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The physical modelling of society: a historical perspective

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Abstract

By seeking to uncover the rules of collective human activities, today's statistical physicists are aiming to return to their roots. Statistics originated in the study of social numbers in the 17th century, and the discovery of statistical invariants in data on births and deaths, crimes and marriages led some scientists and philosophers to conclude that society was governed by immutable "natural" laws beyond the reach of governments, of which the Gaussian "error curve" became regarded as the leitmotif. While statistics flourished as a mathematical tool of all the sciences in the 19th century, it provoked passionate responses from philosophers, novelists and social commentators. Social statistics also guided Maxwell and Boltzmann towards the utilization of probability distributions in the development of the kinetic theory of gases, the foundation of statistical mechanics. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The introduction of probability into the fundamental nature of the quantum world by Bohr, Born and Schrödinger in the 1920s famously scandalized some scholars of science's philosophical foundations. But arguments about chance, probability and determinism were no less heated in the mid-19th century, when statistical ideas entered classical physics.

James Clerk Maxwell (1878) let probabilistic physics bring him to the verge of mysticism: "it is the peculiar function of physical science to lead us to the confines of the incomprehensible, and to bid us behold and receive it in faith, till such time as the

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mystery shall open" [1]. Few scientific issues besides Darwinism (itself made statistical by Darwin's cousin Francis Galton) attracted such debate in the salons, parlours and periodicals of the 19th century.

Yet it was not physicists who began this debate, but social scientists. They found that chance and randomness in the world of people and politics, far from banishing predictability and making social science oxymoronic, seemed to have laws of their own. This appeared to challenge the notion of free will itself—to widespread dismay.

Contemporary efforts to apply the concepts and methods of statistical physics to social phenomena ranging from economics to traffic flow, pedestrian motion, decision making, voting and contact networks are therefore essentially completing a circle whose trajectory commenced centuries previously. Work on social statistics in the 19th century had a direct influence on the founders of statistical physics, who found within it the confidence to abandon a strict Newtonian determinism and instead to trust to a "law of large numbers" in dealing with innumerable particles whose individual behaviours were wholly inscrutable.

2. Why are we so predictable?

Modern physics-based models of social, economic and political behaviour invoke idealizations of human behaviour that might make sociologists blanch. In economics there is, of course, a long history of making mathematical models tractable by gross simplification of human tendencies, leading to the notorious omniscient rational maximizers known as *Homo economicus*. But sociology, while aspiring to become a recognizably scientific discipline, has been loath to abandon a more complex psychological picture of the individual, embedded in a cultural matrix in which behaviour is governed by many diverse influences: custom, religion, economic circumstances, peer pressures and so forth. Evolutionary biologists and sociobiologists such as E.O. Wilson are now calling for behavioural models to be grounded in evolutionary models that assume a strong genetic basis for modes of behaviour [2].

At face value, there might seem to be little room left for statistical physics to make a realistic contribution. But if there is one message that emerges clearly from this discipline, it is that sometimes *the details do not matter*. That, in a nutshell, is what is meant by universality. It does not matter that the Ising model is a ridiculously crude description of a real fluid; they both have the same behaviour at the critical point because in that circumstance only the broad-brush features of the system, such as the dimensionality and range of particle interactions, determine the behaviour.

This is a way of saying that *collective* behaviour tends to be robust, and shared by many apparently different systems. In systems of components that have a tendency both to attract and to repel—to come together or to stay apart—phase transitions are ubiquitous. Models of crime and marriage in which individuals are assumed to gather into certain "camps" according to certain interaction rules [3–5] show classic examples of both first-order and critical phase transitions (Fig. 1). A model of alliance formation preceding the Second World War [6] displays a spinodal point at which a metastable

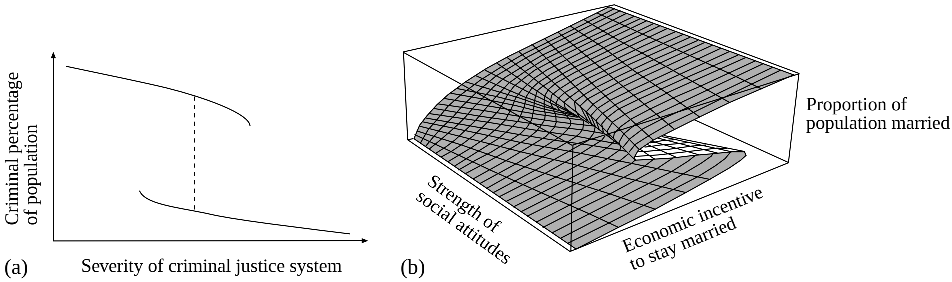

Fig. 1. (a) First-order phase transition between states of high- and low-crime rate as a function of the severity of the criminal justice system. (b) The relation between marriage rates, economic incentives and social pressures. The analogy with the P–V–T surface of a fluid is clear. Both figures are derived from the models in Refs. [4,5].

"energy minimum" vanishes. Europe apparently passed through such a point some time between 1937 and 1938, after which the partitioning of nations into Allied and Axis powers became inevitable. Before this time, an alternative partitioning in which most states allied against the Soviet Union appears also possible, although less likely than that of the eventual historical outcome.

Given the ubiquity of phase transitions in physical science, we can hardly be surprised that they may become manifest in social science. Yet evolutionary biological models of social science fail to anticipate such collective behaviour because they ignore the nonlinearities that interactions can produce. On the whole, such evolutionary models assume that mass human behaviour is a straightforward extrapolation of that of individuals. In this distinction lies the essence of what statistical physics has to offer social science. The political scientist Michael Lind has recently expressed this in an elegant manner:

> A friend of mine who raises dogs tells me that you cannot understand them unless you have half a dozen or more. The behaviour of dogs, when assembled in sufficient numbers, undergoes an astonishing change. They instinctively form a disciplined pack. Traditional political philosophers have been in the position of students of canine behaviour who have observed only individual pet dogs [7].

The jettisoning of a great deal of psychological subtlety in physics-based models need not be seen as a naïve step that ignores human complexity. Rather, it may simply reflect the fact that in many social situations our choices are extremely limited. The psychology of vehicle drivers is no doubt a fertile topic for exploration, but it does not alter the fact that most drivers end up trying to drive on the correct side of the road at a comfortable speed while avoiding collisions. In many social situations it is not a bad approximation to assume that we will tend to do what other people are doing. We make our choices for all sorts of reasons that might be subconsciously motivated statements about how we like to see ourselves—but in the end we have to face the simple decision: PC or Mac?

3. The collective Leviathan

Physics-based social modelling is often perceived as a new idea, but in fact it predates Newton. In some sense a mechanistic view of the world began with the Classical Greek philosophers. It becomes contiguous with modern science from the early 17th century: the time of Descartes, Pierre Gassendi, Francis Bacon and Galileo. The Cartesian mechanistic philosophy can be encapsulated in two principles: all phenomena can be understood on the basis of particles of matter in motion, and these motions can be changed only by direct interaction with other particles [8]. Galileo and Newton uncovered the laws that dictated these motions, although of course Newton's mechanics are not exactly Cartesian.

Applied to living creatures, the mechanistic philosophy looks crude today. But the devices used as analogies for the human body in the 17th century—clocks, pumps, water mills and so forth—were the height of technology in their time. It was no greater insult to Nature for Descartes to say that the body is "a machine made by the hands of God" than it is to regard the human brain now as a fantastically complex computer.

The first person to try to deduce what the mechanical model of the universe meant for human society was the English philosopher Thomas Hobbes (1588–1679) (Fig. 2).

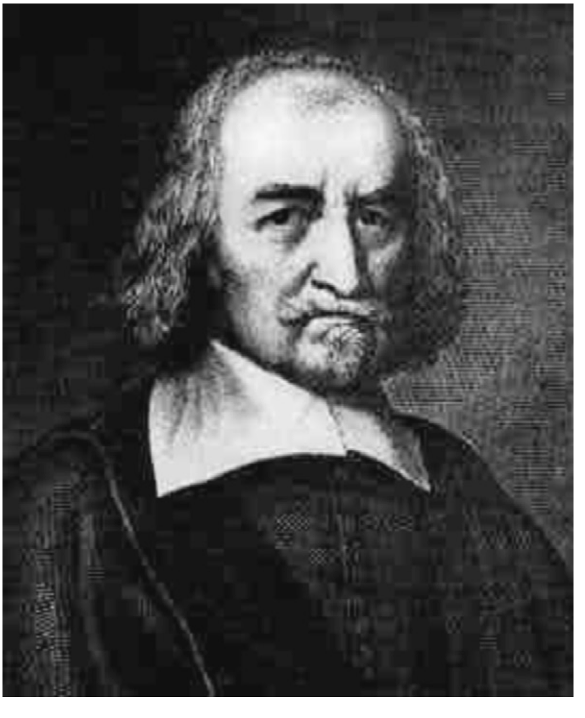

Fig. 2. Thomas Hobbes (1588–1679).

In the 1630s Hobbes became a part of the group of French mechanical empiricists centred around Marin Mersenne, and he travelled to Italy to meet Galileo. During the 1640s he set out to construct a political theory based on Galileo's precept that, as Hobbes saw it, all bodies seek to remain in motion. The result was first *De Cive*, published in 1642, and then his major work *Leviathan,* which appeared shortly after the English Civil War in 1651.

Leviathan seeks to explain how people can escape the anarchic "State of Nature" in which each person exploits their neighbours and life is "solitary, poore, nasty, brutish, and short". Hobbes's answer is that all people must elect a ruler and then relinquish to him all power: basically to create a dictatorship by democratic election. Having just witnessed the horrors of the Civil War (albeit from exile in France), Hobbes appreciated all too well the consequences for a nation that lacked a leader. By inclination a Royalist, Hobbes found an argument that supported absolute monarchy as the only stable way for a nation to govern itself.

The striking aspect of Hobbes's theory from today's perspective is not its questionable conclusions but its methods. By identifying the intrinsic preferences of individuals and the nature of the interactions between them, Hobbes created a theoretical framework that could be recast without too much effort as a lattice model whose Nash equilibrium is that which maximizes the "power" of individuals. Later political philosophers tended to debate or refute Hobbes's precepts and conclusions, but did not pay a great deal of attention to his methods. It would be mistaken to think that Hobbes was wholly objective about the outcome of his model, but nevertheless *Leviathan* is a forerunner of a physics of society in so far as it seeks to progress from definite and mechanical rules of behaviour to make a prediction about modes of collective behaviour.

4. The rise of statistics

Hobbes's Leviathan is the collective "Commonwealth", a kind of organic being comprised of the wishes and actions of the mass of individuals. Admittedly its ruler is a single person, but that person represents the Commonwealth and is invested with all its power. Hobbes essentially invented the notion of a nation state as a collective entity. What is the character, the properties, of this Leviathan? If it is truly an embodiment of the population as a whole, then we need to ask: what is society like? Are there definable characteristics that emerge from the morass of individual behaviours? In the late 17th century, political and natural philosophers, infused with the mechanistic spirit of the dawning Age of Enlightenment, began to wonder whether there was some way of *measuring* society.

Sir William Petty (Fig. 3) was one of those figures that only the 17th century could produce. He was a scientist, associated with both the rationalist school of Descartes and the empiricists Mersenne and Gassendi, as well as being trained in medicine. He became a founding member and Vice President of the Royal Society, and he was highly politically active, a pupil of Thomas Hobbes, a member of Parliament, and an adviser at different times to Charles I, Oliver Cromwell, Charles II and James II. Petty

Fig. 3. William Petty (1623–1687).

recognized that the study of society could hope to emulate the precision of science only if it became quantitative, and he called for a “political arithmetic”.

This induced Petty’s friend, the London haberdasher John Graunt in the 1660s to advocate “social numbers” as a means to guide political policy. Graunt compiled mortality tables, reasoning that good legislation and government is impossible without such demographic data. Petty was one of the first to study the political economy by means of such numbers—although they led this rather unworldly man to such dry and heartless political conclusions that Jonathan Swift was inspired to satirize Petty by making the “rational” suggestion that the Irish poor sustain themselves by eating their children.

Births and deaths were a major preoccupation of the early pioneers of social statistics, including the astronomer Edmund Halley. Astronomers were the principal early beneficiaries of Newton’s mechanics, and enjoyed a sphere of investigation that was almost uniquely explicable via simple, mathematical laws. So it is not surprising that they feature prominently among the pioneers of “social physics”, the search for law-like behaviour in society. Astronomers appreciated how the numbers of Tycho Brahe had led to the laws of Kepler and Newton. Might social numbers likewise reveal “natural” laws?

During the Enlightenment, many philosophers regarded science and rationalism as the key to a utopian era of freedom and equality. Among them was the French mathematician Marquis Marie-Jean-Antoine-Nicolas Caritat de Condorcet (1743–1794) (Fig. 4), whose *Esquise d’un Tableau Historique des progrès de l’Esprit Humain* (*Sketch for a Historical Picture of the Progress of the Human Mind*), written in 1793, is the most optimistic of all science-based utopias. Astonishingly, this was written while Condorcet was on the run from Robespierre’s agents, after he had become convicted of treason for opposing a hastily drafted French Constitution masterminded by

Fig. 4. The Marquis Marie-Jean-Antoine-Nicolas Caritat de Condorcet (1743–1794).

Robespierre. A protégé of Jean d'Alembert and one of the founders of probability theory, Condorcet was captured in 1794 and, condemned to the guillotine, died in his cell, probably by poisoning himself.

The idea that there were laws that stood in relation to society as Newton's mechanics stood in relation to the motion of the planets was shared by many of Condorcet's contemporaries. The Baron de Montesquieu (Charles Louis Secondat de la Brède) asserted as much decades earlier in *The Spirit of the Laws* (1748). In 1784 Immanuel Kant spoke of "universal laws" which,

> However obscure their causes, [permit] us to hope that if we attend to the play of freedom of human will in the large, we may be able to discern a regular movement in it, and that what seems complex and chaotic in the single individual may be seen from the standpoint of the human race as a whole to be a steady and progressive though slow evolution of its original endowment [9].

Claude-Henri de Saint-Simon (1760–1825) shared Condorcet's dream of a society governed by scientific reason, and he imagined that it might lead to the founding of a "Religion of Newton."

On the one hand, this belief in laws of society beyond the reach of governments was a product of the Enlightenment faith in the orderliness of the universe. On the

Fig. 5. Auguste Comte (1798–1857).

other hand, it is not hard to see within it the spectre of the Industrial Revolution with its faceless masses of toiling humanity like so many swarming insects. Before the 19th century, laws that applied to Graunt's "social numbers", such as the approximate (short-term) invariance of mortality averages, were regarded as evidence of divine wisdom and planning. To later commentators such as Malthus and Marx, statistical trends became the preconditions for catastrophe and revolution.

The term "social physics" was first coined by the French political philosopher Auguste Comte (Fig. 5) in the 19th century. In his *System of Positive Philosophy* (1830–1842) he argued that this discipline would complete the scientific description of the world that Galileo, Newton and others had begun:

> Now that the human mind has grasped celestial and terrestrial physics, mechanical and chemical, organic physics, both vegetable and animal, there remains one science, to fill up the series of sciences of observation—social physics. This is what men have now most need of; and this it is the principal aim of the present work to establish [10].

5. The ubiquitous error curve

What were these laws of society? The French astronomer Pierre-Simon Laplace (Fig. 6) began to discern an answer. In 1781 he enumerated male and female births in Paris, explaining their near-equality as merely the expected result of a random process, rather than, as thought previously, a sign of God's wisdom in providing spouses for all.

Laplace showed that the variations in these and other social statistics could be described by a universal "error curve", introduced in 1733 by the mathematician

Fig. 6. Pierre-Simon Laplace (1749–1827).

Abraham De Moivre to describe the probabilities of coin tossing. The ubiquity of this curve, now familiar as the Gaussian, was then regarded as almost miraculous: a natural law that applied as much to human affairs as to coins or the errors in measurement of planetary motions [11].

When the Belgian astronomer Adolphe Quetelet (Fig. 7) came to the French Royal Observatory in 1823 to learn from Laplace and Poisson, he was captivated by the statistical regularities in social data that the two French scientists had uncovered. Quetelet recast Comte's "social physics" as *mécanique sociale*, a mechanical social science based solidly on statistics. In 1832 he wrote that

> whatever concerns the human species, considered *en masse*, belongs to the domain of physical facts; the greater the number of individuals, the more the individual will is submerged beneath the series of general facts which depend on the general causes according to which society exists and is conserved [12].

Quetelet's popularization of Laplace's data impressed people in many fields. The scientist John Herschel spoke approvingly of the work in 1850. Florence Nightingale proposed that Quetelet's social mechanics be taught at Oxford; Karl Marx used Quetelet's statistical laws in developing his labour theory of value. And the utilitarian political philosopher John Stuart Mill felt that Quetelet's work lent support for his conviction

Fig. 7. Adolphe Quetelet (1796–1874).

that society and history were bound by laws as absolute as (if harder to discern than) those of the natural sciences. In *A System of Logic* (1862), Mill had the universal error curve in mind when he wrote,

> very events which in their own nature appear most capricious and uncertain, and which in any individual case no attainable degree of knowledge would enable us to foresee, occur, when considerable numbers are taken into the account, with a degree of regularity approaching to mathematical [13].

But the most visible exposition of these laws was given in the epic (and misnamed) *History of Civilization in England* (1857–1861) by Henry Thomas Buckle (1821–1862), who believed that history had a law-like inevitability. "The great truth", he said, is "that the actions of men... are in reality never inconsistent, but however capricious they may appear only form part of one vast system of universal order" [14]. Buckle, like Adam Smith in the previous century, argued for the principle of political laissez-faire, and for the ability of people to govern themselves. Left to their own devices, he thought, societies automatically produced "order, symmetry, and law", while "lawgivers are nearly always the obstructors of society, instead of its helpers".

Political philosophers had, in Buckle's view, in the past pursued the futile quest of trying to unravel the way society works by asking what makes individuals tick. The empirical science of social statistics avoided such imponderables by discovering laws amongst the numbers.

Buckle laid out his case in his *History*, although he died before he could complete the third and final volume. The book excited intense discussion among social commentators of all sorts. At times, Buckle seemed to imply a compulsion whereby individuals would find themselves acting in a certain way to fulfil statistical quotas. That seemed to undermine the very notion of free will. Ralph Waldo Emerson mocked what he saw as the absurd rigidity of the idea: "Punch makes exactly one capital joke per week; and the journals contrive to furnish one good piece of news every day" [15]. In *Notes From the Underground*, Fyodor Dostoevsky had Buckle in mind when his narrator raves that man would rather make himself mad than be constrained by law-like reason. Friedrich Nietzsche, whose belief in the shaping of history by a few "great men" was second to none, was characteristically acerbic: "so far as there are laws in history, laws are worth nothing and history is worth nothing".

In his more measured way, Tolstoy struggled in *War and Peace* with the questions posed by Buckle's deterministic view of history, with what he called the "relation of free will to necessity". Maurice Evan Hare was more whimsical in 1905:

> There once was a man who said "Damn!"
> It is borne in upon me I am
> An engine that moves
> In predestinate grooves
> I'm not even a bus, I'm a tram [16].

6. Physics becomes statistical

One of the book's earliest readers was more favourably impressed. Although he found it "bumptious", he concluded that nevertheless it contained "a great deal of actually original matter, the result of fertile study, and not mere brainspinning" [17].

This reader was James Clerk Maxwell (Fig. 8). When Maxwell came to study the problem of gases in which the constituent particles were constantly engaging in collisions that no one could hope to observe or to follow, he recognized this as a problem in the same class as those that Buckle had pondered in society, in which the immediate causes of individual behaviour must forever be inscrutable:

> the smallest portion of matter which we can subject to experiment consists of millions of molecules, not one of which ever becomes individually sensible to us. We cannot, therefore, ascertain the actual motions of any one of these molecules; so that we are obliged to abandon the strict historical method, and to adopt the statistical method of dealing with large groups of molecules... In studying the relations between quantities of this kind, we meet with a new kind of regularity, the regularity of averages, which we can depend upon quite sufficiently for all practical purposes [18].

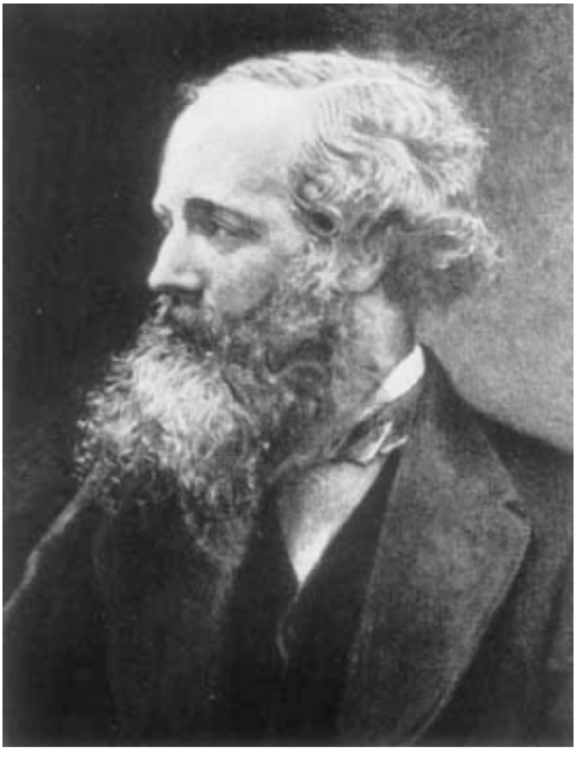

Fig. 8. James Clerk Maxwell (1831–1879).

As he indicated later in 1873, the experiences of "social physicists" lent him confidence that this statistical approach could extract order from the microscopic chaos:

> those uniformities which we observe in our experiments with quantities of matter containing millions of millions of molecules are uniformities of the same kind as those explained by Laplace and wondered at by Buckle arising from the slumping together of multitudes of causes each of which is by no means uniform with the others [19].

Would Maxwell have dared abandon the "strict historical method"—the obligation to explain everything in terms of Newtonian mechanics of particles on fixed trajectories—if studies of society had not shown the presence of laws even in complex systems where the direct causes were obscure? How otherwise might he have found the faith to look for laws in the face of woefully incomplete knowledge about motions?

Maxwell began his work on the kinetic theory of gases shortly after reading Buckle's writings. But in his early work he also drew on the more analytical studies of Quetelet, whose wide application of the error curve came to his attention via John Hershel. Herschel himself alluded to connections between social physics and the early kinetic theory of gases.

Maxwell knew that Rudolf Clausius had used probability laws in 1857 to deduce the effects of molecular collisions on the pressure exerted by a gas on the walls that contained it. But Clausius was interested only in the average velocity of the particles. Maxwell wanted to know how the velocities were distributed around this average. If the error curve worked so well for describing variations from the average in social physics, Maxwell decided that would suffice for him too. So he built the Gaussian distribution into the theory.

Maxwell's velocity distribution was merely an assumption until Ludwig Boltzmann showed in 1872 that any group of randomly moving particles in a gas *must* converge on this distribution. Boltzmann also knew of Buckle's work, and drew analogies between his particles and the individuals in the social censuses that furnished Buckle's statistics:

> The molecules are like so many individuals, having the most various states of motion, and the properties of gases only remain unaltered because the number of these molecules which on the average have a given state of motion is constant [20].

He likened the gas laws, a statement of the invariance of statistical averages, to the uniform profits of insurance companies. In 1886, Maxwell's friend Peter Guthrie Tait compared the statistical approach of the kinetic theory with

> the extraordinary steadiness with which the numbers of such totally unpredictable, though not uncommon phenomena as suicides, twin or triple births, dead letters, & c., in any populous country, are maintained year after year [21].

("Dead letters" are those that remain in the postal system because they are badly addressed. Laplace had commented on how this was a constant fraction of the total turnover of the postal service.)

Statistics, entering physics through the agency of social science, soon came to dominate it. Erwin Schrödinger makes it clear in 1944 that he considers laboratory-scale physics to be statistical rather than deterministic: "physical laws rest on atomic statistics and are therefore only approximate… only in the cooperation of an enormously large number of atoms do statistical laws began to operate… it is in that way that the events acquire truly orderly features" [22]. Schrödinger implies that one can discuss molecules *only* in statistical terms. (With the innovations in single-molecule studies over the past decade, Newtonian mechanics have returned to the microworld.)

The statistical nature of quantum mechanics is different from that of classical physics, as it invokes variables whose values are not just unknown but unknowable. Nonetheless, quantum probability would have had a rockier path if physicists had not already been prepared by the knowledge that a statistical approach does not preclude the existence of precise laws. As early as 1918, the physicist Marian Smoluchowski considered probability to be central to modern physics:

> only Lorentz's equations, electron theory, the energy law, and the principle of relativity have remained unaffected, but it is quite possible that in the course of time exact laws may even here be replaced by statistical regularities [23].

The way to statistical science would have been more tortured if 19th century experience with social statistics had not given scientists the confidence to believe that large-scale order and regularity in nature can arise even when we do not know, or cannot even meaningfully propose, a determining cause for each event. In such situations, we must trust that there are laws within large numbers.

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