IV.

A NEW EXACT SOLUTION IN NON-LINEAR OPTICS (TWO-WAVE-SYSTEM).

[FROM THE DUBLIN INSTITUTE FOR ADVANCED STUDIES.]

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§ 1. INTRODUCTION.

IN any non-linear theory of light the first task is to find the mutual influence of two[1] plane waves of such further specification (as to wave-form and polarization) as may render the answer simplest. When I tackled this problem last year by methods of approximation, at the outset of my investigations of Born's theory,[2] it escaped me that an exact solution is accessible. As usual, it is simpler than the approximate one. I communicate it here by itself. For all that I know it is only the *second* non-trivial exact solution of any problem in any non-linear electrodynamics (the *first* being the centrally symmetric static solution, "Born's electron").

§ 2. SIMPLIFYING TRANSFORMATION.

If it is at all possible to satisfy Born's theory exactly by the superposition of two plane waves crossing each other under an arbitrary angle, a suitable Lorentz-transformation will make the two waves antiparallel. Hence we can simplify our task by investigating the antiparallel case only, being sure that by this specialisation we lose nothing in generality.

§ 3. COMPLYING WITH THE FIELD-EQUATIONS.

Using the notations introduced in sect. 3 of N.O., the analytic expression of a single plane, *circularly polarized* wave is

$$\begin{aligned} \mathfrak{F} &= C\,\mathfrak{a}\,e^{i(\nu t - \mathfrak{k}\mathfrak{r})} \\ \mathfrak{G} &= i\,A\,\mathfrak{F} \end{aligned} \qquad (1)$$

[1] *One* plane Maxwellian wave is almost of necessity an exact solution. See appendix I.

[2] Proc. Roy Irish Acad. (A) **47**, 77; **48**, 91, 1942. The first of these two papers ("Non-linear Optics") is referred to here as N O. Only its first few pages are needed here.

See (3, 1) N.O. But we have here enhanced the setting by the *real constant* A, which (see (2, 1) N.O.) allows for a "dielectric constant" = permeability $= A^{-1}$. We allow it to be negative, when the wave is called "abnormal." The necessity of including here *abnormal* waves, even though we are not interested in them *per se*, will emerge later.

We recall that for the complex polarization-vector $\mathfrak{a}$

$$\mathfrak{a}^2 = \mathfrak{a}^{*2} = 0, \quad (\mathfrak{a}^* \mathfrak{a}) = 2, \quad (\mathfrak{a}\, \mathfrak{k}) = (\mathfrak{a}^* \mathfrak{k}) = 0. \tag{2}$$

From these relations one easily infers[3] that the cross-product

$$[\mathfrak{k}\, \mathfrak{a}] = - i \varepsilon \,|\, \mathfrak{k} \,|\, \mathfrak{a} \tag{3}$$

with $\varepsilon = \pm 1$ (undecided). To satisfy the *field-equations*

$$\operatorname{curl} \mathfrak{G} + \dot{\mathfrak{F}} = 0, \qquad \operatorname{div} \mathfrak{F} = 0, \tag{4}$$

we insert (1) into (4), and find the only further demand.

$$\nu = \varepsilon A \,|\, \mathfrak{k} \,|\,. \tag{5}$$

Hence

$$\varepsilon = \text{sign.}\, A\nu \tag{6}$$

and

$$|\, A \,| = \text{phase velocity.} \tag{7}$$

Since ν, $\mathfrak{k}$, A can independently be replaced by their negatives, *eight* different types of wave are associated with an (ambivalent) direction. The corresponding three geometrical characteristics of the wave are: direction of propagation (±); polarization (R, L); "normality" (n = normal, a = abnormal). The association of geometrical and analytical characteristics is indicated by the following table:—

Geometrical	Analytical Characteristic			
$+ R\ n$	ν	$\mathfrak{k}$	$\mathfrak{a}$	A
$+ L\ n$	$-\nu$	$-\mathfrak{k}$	$\mathfrak{a}$	A
$- R\ n$	ν	$-\mathfrak{k}$	$\mathfrak{a}^*$	A
$- L\ n$	$-\nu$	$\mathfrak{k}$	$\mathfrak{a}^*$	A
$+ R\ a$	$-\nu$	$-\mathfrak{k}$	$\mathfrak{a}^*$	$-A$
$+ L\ a$	ν	$\mathfrak{k}$	$\mathfrak{a}^*$	$-A$
$- R\ a$	$-\nu$	$\mathfrak{k}$	$\mathfrak{a}$	$-A$
$- L\ a$	ν	$-\mathfrak{k}$	$\mathfrak{a}$	$-A$

(8)

[3] Relations which contain $\mathfrak{a}$ within a cross-product are better not taken over from N.O., but established anew, because the *sign of* A interferes with them. The derivation of (3) is given here in appendix II

Explanation: In this table ν and A mean *positive* numbers and $\mathfrak{k}$ and $\mathfrak{a}$ mean certain vectors, the relations (3) and (5) holding between them, with $\varepsilon = +1$. On replacing the quantities ν, $\mathfrak{k}$, $\mathfrak{a}$, A *in equations* (1) by the quantities indicated in the 2nd-5th columns *of the table*, you obtain a wave with the characteristics indicated in the *first* column, where the sign + means propagation *in the direction* of $\mathfrak{k}$. Behold that for *abnormal* waves $\nu > 0$ means *left*-hand-polarization.

§ 4. COMPLYING WITH THE ALGEBRAIC CONDITIONS.

Since the field equations (4) are linear, any number of waves like (1) can be superimposed. But in order to obtain a *solution* we must also satisfy the conditions of conjugateness

$$\mathfrak{G}^* = \frac{2}{(\mathfrak{F}\,\mathfrak{G})}\left(\mathfrak{F} - \tfrac{1}{2}\mathbf{L}\,\mathfrak{G}\right)$$
$$\mathfrak{F}^* = \frac{2}{(\mathfrak{F}\,\mathfrak{G})}\left(-\,\mathfrak{G} - \tfrac{1}{2}\mathbf{L}\,\mathfrak{F}\right). \tag{9}$$

See (2, 4) N.O. For a single wave they demand $A = \pm 1$, and thus $|\mathfrak{k}| = |\nu|$. We shall now show that they can be satisfied for a couple of waves, one of which carries the polarization vector $\mathfrak{a}$, the other $\mathfrak{a}^*$. So we put

$$\mathfrak{F}_1 = C_1\,\mathfrak{a}\,e^{i(\nu_1 t - \mathfrak{k}_1 \mathfrak{r})}, \qquad \mathfrak{F}_2 = C_2\,\mathfrak{a}^*\,e^{i(\nu_2 t - \mathfrak{k}_2 \mathfrak{r})},$$
$$\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2 \qquad \mathfrak{G} = i\,(A_1\,\mathfrak{F}_1 + A_2\,\mathfrak{F}_2). \tag{10}$$

Here $\mathfrak{k}_2$ means just a real numerical multiple of $\mathfrak{k}_1$, the two being either parallel or antiparallel. Consulting the 1st and the 4th column of (8) we see that *the two waves* run antiparallel if they have the *same* "normality," but directly parallel if one is normal, one abnormal.

We have to insert (10) into (9). Paying attention to (2) we easily find

$$\mathfrak{F}^2 - \mathfrak{G}^2 = 2\,(1 + A_1 A_2)\,(\mathfrak{F}_1\,\mathfrak{F}_2)$$
$$(\mathfrak{F}\,\mathfrak{G}) = i\,(A_1 + A_2)\,(\mathfrak{F}_1\,\mathfrak{F}_2)$$
$$\mathbf{L} = \frac{2\,(1 + A_1 A_2)}{i\,(A_1 + A_2)} \tag{11}$$
$$(\mathfrak{F}_1\,\mathfrak{F}_2) = 2\,C_1\,C_2\,e^{i\{(\nu_1 + \nu_2)t - (\mathfrak{k}_1 + \mathfrak{k}_2)\mathfrak{r}\}}$$

Now the salient point is this. When (10) and (11) are inserted into (9) and the denominator $(\mathfrak{F}_1\,\mathfrak{F}_2)$ appearing on the right is removed to the left, then the wave-functions turning up there are precisely the original ones, viz.

$$\mathfrak{F}_1^*\,(\mathfrak{F}_1\,\mathfrak{F}_2) = 2\,|C_1|^2\,\mathfrak{F}_2, \qquad \mathfrak{F}_2^*\,(\mathfrak{F}_1\,\mathfrak{F}_2) = 2\,|C_2|^2\,\mathfrak{F}_1. \tag{12}$$

Equating the coefficients of $\mathfrak{F}_1$ and $\mathfrak{F}_2$ separately, you get from the *first* equ. (9)

$$|C_1|^2 = \frac{1 - A_2^2}{(A_1 + A_2)^2}, \qquad |C_2|^2 = \frac{1 - A_1^2}{(A_1 + A_2)^2}. \tag{13}$$

The *second* equ. (9) only repeats this demand.—Subtracting the two equations (13)

$$|C_1|^2 - |C_2|^2 = \frac{A_1^2 - A_2^2}{(A_1 + A_2)^2} = \frac{A_1 - A_2}{A_1 + A_2}, \tag{14}$$

or

$$\frac{A_2}{A_1} = \frac{1 - |C_1|^2 + |C_2|^2}{1 + |C_1|^2 - |C_2|^2}. \tag{15}$$

To simplify writing put

$$|C_1|^2 = W_1, \qquad |C_2|^2 = W_2, \tag{16}$$

the W's being 4π times the energy-density of a single (normal) wave of that amplitude. Then from (15) and (13)

$$\begin{aligned} \frac{1}{A_1^2} &= 1 + \frac{2W_2}{(1 + W_1 - W_2)^2} > 1 \\ \frac{1}{A_2^2} &= 1 + \frac{2W_1}{(1 - W_1 + W_2)^2} > 1, \end{aligned} \tag{17}$$

showing that both waves move with less than light velocity (see (7))

§ 5. The Signs of the A's. Two Cases.

To find the combination of signs admissible for the A's we make out from (17)

$$\begin{aligned} A_1^2 &= \frac{(1 + W_1 - W_2)^2}{W^2} \\ A_2^2 &= \frac{(1 - W_1 + W_2)^2}{W^2}, \end{aligned} \tag{18}$$

where W stands for the real [1] positive quantity

$$W = \underset{+}{\sqrt{}}(1 + W_1 + W_2)^2 - 4W_1W_2 > 1. \tag{19}$$

Regard to (14)

$$W_1 - W_2 = \frac{A_1 - A_2}{A_2 + A_2} \tag{14 bis}$$

[1] Observe that $W^2 = 1 + (W_1 - W_2)^2 + 2W_1 + 2W_2$.

leaves two ways of extracting the square-roots in (18), to wit, either

$$A_1 = \frac{1 + W_1 - W_2}{W}$$
$$A_2 = \frac{1 - W_1 + W_2}{W}, \tag{20}$$

or (distinguished by ^)

$$\hat{A}_1 = -\frac{1 + W_1 - W_2}{W}$$
$$\hat{A}_2 = -\frac{1 - W_1 + W_2}{W}, \tag{21}$$

giving

$$A_1 + A_2 = \frac{2}{W} > 0, \tag{22}$$

$$\hat{A}_1 + \hat{A}_2 = -\frac{2}{W} < 0, \tag{23}$$

respectively. Hence in the first case at least one of the A's must be positive (normal wave), in the second case at least one of the A's must be negative (abnormal wave).

§ 6. LONGITUDINAL TRANSFORMATIONS.

The two cases remain clearly separated under the aspect of "longitudinal" Lorentz-transformations. Indeed, such a transformation preserves the general form (10) and has therefore, from (11), the *invariants*

$$(1 + A_1 A_2) C_1 C_2, \qquad (A_1 + A_2) C_1 C_2. \tag{24}$$

The second one shows that $A_1 + A_2$ cannot vanish and thus cannot change sign.

Moreover, since the *phases* must be invariants, $\mathfrak{k}_1, \nu_1$ must be a 4-vector, and so must $\mathfrak{k}_2, \nu_2$. Their *invariants*, to wit,

$$\mathfrak{k}_1^2 - \nu_1^2, \qquad \mathfrak{k}_2^2 - \nu_2^2 \tag{25}$$

are positive (under-light-velocity!); hence either of the frequencies can be annihilated and can be made to change sign, which involves a change of sign of the corresponding A and of the direction of propagation of that wave.

Therefore if in a case covered by (20) and (22) one of the A's *is* negative, a Lorentz transformation with gradually increasing parameter will succeed in reversing its sign—and, of course, *before the other A changes sign*, since their sum must remain positive throughout. So we get *both of them positive.*

The same, if in a case covered by (21) and (23) one of the A's *is* positive, you will succeed in making it negative, whilst the other one remains so. And so you get them *both negative.*

Summing up we may say that (20) really refers to a couple of *antiparallel normal waves*, but possibly viewed from a Lorentz-frame in which one of them has become so intense as to impose the features of abnormity on the other one and to reverse its direction.—(21) is the exact counterpart for a couple of *antiparallel abnormal waves.*

Even more striking is the reversal of these considerations. Given e.g. a couple of antiparallel normal waves, however weak, you can always indicate a frame of reference, in which one of them has its direction reversed, dragged along, as it were, by the other one and exhibiting the features of abnormity. In one particular frame it becomes petrified, static, as it were ($\nu = 0$). Moreover you are, in every case, free to choose which of the two you want to subject to such extremity.

It will now be appreciated that we had to include abnormal waves at the outset; not because we are particularly interested in the abnormal couple described by (21), but because the normal couple (20) could otherwise not be exhaustively described.

The quantity W introduced in (19) has a physical meaning, viz. (speaking of the normal couple),

$$\frac{1}{4\pi}(W-1) = \frac{1}{4\pi}\left(\frac{2}{A_1+A_2}-1\right) = \text{energy density}. \qquad (26)$$

A Lorentz-transformation in an arbitrary direction would produce the more general kind of solution with an *arbitrary angle* between the two wave-normals, liquidating the simplification introduced in § 2. There is no reason to follow that up for the moment. But let it be noted that in this case the general features of (10) *are no longer preserved.* E g. the single waves do not remain transversal. (This had already been revealed by the approximate treatment in N.O.)

Appendix I.

I wish to show that a single plane Maxwellian wave is an exact solution of any non-linear electrodynamics of that very general type which Gustav Mie was the first to envisage more than 30 years ago and of which Born's theory is a special case. The general features are these.

Maxwell's equations are formally retained:

$$\begin{aligned} &\text{curl } E + \dot{B} = 0 \qquad && \text{div } B = 0 \\ &\text{curl } H - \dot{D} = 0 \qquad && \text{div } D = 0\,. \end{aligned} \qquad (a_1)$$

The second six-vector H, D is defined by

$$H = \frac{\partial L}{\partial B}, \qquad D = -\frac{\partial L}{\partial E}, \qquad (a_2)$$

where L is an arbitrary function of the fundamental six-vector B, E. Lorentz-invariance is demanded. Moreover it is demanded that in the limit for weak fields

$$H \rightarrow B, \qquad D \rightarrow E. \qquad (a_3)$$

Now, since L is to be an invariant, it can only be a function of

$$J_1 = \tfrac{1}{2}(B^2 - E^2) \quad \text{and} \quad J_2 = (BE). \qquad (a_4)$$

From (a_2) and (a_4)

$$\begin{aligned} H &= \frac{\partial L}{\partial J_1} B + \frac{\partial L}{\partial J_2} E \\ D &= \frac{\partial L}{\partial J_1} E - \frac{\partial L}{\partial J_2} B. \end{aligned} \qquad (a_5)$$

To comply with (a_3), $\frac{\partial L}{\partial J_1}$ and $\frac{\partial L}{\partial J_2}$ must tend to 1 and 0 respectively when both B and E tend to zero. But since those derivatives are functions of J_1 and J_2 only, which tend to zero when B and E do, we must have

$$\left(\frac{\partial L}{\partial J_1}\right)_{\substack{J_1 = 0 \\ J_2 = 0}} = 1, \qquad \left(\frac{\partial L}{\partial J_2}\right)_{\substack{J_1 = 0 \\ J_2 = 0}} = 0. \qquad (a_6)$$

Now for a plane Maxwellian wave $J_1 = J_2 = 0$. Hence in this case, from (a_5) and (a_6)

$$H = B, \qquad D = E.$$

That turns (a_1) into Maxwell's vacuum-equations, which are indeed satisfied by a plane Maxwellian wave. Q.E.D.

APPENDIX II.

(Derivation of equ (3)).

The cross-product $[\mathfrak{k}\,\mathfrak{a}]$, since it is orthogonal to $\mathfrak{k}$, must be a linear combination of $\mathfrak{a}$ and $\mathfrak{a}^*$:

$$[\mathfrak{k}\,\mathfrak{a}] = p\,\mathfrak{a} + q\,\mathfrak{a}^*, \qquad (a_7)$$

p and q being complex numbers. Scalar multiplication by $\mathfrak{a}$, with regard to (2), gives $q = 0$. Thus

$$[\mathfrak{f}\,\mathfrak{a}] = p\,\mathfrak{a} \quad . \tag{a_8}$$

Scalar multiplication by $\mathfrak{a}^*$ gives

$$(\mathfrak{f}\,[\mathfrak{a}\,\mathfrak{a}^*]) = 2p \quad . \tag{a_9}$$

Vectorial multiplication of (a_8) by $\mathfrak{a}^*$ gives

$$-\,2\,\mathfrak{f} = p\,[\mathfrak{a}\,\mathfrak{a}^*] \quad . \tag{a_{10}}$$

The last two equations give

$$p^2 = -\,\mathfrak{f}^2 \,, \qquad \text{thus} \qquad p = \pm\, i\,|\,\mathfrak{f}\,| \,, \tag{a_{11}}$$

which inserted in (a_8) gives equ. (3) Q.E.D.

Behold that the ambivalent sign is genuine, as long as we only use (2). For (2) is symmetric with respect to $\mathfrak{a}$ and $\mathfrak{a}^*$, whereas from (a_8), since $\mathfrak{f}$ is real and p imaginary, follows

$$[\mathfrak{f}\,\mathfrak{a}^*] = -\,p\,\mathfrak{a}^* \quad . \tag{a_{12}}$$