Nuclear Physics B276 (1986) 501-516
North-Holland, Amsterdam

TOWARD STABLE COMPACTIFICATIONS

Frank S. ACCETTA[1], Marcelo GLEISER[2]*, Richard HOLMAN[2] and Edward W. KOLB[1,2]

[1]*Astronomy and Astrophysics Center, Enrico Fermi Institute, The University of Chicago, Chicago, Illinois 60637, USA*
[2]*NASA / Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, Batavia, Illinois 60510, USA*

Received 18 March 1986

We show that compactifications of theories with extra dimensions are unstable if due to monopole configurations of an antisymmetric tensor field balanced against one-loop Casimir corrections. In the case of ten-dimensional supergravity it is possible, at least for a portion of the phase space, to achieve a stable compactification without fine-tuning by including the contribution of fermionic condensates to the monopole configurations.

1. Introduction

In the quest for a unified description of gravity and gauge interactions, extra dimensions are believed by many to play a fundamental role [1]. The idea that we can understand four-dimensional gauge invariance as coming from isometries of higher dimensional gravity is indeed a very attractive one. This can be seen by the now extensive literature on the subject.

Nevertheless, the modern versions of this unification are very different either from Kaluza's original formulation or from the more recent extensions of his model to incorporate non-abelian symmetries. In part, these modifications were motivated for two reasons: first, it has been shown that the dimensionally reduced actions obtained from $4+D$ dimensional gravity could not give rise to chiral fermions as required from phenomenology. Second, the quantum behavior of pure higher dimensional gravity is even more divergent than its four-dimensional counterpart.

It is currently believed that superstring theories [2] offer the best way to cope with the above difficulties. Superstring theories must live in ten dimensions in order to be quantum mechanically consistent. More importantly, they are anomaly-free [3] and one-loop finite [4] if and only if the gauge group is $E_8 \otimes E_8$ or spin(32)/Z_2. This

* On leave of absence from Department of Mathematics, King's College, London WC2R2LS. Address after September 1986: NASA/Fermilab Astrophysics Group.

0550-3213/86/$03.50

intimate relationship between proper quantum behavior, space-time dimensionality and gauge symmetry suggests that superstrings should be taken seriously as a candidate for unification.

Once we assume that higher dimensional theories may be of physical relevance, two questions immediately come to mind; which background configuration offers a stable compactification for the internal manifold and why the scales for the physical space and the internal space differ by roughly 60 orders of magnitude.

These questions are obviously related and are very sensitive to the field content of the theory. Also, the solutions must be consistent with the constancy of the fundamental couplings, which forces the internal radius to be constant or very nearly so before nucleosynthesis [5].

One way of answering these questions is to look for possible compactified solutions of the field equations and follow their cosmological evolution for various toy models to understand the effect of different contributions to the energy-momentum tensor such as monopole configurations, cosmological constant and Casimir corrections coming from bosonic and fermionic degrees of freedom in the theory. In this connection, it has been shown by Maeda [6] that the asymptotic product space of the form (4-dimensional Friedmann) $\times$ (compact internal space) is a classically stable solution or various possible sources of the energy-momentum tensor, including monopole + cosmological constant, Casimir + cosmological constant and also for some supergravity models. These results are somewhat encouraging but, on the other hand, it has also recently been shown [7, 8] that the system with monopole or Casimir plus cosmological constant in $4 + D$ dimensions, with a product space $M^4 \times S^D$ (S^D being the D-dimensional sphere), is semiclassically and, for a range of temperatures (initial entropies), thermally unstable against fluctuations to large values of the internal radius. In this case, the monopole or Casimir term becomes negligible and the cosmological constant dominates the energy density giving rise to a $4 + D$ dimensional isotropic de Sitter space-time.

When we turn our attention to supersymmetric theories, the cosmological constant has to be dropped from Einstein's equations, since it breaks supersymmetry explicitly. Thus, one should try to combine monopole and Casimir effects to look for stable compactifications [9]. The monopole terms are the best known way to obtain spontaneous compactified solutions of the field equations in a natural fashion [10], while the inclusion of Casimir effects is justified by the fact that the compactification scale is very close to the Planck scale [11].

In this paper we show that for pure "Einstein-monopole" configurations with quantum Casimir corrections coming from scalar matter fields, no stable compactification is possible, either for one or a product of internal spheres of d dimensions, d being the rank of the antisymmetric field strength that generates the monopole configuration. In fact, if we picture the internal radius as a scalar field with a potential, 4-dimensional Minkowski space-time is a solution when the value of the field is a maximum of the potential and thus is unstable against perturbations. In addition, the potential is unbounded from below, making the other extremum point, the de Sitter solution for the physical space, unstable against barrier penetration.

By including fermionic contributions, the above conclusions can be avoided. Type I or heterotic string [12] theories contain $N=1$ supersymmetry coupled to $N=1$ super-Yang-Mills [13]. This contains an antisymmetric rank-2 tensor with its accompanying three-index field strength H. We can place H in a monopole configuration on two internal S^3 and we can do the same for fermionic condensates. We find that if the two internal radii are nearly equal, a stable compactification can be achieved. Unfortunately, due to technical difficulties in computing the Casimir contributions of the relevant fields, we cannot say whether this compactification is absolutely stable when the radii are not approximately equal.

The paper is organized as follows: in sect. 2 we develop the basic formalism and assumptions and write the expressions for the monopole configurations and for the Casimir corrections to be used later. In sect. 3 we apply the general formalism to two examples with only gravity and an antisymmetric tensor field. Sect. 4 is devoted to the analysis of the Chapline-Manton action and a possible stable background is found. We conclude with general remarks in sect. 5.

2. Basic formalism

As we mentioned in the introduction, we are interested in studying the stability of different backgrounds with the $4+D$ dimensional geometry being given by a product of the form $M^{4+D} = R \times S^3 \times \prod_{i=1}^{\alpha} S_i^d$ where S^3 and S_i^d are three- and d-dimensional spheres with $D = \alpha d$. Accordingly, we assume that the metric can be written in block-diagonal form as a straightforward generalization of the Robertson-Walker metric in 4 dimensions [14],

$$g_{MN} = \begin{pmatrix} -1 & & & & \\ & a^2(t)\tilde{g}_{ij}(x^k) & & & \\ & & b_1^2(t)\tilde{g}_{m_1 n_1}(y_1^p) & & \\ & & & \ddots & \\ & & & & b_\alpha^2(t)\tilde{g}_{m_\alpha n_\alpha}(y_\alpha^p) \end{pmatrix}, \quad (1)$$

where capital latin indices cover $4+D$ dimensions, $i, j, k = 1, 2, 3$ and m_i, n_i cover each d-dimensional internal sphere. $\tilde{g}_{ij}$ is the maximally symmetric metric of the physical 3-space and $\tilde{g}_{m_i n_i}$ is the unit metric for each internal sphere.

It is easy to generalize this metric to include spheres of different dimensionality, which would be a natural step if there were antisymmetric tensors with different ranks, but this will not be necessary for our purposes.

In order to be consistent with the symmetries of the metric in eq. (1), the energy-momentum tensor must be written as

$$T_{MN} = \begin{pmatrix} \rho & & & & \\ & p g_{ij} & & & \\ & & p^{(1)} g_{m_1 n_1} & & \\ & & & \ddots & \\ & & & & p^{(\alpha)} g_{m_\alpha n_\alpha} \end{pmatrix}, \tag{2}$$

where ρ is the energy density, p is the pressure in the physical 3-space and $p^{(i)}$ is the pressure for each internal space. These can be at most functions of time.

With eqs. (1) and (2), Einstein's equations assume the general form (we take $16\pi G_{4+D} = c = 1$),

$$3\frac{\ddot{a}}{a} + d\sum_{i=1}^{\alpha}\frac{\ddot{b}_i}{b_i} = -\left[\rho + \frac{1}{D+2}\left(-\rho + 3p + d\sum_{i=1}^{\alpha} p^{(i)}\right)\right], \tag{3a}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + d\frac{\dot{a}}{a}\sum_{i=1}^{\alpha}\frac{\dot{b}_i}{b_i} + \frac{2k}{a^2} = p - \frac{1}{D+2}\left[-\rho + 3p + d\sum_{i=1}^{\alpha} p^{(i)}\right], \tag{3b}$$

$$\frac{\ddot{b}_i}{b_i} + (d-1)\frac{\dot{b}_i^2}{b_i^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}_i}{b_i} + d\frac{\dot{b}_i}{b_i}\sum_{j\neq i}\frac{\dot{b}_j}{b_j} + \frac{(d-1)}{b_i^2}$$
$$= p^{(i)} - \frac{1}{D+2}\left[-\rho + 3p + d\sum_{i=1}^{\alpha} p^{(i)}\right], \tag{3c}$$

where the last equation should be written for each internal sphere.

In order to develop the basic formalism to be applied to the supergravity model, we start considering a simpler model with only gravity and an antisymmetric tensor field of rank $d-1$ as the dynamic fields. [10] The field strength, $F_{MN\ldots Q}$, is then a rank d antisymmetric tensor field with energy-momentum tensor given by

$$T^{\mathrm{m}}_{MN} = F_{MP\ldots S}F_N^{\;P\ldots S} - \frac{1}{2d}F_{PQ\ldots S}F^{PQ\ldots S}g_{MN}, \tag{4}$$

where the superscript m refers to monopole contribution.

In this paper we will be interested in the case when the field $F_{M\ldots S}$ takes values only in the internal space. Thus, the most natural ansatz consistent with compactification is given by the Freund-Rubin or "monopole" ansatz,

$$F_{MN\dots Q}=\begin{cases}\sqrt{g^{(d)}}\,\varepsilon_{m_i n_i\dots q_i}f^{(i)}(t), & \text{for each } d\text{-dimensional sphere}\\ 0, & \text{otherwise},\end{cases} \tag{5}$$

where $f^{(i)}(t)$ are functions of time and $g^{(d)}$ is the determinant of the S^d metric. This ansatz guarantees that the field equation for $F_{MN\dots Q}$ is trivially satisfied.

We can use the Bianchi identities for $F_{MN\dots Q}$ to express $f^{(i)}(t)$ in terms of the internal radius $b_i(t)$,

$$f^{(i)}(t)=f_0^{(i)}/b_i^d(t), \qquad f_0^{(i)} \text{ is a constant}. \tag{6}$$

Using (5) and (6) the only non-vanishing components of the energy-momentum tensor are

$$T^{\mathrm{m}}_{\mu\nu}=-\tfrac{1}{2}(d-1)!\sum_{i=1}^{\alpha}\left[\frac{f_0^{(i)}}{b_i^d}\right]^2 g_{\mu\nu}; \qquad \mu,\nu=0,1,2,3, \tag{7a}$$

$$T^{\mathrm{m}}_{m_i n_i}=\tfrac{1}{2}(d-1)!\left[\left[\frac{f_0^{(i)}}{b_i^d}\right]^2-\sum_{j\neq i}\left[\frac{f_0^{(j)}}{b_j^d}\right]^2\right]g_{m_i n_i}. \tag{7b}$$

It is now a simple matter to compare the coefficients in (7) and (2) to express the monopole contribution to the energy-momentum tensor as the energy density ρ and the pressures p and $p^{(i)}$.

The other contribution to be included in the energy-momentum tensor comes from one-loop corrections to the action due to vacuum fluctuations of matter fields [11]. As is by now well known, the fact that we must impose periodic boundary conditions on the quantum fields due to the compactness of the internal space, gives rise to an effect analogous to the Casimir effect of the electromagnetic field.

The Casimir effect has been calculated in a number of different situations within the Kaluza-Klein framework. The calculations are usually restricted to odd dimensional spaces because of restrictions with the ζ-function regularization procedure commonly adopted. In even dimensions there will be an explicit dependence of the free energy on the parameter that sets the scale of the path integral which can, nevertheless, be fixed by imposing certain conditions on the effective potential.

For the particular case of a $4+D$ dimensional manifold $R\times S^3\times S^D$, the free energy is, in the limit where $a\to\infty$ and zero temperature [15],

$$U=V_3\frac{A+A'\ln(2\pi\rho^2)}{b^4}, \tag{8}$$

where A and A' are calculable numerical constants, $\rho^2 = \mu^2 b^2$, μ being the scale mentioned above and $V_3 = 2\pi^2 a^3$ is the volume of S^3. If $4+D$ is odd, the coefficient A' vanishes.

As a first simplifying step, we are going to neglect the logarithmic dependence on the radius. This approach is justified if the internal radius is a slowly-varying function of time, thus allowing us to consider the numerator in (8) as approximately constant. In any case, we believe that the general qualitative picture to be obtained will not be drastically modified by the inclusion of the logarithmic correction.

As we are interested in a general background of the form $R \times S^3 \times \prod_{i=1}^{\alpha} S^d$, the following expression for the free energy due to Casimir corrections will be adopted,

$$U = V_3 \sum_{i=1}^{\alpha} \frac{A^{(i)}}{b_i^4} . \tag{9}$$

This is again an approximate expression which, nevertheless, is supported by some recent calculations for manifolds of product form $M^4 \times S^M \times S^N$ where stable solutions with finite values for the internal radii are found [16]. (Again, this calculation was originally developed for odd dimensional spaces.)

Using the above expression for the free energy, we obtain the energy density and pressures due to the Casimir correction as

$$\rho_c \equiv \frac{1}{V_3 \prod_{i=1}^{\alpha} V_d^{(i)}} U = \frac{1}{\prod_{i=1}^{\alpha} V_d^{(i)}} \sum_{i=1}^{\alpha} \frac{A^{(i)}}{b_i^4} , \tag{10a}$$

$$p_c \equiv -\frac{1}{3 V_3 \prod_{i=1}^{\alpha} V_d^{(i)}} \left(a \frac{\partial U}{\partial a} \right) = -\frac{1}{\prod_{i=1}^{\alpha} V_d^{(i)}} \sum_{i=1}^{\alpha} \frac{A^{(i)}}{b_i^4} , \tag{10b}$$

$$p_c^{(i)} \equiv -\frac{1}{d V_3 \prod_{i=1}^{\alpha} V_d^{(i)}} \left(b_i \frac{\partial U}{\partial b_i} \right) = \frac{4 A^{(i)}}{d \prod_{i=1}^{\alpha} V_d^{(i)}} \frac{1}{b_i^4} , \tag{10c}$$

where

$$V_d^{(i)} = \frac{(2\pi)^{(d+1)/2} b_i^d}{\Gamma\left(\frac{1}{2}(d+1)\right)}$$

is the volume of the d-dimensional sphere. We can see that the Casimir part of the energy-momentum tensor is traceless.

Using eqs. (7) and (10), we obtain the final expressions for the energy density and pressures as

$$\rho=\rho_{\mathrm{m}}+\rho_{\mathrm{c}}=\frac{1}{\prod_{i=1}^{\alpha}V_d^{(i)}}\sum_{i=1}^{\alpha}\frac{A^{(i)}}{b_i^4}+\tfrac{1}{2}(d-1)!\sum_{i=1}^{\alpha}\left[\frac{f_0^{(i)}}{b_i^d}\right]^2, \tag{11a}$$

$$p=p_{\mathrm{m}}+p_{\mathrm{c}}=-\rho, \tag{11b}$$

$$p^{(i)}=p_{\mathrm{m}}^{(i)}+p_{\mathrm{c}}^{(i)}=\frac{4A^{(i)}}{d\prod_{i=1}^{\alpha}V_d^{(i)}}\frac{1}{b_i^4}+\tfrac{1}{2}(d-1)!\left[\left[\frac{f_0^{(i)}}{b_i^d}\right]^2-\sum_{j\neq i}\left[\frac{f_0^{(j)}}{b_j^d}\right]^2\right]. \tag{11c}$$

Given a certain model with geometry $M^{4+D}=\mathrm{R}\times\mathrm{S}^3\times\prod_{i=1}^{\alpha}\mathrm{S}_i^d$, it is easy to obtain the above quantities and substitute them in Einstein's equations (3) to solve for the scale factors. The fact that $\rho=-p$ allows us to define a four-dimensional effective cosmological constant that depends on the internal radius b. The equilibrium value for b will determine if Λ_4 vanishes or not.

In the next section, we are going to apply the above formalism to two specific models, with one D-dimensional internal sphere and two 3-spheres, respectively.

3. Examples

We will start by considering a model with one internal sphere of D-dimensions. This approach naturally generalizes the 6-dimensional Einstein-Maxwell or 7-dimensional Einstein-Kalb-Ramond models.

As we have only one internal space, no summation is necessary in eq. (11). We suppress the i index and call the internal radius $b(t)$. Also, in the Casimir contribution, we redefine the constant A to absorb the numerical factors coming from the definition of the volume of the D-dimensional sphere. For example, if $D=3$, we have $A'\equiv A/2\pi^2$. Thus, Einstein's equations are simply,

$$3\frac{\ddot{a}}{a}+D\frac{\ddot{b}}{b}=-\frac{1}{D+2}\left[\frac{(D+2)A'}{b^{4+D}}+\frac{2(D-1)}{b^{2D}}c\right], \tag{12a}$$

$$\frac{\ddot{a}}{a}+2\frac{\dot{a}^2}{a^2}+D\frac{\dot{a}}{a}\frac{\dot{b}}{b}=-\frac{1}{D+2}\left[\frac{(D+2)A'}{b^{4+D}}+\frac{2(D-1)}{b^{2D}}c\right], \tag{12b}$$

$$\frac{\ddot{b}}{b}+(D-1)\frac{\dot{b}^2}{b^2}+3\frac{\dot{a}}{a}\frac{\dot{b}}{b}=-\frac{D-1}{b^2}+\frac{2}{D+2}\left[\frac{2(D+2)A'}{b^{4+D}}+\frac{3c}{b^{2D}}\right], \tag{12c}$$

where

$$c\equiv\tfrac{1}{2}(D-1)!f_0^2.$$

We are interested in the situation where, for critical values of $a = a_0$, $b = b_0$, the monopole and Casimir contributions will balance each other thus rendering b_0 stable. It is desirable that no net 4-dimensional effective cosmological constant is left over in the physical space-time once the internal radius becomes constant since this would induce a classically stable de Sitter phase. In other words, the ideal equilibrium situation would be given by a constant internal radius and a Minkowski space-time. Radiation contributions to the stress tensor may be added to induce a smooth transition to a Friedmann universe.

These two conditions can be imposed into eq. (12) by requiring that the r.h.s. of both (12a) or (12b) and (12c) vanishes at $a = a_0$, $b = b_0$. We can then express the constants A' and c in terms of the constant value b_0 as

$$A' = \frac{D(D-1)^2}{D-4} b_0^{D+2}, \qquad c = -\frac{D(D+2)(D-1)}{2(D-4)} b_0^{2(D-1)}. \tag{13}$$

As the coefficient A' is, in principle, calculable, the equilibrium value of b_0 can be eventually determined within a realistic framework. Note that for $D > 4$, the monopole coefficient must be negative which is inconsistent with our ansatz. This is perhaps connected with the fact that known theories have at most forms of rank four.

In order to study the stability of the constant value b_0, it is helpful to introduce the logarithmic variable $\phi = \ln(b/b_0)$ in the dynamical equation for the internal radius, eq. (12c). We then obtain,

$$\ddot{\phi} + \left(D\dot{\phi} + 3\frac{\dot{a}}{a} \right)\dot{\phi} = -\frac{D-1}{(D-4)b_0^2}\left[(D-4)e^{-2\phi} - 4(D-1)e^{-(D+4)\phi} + 3D\,e^{-2D\phi} \right], \tag{14}$$

where the r.h.s. can be written $-\partial V/\partial\phi$ with V being interpreted as a potential for ϕ. The next step is to look for the extrema of this potential for $\phi = \phi_c$. We are then forced to solve a polynomial with a degree that varies with D. There is no obvious solution to this unless for some particular cases like $D = 2$. Nevertheless, for the critical value $b = b_0$, we have that $\phi_c = 0$ which is, of course, always a solution for any D. For $b = b_0$, the second derivative of the potential is,

$$\left.\frac{\partial^2 V}{\partial\phi^2}\right|_{\phi_c=0} = -\frac{(D-1)^2}{b_0^2}. \tag{15}$$

Thus, the solution with constant b and 4-dimensional Minkowski space-time is always located at the maximum of the potential for any number of dimensions

($D \neq 1$ by definition). From the general expression for the potential,

$$V(\phi) = -\frac{D-1}{(D-4)b_0^2}\left[\tfrac{1}{2}(D-4)e^{-2\phi} - 4\frac{D-1}{D+4}e^{-(D+4)\phi} + \tfrac{3}{2}e^{-2D\phi} - \frac{(D-1)(D+4)}{2(D+4)}\right], \tag{16}$$

it is easy to check that $\lim_{\phi\to\infty} V(\phi) = (D-1)^2/2(D+4)b_0^2$, where we have fixed the integration constant such that $V(0) = 0$. The reader should note that the effective 4-dimensional cosmological constant is *not* defined by $V(\phi)$, but by the r.h.s. of eq. (12a) or (12b). $V(\phi)$ is *not* the inflationary potential as in 4-dimensional theories. The fixed points of this potential will determine values of b that in turn will define a cosmological term in eq. (12a). This is a common source of confusion in Kaluza-Klein inflationary scenarios.

In fig. 1 we plot the potential for the 6-dimensional case. It is easy to verify the existence of a local minimum at $\phi_c = \ln\sqrt{2}$ or $b_c = \sqrt{2}\, b_0$. This value gives rise to an anti-de Sitter solution for the physical space-time which can be shown to be unstable under semi-classical tunneling [7].

As our next example, we will present the analysis for the case with two internal 3-spheres. This is of particular interest for our analysis of ten-dimensional super-

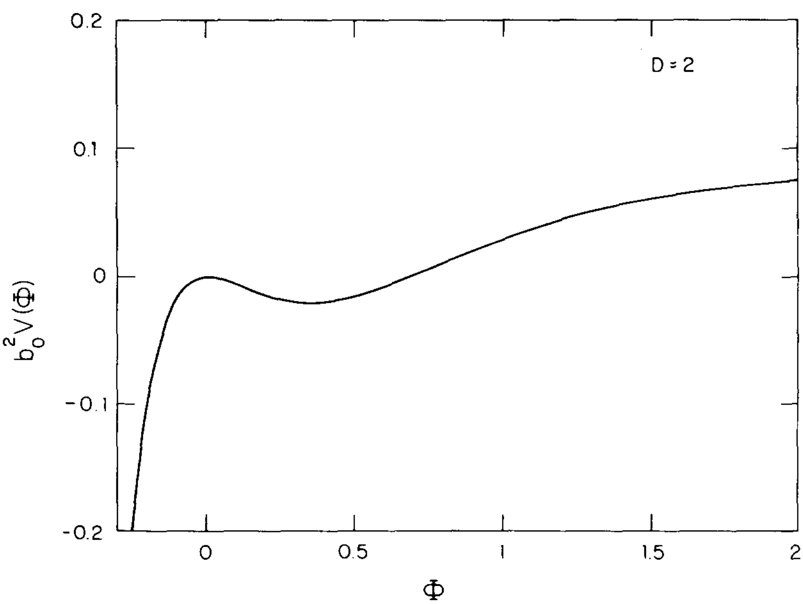

Fig. 1. The potential for the internal radius $b(t) = b_0\exp(\phi)$ for a 6-dimensional Einstein-Maxwell model with an internal 2-sphere. The maximum corresponds to a 4-dimensional Minkowski space-time.

gravity. The S^3 compactification is obtained with a rank-3 tensor F_{MNP}.

From eq. (11), we can write the energy density and the pressure as

$$\rho = \frac{A'^{(1)}}{b_1^7 b_2^3} + \frac{A'^{(2)}}{b_2^7 b_1^3} + \frac{c^{(1)}}{b_1^6} + \frac{c^{(2)}}{b_2^6} = -p\,, \tag{17a}$$

$$p_1 = \frac{4A'^{(1)}}{3b_1^7 b_2^3} + \frac{c^{(1)}}{b_1^6} - \frac{c^{(2)}}{b_2^6}\,, \tag{17b}$$

$$p_2 = \frac{4A'^{(2)}}{3b_2^7 b_1^3} + \frac{c^{(2)}}{b_2^6} - \frac{c^{(1)}}{b_1^6}\,, \tag{17c}$$

where, as before, $A'^{(i)} = A^{(i)}/4\pi^4$; $c^{(i)} = (f_0^{(i)})^2$.

We could now try to generalize our previous analysis to a two-dimensional potential for two "scalar fields" ϕ_1 and ϕ_2 and look for stable configurations for b_1 and b_2. Instead, we are going to consider only the simpler case where the two radii have the same value, $b_1 = b_2 = b$.

With this assumption and taking $A'^{(1)} = A'^{(2)} = A'$, $c^{(1)} = c^{(2)} = c$, Einstein's equations (3) reduce to

$$3\frac{\ddot{a}}{a} + 6\frac{\ddot{b}}{b} = -\left[\frac{2A'}{b^{10}} + \frac{c}{b^6}\right], \tag{18a}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 6\frac{\dot{a}}{a}\frac{\dot{b}}{b} = -\left[\frac{2A'}{b^{10}} + \frac{c}{b^6}\right], \tag{18b}$$

$$\frac{\ddot{b}}{b} + 5\frac{\dot{b}^2}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} = -\frac{2}{b^2} + \frac{4A'}{3b^{10}} + \frac{c}{b^6}\,. \tag{18c}$$

After imposing the vanishing of the r.h.s. of eq. (18), at $a = a_0$, $b = b_0$ and introducing the variable $\phi = \ln(b/b_0)$ as before, we obtain the potential for ϕ from eq. (18c) as

$$V(\phi) = \frac{2}{b_0^2}\left[-\tfrac{1}{2}e^{-2\phi} - \tfrac{1}{5}e^{-10\phi} + \tfrac{1}{2}e^{-6\phi} + \tfrac{1}{5}\right]. \tag{19}$$

This potential has the same shape as the potential of fig. 1, with a maximum at $\phi_c = 0$ or $b_c = b_0$ and a minimum at $\phi_c = \frac{1}{4}\ln 2$.

Thus, for one sphere of arbitrary dimensions or for two 3-spheres for the internal space (and, we believe, any number of them), it is not possible to obtain a configuration which has a stable internal radius and a vanishing effective cosmological constant for the physical space-time.

In the next section we will see that this situation is radically changed for the Chapline-Manton action. We will obtain a potential with a global minimum, compatible with a four-dimensional Minkowski space-time.

4. Stability from ten-dimensional supergravity

We now turn our attention to the $N=1$ ten-dimensional supergravity action which describes the point-like limit of the heterotic and type I superstring models [17]. We will show that we can use the fermionic condensates to provide a stable background geometry. The idea that fermionic condensates have some relationship with the vanishing of the cosmological constant has been used before in Kaluza-Klein supergravity [18].

It has recently been used in the context of ten-dimensional supergravity with a Calabi-Yau compactification in order to provide a mechanism for supersymmetry breaking with zero net cosmological constant at the tree level [19]. The cosmological stability of this model has been studied by Maeda [20], who found a *classically* stable product manifold of (4-dimensional Friedmann) × (Calabi-Yau) for a certain range of initial conditions. The potential for this model closely resembles that obtained for the semi-classically unstable monopole (or Casimir) and cosmological constant since the role of the fermionic condensate is similar to that of Λ.

Here we are going to assume that the condensates take values in the internal space [21], thus contributing to the curvature energy of the monopole configuration that comes from the 3-index field H_{MNP} which is defined in terms of the Kalb-Ramond and the Yang-Mills and Lorentz Chern-Simons 3-forms. This combined monopole configuration is going to be balanced by the Casimir contribution, providing the desired stable background with zero net four-dimensional effective cosmological constant.

The bosonic part of the action including the gluino and subgravitino couplings is given by

$$S=-\tfrac{1}{2}\int \mathrm{d}^{10}z\sqrt{-g^{(10)}}\Big[R+\tfrac{3}{2}\mathrm{e}^{-\sigma}\big[H_{MNP}-\mathrm{e}^{\sigma/2}(\mathrm{Tr}\,\bar{\chi}\Gamma_{MNP}\chi)\big]^2 + \tfrac{1}{2}\mathrm{e}^{-\sigma/2}\big(\mathrm{Tr}\,G_{MN}G^{MN}\big)+\tfrac{1}{2}\,\partial_M\sigma\,\partial^M\sigma+(\mathrm{Tr}\,\bar{\chi}\Gamma_{MNP}\chi)(\bar{\lambda}\Gamma^{MNP}\lambda)\Big], \quad (20)$$

where R is the 10-dimensional curvature scalar, σ is the dilaton field, G_{MN} is the Yang-Mills field strength, H_{MNP} was defined above and χ and λ are the gluino and subgravitino fields, respectively.

Once we assume that the 3-form H and the fermionic condensates will take values in the internal space, the most natural ansatz for compactification is given by the Freund-Rubin ansatz, with the internal geometry given by the product of two 3-spheres. This explains our interest in the second example of sect. 3. We will thus set the Yang-Mills field to zero and take the dilaton field to be constant in space-time. The latter assumption can be relaxed in a more general treatment by taking the dilation field to be a function of time.

By setting the Yang-Mills field to zero, we lose the possibility of obtaining chiral fermions from the model. However, our present concern is with the question of stability of the compactification.

From the action (20), we obtain from the dilation field equation (with $\sigma = \sigma_0$) that

$$e^{-\sigma_0}(H_{MNP})^2 = \tfrac{3}{2} e^{-\sigma_0/2} H_{MNP}(\mathrm{Tr}\,\bar{\chi}\Gamma^{MNP}\chi)\,. \tag{21}$$

Using eq. (21) and introducing $\mathscr{H}_{MNP} \equiv e^{-\sigma_0/2} H_{MNP}$, Einstein's equations become,

$$R_{MN} = \tfrac{9}{2}\mathscr{H}_{MPQ}\mathscr{H}_N^{\ PQ} - \tfrac{1}{2}\mathscr{H}_{PQR}\mathscr{H}^{PQR} g_{MN} - \tfrac{1}{8}(\mathrm{Tr}\,\bar{\chi}\Gamma_{PQR}\chi)(\bar{\lambda}\Gamma^{PQR}\lambda) g_{MN}$$
$$- \tfrac{3}{16}(\mathrm{Tr}\,\bar{\chi}\Gamma_{PQR}\chi)(\mathrm{Tr}\,\bar{\chi}\Gamma^{PQR}\chi) g_{MN} + \tfrac{9}{2}\mathscr{H}_M^{\ PQ}\,\mathrm{Tr}\,\bar{\chi}\Gamma_{NPQ}\chi\,. \tag{22}$$

The Freund-Rubin ansatz for $\mathscr{H}_{MNP}$ and the fermionic condensation is the same as eq. (5) where now $i = 1,2$, $d = 3$, and we introduce the constants h_0, χ_0 and λ_0 for $\mathscr{H}_{MNP}$, χ and λ, respectively.

We could now proceed by substituting this ansatz into Einstein's equations which would describe the evolution of the scale factors $a(t)$, $b_1(t)$, $b_2(t)$ and then study the stability of the dynamical system in the phase plane of $b_1(t)$ and $b_2(t)$. Nevertheless, we again adopt a simpler point of view and study the evolution of the system when the two radii are equal.

Adding the general form of the Casimir contribution to the r.h.s. of eq. (22) and taking $b_1 = b_2 = b$, $\chi_0^{(i)} = \chi_0$, $h_0^{(i)} = h_0$, $\lambda_0^{(i)} = \lambda_0$ for $i = 1,2$, we can write Einstein's equations as,

$$3\frac{\ddot{a}}{a} + 6\frac{\ddot{b}}{b} = -\left[\frac{2A'}{b^{10}} - \frac{c}{b^6}\right], \tag{23a}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 6\frac{\dot{a}}{a}\frac{\dot{b}}{b} = -\left[\frac{2A'}{b^{10}} - \frac{c}{b^6}\right], \tag{23b}$$

$$\frac{\ddot{b}}{b} + 5\frac{\dot{b}^2}{b^2} + 3\frac{\dot{a}}{a}\frac{\dot{b}}{b} = -\frac{2}{b^2} + \frac{4A'}{3b^{10}} + \frac{c'}{b^6}, \tag{23c}$$

where, as before $A' \equiv 4\pi^4 A$. The basic difference to the previous example comes in the two monopole coefficients, c and c', which are given by

$$c \equiv 3\left(2h_0^2 + \tfrac{1}{2}\chi_0\lambda_0 + \tfrac{3}{4}\chi_0^2\right), \tag{24a}$$

$$c' \equiv 3\left(-h_0^2 - 3h_0\chi_0 + \tfrac{1}{2}\chi_0\lambda_0 + \tfrac{3}{4}\chi_0^2\right). \tag{24b}$$

At first glance to eq. (24), one may think that by setting the fermionic fields to zero we should reproduce the previous case and have $c = c'$ (c.f. eq. (18)). The modification in the above was brought in by the use of the dilaton field equation that relates $\mathscr{H}^2$ terms to $\mathscr{H}\bar{\chi}\chi$ terms.

We should point out that the theorem for 10 into 4 compactification [22] will not apply here since it does not hold for non-maximally symmetric four-dimensional space-times and for time-dependent fields.

From eq. (23) we see that now we have three independent constants, h_0, χ_0 and λ_0 (A' is in principle calculable) to relate to the constant value of the radius, b_0, by means of our two stability conditions. Clearly, there is a degree of arbitrariness in the choice of the constants which is unavoidable at this level. To avoid a naive and always convenient fine-tuning, we will obtain the conditions for A', c' and c that would provide a stable internal radius b_0 and a zero net cosmological constant.

The requirement of zero net cosmological constant fixes the value of c in terms of A' and b_0 as

$$c = 2A'b_0^4 . \tag{25}$$

From the requirement that b_0 is a critical point, we obtain,

$$c' = 2b_0^4 - \tfrac{2}{3}c . \tag{26}$$

We can now rewrite Einstein's equation for the internal radius in terms of the logarithmic variable ϕ and try to use the properties of the potential $V(\phi)$ to get a bound for say, c' for a given value of b_0. Following the same procedure as before, we obtain for the derivative of $V(\phi)$,

$$\frac{\partial V}{\partial \phi} = -\frac{1}{b_0^2}\left[-2\,\mathrm{e}^{-2\phi} + \frac{c'}{b_0^4}\mathrm{e}^{-6\phi} + \frac{\left(2b_0^4 - c'\right)}{b_0^4}\mathrm{e}^{-10\phi}\right], \tag{27}$$

which has, in principle, two critical points,

$$\phi_{c_1} = 0 , \qquad \phi_{c_2} = -\tfrac{1}{4}\ln\left[\frac{-2b_0^4}{\left(2b_0^4 - c'\right)}\right], \tag{28}$$

where ϕ_{c_2} is only defined for $2b_0^4 < c'$, $c' > 0$.

For ϕ_{c_1}, we get that the second derivative is

$$\left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi_{c1}=0} = \frac{4}{b_0^6}\left(4b_0^4 - c'\right). \tag{29}$$

Thus, we obtain a minimum for $4b_0^4 > c'$ (for $c' > 0$) or for $c' < 0$. These conditions can be easily rewritten in terms of the constants h_0, χ_0 and λ_0, using the definition of c'.

By comparison with the conditions for ϕ_{c_2}, we see that only an accurate fine-tuning would allow it to occur. In any case, from the expression for the potential, with $V(0)=0$,

$$V(\phi)=\frac{1}{b_0^2}\left[-e^{-2\phi}+\frac{c'}{6b_0^4}e^{-6\phi}+\frac{2b_0^4-c'}{10b_0^4}e^{-10\phi}+\frac{12b_0^4-c'}{15b_0^4}\right]. \qquad (30)$$

It is easy to see that for $2b_0^4 < c'$, we reproduce the same undesirable situation as in the previous section.

An easy way of realizing the correct conditions for stability is to set $\chi_0 = -h_0$ in eq. (24). For this choice, $c = c'$ and from (26) we get that $c = c' = \frac{6}{5}b_0^4$, which is well within the above condition. In fig. 2, we plot the potential $V(\phi)$ for this particular choice of the constants. A similar potential has been found for $N = 2$, 6-dimensional supergravity [23].

Varying the value of c' or of the constant χ_0 will move the minimum up and down until the bound $4b_0^4 = c'$ is reached. For $c' > 4b_0^4$, the situation is inverted and we reproduce the same qualitative features shown in fig. 1.

It is an interesting feature of supergravity that no strict fine-tuning is needed to reach stability but that, instead, there is a range of values for the fields which are

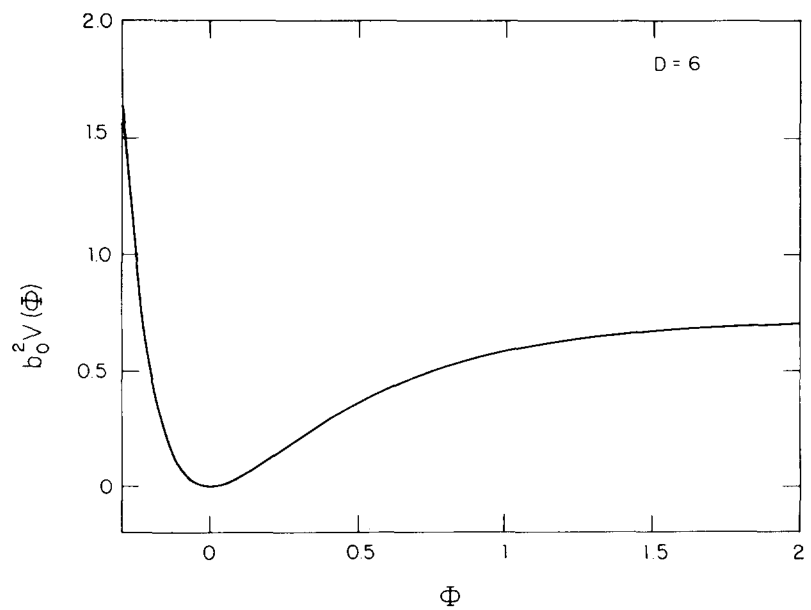

Fig. 2. The potential for 10-dimensional supergravity with fermionic condensates taking values in the internal space. Note that the potential has a global minimum that guarantees stability in the limit where the internal radii are equal.

compatible with the imposed conditions. However as mentioned in the introduction, we do not know whether the system is semi-classically stable away from the $b_1 = b_2$ line phase space.

5. Conclusions

We have shown that by adding Casimir contributions to monopole configurations in general $4 + D$ dimensional models with gravity and an antisymmetric tensor field of rank D for one internal sphere or of rank $d = D/\alpha$ for α internal spheres, it is not possible to obtain a stable compactification where the internal radius is constant with no net effective cosmological constant in four dimensions. The possible critical points for the potential for the internal radius induce either an unstable Minkowski or a semi-classically unstable anti-de Sitter solution for the four-dimensional space-time.

This situation is modified when we consider the action coming from the point-like limit of superstring theories. In this case, we can add fermionic condensates to the monopole configuration which will render the Minkowski solution for the physical space-time stable, at least for a portion of the phase space.

The fact that the potential is flat for large values of the internal space naively suggests that there may be a slow rollover which will produce enough inflation to solve the horizon or flatness problems. One would start far on the right of fig. 2, producing a net cosmological constant as required by inflation, and roll down to the minimum. Some preliminary calculations show that this is not the case* for reasonable choices of the initial value of ϕ. The possibility of reheating is a more viable one, if we associate particle creation with the oscillations of the internal radius about the minimum at b_0. This might induce a smooth transition to a Friedmann cosmology.

This work was supported in part by the Department of Energy and the National Aeronautics and Space Administration. One of us (M.G.) would like to thank CNPq of Brazil for financial support during the completion of this work.

References

[1] E. Witten, Nucl. Phys. B186 (1981) 412;
A. Salam and J. Strathdee, Ann. of Phys. 141 (1982) 316 and references therein
[2] M.B. Green and J.H. Schwarz, Phys. Lett. 109B (1981) 444;
J.H. Schwarz, Phys. Reports 9 (1982) 223
[3] M.B. Green and J.H. Schwarz, Phys. Lett. 149B (1984) 117
[4] M.B. Green and J.H. Schwarz, Phys. Lett. 151B (1985) 21
[5] W.J. Marciano, Phys. Rev. Lett. 52 (1984) 489;
E.W. Kolb, M.J. Perry and T.P. Walker, Phys. Rev. D33 (1985) 869;
M. Gleiser and J.G. Taylor, Phys. Rev. D33 (1986) 570, D31 (1985) 756

* We would like to thank Michael Turner for pointing this out.

[6] K. Maeda, SISSA preprints 34/85/A, and 69/85/A (1985);
E. J. Copeland and D.J. Toms, Phys. Rev. D32 (1985) 1921
[7] J.A. Frieman and E.W. Kolb, Phys. Rev. Lett. 55 (1985) 1435
[8] F.S. Accetta and E.W. Kolb, Fermilab preprint (1986)
[9] D. Bailin, A. Love and J.A. Stein-Schabes, Nucl. Phys. B253 (1985) 387;
Y.Okada, Nucl. Phys. B264 (1986) 197
[10] P.G.O. Freund and M.A. Rubin, Phys. Lett. 97B (1980) 233
[11] T. Appelquist and A. Chodos, Phys. Rev. Lett. 50 (1983) 141;
P. Candelas and S. Weinberg, Nucl. Phys. B237 (1984) 397;
S. Randjbar-Daemi, A. Salam and J. Strathdee, Phys. Lett. 135B (1984) 388
[12] D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. 54 (1985) 502
[13] C.G. Callan, E. Martinec, M.J. Perry and D. Friedan, Nucl. Phys. B262 (1985) 593
[14] P.G.O. Freund, Nucl. Phys. B209 (1982) 146;
M. Gleiser, S. Rajpoot and J.G. Taylor, Ann. of Phys. 160 (1985) 299
[15] E. Myers, Brookhaven National Laboratory preprint BNL 36518 (March 1985)
[16] K. Kikkawa, T. Kubota, S. Sawada and M. Yamasaki, Nucl. Phys. B260 (1985) 429
[17] A.H. Chamseddine, Nucl. Phys. B185 (1981) 403;
G.F. Chapline and N.S. Manton, Phys. Lett. 120B (1983) 105
[18] M.J. Duff and C.A. Orzalesi, Phys. Lett. 122B (1983) 37
[19] M. Dine, R. Rohm, N. Seiberg and E. Witten, Phys. Lett. 156B (1985) 151
[20] K. Maeda, Phys. Lett. 166B (1986) 59
[21] P. Oh, Phys. Lett. 166B (1986) 292
[22] D. Freedman, G. Gibbons and P. West, Phys. Lett. 124B (1983) 491
[23] K. Maeda and H. Nishino, Phys. Lett. 154B (1985) 358