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ON THE DIFFERENTIAL IDENTITIES OF AN AFFINITY

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1. Some time ago I communicated a set of *differential identities*[1] engendered by the mere existence of any scalar density that depends only on the Einstein tensor of an *affine connection*. The derivation followed closely the well-known pattern of the *metrical* case. Indeed the Euler equations of the scalar density were admitted from the outset; the components of the affinity were regarded as those functions of the (secondary) g_{ik} which they become by the Euler equations, and the variation that produces the identities referred only to the g_{ik}. In the following I shall indicate a much simpler and more direct way of establishing identities from a scalar density that depends only on the components of an affinity and its first[2] derivatives. One does not lean on the structural details of the metrical case, but only transfers the general idea from the components g_{ik} to the Γ^i_{kl}. I believe four of *these* identities to embody the relevant aspect of the conservation laws in an affine theory. We shall call our density $\mathfrak{L}$ (reminding of Lagrange), but we do *not* subject the Γ^i_{kl} to any restrictions.

2. The variation of the integral (taken between invariantly fixed limits; $d\tau = dx_1\, dx_2\, dx_3\, dx_4$):

$$I = \int \mathfrak{L}\left(\Gamma^n_{ik}, \frac{\partial \Gamma^n_{ik}}{\partial x_m}\right) d\tau \tag{1}$$

for *any* variation of the Γ^i_{kl} reads:

$$\delta I = \int \mathfrak{L}^{ik}_n\, \delta\Gamma^n_{ik}\, d\tau + \int \frac{\partial}{\partial x_m}\left(\frac{\partial \mathfrak{L}}{\partial \Gamma^n_{ik,\,m}}\, \delta\Gamma^n_{ik}\right) d\tau, \tag{2}$$

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[1] Proc. Roy. Ir. Ac. 52 (A), p. 1, 1948.

[2] Even this restriction could be dropped; but we do not wish to enlarge upon gratuitous generality.

where the components

$$\mathfrak{L}^{ik}{}_n = \frac{\partial \mathfrak{L}}{\partial \Gamma^n{}_{ik}} - \frac{\partial}{\partial x_m}\left(\frac{\partial \mathfrak{L}}{\partial \Gamma^n{}_{ik,\,m}}\right) \tag{3}$$

form a *tensor density.* Now envisage a coordinate transformation that depends on a parameter λ in such a way that for $\lambda = 0$ it becomes identical. We write it, or rather its inverse,

$$x_l = x_l(x'_r) = x'_l + \lambda \phi_l(x'_r) + \lambda^2 \psi_l(x'_r) + \ldots\ldots \tag{4}$$

It is executed in (1) by replacing $\Gamma^n{}_{ik}$ by

$$\Gamma'^n{}_{ik}(x') = \frac{\partial x'_n}{\partial x_m}\frac{\partial x_s}{\partial x'_i}\frac{\partial x_t}{\partial x'_k}\Gamma^m{}_{st}(x) + \frac{\partial x'_n}{\partial x_m}\frac{\partial^2 x_m}{\partial x'_i\,\partial x'_k}, \tag{5}$$

and $\Gamma^n{}_{ik,\,m}$ by the derivative of the aforestanding quantity with respect to x'_m. This substitution, together with the substitution of

$$d\tau' = dx_1'\,dx_2'\,dx_3'\,dx_4'$$

for $d\tau$, and with the appropriate change of the limits of integration, must, of course, leave the invariant I unchanged anyhow. But now we restrict the transformation (4) by prescribing that at the boundary it shall, for any λ, not only become identical ($x' = x$) but also reduce the transformation (5) of the Γ's to identity. (This simply means some obvious demands on the boundary values of the derivatives intervening in (5)). One consequence of this ruling is that the limits of integration are *not* changed; hence—since the *names* of the integration variables are irrelevant—the only *formal* change in (1) is the appearance of $\Gamma'^n{}_{ik}(x')$ instead of $\Gamma^n{}_{ik}(x')$. For finite λ this is a finite change, still it leaves the integral invariant.

Now we develop (5) with respect to λ, not forgetting the arguments of $\Gamma^m{}_{st}$ on the right. We obtain

$$\Gamma'^n{}_{ik}(x') - \Gamma^n{}_{ik}(x') =$$
$$= \lambda\,(\phi_m \Gamma^n{}_{ik,\,m} - \phi_{n\ l}\,\Gamma^l{}_{ik} + \phi_{r,\,i}\,\Gamma^n{}_{rk} + \phi_{s\ k}\,\Gamma^n{}_{is} + \phi_{n,\,i,\,k}) + 0\,(\lambda^2), \tag{6}$$

all functions to be written with x'. According to our assumption about the transformation, the linear term in λ must itself vanish at the boundary. For small λ it amounts to a small variation like the one contemplated in (2). If we use it there, we get $\delta I = 0$, and the second integral on the right vanishes from Gauss' theorem. So we have

$$0 = \int \mathfrak{L}^{ik}{}_n\,(\phi_m \Gamma^n{}_{ik,\,m} - \phi_{n,\,l}\,\Gamma^l{}_{ik} + \phi_{r,\,i}\,\Gamma^n{}_{rk} + \phi_{s,\,k}\,\Gamma^n{}_{is} + \phi_{n,\,i,\,k})\,d\tau\,. \tag{7}$$

(Everything is now written with the x, without a dash; this is not a further licence, it only means omitting the dashes for simplicity.)

We perform the suggested partial integrations, remembering that the ϕ and their derivatives vanish at the boundary. By a change of dummies in the single terms we give the integrand the form: ϕ_m multiplied by a certain expression; the latter must vanish. With a further simplifying change of dummies this gives:

$$\mathfrak{p}_m \equiv \mathfrak{L}^{ik}{}_n \Gamma^n{}_{ik,m} + (\mathfrak{L}^{lk}{}_{m,k} + \mathfrak{L}^{ik}{}_m \Gamma^l{}_{ik} - \mathfrak{L}^{lk}{}_n \Gamma^n{}_{mk} - \mathfrak{L}^{il}{}_n \Gamma^n{}_{im})_{,l} = 0. \quad (8)$$

These are our four principal identities. We have given a name to their first members, and we also put

$$\mathfrak{p}_m \equiv \mathfrak{p}_m{}^{(1)} + \mathfrak{p}_m{}^{(2)} \quad (9)$$

with

$$\mathfrak{p}_m{}^{(1)} \equiv \mathfrak{L}^{ik}{}_n \Gamma^n{}_{ik,m}. \quad (10)$$

3. We delay the proof that $\mathfrak{p}_m$ has the formal build of a vector density.

We first wish to give $\mathfrak{p}_m$ the form of a "plain divergence," which $\mathfrak{p}_m{}^{(2)}$ already has. In some respect $\mathfrak{p}_m{}^{(2)}$ is less interesting. For it is the plain divergence of something that vanishes if the Γ's are subjected to the field equations which result from taking $\mathfrak{L}$ as the Lagrangian, viz. to $\mathfrak{L}^{ik}{}_n = 0$. We hope that $\mathfrak{p}_m{}^{(1)}$ will even in this case be non-trivial. (But we do *not* posit the aforesaid field equations nor any others in what follows !)

If in (2) we choose

$$\delta\Gamma^n{}_{ik} = \epsilon\Gamma^n{}_{ik,l} \quad (11)$$

where ϵ is a small constant, this obviously amounts to replacing every Γ by the value it has in a neighbouring point with the coordinate $x_l + \epsilon$, the other three x remaining the same. Hence, from (1),

$$\delta I = \epsilon \int \frac{\partial \mathfrak{L}}{\partial x_l} d\tau. \quad (12)$$

If you equate this to what you get from (2), you obtain, dropping ϵ,

$$\int \frac{\partial \mathfrak{L}}{\partial x_l} d\tau = \int \mathfrak{L}^{ik}{}_n \Gamma^n{}_{ik,l}\, d\tau + \int \frac{\partial}{\partial x_m} \left(\frac{\partial \mathfrak{L}}{\partial \Gamma^n{}_{ik,m}} \Gamma^n{}_{ik,l} \right) d\tau. \quad (13)$$

Since this must hold for any domain of integration, the combined integrand must vanish. Therefore

$$\mathfrak{p}_m^{(1)} = \mathfrak{L}^{ik}{}_n \Gamma^n{}_{ik,\,m} = \left(\delta^l_m \mathfrak{L} - \frac{\partial \mathfrak{L}}{\partial \Gamma^n{}_{ik,\,l}} \Gamma^n{}_{ik,\,m}\right)_{,\,l} \tag{14}$$

$$= \mathfrak{t}^l_{m\ l}\ ,$$

where we have put

$$\mathfrak{t}^l_m = \delta^l_m \mathfrak{L} - \frac{\partial \mathfrak{L}}{\partial \Gamma^n{}_{ik,\,l}} \Gamma^n{}_{ik\ \ m}\ , \tag{15}$$

(but this is *not* a tensor). So $\mathfrak{p}_m^{(1)}$, and thus the first members of the identities (8) are reduced to "plain divergences."—The relation (14) can also be obtained directly from (3).

4. It is interesting to compute more explicit expressions for the $\mathfrak{t}$-components in the case when $\mathfrak{L}$ contains its arguments only in the form of the Einstein tensor

$$R_{\mu\nu} = -\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x_\alpha} + \frac{\partial \Gamma^\alpha_{\mu\alpha}}{\partial x_\nu} + \Gamma^{\alpha\nu}_\beta\, \Gamma^\alpha_{\mu\beta} - \Gamma^\beta_{\alpha\beta}\, \Gamma^\alpha_{\mu\nu}\ . \tag{16}$$

Now

$$\frac{\partial R_{\mu\nu}}{\partial \Gamma^n{}_{ik,\,m}} = -\ \delta^i_\mu\, \delta^k_\nu\, \delta^m_n + \delta^m_\nu\, \delta^k_n\, \delta^i_\mu\ . \tag{17}$$

Let us use, for the moment just as a convenient abbreviation,

$$\frac{\partial \mathfrak{L}}{\partial R_{\mu\nu}} = \mathfrak{g}^{\mu\nu} \tag{18}$$

(which *is* a tensor, if $\mathfrak{L}$ depends only on the $R_{\mu\nu}$). Then

$$\frac{\partial \mathfrak{L}}{\partial \Gamma^n{}_{ik,\,l}} = \frac{\partial \mathfrak{L}}{\partial R_{\mu\nu}} \frac{\partial R_{\mu\nu}}{\partial \Gamma^n{}_{ik,\,l}} = -\ \mathfrak{g}^{ik}\, \delta^l_n + \mathfrak{g}^{il}\, \delta^k_n\ . \tag{19}$$

Hence

$$\mathfrak{t}^l_m = \delta^l_m \mathfrak{L} + \mathfrak{g}^{ik}\, \Gamma^l{}_{ik,\,m} - \mathfrak{g}^{il}\, \Gamma^k{}_{ik,\,m}\ . \tag{20}$$

I wish to defer to a separate paper the discussion of special cases, in particular of the case when $\mathfrak{L}$ is the square root of the determinant of the $R_{\mu\nu}$. If one thinks of the Γ's as approaching to the Christoffel symbols in a certain limiting case, it seems at first sight unfamiliar, that (20) should contain second derivatives of the metrical tensor in virtue of its containing first derivatives of the Γ's. But, of course, nothing less can be expected under the purely affine aspect. To see that this aspect is not altogether outlandish, let us form the trace of the t-matrix:

$$\begin{aligned} \mathfrak{t}^l_l &= 4\mathfrak{L} + \mathfrak{g}^{ik}\Gamma^l_{ik,\,l} - \mathfrak{g}^{ik}\Gamma^l_{il,\,k} \\ &= 4\mathfrak{L} + \mathfrak{g}^{ik}(-R_{ik} + \Gamma^\beta_{\alpha k}\Gamma^\alpha_{i\beta} - \Gamma^\beta_{\alpha\beta}\Gamma^\alpha_{ik}) \qquad (21) \\ &= 2\mathfrak{L} + \mathfrak{g}^{ik}(\Gamma^\beta_{\alpha k}\Gamma^\alpha_{i\beta} - \Gamma^\beta_{\alpha\beta}\Gamma^\alpha_{ik}) . \end{aligned}$$

In the second line we have used (16), in the third (18) and the homogeneity *any* $\mathfrak{L}(R_{ik})$ exhibits. The second term on the right is of familiar form.

5. We wish to supply the formal proof that the $\mathfrak{p}_n$ explained in (8) is a vector density merely on account of $\mathfrak{L}^{ik}_n$ being a tensor density and Γ^n_{ik} an affinity. The proof is simplified by introducing two special kinds of invariant derivative, only for the present purpose, viz. first

$$\mathfrak{L}^{ik}_{n|s} = \mathfrak{L}^{ik}_{n,\,s} + \mathfrak{L}^{rk}_n\Gamma^i_{rs} + \mathfrak{L}^{ir}_n\underset{\leftrightarrow}{\Gamma^k_{sr}} - \mathfrak{L}^{ik}_r\Gamma^r_{ns} - \mathfrak{L}^{ik}_n\Gamma^r_{rs} . \qquad (22)$$

The pair of subscripts whose order is abnormal has been indicated by $\leftrightarrow$. Contracting we get

$$\mathfrak{L}^{ik}_{n|k} = \mathfrak{L}^{ik}_{n,\,k} + \mathfrak{L}^{rk}_n\Gamma^i_{rk} - \mathfrak{L}^{ik}_r\Gamma^r_{nk} . \qquad (23)$$

This is a density of the type $\mathfrak{l}^i_n$. For it we define

$$\mathfrak{l}^i_{n\|m} = \mathfrak{l}^i_{n,\,m} + \mathfrak{l}^r_n\underset{\leftrightarrow}{\Gamma^i_{mr}} - \mathfrak{l}^i_r\underset{\leftrightarrow}{\Gamma^r_{mn}} - \mathfrak{l}^i_n\Gamma^r_{rm}$$

(with the same meaning of the arrows). Contracting,

$$\mathfrak{l}_{n\|i} = \mathfrak{l}^i_{n,\,i} - \mathfrak{l}^i_r\Gamma^r_{in} . \qquad (24)$$

All these quaint derivatives *are* tensors since the antisymmetric constituent of an affinity is one. It can now be proved by straightforward computation, that

$$\mathfrak{p}_n = \mathfrak{L}^{ik}_{n\,|\,k\,\|\,i} - \mathfrak{L}^{ik}_m B^m_{ikn} ,$$

where B is the curvature tensor of the Γ-affinity.

6. As stated above (in the lines following equ. (2)) the invariant integral (1) retains its value under the transformation (4) even if the latter is not chosen so as to vanish at the boundary. But, of course, the limits of integration in the variables x' are in this case no longer the same as they were in

the x. The vanishing difference $I - I$ is then, even for the infinitesimal transformation, $\lambda \rightarrow 0$, no longer obtained by inserting the linear terms of (6) for $\delta \Gamma^n_{ik}$ into the second member of (2) (where, of course, the *second* term must now be retained), but a *third* term must be added, representing the contribution from the infinitesimal change of the limits of integration. This additional term is easily made out by a separate, independent, auxiliary consideration, viz. by a change of integration variables back from $x' \rightarrow x$ (not to be confused with a change of frame). The said term results from the functional determinant of the primed with respect to the unprimed variables and is found to be the integral of the plain divergence of $(-\lambda \mathfrak{L} \phi_l)$. One thus obtains

$$0 = \int \mathfrak{L}^{ik}{}_n \delta \Gamma^n_{ik} \, d\tau + \int \frac{\partial}{\partial x_m} \left(\frac{\partial \mathfrak{L}}{\partial \Gamma^n_{ik,\,m}} \delta \Gamma^n_{lk} \right) d\tau - \int \frac{\partial}{\partial x_l} (\mathfrak{L} \phi_l) \, d\tau \qquad (25)$$

where $\delta \Gamma^n_{ik}$ now stands as an abbreviation for the expression multiplying λ in (6), written, as everything else here, in the unprimed variables. The identity (25) is the generalization of (7) for *arbitrary* ϕ_l. (It may be noticed that this whole procedure is strictly analogous to the one W. Pauli in his famous article of 1920, Enzyklopädie der Math. Wissensch. Vol. V. 19. sect. 23, adopted in the *metrical* case. Our present *affine* case, while, naturally richer in details, is simpler in principle, because we need *not* employ a non-invariant "substitute" integrand, as has to be done in General Relativity.)

In performing the partial integrations in the first integral of (25) (cp. the meaning of $\delta \Gamma^n_{ik}$ according to (6), as indicated) we must now *retain* the "integrated parts," which we threw away before because they *can* be turned into boundary integrals. In fact they are now the only thing that interests us, and we may leave out the *other* parts, because they vanish according to our previous consideration, from the identity (8). We write the result with a slight re-adjustment of dummies, so as to make it not *too* difficult for the reader, if he so wishes, to ascertain where the single terms come from, and, on the other hand, to compare with (8):

$$0 = \int \left[\left(\mathfrak{L}^{il}{}_m \phi_{m,\,i} \right)_{,\,l} + \left(- \mathfrak{L}^{lk}{}_{m,\,k} - \mathfrak{L}^{ik}{}_m \Gamma^l_{ik} + \mathfrak{L}^{lk}{}_n \Gamma^n_{mk} + \mathfrak{L}^{il}{}_n \Gamma^n_{im} \right) \phi_m \right]_{,\,l} d\tau$$

$$+ \int \left[- \mathfrak{L} \phi_l + \frac{\partial \mathfrak{L}}{\partial \Gamma^n_{ik,\,l}} \left(\Gamma^n_{ik,\,m} \phi_m - \phi_{n,\,l} \Gamma^l_{ik} + \phi_{r,\,i} \Gamma^n_{rk} + \phi_{s,\,k} \Gamma^n_{is} + \phi_{n,\,i,\,k} \right) \right]_{,\,l} d\tau. \qquad (26)$$

This must hold for *any* ϕ_l and for *any* boundary (the same, of course, for the two integrals which might be written as one). Therefore the integrand must vanish for any ϕ_l. Hence the expressions multiplying the four ϕ's, as well as those multiplying their first, second and third derivatives, must all vanish identically.

From the *undifferentiated* ϕ's we get nothing new, only, in view of (8), the relation (14) over again. The identities that spring from the *second* derivatives of ϕ yield alternative expressions for the even parts of the Hamiltonian derivatives (3). In virtue of (3) these identities amount to simple algebraic relations between the derivatives of $\mathfrak{L}$, and are entirely trivial, if $\mathfrak{L}$ depends only on R_{kl}. The same holds for the identities that spring from the *third* derivatives of ϕ.

Only from the *first* derivatives of ϕ comes valuable new information. If you abbreviate the bracket-expression in (8)—though by itself it is not a tensor—by $\mathfrak{T}^l{}_m$, so that (8), in view of (14), reads

$$(\mathfrak{t}^l{}_m + \mathfrak{T}^l{}_m)_{,\,l} = 0\,, \tag{8'}$$

then the 16 identities flowing from the coefficients of $\phi_{m,\,l}$ read

$$\mathfrak{t}^l{}_m + \mathfrak{T}^l{}_m = \left(\mathfrak{L}^{ls} - \frac{\partial\mathfrak{L}}{\partial\Gamma^m{}_{ik,\,s}}\Gamma^l{}_{ik} + \frac{\partial\mathfrak{L}}{\partial\Gamma^n{}_{lk,\,s}}\Gamma^n{}_{mk} + \frac{\partial\mathfrak{L}}{\partial\Gamma^n{}_{il,\,s}}\Gamma^n{}_{im}\right)_{,\,s}. \tag{27}$$

If $\mathfrak{L}$ depends only on R_{kl}, this becomes, from (19),

$$\mathfrak{t}^l{}_m + \mathfrak{T}^l{}_m = (\mathfrak{L}^{ls}{}_m + \delta^s{}_m\mathfrak{g}^{ik}\Gamma^l{}_{ik} - \mathfrak{g}^{lk}\Gamma^s{}_{mk} - \mathfrak{g}^{il}\Gamma^s{}_{im} + \mathfrak{g}^{ls}\Gamma^k{}_{mk})_{,\,l}\,. \tag{28}$$

We recall that $\mathfrak{T}^l{}_m$ vanishes if the Γ's are subjected to the field equations $\mathfrak{L}^{ls}{}_m = 0$. One is inclined to regard $\mathfrak{t}^l{}_m$ (or perhaps $-\mathfrak{t}^l{}_m$) as the pseudo-energy-tensor of such a Γ-field.