International Journal of Modern Physics D
Vol. 13, No. 7 (2004) 1389–1397

ANISTROPIC STARS: EXACT SOLUTIONS AND STABILITY

MARCELO GLEISER
Department of Physics and Astronomy, Dartmouth College,
NH 03755 Hanover, USA
gleiser@dartmouth.edu

KRSNA DEV
Department of Physics, Haverford College Haverford, PA 19041, USA
kdev@haverford.edu

Received 26 February 2004

I report on recent work concerning the existence and stability of self-gravitating spheres with anisotropic pressure. After presenting new exact solutions, Chandrasekhar's variational formalism for radial perturbations is generalized to anisotropic objects and applied to investigate their stability. It is shown that anisotropy can not only support stars of mass M and radius R with $2M/R \geq 8/9$ and arbitrarily large surface redshifts, but that stable configurations exist for values of the adiabatic index γ smaller than the corresponding isotropic value.

Keywords: Relativistic astrophysics; anisotropic stars; stability.

1. Introduction

A common assumption in the study of stellar structure and evolution is that the interior of a star can be modeled as a perfect fluid.[1] This perfect fluid model necessarily requires the pressure in the interior of a star to be isotropic. This approach has been used extensively in the study of polytropes, including white dwarfs, and of compact objects such as neutron stars.[2] However, theoretical advances in the last decades indicate that deviations from local isotropy in the pressure, in particular at very high densities, may play an important role in determining stellar properties.[3]

The physical situations where anisotropic pressure may be relevant are very diverse. By anisotropic pressure we mean that the radial component of the pressure $p_r(r)$ differs from the angular components, $p_\theta(r) = p_\varphi(r) \equiv p_t(r)$. (That $p_\theta(r) = p_\varphi(r)$ is a direct consequence of spherical symmetry.) Of course, spherical symmetry demands both to be strictly a function of the radial coordinate. Boson stars, hypothetical self-gravitating compact objects resulting from the coupling of a complex scalar field to gravity, are systems where anisotropic pressure occurs naturally.[4] In the interior of neutron stars pions may condense. It has been shown that due to the geometry of the π^- modes, anisotropic distributions of pressure could be considered to describe a pion condensed phase configuration.[5] The existence of solid cores and type P superfluidity may also lead to departures from isotropy within the neutron star interior.[2]

Since we still do not have a detailed microscopic formulation of the possible anisotropic stresses emerging in these and other contexts, we take the general approach of finding several exact solutions representing different physical situations, modeled by *ansatze* for the anisotropy factor, $p_t - p_r$. Previous studies have found some exact solutions, assuming certain relations for the anisotropy factor.[6] Our goal here is two fold: first, to find new exact solutions which may better model realistic situations and explore their physical properties;[7] second, to investigate their stability against small radial perturbations.[8] For this, we generalize Chandrasekhar's well-known variational approach to anisotropic objects. We find that not only interesting exact solutions can be found,[7] but that they may have a wider stability range when contrasted with their isotropic counterparts.[8]

2. Relativistic Self-Gravitating Spheres: Basics

We consider a static equilibrium distribution of matter which is spherically symmetric. In Schwarzschild coordinates the metric can be written as

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \,, \tag{1}$$

where all functions depend only on the radial coordinate r. The most general energy–momentum tensor compatible with spherical symmetry is

$$T^\mu_\nu = \mathrm{diag}(\rho, -p_r, -p_t, -p_t) \,. \tag{2}$$

We see that isotropy is not required by spherical symmetry; it is an added assumption. The Einstein field equations for this spacetime geometry and matter distribution are

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi p_r \,; \tag{3}$$

$$e^{-\lambda}\left(\frac{1}{2}\nu'' - \frac{1}{4}\lambda'\nu' + \frac{1}{4}(\nu')^2 + \frac{(\nu' - \lambda')}{2r}\right) = 8\pi p_t \,; \tag{4}$$

$$e^{\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi\rho \,. \tag{5}$$

Note that this is a system of 3 equations with 5 unknowns. Consequently, it is necessary to specify two equations of state, such as $p_r = p_r(\rho)$ and $p_t = p_t(\rho)$.

It is often useful to transform the above equations into a form where the hydrodynamical properties of the system are more evident. For systems with isotropic pressure, this formulation results in the Tolman–Oppenheimer–Volkov (TOV) equation. The generalized TOV equation, including anisotropy, is

$$\frac{dp_r}{dr} = -(\rho + p_r)\frac{\nu'}{2} + \frac{2}{r}(p_t - p_r) \,, \tag{6}$$

with

$$\frac{1}{2}\nu' = \frac{m(r) + 4\pi r^3 p_r}{r(r-2m)}\,, \tag{7}$$

and

$$m(r) = \int_0^r 4\pi r^2 \rho dr\,. \tag{8}$$

Taking $r = R$ in the above expression gives us the Schwarzschild mass, M. (This implicitly assumes that $\rho = 0$ for $r > R$.)

In order to solve the above equations we must impose appropriate boundary conditions. We require that the solution be regular at the origin. This imposes the condition that $m(r) \to 0$ as $r \to 0$. If p_r is finite at the origin then $\nu' \to 0$ as $r \to 0$. The gradient dp_r/dr will be finite at $r = 0$ if $(p_t - p_r)$ vanishes at least as rapidly as r when $r \to 0$. This will be the case in all scenarios examined here.

The radius of the star is determined by the condition $p_r(R) = 0$. It is not necessary for $p_t(R)$ to vanish at the surface. But it is reasonable to assume that all physically interesting solutions will have $p_r,\, p_t \geq 0$ for $r \leq R$.

3. Exact Solutions

In Ref. 7, we obtained several exact solutions for different forms of the pressure anisotropy. Our solutions fall into two classes: (i) $\rho =$ constant, and (ii) $\rho \propto 1/r^2$. Given the limited space, we will restrict the presentation to the latter case. Interested readers should consult Ref. 7 for details.

Consider stars with energy density modeled as

$$\rho = \frac{1}{8\pi}\left(\frac{a}{r^2} + 3b\right), \tag{9}$$

where both a and b are constant. The choice of the values for a and b is dictated by the physical configuration under consideration. For example, $a = 3/7$ and $b = 0$, corresponds to a relativistic Fermi gas, as in the Misner–Zapolsky solution for ultradense cores of neutron stars.[9] If we take $a = 3/7$ and $b \neq 0$ then we have a relativistic Fermi gas core immersed in a constant density background. For large r the constant density term dominates ($r_c^2 \gg a/3b$), and can be thought of as modelling a shell surrounding the core. We also take the pressure anisotropy to be

$$p_t - p_r = \frac{1}{8\pi}\left(\frac{c}{r^2} + d\right), \tag{10}$$

with c and d constant.

We found it convenient to seek solutions for the metric function $\nu(r)$ directly, rather than solving the generalized TOV equation. We then use the known functions

$\lambda(r)$ and $\nu(r)$ to find the radial and tangential pressures. From Eqs. (3), (4), and (5), we find

$$\left(\frac{\nu''}{2}+\frac{(\nu')^2}{4}\right)e^{-\lambda}-\nu'\left(\frac{\lambda'}{4}+\frac{1}{2r}\right)e^{-\lambda}$$
$$-\left(\frac{1}{r^2}+\frac{\lambda'}{2r}\right)e^{-\lambda}+\frac{1}{r^2}=8\pi(p_t-p_r)\,. \tag{11}$$

Introducing a new variable $y=e^{\frac{\nu}{2}}$, Eq. (11) becomes,

$$(y'')e^{-\lambda}-y'\left(\frac{\lambda'}{2}+\frac{1}{r}\right)e^{-\lambda}-y\left[\left(\frac{1}{r^2}+\frac{\lambda'}{2r}\right)e^{-\lambda}-\frac{1}{r^2}\right]=8\pi y(p_t-p_r)\,. \tag{12}$$

Since $e^{-\lambda}=1-2m(r)/r$, using Eq. (9) we find

$$e^{-\lambda}=1-a-br^2\equiv I_b^2(r)\,, \tag{13}$$

where we defined the function $I_b^2(x)\equiv 1-a-bx^2$. When $b=0$, we write $I_0^2\equiv 1-a$. Using the expression for $e^{-\lambda}$ in Eq. (12) and substituting for the pressure anisotropy we find

$$[br^4-(1-a)r^2]y''+(1-a)ry'-(a-c-dr^2)y=0\,. \tag{14}$$

The full solution of Eq. (14) with a, b, c, $d\neq 0$ is in Ref. 7. Here, we will only consider solutions with $b=d=0$. In this case, the total mass is $M=aR/2$, and $\exp[-\lambda]=1-a$. Since for any static spherically-symmetric configuration we expect $(2M/R)_{\rm crit}\leq 1$, we must have $a<1$. (Also, the metric coefficient g_{rr} becomes infinite when $a=1$).

Since we want to construct stars with finite radii and density in the context of anisotropic pressure, we impose boundary conditions such that $p_r(R)=0$. With $b=d=0$, Eq. (14) reduces to an Euler–Cauchy equation,

$$(1-a)r^2y''-(1-a)ry'+(a-c)y=0\,. \tag{15}$$

The solutions of this equation fall into three classes, depending on the value of q (q is real, $q=0$, or q is imaginary), where

$$q\equiv\frac{(1+c-2a)^{\frac{1}{2}}}{(1-a)^{\frac{1}{2}}}\,. \tag{16}$$

We will only show results for q real. (See Ref. 7 for the other cases.)

The solution for y is

$$y=A_+\left(\frac{r}{R}\right)^{1+q}+A_-\left(\frac{r}{R}\right)^{1-q}\,, \tag{17}$$

with the constants A_+ and A_- fixed by boundary conditions. For the case under consideration here ($b=d=0$), the boundary conditions are

$$e^{-\lambda(R)}=e^{\nu(R)}=I_0^2\,,\quad\text{and}\quad e^{\nu(R)}\left.\frac{d\nu}{dr}\right|_R=\frac{a}{R}\,. \tag{18}$$

Applying the boundary conditions we find

$$A_+ = \frac{I_0}{2} + \frac{1 - 3I_0^2}{4qI_0} \quad \text{and} \quad A_- = A_+(q \to -q)\,. \tag{19}$$

The radial pressure for this case, after substituting the expressions for A_+ and A_-, is

$$8\pi p_r = \frac{(3I_0^2 - 1)^2 - 4q^2 I_0^4}{r^2}\left[\frac{R^{2q} - r^{2q}}{(3I_0^2 - 1 + 2qI_0^2)R^{2q} + (1 - 3I_0^2 + 2qI_0^2)r^{2q}}\right]. \tag{20}$$

We note that the boundary conditions automatically guarantee that $p_r(R) = 0$. The radial pressure is always greater than zero provided $a < 2/3$ and $a^2 > 4c(1 - a)$. Since by definition $a > 0$, the second condition implies $c > 0$. Thus, this model does not allow for negative anisotropy. Further, since we are considering the case $q > 0$, we must impose the condition $1 + c < 2a$. Combining the two inequalities for a and c, we obtain, $2a - 1 < c < a^2/4(1 - a)$. Since we have $0 < a < 2/3$ we find that $0 < c < 1/3$. We note that for the anisotropic case the maximum value of a is $2/3$, corresponding to a 33% increase when compared with the isotropic case ($a = 3/7$). In Fig. 1 we plot the radial pressure p_r as a function of the radial coordinate r, for $a = 3/7$ and several values of c. Note that for this choice of a, the inequality $c < a^2/4(1 - a)$ imposes that $c < 0.08$ for positive pressure solutions. This can be seen in the figure. For larger anisotropies, no static self-gravitating stable configuration is possible. For $r \ll R$, we find

$$8\pi r^2 p_r = 3I_0^2 - 1 - 2qI_0^2\,. \tag{21}$$

Choosing $a = 3/7$ we recover, in the limit $c \to 0$, the Misner–Zapolsky solution,[9] with $p_r = 1/(56\pi r^2) = \rho/3$.

4. Stability

For sake of brevity, we will skip most details of our generalization of Chandrasekhar's formalism to anisotropic spheres. Readers can consult Ref. 8 for details. As in Chandrasekhar's original formalism, we limit our study to small radial and baryon-number conserving perturbations. Writing $\rho = \rho_o + \delta\rho$, $p_r = p_{r_o} + \delta p_r$, $p_t = p_{t_o} + \delta p_t$, $\lambda = \lambda_o + \delta\lambda$, and $\nu = \nu_o + \delta\nu$ we find that, to first order in $v = d\xi/dt$ (ξ is the Lagrangian displacement) and using the zeroth-order equations, the perturbation in the radial pressure satisfies,

$$\delta p_r = -{p_{r_o}}' - \gamma p_{r_o}\frac{e^{\nu_o/2}}{r^2}\left(r^2 e^{\nu_o/2}\xi\right)' + \frac{2\xi}{r}\Pi_o\frac{\partial p_{ro}}{\partial \rho_o}\,, \tag{22}$$

with γ being the adiabatic exponent defined as

$$\gamma \equiv \frac{1}{p_r(\partial n/\partial p_r)}\left[n - (\rho + p_r)\frac{\partial n}{\partial p_r}\right], \tag{23}$$

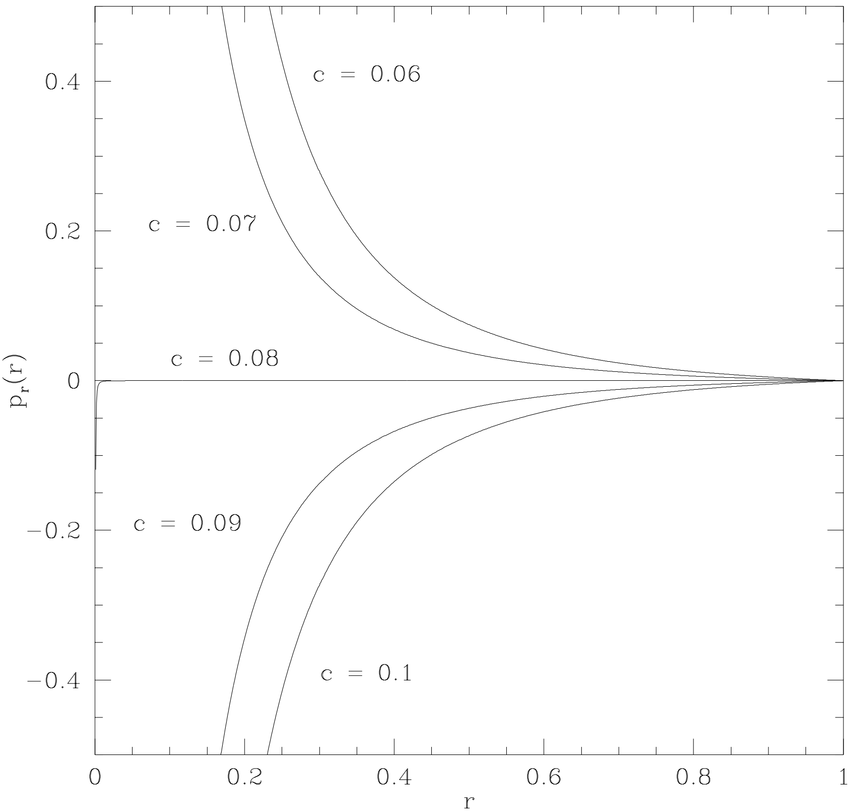

Fig. 1. Radial pressure as a function of r for ρ and $(p_t - p_r) \propto r^{-2}$ and q real.

and $\Pi \equiv p_t - p_r$. We now assume that all perturbations have a time dependence of the form $e^{i\omega t}$. Further, considering $\delta\lambda$, $\delta\nu$, $\delta\rho$, δp_r and $\delta\Pi$ to now represent the amplitude of the various perturbations with the same time dependence we obtain, after using the zeroth-order equations, the pulsation equation governing the radial stability of anisotropic stars[8]

$$\begin{aligned} &\omega^2(\rho_o + p_{r_o})\xi e^{\lambda_o - \nu_o} \\ &\quad = \frac{4}{r} p_{r_o}' \xi - e^{-(\lambda_o + 2\nu_o)/2} \left[e^{(\lambda_o + 3\nu_o)/2} \gamma \frac{p_{r_o}}{r^2} (r^2 e^{-\nu_o/2} \xi)' \right]' \\ &\qquad + 8\pi e^{\lambda_o} (\Pi_o + p_{r_o})(\rho_o + p_{r_o})\xi - \frac{1}{(\rho_o + p_{r_o})} (p_{r_o}')^2 \xi \\ &\qquad + \frac{4 p_{r_o}' \Pi_o \xi}{r(\rho_o + p_{r_o})} - \frac{4\Pi_o^2 \xi}{r^2(\rho_o + p_{r_o})} \\ &\qquad - e^{-(\lambda_o + 2\nu_o)/2} \left[e^{(\lambda_o + 2\nu_o)/2} \frac{2}{r} \xi \Pi_o \left(\frac{\partial p_r}{\partial \rho} + 1 \right) \right]' - \frac{8}{r^2} \Pi_o \xi - \frac{2}{r} \delta\Pi . \end{aligned} \tag{24}$$

The boundary conditions imposed on this equation are

$$\xi = 0 \text{ at } r = 0 \quad \text{and} \quad \delta p_r = 0 \text{ at } r = R\,. \tag{25}$$

The pulsation Eq. (24), together with the boundary conditions Eq. (25), reduce to an eigenvalue problem for the frequency ω and amplitude ξ. One multiplies this equation by $r^2\xi e^{(\lambda+\nu)/2}$ and integrates over the entire range of r, using the orthogonality condition

$$\int_0^R e^{(3\lambda-\nu)/2}(\rho + p_r)r^2\xi^i\xi^j dr = 0 \quad (i \neq j)\,, \tag{26}$$

where ξ^i and ξ^j are the proper solutions belonging to different eigenvalues ω^2.

We now apply this equation to the exact solutions with $\rho \propto 1/r^2$ described above. Using the trial function $\xi = r^2(\rho + p_r)e^{\nu}$ we found that all integrals could be computed exactly. In Table 1 we present results for the frequencies of radial oscillations ω^2 as a function of the anisotropy parameter, c, for given values of the density parameter. Instabilities set in for $\omega^2 < 0$. This can be used to find the critical value for the adiabatic index, $\gamma_c(c)$, and the maximum value for the anisotropy parameter $c_{\max}$. We also give, in Table 2, the values of γ_c above which stable oscillations are possible. Here we see that the effect of a positive anisotropy is to reduce the value of γ, thus giving rise to a more stable configuration when compared with the corresponding isotropic model. In particular, for the Misner–Zapolsky solution ($a = 3/7$), we find that a small positive pressure anisotropy in the equation of state improves the neutron star's core stability.

Table 1. ω^2 versus c for given values of a.

$a = 2/9$	$\omega^2R^2 = 0.95(\gamma - 1.79) + (101.1 - 52.6\gamma)c$
$a = 2/7$	$\omega^2R^2 = 2.3(\gamma - 1.83) + (122.3 - 59.3\gamma)c$
$a = 3/7$	$\omega^2R^2 = 0.57(\gamma - 1.93) + (15.2 - 5.1\gamma)c$
$a = 3.4/7$	$\omega^2R^2 = 0.4(\gamma - 2.6) + (8.9 - 2.3\gamma)c$
$a = 3.49/7$	$\omega^2R^2 = 0.36(\gamma - 2.76) + (8.0 - 1.97\gamma)c$

Table 2. γ_c versus c for given values of a.

$a = 2/9$	$c_{\max} = 0.0016$	$\gamma_c = 1.79 - 6.87c$
$a = 2/7$	$c_{\max} = 0.0028$	$\gamma_c = 1.83 - 13.39c$
$a = 3/7$	$c_{\max} = 0.083$	$\gamma_c = 1.93 - 5.55c$
$a = 3.4/7$	$c_{\max} = 0.11$	$\gamma_c = 2.6 - 2.84c$
$a = 3.47/7$	$c_{\max} = 0.12$	$\gamma_c = 2.75 - 7.29c$

5. Conclusions

We have presented a summary of results concerning the existence of self-gravitating spheres in General Relativity with anisotropic equations of state, aka *anisotropic* *stars*. I have also presented a summary of our investigation of their stability, based on the extension of Chandrasekhar's celebrated variational formalism for isotropic spheres to those with anisotropic energy-momentum tensors.

Although the search for exact solutions restricts the forms of anisotropy we could treat, our results illustrate the fact that, indeed, pressure anisotropy may greatly affect the physical structure of the star, leading to several observational effects. Most importantly, the absolute stability bound $2M/R < 8/9$ can be violated, and the star's surface redshift may be arbitrarily large ($z_s > 2$). It is thus conceivable that objects which are observed at large redshift may actually be closer than we think due to anisotropic distortions. They may also be more stable than we think, especially if pressure anisotropy exists near the stellar core.

Here, I have presented results only for one of the cases we treated, stars with energy density scaling as $1/r^2$. These are of interest as they include ultra-relativistic equations of state used to model the cores of neutron stars. Perhaps the most important lesson of our study is that one must keep an open mind as to whether isotropy is or not a justified assumption to describe stellar matter. Until we have a better microscopic understanding of what truly goes on inside ultra-dense compact objects, isotropy should be taken with a grain of salt. Especially if some of these objects contain bosonic condensates at their cores, as is the case for several models of neutron star interiors and for a whole class of fully-anisotropic hypothetical objects known as boson stars.[4]

Acknowledgments

I'd like to express my gratitude to the organizers of IWARA 2003, and in particular to Hélio Coelho, Sandra Prado, and César Vasconcellos for their invitation and wonderful hospitality. My research is sponsored in part by a NSF grant PHY-0099543.

References

1. D. D. Clayton, in *Principles of Stellar Evolution and Nucleosynthesis* (The University of Chicago Press, Chicago, 1983); R. Kippenhahn and A. Weigert, in *Stellar Structure and Evolution* (Springer-Verlag, Berlin, 1991).
2. N. K. Glendenning, in *Compact Stars*: *Nuclear Physics, Particle Physics and General Relativity* (Springer-Verlag, Berlin, 1997); H. Heiselberg and M. H. Jensen, *Phys. Rep.* **328**, 237 (2000).
3. M. Ruderman, *Annu. Rev. Astron. Astrophys.* **10**, 427 (1972); V. Canuto, *Annu. Rev. Astron. Astrophys.* **12**, 167 (1974).
4. M. Gleiser, *Phys. Rev.* **D38**, 2376 (1988); M. Gleiser and R. Watkins, *Nucl. Phys.* **B319**, 733 (1989). For comprehensive reviews see, A. R. Liddle and M. S. Marsden, *Int. J. Mod. Phys.* **D1**, 101 (1992); P. Jetzer, *Phys. Rep.* **220**, 163 (1992); E. W. Mielke and F. E. Schunck, in *Proceedings of 8th M. Grossmann Meeting*, ed. T. Piran (World Scientific, Singapore, 1998).
5. R. Sawyer and D. Scalapino, *Phys. Rev.* **D7**, 382 (1973).

6. R. L. Bowers and E. P. T. Liang, *Ap. J.* **188**, 657 (1974); J. Ponce de Leon, *J. Math. Phys.* **28**, 1114 (1987); M. Gokhroo and A. Mehra, *Gen. Rel. Grav.* **26**, 75 (1994); H. Bondi, *Mon. Not. R. Astron. Soc.* **259**, 365 (1992); E. S. Corchero, *Class. Quantum Grav.* **15**, 3645 (1998); L. Herrera, *Phys. Lett.* **A165**, 206 (1992); L. Herrera and N. O. Santos, *Phys. Rep.* **286**, 53 (1997).
7. K. Dev and M. Gleiser, *Gen. Rel. Grav.* **24**, 1793 (2002).
8. K. Dev and M. Gleiser, *Gen. Rel. Grav.* **35**, 1435 (2003).
9. C. Misner and H. Zapolsky, *Phys. Rev. Lett.* **12**, 635 (1964).