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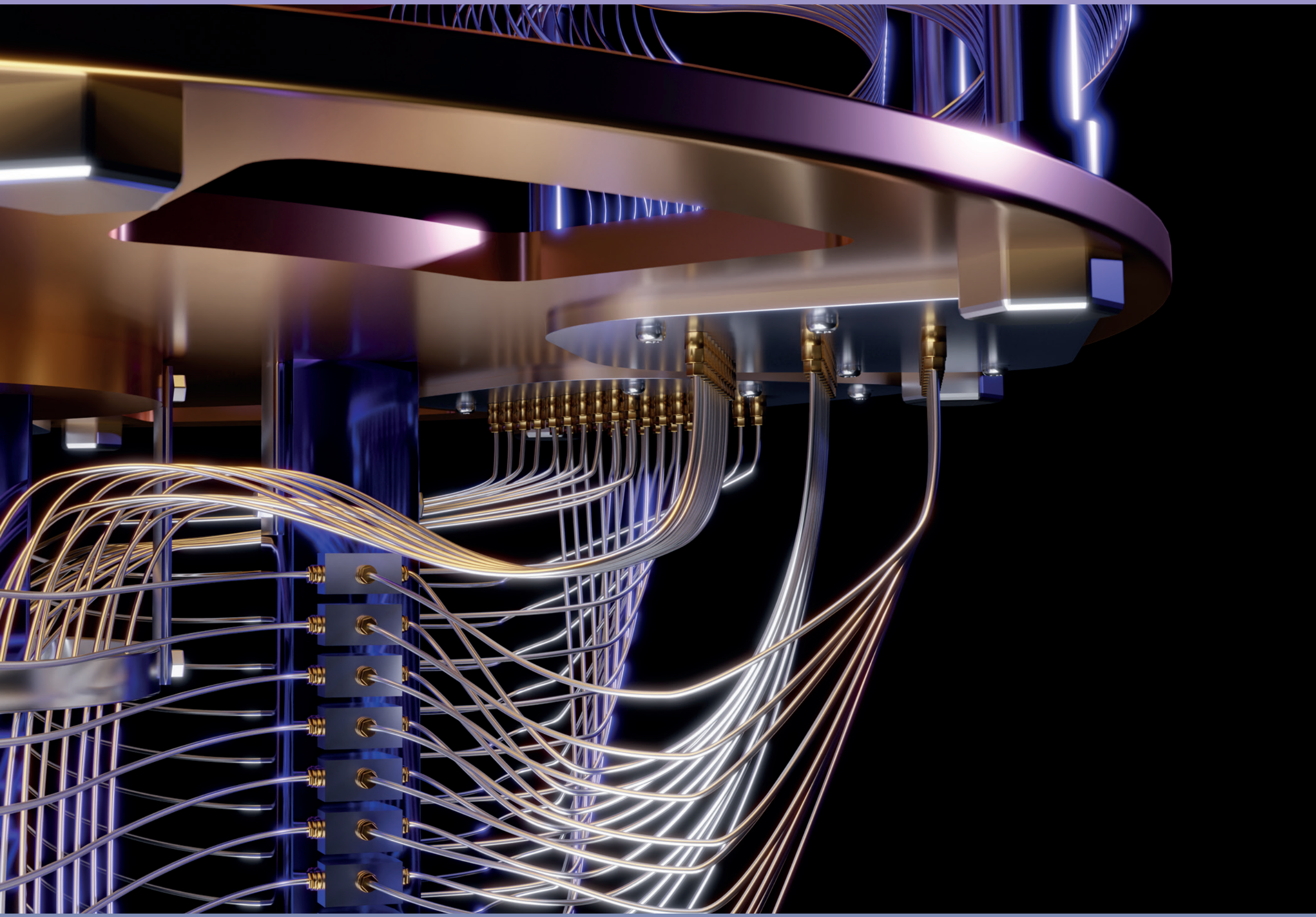
Innovative Quantum Computing

Steven Duplij
Raimund Vogl



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Online at: https://doi.org/10.1088/978-0-7503-5281-9

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About the Editor

F J Duarte is a laser physicist based in Western New York, USA. His career has covered three continents while contributing within the academic, industrial, and defense sectors. Duarte is editor/author of 15 laser optics books and sole author of three books: *Tunable Laser Optics, Quantum Optics for Engineers*, and *Fundamentals of Quantum Entanglement*. Duarte has made original contributions in the fields of coherent imaging, directed energy, high-power tunable lasers, laser metrology, liquid and solid-state organic gain media, narrow-linewidth tunable laser oscillators, organic semiconductor coherent emission, *N*-slit quantum interferometry, polarization rotation, quantum entanglement, and space-to-space secure interferometric communications. He is also the author of the generalized multiple-prism grating dispersion theory and pioneered the use of Dirac's quantum notation in *N*-slit interferometry and classical optics. His contributions have found applications in numerous fields, including astronomical instrumentation, dispersive optics, femtosecond laser microscopy, geodesics, gravitational lensing, heat transfer, laser isotope separation, laser medicine, laser pulse compression, laser spectroscopy, mathematical transforms, nonlinear optics, polarization optics, and tunable diode-laser design. Duarte was elected Fellow of the Australian Institute of Physics in 1987 and Fellow of the Optical Society of America in 1993. He has received various recognitions, including the *Paul F Foreman Engineering Excellence Award* and the *David Richardson Medal* from the Optical Society.

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ISBN 978-0-7503-5281-9 (ebook)
ISBN 978-0-7503-5279-6 (print)
ISBN 978-0-7503-5282-6 (myPrint)
ISBN 978-0-7503-5280-2 (mobi)

DOI 10.1088/978-0-7503-5281-9

Version: 20231101

IOP ebooks

British Library Cataloguing-in-Publication Data: A catalogue record for this book is available from the British Library.

Published by IOP Publishing, wholly owned by The Institute of Physics, London

IOP Publishing, No.2 The Distillery, Glassfields, Avon Street, Bristol, BS2 0GR, UK

US Office: IOP Publishing, Inc., 190 North Independence Mall West, Suite 601, Philadelphia,
PA 19106, USA

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Preface

This book is devoted to the study of exotic and non-standard mathematical methods in quantum computing. The principal ingredients of quantum computation are qubits and their transformations, which can be provided in different ways: first mathematically, and they can then be further realized in hardware.

In this book we consider various extensions of the qubit concept per se, starting from the obscure qubits introduced by the authors, and other fundamental generalizations. We then introduce a new kind of gate, higher braiding gates, which are implemented for topological quantum computations, as well as unconventional computing, when computational complexity is affected by its environment, which needs an additional stage of computation. Other generalizations are also considered and explained in a widely accessible and easy to understand style.

This book will be useful for graduate students and last year students for additional advanced chapters of lecture courses in quantum computer science and information theory.

Steven Duplij and Raimund Vogl
Münster, Germany
August 2023

Author biographies

Steven Duplij

Steven Duplij (Stepan Douplii) is a theoretical and mathematical physicist from the University of Münster, Germany. He was born in Chernyshevsk-Zabaikalsky, Russia, and studied at Kharkov University, Ukraine where he gained his PhD in 1983. While working at Kharkov, he received the title Doctor of Physical and Mathematical Sciences by Habilitation in 1999. Dr Duplij is the editor-compiler of the 'Concise Encyclopedia of Supersymmetry' (2005, Springer), and is the author of more than a hundred scientific publications and several books. He is listed in the World Directory of Mathematicians, Marques Who Is Who In America, the Encyclopedia of Modern Ukraine, the Academic Genealogy of Theoretical Physicists, and the Mathematics Genealogy Project. His scientific interests include supersymmetry and quantum groups, advanced algebraic structures, gravity and nonlinear electrodynamics, constrained systems, and quantum computing.

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Raimund Vogl is the CIO of the University of Münster (Germany) and has been the director of the University's IT center since 2007. He holds a PhD in elementary particle physics from the University of Innsbruck (Austria). After completing his PhD studies in 1995, he joined Innsbruck University Hospital as an IT manager for medical image data solutions and moved on to become the deputy head of IT. He is board member and president of EUNIS (European University Information Systems Organisation), and a member of Deutsche Gesellschaft für Medizinische Informatik, Biometrie und Epidemiologie (GMDS) and the Association for Information Systems (AIS). His current research interest in the field of information systems and information management focuses on the management of complex information infrastructures.

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Chapter 1

Obscure qubits and membership amplitudes

Nowadays, the development of quantum computing technique is governed by theoretical extensions of its ground concepts (Nielsen and Chuang 2000, Kaye *et al* 2007, Williams and Clearwater 1998). One of these extensions is to allow two kinds of uncertainty, sometimes called randomness and vagueness/fuzziness (for a review, see, Goodman and Nguyen 2002), which leads to the formulation of combined probability and possibility theories (Dubois *et al* 2000) (see, also, Bělohlávek 2002, Dubois and Prade 2000, Smith 2008, Zimmermann 2011). Various interconnections between vagueness and quantum probability calculus were considered in Pykacz (2015), Dvurečenskij and Chovanec (1988), Bartková *et al* (2017), and Granik (1994), including the treatment of inaccuracy in measurements (Gudder 1988, 2005), non-sharp amplitude densities (Gudder 1989), and the related concept of partial Hilbert spaces (Gudder 1986).

Relations between truth values and probabilities were also given in Bolotin (2018). The hardware realization of computations with vagueness was considered in Hirota and Ozawa (1989), and Virant (2000). On the fundamental physics side, it was shown that the discretization of space-time at small distances can lead to a discrete (or fuzzy) character for the quantum states themselves.

With a view to applications of these ideas in quantum computing, we introduce a definition of quantum state that is described by both a quantum probability and a membership function (Duplij and Vogl 2021), and thereby incorporate vagueness/fuzziness directly into the formalism. In addition to the probability amplitude, we will define a membership amplitude, and such a state will be called an obscure/fuzzy qubit (or qudit) (Duplij and Vogl 2021).

In general, the Born rule will apply to the quantum probability alone, while the membership function can be taken to be an arbitrary function of all of the amplitudes fixed by the chosen model of vagueness. Two different models of obscure-quantum computations with truth are proposed below: (1) a Product obscure qubit, in which the resulting amplitude is the product (in $\mathbb{C}$) of the quantum

doi:10.1088/978-0-7503-5281-9ch1

amplitude and the membership amplitude; and (2) a Kronecker obscure qubit, for which computations are performed in parallel, so that quantum amplitudes and the membership amplitudes form vectors, which we will call obscure-quantum amplitudes. In the latter case, which we call a double obscure-quantum computation, the protocol of measurement depends on both the quantum and obscure amplitudes. In this case, the density matrix need not be idempotent. We define a new kind of gate, namely, obscure-quantum gates, which are linear transformations in the direct product (not in the tensor product) of spaces: a quantum Hilbert space and a so-called membership space having special fuzzy properties (Duplij and Vogl 2021). We then introduce a new concept of double (obscure-quantum) entanglement, in which vector and scalar concurrences are defined and computed for concrete examples.

1.1 Preliminaries

To establish a notation standard in the literature (see, e.g. Nielsen and Chuang 2000, Kaye *et al* 2007), we present the following definitions. In an underlying d-dimensional Hilbert space, the standard qudit (using the computational basis and Dirac notation) $\mathcal{H}_{\mathrm{q}}^{(d)}$ is given by

$$|\,\psi^{(d)}\rangle = \sum_{i=0}^{d-1} a_i\,|\,i\rangle, \quad a_i \in \mathbb{C},\, |\,i\rangle \in \mathcal{H}_{\mathrm{q}}^{(d)}, \tag{1.1}$$

where a_i is a probability amplitude of the state $|\,i\rangle$. (For a review, see, e.g. Genovese and Traina 2008, Wang *et al* 2020.) The probability $_{pi}$ to measure the ith state is $p_i = F_{p_i}(a_1, \ldots, a_n)$, $0 \leqslant p_i \leqslant 1$, $0 \leqslant i \leqslant d-1$. The shape of the functions F_{p_i} is governed by the Born rule $F_{p_i}(a_1, \ldots, a_d) = |\,a_i\,|^2$, and $\sum_{i=0}^{d} {}_{p_i} = 1$. A one-qudit ($L = 1$) quantum gate is a unitary transformation $U^{(d)}\colon \mathcal{H}_{\mathrm{q}}^{(d)} \to \mathcal{H}_{\mathrm{q}}^{(d)}$ described by unitary $d \times d$ complex matrices acting on the vector (1.1), and for a register containing L qudits quantum gates are unitary $d^L \times d^L$ matrices. The quantum circuit model (Deutsch 1985, Barenco *et al* 1995) forms the basis for the standard concept of quantum computing. Here the quantum algorithms are compiled as a sequence of elementary gates acting on a register containing L qubits (or qudits), followed by a measurement to yield the result (Lloyd 1995, Brylinski and Brylinski 1994).

For further details on qudits and their transformations, see for example the reviews by Genovese and Traina (2008) and Wang *et al* (2020) and the references therein.

1.2 Membership amplitudes

Innovation 1.1. *We define an obscure qudit with d states* via *the following superposition (in place of that given in (1.1))*

$$\left|\,\psi_{\mathrm{ob}}^{(d)}\right\rangle = \sum_{i=0}^{d-1} \alpha_i a_i\,|\,i\rangle, \tag{1.2}$$

where a_i is a (complex) probability amplitude $a_i \in \mathbb{C}$, and we have introduced a (real) membership amplitude α_i, with $\alpha_i \in [0, 1]$, $0 \leqslant i \leqslant d - 1$.

The probability $_{pi}$ to find the *i*th state upon measurement and the membership function μ_i (of truth) for the *i*th state are both functions of the corresponding amplitudes, as follows

$$p_i = F_{p_i}(a_0, \dots, a_{d-1}), \quad 0 \leqslant p_i \leqslant 1, \tag{1.3}$$

$$\mu_i = F_{\mu_i}(\alpha_0, \dots, \alpha_{d-1}), \quad 0 \leqslant \mu_i \leqslant 1. \tag{1.4}$$

The dependence of the probabilities of the *i*th states upon the amplitudes, i.e., the form of the function F_{p_i} is fixed by the Born rule

$$F_{p_i}(a_1, \dots, a_n) = | a_i |^2, \tag{1.5}$$

while the form of F_{μ_i} will vary according to different obscurity assumptions. In this paper we consider only real membership amplitudes and membership functions—complex obscure sets and numbers were considered in Buckley (1989), Ramot *et al* (2002), and Garrido (2012). In this context, the real functions F_{p_i} and F_{μ_i}, $0 \leqslant i \leqslant d - 1$ will contain complete information about the obscure qudit (1.2).

We impose the normalization conditions

$$\sum_{i=0}^{d-1} p_i = 1, \tag{1.6}$$

$$\sum_{i=0}^{d-1} \mu_i = 1, \tag{1.7}$$

where the first condition is standard in quantum mechanics, while the second condition is taken to hold by analogy. Although (1.7) may not be satisfied, we will not consider that case.

For $d = 2$, we obtain for the obscure qubit the general form, instead of that in (1.2),

$$\left| \psi_{\mathrm{ob}}^{(2)} \right\rangle = \alpha_0 a_0 \, | \, 0 \rangle + \alpha_1 a_1 \, | \, 1 \rangle, \tag{1.8}$$

$$F_{p_0}(a_0, a_1) + F_{p_1}(a_0, a_1) = 1, \tag{1.9}$$

$$F_{\mu_0}(\alpha_0, \alpha_1) + F_{\mu_1}(\alpha_0, \alpha_1) = 1. \tag{1.10}$$

The Born probabilities to observe the states $| \, 0 \rangle$ and $| \, 1 \rangle$ are

$$p_0 = F_{p_0}^{\mathrm{Born}}(a_0, a_1) = | a_0 |^2, \quad p_1 = F_{p_1}^{\mathrm{Born}}(a_0, a_1) = | a_1 |^2. \tag{1.11}$$

Innovation 1.2. *The membership functions are*

$$\mu_0 = F_{\mu_0}(\alpha_0, \alpha_1), \quad \mu_1 = F_{\mu_1}(\alpha_0, \alpha_1). \tag{1.12}$$

If we assume the Born rule (1.11) for the membership functions as well

$$F_{\mu_0}(\alpha_0, \alpha_1) = \alpha_0^2, \quad F_{\mu_1}(\alpha_0, \alpha_1) = \alpha_1^2, \tag{1.13}$$

which is one of various possibilities depending on the chosen model, then

$$\mid a_0 \mid^2 + \mid a_1 \mid^2 = 1, \tag{1.14}$$

$$\alpha_0^2 + \alpha_1^2 = 1. \tag{1.15}$$

Using (1.14)–(1.15) we can parameterize (1.8) as

$$\left| \psi_{\mathrm{ob}}^{(2)} \right\rangle = \cos\frac{\theta}{2} \cos\frac{\theta_\mu}{2} \mid 0\rangle + e^{i\varphi} \sin\frac{\theta}{2} \sin\frac{\theta_\mu}{2} \mid 1\rangle, \tag{1.16}$$

$$0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad 0 \leqslant \theta_\mu \leqslant \pi. \tag{1.17}$$

Therefore, obscure qubits (with Born-like rule for the membership functions) are geometrically described by a pair of vectors, each inside a Bloch ball (and not as vectors on the boundary spheres, because ‘ | sin |, | cos |⩽1 ’), where one is for the probability amplitude (an ellipsoid inside the Bloch ball with $\theta_\mu = \mathrm{const}_1$) and the other is for the membership amplitude (which is reduced to an ellipse, being a slice inside the Bloch ball with $\theta = \mathrm{const}_2$, $\varphi = \mathrm{const}_3$). However, the norm of the obscure qubits is not constant because

$$\left\langle \psi_{\mathrm{ob}}^{(2)} \middle| \psi_{\mathrm{ob}}^{(2)} \right\rangle = \frac{1}{2} + \frac{1}{4}\cos\left(\theta + \theta_\mu\right) + \frac{1}{4}\cos\left(\theta - \theta_\mu\right). \tag{1.18}$$

In the case where $\theta = \theta_\mu$, the norm (1.18) becomes $1 - \frac{1}{2}\sin^2\theta$, reaching its minimum $\frac{1}{2}$ when $\theta = \theta_\mu = \frac{\pi}{2}$.

Note that for complicated functions $F_{\mu_{0,1}}(\alpha_0, \alpha_1)$, the condition (1.15) may be not satisfied but the condition (1.7) should nevertheless always be valid. The concrete form of the functions $F_{\mu_{0,1}}(\alpha_0, \alpha_1)$ depends upon the chosen model. In the simplest case, we can identify two arcs on the Bloch ellipse for α_0, α_1 with the membership functions and obtain

$$F_{\mu_0}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arctan\frac{\alpha_1}{\alpha_0}, \tag{1.19}$$

$$F_{\mu_1}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arctan\frac{\alpha_0}{\alpha_1}, \tag{1.20}$$

such that $\mu_0 + \mu_1 = 1$, as in (1.7).

In Mannucci (2006) and Maron *et al* (2013) a two stage special construction of quantum obscure/fuzzy sets was considered. The so-called classical-quantum obscure/fuzzy registers were introduced in the first step (for $n = 2$, the minimal case) as

$$|\,s\rangle_f = \sqrt{1-f}\,|\,0\rangle + \sqrt{f}\,|\,1\rangle, \tag{1.21}$$

$$|\,s\rangle_g = \sqrt{1-g}\,|\,0\rangle + \sqrt{g}\,|\,1\rangle, \tag{1.22}$$

where $f, g \in [0, 1]$ are the relevant classical-quantum membership functions. In the second step their quantum superposition is defined by

$$|\,s\rangle = c_f\,|\,s\rangle_f + c_g\,|\,s\rangle_g, \tag{1.23}$$

where c_f and c_g are the probability amplitudes of the fuzzy states $|\,s\rangle_f$ and $|\,s\rangle_g$, respectively. It can be seen that the state (1.23) is a particular case of (1.8) with

$$\alpha_0 a_0 = c_f\sqrt{1-f} + c_g\sqrt{1-g}, \tag{1.24}$$

$$\alpha_1 a_1 = c_f\sqrt{f} + c_g\sqrt{g}\,. \tag{1.25}$$

This gives explicit connection of our double amplitude description of obscure qubits with the approach (Mannucci 2006, Maron *et al* 2013) which uses probability amplitudes and the membership functions. It is important to note that the use of the membership amplitudes introduced here α_i and (1.2) allows us to exploit the standard quantum gates but not to define new special ones, as in Mannucci (2006) and Maron *et al* (2013).

Another possible form of $F_{\mu_{0,\,1}}(\alpha_0, \alpha_1)$ (1.12), with the corresponding membership functions satisfying the standard fuzziness rules, can be found using a standard homeomorphism between the circle and the square. In Hannachi *et al* (2007b) and Rybalov *et al* (2014), this transformation was applied to the probability amplitudes $a_{0,\,1}$.

Innovation 1.3. *Here we exploit it for the membership amplitudes* $\alpha_{0,\,1}$

$$F_{\mu_0}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arcsin\sqrt{\frac{\alpha_0^2\operatorname{sign}\alpha_0 - \alpha_1^2\operatorname{sign}\alpha_1 + 1}{2}}\,, \tag{1.26}$$

$$F_{\mu_1}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arcsin\sqrt{\frac{\alpha_0^2\operatorname{sign}\alpha_0 + \alpha_1^2\operatorname{sign}\alpha_1 + 1}{2}}\,. \tag{1.27}$$

So for positive $\alpha_{0,\,1}$, we obtain (cf Hannachi *et al* 2007b)

$$F_{\mu_0}(\alpha_0, \alpha_1) = \frac{2}{\pi}\arcsin\sqrt{\frac{\alpha_0^2 - \alpha_1^2 + 1}{2}}, \tag{1.28}$$

$$F_{\mu_1}(\alpha_0, \alpha_1) = 1. \tag{1.29}$$

The equivalent membership functions for the outcome are

$$\max\Big(\min\big(F_{\mu_0}(\alpha_0, \alpha_1), 1 - F_{\mu_1}(\alpha_0, \alpha_1)\big), \min\big(1 - F_{\mu_0}(\alpha_0, \alpha_1)\big), F_{\mu_1}(\alpha_0, \alpha_1)\Big), \tag{1.30}$$

$$\min\Big(\max\big(F_{\mu_0}(\alpha_0, \alpha_1), 1 - F_{\mu_1}(\alpha_0, \alpha_1)\big), \max\big(1 - F_{\mu_0}(\alpha_0, \alpha_1)\big), F_{\mu_1}(\alpha_0, \alpha_1)\Big). \tag{1.31}$$

There are many different models for $F_{\mu_{0,\,1}}(\alpha_0, \alpha_1)$ which can be introduced in such a way that they satisfy the obscure set axioms (Dubois and Prade 2000, Zimmermann 2011).

1.3 Transformations of obscure qubits

Let us consider the obscure qubits in the vector representation, such that

$$|\,0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\,1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1.32}$$

are basis vectors of the two-dimensional Hilbert space $\mathcal{H}_{\mathrm{q}}^{(2)}$. A standard quantum computational process in the quantum register with L obscure qubits (qudits (1.1)) is performed by sequences of unitary matrices U of size $2^L \times 2^L$ ($n^L \times n^L$), $\mathsf{U}^\dagger\mathsf{U} = \mathsf{I}$, which are called quantum gates (I is the unit matrix). Thus, for one obscure qubit, the quantum gates are 2×2 unitary complex matrices.

Innovation 1.4. *In the vector representation, an obscure qubit differs from the standard qubit (1.8) by a 2×2 invertible diagonal (not necessarily unitary) matrix*

$$\left|\,\psi_{\mathrm{ob}}^{(2)}\right\rangle = \mathsf{M}(\alpha_0, \alpha_1)\,|\,\psi^{(2)}\rangle, \tag{1.33}$$

$$\mathsf{M}(\alpha_0, \alpha_1) = \begin{pmatrix} \alpha_0 & 0 \\ 0 & \alpha_1 \end{pmatrix}. \tag{1.34}$$

We call $\mathsf{M}(\alpha_0, \alpha_1)$ a membership matrix which can optionally have the property

$$\mathrm{tr}\mathsf{M}^2 = 1, \tag{1.35}$$

if (1.15) holds.

Let us introduce the orthogonal commuting projection operators

$$\mathsf{P}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{P}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.36}$$

$$\mathsf{P}_0^2 = \mathsf{P}_0, \;\; \mathsf{P}_1^2 = \mathsf{P}_1, \;\; \mathsf{P}_0\mathsf{P}_1 = \mathsf{P}_1\mathsf{P}_0 = \mathsf{0}, \tag{1.37}$$

where $\mathsf{0}$ is the 2×2 zero matrix. Well-known properties of the projections are that

$$\mathsf{P}_0 \mid \psi^{(2)}\rangle = a_0 \mid 0\rangle, \qquad \mathsf{P}_1 \mid \psi^{(2)}\rangle = a_1 \mid 0\rangle, \tag{1.38}$$

$$\langle\psi^{(2)} \mid \mathsf{P}_0 \mid \psi^{(2)}\rangle = \mid a_0 \mid^2, \qquad \langle\psi^{(2)} \mid \mathsf{P}_1 \mid \psi^{(2)}\rangle = \mid a_1 \mid^2. \tag{1.39}$$

Innovation 1.5. *The membership matrix (1.34) can be defined as a linear combination of the projection operators with the membership amplitudes as coefficients*

$$\mathsf{M}(\alpha_0, \alpha_1) = \alpha_0\mathsf{P}_0 + \alpha_1\mathsf{P}_1. \tag{1.40}$$

We compute

$$\mathsf{M}(\alpha_0, \alpha_1) \left| \psi_{\mathrm{ob}}^{(2)} \right\rangle = \alpha_0^2 a_0 \mid 0\rangle + \alpha_1^2 a_1 \mid 1\rangle. \tag{1.41}$$

We can therefore treat the application of the membership matrix (1.33) as providing the origin of a reversible but non-unitary obscure measurement on the standard qubit to obtain an obscure qubit—cf the mirror measurement (Battilotti and Zizzi 2004, Zizzi 2005) and also the origin of ordinary qubit states on the fuzzy sphere (Zizzi and Pessa 2014).

An obscure analog of the density operator (for a pure state) is the following form for the density matrix in the vector representation

$$\rho_{\mathrm{ob}}^{(2)} = \left| \psi_{\mathrm{ob}}^{(2)} \right\rangle \left\langle \psi_{\mathrm{ob}}^{(2)} \right| = \begin{pmatrix} \alpha_0^2 \mid a_0 \mid^2 & \alpha_0 a_0^* \alpha_1 a_1 \\ \alpha_0 a_0 \alpha_1 a_1^* & \alpha_1^2 \mid a_1 \mid^2 \end{pmatrix} \tag{1.42}$$

with the obvious standard singularity property $\det \rho_{\mathrm{ob}}^{(2)} = 0$. But $\mathrm{tr} \rho_{\mathrm{ob}}^{(2)} = \alpha_0^2 \mid a_0 \mid^2 + \alpha_1^2 \mid a_1 \mid^2 \neq 1$, and here there is no idempotence $\left(\rho_{\mathrm{ob}}^{(2)}\right)^2 \neq \rho_{\mathrm{ob}}^{(2)}$, which can distinct $\rho_{\mathrm{ob}}^{(2)}$ from the standard density operator.

1.4 Kronecker obscure qubits

We next introduce an analog of quantum superposition for membership amplitudes, called 'obscure superposition' (cf Cunha *et al* 2019, and also Toffano and Dubois 2017).

Innovation 1.6. *Quantum amplitudes and membership amplitudes will here be considered separately in order to define an obscure qubit taking the form of a double superposition (cf (1.8), and a generalized analog for qudits (1.1) is straightforward)*

$$\mid \boldsymbol{\Psi}_{\mathrm{ob}}\rangle = \frac{\mathsf{A}_0 \mid 0\rangle + \mathsf{A}_1 \mid 1\rangle}{\sqrt{2}}, \tag{1.43}$$

where the two-dimensional vectors

$$\mathsf{A}_{0,\,1} = \begin{bmatrix} a_{0,\,1} \\ \alpha_{0,\,1} \end{bmatrix} \tag{1.44}$$

are the (double) obscure-quantum amplitudes of the generalized states $|\,0\rangle, |\,1\rangle$.

For the conjugate of an obscure qubit we put (informally)

$$\langle \boldsymbol{\Psi}_{\text{ob}}\,| = \frac{\mathsf{A}_0^{\star}\langle 0\,| + \mathsf{A}_1^{\star}\langle 1\,|}{\sqrt{2}}, \tag{1.45}$$

where we denote $\mathsf{A}_{0,\,1}^{\star} = \begin{bmatrix} a_{0,\,1}^{*} & \alpha_{0,\,1} \end{bmatrix}$, such that $\mathsf{A}_{0,\,1}^{\star}\mathsf{A}_{0,\,1} = |\,a_{0,\,1}\,|^2 + \alpha_{0,\,1}^2$. The (double) obscure qubit is normalized in such a way that, if the conditions (1.14)–(1.15) hold, then

$$\langle \boldsymbol{\Psi}_{\text{ob}} | \boldsymbol{\Psi}_{\text{ob}} \rangle = \frac{|\,a_0\,|^2 + |\,a_1\,|^2}{2} + \frac{\alpha_0^2 + \alpha_1^2}{2} = 1. \tag{1.46}$$

Innovation 1.7. *A measurement should be made separately and independently in the probability space and the membership space, which can be represented using an analog of the Kronecker product.*

Indeed, in the vector representation (1.32) for the quantum states and for the direct product amplitudes (1.44) we should have

$$|\,\boldsymbol{\Psi}_{\text{ob}}\rangle_{(0)} = \frac{1}{\sqrt{2}}\mathsf{A}_0 \otimes_{\text{K}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathsf{A}_1 \otimes_{\text{K}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.47}$$

where the (left) Kronecker product is defined by (see (1.32))

$$\begin{gathered} \begin{bmatrix} a \\ \alpha \end{bmatrix} \otimes_{\text{K}} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{bmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ \alpha\begin{pmatrix} c \\ d \end{pmatrix} \end{bmatrix} = \begin{bmatrix} a(c\mathsf{e}_0 + d\mathsf{e}_1) \\ \alpha(c\mathsf{e}_0 + d\mathsf{e}_1) \end{bmatrix}, \\ \mathsf{e}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \mathsf{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \mathsf{e}_{0,\,1} \in \mathcal{H}_{\text{q}}^{(2)}. \end{gathered} \tag{1.48}$$

Informally, the wave function of the obscure qubit, in the vector representation, now lives in the four-dimensional space of (1.48), which has two two-dimensional spaces as blocks. The upper block, the quantum subspace, is the ordinary Hilbert space $\mathcal{H}_{\text{q}}^{(2)}$, but the lower block should have special (fuzzy) properties, if it is treated as an obscure (membership) subspace $\mathcal{V}_{\text{memb}}^{(2)}$. Thus, the four-dimensional space, where lives $\left|\,\Psi_{\text{ob}}^{(2)}\right\rangle$, is not an ordinary tensor product of vector spaces because of (1.48) and the vector A on the lhs has entries of different natures, i.e., the quantum amplitudes $a_{0,\,1}$ and the membership amplitudes $\alpha_{0,\,1}$. Despite the unit vectors in $\mathcal{H}_{\mathrm{q}}^{(2)}$ and $\mathcal{V}_{\mathrm{memb}}^{(2)}$ having the same form (1.32), they belong to different spaces (because they are vector spaces over different fields). Therefore, instead of (1.48), we introduce a Kronecker-like product $\tilde{\otimes}_{\mathrm{K}}$ by

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} \tilde{\otimes}_{\mathrm{K}} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{bmatrix} a(c\mathsf{e}_0 + d\mathsf{e}_1) \\ \alpha(c\varepsilon_0 + d\varepsilon_1) \end{bmatrix}, \tag{1.49}$$

$$\mathsf{e}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathsf{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathsf{e}_{0,\,1} \in \mathcal{H}_{\mathrm{q}}^{(2)}, \tag{1.50}$$

$$\varepsilon_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(\mu)}, \quad \varepsilon_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(\mu)}, \quad \varepsilon_{0,\,1} \in \mathcal{V}_{\mathrm{memb}}^{(2)}. \tag{1.51}$$

In this way, the obscure qubit (1.43) can be presented in the from

$$\begin{aligned} |\,\mathbf{\Psi}_{\mathrm{ob}}\rangle &= \frac{1}{\sqrt{2}} \begin{bmatrix} a_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(\mu)} \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} a_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(\mu)} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} a_0\mathsf{e}_0 \\ \alpha_0\varepsilon_0 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} a_1\mathsf{e}_1 \\ \alpha_1\varepsilon_1 \end{bmatrix}. \end{aligned} \tag{1.52}$$

Therefore, we call the double obscure qubit (1.52) a Kronecker obscure qubit to distinguish it from the obscure qubit (1.8). It can be also presented using the Hadamard product (the element-wise or Schur product)

$$\begin{bmatrix} a \\ \alpha \end{bmatrix} \otimes_{\mathrm{H}} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{bmatrix} ac \\ \alpha d \end{bmatrix} \tag{1.53}$$

in the following form

$$|\,\mathbf{\Psi}_{\mathrm{ob}}\rangle = \frac{1}{\sqrt{2}}\mathsf{A}_0 \otimes_{\mathrm{H}} \mathsf{E}_0 + \frac{1}{\sqrt{2}}\mathsf{A}_1 \otimes_{\mathrm{H}} \mathsf{E}_1, \tag{1.54}$$

where the unit vectors of the total four-dimensional space are

$$\mathsf{E}_{0,\,1} = \begin{bmatrix} \mathsf{e}_{0,\,1} \\ \varepsilon_{0,\,1} \end{bmatrix} \in \mathcal{H}_{\mathrm{q}}^{(2)} \times \mathcal{V}_{\mathrm{memb}}^{(2)}. \tag{1.55}$$

The probabilities $p_{0,\,1}$ and membership functions $\mu_{0,\,1}$ of the states $|\,0\rangle$ and $|\,1\rangle$ are computed through the corresponding amplitudes by (1.11) and (1.12)

$$p_i = |\,a_i\,|^2, \quad \mu_i = F_{\mu_i}(\alpha_0, \alpha_1), \quad i = 0, 1, \tag{1.56}$$

and in the particular case by (1.13) satisfying (1.15).

By way of example, consider a Kronecker obscure qubit (with a real quantum part) with probability p and membership function μ (measure of trust) of the state $|\,0\rangle$, and of the state $|\,1\rangle$ given by $1 - p$ and $1 - \mu$, respectively. In the model (1.19)–(1.20) for μ_i (which is not Born-like) we obtain

$$
\begin{aligned}
|\,\boldsymbol{\Psi}_{\mathrm{ob}}\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix}\begin{pmatrix}\sqrt{p}\\ 0\end{pmatrix}\\ \begin{pmatrix}\cos\dfrac{\pi}{2}\mu\\ 0\end{pmatrix}^{(\mu)}\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}\begin{pmatrix}0\\ \sqrt{1-p}\end{pmatrix}\\ \begin{pmatrix}0\\ \sin\dfrac{\pi}{2}\mu\end{pmatrix}^{(\mu)}\end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}\mathsf{e}_0\sqrt{p}\\ \varepsilon_0\cos\dfrac{\pi}{2}\mu\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}\mathsf{e}_1\sqrt{1-p}\\ \varepsilon_1\sin\dfrac{\pi}{2}\mu\end{bmatrix},
\end{aligned}
\tag{1.57}
$$

where e_i and ε_i are unit vectors defined in (1.50) and (1.51).

This can be compared, e.g., with the classical-quantum approach (1.23), and Mannucci (2006) and Maron *et al* (2013), in which the elements of the columns are multiplied, while we consider them independently and separately.

1.5 Obscure-quantum measurement

Let us consider the case of one Kronecker obscure qubit register $L = 1$ (see (1.47)), or using (1.48) in the vector representation (1.52). The standard (double) orthogonal commuting projection operators, Kronecker projections, are (cf (1.36))

$$
\mathbf{P}_0 = \begin{bmatrix}\mathsf{P}_0 & \mathsf{0}\\ \mathsf{0} & \mathsf{P}_0^{(\mu)}\end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix}\mathsf{P}_1 & \mathsf{0}\\ \mathsf{0} & \mathsf{P}_1^{(\mu)}\end{bmatrix}, \tag{1.58}
$$

where $\mathsf{0}$ is the 2×2 zero matrix, and $\mathsf{P}_{0,\,1}^{(\mu)}$ are the projections in the membership subspace $\mathcal{V}_{\mathrm{memb}}^{(2)}$ (of the same form as the ordinary quantum projections $\mathsf{P}_{0,\,1}$ (1.36))

$$
\mathsf{P}_0^{(\mu)} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}^{(\mu)}, \quad \mathsf{P}_1^{(\mu)} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}^{(\mu)}, \quad \mathsf{P}_0^{(\mu)}, \mathsf{P}_1^{(\mu)} \in \operatorname{End}\mathcal{V}_{\mathrm{memb}}^{(2)}, \tag{1.59}
$$

$$
\mathsf{P}_0^{(\mu)2} = \mathsf{P}_0^{(\mu)}, \ \mathsf{P}_1^{(\mu)2} = \mathsf{P}_1^{(\mu)}, \ \mathsf{P}_0^{(\mu)}\mathsf{P}_1^{(\mu)} = \mathsf{P}_1^{(\mu)}\mathsf{P}_0^{(\mu)} = \mathsf{0}. \tag{1.60}
$$

For the double projections we have (cf (1.37))

$$
\mathbf{P}_0^2 = \mathbf{P}_0, \ \mathbf{P}_1^2 = \mathbf{P}_1, \ \mathbf{P}_0\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_0 = \mathbf{0}, \tag{1.61}
$$

where $\mathbf{0}$ is the 4×4 zero matrix, and $\mathbf{P}_{0,\,1}$ act on the Kronecker qubit (1.58) in the standard way (cf (1.38))

$$\mathbf{P}_0 \mid \boldsymbol{\Psi}_{\mathrm{ob}}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} a_0\begin{pmatrix}1\\0\end{pmatrix} \\ \alpha_0\begin{pmatrix}1\\0\end{pmatrix}^{(\mu)} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} a_0\mathsf{e}_0 \\ \alpha_0\varepsilon_0 \end{bmatrix} = \frac{1}{\sqrt{2}}\mathsf{A}_0 \otimes_{\mathrm{H}} \mathsf{E}_0, \tag{1.62}$$

$$\mathbf{P}_1 \mid \boldsymbol{\Psi}_{\mathrm{ob}}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} a_1\begin{pmatrix}0\\1\end{pmatrix} \\ \alpha_1\begin{pmatrix}0\\1\end{pmatrix}^{(\mu)} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} a_1\mathsf{e}_1 \\ \alpha_1\varepsilon_1 \end{bmatrix} = \frac{1}{\sqrt{2}}\mathsf{A}_1 \otimes_{\mathrm{H}} \mathsf{E}_1. \tag{1.63}$$

Observe that for Kronecker qubits there exist in addition to (1.58) the following orthogonal commuting projection operators

$$\mathbf{P}_{01} = \begin{bmatrix} \mathsf{P}_0 & 0 \\ 0 & \mathsf{P}_1^{(\mu)} \end{bmatrix}, \quad \mathbf{P}_{10} = \begin{bmatrix} \mathsf{P}_1 & 0 \\ 0 & \mathsf{P}_0^{(\mu)} \end{bmatrix}, \tag{1.64}$$

and we call these the crossed double projections. They satisfy the same relations as (1.61)

$$\mathbf{P}_{01}^2 = \mathbf{P}_{01}, \;\; \mathbf{P}_{10}^2 = \mathbf{P}_{10}, \;\; \mathbf{P}_{01}\mathbf{P}_{10} = \mathbf{P}_{10}\mathbf{P}_{01} = \mathbf{0}, \tag{1.65}$$

but act on the obscure qubit in a different (mixing) way than (1.62), i.e.,

$$\mathbf{P}_{01} \mid \boldsymbol{\Psi}_{\mathrm{ob}}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} a_0\begin{pmatrix}1\\0\end{pmatrix} \\ \alpha_1\begin{pmatrix}0\\1\end{pmatrix} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} a_0\mathsf{e}_0 \\ \alpha_1\varepsilon_1 \end{bmatrix}, \tag{1.66}$$

$$\mathbf{P}_{10} \mid \boldsymbol{\Psi}_{\mathrm{ob}}\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} a_1\begin{pmatrix}0\\1\end{pmatrix} \\ \alpha_0\begin{pmatrix}1\\0\end{pmatrix} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} a_1\mathsf{e}_1 \\ \alpha_0\varepsilon_0 \end{bmatrix}. \tag{1.67}$$

The multiplication of the crossed double projections (1.64) and the double projections (1.58) is given by

$$\mathbf{P}_{01}\mathbf{P}_0 = \mathbf{P}_0\mathbf{P}_{01} = \begin{bmatrix} \mathsf{P}_0 & 0 \\ 0 & 0 \end{bmatrix} \equiv \mathbf{Q}_0, \quad \mathbf{P}_{01}\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_{01} = \begin{bmatrix} 0 & 0 \\ 0 & \mathsf{P}_1^{(\mu)} \end{bmatrix} \equiv \mathbf{Q}_1^{(\mu)}, \tag{1.68}$$

$$\mathbf{P}_{10}\mathbf{P}_0 = \mathbf{P}_0\mathbf{P}_{10} = \begin{bmatrix} 0 & 0 \\ 0 & \mathsf{P}_0^{(\mu)} \end{bmatrix} \equiv \mathbf{Q}_0^{(\mu)}, \quad \mathbf{P}_{10}\mathbf{P}_1 = \mathbf{P}_1\mathbf{P}_{10} = \begin{bmatrix} \mathsf{P}_1 & 0 \\ 0 & 0 \end{bmatrix} \equiv \mathbf{Q}_1, \tag{1.69}$$

where the operators $\mathbf{Q}_0$, $\mathbf{Q}_1$ and $\mathbf{Q}_0^{(\mu)}$, $\mathbf{Q}_1^{(\mu)}$ satisfy

$$\mathbf{Q}_0^2 = \mathbf{Q}_0, \;\; \mathbf{Q}_1^2 = \mathbf{Q}_1, \;\; \mathbf{Q}_1\mathbf{Q}_0 = \mathbf{Q}_0\mathbf{Q}_1 = \mathbf{0}, \tag{1.70}$$

$$\mathbf{Q}_0^{(\mu)2} = \mathbf{Q}_0^{(\mu)}, \;\; \mathbf{Q}_1^{(\mu)2} = \mathbf{Q}_1^{(\mu)}, \;\; \mathbf{Q}_1^{(\mu)}\mathbf{Q}_0^{(\mu)} = \mathbf{Q}_0^{(\mu)}\mathbf{Q}_1^{(\mu)} = \mathbf{0}, \tag{1.71}$$

$$\mathbf{Q}_1^{(\mu)}\mathbf{Q}_0 = \mathbf{Q}_0^{(\mu)}\mathbf{Q}_1 = \mathbf{Q}_1\mathbf{Q}_0^{(\mu)} = \mathbf{Q}_0\mathbf{Q}_1^{(\mu)} = \mathbf{0}, \tag{1.72}$$

and we call these ‘half Kronecker (double) projections’.

These relations imply that the process of measurement when using Kronecker obscure qubits (i.e. for quantum computation with truth or membership) is more complicated than in the standard case.

To show this, let us calculate the obscure analogs of expected values for the projections above. Using the notation

$$\bar{\mathbf{A}} \equiv \langle \boldsymbol{\Psi}_{\text{ob}} \mid \mathbf{A} \mid \boldsymbol{\Psi}_{\text{ob}} \rangle. \tag{1.73}$$

Then, using (1.43)–(1.45) for the projection operators $\mathbf{P}_i$, $\mathbf{P}_{ij}$, $\mathbf{Q}_i$, $\mathbf{Q}_i^{(\mu)}$, $i, j = 0, 1$, $i \neq j$, we obtain (cf (1.39))

$$\bar{\mathbf{P}}_i = \frac{\mid a_i \mid^2 + \alpha_i^2}{2}, \quad \bar{\mathbf{P}}_{ij} = \frac{\mid a_i \mid^2 + \alpha_j^2}{2}, \tag{1.74}$$

$$\bar{\mathbf{Q}}_i = \frac{\mid a_i \mid^2}{2}, \qquad \bar{\mathbf{Q}}_i^{(\mu)} = \frac{\alpha_i^2}{2}. \tag{1.75}$$

So follows the relation between the obscure analogs of expected values of the projections

$$\bar{\mathbf{P}}_i = \bar{\mathbf{Q}}_i + \bar{\mathbf{Q}}_i^{(\mu)}, \quad \bar{\mathbf{P}}_{ij} = \bar{\mathbf{Q}}_i + \bar{\mathbf{Q}}_j^{(\mu)}. \tag{1.76}$$

Taking the ket corresponding to the bra Kronecker qubit (1.52) in the form

$$\langle \boldsymbol{\Psi}_{\text{ob}} \mid = \frac{1}{\sqrt{2}}\left[a_0^*(1 \;\; 0), \;\; \alpha_0(1 \;\; 0) \right] + \frac{1}{\sqrt{2}}\left[a_1^*(0 \;\; 1), \;\; \alpha_1(0 \;\; 1) \right], \tag{1.77}$$

a Kronecker (4×4) obscure analog of the density matrix for a pure state is given by (cf (1.42))

$$\rho_{\text{ob}}^{(2)} = \mid \boldsymbol{\Psi}_{\text{ob}} \rangle \langle \boldsymbol{\Psi}_{\text{ob}} \mid = \frac{1}{2}\begin{pmatrix} \mid a_0 \mid^2 & a_0 a_1^* & a_0 \alpha_0 & a_0 \alpha_1 \\ a_1 a_0^* & \mid a_1 \mid^2 & a_1 \alpha_0 & a_1 \alpha_1 \\ \alpha_0 a_0^* & \alpha_0 a_1^* & \alpha_0^2 & \alpha_0 \alpha_1 \\ \alpha_1 a_0^* & \alpha_1 a_1^* & \alpha_0 \alpha_1 & \alpha_1^2 \end{pmatrix}. \tag{1.78}$$

If the Born rule for the membership functions (1.13) and the conditions (1.14)–(1.15) are satisfied, then the density matrix (1.78) is non-invertible because $\det \rho_{\text{ob}}^{(2)} = 0$ and has unit trace $\text{tr}\rho_{\text{ob}}^{(2)} = 1$ but is not idempotent $\left(\rho_{\text{ob}}^{(2)}\right)^2 \neq \rho_{\text{ob}}^{(2)}$ because it holds for the ordinary quantum density matrix (Nielsen and Chuang 2000).

1.6 Kronecker obscure-quantum gates

In general, (double) obscure-quantum computation with L Kronecker obscure qubits (or qudits) can be performed by a product of unitary (block) matrices $\mathbf{U}$ of the (double size to the standard one) size $2 \times (2^L \times 2^L)$ (or $2 \times (n^L \times n^L)$), $\mathbf{U}^\dagger \mathbf{U} = \mathbf{I}$ (here $\mathbf{I}$ is the unit matrix of the same size as $\mathbf{U}$). We can also call such computation a quantum computation with truth (or with membership).

Let us consider obscure-quantum computation with one Kronecker obscure qubit. Informally, we can present the Kronecker obscure qubit (1.52) in the form

$$|\,\boldsymbol{\Psi}_{\mathrm{ob}}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \\ \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}^{(\mu)} \end{bmatrix}. \tag{1.79}$$

Innovation 1.8. *The state* $|\,\boldsymbol{\Psi}_{\mathrm{ob}}\rangle$ *can be interpreted as a vector in the direct product (not tensor product) space* $\mathcal{H}_{\mathrm{q}}^{(2)} \times \mathcal{V}_{\mathrm{memb}}^{(2)}$*, where* $\mathcal{H}_{\mathrm{q}}^{(2)}$ *is the standard two-dimensional Hilbert space of the qubit, and* $\mathcal{V}_{\mathrm{memb}}^{(2)}$ *can be treated as the membership space, which has a different nature from the qubit space and can have a more complex structure.*

For discussion of similar spaces, see for example Dubois *et al* (2000), Bělohlávek (2002), Smith (2008), and Zimmermann (2011). In general, one can consider obscure-quantum computation as a set of abstract computational rules, independently of the introduction of the corresponding spaces.

An obscure-quantum gate will be defined as an elementary transformation on an obscure qubit (1.79) and is performed by unitary (block) matrices of size 4 × 4 (over $\mathbb{C}$) acting in the total space $\mathcal{H}_{\mathrm{q}}^{(2)} \times \mathcal{V}_{\mathrm{memb}}^{(2)}$

$$\mathbf{U} = \begin{pmatrix} \mathsf{U} & 0 \\ 0 & \mathsf{U}^{(\mu)} \end{pmatrix}, \quad \mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{I}, \tag{1.80}$$

$$\mathsf{U}\mathsf{U}^\dagger = \mathsf{U}^\dagger\mathsf{U} = \mathsf{I}, \quad \mathsf{U}^{(\mu)}\mathsf{U}^{(\mu)\dagger} = \mathsf{U}^{(\mu)\dagger}\mathsf{U}^{(\mu)} = \mathsf{I}, \quad \mathsf{U} \in \mathrm{End}\mathcal{H}_{\mathrm{q}}^{(2)},\ \mathsf{U}^{(\mu)} \in \mathrm{End}\,\mathcal{V}_{\mathrm{memb}}^{(2)}, \tag{1.81}$$

where $\mathbf{I}$ is the unit 4 × 4 matrix, I is the unit 2 × 2 matrix, and U and $\mathsf{U}^{(\mu)}$ are unitary 2 × 2 matrices acting on the probability and membership subspaces, respectively. The matrix U (over $\mathbb{C}$) will be called a quantum gate, and we call the matrix $\mathsf{U}^{(\mu)}$ (over $\mathbb{R}$) an obscure gate. We assume that the obscure gates $\mathsf{U}^{(\mu)}$ are of the same shape as the standard quantum gates, but they act in the other (membership) space and have only real elements (see, e.g. Nielsen and Chuang 2000). In this case, an obscure-quantum gate is characterized by the pair $\{\mathsf{U}, \mathsf{U}^{(\mu)}\}$, where the components are known gates (in various combinations), e.g., for one qubit gates: Hadamard, Pauli-X (`NOT`) ,Y,Z (or two qubit gates e.g. `CNOT`, `SWAP`, etc). The transformed qubit then becomes (informally)

$$\mathbf{U} \mid \boldsymbol{\Psi}_{\mathrm{ob}}\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \mathsf{U} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \\ \frac{1}{\sqrt{2}} \mathsf{U}^{(\mu)} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}^{(\mu)} \end{bmatrix}. \tag{1.82}$$

Thus, the quantum and the membership parts are transformed independently for the block diagonal form (1.80). Some examples of this can be found, e.g., in Domenech and Freytes (2006), Mannucci (2006), and Maron *et al* (2013). Differences between the parts were mentioned in Kreinovich *et al* (2011). In this case, an obscure-quantum network is physically realized by a device performing elementary operations in sequence on obscure qubits (by a product of matrices), such that the quantum and membership parts are synchronized in time; for a discussion of the obscure part of such physical devices, see Hirota and Ozawa (1989), Kóczy and Hirota (1990), Virant (2000), and Kosko (1997). Then, the result of the obscure-quantum computation consists of the quantum probabilities of the states together with the calculated level of truth for each of them (see, e.g. Bolotin 2018).

For example, the obscure-quantum gate $\mathbf{U}_{\mathsf{H},\mathtt{NOT}} = \{\text{Hadamard}, \mathtt{NOT}\}$ acts on the state E_0 (1.55) as follows

$$\mathbf{U}_{\mathsf{H},\mathtt{NOT}} \mathsf{E}_0 = \mathbf{U}_{\mathsf{H},\mathtt{NOT}} \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{(\mu)} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{(\mu)} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} (\mathsf{e}_0 + \mathsf{e}_1) \\ \varepsilon_1 \end{bmatrix}. \tag{1.83}$$

It would be interesting to consider the case when $\mathbf{U}$ (1.80) is not block diagonal and try to find possible physical interpretations of the non-diagonal blocks.

1.7 Double entanglement

Let us introduce a register consisting of two obscure qubits ($L = 2$) in the computational basis $\mid \mathsf{ij}'\rangle = \mid \mathsf{i}\rangle \otimes \mid \mathsf{j}'\rangle$, as follows

$$\left| \boldsymbol{\Psi}_{\mathrm{ob}}^{(n=2)}(L = 2) \right\rangle = \mid \boldsymbol{\Psi}_{\mathrm{ob}}(2)\rangle = \frac{\mathsf{B}_{00'} \mid 00'\rangle + \mathsf{B}_{10'} \mid 10'\rangle + \mathsf{B}_{01'} \mid 01'\rangle + \mathsf{B}_{11'} \mid 11'\rangle}{\sqrt{2}}, \tag{1.84}$$

determined by two-dimensional vectors (encoding obscure-quantum amplitudes)

$$\mathsf{B}_{ij'} = \begin{bmatrix} b_{ij'} \\ \beta_{ij'} \end{bmatrix}, \quad i, j = 0, 1, \quad j' = 0', 1', \tag{1.85}$$

where $b_{ij'} \in \mathbb{C}$ are probability amplitudes for a set of pure states and $\beta_{ij'} \in \mathbb{R}$ are the corresponding membership amplitudes. By analogy with (1.43) and (1.46), the normalization factor in (1.84) is chosen so that

$$\langle \boldsymbol{\Psi}_{\mathrm{ob}}(2) | \boldsymbol{\Psi}_{\mathrm{ob}}(2)\rangle = 1, \tag{1.86}$$

if (cf (1.14)–(1.15))

$$|\, b_{00'} \,|^2 + |\, b_{10'} \,|^2 + |\, b_{01'} \,|^2 + |\, b_{11'} \,|^2 = 1, \tag{1.87}$$

$$\beta_{00'}^2 + \beta_{10'}^2 + \beta_{01'}^2 + \beta_{11'}^2 = 1. \tag{1.88}$$

A state of two qubits is entangled if it cannot be decomposed as a product of two one-qubit states, and otherwise it is separable (see, e.g. Nielsen and Chuang 2000).

Innovation 1.9. *We define a product of two obscure qubits (1.43) as*

$$|\, \Psi_{\text{ob}}\rangle \otimes \left|\, \Psi'_{\text{ob}}\right\rangle = \frac{\mathsf{A}_0 \otimes_{\text{H}} \mathsf{A}'_0 \,|\, 00'\rangle + \mathsf{A}_1 \otimes_{\text{H}} \mathsf{A}'_0 \,|\, 10'\rangle + \mathsf{A}_0 \otimes_{\text{H}} \mathsf{A}'_1 \,|\, 01'\rangle + \mathsf{A}_1 \otimes_{\text{H}} \mathsf{A}'_1 \,|\, 11'\rangle}{2}, \tag{1.89}$$

where $\otimes_{\text{H}}$ *is the Hadamard product (1.53).*

Comparing (1.84) and (1.89), we obtain two sets of relations, for probability amplitudes and for membership amplitudes

$$b_{ij'} = \frac{1}{\sqrt{2}} a_i a_{j'}, \tag{1.90}$$

$$\beta_{ij'} = \frac{1}{\sqrt{2}} \alpha_i \alpha_{j'}, \quad i, j = 0, 1, \quad j' = 0', 1'. \tag{1.91}$$

In this case, the relations (1.14)–(1.15) give (1.87)–(1.88).

Two obscure-quantum qubits are entangled if their joint state (1.84) cannot be presented as a product of one qubit states (1.89), and in the opposite case the states are called totally separable. It follows from (1.90)–(1.91) that there are two general conditions for obscure qubits to be entangled

$$b_{00'} b_{11'} \neq b_{10'} b_{01'}, \qquad \text{or } \det \mathbf{b} \neq 0,\, \mathbf{b} = \begin{pmatrix} b_{00'} & b_{01'} \\ b_{10'} & b_{11'} \end{pmatrix}, \tag{1.92}$$

$$\beta_{00'} \beta_{11'} \neq \beta_{10'} \beta_{01'}, \qquad \text{or } \det \boldsymbol{\beta} \neq 0,\, \boldsymbol{\beta} = \begin{pmatrix} \beta_{00'} & \beta_{01'} \\ \beta_{10'} & \beta_{11'} \end{pmatrix}. \tag{1.93}$$

The first equation (1.92) is the entanglement relation for the standard qubit, while the second condition (1.93) is for the membership amplitudes of the two obscure qubit joint state (1.84). The presence of two different conditions (1.92)–(1.93) leads to new additional possibilities (which do not exist for ordinary qubits) for partial entanglement (or partial separability), when only one of them is fulfilled. In this case, the states can be entangled in one subspace (quantum or membership) but not in the other.

The measure of entanglement is numerically characterized by the concurrence. Taking into account the two conditions (1.92)–(1.93), we propose to generalize the notion of concurrence for two obscure qubits in two ways. First, we introduce the vector obscure concurrence

$$\mathsf{C}_{\text{vect}} = \begin{bmatrix} C_{\text{q}} \\ C^{(\mu)} \end{bmatrix} = 2\begin{bmatrix} |\det \mathbf{b}| \\ |\det \beta| \end{bmatrix}, \tag{1.94}$$

where **b** and β are defined in (1.92)–(1.93), and $0 \leqslant C_{\text{q}} \leqslant 1$, $0 \leqslant C^{(\mu)} \leqslant 1$.

Innovation 1.10. *The corresponding scalar obscure concurrence can be defined as*

$$C_{\text{scal}} = \sqrt{\frac{|\det \mathbf{b}|^2 + |\det \beta|^2}{2}}, \tag{1.95}$$

such that $0 \leqslant C_{\text{scal}} \leqslant 1$*. Thus, two obscure qubits are totally separable, if* $C_{\text{scal}} = 0$.

For instance, for an obscure analog of the (maximally entangled) Bell state

$$|\boldsymbol{\Psi}_{\text{ob}}(2)\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}|00'\rangle + \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}|11'\rangle\right) \tag{1.96}$$

we obtain

$$\mathsf{C}_{\text{vect}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_{\text{scal}} = 1. \tag{1.97}$$

A more interesting example is the intermediately entangled two obscure qubit state, e.g.,

$$|\boldsymbol{\Psi}_{\text{ob}}(2)\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \end{bmatrix}|00'\rangle + \begin{bmatrix} \frac{1}{4} \\ \frac{\sqrt{5}}{4} \end{bmatrix}|10'\rangle + \begin{bmatrix} \frac{\sqrt{3}}{4} \\ \frac{1}{2\sqrt{2}} \end{bmatrix}|01'\rangle + \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{4} \end{bmatrix}|11'\rangle\right), \tag{1.98}$$

where the amplitudes satisfy (1.87)–(1.88). If the Born-like rule (as in (1.13)) holds for the membership amplitudes, then the probabilities and membership functions of the states in (1.98) are

$$p_{00'} = \frac{1}{4}, \; p_{10'} = \frac{1}{16}, \; p_{01'} = \frac{3}{16}, \; p_{11'} = \frac{1}{2}, \tag{1.99}$$

$$\mu_{00'} = \frac{1}{2}, \; \mu_{10'} = \frac{5}{16}, \; \mu_{01'} = \frac{1}{8}, \; \mu_{11'} = \frac{1}{16}. \tag{1.100}$$

This means that, e.g., that the state $|10'\rangle$ will be measured with the quantum probability 1/16 and the membership function (truth value) 5/16. For the entangled obscure qubit (1.98) we obtain the concurrences

$$\mathsf{C}_{\mathrm{vect}} = \begin{bmatrix} \frac{1}{2}\sqrt{2} - \frac{1}{8}\sqrt{3} \\ \frac{1}{8}\sqrt{2}\sqrt{5} - \frac{1}{4}\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.491 \\ 0.042 \end{bmatrix}, \tag{1.101}$$
$$C_{\mathrm{scal}} = \sqrt{\frac{53}{128} - \frac{1}{16}\sqrt{5} - \frac{1}{16}\sqrt{2}\sqrt{3}} = 0.348.$$

In the vector representation (1.49)–(1.52), we have

$$|\,\mathsf{ij}'\rangle = |\,\mathsf{i}\rangle \otimes |\,\mathsf{j}'\rangle = \begin{bmatrix} \mathsf{e}_i \otimes_{\mathrm{K}} \mathsf{e}_{j'} \\ \varepsilon_i \otimes_{\mathrm{K}} \varepsilon_{j'} \end{bmatrix}, \quad i, j = 0, 1, \quad j' = 0', 1', \tag{1.102}$$

where $\otimes_{\mathrm{K}}$ is the Kronecker product (1.48), and e_i, ε_i are defined in (1.50)–(1.51). Using (1.85) and the Kronecker-like product (1.49), we put (informally, with no summation)

$$\mathsf{B}_{ij'}\,|\,\mathsf{ij}'\rangle = \begin{bmatrix} b_{ij'}\mathsf{e}_i \otimes_{\mathrm{K}} \mathsf{e}_{j'} \\ \beta_{ij'}\varepsilon_i \otimes_{\mathrm{K}} \varepsilon_{j'} \end{bmatrix}, \quad i, j = 0, 1, \quad j' = 0', 1'. \tag{1.103}$$

To clarify our model, we show here a manifest form of the two obscure qubit state (1.98) in the vector representation

$$|\,\boldsymbol{\Psi}_{\mathrm{ob}}(2)\rangle = \frac{1}{\sqrt{2}}\left(\begin{bmatrix} \frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}^{(\mu)} \end{bmatrix} + \begin{bmatrix} \frac{1}{4}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \\ \frac{\sqrt{5}}{4}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}^{(\mu)} \end{bmatrix} + \begin{bmatrix} \frac{\sqrt{3}}{4}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \\ \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}^{(\mu)} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} \\ \frac{1}{4}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}^{(\mu)} \end{bmatrix}\right). \tag{1.104}$$

Innovation 1.11. *The states above may be called 'symmetric two obscure qubit states'. However, there are more general possibilities, as may be seen from the rhs of (1.103) and (1.104), when the indices of the first and second rows do not coincide. This would allow more possible states, which we call 'non-symmetric two obscure qubit states'. It would be worthwhile to establish their possible physical interpretation.*

These constructions show that quantum computing using Kronecker obscure qubits can involve a rich structure of states, giving a more detailed description with additional variables reflecting vagueness.

1.8 Conclusions

We have proposed a new scheme for describing quantum computation bringing vagueness into consideration, in which each state is characterized by a measure of truth. A membership amplitude is introduced in addition to the probability amplitude in order to achieve this, and we are led thereby to the concept of an obscure qubit. Two kinds of these are considered: the product obscure qubit, in which the total amplitude is the product of the quantum and membership amplitudes; and the Kronecker obscure qubit, where the amplitudes are manipulated separately. In the latter case, the quantum part of the computation is based, as usual, in Hilbert space, while the truth part requires a vague/fuzzy set formalism, which can be performed in the framework of a corresponding fuzzy space. Obscure-quantum computation may be considered as a set of rules (defining obscure-quantum gates) for managing quantum and membership amplitudes independently in different spaces. In this framework, we obtain not only the probabilities of final states but also their membership functions, i.e., how much trust we should assign to these probabilities. Our approach considerably extends the theory of quantum computing by adding the logic part directly to the computation process. Future challenges could lie in the direction of development of the corresponding logic hardware in parallel with the quantum devices.

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Innovative Quantum Computing

Steven Duplij and Raimund Vogl

Chapter 2

Higher braid quantum gates

One of the main problems in the physical realization of quantum computers is the presence of errors, which implies that it is desirable for quantum computations be provided with error correction or that ways be found to make the states more stable, which leads to the concept of topological quantum computation (for reviews, see, e.g., Freedman *et al* 2003, Nayak *et al* 2008, Rowell and Wang 2018, and references therein). In the Turaev approach (Turaev 1988), link invariants can be obtained from the solutions of the constant Yang–Baxter equation (the braid equation). It was realized that the topological entanglement of knots and links is deeply connected with quantum entanglement (Aravind 1997, Kauffman and Lomonaco 2002). Indeed, if the solutions to the constant Yang–Baxter equation (Lambe and Radford 1997) or Yang–Baxter operators/maps (Bukhshtaber 1998, Veselov 2003) are interpreted as a special class of quantum gate, namely braiding quantum gates (Kauffman and Lomonaco 2004, Melnikov *et al* 2018), then the inclusion of non-entangling gates does not change the relevant topological invariants (Alagic *et al* 2016, Kauffman and Mehrotra 2019). For further properties and applications of braiding quantum gates, see Melnikov *et al* (2019), Ballard and Wu (2011b), Kolganov and Morozov (2020), and Kolganov *et al* (2021).

Here we obtain and study (Duplij and Vogl 2021) the solutions to the higher arity (polyadic) braid equations introduced in Duplij (2021b, 2021a) as a polyadic generalization of the constant Yang–Baxter equation, which is considerably different from the generalized Yang–Baxter equation of Rowell *et al* (2010), Kitaev and Wang (2012), Vasquez *et al* (2016), and Padmanabhan *et al* (2020b). We introduce special classes of matrices (star and circle types), to which most of the solutions belong, and find that the so-called magic matrices (Khaneja and Glaser 2001, Kraus and Cirac 2001, Ballard and Wu 2011b) belong to the star class. We investigate their general non-trivial group properties and polyadic generalizations. We then consider the invertible and noninvertible matrix solutions to the higher braid equations as the corresponding higher braiding gates acting on multi-qubit states. It is important for

doi:10.1088/978-0-7503-5281-9ch2

multi-qubit entanglement can speed up quantum key distribution (Epping *et al* 2017) and accelerate various algorithms (Vartiainen *et al* 2004). As an example, we have made detailed computations for the ternary braiding gates as solutions to the ternary braid equations (Duplij 2021b, 2021a). A particular solution to the *n*-ary braid equation is also presented. It is shown that for each multi-qubit state, there exist higher braiding gates that are not entangling and the concrete relations for that are obtained, which can allow us to build non-entangling networks.

2.1 Yang–Baxter operators

Recall here (Kauffman and Lomonaco 2002, 2004) the standard construction of the special kind of gates we will consider, the braiding gates, in terms of solutions to the *constant Yang–Baxter equation* (Lambe and Radford 1997), also called the *algebraic* Yang–Baxter equation (Dye 2003), or the (binary) *braid equation* (Duplij 2021b).

2.1.1 Yang–Baxter maps and braid group

First we consider a general abstract construction of the (binary) braid equation. Let $\mathcal{V}$ be a vector space over a field $\mathbb{K}$ and the mapping $C_{\mathcal{V}^2}\colon \mathcal{V}\otimes\mathcal{V}\to\mathcal{V}\otimes\mathcal{V}$, where $\otimes = \otimes_{\mathbb{K}}$ is the tensor product over $\mathbb{K}$. A linear operator (*braid operator*) $C_{\mathcal{V}^2}$ is called a *Yang–Baxter operator* (denoted by R in Kauffman and Lomonaco 2004 and by B in Lambe and Radford 1997) or *Yang–Baxter map* (Veselov 2003) (denoted by F in Bukhshtaber 1998) if it satisfies the *braid equation* (Drinfeld 1989, 1992, Kassel 1995).

$$(C_{\mathcal{V}^2}\otimes \mathrm{id}_{\mathcal{V}})\circ(\mathrm{id}_{\mathcal{V}}\otimes C_{\mathcal{V}^2})\circ(C_{\mathcal{V}^2}\otimes \mathrm{id}_{\mathcal{V}}) = (\mathrm{id}_{\mathcal{V}}\otimes C_{\mathcal{V}^2})\circ(C_{\mathcal{V}^2}\otimes \mathrm{id}_{\mathcal{V}})\circ(\mathrm{id}_{\mathcal{V}}\otimes C_{\mathcal{V}^2}) \tag{2.1}$$

where $\mathrm{id}_{\mathcal{V}}\colon \mathcal{V}\to\mathcal{V}$ is the identity operator in $\mathcal{V}$. The connection of $C_{\mathcal{V}^2}$ with the R-matrix R is given by $C_{\mathcal{V}^2} = \tau\circ R$, where τ is the flip operation (Drinfeld 1989, Bukhshtaber 1998, Lambe and Radford 1997).

Let us introduce the operators $A_{1,2}\colon \mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}\to\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}$ by

$$A_1 = C_{\mathcal{V}^2}\otimes \mathrm{id}_{\mathcal{V}},\quad A_2 = \mathrm{id}_{\mathcal{V}}\otimes C_{\mathcal{V}^2}, \tag{2.2}$$

It follows from (2.1) that

$$A_1\circ A_2\circ A_1 = A_2\circ A_1\circ A_2. \tag{2.3}$$

If $C_{\mathcal{V}^2}$ is invertible, then $C_{\mathcal{V}^2}^{-1}$ is also the Yang–Baxter map with A_1^{-1} and A_2^{-1}. Therefore, the operators A_i represent the braid group $\mathscr{B}_3 = \{e, \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\}$ by the mapping π_3 as

$$\mathscr{B}_3 \xrightarrow{\pi_3} \mathrm{End}(\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}),\quad \sigma_1 \overset{\pi_3}{\mapsto} A_1,\quad \sigma_2 \overset{\pi_3}{\mapsto} A_2,\quad e \overset{\pi_3}{\mapsto} \mathrm{id}_{\mathcal{V}}. \tag{2.4}$$

The representation π_m of the braid group with m strands

$$\mathscr{B}_m = \left\{ e, \sigma_1, \ldots, \sigma_{m-1} \;\middle|\; \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\quad i = 1, \ldots, m-1, \\ \sigma_i\sigma_j = \sigma_j\sigma_i,\quad |\, i - j \,| \geqslant 2, \end{array} \right\} \tag{2.5}$$

can be obtained using operators $A_i(m)$: V $^{\otimes m}\to$ V $^{\otimes m}$ analogous to (2.2)

$$\begin{aligned} A_i(m) &= \overbrace{\mathrm{id}_{\mathcal{V}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{V}}}^{i-1} \otimes C_{\mathcal{V}^2} \otimes \overbrace{\mathrm{id}_{\mathcal{V}} \otimes \cdots \mathrm{id}_{\mathcal{V}}}^{m-i-1}, \\ A_0(m) &= (\mathrm{id}_{\mathcal{V}})^{\otimes m}, \ \ i = 1, \cdots, m-1, \end{aligned} \tag{2.6}$$

by the mapping π_m: $\mathscr{B}_m \to \mathrm{End}$ V $^{\otimes m}$ in the following way

$$\pi_m(\sigma_i) = A_i(m), \qquad \pi_m(e) = A_0(m). \tag{2.7}$$

In this notation (2.2) is $A_i = A_i(2)$, $i = 1, 2$, and therefore (2.3) represents $\mathscr{B}_3$ by (2.4).

2.1.2 Constant matrix solutions to the Yang–Baxter equation

Consider next a concrete version of the vector space $\mathcal{V}$ that is used in the quantum computation, a d-dimensional Euclidean vector space V_d over complex numbers $\mathbb{C}$ with a basis $\{e_i\}$, $i = 1, \ldots, d$. A linear operator $V_d \to V_d$ is given by a complex $d \times d$ matrix, the identity operator $\mathrm{id}_{\mathcal{V}}$ becomes the identity $d \times d$ matrix I_d, and the Yang–Baxter map $C_{\mathcal{V}^2}$ is a $d^2 \times d^2$ matrix C_{d^2} (denoted by R in Dye 2003) satisfying the matrix algebraic Yang–Baxter equation

$$(C_{d^2} \otimes I_d)(I_d \otimes C_{d^2})(C_{d^2} \otimes I_d) = (I_d \otimes C_{d^2})(C_{d^2} \otimes I_d)(I_d \otimes C_{d^2}), \tag{2.8}$$

being an equality between two matrices of size $d^3 \times d^3$. We use the unified notations, which can be straightforwardly generalized for higher braid operators. In components

$$C_{d^2} \circ (e_{i_1} \otimes e_{i_2}) = \sum_{j_1', j_2' = 1}^{d} c_{i_1 i_2}{}^{j_1' j_2'} \cdot e_{j_1'} \otimes e_{j_2'}, \tag{2.9}$$

the Yang–Baxter equation (2.8) has the shape (where summing is by primed indices)

$$\sum_{j_1', j_2', j_3' = 1}^{d} c_{i_1 i_2}{}^{j_1' j_2'} \cdot c_{j_2' i_3}{}^{j_3' k_3} \cdot c_{j_1' j_3'}{}^{k_1 k_2} = \sum_{l_1', l_2', l_3' = 1}^{d} c_{i_2 i_3}{}^{l_2' l_3'} \cdot c_{i_1 l_2'}{}^{k_1 l_1'} \cdot c_{l_1' l_3'}{}^{k_2 k_3} \equiv q_{i_1 i_2 i_3}{}^{k_1 k_2 k_3}. \tag{2.10}$$

The system (2.10) is highly overdetermined because the matrix C_{d^2} contains d^4 unknown entries, while there are d^6 cubic polynomial equations for them. So for $d = 2$ we have 64 equations for 16 unknowns, while for $d = 3$ there are 729 equations for the 81 unknown entries of C_{d^2}. The unitarity of C_{d^2} imposes a further d^2 quadratic equations, and so for $d = 2$ we have in total 68 equations for 16 unknowns. This makes the direct discovery of solutions for the matrix Yang–Baxter equation (2.10) very cumbersome. Nevertheless, using a conjugation classes method, the unitary solutions and their classification for $d = 2$ were presented in Dye (2003).

In the standard matrix form, (2.9) can be presented by introducing the four-dimensional vector space $\tilde{V}_4 = V \otimes V$ with the natural basis $\tilde{e}_{\tilde{k}} = \{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$, where $\tilde{k} = 1, \ldots, 8$ is a cumulative index.

The linear operator $\tilde{C}_4$: $\tilde{V}_4 \to \tilde{V}_4$ corresponding to (2.9) is given by 4×4 matrix $\tilde{c}_{\tilde{\imath}\tilde{\jmath}}$ as $\tilde{C}_4 \circ \tilde{e}_{\tilde{\imath}} = \sum_{\tilde{\jmath}=1}^{4} \tilde{c}_{\tilde{\imath}\tilde{\jmath}} \cdot \tilde{e}_{\tilde{\jmath}}$. The operators (2.2) become two 8×8 matrices $\tilde{A}_{1,2}$ as

$$\tilde{A}_1 = \tilde{c} \otimes_{\mathrm{K}} I_2, \quad \tilde{A}_2 = I_2 \otimes_{\mathrm{K}} \tilde{c}, \tag{2.11}$$

where $\otimes_{\mathrm{K}}$ is the Kronecker product of matrices and I_2 is the 2×2 identity matrix. In this notation, which is universal and also used for higher braid equations, the operator binary braid equations (2.117) become a single matrix equation

$$\tilde{A}_1 \tilde{A}_2 \tilde{A}_1 = \tilde{A}_2 \tilde{A}_1 \tilde{A}_2, \tag{2.12}$$

which we call the *matrix binary braid equation*, and also the constant Yang–Baxter equation (Dye 2003). In component form, (2.12) is a highly overdetermined system of 64 cubic equations for 16 unknowns, the entries of $\tilde{c}$.

The matrix equation (2.12) has the following gauge invariance, which allows a classification of Yang–Baxter maps (Hietarinta 1993). Introduce an invertible operator Q: $V \to V$ in the two-dimensional vector space $V \equiv V_{d=2}$. In the basis $\{e_1, e_2\}$, its 2×2 matrix q is given by $Q \circ e_i = \sum_{j=1}^{2} q_{ij} \cdot e_j$. In the natural four-dimensional basis $\tilde{e}_{\tilde{k}}$ the tensor product of operators $Q \otimes Q$ is presented by the Kronecker product of matrices $\tilde{q}_4 = q \otimes_{\mathrm{K}} q$. If the 4×4 matrix $\tilde{c}$ is a fixed solution to the Yang–Baxter equation (2.12), then the family of solutions $\tilde{c}(q)$ corresponding to the invertible 2×2 matrix q is the conjugation of $\tilde{c}$ by $\tilde{q}_4$ such that

$$\tilde{c}(q) = \tilde{q}_4 \tilde{c} \tilde{q}_4^{-1} = (q \otimes_{\mathrm{K}} q) \tilde{c} (q^{-1} \otimes_{\mathrm{K}} q^{-1}), \tag{2.13}$$

which follows from conjugating (2.12) by $q \otimes_{\mathrm{K}} q$ and using (2.11). If we include the obvious invariance of (2.12) with respect to an overall factor $t \in \mathbb{C}$, then the general family of solutions becomes (cf the Yang–Baxter equation Hietarinta 1993)

$$\tilde{c}(q, t) = t \tilde{q}_4 \tilde{c} \tilde{q}_4^{-1} = t(q \otimes_{\mathrm{K}} q) \tilde{c} (q^{-1} \otimes_{\mathrm{K}} q^{-1}). \tag{2.14}$$

It follows from (2.13) that the matrix $q \in \mathrm{GL}(2, \mathbb{C})$ is defined up to a complex nonzero factor. In this case we can put

$$q = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}, \tag{2.15}$$

and the manifest form of $\tilde{q}_4$ is

$$\tilde{q}_4 = \begin{pmatrix} a^2 & a & a & 1 \\ ac & ad & c & d \\ ac & c & ad & d \\ c^2 & cd & cd & d^2 \end{pmatrix}. \tag{2.16}$$

The matrix $\tilde{q}_4^{\star} \tilde{q}_4$ (where $\star$ represents Hermitian conjugation) is diagonal (this case is important in a further classification similar to the binary one Dye 2003), when the condition

$$c = -a/d^* \tag{2.17}$$

holds, and so the matrix q takes the special form (depending on two complex parameters)

$$q = \begin{pmatrix} a & 1 \\ -a/d^* & d \end{pmatrix}. \tag{2.18}$$

We call two solutions $\tilde{c}_1$ and $\tilde{c}_2$ of the constant Yang–Baxter equation (2.12) *q-conjugated*, if

$$\tilde{c}_1\tilde{q}_4 = \tilde{q}_4\tilde{c}_2, \tag{2.19}$$

and we will not distinguish between them. The q-conjugation in the form (2.19) does not require the invertibility of the matrix q, and therefore the solutions of different ranks (or invertible and not invertible) can be q-conjugated (for the invertible case, see Hietarinta 1993, Alagic *et al* 2014, Padmanabhan *et al* 2021).

The matrix equation (2.12) does not imply the invertibility of solutions, i.e., matrices $\tilde{c}$ being of full rank (in the binary Yang–Baxter case of rank 4 and $d = 2$). Therefore, below we introduce in a unified way invertible and noninvertible solutions to the matrix Yang–Baxter equation (2.10) for any rank of the corresponding matrices.

2.1.3 Partial identity and unitarity

To be as close as possible to the invertible case, we introduce noninvertible analogs of identity and unitarity. Let M be a diagonal $n \times n$ matrix of rank $r \leqslant n$, and therefore with $n - r$ zeros on the diagonal. If the other diagonal elements are units, such a diagonal M can be reduced by row operations to a block matrix, being a direct sum of the identity matrix $I_{r\times r}$ and the zero matrix $Z_{(n-r)\times(n-r)}$.

Definition 2.1. We call such a diagonal matrix a *block r-partial identity* $I_n^{(\text{block})}(r) = \text{diag}\left\{\overbrace{1, \ldots 1}^{r}, \overbrace{0, \ldots, 0}^{n-r}\right\}$, and without the block reduction a *shuffler-partial identity* $I_n^{(\text{shuffle})}(r)$ (these are connected by conjugation). We will use the term partial identity and $I_n(r)$ to denote any matrix of this form.

Obviously, with the full rank $r = n$ we have $I_n(n) \equiv I_n$, where I_n is the identity $n \times n$ matrix. As with the invertible case and identities, the partial identities (of the corresponding form) are *trivial solutions* of the Yang–Baxter equation.

Innovation 2.2. *If a matrix $M = M(r)$ of size $n \times n$ and rank r satisfies the following r-partial unitarity condition*

$$M(r)^\star M(r) = I_n^{(1)}(r), \tag{2.20}$$

$$M(r)M(r)^{\star} = I_n^{(2)}(r), \tag{2.21}$$

where $M(r)^{\star}$ *is the conjugate-transposed matrix and* $I_n^{(1)}(r)$, $I_n^{(2)}(r)$ *are partial identities (of any kind, they can be different), then* $M(r)$ *is called a r-partial unitary matrix.*

In the case, when $I_n^{(1)}(r) = I_n^{(2)}(r)$, the matrix $M(r)$ is called *normal*. If $M(r)^{\star} = M(r)$, then it is called *r-partial self-adjoint*. In the case of full rank $r = n$, the conditions (2.20)–(2.21) become ordinary unitarity, and $M(n)$ becomes an unitary (and normal) matrix, while a r-partial self-adjoint matrix becomes a self-adjoint matrix or Hermitian matrix.

As an example, we consider a 4 × 4 matrix of rank 3

$$M(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e^{i\beta} & 0 & 0 \\ 0 & 0 & 0 & e^{i\gamma} \\ e^{i\alpha} & 0 & 0 & 0 \end{pmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \tag{2.22}$$

which satisfies the 3-partial unitarity conditions (2.20)–(2.21) with two different 3-partial identities on the rhs

$$M(3)^{\star}M(3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4^{(1)}(3) \neq I_4^{(2)}(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M(3)M(3)^{\star}. \tag{2.23}$$

For a noninvertible matrix $M(r)$, one can define a *pseudoinverse* $M(r)^{+}$ (or the *Moore-Penrose inverse*) (Nashed 1976) by

$$M(r)M(r)^{+}M(r) = M(r), \quad M(r)^{+}M(r)M(r)^{+}=M(r)^{+}, \tag{2.24}$$

and $M(r)M(r)^{+}$, $M(r)^{+}M(r)$ are Hermitian. In the case of (2.22) the partial unitary matrix $M(3)$ coincides with its pseudoinverse

$$M(3)^{\star} = M(3)^{+}, \tag{2.25}$$

which is similar to the standard unitarity $M_{\mathrm{inv}}^{\star} = M_{\mathrm{inv}}^{-1}$ for an invertible matrix M_{inv}. It is important that (2.22) is a solution of the matrix Yang–Baxter equation (2.12)), and so is an example of a noninvertible Yang–Baxter map.

If only the first (second) of the conditions (2.20)–(2.21) holds, then we call such $M(r)$ a *left (right)r-partial unitary matrix*. An example of such a noninvertible Yang–Baxter map of rank 2 is the left 2-partial unitary matrix

$$M(2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & e^{i\alpha} \\ 0 & e^{i\beta} & 0 & 0 \\ 0 & e^{i\beta} & 0 & 0 \\ 0 & 0 & 0 & e^{i\beta} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}, \tag{2.26}$$

which satisfies (2.20), but not (2.21), and so $M(2)$ is not normal

$$M(2)^{\star}M(2)=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\neq\begin{pmatrix}1&0&0&e^{i(\alpha-\beta)}\\0&1&1&0\\0&1&1&0\\e^{i(\beta-\alpha)}&0&0&1\end{pmatrix}=M(2)M(2)^{\star}. \tag{2.27}$$

Nevertheless, the property (2.25) still holds and $M(2)^{\star}=M(2)^{+}$.

2.1.4 Permutation and parameter-permutation 4-vertex Yang–Baxter maps

The system (2.12) with respect to all 16 variables is too cumbersome for direct solution. The classification of all solutions can only be accomplished in special cases, e.g., for matrices over finite fields (Hietarinta 1993) or for fewer than 16 vertices. Here we will start from 4-vertex permutation and parameter-permutation matrix solutions and investigate their group structure. It has been shown by Dye (2003), and Kauffman and Lomonaco (2004) that the special 8-vertex solutions to the Yang–Baxter equation are most important for further applications including braiding gates. We will therefore study the 8-vertex solutions in the most general way: over $\mathbb{C}$ and in various configurations, invertible and not invertible, and also consider their group structure.

First, we introduce the *permutation Yang–Baxter maps* that are presented by the permutation matrices (binary matrices with a single 1 in each row and column), i.e., 4-vertex solutions. In total, there are 64 permutation matrices of size 4 × 4, while only four of them have the full rank 4 and simultaneously satisfy the Yang–Baxter equation (2.12), as follows

$$\tilde{c}^{\,\text{perm}}_{\text{bisymm}}=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix},\begin{pmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{pmatrix},\quad \begin{array}{c}\operatorname{tr}\tilde{c}=2,\ \det\tilde{c}=-1,\\ \text{eigenvalues: }\{1\}^{[2]},\ \{-1\}^{[2]},\end{array} \tag{2.28}$$

$$\tilde{c}^{\,\text{perm}}_{\text{90symm}}=\begin{pmatrix}0&1&0&0\\0&0&0&1\\1&0&0&0\\0&0&1&0\end{pmatrix},\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{pmatrix},\quad \begin{array}{c}\operatorname{tr}\tilde{c}=0,\ \det\tilde{c}=-1,\\ \text{eigenvalues: }1,\ i,\ -1,\ -i.\end{array} \tag{2.29}$$

Here and next we list the eigenvalues to understand which matrices are conjugated, and, after that, if and only if the conjugation matrix is of the form (2.16), then such solutions to the Yang–Baxter equation (2.12) coincide. The traces are important in the construction of corresponding link invariants (Turaev 1988) and local invariants (Balakrishnan and Sankaranarayanan 2010, Sudbery 2001), and the determinants are connected with the concurrence (Jaffali and Oeding 2020, Walter *et al* 2016). Note that the first matrix in (2.28) is the SWAP quantum gate (Nielsen and Chuang 2000).

To understand the symmetry properties of (2.28)–(2.29), we introduce the so-called *reverse matrix* $J\equiv J_n$ of size $n\times n$ by $(J_n)_{ij}=\delta_{i,n+1-i}$. For $n=4$ it is

$$J_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.30}$$

For any $n \times n$ matrix $M \equiv M_n$ the matrix JM is the matrix M reflected vertically, and the product MJ is M reflected horizontally. In addition to the standard *symmetric matrix* satisfying $M = M^{\mathrm{T}}$ (T is the transposition), one can introduce

$$M \text{is persymmetric, if } JM = (JM)^T, \tag{2.31}$$

$$M \text{is } 90^\circ - \text{symmetric, if } M^{\mathrm{T}} = JM. \tag{2.32}$$

Thus, a persymmetric matrix is symmetric with respect to the minor diagonal, while a 90°-symmetric matrix is symmetric under 90°-rotations. A *bisymmetric matrix* is symmetric and persymmetric simultaneously. In this notation, the first family of the permutation solutions (2.28) are bisymmetric but not 90°-symmetric, while the second family of the solutions (2.29) are, oppositely, 90°-symmetric but not symmetric and not persymmetric (which explains their notation).

In the next step, we define the corresponding *parameter-permutation solutions* replacing the units in (2.28) with parameters. We found the following four 4-vertex solutions to the Yang–Baxter equation (2.12)) over $\mathbb{C}$

$$\begin{gathered} \tilde{c}^{\,\mathrm{perm,star}}_{\mathrm{rank}=4}(x, y, z, t) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & y \\ 0 & x & 0 & 0 \\ 0 & 0 & t & 0 \\ z & 0 & 0 & 0 \end{pmatrix}, \\ \operatorname{tr} \tilde{c} = x + t, \\ \det \tilde{c} = -xyzt, \quad x, y, z, t \neq 0, \\ \text{eigenvalues: } x, t, \sqrt{yz}, -\sqrt{yz}, \end{gathered} \tag{2.33}$$

$$\begin{gathered} \tilde{c}^{\,\mathrm{perm,circ}}_{\mathrm{rank}=4}(x, y) = \begin{pmatrix} 0 & 0 & x & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & y & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \\ y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix}, \\ \operatorname{tr} \tilde{c} = 0, \\ \det \tilde{c} = -x^2 y^2, \quad x, y \neq 0, \\ \text{eigenvalues: } \sqrt{xy}, -\sqrt{xy}, i\sqrt{xy}, -i\sqrt{xy}. \end{gathered} \tag{2.34}$$

Innovation 2.3. *The first pair of solutions (2.33) correspond to the bisymmetric permutation matrices (2.28), and we call them star-like solutions, while the second two solutions (2.34) correspond to the 90°-symmetric matrices (2.28), which are called circle-like solutions.*

The first (second) star-like solution in (2.33) with $y = z$ ($x = t$) becomes symmetric (persymmetric), while on the other hand with $x = t$ ($y = z$) it becomes persymmetric (symmetric). They become bisymmetric parameter-permutation solutions if all of the parameters are equal $x = y = z = t$. The circle-like solutions (2.34) are 90°-symmetric when $x = y$.

Using q-conjugation (2.14), one can next get families of solutions depending on the entries of q and the additional complex parameters in (2.15).

2.1.5 Group structure of 4-vertex and 8-vertex matrices

Let us analyze the group structure of 4-vertex matrices (2.33)–(2.34) with respect to matrix multiplication, i.e., which kinds of subgroups in GL(4, $\mathbb{C}$) they can form. For this we introduce four 4-vertex 4×4 matrices over $\mathbb{C}$: two star-like matrices

$$\begin{gathered} N_{\mathrm{star1}} = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}, \\ N_{\mathrm{star2}} = \begin{pmatrix} 0 & 0 & 0 & y \\ 0 & x & 0 & 0 \\ 0 & 0 & t & 0 \\ z & 0 & 0 & 0 \end{pmatrix}, \quad \begin{gathered} \operatorname{tr} N = x + t, \\ \det N = -xyzt, \quad x, y, z, t \neq 0, \\ \text{eigenvalues: } x,\ t,\ \sqrt{yz},\ -\sqrt{yz}, \end{gathered} \end{gathered} \tag{2.35}$$

and two circle-like matrices

$$\begin{gathered} N_{\mathrm{circ1}} = \begin{pmatrix} 0 & 0 & x & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & t & 0 & 0 \end{pmatrix}, \\ N_{\mathrm{circ2}} = \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & 0 & y \\ z & 0 & 0 & 0 \\ 0 & 0 & t & 0 \end{pmatrix}, \quad \begin{gathered} \operatorname{tr} N = 0, \\ \det N = -xyzt, \quad x, y, z, t \neq 0, \\ \text{eigenvalues: } \sqrt[4]{xyzt},\ -\sqrt[4]{xyzt},\ i\sqrt[4]{xyzt},\ -\sqrt[4]{xyzt}, \end{gathered} \quad . \end{gathered} \tag{2.36}$$

Denoting the corresponding sets by $\mathsf{N}_{\mathrm{star1,\,2}} = \{N_{\mathrm{star1,\,2}}\}$ and $\mathsf{N}_{\mathrm{circ1,\,2}} = \{N_{\mathrm{circ1,\,2}}\}$, these do not intersect and are closed with respect to the following multiplications

$$\mathsf{N}_{\mathrm{star1}}\mathsf{N}_{\mathrm{star1}}\mathsf{N}_{\mathrm{star1}} = \mathsf{N}_{\mathrm{star1}}, \tag{2.37}$$

$$\mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{star2}} = \mathsf{N}_{\mathrm{star2}}, \tag{2.38}$$

$$\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}} = \mathsf{N}_{\mathrm{circ1}}, \tag{2.39}$$

$$\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}} = \mathsf{N}_{\mathrm{circ2}}. \tag{2.40}$$

Note that there are no closed binary multiplications among the sets of 4-vertex matrices (2.35)–(2.36).

To give a proper group interpretation of (2.37)–(2.40), we introduce a *k-ary (polyadic) general linear semigroup* $\mathrm{GLS}^{[k]}(n, \mathbb{C}) = \{\mathsf{M}_{\mathrm{full}} | \mu^{[k]}\}$, where $\mathsf{M}_{\mathrm{full}} = \{M_{n\times n}\}$ is the set of $n \times n$ matrices over $\mathbb{C}$ and $\mu^{[k]}$ is an ordinary product of k matrices. The full semigroup $\mathrm{GLS}^{[k]}(n, \mathbb{C})$ is *derived* in the sense that its product can be obtained by repeating the binary products that are (binary) closed at each step.

Innovation 2.4. *The $n \times n$ matrices of special shape can form k-ary subsemigroups of* $\mathrm{GLS}^{[k]}(n, \mathbb{C})$ *that can be closed with respect to the product of at minimum k matrices but not of two matrices, and we call such semigroups k-ary nonderived (or k-nonderived).*

Moreover, we have for the sets $\mathsf{N}_{\mathrm{star1,\,2}}$ and $\mathsf{N}_{\mathrm{circ1,\,2}}$

$$\begin{aligned} &\mathsf{M}_{\mathrm{full}} = \mathsf{N}_{\mathrm{star1}} \bigcup \mathsf{N}_{\mathrm{star2}} \bigcup \mathsf{N}_{\mathrm{circ1}} \bigcup \mathsf{N}_{\mathrm{circ2}}, \\ &\mathsf{N}_{\mathrm{star1}} \bigcap \mathsf{N}_{\mathrm{star2}} \bigcap \mathsf{N}_{\mathrm{circ1}} \bigcap \mathsf{N}_{\mathrm{circ2}} = \varnothing. \end{aligned} \tag{2.41}$$

A simple example of a 3-nonderived subsemigroup of the full semigroup $\mathrm{GLS}^{[k]}(n, \mathbb{C})$ is the set of antidiagonal matrices $\mathsf{M}_{\mathrm{adiag}} = \left\{M_{\mathrm{adiag}}\right\}$ (having nonzero elements on the minor diagonal only): the product $\mu^{[3]}$ of three matrices from $\mathsf{M}_{\mathrm{adiag}}$ is closed, and therefore $\mathsf{M}_{\mathrm{adiag}}$ is a subsemigroup $\mathcal{S}^{[3]}_{\mathrm{adiag}} = \left\{\mathsf{M}_{\mathrm{adiag}} \middle| \mu^{[3]}\right\}$ of the full ternary general linear semigroup $\mathrm{GLS}^{[3]}(n, \mathbb{C})$ with the multiplication $\mu^{[3]}$ as the ordinary triple matrix product.

In the theory of polyadic groups (Dörnte 1929) an analog of the binary inverse M^{-1} is given by the *querelement*, which is denoted by $\bar{M}$ and in the matrix k-ary case is defined by

$$\overbrace{M \cdots M}^{k-1} \bar{M} = M, \tag{2.42}$$

where $\bar{M}$ can be on any place. If each element of the k-ary semigroup $\mathrm{GLS}^{[k]}(n, \mathbb{C})$ (or its subsemigroup) has its querelement $\bar{M}$, then this semigroup is a *k-ary general linear group* $\mathrm{GL}^{[k]}(n, \mathbb{C})$.

In the set of $n \times n$ matrices the binary (ordinary) product is defined (even it is not closed), and for invertible matrices we formally determine the standard inverse M^{-1}, but for arity $k \geqslant 4$ it does not coincide with the querelement $\bar{M}$ because, as follows from (2.42) and cancellativity in $\mathbb{C}$,

$$\bar{M} = M^{2-k}. \tag{2.43}$$

Definition 2.5. The *k-ary (polyadic) identity* $I_n^{[k]}$ in $\mathrm{GLS}^{[k]}(n, \mathbb{C})$ is defined by

$$\overbrace{I_n^{[k]} \cdots I_n^{[k]}}^{k-1} M = M, \tag{2.44}$$

which holds when M in the lhs is on any place.

If M is only on one or another side (but not in the middle places) in (2.44), then $I_n^{[k]}$ is called *left (right) polyadic identity*. For instance, in the subsemigroup (in $\mathrm{GLS}^{[k]}(n, \mathbb{C})$) of antidiagonal matrices $\mathcal{S}_{\mathrm{adiag}}^{[3]}$ the ternary identity $I_n^{[3]}$ can be chosen as the $n \times n$ reverse matrix (2.30) having units on the minor diagonal, while the ordinary $n \times n$ unit matrix I_n is not in $\mathcal{S}_{\mathrm{adiag}}^{[3]}$. It follows from (2.44), that for matrices over $\mathbb{C}$ the (left, right) polyadic identity $I_n^{[k]}$ is

$$\left(I_n^{[k]}\right)^{k-1} = I_n, \tag{2.45}$$

which means that for the ordinary matrix product $I_n^{[k]}$ is a $(k-1)$-root of I_n (or $I_n^{[k]}$ is a reflection of $(k-1)$ degree), while both sides cannot belong to a subsemigroup $\mathcal{S}^{[k]}$ of $\mathrm{GLS}^{[k]}(n, \mathbb{C})$ under consideration (as in $\mathcal{S}_{\mathrm{adiag}}^{[3]}$). Since the solutions of (2.45) are not unique, there can be many k-ary identities in a k-ary matrix semigroup. We denote the set of k-ary identities by $\mathsf{I}_n^{[k]} = \{I_n^{[k]}\}$. In the case of $\mathcal{S}_{\mathrm{adiag}}^{[3]}$ the ternary identity $I_n^{[3]}$ can be chosen as any of the $n \times n$ reverse matrices (2.30) with unit complex numbers $e^{i\alpha_j}$, $j = 1, \ldots, n$ on the minor diagonal, where α_j satisfies additional conditions depending on the semigroup. In the concrete case of $\mathcal{S}_{\mathrm{adiag}}^{[3]}$, the conditions giving (2.45) are $(k-1)\alpha_j = 1 + 2\pi r_j$, $r_j \in \mathbb{Z}$, $j = 1, \ldots, n$.

In the framework of the above definitions, we can interpret the closed products (2.37)–(2.38) as the multiplications $\mu^{[3]}$ of the *ternary semigroups* $\mathcal{S}_{\mathrm{star1,\,2}}^{[3]}(4, \mathbb{C}) = \{\mathsf{N}_{\mathrm{star1,\,2}} | \mu^{[3]}\}$. The corresponding querelements are given by

$$\begin{gathered}
\bar{N}_{\mathrm{star1}} = N_{\mathrm{star1}}^{-1} = \begin{pmatrix} \frac{1}{x} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{z} & 0 \\ 0 & \frac{1}{y} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t} \end{pmatrix}, \\
\bar{N}_{\mathrm{star2}} = N_{\mathrm{star2}}^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{z} \\ 0 & \frac{1}{x} & 0 & 0 \\ 0 & 0 & \frac{1}{t} & 0 \\ \frac{1}{y} & 0 & 0 & 0 \end{pmatrix}, \qquad x, y, z, t \neq 0.
\end{gathered} \tag{2.46}$$

The ternary semigroups having querelements for each element, i.e., the additional operation $\overline{(\,)}$ defined by (2.46), are the *ternary groups* $\mathcal{G}^{[3]}_{\text{star1, 2}}(4, \mathbb{C}) = \{\mathsf{N}_{\text{star1, 2}} | \mu^{[3]}, \overline{(\,)}\}$ which are two (non-intersecting because $\mathsf{N}_{\text{star1}} \cap \mathsf{N}_{\text{star2}} = \varnothing$) subgroups of the ternary general linear group $\mathrm{GL}^{[3]}(4, \mathbb{C})$. The ternary identities in $\mathcal{G}^{[3]}_{\text{star1, 2}}(4, \mathbb{C})$ are the following different continuous sets $\mathsf{I}^{[3]}_{\text{star1, 2}} = \left\{ I^{[3]}_{\text{star1, 2}} \right\}$, where

$$I^{[3]}_{\text{star1}} = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 & 0 \\ 0 & 0 & e^{i\alpha_2} & 0 \\ 0 & e^{i\alpha_3} & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_4} \end{pmatrix}, \quad e^{2i\alpha_1} = e^{2i\alpha_4} = e^{i(\alpha_2+\alpha_3)} = 1, \quad \alpha_j \in \mathbb{R}, \tag{2.47}$$

$$I^{[3]}_{\text{star2}} = \begin{pmatrix} 0 & 0 & 0 & e^{i\alpha_1} \\ 0 & e^{i\alpha_2} & 0 & 0 \\ 0 & 0 & e^{i\alpha_3} & 0 \\ e^{i\alpha_4} & 0 & 0 & 0 \end{pmatrix}, \quad e^{2i\alpha_2} = e^{2i\alpha_3} = e^{i(\alpha_1+\alpha_4)} = 1, \quad \alpha_j \in \mathbb{R}. \tag{2.48}$$

In the particular case $\alpha_j = 0$, $j = 1, 2, 3, 4$, the ternary identities (2.47)–(2.48) coincide with the bisymmetric permutation matrices (2.28).

Next we treat the closed set products (2.39)–(2.40) as the multiplications $\mu^{[5]}$ of the *5-ary semigroups* $\mathcal{S}^{[5]}_{\text{circ1, 2}}(4, \mathbb{C}) = \{\mathsf{N}_{\text{circ1, 2}} | \mu^{[5]}\}$. The querelements are

$$\bar{N}_{\text{circ1}} = N^{-3}_{\text{circ1}} = \begin{pmatrix} 0 & 0 & \frac{1}{yzt} & 0 \\ \frac{1}{xzt} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{xyt} \\ 0 & \frac{1}{xyz} & 0 & 0 \end{pmatrix}, \tag{2.49}$$

$$\bar{N}_{\text{circ2}} = N^{-3}_{\text{circ2}} = \begin{pmatrix} 0 & \frac{1}{yzt} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{xzt} \\ \frac{1}{xyt} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{xyz} & 0 \end{pmatrix}, \quad x, y, z, t \neq 0. \tag{2.50}$$

and the corresponding *5-ary groups* $\mathcal{G}^{[5]}_{\text{circ1, 2}}(4, \mathbb{C}) = \{\mathsf{N}_{\text{circ1, 2}} | \mu^{[5]}, \overline{(\,)}\}$, which are two (non-intersecting because $\mathsf{N}_{\text{circ1}} \cap \mathsf{N}_{\text{circ2}} = \varnothing$) subgroups of the 5-ary general linear group $\mathrm{GL}^{[5]}(n, \mathbb{C})$. We have the following continuous sets of 5-ary identities $\mathsf{I}^{[3]}_{\mathrm{circ}1,\,2} = \left\{ I^{[3]}_{\mathrm{circ}1,\,2} \right\}$ in $\mathscr{G}^{[5]}_{\mathrm{circ}1,\,2}(4, \mathbb{C})$ satisfying

$$I^{[5]}_{\mathrm{circ}1} = \begin{pmatrix} 0 & 0 & e^{i\alpha_1} & 0 \\ e^{i\alpha_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_3} \\ 0 & e^{i\alpha_4} & 0 & 0 \end{pmatrix}, \quad e^{i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} = 1, \quad \alpha_j \in \mathbb{R}, \tag{2.51}$$

$$I^{[5]}_{\mathrm{circ}2} = \begin{pmatrix} 0 & e^{i\alpha_1} & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_2} \\ e^{i\alpha_3} & 0 & 0 & 0 \\ 0 & 0 & e^{i\alpha_4} & 0 \end{pmatrix}, \quad e^{i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} = 1, \quad \alpha_j \in \mathbb{R}. \tag{2.52}$$

In the case $\alpha_j = 0$, $j = 1, 2, 3, 4$, the 5-ary identities (2.51)–(2.52) coincide with the 90°-symmetric permutation matrices (2.29).

Innovation 2.6. *It follows from (2.46)–(2.52) that the 4-vertex star-like (2.35) and circle-like (2.36) matrices form subgroups of the k-ary general linear group* $\mathrm{GL}^{[k]}(4, \mathbb{C})$ *with significantly different properties: they have different querelements and (sets of) polyadic identities, and even the arities of the subgroups* $\mathscr{G}^{[3]}_{\mathrm{star}1,\,2}(4, \mathbb{C})$ *and* $\mathscr{G}^{[5]}_{\mathrm{circ}1,\,2}(4, \mathbb{C})$ *do not coincide (2.37)–(2.40).*

If we take into account that 4-vertex star-like (2.35) and circle-like (2.36) matrices are (binary) additive and distributive, then they form (with respect to the binary matrix addition (+) and the multiplications $\mu^{[3]}$ and $\mu^{[5]}$) the (2, 3)-ring $\mathscr{R}^{[3]}_{\mathrm{star}1,\,2}(4, \mathbb{C}) = \{\mathsf{N}_{\mathrm{star}1,\,2} | +, \mu^{[3]}\}$ and (2, 5)-ring $\mathscr{R}^{[5]}_{\mathrm{circ}1,\,2}(4, \mathbb{C}) = \{\mathsf{N}_{\mathrm{star}1,\,2} | +, \mu^{[5]}\}$.

Next we consider the interaction of the 4-vertex star-like (2.35) and circle-like (2.36) matrix sets, i.e., their exotic module structure. For this, let us recall the ternary (polyadic) module (Duplij 2001) and s-place action (Duplij 2018) definitions, which are suitable for our case. An abelian group $\mathscr{M}$ is a ternary left (middle, right) $\mathscr{R}$-module (or a module over $\mathscr{R}$) if there exists a ternary operation $\mathscr{R} \times \mathscr{R} \times \mathscr{M} \to \mathscr{M}$ ($\mathscr{R} \times \mathscr{M} \times \mathscr{R} \to \mathscr{M}$, $\mathscr{M} \times \mathscr{R} \times \mathscr{R} \to \mathscr{M}$) which satisfies some compatibility conditions (associativity and distributivity) that hold in the matrix case under consideration (and where the module operation is the triple ordinary matrix product) (Duplij 2001). A 5-ary left (right) module $\mathscr{M}$ over $\mathscr{R}$ is a 5-ary operation $\mathscr{R} \times \mathscr{R} \times \mathscr{R} \times \mathscr{R} \times \mathscr{M} \to \mathscr{M}$ ($\mathscr{M} \times \mathscr{R} \times \mathscr{R} \times \mathscr{R} \times \mathscr{R} \to \mathscr{M}$) with analogous conditions (and where the module operation is the pentuple matrix product) (Duplij 2018, 2022).

First, we have the triple relations inside star and circle matrices

$$\mathsf{N}_{\mathrm{star}1}(\mathsf{N}_{\mathrm{star}2})\mathsf{N}_{\mathrm{star}1} = (\mathsf{N}_{\mathrm{star}2}), \quad \mathsf{N}_{\mathrm{circ}1}\mathsf{N}_{\mathrm{circ}2}\mathsf{N}_{\mathrm{circ}1} = \mathsf{N}_{\mathrm{circ}1}, \tag{2.53}$$

$$\mathsf{N}_{\mathrm{star}1}\mathsf{N}_{\mathrm{star}1}(\mathsf{N}_{\mathrm{star}2}) = (\mathsf{N}_{\mathrm{star}2}), \quad \mathsf{N}_{\mathrm{circ}1}\mathsf{N}_{\mathrm{circ}1}\mathsf{N}_{\mathrm{circ}2} = \mathsf{N}_{\mathrm{circ}1}, \tag{2.54}$$

$$(\mathsf{N}_{\mathrm{star2}})\mathsf{N}_{\mathrm{star1}}\mathsf{N}_{\mathrm{star1}} = (\mathsf{N}_{\mathrm{star2}}), \quad \mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}} = \mathsf{N}_{\mathrm{circ1}}, \tag{2.55}$$

$$\mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{star2}}(\mathsf{N}_{\mathrm{star1}}) = (\mathsf{N}_{\mathrm{star1}}), \quad \mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ1}} = \mathsf{N}_{\mathrm{circ2}}, \tag{2.56}$$

$$\mathsf{N}_{\mathrm{star2}}(\mathsf{N}_{\mathrm{star1}})\mathsf{N}_{\mathrm{star2}} = (\mathsf{N}_{\mathrm{star1}}), \quad \mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ2}} = \mathsf{N}_{\mathrm{circ2}}, \tag{2.57}$$

$$(\mathsf{N}_{\mathrm{star1}})\mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{star2}} = (\mathsf{N}_{\mathrm{star1}}), \quad \mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}} = \mathsf{N}_{\mathrm{circ2}}. \tag{2.58}$$

We observe the following module structures on the left-hand column above (elements of the corresponding module are in brackets, and we informally denote modules by their sets): (1) from (2.53)–(2.55), the set $\mathsf{N}_{\mathrm{star2}}$ is a middle, right, and left module over $\mathsf{N}_{\mathrm{star1}}$; (2) from (2.56)–(2.58), the set $\mathsf{N}_{\mathrm{star1}}$ is a middle, right, and left module over $\mathsf{N}_{\mathrm{star2}}$;

$$\mathsf{N}_{\mathrm{star1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{star1}} = \mathsf{N}_{\mathrm{circ2}}, \quad \mathsf{N}_{\mathrm{star1}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{star1}} = \mathsf{N}_{\mathrm{circ1}}, \tag{2.59}$$

$$\mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{star2}} = \mathsf{N}_{\mathrm{circ2}}, \quad \mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{star2}} = \mathsf{N}_{\mathrm{circ1}}, \tag{2.60}$$

$$\mathsf{N}_{\mathrm{star1}}\mathsf{N}_{\mathrm{star1}}(\mathsf{N}_{\mathrm{circ1}}) = (\mathsf{N}_{\mathrm{circ1}}), \quad (\mathsf{N}_{\mathrm{circ1}})\mathsf{N}_{\mathrm{star1}}\mathsf{N}_{\mathrm{star1}} = (\mathsf{N}_{\mathrm{circ1}}), \tag{2.61}$$

$$\mathsf{N}_{\mathrm{star1}}\mathsf{N}_{\mathrm{star1}}(\mathsf{N}_{\mathrm{circ2}}) = (\mathsf{N}_{\mathrm{circ2}}), \quad (\mathsf{N}_{\mathrm{circ2}})\mathsf{N}_{\mathrm{star1}}\mathsf{N}_{\mathrm{star1}} = (\mathsf{N}_{\mathrm{circ2}}), \tag{2.62}$$

$$\mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{star2}}(\mathsf{N}_{\mathrm{circ1}}) = (\mathsf{N}_{\mathrm{circ1}}), \quad (\mathsf{N}_{\mathrm{circ1}})\mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{star2}} = (\mathsf{N}_{\mathrm{circ1}}), \tag{2.63}$$

$$\mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{star2}}(\mathsf{N}_{\mathrm{circ2}}) = (\mathsf{N}_{\mathrm{circ2}}), \quad (\mathsf{N}_{\mathrm{circ2}})\mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{star2}} = (\mathsf{N}_{\mathrm{circ2}}), \tag{2.64}$$

(3) from (2.61)–(2.64), the sets $\mathsf{N}_{\mathrm{circ1,\,2}}$ are a right and left module over $\mathsf{N}_{\mathrm{star1,\,2}}$;

$$\mathsf{N}_{\mathrm{circ1}}(\mathsf{N}_{\mathrm{star1}})\mathsf{N}_{\mathrm{circ1}} = (\mathsf{N}_{\mathrm{star1}}), \quad \mathsf{N}_{\mathrm{circ1}}(\mathsf{N}_{\mathrm{star2}})\mathsf{N}_{\mathrm{circ1}} = (\mathsf{N}_{\mathrm{star2}}), \tag{2.65}$$

$$\mathsf{N}_{\mathrm{circ2}}(\mathsf{N}_{\mathrm{star1}})\mathsf{N}_{\mathrm{circ2}} = (\mathsf{N}_{\mathrm{star1}}), \quad \mathsf{N}_{\mathrm{circ2}}(\mathsf{N}_{\mathrm{star2}})\mathsf{N}_{\mathrm{circ2}} = (\mathsf{N}_{\mathrm{star2}}), \tag{2.66}$$

$$\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{star1}} = \mathsf{N}_{\mathrm{star2}}, \quad \mathsf{N}_{\mathrm{star1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}} = \mathsf{N}_{\mathrm{star2}}, \tag{2.67}$$

$$\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{star2}} = \mathsf{N}_{\mathrm{star1}}, \quad \mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}} = \mathsf{N}_{\mathrm{star1}}, \tag{2.68}$$

$$\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{star1}} = \mathsf{N}_{\mathrm{star2}}, \quad \mathsf{N}_{\mathrm{star1}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}} = \mathsf{N}_{\mathrm{star2}}, \tag{2.69}$$

$$\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{star2}} = \mathsf{N}_{\mathrm{star1}}, \quad \mathsf{N}_{\mathrm{star2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}} = \mathsf{N}_{\mathrm{star1}}, \tag{2.70}$$

(4) from (2.65)–(2.66), the sets $\mathsf{N}_{\mathrm{star1,\,2}}$ are a middle ternary module over $\mathsf{N}_{\mathrm{circ1,\,2}}$;

$$\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}(\mathsf{N}_{\mathrm{star1}}) = (\mathsf{N}_{\mathrm{star1}}), \mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}(\mathsf{N}_{\mathrm{star2}}) = (\mathsf{N}_{\mathrm{star2}}), \tag{2.71}$$

$$(\mathsf{N}_{\mathrm{star1}})\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}} = (\mathsf{N}_{\mathrm{star1}}), (\mathsf{N}_{\mathrm{star2}})\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}} = (\mathsf{N}_{\mathrm{star2}}), \tag{2.72}$$

$$\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}(\mathsf{N}_{\mathrm{star1}}) = (\mathsf{N}_{\mathrm{star1}}), \mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}(\mathsf{N}_{\mathrm{star2}}) = (\mathsf{N}_{\mathrm{star2}}), \tag{2.73}$$

$$(\mathsf{N}_{\mathrm{star1}})\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}} = (\mathsf{N}_{\mathrm{star1}}),\ (\mathsf{N}_{\mathrm{star2}})\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}} = (\mathsf{N}_{\mathrm{star2}}), \tag{2.74}$$

$$\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}(\mathsf{N}_{\mathrm{circ2}}) = (\mathsf{N}_{\mathrm{circ2}}),\ (\mathsf{N}_{\mathrm{circ2}})\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}}\mathsf{N}_{\mathrm{circ1}} = (\mathsf{N}_{\mathrm{circ2}}), \tag{2.75}$$

$$\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}(\mathsf{N}_{\mathrm{circ1}}) = (\mathsf{N}_{\mathrm{circ1}}),\ (\mathsf{N}_{\mathrm{circ1}})\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}}\mathsf{N}_{\mathrm{circ2}} = (\mathsf{N}_{\mathrm{circ1}}). \tag{2.76}$$

(5) from (2.71)–(2.76), the sets $\mathsf{N}_{\mathrm{circ1,\,2}}$ are right and left 5-ary modules over $\mathsf{N}_{\mathrm{circ2,\,1}}$ and $\mathsf{N}_{\mathrm{star1,\,2}}$.

Note that the sum of 4-vertex star solutions of the Yang–Baxter equations (2.33) (with different parameters) gives the shape of 8-vertex matrices, and the same with the 4-vertex circle solutions (2.34). Let us introduce two kind of 8-vertex 4×4 matrices over $\mathbb{C}$: an 8-vertex *star matrix* M_{star} and an 8-vertex *circle matrix* M_{circ} as

$$M_{\mathrm{star}} = \begin{pmatrix} x & 0 & 0 & y \\ 0 & z & s & 0 \\ 0 & t & u & 0 \\ v & 0 & 0 & w \end{pmatrix}, \quad \det M_{\mathrm{star}} = (xw - yv)(st - uz), \quad \operatorname{tr} M_{\mathrm{star}} = x + z + u + w, \tag{2.77}$$

$$M_{\mathrm{circ}} = \begin{pmatrix} 0 & x & y & 0 \\ z & 0 & 0 & s \\ t & 0 & 0 & u \\ 0 & v & w & 0 \end{pmatrix}, \quad \det M_{\mathrm{circ}} = (xw - yv)(st - uz), \quad \operatorname{tr} M_{\mathrm{circ}} = 0. \tag{2.78}$$

If M_{star} and M_{circ} are invertible (the determinants in (2.77)–(2.78) are non-vanishing), then

$$\begin{aligned} M_{\mathrm{star}}^{-1} &= \begin{pmatrix} \frac{w}{xw - yv} & 0 & 0 & -\frac{v}{xw - yv} \\ 0 & -\frac{u}{st - uz} & \frac{t}{st - uz} & 0 \\ 0 & \frac{s}{st - uz} & -\frac{z}{st - uz} & 0 \\ -\frac{y}{xw - yv} & 0 & 0 & \frac{x}{xw - yv} \end{pmatrix}, \\ M_{\mathrm{circ}}^{-1} &= \begin{pmatrix} 0 & \frac{w}{xw - yv} & -\frac{v}{xw - yv} & 0 \\ -\frac{u}{st - uz} & 0 & 0 & \frac{t}{st - uz} \\ \frac{s}{st - uz} & 0 & 0 & -\frac{z}{st - uz} \\ 0 & -\frac{y}{xw - yv} & \frac{x}{xw - yv} & 0 \end{pmatrix}, \end{aligned} \tag{2.79}$$

and therefore the parameter conditions for invertibility are the same in both M_{star} and M_{circ}

$$xw - yv \neq 0, \qquad st - uz \neq 0. \tag{2.80}$$

The corresponding sets $\mathsf{M}_{\mathrm{star}} = \{M_{\mathrm{star}}\}$ and $\mathsf{M}_{\mathrm{circ}} = \{M_{\mathrm{circ}}\}$ are closed under the following multiplications

$$\mathsf{M}_{\mathrm{star}}\mathsf{M}_{\mathrm{star}} = \mathsf{M}_{\mathrm{star}}, \tag{2.81}$$

$$\mathsf{M}_{\mathrm{star}}\mathsf{M}_{\mathrm{circ}} = \mathsf{M}_{\mathrm{circ}}, \qquad \mathsf{M}_{\mathrm{circ}}\mathsf{M}_{\mathrm{star}} = \mathsf{M}_{\mathrm{circ}}, \tag{2.82}$$

$$\mathsf{M}_{\mathrm{circ}}\mathsf{M}_{\mathrm{circ}} = \mathsf{M}_{\mathrm{star}}, \tag{2.83}$$

and in terms of sets we can write $\mathsf{M}_{\mathrm{star}} = \mathsf{N}_{\mathrm{star1}} \cup \mathsf{N}_{\mathrm{star2}}$ and $\mathsf{M}_{\mathrm{circ}} = \mathsf{N}_{\mathrm{circ1}} \cup \mathsf{N}_{\mathrm{circ2}}$, while $\mathsf{N}_{\mathrm{star1}} \cap \mathsf{N}_{\mathrm{star2}} = \varnothing$ and $\mathsf{N}_{\mathrm{circ1}} \cap \mathsf{N}_{\mathrm{circ2}} = \varnothing$ (see (2.41)). Note that, if $\mathsf{M}_{\mathrm{star}}$ and $\mathsf{M}_{\mathrm{circ}}$ are treated as elements of an algebra, then (2.81)–(2.83) are reminiscent of the Cartan decomposition (see, e.g., Helgason 1962), but we will consider them from a more general viewpoint, which will treat such structures as semigroups, ternary groups, and modules.

Innovation 2.7. *The set* $\mathsf{M}_{8\mathrm{vertex}} = \mathsf{M}_{\mathrm{star}} \cup \mathsf{M}_{\mathrm{circ}}$ *is closed and because of the associativity of matrix multiplication,* $\mathsf{M}_{8\mathrm{vertex}}$ *forms a non-commutative semigroup, which we call a 8-vertex matrix semigroup* $\mathscr{S}_{8\mathrm{vertex}}(4, \mathbb{C})$*, which contains the zero matrix* $Z \in \mathscr{S}_{8\mathrm{vertex}}(4, \mathbb{C})$ *and is a subsemigroup of the (binary) general linear semigroup* GLS(4, $\mathbb{C}$).

It follows from (2.81), that $\mathsf{M}_{\mathrm{star}}$ is its subsemigroup $\mathscr{S}^{\mathrm{star}}_{8\mathrm{vertex}}(4, \mathbb{C})$. Moreover, the invertible elements of $\mathscr{S}_{8\mathrm{vertex}}(4, \mathbb{C})$ form a 8-*vertex matrix group* $\mathscr{G}_{8\mathrm{vertex}}(4, \mathbb{C})$ because its identity is a unit 4×4 matrix $I_4 \in \mathsf{M}_{\mathrm{star}}$, and so $\mathsf{M}_{\mathrm{star}}$ is a subgroup $\mathscr{G}^{\mathrm{star}}_{8\mathrm{vertex}}(4, \mathbb{C})$ of $\mathscr{G}_{8\mathrm{vertex}}(4, \mathbb{C})$ and a subgroup of the (binary) general linear group GL(4, $\mathbb{C}$). The structure of $\mathscr{S}_{8\mathrm{vertex}}(4, \mathbb{C})$ (2.81) is similar to that of block-diagonal and block-antidiagonal matrices (of the necessary sizes). So the 8-vertex (binary) matrix semigroup $S_{8\mathrm{vertex}}(4, \mathbb{C})$ in which the parameters satisfy (2.80) is a 8-vertex (binary) matrix group $\mathscr{G}_{8\mathrm{vertex}}(4, \mathbb{C})$, having a subgroup $\mathscr{G}^{\mathrm{star}}_{8\mathrm{vertex}}(4, \mathbb{C}) = \langle \mathsf{M}_{\mathrm{star}} | \cdot, \mathscr{I}_4 \rangle$, where $(\cdot)$ is an ordinary matrix product, and I_4 is its identity.

The group structure of the circle matrices $\mathsf{M}_{\mathrm{circ}}$ (2.78) follows from

$$\mathsf{M}_{\mathrm{circ}}\mathsf{M}_{\mathrm{circ}}\mathsf{M}_{\mathrm{circ}} = \mathsf{M}_{\mathrm{circ}}, \tag{2.84}$$

which means that $\mathsf{M}_{\mathrm{circ}}$ is closed with respect to the product of three matrices (the product of two matrices from $\mathsf{M}_{\mathrm{circ}}$ is outside the set (2.83)). We define a ternary multiplication $\nu^{[3]}$ as the ordinary triple product of matrices.

Innovation 2.8. *Then* $\mathscr{S}^{\mathrm{circ}[3]}_{8\mathrm{vertex}}(4, \mathbb{C}) = \langle \mathsf{M}_{\mathrm{circ}} | \nu^{[3]} \rangle$ *becomes a ternary (3-nonderived) semigroup with the zero* $Z \in \mathsf{M}_{\mathrm{circ}}$*, which is a subsemigroup of the ternary (derived) general linear semigroup* $\mathrm{GLS}^{[3]}(4, \mathbb{C})$*. Instead of the inverse, for each invertible element* $M_{\mathrm{circ}} \in \mathsf{M}_{\mathrm{circ}} \backslash Z$ *we introduce the unique querelement* $\bar{M}_{\mathrm{circ}}$ (Dörnte 1929) *by*

(2.42), *and because the ternary product is the triple ordinary product, we have* $\bar{M}_{\mathrm{circ}} = M_{\mathrm{circ}}^{-1}$ *from (2.43).*

Innovation 2.9. *If the conditions of invertibility (2.80) hold valid, then the ternary semigroup* $\mathcal{S}_{8\mathrm{vertex}}^{\mathrm{circ}(3)}(4, \mathbb{C})$ *becomes the ternary group* $\mathcal{G}_{8\mathrm{vertex}}^{\mathrm{circ}(3)}(4, \mathbb{C}) = \langle \mathsf{M}_{\mathrm{circ}} | \nu^{[3]}, \overline{()} \rangle$ *which does not contain the ordinary (binary) identity, since* $I_4 \notin \mathsf{M}_{\mathrm{circ}}$.

Nevertheless, the ternary group of circle matrices $\mathcal{G}_{8\mathrm{vertex}}^{\mathrm{circ}[3]}(4, \mathbb{C})$ has the following set $\mathsf{I}_{\mathrm{circ}}^{[3]} = \left\{ I_{\mathrm{circ}}^{[3]} \right\}$ of left–right 6-vertex and 8-vertex ternary identities (see (2.44)–(2.45))

$$I_{\mathrm{circ}}^{[3]} = \begin{pmatrix} 0 & \frac{1}{a} & b & 0 \\ a & 0 & 0 & -\frac{ab}{c} \\ 0 & 0 & 0 & \frac{1}{c} \\ 0 & 0 & c & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{ab}{c} & \frac{1}{c} & 0 \\ 0 & 0 & 0 & \frac{1}{b} \\ c & 0 & 0 & a \\ 0 & b & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{ab}{c} & \frac{1-ad}{c} & 0 \\ -\frac{cd}{b} & 0 & 0 & \frac{1-ad}{b} \\ c & 0 & 0 & a \\ 0 & b & d & 0 \end{pmatrix}, \qquad (2.85)$$

which (without additional conditions) depend upon the free parameters $a, b, c, d \in \mathbb{C}$, $b, c \neq 0$, and $\left(I_{\mathrm{circ}}^{[3]}\right)^2 = I_4$, $I_{\mathrm{circ}}^{[3]} \in \mathsf{M}_{\mathrm{circ}}$. In the binary sense, the matrices from (2.85) are mutually similar, but as ternary identities they are different.

If we consider the second operation for matrices (as elements of a general matrix ring), the binary matrix addition (+), then he structure of $\mathsf{M}_{8\mathrm{vertex}} = \mathsf{M}_{\mathrm{star}} \cup \mathsf{M}_{\mathrm{circ}}$ becomes more exotic.

Innovation 2.10. *The set* $\mathsf{M}_{\mathrm{star}}$ *is a* (2, 2)*-ring* $\mathcal{R}_{8\mathrm{vertex}}^{star[2,\,2]} = \langle \mathsf{M}_{\mathrm{star}} | +, \cdot \rangle$ *with the binary addition* (+) *and binary multiplication* ($\cdot$) *from the semigroup* $\mathcal{S}_{8\mathrm{vertex}}^{\mathrm{star}}$, *while* $\mathsf{M}_{\mathrm{circ}}$ *is a* (2, 3)*-ring* $\mathcal{R}_{8\mathrm{vertex}}^{\mathrm{circ}[2,\,3]} = \langle \mathsf{M}_{\mathrm{circ}} | +, \nu^{[3]} \rangle$ *with the binary matrix addition* (+), *the ternary matrix multiplication* $\nu^{[3]}$ *and the zero* Z.

Moreover, because of the distributivity and associativity of binary matrix multiplication, the relations (2.82) mean that the set $\mathsf{M}_{\mathrm{circ}}$ (being an abelian group under binary addition) can be treated as a left and right binary module $\mathcal{M}_{8\mathrm{vertex}}^{\mathrm{circ}}$ over the ring $\mathcal{R}_{8\mathrm{vertex}}^{star(2,\,2)}$ with an operation (*): the module action $M_{\mathrm{circ}} * M_{\mathrm{star}} = M_{\mathrm{circ}}$, $M_{\mathrm{star}} * M_{\mathrm{circ}} = M_{\mathrm{circ}}$ (coinciding with the ordinary matrix product (2.82)). The left and right modules are compatible because the associativity of ordinary matrix multiplication gives the compatibility condition $(M_{\mathrm{circ}} M_{\mathrm{star}}) M'_{\mathrm{circ}} = M_{\mathrm{circ}} (M_{\mathrm{star}} M'_{\mathrm{circ}})$, $M_{\mathrm{star}} \in \mathcal{R}_{8\mathrm{vertex}}^{star(2,\,2)}$, $M_{\mathrm{circ}}, M'_{\mathrm{circ}} \in \mathcal{R}_{8\mathrm{vertex}}^{\mathrm{circ}(2,\,3)}$, and therefore M_{circ} (as an abelian group under the binary addition (+) and the module action (*)) is a $\mathcal{R}_{8\mathrm{vertex}}^{star(2,\,2)}$-bimodule $\mathcal{M}_{8\mathrm{vertex}}^{\mathrm{circ}}$. The last relation (2.83) shows another interpretation of $\mathsf{M}_{\mathrm{circ}}$ as a formal square root of $\mathsf{M}_{\mathrm{star}}$ (as sets).

2.1.6 Star 8-vertex and circle 8-vertex Yang–Baxter maps

Let us consider the star 8-vertex solutions $\tilde{c}$ to the Yang–Baxter equation (2.12), having the shape (2.77), in the most general setting, over $\mathbb{C}$ and for different ranks, i.e., including noninvertible ones. In components, they are determined by

$$\begin{gathered}
vy(u-z)=0,\quad y(t^2-wz-x^2+xz)=0,\quad y(s(x-z)+t(u-x))=0,\\
y(u(w-x)+x^2-s^2)=0,\quad svy-tuz=0,\quad tvy-suz=0,\\
vwy+xz(x-z)-stz=0,\quad y(w^2-wz+xz-s^2)=0,\\
uz(z-u)=0,\quad suz-tvy=0,\quad y(s(w-u)+t(z-w))=0,\\
tuz-svy=0,\quad stz-vxy+wz(z-w)=0,\quad v(s^2-wz-x^2+xz)=0,\\
stu+u^2x-ux^2-vwy=0,\quad v(s(z-w)+t(w-u))=0,\\
uz(u-z)=0,\quad y(t^2+u(w-x)-w^2)=0,\\
v(s(u-x)+t(x-z))=0,\quad v(s^2+u(w-x)-w^2)=0,\\
vy(z-u)=0,\quad v(u(w-x)+x^2-t^2)=0,\\
uw^2+vxy-stu-u^2w=0,\quad v(w^2-t^2-wz+xz)=0.
\end{gathered}\tag{2.86}$$

Solutions from, e.g., Dye (2003) and Hietarinta (1993), etc, should satisfy this overdetermined system of 24 cubic equations for eight variables.

We search for the 8-vertex constant solutions to the Yang–Baxter equation over $\mathbb{C}$ without additional conditions, unitarity, etc (which will be considered in the next sections). We also will need the matrix functions tr and det, which are related to link invariants, as well as the eigenvalues, which help to find similar matrices and q-conjugated solutions to braid equations. Take into account that the Yang–Baxter maps are determined up to a general complex factor $t\in\mathbb{C}$ (2.14). For eigenvalues (which are determined up to the same factor t) we use the notation: $\{\text{eigenvalue}\}^{[\text{algebraic multiplicity}]}$.

We found the following 8-vertex solutions, classified by rank and number of parameters.

- Rank = 4 (invertible star Yang–Baxter maps) are

(1) Quadratic in two parameters

$$\begin{gathered}
\tilde{c}^{\,\text{par}=2}_{\text{rank}=4}(x,y)=\begin{pmatrix} xy & 0 & 0 & y^2\\ 0 & xy & \pm xy & 0\\ 0 & \mp xy & xy & 0\\ -x^2 & 0 & 0 & xy\end{pmatrix},\\
\operatorname{tr}\tilde{c}=4xy,\\
\det\tilde{c}=4x^4y^4,\quad x\neq 0,\ y\neq 0,\\
\text{eigenvalues: }\{(1+i)xy\}^{[2]},\ \{(1-i)xy\}^{[2]},
\end{gathered}\tag{2.87}$$

(2) Quadratic in three parameters

$$\tilde{c}^{\,\mathrm{par}=3}_{\mathrm{rank}=4,\,1}(x,\,y,\,z) = \begin{pmatrix} xy & 0 & 0 & y^2 \\ 0 & zy & \pm xy & 0 \\ 0 & \pm xy & zy & 0 \\ z^2 & 0 & 0 & xy \end{pmatrix}, \tag{2.88}$$
$$\operatorname{tr}\tilde{c} = 2y(x+z),$$
$$\det\tilde{c} = y^4(z^2 - x^2)^2,\ \ z \neq \pm x,\ \ y \neq 0,$$
$$\text{eigenvalues:} y(x-z),\ -y(x-z),\ \{y(x+z)\}^{[2]},$$

(3) Irrational in three parameters

$$\tilde{c}^{\,\mathrm{par}=3}_{\mathrm{rank}=4}(x,\,y,\,z) = \begin{pmatrix} xy & 0 & 0 & y^2 \\ 0 & \dfrac{x+z}{2}y & \pm y\sqrt{\dfrac{x^2+z^2}{2}} & 0 \\ 0 & \pm y\sqrt{\dfrac{x^2+z^2}{2}} & \dfrac{x+z}{2}y & 0 \\ \dfrac{(x+z)^2}{4} & 0 & 0 & yz \end{pmatrix}, \tag{2.89}$$
$$\operatorname{tr}\tilde{c} = 2y(x+z),$$
$$\det\tilde{c} = \frac{1}{16}y^4(x-z)^4,\quad y \neq 0,\, z \neq x,$$
$$\text{eigenvalues}\quad : \left\{\frac{1}{2}y\left(x+z-\sqrt{2}\sqrt{x^2+z^2}\right)\right\}^{[2]},\ \left\{\frac{1}{2}y\left(x+z+\sqrt{2}\sqrt{x^2+z^2}\right)\right\}^{[2]}.$$

Note that only the first and the last cases are genuine 8-vertex Yang–Baxter maps because the three-parameter matrices (2.88) are q-conjugated with the 4-vertex parameter-permutation solutions (2.33). Indeed,

$$\begin{pmatrix} xy & 0 & 0 & y^2 \\ 0 & zy & xy & 0 \\ 0 & xy & zy & 0 \\ z^2 & 0 & 0 & xy \end{pmatrix} = (q \otimes_{\mathrm{K}} q)\begin{pmatrix} y(x+z) & 0 & 0 & 0 \\ 0 & 0 & y(x-z) & 0 \\ 0 & y(x-z) & 0 & 0 \\ 0 & 0 & 0 & y(x+z) \end{pmatrix}(q^{-1} \otimes_{\mathrm{K}} q^{-1}), \tag{2.90}$$

$$q = \begin{pmatrix} \pm\sqrt{\dfrac{y}{z}} & b \\ 1 & \mp b\sqrt{\dfrac{z}{y}} \end{pmatrix}, \tag{2.91}$$

where $b \in \mathbb{C}$ is a free parameter. If $b = \frac{y}{z}$ two matrices q in (2.91) are similar, and we have the unique q-conjugation (2.90), then another solution in (2.88) is q-conjugated to the second 4-vertex parameter-permutation solutions (2.33) such that

$$\begin{pmatrix} xy & 0 & 0 & y^2 \\ 0 & zy & -xy & 0 \\ 0 & -xy & zy & 0 \\ z^2 & 0 & 0 & xy \end{pmatrix} = (q \otimes_{\mathrm{K}} q) \begin{pmatrix} 0 & 0 & 0 & y(x-z) \\ 0 & y(x+z) & 0 & 0 \\ 0 & 0 & y(x+z) & 0 \\ y(x-z) & 0 & 0 & 0 \end{pmatrix} (q^{-1} \otimes_{\mathrm{K}} q^{-1}), \tag{2.92}$$

$$q = \begin{pmatrix} i\sqrt{\dfrac{y}{z}} & \pm i\sqrt{\dfrac{y}{z}} \\ \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} -i\sqrt{\dfrac{y}{z}} & \pm i\sqrt{\dfrac{y}{z}} \\ \pm 1 & 1 \end{pmatrix}, \tag{2.93}$$

where qs are pairwise similar in (2.93), and therefore we have two different q-conjugations.

- Rank = 2 (noninvertible star Yang–Baxter maps) are quadratic in parameters

$$\tilde{c}^{\mathrm{par}=2}_{\mathrm{rank}=2}(x, y) = \begin{pmatrix} xy & 0 & 0 & y^2 \\ 0 & xy & \pm xy & 0 \\ 0 & \pm xy & xy & 0 \\ x^2 & 0 & 0 & xy \end{pmatrix}, \quad \begin{array}{c} \operatorname{tr} \tilde{c} = 4xy, \\ \text{eigenvalues:} \{2xy\}^{[2]}, \{0\}^{[2]}. \end{array} \tag{2.94}$$

There are no star 8-vertex solutions of rank 3. The above two solutions for $\tilde{c}^{\mathrm{par}=2}_{\mathrm{rank}=4}$ with different signs are q-conjugated (2.19), with the matrix q being one of the following

$$q = \begin{pmatrix} 0 & 1 \\ \pm i\dfrac{x}{y} & 0 \end{pmatrix}. \tag{2.95}$$

Further families of solutions can be obtained from (2.87)–(2.94) by applying the general q-conjugation (2.14).

Particular cases of the star solutions are called also X-type operators (Padmanabhan *et al* 2021) or magic matrices (Ballard and Wu 2011b) connected with the Cartan decomposition of $SU(4)$ (Khaneja and Glaser 2001, Kraus and Cirac 2001, Bullock 2004, Bullock and Brennen 2004).

The circle 8-vertex solutions $\tilde{c}$ to the Yang–Baxter equation (2.12) of the shape (2.78) are determined by the following system of 32 cubic equations for eight unknowns over $\mathbb{C}$

$$
\begin{aligned}
&x(ty + z(u - y) - vx) = 0,\ tx^2 + y^2(v - z) - wx^2 = 0,\\
&y(-st + tx + wy - xz) = 0,\ su(x - y) - sxy + uxy = 0,\\
&z(t(y - x) - sz + wx) = 0,\ v(sy + x^2) - z(s^2 + ux) = 0,\\
&swy - s^2v + xy(v - z) = 0,\ swx - s^2w + yz(u - y) = 0,\\
&st^2 - t^2x + z^2(y - u) = 0,\ su(v - z) + x(xz - tu) = 0,\\
&su(w - v) + xy(z - t) = 0,\ s(tu - uw + yz) - ty^2 = 0,\\
&s(tv + z^2) - x(v^2 + wz) = 0,\ svw - vwx + z(xz - wy) = 0,\\
&sw(w - t) + yz(z - v) = 0,\ s(sz + u(v - w) - vy) = 0,\\
&t(tu - vy + z(y - x)) = 0,\ tx(x - s) + u^2v - uvy = 0,\\
&xy(t - w) + u^2w - uvx = 0,\ t(sy + u^2) - w(ux + y^2) = 0,\\
&tz(s - x) - sv^2 + tuv = 0,\ tz(x - y) - svw + uvw = 0,\\
&u(w^2 - tz) - swz + tyz = 0,\ s^2(t - w) + u^2(v - z) = 0,\\
&tx(w - t) + uv(z - v) = 0,\ tvx - t^2y - uvw + vwy = 0,\\
&tvy - t^2u + w(wy - uz) = 0,\ u(s(v - w) - tu + wx) = 0,\\
&twz - tv(w + z) + vwz = 0,\ v(s(w - t) - uw + vx) = 0,\\
&sw^2 - uv^2 + v^2y - w^2x = 0,\ w(sv + u(z - v) - wy) = 0.
\end{aligned}
\tag{2.96}
$$

We found the 8-vertex solutions, classified by rank and number of parameters.

- Rank = 4 (invertible circle Yang–Baxter map) are quadratic in parameters

$$
\tilde{c}^{\,\text{par}=3}_{\text{rank}=4}(x, y, z) = \begin{pmatrix} 0 & xy & yz & 0 \\ z^2 & 0 & 0 & xy \\ xz & 0 & 0 & yz \\ 0 & z^2 & xz & 0 \end{pmatrix},
$$

$$
\begin{gathered}
\operatorname{tr} \tilde{c} = 0,\\
\det \tilde{c} = y^2z^2(z^2 - x^2),\ y \neq 0,\ \ z \neq 0,\ z \neq \pm x,\\
\text{eigenvalues:} \sqrt{-yz}(x - z),\ -\sqrt{-yz}(x - z),\ \sqrt{yz}(x + z),\ -\sqrt{yz}(x + z).
\end{gathered}
\tag{2.97}
$$

- Rank = 2 (noninvertible circle Yang–Baxter map) are linear in parameters

$$
\tilde{c}^{\,\text{par}=2}_{\text{rank}=2}(x, y) = \begin{pmatrix} 0 & -y & -y & 0 \\ -x & 0 & 0 & y \\ -x & 0 & 0 & y \\ 0 & x & x & 0 \end{pmatrix},\quad \text{eigenvalues: } 2\sqrt{xy},\ -2\sqrt{xy},\ \{0\}^{[2]}. \tag{2.98}
$$

There are no circle 8-vertex solutions of rank 3. The corresponding families of solutions can be derived from the above using the q-conjugation (2.14).

A particular case of the 8-vertex circle solution (2.97) was considered in Asaulko and Korablev (2019).

2.1.7 Triangle invertible 9- and 10-vertex solutions

There are some higher vertex solutions to the Yang–Baxter equations that are not in the above star/circle classification. They are determined by the following system of 15 cubic equations for nine unknowns over $\mathbb{C}$

$$\begin{aligned}
&(-py - x(u + w - y) + v(y + z)) + s(v - x)(v + x) = 0,\\
&(-ty + vz + x(y - z)) = 0,\\
&x(t - v) + ty(w - z) + vz(y - u) = 0,\\
&(pz - t(y + z) + x(u + w - z)) + s(x^2 - t^2) = 0,\\
&s(-u + w - y + z) + s(-t(u + z) + x(u - w + y - z) + v(w + y))\\
&uwy - uwz - uyz + wyz = 0,\\
&y(p - t) + u(x - v)) = 0,\ t(pz - t(u + z) + ux) = 0,\\
&{}^2 s + pu(-u + y + z) - t(st + u(u + w)) + u^2 x = 0,\ t(z(p - v) - tw + wx) = 0,\\
&s(t - v) + tw(y - u) + uv(w - z) = 0,\ v(-py + v(w + y) - wx) = 0,\\
&(z(v - p) + tw - wx) = 0,\ v(y(t - p) + u(v - x)) = 0,\\
&{}^2(-s) + pw(w - y - z) + sv^2 + w(v(u + w) - wx) = 0,\\
&(p(w - u) - tw + uv) = 0,
\end{aligned} \tag{2.99}$$

We found the following 9-vertex Yang–Baxter maps

$$\tilde{c}^{\,9\text{-vert},\,1}_{\text{rank}=4} = \begin{pmatrix} x & y & z & s \\ 0 & 0 & x & y \\ 0 & x & 0 & z \\ 0 & 0 & 0 & x \end{pmatrix}, \begin{pmatrix} x & y & y & z \\ 0 & 0 & -x & -y \\ 0 & -x & 0 & -y \\ 0 & 0 & 0 & x \end{pmatrix}, \begin{pmatrix} x & y & -y & z \\ 0 & 0 & x & -\dfrac{zx}{y} \\ 0 & x & 0 & \dfrac{zx}{y} \\ 0 & 0 & 0 & x \end{pmatrix}, \tag{2.100}$$

$$\operatorname{tr} \tilde{c} = 2x, \quad \det \tilde{c} = -x^4, \quad x \neq 0, \quad \text{eigenvalues: } \{x\}^{[3]}, -x. \tag{2.101}$$

The third matrix in (2.100) is conjugated with the 4-vertex parameter-permutation solutions (2.33) of the form (which has the same the same eigenvalues (2.101))

$$\tilde{c}^{\,4-\text{vert}}_{\text{rank}=4}(x) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \tag{2.102}$$

by the conjugated matrix

$$U^{9\text{to}4} = \begin{pmatrix} 1 & -\dfrac{y}{2x} & \dfrac{y}{2x} & 0 \\ 0 & 1 & 0 & -\dfrac{z}{y} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.103}$$

The matrix (2.103) cannot be presented as the Kronecker product $q \otimes_{\mathrm{K}} q$ (2.16), and so the third matrix in (2.100) and (2.102) are different solutions of the Yang–Baxter equation (2.12). Although the first two matrices in (2.100) have the same eigenvalues (2.101), they are not similar because they are different from (2.102) middle Jordan blocks.

Then we have another 3-parameter solutions with fractions

$$\tilde{c}_{\mathrm{rank}=4}^{9-\mathrm{vert},\,2}(x,\,y,\,z) = \begin{pmatrix} x & y & y & z \\ 0 & 0 & -x & y - \dfrac{2xz}{y} \\ 0 & -x & 0 & y - \dfrac{2xz}{y} \\ 0 & 0 & 0 & x\left(\dfrac{4xz}{y^2} - 3\right) \end{pmatrix}, \tag{2.104}$$

$$\operatorname{tr} \tilde{c} = 2x\frac{2xz - y^2}{y^2},$$

$$\det \tilde{c} = x^4\left(3 - \frac{4xz}{y^2}\right), \quad x \neq 0,\ y \neq 0,\ z \neq \frac{3y^2}{4x},$$

$$\text{eigenvalues: } \{x\}^{[2]},\ -x,\ x\left(\frac{4xz}{y^2} - 3\right),$$

and

$$\tilde{c}_{\mathrm{rank}=4}^{9-\mathrm{vert},\,3}(x,\,y,\,z) = \begin{pmatrix} x & y & -y & z \\ 0 & 0 & -x & \dfrac{2zx}{y} + y \\ 0 & 3x & 0 & \dfrac{2zx}{y} - y \\ 0 & 0 & 0 & \dfrac{4zx^2}{y^2} + x \end{pmatrix}, \tag{2.105}$$

$$\operatorname{tr} \tilde{c} = 2x\left(1 + 2\frac{xz}{y^2}\right)$$

$$\det \tilde{c} = 3x^4\left(\frac{4zx}{y^2} + 1\right), \quad x \neq 0,\ y \neq 0,\ z \neq \frac{y^2}{4x}$$

$$\text{eigenvalues: } x,\ i\sqrt{3}x,\ -i\sqrt{3}x,\ x\left(1 + \frac{4zx}{y^2}\right)$$

The 4-parameter 9-vertex solution is

$$
{}^{9-\text{vert,par}=4}_{\text{rank}=4}(x, y, z, s) = \begin{pmatrix} x & y & z & s \\ 0 & 0 & -x & y - \dfrac{2sx}{z} \\ 0 & x - \dfrac{2xy}{z} & 0 & z - \dfrac{2sx}{z} \\ 0 & 0 & 0 & \dfrac{x(4sx - z(2y + z))}{z^2} \end{pmatrix}, \tag{2.106}
$$

$$
\operatorname{tr} \tilde{c} = 2x\frac{2sx - yz}{z^2}
$$

$$
\det \tilde{c} = \frac{x^4(2y - z)(z(2y + z) - 4sx)}{z^3}, \; x \neq 0, \; y \neq \frac{z}{2}, \; z \neq 0,
$$

$$
\text{eigenvalues}: \; x, \; x\sqrt{\frac{2y}{z} - 1}, \; -x\sqrt{\frac{2y}{z} - 1}, \; \frac{x(4sx - z(2y + z))}{z^2}.
$$

We also found a 5-parameter and 9-vertex solution of the form

$$
\tilde{c}^{\,9-\text{vert,par}=5}_{\text{rank}=4}(x, y, z, s, t) = \begin{pmatrix} x & y & z & s \\ 0 & 0 & t & \dfrac{s(t - x)}{z} + y \\ 0 & \dfrac{y(t - x)}{z} + x & 0 & \dfrac{s(t - x)}{z} + z \\ 0 & 0 & 0 & \dfrac{s(t - x)^2 + tz(y + z) - xyz}{z^2} \end{pmatrix}, \tag{2.107}
$$

$$
\operatorname{tr} \tilde{c} = \frac{st^2 + sx^2 + tz^2 + xz^2 - 2stx + tyz - xyz}{z^2},
$$

$$
\det \tilde{c} = \frac{xt(x(y - z) - ty)(s(t - x)^2 + tz(y + z) - xyz)}{z^3},
$$

$$
\text{eigenvalues}: \; x, \; \sqrt{\frac{t}{z}(ty - xy + xz)}, \; -\sqrt{\frac{t}{z}(ty - xy + xz)},
$$

$$
\frac{st^2 - 2stx + tz^2 + ytz + sx^2 - yxz}{z^2}, \; x \neq 0, \; z \neq 0, \; t \neq 0.
$$

Finally, we found the following 3-parameter 10-vertex solution

$$
\tilde{c}^{\,10-\text{vert}}_{\text{rank}=4}(x, y, z) = \begin{pmatrix} x & y & y & \dfrac{y^2}{x} \\ 0 & 0 & -x & -y \\ 0 & -x & 0 & -y \\ z & 0 & 0 & x \end{pmatrix}, \tag{2.108}
$$

$$
\operatorname{tr} \tilde{c} = 2x,
$$

$$
\det \tilde{c} = -x(x^3 + zy^2), \quad x \neq 0,
$$

$$
\text{eigenvalues}: \; \{x\}^{[2]}, \; \sqrt{x^2 + \frac{zy^2}{x}}, \; -\sqrt{x^2 + \frac{zy^2}{x}}.
$$

This solution is conjugated with the 4-vertex parameter-permutation solutions (2.33) of the form (which has the same the same eigenvalues as (2.108))

$$\tilde{c}^{\,4-\mathrm{vert}}_{\mathrm{rank}=4}(x, y, z) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & x + \dfrac{y^2 z}{x^2} & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \tag{2.109}$$

by the conjugated matrix

$$U^{10\mathrm{to}4} = \begin{pmatrix} 0 & \dfrac{x}{z} & -\dfrac{x}{z} & 0 \\ -1 & -\dfrac{x^2}{yz} & \dfrac{x^2}{yz} & -\dfrac{y}{x} \\ 1 & -\dfrac{x^2}{yz} & \dfrac{x^2}{yz} & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{2.110}$$

Since the matrix (2.110) cannot be presented as the Kronecker product $q \otimes_{\mathrm{K}} q$ (2.16), (2.108) and (2.109) are different solutions of the Yang–Baxter equation (2.12).

Further families of the higher vertex solutions to the constant Yang–Baxter equation (2.12) can be obtained from the ones above by using the q-conjugation (2.14).

2.2 Polyadic braid operators and higher braid equations

The polyadic version of the braid equation (2.1) was introduced in Duplij (2021b, 2022). Here we define higher analog of the Yang–Baxter operator and develop its connection with higher braid groups and quantum computations. The whole material of this and the following sections is fully original and innovative.

Let us consider a vector space $\mathcal{V}$ over a field $\mathbb{K}$. A *polyadic (n-ary) braid operator* $C_{\mathcal{V}^n}$ is defined as the mapping (Duplij 2021b)

$$C_{\mathcal{V}^n}\colon \overbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}^{n} \to \overbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}^{n}. \tag{2.111}$$

The polyadic analog of the braid equation (2.1) was introduced in Duplij (2021b) using the associative quiver technique (Duplij 2018).

Let us introduce n operators

$$A_p\colon \overbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}^{2n-1} \to \overbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}^{2n-1}, \tag{2.112}$$

$$A_p = \mathrm{id}_{\mathcal{V}}^{\otimes(p-1)} \otimes C_{\mathcal{V}^n} \otimes \mathrm{id}_{\mathcal{V}}^{\otimes(n-p)}, \quad p = 1, \ldots, n, \tag{2.113}$$

i.e., p is a place of $C_{\mathcal{V}^n}$ instead of one $\mathrm{id}_{\mathcal{V}}$ in $\mathrm{id}_{\mathcal{V}}^{\otimes n}$. A system of $(n-1)$ *polyadic (n-ary) braid equations* is defined by

$$A_1 \circ A_2 \circ A_3 \circ A_4 \circ \cdots \circ A_{n-2} \circ A_{n-1} \circ A_n \circ A_1 \tag{2.114}$$

$$= A_2 \circ A_3 \circ A_4 \circ A_5 \circ \cdots \circ A_{n-1} \circ A_n \circ A_1 \circ A_2 \\ \vdots \tag{2.115}$$

$$= A_n \circ A_1 \circ A_2 \circ A_3 \circ \cdots \circ A_{n-3} \circ A_{n-2} \circ A_{n-1} \circ A_n. \tag{2.116}$$

Example 2.11. In the lowest non-binary case $n = 3$, we have the ternary braid operator $C_{\mathcal{V}^3}\colon \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V} \to \mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$ and two ternary braid equations on $\mathcal{V}^{\otimes 5}$

$$\begin{aligned}&(C_{\mathcal{V}^3} \otimes \mathrm{id}_{\mathcal{V}} \otimes \mathrm{id}_{\mathcal{V}}) \circ (\mathrm{id}_{\mathcal{V}} \otimes C_{\mathcal{V}^3} \otimes \mathrm{id}_{\mathcal{V}}) \circ (\mathrm{id}_{\mathcal{V}} \otimes \mathrm{id}_{\mathcal{V}} \otimes C_{\mathcal{V}^3}) \circ (C_{\mathcal{V}^3} \otimes \mathrm{id}_{\mathcal{V}} \otimes \mathrm{id}_{\mathcal{V}}) \\ &= (\mathrm{id}_{\mathcal{V}} \otimes C_{\mathcal{V}^3} \otimes \mathrm{id}_{\mathcal{V}}) \circ (\mathrm{id}_{\mathcal{V}} \otimes \mathrm{id}_{\mathcal{V}} \otimes C_{\mathcal{V}^3}) \circ (C_{\mathcal{V}^3} \otimes \mathrm{id}_{\mathcal{V}} \otimes \mathrm{id}_{\mathcal{V}}) \circ (\mathrm{id}_{\mathcal{V}} \otimes C_{\mathcal{V}^3} \otimes \mathrm{id}_{\mathcal{V}}) \\ &= (\mathrm{id}_{\mathcal{V}} \otimes \mathrm{id}_{\mathcal{V}} \otimes C_{\mathcal{V}^3}) \circ (C_{\mathcal{V}^3} \otimes \mathrm{id}_{\mathcal{V}} \otimes \mathrm{id}_{\mathcal{V}}) \circ (\mathrm{id}_{\mathcal{V}} \otimes C_{\mathcal{V}^3} \otimes \mathrm{id}_{\mathcal{V}}) \circ (\mathrm{id}_{\mathcal{V}} \otimes \mathrm{id}_{\mathcal{V}} \otimes C_{\mathcal{V}^3}).\end{aligned} \tag{2.117}$$

Note that the higher braid equations presented above differ from the generalized Yang–Baxter equations of Rowell *et al* (2010), Kitaev and Wang (2012), and Chen (2012a).

The higher braid operators (2.111) satisfying the higher braid equations (2.114)–(2.116) can represent the higher braid group (Duplij 2021a) using (2.6) and (2.113). By analogy with (2.6), we introduce m operators by

$$\mathrm{B}_i(m)\colon \overbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}^{m+n-2} \to \overbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}^{m+n-2}, \quad \mathrm{B}_0(m) = (\mathrm{id}_{\mathcal{V}})^{\otimes(m+n-2)}, \tag{2.118}$$

$$\mathrm{B}_i(m) = \mathrm{id}_{\mathcal{V}}^{\otimes(i-1)} \otimes C_{\mathcal{V}^n} \otimes \mathrm{id}_{\mathcal{V}}^{\otimes(m-i-1)}, \quad i = 1, \ldots, m-1. \tag{2.119}$$

The representation $\pi_m^{[n]}$ of the higher braid group $\mathscr{B}_m^{[n+1]}$ (of $(n+1)$-degree in the notation of Duplij 2021a, 2022) (having $m-1$ generators $\boldsymbol{\sigma}_i$ and identity $\mathbf{e}$) is given by

$$\pi_m^{[n]}\colon \mathscr{B}_m^{[n+1]} \longrightarrow \mathrm{End}\, \mathcal{V}^{\otimes(m+n-2)}, \tag{2.120}$$

$$\pi_m^{[n]}(\boldsymbol{\sigma}_i) = \mathrm{B}_i(m), \quad i = 1, \ldots, m-1. \tag{2.121}$$

In this way, the generators $\boldsymbol{\sigma}_i$ of the higher braid group $\mathscr{B}_m^{[n+1]}$ satisfy the relations

- *n higher braid relations*

$$\overbrace{\boldsymbol{\sigma}_i \boldsymbol{\sigma}_{i+1} \cdots \boldsymbol{\sigma}_{i+n-2} \boldsymbol{\sigma}_{i+n-1} \boldsymbol{\sigma}_i}^{n+1} \tag{2.122}$$

$$=\boldsymbol{\sigma}_{i+1}\boldsymbol{\sigma}_{i+2}\cdots\boldsymbol{\sigma}_{i+n-1}\boldsymbol{\sigma}_{i}\boldsymbol{\sigma}_{i+1} \tag{2.123}$$

$$\vdots$$

$$=\boldsymbol{\sigma}_{i+n-1}\boldsymbol{\sigma}_{i}\boldsymbol{\sigma}_{i+1}\boldsymbol{\sigma}_{i+2}\cdots\boldsymbol{\sigma}_{i}\boldsymbol{\sigma}_{i+1}\boldsymbol{\sigma}_{i+n-1}, \tag{2.124}$$

$$i = 1, \dots, m-n, \tag{2.125}$$

- *n-ary far commutativity*

$$\overbrace{\boldsymbol{\sigma}_{i_1}\boldsymbol{\sigma}_{i_2}\cdots\boldsymbol{\sigma}_{i_{n-2}}\boldsymbol{\sigma}_{i_{n-1}}\boldsymbol{\sigma}_{i_n}}^{n} \tag{2.126}$$

$$\vdots$$

$$=\boldsymbol{\sigma}_{\tau(i_1)}\boldsymbol{\sigma}_{\tau(i_2)}\cdots\boldsymbol{\sigma}_{\tau(i_{n-2})}\boldsymbol{\sigma}_{\tau(i_{n-1})}\boldsymbol{\sigma}_{\tau(i_n)}, \tag{2.127}$$

$$\text{if all } \left| i_p - i_s \right| \geqslant n, \quad p, s = 1, \dots, n, \tag{2.128}$$

where τ is an element of the permutation symmetry group $\tau \in S_n$. The relations (2.122)–(2.127) coincide with those from Duplij (2021a, 2022), obtained by another method, i.e., via the polyadic-binary correspondence.

In the case $m = 4$ and $n = 3$, the higher braid group $\mathscr{B}_4^{[4]}$ is represented by (2.117) and generated by three generators $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, $\boldsymbol{\sigma}_3$, which satisfy two braid relations only (without far commutativity)

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3. \tag{2.129}$$

According to (2.126)–(2.127), the far commutativity relations appear when the number of elements of the higher braid groups satisfy

$$m \geqslant m_{\min} = n(n-1) + 2, \tag{2.130}$$

such that all conditions (2.128) should hold. Thus, to have the far commutativity relations in the ordinary (binary) braid group (2.5), we need three generators and $\mathscr{B}_4$, while for $n = 3$ we need at least seven generators $\boldsymbol{\sigma}_i$ and $\mathscr{B}_8^{[4]}$ (see *example* **7.12** in Duplij 2021a).

In the concrete realization of V as a d-dimensional Euclidean vector space V_d over the complex numbers $\mathbb{C}$ and basis $\{e_i\}$, $i = 1, \dots, d$, the polyadic (n-ary) braid operator C_{V^n} becomes a matrix C_{d^n} of size $d^n \times d^n$ which satisfies $n-1$ higher braid equations (2.114)–(2.116) in matrix form. In the components, the matrix braid operator is

$$C_{d^n}\circ(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n}) = \sum_{j_1^{'}, j_2^{'} \cdots j_n^{'}=1}^{d} c_{i_1 i_2 \cdots i_n}{}^{j_1^{'} j_2^{'} \cdots j_n^{'}} \cdot e_{j_1^{'}} \otimes e_{j_2^{'}} \otimes \cdots \otimes e_{j_n^{'}}. \tag{2.131}$$

Thus, we have d^{2n} entries (unknowns) in C_{d^n} satisfying $(n-1)d^{4n-2}$ equations (2.114)–(2.116) in components of polynomial power $n+1$. In the minimal non-binary case $n = 3$, we have $2d^{10}$ equations of power 4 for d^6 unknowns, e.g., even for $d = 2$, we have 2048 for 64 components, and for $d = 3$ there are 118 098 equations for

729 components. Thus, solving the matrix higher braid equations directly is cumbersome and only particular cases are possible to investigate, for instance by using permutation matrices (2.28), or the star and circle matrices (2.77)–(2.78).

2.3 Solutions to the ternary braid equations

Here we consider some special solutions to the minimal ternary version ($n = 3$) of the polyadic braid equation (2.114)–(2.116), the ternary braid equation (2.117).

2.3.1 Constant matrix solutions

Let us consider the following two-dimensional vector space $V \equiv V_{d=2}$ (which is important for quantum computations) and the component matrix realization (2.131) of the ternary braiding operator C_8: $V \otimes V \otimes V \to V \otimes V \otimes V$ as

$$C_8 \circ (e_{i_1} \otimes e_{i_2} \otimes e_{i_3}) = \sum_{j_1', j_2', j_3' = 1}^{2} c_{i_1 i_2 i_3}{}^{j_1' j_2' j_3'} \cdot e_{j_1'} \otimes e_{j_2'} \otimes e_{j_3'}, \quad i_{1,2,3}, j'_{1,2,3} = 1, 2. \tag{2.132}$$

We now turn (2.132) to the standard matrix form (just to fix notations) by introducing the 8-dimensional vector space $\tilde{V}_8 = V \otimes V \otimes V$ with the natural basis $\tilde{e}_{\tilde{k}} = \{e_1 \otimes e_1 \otimes e_1, e_1 \otimes e_1 \otimes e_2, \dots, e_2 \otimes e_2 \otimes e_2\}$, where $\tilde{k} = 1, \dots, 8$ is a cumulative index. The linear operator $\tilde{C}_8$: $\tilde{V}_8 \to \tilde{V}_8$ corresponding to (2.132) is given by the 8×8 matrix $\tilde{c}_{\tilde{i}\tilde{j}}$ as $\tilde{C}_8 \circ \tilde{e}_{\tilde{i}} = \sum_{\tilde{j}=1}^{8} \tilde{c}_{\tilde{i}\tilde{j}} \cdot \tilde{e}_{\tilde{j}}$. The operators (2.112)–(2.113) become three 32×32 matrices $\tilde{A}_{1,2,3}$ as

$$\tilde{A}_1 = \tilde{c} \otimes_{\mathrm{K}} I_2 \otimes_{\mathrm{K}} I_2, \quad \tilde{A}_2 = I_2 \otimes_{\mathrm{K}} \tilde{c} \otimes_{\mathrm{K}} I_2, \quad \tilde{A}_3 = I_2 \otimes_{\mathrm{K}} I_2 \otimes_{\mathrm{K}} \tilde{c}, \tag{2.133}$$

where $\otimes_{\mathrm{K}}$ is the Kronecker product of matrices and I_2 is the 2×2 identity matrix. In this notation, the operator ternary braid equations (2.117) become the matrix equations (cf (2.114)–(2.116)) with $n = 3$

$$\tilde{A}_1 \tilde{A}_2 \tilde{A}_3 \tilde{A}_1 = \tilde{A}_2 \tilde{A}_3 \tilde{A}_1 \tilde{A}_2 = \tilde{A}_3 \tilde{A}_1 \tilde{A}_2 \tilde{A}_3, \tag{2.134}$$

which we call the *total matrix ternary braid equations*. Some weaker versions of ternary braiding are described by the *partial braid equations*

$$\text{partial 12–braid equation } \tilde{A}_1 \tilde{A}_2 \tilde{A}_3 \tilde{A}_1 = \tilde{A}_2 \tilde{A}_3 \tilde{A}_1 \tilde{A}_2, \tag{2.135}$$

$$\text{partial 13–braid equation } \tilde{A}_1 \tilde{A}_2 \tilde{A}_3 \tilde{A}_1 = \tilde{A}_3 \tilde{A}_1 \tilde{A}_2 \tilde{A}_3, \tag{2.136}$$

$$\text{partial 23–braid equation } \tilde{A}_2 \tilde{A}_3 \tilde{A}_1 \tilde{A}_2 = \tilde{A}_3 \tilde{A}_1 \tilde{A}_2 \tilde{A}_3, \tag{2.137}$$

where, obviously, two of them are independent. It follows from (2.114)–(2.116) that the weaker versions of braiding are possible for $n \geqslant 3$, but only for higher than binary braiding (the Yang–Baxter equation (2.8)).

Thus, by comparing (2.134) and (2.129) we conclude that (for each invertible matrix $\tilde{c}$ in (2.133) satisfying (2.134)) the isomorphism $\tilde{\pi}_4^{[4]}$: $\boldsymbol{\sigma}_i \mapsto \tilde{A}_i$, $i = 1, 2, 3$ gives a representation of the braid group $\mathscr{B}_4^{[4]}$ by 32 × 32 matrices over $\mathbb{C}$.

Now we can generate families of solutions corresponding to (2.133)–(2.134) in the following way. Consider an invertible operator Q: $V \to V$ in the two-dimensional vector space $V \equiv V_{d=2}$. In the basis $\{e_1, e_2\}$, its 2×2 matrix q is given by $Q \circ e_i = \sum_{j=1}^{2} q_{ij} \cdot e_j$. In the natural 8-dimensional basis $\tilde{e}_{\tilde{k}}$, the tensor product of operators $Q \otimes Q \otimes Q$ is presented by the Kronecker product of matrices $\tilde{q}_8 = q \otimes_{\mathrm{K}} q \otimes_{\mathrm{K}} q$. Let the 8×8 matrix $\tilde{c}$ be a fixed solution to the ternary braid matrix equations (2.134). Then, the family of solutions $\tilde{c}(q)$ corresponding to the invertible 2×2 matrix q is the conjugation of $\tilde{c}$ by $\tilde{q}_8$ so that

$$\tilde{c}(q) = \tilde{q}_8 \tilde{c} \tilde{q}_8^{-1} = (q \otimes_{\mathrm{K}} q \otimes_{\mathrm{K}} q)\tilde{c}(q^{-1} \otimes_{\mathrm{K}} q^{-1} \otimes_{\mathrm{K}} q^{-1}). \tag{2.138}$$

This also follows directly from the conjugation of the braid equations (2.134)–(2.137) by $q \otimes_{\mathrm{K}} q \otimes_{\mathrm{K}} q \otimes_{\mathrm{K}} q \otimes_{\mathrm{K}} q$ and (2.133). If we include the obvious invariance of the braid equations with the respect of an overall factor $t \in \mathbb{C}$, then the general family of solutions becomes (cf the Yang–Baxter equation Hietarinta 1993)

$$\tilde{c}(q, t) = t\tilde{q}_8 \tilde{c} \tilde{q}_8^{-1} = t(q \otimes_{\mathrm{K}} q \otimes_{\mathrm{K}} q)\tilde{c}(q^{-1} \otimes_{\mathrm{K}} q^{-1} \otimes_{\mathrm{K}} q^{-1}). \tag{2.139}$$

Let

$$q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}), \tag{2.140}$$

and then the manifest form of $\tilde{q}_8$ is

$$\tilde{q}_8 = \begin{pmatrix} a^3 & a^2b & a^2b & ab^2 & a^2b & ab^2 & ab^2 & b^3 \\ a^2c & a^2d & abc & abd & abc & abd & b^2c & b^2d \\ a^2c & abc & a^2d & abd & abc & b^2c & abd & b^2d \\ ac^2 & acd & acd & ad^2 & bc^2 & bcd & bcd & bd^2 \\ a^2c & abc & abc & b^2c & a^2d & abd & abd & b^2d \\ ac^2 & acd & bc^2 & bcd & acd & ad^2 & bcd & bd^2 \\ ac^2 & bc^2 & acd & bcd & acd & bcd & ad^2 & bd^2 \\ c^3 & c^2d & c^2d & cd^2 & c^2d & cd^2 & cd^2 & d^3 \end{pmatrix}. \tag{2.141}$$

It is important that not every conjugation matrix has this very special form (2.141), and that therefore, in general, conjugated matrices are different solutions of the ternary braid equations (2.134). The matrix $\tilde{q}_8^{\star}\tilde{q}_8$ ($\star$ being the Hermitian conjugation) is diagonal (this case is important for further classification similar to the binary one Dye 2003), when the conditions

$$ab^* + cd^* = 0 \tag{2.142}$$

hold, and so the matrix q has the special form (depending on three complex parameters, for $d \neq 0$)

$$q = \begin{pmatrix} a & b \\ -a\dfrac{b^*}{d^*} & d \end{pmatrix}. \tag{2.143}$$

We can present the families (2.138) for different ranks because the conjugation by an invertible matrix does not change rank. To avoid demanding (2.142), due to the cumbersome calculations involved, we restrict ourselves to a triangle matrix for q (2.140).

In general, there are $8 \times 8 = 64$ unknowns (elements of the matrix $\tilde{c}$), and each partial braid equation (2.135)–(2.137) gives $32 \times 32 = 1024$ conditions (of power 4) for the elements of $\tilde{c}$, while the total braid equations (2.134) give twice as many conditions $1024 \times 2 = 2048$ (cf the binary case: 64 cubic equations for 16 unknowns (2.8)). This means that, even in the ternary case, the higher braid system of equations is hugely overdetermined and finding even the simplest solutions is a non-trivial task.

2.3.2 Permutation and parameter-permutation 8-vertex solutions

First we consider the case when $\tilde{c}$ is a binary (or logical) matrix consisting of $\{0, 1\}$ only, and, moreover, it is a permutation matrix (see subsection 2.1.4). In the latter case, $\tilde{c}$ can be considered as a matrix over the field $\mathbb{F}_2$ (Galois field $GF(2)$). In total, there are $8! = 40\,320$ permutation matrices of the size 8×8. All of them are invertible of full rank 8 because they are obtained from the identity matrix by permutation of rows and columns.

We have found the following four invertible 8-vertex permutation matrix solutions to the ternary braid equations (2.134)

$$\begin{gathered}
\tilde{c}^{\,\text{bisymm1}}_{\text{rank}=8} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
\tilde{c}^{\,\text{bisymm2}}_{\text{rank}=8} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{gathered} \operatorname{tr}\tilde{c} = 4, \\ \det\tilde{c} = 1, \\ \text{eigenvalues:}\{1\}^{[4]},\ \{-1\}^{[4]}, \end{gathered}
\end{gathered} \tag{2.144}$$

$$\begin{gathered}
\tilde{c}^{\text{symm1}}_{\text{rank=8}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
\tilde{c}^{\text{symm2}}_{\text{rank=8}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{gathered} \operatorname{tr} \tilde{c} = 4, \\ \det \tilde{c} = 1, \\ \text{eigenvalues:} \{1\}^{[4]}, \ \{-1\}^{[4]}. \end{gathered}
\end{gathered} \tag{2.145}$$

The first two solutions (2.144) are given by bisymmetric permutation matrices (see (2.31)), and we call them 8-vertex bisymm1 and bisymm2, respectively. The second two solutions (2.145) are symmetric matrices only (we call them 8-vertex symm1 and symm2), but one matrix is a reflection of the other with respect to the minor diagonal (making them mutually persymmetric). No 90°-symmetric (see (2.32)) solution for the ternary braid equations (2.134) was found. The bisymmetric and symmetric matrices have the same eigenvalues, and are therefore pairwise conjugate but not q-conjugate because the conjugation matrices do not have the form (2.141). Thus, they are four different permutation solutions to the ternary braid equations (2.134). Note that the bisymm1 solution (2.144) coincides with the three-qubit swap operator introduced in Ballard and Wu (2011b).

All the permutation solutions are reflections (or involutions) $\tilde{c}^2 = I_8$ having $\det \tilde{c} = +1$, eigenvalues $\{1, -1\}$, and are semi-magic squares (the sums in rows and columns are 1, but not the sums in both diagonals). The 8-vertex permutation matrix solutions do not form a binary or ternary group because they are not closed with respect to multiplication.

By analogy with (2.33)–(2.34), we obtain the 8-vertex parameter-permutation solutions from (2.144)–(2.145) by replacing units with parameters and then solving the ternary braid equations (2.134). Each type of the permutation solutions bisymm1, 2 and symm1, 2 from (2.144)–(2.145) will give a corresponding series of parameter-permutation solutions over $\mathbb{C}$. The ternary braid maps are determined up to a general complex factor (see (2.14) for the Yang–Baxter maps and (2.139)), and therefore we can present all the parameter-permutation solutions in polynomial form.

- The bisymm1 series consists of two two-parameter matrices with two two-parameter matrices

$$\tilde{c}^{\text{bisymm1, 1}}_{\text{rank=8}}(x, y) = \begin{pmatrix} xy & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm y^2 & 0 \\ 0 & 0 & xy & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy & 0 & 0 \\ 0 & \pm x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & xy \end{pmatrix}, \tag{2.146}$$

$$\text{tr}\, \tilde{c} = 4xy,$$

$$\det \tilde{c} = x^8 y^8, \quad x, y \neq 0,$$

$$\text{eigenvalues: } \{xy\}^{[6]}, \ \{-xy\}^{[2]},$$

$$\tilde{c}^{\text{bisymm1, 2}}_{\text{rank=8}}(x, y) = \begin{pmatrix} xy & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm y^2 & 0 \\ 0 & 0 & xy & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy & 0 & 0 \\ 0 & \mp x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & xy \end{pmatrix}, \tag{2.147}$$

$$\text{tr}\, \tilde{c} = 4xy,$$

$$\det \tilde{c} = x^8 y^8, \quad x, y \neq 0,$$

$$\text{eigenvalues: } \{xy\}^{[4]}, \ \{ixy\}^{[2]}, \ \{-ixy\}^{[2]}.$$

- The **bisymm2** series consists of four two-parameter matrices

$$\tilde{c}^{\text{bisymm2, 1}}_{\text{rank=8}}(x, y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^6 \\ 0 & \pm x^3 y^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x^4 y^2 & 0 & 0 \\ 0 & 0 & 0 & \pm x^3 y^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm x^3 y^3 & 0 & 0 & 0 \\ 0 & 0 & x^2 y^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm x^3 y^3 & 0 \\ y^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.148}$$

$$\text{tr}\, \tilde{c} = \pm 4 x^3 y^3,$$

$$\det \tilde{c}^{\text{bisymm2}}_{\text{rank=8}}(x, y) = x^{24} y^{24}, \ x, y \neq 0,$$

$$\text{eigenvalues : } \{x^3 y^3\}^{[2]}, \ \{-x^3 y^3\}^{[2]}, \ \{\pm x^3 y^3\}^{[4]},$$

$$\tilde{c}^{\text{bisymm2, 2}}_{\text{rank=8}}(x,\, y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^6 \\ 0 & \pm x^3y^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x^4y^2 & 0 & 0 \\ 0 & 0 & 0 & \pm x^3y^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm x^3y^3 & 0 & 0 & 0 \\ 0 & 0 & -x^2y^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm x^3y^3 & 0 \\ -y^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.149}$$

$$\operatorname{tr} \tilde{c} = \pm 4x^3y^3$$

$$\det \tilde{c}^{\text{bisymm2}}_{\text{rank=8}}(x,\, y) = x^{24}y^{24},\ x,\, y \neq 0,$$

$$\text{eigenvalues}:\ \{ix^3y^3\}^{[2]},\ \{-ix^3y^3\}^{[2]},\ \{\pm x^3y^3\}^{[4]}.$$

- The symm1 series consists of four two-parameter matrices

$$\tilde{c}^{\text{symm1, 1}}_{\text{rank=8}}(x,\, y) = \begin{pmatrix} xy & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm xy & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & 0 & 0 \\ 0 & \pm xy & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & xy & 0 \\ 0 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.150}$$

$$\operatorname{tr} \tilde{c} = 4xy,$$

$$\det \tilde{c} = x^8y^8,\ \ x,\, y \neq 0,$$

$$\text{eigenvalues:}\ \{xy\}^{[6]},\ \{-xy\}^{[2]},$$

$$\tilde{c}^{\text{symm1, 2}}_{\text{rank=8}}(x,\, y) = \begin{pmatrix} xy & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm xy & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & 0 & 0 \\ 0 & \mp xy & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & xy & 0 \\ 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.151}$$

$$\operatorname{tr} \tilde{c} = 4xy,$$

$$\det \tilde{c} = x^8y^8,\ \ x,\, y \neq 0,$$

$$\text{eigenvalues:}\ \{xy\}^{[6]},\ \{-xy\}^{[2]},$$

- The symm2 series consists of four two-parameter matrices

$$\tilde{c}_{\text{rank}=8}^{\text{symm2},\ 1}(x,y)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & xy & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm xy & 0 \\ 0 & 0 & 0 & 0 & xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & xy \end{pmatrix}, \tag{2.152}$$

$$\operatorname{tr}\tilde{c} = 4xy,$$
$$\det\tilde{c} = x^8y^8,\ \ x, y \neq 0,$$
$$\text{eigenvalues: } \{xy\}^{[4]},\ \{ixy\}^{[2]},\ \{-ixy\}^{[2]}.$$

$$\tilde{c}_{\text{rank}=8}^{\text{symm2},\ 2}(x,y)=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & xy & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm xy & 0 \\ 0 & 0 & 0 & 0 & xy & 0 & 0 & 0 \\ -x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & xy \end{pmatrix}, \tag{2.153}$$

$$\operatorname{tr}\tilde{c} = 4xy,$$
$$\det\tilde{c} = x^8y^8,\ \ x, y \neq 0,$$
$$\text{eigenvalues: } \{xy\}^{[4]},\ \{ixy\}^{[2]},\ \{-ixy\}^{[2]}.$$

The above matrices with the same eigenvalues are similar but their conjugation matrices do not have the form of the triple Kronecker product (2.141), and therefore all of them together are 16 different two-parameter invertible solutions to the ternary braid equations (2.134). Further families of solutions can be obtained using ternary q-conjugation (2.139).

2.3.3 Group structure of the star and circle 8-vertex matrices

Here we investigate the group structure of 8×8 matrices by analogy with the star-like (2.35) and circle-like (2.36) 4×4 matrices, which are connected with our 8-vertex constant solutions (2.146)–(2.153) to the ternary braid equations (2.134).

Let us introduce the *star-like* 8×8 *matrices* (cf (2.35)), which correspond to the bisymm series (2.146)–(2.149)

$$V'_{\mathrm{star1}} = \begin{pmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y & 0 \\ 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{pmatrix}, \quad N'_{\mathrm{star2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w & 0 \\ v & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.154}$$

$$\operatorname{tr} N' = x + z + u + w, \quad \det N' = stuvwxyz, \quad s, t, u, v, w, x, y, z \neq 0,$$
$$\text{eigenvalues: } x, z, u, w, -\sqrt{yv}, \sqrt{yv}, -\sqrt{st}, \sqrt{st},$$

and the *circle-like* 8×8 *matrices* (cf (2.36)), which correspond to the symm series (2.150)–(2.153)

$$N'_{\mathrm{circ1}} = \begin{pmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & s & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & w & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad N'_{\mathrm{circ2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & y & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & u & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v \end{pmatrix}, \tag{2.155}$$

$$\begin{gathered} \operatorname{tr} N' = x + s + u + v, \quad \det N' = stuvwxyz, \quad s, t, u, v, w, x, y, z \neq 0, \\ \text{eigenvalues: } x, s, u, v, -\sqrt{ty}, \sqrt{ty}, -\sqrt{wz}, \sqrt{wz}. \end{gathered} \tag{2.156}$$

We denote the corresponding sets by $\mathsf{N}'_{\mathrm{star1,\,2}} = \left\{N'_{\mathrm{star1,\,2}}\right\}$ and $\mathsf{N}'_{\mathrm{circ1,\,2}} = \left\{N'_{\mathrm{circ1,\,2}}\right\}$, and then we have for them (which differs from 4×4 matrix sets (2.41))

$$\mathsf{M}'_{\mathrm{full}} = \mathsf{N}'_{\mathrm{star1}} \cup \mathsf{N}'_{\mathrm{star2}} \cup \mathsf{N}'_{\mathrm{circ1}} \cup \mathsf{N}'_{\mathrm{circ2}}, \quad \mathsf{N}'_{\mathrm{star1}} \cap \mathsf{N}'_{\mathrm{star2}} \cap \mathsf{N}'_{\mathrm{circ1}} \cap \mathsf{N}'_{\mathrm{circ2}} = \mathsf{D}, \tag{2.157}$$

where D is the set of diagonal 8×8 matrices. Again, as for 4×4 star-like and circle-like matrices, there are no closed binary multiplications among the sets of 8-vertex matrices (2.154)–(2.155). Nevertheless, we have the following triple set products

$$\mathsf{N}'_{\mathrm{star1}}\mathsf{N}'_{\mathrm{star1}}\mathsf{N}'_{\mathrm{star1}} = \mathsf{N}'_{\mathrm{star1}}, \tag{2.158}$$

$$\mathsf{N}'_{\mathrm{star2}}\mathsf{N}'_{\mathrm{star2}}\mathsf{N}'_{\mathrm{star2}} = \mathsf{N}'_{\mathrm{star2}}, \tag{2.159}$$

$$\mathsf{N}'_{\mathrm{circ1}}\mathsf{N}'_{\mathrm{circ1}}\mathsf{N}'_{\mathrm{circ1}} = \mathsf{N}'_{\mathrm{circ1}}, \tag{2.160}$$

$$\mathsf{N}'_{\mathrm{circ2}}\mathsf{N}'_{\mathrm{circ2}}\mathsf{N}'_{\mathrm{circ2}} = \mathsf{N}'_{\mathrm{circ2}}, \tag{2.161}$$

which should be compared with the analogous 4×4 matrices (2.39)–(2.40): note that now we do not have pentuple products.

Using the definitions (2.42)–(2.45), we interpret the closed products (2.158)–(2.159) and (2.160)–(2.161) as the multiplications $\mu^{[3]}$ (being the ordinary triple matrix product) of the *ternary semigroups* $\mathcal{S}^{[3]}_{\text{star1, 2}}(8, \mathbb{C}) = \left\{ \mathsf{N}'_{\text{star1, 2}} \middle| \mu^{[3]} \right\}$ and $\mathcal{S}^{[3]}_{\text{circ1, 2}}(8, \mathbb{C}) = \left\{ \mathsf{N}'_{\text{circ1, 2}} \middle| \mu^{[3]} \right\}$, respectively. The corresponding querelements (2.42) are given by

$$\bar{N}'_{\text{star1}} = N'^{-1}_{\text{star1}} = \begin{pmatrix} \frac{1}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v} & 0 \\ 0 & 0 & \frac{1}{z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{u} & 0 & 0 \\ 0 & \frac{1}{y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{w} \end{pmatrix}, \tag{2.162}$$

and

$$\bar{N}'_{\text{star2}} = N'^{-1}_{\text{star2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v} \\ 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{t} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{u} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{w} & 0 \\ \frac{1}{y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, s, t, u, v, w, x, y, z \neq 0, \tag{2.163}$$

and

$$\bar{N}'_{\text{circ1}} = N'^{-1}_{\text{circ1}} = \begin{pmatrix} \frac{1}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{w} \\ 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{u} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v} & 0 \\ 0 & 0 & \frac{1}{z} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.164}$$

$$\bar{N}'_{\text{circ2}} = N'^{-1}_{\text{circ2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{t} & 0 & 0 \\ 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{w} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{u} & 0 & 0 & 0 \\ \frac{1}{y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v} \end{pmatrix}, s, t, u, v, w, x, y, z \neq 0. \tag{2.165}$$

The ternary semigroups $\mathcal{S}^{[3]}_{\text{star1, 2}}(8, \mathbb{C}) = \left\{\mathsf{N}'_{\text{star1, 2}} \middle| \mu^{[3]}\right\}$ and $\mathcal{S}^{[3]}_{\text{circ1, 2}}(8, \mathbb{C}) = \left\{\mathsf{N}'_{\text{circ1, 2}} \middle| \mu^{[3]}\right\}$ in which every element has its querelement given by (2.162)–(2.164) become the *ternary groups* $\mathcal{G}^{[3]}_{\text{star1, 2}}(8, \mathbb{C}) = \left\{\mathsf{N}'_{\text{star1, 2}} \middle| \mu^{[3]}, \overline{(\,)}\right\}$ and $\mathcal{G}^{[3]}_{\text{circ1, 2}}(8, \mathbb{C}) = \left\{\mathsf{N}'_{\text{circ1, 2}} \middle| \mu^{[3]}, \overline{(\,)}\right\}$, which are four different (3-nonderived) ternary subgroups of the derived ternary general linear group $\mathrm{GL}^{[3]}(8, \mathbb{C})$. The ternary identities in $\mathscr{G}^{[3]}_{\mathrm{star1,\,2}}(8,\,\mathbb{C})$ and $\mathscr{G}^{[3]}_{\mathrm{circ1,\,2}}(8,\,\mathbb{C})$ are the following different continuous sets $\mathbf{I}^{\prime[3]}_{\mathrm{star1,\,2}} = \left\{I^{\prime[3]}_{\mathrm{star1,\,2}}\right\}$ and $\mathbf{I}^{\prime[3]}_{\mathrm{circ1,\,2}} = \left\{I^{\prime[3]}_{\mathrm{circ1,\,2}}\right\}$, where

$$I^{\prime[3]}_{\mathrm{star1}} = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{i\alpha_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\alpha_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\alpha_6} & 0 & 0 \\ 0 & e^{i\alpha_7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_8} \end{pmatrix},$$
$$I^{\prime[3]}_{\mathrm{star2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_2} \\ 0 & e^{i\alpha_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\alpha_4} & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\alpha_6} & 0 & 0 & 0 \\ 0 & 0 & e^{i\alpha_5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_8} & 0 \\ e^{i\alpha_7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.166}$$

$$e^{2i\alpha_1} = e^{2i\alpha_3} = e^{2i\alpha_6} = e^{2i\alpha_8} = e^{i(\alpha_2+\alpha_7)} = e^{i(\alpha_4+\alpha_5)} = 1,\ \ \alpha_1,\,\ldots\,,\,\alpha_8 \in \mathbb{R}, \tag{2.167}$$

and

$$I^{\prime[3]}_{\mathrm{circ1}} = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\alpha_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_3} \\ 0 & 0 & 0 & e^{i\alpha_4} & 0 & 0 & 0 & 0 \\ 0 & e^{i\alpha_5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\alpha_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_7} & 0 \\ 0 & 0 & e^{i\alpha_8} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$I^{\prime[3]}_{\mathrm{circ2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e^{i\alpha_2} & 0 & 0 \\ 0 & e^{i\alpha_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\alpha_4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_3} & 0 \\ 0 & 0 & 0 & 0 & e^{i\alpha_6} & 0 & 0 & 0 \\ e^{i\alpha_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_7} \end{pmatrix}, \tag{2.168}$$

$$e^{2i\alpha_1} = e^{2i\alpha_4} = e^{2i\alpha_6} = e^{2i\alpha_7} = e^{i(\alpha_3+\alpha_8)} = e^{i(\alpha_2+\alpha_5)} = 1,\ \ \alpha_1,\,\ldots\,,\,\alpha_8 \in \mathbb{R}, \tag{2.169}$$

such that all the identities are the 8×8 matrix reflections $(I^{\prime[3]})^2 = I_8$ (see (2.45)). If $\alpha_j = 0$, $j = 1, \ldots, 8$, then the ternary identities (2.167)–(2.169) coincide with the 8×8 permutation matrices (2.144)–(2.145), which are solutions to the ternary braid equations (2.134).

The module structure of the 8-vertex star-like (2.154) and circle-like (2.155) 8×8 matrix sets differs from the 4×4 matrix sets (2.53)–(2.76). First, because of the absence of pentuple matrix products (2.71)–(2.76), and second through some differences in the ternary closed products of sets.

We have the following triple relations between star and circle matrices separately (the sets corresponding to modules are in brackets, and we informally denote modules by their sets)

$$\mathsf{N}'_{\mathrm{star1}}(\mathsf{N}'_{\mathrm{star2}})\mathsf{N}'_{\mathrm{star1}} = (\mathsf{N}'_{\mathrm{star2}}), \quad \mathsf{N}'_{\mathrm{circ1}}(\mathsf{N}'_{\mathrm{circ2}})\mathsf{N}'_{\mathrm{circ1}} = (\mathsf{N}'_{\mathrm{circ2}}), \tag{2.170}$$

$$\mathsf{N}'_{\mathrm{star1}}\mathsf{N}'_{\mathrm{star1}}(\mathsf{N}'_{\mathrm{star2}}) = (\mathsf{N}'_{\mathrm{star2}}), \quad \mathsf{N}'_{\mathrm{circ1}}\mathsf{N}'_{\mathrm{circ1}}(\mathsf{N}'_{\mathrm{circ2}}) = \mathsf{N}'_{\mathrm{circ2}}, \tag{2.171}$$

$$(\mathsf{N}'_{\mathrm{star2}})\mathsf{N}'_{\mathrm{star1}}\mathsf{N}'_{\mathrm{star1}} = (\mathsf{N}'_{\mathrm{star2}}), \quad (\mathsf{N}'_{\mathrm{circ2}})\mathsf{N}'_{\mathrm{circ1}}\mathsf{N}'_{\mathrm{circ1}} = (\mathsf{N}'_{\mathrm{circ2}}), \tag{2.172}$$

$$\mathsf{N}'_{\mathrm{star2}}\mathsf{N}'_{\mathrm{star2}}(\mathsf{N}'_{\mathrm{star1}}) = (\mathsf{N}'_{\mathrm{star1}}), \quad \mathsf{N}'_{\mathrm{circ2}}\mathsf{N}'_{\mathrm{circ2}}(\mathsf{N}'_{\mathrm{circ1}}) = (\mathsf{N}'_{\mathrm{circ1}}), \tag{2.173}$$

$$\mathsf{N}'_{\mathrm{star2}}(\mathsf{N}'_{\mathrm{star1}})\mathsf{N}'_{\mathrm{star2}} = (\mathsf{N}'_{\mathrm{star1}}), \quad \mathsf{N}'_{\mathrm{circ2}}(\mathsf{N}'_{\mathrm{circ1}})\mathsf{N}'_{\mathrm{circ2}} = (\mathsf{N}'_{\mathrm{circ1}}), \tag{2.174}$$

$$(\mathsf{N}'_{\mathrm{star1}})\mathsf{N}'_{\mathrm{star2}}\mathsf{N}'_{\mathrm{star2}} = (\mathsf{N}'_{\mathrm{star1}}), \quad (\mathsf{N}'_{\mathrm{circ1}})\mathsf{N}'_{\mathrm{circ2}}\mathsf{N}'_{\mathrm{circ2}} = (\mathsf{N}'_{\mathrm{circ1}}). \tag{2.175}$$

So we may observe the following module structures: (1) from (2.170)–(2.172), the sets $\mathsf{N}'_{\mathrm{star2}}$ $(\mathsf{N}'_{\mathrm{circ2}})$ are the middle, right, and left ternary modules over $\mathsf{N}'_{\mathrm{star1}}$ $(\mathsf{N}'_{\mathrm{circ1}})$; (2) from (2.173)–(2.175), the set $\mathsf{N}'_{\mathrm{star1}}$ $(\mathsf{N}'_{\mathrm{circ1}})$ are middle, right, and left ternary modules over $\mathsf{N}'_{\mathrm{star2}}$ $(\mathsf{N}'_{\mathrm{circ2}})$;

$$\mathsf{N}'_{\mathrm{star1}}\mathsf{N}'_{\mathrm{star1}}(\mathsf{N}'_{\mathrm{circ1}}) = (\mathsf{N}'_{\mathrm{circ1}}), \quad (\mathsf{N}'_{\mathrm{circ1}})\mathsf{N}'_{\mathrm{star1}}\mathsf{N}'_{\mathrm{star1}} = (\mathsf{N}'_{\mathrm{circ1}}), \tag{2.176}$$

$$\mathsf{N}'_{\mathrm{star1}}\mathsf{N}'_{\mathrm{star1}}(\mathsf{N}'_{\mathrm{circ2}}) = (\mathsf{N}'_{\mathrm{circ2}}), \quad (\mathsf{N}'_{\mathrm{circ2}})\mathsf{N}'_{\mathrm{star1}}\mathsf{N}'_{\mathrm{star1}} = (\mathsf{N}'_{\mathrm{circ2}}), \tag{2.177}$$

$$\mathsf{N}'_{\mathrm{star2}}\mathsf{N}'_{\mathrm{star2}}(\mathsf{N}'_{\mathrm{circ1}}) = (\mathsf{N}'_{\mathrm{circ1}}), \quad (\mathsf{N}'_{\mathrm{circ1}})\mathsf{N}'_{\mathrm{star2}}\mathsf{N}'_{\mathrm{star2}} = (\mathsf{N}'_{\mathrm{circ1}}), \tag{2.178}$$

$$\mathsf{N}'_{\mathrm{star2}}\mathsf{N}'_{\mathrm{star2}}(\mathsf{N}'_{\mathrm{circ2}}) = (\mathsf{N}'_{\mathrm{circ2}}), \quad (\mathsf{N}'_{\mathrm{circ2}})\mathsf{N}'_{\mathrm{star2}}\mathsf{N}'_{\mathrm{star2}} = (\mathsf{N}'_{\mathrm{circ2}}), \tag{2.179}$$

(3) from (2.176)–(2.179), the sets $\mathsf{N}'_{\mathrm{circ1,\,2}}$ are right and left ternary modules over $\mathsf{N}'_{\mathrm{star1,\,2}}$;

$$\mathsf{N}'_{\mathrm{circ1}}\mathsf{N}'_{\mathrm{circ1}}(\mathsf{N}'_{\mathrm{star1}}) = (\mathsf{N}'_{\mathrm{star1}}), \quad (\mathsf{N}'_{\mathrm{star1}})\mathsf{N}'_{\mathrm{circ1}}\mathsf{N}'_{\mathrm{circ1}} = (\mathsf{N}'_{\mathrm{star1}}), \tag{2.180}$$

$$\mathsf{N}'_{\mathrm{circ1}}\mathsf{N}'_{\mathrm{circ1}}(\mathsf{N}'_{\mathrm{star2}}) = (\mathsf{N}'_{\mathrm{star2}}), \quad (\mathsf{N}'_{\mathrm{star2}})\mathsf{N}'_{\mathrm{circ1}}\mathsf{N}'_{\mathrm{circ1}} = (\mathsf{N}'_{\mathrm{star2}}), \tag{2.181}$$

$$\mathsf{N}'_{\mathrm{circ2}}\mathsf{N}'_{\mathrm{circ2}}(\mathsf{N}'_{\mathrm{star1}}) = (\mathsf{N}'_{\mathrm{star1}}), \quad (\mathsf{N}'_{\mathrm{star1}})\mathsf{N}'_{\mathrm{circ2}}\mathsf{N}'_{\mathrm{circ2}} = (\mathsf{N}'_{\mathrm{star1}}), \tag{2.182}$$

$$\mathsf{N}'_{\mathrm{circ2}}\mathsf{N}'_{\mathrm{circ2}}(\mathsf{N}'_{\mathrm{star2}}) = (\mathsf{N}'_{\mathrm{star2}}), \quad (\mathsf{N}'_{\mathrm{star2}})\mathsf{N}'_{\mathrm{circ2}}\mathsf{N}'_{\mathrm{circ2}} = (\mathsf{N}'_{\mathrm{star2}}), \tag{2.183}$$

(4) from (2.180)–(2.183), the sets $\mathsf{N}'_{\mathrm{star1,\,2}}$ are right and left ternary modules over $\mathsf{N}'_{\mathrm{circ1,\,2}}$.

2.3.4 Group structure of the star and circle 16-vertex matrices

Next we will introduce 8 × 8 matrices of a special form similar to the star 8-vertex matrices (2.77) and the circle 8-vertex matrices (2.78), analyze their group structure, and establish which ones could be 16-vertex solutions to the ternary braid equations (2.134). We will derive the solutions in the opposite way to that for the 8-vertex Yang–Baxter maps, following the note after (2.34). Indeed, the sum of the permutation bisymm solutions (2.144) gives the shape of the 8 × 8 star matrix M'_{star} (as in (2.77)), while the sum of symm solutions (2.145) gives the 8 × 8 circle matrix M'_{circ} (as in (2.78))

$$M'_{\mathrm{star}} = \begin{pmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & z & 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & t & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & v & w & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & d & 0 & 0 \\ 0 & f & 0 & 0 & 0 & 0 & g & 0 \\ h & 0 & 0 & 0 & 0 & 0 & 0 & p \end{pmatrix}, \tag{2.184}$$

$$M'_{\mathrm{circ}} = \begin{pmatrix} x & 0 & 0 & 0 & 0 & y & 0 & 0 \\ 0 & z & 0 & 0 & s & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & u \\ 0 & 0 & 0 & v & 0 & 0 & w & 0 \\ 0 & f & 0 & 0 & g & 0 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & d \end{pmatrix}, \tag{2.185}$$

$$\begin{aligned} \operatorname{tr} M' &= x + z + t + v + b + d + g + p, \\ \det M' &= (bv - aw)(cu - dt)(fs - gz)(px - hy), \end{aligned} \tag{2.186}$$

$$\begin{gathered} \text{eigenvalues}: \frac{1}{2}\left(d + t - \sqrt{4cu + (d-t)^2}\right), \frac{1}{2}\left(d + t + \sqrt{4cu + (d-t)^2}\right), \\ \frac{1}{2}\left(b + v - \sqrt{4aw + (b-v)^2}\right), \frac{1}{2}\left(b + v + \sqrt{4aw + (b-v)^2}\right), \\ \frac{1}{2}\left(p + x + \sqrt{4hy + (p-x)^2}\right), \frac{1}{2}\left(p + x - \sqrt{4hy + (p-x)^2}\right), \\ \frac{1}{2}\left(g + z - \sqrt{4fs + (g-z)^2}\right), \frac{1}{2}\left(g + z + \sqrt{4fs + (g-z)^2}\right). \end{gathered} \tag{2.187}$$

The 16-vertex matrices are invertible, if $\det M'_{\text{star}} \neq 0$ and $\det M'_{\text{circ}} \neq 0$, which give the following joint conditions on the parameters (cf (2.80))

$$bv - aw \neq 0, \;\; cu - dt \neq 0, \;\; fs - gz \neq 0, \;\; px - hy \neq 0. \tag{2.188}$$

Only in this concrete parametrization (2.184) and (2.185) do the matrices M'_{star} and M'_{circ} have the same trace, determinant, and eigenvalues, and they are diagonalizable and conjugate via

$$U' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{2.189}$$

The matrix U' cannot be presented in the form of a triple Kronecker product (2.141), and so two matrices M'_{star} and M'_{circ} are not q-conjugate in the parametrization (2.184) and (2.185), and can lead to different solutions to the ternary braid equations (2.134). It follows from (2.188) that 16-vertex matrices with all nonzero entries equal to 1 are noninvertible, having vanishing determinant and rank 4 (despite each one being a sum of two permutation matrices). In the case all the conditions (2.188) holding, the inverse matrices become

$$M'^{-1}_{\text{star}} = \begin{pmatrix} \frac{p}{px-hy} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{y}{px-hy} \\ 0 & -\frac{g}{fs-gz} & 0 & 0 & 0 & 0 & \frac{s}{fs-gz} & 0 \\ 0 & 0 & -\frac{d}{cu-dt} & 0 & 0 & \frac{u}{cu-dt} & 0 & 0 \\ 0 & 0 & 0 & \frac{b}{bv-aw} & -\frac{w}{bv-aw} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{bv-aw} & \frac{v}{bv-aw} & 0 & 0 & 0 \\ 0 & 0 & \frac{c}{cu-dt} & 0 & 0 & -\frac{t}{cu-dt} & 0 & 0 \\ 0 & \frac{f}{fs-gz} & 0 & 0 & 0 & 0 & -\frac{z}{fs-gz} & 0 \\ -\frac{h}{px-hy} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{x}{px-hy} \end{pmatrix}, \tag{2.190}$$

$$M'^{-1}_{\mathrm{circ}} = \begin{pmatrix} \frac{p}{px-hy} & 0 & 0 & 0 & 0 & -\frac{y}{px-hy} & 0 & 0 \\ 0 & -\frac{g}{fs-gz} & 0 & 0 & \frac{s}{fs-gz} & 0 & 0 & 0 \\ 0 & 0 & -\frac{d}{cu-dt} & 0 & 0 & 0 & 0 & \frac{u}{cu-dt} \\ 0 & 0 & 0 & \frac{b}{bv-aw} & 0 & 0 & -\frac{w}{bv-aw} & 0 \\ 0 & \frac{f}{fs-gz} & 0 & 0 & -\frac{z}{fs-gz} & 0 & 0 & 0 \\ -\frac{h}{px-hy} & 0 & 0 & 0 & 0 & \frac{x}{px-hy} & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{bv-aw} & 0 & 0 & \frac{v}{bv-aw} & 0 \\ 0 & 0 & \frac{c}{cu-dt} & 0 & 0 & 0 & 0 & -\frac{t}{cu-dt} \end{pmatrix}. \quad (2.191)$$

Denoting the sets of matrices corresponding to (2.184) and (2.185) by $\mathsf{M}'_{\mathrm{star}}$ and $\mathsf{M}'_{\mathrm{circ}}$, their multiplications are

$$\mathsf{M}'_{\mathrm{star}}\mathsf{M}'_{\mathrm{star}} = \mathsf{M}'_{\mathrm{star}}, \qquad \mathsf{M}'_{\mathrm{circ}}\mathsf{M}'_{\mathrm{circ}} = \mathsf{M}'_{\mathrm{circ}}, \tag{2.192}$$

and in term of sets $\mathsf{M}'_{\mathrm{star}} = \mathsf{N}'_{\mathrm{star}1} \cup \mathsf{N}'_{\mathrm{star}2}$ and $\mathsf{M}'_{\mathrm{circ}} = \mathsf{N}'_{\mathrm{circ}1} \cup \mathsf{N}'_{\mathrm{circ}2}$, and $\mathsf{N}'_{\mathrm{star}1} \cap \mathsf{N}'_{\mathrm{star}2} = \mathsf{D}$ and $\mathsf{N}'_{\mathrm{circ}1} \cap \mathsf{N}'_{\mathrm{circ}2} = \mathsf{D}$ (see (2.157)). Note that the structure (2.192) is considerably different from the binary case (2.81)–(2.83), and therefore it may not necessarily be related to the Cartan decomposition.

The products (2.192) mean that both $\mathsf{M}'_{\mathrm{star}}$ and $\mathsf{M}'_{\mathrm{circ}}$ are separately closed with respect to binary matrix multiplication $(\cdot)$, and therefore $\mathscr{S}^{\mathrm{star}}_{16\mathrm{vert}} = \langle \mathsf{M}'_{\mathrm{star}} | \cdot \rangle$ and $\mathscr{S}^{\mathrm{circ}}_{16\mathrm{vert}} = \langle \mathsf{M}'_{\mathrm{circ}} | \cdot \rangle$ are semigroups. We denote their intersection by $\mathscr{S}^{\mathrm{diag}}_{8vert} = \mathscr{S}^{\mathrm{star}}_{16\mathrm{vert}} \cap \mathscr{S}^{\mathrm{circ}}_{16\mathrm{vert}}$ which is a semigroup of diagonal 8-vertex matrices. In case, the invertibility conditions (2.188) are fulfilled, the sets $\mathsf{M}'_{\mathrm{star}}$ and $\mathsf{M}'_{\mathrm{circ}}$ form subgroups $\mathscr{G}^{\mathrm{star}}_{16\mathrm{vert}} = \langle \mathsf{M}'_{\mathrm{star}} | \cdot, (\_\_)^{-1}, \mathscr{I}_8 \rangle$ and $\mathscr{G}^{\mathrm{circ}}_{16\mathrm{vert}} = \langle \mathsf{M}'_{\mathrm{circ}} | \cdot, (\_\_)^{-1}, \mathscr{I}_8 \rangle$ (where I_8 is the 8×8 identity matrix) of $\mathrm{GL}(8, \mathbb{C})$ with the inverse elements given explicitly by (2.190)–(2.191). Because the elements M'_{star} and M'_{circ} in (2.184) and (2.185) are conjugates by the invertible matrix U' (2.189), the subgroups $\mathscr{G}^{\mathrm{star}}_{16\mathrm{vert}}$ and $\mathscr{G}^{\mathrm{circ}}_{16\mathrm{vert}}$ (as well as the semigroups $\mathscr{S}^{\mathrm{star}}_{16\mathrm{vert}}$ and $\mathscr{S}^{\mathrm{circ}}_{16\mathrm{vert}}$) are isomorphic by the obvious isomorphism

$$M'_{\mathrm{star}} \mapsto U' M'_{\mathrm{circ}} U'^{-1}, \tag{2.193}$$

where U' is in (2.189).

The interaction between $\mathsf{M}'_{\mathrm{star}}$ and $\mathsf{M}'_{\mathrm{circ}}$ also differs from the binary case (2.82), because

$$\mathsf{M}'_{\mathrm{star}}\mathsf{M}'_{\mathrm{circ}} = \mathsf{M}'_{\mathrm{quad}}, \quad \mathsf{M}'_{\mathrm{circ}}\mathsf{M}'_{\mathrm{star}} = \mathsf{M}'_{\mathrm{quad}}, \tag{2.194}$$

$$\mathsf{M}'_{\mathrm{quad}}\mathsf{M}'_{\mathrm{quad}} = \mathsf{M}'_{\mathrm{quad}}, \tag{2.195}$$

where $\mathsf{M}'_{\mathrm{quad}}$ is a set of 32-vertex so-called *quad-matrices* of the form

$$M'_{\mathrm{quad}} = \begin{pmatrix} x_1 & 0 & y_1 & 0 & 0 & z_1 & 0 & s_1 \\ 0 & t_1 & 0 & u_1 & v_1 & 0 & w_1 & 0 \\ a_1 & 0 & b_1 & 0 & 0 & c_1 & 0 & d_1 \\ 0 & f_1 & 0 & g_1 & h_1 & 0 & p_1 & 0 \\ 0 & x_2 & 0 & y_2 & z_2 & 0 & s_2 & 0 \\ t_2 & 0 & u_2 & 0 & 0 & v_2 & 0 & w_2 \\ 0 & a_2 & 0 & b_2 & c_2 & 0 & d_2 & 0 \\ f_2 & 0 & g_2 & 0 & 0 & h_2 & 0 & p_2 \end{pmatrix}. \qquad (2.196)$$

Because of (2.195), the set $\mathsf{M}'_{\mathrm{quad}}$ is closed with respect to matrix multiplication, and therefore (for invertible matrices M'_{quad}) the group $\mathscr{G}^{\mathrm{quad}}_{32\mathrm{vert}} = \left\langle \mathsf{M}'_{\mathrm{quad}} \middle| \cdot, (\_\_)^{-1}, \mathscr{I}_8 \right\rangle$ is a subgroup of GL(8, $\mathbb{C}$). So, in trying to find higher 32-vertex solutions (having at most half as many unknown variables as a general 8 × 8 matrix) to the ternary braid equations (2.134), it is worthwhile to search within the class of quad-matrices (2.196).

Innovation 2.12. *The group structure of the above 16-vertex 8 × 8 matrices (2.192)–(2.195) is considerably different to that of 8-vertex 4 × 4 matrices (2.77)–(2.78) because the former contains two isomorphic binary subgroups $\mathscr{G}^{\mathrm{star}}_{16\mathrm{vert}}$ and $\mathscr{G}^{\mathrm{circ}}_{16\mathrm{vert}}$ of* GL(8, $\mathbb{C}$) *(cf (2.81)–(2.83) and (2.192)).*

The sets $\mathsf{M}'_{\mathrm{star}}$, $\mathsf{M}'_{\mathrm{circ}}$ and $\mathsf{M}'_{\mathrm{quad}}$ are also closed with respect to matrix addition, and therefore (because of the distributivity of $\mathbb{C}$) they are the matrix rings $\mathscr{R}^{\mathrm{star}}_{16\mathrm{vert}}$, $\mathscr{R}^{\mathrm{circ}}_{16\mathrm{vert}}$ and $\mathscr{R}^{\mathrm{quad}}_{32vert}$, respectively. In the invertible case (2.188) and $\det M'_{\mathrm{quad}} \neq 0$, these become matrix fields.

2.3.5 Pauli matrix presentation of the star and circle 16-vertex constant matrices

The main peculiarity of the 16-vertex 8 × 8 matrices (2.192)–(2.195) is the fact that they can be expressed as special tensor (Kronecker) products of the Pauli matrices (see, also, Khaneja and Glaser 2001, Ballard and Wu 2011b). Indeed, let

$$\Sigma_{ijk} = \rho_i \otimes_{\mathrm{K}} \rho_j \otimes_{\mathrm{K}} \rho_k, \qquad i, j, k = 1, 2, 3, 4, \qquad (2.197)$$

where ρ_i are Pauli matrices (we have already used the letter 'σ' for the braid group generators (2.5))

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_4 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (2.198)$$

Among the total of 64 8 × 8 matrices Σ_{ijk} (2.197), there are 24 which generate M'_{star} (2.184) and M'_{circ} (2.185):

- Eight diagonal matrices: $\Sigma_{\mathrm{diag}} = \{\Sigma_{333}, \Sigma_{334}, \Sigma_{343}, \Sigma_{344}, \Sigma_{433}, \Sigma_{434}, \Sigma_{443}, \Sigma_{444}\}$;

- Eight antidiagonal matrices: $\Sigma_{adiag} = \{\Sigma_{111}, \Sigma_{112}, \Sigma_{121}, \Sigma_{122}, \Sigma_{211}, \Sigma_{212}, \Sigma_{221}, \Sigma_{222}\}$;
- Eight circle-like matrices (M'_{circ} with 0s on the diagonal): $\Sigma_{circ} = \{\Sigma_{131}, \Sigma_{132}, \Sigma_{141}, \Sigma_{142}, \Sigma_{231}, \Sigma_{232}, \Sigma_{241}, \Sigma_{242}\}$.

Thus, in general we have the following set structure for the star and circle 16-vertex matrices (2.184) and (2.185)

$$\mathrm{M}'_{star} = \Sigma_{diag} \bigcup \Sigma_{adiag}, \tag{2.199}$$

$$\mathrm{M}'_{circ} = \Sigma_{diag} \bigcup \Sigma_{circ}, \tag{2.200}$$

$$\mathrm{M}'_{star} \bigcap \mathrm{M}'_{circ} = \Sigma_{diag}. \tag{2.201}$$

In particular, for the 8-vertex permutation solutions (2.144)–(2.145) of the ternary braid equations (2.134), we have

$$\tilde{c}^{\,bisymm1,\,2}_{rank=8} = \frac{1}{2}(\Sigma_{111} + \Sigma_{444} \pm \Sigma_{212} \pm \Sigma_{343}), \tag{2.202}$$

$$\tilde{c}^{\,symm1,\,2}_{rank=8} = \frac{1}{2}(\Sigma_{141} + \Sigma_{444} \pm \Sigma_{232} \pm \Sigma_{333}). \tag{2.203}$$

The noninvertible 16-vertex solutions M'_{star} (2.184) and M'_{circ} (2.185) having 1s on nonzero places are of *rank* = 4 and can be presented by (2.197) as follows

$$M'_{star}(1) = \begin{pmatrix} 1&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&1&0\\ 0&0&1&0&0&1&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&0&1&1&0&0&0\\ 0&0&1&0&0&1&0&0\\ 0&1&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&1 \end{pmatrix} = \Sigma_{111} + \Sigma_{444}, \tag{2.204}$$

$$M'_{circ}(1) = \begin{pmatrix} 1&0&0&0&0&1&0&0\\ 0&1&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&1\\ 0&0&0&1&0&0&1&0\\ 0&1&0&0&1&0&0&0\\ 1&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&1&0\\ 0&0&1&0&0&0&0&1 \end{pmatrix} = \Sigma_{141} + \Sigma_{444}. \tag{2.205}$$

Similarly, one can obtain the Pauli matrix presentation for the general star and circle 16-vertex matrices (2.184) and (2.185) which will contain linear combinations of the 16 parameters as coefficients before the Σs.

2.3.6 Invertible and noninvertible 16-vertex solutions to the ternary braid equations

First, consider the 16-vertex solutions to (2.134) having the star matrix shape (2.184). We found the following two one-parameter invertible solutions

$$\tilde{c}^{16-\mathrm{vert,star}}_{\mathrm{rank}=8}(x) = \begin{pmatrix} x^3 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & x^3 & 0 & 0 & 0 & 0 & \mp x^2 & 0 \\ 0 & 0 & x^3 & 0 & 0 & -x^2 & 0 & 0 \\ 0 & 0 & 0 & x^3 & \mp x^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm x^2 & x^3 & 0 & 0 & 0 \\ 0 & 0 & x^4 & 0 & 0 & x^3 & 0 & 0 \\ 0 & \pm x^4 & 0 & 0 & 0 & 0 & x^3 & 0 \\ x^6 & 0 & 0 & 0 & 0 & 0 & 0 & x^3 \end{pmatrix}, \tag{2.206}$$

$$\mathrm{tr}\,\tilde{c} = 8x^3,$$

$$\det \tilde{c} = 16x^{24}, \quad x \neq 0,$$

$$\text{eigenvalues:} \{(1 + i)x^3\}^{[4]}, \{(1 - i)x^3\}^{[4]}.$$

Both matrices in (2.206) are diagonalizable and are conjugates via

$$U_{\mathrm{star}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.207}$$

which cannot be presented in the form of a triple Kronecker product (2.141). Therefore, the two solutions in (2.206) are not q-conjugate and become different 16-vertex one-parameter invertible solutions of the braid equations (2.134).

In search of 16-vertex solutions to the total braid equations (2.134) of the circle matrix shape (2.185), we found that only noninvertible ones exist. They are the following two 2-parameter solutions of rank 4

$$\tilde{c}^{16-\mathrm{vert,circ}}_{\mathrm{rank}=4}(x, y) = \begin{pmatrix} \pm xy & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & \pm xy & 0 & 0 & xy & 0 & 0 & 0 \\ 0 & 0 & \pm xy & 0 & 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & \pm xy & 0 & 0 & xy & 0 \\ 0 & xy & 0 & 0 & \pm xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & \pm xy & 0 & 0 \\ 0 & 0 & 0 & xy & 0 & 0 & \pm xy & 0 \\ 0 & 0 & x^2 & 0 & 0 & 0 & 0 & \pm xy \end{pmatrix}, \tag{2.208}$$

$$\mathrm{tr}\,\tilde{c} = \pm 8xy,$$

$$\text{eigenvalues: } \{2xy\}^{[4]}, \{0\}^{[4]}.$$

Two matrices in (2.208) are not even conjugates in the standard way, and so they are different 16-vertex two-parameter noninvertible solutions to the braid equations (2.134).

For the only partial 13-braid equation (2.136), there are four polynomial 16-vertex two-parameter invertible solutions

$$\tilde{c}^{16-\text{vert, 13circ}}_{\text{rank}=8}(x,y)=\begin{pmatrix} x & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & \pm y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & \pm x & 0 \\ 0 & x & 0 & 0 & xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & \pm x & 0 & 0 & xy & 0 \\ 0 & 0 & \pm x^2 & 0 & 0 & 0 & 0 & x \end{pmatrix}, \tag{2.209}$$

$$\begin{pmatrix} x & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & -x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & \pm y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & \mp x & 0 \\ 0 & -x & 0 & 0 & xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & \mp x & 0 & 0 & xy & 0 \\ 0 & 0 & \pm x^2 & 0 & 0 & 0 & 0 & x \end{pmatrix}, \tag{2.210}$$

$$\begin{gathered} \operatorname{tr}\tilde{c}=4x(y+1), \\ \det\tilde{c}=x^8(y^2-1)^4,\ x\neq 0,\ y\neq 1, \\ \text{eigenvalues}:\{x(y+1)\}^{[4]},\ \{x(y-1)\}^{[2]},\ \{-x(y-1)\}^{[2]}. \end{gathered} \tag{2.211}$$

Also, for the partial 13-braid equation (2.136), we found four exotic irrational (an analog of (2.89) for the Yang–Baxter equation (2.12)) 16-vertex, two-parameter invertible solutions of rank 8 of the form

$$\begin{aligned} &\tilde{c}^{16-\text{vert, 13circ, 1}}_{\text{rank}=8}(x,y) \\ &=\begin{pmatrix} x(2y-1) & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & x\sqrt{2(y-1)y+1} & 0 & 0 & 0 \\ 0 & 0 & x(2y-1) & 0 & 0 & 0 & 0 & \pm y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & \pm x\sqrt{2(y-1)y+1} & 0 \\ 0 & x\sqrt{2(y-1)y+1} & 0 & 0 & xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & \pm x\sqrt{2(y-1)y+1} & 0 & 0 & xy & 0 \\ 0 & 0 & \pm x^2 & 0 & 0 & 0 & 0 & x \end{pmatrix}, \end{aligned} \tag{2.212}$$

$$\tilde{c}^{16-\text{vert, 13circ, 2}}_{\text{rank}=8}(x, y)$$

$$= \begin{pmatrix} x(2y-1) & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & -x\sqrt{2(y-1)y+1} & 0 & 0 & 0 \\ 0 & 0 & x(2y-1) & 0 & 0 & 0 & 0 & \pm y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & \mp x\sqrt{2(y-1)y+1} & 0 \\ 0 & -x\sqrt{2(y-1)y+1} & 0 & 0 & xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & \mp x\sqrt{2(y-1)y+1} & 0 & 0 & xy & 0 \\ 0 & 0 & \pm x^2 & 0 & 0 & 0 & 0 & x \end{pmatrix}, \quad (2.213)$$

$$\operatorname{tr} \tilde{c} = 8xy, \quad \det \tilde{c} = x^8(y-1)^8, \qquad x \neq 0, \quad y \neq 1, \qquad (2.214)$$

$$\text{eigenvalues}: \left\{ x\left(y + \sqrt{2(y-1)y+1}\right) \right\}^{[4]}, \left\{ x\left(y - \sqrt{2(y-1)y+1}\right) \right\}^{[4]}. \quad (2.215)$$

The matrices in (2.209)–(2.213) are diagonalizable, have the same eigenvalues (2.215), and are pairwise conjugate by

$$U_{\text{circ}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (2.216)$$

Because U_{circ} cannot be presented in the form (2.141), all solutions in (2.209)–(2.213) are not mutually q-conjugate and become eight different 16-vertex two-parameter invertible solutions to the partial 13-braid equation (2.136). If $y = 1$, then the matrices (2.209)–(2.213) become of rank 4 with vanishing determinants (2.211), (2.214), and therefore in this case they are a 16-vertex one-parameter circle of noninvertible solutions to the total braid equations (2.134).

Further families of solutions could be constructed using additional parameters: the scaling parameter t in (2.139) and the complex elements of the matrix q (2.140).

2.3.7 Higher 2^n-vertex constant solutions to n-ary braid equations

Next we considered the 4-ary constant braid equations (2.114)–(2.116) and found the following 32-vertex star solution

$$\tilde{c}_{16}=\begin{pmatrix}1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1\\0&1&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\0&0&1&0&0&0&0&0&0&0&0&0&0&-1&0&0\\0&0&0&1&0&0&0&0&0&0&0&0&-1&0&0&0\\0&0&0&0&1&0&0&0&0&0&0&-1&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&-1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&-1&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&-1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0\\0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0\\0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\end{pmatrix}. \tag{2.217}$$

We may compare (2.217) with particular cases of the star solutions to the Yang–Baxter equation (2.87) and the ternary braid equation (2.206)

$$\tilde{c}_4=\begin{pmatrix}1&0&0&-1\\0&1&-1&0\\0&1&1&0\\1&0&0&1\end{pmatrix},\quad \tilde{c}_8=\begin{pmatrix}1&0&0&0&0&0&0&-1\\0&1&0&0&0&0&-1&0\\0&0&1&0&0&-1&0&0\\0&0&0&1&-1&0&0&0\\0&0&0&1&1&0&0&0\\0&0&1&0&0&1&0&0\\0&1&0&0&0&0&1&0\\1&0&0&0&0&0&0&1\end{pmatrix}. \tag{2.218}$$

Informally we call such solutions the Minkowski star solutions because their legs have the Minkowski signature. Thus, we assume that in the general case for the n-ary braid equation there exist 2^{n+1}-vertex $2^n \times 2^n$ matrix Minkowski star invertible solutions of the above form

$$\tilde{c}_{2^n}=\begin{pmatrix}1&0&0&0&0&-1\\0&\ddots&0&0&⋰&0\\0&0&1&-1&0&0\\0&0&1&1&0&0\\0&⋰&0&0&\ddots&0\\1&0&0&0&0&1\end{pmatrix}. \tag{2.219}$$

This allows us to use the general solution (2.219) as n-ary braiding quantum gates with an arbitrary number of qubits.

2.4 Invertible and noninvertible quantum gates

Informally, quantum computing consists of preparation (setting up an initial quantum state), evolution (by a quantum circuit), and measurement (projection onto the final state). Mathematically (in the computational basis), the initial state is a vector in a Hilbert space (multi-qubit state), the evolution is governed by successive (quantum circuit) invertible linear transformations (unitary matrices called quantum gates), and the measurement is made by noninvertible projection matrices to leave only one final quantum (multi-qubit) state. So, quantum computing is noninvertible overall, and we may consider noninvertible transformations at each step. It was then realized that one can invite the Yang–Baxter operators (solutions of the constant Yang–Baxter equation) into quantum gates, providing a means of entangling otherwise non-entangled states. This insight uncovered a deep connection between quantum and topological computation (for details, see e.g. Kauffman and Lomonaco 2002, 2004).

Here we propose extending the above picture in two directions. First, we can treat higher braided operators as higher braiding gates. Second, we will analyze the possible role of noninvertible linear transformations (described by the partial unitary matrices introduced in (2.20)–(2.21)), which can be interpreted as a property of some hypothetical quantum circuit, e.g., with specific loss of information, some kind of dissipativity or vagueness. This can be considered as an intermediate case between standard unitary computing and the measurement only computing of Bonderson *et al* (2008).

To establish notation recall (Nielsen and Chuang 2000) that in the computational basis (vector representation) and Dirac notation, a (pure) one-qubit state is described by a vector in two-dimensional Hilbert space $V = \mathbb{C}^2$

$$
\begin{aligned}
&|\psi\rangle \equiv |\psi^{(1)}\rangle = a_0 |0\rangle + a_1 |1\rangle, \ |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
&|a_0|^2 + |a_1|^2 = 1, \quad a_i \in \mathbb{C}, \ i = 1, 2,
\end{aligned} \tag{2.220}
$$

where a_i is a probability amplitude of $|i\rangle$. Sometimes, for a one-qubit state it is convenient to use the Bloch unit sphere representation (normalized up to a general unimportant and unmeasurable phase)

$$
|\psi(\theta, \gamma)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\gamma}\sin\frac{\theta}{2}|1\rangle, \quad 0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \gamma \leqslant 2\pi. \tag{2.221}
$$

A (pure) state of L-qubits $|\psi^{(L)}\rangle$ is described by 2^L amplitudes, and so is a vector in 2^L-dimensional Hilbert space. If $|\psi^{(L)}\rangle$ cannot be presented as a tensor product of L one-qubit states (2.220), then it is called *entangled*. For instance, a two-qubit pure state

$$
\begin{aligned}
&|\psi^{(2)}\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle, \\
&|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1, \\
&a_{ij} \in \mathbb{C}, \ i, j = 1, 2,
\end{aligned} \tag{2.222}
$$

is entangled, if $\det(a_{ij}) \neq 0$, and the *concurrence*

$$C^{(2)} \equiv C^{(2)}(|\,\psi^{(2)}\rangle) = 2\,\left|\,\det(a_{ij})\,\right| \tag{2.223}$$

is the measure of entanglement $0 \leqslant C^{(2)} \leqslant 1$. It follows from (2.220) that the tensor product of states has vanishing concurrence $C^{(2)}(|\,\psi_1\rangle \otimes |\,\psi_2\rangle) = 0$. An example of the maximally entangled ($C^{(2)} = 1$) two-qubit states is the (first) Bell state $|\,\psi^{(2)}\rangle_{\text{Bell}} = (|\,00\rangle + |\,11\rangle)/\sqrt{2}$.

The concurrence of the three-qubit state

$$|\,\psi^{(3)}\rangle = \sum_{i,j,k=0}^{1} a_{ijk}\,|\,ijk\rangle, \qquad \sum_{i,j,k=0}^{1} \left|\,a_{ijk}\,\right|^2 = 1, \;\; a_{ijk} \in \mathbb{C}, \tag{2.224}$$

is determined by the Cayley's 2 × 2 × 2 hyperdeterminant

$$C^{(3)} = 4\,\left|\,\text{hdet}(a_{ijk})\,\right|, \qquad 0 \leqslant C^{(3)} \leqslant 1, \tag{2.225}$$

$$\begin{aligned} \text{hdet}(a_{ijk}) &= a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 - 2a_{000}a_{001}a_{110}a_{111} \\ &- 2a_{000}a_{010}a_{101}a_{111} - 2a_{000}a_{011}a_{100}a_{111} - 2a_{001}a_{010}a_{101}a_{111} - 2a_{001}a_{011}a_{100}a_{110} \\ &- 2a_{010}a_{011}a_{100}a_{101} + 4a_{000}a_{011}a_{101}a_{110} + 4a_{001}a_{010}a_{100}a_{111}. \end{aligned} \tag{2.226}$$

Thus, if the three-qubit state (2.224) is not entangled, then $C^{(3)} = 0$ (for the tensor product of one-qubit states). One of the maximally entangled ($C^{(3)} = 1$) three-qubit states is the GHZ state $|\,\psi^{(3)}\rangle_{\text{GHZ}} = (|\,000\rangle + |\,111\rangle)/\sqrt{2}$.

A quantum L-qubit gate is a linear transformation of 2^L-dimensional Hilbert space $(\mathbb{C}^2)^{\otimes L} \to (\mathbb{C}^2)^{\otimes L}$ which in the computational basis (2.220) is described of the $2^L \times 2^L$ matrix $U^{(L)}$ such that the L-qubit state transforms as $|\,\psi^{'(L)}\rangle = U^{(L)}\,|\,\psi^{(L)}\rangle$. In this way, a *quantum circuit* is described as the successive application of elementary gates to an initial quantum state, which is the product of the corresponding matrices (for details, see, e.g., Nielsen and Chuang 2000). It is a standard assumption that each elementary L-qubit transformation is *unitary*, which implies the following strong restriction on the corresponding matrix $U \equiv U^{(L)}$ as

$$U^{\star}U = UU^{\star} = I \equiv I_{2^L \times 2^L}, \tag{2.227}$$

where I is the $2^L \times 2^L$ identity matrix for L-qubit state and the operation ($\star$) is the conjugate-transposition. The first equality in (2.227) means that the matrix $U^{(L)}$ is *normal* (cf (2.20)–(2.21)). The equations (2.227) follow from the definition of the *adjoint operator*

$$\langle U\psi^{(L)} | I\varphi^{(L)} \rangle = \langle I\psi^{(L)} | U^{\star}\varphi^{(L)} \rangle \tag{2.228}$$

applied to this simplest example of L-qubits in the 2^L-dimensional Hilbert space $(\mathbb{C}^2)^{\otimes L}$ (for the general case the derivation almost literally coincides), which we write in the following special form (in Dirac notation with bra- and ket- vectors) with explicitly added identities. Then, (2.227) follows from (2.228) as

$$\langle U^{\star}U\psi^{(L)}|I\varphi^{(L)}\rangle = \langle I\psi^{(L)}|UU^{\star}\varphi^{(L)}\rangle = \langle I\psi^{(L)}|I\varphi^{(L)}\rangle, \tag{2.229}$$

and any unitary matrix preserves the inner product

$$\langle U\psi^{(L)}|U\varphi^{(L)}\rangle = \langle I\psi^{(L)}|I\varphi^{(L)}\rangle, \tag{2.230}$$

which means that unitary operators satisfying (2.227) are bounded operators (bounded matrices in our case) and invertible with the inverse $U^{-1} = U^{\star}$.

Let us consider a possibility of noninvertible intermediate transformations of L-qubit states, i.e., *noninvertible gates*, which are described by the $2^L \times 2^L$ matrices $U(r)$ of (possibly) less than full rank $1 \leqslant r \leqslant 2^L$. This can be related to the production of degenerate states (see, e.g. Jaffali and Oeding 2020), particle loss (Neven *et al* 2018, Fraïsse and Braun 2016, Zangi and Qiao 2021), and the role of ranks in multiparticle entanglement (Chong *et al* 2005, Bruzda *et al* 2019).

In the limited cases $U(r = 2^L) \equiv U = U^{(L)}$, and $U(1)$ corresponds to the measurement matrix being the projection to one final vector $|\, i_{\mathrm{final}}\rangle$. In this case, for noninvertible transformations with $r < 2^L$ instead of unitarity (2.227), we consider partial unitarity (2.20)–(2.21) as

$$U(r)^{\star}U(r) = I_1(r), \tag{2.231}$$

$$U(r)U(r)^{\star} = I_2(r), \tag{2.232}$$

where $I_1(r)$ and $I_2(r)$ are (or may be) different partial shuffle identities having r units on the diagonal. There is an exotic limiting case, which is impossible for the identity I: we call two partial identities *orthogonal*, if

$$I_1(r)I_2(r) = Z, \tag{2.233}$$

where $Z = Z_{2^L\times 2^L}$ is the zero $2^L \times 2^L$ matrix.

We propose corresponding noninvertible analogs of (2.228)–(2.230) as follows. The *partial adjoint operator* $U(r)^{\star}$ in the 2^L-dimensional Hilbert space $(\mathbb{C}^2)^{\otimes L}$ is defined by

$$\langle U(r)\psi^{(L)}|I_2(r)\varphi^{(L)}\rangle = \langle I_1(r)\psi^{(L)}|U(r)^{\star}\varphi^{(L)}\rangle, \tag{2.234}$$

such that (see (2.231)–(2.232))

$$\langle U(r)^{\star}U(r)\psi^{(L)}|I_2(r)\varphi^{(L)}\rangle = \langle I_1(r)\psi^{(L)}|U(r)U(r)^{\star}\varphi^{(L)}\rangle = \langle I_1(r)\psi^{(L)}|I_2(r)\varphi^{(L)}\rangle. \tag{2.235}$$

We call the rhs of (2.235) the *partial inner product*. So instead of (2.230), we define $U(r)$ as the *partially bounded operator*

$$\langle U(r)\psi^{(L)}|U(r)\varphi^{(L)}\rangle = \langle I_1(r)\psi^{(L)}|I_2(r)\varphi^{(L)}\rangle. \tag{2.236}$$

Thus, if the partial identities are orthogonal (2.233), then the partial inner product vanishes identically, and the operator $U(r)$ becomes a zero norm operator in the sense of (2.236), although (2.231)–(2.232) are not zero.

In case the rank r is fixed, there can be $(2^L!/r!(2^L - r)!)^2$ partial unitary matrices $U(r)$ satisfying (2.231)–(2.232).

We define a *general unitary semigroup* as a semigroup of matrices $U(r)$ of rank r satisfying partial regularity (2.231)–(2.232) (in the symmetric case $I_1(r) = I_2(r) \equiv I(r)$).

As an example, we consider two 2-qubit states (2.222) $|\,\psi^{(2)}\rangle$ and $|\,\varphi^{(2)}\rangle$ (with a'_{ij} and $|\,i'j'\rangle$) and the noninvertible transformation described by three-parameter 4×4 matrices of rank 3 (but which are not nilpotent)

$$U(3) = U^{(L=2)}(r = 3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e^{i\beta} & 0 & 0 \\ 0 & 0 & 0 & e^{i\gamma} \\ e^{i\alpha} & 0 & 0 & 0 \end{pmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}. \tag{2.237}$$

The partial unitarity (2.231)–(2.232) and partial identities now become

$$U(3)^{\star} U(3) = I_1(3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.238}$$

$$U(3) U(3)^{\star} = I_2(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.239}$$

The partial identities (2.238)–(2.239) are not orthogonal (2.233), because

$$I_1(3) I_2(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \neq Z, \tag{2.240}$$

which directly gives the signature of the partial inner product (2.235), in our case of the Hilbert space $(\mathbb{C}^2)^{\otimes 2}$.

The definition of a partial adjoint operator (2.234) is satisfied with both sides being equal to $a_{00} a'_{11} e^{i\alpha} \langle 00|1'1'\rangle + a_{01} a'_{01} e^{i\beta} \langle 01|0'1'\rangle + a_{11} a'_{10} e^{i\gamma} \langle 11|1'0'\rangle$. The partial boundedness condition (2.236) holds with the partial inner product (2.235) becoming $a_{01} a'_{01} \langle 01|0'1'\rangle + a_{11} a'_{11} \langle 11|1'1'\rangle$, thus $U(3)$ (2.237), which is a bounded partial unitary operator.

An example of a zero norm (in our sense (2.236)) operator is the two-parameter partial unitary rank 2 matrix

$$U_{\text{nil}}(2) = U^{(L=2)}(r = 2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\beta} \\ e^{i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}. \tag{2.241}$$

The partial unitarity relations for $U_{\text{nil}}(2)$ have the form

$$U_{\text{nil}}(2)^{\star} U_{\text{nil}}(2) = I_{\text{nil},1}(2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.242}$$

$$U_{\text{nil}}(2) U_{\text{nil}}(2)^{\star} = I_{\text{nil},2}(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.243}$$

It may be seen that the partial identities $I_{\text{nil},1}(2)$ and $I_{\text{nil},2}(2)$ are now orthogonal (2.233), and the partial inner product (2.235) vanishes identically, and also the boundedness condition (2.236) holds with the rhs vanishing, despite $U_{\text{nil}}(2)$ being a nonzero nilpotent matrix (2.241).

2.5 Binary braiding quantum gates

Let us consider those Yang–Baxter maps that could be linear transformations of two-qubit spaces. We will pay attention to the most general 8-vertex solutions to the Yang–Baxter equations (2.87)–(2.94) and (2.97)–(2.98), which are unitary (and invertible) or partial unitary (2.20)–(2.21) (and noninvertible).

Consider the unitary version of the invertible star 8-vertex solutions (2.87)–(2.89) to the matrix Yang–Baxter equation (2.12). We use the exponential form of the parameters

$$\begin{gathered} x = r_x e^{i\alpha}, \quad y = r_y e^{i\beta}, \quad z = r_z e^{i\gamma}, \\ r_{x,y,z}, \alpha, \beta, \gamma \in \mathbb{R}, \quad r_{x,y,z} \geqslant 0, \quad |\,\alpha\,|, |\,\beta\,|, |\,\gamma\,| \leqslant 2\pi. \end{gathered} \tag{2.244}$$

For (2.87), exploiting unitarity (2.227) we obtain

$$U_{\text{rank}=4}^{8-\text{vert,star}}(\alpha, \beta) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(\alpha+\beta)} & 0 & 0 & e^{2i\beta} \\ 0 & e^{i(\alpha+\beta)} & \pm e^{i(\alpha+\beta)} & 0 \\ 0 & \mp e^{i(\alpha+\beta)} & e^{i(\alpha+\beta)} & 0 \\ -e^{2i\alpha} & 0 & 0 & e^{i(\alpha+\beta)} \end{pmatrix}, \tag{2.245}$$

$$\begin{aligned} \operatorname{tr} U &= 2\sqrt{2}\, e^{i(\alpha+\beta)}, \\ \det U &= e^{4i(\alpha+\beta)}, \end{aligned}, \tag{2.246}$$

$$\text{eigenvalues}: \ \{-(-1)^{3/4} e^{i(\alpha+\beta)}\}^{[2]}, \{(-1)^{1/4} e^{i(\alpha+\beta)}\}^{[2]}. \tag{2.247}$$

With the particular choice of parameters $\alpha = \beta = 0$ and lower signs, the solution (2.245) coincides with the 8-vertex braiding gate of Kauffman and Lomonaco (2004).

Next we search for unitary solutions among the invertible circle of 8-vertex traceless solutions (2.97) to the matrix Yang–Baxter equation (2.12) with parameters in the exponential form (2.244). The unitarity conditions (2.227) give the following equations on the parameters (2.244)

$$r = r_y = r_z, \quad r^2(r_x^2 + r^2) = 1, \quad r^8 + r^6 - 2r^4 + 1 = r^2 \tag{2.248}$$

$$\alpha - \beta = \frac{\pi}{2}. \tag{2.249}$$

The system of equations (2.248) has two real positive (or zero) solutions

$$(1) \quad r_x = 1, \quad r = \sqrt{\frac{\sqrt{5} - 1}{2}}, \tag{2.250}$$

$$(2) \quad r_x = 0, \quad r = 1. \tag{2.251}$$

Thus, only the first solution leads to an 8-vertex two-parameter unitary braiding quantum gate of the form (we put $\gamma \mapsto \beta$ in (2.244))

$$U_{\mathrm{rank}=4}^{8-\mathrm{vert,circ}}(\alpha, \beta)$$
$$= \sqrt{\frac{\sqrt{5}-1}{2}} \begin{pmatrix} 0 & e^{i(\alpha+\beta)} & ie^{i(\alpha+\beta)}\sqrt{\frac{\sqrt{5}-1}{2}} & 0 \\ -e^{2i\alpha}\sqrt{\frac{\sqrt{5}-1}{2}} & 0 & 0 & e^{i(\alpha+\beta)} \\ ie^{2i\alpha} & 0 & 0 & ie^{i(\alpha+\beta)}\sqrt{\frac{\sqrt{5}-1}{2}} \\ 0 & -e^{2i\alpha}\sqrt{\frac{\sqrt{5}-1}{2}} & ie^{2i\alpha} & 0 \end{pmatrix}, \tag{2.252}$$

$$\det U = e^{2i(3\alpha+\beta)}. \tag{2.253}$$

The second solution (2.251) gives 4-vertex two-parameter unitary braiding quantum gate (we also put $\gamma \mapsto \beta$ in (2.244))

$$U_{\mathrm{rank}=4}^{4-\mathrm{vert,circ}}(\alpha, \beta) = \begin{pmatrix} 0 & 0 & e^{i(\alpha+\beta)} & 0 \\ e^{2i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i(\alpha+\beta)} \\ 0 & e^{2i\alpha} & 0 & 0 \end{pmatrix}, \quad \det U = -e^{2i(3\alpha+\beta)}. \tag{2.254}$$

The noninvertible 8-vertex circle solution (2.98) to the Yang–Baxter equation (2.12) cannot be partial unitary (2.231)–(2.232) with any values of its parameters.

2.6 Higher braiding quantum gates

In general, only special linear transformations of 2^L-dimensional Hilbert space can be treated as elementary quantum gates for an L-qubit state (Nielsen and Chuang 2000). First, in the invertible case, the transformations should be unitary (2.227), and in the hypothetical noninvertible case they can satisfy partial unitarity (2.231)–

(2.232). Second, the braiding gates have to be $2^L \times 2^L$ matrix solutions to the constant Yang–Baxter equation (Kauffman and Lomonaco 2004) or higher braid equations (2.114)–(2.116). Here we consider (as a lowest case higher example) the ternary braiding gates acting on 3-qubit quantum states, i.e., 8×8 matrix solutions to the ternary braid equations (2.134), which satisfy unitarity (2.227) or partial unitarity (2.231)–(2.232).

Note that all the permutation solutions (2.144)–(2.145) are by definition unitary, and are therefore ternary braiding gates automatically, and we call them *permutation8-vertex ternary braiding quantum gates* $U_{\text{perm}}^{8-\text{vertex}}$. By the same reasoning the unitary version of the invertible star 8-vertex parameter-permutation solutions (2.146)–(2.153) to the ternary braid equations (2.134) will contain the complex numbers of unit magnitude as parameters.

Indeed, for the bisymmetric series (2.146)–(2.147) of star-like solutions we have four two-real parameter unitary ternary braiding quantum gates ($\varkappa = \pm 1$)

$$U_{\text{bisymm1}}^{8-\text{vertex}}(\alpha, \beta) = \begin{pmatrix} e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varkappa e^{2i\beta} & 0 \\ 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varkappa e^{2i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\varkappa e^{2i\beta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 \\ 0 & \pm\varkappa e^{2i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} \end{pmatrix}, \tag{2.255}$$

$$\alpha, \beta \in \mathbb{R}, \quad |\,\alpha\,|, |\,\beta\,| \leqslant 2\pi,$$

which is a ternary analog of the first parameter-permutation solution to the Yang–Baxter equation from (2.33). The ternary analog of the second star solution is the following unitary version of the bisymmetric series (2.148)–(2.149)

$$U_{\text{bisymm2}}^{8-\text{vertex}}(\alpha, \beta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{6i\alpha} \\ 0 & \varkappa e^{3i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{2i(2\alpha+\beta)} & 0 & 0 \\ 0 & 0 & 0 & \varkappa e^{3i(\alpha+\beta)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varkappa e^{3i(\alpha+\beta)} & 0 & 0 & 0 \\ 0 & 0 & \pm e^{2i(\alpha+2\beta)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varkappa e^{3i(\alpha+\beta)} & 0 \\ \pm e^{6i\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.256}$$

The same unitary ternary analogs of the symmetric series (2.150)–(2.153) for the first and the second circle-like solutions from (2.34) are

$$U^{8-\mathrm{vertex}}_{\mathrm{symm1}}(\alpha, \beta) = \begin{pmatrix} e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varkappa e^{i(\alpha+\beta)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{2i\beta} \\ 0 & 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 \\ 0 & \pm\varkappa e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} & 0 \\ 0 & 0 & \pm e^{2i\alpha} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.257}$$

and

$$U^{8-\mathrm{vertex}}_{\mathrm{symm2}}(\alpha, \beta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e^{2i\beta} & 0 & 0 \\ 0 & e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varkappa e^{i(\alpha+\beta)} & 0 \\ 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 & 0 \\ \pm e^{2i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\varkappa e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} \end{pmatrix}, \tag{2.258}$$

respectively.

The invertible 16-vertex star-like solutions (2.206) to the ternary braid equations (2.134) lead to the following two unitary one-parameter ternary braiding quantum gates (cf the binary case (2.245))

$$U^{16-\mathrm{vertex}}_{3-\mathrm{qubits}\pm}(\alpha) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{3i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & e^{3i\alpha} & 0 & 0 & 0 & 0 & \mp e^{2i\alpha} & 0 \\ 0 & 0 & e^{3i\alpha} & 0 & 0 & -e^{2i\alpha} & 0 & 0 \\ 0 & 0 & 0 & e^{3i\alpha} & \mp e^{4i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm e^{2i\alpha} & e^{3i\alpha} & 0 & 0 & 0 \\ 0 & 0 & e^{4i\alpha} & 0 & 0 & e^{3i\alpha} & 0 & 0 \\ 0 & \pm e^{4i\alpha} & 0 & 0 & 0 & 0 & e^{3i\alpha} & 0 \\ e^{6i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & e^{3i\alpha} \end{pmatrix}. \tag{2.259}$$

The braiding gate (2.259) is a ternary analog of (2.245), and therefore with $\alpha = 0$ it can be treated as a ternary analog of the 8-vertex braiding gate considered in Kauffman and Lomonaco (2004). Note that the solution $U^{16-\mathrm{vertex}}_{3-\mathrm{qubits}+}(0)$ is transpose to the so-called generalized Bell matrix (Rowell *et al* 2010). Comparing (2.184) and (2.259), we observe that the ternary braiding quantum gates (acting on 3 qubits) are those elements of the 16-vertex star semigroup $\mathcal{G}^{\mathrm{star}}_{16\mathrm{vert}}$ (2.192), which satisfy unitarity (2.227).

In the same way, the 32-vertex analog the 8-vertex binary braiding gate of Kauffman and Lomonaco (2004) (now acting on four qubits) is the following constant 4-ary braiding unitary quantum gate

$$U^{32-\mathrm{vertex}}_{4-\mathrm{qubits}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.260}$$

Thus, in general, the Minkowski star solutions for n-ary braid equations correspond to 2^n-vertex braiding unitary quantum gates as $2^L \times 2^L$ matrices acting on $L = n$ qubits

$$U^{2^L-\mathrm{vertex}}_{L-\mathrm{qubits}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & \ddots & 0 & 0 & \iddots & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & \iddots & 0 & 0 & \ddots & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.261}$$

The braiding gate (2.261) can be treated as a polyadic (n-ary) generalization of the GHZ generator (see, e.g., Rowell *et al* 2010, Ballard and Wu 2011b) acting on $L = n$ qubits.

2.7 Entangling braiding gates

Entangled quantum states are obtained from separable states by acting with special quantum gates on two-qubit states and multi-qubit states (Jaffali and Oeding 2020, 2016). Here we consider the concrete form of braiding gates, which can be entangling or not entangling. There are general considerations on these subjects for the Yang–Baxter maps (Kauffman and Lomonaco 2004, Balakrishnan and Sankaranarayanan 2010, Padmanabhan *et al* 2021) and generalized Yang–Baxter maps (Chen 2012a, Vasquez *et al* 2016, Rowell *et al* 2010, Padmanabhan *et al*

2020b). We present the solutions for the binary and ternary braid maps introduced above, which connect the parameters of the gate and the state.

2.7.1 Entangling binary braiding gates

Let us first examine how the 8-vertex star binary braiding gate $U_s(\alpha, \beta) \equiv U^{8-\text{vert,star}}_{\text{rank}=4}(\alpha, \beta)$ (2.245) acts on the product of one-qubit states concretely. We use the Bloch representation (2.221) to obtain the expression for the transformed concurrence (2.223)

$$\begin{aligned} &C^{(2)}_{s\pm}(U_s(\alpha, \beta) \mid \psi(\theta_1, \gamma_1)\rangle \otimes \mid \psi(\theta_2, \gamma_2)\rangle) \\ &= \left| \left(e^{i(\beta+2\gamma_1)} \sin^2 \frac{\theta_1}{2} \pm e^{i\alpha} \cos^2 \frac{\theta_1}{2}\right)\left(e^{i(\beta+2\gamma_2)} \sin^2 \frac{\theta_2}{2} \mp e^{i\alpha} \cos^2 \frac{\theta_2}{2}\right) \right|. \end{aligned} \tag{2.262}$$

In general, a braiding gate is *entangling* if the transformed concurrence (2.262) does not vanish, and its roots give the values of the gate parameters $U(\alpha, \beta)$ for which the gate is *not entangling* for a given two-qubit state. In search of the solutions for the transformed concurrence $C^{(2)}_{s\pm} = 0$, we observe that one of the qubits has to be on the Bloch sphere equator $\theta_1 = \frac{\pi}{2}$ (or $\theta_2 = \frac{\pi}{2}$). Only in this case can the first (or second) bracket in (2.262) vanish, and we obtain

$$(1) C^{(2)}_{s+} = 0, \text{ if } \theta_1 = \frac{\pi}{2} \text{and } \alpha - \beta = 2\gamma_1 - \pi, \text{ or } \theta_2 = \frac{\pi}{2} \text{and } \alpha - \beta = 2\gamma_2; \tag{2.263}$$

$$(2) C^{(2)}_{s-} = 0, \text{ if } \theta_1 = \frac{\pi}{2} \text{and } \alpha - \beta = 2\gamma_1, \text{ or } \theta_2 = \frac{\pi}{2} \text{and } \alpha - \beta = 2\gamma_2 - \pi. \tag{2.264}$$

Therefore, the 8-vertex star binary braiding gates (2.245) with the parameters fixed by (2.263)–(2.264) are not entangling.

For the 8-vertex circle binary braiding gate $U_c(\alpha, \beta) \equiv U^{8-\text{vert,circ}}_{\text{rank}=4}(\alpha, \beta)$ (2.252), we obtain

$$\begin{aligned} &C^{(2)}_{c}(U_c(\alpha, \beta) \mid \psi(\theta_1, \gamma_1)\rangle \otimes \mid \psi(\theta_1, \gamma_1)\rangle) \\ &= W \left| \left(e^{i(\beta+2\gamma_1)} \sin^2 \frac{\theta_1}{2} - ie^{i\alpha} \cos^2 \frac{\theta_1}{2}\right)\left(e^{i(\beta+2\gamma_2)} \sin^2 \frac{\theta_2}{2} - ie^{i\alpha} \cos^2 \frac{\theta_2}{2}\right) \right|, \end{aligned} \tag{2.265}$$

$$W = \frac{(\sqrt{5} - 1)^{\frac{3}{2}}}{\sqrt{2}} = 0.971\,74. \tag{2.266}$$

Analogously to (2.263)–(2.264), the concurrence of the states transformed by the 8-vertex circle binary braiding gate (2.252) can vanish if

$$C^{(2)}_{c} = 0, \text{ if } \theta_1 = \frac{\pi}{2} \text{ and } \alpha - \beta = 2\gamma_1 - \frac{\pi}{2}, \text{ or } \theta_2 = \frac{\pi}{2} \text{ and } \alpha - \beta = 2\gamma_2 - \frac{\pi}{2}. \tag{2.267}$$

Thus, the 8-vertex circle binary braiding gates (2.252) are not entangling if the parameters satisfy (2.267).

In the case of the 4-vertex circle binary braiding gate (2.254), the transformed concurrence vanishes identically, and therefore this gate is not entangling for any values of its parameters.

2.7.2 Entangling ternary braiding gates

Let us consider the tensor product of three qubit pure states $|\psi(\theta_1, \gamma_1)\rangle \otimes |\psi(\theta_2, \gamma_2)\rangle \otimes |\psi(\theta_3, \gamma_3)\rangle$ (in the Bloch representation (2.221)), which obviously has zero concurrence $C^{(3)}$ (2.225) because of the vanishing of the hyperdeterminant (2.226). After transforming by the 16-vertex star ternary braiding gates $U_{16}(\alpha) \equiv U_{3-\text{qubits}}^{16-\text{vertex}}(\alpha)$ (2.259), the concurrence becomes

$$\begin{aligned}&C_{16\pm}^{(3)}(U_{16}(\alpha)\,|\psi(\theta_1, \gamma_1)\rangle \otimes |\psi(\theta_2, \gamma_2)\rangle \otimes |\psi(\theta_3, \gamma_3)\rangle)\\ &= \frac{1}{64}\,|\,e^{2i\alpha} \pm e^{2i\gamma_1} + (e^{2i\alpha} \mp e^{2i\gamma_1})\cos\theta_1\,|^2\,|\,e^{2i\alpha} - e^{2i\gamma_2} + (e^{2i\alpha} + e^{2i\gamma_2})\cos\theta_2\,|^2\\ &|\,e^{2i\alpha} \mp e^{2i\gamma_3} + (e^{2i\alpha} \pm e^{2i\gamma_3})\cos\theta_3\,|^2.\end{aligned} \tag{2.268}$$

We observe that the ternary concurrence (2.268) vanishes if any of the brackets is equal to zero. Because the domain of all angles is $\mathbb{R}$, we only have solutions for fixed discrete $\theta_k = \pi, -\pi, \pi/2$, $k = 1, 2, 3$, which means that on the Bloch sphere the quantum states should be on the equator (as in the binary case), or additionally at the poles. In this case, $e^{i\alpha} = \pm e^{i\gamma_k}$, and

$$\alpha = \begin{cases} \gamma_k \\ \gamma_k + \pi \end{cases}, \quad k = 1, 2, 3. \tag{2.269}$$

Thus, for a fixed three-qubit product state one (or more) of which is at a pole or the equator of the Bloch sphere, those ternary braiding gates $U_{16}(\alpha)$ satisfying the conditions (2.269) are not entangling $C_{16\pm}^{(3)} = 0$, whereas in other cases they are entangling $C_{16\pm}^{(3)} \neq 0$.

By analogy, a similar action of the 8-vertex bisymmetric (star-like) ternary braiding gates $U_{8b1,\,2}(\alpha, \beta) \equiv U_{\text{bisymm1},\,2}^{8-\text{vertex}}(\alpha, \beta)$ (2.255)–(2.256) gives

$$\begin{aligned}&C_{8b1}^{(3)}(U_{8b1}(\alpha, \beta)\,|\psi(\theta_1, \gamma_1)\rangle \otimes |\psi(\theta_2, \gamma_2)\rangle \otimes |\psi(\theta_3, \gamma_3)\rangle)\\ &= \left|\sin^2\theta_1 \sin^2\theta_3\left(e^{2i(\beta+\gamma_2)}\sin^2\frac{\theta_2}{2} - e^{2i\alpha}\cos^2\frac{\theta_2}{2}\right)^2\right|,\end{aligned} \tag{2.270}$$

$$\begin{aligned}&C_{8b2}^{(3)}(U_{8b2}(\alpha, \beta)\,|\psi(\theta_1, \gamma_1)\rangle \otimes |\psi(\theta_2, \gamma_2)\rangle \otimes |\psi(\theta_3, \gamma_3)\rangle)\\ &= \left|\sin^2\theta_1 \sin^2\theta_3\left(e^{2i(\alpha+\gamma_2)}\sin^2\frac{\theta_2}{2} - e^{2i\beta}\cos^2\frac{\theta_2}{2}\right)^2\right|.\end{aligned} \tag{2.271}$$

Their solutions coincide with the binary case (2.263)–(2.264) applied to the middle qubit $|\psi(\theta_2, \gamma_2)\rangle$ and $\gamma_2 \to 2\gamma_2$.

The action of the 8-vertex symmetric (circle-like) ternary braiding gates $U_{8s}(\alpha, \beta) \equiv U_{\text{symm}1,\,2}^{8-\text{vertex}}(\alpha, \beta)$ (2.257)–(2.258) leads to the transformed concurrence

$$C_{8s}^{(3)}(U_{8s}(\alpha, \beta) \mid \psi(\theta_1, \gamma_1)\rangle \otimes \mid \psi(\theta_2, \gamma_2)\rangle \otimes \mid \psi(\theta_3, \gamma_3)\rangle) \\ = \left| \sin^2 \theta_2 \left(e^{i(\beta+2\gamma_1)} \sin^2 \frac{\theta_1}{2} - e^{i\alpha} \cos^2 \frac{\theta_1}{2} \right) \left(e^{i(\beta+2\gamma_3)} \sin^2 \frac{\theta_3}{2} - e^{i\alpha} \cos^2 \frac{\theta_3}{2} \right) \right|. \tag{2.272}$$

The conditions for this to vanish, i.e., when the gate $U_{8s}(\alpha, \beta)$ becomes not entangling, coincide with those for the binary case (2.263)–(2.264), applied here to the first or the third qubit.

Thus we have shown that the braiding binary and ternary quantum gates can be either entangling or not entangling, depending on how their parameters are related to the concrete quantum state on which they act.

The constructions presented here (Duplij and Vogl 2021) could be used, e.g., in the entanglement-free protocols (de Burgh and Bartlett 2005, Rehman and Shin 2021) and some experiments (Almeida *et al* 2014, Higgins *et al* 2007). This can also allow us to build quantum networks without any entangling at all *non-entangling networks*, when the next gate depends upon the previous state in such a way that at each step there is no entangling because the separable, but different, final state is received from a separable initial state.

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Innovative Quantum Computing

Steven Duplij and Raimund Vogl

Chapter 3

Supersymmetry and quantum computing

It is well-known that quantum computation is based on the algebraic structure of its constituents, qubits and qudits, living in some Hilbert space. Therefore, possible improvements could be connected with some special generalizations of the Hilbert space. A promising direction is supersymmetric generalization of the ordinary Hilbert space (De Witt 1992, Constantinescu 2002) and consideration of various super analogs of quantum states in it, with simultaneous passing from corresponding groups to supergroups.

3.1 Superspaces and supermatrices

Let us consider the main ideas in the supersymmetrization of qubits (Borsten *et al* 2010, 2014, 2015). The principal statement changes the Hilbert space to super Hilbert space (in sense of Rudolph 2000) and considers quantum states as (even) supervectors in the latter, i.e., taking values in the corresponding Grassmann algebra (or some more general supercommutative superalgebra). In this approach, the inner product and probabilities contain Grassmann algebra parts. In the same way, the bra/ket formalism of quantum mechanics (Dirac 1939, van Eijndhoven and de Graaf 1985, Gieres 2000) transforms to super-bra/super-ket formalism with additional parity rules. Here we will point out the foremost relations and statements concisely (only needed), while we refer to the details and further notations to the standard supermathematics sources (Berezin 1987, Leites 1980, De Witt 1992). To clarify the structure of variables, we present some formulas in two columns: ordinary (left) and supersymmetric (right) cases, and moreover we use different notations for them (the latter will be marked in bold).

Let $\Lambda_N(\mathbb{C})$ be a complex Grassmann algebra having N anticommuting generators θ_i. The nilpotence of θs (which follows from their anticommutativity) leads to its finiteness (with dimension 2^N) and to the decompositions of any element $z \in \Lambda_N(\mathbb{C})$ (informally)

doi:10.1088/978-0-7503-5281-9ch3

$$z = \overbrace{z_{\text{body}}}^{\text{no } \theta' s} + \overbrace{z_{\text{soul}}}^{\text{with } \theta' s} = \overbrace{z_{\text{invert}}}^{\text{no } \theta' s \textit{ and } \text{no } 0} + \overbrace{z_{\text{noninvert}}}^{\text{with } \theta' s \textit{ and } 0} = \overbrace{z_{\text{even}}}^{\text{even } \theta' s} + \overbrace{z_{\text{odd}}}^{\text{odd } \theta' s}, \tag{3.1}$$

where $z_{\text{body}} \in \mathbb{C}$, $z_{\text{invert}} \in \mathbb{C}\backslash 0$, $z_{\text{soul}} \in \Lambda_N(\mathbb{C})\backslash\mathbb{C}$, $z_{\text{noninvert}} \in \Lambda_N(\mathbb{C})\backslash\mathbb{C} \cup \{0\}$. The last decomposition allows us to introduce the degree by deg $z_{\text{even}} = \bar{0}$ and deg $z_{\text{odd}} = \bar{1}$, and elements with the fixed degree are homogeneous. Obviously, that for homogeneous elements deg $yz =$ deg $y+$ deg $z(\text{mod } 2)$. Another name of deg is parity (or grade), in special cases they are fine different (Bernstein *et al* 2013) but in the superqubit context all of them are interchangable. Thus, the mapping deg: $\Lambda_N(\mathbb{C}) \to \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$, gives the direct sum decomposition of the Grassmann algebra $\Lambda_N(\mathbb{C}) = \Lambda_N^{(\text{even})}(\mathbb{C}) \oplus \Lambda_N^{(\text{odd})}(\mathbb{C}) = \Lambda_N^{(\bar{0})}(\mathbb{C}) \oplus \Lambda_N^{(\bar{1})}(\mathbb{C})$, which means that $\Lambda_N(\mathbb{C})$ is the simplest example of $\mathbb{Z}_2$-graded algebra. The analog of the ordinary commutator for $\Lambda_N(\mathbb{C})$ is the supercommutator

$$[y, z] = yz - zy, \;\; y, z \in \mathbb{C}, \overset{\text{susy}}{\Longrightarrow} [y, z]_{\text{deg}} = yz - (-1)^{\deg y \deg z} zy, \; y, z \in \Lambda_N(\mathbb{C}). \tag{3.2}$$

If $[y, z]_{\text{deg}} = 0$ for all elements of a superalgebra, then it is supercommutative, which is indeed the case of the Grassmann algebra $\Lambda_N(\mathbb{C})$. The same rule will be implied for all of the other $\mathbb{Z}_2$-graded homogeneous variables.

The ordinary involution * and the grade involution $\sharp$ (superstar or superinvolution (Bernstein *et al* 2013, Borsten *et al* 2010)) can be defined on $\Lambda_N(\mathbb{C})$ as follows

$$(xy)^* = \bar{x}y, \;\; (yz)^* = z^* y^*, \;\; (y)^{**} = y, \tag{3.3}$$

$$(xy)^\sharp = \bar{x}y^\sharp, \;\; (yz)^\sharp = y^\sharp z^\sharp, \;\; (y)^{\sharp\sharp} = (-1)^{\deg y} y, \quad x \in \mathbb{C}, \;\; y, z \in \Lambda_N(\mathbb{C}), \tag{3.4}$$

such that $z^* \in \Lambda_N^{(\deg z)}(\mathbb{C})$, $z^\sharp \in \Lambda_N^{(\deg z)}(\mathbb{C})$.

The superqubits live in a finite-dimensional $\mathbb{Z}_2$-graded linear vector space (or superspace) $\mathscr{V}$ over $\mathbb{C}$ (or any other field $\mathbb{K}$), which has the same decomposition on the even and odd parts as the Grassmann algebra above $\mathscr{V} = \mathcal{V}^{(\bar{0})} \oplus \mathcal{V}^{(\bar{1})}$. If the dimensions of the component spaces $\dim \mathcal{V}^{(\bar{0})} = p$ and $\dim \mathcal{V}^{(\bar{1})} = q$, then we denote the $\mathbb{Z}_2$-graded vector space $\mathscr{V} = \mathbb{C}^{p|q}$, and its graded dimension $\dim \mathbb{C}^{p|q} = p + q$. The $\mathbb{Z}_2$-graded direct product of superspaces $\hat{\otimes}$ (which is used for superqubit constructions) is crucially different from the ordinary direct product of spaces $\otimes$ (exploited for qubits). Indeed, we have the ordinary direct product

$$\mathscr{V} \otimes \mathscr{W} = \mathcal{V}^{(\bar{0})} \otimes \mathcal{W}^{(\bar{0})} + \mathcal{V}^{(\bar{1})} \otimes \mathcal{W}^{(\bar{1})} + \mathcal{V}^{(\bar{1})} \otimes \mathcal{W}^{(\bar{0})} + \mathcal{V}^{(\bar{0})} \otimes \mathcal{W}^{(\bar{1})}, \tag{3.5}$$

which does not allow us to introduce the $\mathbb{Z}_2$-graded structure without additional assumptions.

Innovation 3.1. *Only the definition of a new operation, the $\mathbb{Z}_2$-graded direct product $\hat{\otimes}$, gives the consistent superspace structure of product (by endowing two last terms in (3.5) with odd grading of the product)*

$$(\mathscr{V} \hat{\otimes} \mathscr{W})^{(\mathsf{k})} = \bigoplus_{\mathsf{k}=\mathsf{r}\boxplus\mathsf{s}} \mathcal{V}^{(\mathsf{r})} \otimes \mathcal{W}^{(\mathsf{s})}, \qquad \mathsf{k}, \mathsf{r}, \mathsf{s} \in \mathbb{Z}_2, \quad \mathsf{r}\boxplus\mathsf{s} \equiv \mathsf{r} + \mathsf{s} (\mathrm{mod}\, 2), \tag{3.6}$$

or simply

$$(\mathscr{V} \hat{\otimes} \mathscr{W})^{(\bar{0})} = \mathcal{V}^{(\bar{0})} \otimes \mathcal{W}^{(\bar{0})} + \mathcal{V}^{(\bar{1})} \otimes \mathcal{W}^{(\bar{1})}, \tag{3.7}$$

$$(\mathscr{V} \hat{\otimes} \mathscr{W})^{(\bar{1})} = \mathcal{V}^{(\bar{1})} \otimes \mathcal{W}^{(\bar{0})} + \mathcal{V}^{(\bar{0})} \otimes \mathcal{W}^{(\bar{1})}. \tag{3.8}$$

Usually, the (different) operations $\hat{\otimes}$ *and* $\otimes$ *are denoted by the same symbol, but they should be used with care and taking account the actual distinction of (3.5) and (3.6).*

In the consideration of mappings between superspaces and trying to introduce $\mathbb{Z}_2$-graded structure for them, we also note some peculiarities (important for superqubit constructions). Indeed, the set of homomorphisms $\{\mathbf{T}\}$ from superspace $\mathscr{V}$ to superspace $\mathscr{W}$ is defined in the standard way

$$\mathrm{Hom}(\mathscr{V}, \mathscr{W}) = \{\mathbf{T} | \mathbf{T}\, \mathscr{V} \subset \mathscr{W}\}. \tag{3.9}$$

We could assume that the $\mathbb{Z}_2$-graded structure is analogous to (3.6)

$$\mathrm{Hom}^{(\mathsf{k})}(\mathscr{V}, \mathscr{W}) = \{\mathbf{T} \in \mathrm{Hom}(\mathscr{V}, \mathscr{W}) | \mathbf{T}\, \mathcal{V}^{(\mathsf{r})} \subset \mathcal{W}^{(\mathsf{r}\boxplus\mathsf{k})}\}, \qquad \mathsf{k}, \mathsf{r} \in \mathbb{Z}_2. \tag{3.10}$$

Innovation 3.2. *Only even mappings* $\mathbf{T}^{(\bar{0})} = \mathrm{Hom}^{(\bar{0})}(\mathscr{V}, \mathscr{W})$ *(*$\deg \mathbf{T} = \bar{0} \in \mathbb{Z}_2$*) are homomorphisms. Only odd mappings* $\mathbf{T}^{(\bar{1})}$ *(*$\deg \mathbf{T} = \bar{1} \in \mathbb{Z}_2$*) are not morphisms at all because they cannot be composed:* $\mathbf{T}^{(\bar{1})} \circ \mathbf{T}^{(\bar{1})}$ *is not odd, but is even mapping.*

The same observation can be made for the linear operators in a $\mathbb{Z}_2$-graded linear vector space $\mathbb{C}^{p|q}$ given by (super)matrices. In the standard basis, a linear operator $\mathbf{T} \in \mathrm{End}(\mathbb{C}^{p|q})$ can be represented by the square block $(p+q) \times (p+q)$ supermatrix over $\Lambda_N(\mathbb{C})$ (Berezin 1987, Leites 1980) (other representations are also possible Bernstein *et al* 2013)

$$\mathrm{M} = \begin{pmatrix} A_{p\times p} & B_{p\times q} \\ C_{q\times p} & D_{q\times q} \end{pmatrix} \in \mathrm{Mat}(p|q, \Lambda_N(\mathbb{C})), \tag{3.11}$$

where the even (ordinary) matrices $A_{p\times p}$, $D_{q\times q}$ are over $\Lambda_N^{(\mathrm{even})}(\mathbb{C}) = \Lambda_N^{(\bar{0})}(\mathbb{C})$, and the odd (ordinary) matrices $B_{p\times q}$, $C_{q\times p}$ are over $\Lambda_N^{(\mathrm{odd})}(\mathbb{C}) = \Lambda_N^{(\bar{1})}(\mathbb{C})$. This full supermatrix has the total parity (degree) $\deg \mathrm{M} = 0$ and

$$\mathrm{M}_{\deg M=0} = \mathrm{M}_{\mathrm{body}} + \mathrm{M}_{\mathrm{soul}} = \begin{pmatrix} \left(A_{p\times p}\right)_{\mathrm{body}} & 0_{p\times q} \\ 0_{q\times p} & \left(D_{q\times q}\right)_{\mathrm{body}} \end{pmatrix} + \mathrm{M}_{\mathrm{soul}}. \tag{3.12}$$

If oppositely, $A_{p\times p}$, $D_{q\times q}$ are (ordinary) matrices over $\Lambda_N^{(\bar{1})}(\mathbb{C})$, and $B_{p\times q}$, $C_{q\times p}$ are (ordinary) matrices over $\Lambda_N^{(\bar{0})}(\mathbb{C})$, then $\deg \mathrm{M}=1$, and

$$\mathrm{M}_{\deg \mathrm{M}=1} = \mathrm{M}_{\mathrm{body}} + \mathrm{M}_{\mathrm{soul}} = \begin{pmatrix} 0_{p\times p} & \left(B_{p\times q}\right)_{\mathrm{body}} \\ \left(C_{q\times p}\right)_{\mathrm{body}} & 0_{q\times q} \end{pmatrix} + \mathrm{M}_{\mathrm{soul}}. \tag{3.13}$$

Innovation 3.3. The supermatrices with $\deg \mathrm{M}=1$ are not morphisms of $\mathbb{C}^{p|q}$ because their product gives the first ones having $\deg \mathrm{M}=0$, and therefore

$$\{\mathrm{M}\}_{\deg \mathrm{M}=1} \notin \mathrm{End}(\mathbb{C}^{p|q}). \tag{3.14}$$

After the decomposition of the matrices (3.11) with $\deg \mathrm{M}=0$ (reminding (3.12), (3.13))

$$\mathrm{M} = \mathrm{M}^{(\mathrm{even})} + \mathrm{M}^{(\mathrm{odd})} = \mathrm{M}^{(0)} + \mathrm{M}^{(1)}, \tag{3.15}$$

$$\mathrm{M}^{(0)} = \begin{pmatrix} A_{p\times p} & 0_{p\times q} \\ 0_{q\times p} & D_{q\times q} \end{pmatrix}, \tag{3.16}$$

$$\mathrm{M}^{(1)} = \begin{pmatrix} 0_{p\times p} & B_{p\times q} \\ C_{q\times p} & 0_{q\times q} \end{pmatrix}, \tag{3.17}$$

we observe that $\mathrm{M}^{(\bar{0})}\mathrm{M}'^{(\bar{0})} = \mathrm{M}''^{(\bar{0})}$, and therefore the corresponding operators are even endomorphisms of $\mathbb{C}^{p|q}$

$$\{\mathbf{T}^{(\bar{0})}\} \in \mathrm{End}(\mathbb{C}^{p|q}), \tag{3.18}$$

but

$$\mathrm{M}^{(\bar{1})}\mathrm{M}'^{(\bar{1})} = \mathrm{M}''^{(\bar{0})}. \tag{3.19}$$

Innovation 3.4. *The set* $\{\mathrm{M}^{(\bar{1})}\}$ *is not closed under composition (matrix multiplication), therefore the corresponding odd operators* $\mathbf{T}^{(1)}$ *are not morphisms by definition at all*

$$\{\mathbf{T}^{(\bar{1})}\} \notin \mathrm{End}(\mathbb{C}^{p|q}). \tag{3.20}$$

Both "even" and "odd" superoperators (considered together) are morphisms.

These considerations should be taken into account during consistent calculations with superqubits and supersymmetric quantum gates.

We remind some common notions in present notations for self-consistency. First, in contrast to the standard transpose operator $\mathsf{T}: \mathrm{Mat}(p, \mathbb{C}) \to \mathrm{Mat}(p, \mathbb{C})$, the supertranspose operator sT: $\mathrm{Mat}(p|q,\, \Lambda_N(\mathbb{C})) \to \mathrm{Mat}(p|q,\, \Lambda_N(\mathbb{C}))$ is double-valued depending of the parity of supermatrix

$$\begin{pmatrix} A_{p\times p} & B_{p\times q} \\ C_{q\times p} & D_{q\times q} \end{pmatrix}^{\mathsf{sT}} = \begin{cases} \begin{pmatrix} A^{\mathsf{T}}_{p\times p} & C^{\mathsf{T}}_{q\times p} \\ -B^{\mathsf{T}}_{p\times q} & D^{\mathsf{T}}_{q\times q} \end{pmatrix}, & \text{if } \deg \mathrm{M} = \bar{0}, \\ \begin{pmatrix} A^{\mathsf{T}}_{p\times p} & -C^{\mathsf{T}}_{q\times p} \\ B^{\mathsf{T}}_{p\times q} & D^{\mathsf{T}}_{q\times q} \end{pmatrix}, & \text{if } \deg \mathrm{M} = \bar{1}. \end{cases} \tag{3.21}$$

It is seen that $(\mathsf{sT})^{\circ 2} \neq \mathrm{id}$ (while $\mathsf{T}^{\circ 2} = \mathrm{id}$) but $(\mathsf{sT})^{\circ 4} = \mathrm{id}$, and therefore the supertranspose operator is the reflection of order 4, while the transpose is the ordinary reflection (of order 2). For two supermatrices of the same shape $\mathrm{M}, \mathrm{N} \in \mathrm{Mat}(p|q,\, \Lambda_N(\mathbb{C}))$, we have

$$(\mathrm{MN})^{\mathsf{sT}} = (-1)^{\deg \mathrm{M} \deg N} \mathrm{N}^{\mathsf{sT}} \mathrm{M}^{\mathsf{sT}}, \tag{3.22}$$

and in particular

$$(a\mathrm{M})^{\mathsf{sT}} = a\mathrm{M}^{\mathsf{sT}}, \quad \forall\, a \in \Lambda_N(\mathbb{C}), \tag{3.23}$$

which means that supertranspose sT is a $\Lambda_N(\mathbb{C})$-module, e.g., in case of the ordinary transpose operator T, which is a $\mathbb{C}$-module.

The supertrace is the homomorphism str: $\mathrm{Mat}(p|q,\, \Lambda_N(\mathbb{C})) \to \Lambda_N(\mathbb{C})$ that is also double-valued (depending of parity of supermatrix) mapping (for the supermatrix of the standard format (3.11))

$$\mathrm{str}\begin{pmatrix} A_{p\times p} & B_{p\times q} \\ C_{q\times p} & D_{q\times q} \end{pmatrix} = \begin{cases} \mathrm{tr}\, A_{p\times p} - \mathrm{tr}\, D_{q\times q}, & \text{if } \deg \mathrm{M} = \bar{0}, \\ \mathrm{tr}\, A_{p\times p} + \mathrm{tr}\, D_{q\times q}, & \text{if } \deg \mathrm{M} = \bar{1}, \end{cases} \tag{3.24}$$

which is additive and has the supercommutativity property, where M, N are ordinary matrices, and str is invariant with respect to supertranspose (analogous to ordinary trace)

$$\mathrm{tr}\, \mathrm{M}^{\mathsf{T}} = \mathrm{tr}\, \mathrm{M} \overset{\text{susy}}{\Longrightarrow} \mathrm{str}\, \mathrm{M}^{\mathsf{sT}} = \mathrm{str}\, \mathrm{M}. \tag{3.25}$$

The standard superdeterminant (Pakhomov 1974, Berezin and Leites 1975) (or Berezinian Berezin 1987, Leites (1980)) in our notation is

$$\mathrm{Ber}\, \mathrm{M} = \mathrm{Ber}\begin{pmatrix} A_{p\times p} & B_{p\times q} \\ C_{q\times p} & D_{q\times q} \end{pmatrix} = \det\left(A_{p\times p} - B_{p\times q} D^{-1}_{q\times q} C_{q\times p}\right)\left(\det D_{q\times q}\right)^{-1}, \tag{3.26}$$

which differs from the ordinary determinant by the power (-1) in the last multiplier. The mapping Ber is a homomorphism of $\mathrm{Mat}(p|q,\, \Lambda_N(\mathbb{C}))$ and invariant with respect to supertranspose sT (3.21)

$$\det(\mathrm{M}^{\mathsf{T}}) = \det \mathrm{M} \overset{\text{susy}}{\Longrightarrow} \mathrm{Ber}(\mathrm{M}^{\mathsf{sT}}) = \mathrm{Ber}\, \mathrm{M}. \tag{3.27}$$

The connection of Ber and str is similar to the ordinary case

$$\det \mathrm{M} = e^{\mathrm{tr}(\ln \mathrm{M})} \overset{\mathrm{susy}}{\Longrightarrow} \mathrm{Ber}\, \mathrm{M} = e^{\mathrm{str}(\ln \mathrm{M})}, \tag{3.28}$$

$$\det e^{\mathrm{M}} = e^{\mathrm{tr}\, \mathrm{M}} \overset{\mathrm{susy}}{\Longrightarrow} \mathrm{Ber}\, e^{\mathrm{M}} = e^{\mathrm{str}\, \mathrm{M}}, \tag{3.29}$$

where $\mathrm{M} \in \mathrm{Mat}(p, \mathbb{C})$ and $\mathrm{M} \in \mathrm{Mat}(p|q, \Lambda_N(\mathbb{C}))$.

The Berezinian (3.26) has the inconvenient property for characterizing the entanglement: Ber is not defined for noninvertible $D_{q\times q}$. Therefore, in Borsten *et al* (2010) it was proposed to use another possible function for the entanglement measure, which has many properties of Berezinian (but not all of them), and which satisfies (3.27) and has the ordinary det as the nonsupersymmetric limit (when odd variables vanish). Because the notion sdet is widely used for Ber (Berezin 1987, Leites 1980), we denote this function sdTr, which can be defined by the following informal analogy

$$\det \mathrm{M} = \frac{1}{2}\mathrm{tr}((\mathrm{M}E_{\mathrm{sl}})^{\mathsf{T}}(\mathrm{M}E_{\mathrm{sl}})) \overset{\mathrm{susy}}{\Longrightarrow} \mathrm{sdTr}\, \mathrm{M} = \frac{1}{2}\mathrm{str}\Big(\big(\mathrm{M}E_{\mathrm{osp}}\big)^{\mathrm{s}\mathsf{T}}\big(\mathrm{M}E_{\mathrm{osp}}\big)\Big), \tag{3.30}$$

where E_{sl} and $\mathrm{E}_{\mathrm{osp}}$ are $SL(2)$ and $OSp(1|2)$ invariant tensors (Bernstein *et al* 2013) (determining the corresponding group and supergroup in the standard way $M^{\mathsf{T}}E_{\mathrm{sl}}M = M$ and $\mathrm{M}^{\mathrm{s}\mathsf{T}}\mathrm{E}_{\mathrm{osp}}\mathrm{M} = \mathrm{M}$). The main property of sdTr is vanishing on the direct product states, and therefore it can measure whether a quantum (two superqubit) state is unentangled or entangled (see below).

3.2 Super Hilbert spaces and operators

Let us denote vectors (quantum states) in the r-dimensional complex Hilbert space $\mathcal{H}_r$ by kets $|\,\psi\rangle$ and the inner product by $\langle|\rangle$: $\mathcal{H}_r \times \mathcal{H}_r \to \mathbb{C}$, which is a nondegenerate Hermitean and positive form that is linear in the first argument and antilinear (conjugate linear) in the second argument. The bra $\langle\varphi\,|$ is defined as an element of the dual space $\mathcal{H}_r^{\dagger}$ (in the notation of Borsten *et al* (2010), for a inner product vector space $\mathcal{V}$ the notation $\mathcal{V}^*$ is also used for its dual), which is the functional $\langle\varphi|\rangle$: $\mathcal{H}_r \to \mathbb{C}$, such that the action on a ket is denoted by $\langle\varphi\,|\,(|\,\psi\rangle) \coloneqq \langle\varphi|\psi\rangle$ and coincides with the inner product after the identification of the Hilbert space with its dual (Riesz representation theorem Rudin 1991). Informally, one can write the injection $(|\,\psi\rangle)^{\dagger} = \langle\psi\,|$, which in the matrix representation (and finite-dimensional) standardly coincides with Hermitean adjoint, when $|\,\psi\rangle$ becomes a matrix-column, $\langle\psi\,|$ is a matrix-row, and the inner product is a scalar product, the obvious property $\langle\varphi|\psi\rangle^{\dagger} = \langle\psi|\varphi\rangle$ also holds valid.

In a similar way, we consider the $(r|s)$-dimensional super Hilbert space $\mathscr{H}_{(r|s)}$ over $\Lambda_N(\mathbb{C})$ as the $\mathbb{Z}_2$-graded space $\mathscr{H}_{(r|s)} = \mathscr{H}^{(\bar{0})}_{(r|s)} \oplus \mathscr{H}^{(\bar{1})}_{(r|s)}$, such that the supersymmetric quantum states, if homogenous, carry $\mathbb{Z}_2$-grading $\pi_\psi = \bar{0}, \bar{1} \in \mathbb{Z}_2$, and they are denoted by super-kets $||\boldsymbol{\psi}^{(\pi_\psi)}\rangle \in \mathcal{H}^{(\pi_\psi)}_{(r|s)}$ with $\pi_\psi = \deg \boldsymbol{\psi}$, while the body of even super states are the ordinary kets

$$||\boldsymbol{\psi}^{(\bar{0})}\rangle_{\text{body}} = |\,\psi\rangle \in \mathcal{H}_r. \tag{3.31}$$

We denote the super inner product by $\langle||\rangle$: $\mathscr{H}_{(r|s)} \times \mathscr{H}_{(r|s)} \to \Lambda_N(\mathbb{C})$ obeying the property

$$\langle.||.\rangle_{\text{body}} = \langle.|.\rangle \in \mathbb{C}. \tag{3.32}$$

The super dual Hilbert space $\mathscr{H}^{\ddagger}_{(r|s)}$ is defined as the space of the functionals $\langle\boldsymbol{\phi}^{(\pi_\varphi)}||\rangle$: $\mathscr{H}^{(\pi_\psi)}_{(r|s)} \to \Lambda_N(\mathbb{C})$, and the super bra $\langle\boldsymbol{\phi}^{(\pi_\varphi)}||$ with $\pi_\varphi = \deg\boldsymbol{\phi} \in \mathbb{Z}_2$ is given by the action

$$\langle\varphi \,|\, (|\,\psi\rangle) = \langle\varphi|\psi\rangle \in \mathbb{C} \overset{\text{susy}}{\Longrightarrow} \langle\boldsymbol{\phi}^{(\pi_\varphi)}||(||\boldsymbol{\psi}^{(\pi_\psi)}\rangle) = \delta_{\pi_\varphi\pi_\psi}\langle\boldsymbol{\phi}^{(\pi_\varphi)}||\boldsymbol{\psi}^{(\pi_\psi)}\rangle \in \Lambda_N(\mathbb{C}). \tag{3.33}$$

The presence of the delta function in (3.33) means that the commonly used agreement that the graded super vectors of opposite parity are mutually orthogonal

$$\langle\boldsymbol{\phi}^{(\pi_\varphi)}||\boldsymbol{\psi}^{(\pi_\psi)}\rangle = 0, \text{ if } \pi_\varphi \neq \pi_\psi. \tag{3.34}$$

Therefore, in similar expressions we will put

$$\pi_\varphi = \pi_\psi = \pi = \bar{0},\ \bar{1} \in \mathbb{Z}_2. \tag{3.35}$$

In this case, we have

$$\langle\boldsymbol{\phi}^{(\pi)}||\boldsymbol{\psi}^{(\pi)}\rangle^{\sharp} = \langle\boldsymbol{\psi}^{(\pi)}||\boldsymbol{\phi}^{(\pi)}\rangle. \tag{3.36}$$

Thus, informally, one can write

$$(||\boldsymbol{\psi}^{(\pi)}\rangle)^{\ddagger} = \langle\boldsymbol{\psi}^{(\pi)}||, \tag{3.37}$$

which means that ($\ddagger$) does not change parity π, and it is the reflection of order 4, because

$$(||\boldsymbol{\psi}^{(\pi)}\rangle)^{\ddagger\ddagger} = (-1)^{\pi}||\boldsymbol{\psi}^{(\pi)}\rangle, \tag{3.38}$$

$$(||\boldsymbol{\psi}^{(\pi)}\rangle)^{\ddagger\ddagger\ddagger\ddagger} = ||\boldsymbol{\psi}^{(\pi)}\rangle. \tag{3.39}$$

If $z \in \Lambda_N(\mathbb{C})$ has a fixed parity, then its product with super ket and super bra behaves with respect to ($\ddagger$) differently

$$(||\boldsymbol{\psi}^{(\pi)}\rangle z)^{\ddagger} = (-1)^{\pi \deg z} z^{\sharp}\langle\boldsymbol{\psi}^{(\pi)}||, \tag{3.40}$$

$$(z\langle\boldsymbol{\psi}^{(\pi)}||)^{\ddagger} = (-1)^{\pi(\deg z+1)}||\boldsymbol{\psi}^{(\pi)}\rangle z^{\sharp}, \tag{3.41}$$

where ($\sharp$) is the graded involution or superstar (3.4).

Innovation 3.5. *We can omit the δ-function in (3.33), and this will lead to a new kind of Hilbert spaces that allow mixing of gradings, such that all above formulas should be changed.*

In the super Hilbert space $\mathscr{H}_{(r|s)}$ the superadjoint (‡) of the superoperator (3.9) with the standard graded structure (3.10) is defined by

$$\langle T\ \varphi|\psi\rangle = \langle\varphi|T^{\dagger}\ \psi\rangle \overset{\text{susy}}{\Longrightarrow} \langle\mathbf{T}^{(\pi_T)}\ \boldsymbol{\phi}^{(\pi_\varphi)}||\boldsymbol{\psi}^{(\pi_\psi)}\rangle = (-1)^{\pi_\varphi\pi_T}\langle\boldsymbol{\phi}^{(\pi_\varphi)}||\mathbf{T}^{(\pi_T)\ddagger}\ \boldsymbol{\psi}^{(\pi_\psi)}\rangle, \tag{3.42}$$

where T and $T^{\dagger}$ are an operator and its adjoint in the Hilbert space $\mathcal{H}_r$, $|\ \psi\rangle, |\ \varphi\rangle \in \mathcal{H}_r$, and $\boldsymbol{\phi}^{(\pi_\varphi)} \in \mathcal{H}_{(r|s)}^{(\pi)}$, $\boldsymbol{\psi}^{(\pi_\psi)} \in \mathcal{H}_{(r|s)}^{(\pi_\psi)}$, $\pi_T = \deg \mathbf{T} \in \mathbb{Z}_2$. Here we do not have the restriction (3.35) because the superoperator $\mathbf{T}^{(\pi_T)}$ with $n_T = \bar{1}$ can change the parity of quantum states. The superadjoint of the action on the quantum state is

$$(T\ |\ \psi\rangle)^{\dagger} = \langle\psi\ |\ \ T^{\dagger} \overset{\text{susy}}{\Longrightarrow} (\mathbf{T}^{(\pi_T)}||\boldsymbol{\psi}^{(\pi_\psi)}\rangle)^{\ddagger} = (-1)^{\pi_\psi\pi_T}\langle\boldsymbol{\psi}^{(\pi_\psi)}||\mathbf{T}^{(\pi_T)\ddagger}, \tag{3.43}$$

The definition (3.42) is equivalent to (see (4.12))

$$\langle\boldsymbol{\phi}^{(\pi_\varphi)}||\mathbf{T}^{(\pi_T)\ddagger}||\boldsymbol{\psi}^{(\pi_\psi)}\rangle = (-1)^{\pi_\varphi\pi_\psi+\pi_\psi+(\pi_\varphi+\pi_\psi)\pi_T}\langle\boldsymbol{\psi}^{(\pi_\psi)}||\mathbf{T}^{(\pi_T)}||\boldsymbol{\phi}^{(\pi_\varphi)}\rangle^{\sharp}. \tag{3.44}$$

If the superoperator $\mathbf{T}$ has a supermatrix representation in $\mathrm{Mat}(p|q,\ \Lambda_N(\mathbb{C}))$, then its superadjoint is represented by composition of supertranspose (3.21) and the graded involution (superstar) (3.4) as

$$\mathrm{M}^{\dagger} = \bar{\mathrm{M}}^{\mathsf{T}},\ \ \mathrm{M} \in \mathrm{Mat}(p,\ \mathbb{C}) \overset{\text{susy}}{\Longrightarrow} \mathrm{M}^{\ddagger} = (\mathrm{M}^{\sharp})^{\mathsf{sT}},\ \mathrm{M} \in \mathrm{Mat}(p|q,\ \Lambda_N(\mathbb{C})), \tag{3.45}$$

which is the superanalog of the Hermitean conjugation (conjugate transpose).

3.3 Qubits and superqubits

Mathematically, qubits (or d-qudits) and superqubits (or $(r|s)$-superqudits) are normalized vectors in the r-dimensional Hilbert space and $(r|s)$-dimensional super Hilbert space, respectively, which are presented in the Dirac bra-ket notation (see previous section). They are written in the computational basis to thoroughly study various symmetries and introduce suitable variables that can consistently measure entanglement. Because the super Hilbert space is $\mathbb{Z}_2$-graded, there can exist even and odd vectors (as for the general quantum states in the previous section) that can correspond to even and odd superqubits, respectively.

The definitions of a single qudit in the complex Hilbert space $\mathcal{H}_d$ and a single superqudit in super Hilbert space $\mathscr{H}_{(r|s)}$ (over $\Lambda_N(\mathbb{C})$) can be written, in general, as the expansions of the (pure) quantum states on the computational (super) basis as follows

$$|\ \Psi\rangle = |\ \Psi\rangle_{(d)} = x_0\ |\ 0\rangle + x_1\ |\ 1\rangle + \cdots + x_{d-1}\ |\ d-1\rangle, \tag{3.46}$$

$$\begin{gathered} |\ x\ |_0^2 + |\ x\ |_1^2 + \cdots + |\ x\ |_{d-1}^2 = 1, \quad x_i \in \mathbb{C},\ |\ i\rangle \in \mathcal{H}_d,\ i = 0,\ \ldots\ ,\ d-1, \\ \Downarrow_{\text{susy}} \end{gathered} \tag{3.47}$$

$$\|\boldsymbol{\Psi}^{(\bar{0})}\rangle = \|\mathbf{0}\rangle x_0 + \|\mathbf{1}\rangle x_1 + \cdots + \|\mathbf{r}-1\rangle x_{r-1} + \|\text{œ}_0\rangle \text{æ}_0 + \cdots + \|\text{œ}_{s-1}\rangle \text{æ}_{s-1}, \tag{3.48}$$

$$x_0^{\#} x_0 + x_1^{\#} x_1 + \cdots + x_{r-1}^{\#} x_{r-1} - \text{æ}_0^{\#} \text{æ}_0 - \cdots \text{æ}_{s-1}^{\#} \text{æ}_{s-1} = 1, \tag{3.49}$$

$$\begin{gathered} \|\boldsymbol{\Psi}^{(\bar{1})}\rangle = \|\mathbf{0}\rangle \text{æ}_0 + \|\mathbf{1}\rangle \text{æ}_1 + \cdots + \|\mathbf{r}-1\rangle \text{æ}_{r-1} + \|\text{œ}_0\rangle x_0 + \cdots + \|\text{œ}_{s-1}\rangle x_{s-1}, \\ x_i \in \Lambda_N^{(0)}(\mathbb{C}),\ \|\mathbf{j}\rangle \in \mathscr{H}_{(r|s)}^{(\bar{0})},\ \text{æ}_\alpha \in \Lambda_N^{(\bar{1})}(\mathbb{C}),\ \|\text{œ}_\alpha\rangle \in \mathscr{H}_{(r|s)}^{(\bar{1})}. \end{gathered} \tag{3.50}$$

We assume that $\deg \|\mathbf{i}\rangle = \deg \mathbf{x}_i = \bar{0}$, $\deg \|\text{œ}_\alpha\rangle = \deg \text{æ}_\alpha = \bar{1}$, and therefore the superqudit (3.48) has the even parity $\pi_\Psi = \deg \|\boldsymbol{\Psi}\rangle = \bar{0}$, and we call it the even superqudit $\|\boldsymbol{\Psi}^{(\bar{0})}\rangle = \|\boldsymbol{\Psi}\rangle^{\text{even}}$, while the superqudit (3.50) has the odd parity $\pi_\Psi = \deg \|\boldsymbol{\Psi}\rangle = \bar{1}$, and we call it the odd superqudit $\|\boldsymbol{\Psi}^{(\bar{1})}\rangle = \|\boldsymbol{\Psi}\rangle^{\text{odd}}$, denoting both of them $\|\boldsymbol{\Psi}^{(\pi)}\rangle = \|\boldsymbol{\Psi}^{(\pi_\Psi)}\rangle$. The normalization of the odd superqudit can be done using some special Grassmann norms considered in Rudolph (2000), Rogers (2007) and Haba and Kupsch (1995).

Definition 3.6. The qudits $|\ \Psi\rangle$ and superqudits $\|\boldsymbol{\Psi}\rangle$ are

1. Linear spans of the corresponding subspace $\operatorname{span}(\{|\ i\rangle\}) \subseteq \mathcal{H}_d$ and sub-superspace $\operatorname{span}(\{\|\mathbf{i}\rangle\}|\{\|\text{œ}_\alpha\rangle\}) \subseteq \mathscr{H}_{(r|s)}$, respectively,
2. Having the normalization conditions (3.47), (3.49).

For consistency, it is natural to assume that the superqudit (3.48) has the Grassmannless limit, body map (Rogers 1980), as the ordinary qudit (3.46)

$$\|\boldsymbol{\Psi}^{(\bar{0})}\rangle_{\text{body}} = |\ \Psi\rangle_{(r)}. \tag{3.51}$$

The normalization conditions (3.47) and (3.49) distinguish (super)qudits among general span subspaces, which allows us to endow them probablistic interpretation. If the limit (3.51) is accepted, then (3.47) and (3.49), as well as the bases $\{|\ i\rangle\} \in \mathcal{H}_r$ and $\{\|\mathbf{i}\rangle\} \in \mathscr{H}_{(r|s)}$ are connected with the body map.

The (super)qudits in minimum dimensions $d = 2$, $r = 2$, $s = 1$ are called (super) qubits (Borsten *et al* 2010) and have the form[1]

$$\begin{array}{ccc} & & \|\boldsymbol{\Psi}^{(\bar{0})}\rangle = \|\mathbf{0}\rangle x_0 + \|\mathbf{1}\rangle x_1 + \|\text{œ}\rangle \text{æ}, \\ |\ \Psi\rangle = |\ \Psi\rangle_{(2)} = x_0\,|\ 0\rangle + x_1\,|\ 1\rangle, & & x_0^{\#} x_0 + x_1^{\#} x_1 - \text{æ}^{\#} \text{æ} = 1, \\ |\ x_0\ |^2 + |\ x_1\ |^2 = 1, & \overset{\text{susy}}{\Longrightarrow} & \|\boldsymbol{\Psi}^{(\bar{1})}\rangle = \|\mathbf{0}\rangle \text{æ}_0 + \|\mathbf{1}\rangle \text{æ}_1 + \|\text{œ}\rangle x, \\ x_0, x_1 \in \mathbb{C}, \quad |\ 0\rangle, |\ 1\rangle \in \mathcal{H}_2, & & x, x_0, x_1 \in \Lambda_N^{(\bar{0})}(\mathbb{C}), \quad \|\mathbf{0}\rangle, \|\mathbf{1}\rangle \in \mathscr{H}_{(2|1)}^{(\bar{0})}, \\ & & \text{æ}, \text{æ}_0, \text{æ}_1 \in \Lambda_N^{(\bar{1})}(\mathbb{C}), \quad \|\text{œ}\rangle \in \mathscr{H}_{(2|1)}^{(\bar{1})}. \end{array} \tag{3.52}$$

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[1] For clarity and convenience for applications, we use the manifest presentation of different variables. The right coordinates are used in superqubits according to the sign agreement of Borsten *et al* (2010).

There are four main operations between two single (super)qubits.

(1) *Inner product* of bra (super)qubit and ket (super)qubit

$$\langle\Psi|\Psi'\rangle = \bar{x}_0 x_0' + \bar{x}_1 x_1' \in \mathbb{C} \overset{\text{susy}}{\Longrightarrow} \begin{array}{l} \langle \boldsymbol{\Psi}^{(\bar{0})}||\boldsymbol{\Psi}^{(\bar{0})\prime}\rangle = x_0^{\sharp} x_0' + x_1^{\sharp} x_1' - \text{æ}^{\sharp}\text{æ}', \\ \langle \boldsymbol{\Psi}^{(\bar{1})}||\boldsymbol{\Psi}^{(\bar{1})\prime}\rangle = \text{æ}_0^{\sharp}\text{æ}_0' + \text{æ}_1^{\sharp}\text{æ}_1' + x^{\sharp} x' \end{array} \tag{3.53}$$

where $\overline{()}$ is complex conjugation and $(\sharp)$ is the grade involution (3.4).

If the states coincide, $|\Psi'\rangle = |\Psi\rangle$ and $||\boldsymbol{\Psi}'\rangle = ||\boldsymbol{\Psi}\rangle$, then (3.53) are square norms of $|\Psi\rangle$ and $||\boldsymbol{\Psi}^{(\bar{0})}\rangle$ becoming unity for normalized (super)qubits. For physical states, the square norm of the even superqubit is positive

$$||\boldsymbol{\Psi}^{(\bar{0})}||^2_{\text{body}} = \langle \boldsymbol{\Psi}^{(\bar{0})}||\boldsymbol{\Psi}^{(\bar{0})\prime}\rangle_{\text{body}} > 0. \tag{3.54}$$

(2) *Outer product* of ket and bra gives the density (super)matrix of (super)qubit

$$\rho = |\Psi\rangle\langle\Psi| = \begin{pmatrix} x_0\bar{x}_0 & x_1\bar{x}_0 \\ x_0\bar{x}_1 & x_1\bar{x}_1 \end{pmatrix} \tag{3.55}$$

$$\Downarrow_{\text{susy}} \tag{3.56}$$

$$\rho^{(\bar{0})} = ||\boldsymbol{\Psi}^{(\bar{0})}\rangle\langle\boldsymbol{\Psi}^{(\bar{0})}|| = \begin{pmatrix} x_0 x_0^{\sharp} & x_1 x_0^{\sharp} & \text{æ} x_0^{\sharp} \\ x_0 x_1^{\sharp} & x_1 x_1^{\sharp} & \text{æ} x_1^{\sharp} \\ -x_0\text{æ}^{\sharp} & -x_1\text{æ}^{\sharp} & -\text{æ}\text{æ}^{\sharp} \end{pmatrix}, \tag{3.57}$$

$$\rho^{(\bar{1})} = ||\boldsymbol{\Psi}^{(\bar{1})}\rangle\langle\boldsymbol{\Psi}^{(\bar{1})}|| = \begin{pmatrix} -\text{æ}_0\text{æ}_0^{\sharp} & -\text{æ}_1\text{æ}_0^{\sharp} & -x\text{æ}_0^{\sharp} \\ -\text{æ}_0\text{æ}_1^{\sharp} & -\text{æ}_1\text{æ}_1^{\sharp} & -x\text{æ}_1^{\sharp} \\ \text{æ}_0 x^{\sharp} & \text{æ}_1 x^{\sharp} & x x^{\sharp} \end{pmatrix}, \tag{3.58}$$

and the body map limit for $\rho^{(\bar{0})}$ is similar to (3.51). The standard connection of the inner product with the (super)trace of density matrix for a given (super)qubit holds valid (taking into account gradings)

$$\operatorname{tr}\rho = \langle\Psi|\Psi\rangle \in \mathbb{C} \overset{\text{susy}}{\Longrightarrow} \begin{array}{l} \operatorname{str}\rho^{(\bar{0})} = \langle\boldsymbol{\Psi}^{(\bar{0})}||\boldsymbol{\Psi}^{(\bar{0})}\rangle \in \Lambda_N^{(\bar{0})}(\mathbb{C}), \\ \operatorname{str}\rho^{(\bar{1})} = -\langle\boldsymbol{\Psi}^{(\bar{1})}||\boldsymbol{\Psi}^{(\bar{1})}\rangle \in \Lambda_N^{(\bar{0})}(\mathbb{C}). \end{array} \tag{3.59}$$

(3) *Tensor product* of two ket (super)qubits (or two bra (super)qubits) (3.52) can be presented as the following manifest expansions on elementary tensors (all gradings appear and are shown for clarity and direct usage in computations)

$$\begin{gathered}
\mid \Psi\rangle \otimes \mid \Psi'\rangle = x_0x_0' \mid 0\rangle \otimes \mid 0'\rangle + x_0x_1' \mid 0\rangle \otimes \mid 1'\rangle + x_1x_0' \mid 1\rangle \otimes \mid 0'\rangle + x_1x_1' \mid 1\rangle \otimes \mid 1'\rangle \\
\Downarrow_{\text{susy}} \\
\begin{pmatrix} ||\mathbf{\Psi}^{(\bar{0})}\rangle \otimes ||\mathbf{\Psi}^{(\bar{0})'}\rangle \\ ||\mathbf{\Psi}^{(\bar{1})}\rangle \otimes ||\mathbf{\Psi}^{(\bar{1})'}\rangle \\ ||\mathbf{\Psi}^{(\bar{0})}\rangle \otimes ||\mathbf{\Psi}^{(\bar{1})'}\rangle \\ ||\mathbf{\Psi}^{(\bar{1})}\rangle \otimes ||\mathbf{\Psi}^{(\bar{0})'}\rangle \end{pmatrix} = ||\mathbf{0}\rangle \otimes ||\mathbf{0}'\rangle \begin{pmatrix} x_0x_0' \\ \text{æ}_0\text{æ}_0' \\ x_0\text{æ}_0' \\ \text{æ}_0x_0' \end{pmatrix} + ||\mathbf{0}\rangle \otimes ||\mathbf{1}'\rangle \begin{pmatrix} x_0x_1' \\ \text{æ}_0\text{æ}_1' \\ x_0\text{æ}_1' \\ \text{æ}_0x_1' \end{pmatrix} \\
+ ||\mathbf{0}\rangle \otimes ||\text{œ}'\rangle \begin{pmatrix} x_0\text{æ}' \\ \text{æ}_0x' \\ x_0x' \\ \text{æ}_0\text{æ}' \end{pmatrix} + ||\mathbf{1}\rangle \otimes ||\mathbf{0}'\rangle \begin{pmatrix} x_1x_0' \\ \text{æ}_1\text{æ}_0' \\ x_1\text{æ}_0' \\ \text{æ}_1x_0' \end{pmatrix} + ||\mathbf{1}\rangle \otimes ||\mathbf{1}'\rangle \begin{pmatrix} x_1x_1' \\ \text{æ}_1\text{æ}_1' \\ x_1\text{æ}_1' \\ \text{æ}_1x_1' \end{pmatrix} \\
+ ||\mathbf{1}\rangle \otimes ||\text{œ}'\rangle \begin{pmatrix} x_1\text{æ}' \\ \text{æ}_1x' \\ x_1x' \\ \text{æ}_1\text{æ}' \end{pmatrix} + ||\text{œ}\rangle \otimes ||\mathbf{0}'\rangle \begin{pmatrix} \text{æ}x_0' \\ x\text{æ}_0' \\ \text{ææ}_0' \\ xx_0' \end{pmatrix} + ||\text{œ}\rangle \otimes ||\mathbf{1}'\rangle \begin{pmatrix} \text{æ}x_1' \\ x\text{æ}_1' \\ \text{ææ}_1' \\ xx_1' \end{pmatrix} \\
+ ||\text{œ}\rangle \otimes ||\text{œ}'\rangle \begin{pmatrix} \text{ææ}' \\ xx' \\ \text{æ}x' \\ x\text{æ}' \end{pmatrix} \in \begin{pmatrix} \left(\mathscr{H}_{(2|1)} \otimes \mathscr{H}_{(2|1)}\right)^{(\bar{0})} \\ \left(\mathscr{H}_{(2|1)} \otimes \mathscr{H}_{(2|1)}\right)^{(\bar{0})} \\ \left(\mathscr{H}_{(2|1)} \otimes \mathscr{H}_{(2|1)}\right)^{(\bar{1})} \\ \left(\mathscr{H}_{(2|1)} \otimes \mathscr{H}_{(2|1)}\right)^{(\bar{1})} \end{pmatrix}.
\end{gathered} \tag{3.60}$$

Thus, there are four different superqubit tensor products, depending of their parity.

Definition 3.7. The (pure) quantum state that can be obtained as a tensor product is called a separable state.

(4) *Cross product* of two ket qutrits ((3.46) with $d = 3$) of the form

$$\mid \Psi\rangle = \mid \Psi\rangle_{(3)} = x_0 \mid 0\rangle + x_1 \mid 1\rangle + x_2 \mid 2\rangle, \tag{3.61}$$

$$\begin{gathered} \mid x_0 \mid^2 + \mid x_1 \mid^2 + \mid x_2 \mid^2 = 1, \\ x_0, x_1, x_2 \in \mathbb{C}, \quad \mid 0\rangle, \mid 1\rangle, \mid 2\rangle \in \mathcal{H}_3, \end{gathered} \tag{3.62}$$

can be defined by analogy with ordinary cross product of vectors

$$\mid \Phi\rangle_{\text{cross}} = \mid \Psi\rangle \times \mid \Psi'\rangle = \sum_{i,j,k=0,1,2} \epsilon_{ijk}x_jx_k' \mid i\rangle \tag{3.63}$$

$$= \det \begin{pmatrix} \mid 0\rangle & \mid 1\rangle & \mid 2\rangle \\ x_0 & x_1 & x_2 \\ x_0' & x_1' & x_2' \end{pmatrix} = \det M_{\mid 0\rangle} \mid 0\rangle + \det M_{\mid 1\rangle} \mid 1\rangle + \det M_{\mid 2\rangle} \mid 2\rangle \tag{3.64}$$

$$= (x_1 x_2' - x_2 x_1')\,|\,0\rangle - (x_0 x_2' - x_2 x_0')\,|\,1\rangle + (x_0 x_1' - x_1 x_0')\,|\,2\rangle,\quad x_i,\, x_i' \in \mathbb{C}, \quad (3.65)$$

where $M_{|i\rangle}$ is the minor of element $|\,i\rangle$, and ϵ_{ijk} fully antisymmetric tensor. The last expanded form (3.65) is convenient to use for superqubits as well.

Definition 3.8. We call the qutrit $|\,\Phi\rangle_{\text{cross}}$ that is built as the cross product (3.65) a cross-qutrit.

The square norm of the cross-qutrit is

$$\begin{aligned} \|\,|\,\Phi\rangle_{\text{cross}}\,\|^2 &= \|\Psi\,\|^2\|\Psi'\|^2 - |\,\langle\Psi|\Psi'\rangle\,|^2 \\ &= \left|\,\det \mathrm{M}_{|0\rangle}\,\right|^2 + \left|\,\det \mathrm{M}_{|1\rangle}\,\right|^2 + \left|\,\det \mathrm{M}_{|2\rangle}\,\right|^2. \end{aligned} \quad (3.66)$$

Therefore, for the normalized qutrit $|\,\Phi\rangle_{\text{cross}}$, we have the additional condition (together with two conditions (3.62) for $|\,\Psi\rangle$ and $|\,\Psi'\rangle$)

$$\left|\,\det \mathrm{M}_{|0\rangle}\,\right|^2 + \left|\,\det \mathrm{M}_{|1\rangle}\,\right|^2 + \left|\,\det \mathrm{M}_{|2\rangle}\,\right|^2 = 1. \quad (3.67)$$

Definition 3.9. The (pure) quantum state which can be obtained as a cross product is called a cross-separable state.

The cross-qutrits have special properties and can be connected with the concurrence measure in considering entanglement (see below).

3.4 Multi-(super)qubit states

The multi-(super)qudit quantum states are vectors in the tensor product of n (super) Hilbert spaces $\mathcal{H}_d^{\otimes n} = \overbrace{\mathcal{H}_d \otimes \cdots \otimes \mathcal{H}_d}^{n}$ (resp. $\mathscr{H}_{(r|s)}^{\otimes n} = \overbrace{\mathscr{H}_{(r|s)} \otimes \cdots \otimes \mathscr{H}_{(r|s)}}^{\mathrm{n}}$). On first sight, a straightforward way to obtain such vectors is to use the tensor product (3.60) repeatedly $n - 1$ times. However, this procedure is too restricted and can only give separable states. The consequent definition should be made in terms of spans, as in **Definition** 3.6.

Definition 3.10. The multi-qudits (n-qudit states) are

(1) Linear span of the Hilbert subspace

$$\begin{aligned} &\{|\,\Psi(n)\rangle\} = \operatorname{span}(|\,i_1\rangle \otimes \cdots \otimes |\,i_n\rangle) \subseteq \overbrace{\mathcal{H}_d \otimes \cdots \otimes \mathcal{H}_d}^{n}, \quad |\,i_k\rangle \in \mathcal{H}_d, \quad k = 1, \ldots, n, \\ &|\,\Psi(n)\rangle = \sum_{i_1=0}^{d-1} \cdots \sum_{i_n=0}^{d-1} x_{i_1\cdots i_n}\,|\,i_1\rangle \otimes \cdots \otimes |\,i_n\rangle, \quad x_{i_1\cdots i_n} \in \mathbb{C}, \end{aligned} \quad (3.68)$$

(2) With the normalization (3.46)

$$\sum_{i_1=0}^{d-1}\cdots\sum_{i_n=0}^{d-1} | x |^2_{i_1\cdots i_n} = 1. \tag{3.69}$$

Definition 3.11. The multi-superqudits (*n*-superqudit states) are

(1) Linear span of the super Hilbert subspace

$$\begin{gathered}\mathrm{span}(||\mathbf{I}^{\cdot}_1\rangle \otimes \cdots \otimes ||\mathbf{I}^{\cdot}_n\rangle) \subseteq \overbrace{\mathscr{H}_{(r|s)} \otimes \cdots \otimes \mathscr{H}_{(r|s)}}^{n}, \quad ||\mathbf{I}^{\cdot}_k\rangle = (||\mathbf{i}_k\rangle, ||\text{œ}_k\rangle), \\ ||\mathbf{i}_k\rangle \in \mathscr{H}^{(\bar{0})}_{(r|s)}, ||\text{œ}_k\rangle \in \mathscr{H}^{(\bar{1})}_{(r|s)}, \quad \mathtt{k} = 1, \ldots, \mathtt{n},\end{gathered} \tag{3.70}$$

which respect parities of variables, such that we have even and odd superqubits (see (3.6))

$$\begin{gathered}||\mathbf{\Psi}^{(\mathsf{k})}(n)\rangle = \sum_{j_1=0}^{n-1}\cdots\sum_{j_n=0}^{n-1} \mathbf{y}_{j_1\cdots j_n} ||\mathbf{I}^{\cdot}_1\rangle \otimes \cdots \otimes ||\mathbf{I}^{\cdot}_n\rangle, \quad \mathbf{y}_{j_1\cdots j_n} = (x_{j_1\cdots j_n}, \text{æ}_{j_1\cdots j_n}), \\ \mathsf{k} = \deg \mathbf{y}_{j_1\cdots j_n} \boxplus \deg ||\mathbf{I}^{\cdot}_1\rangle \boxplus \cdots \deg ||\mathbf{I}^{\cdot}_n\rangle = \bar{0},\, \bar{1} \in \mathbb{Z}_2,\end{gathered} \tag{3.71}$$

(2) Normalization can be made for the even multi-superqubit $||\mathbf{\Psi}^{(\bar{0})}(n)\rangle$ only, as for the single superqubit (3.52).

To clarify the difference between the separable (3.60) and nonseparable (3.68), (3.71) states, we consider the example of two (super)qubits. Thus, for $n = 2$ (two-party states), we obtain

$$\begin{aligned} &| \Psi(2)\rangle = x_{00} | 0\rangle \otimes | 0'\rangle + x_{01} | 0\rangle \otimes | 1'\rangle + x_{10} | 1\rangle \otimes | 0'\rangle + x_{11} | 1\rangle \otimes | 1'\rangle \\ &\Downarrow_{\mathrm{susy}} \\ &\begin{pmatrix} ||\mathbf{\Psi}^{(\bar{0})}(2)\rangle \\ ||\mathbf{\Psi}^{(\bar{1})}(2)\rangle \end{pmatrix} = ||\mathbf{0}\rangle \otimes ||\mathbf{0}'\rangle \begin{pmatrix} x_{00} \\ \text{æ}_{00} \end{pmatrix} + ||\mathbf{0}\rangle \otimes ||\mathbf{1}'\rangle \begin{pmatrix} x_{01} \\ \text{æ}_{01} \end{pmatrix} \\ &+ ||\mathbf{0}\rangle \otimes ||\text{œ}'\rangle \begin{pmatrix} \text{æ}_{02} \\ x_{02} \end{pmatrix} + ||\mathbf{1}\rangle \otimes ||\mathbf{0}'\rangle \begin{pmatrix} x_{10} \\ \text{æ}_{10} \end{pmatrix} + ||\mathbf{1}\rangle \otimes ||\mathbf{1}'\rangle \begin{pmatrix} x_{11} \\ \text{æ}_{11} \end{pmatrix} \\ &+ ||\mathbf{1}\rangle \otimes ||\text{œ}'\rangle \begin{pmatrix} \text{æ}_{12} \\ x_{12} \end{pmatrix} + ||\text{œ}\rangle \otimes ||\mathbf{0}'\rangle \begin{pmatrix} \text{æ}_{20} \\ x_{20} \end{pmatrix} + ||\text{œ}\rangle \otimes ||\mathbf{1}'\rangle \begin{pmatrix} \text{æ}_{21} \\ x_{21} \end{pmatrix} \\ &+ ||\text{œ}\rangle \otimes ||\text{œ}'\rangle \begin{pmatrix} x_{22} \\ \text{æ}_{22} \end{pmatrix} \in \begin{pmatrix} \left(\mathscr{H}_{(2|1)} \otimes \mathscr{H}'_{(2|1)}\right)^{(\bar{0})} \\ \left(\mathscr{H}_{(2|1)} \otimes \mathscr{H}'_{(2|1)}\right)^{(\bar{1})} \end{pmatrix}, \end{aligned} \tag{3.72}$$

where $| \Psi(2)\rangle$ has four bosons, $||\mathbf{\Psi}^{(\bar{0})}(2)\rangle$ has five bosons and four fermions, $||\mathbf{\Psi}^{(\bar{1})}(2)\rangle$ has four bosons and five fermions. Comparing the tensor product (3.60) and (3.72), we observe that for separable states all the amplitudes (coordinates) in (3.72) can be composed

$$\begin{array}{c} x_{ij} \overset{\text{sep}}{=} x_i x'_j, \\ x_i \in \mathbb{C}, \\ i, j = 0, 1 \end{array} \overset{\text{susy}}{\Longrightarrow} \begin{array}{c} x_{ij} \overset{\text{sep}}{=} \begin{cases} x_i x'_j \\ \text{æ}_i \text{æ}'_j \end{cases}, \; x_{i2} \overset{\text{sep}}{=} \begin{cases} x_i x' \\ \text{æ}_i \text{æ}' \end{cases}, \; x_{2i} \overset{\text{sep}}{=} \begin{cases} \text{æ}\text{æ}'_i \\ x x'_i \end{cases}, \; x_{22} \overset{\text{sep}}{=} \begin{cases} \text{æ}\text{æ}' \\ x x' \end{cases} \\ \text{æ}_{ij} \overset{\text{sep}}{=} \begin{cases} x_i \text{æ}'_j \\ \text{æ}_i x'_j \end{cases}, \; \text{æ}_{i2} \overset{\text{sep}}{=} \begin{cases} x_i \text{æ}' \\ \text{æ}_i x' \end{cases}, \; \text{æ}_{2i} \overset{\text{sep}}{=} \begin{cases} \text{æ} x'_i \\ x \text{æ}'_i \end{cases}, \; \text{æ}_{22} \overset{\text{sep}}{=} \begin{cases} \text{æ} x' \\ x \text{æ}' \end{cases} \\ x, x_i,\, x_{ij},\, x_{i2},\, x_{2i},\, x',\, x'_i \in \Lambda_N^{(\bar{0})}(\mathbb{C}), \\ \text{æ},\, \text{æ}_i,\, \text{æ}_{ij},\, \text{æ}_{i2},\, \text{æ}_{2i},\, \text{æ}',\, \text{æ}'_i \in \Lambda_N^{(\bar{1})}(\mathbb{C}). \end{array} \tag{3.73}$$

Remark 3.12. The separability of two-party superqubit even $||\mathbf{\Psi}^{(\bar{0})}(2)\rangle$ and odd $||\mathbf{\Psi}^{(\bar{1})}(2)\rangle$ states (3.72) is determined in the nonunique way (3.73).

Definition 3.13. Multi-(super)qubit states are called entangled (inseparable) if at least one of their amplitudes ($\mathbf{y}_{j_1 \cdots j_n}$ in (3.71)) cannot be presented in the composite factorized form (3.73).

A suitable function that can measure entanglement should have the main property: vanishing for the separable states (3.60). The simplest such function for two qubits (without other requirements) is the determinant. Indeed, for the separable two party (super)qubit system, we have from the factorization (3.73)

$$\begin{array}{l} |\,\Psi(2)\rangle : f(x) = \det x_{ij} \overset{\text{sep}}{=} \det\left(x_i x'_j\right) \equiv 0, \quad \forall\, x_i,\, x'_j \in \mathbb{C}, \;\; i, j = 0, 1. \\ \qquad\quad \Downarrow_{\text{susy}} \end{array} \tag{3.74}$$

$$||\mathbf{\Psi}^{(\bar{0})}(2)\rangle \colon f^{(\bar{0})}(y) = \det\left(x_{ij} x_{22} + \text{æ}_{i2} \text{æ}_{2j}\right) \overset{\text{sep}}{=} 0, \;\; y_{ij} = (x_{ij},\, \text{æ}_{ij}), \tag{3.75}$$

$$||\mathbf{\Psi}^{(\bar{1})}(2)\rangle \colon f^{(\bar{1})}(y) = \det\left(\text{æ}_{ij} \text{æ}_{22} - x_{i2} x_{2j}\right) \overset{\text{sep}}{=} 0, \, x_{ij} \in \Lambda_N^{(\bar{0})}(\mathbb{C}), \, \text{æ}_{ij} \in \Lambda_N^{(\bar{1})}(\mathbb{C}). \tag{3.76}$$

Further requirements can be imposed, for ordinary qubits they are positivity, monotonicity, and the range in $\{0, 1\}$, as probability, which gives the concurrence (Hill and Wootters 1997, Wootters 1998, Horodecki *et al* 2009)

$$C_2(x) = C(|\,\Psi(2)\rangle) = 2\,|\,f(x)\,| = 2\,\left|\,\det x_{ij}\,\right|, \tag{3.77}$$

such that for maximally entangled states, e.g., the Bell state $x_{11} = x_{22} = \frac{1}{\sqrt{2}}$, $x_{01} = x_{10} = 0$, to get $C(x) = 1$.

Innovation 3.14. *We define the even and odd superconcurrences by*

$$C^{(\bar{0})}(y) = C(|\Psi^{(\bar{0})}(2)\rangle) = 2\,\Big\|\det\,(x_{ij}\mathscr{x}_{22} + \text{æ}_{i2}\text{æ}_{2j})\,\Big\|_R, \tag{3.78}$$

$$C^{(\bar{1})}(y) = C(|\Psi^{(\bar{1})}(2)\rangle) = 2\,\Big\|\det\,(\text{æ}_{ij}\text{æ}_{22} - x_{i2}\mathscr{x}_{2j})\,\Big\|_R, \tag{3.79}$$

where $\|\|_R$ *is one of the Grassmann norms* (De Witt 1992, Rogers 2007, Rudolph 2000).

The square of the concurrence is called tangle (Borsten *et al* 2010), which can be written for two qubits in the form

$$\tau(x) = 4f(x)\overline{f(x)} = 4\det x_{ij}\det \bar{x}_{ij}, \tag{3.80}$$

where $(\overline{\ })$ is the complex conjugation.

Innovation 3.15. *For two even/odd superqubits, by analogy with (3.80) and taking into account possible noninvertibilities, we can define the even supertangle* $\boldsymbol{\tau}^{(\bar{0})}(x)$ *and odd supertangle* $\boldsymbol{\tau}^{(\bar{1})}(x)$ *in the following way*

$$\boldsymbol{\tau}^{(\bar{0})}(y)\mathscr{x}_{22}(\mathscr{x}_{22})^{\sharp} = 4f^{(\bar{0})}(y)(f^{(\bar{0})}(y))^{\sharp}, \tag{3.81}$$

$$\boldsymbol{\tau}^{(\bar{1})}(y)\text{æ}_{22}(\text{æ}_{22})^{\sharp} = 4f^{(\bar{1})}(y)(f^{(\bar{1})}(y))^{\sharp}, \tag{3.82}$$

where $f^{(\bar{0})}(y)$ *and* $f^{(\bar{1})}(y)$ *are defined in (3.75) and (3.76), respectively.*

In case of invertible $\mathscr{x}_{22}$, the even superconcurrence $C^{(\bar{0})}(y)$ (3.78) and even supertangle $\boldsymbol{\tau}^{(\bar{0})}(x)$ (3.81) can be connected with the Berezinian (3.26).

There are many other entanglement measures, e.g., entropy of entanglement, positive partial transpose, quantum discord, entanglement of formation, distillable entanglement, entanglement cost, squashed entanglement, and entanglement witnesses (Horodecki *et al* 2009). Some of them can also be applied for multi-(super) qudits, for superqubits, see, e.g. Borsten *et al* (2010).

The entanglement classification and manipulation can be provided by considering various local symmetries of multi-(super)qubit systems. The main paradigm is local operations and classical communication (LOCC), which was proposed in Bennett *et al* (1996): it is not possible to change the quantum property of a many party state (e.g. increase its entanglement) using local operations (e.g. on one party qubits) and classical channels only. Thus, the many party quantum states can be classified in such a way that each class contains the representative state with maximum entanglement. If some operations can be performed using LOCC, but may fail, they are called stochastic local operations and classical communication (SLOCC) (Vidal 2000, Verstraete *et al* 2001). Quantum states that can be transformed into one another are called SLOCC equivalent and the corresponding equivalence classes are called entanglement (SLOCC) classes, which are invariant under invertible unitary transformations (Eltschka and Siewert 2014, Verstraete *et al* 2001).

A single qubit $|\,\Psi\rangle$ (3.52) carries the fundamental representation of $SU(2)$ group, and therefore for the n-qubit state the LOCC equivalence group is $[SU(2, \mathbb{C})]^{\otimes n}$ (Vidal 2000, Verstraete *et al* 2001), while the SLOCC equivalence group is $[SL(2, \mathbb{C})]^{\otimes n}$ (Borsten *et al* 2010). Thus, any separable n-qubit state will remain separable under all $[SU(2)]^{\otimes n}$ operations. In a similar way, the even superqubit $||\mathbf{\Psi}^{(\bar{0})}\rangle$ (3.52) carries the fundamental representation of the local operation (unitary orthosymplectic) group $uOSp(2|1)$, and so for n-superqubit state the LOCC equivalence group is $[uOSp(2|1)]^{\otimes n}$ and the SLOCC equivalence group is $[OSp(2|1)]^{\otimes n}$ (Borsten *et al* 2015).

The supersymmetrization of (S)LOCC groups is different from supersymmetrization of the Poincaré group, and therefore artificially adding the superpartners of the electron and photon does not give a superqubit (Brádler 2012). Nevertheless, supersymmetric extension of quantum mechanics based on superqubits may be a candidate for a superquantum theory that lies in the gap between the ordinary quantum theory and nonlocal boxes (Popescu and Rohrlich 1994, Borsten *et al* 2014). There can be applications of superqubits in condensed matter physics where the orthosymplectic Lie superalgebras play an important role (Efetov 1997).

3.5 Innovations

Here we consider the following generalizations of superqubits.

Innovation 3.16. (Odd superqubits). *The odd superqubits* $||\mathbf{\Psi}^{(\bar{1})}\rangle$ *were introduced in (3.52) by analogy with the odd superfields. We suppose that the SLOCC equivalence group for odd superqubits could be connected with the periplectic group, a subgroup of the general linear supergroup over* $\Lambda_N(\mathbb{C})$*, which preserves the odd bilinear form* (Leites and Serganova 1991, Deligne *et al* 2018).

Innovation 3.17. [Tensor product of qubit and superqubit] *From the first glance, one can think that the tensor product of* $|\,\Psi\rangle \in \mathcal{H}_2$ *and* $||\mathbf{\Psi}'\rangle \in \mathscr{H}_{(2|1)}$ *is a particular case of* $||\mathbf{\Psi}\rangle \otimes ||\mathbf{\Psi}'\rangle$ *(3.60) where one multiplier is the body map (3.51). However, the consistent construction is more complicated because the spaces* $\mathcal{H}_d$ *and* $\mathscr{H}_{(r|s)}$ *are over different fields. In general, the tensor product of the vector space* $\mathcal{V}_1$ *over* $\mathbb{k}_1$ *and* $\mathbb{V}_2$ *over* $\mathbb{k}_2$ *can be built as*

$$\mathcal{V}_1 \otimes_{\mathbb{k}_2} \mathcal{V}_2 := \mathcal{V}_1 \otimes_{\mathbb{k}_2} \left(\mathbb{k}_2 \otimes_{\mathbb{k}_1} \mathcal{V}_2\right). \tag{3.83}$$

The same construction can be provided for the corresponding Hilbert spaces by consideration their inner products. Moreover, the properties of the qubit tensor product *with even* $||\mathbf{\Psi}^{(\bar{0})}\rangle$ *and odd* $||\mathbf{\Psi}^{(\bar{1})}\rangle$ *superqubits are fully different. Indeed, from (3.52) we have the mixed qubit-superqubit tensor product (using informally the same its sign)*

$$\begin{pmatrix} |\ \Psi\rangle \otimes ||\mathbf{\Psi}^{(\bar{0})'}\rangle \\ |\ \Psi\rangle \otimes ||\mathbf{\Psi}^{(\bar{1})'}\rangle \end{pmatrix} = |\ 0\rangle \otimes ||\mathbf{0}'\rangle \begin{pmatrix} x_0 x_0' \\ x_0 \text{æ}_0' \end{pmatrix} + |\ 0\rangle \otimes ||\mathbf{1}'\rangle \begin{pmatrix} x_0 x_1' \\ x_0 \text{æ}_1' \end{pmatrix}$$
$$+ |\ 0\rangle \otimes ||\text{œ}'\rangle \begin{pmatrix} x_0 \text{æ}' \\ x_0 x' \end{pmatrix} + |\ 1\rangle \otimes ||\mathbf{0}'\rangle \begin{pmatrix} x_1 x_0' \\ x_1 \text{æ}_0' \end{pmatrix} + |\ 1\rangle \otimes ||\mathbf{1}'\rangle \begin{pmatrix} x_1 x_1' \\ x_1 \text{æ}_1' \end{pmatrix} \tag{3.84}$$
$$+ |\ 1\rangle \otimes ||\text{œ}'\rangle \begin{pmatrix} x_1 \text{æ}' \\ x_1 x' \end{pmatrix} \in \begin{pmatrix} \left(\mathcal{H}_2 \otimes \mathscr{H}_{(2|1)}\right)^{(\bar{0})} \\ \left(\mathcal{H}_2 \otimes \mathscr{H}_{(2|1)}\right)^{(\bar{1})} \end{pmatrix}.$$

Such mixed tensor products and corresponding qubit-superqubit nonseparable quantum states would be worthwhile to investigate in detail from the viewpoint of entanglement and constructing SLOCC equivalence groups for them.

Innovation 3.18. [Concurrence through cross product of qutrits] *Here we show that concurrence of 2-qutrit states can be expressed through the cross product of qutrits. Let us consider a general nonseparable 2-qutrit state which is not the tensor product of two qutrits (3.61)*

$$|\ \Psi(2)\rangle_{(3)} = \sum_{i,j'=0,1,2} x_{ij'}\ |\ i\rangle \otimes |\ j'\rangle, \tag{3.85}$$

$$\sum_{i,j'=0,1,2} |\ x\ |_{ij'}^2 = 1, \quad x_{ij'} \in \mathbb{C}, \quad |\ i\rangle \in \mathcal{H}_3, |\ j'\rangle \in \mathcal{H}_3'. \tag{3.86}$$

Now we present 2-qutrit state (3.85) as some special kind of superposition by introducing three (ancilla) qutrits $|\ \Phi_i\rangle_{(3)}$ *and call it the semi-separable form of 2-qutrit state*

$$|\ \Psi(2)\rangle_{(3)} = a_0\ |\ \Phi_0\rangle \otimes |\ 0'\rangle + a_1\ |\ \Phi_1\rangle \otimes |\ 1'\rangle + a_2\ |\ \Phi_2\rangle \otimes |\ 2'\rangle, \tag{3.87}$$

$$|\ \Phi_i\rangle = |\ \Phi_i\rangle_{(3)} = y_{i0}\ |\ 0\rangle + y_{i1}\ |\ 1\rangle + y_{i2}\ |\ 2\rangle, \quad a_i, y_{ij} \in \mathbb{C}, \ i, j = 0, 1, 2, \tag{3.88}$$

$$\sum_{j=0,1,2} |\ y\ |_{ij}^2 = 1, \textit{for each } i = 0, 1, 2, \quad |\ i\rangle \in \mathcal{H}_3, |\ j'\rangle \in \mathcal{H}_3'. \tag{3.89}$$

Both normalizatons (3.86) and (3.89) lead to the restriction on the coefficients a_i *in the expansion (3.87), and if they are real* $a_i \in \mathbb{R}$*, then* $a_0^2 + a_1^2 + a_2^2 = 1$*, moreover, in the simplest case we can choose*

$$a_0 = a_1 = a_2 = \frac{1}{\sqrt{3}}. \tag{3.90}$$

Thus, we obtain the relation between amplitudes

$$y_{ij} = x_{ij}\sqrt{3}. \tag{3.91}$$

Let us construct three cross products (3.65) of the ancilla qutrits (3.88) $\left|\Phi_i \times \Phi_j\right\rangle \equiv |\Phi_i\rangle \times \left|\Phi_j\right\rangle$, $(i, j) = (0, 1), (1, 2), (2, 0)$, *which have the square norms (3.66) (given in terms of their amplitudes (3.88) in the form which is convenient for application to superqubits)*

$$||\Phi_0 \times \Phi_1||^2 = \left|y_{00}y_{11} - y_{01}y_{10}\right|^2 + \left|y_{00}y_{12} - y_{02}y_{10}\right|^2 + \left|y_{01}y_{12} - y_{02}y_{11}\right|^2, \tag{3.92}$$

$$||\Phi_1 \times \Phi_2||^2 = \left|y_{10}y_{21} - y_{11}y_{20}\right|^2 + \left|y_{10}y_{22} - y_{12}y_{20}\right|^2 + \left|y_{11}y_{22} - y_{12}y_{21}\right|^2, \tag{3.93}$$

$$||\Phi_2 \times \Phi_0||^2 = \left|y_{00}y_{21} - y_{01}y_{20}\right|^2 + \left|y_{00}y_{22} - y_{02}y_{20}\right|^2 + \left|y_{01}y_{22} - y_{02}y_{21}\right|^2. \tag{3.94}$$

Observe, that the sum of the square norms after the substitution (3.91) coincides with the concurrence for 2-qutrits (Cereceda 2003) (*in* Pashaev 2023 *the coefficient* $\sqrt{3}$ *was lost). Thus, we obtain the expression for the 2-qutrit concurrence*

$$C_3\left(|\Psi(2)\rangle_{(3)}\right) = \sqrt{||\Phi_0 \times \Phi_1||^2 + ||\Phi_1 \times \Phi_2||^2 + ||\Phi_2 \times \Phi_0||^2}\,|_{y_{ij}=x_{ij}\sqrt{3}}. \tag{3.95}$$

Definition 3.19. The concurrence of 2-qutrit state (3.95) can informally be treated as the space diagonal of the rectangular parallelepiped (cuboid) built on three ancilla qutrit cross product vectors $|\Phi_i\rangle \times \left|\Phi_j\right\rangle$, $(i, j) = (0, 1), (1, 2), (2, 0)$ with the further substitution (3.91). We call this procedure a cross product concurrence computation.

Innovation 3.20. (Cross product of 7-qudits). *The cross product of two vectors (without modifications or extensions of its standard definition) exists in three and seven dimensions only* (Brown and Gray 1967). *Therefore, the above general method of concurrence construction can be provided in a similar way for two 7-qudit state using 7 ancilla 7-qudit cross products. The final formula* $C_7\left(|\Psi(2)\rangle_{(7)}\right)$ *will have possibly the same shape as that of 2-qutrit state (3.95), but with 7 summands.*

Innovation 3.21. (Cross product of n-qudits). *The cross product can be defined in n dimensions if we wish to modify it by an additional cross term* (Silagadze 2002, Tian et al 2013). *In the same way, the concurrence for n-qudit state* $C_n\left(|\Psi(2)\rangle_{(n)}\right)$ *can be computed through their cross products using the above procedure of introducing n ancilla n-qudits (3.87) and considering the space diagonal of the rectangular parallelepiped in n dimensions, similarly to (3.95).*

Innovation 3.22. (Cross product of 7-superqudits). *The cross product concurrence computation procedure can also be applied for superqubits because they are effectively* *defined in three-dimensional space (superspace* $\mathscr{H}_{(2|1)}$*), as well as for 7-superqubits in effective seven-dimensional superspace* $\mathscr{H}_{(r|s)}$, $r + s = 7$. *We can introduce a super-analog of (3.63), at least informally, taking into account* $\mathbb{Z}_2$ *sign rule in further calculations and use the graded involution (3.4) instead of the complex conjugation.*

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Innovative Quantum Computing

Steven Duplij and Raimund Vogl

Chapter 4

Duality quantum computing

The duality (quantum) computer is based on the interference principle of any quantum system, but in a special way (Long 2006a, Long and Liu 2008). The main idea is to consider an undisturbed quantum system from the wave viewpoint, while on the measurement stage it is treated from the particle viewpoint. In this way, the initial quantum state (as wave) can be (1) decomposed into subwaves moving along separate paths and (2) combined at some point where they interfere (Gudder 2007). These two operations provide the additional duality parallelism, which can improve the calculational characteristics and the possible superiority of a duality computer (Long 2006a, Gudder 2008). The corresponding two additional operations are quantum operators of a new kind (duality gates): (quantum wave) divider and (quantum wave) combiner. The subwaves pass through a set of quantum gates and are collected by the combiner. The measurement is then performed on the joint final state. This procedure is a division of state of the same particle but is not a clone of the state of one particle onto another particle, and therefore this does not violate the no cloning theorem (Long 2011). The connection of the duality computer concept with the interference principle and computational applications was given in Long and Liu (2008), and the experimental realization was given in Wei *et al* (2017). Here we outline general mathematical constructions of a duality computer and present a new interpretation based on analogy with a convolution product in the polyadic Hopf algebra theory (Duplij 2022), which can be interesting by itself.

4.1 Duality computing and polyadic operations

Let us consider the complex Hilbert space $\mathcal{H}$ with the inner product $\langle\ |\rangle$ and denote the direct sum of n its copies by $\mathcal{H}^{\oplus n} = \oplus_{i=1}^{n}\mathcal{H}_i = \overbrace{\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n}^{n}$, where $\mathcal{H}_i = \mathcal{H}$, $1 \leqslant i \leqslant n$. For short, we also use the vector-like notation $\vec{\mathcal{H}} = \mathcal{H}^{\oplus n}$, such that the total quantum state $\oplus_{i=1}^{n}|\psi_i\rangle$ becomes

doi:10.1088/978-0-7503-5281-9ch4

$$\left|\vec{\Psi}\right\rangle = \left|\vec{\Psi}(n)\right\rangle = \begin{pmatrix} |\,\psi_1\rangle \\ |\,\psi_2\rangle \\ \vdots \\ |\,\psi_n\rangle \end{pmatrix} \in \vec{\mathcal{H}}, \quad |\,\psi_i\rangle \in \mathcal{H}_i, \;\; 1 \leqslant i \leqslant n, \tag{4.1}$$

where we use number of slits n in the vector state (or operators below) manifestly when it will be needed.

If all the states in (4.1) are the same $|\,\psi_i\rangle = |\,\psi\rangle$, then we place the subscript (=) as follows $\left|\vec{\Psi}\right\rangle = \left|\vec{\Psi}_=\right\rangle$, and this state will be called symmetric. A similar brief notation will be used for other variables taking values in the direct sum.

The total inner product $\langle\;|\rangle^{\rightarrow}$ of two vectors $\left|\vec{\Psi}\right\rangle$ and $\left|\vec{\Phi}\right\rangle$ is defined by (in the bra-ket notation)

$$\left\langle \vec{\Phi}\middle|\vec{\Psi}\right\rangle^{\rightarrow} = \left\langle \varphi_1\middle|\psi_1\right\rangle + \left\langle \varphi_2\middle|\psi_2\right\rangle + \cdots + \left\langle \varphi_n\middle|\psi_n\right\rangle \in \mathbb{C}, \;\; \left|\,\varphi_i\right\rangle, \left|\,\psi_i\right\rangle \in \mathcal{H}_i = \mathcal{H}, \;\; 1 \leqslant i \leqslant n. \tag{4.2}$$

The space $\vec{\mathcal{H}}$ endowed with the total inner product $\langle\;|\rangle^{\rightarrow}$ (4.2) becomes a complex Hilbert space. The norm of the total space $\|\;\;\|^{\rightarrow}$ is induced by (4.2)

$$\left\| \vec{\Psi} \right\|^{\rightarrow} = \sqrt{\|\,\psi_1\,\|^2 + \|\,\psi_2\,\|^2 + \cdots + \|\,\psi_n\,\|^2}\,, \tag{4.3}$$

where $\|\,\psi_i\,\| = \sqrt{\langle\psi_i|\psi_i\rangle}$ is the norm in $\mathcal{H}_i$.

Thus, we have four different mappings of Hilbert spaces $\mathcal{H}$ and $\vec{\mathcal{H}}$

$$\mathcal{H} \rightarrow \mathcal{H}, \tag{4.4}$$

$$\vec{\mathcal{H}} \rightarrow \vec{\mathcal{H}}, \tag{4.5}$$

$$\mathcal{H} \rightarrow \vec{\mathcal{H}}, \tag{4.6}$$

$$\vec{\mathcal{H}} \rightarrow \mathcal{H}. \tag{4.7}$$

The first two mappings (4.4)–(4.5) are the standard (bounded linear) operators in the Hilbert spaces, such that $\mathbf{T}$: $\mathcal{H} \rightarrow \mathcal{H}$ and $\vec{\mathbf{T}}: \vec{\mathcal{H}} \rightarrow \vec{\mathcal{H}}$, where

$$\vec{\mathbf{T}} = \vec{\mathbf{T}}(n) = \begin{pmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \\ \vdots \\ \mathbf{T}_n \end{pmatrix}, \quad \mathbf{T}_i\colon \mathcal{H} \rightarrow \mathcal{H}, \tag{4.8}$$

The action on the total quantum state becomes

$$\vec{\mathbf{T}}\left|\vec{\Psi}\right\rangle = \begin{pmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \\ \vdots \\ \mathbf{T}_n \end{pmatrix} \bullet \begin{pmatrix} |\,\psi_1\rangle \\ |\,\psi_2\rangle \\ \vdots \\ |\,\psi_n\rangle \end{pmatrix} = \begin{pmatrix} \mathbf{T}_1\,|\,\psi_1\rangle \\ \mathbf{T}_2\,|\,\psi_2\rangle \\ \vdots \\ \mathbf{T}_n\,|\,\psi_n\rangle \end{pmatrix}, \tag{4.9}$$

and we informally define $\mathbf{T}_i \mid \psi_i\rangle := \mid \mathbf{T}_i\, \psi_i\rangle$, such that $\vec{\mathbf{T}} \left| \vec{\Psi} \right\rangle := \left| \vec{\mathbf{T}}\, \vec{\Psi} \right\rangle$, and ($\bullet$) is the Hadamard product (here it is the componentwise action). The norm of the total operator $\vec{\mathbf{T}}$ is defined by analogy with (4.3)

$$\left\| \vec{\mathbf{T}} \right\| = \sqrt{\| \mathbf{T}_1 \|^2 + \| \mathbf{T}_2 \|^2 + \cdots + \| \mathbf{T}_n \|^2}, \tag{4.10}$$

where $\| \mathbf{T}_i \| = \sup \left\{ \frac{\| \mathbf{T}_i \psi_i \|}{\| \psi_i \|}, \forall \mid \psi_i\rangle \in \mathcal{H}_i, \mid \psi_i\rangle \neq 0 \right\}$.

The inner product in $\mathcal{H}$ of any quantum state $\mid \varphi\rangle \in \mathcal{H}$ with the transformed state $\mathbf{T} \mid \psi\rangle$ can be written as the functional $(\langle \varphi \mid)(\mathbf{T} \mid \psi\rangle) = \langle \varphi | \mathbf{T}\, \psi\rangle = \langle \varphi \mid \mathbf{T} \mid \psi\rangle$, which is the convolution (in $\mathbb{C}$) of the operator $\mathbf{T}$ with the states $\mid \varphi\rangle$ and $\mid \psi\rangle$. The convolution of the operator $\vec{\mathbf{T}}$ in the total Hilbert space $\vec{\mathcal{H}}$ should be written with respect to the total inner product $\langle \; \mid \; \rangle^{\rightarrow}$ (4.2) in the following way

$$\left\langle \vec{\Phi} \middle| \vec{\mathbf{T}} \middle| \vec{\Psi} \right\rangle^{\rightarrow} = \langle \varphi_1 \mid \mathbf{T}_1 \mid \psi_1\rangle + \langle \varphi_2 \mid \mathbf{T}_2 \mid \psi_2\rangle + \cdots + \langle \varphi_n \mid \mathbf{T}_n \mid \psi_n\rangle \in \mathbb{C}. \tag{4.11}$$

If the states are the same $\mid \varphi\rangle = \mid \psi\rangle$, then the operator convolution $\langle \psi \mid \mathbf{T} \mid \psi\rangle$ is called the expectation value of the operator $\mathbf{T}$, and the total expectation value of $\vec{\mathbf{T}} \in \vec{\mathcal{H}}$ is determined using (4.11) with $\left| \vec{\Phi} \right\rangle = \left| \vec{\Psi} \right\rangle$.

The adjoint operator $\mathbf{T}^*$ with respect to the inner product $\langle \; \mid \; \rangle$ in $\mathcal{H}$ is defined by

$$\langle \varphi \mid \mathbf{T}^* \mid \psi\rangle = \overline{\langle \psi \mid \mathbf{T} \mid \varphi\rangle} \in \mathbb{C}, \tag{4.12}$$

where $\overline{()}$ is the complex conjugation. The corresponding adjoint operator $\overrightarrow{\mathbf{T}^*}$ in the total Hilbert space $\vec{\mathcal{H}}$ is defined in the similar way with respect to the total inner product

$$\left\langle \vec{\Phi} \middle| \vec{\mathbf{T}}^* \middle| \vec{\Psi} \right\rangle^{\rightarrow} = \overline{\left\langle \vec{\Psi} \middle| \mathbf{T} \middle| \vec{\Phi} \right\rangle^{\rightarrow}}, \tag{4.13}$$

which can be written using (4.11) as

$$\left\langle \vec{\Phi} \middle| \vec{\mathbf{T}}^* \middle| \vec{\Psi} \right\rangle^{\rightarrow} = \overline{\langle \psi_1 \mid \mathbf{T}_1 \mid \varphi_1\rangle} + \overline{\langle \psi_2 \mid \mathbf{T}_2 \mid \varphi_2\rangle} + \cdots + \overline{\langle \psi_n \mid \mathbf{T}_n \mid \varphi_n\rangle} \in \mathbb{C}. \tag{4.14}$$

The convolution can be written as $\langle \varphi \mid \mathbf{T} \mid \psi\rangle = (\langle \varphi \mid \mathbf{T})(\mid \psi\rangle)$, where the bra vector $(\langle \varphi \mid \mathbf{T})$ in the matrix notation corresponds to $\varphi^\dagger \mathbf{T}$, which is equal to $(\mathbf{T}^\dagger \varphi)^\dagger$ (because $\mathbf{T}^* \longrightarrow \mathbf{T}^\dagger$ in the matrix notation), where ($\dagger$) is the Hermitean conjugation. Thus, informally we can define $\langle \varphi \mid \mathbf{T} := \langle \mathbf{T}^*\, \varphi \mid$ to get the conventional relation for the adjoint operator

$$\langle \varphi | \mathbf{T}\, \psi\rangle = \langle \mathbf{T}^*\, \varphi | \psi\rangle. \tag{4.15}$$

In the total space we informally define by analogy $\left\langle \vec{\Phi} \right| \vec{\mathbf{T}} := \left\langle \vec{\mathbf{T}}^*\; \vec{\Phi} \right|$, and we have with respect to the total inner product in the total Hilbert space $\vec{\mathcal{H}}$

$$\left\langle \vec{\boldsymbol{\Phi}} \middle| \vec{\mathbf{T}}\, \vec{\boldsymbol{\Psi}} \right\rangle^{\rightarrow} = \left\langle \vec{\mathbf{T}}^{*}\; \vec{\boldsymbol{\Phi}} \middle| \vec{\boldsymbol{\Psi}} \right\rangle^{\rightarrow} \in \mathbb{C}, \tag{4.16}$$

which can be expanded using (4.11)–(4.14) to obtain

$$\left\langle \varphi_1 \middle| \mathbf{T}_1\, \psi_1 \right\rangle + \left\langle \varphi_2 \middle| \mathbf{T}_2\, \psi_2 \right\rangle + \cdots + \left\langle \varphi_n \middle| \mathbf{T}_n\, \psi_n \right\rangle = \left\langle \mathbf{T}_1^*\, \varphi_1 \middle| \psi_1 \right\rangle + \left\langle \mathbf{T}_2^*\, \varphi_2 \middle| \psi_2 \right\rangle + \cdots + \left\langle \mathbf{T}_n^*\, \varphi_n \middle| \psi_n \right\rangle. \tag{4.17}$$

It follows from (4.11), (4.17), and the commutativity of $\mathbb{C}$ that knowing the operator convolutions (which are in $\mathbb{C}$) in each subspace determines the total convolutions uniquely, but not vice versa.

Recall that the unitary operator $\mathbf{T} = \mathbf{U}$ preserves the inner product in the Hilbert space

$$\langle \mathbf{U}\, \varphi | \mathbf{U}\, \psi \rangle = \langle \varphi | \psi \rangle \in \mathbb{C}. \tag{4.18}$$

Using (4.15), we standardly obtain that unitary operators satisfy

$$\mathbf{U}^* \circ \mathbf{U} = \mathbf{U} \circ \mathbf{U}^* = \mathrm{id}. \tag{4.19}$$

In the direct sum of spaces $\vec{\mathcal{H}}$ we have the definition of $\vec{\mathbf{U}}$ with the respect of the total inner product

$$\left\langle \vec{\mathbf{U}}\, \vec{\boldsymbol{\Phi}} \middle| \vec{\mathbf{U}}\, \vec{\boldsymbol{\Psi}} \right\rangle^{\rightarrow} = \left\langle \vec{\boldsymbol{\Phi}} \middle| \vec{\boldsymbol{\Psi}} \right\rangle \in \mathbb{C}, \tag{4.20}$$

and by means of (4.16) we obtain

$$\vec{\mathbf{U}}^{*} \circ \vec{\mathbf{U}} = \vec{\mathbf{U}} \circ \vec{\mathbf{U}}^{*} = \vec{\mathrm{id}}, \quad \vec{\mathrm{id}} = \overbrace{\mathrm{id} \oplus \mathrm{id} \oplus \cdots \oplus \mathrm{id}}^{n}. \tag{4.21}$$

The unitary operators $\{\mathbf{U}\}$ acting in $\mathcal{H}$ are widely used as quantum gates in quantum computers, while the vector operators $\{\vec{\mathbf{U}}\}$ act in the total space $\vec{\mathcal{H}}$ and are exploited as vector quantum gates in duality computing (Long and Liu 2008).

The second two mappings (4.6)–(4.7) can have another meaning (than the previous ordinary operators acting in some Hilbert space) because they are multiary operations between Hilbert spaces: $\mathcal{H}$ and $\vec{\mathcal{H}}$. We propose to treat them as n-ary comultiplication (4.6) and n-ary multiplication (4.7) of the special kind.

Definition 4.1. Let $|\,\psi\rangle \in \mathcal{H}$ be a quantum state in the Hilbert space $\mathcal{H}$, then the divider $\mathbf{D}_{\mathrm{p}}$ is the operation which splits the quantum wave into n weighted subwaves, describing multi-slits, as

$$\mathbf{D}_{\mathrm{p}}(n)\colon \mathcal{H} \to \vec{\mathcal{H}} = \mathcal{H}^{\oplus n}, \tag{4.22}$$

$$\mathbf{D}_{\mathrm{p}}(n)\,|\,\psi\rangle = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \bullet \begin{pmatrix} |\,\psi\rangle \\ |\,\psi\rangle \\ \vdots \\ |\,\psi\rangle \end{pmatrix} = \begin{pmatrix} p_1\,|\,\psi\rangle \\ p_2\,|\,\psi\rangle \\ \vdots \\ p_n\,|\,\psi\rangle \end{pmatrix} = \vec{P} \bullet \left|\, \vec{\Psi}_{=} \right\rangle, \quad p_i \in \mathbb{C}, \tag{4.23}$$

where the probability distribution $\left\{\vec{P}\right\}$ is called the divider structure. The divider structure is normalized, if

$$\left(\vec{P}\right)^{\dagger}\vec{P} = (\bar{p}_1, \bar{p}_2,\ldots,\bar{p}_n)\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} = \sum_{i=1}^{n} |\, p_i\,|^2 = 1, \quad p_i \in \mathbb{C}. \tag{4.24}$$

Definition 4.2. The divider structure is called uniform, if

$$p_1 = p_2 = \cdots = p_n = \frac{1}{n}. \tag{4.25}$$

The divider operation $\mathbf{D}_{\mathrm{p}}(n)$ (4.22) can be treated a special analog of n-ary comultiplication (deformed n-ary coaddi) map ($\Delta^{(n)}$) in the polyadic Hopf algebra theory (see Duplij 2022, chapter 9). The analog of polyadic total coassociativity for $\mathbf{D}_{\mathrm{p}}(n)$ is

$$\begin{gathered}\Bigg(\overbrace{\mathrm{id}\oplus\cdots\oplus\mathrm{id}}^{n-1-i}\oplus\mathbf{D}_{\mathrm{p}}(n)\oplus\overbrace{\mathrm{id}\oplus\cdots\oplus\mathrm{id}}^{i}\Bigg)\circ\mathbf{D}_{\mathrm{p}}(n)\\ =\Bigg(\overbrace{\mathrm{id}\oplus\cdots\oplus\mathrm{id}}^{n-1-j}\oplus\mathbf{D}_{\mathrm{p}}(n)\oplus\overbrace{\mathrm{id}\oplus\cdots\oplus\mathrm{id}}^{j}\Bigg)\circ\mathbf{D}_{\mathrm{p}}(n),\\ \forall\, i, j = 0,\ldots,n-1,\;\; i\neq j,\;\; \mathrm{id}\equiv\mathrm{id}_{\mathrm{H}}\colon \mathcal{H}\to\mathcal{H},\;\; \mathrm{id}\circ|\,\psi\rangle = |\,\psi\rangle.\end{gathered} \tag{4.26}$$

There are $(n-1)^2$ relations in (4.26). If not all of them satisfied the polyadic coassociativity, then this is called partial (see, e.g. Thurston 1949, Belousov 1972, Sokhatsky 1997).

Definition 4.3. Let $|\,\vec{\Phi}\rangle\in\vec{\mathcal{H}}$ be a direct sum of quantum states (4.1), then the combiner operation that gathers multi-slits in one quantum state as follows

$$\mathbf{C}_{\mathrm{q}}(n)\colon \vec{\mathcal{H}}\to\mathcal{H}, \tag{4.27}$$

$$\mathbf{C}_{\mathrm{q}}(n)\left|\,\vec{\Psi}\right\rangle=\mathbf{C}_{\mathrm{q}}\begin{pmatrix} |\,\psi_1\rangle \\ |\,\psi_2\rangle \\ \vdots \\ |\,\psi_n\rangle \end{pmatrix} = q_1\,|\,\psi_1\rangle + q_2\,|\,\psi_1\rangle + \cdots + q_n\,|\,\psi_n\rangle = \vec{Q}^{T}\left|\,\vec{\Psi}\right\rangle\in\mathcal{H}, \quad q_i \in \mathbb{C}, \tag{4.28}$$

where the probability distribution $\vec{Q}^{T} = \{q_1, q_2,\ldots,q_n\}$ is called the combiner structure, and it is normalized, if

$$\left(\vec{Q}\right)^T\vec{\bar{Q}} = (q_1, q_2,\ldots,q_n)\begin{pmatrix}\bar{q}_1\\ \bar{q}_2\\ \vdots\\ \bar{q}_n\end{pmatrix} = \sum_{i=1}^{n}|\,q_i\,|^2 = 1, \quad q_i \in \mathbb{C}. \tag{4.29}$$

The combiner operation $\mathbf{C}_{\mathrm{q}}(n)$ (4.27) can be treated as a deformed (by the probability distribution $\vec{Q}$) analog of n-ary multiplication (or more exactly addition) map ($\mu^{(n)}$) in the polyadic algebra theory (Duplij 2022, chapter 5). The analog of polyadic total associativity for $\mathbf{C}_{\mathrm{q}}$ is

$$\begin{gathered}\mathbf{C}_{\mathrm{q}}(n)\circ\left(\overbrace{\mathrm{id}\oplus\cdots\oplus\mathrm{id}}^{n-1-i}\oplus\mathbf{C}_{\mathrm{q}}(n)\oplus\overbrace{\mathrm{id}\oplus\cdots\oplus\mathrm{id}}^{i}\right)\\ =\mathbf{C}_{\mathrm{q}}(n)\circ\left(\overbrace{\mathrm{id}\oplus\cdots\oplus\mathrm{id}}^{n-1-j}\oplus\mathbf{C}_{\mathrm{q}}(n)\oplus\overbrace{\mathrm{id}\oplus\cdots\oplus\mathrm{id}}^{j}\right),\\ \forall\, i, j = 0,\ldots,n-1,\ \ i\neq j,\ \ \mathrm{id}\equiv\mathrm{id}_{\mathrm{H}}\colon \mathcal{H}\to\mathcal{H},\ \ \mathrm{id}\circ|\,\psi\rangle = |\,\psi\rangle.\end{gathered} \tag{4.30}$$

There are $(n-1)^2$ relations in (4.26). If not all of them satisfied the polyadic associativity, then this is called partial (see, e.g. Thurston 1949, Belousov 1972).

Proposition 4.4. The combiner operation $\mathbf{C}_{\mathrm{q}}(n)$ is totally polyadic associative, if the probability distribution $\left\{\vec{Q}\right\}$ is idempotent

$$\vec{Q}\bullet\vec{Q} = \vec{Q}, \quad \text{or} \ \ q_i^2 = q_i, \ \ i = 1,\ldots,n. \tag{4.31}$$

Proof. Make the action of both sides of (4.30) on $|\,\psi\rangle$ and insert (4.28) into each of n places consequently to get (4.31).

Corollary 4.5. Because $q_i \in \mathbb{C}$, and in $\mathbb{C}$ there only two idempotents, that are 0 and 1, the total associativity of $\mathbf{C}_{\mathrm{q}}$ for invertible q_i implies that all $q_i = 1$, $i = 1,\ldots,n$.

Corollary 4.6. If some q_i in (4.28) are not idempotent (4.31), then the combiner operation is a polyadic operator (n-ary multiplication) $\mathbf{C}_{\mathrm{q}}\colon \vec{\mathcal{H}} \to \mathcal{H}$ that is not totally associative.

Definition 4.7. The combiner structure $\left\{\vec{Q}\right\}$ is called uniform, if

$$q_1 = q_2 = \cdots = q_n = \frac{1}{n}. \tag{4.32}$$

The nonassociative binary operators are widely used in quantum mechanics and quantum field theory (Løhmus *et al* 1994).

Let us consider possible relations between divider and combiner operations. Initially, we will not fix the probability distributions (in our approach, deformation parameters) $\left\{\vec{P}\right\}$ and $\left\{\vec{Q}\right\}$, trying to find their connections in special cases.

Proposition 4.8. The composition ($\circ$) of divider and combiner is the identity operator in $\mathcal{H}$, if

$$\mathbf{C}_{\mathrm{q}}(n)\circ\mathbf{D}_{\mathrm{p}}(n) = \mathrm{id} \Longleftrightarrow q_1p_1 + q_2p_2 + \cdots + q_np_n = 1, \qquad q_i, p_i \in \mathbb{C}. \tag{4.33}$$

Proof. It follows directly from consequent acting first of $\mathbf{D}_{\mathrm{p}}$ and then $\mathbf{C}_{\mathrm{q}}$ on the quantum state $|\psi\rangle$ and then using the definitions (4.23) and (4.28).

Corollary 4.9. A particular case

$$q_i = \bar{p}_i \tag{4.34}$$

corresponds to the complex duality computing (Cao *et al* 2012), in this choice the condition (4.33) leads to uniformity of both $\mathbf{C}_{\mathrm{q}}(n)$ (4.32) and $\mathbf{D}_{\mathrm{p}}(n)$ (4.25).

Without the restrictions (4.34) the equation (4.33) has an infinite number of solutions, even when both divider and combiner operations are uniform.

Definition 4.10. We say that the divider and combiner are consistent if they are similar to n-ary coalgebra map and n-ary algebra map, respectively (Duplij 2022, chapter 9), i.e., $\mathbf{D}_{\mathrm{p}}(n)$ and $\mathbf{C}_{\mathrm{q}}(n)$ satisfy

$$\mathbf{C}_{\mathrm{q}}(n)\circ\left(\overbrace{\mathbf{D}_{\mathrm{p}}(n) \oplus \cdots \oplus \mathbf{D}_{\mathrm{p}}(n)}^{n}\right)\left|\vec{\Psi}\right\rangle = \mathbf{D}_{\mathrm{p}}(n)\circ\mathbf{C}_{\mathrm{q}}(n)\left|\vec{\Psi}\right\rangle. \tag{4.35}$$

In the Hopf algebra theory, the consistency condition is the part of polyadic bialgebra definition in terms of n-ary multiplication and n-ary comultiplication (Duplij 2022, chapter 9).

Proposition 4.11. If the total state is symmetric $\rangle\left|\vec{\Psi}\right\rangle = \left|\vec{\Psi}_{=}\right\rangle$ (4.1), then the divider and combiner are consistent, when the probability distributions are connected by n equations

$$q_i(p_1 + p_2 + \cdots + p_n) = p_i(q_1 + q_2 + \cdots + q_n), \quad i = 1,\ldots,n, \quad p_i, q_i \in \mathbb{C}. \tag{4.36}$$

Proof. It follows directly from the definitions (4.23), (4.28) and the consistency condition (4.36).

In case $n = 2$, we have only one condition $q_1 p_2 = q_2 p_1$.

Definition 4.12. We introduce an analog of the polyadic ith partial antipode $\mathbf{S}_i$ (Duplij 2022, chapter 9) by

$$\mathbf{C}_{\mathrm{q}}(n)\circ\left(\overbrace{\mathrm{id}\oplus\cdots\oplus\mathrm{id}}^{n-1-i}\oplus\mathbf{S}_i(n)\oplus\overbrace{\mathrm{id}\oplus\cdots\oplus\mathrm{id}}^{i}\right)\circ\mathbf{D}_{\mathrm{p}}(n)\,|\,\psi\rangle = |\,\psi\rangle,\quad |\,\psi\rangle\in\mathcal{H}. \tag{4.37}$$

The full antipode is defined as $\mathbf{S}_{\mathrm{pq}} = \mathbf{S}_i(n)$, $i = 1,\ldots,n$ if all partial antipodes are equal.

For ith partial antipode from the definitions (4.23), (4.28) and (4.37) we obtain

$$\mathbf{S}_i(n) = \frac{1}{q_i p_i}\left(1 - \sum_{k=1,k\neq i}^{n} q_k p_k\right),\quad p_i, q_i \in \mathbb{C}. \tag{4.38}$$

To finally understand the general algebraic structure of the duality computing, we turn to further similarity with the polyadic Hopf algebra theory (Duplij 2022, chapter 9) and introduce an analog of the n-ary convolution for gates. Recall that the binary convolution product of two operators $\mathbf{T}_1$ and $\mathbf{T}_2$ in the bialgebra having multiplication μ and comultiplication Δ is defined by $\mu\circ(\mathbf{T}_1\otimes\mathbf{T}_2)\circ\Delta$ (Abe 1980, Sweedler 1969, Radford 2012).

Let $\vec{\mathbf{T}}$ be an operator (4.8) in the total Hilbert space $\vec{\mathcal{H}}$ (in the vector notation), $\mathbf{D}_{\mathrm{p}}$ and $\mathbf{C}_{\mathrm{q}}$ be the divider (4.23) and combiner (4.28).

Definition 4.13. The duality n-ary convolution of the vector operator $\vec{\mathbf{T}}$ is the composition

$$\mathbf{Q}_{\mathrm{dual}}^{(n),q,p}\left(\vec{\mathbf{T}}\right)=\mathbf{C}_{\mathrm{q}}(n)\circ\left(\overbrace{\mathbf{T}_1\oplus\mathbf{T}_2\oplus\cdots\oplus\mathbf{T}_n}^{n}\right)\circ\mathbf{D}_{\mathrm{p}}(n) = \mathbf{C}_{\mathrm{q}}(n)\circ\vec{\mathbf{T}}\circ\mathbf{D}_{\mathrm{p}}(n). \tag{4.39}$$

In this way, the antipode (4.37) can be treated as the polyadic inverse of the identity with respect to the n-ary convolution product (4.39)

Informally, the ordinary quantum computation process (on pure states) consists of:

(1) Preparation of the initial state $|\,\psi\rangle_{\mathrm{init}}$.

(2) Computation as consequent action on $|\,\psi\rangle$ with the set of k quantum gates being unitary operators $\{\mathbf{U}^{(1)}, \mathbf{U}^{(2)},\ldots,\mathbf{U}^{(k)}\}$ by their composition to obtain the final (still not measured) quantum state

$$|\,\psi\rangle_{\mathrm{fin}} = \mathbf{U}^{(k)}\circ\mathbf{U}^{(k-1)}\circ\cdots\circ\mathbf{U}^{(2)}\circ\mathbf{U}^{(1)}\,|\,\psi\rangle_{\mathrm{init}} = \mathbf{U}\,|\,\psi\rangle_{\mathrm{init}}. \tag{4.40}$$

(3) Measurement $\mathbf{M}$: $|\,\psi\rangle_{\mathrm{fin}}\longrightarrow|\,\psi\rangle_{\mathrm{measured}}$.

In the duality computation processing, one changes *(2)* and replaces (4.40) with the more complicated set of gates acting in subspaces. Indeed, if in each ith subspace we have k unitary gates

$$\vec{\mathbf{U}} = \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \vdots \\ \mathbf{U}_n \end{pmatrix}, \ \mathbf{U}_i = \mathbf{U}_i^{(k)} \circ \mathbf{U}_i^{(k-1)} \circ \cdots \circ \mathbf{U}_i^{(2)} \circ \mathbf{U}_i^{(1)}, \quad \mathbf{U}_i \colon \mathcal{H} \to \mathcal{H}, \quad i = 1, \ldots, n, \qquad (4.41)$$

then we obtain in total kn unitary gates.

Definition 4.14. Duality computation with n sub-slits is defined by kn unitary gates, vector unitary gates $\vec{\mathbf{U}}$ (instead of the standard unitary gate $\mathbf{U}$ (4.40) composed from k unitary gates)

$$|\,\psi\rangle_{\text{fin}} = \mathbf{Q}_{\text{dual}}^{(n),q,p}\left(\vec{\mathbf{U}}\right) |\,\psi\rangle_{\text{init}}, \qquad (4.42)$$

where the duality operator $\mathbf{Q}_{\text{dual}}^{(n),q,p}\left(\vec{\mathbf{U}}\right)$ is defined in (4.39).

Effectively, any operator connecting initial and final quantum states can be called a generalized quantum gate, but duality quantum gates are special nonunitary combinations of the given unitary operators.

Theorem 4.15. The duality quantum gate $\mathbf{Q}_{\text{dual}}^{(n),q,p}\left(\vec{\mathbf{U}}\right)$ is a nonunitary operator of the form

$$\mathbf{T}_{\text{dual}} = \mathbf{Q}_{\text{dual}}^{(n),q,p}\left(\vec{\mathbf{U}}\right) = q_1 p_1 \mathbf{U}_1 + q_2 p_2 \mathbf{U}_2 + \cdots + q_n p_n \mathbf{U}_n, \quad q_i, p_i \in \mathbb{C}. \qquad (4.43)$$

Proof. It follows directly from (4.42) the divider (4.23) and combiner (4.28).

Remark 4.16. The duality gate (4.43) is nonunitary, but with the special choice of probability distributions $\left\{\vec{Q}\right\}$ and $\left\{\vec{P}\right\}$ one could obtain the unitary $\mathbf{T}_{\text{dual}}$.

If the probability distributions satisfy (4.33), then the product $q_i p_i$ can be treated as the probability of the quantum wave to pass through the ith slit.

Corollary 4.17. If all components of $\vec{\mathbf{U}}$ are equal and (4.33) satisfied, then the duality quantum computer reduces to the ordinary quantum computer.

The case $q_i = \bar{p}_i$ was considered in Cao *et al* (2012), and such $\mathbf{T}_{\text{dual}}$ was called generalized duality quantum gate.

4.2 Higher duality computing

Now we propose another generalization of the duality computation (4.42), i.e., the higher duality one, by using higher powers of the divider (4.23) and combiner (4.28). Because they are polyadic (multiary) operations, to be consistent with arities and number of entries, we should use the polyadic powers for them (Duplij 2022). The main consequence of this would be the possibility of having different compositions of the divider $\mathbf{D}_{\mathrm{p}}(n_p)$ (4.23) and combiner $\mathbf{C}_{\mathrm{q}}(n_{\mathrm{q}})$ (4.28) with different arities $n_p \neq n_{\mathrm{q}}$, such that the number of slits remains to be equal to n.

Definition 4.18. The polyadic power ℓ_{p} of the divider $\mathbf{D}_{\mathrm{p}}(n_{\mathrm{p}})$ (4.23) as n_{p}-ary comultiplication (coaddition) is defined by the coiterated coaction (or ℓ_{p} compositions)

$$\left(\mathbf{D}_{\mathrm{p}}(n_{\mathrm{p}})\right)^{\circ\ell_{\mathrm{p}}} = \Bigg(\Bigg(\Bigg(\overbrace{\mathrm{id} \oplus \cdots \oplus \mathrm{id}}^{n_{\mathrm{p}}-1} \oplus \cdots \Bigg(\overbrace{\mathrm{id} \oplus \cdots \oplus \mathrm{id}}^{n_{\mathrm{p}}-1} \oplus \overbrace{\mathbf{D}_{\mathrm{p}}(n_{\mathrm{p}})\big)\cdots\circ\mathbf{D}_{\mathrm{p}}(n_{\mathrm{p}})\big)\circ\mathbf{D}_{\mathrm{p}}(n_{\mathrm{p}})}^{\ell_{\mathrm{p}}}, \tag{4.44}$$

$$\mathrm{id}\colon \mathcal{H} \to \mathcal{H}, \ \ \mathbf{D}_{\mathrm{p}}(n_{\mathrm{p}})\colon \mathcal{H} \to \mathcal{H}^{\oplus n_{\mathrm{p}}},$$

and the arity n'_{q} of the composed operation (4.46) is equal to

$$n'_{\mathrm{p}} = \ell_{\mathrm{p}}\big(n_{\mathrm{p}} - 1\big) + 1. \tag{4.45}$$

Definition 4.19. The polyadic power ℓ_{q} of the combiner (4.28) as n_{q}-ary multiplication (addition) is defined by the iterated action (or ℓ_{q} compositions)

$$\left(\mathbf{C}_{\mathrm{q}}(n_{\mathrm{q}})\right)^{\circ\ell_{\mathrm{q}}} = \overbrace{\mathbf{C}_{\mathrm{q}}(n_{\mathrm{q}})\circ\big(\mathbf{C}_{\mathrm{q}}(n_{\mathrm{q}})\circ\cdots\big(\mathbf{C}_{\mathrm{q}}(n_{\mathrm{q}})}^{\ell_{\mathrm{q}}} \oplus \overbrace{\mathrm{id} \oplus \cdots \oplus \mathrm{id}}^{n_{\mathrm{q}}-1}\Big)\cdots\oplus\overbrace{\mathrm{id} \oplus \cdots \oplus \mathrm{id}}^{n_{\mathrm{q}}-1}\Big)\Bigg|, \tag{4.46}$$

$$\mathrm{id}\colon \mathcal{H} \to \mathcal{H}, \ \ \mathbf{C}_{\mathrm{q}}(n_{\mathrm{q}})\colon \mathcal{H}^{\oplus n_{\mathrm{q}}} \to \mathcal{H},$$

and the arity n'_{q} of the composed operation (4.46) is equal to

$$n'_{\mathrm{q}} = \ell_{\mathrm{q}}\big(n_{\mathrm{q}} - 1\big) + 1. \tag{4.47}$$

Note that the brackets in the polyadic powers (4.46) and (4.44) can be omitted if operations $\mathrm{C}_{\mathrm{q}}(n_{\mathrm{q}})$ and $\mathbf{D}_{\mathrm{p}}(n_{\mathrm{p}})$ are totally associative and coassociative, respectively.

Now, by analogy with (4.33), the composition of ℓ_{p} power of divider and ℓ_{q} power of combiner can be the identity operator in $\mathcal{H}$ but not for all divider and combiner arities n_{p} and n_{q}.

Proposition 4.20. The composition of ℓ_{p} power of divider and ℓ_{q} power of combiner can be the identity operator in $\mathcal{H}$ if their arities satisfy

$$\mathbf{C}_{\mathrm{q}}(n_{\mathrm{q}})^{\circ\ell_{\mathrm{q}}}\circ\mathbf{D}_{\mathrm{p}}(n_{\mathrm{p}})^{\circ\ell_{\mathrm{p}}} = \mathrm{id}, \tag{4.48}$$

$$\ell_{\mathrm{p}}\left(n_{\mathrm{p}} - 1\right) = \ell_{\mathrm{q}}\left(n_{\mathrm{q}} - 1\right). \tag{4.49}$$

Proof. It follows directly from the definitions (4.23) and (4.28) that the arities of the powers should coincide $n'_{\mathrm{p}} = n'_{\mathrm{q}}$ and the arity formulas (4.45), (4.47).

Equation (4.49) can be treated as so called quantization of arities, and therefore the analog of (4.43) depends of their concrete values.

Example 4.21. Let us consider the minimal case of nonbinary ($n_{\mathrm{p,q}} > 2$) and unequal arities $n_{\mathrm{p}} = 3$, $\ell_{\mathrm{p}} = 3$ and $n_{\mathrm{q}} = 4$, $\ell_{\mathrm{q}} = 2$, and the total number of slits $\ell_{\mathrm{p}}\left(n_{\mathrm{p}} - 1\right) + 1 = \ell_{\mathrm{q}}\left(n_{\mathrm{q}} - 1\right) + 1 = 7$. Then for the values of the polyadic powers of divider (4.44) and combiner (4.46) we have manifestly using (4.27) and (4.22)

$$\mathbf{D}_{\mathrm{p}}(3)^{\circ 3} \mid \psi\rangle = \begin{pmatrix} p_1 \mid \psi\rangle \\ p_2 \mid \psi\rangle \\ p_1 p_3 \mid \psi\rangle \\ p_2 p_3 \mid \psi\rangle \\ p_1 p_3^2 \mid \psi\rangle \\ p_2 p_3^2 \mid \psi\rangle \\ p_3^3 \mid \psi\rangle \end{pmatrix}, \tag{4.50}$$

$$\mathbf{C}_{\mathrm{q}}(4)^{\circ 2} \left| \vec{\Psi}(7) \right\rangle = q_1(q_1 \mid \psi_1\rangle + q_2 \mid \psi_2\rangle + q_3 \mid \psi_3\rangle + q_4 \mid \psi_4\rangle) + q_2 \mid \psi_5\rangle + q_3 \mid \psi_6\rangle + q_4 \mid \psi_7\rangle. \tag{4.51}$$

The condition that the composition of the powers (4.50) and (4.51) to be the identity (4.48) gives the equation for the probability distributions $\left\{\vec{Q}(4)\right\} = \{q_1, q_2, q_3, q_4\}$ and $\left\{\vec{P}(3)\right\} = \{p_1, p_2, p_3\}$

$$q_1^2 p_1 + q_1 q_2 p_2 + q_1 q_3 p_1 p_3 + q_1 q_4 p_2 p_3 + q_2 p_1 p_3^2 + q_3 p_2 p_3^2 + q_4 p_3^3 = 1, \tag{4.52}$$

which is nonlinear in q_i, $_{pi}$ and should be compared with the standard linear case of unity powers (4.33).

Let us introduce the higher analog of the duality n-ary operator $\mathbf{Q}_{\mathrm{dual}}^{(n),\mathrm{q,p}}$ (4.39).

Definition 4.22. The higher duality n-ary convolution of the vector operator $\vec{\mathbf{T}} \in \vec{\mathcal{H}}$ is the composition of the ℓ_{p} dividers (4.44) and ℓ_{q} combiners (4.46) which maps $\mathcal{H} \to \mathcal{H}$

$$\mathbf{Q}_{\mathrm{hdual},\, \ell_{\mathrm{q}},\, \ell_{\mathrm{p}}}^{(n),\mathrm{q,p}}\left(\vec{\mathbf{T}}\right) = \mathbf{C}_{\mathrm{q}}^{\circ \ell_{\mathrm{q}}}(n_{\mathrm{q}}) \circ \left(\overbrace{\mathbf{T}_1 \oplus \mathbf{T}_2 \oplus \cdots \oplus \mathbf{T}_n}^{n} \right) \circ \mathbf{D}_{\mathrm{p}}^{\circ \ell_{\mathrm{p}}}(n_{\mathrm{p}}) = \mathbf{C}_{\mathrm{q}}^{\circ \ell_{\mathrm{q}}}(n_{\mathrm{q}}) \circ \vec{\mathbf{T}} \circ \mathbf{D}_{\mathrm{p}}^{\circ \ell_{\mathrm{p}}}(n_{\mathrm{p}}), \tag{4.53}$$

where the number of slits n is equal to

$$n = \ell_{\mathrm{p}}\left(n_{\mathrm{p}} - 1\right) = \ell_{\mathrm{q}}\left(n_{\mathrm{q}} - 1\right). \tag{4.54}$$

Definition 4.23. The higher duality computation with n sub-slits is defined by kn unitary gates, vector unitary gates $\vec{\mathbf{U}}$ (4.41)

$$\mid \psi\rangle_{\mathrm{fin}} = \mathbf{Q}^{(n),\mathrm{q,p}}_{\mathrm{hdual},\, \ell_{\mathrm{q}},\, \ell_{\mathrm{p}}}\left(\vec{\mathbf{U}}\right) \mid \psi\rangle_{\mathrm{init}}, \tag{4.55}$$

where the duality operator $\mathbf{Q}^{(n),\mathrm{q,p}}_{\mathrm{hdual},\, \ell_{\mathrm{q}},\, \ell_{\mathrm{p}}}\left(\vec{\mathbf{U}}\right)$ is defined in (4.53).

Theorem 4.24. The $\left(\ell_{\mathrm{q}}, \ell_{\mathrm{p}}\right)$-higher duality quantum gate $\mathbf{Q}^{(n),\mathrm{q,p}}_{\mathrm{hdual},\, \ell_{\mathrm{q}},\, \ell_{\mathrm{p}}}\left(\vec{\mathbf{U}}\right)$ is a non-unitary operator of the form

$$\mathbf{T}_{\mathrm{hdual}} = \mathbf{Q}^{(n),\mathrm{q,p}}_{\mathrm{hdual},\, \ell_{\mathrm{q}},\, \ell_{\mathrm{p}}}\left(\vec{\mathbf{U}}\right) = \mathbf{C}^{\circ \ell_{\mathrm{q}}}_{\mathrm{q}}(n) \circ \vec{\mathbf{U}} \circ \mathbf{D}^{\circ \ell_{\mathrm{p}}}_{\mathrm{p}}(n), \quad \mathbf{T}_{\mathrm{hdual}}\colon \mathcal{H} \to \mathcal{H}. \tag{4.56}$$

It is important that not all possible values of arities and powers are allowed, but only those which satisfy the quantization condition (4.54). The allowed number of slits n and corresponding n_{q}, n_{p} and ℓ_{q}, ℓ_{p} are presented in table 4.1. Note that the unusual peculiarity comes from the nondiagonal entries, which correspond to unequal arities of divider and combiner $n_{\mathrm{p}} \neq n_{\mathrm{q}}$. The table is symmetric, which means that the arity n (number of slits) is invariant under the exchange $\left(n_{\mathrm{p}}, \ell_{\mathrm{p}}\right) \longleftrightarrow \left(n_{\mathrm{q}}, \ell_{\mathrm{q}}\right)$ following from (4.54).

Table 4.1. The allowed valued of slits for given arities n_{p}, n_{q} and polyadic powers (or numbers of divider and combiner compositions) ℓ_{p}, ℓ_{q}. The framed box corresponds to the binary standard duality convolution (4.39) with two slits $n = 2$.

n_q n_p	$\mathbf{C}_q$	$n_q = 2$			$n_q = 3$			$n_q = 4$			$n_q = 5$		
$\mathbf{D}_p$	ℓ_q ℓ_p	$\ell_q = 1$	$\ell_q = 2$	$\ell_q = 3$	$\ell_q = 1$	$\ell_q = 2$	$\ell_q = 3$	$\ell_q = 1$	$\ell_q = 2$	$\ell_q = 3$	$\ell_q = 1$	$\ell_q = 2$	$\ell_q = 3$
$n_p = 2$	$\ell_p = 1$	$\boxed{\mathbf{2}}$											
	$\ell_p = 2$		**3**		**3**								
	$\ell_p = 3$			**4**				**4**					
$n_p = 3$	$\ell_p = 1$		**3**		**3**								
	$\ell_p = 2$					**5**					**5**		
	$\ell_p = 3$						**7**		**7**				
$n_p = 4$	$\ell_p = 1$			**4**				**4**					
	$\ell_p = 2$						**7**		**7**				
	$\ell_p = 3$									**10**			
$n_p = 5$	$\ell_p = 1$					**5**					**5**		
	$\ell_p = 2$											**9**	
	$\ell_p = 3$												**13**

Example 4.25. (*Example* 4.21 continued). With the concrete parameters (4.50) and (4.51), we have the 7-ary convolution product with three dividers and two combiners

$$\begin{aligned}\mathbf{T}_{\mathrm{hdual}} &= \mathbf{Q}^{(n),\mathrm{q},\mathrm{p}}_{\mathrm{hdual},\,\ell_{\mathrm{q}},\,\ell_{\mathrm{p}}}\left(\vec{\mathbf{U}}\right) = \mathbf{C}_{\mathrm{q}}(4)^{\circ 2}\circ\vec{\mathbf{U}}(7)\circ\mathbf{D}_{\mathrm{p}}(3)^{\circ 3}\mid\psi\rangle \\ &= q_1^2 p_1\mathbf{U}_1 + q_1 q_2 p_2\mathbf{U}_2 + q_1 q_3 p_1 p_3\mathbf{U}_3 + q_1 q_4 p_2 p_3\mathbf{U}_4 + q_2 p_1 p_3^2\mathbf{U}_5 + q_3 p_2 p_3^2\mathbf{U}_6 + q_4 p_3^3\mathbf{U}_7.\end{aligned} \tag{4.57}$$

Effectively, a nonunitary operator connecting initial and final quantum states can be called a higher generalized quantum gate, i.e., higher duality quantum gate. Indeed, to continue *example* 4.21 and 4.25, and consider a nonunitary generalized quantum gate

$$\mathbf{T}(r) = r_1\mathbf{U}_1 + r_2\mathbf{U}_2 + r_3\mathbf{U}_3 + r_4\mathbf{U}_4 + r_5\mathbf{U}_5 + r_6\mathbf{U}_6 + r_7\mathbf{U}_7, \tag{4.58}$$

$$\mid r_1\mid + \mid r_2\mid + \mid r_3\mid + \mid r_4\mid + \mid r_5\mid + \mid r_6\mid + \mid r_7\mid = 1, \qquad r_i\in\mathbb{C}. \tag{4.59}$$

If the parameters r_i are given, we can find the corresponding higher duality $n = 7$ slits quantum gate (4.57) with the composition of three (ternary) dividers (4.50) and two (4-ary) combiners (4.51), which have the following probability distributions $\left\{\vec{P}(3)\right\}$ and $\left\{\vec{Q}(4)\right\}$, where, e.g.,

$$p_1 = \frac{1}{r_2}r_4\frac{r_5}{r_7},\quad p_2 = \frac{r_1}{r_2 r_3^2}r_4 r_5\frac{r_6}{r_7}\sqrt[6]{\frac{1}{r_1^5}r_2^3\frac{r_3^8}{r_4 r_5^3 r_6^2}r_7},\quad p_3 = \frac{r_1^2}{r_2 r_3^3}r_5 r_6\sqrt[3]{\frac{1}{r_1^5}r_2^3\frac{r_3^8}{r_4 r_5^3 r_6^2}r_7}, \tag{4.60}$$

$$q_1 = \frac{r_1^3}{r_2 r_3^4}r_5 r_6\sqrt{\frac{1}{r_1^5}r_2^3\frac{r_3^8}{r_4 r_5^3 r_6^2}r_7},\qquad q_2 = \frac{r_1}{r_3^2}r_5\sqrt[3]{\frac{1}{r_1^5}r_2^3\frac{r_3^8}{r_4 r_5^3 r_6^2}r_7}, \tag{4.61}$$

$$q_3 = \sqrt[6]{\frac{1}{r_1^5}r_2^3\frac{r_3^8}{r_4 r_5^3 r_6^2}r_7},\qquad q_4 = \frac{1}{r_1}r_3\frac{r_4}{r_6}, \tag{4.62}$$

and there are five other more cumbersome solutions. We should also take (4.59) into account, which gives (4.52).

Let us consider the reverse convolution (with respect to (4.39)) by the divider and combiner of an operator in $\mathcal{H}$.

Definition 4.26. The duality reverse n-ary convolution of the operator $\mathbf{T}$: $\mathcal{H}\to\mathcal{H}$ by the divider $\mathbf{D}_{\mathrm{p}}(n)$ and combiner $\mathbf{C}_{\mathrm{q}}(n)$ is the composition

$$\mathbf{P}^{(n),\mathrm{q},\mathrm{p}}_{\mathrm{rdual}}(\mathbf{T}) = \mathbf{D}_{\mathrm{p}}(n)\circ\mathbf{T}\circ\mathbf{C}_{\mathrm{q}}(n). \tag{4.63}$$

Proposition 4.27. The action of the duality reverse n-ary convolution with the probability distributions $\left\{\vec{Q}\right\}$ and $\left\{\vec{P}\right\}$ on the vector quantum state $\left|\vec{\Psi}_{=}(n)\right\rangle \in \mathcal{H}$ is

$$\mathbf{P}^{(n),\mathrm{q,p}}_{\mathrm{rdual}}(\mathbf{T})\left|\vec{\Psi}_{=}(n)\right\rangle = \left(\vec{P}\vec{Q}^{T}\right)\otimes_{\mathrm{K}}\hat{T}\left|\vec{\Psi}_{=}(n)\right\rangle, \tag{4.64}$$

where $\otimes_{\mathrm{K}}$ is the Kronecker product of the matrix $\vec{P}\vec{Q}^{T} \in M_{n\times n}(\mathbb{C})$ and $\hat{T}$ is the matrix of the operator $\mathbf{T}$ in its matrix representation.

Proof. It follows from the manifest form of the divider (4.22) and combiner (4.27) and their linearity.

Note that if the duality reverse n-ary convolution (4.63) could be identity (for $\mathbf{T} = \mathrm{id}$), then together with (4.33) the divider $\mathbf{D}_{\mathrm{p}}(n)$ and combiner $\mathbf{C}_{\mathrm{q}}(n)$ become an n-ary analog of the biproduct in category theory (Mac Lane 1971). However, the condition that is needed for this condition ($\vec{P}\vec{Q}^{T}$ is the identity matrix) is never satisfied for nonvanishing probability distributions.

4.3 Duality quantum mode

The duality computer can be simulated by the ordinary quantum computer in a special work mode, i.e., having an additional/auxiliary qubit (or qudit) (Wei *et al* 2016). The main idea of the duality quantum mode computer is to provide the one-to-one correspondence of the auxiliary qudit state with the unitary operations on the slits (Long 2011).

The total state of the k-qubits $|\psi\rangle_{\mathrm{init}} \in \mathcal{H}$ and one auxiliary qudit (n-dit) representing n-slits $|\varphi\rangle_{\mathrm{aux}} \in \mathcal{H}$ is the direct product $|\psi\rangle_{\mathrm{init}} \otimes |\varphi\rangle_{\mathrm{aux}}$. The divider operation is represented by the unitary operator $\mathbf{V}$ acting on qudit, while the combider operation corresponds to the unitary operator $\mathbf{W}$ acting on qudit $|\varphi\rangle_{\mathrm{aux}}$. Between $\mathbf{V}$ and $\mathbf{W}$ there are n controlled operations corresponding to $\mathbf{U}_1, \mathbf{U}_2\cdots\mathbf{U}_n$ one-to-one related to the states of qudit $\mathbf{U}_i\longleftrightarrow|i-1\rangle$. The whole duality quantum mode process can be presented as four consequent steps.

(1) Action of $\mathbf{V}$ on $|0\rangle_{\mathrm{aux}}$. Prepare the initial quantum state, e.g., with $|\varphi\rangle_{\mathrm{aux}} = |0\rangle_{\mathrm{aux}} = |0\rangle$, as $|\psi\rangle_{\mathrm{init}} \otimes |0\rangle$. The divider operation for n slits corresponds to the acting of the unitary operator $\mathbf{V}$ of the qudit (n-dit) state $|0\rangle$ as

$$|0\rangle \overset{\mathbf{V}}{\mapsto} \mathbf{V}|0\rangle = \overbrace{\left(\sum_{i=1}^{n}|i-1\rangle\langle i-1|\right)}^{\mathrm{id}}\mathbf{V}|0\rangle = \sum_{i=1}^{n}V_{i,1}|i-1\rangle, \tag{4.65}$$

where $V_{i,1} = \langle i-1|\mathbf{V}|0\rangle \in \mathbb{C}$ is the convolution of operator $\mathbf{V}\colon \mathcal{H}\to\mathcal{H}$ between the states $|i-1\rangle$ and $|0\rangle$, i.e., its matrix element, which represents the divider n-ary structure $\left\{\vec{P}(n)\right\}$ (4.23) after the identification

$$p_i = V_{i,1} = \langle i-1 \mid \mathbf{V} \mid 0\rangle. \tag{4.66}$$

Obviously, by definition

$$\sum_{i=1}^{n} \mid V_{i,1} \mid = \sum_{i=1}^{n} \mid p_i \mid^2 = 1, \tag{4.67}$$

as it should be for probabilities. Thus, using (4.65) and (4.66) the final sub-state $\mid \psi\rangle_{\text{init}} \otimes \mid i-1\rangle$ corresponds to the ith slit sub-wave, and

$$\mid \psi\rangle_{\text{init}} \otimes \mid 0\rangle \overset{\text{id}\otimes\mathbf{V}}{\mapsto} \mid \psi\rangle_{\text{init}} \otimes \sum_{i=1}^{n} p_i \mid i-1\rangle = \sum_{i=1}^{n} p_i \mid \psi\rangle_{\text{init}} \otimes \mid i-1\rangle. \tag{4.68}$$

(2) Action of $\mathbf{U}_i$ on $\mid \psi\rangle_{\text{init}}$. The auxiliary controlled operation means that the action of the unitary operator $\mathbf{U}_i$ on $\mid \psi\rangle_{\text{init}}$ will be applied to the ith slit only, i.e., to the ith summand inside the last term (4.68). This is the reason why the same (for each i) initial state $\mid \psi\rangle_{\text{init}}$ was inserted into the sum. Therefore,

$$\sum_{i=1}^{n} p_i \mid \psi\rangle_{\text{init}} \otimes \mid i-1\rangle \overset{\mathbf{U}_i\otimes\text{id}}{\mapsto} \sum_{i=1}^{n} p_i (\mathbf{U}_i \mid \psi\rangle_{\text{init}}) \otimes \mid i-1\rangle. \tag{4.69}$$

(3) Action of $\mathbf{W}$ on the state $\mid i-1\rangle$. By analogy with (4.65) for each ith slit we have

$$\mid i-1\rangle \overset{\mathbf{W}}{\mapsto} \mathbf{W} \mid i-1\rangle = \overbrace{\left(\sum_{j=1}^{n} \mid j-1\rangle\langle j-1 \mid\right)}^{\text{id}} \mathbf{W} \mid i-1\rangle = \sum_{j=1}^{n} W_{j,i} \mid j-1\rangle, \tag{4.70}$$

where $W_{j,i} = \langle j-1 \mid \mathbf{W} \mid i-1\rangle \in \mathbb{C}$ is the convolution of operator (being a representative of the combiner) $\mathbf{W}\colon \mathcal{H} \to \mathcal{H}$ between the states $\mid j-1\rangle$ and $\mid i-1\rangle$, i.e., its matrix element, which represents (for fixed ith slit) the probabilities

$$q_j^{(i)} = W_{j,i} = \langle j-1 \mid \mathbf{W} \mid i-1\rangle \in \mathbb{C}. \tag{4.71}$$

Now the normalization condition is

$$\sum_{i=1}^{n} \mid W_{j,i} \mid = \sum_{i=1}^{n} \left| q_j^{(i)} \right|^2 = 1, \quad \forall j = 1,\ldots,n, \tag{4.72}$$

as it should be by definition of probability. Then, using (4.69) and (4.70), the final quantum state will take the form

$$
\begin{aligned}
|\psi\rangle_{\text{init}} \otimes |0\rangle &\mapsto \sum_{i=1}^{n} p_i \mathbf{U}_i |\psi\rangle_{\text{init}} \otimes \sum_{j=1}^{n} q_j^{(i)} |j-1\rangle \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} p_i q_j^{(i)} \mathbf{U}_i (|\psi\rangle_{\text{init}} \otimes |j-1\rangle) = \sum_{j=1}^{n} \mathbf{T}_{\text{qdual}}^{(j)} |\psi\rangle_{\text{init}} \otimes |j-1\rangle,
\end{aligned}
\tag{4.73}
$$

where

$$\mathbf{T}_{\text{qdual}}^{(j)} = \sum_{i=1}^{n} q_j^{(i)} p_i \mathbf{U}_i \tag{4.74}$$

represents the duality gate in ordinary quantum computer.

(4) Complete measurement. After the previous three steps the auxiliary qubit arrives into the superposition state; therefore, the ith detector is placed at ith slit, when the qudit wave function is in the final state $|i-1\rangle$.

Consider the properties of the duality gate operator $\mathbf{T}_{\text{qdual}}^{(j)}$ (4.74). The condition

$$\left| \sum_{i=1}^{n} q_j^{(i)} p_i \right| \leqslant 1, \quad \forall j = 1, \ldots, n, \tag{4.75}$$

leads to the allowable duality gates (Long *et al* 2009). Because of the Cauchy–Bunyakovsky–Schwarz inequality and unitarity of the operators **V** and **W**, together with (4.67) and (4.72), we have

$$\sum_{i=1}^{n} \left| q_j^{(i)} p_i \right| \leqslant 1, \quad \forall j = 1, \ldots, n. \tag{4.76}$$

The duality gate (4.74) with the condition (4.76) is called the restricted allowable generalized quantum gate (Long *et al* 2009, Cao *et al* 2010).

Now we present the higher duality computation on the ordinary quantum computer. To model the polyadic power of the divider (4.44), we introduce the higher analog of the unitary operator action on the qudit state (4.65) and call it the duality power denoted by $(\ell_{\circledast})$. In this case the number of slits n does not coincide with the number of vectors of qudit n_d, and they are related by the formula of polyadic power (4.54) as follows

$$n = \ell_{\circledast}(n_d - 1) + 1. \tag{4.77}$$

We propose the following definition of the duality power, which is consistent with the polyadoc power (4.44). For instance, in case of quadratic duality power, instead of (4.65), we have for the n slits and n_d-dit the general formula

$$|0\rangle \overset{\mathbf{V}^{\circledast 2}}{\mapsto} \mathbf{V}^{\circledast 2} |0\rangle = \sum_{i=1}^{n_d} \left[\delta_{i,1} V_{i,1} \left(\sum_{j=1}^{n_d} V_{j,1} |j-1\rangle \right) + (1 - \delta_{i,1}) V_{i,1} |i-1\rangle \right], \tag{4.78}$$

$$n = 2n_d - 1, \tag{4.79}$$

where $\delta_{i,j}$ is the ordinary delta function, and $V_{i,1}$ is defined in (4.66). For the ternary duality power case $\ell_{\circledast} = 3$, to avoid cumbersome formulas, we give the concrete example $n_d = 3$, $n = 7$ from which the general pattern is clearly seen (cf (4.50))

$$\mid 0 \rangle \overset{\mathbf{V}^{\circledast 3}}{\mapsto} \mathbf{V}^{\circledast 3} \mid 0 \rangle = p_1 \big[p_1 (p_1 \mid 0 \rangle + p_2 \mid 1 \rangle + p_3 \mid 2 \rangle) + p_2 \mid 1 \rangle + p_3 \mid 2 \rangle \big] + p_2 \mid 1 \rangle + p_3 \mid 2 \rangle, \tag{4.80}$$

where $p_i = V_{i,1}$ and $V_{i,j} = \langle i - 1 \mid \mathbf{V} \mid j - 1 \rangle$ is the matrix element of the unitary operator $\mathbf{V}$, $i, j = 1,\ldots,n_d$.

The other steps *(2)–(4)* in the duality quantum mode computation can be taken to be the same. This will lead to the higher nonlinear (in terms of the matrix elements of the operators $\mathbf{V}$ and $\mathbf{W}$) duality version of computation with the generalized nonunitary quantum gates of the form (4.74).

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Chapter 5

Measurement-based quantum computing

The measurement-based quantum computation model (Raussendorf and Briegel 2001, Raussendorf *et al* 2003) is a counterpart of the standard circuit model grounded on the unitary evolution (Deutsch 1989, Deutsch and Jozsa 1992). In the latter, the measurement is provided at the end of the whole computation to get the classical output, while in the measurement-based computation the principal operation is the measurement itself. Informally, the computation starts with several entangled qubits and measurements act on each qubit separately. The result is then exploited for next measurements. To avoid measurement indeterminacy, local unitary operations (named corrections) are implemented, which give the one-way computation (Raussendorf and Briegel 2001).

In general, the computation or the measurement pattern (program) consists of input and output sets of qubits connected with the sequence of basic commands. The patterns are then merged using tensor products and compositions (Danos *et al* 2007). The one-qubit measurement-based basic commands are:

1. *Preparation* N_i of qubit i in the state $|+\rangle_i = (|0\rangle + |1\rangle)/\sqrt{2}$ (set N);
2. *Entanglement* $E_{ij} = \wedge Z_{ij}$ (controlled-Z) of two qubits i and j (set E);
3. *Measurement* $M_i(\alpha)$ of qubit i defined by projections on $|\pm_\alpha\rangle_i = (|0\rangle \pm e^{i\alpha}|1\rangle)/\sqrt{2} = P(\alpha)|\pm\rangle$, where $P(\alpha) = \operatorname{diag}(1, e^{i\alpha})$ is the phase operator, $0 \leqslant \alpha \leqslant 2\pi$ is the angle of measurement (set M);
4. *Corrections* being the Pauli operators X_i and Z_i (set C).

The result of the measurement provided at qubit i is presented by *outcomes* $s_i = 0, 1 \in \mathbb{Z}_2$, where the convention is $s_i = 0$, if the initial qubit $|+\rangle_i$ after the measurement becomes $|+_\alpha\rangle_i$, while $s_i = 1$, if $|+\rangle_i$ collapses into $|-_\alpha\rangle_i$. After the measurement, the set of initial qubits $\{i\}$ produces the sum (in $\mathbb{Z}_2$) of individual outcomes $\sum s_i = s$, which is named *signal*, while the set $\{i\}$ is called the *domain* of the signal.

doi:10.1088/978-0-7503-5281-9ch5

The main idea of this method is some additional functional dependence of the corrections $X_i \to X_i^{(s)}$, $Z_i \to Z_i^{(s)}$ and measurements $M_i(\alpha) \to M_i^{(s,t)}(\alpha)$ from signals s, t. Because the signals are in $\mathbb{Z}_2$, the dependences from them become simple discrete functions

$$X_i^{(s)} = \begin{cases} I, & s = 0, \\ X_i, & s = 1 \end{cases}, \qquad Z_i^{(s)} = \begin{cases} I, & s = 0, \\ Z_i, & s = 1 \end{cases}, \tag{5.1}$$

$$M_i^{(s,t)}(\alpha) = M_i((-1)^s\alpha + \pi t) = \begin{cases} M_i(\alpha), & s = 0,\ t = 0, \\ M_i(-\alpha), & s = 1,\ t = 0, \\ M_i(\alpha + \pi), & s = 0,\ t = 1, \\ M_i(-\alpha + \pi), & s = 1,\ t = 1 \end{cases}. \tag{5.2}$$

The signal modification of measurements (5.2) can be expressed by the X- and Z-actions $\mathbf{A}_X$ and $\mathbf{A}_Z$ of conjugation with the Pauli matrices defined by (no summation)

$$\mathbf{A}_X \circ M_i(\alpha) \equiv X_i M_i(\alpha) X_i = M_i(-\alpha) = M_i^{(1,\,0)}(\alpha), \tag{5.3}$$

$$\mathbf{A}_Z \circ M_i(\alpha) \equiv Z_i M_i(\alpha) Z_i = M_i(\alpha + \pi) = M_i^{(0,\,1)}(\alpha). \tag{5.4}$$

The actions (5.3) and (5.4) commute because the addition of angles α is mod 2π. Since the measurements are destructive, the actions (5.3) and (5.4) can be simplified as follows

$$M_i(\alpha) X_i = M_i(-\alpha), \tag{5.5}$$

$$M_i(\alpha) Z_i = M_i(\alpha - \pi). \tag{5.6}$$

The signal domains of dependent commands give the set of such measurements that should be made before determination of the actual command value.

In general, the measurement pattern $\mathscr{P}$ is defined as

1. Three sets:
    - a. The computation space V of qubits and the associated quantum state space $\mathcal{H}$, which is $\otimes_{i\in \mathsf{V}}\mathbb{C}^2$.
    - b. The pattern inputs $\mathsf{In} \in \mathsf{V}$ and outputs $\mathsf{Out} \in \mathsf{V}$ sets, together with their associated quantum state spaces $\mathcal{H}_{\mathsf{In}}$ and $\mathcal{H}_{\mathsf{Out}}$, correspondingly.
2. Two injective maps $\mathbf{i}$: $\mathsf{In} \to \mathsf{V}$ and $\mathbf{o}$: $\mathsf{Out} \in \mathsf{V}$.
3. The finite sequence of n commands $A_1, A_2, \ldots, A_{n-1}, A_n$ which act from the left to the right as $A_n A_{n-1} \cdots A_2 A_1$ on the pattern inputs from the set In.

In this notation, the measurement pattern becomes the map $\mathbf{P}$: $\mathsf{In} \to \mathsf{Out}$ and the pattern type is denoted by (V, In, Out). To simplify the notation, the sets of states

i(In) and **o**(Out) are denoted by the same letters In and Out, correspondingly. Providing a consequent pattern computation needs four conditions of definiteness

- **Def0** If outcome state is not measured, then no commands depend on it.
  Otherwise, one tries to apply a command depending on an outcome that is not known.
- **Def1** If a qubit is measured, then no commands act on it.
  If not, then one tries to execute a command on an already measured, and therefore changed, qubit.
- **Def2** If a qubit is not the input one but not prepared, then no commands act on it.
  Otherwise, one tries to apply a command to a not existing qubit.
- **Def3** If a qubit is not output, then it can be measured.
  In other words, since measurement consumes the qubits, this statement makes sure that the final state is in the output state.

If all of the statements are satisfied, then the conjunction **Def** = **Def0** ∧**Def1** ∧ **Def2** ∧ **Def3** will be used. It is important that a given pattern should satisfy **Def**, in general. The case when one exploits neither input not output qubits in a pattern corresponds to the auxiliary qubits, which considerably enlarges the space computation complexity. To avoid this, one should use as small number of the auxiliary qubits as possible. Moreover, one assumes that the inputs In and outputs Out can intersect, which can lead to simple unitaries implementations (Danos *et al* 2005).

The combination of patterns can be provided in two ways:

Composition If for two patterns defined by (V_1, In_1, Out_l) and (V_2, In_2, Out_2) we have $V_1 \cap V_2 = Out_1 = In_2$, then the composition $\mathscr{P} = \mathscr{P}_2 \circ \mathscr{P}_1$ can be given by (V, In, Out), where

$$V = V_1 \bigcup V_2, \quad In = In_1, \quad Out = Out_2, \tag{5.7}$$

and the commands are concatenated consequently.

Tensor product If the sets do not intersect $V_1 \cap V_2 = \varnothing$, then we can construct the tensor product pattern $\mathscr{P} = \mathscr{P}_2 \otimes \mathscr{P}_1$ which is defined by

$$V = V_1 \bigcup V_2, \quad In = In_1 \bigcup In_2, \quad Out = Out_1 \bigcup Out_2. \tag{5.8}$$

Since the sets V_1 and V_2 are disjoint, the commands from different patterns commute and are applied for qubits from different sets independently.

If V is initially not measured qubits (and then still active), then we denote the set V* as the measured qubits, which become classical bits. Therefore, the computation state space is

$$S = \Sigma_{VV_*} \mathcal{H}_V \times \mathbb{Z}_2^{V*}, \tag{5.9}$$

where $\mathbb{Z}_2^{\mathsf{V}^*}$ is outcome space of bits. We denote $\mathbf{\Gamma}\colon \mathsf{V}^*\rightarrow\mathbb{Z}_2$, and then the space (5.9) becomes the set of the quadruples

$$\mathsf{S} = \{(\mathsf{V}, \mathsf{V}^*, q, \mathbf{\Gamma})\} \tag{5.10}$$

(for short notation, the set of pairs $\mathsf{S} = \{(q, \mathbf{\Gamma})\}$), where $q \in \mathcal{H}_{\mathsf{V}}$ is a quantum state. The value of signals given by $\mathbf{\Gamma}$ are defined as $s_\Gamma = \sum_{i\in\mathsf{In}}\mathbf{\Gamma}(i)$ with the sum in $\mathbb{Z}_2$. If the outcome is empty, then the notation $\mathbb{Z}_2^{\varnothing}$ will be used. Some modification of the outcome map can be defined as follows

$$\mathbf{\Gamma}[k/i](j) = \begin{cases} k, & i = j \\ \mathbf{\Gamma}(j), & i \neq j, \end{cases}, \quad i, j, k \in \mathbb{Z}_2. \tag{5.11}$$

which maps $\mathbb{Z}_2^{\mathsf{V}^*\cup\{i\}} \rightarrow \mathbb{Z}_2$. We can then write the action of the commands on the computation space (5.9) as (suppressing V, V^*)

$$\begin{tikzcd}
 & (q \otimes |+\rangle_i, \mathbf{\Gamma}) & \\
(Y_i^{(s_\Gamma)}q, \mathbf{\Gamma}) & (q, \mathbf{\Gamma}) \arrow[l, "Y_i^{(s_\Gamma)}"'] \arrow[u, "N_i"] \arrow[r, "X_i^{(s_\Gamma)}"] \arrow[d, "E_{ij}"] & (X_i^{(s_\Gamma)}q, \mathbf{\Gamma}) \\
 & (\wedge Z_{ij}q, \mathbf{\Gamma}) &
\end{tikzcd} \tag{5.12}$$

and the action of the measurements (5.2) on the quadruples (5.10) as

$$(\mathsf{V} \bigcup \{i\}, \mathsf{V}^*, q, \mathbf{\Gamma}) \xrightarrow{M_i^{(s,t)}(\alpha)} \begin{cases} \left(\mathsf{V}, \mathsf{V}^*\bigcup\{i\}, \langle +_{\alpha_\Gamma}|_i\ q, \mathbf{\Gamma}[0/i]\right), \\ \left(\mathsf{V}, \mathsf{V}^*\bigcup\{i\}, \langle -_{\alpha_\Gamma}|_i\ q, \mathbf{\Gamma}[1/i]\right), \end{cases}, \tag{5.13}$$

using the modified α from (5.2) in the form $\alpha_\Gamma = (-1)^{s_\Gamma}\alpha + t_\Gamma\pi$.

For instance, the structure of the patterns with Pauli corrections $X,\ Y$ in terms of the above commands can be formally (with suppressing all indices) written as (Mhalla *et al* 2022) $(\prod XYM)EN$.

In general, the execution of a pattern can be presented in the diagrammatic form (schematically)

$$\mathcal{H}_{\mathsf{In}} \longrightarrow \mathcal{H}_{\mathsf{In}} \times \mathbb{Z}_2^{\varnothing} \xrightarrow{\text{preparation}} \mathcal{H}_{\mathsf{V}} \times \mathbb{Z}_2^{\varnothing} \xrightarrow{A_1, A_2, \ldots, A_{n-1}, A_n} \mathcal{H}_{\mathsf{Out}} \times \mathbb{Z}_2^{\mathsf{V}*\mathsf{Out}} \longrightarrow \mathcal{H}_{\mathsf{Out}}. \tag{5.14}$$

The Brach map denoted by $\mathbf{B_s}$ can be formally written as

$$\mathbf{B_s} = \mathbf{C_s}\circ\mathbf{M_s}\circ\mathbf{U}, \tag{5.15}$$

where $\mathbf{U}\colon \mathcal{H}_{\mathsf{In}} \hookrightarrow \mathcal{H}_{\mathsf{V}}$ is a unitary embedding which is branch independent, $\mathbf{M_s}\colon \mathcal{H}_{\mathsf{V}} \rightarrow \mathcal{H}_{\mathsf{Out}}$ is a projection being collection of measurements along the branch, and $\mathbf{C_s}$ is a map corresponding to corrections on the output. Because all of the above maps are unitary, the resulting branch map is also unitary

$$\sum_{\mathbf{s}} \mathbf{B}_{\mathbf{s}}^{\dagger} \mathbf{B}_{\mathbf{s}} = \mathbf{I}, \tag{5.16}$$

where $\mathbf{I}$ is the identity matrix, and $\mathbf{s} \in \mathbb{Z}_2^n$. It follows from (5.16), that each pattern is presented by a positive map that preserves the trace (Danos *et al* 2007). A pattern is called deterministic if it is a positive trace preserving map and sends pure states to pure states.

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Innovative Quantum Computing

Steven Duplij and Raimund Vogl

Chapter 6

Quantum walks

Quantum walks are the quantum counterpart of the classical random walks and they play an important role in the modelling of many phenomena, e.g., information spreading in complex networks (Noh and Rieger 2004), optimal search strategies (Lv *et al* 2002), genetic sequence location (van den Engh *et al* 1992), and chemical reactions (Gillespie 1977). The term 'quantum walks' was introduced in Aharonov *et al* (1993), but the idea to incorporate quantum effects to stochastic calculus appeared in Iche and Nozieres (1978), while the coherence effects in evolution of Brownian quantum particle were first considered in Schwinger (1961). The quantum analogies of classical random walks in discrete time and space were investigated in Godoy and Fujita (1992). The quantum cellular automata were introduced in Grössing and Zeilinger (1988), which appeared to be equivalent to the construction of Aharonov *et al* (1993), and which can be considered as one particle sector of the former; for a review, see (Arrighi 2019) and more general (Venegas-Andraca 2012). The connections between correlated classical random walks and quantum walks were given in Konno (2009) using matrix methods.

There are two models of quantum walks:

(1) Discrete quantum walks consist of two systems, called a walker and a coin, and the evolution unitary operator acts on them in discrete time steps.
(2) Continuous quantum walks consists of one quantum system called a walker, which 'walks' without time restrictions, which is described by the evolution operator (Hamiltonian) and the Schrödinger equation (Childs *et al* 2002).

The general topology in both cases can be described by discrete graphs.

6.1 Discrete quantum walks

In the case of discrete quantum walks on a line, the total quantum state consists of quantum states of the walker and the coin, i.e., the total Hilbert state $\mathcal{H}_{\text{tot}}$ becomes the direct product

doi:10.1088/978-0-7503-5281-9ch6

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{coin}} \otimes \mathcal{H}_{\text{walk}}. \tag{6.1}$$

The position of the walker is described by the vector from the computational basis of the walker Hilbert space $\left|\psi_{\text{walk}}\right\rangle \in \mathcal{H}_{\text{walk}}$, which is infinite-dimensional and countable, such that the walker state $\left|\psi_{\text{walk}}\right\rangle$ is the quantum superposition

$$|\psi_{\text{walk}}\rangle = \sum_{\ell\in\mathbb{Z}} w_\ell\,|\ell\rangle_{\text{w}}, \quad \sum_{\ell\in\mathbb{Z}} w_\ell^2 = 1, \quad w_\ell \in \mathbb{C}. \tag{6.2}$$

In distinction to the classical coin, which can be in two states, the quantum s-state coin can be not only in s canonical basis states $|\mathbf{0}\rangle_{\text{c}}, |\mathbf{1}\rangle_{\text{c}},\ldots,|\mathbf{s-1}\rangle_{\text{c}}$, but also in their quantum superposition

$$|\boldsymbol{\psi}_{\text{coin}}\rangle = \sum_{j=0}^{s-1} c_j\,|\mathbf{j}\rangle_{\text{c}}, \quad \sum_{j=0}^{s-1} c_j^2 = 1, \quad c_j \in \mathbb{C}. \tag{6.3}$$

Usually, to be closer to the classical case, one puts $s = 2$. The total state of the quantum walk is given by

$$|\Psi_{\text{tot}}\rangle = |\boldsymbol{\psi}_{\text{coin}}\rangle \otimes |\psi_{\text{walk}}\rangle, \tag{6.4}$$

and the initial total state, if to take $\left|\psi_{\text{walk}}\right\rangle_{\text{initial}} = |0\rangle_{\text{w}}$, becomes

$$|\Psi_{\text{tot}}\rangle_{\text{initial}} = |\boldsymbol{\psi}_{\text{coin}}\rangle_{\text{initial}} \otimes |0\rangle_{\text{w}}. \tag{6.5}$$

In general, the total state can be written as

$$|\Psi_{\text{tot}}\rangle = \sum_{\ell\in\mathbb{Z}} \left(\varphi_{0,\ell}\,|\mathbf{0}\rangle_{\text{c}} \otimes |\ell\rangle_{\text{w}} + \varphi_{1,\ell}\,|\mathbf{1}\rangle_{\text{c}} \otimes |\ell\rangle_{\text{w}}\right), \tag{6.6}$$

$$\sum_{\ell\in\mathbb{Z}} \left(\left|\varphi_{0,\ell}\right|^2 + \left|\varphi_{1,\ell}\right|^2\right) = 1 \quad \varphi_{0,\ell}, \varphi_{1,\ell} \in \mathbb{C}. \tag{6.7}$$

It follows from (6.2)–(6.3) that

$$\varphi_{j,\ell} = c_j w_\ell, \quad \ell \in \mathbb{Z}, \quad j = 0, 1, \tag{6.8}$$

and so the normalization condition (6.7) reduces one parameter from the set of ones describing the total state (6.6).

By analogy with the classical random walk, we need one operator to move the walker on the line and one operator to play the same role as the coin toss. In contrast to the classic case, where such an operator is represented by a stochastic matrix, in the case of the quantum walk evolution there is no room for randomness before measurement and it is represented by a unitary matrix, which acts as an internal rotation in the internal state space. The goal of the coin operator is to render the coin state in a superposition, while the randomness is introduced by making a measurement on the system after both evolution operators have been applied to the total quantum system for many times.

Thus, the evolution of a quantum walk is driven by the special composite action of two unitary operators: (1) in the first, a shift operator **S** acts in a combined total position-coin space $\mathcal{H}_{\text{tot}}$; (2) in the second, the coin operator **C** acts in the coin space $\mathcal{H}_{\text{coin}}$. In this way, the total evolution is described by the unitary operator **U**, which is defined by the main formula of the coined quantum walk concept

$$\mathbf{U} = \mathbf{S}\circ(\mathbf{C} \otimes \mathbf{I}_{\text{w}}), \tag{6.9}$$

$$\mathbf{S}\colon \mathcal{H}_{\text{coin}} \otimes \mathcal{H}_{\text{walk}} \to \mathcal{H}_{\text{coin}} \otimes \mathcal{H}_{\text{walk}},\quad \mathbf{C}\colon \mathcal{H}_{\text{coin}} \to \mathcal{H}_{\text{coin}},\quad \mathbf{U}\colon \mathcal{H}_{\text{tot}} \to \mathcal{H}_{\text{tot}}, \tag{6.10}$$

where $\mathbf{I}_{\text{w}} \in \mathcal{H}_{\text{walk}}$ is the unity of the walker space $\mathcal{H}_{\text{walk}}$.

If we consider the two-state coin $s = 2$ (6.6), then the operator **S** should act on the total quantum state (6.4) by shifts that are dependent from the coin state

$$\mathbf{S}\circ\big(|\,\mathbf{0}\rangle_{\text{c}} \otimes |\,\ell\rangle_{\text{w}}\big) = |\,\mathbf{0}\rangle_{\text{c}} \otimes |\,\ell + 1\rangle_{\text{w}}, \tag{6.11}$$

$$\mathbf{S}\circ\big(|\,\mathbf{1}\rangle_{\text{c}} \otimes |\,\ell\rangle_{\text{w}}\big) = |\,\mathbf{1}\rangle_{\text{c}} \otimes |\,\ell - 1\rangle_{\text{w}}. \tag{6.12}$$

This can be written in the unified form

$$\mathbf{S}\circ\big(|\,\mathbf{j}\rangle_{\text{c}} \otimes |\,\ell\rangle_{\text{w}}\big) = |\,\mathbf{j}\rangle_{\text{c}} \otimes |\,\ell + (-1)^{j}\rangle_{\text{w}}, \tag{6.13}$$

i.e., the shift operator depends on the coin state $\mathbf{S} = \mathbf{S}_j$. Therefore, in the computational basis, **S** can be presented using two projections in $\mathcal{H}_{\text{c}}$ as (the outer product representation)

$$\mathbf{S} = |\,\mathbf{0}\rangle_{\text{c}}\langle\mathbf{0}\,|_{\text{c}} \otimes \sum_{\ell\in\mathbb{Z}} |\,\ell + 1\rangle_{\text{w}}\langle\ell\,|_{\text{w}} + |\,\mathbf{1}\rangle_{\text{c}}\langle\mathbf{1}\,|_{\text{c}} \otimes \sum_{\ell\in\mathbb{Z}} |\,\ell - 1\rangle_{\text{w}}\langle\ell\,|_{\text{w}}, \tag{6.14}$$

which satisfies the required shifting properties in the walker space (6.11)–(6.12).

The coin operator **C** is an arbitrary element of the unitary group $\mathcal{U}(s)$, and for the two-state coin $s = 2$, and it can be represented by the four real parameter 2×2 complex matrix $\hat{C}$ of the form

$$\hat{C} = \hat{C}_{\alpha,\beta,\gamma,\theta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{i\gamma}\begin{pmatrix} e^{i\alpha}\cos\theta & e^{i\beta}\sin\theta \\ -\,e^{-i\beta}\sin\theta & e^{-i\alpha}\cos\theta \end{pmatrix},\quad a, b, c, d \in \mathbb{C},\quad \alpha, \beta, \gamma, \theta \in \mathbb{R}. \tag{6.15}$$

In most cases, for quantum walks with two-state coin the Hadamard operator is widely used

$$\mathbf{C}_{\text{H}} = \frac{1}{\sqrt{2}}\big(|\,\mathbf{0}\rangle_{\text{c}}\langle\mathbf{0}\,|_{\text{c}} + |\,\mathbf{0}\rangle_{\text{c}}\langle\mathbf{1}\,|_{\text{c}} + |\,\mathbf{1}\rangle_{\text{c}}\langle\mathbf{0}\,|_{\text{c}} - |\,\mathbf{1}\rangle_{\text{c}}\langle\mathbf{1}\,|_{\text{c}}\big), \tag{6.16}$$

or in the matrix representation (6.15)

$$\hat{C}_{\text{H}} = \hat{C}_{\alpha=\frac{\pi}{2},\beta=\frac{\pi}{2},\gamma=\frac{\pi}{2},\theta=\frac{\pi}{4}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{6.17}$$

The evolution of the total state (6.4) during the discrete time (=t) quantum walk after t steps $|\,\Psi_{\text{tot}}(t)\rangle$ is given by the application of the unitary operator (6.9) t times in the following way

$$|\,\Psi_{\text{tot}}(t)\rangle = \mathbf{U}^t\,|\,\Psi_{\text{tot}}(0)\rangle, \tag{6.18}$$

where $|\,\Psi_{\text{tot}}(0)\rangle = |\,\Psi_{\text{tot}}\rangle_{\text{initial}}$ (6.5).

Example 6.1. Using (6.9) and (6.16), we can get the first three steps for the Hadamard quantum walk with the two-state coin as

$$|\,\Psi_{\text{tot}}(1)\rangle = \frac{1}{\sqrt{2}}\,|\,\mathbf{0}\rangle_{\text{c}} \otimes |\,1\rangle_{\text{w}} + \frac{1}{\sqrt{2}}\,|\,\mathbf{1}\rangle_{\text{c}} \otimes |{-1}\rangle_{\text{w}}, \tag{6.19}$$

$$|\,\Psi_{\text{tot}}(2)\rangle = -\frac{1}{2}\,|\,\mathbf{1}\rangle_{\text{c}} \otimes |{-2}\rangle_{\text{w}} + \frac{1}{2}(|\,\mathbf{0}\rangle_{\text{c}} + |\,\mathbf{1}\rangle_{\text{c}}) \otimes |\,0\rangle_{\text{w}} + \frac{1}{2}\,|\,\mathbf{0}\rangle_{\text{c}} \otimes |\,2\rangle_{\text{w}} \tag{6.20}$$

$$= \frac{1}{2}\,|\,\mathbf{0}\rangle_{\text{c}} \otimes (|\,0\rangle_{\text{w}} + |\,2\rangle_{\text{w}}) + \frac{1}{2}\,|\,\mathbf{1}\rangle_{\text{c}} \otimes (|\,0\rangle_{\text{w}} - |{-2}\rangle_{\text{w}}), \tag{6.21}$$

$$|\,\Psi\text{tot}(3)\rangle = \frac{1}{2\sqrt{2}}\,|\,\mathbf{1}\rangle\text{c} \otimes |{-3}\rangle\text{w} - \frac{1}{2\sqrt{2}}\,|\,\mathbf{0}\rangle\text{c} \otimes |{-1}\rangle\text{w} + \frac{1}{2\sqrt{2}}(2\,|\,\mathbf{0}\rangle\text{c} + |\,\mathbf{1}\rangle\text{c}) \otimes |\,1\rangle\text{w} + \frac{1}{2\sqrt{2}}\,|\,\mathbf{0}\rangle\text{c} \otimes |\,3\rangle\text{w} \tag{6.22}$$

$$= \frac{1}{2\sqrt{2}}\,|\,\mathbf{0}\rangle_{\text{c}} \otimes (-|{-1}\rangle_{\text{w}} + 2\,|\,1\rangle_{\text{w}} + |\,3\rangle_{\text{w}}) + \frac{1}{2\sqrt{2}}\,|\,\mathbf{1}\rangle_{\text{c}} \otimes (|\,1\rangle_{\text{w}} + |{-3}\rangle_{\text{w}}). \tag{6.23}$$

If the final state at the time t is known $\Psi_{\text{tot}}(t)$, then the standard way to describe the quantum walk is the partial measurement of the walker state probabilities (see, e.g. Portugal 2013).

However, we now have the tensor product of two spaces (6.1). Therefore, to have the complete description of the quantum walk, we propose to also consider the partial measurement of the (s-) coin state probabilities.

Let the total state at the time t (6.18) have the general form (see (6.6)–(6.8))

$$|\,\Psi_{\text{tot}}(t)\rangle = \sum_{\ell\in\mathbb{Z}}\sum_{j=0}^{s-1}\varphi_{j,\ell}(t)\,|\,\mathbf{j}\rangle_{\text{c}} \otimes |\,\ell\rangle_{\text{w}}, \tag{6.24}$$

$$\sum_{\ell\in\mathbb{Z}}\left|\,\varphi_{j,\ell}(t)\,\right|^2 = 1 \quad \varphi_{j,\ell} \in \mathbb{C}. \tag{6.25}$$

We denote the doubly partial probability of the state $|\,\mathbf{j}\rangle_{\text{c}} \otimes |\,\ell\rangle_{\text{w}}$ at time t by

$$p_{j,\ell}(t) = \left|\,\varphi_{j,\ell}(t)\,\right|^2, \quad \sum_{\ell\in\mathbb{Z}}\sum_{j=0}^{s-1}p_{j,\ell}(t) = 1. \tag{6.26}$$

Now we propose to characterize the quantum walk by two partial probability distributions:

(1) The walker probability distribution

$$p_\ell^{\text{walk}}(t) = \sum_{j=0}^{s-1} \left| \varphi_{j,\ell}(t) \right|^2, \tag{6.27}$$

$$\sum_{\ell \in \mathbb{Z}} p_\ell^{\text{walk}}(t) = 1. \tag{6.28}$$

(2) The coin probability distribution

$$p_j^{\text{coin}}(t) = \sum_{\ell \in \mathbb{Z}} \left| \varphi_{j,\ell}(t) \right|^2, \tag{6.29}$$

$$\sum_{j=0}^{s-1} p_j(t) = 1. \tag{6.30}$$

In the standard approach (Portugal 2013), only the first (walker) distribution (6.27) is usually considered: the time is fixed by $t = t_0$, and the graph $\left\{\ell,\, p_\ell^{\text{walk}}(t_0)\right\}$ is plotted. Nevertheless, the coin probability distribution (6.29) gives additional information about the quantum walk. To observe the difference between (6.27) and (6.29) concretely, we continue *example* 6.1 in detail.

Example 6.2. (*Example* 6.1 continued) Here we compute the walker and coin probabilities (6.27) and (6.29) for three steps $t = 1, 2, 3$ of the Hadamard walk $\Psi_{\text{tot}}(t)$ in (6.19)–(6.23).The formulas (6.19), (6.20), and (6.22) are convenient to use for the walker probabilities, and the formulas (6.19), (6.21), and (6.23) can be used for the coin probabilities. We derive the walker probabilities $p_\ell^{\text{walk}}(t)$ from (6.19)

$$p_{\ell=1}^{\text{walk}}(t=1) = p_{\ell=|1\rangle_{\text{w}}}^{\text{walk}}(t=1) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}, \tag{6.31}$$

$$p_{\ell=-1}^{\text{walk}}(t=1) = p_{\ell=|-1\rangle_{\text{w}}}^{\text{walk}}(t=1) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}, \tag{6.32}$$

and from (6.20) we obtain the symmetric distribution

$$p_{\ell=-2}^{\text{walk}}(t=2) = p_{\ell=|-2\rangle_{\text{w}}}^{\text{walk}}(t=2) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}, \tag{6.33}$$

$$p_{\ell=0}^{\text{walk}}(t=2) = p_{\ell=|0\rangle_{\text{w}}}^{\text{walk}}(t=2) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}, \tag{6.34}$$

$$p_{\ell=2}^{\text{walk}}(t=2) = p_{\ell=|2\rangle_{\text{w}}}^{\text{walk}}(t=2) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}. \tag{6.35}$$

The probability distribution $p_\ell^{\text{walk}}(t)$ for the third step $t = 3$ is nonsymmetric (6.22)

$$p_{\ell=-3}^{\text{walk}}(t=3) = p_{\ell=|-3\rangle_{\text{w}}}^{\text{walk}}(t=3) = \left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{8}, \tag{6.36}$$

$$p_{\ell=-1}^{\text{walk}}(t=3) = p_{\ell=|-1\rangle_{\text{w}}}^{\text{walk}}(t=3) = \left(-\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{8}, \tag{6.37}$$

$$p_{\ell=1}^{\text{walk}}(t=3) = p_{\ell=|1\rangle_{\text{w}}}^{\text{walk}}(t=3) = \left(2\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{5}{8}, \tag{6.38}$$

$$p_{\ell=3}^{\text{walk}}(t=3) = p_{\ell=|3\rangle_{\text{w}}}^{\text{walk}}(t=3) = \left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{8}, \tag{6.39}$$

as well as for further steps (times) $t > 3$.

For the coin probabilities $p_\ell^{\text{coin}}(t)$ we have from (6.19)

$$p_{j=0}^{\text{coin}}(t=1) = p_{j=|\mathbf{0}\rangle_{\text{c}}}^{\text{coin}}(t=1) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}, \tag{6.40}$$

$$p_{j=1}^{\text{coin}}(t=1) = p_{j=|\mathbf{1}\rangle_{\text{c}}}^{\text{coin}}(t=1) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}, \tag{6.41}$$

and from (6.21) we have for the second step $t = 2$ the symmetric distribution

$$p_{j=0}^{\text{coin}}(t=2) = p_{j=|\mathbf{0}\rangle_{\text{c}}}^{\text{coin}}(t=2) = \left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right) = \frac{1}{2}, \tag{6.42}$$

$$p_{j=1}^{\text{coin}}(t=2) = p_{j=|\mathbf{1}\rangle_{\text{c}}}^{\text{coin}}(t=2) = \left(\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2\right) = \frac{1}{2}, \tag{6.43}$$

The probability distribution $p_j^{\text{coin}}(t)$ for the third step $t = 3$ is also nonsymmetric as $p_\ell^{\text{walk}}(t = 3)$, so from (6.23) we get

$$p_{j=0}^{\text{coin}}(t=3) = p_{j=|\mathbf{0}\rangle_c}^{\text{coin}}(t=3) = \left(\left(-\frac{1}{2\sqrt{2}}\right)^2 + \left(2\frac{1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2\right) = \frac{3}{4}, \quad (6.44)$$

$$p_{j=1}^{\text{coin}}(t=3) = p_{j=|\mathbf{1}\rangle_c}^{\text{coin}}(t=3) = \left(\left(\frac{1}{2\sqrt{2}}\right)^2 + \left(2\frac{1}{2\sqrt{2}}\right)^2\right) = \frac{1}{4}, \quad (6.45)$$

and in the similar way for further steps (discrete times) $t > 3$.

As it should be, both the above walker and coin probability distributions are correctly normalized satisfying (6.28) and (6.30) at each discrete time t.

6.1.1 Polyander visualization of quantum walks

The coin probability distribution $p_j^{\text{coin}}(t)$ introducted in (6.29), from the first glance, can be also characterized at the fixed time $t = t_0$ by the graph $\left\{j, p_j^{\text{coin}}(t_0)\right\}$ as the walker probability distribution $p_\ell^{\text{walk}}(t_0)$. However, because the coin has a specific physical sense, we propose here another way of the quantum walk description, which originates from genome landscapes (Azbel' M Y 1973, 1995, Lobry 1996) and one-dimensional DNA walks (Cebrat and Dudek 1998) and trianders (Duplij and Duplij 2005).

Innovation 6.3. *We can consider the time evolution of the probability for the concrete quantum state when we provide the corresponding measurements in the coin or walker subspaces, i.e., we fix the states $\ell = \ell_0$ or $j = j_0$ and introduce the following time evolution graphs $\left\{t, p_{\ell=\ell_0}^{\text{walk}}(t)\right\}$ or $\left\{t, p_{j=j_0}^{\text{coin}}(t)\right\}$.*

Definition 6.4. The polyander visualization of a quantum walk is its description by the time evolution graphs $\left\{t, p_\ell^{\text{walk}}(t)\right\}$ or $\left\{t, p_j^{\text{coin}}(t)\right\}$. Each line of the graph describing the probability evolution of the fixed quantum state $\ell = \ell_0$ for $|\,\ell_0\rangle_w$ or $j = j_0$ for $|\,\mathbf{j}\rangle_c$ is called a leg of the polyander.

It is obvious that the walker polyander has a finitely increasing number of legs and corresponding quantum states, while the s-side coin polyander has exactly s legs.

For *example* 6.1, we obtain the following.

Example 6.5. (*Example* 6.1 continued) The walker polyander $p_\ell^{\text{walk}}(t)$ in the time range $1 \leqslant t \leqslant 3$ has seven legs (quantum states) $-3 \leqslant \ell \leqslant 3$, which have the following probability evolutions

$\|\ell\rangle$-leg\time t	1	2	3
$\|-3\rangle_w$	0	0	$\frac{1}{8}$
$\|-2\rangle_w$	0	$\frac{1}{4}$	0
$\|-1\rangle_w$	$\frac{1}{2}$	0	$\frac{1}{8}$
$\|0\rangle_w$	0	$\frac{1}{2}$	0
$\|1\rangle_w$	$\frac{1}{2}$	0	$\frac{5}{8}$
$\|2\rangle_w$	0	$\frac{1}{4}$	0
$\|3\rangle_w$	0	0	$\frac{1}{8}$

(6.46)

The coin polyander $p_j^{\text{coin}}(t)$ in the time range $1 \leqslant t \leqslant 3$ has two legs (quantum states), $j = 0, 1$, which have the following probability evolutions

$\|\mathbf{j}\rangle$-leg\time t	1	2	3
$\|\mathbf{0}\rangle_c$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$
$\|\mathbf{1}\rangle_c$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$

(6.47)

Each leg can be presented as a horizontal strip of the width 1 on which the points corresponding to the probabilities $0 \leqslant p(t) \leqslant 1$ at times $t = 1, 2, 3\ldots$ are indicated. The probability behaviour of each quantum state can then be visually seen and mutually compared in the same time points.

For the coin polyander, it is important to consider the probability differences because of the following

Definition 6.6. The total quantum state is called trivial at the time $t = t_{\text{triv}}$ if all the s-side coin states have equal probabilities $p_j^{\text{coin}}(t_{\text{triv}}) = \frac{1}{s}$, $j = 0, 1, \ldots, s - 1$, $s \geqslant 2$.

Definition 6.7. The quantum walk is called trivial if the s-side coin states are trivial at all times.

In the case of the standard coin $s = 2$, the triviality means that the measurements of both sides give the same probability at the $t = t_{\text{triv}}$. Therefore, to describe triviality in detail, we should introduce the differences and search for nonzero ones.

Definition 6.8. The bias s-side coin polyander has $(s - 1)$ legs, which are defined by

$$\Delta p_j^{\text{coin}}(t) = p_j^{\text{coin}}(t) - p_{j+1}^{\text{coin}}(t), \quad j = 0, \ldots, s - 2. \tag{6.48}$$

Example 6.9. (*Example* 6.1 continued) The 2-side coin bias polyander in the time range $1 \leqslant t \leqslant 3$ has one leg which has the following probability evolution $\Delta p_0^{\text{coin}}(t) = \Delta p_{j=|\mathbf{0}\rangle_c}^{\text{coin}}(t) - \Delta p_{j=|\mathbf{1}\rangle_c}^{\text{coin}}(t)$ (see (6.47))

$j\backslash$time t	1	2	3
0	0	0	$\frac{1}{2}$

(6.49)

which can be nontrivial after the time $t = 3$ only.

In the higher times, the walker and coin polyanders, as well as the bias coin polyander, will have more complicated behavior, which in any case needs the manifest form of the total quantum state (6.18). In *examples* 6.5 and 6.9, we considered for clarity only the time range $1 \leqslant t \leqslant 3$ and the 2-side coin to show in detail how to compute probability polyanders for finite times. The physical sense of the bias polyander is in the following: its nonzero values show nontriviality evolution along the quantum walk.

Thus, polyanders allow us to further study the fine structure, and thoroughly characterize and visually present quantum walks from different viewpoints.

6.1.2 Methods of final states computation

The main goal of studying the quantum walks is to obtain the analytical expression for the final quantum state (6.18) in discrete finite times $t \in \mathbb{Z}$, and then calculate the dynamical and statistical properties of various probability distributions and characteristics.

The main computational methods to find the total quantum state (6.18) are

1. **The Schrödinger approach**. Starting from an arbitrary state of the quantum walk with a certain walker position, to provide the discrete time Fourier transform (Ambainis *et al* 2001) and obtain the closed form of total amplitudes.
2. **The combinatorial approach**. The amplitude at any discrete time is derived as a sum of amplitudes of all paths starting from the initial state and ending up in the final state. This can be treated as reminiscent of the standard path integral technique.

In Carteret *et al* (2005), it was shown that both Schrödinger and combinatorial approaches are equivalent. Among less known methods, we can mention the alternative description of quantum walks based on the scattering theory (Feldman and Hillery 2007) and the analytic formulation of probability densities and moments (Fuss *et al* 2007).

6.1.2.1 Fourier transform and analytic solutions

In general, the usage of the Fourier transform is the standard way to simplify computations by turning equations to algebraic ones. In its application to quantum works and analysing the evolution (6.18), there two peculiarities:

(1) The Fourier transform is applied to one subspace from the product (6.4), i.e., the walker one $\mathcal{H}_{\text{walk}}$.
(2) Sometimes it is simpler to turn from transforming functions to transform the computational basis of the walker subspace.

Following *(2)*, we transform the computational basis of the walker space $\mathcal{H}_{\text{walk}}$ as

$$\| \mathsf{k}\rangle\rangle_{\text{w}} = \sum_{\ell \in \mathbb{Z}} e^{i\mathsf{k}\ell} \,|\, \ell\rangle_{\text{w}}, \quad \ell \in \mathbb{Z}, \; |\, \ell\rangle_{\text{w}}, \; \| \mathsf{k}\rangle\rangle_{\text{w}} \in \mathcal{H}_{\text{walk}}, \tag{6.50}$$

where the Fourier transformed vectors $\| \mathsf{k}\rangle\rangle_{\text{w}}$ are denoted by the double brackets and depend on the continuous real wave number $\mathsf{k} \in \mathbb{R}$, $-\pi \leqslant \mathsf{k} \leqslant \pi$. The inverse transformation is

$$|\, \ell\rangle_{\text{w}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\mathsf{k} e^{-i\mathsf{k}\ell} \,\| \mathsf{k}\rangle\rangle_{\text{w}}. \tag{6.51}$$

Let us introduce the Fourier transformation of the amplitudes $\varphi_{j,\ell}(t)$ at time t from the decomposition (6.24) in the standard way by

$$\Phi_{j,\mathsf{k}}(t) = \sum_{\ell \in \mathbb{Z}} e^{-i\mathsf{k}\ell} \varphi_{j,\ell}(t), \quad -\pi \leqslant \mathsf{k} \leqslant \pi. \tag{6.52}$$

The inverse Fourier transform becomes

$$\varphi_{j,\ell}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\mathsf{k} e^{i\mathsf{k}\ell} \Phi_{j,\mathsf{k}}(t). \tag{6.53}$$

Then, instead of the computational basis $|\, \mathbf{j}\rangle_{\text{c}} \otimes |\, \ell\rangle_{\text{w}}$ in (6.24), using (6.50) and (6.53) and cancelling exponents, we can present the total state in the Fourier basis $|\, \mathbf{j}\rangle_{\text{c}} \otimes \| \mathsf{k}\rangle\rangle_{\text{w}}$ as follows

$$|\, \Psi_{\text{tot}}(t)\rangle = \frac{1}{2\pi} \sum_{j=0}^{s-1} \int_{-\pi}^{\pi} \Phi_{j,\mathsf{k}}(t) \,|\, \mathbf{j}\rangle_{\text{c}} \otimes \| \mathsf{k}\rangle\rangle_{\text{w}}. \tag{6.54}$$

The action of the shift operator $\mathbf{S}$ on the Fourier basis can be derived from (6.13) and using (6.50), as follows

$$\begin{aligned}
\mathbf{S}\circ(|\, \mathbf{j}\rangle_{\text{c}} \otimes \| \mathsf{k}\rangle\rangle_{\text{w}}) &= \sum_{\ell \in \mathbb{Z}} e^{i\mathsf{k}\ell} \mathbf{S}\circ(|\, \mathbf{j}\rangle_{\text{c}} \otimes |\, \ell\rangle_{\text{w}}) = \sum_{\ell \in \mathbb{Z}} e^{i\mathsf{k}\ell} \mathbf{S}\circ(|\, \mathbf{j}\rangle_{\text{c}} \otimes |\, \ell\rangle_{\text{w}}) \\
&= \sum_{\ell \in \mathbb{Z}} e^{i\mathsf{k}\ell} \left(|\, \mathbf{j}\rangle_{\text{c}} \otimes |\, \ell + (-1)^{j}\rangle_{\text{w}}\right) = \sum_{\ell' \in \mathbb{Z}} e^{i\mathsf{k}(\ell' - (-1)^{j})} (|\, \mathbf{j}\rangle_{\text{c}} \otimes |\, \ell'\rangle_{\text{w}}) \\
&= e^{-i\mathsf{k}(-1)^{j}} \sum_{\ell' \in \mathbb{Z}} e^{i\mathsf{k}\ell'} (|\, \mathbf{j}\rangle_{\text{c}} \otimes |\, \ell'\rangle_{\text{w}}) = e^{-i\mathsf{k}(-1)^{j}} \,|\, \mathbf{j}\rangle_{\text{c}} \otimes \| \mathsf{k}\rangle\rangle_{\text{w}},
\end{aligned} \tag{6.55}$$

where we used the substitution $\ell' = \ell + (-1)^j$ and the translation symmetry of the infinite sum.

In the case of the two-side coin $j = 0, 1$ and the Hadamard quantum walk (6.16)–(6.17), the action of operators can be expressed in the matrix form.

So we apply the total evolution operator **U** (6.9) in the matrix form to the Fourier basis $|\,\mathbf{j}\rangle_c \otimes ||\,\mathsf{k}\rangle\rangle_w$ using (6.17) to get

$$
\begin{aligned}
\hat{U}(|\,\mathbf{j}'\rangle_c \otimes ||\,\mathsf{k}\rangle\rangle_w) &= \hat{S}\left(\left(\sum_{j=0}^{1} \hat{C}_{jj'}\,|\,\mathbf{j}\rangle_c\right) \otimes ||\,\mathsf{k}\rangle\rangle_w\right) \\
&= \left(\sum_{j=0}^{1} e^{-i\mathsf{k}(-1)^j} \hat{C}_{jj'}\,|\,\mathbf{j}\rangle_c\right) \otimes ||\,\mathsf{k}\rangle\rangle_w = \sum_{j=0}^{1} \bar{\mathbf{C}}_{jj'}(\mathsf{k})\,|\,\mathbf{j}\rangle_c \otimes ||\,\mathsf{k}\rangle\rangle_w,
\end{aligned}
\tag{6.56}
$$

where

$$
\bar{\mathbf{C}}(\mathsf{k}) = \begin{pmatrix} e^{-i\mathsf{k}} & 0 \\ 0 & e^{i\mathsf{k}} \end{pmatrix} \hat{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\mathsf{k}} & e^{-i\mathsf{k}} \\ e^{i\mathsf{k}} & -\,e^{i\mathsf{k}} \end{pmatrix}. \tag{6.57}
$$

It follows from (6.56) that diagonalization of $\bar{\mathbf{C}}(\mathsf{k})$ leads to the spectral decomposition of the total operator $\hat{U}$. Indeed, if $\lambda(\mathsf{k})$ is the eigenvalue of the matrix $\bar{\mathbf{C}}(\mathsf{k})$, then it is also the eigenvalue of $\hat{U}$, as is seen from (6.56). We denote the corresponding $\lambda(\mathsf{k})$ eigenvector by $\left|\left|\,\mathsf{v}_{\lambda(\mathsf{k})}\right\rangle\right\rangle_c$, such that

$$
\hat{U}\circ\left(\left|\left|\,\mathsf{v}_{\lambda(\mathsf{k})}\right\rangle\right\rangle_c \otimes ||\,\mathsf{k}\rangle\rangle_w\right) = \left(\bar{\mathbf{C}}(\mathsf{k})\circ\left|\left|\,\mathsf{v}_{\lambda(\mathsf{k})}\right\rangle\right\rangle_c\right) \otimes ||\,\mathsf{k}\rangle\rangle_w = \lambda(\mathsf{k})\left|\left|\,\mathsf{v}_{\lambda(\mathsf{k})}\right\rangle\right\rangle_c \otimes ||\,\mathsf{k}\rangle\rangle_w. \tag{6.58}
$$

The matrix $\bar{\mathbf{C}}(\mathsf{k})$ (6.57) has two eigenvalues

$$
\lambda_1(\mathsf{k}) = e^{-i\alpha(\mathsf{k})}, \quad \lambda_2(\mathsf{k}) = -e^{i\alpha(\mathsf{k})}, \tag{6.59}
$$

$$
\alpha(\mathsf{k}) = \arcsin\left(\frac{1}{\sqrt{2}} \sin \mathsf{k}\right), \ -\frac{\pi}{2} \leqslant \alpha(\mathsf{k}) \leqslant \frac{\pi}{2}, \tag{6.60}
$$

and two corresponding normalized eigenvectors

$$
\left|\left|\,\mathsf{v}_{\lambda_{1,2}(\mathsf{k})}\right\rangle\right\rangle_c = \frac{1}{\sqrt{r_{1,2}}} \begin{pmatrix} e^{-i\mathsf{k}} \\ \pm\sqrt{2}\,e^{-i\alpha(\mathsf{k})} - e^{-i\mathsf{k}} \end{pmatrix}, \tag{6.61}
$$

$$
r_{1,2} = 2\left(1 + \cos^2 \mathsf{k} \mp \cos \mathsf{k}\sqrt{1 + \cos^2 \mathsf{k}}\right). \tag{6.62}
$$

Thus, in the total evolution the operator can be written in terms of eigenvalues and eigenvectors of $\bar{\mathbf{C}}(\mathsf{k})$ (6.57)

$$
\hat{U} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\mathsf{k}\left[\left(e^{-i\alpha(\mathsf{k})}\left|\left|\,\mathsf{v}_{\lambda_1(k)}\right\rangle\right\rangle_c\left\langle\left\langle \mathsf{v}_{\lambda_1(k)}\right|\right|_c - e^{i\alpha(\mathsf{k})}\left|\left|\,\mathsf{v}_{\lambda_2(k)}\right\rangle\right\rangle_c\left\langle\left\langle \mathsf{v}_{\lambda_2(k)}\right|\right|_c\right) \otimes ||\,\mathsf{k}\rangle\rangle_w\langle\langle \mathsf{k}\,||_w\right]. \tag{6.63}
$$

Using orthogonality of the basis eigenvectors, the power of the evolution operator can be presented as

$$\hat{U}^t = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\mathsf{k}\left[\left(e^{-i\alpha(\mathsf{k})t} \left\| \mathsf{v}_{\lambda_1(k)}\right\rangle\right\rangle_{\mathrm{c}} \left\langle\left\langle \mathsf{v}_{\lambda_1(k)} \right\|_{\mathrm{c}} + (-1)^t e^{i\alpha(\mathsf{k})t} \left\| \mathsf{v}_{\lambda_2(k)}\right\rangle\right\rangle_{\mathrm{c}} \left\langle\left\langle \mathsf{v}_{\lambda_2(k)} \right\|_{\mathrm{c}}\right) \otimes \| \mathsf{k}\rangle\rangle_{\mathrm{w}}\langle\langle \mathsf{k} \|_{\mathrm{w}}\right]. \quad (6.64)$$

Now we can use the main quantum evolution formula (6.18) to obtain the total quantum state at any time from an initial quantum state (6.5). For instance, if $| \Psi_{\mathrm{tot}}\rangle_{initial} = | \mathbf{0}\rangle_{\mathrm{c}} \otimes | 0\rangle_{\mathrm{w}}$, then using (6.54) and (6.61), we derive the Fourier transformed amplitudes

$$\begin{aligned}\Phi_{j=0,\mathsf{k}}(t) &= \frac{1}{2\sqrt{1 + \cos^2 \mathsf{k}}}\left[\left(\sqrt{1 + \cos^2 \mathsf{k}} + \cos \mathsf{k}\right)e^{-i\alpha(\mathsf{k})t} + \left(\sqrt{1 + \cos^2 \mathsf{k}} - \cos \mathsf{k}\right)e^{i(\pi+\alpha(\mathsf{k}))t}\right],\\ \Phi_{j=1,\mathsf{k}}(t) &= \frac{e^{i\mathsf{k}}}{2\sqrt{1 + \cos^2 \mathsf{k}}}(e^{-i\alpha(\mathsf{k})t} - e^{i(\pi+\alpha(\mathsf{k}))t}).\end{aligned} \quad (6.65)$$

Then by applying the reverse Fourier transform (6.53) and taking into account symmetries of integrand, we get the amplitudes in the computational basis at the arbitrary time t as

$$\varphi_{j=0,\ell}(t) = \begin{cases}\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} d\mathsf{k}e^{i(\mathsf{k}\ell-\alpha(\mathsf{k})t)}\left(\frac{\cos \mathsf{k}}{\sqrt{1 + \cos^2 \mathsf{k}}} + 1\right), & t + \ell = \text{even},\\ 0, \quad t + \ell = \text{odd}, \end{cases} \quad (6.66)$$

$$\varphi_{j=1,\ell}(t) = \begin{cases}\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} d\mathsf{k}e^{i(\mathsf{k}\ell-\alpha(\mathsf{k})t+\mathsf{k})}\frac{1}{\sqrt{1 + \cos^2 \mathsf{k}}}, & t + \ell = \text{even},\\ 0, \quad t + \ell = \text{odd}. \end{cases} \quad (6.67)$$

Finally, using the partial probability formulas (6.27) and (6.29), one can plot the time evolution graphs $\left\{t, p_{\ell=\ell_0}^{\mathrm{walk}}(t)\right\}$ and $\left\{t, p_{j=j_0}^{\mathrm{coin}}(t)\right\}$, i.e., to provide the polyander visualization (see section 6.1.1).

6.1.3 Generalizations of discrete-time quantum walks

There are plenty of generalizations of the above constructions. Nevertheless, the main procedures remain nearly the same.

- **Coin operator.** The most general form of the two-sided ($s = 2$) coin operator **C** is given by the complex matrix (6.15) from the unitary group $\mathcal{U}(2)$, i.e., other than the Hadamard matrix (6.17) can be considered, such as the Fourier coin (Portugal 2013).
- **Higher dimensions.** The main quantum walk equation (6.9) can be extended to a higher dimension of the s-sided coin when $\mathcal{H}_{\mathrm{coin}}$ is $2s$-dimensional Hilbert space and $\mathcal{H}_{\mathrm{walk}}$ is the Hilbert space corresponding to the direct product $\overbrace{\mathbb{Z} \otimes \cdots \otimes \mathbb{Z}}^{s}$. The common choice for an s-sided coin is the Grover operator

described by the corresponding $2s$-dimensional matrix $\hat{C}_{\mathrm{Grover}}$ that was proposed in Moore and Russell (2002).

- **Anyonic quantum walks.** To include the braiding interaction, one includes the additional Hilbert space (fusion space) $\mathcal{H}_{\mathrm{fusion}}$ where the generators of the braid group act. Then the total space becomes $\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_{\mathrm{coin}} \otimes \mathcal{H}_{\mathrm{fusion}} \otimes \mathcal{H}_{\mathrm{walk}}$, and the time evolution contains the additional braid operator in some representation (Lehman *et al* 2011).

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