex, lower semicontinuous, and proper. Then*

$$\boxed{(\partial f)^{-1} = \partial f^*.}$$

19.2 ▪ Basic Conjugate Calculus

Proposition 19.7 *Let $f_1, \dots, f_m$ be proper functions on Euclidean spaces $X_1, \dots, X_m$, respectively. Define the separable sum*

$$f\colon X_1 \times \cdots \times X_m \to \left]-\infty, +\infty\right] : (x_1, \dots, x_m) \mapsto f_1(x_1) + \cdots + f_m(x_m).$$

Then

$$f^*\colon X_1 \times \cdots \times X_m \to \left]-\infty, +\infty\right] : (y_1, \dots, y_m) \mapsto f_1^*(y_1) + \cdots + f_m^*(y_m).$$

Proposition 19.8 *Let $f\colon X \to \left]-\infty, +\infty\right]$ be proper, let $b \in X$, and let $\gamma \in \mathbb{R}$. Set $g(x) := f(x) + \langle x, b\rangle + \gamma$. Then $g^*(y) = f^*(y - b) - \gamma$.*

Example 19.9 Suppose that $g(x) = \iota_{\mathbb{R}^n_+}(x) + \langle x, c\rangle$, where $c \in X$. Then $g^*(y) = \iota_{\mathbb{R}^n_-}(y - c)$ by Example 18.5 and Proposition 19.8.

Proposition 19.10 *Let $f\colon X \to \left]-\infty, +\infty\right]$ be proper, and let $A\colon X \to X$ be linear and bijective. Then $(f \circ A)^* = f^* \circ A^{-*}$, where $A^{-*} := (A^{-1})^* = (A^*)^{-1}$.*

Proof. Indeed,

$$\begin{aligned}
(f \circ A)^*(y) &= \sup_{x \in X} \big(\langle x, y\rangle - f(Ax)\big) = \sup_{x \in X} \big(\langle A^{-1}Ax, y\rangle - f(Ax)\big) \\
&= \sup_{x \in X} \big(\langle Ax, A^{-*}y\rangle - f(Ax)\big) = \sup_{z \in X} \big(\langle z, A^{-*}y\rangle - f(z)\big) \\
&= f^*(A^{-*}y),
\end{aligned}$$

as claimed. □

Proposition 19.11 *Let $f\colon X \to \left]-\infty, +\infty\right]$ be proper, and let $\alpha > 0$. Then $(\alpha f)^* = \alpha f^*(\cdot/\alpha)$ and $(\alpha f(\cdot/\alpha))^* = \alpha f^*$.*

Exercises

Exercise 19.1 Provide a function f that is convex and lower semicontinuous such that $f^{**} \neq f$.

Exercise 19.2 (negative entropy) Consider the function

$$g(x) := \begin{cases} +\infty & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ x\ln(x) - x & \text{if } x > 0. \end{cases}$$

Determine g^*. (This is easy if you recall Theorem 19.2 and Example 18.12!)

Exercise 19.3 Provide an example of a proper convex function f such that $x \in \partial f^*(y)$ but $y \notin \partial f(x)$ — thus illustrating the importance of lower semicontinuity in Theorem 19.5.

Exercise 19.4 Provide the details of the proof for Proposition 19.7.

Exercise 19.5 Provide the details of the proof for Proposition 19.8.

Exercise 19.6 Provide the details of the proof for Proposition 19.11.

Exercise 19.7 Let $A \in \mathbb{R}^{n \times n}$ be positive definite, let $b \in \mathbb{R}^n$, and let $\gamma \in \mathbb{R}$. Set $g(x) := \frac{1}{2}\langle x, Ax\rangle + \langle x, b\rangle + \gamma$. Determine g^*.

Exercise 19.8 (energy is the only self-conjugate function) Show that $f = f^*$ if and only if $f = \frac{1}{2}\|\cdot\|^2$.

Chapter 20

Fenchel–Rockafellar Duality

In this chapter, we assume the following:

Y is a Euclidean space; $A\colon X \to Y$ is linear; f is convex, lower semicontinuous, and proper on X; and g is convex, lower semicontinuous, and proper on Y.

20.1 ▪ Primal and Dual Problems, and Weak Duality

Consider the triple (f, A, g). The associated *primal problem* asks to

$$\underset{x\in X}{\text{minimize}} \qquad f(x) + g(Ax) \tag{20.1}$$

and the associated *optimal primal value* is

$$\mu := \inf_{x\in X} \big(f(x) + g(Ax)\big). \tag{20.2}$$

A point $x \in X$ is a *primal solution* if $f(x) + g(Ax) = \mu$. The associated *(Fenchel–Rockafellar) dual problem* asks to

$$\underset{y\in Y}{\text{minimize}} \qquad f^*(-A^*y) + g^*(y), \tag{20.3}$$

while the associated *optimal dual value* is

$$\mu^* := \inf_{y\in Y} \big(f^*(-A^*y) + g^*(y)\big). \tag{20.4}$$

A point $y \in Y$ is a *dual solution* if $f^*(-A^*y) + g^*(y) = \mu^*$.

We note that the dual problem is the same as the primal problem for the triple $(g^*, -A^*, f^*)$, thus we define the dual triple by $(f, A, g)^* := (g^*, -A^*, f^*)$. Note that if f and g are convex, lower semicontinuous, and proper, then the Biconjugate Theorem (Theorem 19.2) and Proposition 19.10 yield $f^{**} = f$ and $g^{**} = g$. Because $-(-A^*)^* = A$, we have $(f, A, g)^{**} = (f, A, g)$, i.e., *the dual of the dual problem is the primal problem!* If $X = Y$ and $A = \mathrm{Id}$, then one obtains *Fenchel Duality*. The primal and dual problems are always related by the following result:

Proposition 20.1 (weak duality) *For all $x \in X$ and $y \in Y$, we have*

$$f(x) + g(Ax) \geqslant \mu \geqslant -\mu^* \geqslant -f^*(-A^*y) - g^*(y). \tag{20.5}$$

Proof. Let $(x, y) \in X \times Y$. The left and right inequalities in (20.5) are clear. Next, by Fenchel–Young, $f(x) + f^*(-A^*y) \geqslant \langle x, -A^*y\rangle = -\langle x, A^*y\rangle$ and $g(Ax) + g^*(y) \geqslant \langle Ax, y\rangle$. Adding and recalling that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ yields $f(x) + f^*(-A^*y) + g(Ax) + g^*(y) \geqslant 0$. Now rearrange to learn that

$$f(x) + g(Ax) \geqslant -f^*(-A^*y) - g^*(y).$$

Taking the infimum and supremum now yields

$$\begin{aligned}\mu = \inf_{x\in X} \big(f(x) + g(Ax)\big) &\geqslant \sup_{y\in Y} \big(- f^*(-A^*y) - g^*(y)\big)\\ = -\inf_{y\in Y} \big(f^*(-A^*y) + g^*(y)\big) &= -\mu^*\end{aligned}$$

and we are done! □

Definition 20.2 (duality gap) Consider Proposition 20.1. If $\mu = -\mu^* \in [-\infty, +\infty]$, then we say there is *no duality gap*; if $\mu > -\mu^*$, then we have a *positive duality gap* $\mu + \mu^*$.

Example 20.3 (linear programming revisited) Suppose that $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$. Set $K := \mathbb{R}^n_+$ and suppose that $f(x) = \langle x, c\rangle + \iota_K(x)$ and $g(y) = \iota_{\{b\}}(y)$. Then the primal problem turns into

$$\text{minimize } \langle x, c\rangle \text{ subject to } x \geqslant 0 \text{ and } Ax = b. \tag{20.6}$$

We have $f^*(x) = \iota_{-K}(x - c) = \iota_K(c - x)$ (see Example 19.9) and $g^*(y) = \langle y, b\rangle$ (see Example 18.7). The dual problem thus is

$$\text{minimize } \langle y, b\rangle \text{ subject to } c + A^*y \geqslant 0. \tag{20.7}$$

Then weak duality (Proposition 20.1) turns this into

$$\begin{aligned}\inf\big\{\langle x, c\rangle \;\big|\; x \geqslant 0,\ Ax = b\big\} = \mu&\\ &\geqslant -\mu^* = -\inf\big\{\langle y, b\rangle \;\big|\; c + A^*y \geqslant 0\big\}\\ &= \sup\big\{\langle -y, b\rangle \;\big|\; A^*(-y) \leqslant c\big\},\end{aligned}$$

which is—up to a minus sign in the variable y—the well-known weak duality from linear programming!

20.2 ▪ Fenchel–Rockafellar Duality

The simultaneous existence of primal/dual solutions can be very nicely characterized:

Theorem 20.4 *Let $x \in \operatorname{dom} f$ and $y \in \operatorname{dom} g$. Then the following are equivalent:*

(i) *x is a primal solution, y is a dual solution, and $\mu = -\mu^*$.*

(ii) $f(x) + g(Ax) = -f^*(-A^*y) - g^*(y)$.

(iii) *$-A^*y \in \partial f(x)$ and $Ax \in \partial g^*(y)$.*

(iv) *$x \in \partial f^*(-A^*y)$ and $y \in \partial g(Ax)$.*

(v) *$x \in (\partial f)^{-1}(-A^*y)$ and $y \in \partial g(Ax)$.*

Proof. "(i)⇒(ii)": Suppose that (i) holds. Then $f(x) + g(Ax) = \mu = -\mu^* = -f^*(-A^*y) - g^*(y)$.

"(i)⇐(ii)": Suppose that (ii) holds. Then the chain of inequalities in (20.5) turns into one of equalities and we deduce that $f(x) + g(Ax) = \mu$, i.e., x is a primal solution; $\mu = -\mu^*$; $-\mu^* = -f^*(-A^*y) - g^*(y)$, i.e., y is a dual solution.

"(ii)⇒(iii)": Suppose that (ii) holds. Then this and the Fenchel–Young inequality applied twice give

$$0 = \big(f(x) + f^*(-A^*y)\big) + \big(g^*(y) + g(Ax)\big) \geqslant \langle x, -A^*y\rangle + \langle y, Ax\rangle = 0.$$

We have equality throughout and therefore $f(x)+f^*(-A^*y) = \langle x, -A^*y\rangle$ and $g^*(y)+g(Ax) = \langle y, Ax\rangle$. Now Theorem 19.5 gives $-A^*y \in \partial f(x)$ and $Ax \in \partial g^*(y)$.

"(ii)⇐(iii)": Suppose that (iii) holds. By Theorem 19.5, $f(x) + f^*(-A^*y) = \langle x, -A^*y\rangle$ and $g^*(y) + g(Ax) = \langle y, Ax\rangle$. The conclusion follows because $\langle x, A^*y\rangle = \langle y, Ax\rangle$.

"(iii)⇔(iv)⇔(v)": This follows from Corollary 19.6. ☐

Corollary 20.5 (primal solutions via one dual solution) *Suppose that the dual problem has a solution, say y, and that $\mu = -\mu^*$. Then the (possibly empty)*

$$\boxed{\textit{set of all primal solutions} \;=\; \partial f^*(-A^*y) \cap A^{-1}\big(\partial g^*(y)\big).}$$

Example 20.6 Assume that $f(x) = \exp(x) = g(x)$. Then $\mu = 0$ and the primal has no solution. The dual problem asks to minimize $f^*(-y) + f^*(y)$. This problem has a unique dual solution: $y = 0$. We have $\mu^* = 0$ as well. Hence Corollary 20.5 yields that the set of primal solutions is empty, which we already observed directly. The duality gap $\mu + \mu^*$ is equal to 0.

Corollary 20.5 illustrates the need to derive conditions sufficient to guarantee lack of a duality gap and the existence of dual solutions. The next result provides these.

Fact 20.7 [3, Section 5.3] *Suppose that one of the following holds:*

(i) $(\operatorname{ri}\operatorname{dom} g) \cap A(\operatorname{ri}\operatorname{dom} f) \neq \varnothing$.

(ii) $(\operatorname{int}\operatorname{dom} g) \cap A(\operatorname{dom} f) \neq \varnothing$.

(iii) *g is polyhedral and* $(\operatorname{dom} g) \cap \operatorname{ri} A(\operatorname{dom} f) \neq \varnothing$.

(iv) *f and g are polyhedral and* $(\operatorname{dom} g) \cap A(\operatorname{dom} f) \neq \varnothing$.

Then the Fenchel–Rockafellar dual problem has solutions and $\mu = -\mu^*$.

Example 20.8 (decomposition with respect to a cone) Let $z \in X$, and let K be a nonempty closed convex cone in X. Consider the problem (P), which asks to

$$\text{minimize } \tfrac{1}{2}\|x - z\|^2 \text{ subject to } x \in K.$$

We model this as a Fenchel primal problem with $f = \frac{1}{2}\|\cdot - z\|^2$ and $g = \iota_K$. Then the primal optimal value is $\mu = \frac{1}{2}d_K^2(z)$ and we know from the Projection Theorem that the unique primal solution is $x = \mathrm{P}_K(z)$. Using Exercise 18.8, Example 18.3, and (7.2), we compute $g^* = \sigma_K = \iota_{K^\ominus}$ and $f^*(y) = \langle y, z\rangle + \frac{1}{2}\|y\|^2 = \frac{1}{2}\|z + y\|^2 - \frac{1}{2}\|z\|^2$ with $\nabla f^*(y) = y + z$. Thus $f^*(-y) = \frac{1}{2}\|y - z\|^2 - \frac{1}{2}\|z\|^2$ and $\nabla f^*(-y) = z - y$. The Fenchel dual problem asks to minimize $f^*(-y) + g^*(y)$, i.e., to

$$\text{minimize } \tfrac{1}{2}\|y - z\|^2 - \tfrac{1}{2}\|z\|^2 \text{ subject to } y \in K^\ominus.$$

We find that the unique dual solution is $y = \mathrm{P}_{K^\ominus}(z)$ and that the dual optimal value is $\mu^* = \frac{1}{2}d^2_{K^\ominus}(z) - \frac{1}{2}\|z\|^2$. By Fact 20.7(i), $\mu = -\mu^*$. It now follows from Corollary 20.5 that $x = \nabla f^*(-y) = z - y$, i.e., $z = x + y$ and we recover the *Moreau decomposition*

$$\boxed{z = \mathrm{P}_K(z) + \mathrm{P}_{K^\ominus}(z).}$$

Moreover, our knowledge that $\mu = -\mu^*$ turns into $\mu + \mu^* = 0 \Leftrightarrow 0 = d^2_K(z) + d^2_{K^\ominus}(z) - \|z\|^2 = \|z - \mathrm{P}_K(z)\|^2 + \|z - \mathrm{P}_{K^\ominus}(z)\|^2 - \|z\|^2 = \|\mathrm{P}_{K^\ominus}(z)\|^2 + \|\mathrm{P}_K(z)\|^2 - \|\mathrm{P}_K(z) + \mathrm{P}_{K^\ominus}(z)\|^2 = -2\left\langle \mathrm{P}_K(z), \mathrm{P}_{K^\ominus}(z)\right\rangle$ and so

$$\mathrm{P}_K(z) \perp \mathrm{P}_{K^\ominus}(z).$$

Exercises

Exercise 20.1 Provide details for Example 20.6.

Exercise 20.2 Find two functions f, g which are convex, lower semicontinuous, and proper on $\mathbb{R}$ such that the Fenchel primal problem has a unique solution, the Fenchel dual problem has no solution, and the duality gap is 0.

Exercise 20.3 Consider the closed convex cone

$$C := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; x_1 - \sqrt{x_1^2 + x_2^2} \geqslant x_3\right\}.$$

Show that $\iota_C^* = \iota_D$, where D is the closed convex cone

$$D := \left\{(y_1, y_2, y_3) \in \mathbb{R}^3 \;\middle|\; y_1^2 + 2y_1y_3 + y_2^2 \leqslant 0 \wedge y_3 \geqslant 0\right\}.$$

(Hence $D = C^\ominus$.)

Exercise 20.4 (finite positive duality gap) Suppose that $X = \mathbb{R}^3$, $f = \iota_C$, where C is as in Exercise 20.3, and $g(x) = \exp(x_2) + \iota_{\{0\}}(x_3)$. Set up the Fenchel dual, find μ and μ^*, and enjoy checking the fact that the duality gap $\mu + \mu^*$ is equal to 1.

Exercise 20.5 (infinite duality gap) Suppose that $X = \mathbb{R}$, set $f(x) = -x + \iota_{\mathbb{R}_+}$, $g = \iota_{\{1\}}$, and $A = 0$. Compute μ and μ^* in this case.

Chapter 21

Infimal Convolution and Conjugacy

Recall that the infimal convolution (Definition 6.8) of two proper functions f, g on X is given by

$$(f \,\square\, g)(y) = \inf_x \big(f(x) + g(y - x)\big).$$

21.1 ▪ Fenchel Conjugate of the Infimal Convolution

Proposition 21.1 *Let f, g be proper on X. Then*

$$(f \,\square\, g)^* = f^* + g^*.$$

Proof. Let $y \in X$. We obtain (using the change of variable $z = x - u$ in the penultimate equality)

$$\begin{aligned}
(f \,\square\, g)^*(y) &= \sup_{x \in X} \big(\langle x, y \rangle - (f \,\square\, g)(x) \big) \\
&= \sup_{x \in X} \big(\langle x, y \rangle - \inf_{u \in X} (f(u) + g(x - u)) \big) \\
&= \sup_{x \in X} \sup_{u \in X} \big(\langle x, y \rangle - f(u) - g(x - u) \big) \\
&= \sup_{u \in X} \sup_{x \in X} \big(\langle u, y \rangle - f(u) + \langle x - u, y \rangle - g(x - u) \big) \\
&= \sup_{u \in X} \big(\langle u, y \rangle - f(u) \big) + \sup_{z \in X} \big(\langle z, y \rangle - g(z) \big) \\
&= f^*(y) + g^*(y),
\end{aligned}$$

as announced. □

Corollary 21.2 *Let f, g be convex and proper on X, and assume that $f \,\square\, g$ is lower semicontinuous and proper. Then $f \,\square\, g = (f^* + g^*)^*$.*

Proof. By Proposition 21.1, we have $f^* + g^* = (f \,\square\, g)^*$. Taking the conjugate again gives $(f^* + g^*)^* = (f \,\square\, g)^{**}$. But $f \,\square\, g$ is convex (Proposition 6.9), lower semicontinuous, and proper (by assumption); hence, $f \,\square\, g = (f \,\square\, g)^{**}$ by the Biconjugate Theorem (Theorem 19.2). □

21.2 ▪ Fenchel Conjugate of the Sum

The following result is "dual" to Proposition 21.1:

Fact 21.3 [3, Proposition 15.5 and Theorem 15.3] *Let f, g be convex, lower semicontinuous, and proper on X. Assume that* $(\operatorname{ri}\operatorname{dom} f) \cap (\operatorname{ri}\operatorname{dom} g) \neq \varnothing$. *Then*

$$(f+g)^* = f^* \,\square\, g^*$$

and the infimum in the definition of the infimal convolution is attained, i.e., a minimum.

Remark 21.4 (Fenchel vs. Fourier) In harmonic analysis, the Fourier transform of a function,

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)\exp(-2\pi \mathrm{i} y)\,\mathrm{d}x,$$

is of central importance. The Fenchel conjugate is the Fourier transform of convex analysis. The Biconjugate Theorem discusses when $f^{**} = f$, while in harmonic analysis we have $\hat{\hat{f}}(x) = f(-x)$. The counterpart to the infimal convolution is the classical convolution

$$(f \star g)(y) = \int_{\mathbb{R}} f(x)g(y-x)\,\mathrm{d}x.$$

The classical convolution is important because $\widehat{f \star g} = \hat{f}\cdot\hat{g}$ and $\widehat{f \cdot g} = \hat{f}\star\hat{g}$; these results have counterparts in $(f \,\square\, g)^* = f^* + g^*$ and $(f+g)^* = f^* \,\square\, g^*$.

Exercises

Exercise 21.1 Let C and D be subsets of X. Show that $\iota_C \,\square\, \iota_D = \iota_{C+D}$.

Exercise 21.2 Let f be a proper function on X, and let $b \in X$. Simplify $f \,\square\, \iota_{\{b\}}$.

Exercise 21.3 Compute $f \,\square\, g$ when $f(x) = x$ and $g(x) = -x$, where $x \in \mathbb{R}$.

Exercise 21.4 Let f, g be functions from X to $\left]-\infty, +\infty\right]$. Show that $\operatorname{epi} f + \operatorname{epi} g \subseteq \operatorname{epi}(f \,\square\, g)$.

Exercise 21.5 Let f, g be functions from X to $\left]-\infty, +\infty\right]$. Assume that the infimal convolution is everywhere exact, i.e., if $y \in X$, then there exists $x \in X$ such that $(f \,\square\, g)(y) = f(x) + g(y-x)$. Show that $\operatorname{epi} f + \operatorname{epi} g = \operatorname{epi}(f \,\square\, g)$. (For this reason, the infimal convolution is also referred to as the *episum*.)

Exercise 21.6 Suppose that $X = \mathbb{R}^2$ and consider the proper lower semicontinuous function $f(x) = -\sqrt{x_1 x_2}$ with $\operatorname{dom} f = \mathbb{R}^2_+$. Show that

$$f^*(y_1, y_2) = \begin{cases} 0 & \text{if } y_1 \leqslant \dfrac{1}{4y_2} < 0; \\ +\infty & \text{otherwise.} \end{cases}$$

Exercise 21.7 (Attouch–Brezis) Suppose that $X = \mathbb{R}^2$ and consider the two proper lower semicontinuous convex functions $f(x) = -\sqrt{x_1 x_2}$ with $\operatorname{dom} f = \mathbb{R}^2_+$ and $g = \iota_{\{0\}\times\mathbb{R}}$. Show that $(f+g)^* = \iota_{\mathbb{R}\times\mathbb{R}_-} \neq \iota_{\mathbb{R}\times\mathbb{R}_{--}} = f^* \,\square\, g^*$.

Chapter 22

Nonexpansive Mappings

A mapping T from a subset D of X to Y is *Lipschitz continuous with constant* $L \geqslant 0$, or simply *L-Lipschitz*, if $\|T(x) - T(y)\|_Y \leqslant L\|x - y\|_X$ for all x, y in D, where $\|\cdot\|_X$ denotes a norm on X and $\|\cdot\|_Y$ denotes a norm on Y and $L \geqslant 0$. We will often write Tx instead of $T(x)$, etc. If $L = 1$, we say that T is *nonexpansive*, while if $L < 1$, we say that T is a *(Banach) contraction*. If $Y = \mathbb{R}$, we use $\|\cdot\|_Y = |\cdot|$. If we don't single the norm out, the Euclidean norm is used.

In this chapter, we focus on nonexpansive mappings, while the next one will consider Lipschitz operators.

22.1 ▪ Firmly Nonexpansive Mappings

Definition 22.1 Let D be a subset of X, and let $T\colon D \to X$. Then T is *firmly nonexpansive* if

$$(\forall x \in D)(\forall y \in D) \quad \|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \leqslant \|x - y\|^2.$$

It is clear that every firmly nonexpansive map is also nonexpansive. The converse fails, however:

Example 22.2 Clearly, Id is firmly nonexpansive. However, $-\mathrm{Id}$ is nonexpansive but not firmly nonexpansive: Indeed, $\|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 = 5\|x - y\|^2 > \|x - y\|^2$ whenever $x \neq y$.

Theorem 22.3 *Let $T\colon D \to X$. Then the following are equivalent:*

(i) *T is firmly nonexpansive.*

(ii) *$\mathrm{Id} - T$ is firmly nonexpansive.*

(iii) *$(\forall x \in D)(\forall y \in D)$ $\|Tx - Ty\|^2 \leqslant \langle x - y, Tx - Ty\rangle$.*

(iv) *$2T - \mathrm{Id}$ is nonexpansive.*

Proof. Life is easier if we write $a := x - y$ and $b := Tx - Ty$.

"(i)⇔(ii)": Clear since $\mathrm{Id} - (\mathrm{Id} - T) = T$.

"(i)⇔(iii)": Note that

$$\|a\|^2 - \big(\|b\|^2 + \|a - b\|^2\big) = 2\big(\langle a, b\rangle - \|b\|^2\big),$$

which implies (i) $\Leftrightarrow$ $\|b\|^2 + \|a - b\|^2 \leqslant \|a\|^2 \Leftrightarrow 0 \leqslant \|a\|^2 - (\|b\|^2 + \|a - b\|^2) \Leftrightarrow 0 \leqslant 2(\langle a, b\rangle - \|b\|^2) \Leftrightarrow \|b\|^2 \leqslant \langle a, b\rangle \Leftrightarrow$ (iii).

"(iii)⇔(iv)": Note that

$$\|a\|^2 - \|2b-a\|^2 = 4\big(\langle a,b\rangle - \|b\|^2\big),$$

which implies (iv) ⇔ $\|2b-a\| \leqslant \|a\| \Leftrightarrow 0 \leqslant \|a\|^2 - \|2b-a\|^2 \Leftrightarrow 0 \leqslant 4(\langle a,b\rangle - \|b\|^2) \Leftrightarrow \|b\|^2 \leqslant \langle a,b\rangle$ ⇔ (iii). □

Example 22.4 Let C be a nonempty closed convex subset of X. Then both P_C and $\mathrm{Id} - \mathrm{P}_C$ are firmly nonexpansive (and hence nonexpansive). Moreover, $2\mathrm{P}_C - \mathrm{Id}$ is nonexpansive.

Proof. Combine Theorem 22.3 with Corollary 8.5. □

Proposition 22.5 *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then the following hold:*

(i) *g is nonexpansive* ⇔ $(\forall x \in \mathbb{R})\ |g'(x)| \leqslant 1$.

(ii) *g is firmly nonexpansive* ⇔ $(\forall x \in \mathbb{R})\ 0 \leqslant g'(x) \leqslant 1$.

Proposition 22.6 *Let $A\colon X \to X$ be linear. Then the following hold:*

(i) *A is nonexpansive* ⇔ $\mathrm{Id} - A^*A$ *is positive semidefinite.*

(ii) *A is firmly nonexpansive* ⇔ $A + A^* - 2A^*A$ *is positive semidefinite.*

It is clear that the composition of nonexpansive mappings is nonexpansive; however, this nice property fails for firmly nonexpansive mappings:

Example 22.7 Consider

$$A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B := \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then both A and B are firmly nonexpansive but BA is not.

On the positive side, we have the following:

Proposition 22.8 *Let T_1 and T_2 be firmly nonexpansive on X, and let x, y be in X. Then*

$$\tfrac{1}{2}\|(\mathrm{Id} - T_2T_1)x - (\mathrm{Id} - T_2T_1)y\|^2 \leqslant \|x-y\|^2 - \|T_2T_1x - T_2T_1y\|^2.$$

Proof. Set $a := x - y$, $b := T_1x - T_1y$, and $c := T_2T_1x - T_2T_1y$. Using the definition twice, we have

$$\begin{aligned} \|T_2T_1x - T_2T_1y\|^2 &\leqslant \|T_1x - T_1y\|^2 - \|(\mathrm{Id} - T_2)T_1x - (\mathrm{Id} - T_2)T_1y\|^2 \\ &\leqslant \|x-y\|^2 - \|(\mathrm{Id} - T_1)x - (\mathrm{Id} - T_1)y\|^2 \\ &\quad - \|(\mathrm{Id} - T_2)T_1x - (\mathrm{Id} - T_2)T_1y\|^2; \end{aligned}$$

equivalently, in terms of a, b, c: $\|c\|^2 \leqslant \|a\|^2 - \|a-b\|^2 - \|b-c\|^2$, i.e.,

$$\|a-b\|^2 + \|b-c\|^2 \leqslant \|a\|^2 - \|c\|^2.$$

On the other hand, $2\|a-b\|^2 + 2\|b-c\|^2 = \|a-c\|^2 + \|a+c-2b\|^2 \geqslant \|a-c\|^2$ and thus

$$\tfrac{1}{2}\|a-c\|^2 \leqslant \|a-b\|^2 + \|b-c\|^2.$$

Altogether,

$$\tfrac{1}{2}\|a-c\|^2 \leqslant \|a\|^2 - \|c\|^2,$$

which is precisely the desired conclusion. □

Proposition 22.9 (fixed point set) *Let $T\colon X \to X$ be nonexpansive. Then the set of fixed points* $\operatorname{Fix} T := \{x \in X \mid x = Tx\}$ *is closed and convex.*

Proof. The result is trivial if $\operatorname{Fix} T$ is empty; thus, we assume that x_0, x_1 belong to $\operatorname{Fix} T$ and that $\lambda \in [0,1]$. Set $x_\lambda := (1-\lambda)x_0 + \lambda x_1$. Then, using (1.4) twice, we obtain

$$\begin{aligned}
&\|x_\lambda - Tx_\lambda\|^2 \\
&\quad = \|(1-\lambda)(x_0 - Tx_\lambda) + \lambda(x_1 - Tx_\lambda)\|^2 \\
&\quad = (1-\lambda)\|x_0 - Tx_\lambda\|^2 + \lambda\|x_1 - Tx_\lambda\|^2 - (1-\lambda)\lambda\|x_0 - x_1\|^2 \\
&\quad = (1-\lambda)\|Tx_0 - Tx_\lambda\|^2 + \lambda\|Tx_1 - Tx_\lambda\|^2 - (1-\lambda)\lambda\|x_0 - x_1\|^2 \\
&\quad \leqslant (1-\lambda)\|x_0 - x_\lambda\|^2 + \lambda\|x_1 - x_\lambda\|^2 - (1-\lambda)\lambda\|x_0 - x_1\|^2 \\
&\quad = \|(1-\lambda)(x_0 - x_\lambda) + \lambda(x_1 - x_\lambda)\|^2 \\
&\quad = 0,
\end{aligned}$$

which yields $x_\lambda = Tx_\lambda$ and hence the convexity of $\operatorname{Fix} T$. Closedness follows easily from the continuity of T. □

22.2 ▪ Averaged Mappings

We saw in Example 22.7 that the class of firmly nonexpansive operators is not closed under composition. A useful larger class does have this property: averaged mappings, which we introduce next!

Definition 22.10 Let $T\colon X \to X$, and let $\alpha \in \,]0,1[$. Then T is *α-averaged* if there exists $N\colon X \to X$ such that N is nonexpansive and $T = (1-\alpha)\mathrm{Id} + \alpha N$.

Theorem 22.11 *Let $T\colon X \to X$ be nonexpansive, and let $\alpha \in \,]0,1[$. Then T is α-averaged if and only if for all x, y, we have*

$$\|Tx - Ty\|^2 \leqslant \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|(\mathrm{Id}-T)x - (\mathrm{Id}-T)y\|^2. \tag{22.1}$$

Proof. Write $T = (1-\alpha)\mathrm{Id} + \alpha N$ for $N\colon X \to X$. Then $N = (1-\alpha^{-1})\mathrm{Id} + \alpha^{-1}T$. Write $u = x - y$ and $v = Tx - Ty$. Using $\alpha(1.5)$, we obtain

$$\begin{aligned}
&\alpha^2\big(\|x-y\|^2 - \|Nx - Ny\|^2\big) \\
&\quad = \alpha^2\big(\|x-y\|^2 - \|(1-\alpha^{-1})(x-y) + \alpha^{-1}(Tx - Ty)\|^2\big) \\
&\quad = \alpha^2\big(\|u\|^2 - \|(1-\alpha^{-1})u + \alpha^{-1}v\|^2\big) \\
&\quad = \alpha\big(\|u\|^2 - \tfrac{1-\alpha}{\alpha}\|u-v\|^2 - \|v\|^2\big) \\
&\quad = \alpha\big(\|x-y\|^2 - \tfrac{1-\alpha}{\alpha}\|(\mathrm{Id}-T)x - (\mathrm{Id}-T)y\|^2 - \|Tx-Ty\|^2\big).
\end{aligned}$$

This identity shows that (22.1) $\Leftrightarrow$ N is nonexpansive and we are done. □

Remark 22.12 Note that an operator is $\frac{1}{2}$-averaged if and only if it is firmly nonexpansive.

If T is α-averaged and $\alpha < \beta < 1$, then T is also β-averaged. Loosely speaking, α is a measure of distance to the identity. Notice that the smaller the α, the better, as then the inequality (22.1) becomes tighter.

Corollary 22.13 *Let T_1 and T_2 be firmly nonexpansive on X. Then T_2T_1 is $\frac{2}{3}$-averaged.*

Proof. Combine Proposition 22.8 with Theorem 22.11. □

Proposition 22.14 *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function, and let $\alpha \in \,]0,1[$. Then g is α-averaged $\Leftrightarrow$ $(\forall x \in \mathbb{R})\ 1 - 2\alpha \leqslant g'(x) \leqslant 1$. Consequently, g is averaged $\Leftrightarrow$ $-1 < \mu := \inf g'(\mathbb{R}) \leqslant \sup g'(\mathbb{R}) \leqslant 1$, in which case g is $(1-\mu)/2$-averaged.*

Proposition 22.15 *Let $A\colon X \to X$ be linear, and let $\alpha \in \,]0,1[$. Then A is α-averaged $\Leftrightarrow$ $(2\alpha - 1)\mathrm{Id} - (A^*A - (1-\alpha)(A + A^*))$ is positive semidefinite.*

Remark 22.16 Let $M \in \mathbb{R}^{n\times n}$. Assume that M is nonexpansive; equivalently, $M^{\mathsf{T}}M \preceq \mathrm{Id}$. Then T is α-averaged if $\alpha \in \,]0,1[$ and

$$0 \preceq (\mathrm{Id} - M)^{\mathsf{T}}(\mathrm{Id} - M) = \mathrm{Id} + M^{\mathsf{T}}M - (M + M^{\mathsf{T}}) \preceq \alpha(2\mathrm{Id} - (M + M^{\mathsf{T}})) \tag{22.2}$$

by Proposition 22.15. Note that (22.2) holds with $\alpha = 1$. To decide whether or not M is averaged and also find the best possible constant in the affirmative case, we set

$$A := (\mathrm{Id} - M)^{\mathsf{T}}(\mathrm{Id} - M) \quad \text{and} \quad B := 2\mathrm{Id} - (M + M^{\mathsf{T}}).$$

The problem is to find the smallest possible $\gamma \geqslant 0$ such that

$$A \preceq \gamma B.$$

To find the optimal γ, we can employ the procedure outlined in Proposition 1.1. If $\gamma = 1$, then M is not averaged. If $\gamma < 1$, then M is γ-averaged and the constant is the smallest possible. (If $\gamma = 0$, then M was the identity.)

Fact 22.17 [3, Proposition 4.46] *Let T_i be α_i-averaged for $i \in I := \{1, \ldots, m\}$. Then*

$$T := T_m T_{m-1} \cdots T_1 \ \text{ is } \beta\text{-averaged with } \ \beta := \frac{\sum_{i\in I} \frac{\alpha_i}{1-\alpha_i}}{1 + \sum_{i\in I} \frac{\alpha_i}{1-\alpha_i}}.$$

22.3 ▪ Sequential Results

Definition 22.18 (Fejér monotonicity) A sequence $(x_k)_{k\in\mathbb{N}}$ in X is *Fejér-monotone* with respect to a nonempty subset C of X if

$$(\forall c \in C)(\forall k \in \mathbb{N}) \quad \|x_{k+1} - c\| \leqslant \|x_k - c\|.$$

Theorem 22.19 *Let $(x_k)_{k\in\mathbb{N}}$ be Fejér-monotone with respect to a nonempty subset C of X. Then $(x_k)_{k\in\mathbb{N}}$ is bounded; moreover, $(x_k)_{k\in\mathbb{N}}$ converges to some point in C $\Leftrightarrow$ $(x_k)_{k\in\mathbb{N}}$ has a cluster point in C.*

Proof. Note that $\|x_{k+1} - c\| \leqslant \|x_k - c\| \leqslant \cdots \leqslant \|x_0 - c\|$, so $\lim_{k\in\mathbb{N}} \|x_k - c\|$ exists for every $c \in C$ and hence $(x_k)_{k\in\mathbb{N}}$ is bounded. We now prove the "moreover" part. The implication "$\Rightarrow$" is clear. Now assume that $\bar{c} \in C$ is some cluster point, say $x_{n_k} \to \bar{c}$. Then $\lim_{k\in\mathbb{N}} \|x_k - \bar{c}\| = \lim_{k\in\mathbb{N}} \|x_{n_k} - \bar{c}\| = 0$, and hence the entire sequence $(x_k)_{k\in\mathbb{N}}$ converges to $\bar{c}$ as well. ☐

Corollary 22.20 *Let $T\colon X \to X$ be α-averaged with $\operatorname{Fix} T \neq \varnothing$. Then $(T^k x_0)_{k\in\mathbb{N}}$ is Fejér-monotone with respect to $\operatorname{Fix} T$, and it converges to some point in $\operatorname{Fix} T$.*

Proof. Set $(\forall k \in \mathbb{N})\ x_k = T^k x_0$. Let $y \in \operatorname{Fix} T$. Then $Ty = y$ and hence Theorem 22.11 yields

$$\frac{1-\alpha}{\alpha}\|x_k - x_{k+1}\|^2 \leqslant \|x_k - y\|^2 - \|x_{k+1} - y\|^2.$$

Hence $(x_k)_{k\in\mathbb{N}}$ is Fejér-monotone with respect to $\operatorname{Fix} T$ and $\sum_{k\in\mathbb{N}} \|x_k - x_{k+1}\|^2 < +\infty$. Thus, $x_k - x_{k+1} = x_k - Tx_k \to 0$. By Theorem 22.19, $(x_k)_{k\in\mathbb{N}}$ is bounded. Let $\bar{x}$ be a cluster point of $(x_k)_{k\in\mathbb{N}}$, say $x_{n_k} \to \bar{x}$. Then the continuity of T yields $0 \leftarrow x_{n_k} - Tx_{n_k} \to \bar{x} - T\bar{x}$, i.e., $\bar{x} \in \operatorname{Fix} T$. By Theorem 22.19, the entire sequence $(x_k)_{k\in\mathbb{N}}$ converges to $\bar{x} \in \operatorname{Fix} T$. ☐

Exercises

Exercise 22.1 Let $T\colon X \to X$, and let x and y be in D. Show that the following are equivalent:

(i) $\|Tx - Ty\|^2 + \|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 = \|x - y\|^2$.

(ii) $\|Tx - Ty\|^2 = \langle x - y, Tx - Ty\rangle$.

(iii) $\langle Tx - Ty, (\mathrm{Id} - T)x - (\mathrm{Id} - T)y\rangle = 0$.

(iv) $(\forall \alpha \in \mathbb{R})\ \|Tx - Ty\| \leqslant \|\alpha(x - y) + (1 - \alpha)(Tx - Ty)\|$.

(v) $\|(2T - \mathrm{Id})x - (2T - \mathrm{Id})y\| = \|x - y\|$.

Exercise 22.2 Provide the details for Proposition 22.5.

Exercise 22.3 Provide the details for Proposition 22.6.

Exercise 22.4 Provide the details for Example 22.7.

Exercise 22.5 Suppose $T\colon X \to X$ is firmly nonexpansive and $T \circ T = T$. Show that $T = \mathrm{P}_C$, where $C = \operatorname{ran} T$.

Exercise 22.6 Let T_1, T_2 be firmly nonexpansive on X, let λ_1, λ_2 be in $[0, 1]$ such that $\lambda_1 + \lambda_2 = 1$, and set $T := \lambda_1 T_1 + \lambda_2 T_2$. Show that T is firmly nonexpansive.

Exercise 22.7 Provide the details for Remark 22.12.

Exercise 22.8 Provide the details for Proposition 22.14.

Exercise 22.9 Provide the details for Proposition 22.15.

Exercise 22.10 Show that $g(x) := (2x \arctan(x) - \ln(x^2 + 1))/\pi$ is nonexpansive but not averaged.

Exercise 22.11 Show that $g(x) := \sqrt{x^2 + 1}$ is nonexpansive but not averaged.

Exercise 22.12 Show that $g(x) := 2\ln(1 + e^x) - x$ is nonexpansive but not averaged.

Exercise 22.13 Show that $A\colon \mathbb{R}^2 \to \mathbb{R}^2\colon (x, y) \mapsto (-y, x)$ is nonexpansive but not averaged by using Proposition 22.15.

Exercise 22.14 Show that $M\colon \mathbb{R}^2 \to \mathbb{R}^2\colon (x, y) \mapsto (-y, x)$ is nonexpansive but not averaged by using the strategy outlined in Remark 22.16.

Exercise 22.15 Show that $M\colon \mathbb{R}^2 \to \mathbb{R}^2\colon (x, y) \mapsto \frac{1}{2}(x, x)$ is nonexpansive but not averaged by using the strategy outlined in Remark 22.16.

Exercise 22.16 Provide a sequence that is Fejér-monotone with respect to a nonempty subset C of X but that does not converge to a point in C.

Exercise 22.17 Provide a nonexpansive operator T with $\operatorname{Fix} T \neq \varnothing$ and a point $x_0 \in X$ such that $(T^k x_0)_{k\in\mathbb{N}}$ does not converge to a fixed point of T.

Chapter 23

Lipschitz Continuity and Smoothness

We recall from the last chapter the following: A mapping T from a subset D of X to Y is *Lipschitz continuous with constant* $L \geqslant 0$, or simply *L-Lipschitz*, if $\|T(x) - T(y)\|_Y \leqslant L\|x - y\|_X$ for all x, y in D, where $\|\cdot\|_X$ denotes a norm on X and $\|\cdot\|_Y$ denotes a norm on Y and $L \geqslant 0$. We will often write Tx instead of $T(x)$, etc. If $L = 1$, we say that T is *nonexpansive*, while if $L < 1$, we say that T is a *(Banach) contraction*. If $Y = \mathbb{R}$, we use $\|\cdot\|_Y = |\cdot|$, and if $Y = X$, we often use the Euclidean norm.

While we discussed nonexpansiveness in the last chapter, we now focus on Lipschitz properties.

23.1 ▪ Lipschitz Functions

Theorem 23.1 *Let f be convex and proper on X. Suppose that C is a nonempty subset of* $\operatorname{int}\operatorname{dom} f$. *Then the following hold:*

(i) *If $L := \sup\|\partial f(C)\| < +\infty$, then f is L-Lipschitz on C.*

(ii) *If C is open and f is L-Lipschitz on C for some constant L, then* $\sup\|\partial f(C)\| \leqslant L$.

Proof. Note that $C \subseteq \operatorname{dom}\partial f$ by Fact 9.12.

(i): Take x, y in C and $u \in \partial f(x)$ and $v \in \partial f(y)$. Then

$$\begin{aligned} f(x) - f(y) &\leqslant \langle x - y, u\rangle \leqslant \|x - y\|\|u\| \leqslant L\|x - y\|, \\ f(y) - f(x) &\leqslant \langle y - x, v\rangle \leqslant \|y - x\|\|v\| \leqslant L\|x - y\|. \end{aligned}$$

Hence $|f(x) - f(y)| = \max\{f(x) - f(y), f(y) - f(x)\} \leqslant L\|x - y\|$.

(ii): Let $x \in C$ and $u^* \in \partial f(x)$. Pick $u \in X$ such that $\|u\| \leqslant 1$ and $\langle u, u^*\rangle = \|u^*\|$. Because C is open, there exists $\delta > 0$ such that $B[x; \delta] \subseteq C$. Hence

$$\begin{aligned} f(x) + \delta\langle u, u^*\rangle &= f(x) + \langle (x + \delta u) - x, u^*\rangle \\ &\leqslant f(x + \delta u). \end{aligned}$$

Thus $\delta\|u^*\| = \delta\langle u, u^*\rangle \leqslant f(x+\delta u) - f(x) \leqslant L\|(x+\delta u) - x\| = L\delta\|u\| \leqslant L\delta$, which implies $\|u^*\| \leqslant L$. □

Corollary 23.2 *Let f be convex and proper on X, and suppose that C is a nonempty compact subset of* $\operatorname{int}\operatorname{dom} f$. *Then there exists $L \geqslant 0$ such that $f|_C$ is L-Lipschitz.*

Proof. By Fact 9.14, $\partial f(C)$ is bounded. Now apply Theorem 23.1(i). □

Corollary 23.3 *Let $f\colon X \to \mathbb{R}$ be convex, and let $L \geqslant 0$. Then f is globally L-Lipschitz, i.e.,*

$$(\forall x \in X)(\forall y \in X) \quad |f(x) - f(y)| \leqslant L\|x - y\|$$

if and only if $\operatorname{ran} \partial f \subseteq B[0; L]$.

23.2 ▪ Smoothness and the Descent Lemma

We now turn to functions whose gradients are Lipschitz continuous.

Definition 23.4 (Lipschitz smoothness) Let $f\colon X \to \left]-\infty, +\infty\right]$ be proper and assume that D is a nonempty open subset of $\operatorname{dom} f$ and that f is differentiable on D. Let $L \geqslant 0$. Then f is *L-smooth* on D if ∇f is L-Lipschitz on D, i.e.,

$$(\forall x \in D)(\forall y \in D) \quad \|\nabla f(x) - \nabla f(y)\| \leqslant L\|x - y\|.$$

Example 23.5 Let C be a nonempty closed convex subset of X. Then the functions $f(x) := \frac{1}{2}\|x\|^2 - \frac{1}{2}d_C^2(x)$ and $g(x) := \frac{1}{2}d_C^2(x)$ are both 1-smooth.

The next result is known as the "Descent Lemma":

Lemma 23.6 (Descent Lemma) *Let $f\colon X \to \left]-\infty, +\infty\right]$ be proper and assume that D is a nonempty open convex subset of* $\operatorname{dom} f$ *and that f is differentiable on D. Suppose that f is L-smooth on D. Then*

$$(\forall x \in D)(\forall y \in D) \;\; f(y) \leqslant f(x) + \langle y - x, \nabla f(x)\rangle + \frac{L}{2}\|x - y\|^2;$$

put differently (and switching x and y),

$$(\forall x \in D)(\forall y \in D) \;\; D_f(x, y) \leqslant \frac{L}{2}\|x - y\|^2.$$

Proof. Let x, y be in D, and set $h(t) := f(x+t(y-x))$. Then $h'(t) = \langle y - x, \nabla f(x + t(y - x))\rangle$. Hence the Fundamental Theorem of Calculus yields

$$f(y) - f(x) = h(1) - h(0) = \int_0^1 h'(t)\, dt = \int_0^1 \langle y - x, \nabla f(x + t(y - x))\rangle\, dt.$$

Using this, $\langle y - x, \nabla f(x)\rangle = \int_0^1 \langle y - x, \nabla f(x)\rangle\, dt$, the Cauchy–Schwarz inequality, and L-smoothness of f, we thus obtain

$$\begin{aligned}
f(y) - f(x) - \langle y - x, \nabla f(x)\rangle &\leqslant |f(y) - f(x) - \langle y - x, \nabla f(x)\rangle| \\
&= \left|\int_0^1 \langle y - x, \nabla f(x + t(y - x)) - \nabla f(x)\rangle\, dt\right| \\
&\leqslant \int_0^1 \|y - x\|\|\nabla f(x + t(y - x)) - \nabla f(x)\|\, dt \\
&\leqslant \int_0^1 L\|y - x\|^2 t\, dt = L\|y - x\|^2 \tfrac{1}{2}t^2\Big|_0^1 \\
&= \frac{L}{2}\|y - x\|^2,
\end{aligned}$$

as announced. ☐

Remark 23.7 Suppose that f is as in Lemma 23.6, and let $0 \leqslant \alpha < 2/L$. If $x \in D$ and $x - \alpha\nabla f(x) \in D$, then

$$f\big(x - \alpha\nabla f(x)\big) \leqslant f(x) + \alpha\Big(\frac{L\alpha}{2} - 1\Big)\|\nabla f(x)\|^2 \leqslant f(x),$$

which is why we refer to Lemma 23.6 as the Descent Lemma.

23.3 ▪ Characterizations of Smoothness

Fact 23.8 [5, Theorem 5.8] *Let $f\colon X \to \mathbb{R}$ be convex and differentiable and let $L > 0$. Then the following are equivalent:*

(i) *f is L-smooth, i.e., $(\forall x \in X)(\forall y \in X)\ \|\nabla f(x) - \nabla f(y)\| \leqslant L\|x - y\|$.*

(ii) $(\forall x \in X)(\forall y \in X)\ f(y) \leqslant f(x) + \langle y - x, \nabla f(x)\rangle + \frac{L}{2}\|x - y\|^2$.

(iii) $(\forall x \in X)(\forall y \in X)\ f(y) \geqslant f(x) + \langle y - x, \nabla f(x)\rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|^2$.

(iv) $(\forall x \in X)(\forall y \in X)\ \langle x - y, \nabla f(x) - \nabla f(y)\rangle \geqslant \frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2$.

(v) $(\forall x \in X)(\forall y \in X)(\forall \lambda \in [0,1])$
$(1-\lambda)f(x) + \lambda f(y) \leqslant f((1-\lambda)x + \lambda y) + \frac{L}{2}(1-\lambda)\lambda\|x - y\|^2$.

If $X = \mathbb{R}^n$ and f is twice continuously differentiable, then we may add the following items to this list:

(vi) $(\forall x \in \mathbb{R}^n)\ \nabla^2 f(x) \preceq L\mathrm{Id}$.

(vii) $(\forall x \in \mathbb{R}^n)\ \lambda_{\max}(\nabla^2 f(x)) \leqslant L$.

(viii) $(\forall x \in \mathbb{R}^n)\ \|\nabla^2 f(x)\| \leqslant L$.

Proof. "(i)⇒(ii)": Lemma 23.6.

"(ii)⇒(iii)": The hypothesis (ii) states that $D_f \leqslant \frac{L}{2}\|\cdot - \cdot\|^2$.

Now fix $x \in X$. Set

$$h := y \mapsto D_f(y, x) = f(y) - f(x) - \langle y - x, \nabla f(x)\rangle\,.$$

Because $h = f + \langle\cdot, -\nabla f(x)\rangle + (\langle x, \nabla f(x)\rangle - f(x))$ is an affine perturbation of f, it follows from Example 12.11 that $D_h = D_f$. Hence

$$(\forall z \in X)(\forall y \in X) \quad h(z) - h(y) - \langle z - y, \nabla h(y)\rangle = D_h(z, y) \leqslant \frac{L}{2}\|z - y\|^2.$$

Next, by Proposition 12.8, the function $h = D_f(\cdot, x)$ has x as a minimizer with $h(x) = 0$, and also $\nabla h(y) = \nabla f(y) - \nabla f(x)$. Hence

$$(\forall z \in X) \quad h(x) \leqslant h(z).$$

Now fix also $y \in X$. If $\nabla f(y) = \nabla f(x)$, i.e., $\nabla h(y) = 0$, then the desired conclusion follows from Theorem 11.10(ii). So we assume that $\nabla h(y) = \nabla f(y) - \nabla f(x) \neq 0$. Next, obtain a vector $w \in X$ such that $\|w\| = 1$ and $\langle w, \nabla h(y)\rangle = \|\nabla h(y)\|$. Set

$$z := y - \frac{\|\nabla h(y)\|}{L}w.$$

Then

$$
\begin{aligned}
0 = h(x) &\leqslant h(z) \\
&\leqslant h(y) + \langle z - y, \nabla h(y) \rangle + \frac{L}{2}\|z - y\|^2 \\
&= h(y) - \frac{\|\nabla h(y)\|}{L}\langle w, \nabla h(y) \rangle + \frac{L}{2}\frac{\|\nabla h(y)\|^2}{L^2}\|w\|^2 \\
&= h(y) - \frac{\|\nabla h(y)\|}{L}\|\nabla h(y)\| + \frac{\|\nabla h(y)\|^2}{2L} \\
&= h(y) - \frac{\|\nabla h(y)\|^2}{2L} \\
&= f(y) - f(x) - \langle y - x, \nabla f(x) \rangle - \frac{1}{2L}\|\nabla f(y) - \nabla f(x)\|^2
\end{aligned}
$$

and we obtain (iii).

"(iv)⇒(i)": We have, using the Cauchy–Schwarz inequality,

$$
\begin{aligned}
\frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2 &\leqslant \langle x - y, \nabla f(x) - \nabla f(y) \rangle \\
&\leqslant \|x - y\|\|\nabla f(x) - \nabla f(y)\|,
\end{aligned}
$$

from which the result follows.

"(ii)⇒(v)": Let x_0, x_1 be in X, and set $x_\lambda := (1-\lambda)x_0 + \lambda x_1$ for $\lambda \in \,]0,1[$. By (ii),

$$
\begin{aligned}
f(x_0) &\leqslant f(x_\lambda) + \langle x_0 - x_\lambda, \nabla f(x_\lambda) \rangle + \frac{L}{2}\|x_0 - x_\lambda\|^2, \\
f(x_1) &\leqslant f(x_\lambda) + \langle x_1 - x_\lambda, \nabla f(x_\lambda) \rangle + \frac{L}{2}\|x_1 - x_\lambda\|^2,
\end{aligned}
$$

which is the same as

$$
f(x_0) \leqslant f(x_\lambda) + \lambda \langle x_0 - x_1, \nabla f(x_\lambda) \rangle + \lambda^2\frac{L}{2}\|x_0 - x_1\|^2, \tag{23.1a}
$$

$$
f(x_1) \leqslant f(x_\lambda) + (1-\lambda) \langle x_1 - x_0, \nabla f(x_\lambda) \rangle + (1-\lambda)^2\frac{L}{2}\|x_0 - x_1\|^2. \tag{23.1b}
$$

Now $(1-\lambda)$(23.1a) $+ \lambda$(23.1b) turns into

$$
(1-\lambda)f(x_0) + \lambda f(x_1) \leqslant f(x_\lambda) + (1-\lambda)\lambda\frac{L}{2}\|x_0 - x_1\|^2,
$$

which yields (v).

"(v)⇒(ii)": Let x, y be in X and $\lambda \in \,]0,1[$. By assumption, we have $\lambda f(y) \leqslant \lambda f(x) + (f((1-\lambda)x + \lambda y) - f(x)) + \frac{L}{2}(1-\lambda)\lambda\|x - y\|^2$; thus, dividing by $\lambda > 0$ yields

$$
\begin{aligned}
f(y) &\leqslant f(x) + \frac{f(x + \lambda(y - x)) - f(x)}{\lambda} + \frac{L}{2}(1-\lambda)\|x - y\|^2 \\
&\to f(x) + f'(x; y - x) + \frac{L}{2}\|x - y\|^2 \\
&= f(x) + \langle y - x, \nabla f(x) \rangle + \frac{L}{2}\|x - y\|^2
\end{aligned}
$$

as $\lambda \to 0^+$ and by Proposition 11.8.

From now on, we assume that $X = \mathbb{R}^n$ and that f is twice continuously differentiable.

"(vi)⇔(vii)⇔(viii)": Clear because for a positive semidefinite matrix A, we have $\lambda_{\max}(A) = \|A\| = \max\{\langle x, Ax \rangle \mid \|x\| \leqslant 1\}$ by, e.g., [8, Theorem 4.21].

Let x, y be in X.

"(i)⇒(viii)": Let $\alpha > 0$. Then $\|\nabla f(x+\alpha y) - \nabla f(x)\| \leqslant \alpha L\|y\|$. Dividing by α and then letting $\alpha \to 0^+$ yields $\|(\nabla^2 f(x))y\| \leqslant L\|y\|$. Hence $\|\nabla^2 f(x)\| \leqslant L$.

"(viii)⇒(i)": The derivative of $G(t) := \nabla f(x + t(y-x))$ is $\nabla G(t) = \nabla^2 f(x + t(y - x))(y - x)$. The Fundamental Theorem of Calculus now yields

$$\nabla f(y) - \nabla f(x) = G(1) - G(0) = \int_0^1 \nabla^2 f(x + t(y-x))(y-x)\, dt;$$

thus

$$\|\nabla f(y) - \nabla f(x)\| = \left\| \int_0^1 \nabla^2 f(x + t(y-x))(y-x)\, dt \right\| \leqslant \int_0^1 L\, dt \|y - x\| = L\|y-x\|,$$

and so f is L-smooth. □

Corollary 23.9 (Baillon–Haddad Theorem) *Let $f\colon X \to \mathbb{R}$ be convex and differentiable. Then ∇f is nonexpansive if and only if ∇f is firmly nonexpansive.*

Corollary 23.10 *Let f be convex and L-smooth on X with respect to the Euclidean norm. Then both $\frac{1}{L}\nabla f$ and $\mathrm{Id} - \frac{1}{L}\nabla f$ are firmly nonexpansive.*

Proof. $\frac{1}{L}\nabla f$ is clearly nonexpansive. By Corollary 23.9, $\frac{1}{L}\nabla f$ is firmly nonexpansive. Now apply Theorem 22.3 to obtain firm nonexpansiveness of $\mathrm{Id} - \frac{1}{L}\nabla f$. □

Example 23.11 (log-sum-exp is 1-smooth) On $X = \mathbb{R}^n$, recall that the log-sum-exp function is defined by

$$f(x) = f(x_1, \ldots, x_n) = \ln\big(\exp(x_1) + \cdots + \exp(x_n)\big).$$

Setting $u := u(x) := [u_1, \ldots, u_n]^\mathsf{T}$ with $u_i := \exp(x_i)/\sum_{j=1}^n \exp(x_j) \in \left]0,1\right[$, we have

$$\nabla^2 f(x) = \begin{bmatrix} u_1 & 0 & 0 & \cdots & 0 \\ 0 & u_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & & u_n \end{bmatrix} - uu^\mathsf{T} \preccurlyeq \mathrm{Id}, \tag{23.2}$$

and so f is 1-smooth.

Exercises

Exercise 23.1 Use Corollary 23.3 to find $L \geqslant 0$ such that the function $f(x) = \sqrt{1+x^2}$ is L-Lipschitz.

Exercise 23.2 Use Fact 23.8 to show that there is no $L \geqslant 0$ such that the exponential function is L-smooth.

Exercise 23.3 Let f be proper on X, and assume that D is a nonempty open convex subset of $\operatorname{dom} f$ and that f is differentiable on D. Additionally assume that f is concave. Show that $D_f(x,y) \leqslant \frac{L}{2}\|x-y\|^2$ for every $L \geqslant 0$ and for all x, y in D.

Exercise 23.4 Provide the details for Example 23.5.

Exercise 23.5 Provide the details for Remark 23.7.

Exercise 23.6 Consider $f(x) = -\frac{1}{2}\|x\|^2$ on $X = \mathbb{R}^n$. Show that for $L = 1$, Fact 23.8(i) holds but Fact 23.8(iv) fails.

Exercise 23.7 Show that (iii)⇒(iv) in Fact 23.8.

Exercise 23.8 Provide the details for Example 23.11.

Chapter 24

Strong Convexity

A long time ago, we discussed strict convexity (see Definition 3.10). Now we present a stronger, more quantitative version: strong convexity.

> Throughout this chapter, $\|\cdot\|$ denotes the Euclidean norm.

24.1 ▪ Characterizations of Strong Convexity

Definition 24.1 (strong convexity) Let f be proper on X, and let $\beta > 0$. Then f is *β-strongly convex* if $(\forall x \in X)(\forall y \in X)(\forall \lambda \in \,]0,1[)$ we have

$$f((1-\lambda)x + \lambda y) \leqslant (1-\lambda)f(x) + \lambda f(y) - \beta\frac{(1-\lambda)\lambda}{2}\|x-y\|^2.$$

If we don't care about the specific value of the parameter β, then we simply say that f is strongly convex.

Proposition 24.2 *Let f be proper and β-strongly convex for some $\beta > 0$. Then f is strictly convex.*

Proposition 24.3 *Let f be proper and β-strongly convex for some $\beta > 0$, let g be convex, and let $\alpha > 0$. Suppose that* $\operatorname{dom} f \cap \operatorname{dom} g \neq \varnothing$. *Then $\alpha f + g$ is $\alpha\beta$-strongly convex.*

Proof. Let x, y be in X, and let $\lambda \in \,]0,1[$. The assumptions imply

$$f\big((1-\lambda)x + \lambda y\big) \leqslant (1-\lambda)f(x) + \lambda f(y) - \beta\frac{(1-\lambda)\lambda}{2}\|x-y\|^2, \tag{24.1a}$$

$$g\big((1-\lambda)x + \lambda y\big) \leqslant (1-\lambda)g(x) + \lambda g(y). \tag{24.1b}$$

Considering α(24.1a) + (24.1b) yields the result. □

Fact 24.4 [5, Theorem 5.24] *Let f be convex, lower semicontinuous, and proper on X, and let $\beta > 0$. Then the following are equivalent:*

(i) *f is β-strongly convex.*

(ii) $(\forall (x,u) \in \operatorname{gra}\partial f)(\forall y \in \operatorname{dom} f)$ $f(x) + \langle u, y-x\rangle \leqslant f(y) - \frac{\beta}{2}\|x-y\|^2$.

(iii) $(\forall(x,u)\in \operatorname{gra}\partial f)(\forall(y,v)\in \operatorname{gra}\partial f)\ \langle x-y,u-v\rangle \geqslant \beta\|x-y\|^2$.

(iv) $f-\beta\frac{1}{2}\|\cdot\|^2$ *is convex.*

Proof. "(i)⇔(iv)": Let x,y be in X, and let $\lambda\in\,]0,1[$. Then, using (1.4), we see that the inequality

$$\begin{aligned} f\big((1-\lambda)x+\lambda y\big) &\leqslant (1-\lambda)f(x)+\lambda f(y)-\beta\frac{(1-\lambda)\lambda}{2}\|x-y\|^2 \\ &= (1-\lambda)f(x)+\lambda f(y)-\tfrac{\beta}{2}\big((1-\lambda)\|x\|^2+\lambda\|y\|^2-\|(1-\lambda)x+\lambda y\|^2\big) \end{aligned}$$

is equivalent to

$$f\big((1-\lambda)x+\lambda y\big)-\tfrac{\beta}{2}\|(1-\lambda)x+\lambda y\|^2 \leqslant (1-\lambda)\big(f(x)-\tfrac{\beta}{2}\|x\|^2\big)+\lambda\big(f(y)-\tfrac{\beta}{2}\|y\|^2\big),$$

and this yields the equivalence. □

Example 24.5 Suppose that $X=\mathbb{R}^n$ and let $f(x)=\frac{1}{2}\langle x,Ax\rangle+\langle x,b\rangle+\gamma$, where $A\in\mathbb{S}^n$, $b\in\mathbb{R}^n$, and $\gamma\in\mathbb{R}$. Then f is strongly convex ⇔ $A\succ 0$, in which case $\lambda_{\min}(A)$ is the largest possible constant of strong convexity of f.

Example 24.6 The function $\beta\frac{1}{2}\|\cdot\|^2$, where $\beta>0$, is β-strongly convex.

24.2 ▪ Strong Convexity and Optimization

Minimizing a strongly convex function is a joyous task because of the following result:

Theorem 24.7 *Let f be β-strongly convex and lower semicontinuous, where $\beta>0$. Then the following hold:*

(i) *f is supercoercive and hence coercive.*

(ii) *f has a unique minimizer, say $\bar{x}$.*

(iii) $(\forall x\in X)\ f(x)-f(\bar{x})\geqslant \frac{\beta}{2}\|x-\bar{x}\|^2$, *where $\bar{x}$ is as in* (ii).

Proof. Set

$$g := f-\beta\tfrac{1}{2}\|\cdot\|^2.$$

Then $\operatorname{dom} g=\operatorname{dom} f\neq\varnothing$ and so g is proper. Because f is lower semicontinuous and $-\beta\|\cdot\|^2$ is continuous, the function g is lower semicontinuous. By Fact 24.4(iv), g is convex. By Fact 9.12, $\operatorname{gra}\partial g\neq\varnothing$, say $u\in\partial g(x_0)$. Then

$$(\forall x\in X)\quad g(x)\geqslant g(x_0)+\langle x-x_0,u\rangle\geqslant \gamma_0-\|x\|\|u\|, \tag{24.2}$$

where we used Cauchy–Schwarz and set $\gamma_0:=g(x_0)-\langle x_0,u\rangle$.

(i): Adding back $\beta\frac{1}{2}\|x\|^2$ in (24.2) yields

$$(\forall x\in X)\quad f(x)\geqslant \beta\tfrac{1}{2}\|x\|^2-\|x\|\|u\|+\gamma_0. \tag{24.3}$$

Hence

$$\varliminf_{\|x\|\to+\infty}\frac{f(x)}{\|x\|}\geqslant \varliminf_{\|x\|\to+\infty}\Big(\beta\tfrac{1}{2}\|x\|-\|u\|+\frac{\gamma_0}{\|x\|}\Big)=+\infty.$$

It follows that f is supercoercive and hence coercive.

(ii): By Theorem 5.4 and Proposition 3.13, f has exactly one minimizer, which we call $\bar{x}$.

(iii): By (ii) and Fermat's rule (Lemma 9.2), we have $0 \in \partial f(\bar{x})$. Let $x \in \operatorname{dom} f$. Then Fact 24.4(ii) yields $f(\bar{x}) = f(\bar{x}) + \langle 0, x - \bar{x}\rangle \leqslant f(x) - \frac{\beta}{2}\|\bar{x} - x\|^2$. The proof is complete. $\square$

Corollary 24.8 *Let f be β-strongly convex and lower semicontinuous, where $\beta > 0$. Then* $\operatorname{dom} f^* = X$, *$f^*$ is differentiable on X, and the supremum in the definition of f^* is always a maximum.*

Proof. Let $y \in X$. Then

$$f^*(y) = \sup_{x \in X} \big(\langle x, y\rangle - f(x)\big) = -\inf_{x\in X} \big(f(x) + \langle x, -y\rangle\big).$$

Now f is strongly convex, hence so is $f + \langle \cdot, -y\rangle$ by Proposition 24.3. Next, Theorem 24.7(ii) shows that the infimum above is *uniquely* attained, and therefore so is the supremum, say, at x_y. By Theorem 19.5, $f^*(y) = \langle x_y, y\rangle - f(x_y)$; hence, $\partial f^*(y) = \{x_y\}$ is a singleton. Now apply Fact 11.9. $\square$

24.3 ▪ Duality between Smoothness and Strong Convexity

Theorem 24.9 *Let $f\colon X \to \mathbb{R}$ be convex and L-smooth where $L > 0$. Then f^* is $(1/L)$-strongly convex.*

Proof. First, f^* is convex, lower semicontinuous, and proper by Theorem 18.15. Let $(u, x), (v, y)$ be in $\operatorname{gra}\partial f^*$. The differentiability assumption, Fact 11.9, and Corollary 19.6 give $\{(x, u), (y, v)\} \subseteq \operatorname{gra}\partial f = \operatorname{gra}\nabla f$. The equivalence (i)⇔(iv) of Fact 23.8 yields $\langle x - y, u - v\rangle \geqslant \frac{1}{L}\|u - v\|^2$. In turn, the equivalence (iii)⇔(i) in Fact 24.4 now yields the $(1/L)$-strong convexity of f^*. $\square$

Theorem 24.10 *Let f be β-strongly convex and lower semicontinuous where $\beta > 0$. Then f^* is $(1/\beta)$-smooth.*

Proof. By Corollary 24.8, f^* is differentiable everywhere. Let u, v be in X and set $x := \nabla f^*(u)$ and $y := \nabla f^*(v)$. Then $u \in \partial f(x)$ and $v \in \partial f(y)$. The β-strong convexity of f coupled with the equivalence (i)⇔(iii) of Fact 24.4 yields $\langle x - y, u - v\rangle \geqslant \beta\|x - y\|^2$. Put differently, $\langle u - v, \nabla f^*(u) - \nabla f^*(v)\rangle \geqslant \beta\|\nabla f^*(u) - \nabla f^*(v)\|^2$. On the other hand, the Cauchy–Schwarz inequality yields $\langle u - v, \nabla f^*(u) - \nabla f^*(v)\rangle \leqslant \|u-v\|\|\nabla f^*(u) - \nabla f^*(v)\|$. Altogether, $\beta\|\nabla f^*(u) - \nabla f^*(v)\|^2 \leqslant \|u - v\|\|\nabla f^*(u) - \nabla f^*(v)\|$, and this implies $\|\nabla f^*(u) - \nabla f^*(v)\| \leqslant (1/\beta)\|u - v\|$. $\square$

Example 24.11 Consider the continuous convex function $f(x) = \sqrt{1 + \|x\|^2}$ on $\mathbb{R}^n$, which has gradient $\nabla f(x) = x/f(x)$ and Hessian

$$0 \preceq \nabla^2 f(x) = \frac{1}{f(x)}\mathrm{Id} - \frac{1}{(f(x))^3}xx^{\mathsf{T}} \preceq \mathrm{Id}, \tag{24.4}$$

and whose Fenchel conjugate is

$$f^*(y) = \begin{cases} -\sqrt{1 - \|y\|^2} & \text{if } \|y\| \leqslant 1;\\ +\infty & \text{if } \|y\| > 1,\end{cases}$$

i.e., the ball pen function. Then f is 1-smooth and f^* is 1-strongly convex.

24.4 ▪ Smoothness of Infimal Convolutions

Fact 24.12 [5, Theorem 5.30] *Let f be convex, lower semicontinuous, and proper on X. Let g be L-smooth where $L > 0$. Suppose that* $\operatorname{ran}(f \,\square\, g) \subseteq \mathbb{R}$. *Then $f \,\square\, g$ is L-smooth. Let $y \in X$. If x minimizes $f + g(y - \cdot)$, i.e., $(f \,\square\, g)(y) = f(x) + g(y - x)$, then $\nabla(f \,\square\, g)(y) = \nabla g(x - y)$.*

Proof. By Corollary 21.2, we have $f \,\square\, g = (f^* + g^*)^*$. Hence $f^* + g^*$ is proper (for otherwise $(f^* + g^*)^* = -\infty$, which is absurd). Note that g^* is $(1/L)$-strongly convex by Theorem 24.9. Hence $f^* + g^*$ also is $(1/L)$-strongly convex by Proposition 24.3. Next, by Theorem 24.10, $(f^* + g^*)^*$ is L-smooth. Altogether, $f \,\square\, g$ is L-smooth. $\square$

Corollary 24.13 *Let f be convex, lower semicontinuous, and proper on X, let $L > 0$, and set $q := \frac{1}{2}\|\cdot\|^2$. Then*

$$f \,\square\, (Lq)$$

is L-smooth. Let $y \in X$. Then there exists a unique vector $x_y \in X$ such that $(f \,\square\, (Lq))(y) = \min_{x \in X}(f(x) + Lq(y - x)) = f(x_y) + Lq(y - x_y)$; moreover,

$$\nabla\big(f \,\square\, (Lq)\big)(y) = L(y - x_y) \tag{24.5}$$

and

$$(\forall x \in X) \quad f(x) + Lq(y - x) \geqslant (f \,\square\, (Lq))(y) + \frac{L}{2}\|x - x_y\|^2. \tag{24.6}$$

Proof. Because $\nabla(Lq) = L\mathrm{Id}$, this follows from Fact 24.12 except for the existence and uniqueness of x_y, which we establish now:

First, Lq is L-strongly convex by Example 24.6. Next, $h\colon x \mapsto f(x) + Lq(y - x) = Lq(x) + (f(x) + L\langle x, -y\rangle + Lq(y))$ is L-strongly convex by Proposition 24.3. By Theorem 24.7(ii), h has a unique minimizer x_y for which

$$(\forall x \in X) \quad h(x) - h(x_y) \geqslant \frac{L}{2}\|x - x_y\|^2,$$

which gives (24.6). $\square$

Example 24.14 Let C be a nonempty closed convex subset of X, and let $y \in X$. Then $\frac{1}{2}d_C^2(y) = (\iota_C \,\square\, \frac{1}{2}\|\cdot\|^2)(y) = \iota_C(\mathrm{P}_C(y)) + \frac{1}{2}\|\mathrm{P}_C(y) - y\|^2$, and so if $f = \iota_C$, then $x_y = \mathrm{P}_C(y)$ in Corollary 24.13, and so $\nabla\frac{1}{2}d_C^2(y) = y - \mathrm{P}_C(y)$, as we saw directly in Example 12.6.

Exercises

Exercise 24.1 Provide the details for Proposition 24.2.

Exercise 24.2 Suppose that each f_i is β_i-strongly convex with respect to the Euclidean norm, and that $f := f_1 + \cdots + f_m$ is proper. Then f is β-strongly convex, where $\beta = \beta_1 + \cdots + \beta_m$.

Exercise 24.3 Provide the details for Example 24.5.

Exercise 24.4 Provide the details for Example 24.6.

Exercise 24.5 Provide the details for Example 24.11.

Exercise 24.6 Suppose that $p \in\,]1, +\infty[\, \smallsetminus \{2\}$ and set $f(x) = \frac{1}{p}|x|^p$, which is a convex function. Show that f is neither strongly convex nor L-smooth for any $L \geq 0$.

Chapter 25

Proximal Mappings

Proximal mappings are generalizations of projection mappings, which we studied earlier in Chapter 8.

Throughout this chapter, $\|\cdot\|$ denotes the Euclidean norm.

25.1 ▪ Characterizations and Examples

Corollary 24.13 makes the following definition well defined.

Definition 25.1 (proximal mapping) Let f be convex, lower semicontinuous, and proper on X. Then the *proximal mapping* (a.k.a. *prox operator*) is

$$\mathrm{P}_f \colon X \to X : y \mapsto \operatorname*{Argmin}_{x \in X} \left(f(x) + \tfrac{1}{2}\|x-y\|^2 \right),$$

where Argmin denotes the set of minimizers. In the literature, one also finds the more common notation Prox_f or prox_f.

Theorem 8.4 yields the following:

Example 25.2 Let C be a nonempty closed convex subset of X. Then

$$\mathrm{P}_C = \mathrm{P}_{\iota_C}.$$

Useful for finding the proximal mapping is the following characterization which generalizes Theorem 8.4:

Theorem 25.3 *Let f be convex, lower semicontinuous, and proper on X. Let $x \in \operatorname{dom} f$ and $y \in X$. Then the following are equivalent:*

(i) $x = \mathrm{P}_f(y)$.

(ii) $y - x \in \partial f(x)$.

(iii) $(\forall z \in X)\ \langle z-x, y-x\rangle \leqslant f(z) - f(x)$.

In particular, $\operatorname{ran} \mathrm{P}_f \subseteq \operatorname{dom} \partial f$ *and*

$$\boxed{\mathrm{P}_f = (\mathrm{Id} + \partial f)^{-1}. \qquad (25.1)}$$

Proof. We have (i) $\Leftrightarrow$ x minimizes $f + \frac{1}{2}\|\cdot - y\|^2 \Leftrightarrow 0 \in \partial(f + \frac{1}{2}\|\cdot - y\|^2)(x) = \partial f(x) + x - y$ $\Leftrightarrow$ (ii). The equivalence (ii)$\Leftrightarrow$(iii) is clear from the definition of the subdifferential. The "in particular" part follows from (ii). $\square$

Corollary 25.4 *Let C be a nonempty closed convex subset of X. Let $x \in C = \operatorname{dom} \iota_C$ and $y \in X$. Because $\mathrm{P}_{\iota_C} = \mathrm{P}_C$ and $\partial \iota_C(x) = N_C(x)$, we learn from Theorem* 25.3 *that $x = \mathrm{P}_C(y)$ $\Leftrightarrow y - x \in N_C(x) \Leftrightarrow (\forall c \in C)\ \langle c - x, y - x\rangle \leqslant 0$. This sheds further light on the Projection Theorem (Theorem* 8.4*).*

Remark 25.5 Consider Theorem 25.3 and its notation. If one has a guess x for what $\mathrm{P}_f(y)$ is, then one can justify that guess by simply checking that $y - x \in \partial f(x)$.

Example 25.6 (convex quadratic) Suppose that $X = \mathbb{R}^n$, let $A \in \mathbb{S}^n$ be positive semidefinite, let $b \in \mathbb{R}^n$, and let $\gamma \in \mathbb{R}$. Set

$$f(x) = \tfrac{1}{2}\langle x, Ax\rangle + \langle x, b\rangle + \gamma.$$

Then

$$\mathrm{P}_f(y) = (\mathrm{Id} + A)^{-1}(y - b),$$

where the inverse is taken in the sense of linear algebra.

Proof. We have $\nabla f(x) = Ax + b$. So $y - x = \nabla f(x) \Leftrightarrow y - x = Ax + b \Leftrightarrow y - b = (\mathrm{Id} + A)x$ and we are done by Theorem 25.3. $\square$

Example 25.7 Suppose that $X = \mathbb{R}$, let $\lambda > 0$, and let $y \in \mathbb{R}$. Then

$$\mathrm{P}_{\lambda|\cdot|}(y) = \operatorname{sign}(y)\max\{|y| - \lambda, 0\} = \begin{cases} y + \lambda & \text{if } y < -\lambda; \\ 0 & \text{if } |y| \leqslant \lambda; \\ y - \lambda & \text{if } y > \lambda. \end{cases} \qquad (25.2)$$

Proof. Set $f := \lambda|\cdot|$, and let $x \in \mathbb{R}$. From Theorem 25.3, Example 9.3, and (13.1), it follows that

$$x = \mathrm{P}_f(y) \Leftrightarrow y - x \in \partial f(x) = \begin{cases} \{-\lambda\} & \text{if } x < 0; \\ [-\lambda, \lambda] & \text{if } x = 0; \\ \{\lambda\} & \text{if } x > 0. \end{cases}$$

We now the employ the strategy outlined in Remark 25.5 for proving (25.2).
Case 1: $y < -\lambda$. If $x = y + \lambda$, then $x < 0$ and so $y - x = -\lambda \in \partial f(x)$.
Case 2: $|y| \leqslant \lambda$. If $x = 0$, then $y - x = y \in \partial f(x)$.
Case 3: $y > \lambda$. If $x = y - \lambda$, then $x > 0$ and so $y - x = \lambda \in \partial f(x)$. $\square$

Example 25.8 Suppose that $X = \mathbb{R}$, let $\alpha > 0$, and set

$$f(x) := \begin{cases} -\alpha \ln(x) & \text{if } x > 0; \\ +\infty & \text{if } x \leqslant 0. \end{cases}$$

Then

$$\mathrm{P}_f(y) = \frac{y + \sqrt{y^2 + 4\alpha}}{2}.$$

25.2 ▪ The Proximal Mapping and Optimization

The importance of the prox operator in optimization becomes clear from the following result:

Theorem 25.9 *Let f be convex, lower semicontinuous, and proper on X, and let $y \in X$. Then*

$$f\big(\mathrm{P}_f(y)\big) \leqslant f(y) - \tfrac{1}{2}\|y - \mathrm{P}_f(y)\|^2 \tag{25.3}$$

and

$$y = \mathrm{P}_f(y) \quad \Leftrightarrow \quad y \text{ is a minimizer of } f. \tag{25.4}$$

Proof. Indeed, $f(\mathrm{P}_f(y)) + \frac{1}{2}\|y - \mathrm{P}_f(y)\|^2 = \min_{x \in X}(f(x) + \frac{1}{2}\|x - y\|^2) \leqslant f(y) + \frac{1}{2}\|y - y\|^2 = f(y)$, which gives (25.3). In view of Theorem 25.3 and Fermat's rule, we have $y = \mathrm{P}_f(y) \Leftrightarrow 0 = y - y \in \partial f(y) \Leftrightarrow y$ is a minimizer of f. □

In fact, more can be said:

Fact 25.10 [3, Proposition 12.33] *Let f be convex, lower semicontinuous, and proper on X, let $0 < \mu < \nu$, and let $y \in X$. Then the following hold:*

(i) $\inf f(X) \leqslant f(\mathrm{P}_{\nu f}(y)) \leqslant f(\mathrm{P}_{\mu f}(y)) \leqslant f(y)$.

(ii) $\lim_{\mu \to +\infty} f(\mathrm{P}_{\mu f}(y)) = \inf f(X)$.

(iii) $\lim_{\mu \to 0^+} f(\mathrm{P}_{\mu f}(y)) = f(y)$.

(iv) *If* $y \in \operatorname{dom} f$, *then* $\lim_{\mu \to 0^+} \frac{1}{\mu}\|y - \mathrm{P}_{\mu f} y\|^2 = 0$.

25.3 ▪ Prox Calculus

Proposition 25.11 (separability) *Let $X_1, \ldots, X_m$ be finite-dimensional Euclidean spaces. Suppose each f_i is convex, lower semicontinuous, and proper on X_i, where $i \in \{1, \ldots, m\}$. Set*

$$f\colon X_1 \times \cdots \times X_m \to \,]-\infty, +\infty] : (x_1, \ldots, x_m) \mapsto f_1(x_1) + \cdots + f_m(x_m).$$

Let $(y_1, \ldots, y_m) \in X_1 \times \cdots \times X_m$. Then

$$\mathrm{P}_f(y_1, \ldots, y_m) = \big(\mathrm{P}_{f_1}(y_1), \ldots, \mathrm{P}_{f_m}(y_m)\big).$$

Proof. We have $(x_1, \ldots, x_m) = \mathrm{P}_f(y_1, \ldots, y_m) \Leftrightarrow (x_1, \ldots, x_m)$ minimizes $(z_1, \ldots, z_m) \mapsto \sum_{i=1}^m (f_i(z_i) + \frac{1}{2}\|z_i - y_i\|^2) \Leftrightarrow$ each x_i minimizes $z_i \mapsto f_i(z_i) + \frac{1}{2}\|z_i - y_i\|^2 \Leftrightarrow$ each $x_i = \mathrm{P}_{f_i}(y_i)$. □

Combining Proposition 25.11 and (25.2) yields the following:

Example 25.12 (soft thresholder) On $X = \mathbb{R}^n$, define the function f by $f(x) := f(x_1, \ldots, x_n) = \lambda\|x\|_1 = \sum_{i=1}^n \lambda|x_i|$, where $\lambda > 0$. Then we obtain the *soft thresholder*

$$\mathrm{P}_f(y_1, \ldots, y_n) = \big(\operatorname{sign}(y_i)\max\{|y_i| - \lambda, 0\}\big)_{i=1}^n.$$

Other calculus rules are somewhat nonintuitive:

Proposition 25.13 *Let g be convex, lower semicontinuous, and proper on X. Suppose $A\colon X \to X$ is linear and $A^* = \alpha A^{-1}$ for some $\alpha > 0$. Let $b \in X$, and set $f(x) := g(Ax + b)$. Then*

$$\mathrm{P}_f(y) = \frac{A^*\big(\mathrm{P}_{\alpha g}(Ay + b) - b\big)}{\alpha}.$$

Proof. Note that $AA^* = A^*A = \alpha\mathrm{Id}$. We have $x = \mathrm{P}_f(y) \Leftrightarrow y \in x + \partial f(x) = x + A^*\partial g(Ax + b) \Leftrightarrow Ay + b \in Ax + b + AA^*\partial g(Ax + b) = (\mathrm{Id} + \partial(\alpha g))(Ax + b) \Leftrightarrow Ax + b = \mathrm{P}_{\alpha g}(Ay + b) \Leftrightarrow \alpha x + A^*b = A^*Ax + A^*b = A^*\mathrm{P}_{\alpha g}(Ay + b)$, which yields the result. □

Proposition 25.14 *Let g be convex, lower semicontinuous, and proper on X. Set $f(x) := g(\alpha x + b)$, where $\alpha \in \mathbb{R} \smallsetminus \{0\}$ and $b \in X$. Then*

$$\mathrm{P}_f(y) = \frac{\mathrm{P}_{\alpha^2 g}(\alpha y + b) - b}{\alpha}.$$

Proof. We have $x = \mathrm{P}_f(y) \Leftrightarrow y \in x + \partial f(x) = x + \alpha\partial g(\alpha x + b) \Leftrightarrow \alpha y + b \in (\alpha x + b) + \alpha^2\partial g(\alpha x + b) = (\mathrm{Id} + \partial(\alpha^2 g))(\alpha x + b) \Leftrightarrow \alpha x + b = \mathrm{P}_{\alpha^2 g}(\alpha y + b)$ and this concludes the proof. □

Example 25.15 Let C be a nonempty closed convex subset of X, and let $u \in X$. Then $\mathrm{P}_{u+C}(y) = u + \mathrm{P}_C(y - u)$.

Proposition 25.16 *Let g be convex, lower semicontinuous, and proper on X. Set $f(x) := \alpha g(x/\alpha)$, where $\alpha > 0$. Then*

$$\mathrm{P}_f(y) = \alpha\mathrm{P}_{\alpha^{-1}g}(\alpha^{-1}y).$$

Proof. We have $x = \mathrm{P}_f(y) \Leftrightarrow y \in x + \partial f(x) = x + \partial g(\alpha^{-1}x) \Leftrightarrow \alpha^{-1}y \in \alpha^{-1}x + \alpha^{-1}\partial g(\alpha^{-1}x) = (\mathrm{Id} + \partial(\alpha^{-1}g))(\alpha^{-1}x) \Leftrightarrow \alpha^{-1}x = \mathrm{P}_{\alpha^{-1}g}(\alpha^{-1}y)$, yielding the result. □

Proposition 25.17 *Let g be convex, lower semicontinuous, and proper on X. Set $f(x) := g(x) + \alpha\frac{1}{2}\|x\|^2 + \langle x, b\rangle + \gamma$, where $\alpha \geqslant 0$, $b \in X$, and $\gamma \in \mathbb{R}$. Then*

$$\mathrm{P}_f(y) = \mathrm{P}_{(1+\alpha)^{-1}g}\big((1 + \alpha)^{-1}(y - b)\big).$$

Proof. We have $x = \mathrm{P}_f(y) \Leftrightarrow y \in x + \partial f(x) = x + \partial g(x) + \alpha x + b = (1 + \alpha)x + b + \partial g(x) \Leftrightarrow (1 + \alpha)^{-1}(y - b) \in (\mathrm{Id} + \partial((1 + \alpha)^{-1}g))(x)$, which yields the result. □

Example 25.18 Suppose that $X = \mathbb{R}$, let $\beta \geqslant 0$, $b \in \mathbb{R}$, and set

$$f(x) := \begin{cases} bx & \text{if } 0 \leqslant x \leqslant \beta; \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\mathrm{P}_f(y) = \min\{\max\{y - b, 0\}, \beta\}$.

Proof. Set $C := [0, \beta]$. Then $\mathrm{P}_{\iota_C}(y) = \mathrm{P}_C(y) = \min\{\max\{y, 0\}, \beta\}$ by (8.4). Note that $f(x) = \iota_C(x) + bx$. Using Proposition 25.17, we obtain $\mathrm{P}_f(y) = \mathrm{P}_C(y - b)$, which gives the result. □

Fact 25.19 *Let C be a nonempty closed convex subset of X, and let $\phi\colon \mathbb{R} \to \mathbb{R}$ be convex, even, and differentiable on $\mathbb{R} \smallsetminus \{0\}$. Set $f := \phi \circ d_C$. Then*

$$\mathrm{P}_f(y) = \begin{cases} \mathrm{P}_C(y) & \text{if } d_C(y) \leqslant \phi'_+(0); \\ y + \dfrac{\mathrm{P}_{\phi^*}(d_C(y))}{d_C(y)}(\mathrm{P}_C(y) - y) & \text{if } d_C(y) > \phi'_+(0). \end{cases}$$

Proof. This follows by combining [3, Proposition 24.27] with (12.1). □

For more proximal calculus rules, see [3] and [5]. Unfortunately, no general rules are available for P_{f+g}, $\mathrm{P}_{\alpha f}$, and $\mathrm{P}_{f\circ A}$.

Exercises

Exercise 25.1 Explain why the matrix $\mathrm{Id} + A$ in Example 25.6 is invertible in the sense of linear algebra.

Exercise 25.2 Provide the details for Example 25.8.

Exercise 25.3 Suppose that $X = \mathbb{R}^n$, let $\alpha > 0$, and set

$$f(x) := \begin{cases} -\alpha \sum_{i=1}^{n} \ln(x_i) & \text{if } x \in \mathbb{R}^n_{++}; \\ +\infty & \text{otherwise.} \end{cases}$$

Determine P_f.

Exercise 25.4 Use Proposition 25.14 to show Example 25.15.

Exercise 25.5 Suppose that $X = \mathbb{R}$, let $\beta \geqslant 0$, $b \in \mathbb{R}$, and set

$$f(x) := \begin{cases} bx & \text{if } |x| \leqslant \beta; \\ +\infty & \text{otherwise.} \end{cases}$$

Show that $\mathrm{P}_f(y) = \min\{\max\{y - b, -\beta\}, \beta\}$.

Exercise 25.6 Let $g\colon \mathbb{R} \to \,]-\infty, +\infty]$ be convex, lower semicontinuous, and proper such that $\operatorname{dom} g \subseteq \mathbb{R}_+$, and set $f(x) := g(\|x\|)$. Show that

$$\mathrm{P}_f(y) = \begin{cases} \mathrm{P}_g(\|y\|)y/\|y\| & \text{if } y \neq 0; \\ \{u \in X \mid \|u\| = \|\mathrm{P}_g(0)\|\} & \text{if } y = 0. \end{cases}$$

Exercise 25.7 Suppose that $X = \mathbb{R}$, let $\beta \geqslant 0$, $b \geqslant 0$, and set

$$f(x) := \begin{cases} b|x| & \text{if } |x| \leqslant \beta; \\ +\infty & \text{otherwise.} \end{cases}$$

Show that $\mathrm{P}_f(y) = \operatorname{sign}(y) \min\{\max\{y - b, 0\}, \beta\}$.

Exercise 25.8 Suppose that $X = \mathbb{R}$, let $\lambda > 0$, and set

$$f(x) := \begin{cases} 0 & \text{if } x \neq 0; \\ -\lambda & \text{if } x = 0. \end{cases}$$

Determine P_f.

Exercise 25.9 Suppose that $X = \mathbb{R}$, let $\lambda > 0$, and set

$$f(x) := \begin{cases} 0 & \text{if } x \neq 0; \\ \lambda & \text{if } x = 0. \end{cases}$$

Determine P_f.

Exercise 25.10 Suppose that C is a nonempty closed convex subset of X, and let $\lambda > 0$. Use Fact 25.19 to show that

$$\mathrm{P}_{\lambda d_C}(y) = \begin{cases} \mathrm{P}_C(y) & \text{if } d_C(y) \leqslant \lambda; \\ y + \lambda \dfrac{\mathrm{P}_C(y) - y}{d_C(y)} & \text{if } d_C(y) > \lambda. \end{cases}$$

Exercise 25.11 Let C be a nonempty closed convex subset of X, let $\lambda > 0$, and let $\mu > 0$. Set $f := \lambda d_C + \mu\frac{1}{2}\|\cdot\|^2$. Use Exercise 25.10 and Proposition 25.17 to show that

$$\mathrm{P}_f(y) = \begin{cases} \mathrm{P}_C\Big(\dfrac{y}{1+\mu}\Big) & \text{if } d_C\Big(\dfrac{y}{1+\mu}\Big) \leqslant \dfrac{\lambda}{1+\mu}; \\ \dfrac{y}{1+\mu} + \dfrac{\lambda}{1+\mu}\dfrac{\mathrm{P}_C\Big(\dfrac{y}{1+\mu}\Big) - \dfrac{y}{1+\mu}}{d_C\Big(\dfrac{y}{1+\mu}\Big)} & \text{if } d_C\Big(\dfrac{y}{1+\mu}\Big) > \dfrac{\lambda}{1+\mu}. \end{cases}$$

Exercise 25.12 Let $c \in X$ and $\lambda > 0$. Set $f(x) := \lambda\|x - c\| + \frac{1}{2}\|x\|^2$. Use Exercise 25.11 to show that

$$\mathrm{P}_f(y) = \begin{cases} c & \text{if } \|y - 2c\| \leqslant \lambda; \\ \dfrac{y}{2} + \dfrac{\lambda}{2}\dfrac{2c - y}{\|y - 2c\|} & \text{if } \|y - 2c\| > \lambda. \end{cases}$$

Chapter 26

Prox Decomposition

Throughout this chapter, $\|\cdot\|$ again denotes the Euclidean norm.

26.1 ▪ Decomposition and the Proximal Point Algorithm

Theorem 26.1 (prox decomposition) *Let f be convex, lower semicontinuous, and proper on X. Then both P_f and $\mathrm{Id} - \mathrm{P}_f$ are firmly nonexpansive, and*

$$\boxed{\mathrm{P}_f + \mathrm{P}_{f^*} = \mathrm{Id}. \qquad (26.1)}$$

Proof. Let x, y be in X, and set $p := \mathrm{P}_f(x), q := \mathrm{P}_f(y)$. By Theorem 25.3, $x - p \in \partial f(p)$ and $y - q \in \partial f(q)$. On the other hand, ∂f is monotone (see Proposition 9.8). Altogether, $\langle p - q, (x - p) - (y - q)\rangle \geqslant 0$. Thus $\|p - q\|^2 \leqslant \langle x - y, p - q\rangle$. By Theorem 22.3, both P_f and $\mathrm{Id} - \mathrm{P}_f$ are firmly nonexpansive.

Using Theorem 25.3 and Corollary 19.6, we have

$$
\begin{aligned}
& p = \mathrm{P}_f(x) && (26.2\mathrm{a})\\
\Leftrightarrow\ & x - p \in \partial f(p) && (26.2\mathrm{b})\\
\Leftrightarrow\ & p \in (\partial f)^{-1}(x - p) && (26.2\mathrm{c})\\
\Leftrightarrow\ & p \in \partial f^*(x - p) && (26.2\mathrm{d})\\
\Leftrightarrow\ & x - (x - p) \in \partial f^*(x - p) && (26.2\mathrm{e})\\
\Leftrightarrow\ & x - p = \mathrm{P}_{f^*}(x). && (26.2\mathrm{f})
\end{aligned}
$$

Adding (26.2a) and (26.2f) yields $x = \mathrm{P}_f(x) + \mathrm{P}_{f^*}(x)$, i.e., (26.1). ☐

We will study algorithms in greater detail later; however, we can quickly derive convergence of the famous Proximal Point Algorithm:

Corollary 26.2 (Proximal Point Algorithm) *Let f be convex, lower semicontinuous, and proper on X. Suppose that the set of minimizers of f is nonempty, and let $x_0 \in X$. Then the sequence generated by*

$$(\forall k \in \mathbb{N}) \quad x_{k+1} := \mathrm{P}_f(x_k)$$

converges to a minimizer of f.

Proof. Denote by C the set of minimizers of f. By (25.4), $C = \operatorname{Fix} \mathrm{P}_f$. By Theorem 26.1, P_f is firmly nonexpansive, hence $\frac{1}{2}$-averaged. Therefore, by Corollary 22.20, $(x_k)_{k\in\mathbb{N}}$ converges to some point in C. ☐

Less memorable but more general is the following refinement of Theorem 26.1.

Corollary 26.3 *Let g be convex, lower semicontinuous, and proper on X, and let $\alpha > 0$. Then*

$$\mathrm{Id} = \mathrm{P}_{\alpha g} + \alpha \mathrm{P}_{(1/\alpha)g^*}\left(\tfrac{\cdot}{\alpha}\right).$$

Proof. Set $f := \alpha g$. Note that $f^* = \alpha g^*(\cdot/\alpha)$ by Proposition 19.11. By Proposition 25.16, $\mathrm{P}_{f^*} = \alpha \mathrm{P}_{(1/\alpha)g^*}(\cdot/\alpha)$. The result now follows from Theorem 26.1. ☐

26.2 ▪ Examples

Corollary 26.3 has very nice consequences.

Example 26.4 Let C be a nonempty closed convex subset of X, and let $\alpha > 0$. Then

$$\mathrm{P}_{\alpha\sigma_C}(y) = y - \alpha \mathrm{P}_C(y/\alpha).$$

Example 26.5 Let $\|\cdot\|$ be any norm on X, with dual norm $\|\cdot\|_*$, let C be the unit ball with respect to $\|\cdot\|$, and let $\alpha > 0$. Because $\sigma_C = \|\cdot\|_*$ (see Example 7.3), it follows from Example 26.4 that

$$\mathrm{P}_{\alpha\|\cdot\|_*}(y) = y - \alpha \mathrm{P}_C(y/\alpha).$$

Example 26.6 Let $\|\cdot\|$ denote the Euclidean norm, and let $\alpha > 0$. Then

$$\mathrm{P}_{\alpha\|\cdot\|}(y) = \left(1 - \frac{\alpha}{\max\{\|y\|, \alpha\}}\right) y.$$

Example 26.7 (conical decomposition) Let K be a nonempty closed convex cone in X. Then specializing Example 26.4 (with $\alpha = 1$) yields

$$\mathrm{Id} = \mathrm{P}_K + \mathrm{P}_{K^\ominus} \tag{26.3}$$

because $\sigma_K = \iota_{K^\ominus}$ by (7.2). If $K = Y$, where Y is a linear subspace of X, then $Y^\ominus = Y^\perp$ (see Example 7.11), and thus we obtain the Pythagorean decomposition

$$\mathrm{Id} = \mathrm{P}_Y + \mathrm{P}_{Y^\perp}. \tag{26.4}$$

Example 26.8 We have $\mathrm{Id} = \mathrm{P}_{\mathbb{R}^n_+} + \mathrm{P}_{\mathbb{R}^n_-}$.

Exercises

Exercise 26.1 Provide the details for Example 26.4.

Exercise 26.2 Provide the details for Example 26.6.

Exercise 26.3 Specialize Example 26.5 when the norm considered is the max-norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$.

Exercise 26.4 Provide the details for Example 26.8.

Exercise 26.5 Discuss and compare the behavior of the Proximal Point Algorithm when (i) $f(x) = \iota_C$ and (ii) $f(x) = d_C$, where C is the closed unit ball with respect to the Euclidean norm.

Chapter 27

Envelopes

Throughout this chapter, $\|\cdot\|$ again denotes the Euclidean norm.

27.1 ▪ Basic Properties and Examples

Using Proposition 6.9 and Fact 6.13, we see that the following definition is well defined.

Definition 27.1 (envelope) Let f be convex, lower semicontinuous, and proper on X. Then the *(Moreau) envelope, with smoothing parameter* $\mu > 0$, is the full-domain continuous convex function $\mathrm{env}_\mu f$ defined by

$$\mathrm{env}_\mu f(y) := \min_{x \in X} \big(f(x) + \tfrac{1}{2\mu}\|x - y\|^2\big) = \big(f \,\square\, \tfrac{1}{\mu}\tfrac{1}{2}\|\cdot\|^2\big)(y).$$

If $\mu = 1$, we will also simply write $\mathrm{env} f$ instead of $\mathrm{env}_1 f$.

Theorem 27.2 *Let f be convex, lower semicontinuous, and proper on X, and let $\mu > 0$. Then the following hold:*

(i) *For every $y \in X$, we have*

$$\mathrm{env}_\mu f(y) = f\big(\mathrm{P}_{\mu f}(y)\big) + \frac{1}{2\mu}\big\|y - \mathrm{P}_{\mu f}(y)\big\|^2. \tag{27.1}$$

(ii) *The envelope $\mathrm{env}_\mu f$ is $(1/\mu)$-smooth and its gradient*

$$\nabla \,\mathrm{env}_\mu f(y) = \frac{y - \mathrm{P}_{\mu f}(y)}{\mu} \tag{27.2}$$

is therefore $(1/\mu)$-Lipschitz.

Proof. (i): This is clear from the definition of the envelope and the proximal mapping. (ii): Corollary 24.13. ☐

We now turn to approximation properties of the envelope.

Fact 27.3 [3, Chapter 12] *Let f be convex, lower semicontinuous, and proper on X, let $0 < \mu < \nu$, and let $y \in X$. Then the following hold:*

(i) $\inf f(X) \leqslant \operatorname{env}_\nu f(y) \leqslant \operatorname{env}_\mu f(y) \leqslant f(y)$.

(ii) $\inf f(X) = \inf \operatorname{env}_\mu f(X)$.

(iii) $\lim_{\mu\to+\infty} \operatorname{env}_\mu f(y) = \inf f(X)$.

(iv) $\lim_{\mu\to 0+} \operatorname{env}_\mu f(y) = f(y)$.

Example 27.4 Let C be a nonempty closed convex subset of X, and let $\mu > 0$. Then $\mu\iota_C = \iota_C$ and so $\mathrm{P}_{\mu\iota_C} = \mathrm{P}_{\iota_C} = \mathrm{P}_C$. By (27.1),

$$\operatorname{env}_\mu \iota_C(y) = \iota_C(\mathrm{P}_C(y)) + \frac{1}{2\mu}\|y - \mathrm{P}_C(y)\|^2 = \frac{1}{2\mu}d_C^2(y),$$

and by (27.2),

$$\nabla \operatorname{env}_\mu \iota_C(y) = \frac{y - \mathrm{P}_C(y)}{\mu}$$

is $(1/\mu)$-Lipschitz, which we also can deduce directly from Example 22.4.

Example 27.5 (Huber loss) Let $\mu > 0$. We saw earlier (Example 26.6) that

$$\mathrm{P}_{\mu\|\cdot\|}(y) = \Big(1 - \frac{\mu}{\max\{\|y\|, \mu\}}\Big)y,$$

where we reiterate that $\|\cdot\|$ denotes the Euclidean norm. Then

$$\begin{aligned}
\operatorname{env}_\mu\|\cdot\|(y) &= \|\mathrm{P}_{\mu f}(y)\| + \frac{1}{2\mu}\|x - \mathrm{P}_{\mu f}(y)\|^2\\
&= \Big\|\Big(1 - \frac{\mu}{\max\{\|y\|, \mu\}}\Big)y\Big\| + \frac{1}{2\mu}\Big\|y - \Big(1 - \frac{\mu}{\max\{\|y\|, \mu\}}\Big)y\Big\|^2\\
&= \begin{cases} \dfrac{1}{2\mu}\|y\|^2 & \text{if } \|y\| \leqslant \mu;\\ \|y\| - \dfrac{\mu}{2} & \text{if } \|y\| > \mu. \end{cases}
\end{aligned}$$

The function $\mu \operatorname{env}_\mu\|\cdot\|$ is known as the *Huber loss function.*

27.2 ▪ Envelope Calculus

Proposition 27.6 *Let f be convex, lower semicontinuous, and proper on X, let $\mu > 0$, and let $\alpha > 0$. Then the following hold:*

(i) $(\operatorname{env}_\mu f)^* = f^* + \mu\frac{1}{2}\|\cdot\|^2$.

(ii) $\alpha \operatorname{env}_\mu f = \operatorname{env}_{\alpha^{-1}\mu}(\alpha f)$.

Proof. (i): Indeed, using Proposition 21.1 and Exercise 18.6, we have $(\operatorname{env}_\mu f)^* = (f \,\square\, (\frac{1}{\mu}\frac{1}{2}\|\cdot\|^2))^* = f^* + (\frac{1}{\mu}\frac{1}{2}\|\cdot\|^2)^* = f^* + \mu\frac{1}{2}\|\cdot\|^2$. (ii): Indeed, $(\alpha \operatorname{env}_\mu f)(y) = \alpha \min_{x\in X}(f(x) + \frac{1}{2\mu}\|x - y\|^2) = \min_{x\in X}((\alpha f)(x) + \frac{1}{2\alpha^{-1}\mu}\|x - y\|^2) = (\operatorname{env}_{\alpha^{-1}\mu}(\alpha f))(y)$. $\square$

As always, separability yields a pleasant result:

Proposition 27.7 *Let each f_i be convex, lower semicontinuous, and proper on X_i, where $i \in I := \{1, \ldots, m\}$, and let $\mu > 0$. Set $f(x_1, \ldots, x_m) := \sum_{i \in I} f_i(x_i)$. Then*

$$\mathrm{env}_\mu f(y_1, \ldots, y_m) = \sum_{i \in I} \mathrm{env}_\mu f_i(y_i).$$

Example 27.8 Let $f(x) = \|x\|_1$ be the 1-norm on $X = \mathbb{R}^n$, and let $\mu > 0$. Combining Proposition 27.7 with Example 27.5, we obtain

$$\mathrm{env}_\mu f(y_1, \ldots, y_n) = \sum_{i=1}^{n} H_\mu(y_i),$$

where

$$H_\mu(\eta) := \begin{cases} \dfrac{1}{2\mu}\eta^2 & \text{if } |\eta| \leqslant \mu; \\ |\eta| - \dfrac{\mu}{2} & \text{if } |\eta| > \mu. \end{cases}$$

Theorem 27.9 *Let f be convex, lower semicontinuous, and proper on X, let $\mu > 0$, and let $\alpha > 0$. Then*

$$\mathrm{P}_{\alpha\, \mathrm{env}_\mu f} = \frac{\mu}{\mu + \alpha}\mathrm{Id} + \frac{\alpha}{\mu + \alpha}\mathrm{P}_{(\mu+\alpha) f}. \tag{27.3}$$

Proof. Let $y \in X$. Assume first that

$$\alpha = 1.$$

Set $g := \mathrm{env}_\mu f$, $q := \mathrm{P}_{(\mu+1)f}(y)$ and $x := \frac{\mu}{\mu+1}y + \frac{1}{\mu+1}q$. Then

$$(\mu + 1)x - \mu y = q \tag{27.4}$$

and $x - q = \frac{\mu}{\mu+1}y + \frac{1}{\mu+1}q - \frac{\mu+1}{\mu+1}q = \frac{\mu}{\mu+1}(y - q)$ and so

$$\frac{x - q}{\mu} = \frac{y - q}{\mu + 1}. \tag{27.5}$$

By the characterization of the prox operator (Theorem 25.3), we have

$$\begin{aligned} q = \mathrm{P}_{(\mu+1)f}(y) &\Leftrightarrow y - q \in \partial\big((\mu + 1)f\big)(q) \\ &\Leftrightarrow \frac{y - q}{\mu + 1} \in \partial f(q) \\ &\Leftrightarrow \frac{x - q}{\mu} \in \partial f(q) && \text{(using (27.5))} \\ &\Leftrightarrow x \in q + \partial(\mu f)(q) \\ &\Leftrightarrow q = \mathrm{P}_{\mu f}(x) \\ &\Leftrightarrow \mathrm{P}_{\mu f}(x) = (\mu + 1)x - \mu y && \text{(using (27.4))} \\ &\Leftrightarrow \mu y - \mu x = x - \mathrm{P}_{\mu f}(x) \\ &\Leftrightarrow y - x = \frac{x - \mathrm{P}_{\mu f}(x)}{\mu} = \nabla g(x) && \text{(using (27.2))} \\ &\Leftrightarrow x = \mathrm{P}_g(y). \end{aligned}$$

This verifies (27.3) when $\alpha = 1$.

Now assume the general case, i.e.,

$$\alpha > 0.$$

Using Proposition 27.6(ii) and the formula just established when $\alpha = 1$, we have

$$\begin{aligned}
\mathrm{P}_{\alpha\,\mathrm{env}_\mu f} &= \mathrm{P}_{1\cdot \mathrm{env}_{\alpha^{-1}\mu}(\alpha f)} = \frac{\alpha^{-1}\mu}{\alpha^{-1}\mu+1}\mathrm{Id} + \frac{1}{\alpha^{-1}\mu+1}\mathrm{P}_{(\alpha^{-1}\mu+1)(\alpha f)} \\
&= \frac{\mu}{\mu+\alpha}\mathrm{Id} + \frac{\alpha}{\mu+\alpha}\mathrm{P}_{(\mu+\alpha)f}
\end{aligned}$$

and we are done! ☐

Example 27.10 Let C be a nonempty closed convex subset of X, let $\mu > 0$, and let $\alpha > 0$. Then $(\mu+\alpha)\iota_C = \iota_C$, $\mathrm{P}_{(\mu+\alpha)\iota_C} = \mathrm{P}_C$, and $\mathrm{env}_\mu \iota_C = \frac{1}{2\mu} d_C^2$. Consequently, (27.3) yields

$$\mathrm{P}_{\alpha \frac{1}{2\mu} d_C^2} = \frac{\mu}{\mu+\alpha}\mathrm{Id} + \frac{\alpha}{\mu+\alpha}\mathrm{P}_C.$$

Let us summarize various prox operators associated with a convex set:

Theorem 27.11 *Let C be a nonempty closed convex subset of X, let $\alpha > 0$, and let $y \in X$. Then the following hold:*

(i) $\mathrm{P}_{\alpha\iota_C}(y) = \mathrm{P}_C(y)$.

(ii) $\mathrm{P}_{\alpha d_C}(y) = \begin{cases} \mathrm{P}_C(y) & \textit{if } d_C(y) \leqslant \alpha; \\ \dfrac{d_C(y)-\alpha}{d_C(y)} y + \dfrac{\alpha}{d_C(y)}\mathrm{P}_C(y) & \textit{if } d_C(y) > \alpha. \end{cases}$

(iii) $\mathrm{P}_{\alpha \frac{1}{2} d_C^2}(y) = \dfrac{1}{1+\alpha} y + \dfrac{\alpha}{1+\alpha}\mathrm{P}_C(y).$

Proof. (i): Clear because $\alpha\iota_C = \iota_C$ and $\mathrm{P}_{\iota_C} = \mathrm{P}_C$.

(ii): Set $\phi := \alpha|\cdot|$, which is an even convex function, differentiable on $\mathbb{R}\smallsetminus\{0\}$, with $\phi'_+(0) = \alpha$ and $\phi^* = \iota_{[-\alpha,\alpha]}$ (see Example 18.4). Note that $\mathrm{P}_{\phi^*} = \mathrm{P}_{[-\alpha,\alpha]}$. Because $\alpha d_C = \phi \circ d_C$, the result follows from Fact 25.19.

(iii): This follows from Example 27.10 with $\mu = 1$. ☐

We conclude by stating a result that sharpens the Moreau decomposition (Theorem 26.1):

Fact 27.12 [3, Theorem 14.3 and Remark 14.4] *Let f be convex, lower semicontinuous, and proper on X, and let $\mu > 0$. Then*

$$\mathrm{env} f + \mathrm{env} f^* = \tfrac{1}{2}\|\cdot\|^2; \tag{27.6}$$

more generally,

$$\mathrm{env}_\mu f(y) + \mathrm{env}_{\mu^{-1}} f^*(y/\mu) = \frac{1}{2\mu}\|y\|^2.$$

Remark 27.13 We know that $\mathrm{P}_f + \mathrm{P}_{f^*} = \mathrm{Id}$ ((26.1)). On the other hand, $\nabla\,\mathrm{env} f = \mathrm{Id} - \mathrm{P}_f = \mathrm{P}_{f^*}$ and $\nabla\,\mathrm{env} f^* = \mathrm{P}_f$ ((27.2)). Thus, (27.6) may be viewed as the integrated version of (26.1).

Exercises

Exercise 27.1 Compute $\operatorname{env}_\mu f$ when f is defined as in Example 25.6.

Exercise 27.2 Provide the details for Fact 27.3(i)&(ii).

Exercise 27.3 Provide the details for Fact 27.3(iii).

Exercise 27.4 Suppose that $f = \frac{1}{2}\|\cdot\|^2$. What does (27.6) turn into in this case?

Exercise 27.5 Suppose that f, g are convex, lower semicontinuous, and proper on X. Show that $f = g \Leftrightarrow \operatorname{env} f = \operatorname{env} g$.

Chapter 28

Subgradient Methods

In this chapter, we discuss algorithms for minimizing a function using gradient or subgradient information.

28.1 ▪ Descent Direction and Classical Steepest Descent

Definition 28.1 (descent direction) Let $f\colon X \to \left]-\infty,+\infty\right]$ be proper, let $x \in \operatorname{dom} f$, and let $d \in X \smallsetminus \{0\}$. Then d is a *descent direction* of f at x if the directional derivative of f at x in direction d not only exists but is also negative:

$$f'(x;d) = \lim_{t\to 0^+} \frac{f(x+td)-f(x)}{t} < 0.$$

Remark 28.2 Consider Definition 28.1.

(i) If d is a descent direction of f at x, then there exists $\delta > 0$ such that if $0 < t < \delta$, then $f(x+td) < f(x)$. This is why d is called a descent direction — the function value has decreased!

(ii) If f is differentiable at x and $\nabla f(x) \neq 0$, then $-\nabla f(x)$ is a descent direction because

$$f'\big(x; -\nabla f(x)\big) = \langle -\nabla f(x), \nabla f(x)\rangle = -\|\nabla f(x)\|^2 < 0.$$

In fact, we obtained δ from (i) in the smooth case explicitly in Remark 23.7.

(iii) In contrast to (ii), a negative subgradient need not be a descent direction; see Exercise 28.1.

The literature on using descent directions for finding minimizers is vast. We record a prototypical result.

Fact 28.3 (classical steepest descent) [37, Corollary 3.2.7] *Suppose that $f\colon X \to \mathbb{R}$ is strictly convex, coercive, and differentiable. Let $x_0 \in X$ and update*

$$(\forall k \in \mathbb{N}) \quad x_{k+1} := x_k - t_k \nabla f(x_k),$$

where t_k minimizes $\mathbb{R}_+ \to \mathbb{R}\colon t \mapsto f(x_k - t\nabla f(x_k))$. Then $(x_k)_{k\in\mathbb{N}}$ converges to the unique minimizer of f.

Somewhat shockingly, if one drops the differentiability assumption, then Fact 28.3 may fail — this is a classical example due to Wolfe, which is detailed in [5, Section 8.1.2].

In the next section, we turn to the more challenging case where a constraint set is present and the function is not necessarily differentiable.

28.2 ▪ Projected Subgradient Method

In this section, we assume that

$$f \text{ is convex, lower semicontinuous, and proper on } X, \tag{28.1a}$$

$$C \text{ is a nonempty closed convex subset of } \operatorname{dom} \partial f, \tag{28.1b}$$

$$\sup \|\partial f(C)\| \leqslant L < +\infty. \tag{28.1c}$$

We also assume and set

$$S := \operatorname*{Argmin}_{x \in C} f(x) \neq \varnothing, \tag{28.1d}$$

$$\mu := \min_{x \in C} f(x). \tag{28.1e}$$

Our aim is to find a point in S directly or approximately. To this end, let $x \in C$ and evaluate $f(x)$. If $f(x) = \mu$, then $x \in S$ and we are done. So we assume that $f(x) > \mu$. We now take

$$f'(x) \in \partial f(x), \tag{28.2}$$

which is possible because of (28.1b). Note that by Fermat's rule we have that

$$f'(x) \neq 0. \tag{28.3}$$

Let us consider, for $t \geqslant 0$, the update

$$x_+ := \mathrm{P}_C\big(x - tf'(x)\big). \tag{28.4}$$

Let $s \in S$. Then $s \in C$; so, $\mathrm{P}_C(s) = s$ and

$$\langle s - x, f'(x) \rangle \leqslant f(s) - f(x) = \mu - f(x). \tag{28.5}$$

We now estimate

$$\begin{aligned}
\|x_+ - s\|^2 &= \|\mathrm{P}_C(x - tf'(x)) - \mathrm{P}_C(s)\|^2 && (28.6a)\\
&\leqslant \|(x - tf'(x)) - s\|^2 && (28.6b)\\
&= \|(x - s) - tf'(x)\|^2 && (28.6c)\\
&= \|x - s\|^2 - 2t\,\langle x - s, f'(x)\rangle + t^2\|f'(x)\|^2 && (28.6d)\\
&\leqslant \|x - s\|^2 - 2t(f(x) - \mu) + t^2\|f'(x)\|^2, && (28.6e)
\end{aligned}$$

where (28.6b) follows from the (firm) nonexpansiveness of P_C (see Example 22.4), while (28.6e) follows from (28.5).

Ideally, we want to have the update x_+ as close as possible to s. While it is not clear what the best choice for t is to achieve this, we can and do instead minimize the quadratic expression in (28.6e) over $t \geqslant 0$. The derivative of this expression with respect to t is $-2(f(x) - \mu) + 2t\|f'(x)\|^2$, which upon setting this derivative equal to 0 yields

$$t = \frac{f(x) - \mu}{\|f'(x)\|^2}. \tag{28.7}$$

The choice for t presented in (28.7) is known as *Polyak's rule*; however, it does require a priori knowledge of μ for its implementation. Taking this choice of t from (28.7) and plugging it into (28.6) yields the very powerful estimate

$$\|x_+ - s\|^2 \leqslant \|x - s\|^2 - \frac{(f(x) - \mu)^2}{\|f'(x)\|^2}. \tag{28.8}$$

Having carefully analyzed the update from x to x_+, we now record two convergence results. The first one assumes that we know μ so we can work with Polyak's rule, while the second one is applicable even when μ is not known.

Theorem 28.4 (Polyak's projected subgradient method) *Suppose that* (28.1) *holds, and let* $x_0 \in C$. *Given* $k \in \mathbb{N}$ *and* $x_k \in C$, *evaluate* $f(x_k)$. *If* $f(x_k) = \mu$, *then* $x_k \in S$ *and we are done. Otherwise, pick* $f'(x_k) \in \partial f(x_k)$ *and generate the next iterate via*

$$x_{k+1} := \mathrm{P}_C\big(x_k - t_k f'(x_k)\big), \quad \textit{where } t_k := \frac{f(x_k) - \mu}{\|f'(x_k)\|^2}. \tag{28.9}$$

If we haven't terminated after finitely many steps, then we also set $\mu_k := \min\{f(x_0), \ldots, f(x_k)\}$ *and the following hold:*

(i) **(Fejér monotonicity)** $(\forall s \in S)(\forall k \in \mathbb{N})\ \|x_{k+1} - s\| \leqslant \|x_k - s\|$.

(ii) **(convergence of the function values)** $f(x_k) \to \mu$.

(iii) $\mu_k - \mu \leqslant \frac{L d_S(x_0)}{\sqrt{k+1}}$.

(iv) *If* $\varepsilon > 0$ *and* $k \geqslant \frac{L^2 d_S^2(x_0)}{\epsilon^2} - 1$, *then* $\mu_k \leqslant \mu + \varepsilon$.

(v) **(convergence of the iterates)** $x_k \to \bar{x} \in S$.

Proof. Observe that (28.8) yields

$$\|x_{k+1} - s\|^2 \leqslant \|x_k - s\|^2 - \frac{(f(x_k) - \mu)^2}{\|f'(x_k)\|^2}, \tag{28.10}$$

which yields not only (i) but also

$$\begin{aligned}
\frac{(k+1)(\mu_k - \mu)^2}{L^2} &\leqslant \frac{1}{L^2} \sum_{m=0}^{k} (f(x_m) - \mu)^2 && \text{(28.11a)}\\
&\leqslant \sum_{m=0}^{k} \frac{(f(x_m) - \mu)^2}{\|f'(x_m)\|^2} && \text{(28.11b)}\\
&\leqslant \sum_{m=0}^{k} \big(\|x_m - s\|^2 - \|x_{m+1} - s\|^2\big) && \text{(28.11c)}\\
&= \|x_0 - s\|^2 - \|x_{k+1} - s\|^2 && \text{(28.11d)}\\
&\leqslant \|x_0 - s\|^2. && \text{(28.11e)}
\end{aligned}$$

It follows that $\sum_{m=0}^{\infty}(f(x_m) - \mu)^2 < +\infty$, and thus $f(x_m) - \mu \to 0$ and so (ii) holds. Moreover, (28.11) with $s = \mathrm{P}_S(x_0)$ also yields $(k+1)(\mu_k - \mu)^2/L^2 \leqslant d_S^2(x_0)$, which rearranges to (iii). Note that (iv) is a consequence of (iii).

We finally turn to (v). It is clear from (i) that $(x_k)_{k\in\mathbb{N}}$ is bounded. Let $\bar{x}$ be a cluster point of $(x_k)_{k\in\mathbb{N}}$, say $x_{n_k} \to \bar{x}$. Because $(x_k)_{k\in\mathbb{N}}$ lies in C, which is a closed set, we deduce that $\bar{x} \in C$.

From the lower semicontinuity of f and (ii), we obtain

$$\mu = \min f(C) \leqslant f(\bar{x}) \leqslant \varliminf_{k\in\mathbb{N}} f(x_{n_k}) = \mu.$$

Therefore, $f(\bar{x}) = \mu$ and so $\bar{x} \in S$. Finally, combining this with (i) and Theorem 22.19, we deduce that $x_k \to \bar{x}$. ☐

Theorem 28.5 *Suppose that* (28.1) *holds, and pick a sequence* $(t_k)_{k\in\mathbb{N}}$ *in* $\mathbb{R}_{++}$ *such that*

$$\sum_{k\in\mathbb{N}} t_k^2 < +\infty \quad \textit{and} \quad \sum_{k\in\mathbb{N}} t_k = +\infty. \tag{28.12}$$

Let $x_0 \in C$. *Given* $k \in \mathbb{N}$ *and* $x_k \in C$, *pick* $f'(x_k) \in \partial f(x_k)$. *If* $f'(x_k) = 0$, *then* $x_k \in S$ *and we are done. Otherwise, generate the next iterate via*

$$x_{k+1} := \mathrm{P}_C\big(x_k - t_k f'(x_k)\big). \tag{28.13}$$

If we haven't terminated after finitely many steps, then

$$\mu_k := \min\{f(x_0), \ldots, f(x_k)\} \to \mu.$$

Proof. Set $s = \mathrm{P}_S(x_0)$. Then $\|x_0 - s\|^2 = d_S^2(x_0)$. It follows from (28.6) that for every $m \in \{0, 1, \ldots, k\}$ we have

$$\begin{aligned} 2t_m(\mu_k - \mu) &\leqslant 2t_m(f(x_m) - \mu) \\ &\leqslant \|x_m - s\|^2 - \|x_{m+1} - s\|^2 + t_m^2\|f'(x_m)\|^2 \\ &\leqslant \|x_m - s\|^2 - \|x_{m+1} - s\|^2 + L^2 t_m^2. \end{aligned}$$

Summing and telescoping yields the estimate $2(\mu_k - \mu)\sum_{m=0}^k t_m \leqslant \|x_0 - s\|^2 + L^2\sum_{m=0}^k t_m^2$. Therefore,

$$\mu_k - \mu \leqslant \frac{\|x_0 - s\|^2}{2\sum_{m=0}^k t_m} + L^2\frac{\sum_{m=0}^k t_m^2}{2\sum_{m=0}^k t_m}, \tag{28.14}$$

and the conclusion now follows from (28.12). ☐

We note that a standard choice for the sequence $(t_k)_{k\in\mathbb{N}}$ in (28.12) is $t_k = 1/(k+1)$.

28.3 ▪ Remotest-Set and Alternating Projections

Let $m \in \{2, 3, \ldots\}$, set $I := \{1, 2, \ldots, m\}$, and let $(S_i)_{i\in I}$ be a family of nonempty closed convex subsets of X. Our goal is to solve the *convex feasibility problem*

$$\text{find } x \in S := S_1 \cap \cdots \cap S_m, \quad \text{where we assume that } S \neq \varnothing. \tag{28.15}$$

To this end, we set

$$f(x) := \max\{d_{S_1}(x), \ldots, d_{S_m}(x)\}. \tag{28.16}$$

Note that f is convex and continuous on X by Example 6.10 and Fact 6.13. Moreover, $\operatorname{dom}\partial f = X$ by Corollary 9.13. Hence (28.1a) holds and so does (28.1b) with $C := X$. Clearly, $(\forall x \in X)$ $f(x) \geqslant 0$ and $f(x) = 0 \Leftrightarrow (\forall i \in I)\ d_{S_i}(x) = 0 \Leftrightarrow (\forall i \in I)\ x \in S_i \Leftrightarrow x \in S$. Thus $S = \operatorname{Argmin}_{x\in X} f(x)$ and

$$\mu := \min f(X) = 0. \tag{28.17}$$

This takes care of (28.1d) and (28.1e). Next, it follows from Example 14.2 and Fact 14.8 that

$$(\forall x \in S) \quad \partial f(x) = \operatorname{conv}\bigcup_{i\in I}\big(B[0;1] \cap N_{S_i}(x)\big) \tag{28.18}$$

and

$$(\forall x \in X \smallsetminus S) \quad \partial f(x) = \operatorname{conv}\left\{ \frac{x - \mathrm{P}_{S_i}(x)}{d_{S_i}(x)} \;\middle|\; d_{S_i}(x) = f(x) \right\}. \tag{28.19}$$

In either case, $\partial f(x) \subseteq B[0;1]$ so (28.1c) holds with $L := 1$.

Assume that $x \in X \smallsetminus S$, and pick a *remotest-set* index $i \in I$ such that

$$f(x) = d_{S_i}(x) > 0, \text{ and set } f'(x) := \frac{x - \mathrm{P}_{S_i}(x)}{d_{S_i}(x)} \in \partial f(x), \tag{28.20}$$

where the inclusion follows from (28.19). Hence, in view of (28.17), Polyak's stepsize simplifies to

$$t = \frac{f(x) - \mu}{\|f'(x)\|^2} = \frac{d_{S_i}(x) - 0}{\left\| \frac{x - \mathrm{P}_{S_i}(x)}{d_{S_i}(x)} \right\|^2} = d_{S_i}(x). \tag{28.21}$$

Consequently, the update x_+ from (28.4) turns simply into

$$x_+ = \mathrm{P}_X(x - tf'(x)) = x - d_{S_i}(x) \frac{x - \mathrm{P}_{S_i}(x)}{d_{S_i}(x)} \tag{28.22a}$$

$$= \mathrm{P}_{S_i}(x). \tag{28.22b}$$

Specializing now Theorem 28.4 to the setting of this section, we obtain the following:

Theorem 28.6 (remotest-set projections) *Let $x_0 \in S$. Given $k \in \mathbb{N}$ and $x_k \in X$, evaluate $f(x_k)$. If $f(x_k) = 0$, then $x_k \in S$ and we are done. Otherwise, pick a remotest-set index $i_k \in I$ such that $f(x) = d_{S_{i_k}}(x_k) > 0$ and update via*

$$x_{k+1} := \mathrm{P}_{S_{i_k}}(x_k). \tag{28.23}$$

If we haven't terminated after finitely many steps, then $f(x_k) \to 0$ and $x_k \to \bar{x} \in S$.

Specializing further to $m = 2$ yields the following (see also Figure 28.1):

Corollary 28.7 (method of alternating projections (MAP)) *Given a starting point $x_0 \in X$, the sequence of alternating projections*

$$(x_k)_{k\in\mathbb{N}} = (x_0, \mathrm{P}_{S_1}x_0, \mathrm{P}_{S_2}\mathrm{P}_{S_1}x_0, \mathrm{P}_{S_1}\mathrm{P}_{S_2}\mathrm{P}_{S_1}x_0, \ldots)$$

converges to some point in $S_1 \cap S_2$.

Example 28.8 Let $A \in \mathbb{R}^{m\times n}$, let $b \in \mathbb{R}^m$, and assume that

$$S := \mathbb{R}^n_+ \cap A^{-1}(b) \neq \varnothing.$$

Let $x_0 = [x_{1,0}, \ldots, x_{n,0}]^\mathsf{T} \in \mathbb{R}^n$. Given $k \in \mathbb{N}$, update via

$$x_{k+1} = \big(x_k - A^\dagger(Ax_k - b)\big)^+,$$

where $y^+ = [\max\{y_1, 0\}, \ldots, \max\{y_n, 0\}]^\mathsf{T} \in \mathbb{R}^n$ and $A^\dagger$ denotes the Moore–Penrose inverse (see Fact 8.12) of A. Then $(x_k)_{k\in\mathbb{N}}$ converges to some point in S. See Figure 28.2 for an implementation in `Julia`.

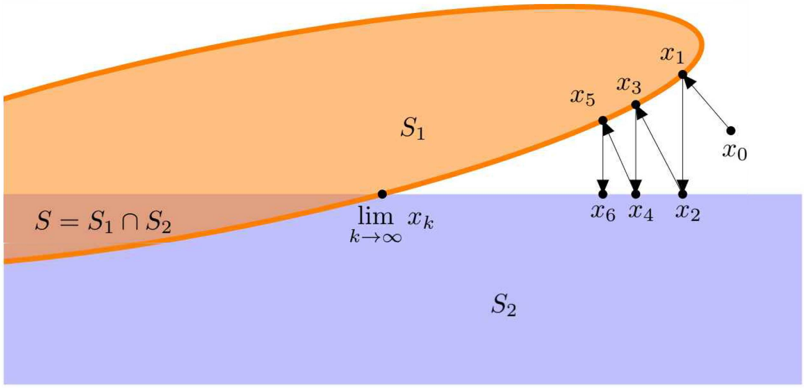


Figure 28.1. *Illustrating the first few iterates and the limit of the MAP (see Corollary* 28.7*).*

In [1]:
```
using LinearAlgebra; import Random;  Random.seed!(1234);
```

We implement the method of alternating projections (MAP) to find a solution to $Ax = b$ and $x \geq 0$ (coordinatewise). Here $A \in \mathbb{R}^{15\times 20}$, $S_1 = A^{-1}(b)$, and $S_2 = \mathbb{R}^{20}_{+}$. We start by generating the data.

In [2]:
```
n=20; m=15; A = randn(m,n); sol = abs.(randn(n)); b = A*sol;
```

We now define the projections w.r.t. S_1 and S_2, which we denote by `P1` and `P2`, respectively.

In [3]:
```
P1(x) = x - pinv(A)*(A*x-b); # pinv gives the Moore-Penrose/pseudo
inverse
P2(x) = max.(x,0);
```

Let's run MAP until the error measure $\max\{d_{S_1}(x), d_{S_2}(x)\}$ is less than $\varepsilon := 10^{-8}$.

In [4]:
```
errormeasure(x) = max.(norm(x-P1(x)),norm(x-P2(x))); epsilon = 1/10^8;
```

In [5]:
```
x = zeros(n,1); println("The starting vector of all zeros has error
measure: ",errormeasure(x))
```

Out[5]:
```
The starting vector of all zeros has error measure: 3.8164960360560993
```

In [6]:
```
k=0;
while errormeasure(x) > epsilon
    k = k+1; x = P2(P1(x));
end;
```

In [7]:
```
println("We needed $k iterations after which the error measure is:
",errormeasure(x));
```

Out[7]:
```
We needed 67 iterations after which the error measure is:
8.029976779957408e-9
```

Figure 28.2. *Employing MAP to find a nonnegative solution of $Ax = b$.*

Exercises

Exercise 28.1 (Vandenberghe) Consider the continuous convex function $f(x_1, x_2) := |x_1| + 2|x_2|$ on $\mathbb{R}^2$. Show that $(1, 2) \in \partial f(1, 0)$ yet $-(1, 2)$ is *not* a descent direction of f at $(1, 0)$.

Exercise 28.2 Consider the function $f(x) = \frac{1}{2}\|x\|^2$ and $C := B[0; 1]$. Compute the sequence $(x_k)_{k\in\mathbb{N}}$ generated by Polyak's projected subgradient method when $x_0 \in C \smallsetminus \{0\}$.

Exercise 28.3 Consider the function $f(x) = \frac{1}{2}\|x\|^2$ and $C := B[0; 1]$. Compute the sequence $(x_k)_{k\in\mathbb{N}}$ generated in Theorem 28.5 when $t_k = 1/(k + 2)$.

Exercise 28.4 Provide the details for Example 28.8.

Exercise 28.5 In Section 28.3, suppose that $X = \mathbb{R}$, $m = 2$, $S_1 = \{-1\}$, and $S_2 = \{1\}$. Note that $S = \varnothing$ in this case. What will the sequence $(\mu_k)_{k\in\mathbb{N}}$ from Theorem 28.5 with f given by (28.16) converge to?

Chapter 29
The Proximal Gradient Method

In this chapter, $\|\cdot\|$ denotes the Euclidean norm and we assume the following:

$$
\begin{aligned}
&f\colon X \to \mathbb{R} \text{ is convex and } L\text{-smooth, where } L \in \mathbb{R}_{++}, && (29.1a)\\
&g\colon X \to \left]-\infty, +\infty\right] \text{ is convex and lower semicontinuous}, && (29.1b)\\
&F := f + g, && (29.1c)\\
&S := \operatorname{Argmin} F \neq \varnothing, \quad \mu := \min F(X), && (29.1d)\\
&T := \mathrm{P}_{\frac{1}{L}g} \circ \big(\mathrm{Id} - \tfrac{1}{L}\nabla f\big). && (29.1e)
\end{aligned}
$$

We will study the proximal gradient algorithm, which uses ∇f and $\mathrm{P}_{\frac{1}{L}g}$ combined as in (29.1e) to find a minimizer of F.

29.1 ▪ The Proximal Gradient Operator

The operator T from (29.1e) is known as the *proximal gradient operator*; its relevance to finding points in S becomes clear in the following result:

Proposition 29.1 *We have*

$$S = \operatorname{Fix} T \quad \textit{is convex, closed, and nonempty,} \tag{29.2}$$

and T is $\frac{2}{3}$-averaged, i.e., for all x, y in X, we have

$$\tfrac{1}{2}\|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \leq \|x - y\|^2 - \|Tx - Ty\|^2. \tag{29.3}$$

Proof. Let $x \in X$. Then

$$
\begin{aligned}
x \in S &\Leftrightarrow 0 \in \partial(f + g)(x) && \text{(Fermat's rule (Lemma 9.2))}\\
&\Leftrightarrow 0 \in \nabla f(x) + \partial g(x) && \text{(sum rule (Fact 13.7))}\\
&\Leftrightarrow 0 \in \tfrac{1}{L}\nabla f(x) + \tfrac{1}{L}\partial g(x) &&\\
&\Leftrightarrow 0 \in \tfrac{1}{L}\nabla f(x) + \partial\big(\tfrac{1}{L}g\big)(x) && \text{(positive-multiple rule (13.1))}\\
&\Leftrightarrow x - \tfrac{1}{L}\nabla f(x) \in \big(\mathrm{Id} + \partial\big(\tfrac{1}{L}g\big)\big)(x) &&\\
&\Leftrightarrow x = \mathrm{P}_{\frac{1}{L}g}(\mathrm{Id} - \tfrac{1}{L}\nabla f)(x) && \text{(by (25.1))}\\
&\Leftrightarrow x = Tx &&\\
&\Leftrightarrow x \in \operatorname{Fix} T,
\end{aligned}
$$

which proves $S = \operatorname{Fix} T$. Note that S is nonempty by (29.1d), while closedness and convexity follow from Proposition 22.9. We've verified (29.2).

Next, on the one hand, $\mathrm{Id} - \frac{1}{L}\nabla f$ is firmly nonexpansive by Corollary 23.10. On the other hand, Theorem 26.1 (applied with f replaced by $\frac{1}{L}g$) implies that $\operatorname{Prox}_{\frac{1}{L}g}$ is firmly nonexpansive. Altogether, (29.3) now follows from Proposition 22.8 and Corollary 22.13. ☐

Lemma 29.2 *Let x, y be in X, and set $y_+ := Ty$. Then*

$$\begin{aligned} F(x) - F(y_+) &\geqslant \frac{L}{2}\|x - y_+\|^2 - \frac{L}{2}\|x - y\|^2 + D_f(x,y) && (29.4a)\\ &\geqslant \frac{L}{2}\|x - y_+\|^2 - \frac{L}{2}\|x - y\|^2, && (29.4b)\end{aligned}$$

where $D_f(x,y)$ is the Bregman distance between x and y (see Definition 12.7*).*

Proof. Define an auxiliary function h by

$$h(z) := f(y) + \langle z - y, \nabla f(y)\rangle + g(z) + \frac{L}{2}\|z - y\|^2, \tag{29.5}$$

and observe that h is L-strongly convex. If $z \in X$, then

$$\begin{aligned} z \text{ minimizes } h &\Leftrightarrow 0 \in \partial h(z) = \nabla f(y) + \partial g(z) + L(z - y)\\ &\Leftrightarrow y - \tfrac{1}{L}\nabla f(y) \in \big(\mathrm{Id} + \partial(\tfrac{1}{L}g)\big)(z)\\ &\Leftrightarrow z = \mathrm{P}_{\frac{1}{L}g}(\mathrm{Id} - \tfrac{1}{L}\nabla f)(y) = Ty = y_+.\end{aligned}$$

Thus

$$\operatorname{Argmin} h = \{y_+\},$$

and now Theorem 24.7(iii), implies that

$$h(x) - h(y_+) \geqslant \frac{L}{2}\|x - y_+\|^2. \tag{29.6}$$

On the other hand, it follows from the descent lemma (Lemma 23.6) that

$$f(y_+) \leqslant f(y) + \langle y_+ - y, \nabla f(y)\rangle + \frac{L}{2}\|y_+ - y\|^2. \tag{29.7}$$

Next, (29.5) and (29.7) yield

$$\begin{aligned} h(y_+) &= f(y) + \langle y_+ - y, \nabla f(y)\rangle + g(y_+) + \tfrac{L}{2}\|y_+ - y\|^2 && (29.8a)\\ &\geqslant f(y_+) + g(y_+) = F(y_+). && (29.8b)\end{aligned}$$

Using (29.6) and (29.8), we get

$$\begin{aligned} \frac{L}{2}\|x - y_+\|^2 &\leqslant h(x) - h(y_+)\\ &= f(y) + \langle x - y, \nabla f(y)\rangle + g(x) + \frac{L}{2}\|x - y\|^2 - h(y_+)\\ &\leqslant f(y) + \langle x - y, \nabla f(y)\rangle + g(x) + \frac{L}{2}\|x - y\|^2 - F(y_+)\\ &= f(x) + g(x) - \big(f(x) - f(y) - \langle x - y, \nabla f(y)\rangle\big) + \frac{L}{2}\|x - y\|^2 - F(y_+)\\ &= F(x) - D_f(x,y) + \frac{L}{2}\|x - y\|^2 - F(y_+).\end{aligned}$$

Rearranging yields (29.4a), and (29.4b) follows from Proposition 12.8. □

Corollary 29.3 *Let $y \in X$, and set $y_+ := Ty$. Then*

$$F(y_+) \leqslant F(y) - \frac{L}{2}\|y - y_+\|^2. \tag{29.9}$$

Proof. Apply Lemma 29.2 with x replaced by y. □

29.2 ▪ The Proximal Gradient Method

Iterating the proximal gradient operator yields a sequence with very nice properties:

Theorem 29.4 (Proximal Gradient Method (PGM)) *Recall our assumptions* (29.1) *and let $x_0 \in X$. Generate the sequence $(x_k)_{k\in\mathbb{N}}$ via*

$$x_{k+1} := T(x_k). \tag{29.10}$$

Then the following hold:

(i) **(Fejér monotonicity)** $(\forall s \in S)(\forall k \in \mathbb{N})\ \|x_{k+1} - s\| \leqslant \|x_k - s\|$.

(ii) **(convergence of the iterates)** $(x_k)_{k\in\mathbb{N}}$ *converges to a point in S.*

(iii) **(monotone function value convergence)** *For $k \geqslant 1$, we have*

$$0 \leqslant F(x_{k+1}) - \mu \leqslant F(x_k) - \mu \leqslant \frac{L d_S^2(x_0)}{2k} = \mathcal{O}\big(\tfrac{1}{k}\big). \tag{29.11}$$

(iv) **(asymptotic regularity)** *For $k \geqslant 1$, we have*

$$\|x_{k-1} - x_k\| \leqslant \frac{\sqrt{2}d_S(x_0)}{\sqrt{k}} = \mathcal{O}\big(\tfrac{1}{\sqrt{k}}\big). \tag{29.12}$$

Proof. (i)&(ii): By Proposition 29.1, the operator T is $\frac{2}{3}$-averaged with $\operatorname{Fix} T = S$. Now apply Corollary 22.20.

(iii): Corollary 29.3 (applied with y replaced by x_k) yields

$$F(x_{k+1}) \leqslant F(x_k) \tag{29.13}$$

and we have a monotonically decreasing sequence of function values.

Now let $s \in S$. Then (29.4) (applied with x replaced by s and y replaced by x_m) yields

$$0 \leqslant \frac{2}{L}\big(F(x_{m+1}) - \mu\big) = \frac{2}{L}\big(F(x_{m+1}) - F(s)\big) \leqslant \|x_m - s\|^2 - \|x_{m+1} - s\|^2.$$

Summing the above inequalities from $m = 0$ to $m = k - 1$ and telescoping yields

$$0 \leqslant \frac{2}{L}\sum_{m=0}^{k-1}\big(F(x_{m+1}) - \mu\big) \leqslant \|x_0 - s\|^2 - \|x_k - s\|^2 \leqslant \|x_0 - s\|^2. \tag{29.14}$$

Recall from (29.2) that S is convex, closed, and nonempty. Hence, with the choice $s = \mathrm{P}_S(x_0)$, we learn from (29.13) and (29.14) that

$$\frac{2}{L}k\big(F(x_k)-\mu\big) \leqslant \frac{2}{L}\sum_{m=0}^{k-1}\big(F(x_{m+1})-\mu\big) \leqslant \|x_0-P_S(x_0)\|^2 = d_S^2(x_0),$$

and this yields (iii).

(iv): Let $k \geqslant 1$. Applying (29.3) with x replaced by x_k and y replaced by $s := \mathrm{P}_S(x_0) \in S = \operatorname{Fix} T$ (see (29.2)) yields

$$\tfrac{1}{2}\|x_k - x_{k+1}\|^2 \leqslant \|x_k - s\|^2 - \|x_{k+1}-s\|^2.$$

On the other hand, the nonexpansiveness of T (see Proposition 29.1) results in

$$\|x_k - x_{k+1}\| \leqslant \|x_{k-1}-x_k\| \leqslant \cdots \leqslant \|x_0 - x_1\|.$$

Altogether,

$$\begin{aligned}\frac{k}{2}\|x_{k-1}-x_k\|^2 &\leqslant \frac{1}{2}\sum_{m=0}^{k-1}\|x_m - x_{m+1}\|^2 \leqslant \|x_0-s\|^2 - \|x_k - s\|^2\\ &\leqslant \|x_0 - s\|^2 = d_S^2(x_0),\end{aligned}$$

and this completes the proof. □

Corollary 29.5 (Proximal Point Algorithm revisited) *Assume that $C := \operatorname{Argmin} g \neq \varnothing$, let $\rho > 0$, let $x_0 \in X$, and generate the sequence $(x_k)_{k\in\mathbb{N}}$ via*

$$x_{k+1} := \mathrm{P}_{\rho g} x_k.$$

Then $(x_k)_{k\in\mathbb{N}}$ converges to a point in C and $g(x_k) - \min g(X) \leqslant d_C^2(x_0)/(2\rho k)$.

Corollary 29.6 *Assume that $C := \operatorname{Argmin} f \neq \varnothing$, let $x_0 \in X$, and generate the sequence $(x_k)_{k\in\mathbb{N}}$ via*

$$x_{k+1} := x_k - \tfrac{1}{L}\nabla f(x_k).$$

Then $(x_k)_{k\in\mathbb{N}}$ converges to a point in C and $f(x_k) - \min f(X) \leqslant L d_C^2(x_0)/(2k)$.

29.3 ▪ Regularized Least Squares

In this section, we discuss an application of the PGM that is known as "ISTA" (which stands for "Iterative Shrinkage Thresholding Algorithm").

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $\lambda > 0$. The problem of interest is to

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad \tfrac{1}{2}\|Ax-b\|^2 + \lambda\|x\|_1. \tag{29.15}$$

The problem (29.15) fits the framework considered (see (29.1)) perfectly: Set $f(x) := \frac{1}{2}\|Ax-b\|^2$ and $g(x) := \lambda\|x\|_1$. Because g is coercive and f is bounded below, it is clear that $F := f + g$ has at least one minimizer by Corollary 5.5: $S := \operatorname{Argmin}_{x\in\mathbb{R}^n} F(x) \neq \varnothing$. Moreover,

$$\nabla f(x) = A^{\mathsf{T}}(Ax-b) \quad\text{and}\quad \nabla^2 f(x) = A^{\mathsf{T}}A. \tag{29.16}$$

By Fact 23.8, ∇f is Lipschitz continuous with (optimal) constant

$$\lambda_{\max}(A^{\mathsf{T}}A). \tag{29.17}$$

If the computation of $\lambda_{\max}(A^\mathsf{T}A)$ is too costly, one may use

$$\|A\|_\mathsf{F}^2 = \sum_{1\leqslant i\leqslant m,\, 1\leqslant j\leqslant n} A_{i,j}^2 \tag{29.18}$$

instead — note that this constant is also equal to $\operatorname{tr}(A^\mathsf{T}A) = \sum_{i=1}^n \lambda_i(A^\mathsf{T}A)$ but it completely avoids the computation of eigenvalues.

Thus picking $L \in \{\lambda_{\max}(A^\mathsf{T}A), \|A\|_\mathsf{F}^2\}$, we see that all assumptions (29.1) are in place. The gradient step is

$$x - \tfrac{1}{L}\nabla f(x) = x - \tfrac{1}{L}A^\mathsf{T}(Ax-b), \tag{29.19}$$

while the proximal step is, thanks to Example 25.12,

$$\operatorname{Prox}_{\frac{1}{L}g}(x) = \big(\operatorname{sign}(x_i)\max\{|x_i| - \lambda/L, 0\}\big)_{i=1}^n. \tag{29.20}$$

Thus T can be computed and we are then able to apply either the PGM considered in this chapter (or "FISTA" from Chapter 30 below).

Here is an implementation in `Julia`:

In [1]:
```
using Plots; using LinearAlgebra; import Random:seed!; seed!(1357);
```

We start with the setup: We have $A \in \mathbb{R}^{100\times 110}$ and $b \in \mathbb{R}^{100}$.
We set up the problem so that it is consistent, with one solution at least being sparse.
Note that the problem is *underdetermined*. We choose $\lambda = 1$.

In [2]:
```
m=100; d=110; A=randn(m,d);
xtrue=zeros(d,1);
xtrue[1]=1; xtrue[2]=1; xtrue[3]=1; xtrue[4]=-1; xtrue[5]=-1;
xtrue[6]=-1;
b=A*xtrue;
Lsmallest=eigmax(A'*A);
println("Smallest Lipschitz constant (our L) is ",Lsmallest);
L=Lsmallest;  lambda = 1;
```

Out[2]:
```
Smallest Lipschitz constant (our L) is 403.1089636391968
```

The prox-grad operator T is the composition $T = T_2 \circ T_1$,
where $T_1(x) = x - \frac{1}{L}\nabla f(x) = x - \frac{1}{L}A^T(Ax-b)$
and T_2 is the prox operator of λ/L times the ℓ_1 norm.

In [3]:
```
T1(x)=x-(1/L)*A'*(A*x-b);
T2(x)=sign.(x).*max.(0,abs.(x).-lambda/L);
T(x)=T2(T1(x));
```

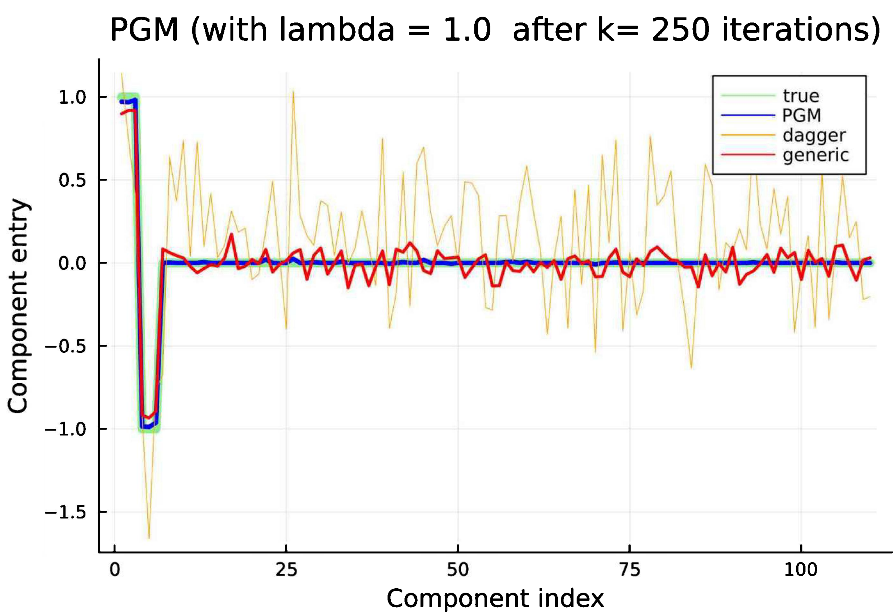
We also record the Moore-Penrose and a generic solution.

In [4]:
```
allones = ones(d,1); # the vector of all ones:
xdagger = allones - pinv(A)*(A*allones-b); # the closest solution to
allones of Ax=b
xgeneric = A\b;  # the generic solution = the closest to the origin
```

We now start the Prox Gradient Method (PGM) with a starting point of all ones.

In [5]:
```
# now we run PGM

x = allones;
lambda = 1.0;
kcounter = 250; # number of iterations

k=0; # counter
while k < kcounter
    x = T(x)
    k = k + 1;
end;


# plotting
xx=1:d;yy=[xtrue,x,xdagger,xgeneric];
plot(xx,yy,title="PGM (with lambda = $lambda  after k= $k
iterations)", xlabel="Component index",
    ylabel="Component entry",
label=["true" "PGM" "dagger" "generic"],color=["lightgreen" "blue"
"orange" "red"],
linestyle=[:solid :solid :solid :solid], lw=[6 3 0.5 2])
```

Out[5]:



Let's see how the various answers differ with respect to the objective function.

In [6]:
```
F(x)=1/2*norm(A*x-b)^2+lambda*norm(x,1); # set up F
[F(xtrue) F(x) F(xdagger) F(xgeneric)]
```

Out[6]:
```
1×4 Matrix{Float64}:
 6.0  6.09182  38.3453  11.1815
```

Exercises

Exercise 29.1 Explain why the function h defined in (29.5) is L-strongly convex.

Exercise 29.2 Provide the details for Corollary 29.5.

Exercise 29.3 Provide the details for Corollary 29.6.

Exercise 29.4 Consider Corollary 28.7 and its notation. One key operator of MAP is $T := \mathrm{P}_{S_2}\mathrm{P}_{S_1}$. Explain how T can be interpreted as a proximal gradient operator.

Exercise 29.5 Let S_1, S_2 be nonempty closed convex subsets of X. It turns out that $T := \mathrm{P}_{S_2}(\mathrm{Id} - \mathrm{P}_{S_1})$ is a proximal gradient operator. Determine an optimization problem that gives rise to T as its proximal gradient operator.

Exercise 29.6 Define the function $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) := \frac{1}{3}|x|^3$, and let $0 < L < +\infty$. Define the gradient operator by

$$T(x) := x - \tfrac{1}{L}f'(x).$$

(i) Compute f'' and deduce that f is convex but not L-smooth.

(ii) Show that $|T(x)| = |x| \cdot \left|1 - \frac{|x|}{L}\right|$ and deduce that if $|x| \geqslant 3L$, then $|T(x)| \geqslant 2|x|$.

(iii) Deduce that the gradient method of Corollary 29.6 may produce an unbounded sequence illustrating that the assumption on L-smoothness of f is important.

Chapter 30

The Fast Iterative Soft Thresholding Algorithm (FISTA)

As in the previous chapter, $\|\cdot\|$ denotes the Euclidean norm and we assume the following:

$$
\begin{aligned}
&f\colon X \to \mathbb{R} \text{ is convex and } L\text{-smooth, where } L \in \mathbb{R}_{++}, && \text{(30.1a)}\\
&g\colon X \to \left]-\infty, +\infty\right] \text{ is convex and lower semicontinuous}, && \text{(30.1b)}\\
&F := f + g, && \text{(30.1c)}\\
&S := \operatorname{Argmin} F \neq \varnothing, \quad \mu := \min F(X), && \text{(30.1d)}\\
&T := \mathrm{P}_{\frac{1}{L}g}(\mathrm{Id} - \tfrac{1}{L}\nabla f). && \text{(30.1e)}
\end{aligned}
$$

We will study a modern variant of the proximal gradient algorithm, which uses the proximal gradient operator defined in (30.1e) to find a minimizer of F "more quickly" than the PGM.

30.1 ▪ Parameter Sequences for FISTA

Definition 30.1 We say a sequence $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_{++}$ is a parameter sequence for the Fast Iterative Soft Thresholding Algorithm (FISTA) if the following hold for every $k \in \mathbb{N}$:

$$
\begin{aligned}
t_k &\geqslant \frac{k+2}{2} \geqslant 1 = t_0, && \text{(30.2a)}\\
t_k^2 &\geqslant t_{k+1}^2 - t_{k+1}. && \text{(30.2b)}
\end{aligned}
$$

Example 30.2 The sequence defined by

$$t_0 := 1 \quad \text{and} \quad (\forall k \in \mathbb{N})\ \ t_{k+1} := \frac{1 + \sqrt{1 + 4t_k^2}}{2}$$

is a parameter sequence for FISTA.

Example 30.3 The sequence defined by

$$(\forall k \in \mathbb{N})\ \ t_k := \frac{k+2}{2}$$

is a parameter sequence for FISTA.

30.2 ▪ FISTA

Theorem 30.4 (FISTA) *Recall our assumptions* (30.1), *and let* $(t_k)_{k\in\mathbb{N}}$ *be a parameter sequence for FISTA, i.e.,* (30.2) *holds. Let* $x_0 \in X$, *and set* $y_0 := x_0$. *Given* $k \in \mathbb{N}$, *update via*

$$x_{k+1} := Ty_k = \mathrm{P}_{\frac{1}{L}g}\big(y_k - \tfrac{1}{L}\nabla f(y_k)\big), \tag{30.3a}$$

$$y_{k+1} := x_{k+1} + \frac{t_k - 1}{t_{k+1}}(x_{k+1} - x_k). \tag{30.3b}$$

Then

$$0 \leqslant F(x_k) - \mu \leqslant \frac{2Ld_S^2(x_0)}{(k+1)^2} = \mathcal{O}\Big(\frac{1}{k^2}\Big). \tag{30.4}$$

Proof. Set $s := \mathrm{P}_S(x_0) \in S$ and

$$\delta_k := F(x_k) - \mu = F(x_k) - F(s) \geqslant 0 \quad \text{for } k \geqslant 0, \tag{30.5a}$$

$$z_k := s + (t_{k-1} - 1)x_{k-1} - t_{k-1}x_k \quad \text{for } k \geqslant 1. \tag{30.5b}$$

Let $k \geqslant 1$. Then (30.3b) and (30.5b) yield

$$\begin{aligned} s + (t_k - 1)x_k - t_k y_k &= s + (t_k - 1)x_k - t_k x_k - (t_{k-1} - 1)(x_k - x_{k-1}) \\ &= s + (t_{k-1} - 1)x_{k-1} - t_{k-1}x_k \\ &= z_k; \end{aligned}$$

therefore,

$$\|s + (t_k - 1)x_k - t_k y_k\|^2 = \|z_k\|^2. \tag{30.6}$$

Next,

$$\begin{aligned} &t_{k-1}^2\delta_k - t_k^2\delta_{k+1} \\ &\quad\geqslant (t_k^2 - t_k)\delta_k - t_k^2\delta_{k+1} && \text{(by (30.2b))} \\ &\quad= t_k^2\Big(\Big(1 - \frac{1}{t_k}\Big)\delta_k - \delta_{k+1}\Big) \\ &\quad= t_k^2\Big(\Big(1 - \frac{1}{t_k}\Big)\big(F(x_k) - F(s)\big) - \big(F(x_{k+1}) - F(s)\big)\Big) && \text{(by (30.5a))} \\ &\quad= t_k^2\Big(\Big(1 - \frac{1}{t_k}\Big)F(x_k) + \frac{1}{t_k}F(s) - F(x_{k+1})\Big) \\ &\quad\geqslant t_k^2\Big(F\Big(\frac{1}{t_k}s + \Big(1 - \frac{1}{t_k}\Big)x_k\Big) - F(x_{k+1})\Big) && \text{((30.2a) and } F \text{ is convex)} \\ &\quad\geqslant t_k^2\frac{L}{2}\Big(\Big\|\frac{1}{t_k}s + \Big(1 - \frac{1}{t_k}\Big)x_k - x_{k+1}\Big\|^2 - \Big\|\frac{1}{t_k}s + \Big(1 - \frac{1}{t_k}\Big)x_k - y_k\Big\|^2\Big) \\ &&& \text{(by Lemma 29.2 and } x_{k+1} = Ty_k) \\ &\quad= \frac{L}{2}\big(\|s + (t_k - 1)x_k - t_k x_{k+1}\|^2 - \|s + (t_k - 1)x_k - t_k y_k\|^2\big) \\ &\quad= \frac{L}{2}\big(\|z_{k+1}\|^2 - \|z_k\|^2\big). && \text{(by (30.5b) and (30.6))} \end{aligned}$$

Hence $t_k^2\delta_{k+1} + (L/2)\|z_{k+1}\|^2 \leqslant t_{k-1}^2\delta_k + (L/2)\|z_k\|^2$ and thus

$$
\begin{aligned}
\frac{2}{L}t_{k-1}^2\delta_k &\leqslant \frac{2}{L}t_{k-1}^2\delta_k + \|z_k\|^2 && (30.7a)\\
&\leqslant \frac{2}{L}t_{k-2}^2\delta_{k-1} + \|z_{k-1}\|^2 && (30.7b)\\
&\ \ \vdots \\
&\leqslant \frac{2}{L}t_0^2\delta_1 + \|z_1\|^2 && (30.7c)\\
&= \frac{2}{L}\big(F(x_1) - \mu\big) + \|x_1 - s\|^2 && (30.7d)
\end{aligned}
$$

because $z_1 = s + (t_0 - 1)x_0 - t_0x_1 = s + (1-1)x_0 - (1)x_1 = s - x_1$.

On the other hand, using Lemma 29.2 again and recalling that $Tx_0 = Ty_0 = x_1$, we estimate $\mu - F(x_1) = F(s) - F(x_1) \geqslant (L/2)\|s - x_1\|^2 - (L/2)\|s - x_0\|^2$; equivalently,

$$\frac{2}{L}\big(F(x_1) - \mu\big) \leqslant \|s - x_0\|^2 - \|s - x_1\|^2. \tag{30.8}$$

Altogether, recalling (30.7) and (30.8), we get $(2/L)t_{k-1}^2\delta_k \leqslant \|x_0 - s\|^2 = d_S^2(x_0)$. Finally, combining this with (30.2a), we obtain

$$F(x_k) - \mu = \delta_k \leqslant \frac{L}{2t_{k-1}^2}d_S^2(x_0) \leqslant \frac{L}{2}\left(\frac{2}{k+1}\right)^2 d_S^2(x_0) = \frac{2Ld_S^2(x_0)}{(k+1)^2}$$

and we're done. ☐

30.3 ▪ Regularized Least Squares Revisited

Recall the setup from Section 29.3. With FISTA now at our disposal, we illustrate and compare using the following `Julia` implementation:

In [1]:
```
using Plots; using LinearAlgebra; using Random;
import Random:seed!; seed!(1357);
```

We start with the setup. We have $A \in \mathbb{R}^{100\times 110}$ and $b \in \mathbb{R}^{100}$.
We set up the problem so that it is consistent, with one solution at least being sparse.
Note that the problem is *underdetermined*. We choose $\lambda = 1$.

In [2]:
```
m=100; d=110; A=randn(m,d);
xtrue=zeros(d,1);
xtrue[1]=1; xtrue[2]=1; xtrue[3]=1; xtrue[4]=-1;
xtrue[5]=-1;xtrue[6]=-1;
b=A*xtrue; Lsmallest=eigmax(A'*A);
println("Smallest Lipschitz constant (our L) is ",Lsmallest);
L=Lsmallest; lambda = 1;

F(x)=1/2*norm(A*x-b)^2+lambda*norm(x,1); # set up F
```

Out[2]:
```
Smallest Lipschitz constant (our L) is 403.1089636391968

F (generic function with 1 method)
```

The prox-grad operator T is the composition $T = T_2 \circ T_1$,
where $T_1(x) = x - \frac{1}{L}\nabla f(x) = x - \frac{1}{L}A^T(Ax - b)$
and T_2 is the prox operator of λ/L times the ℓ_1 norm.

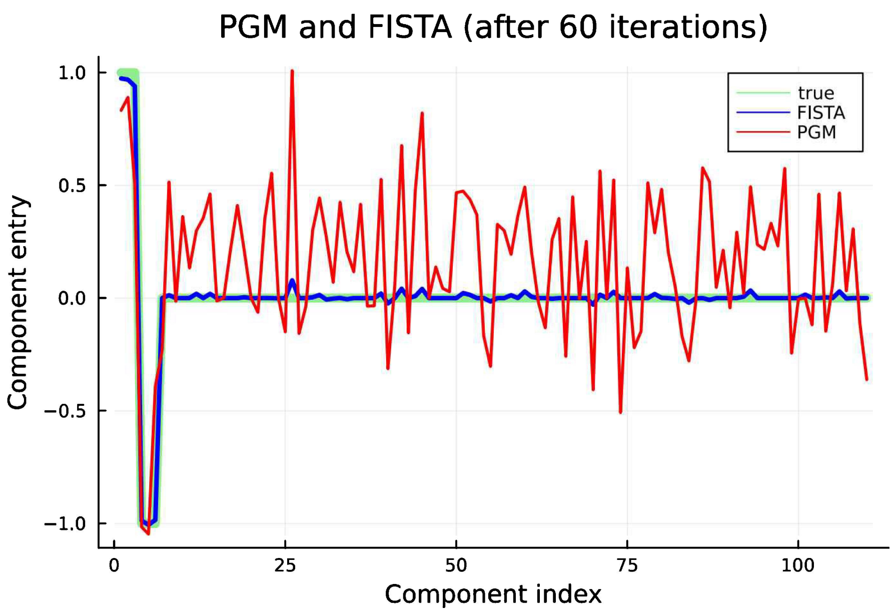
In [3]:
```
T1(x)=x-(1/L)*A'*(A*x-b);
T2(x)=sign.(x).*max.(0,abs.(x).-lambda/L);
T(x)=T2(T1(x));
```

In [4]:
```
allones = ones(d,1); # the vector of all ones will be our starting point:
```

We now start the Prox Gradient Method (PGM) and FISTA with the starting point of all ones.

In [5]:
```
# now we run PGM and Fast PGM (FISTA)

xpgm = allones; # xpgm will be PGM iterate

x = allones; # x_0. x will be the FISTA iterate
t = 1; # t_0 = 1;
y = x; # y_0 = x_0;
lambda = 1.0;

kcounter = 60; # 25, 50, 100, 250 are good choices

# container for function values
Fpgm = ones(kcounter,1);
Ffista = ones(kcounter,1);

k=0; # counter
while k < kcounter
    # update iterates
    xpgm = T(xpgm); # PGM update
    xold = x; # temporary variable
    told = t; # temporary variable
    x = T(y); # x_{n+1} update
    t = (1+sqrt(1+4*told^2))/2; # t_{n+1} update
    y = x + (told-1)/t*(x-xold); # y_{n+1} update
    k = k + 1;

    # record function values
    Fpgm[k]=F(xpgm);
    Ffista[k] = F(x);
end;

# plotting last iterate
xx=1:d;yy=[xtrue,x,xpgm];
plot(xx,yy,title="PGM and FISTA (after $k iterations)",
    xlabel="Component index", ylabel="Component entry",
label=["true" "FISTA" "PGM"],color=["lightgreen" "blue" "red"],
linestyle=[:solid :solid :solid], lw=[6 3 2])
```

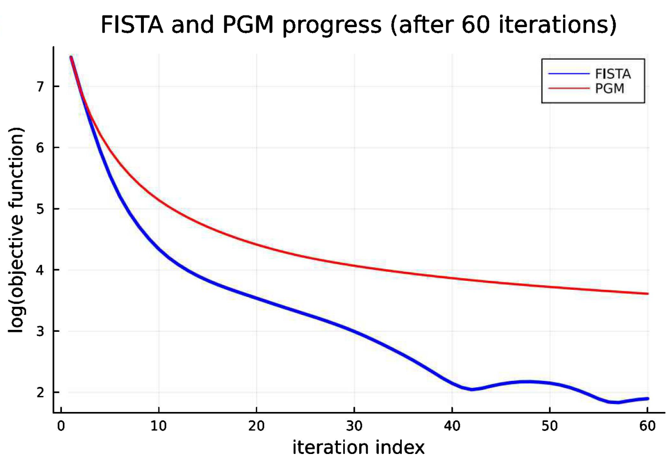
Out[5]:



In [6]:

```
# now plotting progression of function values

lFfista = log.(Ffista);
lFpgm = log.(Fpgm);

xx=1:kcounter;yy=[lFfista,lFpgm];
plot(xx,yy,title="FISTA and PGM progress (after $k iterations)",
    xlabel="iteration index", ylabel="log(objective function)",
label=["FISTA" "PGM"],color=["blue" "red"],
linestyle=[:solid :solid], lw=[3 2])
```

Out[6]:



It is clear that FISTA is indeed faster than PGM; note also that $(F(x_k))_{k\in\mathbb{N}}$ is no longer decreasing for FISTA.

Exercises

Exercise 30.1 Provide the details for Example 30.2.

Exercise 30.2 Provide the details for Example 30.3.

Exercise 30.3 Suppose that $(t_k)_{k\in\mathbb{N}}$ is given either by Example 30.2 or by Example 30.3. Show that in either case, we have $(\forall k \in \mathbb{N})\ t_k \leqslant k+1$.

Exercise 30.4 The quotient $(t_k - 1)/t_{k+1}$ plays a key role in the update step of FISTA (see (30.3b)). What is this quotient when we employ the parameter sequence from Example 30.3? And does it converge when $k \to +\infty$?

Exercise 30.5 The quotient $(t_k - 1)/t_{k+1}$ plays a key role in the update step of FISTA (see (30.3b)). Does this quotient have a limit as when $k \to +\infty$ when we employ the parameter sequence from Example 30.2?

Exercise 30.6 We have seen in Exercise 29.4 that MAP can be interpreted as the PGM. Design a numerical experiment that relies on the setting of Example 28.8 and compare this to the FISTA variant.

Exercise 30.7 Let T be the proximal gradient operator from (30.1e), let $y_0 := x_0 \in X$ and consider the sequences generated by $x_{k+1} := Ty_k$ and $y_{k+1} := 2x_{k+1} - x_k$. Show the following: (i) If $x_k \to x$, then $x \in \operatorname{Fix} T$ and $y_k \to x$. (ii) If $y_k \to y$, then $y \in \operatorname{Fix} T$ and $x_k \to y$.

Chapter 31

Douglas–Rachford Algorithm

In this chapter,

$$f \text{ and } g \text{ are convex, lower semicontinuous, and proper on } X, \tag{31.1a}$$

$$S := \operatorname{Argmin}(f+g), \;\; \mu := \inf(f+g)(X). \tag{31.1b}$$

31.1 ▪ Alternating Proximal Mappings

We have seen that iterating forward gradient steps (see Corollary 29.6) or proximal steps (see Corollary 29.5) leads to minimizers of a single function; moreover, the PGM (see Theorem 29.4) combines these operators to find a minimizer of the sum of a convex and a smooth function.

It is thus natural to expect that iterating proximal operators could lead to minimizers of the sum of two general convex functions. Unfortunately, this is not true, as we show now:

Suppose that $X = \mathbb{R}$ and consider the two *nonsmooth* convex functions defined by

$$f(x) := 4|x+1| + \tfrac{1}{2}x^2 \quad \text{and} \quad g(x) := 4|x-1| + \tfrac{1}{2}x^2$$

which have proximal mappings

$$\mathrm{P}_f(y) = \begin{cases} -1 & \text{if } |y+2| \leqslant 4; \\ \dfrac{y}{2} - 2\dfrac{y+2}{|y+2|} & \text{if } |y+2| > 4 \end{cases} \tag{31.2a}$$

and

$$\mathrm{P}_g(y) = \begin{cases} 1 & \text{if } |y-2| \leqslant 4; \\ \dfrac{y}{2} - 2\dfrac{y-2}{|y-2|} & \text{if } |y-2| > 4. \end{cases} \tag{31.2b}$$

The minimizers of the function $4|x+1| + 4|x-1|$ are $[-1,1]$; thus, the minimizer of $f(x) + g(x) = 4|x+1| + 4|x-1| + x^2$ is unique and equal to 0:

$$S = \operatorname{Argmin}(f+g) = \{0\}. \tag{31.3}$$

Let us now employ alternating proximal mappings, starting at the minimizer 0: Because $|0+2| = 2 < 4$, it follows from (31.2a) that $\mathrm{P}_f(0) = -1$. In turn, because $|-1-2| = 3 < 4$, it follows from (31.2b) that $\mathrm{P}_g(-1) = 1$. Next, because $|1 + 2| = 3 < 4$, (31.2a) yields $\mathrm{P}_f(1) = -1$. To sum up, the alternating proximal mappings sequence is

$$\big(0, \mathrm{P}_f(0), \mathrm{P}_g\mathrm{P}_f(0), \mathrm{P}_f\mathrm{P}_g\mathrm{P}_f(0), \dots\big) = (0, -1, 1, -1, 1, \dots).$$

Thus,

$$(\mathrm{P}_g\mathrm{P}_f)^k(0) \to 1 \quad \text{and} \quad (\mathrm{P}_f\mathrm{P}_g)^k(\mathrm{P}_f(0)) \to -1.$$

This clearly illustrates that iterating the proximal mappings P_f and P_g does *not* yield a minimizer of $f + g$!

In fact, the behavior of this iteration is explained by the following result:

Proposition 31.1 *Suppose that the function* $g + \operatorname{env} f$ *has a minimizer, and let* $x_0 \in X$. *Then the sequence* $((\mathrm{P}_g\mathrm{P}_f)^k(x_0))_{k\in\mathbb{N}}$ *converges to a minimizer of* $g + \operatorname{env} f$.

31.2 ▪ Reflected Proximal Mappings

Definition 31.2 The *reflected proximal mapping* or *reflectant* of f is defined by

$$\boxed{\mathrm{R}_f := 2\mathrm{P}_f - \mathrm{Id}. \qquad (31.4)}$$

From now on, we also set

$$\boxed{Z := \operatorname{zer}(\partial f + \partial g) = \big\{x \in X \;\big|\; 0 \in \partial f(x) + \partial g(x)\big\}. \qquad (31.5)}$$

Reflectants are intimately connected to Z, as the next result illustrates:

Proposition 31.3 *We have*

$$Z = \mathrm{P}_f\big(\operatorname{Fix}(\mathrm{R}_g\mathrm{R}_f)\big). \tag{31.6}$$

Proof. "$\subseteq$": Suppose that $z \in Z$. Then $0 \in \partial f(z) + \partial g(z)$ and so there exists $w \in \partial f(z)$ such that $-w \in \partial g(z)$. In turn, $z + w \in (\mathrm{Id} + \partial f)(z)$ and $z - w \in (\mathrm{Id} + \partial g)(z)$; equivalently,

$$z = \mathrm{P}_f(z + w) \;\text{ and }\; z = \mathrm{P}_g(z - w). \tag{31.7}$$

It now suffices to show that $z + w \in \operatorname{Fix}(\mathrm{R}_g\mathrm{R}_f)$. Indeed, $\mathrm{R}_f(z+w) = 2\mathrm{P}_f(z+w) - (z+w) = 2z - (z + w) = z - w$ and then $\mathrm{R}_g(z - w) = 2\mathrm{P}_g(z - w) - (z - w) = 2z - (z - w) = z + w$ and we are done.

The inclusion "$\supseteq$" is proved similarly. ☐

Proposition 31.4 *The reflectants* $\mathrm{R}_f, \mathrm{R}_g$ *are nonexpansive, and so is their composition* $\mathrm{R}_g\mathrm{R}_f$.

Proof. By Theorem 26.1, P_f and P_g are firmly nonexpansive. Now Theorem 22.3 yields the nonexpansiveness of R_f and R_g. Finally, the composition of nonexpansive mappings is nonexpansive; in particular, $\mathrm{R}_g\mathrm{R}_f$ is nonexpansive. ☐

31.3 ▪ The Douglas–Rachford Operator

Definition 31.5 The *Douglas–Rachford operator* of (f, g) is defined by

$$T := \mathrm{T}_{\mathrm{DR}}(f,g) := \mathrm{Id} - \mathrm{P}_f + \mathrm{P}_g\mathrm{R}_f. \tag{31.8}$$

Lemma 31.6 *The following hold:*

(i) $T = \frac{1}{2}(\mathrm{Id} + \mathrm{R}_g\mathrm{R}_f)$.

(ii) T *is firmly nonexpansive.*

(iii) $\mathrm{Fix}\, T = \mathrm{Fix}\, \mathrm{R}_g\mathrm{R}_f$.

(iv) $\mathrm{P}_f(\mathrm{Fix}\, T) = Z$.

Proof. (i): Indeed, we have

$$\begin{aligned}
\tfrac{1}{2}(\mathrm{Id} + \mathrm{R}_g\mathrm{R}_f) &= \tfrac{1}{2}(\mathrm{Id} + (2\mathrm{P}_g - \mathrm{Id})(2\mathrm{P}_f - \mathrm{Id})) \\
&= \tfrac{1}{2}(\mathrm{Id} + (2\mathrm{P}_g(2\mathrm{P}_f - \mathrm{Id}) - 2\mathrm{P}_f + \mathrm{Id}) \\
&= \tfrac{1}{2}(2\mathrm{Id} + 2\mathrm{P}_g(2\mathrm{P}_f - \mathrm{Id}) - 2\mathrm{P}_f) \\
&= \mathrm{Id} - \mathrm{P}_f + \mathrm{P}_g(2\mathrm{P}_f - \mathrm{Id}) \\
&= \mathrm{Id} - \mathrm{P}_f + \mathrm{P}_g\mathrm{R}_f,
\end{aligned}$$

as claimed. (ii): By Proposition 31.4, the operator $\mathrm{R}_g\mathrm{R}_f$ is nonexpansive. It now follows from Theorem 22.3 that $\frac{1}{2}(\mathrm{Id} + \mathrm{R}_g\mathrm{R}_f)$ is firmly nonexpansive. However, by (i), this is precisely the Douglas–Rachford operator T.

(iii): This is an easy consequence of (i).

(iv): Combine (iii) with (31.6). ☐

31.4 ▪ The Douglas–Rachford Algorithm

We are now ready for the main result of this chapter.

Theorem 31.7 (Douglas–Rachford Algorithm) *Suppose that $Z \neq \varnothing$. Let $x_0 \in X$ and generate the* governing sequence $(x_k)_{k\in\mathbb{N}}$ *via*

$$x_{k+1} := Tx_k = x_k - \mathrm{P}_f x_k + \mathrm{P}_g(2\mathrm{P}_f x_k - x_k). \tag{31.9}$$

Then there exists $\overline{x} \in X$ such that the following hold:

(i) $x_k \to \overline{x} \in \mathrm{Fix}\, T$.

(ii) $\mathrm{P}_f x_k \to \mathrm{P}_f\overline{x} \in Z = S$, *i.e., the* shadow sequence $(\mathrm{P}_f x_k)_{k\in\mathbb{N}}$ *finds a minimizer.*

(iii) $\mathrm{P}_g(2\mathrm{P}_f x_k - x_k) \to \mathrm{P}_f\overline{x}$.

Proof. By Lemma 31.6(iv),

$$\mathrm{P}_f(\mathrm{Fix}\, T) = Z.$$

The assumption that $Z \neq \varnothing$ yields $\mathrm{Fix}\, T \neq \varnothing$.

(i): Lemma 31.6(ii) states that T is firmly nonexpansive; hence, T is $\frac{1}{2}$-averaged (Remark 22.12). By Corollary 22.20, $(x_k)_{k\in\mathbb{N}}$ converges to some point in $\operatorname{Fix} T$, say $\overline{x}$.

(ii): On the one hand, $x_k \to \overline{x} \in \operatorname{Fix} T$ by (i). On the other hand, P_f is firmly nonexpansive and hence continuous. Altogether, $\mathrm{P}_f(x_k) \to \mathrm{P}_f(\overline{x}) \in \mathrm{P}_f(\operatorname{Fix} T)$. Finally, $\mathrm{P}_f(\operatorname{Fix} T) = Z$ (see Lemma 31.6(iv)) and $Z = S$ (see Theorem 13.13(ii)).

(iii): By (i), $x_k - x_{k+1} = \mathrm{P}_f x_k - \mathrm{P}_g(2\mathrm{P}_f x_k - x_k) \to 0$. On the other hand, $\mathrm{P}_f x_k \to \mathrm{P}_f \overline{x}$ and the result follows. ☐

Remark 31.8 Consider Theorem 31.7 and its notation. The assumption that $Z \neq \varnothing$ is implied by requiring that $f + g$ have minimizers, i.e., $S \neq \varnothing$, and that a constraint qualification such as $\operatorname{ri} \operatorname{dom} f \cap \operatorname{ri} \operatorname{dom} g \neq \varnothing$ hold — see also Remark 13.14.

Example 31.9 Let us revisit and adopt the scenario considered in Section 31.1. Set $x_0 := 0$. Then $\mathrm{P}_f(x_0) = -1$ and

$$(\forall k \geqslant 1) \quad x_k = 4 - \frac{1}{2^{k-2}} \ \text{ and } \ \mathrm{P}_f(x_k) = \frac{-1}{2^{k-1}}. \tag{31.10}$$

By Theorem 31.7, $x_k \to 4 \in \operatorname{Fix} T$ and $\mathrm{P}_f(x_k) \to 0 \in S$.

Example 31.10 (convex feasibility revisited) As in Example 28.8, we let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and we assume that $S := \mathbb{R}^n_+ \cap A^{-1}(b) \neq \varnothing$. We model this feasibility problem with $f = \iota_{A^{-1}(b)}$ and $g = \iota_{\mathbb{R}^n_+}$. Then $\mathrm{P}_f(x) = \mathrm{P}_{A^{-1}(b)}(x) = x - A^\dagger(Ax - b)$ and $\mathrm{P}_g = \mathrm{P}_{\mathbb{R}^n_+}(x) = x^+$. With the Douglas–Rachford operator

$$T(x) = x - \mathrm{P}_f(x) + \mathrm{P}_g(2\mathrm{P}_f(x) - x),$$

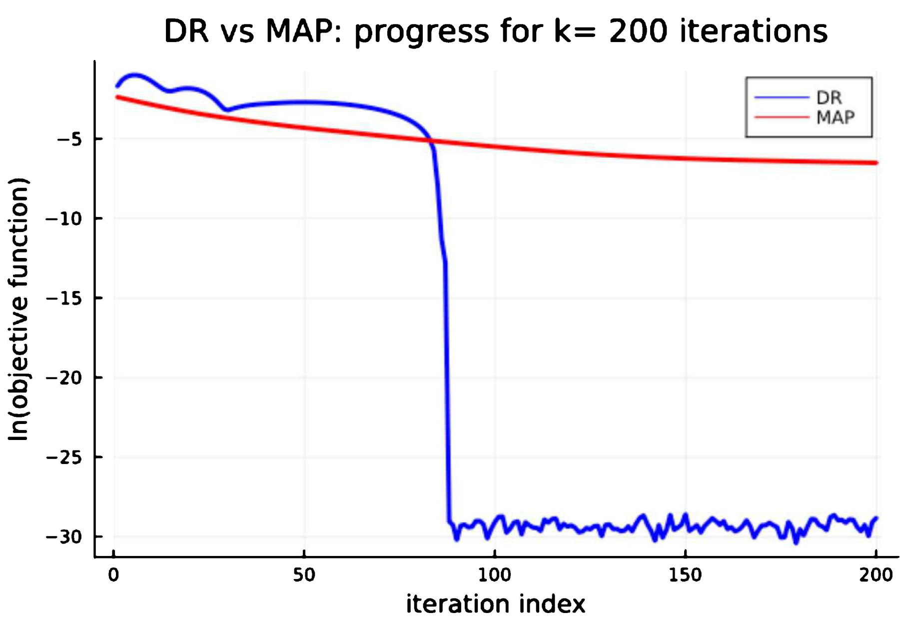
we generate the sequence $(x_k)_{k\in\mathbb{N}} = (T^k x_0)_{k\in\mathbb{N}}$ for which $\mathrm{P}_f(x_k) \to$ some point in S. In the following `Julia` code, we implemented this and compare its performance to MAP from Example 28.8.

In [1]:
```
using Plots;
using LinearAlgebra;
using Random;
import Random:seed!; seed!(1357);
```

In [2]:
```
Porthant(x) = max.(x,0); # projection on orthant
dorthant(x) = norm(x-Porthant(x)); # distance to orthant

m = 400;
d = 402;
A = rand(m,d).-0.5; xsol = rand(d);  b = A*xsol;
Adagger = pinv(A);
Pset(x) = x - Adagger*(A*x-b);
dset(x) = norm(x-Pset(x));

F(x) = max(dorthant(x),dset(x));
MAP(x) = Porthant(Pset(x));
DR(x) = x - Pset(x) + Porthant(2*Pset(x)-x);
allzeros = zeros(d,1); # starting point for DR and MAP
```

In [3]:
```
kcounter = 200;

xMAP = allzeros;
xDR = allzeros;
FMAP = ones(kcounter,1);
FDR = ones(kcounter,1);

k=0;
while k < kcounter
    xMAP = MAP(xMAP); # Method of Alt Proj update
    xDR = DR(xDR); # DR governing sequence
    k = k + 1;
    FMAP[k]=F(xMAP);
    FDR[k] = F(Porthant(xDR)); # we monitor shadow sequence!
end;

lFMAP = log.(FMAP);
lFDR = log.(FDR);

xx=1:kcounter;yy=[lFDR,lFMAP];
plot(xx,yy,title="DR vs MAP: progress for k= $k iterations",
    xlabel="iteration index",
    ylabel="ln(objective function)",
    label=["DR" "MAP"],color=["blue" "red"],
    linestyle=[:solid :solid], lw=[3 3])
```

Out[3]:



(In passing, we note that the convergence of Douglas–Rachford is *finite* if $\mathbb{R}^n_{++} \cap A^{-1}(b) \neq \varnothing$.)

Remark 31.11 (ADMM) A related algorithm is the so-called *Alternating Direction Method of Multipliers (ADMM)*, which aims to solve the problem

$$\underset{\substack{x\in X, y\in Y,\\ Ax+By=c}}{\text{minimize}} \; f(x)+g(y), \tag{31.11}$$

where $A\colon X \to Z$ and $B\colon Y \to Z$ are linear and $c \in Z$. Given a starting point $(x_0, y_0, z_0) \in X \times Y \times Z$, $\rho > 0$, and a current iterate $(x_k, y_k, z_k) \in X \times Y \times Z$, the next iterate of ADMM is obtained by updating

$$\begin{aligned}
x_{k+1} &:= \underset{x\in X}{\operatorname{Argmin}}\, f(x) + \langle Ax + By_k - c, z_k\rangle + \tfrac{\rho}{2}\|Ax + By_k - c\|^2,\\
y_{k+1} &:= \underset{y\in Y}{\operatorname{Argmin}}\, g(y) + \langle Ax_{k+1} + By - c, z_k\rangle + \tfrac{\rho}{2}\|Ax_{k+1} + By - c\|^2,\\
z_{k+1} &:= z_k + \rho\big(Ax_{k+1} + By_{k+1} - c\big).
\end{aligned}$$

Now assume $\rho = 1$, $Z = Y = X$, $A = \mathrm{Id}$, $B = -\mathrm{Id}$, and $c = 0$. Then the problem (31.11) asks to find a minimizer of $f + g$, and the update simplifies to

$$\begin{aligned}
x_{k+1} &= \underset{x\in X}{\operatorname{Argmin}}\, f(x) + \langle x - y_k, z_k\rangle + \tfrac{1}{2}\|x - y_k\|^2,\\
y_{k+1} &= \underset{y\in X}{\operatorname{Argmin}}\, g(y) + \langle x_{k+1} - y, z_k\rangle + \tfrac{1}{2}\|x_{k+1} - y\|^2,\\
z_{k+1} &= z_k + x_{k+1} - y_{k+1},
\end{aligned}$$

and further to

$$\begin{aligned}
x_{k+1} &= \underset{x\in X}{\operatorname{Argmin}}\, f(x) + \tfrac{1}{2}\|x - y_k + z_k\|^2,\\
y_{k+1} &= \underset{y\in X}{\operatorname{Argmin}}\, g(y) + \tfrac{1}{2}\|x_{k+1} - y + z_k\|^2,\\
z_{k+1} &= z_k + x_{k+1} - y_{k+1}.
\end{aligned}$$

In terms of proximal mappings, this means

$$\begin{aligned}
x_{k+1} &= \mathrm{P}_f(y_k - z_k),\\
y_{k+1} &= \mathrm{P}_g(x_{k+1} + z_k),\\
z_{k+1} &= z_k + x_{k+1} - y_{k+1}.
\end{aligned}$$

Now set $w_k = x_k + z_{k-1}$ for all $k \geqslant 1$. Then $y_k = \mathrm{P}_g(x_k + z_{k-1}) = \mathrm{P}_g(w_k)$. Next, $y_k - z_k = y_k - (z_{k-1} + x_k - y_k) = 2y_k - w_k = \mathrm{R}_g(w_k)$ and so $x_{k+1} = \mathrm{P}_f\mathrm{R}_g(w_k)$. Finally, $\mathrm{T}_{\mathrm{DR}}(g, f)(w_k) = w_k - \mathrm{P}_g(w_k) + \mathrm{P}_f\mathrm{R}_g(w_k) = w_k - y_k + x_{k+1} = -y_k + x_{k+1} = x_{k+1} + z_k = x_{k+1} + z_k = w_{k+1}$.

It follows that $(w_k)_{k\in\mathbb{N}}$ is the sequence of iterates of the Douglas–Rachford operator for the pair (g, f)! Consequently, the shadow sequence $(\mathrm{P}_g(w_k))_{k\in\mathbb{N}} = (y_k)_{k\in\mathbb{N}}$ converges to a minimizer of $f + g$ and so does the sequence $(\mathrm{P}_f\mathrm{R}_g(w_k))_{k\in\mathbb{N}} = (x_k)_{k\in\mathbb{N}}$. This shows that the Douglas–Rachford algorithm is a special case of ADMM. Remarkably and conversely, it is possible to study ADMM as a special case of the Douglas–Rachford algorithm; see, e.g., [41, Section 3.1].

31.5 ▪ Douglas–Rachford and Fenchel Duality

The Douglas–Rachford operator has attractive properties with respect to duality. Recall that the set S defined in (31.1b) consists of the solutions of the primal problem of the ordered pair

(f, g) or (g, f) which asks to

$$\underset{x \in X}{\text{minimize}} \quad f(x) + g(x). \tag{31.12}$$

We discussed Fenchel duality in Chapter 20. It will be convenient to define the notation

$$h^\vee(y) := h(-y);$$

this allows us to write (see (20.3)) the Fenchel dual of (g, f) as

$$\underset{y \in X}{\text{minimize}} \quad f^*(y) + g^{*\vee}(y), \tag{31.13}$$

with associated set of minimizers

$$S^* = \operatorname{Argmin}(f^* + g^{*\vee}). \tag{31.14}$$

Remark 31.12 We chose to work with (g, f) instead of (f, g) because this leads to the beautiful Proposition 31.14 below.

Let us denote the counterpart of the set Z (see (31.5)) for the dual problem by

$$Z^* := \operatorname{zer}\big(\partial f^* + \partial(g^{*\vee})\big) = \big\{k \in X \;\big|\; 0 \in \partial f^*(k) - \partial g^*(-k)\big\}. \tag{31.15}$$

Next, the Douglas–Rachford operator for the dual problem (31.13) is

$$\begin{aligned} \mathrm{T}_{\mathrm{DR}}(f^*, g^{*\vee}) &= \mathrm{Id} - \mathrm{P}_{f^*} + \mathrm{Prox}_{g^{*\vee}}(2\mathrm{P}_{f^*} - \mathrm{Id}) && (31.16a) \\ &= \tfrac{1}{2}\big(\mathrm{Id} + \mathrm{R}_{g^{*\vee}}\mathrm{R}_{f^*}\big), && (31.16b) \end{aligned}$$

where we used Lemma 31.6(i) to get (31.16b). We will need the following result to simplify $\mathrm{T}_{\mathrm{DR}}(f^*, g^{*\vee})$ further:

Lemma 31.13 *The following hold:*

(i) $\mathrm{P}_{f^\vee} = -\mathrm{P}_f \circ (-\mathrm{Id})$.

(ii) $\mathrm{R}_{f^*} = -\mathrm{R}_f$.

(iii) $\mathrm{R}_{g^{*\vee}} = \mathrm{R}_g \circ (-\mathrm{Id})$.

We now obtain the following beautiful duality result:

Proposition 31.14 (self-duality of the Douglas–Rachford operator) *We have*

$$\mathrm{R}_{g^{*\vee}}\mathrm{R}_{f^*} = \mathrm{R}_g\mathrm{R}_f \;\; \textit{and} \;\; \mathrm{T}_{\mathrm{DR}}(f^*, g^{*\vee}) = \mathrm{T}_{\mathrm{DR}}(f, g). \tag{31.17}$$

Proof. Using Lemma 31.13(iii)&(ii), we have $\mathrm{R}_{g^{*\vee}}\mathrm{R}_{f^*} = (\mathrm{R}_g \circ (-\mathrm{Id}))(-\mathrm{R}_f) = \mathrm{R}_g\mathrm{R}_f$. In turn, combining this with Lemma 31.6(i), we obtain $\mathrm{T}_{\mathrm{DR}}(f^*, g^{*\vee}) = \mathrm{T}_{\mathrm{DR}}(f, g)$. ☐

We end this chapter with the following sharpened primal-dual version of Theorem 31.7.

Theorem 31.15 (Douglas–Rachford Algorithm is primal-dual!) *Suppose that $Z \neq \varnothing$, or equivalently that $Z^* \neq \varnothing$. Let $x_0 \in X$ and generate the sequence $(x_k)_{k \in \mathbb{N}}$ via*

$$x_{k+1} := Tx_k = x_k - \mathrm{P}_f x_k + \mathrm{P}_g(2\mathrm{P}_f x_k - x_k). \tag{31.18}$$

Then there exists $\overline{x} \in X$ such that the following hold:

(i) $x_k \to \overline{x} \in \operatorname{Fix} T$.

(ii) $\mathrm{P}_f x_k \to \mathrm{P}_f \overline{x} \in Z = S$.

(iii) $x_k - \mathrm{P}_f x_k \to \overline{x} - \mathrm{P}_f \overline{x} \in Z^* = S^*$.

Proof. We saw in Proposition 31.14 that $T = \mathrm{T}_{\mathrm{DR}}(f, g) = \mathrm{T}_{\mathrm{DR}}(f^*, g^{*\vee})$. Hence Lemma 31.6(iv) and Theorem 26.1 yield

$$Z = \mathrm{P}_f(\operatorname{Fix} T) \quad \text{and} \quad Z^* = \mathrm{P}_{f^*}(\operatorname{Fix} T) = (\mathrm{Id} - \mathrm{P}_f)(\operatorname{Fix} T).$$

Hence $Z \neq \varnothing \Leftrightarrow \operatorname{Fix} T \neq \varnothing \Leftrightarrow Z^* \neq \varnothing$. Now items (i) and (ii) were already proved in Theorem 31.7. Recalling that $\mathrm{P}_{f^*} = \mathrm{Id} - \mathrm{P}_f$, it is now clear that (ii) — when applied to the dual problem — yields (iii). ☐

Exercises

Exercise 31.1 Prove (31.2).

Exercise 31.2 Prove (31.3).

Exercise 31.3 Prove Proposition 31.1.

Exercise 31.4 Prove the inclusion "$\supseteq$" in (31.6).

Exercise 31.5 Provide the details for (31.10).

Exercise 31.6 Provide the details for Lemma 31.13.

Exercise 31.7 Let f and g be proper lower semicontinuous convex functions on X. Recall that $\mathrm{R}_f = 2\mathrm{P}_f - \mathrm{Id}$ and $\mathrm{R}_g = 2\mathrm{P}_g - \mathrm{Id}$. Let $\alpha \in \left]0, 1\right[$ and define $\widetilde{T} = (1-\alpha)\mathrm{Id} + \alpha \mathrm{R}_g \mathrm{R}_f$. Observe that when $\alpha = \frac{1}{2}$ we recover the Douglas–Rachford operator.

(i) Prove that $\widetilde{T}$ is α-averaged.

(ii) Prove that $\operatorname{Fix} \widetilde{T} = \operatorname{Fix} \mathrm{R}_g \mathrm{R}_f$.

(iii) Suppose that $\operatorname{Argmin}(f+g) \neq \varnothing$ and that $\operatorname{ri} \operatorname{dom} f \cap \operatorname{ri} \operatorname{dom} g \neq \varnothing$. This ensures that $\operatorname{Fix} \widetilde{T} \neq \varnothing$ (you do not need to prove this). Let $x_0 \in X$, and set $(\forall k \in \mathbb{N})\ x_{k+1} := \widetilde{T}^{k+1} x_0$. Prove that $(x_k)_{k \in \mathbb{N}}$ converges to a point in $\operatorname{Fix} \widetilde{T}$.

(iv) Prove that $(\mathrm{P}_f(x_k))_{k \in \mathbb{N}}$ converges to a minimizer of $f + g$.

Chapter 32

Peaceman–Rachford Algorithm

In this chapter,

$$
\begin{aligned}
&f \text{ and } g \text{ are convex, lower semicontinuous, and proper on } X, && \text{(32.1a)}\\
&S := \operatorname{Argmin}(f+g),\ \ \mu := \inf(f+g)(X). && \text{(32.1b)}
\end{aligned}
$$

32.1 ▪ Alternating Reflectants

We've seen in Section 31.1 that iterating the proximal mappings P_f and P_g does not lead to a point in S. In view of (31.6), it is tempting to iterate the reflectants $\mathrm{R}_f = 2\mathrm{P}_f - \mathrm{Id}$ and $\mathrm{R}_g = 2\mathrm{P}_g - \mathrm{Id}$.

Let's look at the following special case: Assume that

$$f \equiv 0 \ \text{ and } \ g = \iota_{\{0\}}.$$

Then $S = \{0\}$,

$$\mathrm{R}_f = \mathrm{Id} \ \text{ and } \ \mathrm{R}_g = -\mathrm{Id}; \tag{32.2}$$

hence, $\mathrm{R}_g\mathrm{R}_f = -\mathrm{Id}$. Iterating the reflectants in the form $((\mathrm{R}_g\mathrm{R}_f)^k x_0)_{k\in\mathbb{N}}$ won't lead to a convergent sequence — unless $x_0 = 0$, the only element in S. Our hopes are dashed even if we consider the associated shadow sequence $(\mathrm{P}_f(\mathrm{R}_g\mathrm{R}_f)^k x_0)_{k\in\mathbb{N}}$ because this is the same sequence as $((\mathrm{R}_g\mathrm{R}_f)^k x_0)_{k\in\mathbb{N}}$.

Something very interesting happens when we switch the roles of f and g: assume now that

$$f = \iota_{\{0\}} \ \text{ and } \ g \equiv 0.$$

Then again $S = \{0\}$, but this time

$$\mathrm{R}_f = -\mathrm{Id} \ \text{ and } \ \mathrm{R}_g = \mathrm{Id};$$

and again $\mathrm{R}_g\mathrm{R}_f = -\mathrm{Id}$ and the sequence $((\mathrm{R}_g\mathrm{R}_f)^k x_0)_{k\in\mathbb{N}}$ won't be of use unless we started at $x_0 = 0$. However, this time the associated shadow sequence $(\mathrm{P}_f(\mathrm{R}_g\mathrm{R}_f)^k x_0)_{k\in\mathbb{N}}$ is just the sequence $(0, 0, \ldots)$ which does converge to 0, the unique point in S. It turns out that this is not a coincidence — it is guaranteed because f is strongly convex! We shall derive the corresponding general convergence result in the next section.

32.2 ▪ The Peaceman–Rachford Algorithm

Proposition 32.1 *Assume that f is α-strongly convex for some $\alpha > 0$. Then for every x, y in X, we have*

$$\langle x - y, \mathrm{P}_f(x) - \mathrm{P}_f(y)\rangle \geqslant (1+\alpha)\|\mathrm{P}_f(x) - \mathrm{P}_f(y)\|^2. \tag{32.3}$$

Proof. Set $p := \mathrm{P}_f(x)$ and $q := \mathrm{P}_f(y)$. It follows from Theorem 25.3 that $x - p \in \partial f(p)$ and $y - q \in \partial f(q)$. Using Fact 24.4 we have $\langle p - q, (x-p) - (y-q)\rangle \geqslant \alpha\|p-q\|^2$; equivalently $\langle p - q, x - y\rangle - \|p-q\|^2 \geqslant \alpha\|p-q\|^2$. Rearranging yields the desired result. □

Definition 32.2 The *Peaceman–Rachford operator* of (f, g) is defined by

$$\boxed{T := \mathrm{T}_{\mathrm{PR}}(f,g) := \mathrm{R}_g\mathrm{R}_f. \tag{32.4}}$$

Theorem 32.3 (Peaceman–Rachford Algorithm) *Suppose that $Z \neq \varnothing$ and that f is α-strongly convex, where $\alpha > 0$. Then $Z = S$ is a singleton, say $S = \{s\}$. Now let $x_0 \in X$ and generate the sequence $(x_k)_{k\in\mathbb{N}}$ via*

$$x_{k+1} := T(x_k). \tag{32.5}$$

Then $\mathrm{P}_f(x_k) \to s$.

Proof. First, by Theorem 13.13(ii), $S = Z$. Clearly, $f + g$ is also α-strongly convex; hence, by Theorem 24.7(ii), S is a singleton, say $S = \{s\}$. Because $Z \neq \varnothing$, we have $\mathrm{Fix}(\mathrm{R}_g\mathrm{R}_f) \neq \varnothing$ and also $Z = \mathrm{P}_f(\mathrm{Fix}(\mathrm{R}_g\mathrm{R}_f))$ by (31.6). Let $x \in \mathrm{Fix}(\mathrm{R}_g\mathrm{R}_f)$, i.e., $x = Tx$. Using the nonexpansiveness of R_g (Proposition 31.4) as well as Proposition 32.1, we have for all $k \in \mathbb{N}$,

$$\begin{aligned}
\|x_{k+1} - x\|^2 &= \|Tx_k - Tx\|^2 \\
&= \|\mathrm{R}_g\mathrm{R}_f x_k - \mathrm{R}_g\mathrm{R}_f x\|^2 \\
&\leqslant \|\mathrm{R}_f x_k - \mathrm{R}_f x\|^2 \\
&= \|2\mathrm{P}_f x_k - x_k - (2\mathrm{P}_f(x) - x)\|^2 \\
&= \|2(\mathrm{P}_f x_k - \mathrm{P}_f x) - (x_k - x)\|^2 \\
&= \|x_k - x\|^2 - 4\langle \mathrm{P}_f x_k - \mathrm{P}_f x, x_k - x\rangle + 4\|\mathrm{P}_f x_k - \mathrm{P}_f x\|^2 \\
&\leqslant \|x_k - x\|^2 - 4\alpha\|\mathrm{P}_f x_k - \mathrm{P}_f x\|^2.
\end{aligned}$$

Rearranging and telescoping, we obtain

$$4\alpha\sum_{k\in\mathbb{N}}\|\mathrm{P}_f x_k - \mathrm{P}_f x\|^2 \leqslant \|x_0 - x\|^2 < +\infty. \tag{32.6}$$

Hence $\mathrm{P}_f x_k \to \mathrm{P}_f x$. On the other hand, $\mathrm{P}_f x \in \mathrm{P}_f(\mathrm{Fix}\, T) = Z = S = \{s\}$. Altogether, hence $\mathrm{P}_f x_k \to s$. □

Example 32.4 (best approximation via Peaceman–Rachford) Suppose that A, B are closed convex subsets of X such that $A \cap B \neq \varnothing$. Let $z \in X$. Our problem is to find

$$P_{A\cap B}(z).$$

We model this by assuming that

$$f(x) := \alpha\tfrac{1}{2}\|x - z\|^2 + \iota_A(x) \quad\text{and}\quad g(x) := \iota_B(x).$$

Note that f is α-strongly convex and that $S = \operatorname{Argmin}(f+g) = \{P_{A\cap B}(z)\}$. Moreover,

$$\operatorname{Prox}_f(y) = P_A\Big(\frac{y+\alpha z}{1+\alpha}\Big) \tag{32.7a}$$

and

$$T(y) = R_g R_f(y) = y + 2P_B\bigg(2P_A\Big(\frac{y+\alpha z}{1+\alpha}\Big) - y\bigg) - 2P_A\Big(\frac{y+\alpha z}{1+\alpha}\Big). \tag{32.7b}$$

Given $x_0 \in X$ and $(x_k)_{k\in\mathbb{N}}$ generated by $x_{k+1} = T(x_k)$, it follows from Theorem 32.3 that $\operatorname{Prox}_f(x_k) \to P_{A\cap B}(z)$.

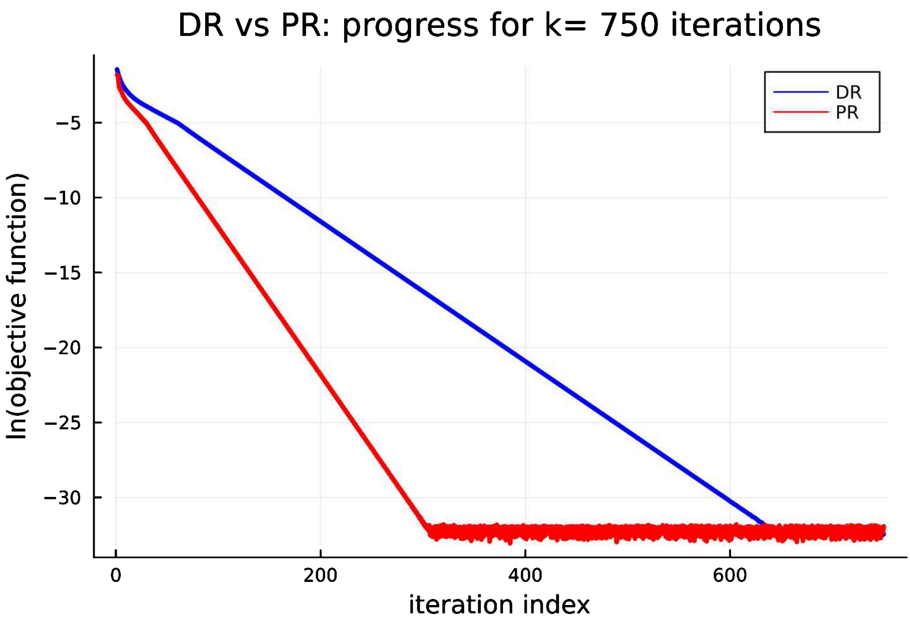
Below is a numerical illustration in `Julia`.

In [1]:
```
using Plots; using LinearAlgebra;
using Random; import Random:seed!; seed!(1357);
```

In [2]:
```
m = 100; d = 120;  A = rand(m,d).-0.5;
allzeros = zeros(d,1);
z = allzeros; # point we wish to project
xsol = rand(d);  b = A*xsol;
Adagger = pinv(A); Pset(x) = x - Adagger*(A*x-b);

P1(x,z) = Pset((x+z)/2); # P_f
R1(x,z) = 2*P1(x,z)-x; # R_f
d1(x) = norm(x-Pset(x)); # distance to set

P2(x) = max.(x,0); # projection on orthant, i.e., P_g
R2(x) = 2*P2(x)-x; # R_g
d2(x) = norm(x-P2(x)); # distance to orthant

DR(x,z) = x - P1(x,z) + P2(R1(x,z)); # Douglas-Rachford operator
PR(x,z) = R2(R1(x,z)); # Peaceman-Rachford operator
```

In [3]:
```
kcounter = 750;

F(x) = max(d1(x),d2(x));
xDR = allzeros;
xPR = allzeros;
FDR = ones(kcounter,1);
FPR = ones(kcounter,1);

k=0;
while k < kcounter
    xDR = DR(xDR,z); # DR governing sequence
    xPR = PR(xPR,z); # PR governing sequence
    k = k + 1;
    FDR[k] = F(P1(xDR,z)); # monitor shadows
    FPR[k] = F(P1(xPR,z)); # monitor shadows
end;

lFDR = log.(FDR);
lFPR = log.(FPR);
xx=1:kcounter;yy=[lFDR,lFPR];

plot(xx,yy,title="DR vs PR: progress for k= $k iterations",
    xlabel="iteration index",
    ylabel="ln(objective function)",
    label=["DR" "PR"],color=["blue" "red"],
    linestyle=[:solid :solid], lw=[3 3])
```

Out[3]:



Note that Peaceman–Rachford is faster than Douglas–Rachford.

Exercises

Exercise 32.1 Provide the details for (32.2).

Exercise 32.2 Provide examples of (f, g) where (i) $\mathrm{Fix}(\mathrm{R}_g\mathrm{R}_f) = \{0\}$ or (ii) $\mathrm{Fix}(\mathrm{R}_g\mathrm{R}_f) = X$.

Exercise 32.3 Consider Theorem 32.3. Give an example where $(x_k)_{k\in\mathbb{N}}$ fails to converge to a point in $\mathrm{Fix}(\mathrm{R}_g\mathrm{R}_f)$. Must the cluster points of $(x_{k\in\mathbb{N}})$ lie in $\mathrm{Fix}\,T$?

Exercise 32.4 Provide the details for (32.7).

Chapter 33

The Product Space Trick

In the past chapters, we focused on minimizing the sum of *two* functions — thus, a natural question arises: How do we deal with a sum of more than two functions? In this chapter, we focus on a straightforward approach to dealing with this situation as well as with a more recent one. We assume that

$$f_1, \ldots, f_m \text{ are convex, lower semicontinuous, and proper on } X, \tag{33.1a}$$
$$S := \operatorname{Argmin}(f_1 + \cdots + f_m). \tag{33.1b}$$

33.1 ▪ The Standard Product Space Approach

In this section, we set

$$\mathbf{X} := X^m, \tag{33.2a}$$
$$\mathbf{D} := \big\{(x, \ldots, x) \in \mathbf{X} \,\big|\, x \in X\big\}, \tag{33.2b}$$
$$\mathbf{f}(\mathbf{x}) := \mathbf{f}(x_1, \ldots, x_m) = f_1(x_1) + \cdots + f_m(x_m), \tag{33.2c}$$
$$\mathbf{S} := \operatorname{Argmin}(\mathbf{f} + \iota_{\mathbf{D}}). \tag{33.2d}$$

Recall that the inner product in $\mathbf{X}$ is defined by $\langle \mathbf{x}, \mathbf{y} \rangle := \langle x_1, y_1 \rangle + \cdots + \langle x_m, y_m \rangle$ if $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$.

Proposition 33.1 *We have the following:*

(i) *If $x \in S$, then the m-fold copy $(x, \ldots, x)$ lies in* $\mathbf{S}$.

(ii) *If $\mathbf{x} \in \mathbf{S}$, then $\mathbf{x} = (x, \ldots, x)$ for some $x \in S$.*

The last result reduces the problem of finding minimizers of the sum $f_1 + \cdots + f_m$ to that of finding minimizers of $\mathbf{f} + \iota_{\mathbf{D}}$. The latter problem can certainly be tackled using, for instance, the Douglas–Rachford algorithm, *provided we can compute the proximal mappings* of $\mathbf{f}$ and $\iota_{\mathbf{D}}$. Fortunately, these can be found fairly easily, provided we have access to each P_{f_i}:

Proposition 33.2 *Let $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbf{X}$. Then*

$$\mathrm{P}_{\mathbf{f}}(\mathbf{x}) = \big(\mathrm{P}_{f_1}(x_1), \ldots, \mathrm{P}_{f_m}(x_m)\big) \tag{33.3}$$

and

$$\mathrm{P}_{\iota_{\mathbf{D}}}(\mathbf{x}) = \mathrm{P}_{\mathbf{D}}(\mathbf{x}) = (y, \dots, y), \quad \textit{where } y := \tfrac{1}{m}\big(x_1 + \cdots + x_m\big). \tag{33.4}$$

Proof. The formula (33.3) is a consequence of Proposition 25.11. The projection of $\mathbf{x}$ onto $\mathbf{D}$ is the point $\mathbf{z} = (z, \dots, z)$, where z minimizes the function $w \mapsto \sum_{i=1}^{m} \|w - x_i\|^2$. This last function is convex and differentiable; finding its critical point leads directly to (33.4). $\square$

Armed with Proposition 33.2, one can apply various algorithms in $\mathbf{X}$ to find a point in S. Let us record the Douglas–Rachford algorithm applied to the pair $(\iota_{\mathbf{D}}, \mathbf{f})$, but written in a form that does not mention $\mathbf{X}$ explicitly:

Remark 33.3 (Douglas–Rachford, standard approach) Pick starting points $x_{0,1}, \dots, x_{0,m}$ in X, and set $\bar{x}_0 := \frac{1}{m}\sum_{i=1}^{m} x_{0,i}$. For every $k \in \mathbb{N}$ and $i \in \{1, \dots, m\}$, update via

$$x_{k+1,i} := x_{k,i} - \bar{x}_k + \mathrm{P}_{f_i}(2\bar{x}_k - x_{k,i}), \tag{33.5a}$$

$$\bar{x}_{k+1} := \tfrac{1}{m}\textstyle\sum_{i=1}^{m} x_{k+1,i}. \tag{33.5b}$$

Provided suitable assumptions hold, the sequence $(\bar{x}_k)_{k\in\mathbb{N}}$ will converge to a point in S by Theorem 31.7(ii).

33.2 ▪ The Campoy–Kruger Approach

In this section, we set

$$\mathbf{X} := X^{m-1}, \tag{33.6a}$$

$$\mathbf{D} := \big\{(x, \dots, x) \in \mathbf{X} \;\big|\; x \in X\big\}, \tag{33.6b}$$

$$\mathbf{f}(\mathbf{x}) := \mathbf{f}(x_1, \dots, x_{m-1}) = f_1(x_1) + \cdots + f_{m-1}(x_{m-1}), \tag{33.6c}$$

$$\mathbf{g}(\mathbf{x}) := \mathbf{g}(x_1, \dots, x_{m-1}) = \tfrac{1}{m-1}\big(f_m(x_1) + \cdots + f_m(x_{m-1})\big), \tag{33.6d}$$

$$\mathbf{S} := \operatorname{Argmin}(\mathbf{f} + \mathbf{g} + \iota_{\mathbf{D}}). \tag{33.6e}$$

Note that these objects reside in X^{m-1}, unlike the standard approach, which operates in X^m.

Proposition 33.4 *We have the following:*

(i) *If $x \in S$, then the $(m-1)$-fold copy $(x, \dots, x)$ lies in $\mathbf{S}$.*

(ii) *If $\mathbf{x} \in \mathbf{S}$, then $\mathbf{x} = (x, \dots, x)$ for some $x \in S$.*

Proposition 33.5 *Let $\mathbf{x} = (x_1, \dots, x_{m-1}) \in \mathbf{X}$. Then*

$$\mathrm{P}_{\mathbf{f}}(\mathbf{x}) = \big(\mathrm{P}_{f_1}(x_1), \dots, \mathrm{P}_{f_{m-1}}(x_{m-1})\big) \tag{33.7}$$

and

$$\mathrm{P}_{\mathbf{g}+\iota_{\mathbf{D}}}(\mathbf{x}) = (y, \dots, y), \quad \textit{where } y := \mathrm{P}_{\frac{1}{m-1}f_m}\big(\tfrac{1}{m-1}(x_1 + \cdots + x_{m-1})\big). \tag{33.8}$$

Proof. Again, the formula (33.7) is a consequence of Proposition 25.11.

Denote the embedding operator that sends $y \in X$ to its $(m-1)$-fold copy $(y, \dots, y) \in \mathbf{X}$ by E. Because the Argmin does not change when we add a constant to its argument or multiply by a positive number, we deduce that

$$\begin{aligned}
\mathrm{P}_{\mathbf{g}+\iota_{\mathbf{D}}}(\mathbf{x}) &= \operatorname*{Argmin}_{\mathbf{y}\in\mathbf{X}} \big(\mathbf{g}(\mathbf{y}) + \iota_{\mathbf{D}}(\mathbf{y}) + \tfrac{1}{2}\|\mathbf{y}-\mathbf{x}\|^2\big)\\
&= E \operatorname*{Argmin}_{y\in X} \big(f_m(y) + \tfrac{1}{2}\textstyle\sum_{i=1}^{m-1}\|y-x_i\|^2\big)\\
&= E \operatorname*{Argmin}_{y\in X} \big(f_m(y) + \tfrac{1}{2}\textstyle\sum_{i=1}^{m-1}(\|y\|^2 - 2\langle y, x_i\rangle)\big)\\
&= E \operatorname*{Argmin}_{y\in X} \big(\tfrac{1}{m-1}f_m(y) + \tfrac{1}{2}\|y\|^2 - \tfrac{1}{m-1}\textstyle\sum_{i=1}^{m-1}\langle y, x_i\rangle\big)\\
&= E \operatorname*{Argmin}_{y\in X} \big(\tfrac{1}{m-1}f_m(y) + \tfrac{1}{2}\big\|y - \tfrac{1}{m-1}\textstyle\sum_{i=1}^{m-1}x_i\big\|^2\big)\\
&= E\,\mathrm{P}_{\frac{1}{m-1}f_m}\big(\tfrac{1}{m-1}(x_1+\cdots+x_{m-1})\big),
\end{aligned}$$

and this verifies (33.8). □

Armed with Proposition 33.5, one can apply various algorithms in $\mathbf{X}$ to find a point in S. Let us record the Douglas–Rachford algorithm applied to the pair $(\mathbf{g}+\iota_{\mathbf{D}}, \mathbf{f})$, but written in a form that does not mention $\mathbf{X}$ explicitly:

Remark 33.6 (Douglas–Rachford, Campoy–Kruger approach) Pick points $x_{0,1},\ldots,x_{0,m-1}$ in X, and set $\bar{x}_0 := \mathrm{P}_{\frac{1}{m-1}f_m}\big(\frac{1}{m-1}\sum_{i=1}^{m-1}x_{0,i}\big)$. For every $k\in\mathbb{N}$ and $i\in\{1,\ldots,m-1\}$, update

$$x_{k+1,i} := x_{k,i} - \bar{x}_k + \mathrm{P}_{f_i}(2\bar{x}_k - x_{k,i}), \tag{33.9a}$$

$$\bar{x}_{k+1} := \mathrm{P}_{\frac{1}{m-1}f_m}\big(\tfrac{1}{m-1}\textstyle\sum_{i=1}^{m-1}x_{k+1,i}\big). \tag{33.9b}$$

Provided suitable assumptions hold, the sequence $(\bar{x}_k)_{k\in\mathbb{N}}$ will converge to a point in S by Theorem 31.7(ii).

Exercises

Exercise 33.1 Provide the details for Proposition 33.1.

Exercise 33.2 Provide the details as well as some suitable assumptions for Remark 33.3.

Exercise 33.3 Provide the details for Proposition 33.4.

Exercise 33.4 Provide the details as well as some suitable assumptions for Remark 33.6.

Exercise 33.5 Suppose that $m=2$. Explain how Remark 33.6 reduces to the Douglas–Rachford algorithm applied to the pair (f_2, f_1).

Bibliographical Pointers

The following nice books aim at a similar audience while emphasizing somewhat different complementary topics: [1], [4], [34], and [37].

Congratulations on completing the book in your hand! If you wish to move on to more advanced topics but need an introduction to infinite-dimensional spaces, we recommend [21], [25], and [29] for background reading.

Ready to dive into advanced topics? In addition to [3], [5], and [39], which served as guides to preparing this book, we recommend [7], [9], [10], [11], [12], [13], [14], [15], [17], [23], [24], [30], [31], [32], [33], [36], [38], [40], [41], and [42].

We conclude by providing selected additional references for each chapter.

Chapter 1: This material is standard; see also [5] and [25]. Chapter 2, Chapter 3, and Chapter 4: [39]. Chapter 5: The material is standard; see also [3]. Chapter 6 and Chapter 7: [39]. Chapter 8: The material is fairly standard; see also [3] and [5]. Chapter 9: [39]. Chapter 10: [5] and [39]. Chapter 11: [39]; see also [3] and [34]. Chapter 12 [39]; see also [3] and [5]. Chapter 13: See [39], [3], and [5]. For Section 13.3, see Chapter 14: [3], [30], and [39]. Chapter 15, Chapter 16, and Chapter 17: [5] and [7]. Chapter 18, Chapter 19, Chapter 20, and Chapter 21: [39]. Chapter 22: [2], [3], [18], and [20]. Chapter 23, Chapter 24, and Chapter 25: [3] and [5]. Chapter 26 and Chapter 27: [3], [5], and [35]. Chapter 28: [5] and [38]. Chapter 29: [3], [5], and [19]. Chapter 30: [5] and [6]. Chapter 31 and Chapter 32: [3], [22], [28], and [41]. Chapter 33: [16], [26], and [27].

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