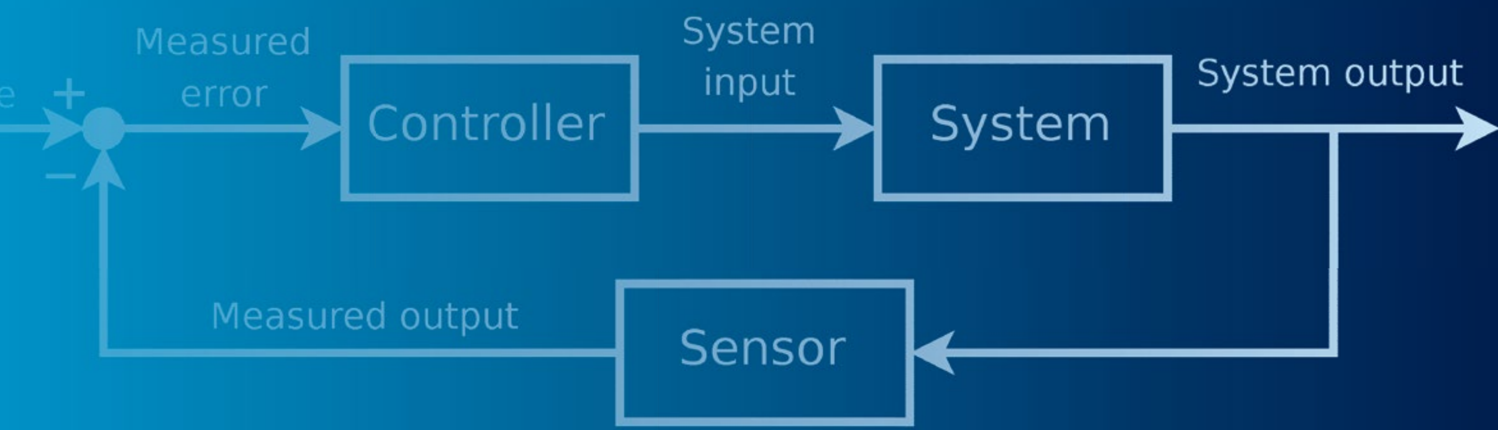
Mehdi Rahmani-Andebili



Feedback Control Systems Analysis and Design

Practice Problems, Methods, and Solutions

Springer

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Preface

Feedback control systems analysis and design is one of the main courses of electrical engineering and the fundamental course of control engineering major which is taught for junior students. The subjects include different representations of linear time-invariant (LTI) systems, stability analysis of LTI systems, analysis of transient response, analysis of steady state response, graphical analysis and design, and controller design.

In the chapters concerning with the graphical analysis and design in time domain as well as the controller design in time domain, MATLAB has been applied to accurately plot the root locus of the control systems and design the related controllers.

Like the previously published textbooks, the textbook includes very detailed and multiple methods of problem solutions. It can be used as a practicing textbook by students and as a supplementary teaching source by instructors.

To help students study the textbook in the most efficient way, the exercises have been categorized in nine different levels. In this regard, for each problem of the textbook a difficulty level (easy, normal, or hard) and a calculation amount (small, normal, or large) have been assigned. Moreover, in each chapter, problems have been ordered from the easiest problem with the smallest calculations to the most difficult problems with the largest calculations. Therefore, students are suggested to start studying the textbook from the easiest problems and continue practicing until they reach the normal and then the hardest ones. On the other hand, this classification can help instructors choose their desirable problems to conduct a quiz or a test. Moreover, the classification of computation amount can help students manage their time during future exams and instructors give the appropriate problems based on the exam duration.

Since the problems have very detailed solutions and some of them include multiple methods of solution, the textbook can be useful for the under-prepared students. In addition, the textbook is beneficial for knowledgeable students because it includes advanced exercises.

In the preparation of problem solutions, it has been tried to use typical methods to present the textbook as an instructor-recommended one. In other words, the heuristic methods of problem solution have never been used as the first method of problem solution. By considering this key point, the textbook will be in the direction of instructors' lectures, and the instructors will not see any untaught problem solutions in their students' answer sheets.

The Iranian University Entrance Exams for the master's and PhD degrees in electrical engineering major is the main reference of the textbook; however, all the problem solutions have been provided by me. The Iranian University Entrance Exams is one of the most competitive university entrance exams in the world that allows only 10% of the applicants to get into prestigious and tuition-free Iranian universities.

Butte, MT, USA Mehdi Rahmani-Andebili

The Other Works Published by the Author

The author has already published the books and textbooks below with Springer Nature.

Textbooks

Power System Analysis: Practice Problems, Methods, and Solutions
Advanced Electrical Circuit Analysis: Practice Problems, Methods
AC Electrical Circuit Analysis: Practice Problems, Methods
Calculus: Practice Problems, Methods, and Solutions
Precalculus: Practice Problems, Methods, and Solutions
DC Electrical Circuit Analysis: Practice Problems, Methods, and Solutions

Books

Applications of Artificial Intelligence in Planning and Operation of Smart Grids
Design, Control, and Operation of Microgrids in Smart Grids
Applications of Fuzzy Logic in Planning and Operation of Smart Grids
Operation of Smart Homes
Planning and Operation of Plug-in Electric Vehicles: Technical, Geographical, and Social Aspects

Contents

About the Author

Mehdi Rahmani-Andebili is an assistant professor in the Electrical Engineering Department at Montana Technological University, MT, USA. Before that, he was also an assistant professor in the Engineering Technology Department at State University of New York, Buffalo State, NY, USA, during 2019–2021. He received his first MSc and PhD degrees in electrical engineering (power system) from Tarbiat Modares University and Clemson University in 2011 and 2016, respectively, and his second MSc degree in physics and astronomy from the University of Alabama in Huntsville in 2019. Moreover, he was a postdoctoral fellow at Sharif University of Technology during 2016–2017. As a professor, he has taught many courses such as Essentials of Electrical Engineering Technology, Electrical Circuits Analysis I, Electrical Circuits Analysis II, Electrical Circuits and Devices, Industrial Electronics, Renewable Distributed Generation and Storage, Feedback Controls, DC and AC Electric Machines, and Power System Analysis. Dr. Rahmani-Andebili has more than 200 single-author and first-author publications including journal papers, conference papers, textbooks, books, and book chapters. He is an IEEE Senior Member and the permanent reviewer of many credible journals. His research areas include smart grid, power system operation and planning, integration of renewables and energy storages into power system, energy scheduling and demand-side management, plug-in electric vehicles, distributed generation, and advanced optimization techniques in power system studies.

1 Problems: Different Representations of Linear Time-Invariant (LTI) Systems

Abstract

In this chapter, different representations of linear time-invariant (LTI) systems, including differential equation representation, impulse response representation, transfer function representation, block-diagram representation, signal flow graph (SFG) representation, and state space representation, are studied. Herein, Mason's gain formula will be applied to determine the transfer function of system from its block-diagram or signal flow graph (SFG). In this chapter, the problems are categorized in different levels based on their difficulty levels (easy, normal, and hard) and calculation amounts (small, normal, and large). Additionally, the problems are ordered from the easiest problem with the smallest computations to the most difficult problems with the largest calculations.

1.1 Determine the characteristic equation of a control system with the block-diagram shown in Fig. 1.1.

Difficulty level ● Easy ○ Normal ○ Hard
Calculation amount ● Small ○ Normal ○ Large

1) $1 + G_2H_2 - G_1G_2G_3H_1H_2$
2) $1 + G_1G_2H_2 - G_1G_2G_3H_1H_2$
3) $1 + G_2H_2 + G_1G_2H_1$
4) $1 + G_2H_2 + G_1G_2H_1 - G_1G_2G_3H_1H_2$

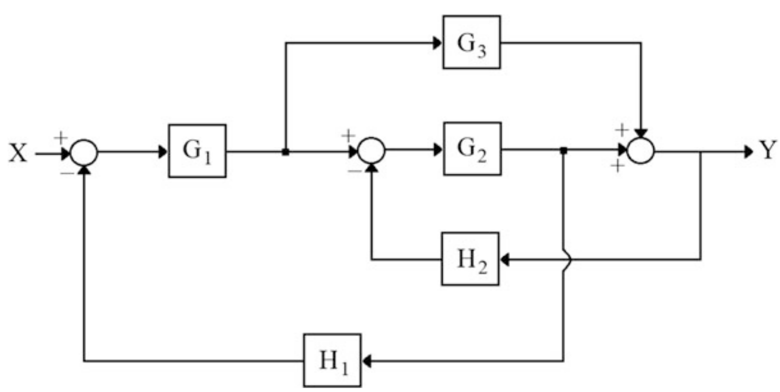

Figure 1.1 The control system of problem 1.1

1.2 Figure 1.2 illustrates the signal flow graph (SFG) of a control system. Determine its transfer function.

Difficulty level ● Easy ○ Normal ○ Hard
Calculation amount ● Small ○ Normal ○ Large

1) $\frac{G_1+G_2}{1-G_2H+G_1G_2}$
2) $\frac{G_1+G_2}{1+G_2H-G_1G_2}$
3) $\frac{G_1+G_2}{1-G_2H}$
4) $\frac{G_1+G_2}{1+G_2H}$

M. Rahmani-Andebili, *Feedback Control Systems Analysis and Design*, https://doi.org/10.1007/978-3-030-95277-8_1

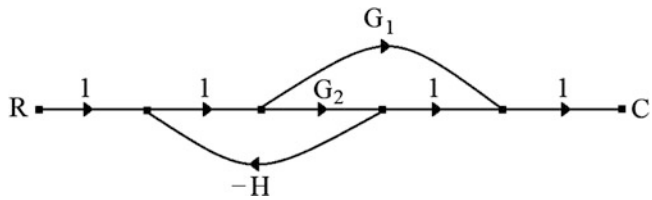

Figure 1.2 The control system of problem 1.2

1.3 The state transition matrix of a control system with the state equation of $[\dot{x}(t)] = [A][x(t)] + [B][u(t)]$ is as follows. Determine $[A]$.

$$[\varphi(t)] = \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (11t)e^{-t} \end{bmatrix}$$

Difficulty level ● Easy ○ Normal ○ Hard
Calculation amount ● Small ○ Normal ○ Large

1) $\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$
2) $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$
3) $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$
4) $\begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}$

1.4 In the block-diagram, shown in Fig. 1.3, determine the transfer function of $\frac{Y(s)}{X(s)}$.

Difficulty level ● Easy ○ Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $\frac{5}{s(s+2)}$
2) $\frac{5}{s^2+22s+5}$
3) $\frac{4s+1}{s^2+22s+5}$
4) $1 + \frac{5(4s+1)}{s(s+2)}$

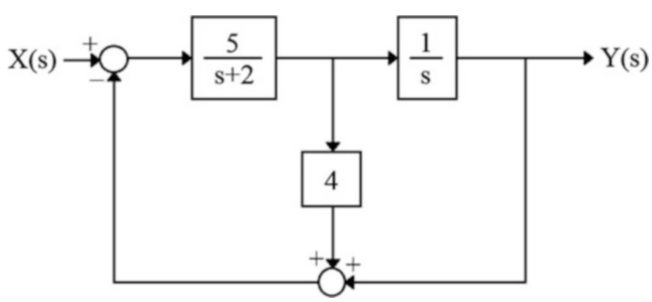

Figure 1.3 The control system of problem 1.4

1.5 The state equations of a LTI control system, which is in zero-state, are as follows. Determine the steady-state value of its output.

$$\dot{\mathbf{X}} = \begin{bmatrix} -5 & 1 \\ -3 & -1 \end{bmatrix}\mathbf{X} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t), \quad \mathbf{Y} = \mathbf{X}$$

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	● Small	○ Normal	○ Large

1) $\begin{bmatrix} -\frac{1}{8} \\ -\frac{5}{8} \end{bmatrix}$

2) $\begin{bmatrix} -\frac{5}{8} \\ -\frac{1}{8} \end{bmatrix}$

3) $\begin{bmatrix} \frac{1}{8} \\ \frac{5}{8} \end{bmatrix}$

4) $\begin{bmatrix} \frac{5}{8} \\ \frac{1}{8} \end{bmatrix}$

1.6 In the block-diagram shown in Fig. 1.4, determine the value of k_1, k_2, k_3, so that the transfer function is as follows:

$$T(s) = \frac{6}{(s+2)(s+3)}$$

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) $k_1 = 1, k_2 = \frac{2}{3}, k_3 = 6$
2) $k_1 = \frac{2}{3}, k_2 = 1, k_3 = 6$
3) $k_1 = \frac{3}{2}, k_2 = \frac{3}{2}, k_3 = 6$
4) $k_1 = 1, k_2 = \frac{3}{2}, k_3 = 6$

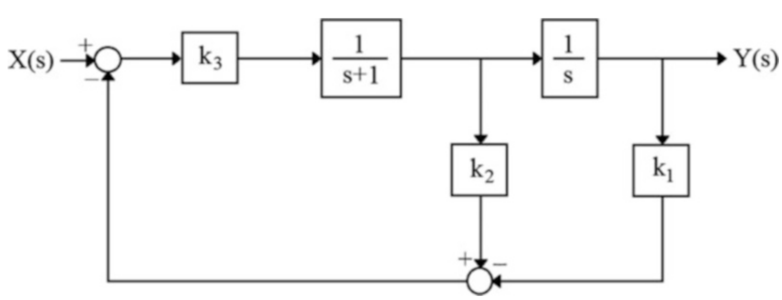

Figure 1.4 The control system of problem 1.6

1.7 The differential equation of a control system is as follows:

$$\frac{d^3}{dt^3}y(t) + 3\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 4y(t) = u(t)$$

Determine the state and output equations of the system in the matrices form.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t), y(t) = [-1 \ \ 0 \ \ 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

2) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 4 & 6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t), y(t) = [1 \ \ 0 \ \ 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

3) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t), y(t) = [1 \ \ 0 \ \ 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

4) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 6 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}u(t), y(t) = [1 \ \ 0 \ \ 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

1.8 Determine matrix **A** in the state equations ($\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{B}u$) for the block-diagram of Fig. 1.5 if $\mathbf{X} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $\begin{bmatrix} 0 & -\beta \\ 1 & -\alpha \end{bmatrix}$

2) $\begin{bmatrix} -\alpha & -\beta \\ 1 & 0 \end{bmatrix}$

3) $\begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix}$

4) $\begin{bmatrix} -\alpha & 1 \\ -\beta & 0 \end{bmatrix}$

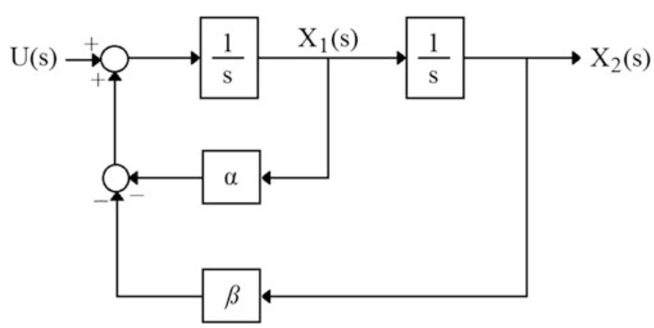

Figure 1.5 The control system of problem 1.8

1.9 Determine the transfer function of a control system with the following state equations:

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{AX} + \mathbf{B}u \\ \mathbf{Y} = \mathbf{CX} \end{cases}, \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix}, \mathbf{C} = [1 \quad 0]$$

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $\frac{m}{s^2+bs+k}$
2) $\frac{k}{bs^2+ms+k}$
3) $\frac{b}{ms^2+bs+k}$
4) $\frac{1}{ms^2+bs+k}$

1.10 The state equations of a control system are as follows. Determine the state-transition matrix of the system ($\boldsymbol{\varphi}(t)$).

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} r(t)$$

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $\begin{bmatrix} (1+t)e^{-2t} & te^{-2t} \\ te^{-2t} & (1-t)e^{-2t} \end{bmatrix}$
2) $\begin{bmatrix} 2e^{-2t} & e^{-2t} \\ -e^{-2t} & -2e^{-2t} \end{bmatrix}$
3) $\begin{bmatrix} (1+t)e^{-2t} & te^{-2t} \\ -te^{-2t} & (1-t)e^{-2t} \end{bmatrix}$
4) The value of b is needed to determine the state-transition matrix.

1.11 Consider the LTI control system below.

$$\dot{\mathbf{X}} = \mathbf{AX}, \quad y = \mathbf{CX}$$

Determine the output of the system based on the following information:

$$\mathbf{A} = \begin{bmatrix} -\dfrac{3}{2} & \dfrac{3}{2} \\ \dfrac{1}{6} & -\dfrac{3}{2} \end{bmatrix}, \mathbf{C} = [1 \quad 0], \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $e^{-t} + e^{-2t}$
2) $e^{-t} + 2e^{-2t}$
3) $e^{-t} + 1.5e^{-2t}$
4) $1.5e^{-t} + e^{-2t}$

1.12 Determine the state equations of the control system shown in Fig. 1.6.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u(t)$

2) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u(t)$

3) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u(t)$

4) $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u(t)$

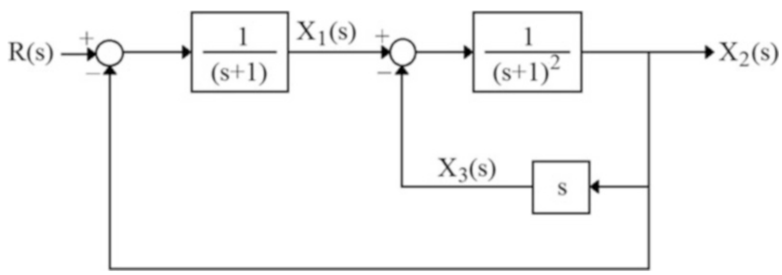

Figure 1.6 The control system of problem 1.12

1.13 In the rotational mechanical system shown in Fig. 1.7, determine the transfer function of $\frac{\theta_2(s)}{T(s)}$.

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) $\frac{J_1 s^2 + k}{s^2(J_1 J_2 s^2 + k(J_1 + J_2))}$

2) $\frac{J_2 s^2 + k}{s^2(J_1 J_2 s^2 + k(J_1 + J_2))}$

3) $\frac{k}{s^2(J_1 J_2 s^2 + k(J_1 + J_2))}$

4) $\frac{k}{J_1 J_2 s^4 + k(J_1 + J_2)s^2 + 2k^2}$

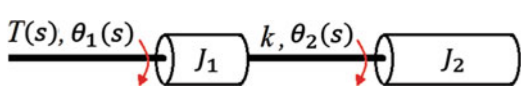

Figure 1.7 The control system of problem 1.13

2 Solutions of Problems: Different Representations of Linear Time-Invariant (LTI) Systems

Abstract

In this chapter, the problems of the first chapter are fully solved, in detail, step-by-step, and with different methods.

2.1 To determine the characteristic equation of the system (Δ), we can use Mason's gain formula, as follows:

$$\Delta = 1 - \sum_{a} L_a + \sum_{a,b} L_a L_b - \sum_{a,b,c} L_a L_b L_c + \ldots$$

where:

$\sum_{a} L_a$: The sum of gains of loops

$\sum_{a,b} L_a L_b$: The sum of product of gains of any two non-touching loops (without any common nodes)

$\sum_{a,b,c} L_a L_b L_c$: The sum of product of gains of any three pairwise non-touching loops (without any common nodes)

Now, for the system shown in Fig. 2.1, we have:

$$\sum_{a} L_a = (-G_1G_2H_1) + (-G_2H_2) + (G_1G_3H_2G_2H_1)$$

$$\sum_{a,b} L_a L_b = 0$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta = 1 - ((-G_1G_2H_1) + (-G_2H_2) + (G_1G_3H_2G_2H_1)) + 0 - 0$$

$$\Rightarrow \Delta = 1 + G_1G_2H_1 + G_2H_2 - G_1G_3H_2G_2H_1$$

Choice (4) is the answer.

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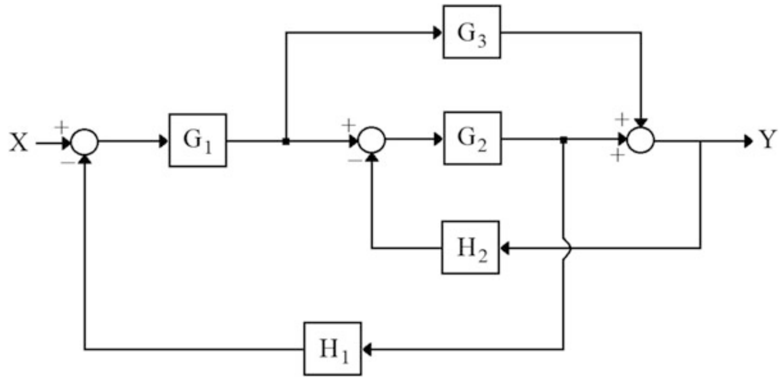

Figure 2.1 The control system of solution of problem 2.1

2.2 To determine the transfer function of a system, we can use Mason's gain formula, as follows:

$$P = \frac{1}{\Delta}\sum_{k=1}^{N} p_k \Delta_k$$

where:

$$\Delta = 1 - \sum_{a} L_a + \sum_{a,b} L_a L_b - \sum_{a,b,c} L_a L_b L_c + \ldots$$

P: The total gain from the input point to the output one

Δ: The determinant of the graph which is the same as the characteristic equation of the system

N: The number of forward paths from the input point to the output one

k: The index of forward path from the input point to the output

p_k: The gain of the k'th forward path from the input point to the output one

Δ_k: The determinant of the graph if the k'th forward path is removed

$\sum_{a} L_a$: The sum of gains of loops

$\sum_{a,b} L_a L_b$: The sum of product of gains of any two non-touching loops (without any common nodes)

$\sum_{a,b,c} L_a L_b L_c$: The sum of product of gains of any three pairwise non-touching loops (without any common nodes)

Now, for the system shown in Fig. 2.2, we have:

$$N = 2$$

$$p_1 = 1 \times 1 \times G_1 \times 1 = G_1$$

$$p_1 = 1 \times 1 \times G_1 \times 1 \times 1 = G_2$$

$$\Delta_1 = 1$$

$$\Delta_2 = 1$$

$$\Rightarrow \sum_{k=1}^{N} p_k \Delta_k = p_1 \Delta_1 + p_2 \Delta_2 = G_1 + G_2$$

$$\sum_{a} L_a = 1 \times G_2 \times (-H) = -G_2 H$$

$$\sum_{a,b} L_a L_b = 0$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta = 1 - (-G_2 H) + 0 - 0 = 1 + G_2 H$$

$$\Rightarrow T(s) = \frac{R(s)}{C(s)} = \frac{G_1 + G_2}{1 + G_2 H}$$

Choice (4) is the answer.

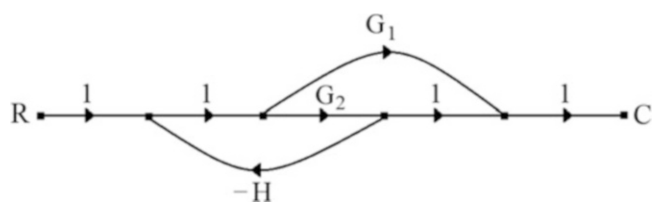

Figure 2.2 The control system of solution of problem 2.2

2.3 Based on the information given in the problem, the state transition matrix of the control system with the state equation of $[\dot{x}(t)] = [A][x(t)] + [B][u(t)]$ is as follows:

$$[\varphi(t)] = \begin{bmatrix} (1+t)e^{-t} & te^{-t} \\ -te^{-t} & (11t)e^{-t} \end{bmatrix} \tag{1}$$

From one of the properties of state transition matrix, we know that:

$$[A] = \frac{d}{dt}[\varphi(t)]\Big|_{t=0} \tag{2}$$

Solving (1) and (2):

$$[A] = \begin{bmatrix} -te^{-t} & (1-t)e^{-t} \\ (t-1)e^{-t} & (t-2)e^{-t} \end{bmatrix}\Bigg|_{t=0}$$

$$\Rightarrow [A] = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

Choice (3) is the answer.

2.4 To determine the transfer function of the system, we can use Mason's gain formula, as follows:

$$P = \frac{1}{\Delta}\sum_{k=1}^{N} p_k \Delta_k$$

where:

$$\Delta = 1 - \sum_a L_a + \sum_{a,b} L_a L_b - \sum_{a,b,c} L_a L_b L_c + \ldots$$

P: The total gain from the input point to the output one

Δ: The determinant of the graph which is the same as the characteristic equation of the system

N: The number of forward paths from the input point to the output one

k: The index of forward path from the input point to the output

p_k: The gain of the k'th forward path from the input point to the output one

Δ_k: The determinant of the graph if the k'th forward path is removed

$\sum_a L_a$: The sum of gains of loops

$\sum_{a,b} L_a L_b$: The sum of product of gains of any two non-touching loops (without any common nodes)

$\sum_{a,b,c} L_a L_b L_c$: The sum of product of gains of any three pairwise non-touching loops (without any common nodes)

Now, for the system shown in Fig. 2.3, we have:

$$N = 1$$

$$p_1 = \frac{5}{s+2} \times \frac{1}{s} = \frac{5}{s(s+2)}$$

$$\Delta_1 = 1$$

$$\Rightarrow \sum_{k=1}^{N} p_k \Delta_k = p_1 \Delta_1 = \frac{5}{s(s+2)} \times 1 = \frac{5}{s(s+2)}$$

$$\sum_a L_a = \left(-\frac{5}{s+2} \times \frac{1}{s}\right) + \left(-\frac{5}{s+2} \times 4\right) = -\frac{5}{s(s+2)} - \frac{20}{s+2}$$

$$\sum_{a,b} L_a L_b = 0$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta = 1 - \left(-\frac{5}{s(s+2)} - \frac{20}{s+2}\right) + 0 - 0 = 1 + \frac{5}{s(s+2)} + \frac{20}{s+2}$$

$$\Rightarrow T(s) = \frac{Y(s)}{X(s)} = \frac{\frac{5}{s(s+2)}}{1 + \frac{5}{s(s+2)} + \frac{20}{s+2}} = \frac{\frac{5}{s(s+2)}}{\frac{s(s+2)+5+20s}{s(s+2)}} = \frac{5}{s^2 + 22s + 5}$$

$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{5}{s^2 + 22s + 5}$$

Choice (2) is the answer.

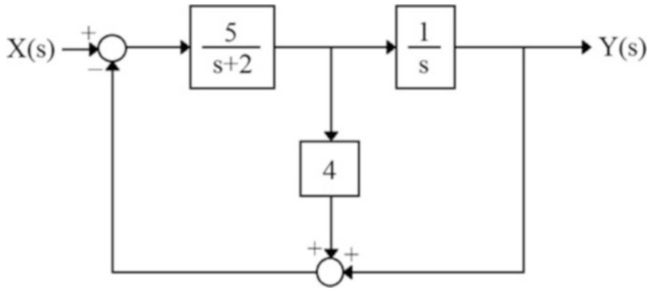

Figure 2.3 The control system of solution of problem 2.4

2.5 Based on the information given in the problem, we have:

$$\dot{\mathbf{X}} = \begin{bmatrix} -5 & 1 \\ -3 & -1 \end{bmatrix}\mathbf{X} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \quad (1)$$

$$\mathbf{Y} = \mathbf{X} \quad (2)$$

When a system is in its steady-state condition, its state variables are constant. In other words, the first time derivate of the state variables is zero, as can be seen in the following:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3)$$

Solving (1) and (3) for unit step input ($u(t) = 1$):

$$\begin{bmatrix} -5 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -5x_1(t) + x_2(t) \\ -3x_1(t) - x_2(t) + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1(t) = \frac{1}{8}, x_2(t) = \frac{5}{8} \quad (4)$$

Solving (2) and (4):

$$\mathbf{Y} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ \frac{5}{8} \end{bmatrix}$$

Choice (3) is the answer.

2.6 Based on the information given in the problem, we know that:

$$T(s) = \frac{6}{(s+2)(s+3)} = \frac{6}{s^2 + 5s + 6} \quad (1)$$

To determine the transfer function of the system, we can use Mason's gain formula, as follows:

$$P = \frac{1}{\Delta}\sum_{k=1}^{N} p_k \Delta_k$$

where:

$$\Delta = 1 - \sum_a L_a + \sum_{a,b} L_a L_b - \sum_{a,b,c} L_a L_b L_c + \ldots$$

P: The total gain from the input point to the output one

Δ: The determinant of the graph which is the same as the characteristic equation of the system

N: The number of forward paths from the input point to the output one

k: The index of forward path from the input point to the output

p_k: The gain of the k'th forward path from the input point to the output one

Δ_k: The determinant of the graph if the k'th forward path is removed

$\sum_a L_a$: The sum of gains of loops

$\sum_{a,b} L_a L_b$: The sum of product of gains of any two non-touching loops (without any common nodes)

$\sum_{a,b,c} L_a L_b L_c$: The sum of product of gains of any three pairwise non-touching loops (without any common nodes)

Now, for the system shown in Fig. 2.4, we have:

$$N = 1$$

$$p_1 = k_3 \times \frac{1}{s+1} \times \frac{1}{s} = \frac{k_3}{s(s+1)}$$

$$\Delta_1 = 1$$

$$\Rightarrow \sum_{k=1}^{N} p_k \Delta_k = p_1 \Delta_1 = \frac{k_3}{s(s+1)} \times 1 = \frac{k_3}{s(s+1)}$$

$$\sum_a L_a = \left(-k_3 \times \frac{1}{s+1} \times k_2\right) + \left(-k_3 \times \frac{1}{s+1} \times \frac{1}{s} \times k_1\right) = -\frac{k_3 k_2}{s+1} - \frac{k_3 k_1}{s(s+1)}$$

$$\sum_{a,b} L_a L_b = 0$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta = 1 - \left(-\frac{k_3 k_2}{s+1} - \frac{k_3 k_1}{s(s+1)}\right) + 0 - 0 = 1 + \frac{k_3 k_2}{s+1} + \frac{k_3 k_1}{s(s+1)}$$

$$\Rightarrow T(s) = \frac{\frac{k_3}{s(s+1)}}{1 + \frac{k_3 k_2}{s+1} + \frac{k_3 k_1}{s(s+1)}} = \frac{\frac{k_3}{s(s+1)}}{\frac{s(s+1) + k_3 k_2 s + k_3 k_1}{s(s+1)}}$$

$$\Rightarrow T(s) = \frac{k_3}{s^2 + (1 + k_3 k_2)s + k_3 k_1} \tag{2}$$

Solving (1) and (2):

$$\frac{k_3}{s^2 + (1 + k_3 k_2)s + k_3 k_1} = \frac{6}{s^2 + 5s + 6} \Rightarrow \begin{cases} k_3 = 6 \\ k_3 k_1 = 6 \Rightarrow k_1 = 1 \\ 1 + k_3 k_2 = 5 \Rightarrow k_2 = \frac{2}{3} \end{cases}$$

Choice (1) is the answer.

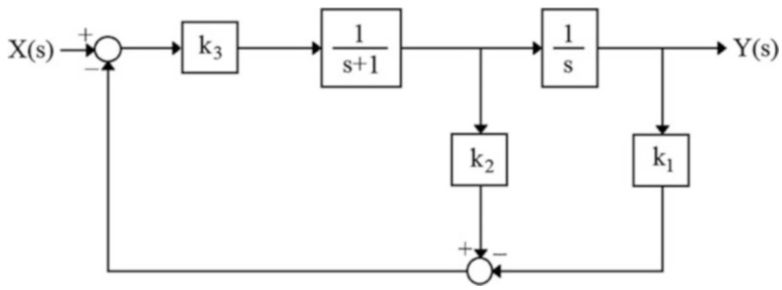

Figure 2.4 The control system of solution of problem 2.6

2.7 Based on the information given in the problem, we know the differential equation of the control system.

$$\frac{d^3}{dt^3}y(t) + 3\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 4y(t) = u(t) \tag{1}$$

By choosing the variables of $x_1(t)$, $x_2(t)$, and $x_3(t)$ as the state variables, we can write:

$$\begin{cases} x_1(t) \triangleq y(t) & (2) \\ x_2(t) \triangleq \frac{d}{dt}y(t) & (3) \\ x_3(t) \triangleq \frac{d^2}{dt^2}y(t) & (4) \end{cases} \Rightarrow \begin{cases} \frac{d}{dt}x_1(t) = \frac{d}{dt}y(t) & (5) \\ \frac{d}{dt}x_2(t) = \frac{d^2}{dt^2}y(t) & (6) \\ \frac{d}{dt}x_3(t) = \frac{d^3}{dt^3}y(t) & (7) \end{cases}$$

Solving (3) and (5):

$$\frac{d}{dt}x_1(t) = x_2(t) \tag{8}$$

Solving (4) and (6):

$$\frac{d}{dt}x_2(t) = x_3(t) \tag{9}$$

Solving (1) and (7):

$$\frac{d}{dt}x_3(t) = -3\frac{d^2}{dt^2}y(t) - 6\frac{d}{dt}y(t) - 4y(t) + u(t)$$

$$\xrightarrow{(2),\,(3),\,(4)} \frac{d}{dt}x_3(t) = -3x_3(t) - 6x_2(t) - 4x_1(t) + u(t) \tag{10}$$

By arranging (8), (9), and (10) in the form of matrices, we will achieve the state equations of the system, as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t)$$

Moreover, from (2), we can achieve the output equation of the system, as follows:

$$y(t) = [1 \ \ 0 \ \ 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Choice (3) is the answer.

2.8 Based on the information given in the problem, we have:

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{B}u \tag{1}$$

$$\mathbf{X} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2}$$

From the block-diagram, shown in Fig. 2.5, we can write:

$$\begin{cases} X_2(s) = \frac{1}{s}X_1(s) & (3) \\ \dfrac{X_1(s)}{\frac{1}{s}} = U(s) - \alpha X_1(s) - \beta X_2(s) & (4) \end{cases} \Rightarrow \begin{cases} sX_2(s) = X_1(s) & (5) \\ sX_1(s) = -\alpha X_1(s) - \beta X_2(s) + U(s) & (6) \end{cases}$$

By transferring from Laplace domain to time domain, we have:

$$\begin{cases} \dot{x}_2(t) = x_1(t) & (7) \\ \dot{x}_1(t) = -\alpha x_1(t) - \beta x_2(t) + u(t) & (8) \end{cases}$$

By arranging (7) and (8) in the form of matrices, we will achieve the state equations of the system, as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\alpha & -\beta \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)$$

$$\Rightarrow \mathbf{A} = \begin{bmatrix} -\alpha & -\beta \\ 1 & 0 \end{bmatrix}$$

Choice (2) is the answer.

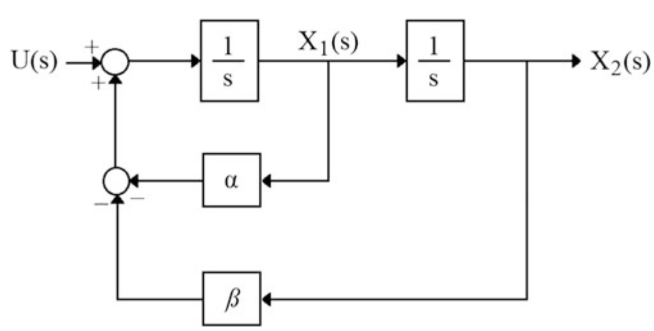

Figure 2.5 The control system of solution of problem 2.8

2.9 Based on the information given in the problem, we have:

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{AX} + \mathbf{B}u \\ \mathbf{Y} = \mathbf{CX} \end{cases}, \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (1)$$

The transfer function of a system can be determined from its state equations, as follows:

$$T(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Therefore:

$$T(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} + 0 = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$\Rightarrow T(s) = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

$$\Rightarrow T(s) = \frac{1}{ms^2 + bs + k}$$

Choice (4) is the answer.

2.10 Based on the information given in the problem, the state equations of the control system are as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} r(t) \quad (1)$$

The state-transition matrix of a system ($\boldsymbol{\varphi}(t)$) can be determined by using the following relation:

$$\boldsymbol{\varphi}(t) = L^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\} \quad (2)$$

Solving (1) and (2):

$$\boldsymbol{\varphi}(t) = L^{-1}\left\{\begin{bmatrix} s+1 & -1 \\ 1 & s+3 \end{bmatrix}^{-1}\right\} = L^{-1}\left\{\frac{1}{s^2 + 4s + 4}\begin{bmatrix} s+3 & 1 \\ -1 & s+1 \end{bmatrix}\right\} = L^{-1}\left\{\frac{1}{(s+2)^2}\begin{bmatrix} s+3 & 1 \\ -1 & s+1 \end{bmatrix}\right\}$$

$$\boldsymbol{\varphi}(t) = L^{-1}\left\{\begin{bmatrix} \frac{s+3}{(s+2)^2} & \frac{1}{(s+2)^2} \\ -\frac{1}{(s+2)^2} & \frac{s+1}{(s+2)^2} \end{bmatrix}\right\} = L^{-1}\left\{\begin{bmatrix} \frac{1}{s+2} + \frac{1}{(s+2)^2} & \frac{1}{(s+2)^2} \\ -\frac{1}{(s+2)^2} & \frac{1}{s+2} - \frac{1}{(s+2)^2} \end{bmatrix}\right\}$$

$$\boldsymbol{\varphi}(t) = \begin{bmatrix} (1+t)e^{-2t} & te^{-2t} \\ -te^{-2t} & (1-t)e^{-2t} \end{bmatrix}$$

Choice (3) is the answer.

2.11 Based on the information given in the problem, we have:

$$\dot{\mathbf{X}} = \mathbf{AX}, \quad y = \mathbf{CX} \tag{1}$$

$$\mathbf{A} = \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} \\ \frac{1}{6} & -\frac{3}{2} \end{bmatrix}, \mathbf{C} = [1 \quad 0], \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{2}$$

The output of a system can be determined from its state equations, as follows:

$$Y(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \tag{3}$$

Solving (1), (2), and (3):

$$Y(s) = [1 \quad 0] \begin{bmatrix} s + \frac{3}{2} & -\frac{3}{2} \\ -\frac{1}{6} & s + \frac{3}{2} \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = [1 \quad 0] \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s + \frac{3}{2} & \frac{3}{2} \\ \frac{1}{6} & s + \frac{3}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$Y(s) = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s + \frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{2s + 3}{(s + 1)(s + 2)} = \frac{1}{s + 1} + \frac{1}{s + 2}$$

$$y(t) = L^{-1}\left(\frac{1}{s + 1} + \frac{1}{s + 2}\right) \Rightarrow y(t) = e^{-t} + e^{-2t}$$

Choice (1) is the answer.

2.12 As can be seen from Fig. 2.6, x_1, x_2, and x_3 have been chosen as the state variables.

Now, we can write:

$$\frac{X_1(s)}{\frac{1}{s+1}} = R(s) - X_2(s) \Rightarrow (s + 1)X_1(s) = R(s) - X_2(s) \tag{1}$$

$$\frac{X_2(s)}{\frac{1}{(s+1)^2}} = X_1(s) - X_3(s) \Rightarrow \left(s^2 + 2s + 1\right)X_2(s) - X_1(s) + X_3(s) = 0 \tag{2}$$

$$X_3(s) = sX_2(s) \tag{3}$$

Solving (2) and (3):

$$sX_3(s) + 2X_3(s) + X_2(s) - X_1(s) + X_3(s) = 0 \Rightarrow sX_3(s) + 3X_3(s) + X_2(s) - X_1(s) = 0 \tag{4}$$

By transferring the equations of (1), (3), and (4) from Laplace domain to time domain, we have:

$$\dot{x}_1(t) = -x_1(t) + r(t) - x_2(t) = 0 \tag{5}$$

$$\dot{x}_2(t) = x_3(t) \tag{6}$$

$$\dot{x}_3(t) = -3x_3(t) - x_2(t) + x_1(t) \tag{7}$$

By arranging (5), (6), and (7) in the form of matrices, the state equations of the system are as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

Choice (1) is the answer.

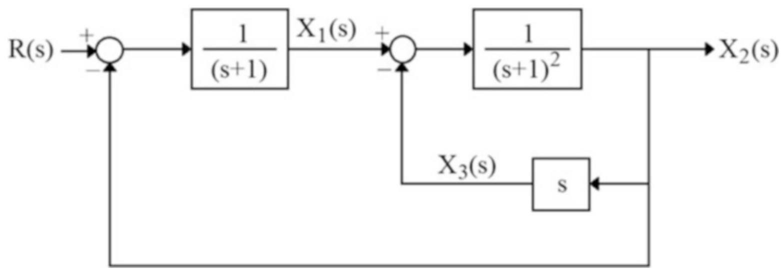

Figure 2.6 The control system of solution of problem 2.12

2.13 Based on Newton's second law for a rotational system, we have:

$$\sum_i T_i(t) = J\ddot{\theta}(t) \tag{1}$$

where $T(t)$, J, and $\theta(t)$ are torque, rotational inertia or moment of inertia, and angular position, respectively.

By applying Newton's second law on the first mass of the system shown in Fig. 2.7, we can write

$$T(t) - k(\theta_1(t) - \theta_2(t)) = J_1\ddot{\theta}_1(t) \tag{2}$$

Applying Newton's second law on the second mass:

$$0 - k(\theta_2(t) - \theta_1(t)) = J_2\ddot{\theta}_2(t) \tag{3}$$

By transferring the equations of (2) and (3) from time domain to Laplace domain, we have:

$$T(s) - \left(J_1 s^2 + k\right)\theta_1(s) + k\theta_2(s) = 0 \tag{4}$$

$$k\theta_1(s) - \left(J_2 s^2 + k\right)\theta_2(s) = 0 \tag{5}$$

Solving (4) and (5):

$$T(s) = \left(J_1 s^2 + k\right)\left(\frac{J_2 s^2 + k}{k}\right)\theta_2(s) - k\theta_2(s) \Rightarrow \frac{T(s)}{\theta_2(s)} = \left(J_1 s^2 + k\right)\left(\frac{J_2 s^2 + k}{k}\right) - k$$

$$\Rightarrow \frac{T(s)}{\theta_2(s)} = \frac{J_1 J_2 s^4 + k(J_1 + J_2)s^2}{k} \Rightarrow \frac{\theta_2(s)}{T(s)} = \frac{k}{s^2\left(J_1 J_2 s^2 + k(J_1 + J_2)\right)}$$

Choice (3) is the answer.

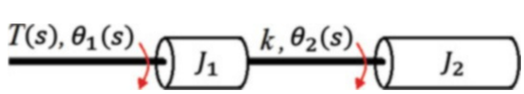

Figure 2.7 The control system of solution of problem 2.13

3 Problems: Stability Analysis of Linear Time-Invariant (LTI) Systems

Abstract

In this chapter, the stability of linear time-invariant (LTI) systems is studied. Herein, Routh-Hurwitz table is applied to determine the stability status of the closed-loop system. In this chapter, the problems are categorized in different levels based on their difficulty levels (easy, normal, and hard) and calculation amounts (small, normal, and large). Additionally, the problems are ordered from the easiest problem with the smallest computations to the most difficult problems with the largest calculations.

3.1. The equation below shows the characteristic equation of a closed-loop control system. Determine its stability status.

$$s^4 + 2s^3 + s^2 + 4s + 4 = 0$$

Difficulty level	● Easy	○ Normal	○ Hard
Calculation amount	● Small	○ Normal	○ Large

1) The system is stable.
2) The system has one unstable root.
3) The system has two unstable roots.
4) The system has three unstable roots.

3.2. Which one of the transfer functions below has a non-zero primary time response?

Difficulty level	● Easy	○ Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) $\frac{1}{s^2+2s+2}$
2) $\frac{s}{s^2+2s+2}$
3) $\frac{s+1}{s^2+2s+2}$
4) $\frac{s^2+2s+1}{s^2+2s+2}$

3.3. Which one of the following choices is correct about a closed-loop control system with the characteristic equation of $4s^3 + 2s^2 + ks + 1 = 0$?

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	● Small	○ Normal	○ Large

1) For $k = 2$, it oscillates with the angular frequency of $\frac{\sqrt{2}}{2}$ rad/sec.
2) For $k = 2$, it oscillates with the angular frequency of 1 rad/sec.
3) For $k > 2$, it is stable without any oscillation.
4) For $k = 4$, it oscillates with the angular frequency of 2 rad/sec.

M. Rahmani-Andebili, *Feedback Control Systems Analysis and Design*, https://doi.org/10.1007/978-3-030-95277-8_3

3.4. The open-loop transfer function of a control system with a negative unity feedback is as follows:

$$G(s) = \frac{k}{(s-1)(s+3)(s+5)}$$

For what value of k, does the closed-loop system response oscillate?

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ● Small ○ Normal ○ Large

1) -15
2) 15
3) 34
4) 64

3.5. Determine the period of oscillations of the closed-loop control system's response illustrated in Fig. 3.1.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ● Small ○ Normal ○ Large

1) 11.2 *sec*
2) 6.5 *sec*
3) 2.2 *sec*
4) $2\sqrt{10}$ *sec*

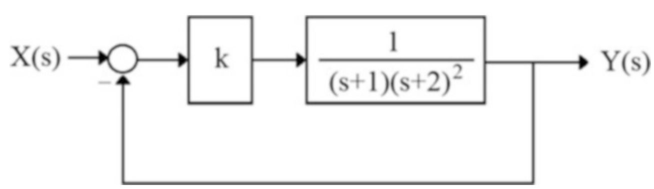

Figure 3.1 The control system of problem 3.5

3.6. The differential equations of a control system are as follows:

$$\begin{cases} \dot{x}_1(t) + x_1(t) - 2u(t) + ax_2(t) = 0 \\ \dot{x}_2(t) - bx_1(t) + 4u(t) = 0 \end{cases}$$

For what value of "a" and "b", the system is stable?

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ● Small ○ Normal ○ Large

1) $ab \geq 0$
2) $a > 0,\ b < 0$
3) $a > 0,\ b = 0$
4) $a < 0,\ b < 0$

3.7. The state equations of a control system are as follows. For what value of "k", the system is stable?

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -k-2 & -2k-3 \\ k+1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \ \ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $k > -2$
2) $k > -1$
3) $-2 < k < -1.5$
4) $(-2, -1.5) \cup (-1, \infty)$

3.8. In the control system shown in Fig. 3.2, determine the range of "p", so that the system is stable.

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) $p > 0$
2) $p > -1$
3) $-3 < p < -1$
4) $-3 < p < 1$

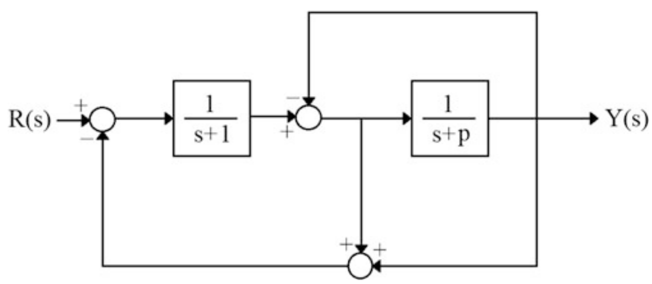

Figure 3.2 The control system of problem 3.8

3.9. For a control system with the signal flow graph (SFG), shown in Fig. 3.3, and the transfer function of $\frac{C(s)}{R(s)}$, which one of the following choices is correct?

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) The system is always unstable.
2) For $k < 1$, the system is stable.
3) For $k = -1$, the system has an undamped response.
4) For $k < 1$, the system is unstable.

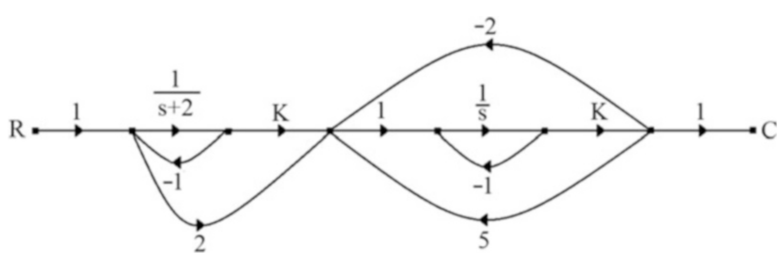

Figure 3.3 The control system of problem 3.9

3.10. For the control system, shown in Fig. 3.4, determine the hidden modes of the system.

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) $1, \pm j$
2) 0
3) $-1, 0$
4) $1, 0$

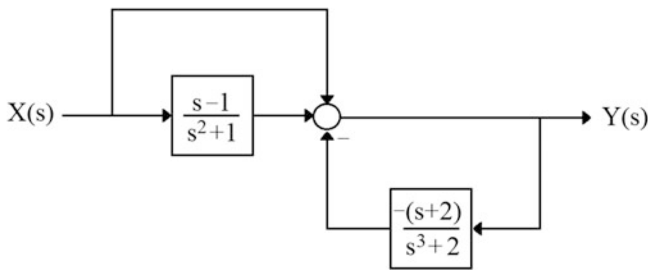

Figure 3.4 The control system of problem 3.10

3.11. For what range of k, the control system, shown in Fig. 3.5, is stable?

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $-\frac{4}{3} < k < 0$
2) $-\frac{1}{3} < k < 0$
3) $-\frac{5}{3} < k < 0$
4) $-\frac{2}{3} < k < 0$

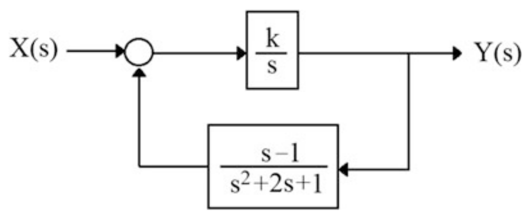

Figure 3.5 The control system of problem 3.11

3.12. The equation below shows the characteristic equation of a control system. How many unstable poles does it have?

$$s^5 + s^4 + 5s^3 + 5s^2 + 12s + 10 = 0$$

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ● Small ○ Normal ○ Large

1) 1
2) 2
3) 3
4) 0

3.13. The differential equations of a control system are as follows:

$$\begin{cases} \dot{x}_1(t) = ax_1(t) + x_2(t) + u(t) \\ \dot{x}_2(t) = -2x_1(t) + x_2(t) + u(t) \\ \dot{x}_3(t) = -x_3(t) \end{cases}$$

For what value of "a" the system is stable?

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ● Normal ○ Large

1) $a < -1$
2) $a > -2$
3) $-2 < a < -1$
4) $1 < a < 2$

3.14. Determine the transfer function of $\frac{C(s)}{R(s)}$ for the control system, shown in Fig. 3.6. Is this system internally stable or unstable?

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ● Normal ○ Large

1) $\frac{3}{(s+1)(s+2)}$, stable
2) $\frac{3}{(s+1)(s-4)}$, unstable
3) $\frac{3}{(s+2)(s-4)}$, unstable
4) $\frac{3}{(s+1)(s+2)}$, unstable

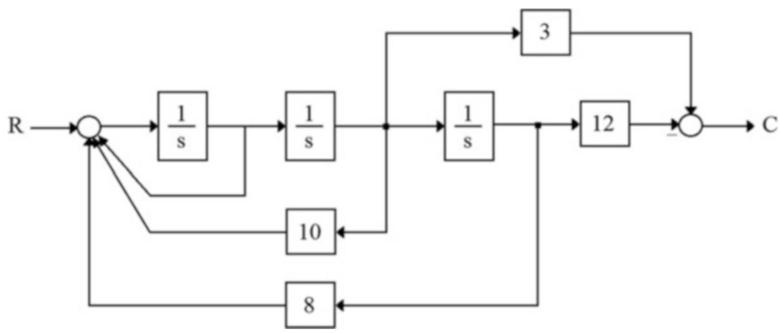

Figure 3.6 The control system of problem 3.14

3.15. In the control system, shown in Fig. 3.7, the controller is in the form of $G_c(s) = k_P + \frac{k_I}{s}$. Which one of the choices, illustrated in Fig. 3.8, graphically shows the stability area of the closed-loop system?

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ● Normal ○ Large

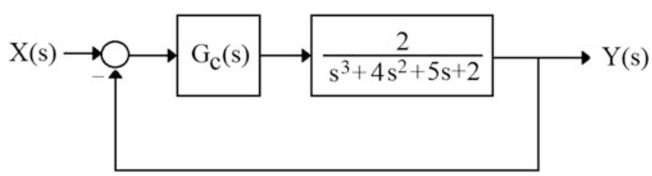

Figure 3.7 The control system of problem 3.15

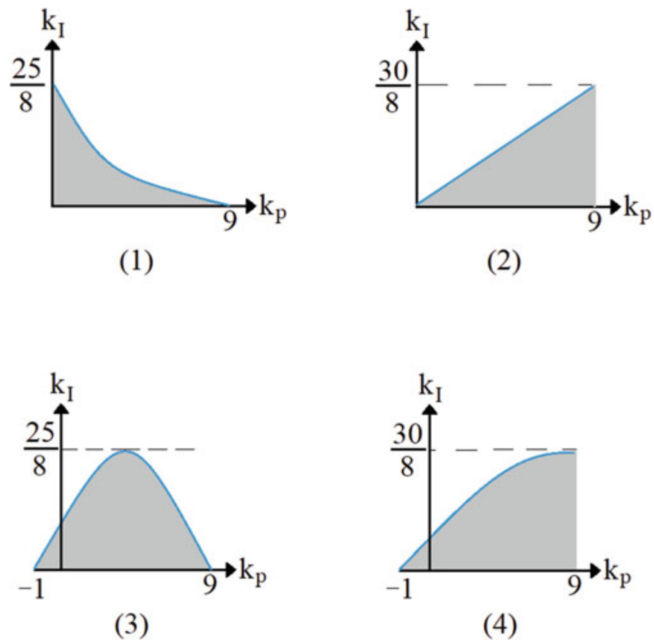

Figure 3.8 The control system of problem 3.15

3.16. For a control system with a negative unity feedback and the following open-loop transfer function, which one of the choices, shown in Fig. 3.9, graphically shows the stability area of both open-loop and closed-loop systems?

$$G(s) = \frac{k_1}{s^3 + 2s^2 + 2s + k_2}$$

Difficulty level	○ Easy	○ Normal	● Hard
Calculation amount	○ Small	○ Normal	● Large

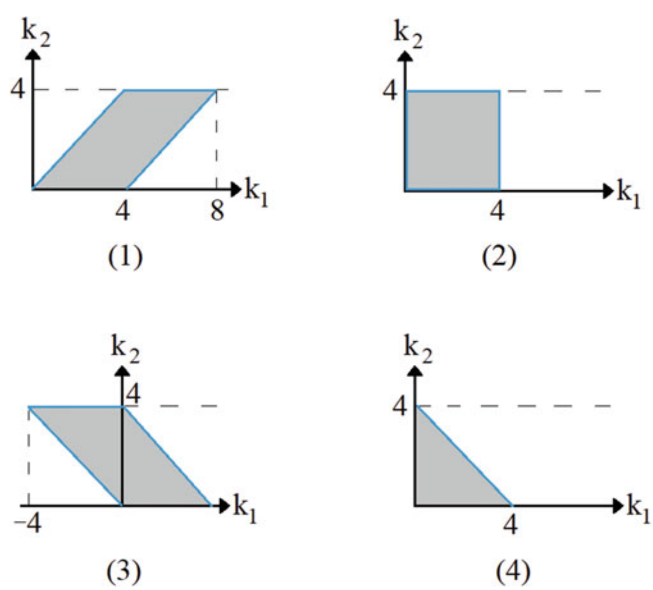

Figure 3.9 The control system of problem 3.16

4 Solutions of Problems: Stability Analysis of Linear Time-Invariant (LTI) Systems

Abstract

In this chapter, the problems of the third chapter are fully solved, in detail, step-by-step, and with different methods.

4.1 To determine the stability status of a control system, we can use Routh-Hurwitz table. Suppose that the characteristic equation of a system is as follows:

$$\Delta(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_2 s^2 + a_1 s^1 + a_0 s^0 \tag{1}$$

The structure of Routh-Hurwitz table is presented in the following. As can be seen, the coefficients of the characteristic equation are placed on the first two rows of the table with the specific pattern. However, the coefficients of the next rows need to be determined by using (2) and (3), until the last row (s^0) is filled.

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ s^1 & & & & \\ s^0 & & & & \end{array}$$

$$b_{n-1} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}} \tag{2}$$

$$b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}} \tag{3}$$

Based on Routh-Hurwitz table rule, the number of sign changes in the first column of the table determines the number of poles in the right-half plane (RHP) or the number of unstable poles.

Based on the information given in the problem, the characteristic equation of the closed-loop control system is as follows:

$$\Delta(s) = s^4 + 2s^3 + s^2 + 4s + 4$$

Therefore, for this problem, we have:

M. Rahmani-Andebili, *Feedback Control Systems Analysis and Design*, https://doi.org/10.1007/978-3-030-95277-8_4

$$\begin{array}{c|ccc} s^4 & 1 & 1 & 4 \\ s^3 & 2 & 4 & \\ s^2 & -1 & 4 & \\ s^1 & 12 & & \\ s^0 & 4 & & \end{array}$$

As can be seen, there are two sign changes in the first column of the table. Therefore, the system has two unstable roots. Choice (3) is the answer.

4.2 Unit step function is the input of the system. Therefore:

$$x(t) = u(t) \Rightarrow X(s) = \frac{1}{s}$$

The output of the system can be determined by using its transfer function as follows:

$$T(s) = \frac{Y(s)}{X(s)} \Rightarrow Y(s) = T(s)X(s) \xrightarrow{X(s) = \frac{1}{s}} Y(s) = \frac{1}{s} \times T(s)$$

From initial value theorem, we know that:

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} sY(s)$$

Applying the theorem on choice (1):

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} s \times \frac{1}{s} \times \frac{1}{s^2 + 2s + 2} = 0$$

Applying the theorem on choice (2):

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} s \times \frac{1}{s} \times \frac{s}{s^2 + 2s + 2} = 0$$

Applying the theorem on choice (3):

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} s \times \frac{1}{s} \times \frac{s+1}{s^2 + 2s + 2} = 0$$

Applying the theorem on choice (4):

$$\lim_{t \to 0} y(t) = \lim_{s \to \infty} s \times \frac{1}{s} \times \frac{s^2 + 2s + 1}{s^2 + 2s + 2} = 1$$

Choice (4) is the answer.

4.3 Based on the information given in the problem, the characteristic equation of the closed-loop control system is as follows:

$$\Delta(s) = 4s^3 + 2s^2 + ks + 1$$

The Routh-Hurwitz table for the system is as follows:

$$\begin{array}{c|cc} s^3 & 4 & k \\ s^2 & 2 & 1 \\ s^1 & 2k-4 & \\ s^0 & 1 & \end{array}$$

To have an oscillating system, all the components in one of the rows of the Routh-Hurwitz table corresponding to an odd exponent must be zero. For $k = 2$, the row corresponding to s^1 becomes zero.

Moreover, the angular frequency of the oscillations can be determined by solving the equation of the previous row ($A(s^2)$, as the auxiliary equation), as follows:

$$A\left(s^2\right) = 2s^2 + 1 = 0 \Rightarrow s = \pm j\frac{\sqrt{2}}{2} \Rightarrow \omega = \frac{\sqrt{2}}{2}\ rad/sec$$

Choice (1) is the answer.

4.4 Based on the information given in the problem, the open-loop transfer function of the control system is as follows:

$$G(s) = \frac{k}{(s-1)(s+3)(s+5)}$$

The characteristic equation of the closed-loop control system with a negative unity feedback can be determined as follows:

$$\Delta(s) = 1 + G(s) = 0$$

$$1 + \frac{k}{(s-1)(s+3)(s+5)} = 0 \Rightarrow \Delta(s) = s^3 + 7s^2 + 7s + k - 15$$

The Routh-Hurwitz table for this system is as follows:

$$\begin{array}{c|cc} s^3 & 1 & 7 \\ s^2 & 7 & k-15 \\ s^1 & 64-k & \\ s^0 & k-15 & \end{array}$$

To have an oscillating system, all the elements in one of the rows of the Routh-Hurwitz table corresponding to an odd exponent must be zero. In this control system, for $k = 64$, one of the rows, that is, the row corresponding to s^1, becomes zero. Choice (4) is the answer.

4.5 The characteristic equation of the closed-loop control system, shown in Fig. 4.1, can be determined as follows:

$$\Delta(s) = 1 + G(s)H(s) = 0$$

$$1 + \frac{k}{(s+1)(s+2)^2} = 0 \Rightarrow \Delta(s) = s^3 + 5s^2 + 8s + k + 4$$

The Routh-Hurwitz table for the system is as follows:

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 5 & k+4 \\ s^1 & 36-k & \\ s^0 & k+4 & \end{array}$$

To have an oscillating system, all the elements in one of the rows of the Routh-Hurwitz table corresponding to an odd exponent must be zero. In this control system, for $k = 36$, one of the rows, that is, the row corresponding to s^1, becomes zero.

The angular frequency of the oscillations can be determined by solving the equation of the previous row ($A(s^2)$, as the auxiliary equation), as follows:

$$A(s^2) = 5s^2 + (k+4) = 0 \xrightarrow{k=36} A(s^2) = 5s^2 + (36+4) = 0$$

$$\Rightarrow s^2 + 8 = 0 \Rightarrow s = \pm j2\sqrt{2} \Rightarrow \omega = 2\sqrt{2}\ rad/\ sec$$

The period of the oscillations of the closed-loop control system's response can be calculated as follows:

$$\Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{2\sqrt{2}} \Rightarrow T \approx 2.2\ sec$$

Choice (3) is the answer.

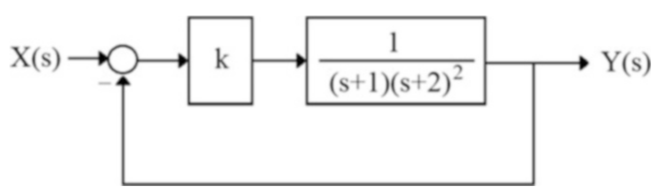

Figure 4.1 The control system of solution of problem 4.5

4.6 Based on the information given in the problem, the differential equations of the control system are as follows:

$$\begin{cases} \dot{x}_1(t) + x_1(t) - 2u(t) + ax_2(t) = 0 \\ \dot{x}_2(t) - bx_1(t) + 4u(t) = 0 \end{cases}$$

By choosing $x_1(t)$ and $x_2(t)$ as the state variables, the state equations of the system in the form of matrices are as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & -a \\ b & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix} u(t)$$

The characteristic equation of the system can be determined as follows:

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s+1 & a \\ -b & s \end{vmatrix} = s(s+1) + ab = s^2 + s + ab \tag{1}$$

For a second-order system with the characteristic equation of $a_2s^2 + a_1s + a_0$, the system is stable if and only if all the coefficients are non-zero and have the same sign. In other words:

$$a_2, a_1, a_0 > 0 \tag{2}$$

Solving (1) and (2):

$$ab > 0 \Rightarrow \begin{cases} a, b > 0 \\ a, b < 0 \end{cases}$$

Choice (4) is the answer.

4.7 Based on the information given in the problem, the state equations of the control system are as follows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -k-2 & -2k-3 \\ k+1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \;\; 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The characteristic equation of the system can be determined as follows:

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s+k+2 & 2k+3 \\ -k-1 & s \end{vmatrix} = s^2 + (k+2)s + (k+1)(2k+3) \tag{1}$$

For a second-order system with the characteristic equation of $a_2s^2 + a_1s + a_0$, the system is stable if and only if all the coefficients are non-zero and have the same sign. In other words:

$$a_2, a_1, a_0 > 0 \tag{2}$$

Solving (1) and (2):

$$\begin{cases} k+2 > 0 \\ (k+1)(2k+3) > 0 \end{cases} \Rightarrow \begin{cases} k > -2 \\ k > -1, k < -1.5 \end{cases} \Rightarrow \{k > -2\} \cap \{\{k < -1.5\} \cup \{k > -1\}\}$$

$$\Rightarrow k \in \{(-2, -1.5) \cup (-1, \infty)\}$$

Choice (4) is the answer.

4.8 To determine the characteristic equation of the system (Δ), we can use Mason's gain formula, as follows:

$$\Delta = 1 - \sum_a L_a + \sum_{a,b} L_a L_b - \sum_{a,b,c} L_a L_b L_c + \ldots$$

where:
$\sum_a L_a$: The sum of gains of loops
$\sum_{a,b} L_a L_b$: The sum of product of gains of any two non-touching loops (without any common nodes)
$\sum_{a,b,c} L_a L_b L_c$: The sum of product of gains of any three pairwise non-touching loops (without any common nodes)

Now, for the system, shown in Fig. 4.2, we have:

$$\sum_a L_a = -\frac{1}{s+1} + \left(-\frac{1}{s+p}\right) + \left(-\frac{1}{s+1}\frac{1}{s+p}\right) = -\left(\frac{s+p+s+1+1}{(s+1)(s+p)}\right) = -\frac{2s+p+2}{(s+1)(s+p)}$$

$$\sum_{a,b} L_a L_b = 0$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta = 1 - \left(-\frac{2s+p+2}{(s+1)(s+p)}\right) = 0 \Rightarrow \Delta = s^2 + (p+3)s + 2p + 2 \qquad (1)$$

As we know, for a second-order system with the characteristic equation of $a_2s^2 + a_1s + a_0$, the system is stable if and only if all the coefficients are non-zero and have the same sign. In other words:

$$a_2, a_1, a_0 > 0 \qquad (2)$$

Solving (1) and (2):

$$\begin{cases} p > -3 \\ p > -1 \end{cases} \Rightarrow p > -1$$

Choice (2) is the answer.

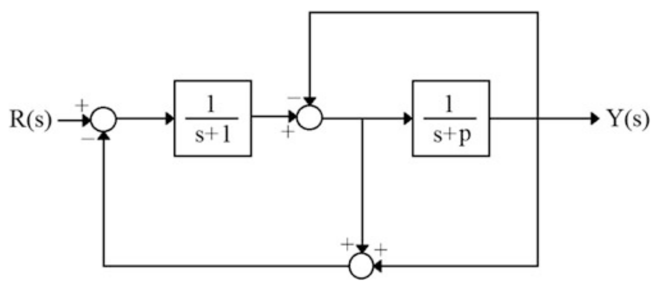

Figure 4.2 The control system of solution of problem 4.8

4.9 As can be noticed form the choices, we need to determine the stability status of the system that can be done by using the characteristic equation of the control system. To determine the characteristic equation of the system (Δ), we can use Mason's gain formula, as follows:

$$\Delta = 1 - \sum_a L_a + \sum_{a,b} L_a L_b - \sum_{a,b,c} L_a L_b L_c + \ldots$$

where:

$\sum_a L_a$: The sum of gains of loops

$\sum_{a,b} L_a L_b$: The sum of product of gains of any two non-touching loops (without any common nodes)

$\sum_{a,b,c} L_a L_b L_c$: The sum of product of gains of any three pairwise non-touching loops (without any common nodes)

Now, for the system shown in Fig. 4.3, we have:

$$\sum_a L_a = \left(-\frac{1}{s+1}\right) + \left(-\frac{1}{s}\right) + \left(\frac{5k}{s}\right) + \left(-\frac{2k}{s}\right) = \frac{-s-s-1+5k(s+1)-2k(s+1)}{s(s+1)} = \frac{s(-2+3k)-1+3k}{s(s+1)}$$

$$\sum_{a,b} L_a L_b = \left(-\frac{1}{s+1}\right) \times \left(-\frac{1}{s}\right) + \left(-\frac{1}{s+1}\right) \times \left(\frac{5k}{s}\right) + \left(-\frac{1}{s+1}\right) \times \left(-\frac{2k}{s}\right) = \frac{1-5k+2k}{s(s+1)} = \frac{1-3k}{s(s+1)}$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta = 1 - \left(\frac{s(-2+3k)-1+3k}{s(s+1)} + \frac{1-3k}{s(s+1)}\right) = \frac{s^2+s+2s-3ks}{s(s+1)} = \frac{s(s+3-3k)}{s(s+1)} = 0$$

$$\Rightarrow s(s+3-3k) = 0 \Rightarrow s = 0, 3k-3$$

For stability, all the poles must be in left-half plane (LHP). Therefore:

$$3k - 3 < 0 \Rightarrow k < 1$$

Choice (2) is the answer.

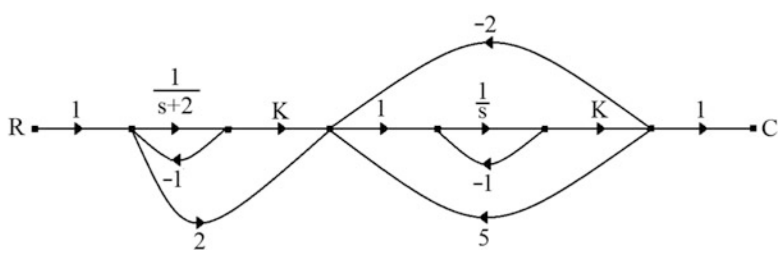

Figure 4.3 The control system of solution of problem 4.9

4.10 The hidden modes of the control system are those poles that are cancelled by the same zeros. Now, we need to determine the transfer function of the system illustrated in Fig. 4.4, as follows. Herein, we assume that the main system includes two cascaded subsystems.

$$\frac{Y(s)}{X(s)} = \left(1 + \frac{s-1}{s^2+1}\right) \times \frac{1}{1 - \frac{s+2}{s^3+2}} = \frac{s^2+1+s-1}{s^2+1} \times \frac{s^3+2}{s^3+2-s-2} = \frac{s(s+1)}{s^2+1} \times \frac{s^3+2}{s(s^2-1)} \quad (1)$$

$$\frac{Y(s)}{X(s)} = \frac{s^3+2}{(s^2+1)(s-1)} \quad (2)$$

As can be noticed from (1) and (2), the term of $s(s+1)$, corresponding to the poles of $s = 0, -1$, has been cancelled by the same zeros; thus they are the hidden modes of the system.

Choice (3) is the answer.

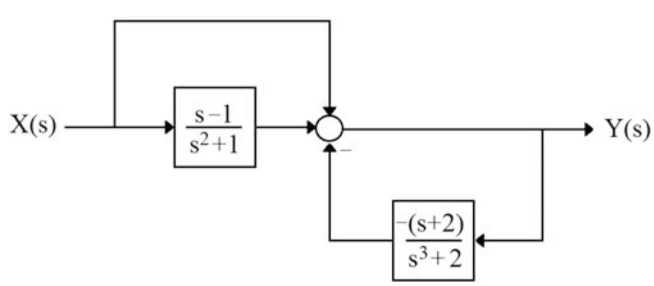

Figure 4.4 The control system of solution of problem 4.10

4.11 The characteristic equation of the positive unity feedback system, shown in Fig. 4.5, can be determined as follows:

$$\Delta(s) = 1 - G(s)H(s) = 0$$

$$1 - \left(-\frac{k}{s}\right)\left(\frac{s-1}{s^2+2s+1}\right) = 0 \Rightarrow \Delta(s) = s^3 + 2s^2 + (k+1)s - k$$

The Routh-Hurwitz table for the system is as follows:

$$\begin{array}{c|cc} s^3 & 1 & k+1 \\ s^2 & 2 & -k \\ s^1 & \dfrac{3k+2}{2} & \\ s^0 & -k & \end{array}$$

For the stability of the closed-loop system, the constraints below must be held.

$$\begin{cases} \dfrac{3k+2}{2} > 0 \\ -k > 0 \end{cases} \Rightarrow -\frac{2}{3} < k < 0$$

Choice (4) is the answer.

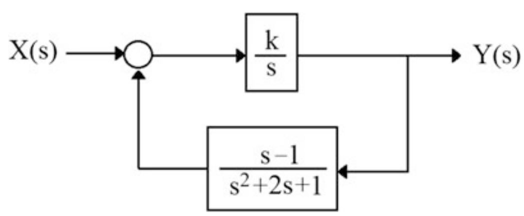

Figure 4.5 The control system of solution of problem 4.11

4.12 Based on the information given in the problem, the characteristic equation of the control system is as follows:

$$\Delta(s) = s^5 + s^4 + 5s^3 + 5s^2 + 12s + 10$$

The Routh-Hurwitz table for the system is as follows. In the first column of the third row, the quantity is zero. Therefore, determining the quantity in the first column of the fourth row is impossible, as it needs to be divided by zero. Based on the rule, the zero needs to be replaced by a very small positive quantity (ε), and then the process is continued.

$$\begin{array}{c|ccc} s^5 & 1 & 5 & 12 \\ s^4 & 1 & 5 & 10 \\ s^3 & 0 \rightarrow \varepsilon & 2 & \\ s^2 & A = \dfrac{5\varepsilon - 2}{\varepsilon} & 10 & \\ s^1 & B = \dfrac{2A - 10\varepsilon}{A} & & \\ s^0 & 10 & & \end{array}$$

The value of A is negative, but the value of B is positive. Therefore, the table includes two sign changes in its first column. Consequently, the system has two unstable poles. Choice (2) is the answer.

4.13 Based on the information given in the problem, the differential equations of the control system are as follows:

$$\begin{cases} \dot{x_1}(t) = ax_1(t) + x_2(t) + u(t) \\ \dot{x_2}(t) = -2x_1(t) + x_2(t) + u(t) \\ \dot{x_3}(t) = -x_3(t) \end{cases}$$

By choosing $x_1(t)$, $x_2(t)$, and $x_3(t)$ as the state variables, the state equations of the system in the form of matrices are as follows:

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{x_3}(t) \end{bmatrix} = \begin{bmatrix} a & 1 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t)$$

The characteristic equation of the system can be determined as follows:

$$\Delta(s) = |s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s-a & -1 & 0 \\ 2 & s-1 & 0 \\ 0 & 0 & s+1 \end{vmatrix} = (s+1)\left(s^2 - (a+1) + a + 2\right) \qquad (1)$$

In (1), $(s + 1)$ is a stable pole. For a second-order system with the characteristic equation of $a_2s^2 + a_1s + a_0$, the system is stable if and only if all the coefficients are non-zero and have the same sign. In other words:

$$a_2, a_1, a_0 > 0 \qquad (2)$$

Solving (1) and (2):

$$\begin{cases} -(a+1) > 0 \\ a+2 > 0 \end{cases} \Rightarrow -2 < a < -1$$

Choice (3) is the answer.

4.14 To determine the transfer function of the system, we can use Mason's gain formula, as follows:

$$P = \frac{1}{\Delta} \sum_{k=1}^{N} p_k \Delta_k$$

where:

$$\Delta = 1 - \sum_a L_a + \sum_{a,b} L_a L_b - \sum_{a,b,c} L_a L_b L_c + \ldots$$

P: The total gain from the input point to the output one

Δ: The determinant of the graph which is the same as the characteristic equation of the system

N: The number of forward paths from the input point to the output one

k: The index of forward path from the input point to the output

p_k: The gain of the k'th forward path from the input point to the output one

Δ_k: The determinant of the graph if the k'th forward path is removed

$\sum_a L_a$: The sum of gains of loops

$\sum_{a,b} L_a L_b$: The sum of product of gains of any two non-touching loops (without any common nodes)

$\sum_{a,b,c} L_a L_b L_c$: The sum of product of gains of any three pairwise non-touching loops (without any common nodes)

Now, for the system shown in Fig. 4.6, we can have:

$$N = 2$$

$$p_1 = -\frac{1}{s} \times \frac{1}{s} \times \frac{1}{s} \times 12 = -\frac{12}{s^3}$$

$$\Delta_1 = 1$$

$$p_2 = \frac{1}{s} \times \frac{1}{s} \times 3 = \frac{3}{s^2}$$

$$\Delta_2 = 1$$

$$\Rightarrow \sum_{k=1}^{N} p_k \Delta_k = -\frac{12}{s^3} \times 1 + \frac{3}{s^2} \times 1 = \frac{3s - 12}{s^3}$$

$$\sum_a L_a = \frac{1}{s} + \frac{1}{s} \times \frac{1}{s} \times 10 + \frac{1}{s} \times \frac{1}{s} \times \frac{1}{s} \times 8 = \frac{s^2 + 10s + 8}{s^3}$$

$$\sum_{a,b} L_a L_b = 0$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta = 1 - \frac{s^2 + 10s + 8}{s^3} = \frac{s^3 - s^2 - 10s - 8}{s^3}$$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{\frac{3s-12}{s^3}}{\frac{s^3 - s^2 - 10s - 8}{s^3}} = \frac{3s - 12}{s^3 - s^2 - 10s - 8} = \frac{s(s-4)}{(s-4)(s+1)(s+2)}$$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{s}{(s+1)(s+2)}$$

Although no unstable pole is seen in the transfer function, the system is internally unstable because one unstable pole and zero, that is, $(s - 4)$ has been canceled from the transfer function. Herein, $s = 4$ is called the hidden mode of the system.

Choice (4) is the answer.

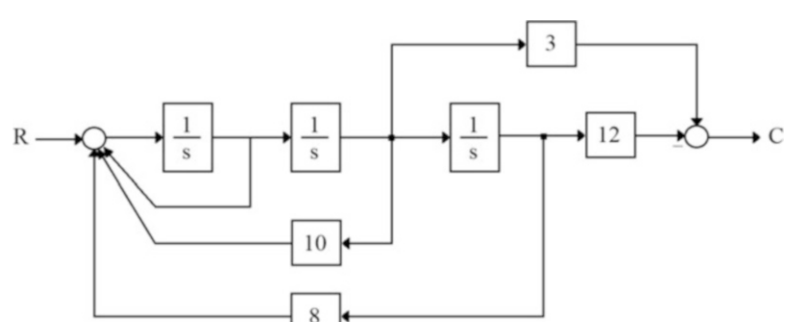

Figure 4.6 The control system of solution of problem 4.14

4.15 First, we need to determine the characteristic equation of the control system (shown in Fig. 4.8) as follows:

$$1 + G_c(s)G(s) = 0 \Rightarrow 1 + \left(k_P + \frac{k_I}{s}\right)\left(\frac{2}{s^3 + 4s^2 + 5s + 2}\right) = 0$$

$$\Rightarrow 1 + \frac{2(sk_P + k_I)}{s(s^3 + 4s^2 + 5s + 2)} = 0 \Rightarrow \Delta(s) = s^4 + 4s^3 + 5s^2 + 2(1 + k_P)s + 2k_I$$

For the given system, the table will be as follows:

$$\begin{array}{c|ccc} s^4 & 1 & 5 & 2k_I \\ s^3 & 4 & 2(1+k_P) & \\ s^2 & \dfrac{20-2(1+k_P)}{4} & 2k_I & \\ s^1 & A & & \\ s^0 & 2k_I & & \end{array}$$

where:

$$A = 2(1 + k_P) - \frac{8k_I}{\frac{20-2(1+k_P)}{4}} = \frac{2(1 + k_P)(18 - 2k_P) - 32k_I}{(18 - 2k_P)} = \frac{-(k_P)^2 + 8k_P + 9 - 8k_I}{4.5 - 0.5k_P}$$

For the stability of the system, all the elements in the first column of the table must be positive. Therefore, the constraints below must be held.

$$\begin{cases} \dfrac{20 - 2(1 + k_P)}{4} > 0 \\ \dfrac{-(k_P)^2 + 8k_P + 9 - 8k_I}{4.5 - 0.5k_P} > 0 \\ 2k_I > 0 \end{cases} \Rightarrow \begin{cases} 18 - 2k_P > 0 \\ -(k_P)^2 + 8k_P + 9 - 8k_I > 0 \\ 2k_I > 0 \end{cases} \Rightarrow \begin{cases} k_P < 9 & (1) \\ k_I < -\dfrac{1}{8}(k_P)^2 + k_P + \dfrac{9}{8} & (2) \\ k_I > 0 & (3) \end{cases}$$

From (2), it can be noticed that the graph is a parabola that opens downward (the vertex is a maximum point). Cases 3 and 4 have this feature. However, in (2), $k_I = 0$ is achieved for $k_P = 9$. Choice (3) is the answer.

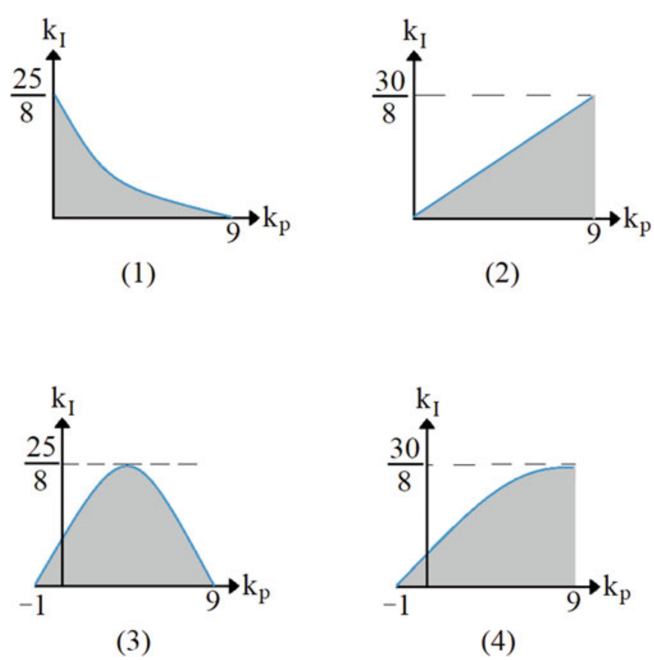

Figure 4.7 The control system of solution of problem 4.15

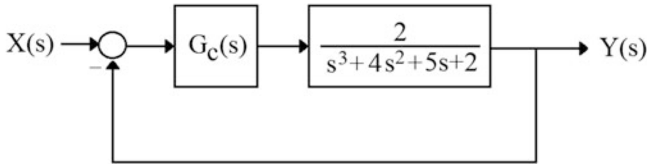

Figure 4.8 The control system of solution of problem 4.15

4.16 Based on the information given in the problem, the transfer function of the open-loop system is as follows:

$$G(s) = \frac{k_1}{s^3 + 2s^2 + 2s + k_2}$$

Based on the problem, both open-loop and closed-loop systems must be stable. Therefore, we need to evaluate the characteristic equations of the open-loop and closed-loop systems by using Routh-Hurwitz table.

The characteristic equation of the open-loop system is as follows:

$$\Delta_1(s) = s^3 + 2s^2 + 2s + k_2$$

Moreover, the characteristic equation of the close-loop system with a negative unity feedback can be determined as follows:

$$1 + G(s) = 0$$

$$1 + \frac{k_1}{s^3 + 2s^2 + 2s + k_2} = 0 \Rightarrow \Delta_2(s) = s^3 + 2s^2 + 2s + k_2 + k_1$$

The Routh-Hurwitz table for the open-loop system is as follows:

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 2 & k_2 \\ s^1 & \dfrac{4-k_2}{2} & \\ s^0 & k_2 & \end{array}$$

For the stability of the open-loop system, the constraints below must be held.

$$\begin{cases} \dfrac{4-k_2}{2} > 0 \\ k_2 > 0 \end{cases} \Rightarrow 0 < k_2 < 4 \tag{4}$$

In addition, the Routh-Hurwitz table for the closed-loop system is as follows:

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 2 & k_1 + k_2 \\ s^1 & \dfrac{4-(k_1+k_2)}{2} & \\ s^0 & k_1 + k_2 & \end{array}$$

For the stability of the closed-loop system, the constraints below must be held.

$$\begin{cases} \dfrac{4-(k_1+k_2)}{2} > 0 \\ k_1+k_2 > 0 \end{cases} \Rightarrow 0 < k_1+k_2 < 4 \tag{5}$$

From (5), we have the equations below that are the equations of straight lines.

$$k_2 > -k_1 \tag{6}$$

$$k_2 < 4 - k_1 \tag{7}$$

Considering (4), (6), and (7) and drawing them on a plot result in the graph shown in choice 2. Choice (3) is the answer.

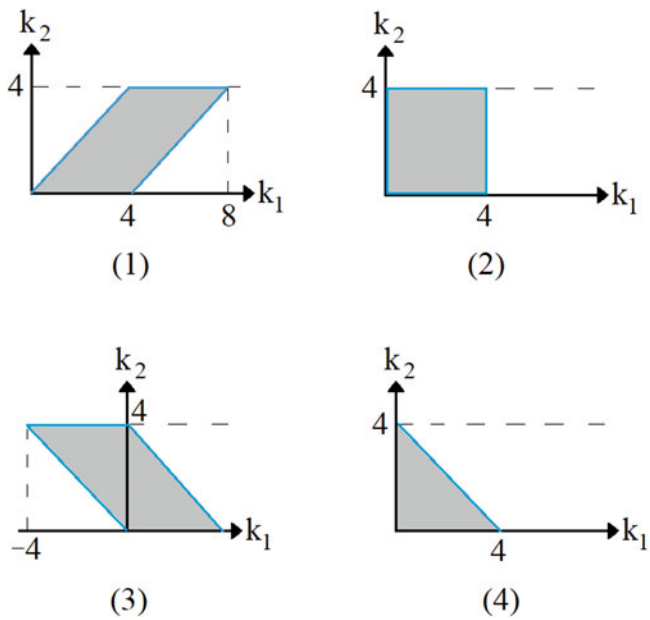

Figure 4.9 The control system of solution of problem 4.16

5 Problems: Analysis of Transient Response

Abstract

In this chapter, the transient response of second-order control systems is analyzed based on several parameters such as damping ratio, overshoot percentage, rise time, settling time, and peak time. Herein, the transient response of the second-order systems is categorized in different classes, including overdamped response, critically damped response, underdamped response, and undamped response. In this chapter, the problems are categorized in different levels based on their difficulty levels (easy, normal, and hard) and calculation amounts (small, normal, and large). Additionally, the problems are ordered from the easiest problem with the smallest computations to the most difficult problems with the largest calculations.

5.1 The open-loop transfer function of a control system with a negative unity feedback is as follows:

$$G(s) = \frac{10}{s(s+1)(s+10)}$$

Which one of the following choices is a good approximation for the open-loop transfer function?

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	● Small	○ Normal	○ Large

1) $\widetilde{G}(s) = \frac{1}{s(s+10)}$
2) $\widetilde{G}(s) = \frac{10}{s(s+1)}$
3) $\widetilde{G}(s) = \frac{10}{s(s+10)}$
4) $\widetilde{G}(s) = \frac{1}{s(s+1)}$

5.2 The control system, shown in Fig. 5.1, has the following state equations in matrix form:

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{C} = [k \;\; 0 \;\; 0]$$

Which one of the following choices is correct about the system?

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) The system is unstable for $k > 1$.
2) The system is unstable for $k > 0$.
3) The system's transient response is overdamped for $k > 0$.
4) The system's transient response can be overdamped, critically damped, or underdamped for $k > 0$.

M. Rahmani-Andebili, *Feedback Control Systems Analysis and Design*, https://doi.org/10.1007/978-3-030-95277-8_5

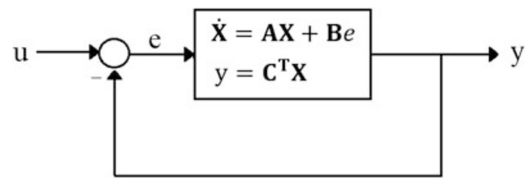

Figure 5.1 The control system of problem 5.2

5.3 Determine the value of parameters "a" and "b," so that the control system, shown in Fig. 5.2, has the fastest response without any damping oscillation to a unit step function.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $\begin{cases} a^2 - 4b < 28 \\ a > 0 \end{cases}$
2) $\begin{cases} a^2 - 4b = 28 \\ a > 0 \end{cases}$
3) $\begin{cases} a^2 - 4b < 28 \\ a > 0 \\ b > -7 \end{cases}$
4) $\begin{cases} a^2 - 4b = 28 \\ a > 0 \\ b > -7 \end{cases}$

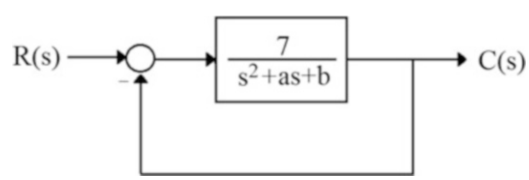

Figure 5.2 The control system of problem 5.3

5.4 In the control system, shown in Fig. 5.3, the value of k has been designed to have the fastest response but without any overshooting. In this condition, determine the settling time of the system response.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) 4 *sec*
2) 2 *sec*
3) 3 *sec*
4) 0.5 *sec*

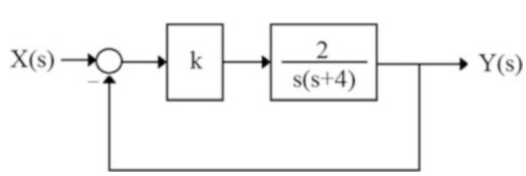

Figure 5.3 The control system of problem 5.4

5.5 The open-loop transfer function of a control system is as follows:

$$G(s) = \frac{k}{s(\tau s + 1)}$$

Determine the value of k so that the closed-loop system has an underdamped response to a unit step input. Moreover, determine the damping ratio of the system response.

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) $k > \frac{1}{4\tau}, \xi = \frac{1}{2\sqrt{\tau k}}$
2) $k < \frac{1}{4\tau}, \xi = \frac{1}{2\tau\sqrt{k}}$
3) $k > \frac{1}{4\tau}, \xi = \frac{1}{2\tau\sqrt{k}}$
4) $k < \frac{1}{4\tau}, \xi = \frac{1}{2\sqrt{\tau k}}$

5.6 In the control system, shown in Fig. 5.4, determine the value of "k_1" and "k_2," so that the damping ratio and the settling time (5% criterion) of the closed-loop system are 0.5 and 2 seconds, respectively.

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	○ Normal	● Large

1) $k_1 = 4, k_2 = 3$
2) $k_1 = 16, k_2 = 3$
3) $k_1 = 9, k_2 = 2$
4) $k_1 = 3, k_2 = 2$

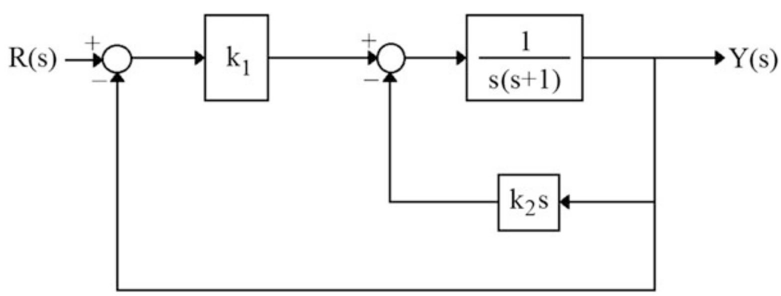

Figure 5.4 The control system of problem 5.6

5.7 In the control system, shown in Fig. 5.5, determine the value of "k_1" and "k_2," so that the settling time (2% criterion) and the peak time of the closed-loop system are $\frac{4}{3}$ seconds and $\frac{\pi}{4}$ seconds, respectively.

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	○ Normal	● Large

1) $k_1 = 25, k_2 = 0.02$
2) $k_1 = 25, k_2 = 0.04$
3) $k_1 \approx 16, k_2 = 0.02$
4) $k_1 \approx 16, k_2 \approx \frac{1}{16}$

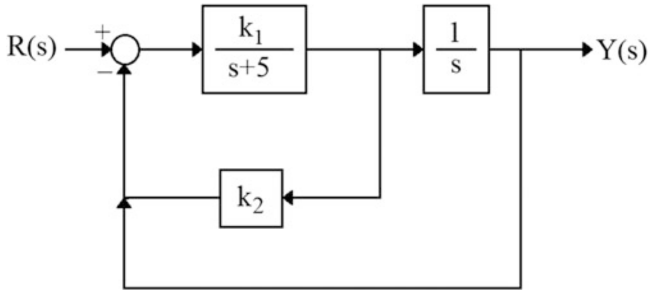

Figure 5.5 The control system of problem 5.7

5.8 The differential equation of a control system with the zero-primary condition is as follows:

$$y''(t) + 4y'(t) + 20y(t) = r(t)$$

Determine the time that the second peak in the system response occurs if the input is $r(t) = 4u(t)$.

Difficulty level	○ Easy	○ Normal	● Hard
Calculation amount	○ Small	● Normal	○ Large

1) $\frac{3\pi}{4}$
2) $\frac{\pi}{3}$
3) $\frac{\pi}{2}$
4) $\frac{\pi}{20}$

5.9 A unit step function ($f(t) = u(t)$) is applied on the mechanical system shown in Fig. 5.6.1. The output is the horizontal position of the mass which is illustrated in Fig. 5.6.2. Determine the damping ratio of the system response if $M = 1\ kg$, $B = 1$.

Difficulty level	○ Easy	○ Normal	● Hard
Calculation amount	○ Small	○ Normal	● Large

1) $\frac{1}{4}$
2) $\frac{1}{2}$
3) $\frac{1}{\sqrt{k}}$
4) $\frac{1}{2k}$

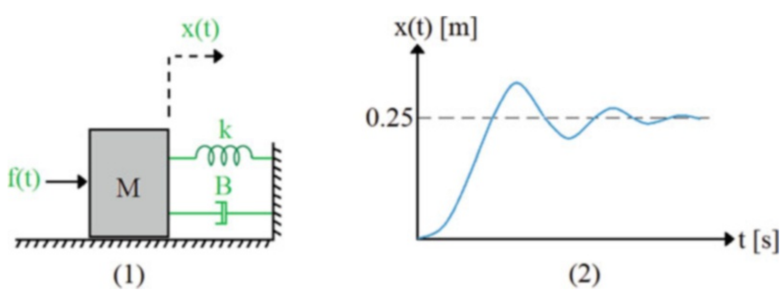

Figure 5.6 The control system of problem 5.9

5.10 Figure 5.7 illustrates the unit step response of a second-order control system. Determine its approximate closed-loop transfer function.

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ○ Normal ● Large

1) $\frac{240}{s^2+136s+240}$
2) $\frac{240^2}{s^2+136s+240^2}$
3) $\frac{336}{s^2+240s+336}$
4) $\frac{336^2}{s^2+240s+336^2}$

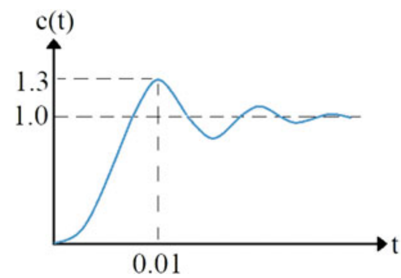

Figure 5.7 The control system of problem 5.10

6 Solutions of Problems: Analysis of Transient Response

Abstract
In this chapter, the problems of the fifth chapter are fully solved, in detail, step-by-step, and with different methods.

6.1 Based on the information given in the problem, the open-loop transfer function of the control system with a negative unity feedback is as follows:

$$G(s) = \frac{10}{s(s+1)(s+10)} \quad (1)$$

As can be noticed from (1), the pole of $(s + 10)$ is nondominant as it is very far from the origin compared to the other poles. Thus, this pole can be ignored to decrease the order of the open-loop transfer function.

$$\tilde{G}(s) = \frac{k}{s(s+1)} \quad (2)$$

However, the steady-state gain (DC value) of the transfer function must be left intact. Therefore, the value of k can be determined as follows:

$$\left.\frac{10}{s(s+1)(s+10)}\right|_{s=0} \approx \left.\frac{k}{s(s+1)}\right|_{s=0} \Rightarrow \left.\frac{10}{(s+10)}\right|_{s=0} \approx \left.\frac{k}{1}\right|_{s=0} \Rightarrow \frac{10}{10} = \frac{k}{1} \Rightarrow k = 1 \quad (3)$$

Solving (2) and (3):

$$\tilde{G}(s) = \frac{1}{s(s+1)}$$

Choice (4) is the answer.

6.2 Based on the information given in the problem, the state equations of the open-loop system are as follows:

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{C} = [k \;\; 0 \;\; 0] \quad (1)$$

The transfer function of a system can be determined from its state equations, as follows:

M. Rahmani-Andebili, *Feedback Control Systems Analysis and Design*, https://doi.org/10.1007/978-3-030-95277-8_6

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{2}$$

By solving (1) and (2), the transfer function of the open-loop system is determined as follows:

$$G(s) = [k \quad 0 \quad 0]\begin{bmatrix} s+2 & -1 & -1 \\ -1 & s+1 & 0 \\ -1 & 0 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{k(s+1)}{s(s+3)} \tag{3}$$

Moreover, the transfer function of the closed-loop system can be determined as follows.

$$T(s) = \frac{Y(s)}{U(s)} = \frac{\frac{k(s+1)}{s(s+3)}}{1 + \frac{k(s+1)}{s(s+3)}} = \frac{k(s+1)}{s^2 + (3+k)s + k} \tag{4}$$

The denominator of a transfer function is the characteristic equation of the system. Thus:

$$\Delta(s) = s^2 + (3+k)s + k \tag{5}$$

A second-order system with the characteristic equation of $a_2 s^2 + a_1 s + a_0$ is stable if and only if all the coefficients are non-zero and have the same sign. In other words:

$$a_2, a_1, a_0 > 0 \tag{6}$$

Solving (5) and (6):

$$\begin{cases} k > -3 \\ k > 0 \end{cases} \overset{\cap}{\Longrightarrow} k > 0 \tag{7}$$

Therefore, Cases 1 and 2 are incorrect.

In a system with the characteristic equation of $\Delta(s) = a_2 s^2 + a_1 s + a_0$, the system is in the overdamped status if and only if:

$$\text{Discriminant} = (a_1)^2 - 4a_2 a_0 > 0 \tag{8}$$

Therefor:

$$\text{Discriminant} = (3+k)^2 - 4k = k^2 + 2k + 9 \tag{9}$$

Solving (7) and (9):

$$\text{Discriminant} > 0 \tag{10}$$

Therefore, the system's transient response is overdamped for $k > 0$. Choice (3) is the answer.

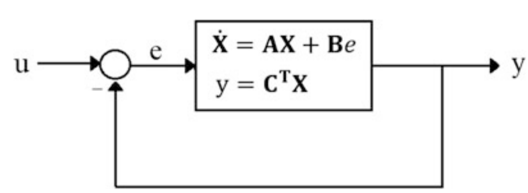

Figure 6.1 The control system of solution of problem 6.2

6.3 Critically damped response is the fastest response without any damping oscillation. In a system with the characteristic equation of $\Delta(s) = a_2s^2 + a_1s + a_0$, the system has a critically damped response if and only if:

$$(a_1)^2 - 4a_2a_0 = 0 \quad (1)$$

The characteristic equation of the closed-loop control system can be determined as follows:

$$1 + G(s)H(s) = 0 \Rightarrow 1 + \frac{7}{s^2 + as + b} = 0$$

$$\Rightarrow \Delta(s) = s^2 + as + b + 7 \quad (2)$$

Solving (1) and (2):

$$(a)^2 - 4 \times 1 \times (b + 7) = 0 \Rightarrow a^2 - 4b = 28 \quad (3)$$

Moreover, the system must be stable. A second-order system with the characteristic equation of $a_2s^2 + a_1s + a_0$ is stable if and only if all the coefficients are non-zero and have the same sign. In other words:

$$a_2, a_1, a_0 > 0 \quad (4)$$

Solving (2) and (4):

$$a > 0 \& b > -7 \quad (5)$$

Solving (3) and (5):

$$\begin{cases} a^2 - 4b = 28 \\ a > 0 \\ b > -7 \end{cases}$$

Choice (4) is the answer.

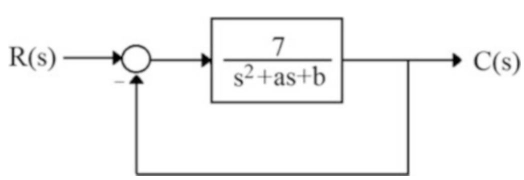

Figure 6.2 The control system of solution of problem 6.3

6.4 Critically damped response is the fastest response without any damping oscillation. In a system with the characteristic equation of $\Delta(s) = a_2s^2 + a_1s + a_0$, the system has a critically damped response if and only if:

$$(a_1)^2 - 4a_2a_0 = 0 \quad (1)$$

The characteristic equation of the closed-loop control system can be determined as follows:

$$1 + G(s)H(s) = 0 \Rightarrow 1 + \frac{2k}{s(s + 4)} = 0$$

$$\Rightarrow \Delta(s) = s^2 + 4s + 2k \quad (2)$$

Solving (1) and (2):

$$4^2 - 4 \times 1 \times 2k = 0 \Rightarrow k = 2 \quad (3)$$

Therefore, the characteristic equation of the closed-loop system is as follows:

$$\Delta(s) = s^2 + 4s + 4 \quad (4)$$

The standard second-order characteristic equation of a control system is presented as follows:

$$\Delta(s) = s^2 + 2\xi\omega_n s + {\omega_n}^2 \quad (5)$$

where, its settling time can be determined as follows:

$$t_s = \frac{4}{\xi\omega_n} \quad (6)$$

By comparing (4) and (5), it is concluded that:

$$\omega_n = 2, \xi = 1 \quad (7)$$

Solving (6) and (7):

$$t_s = \frac{4}{\xi\omega_n} = \frac{4}{1 \times 2} = 2 \ sec$$

Choice (2) is the answer.

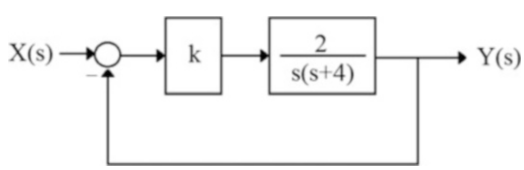

Figure 6.3 The control system of solution of problem 6.4

6.5 Based on the information given in the problem, the open-loop transfer function of the control system is as follows:

$$G(s) = \frac{k}{s(\tau s + 1)}$$

The characteristic equation of the closed-loop control system can be determined as follows:

$$1 + G(s) = 0 \Rightarrow 1 + \frac{k}{s(\tau s + 1)} = 0$$

$$\Rightarrow \Delta(s) = \tau s^2 + s + k \Rightarrow \Delta(s) = s^2 + \frac{1}{\tau}s + \frac{k}{\tau} \quad (1)$$

In a system with the characteristic equation of $\Delta(s) = a_2 s^2 + a_1 s + a_0$, the system has an underdamped response if:

$$(a_1)^2 - 4a_2a_0 < 0 \quad (2)$$

Solving (1) and (2):

$$\left(\frac{1}{\tau}\right)^2 - 4 \times 1 \times \frac{k}{\tau} < 0 \Rightarrow 1 - 4k\tau < 0 \Rightarrow k > \frac{1}{4\tau}$$

Moreover, the damping ratio of the unit step response can be determined by comparing the characteristic equation of the system (see (1)) with the standard second-order characteristic equation, that is, $\Delta(s) = s^2 + 2\xi\omega_n s + \omega_n{}^2$, as follows:

$$\xi = \frac{\frac{1}{\tau}}{2\omega_n} = \frac{\frac{1}{\tau}}{2\sqrt{\frac{k}{\tau}}} \Rightarrow \xi = \frac{1}{2\sqrt{\tau k}}$$

Choice (1) is the answer.

6.6 Based on the information given in the problem, the damping ratio and the settling time (5% criterion) of the closed-loop system are 0.5 and 2 seconds, respectively. In other words:

$$\xi = 0.5 \quad (1)$$

$$t_s = 2 \ sec \quad (2)$$

The settling time (5% criterion) of a second-order system can be determined as follows:

$$t_s = \frac{3}{\xi\omega_n} \quad (3)$$

Solving (1), (2), and (3):

$$2 = \frac{3}{0.5 \times \omega_n} \Rightarrow \omega_n = 3 \ rad/sec \quad (4)$$

The transfer function of the inner closed-loop system, shown in Fig. 6.4, can be determined as follows:

$$G(s) = \frac{\frac{1}{s(s+1)}}{1 + \frac{k_2 s}{s(s+1)}} = \frac{1}{s^2 + (k_2 + 1)s} \quad (5)$$

Then, the transfer function of the whole closed-loop system can be determined as follows:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{k_1 \times G(s)}{1 + k_1 \times G(s)} = \frac{k_1 \times \frac{1}{s^2 + (k_2+1)s}}{1 + k_1 \times \frac{1}{s^2 + (k_2+1)s}} = \frac{k_1}{s^2 + (k_2 + 1)s + k_1} \quad (6)$$

The denominator of a transfer function is its characteristic equation. Thus, by comparing (6) with the standard second-order characteristic equation, that is, $\Delta(s) = s^2 + 2\xi\omega_n s + \omega_n{}^2$, we have:

$$\begin{cases} k_1 = \omega_n{}^2 & (7) \\ k_2 + 1 = 2\xi\omega_n & (8) \end{cases}$$

Solving (4) and (7):

$$k_1 = 3^2 \Rightarrow k_1 = 9$$

Solving (1), (4), and (8):

$$k_2 + 1 = 2 \times 0.5 \times 3 \Rightarrow k_2 = 2$$

Choice (3) is the answer.

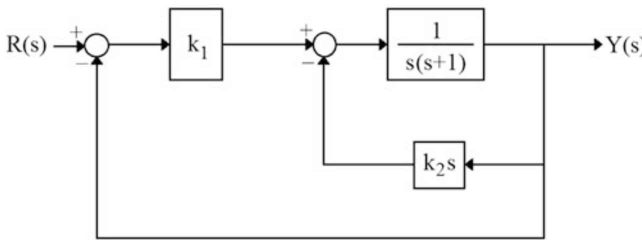

Figure 6.4 The control system of solution of problem 6.6

6.7 Based on the information given in the problem, the settling time (2% criterion) and the peak time of the closed-loop system are $\frac{4}{3}$ seconds and $\frac{\pi}{4}$ seconds, respectively. In other words:

$$t_s = \frac{4}{3} \ sec \tag{1}$$

$$t_p = \frac{\pi}{4} \ sec \tag{2}$$

The settling time (2% criterion) of a second-order system can be determined as follows:

$$t_s = \frac{4}{\xi \omega_n} = \frac{4}{\sigma} \tag{3}$$

Moreover, the peak time of a second-order system can be determined as follows:

$$t_p = \frac{\pi}{\omega_d} \tag{4}$$

Solving (1) and (3):

$$\frac{4}{3} = \frac{4}{\sigma} \Rightarrow \sigma = 3 \tag{5}$$

Solving (2) and (4):

$$\frac{\pi}{4} = \frac{\pi}{\omega_d} \Rightarrow \omega_d = 4 \tag{6}$$

The undamped natural angular frequency of a second-order control system can be determined as follows:

$$\omega_n = \sqrt{{\omega_d}^2 + \sigma^2} \tag{7}$$

Solving (5), (6), and (7):

$$\omega_n = \sqrt{{\omega_d}^2 + \sigma^2} = \sqrt{4^2 + 3^2} = 5 \tag{8}$$

The transfer function of the inner closed-loop system, illustrated Fig. 6.5, can be determined as follows:

$$G(s) = \frac{\frac{k_1}{s+5}}{1 + \frac{k_1 k_2}{s+5}} = \frac{k_1}{s + 5 + k_1 k_2} \tag{9}$$

Then, the transfer function of the whole closed-loop system can be determined as follows:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s) \times \frac{1}{s}}{1 + G(s) \times \frac{1}{s}} = \frac{\frac{k_1}{s+5+k_1k_2} \times \frac{1}{s}}{1 + \frac{k_1}{s+5+k_1k_2} \times \frac{1}{s}} = \frac{k_1}{s^2 + (5 + k_1 k_2)s + k_1} \tag{10}$$

The denominator of a transfer function is its characteristic equation. Thus, by comparing (10) with the standard second-order characteristic equation, that is, $\Delta(s) = s^2 + 2\sigma s + {\omega_n}^2$, we have:

$$\begin{cases} k_1 = {\omega_n}^2 & (11) \\ 5 + k_1 k_2 = 2\sigma & (12) \end{cases}$$

Solving (8) and (11):

$$k_1 = 25 \tag{13}$$

Solving (5), (12), and (13):

$$5 + 25k_2 = 2 \times 3 \Rightarrow k_2 = 0.04$$

Choice (2) is the answer.

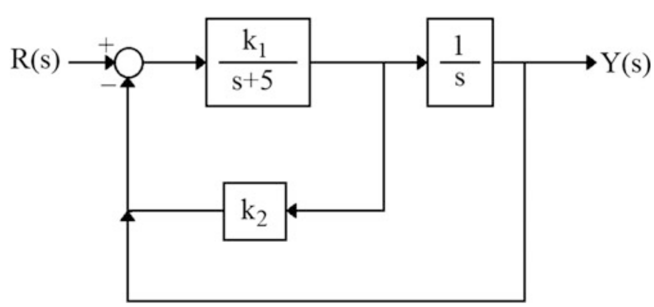

Figure 6.5 The control system of solution of problem 6.7

6.8 Based on the information given in the problem, the differential equation of the control system with the zero-primary condition is as follows:

$$y''(t) + 4y'(t) + 20y(t) = r(t) \tag{1}$$

The transfer function of the system can be determined, as follows:

$$y''(t) + 4y'(t) + 20y(t) = r(t) \xrightarrow{\text{Laplace Trans.}} \left(s^2 + 4s + 20\right)Y(s) = R(s)$$

$$\Rightarrow T(s) = \frac{Y(s)}{R(s)} = \frac{1}{s^2 + 4s + 20} \tag{2}$$

In an underdamped system, the time that different peaks in the system response occur can be determined as follows:

$$t_{pn} = \frac{(2n-1)\pi}{\omega_d} \tag{3}$$

Therefore, the time of the second peak is as follows:

$$t_{p2} = \frac{(2 \times 2 - 1)\pi}{\omega_d} = \frac{3\pi}{\omega_d} \tag{4}$$

The denominator of a transfer function is the characteristic equation of the system. Thus, by comparing (2) with the standard second-order characteristic equation, that is, $\Delta(s) = s^2 + 2\sigma s + {\omega_n}^2$, we have:

$$\begin{cases} {\omega_n}^2 = 20 & (5) \\ 2\sigma = 4 \Rightarrow \sigma = 2 & (6) \end{cases}$$

The damped angular frequency of a second-order control system can be determined as follows:

$$\omega_d = \sqrt{{\omega_n}^2 - \sigma^2} \tag{7}$$

Solving (5), (6), and (7):

$$\omega_d = \sqrt{20 - 2^2} = 4 \tag{8}$$

Solving (4) and (8):

$$t_{p2} = \frac{3\pi}{\omega_d} = \frac{3\pi}{4}$$

Choice (1) is the answer.

6.9 Based on the information given in the problem, we have:

$$f(t) = u(t) \tag{1}$$

$$M = 1\ kg, B = 1 \tag{2}$$

Moreover, from the graph, shown in Fig. 6.6.2, it is noticed that:

$$\lim_{t \to \infty} x(t) = \frac{1}{4} \tag{3}$$

Based on Newton's second law for a translational system, we have:

$$\sum f(t) = M\ddot{x}(t) \tag{4}$$

where $f(t)$, M, and $x(t)$ are force, mass, and position, respectively.

Therefore, by applying Newton's second law on the system (see Fig. 6.6.1), we have:

$$f(t) - f_k(t) - f_B(t) = M\ddot{x}(t) \Rightarrow f(t) - kx(t) - B\dot{x}(t) = M\ddot{x}(t) \tag{5}$$

Solving (2) and (5):

$$f(t) - kx(t) - \dot{x}(t) = \ddot{x}(t) \xrightarrow{Laplace\ Trans.} F(s) - kX(s) - sX(x) = s^2X(t)$$

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + s + k} \tag{6}$$

Equation (6) shows the transfer function of the system.

$$\Rightarrow X(s) = \frac{1}{s^2 + s + k} F(s) \tag{7}$$

From final value theorem, we know that:

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s) \tag{8}$$

Solving (1), (7), and (8), and knowing that $F(s) = L(f(t)) = \frac{1}{s}$:

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s \times \frac{1}{s^2 + s + k} \times \frac{1}{s} \Rightarrow \lim_{t \to \infty} x(t) = \frac{1}{k} \tag{9}$$

Solving (3) and (9):

$$\frac{1}{k} = \frac{1}{4} \Rightarrow k = 4 \tag{10}$$

Solving (6) and (10):

$$T(s) = \frac{1}{s^2 + s + 4} \tag{11}$$

The denominator of a transfer function is its characteristic equation. Thus, by comparing (11) with the standard second-order characteristic equation, that is, $\Delta(s) = s^2 + 2\xi\omega_n s + \omega_n{}^2$, the damping ratio of the unit step response can be determined, as follows:

$$\begin{cases} \omega_n{}^2 = 4 \Rightarrow \omega_n = 2 \\ 2\xi\omega_n = 1 \end{cases} \Rightarrow \xi = \frac{1}{4}$$

Choice (1) is the answer.

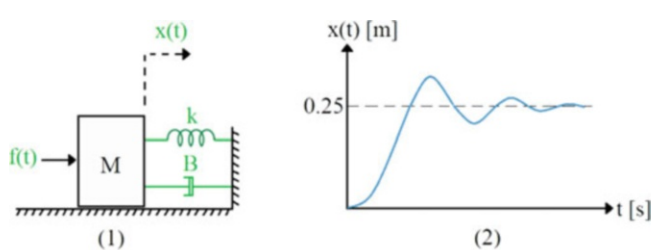

Figure 6.6 The control system of solution of problem 6.9

6.10 The information below can be extracted from the graph, shown in Fig. 6.7.

$$r(t) = u(t) \tag{1}$$

$$\lim_{t\to\infty} c(t) = 1 \tag{2}$$

$$t_p = 0.01 \tag{3}$$

$$O.S.\% = 30\% \tag{4}$$

The peak time of a second-order control system can be determined as follows:

$$t_p \approx \frac{\pi}{\omega_d} \tag{5}$$

Solving (3) and (5):

$$0.01 = \frac{\pi}{\omega_d} \Rightarrow \omega_d \approx 314\ rad/sec \tag{6}$$

The damping ratio of a second-order control system can be determined as follows:

$$\xi \approx \frac{\ln\left(\frac{O.S.\%}{100}\right)}{\sqrt{\pi^2 + \left(\frac{O.S.\%}{100}\right)^2}} \tag{7}$$

Solving (4) and (7):

$$\xi \approx \frac{\ln\left(\frac{30}{100}\right)}{\sqrt{\pi^2 + \left(\frac{30}{100}\right)^2}} \approx 0.35 \tag{8}$$

The undamped natural angular frequency of a second-order control system can be determined as follows:

$$\omega_n = \frac{\omega_d}{\sqrt{1-\xi^2}} \tag{9}$$

Solving (6), (8), and (9):

$$\omega_n = \frac{314}{\sqrt{1-0.35^2}} \approx 336 \tag{10}$$

The standard transfer function of a second-order control system has the following from:

$$T(s) = \frac{C(s)}{R(s)} \approx \frac{A}{s^2 + 2\xi\omega_n s + {\omega_n}^2} \tag{11}$$

Solving (6), (8), (10), and (11):

$$T(s) = \frac{C(s)}{R(s)} \approx \frac{A}{s^2 + 240s + 336^2} \tag{12}$$

From final value theorem, we know that:

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) \tag{13}$$

Solving (12) and (13), and knowing that $R(s) = L(r(t)) = \frac{1}{s}$:

$$\lim_{t\to\infty} c(t) \approx \lim_{s\to 0} s \times \frac{A}{s^2 + 240s + 336^2} \times \frac{1}{s} = \frac{A}{336^2} \tag{14}$$

Solving (3) and (14):

$$\frac{A}{336^2} = 1 \Rightarrow A \approx 336^2 \tag{15}$$

Solving (12) and (15):

$$T(s) \approx \frac{336^2}{s^2 + 240s + 336^2}$$

Choice (4) is the answer.

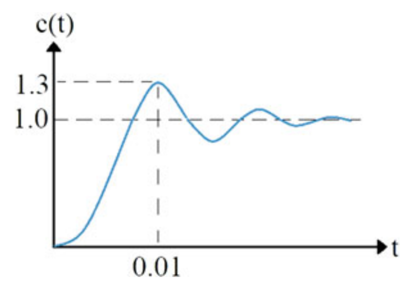

Figure 6.7 The control system of solution of problem 6.10

7 Problems: Analysis of Steady-State Response

Abstract

In this chapter, the steady-state error of a closed-loop control system to a reference input and a disturbance or noise is studied. In this chapter, the problems are categorized in different levels based on their difficulty levels (easy, normal, and hard) and calculation amounts (small, normal, and large). Additionally, the problems are ordered from the easiest problem with the smallest computations to the most difficult problems with the largest calculations.

7.1 Consider the system shown in Fig. 7.1. Determine the type of the system and the steady-state error of the closed-loop control system to a unit ramp function. Assume that the closed-loop system is stable.

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) 1, $\frac{k_1}{k_2}$
2) 0, 0
3) 0, ∞
4) 1, $\frac{k_2}{k_1}$

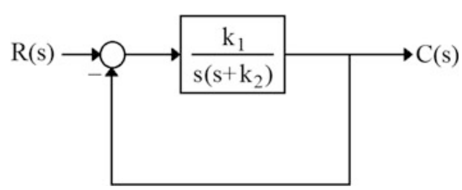

Figure 7.1 The control system of problem 7.1

7.2 In the closed-loop control system shown in Fig. 7.2, calculate the steady-state error to a unit step function.

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) ∞
2) 5
3) $\frac{1}{6}$
4) 0

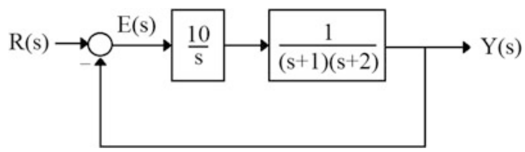

Figure 7.2 The control system of problem 7.2

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7.3 Determine the static error constant to a unit ramp function if the transfer function of the closed-loop control system, shown in Fig. 7.3, is as follows:

$$T(s) = \frac{s+6}{(s+1)(s+2)(s+3)}$$

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) $k_v = \frac{3}{5}$
2) $k_v = \frac{5}{6}$
3) $k_v = \frac{5}{3}$
4) $k_v = \frac{6}{5}$

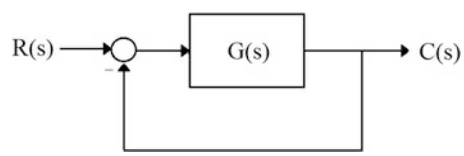

Figure 7.3 The control system of problem 7.3

7.4 Determine the steady-state error to a unit step function if the closed-loop control system includes a negative unity feedback and the open-loop transfer function is as follows:

$$G(s) = \frac{10(s+4)}{s^2(s+1)}$$

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) 0
2) 40
3) $\frac{1}{40}$
4) ∞

7.5 Determine the steady-state error to a unit ramp function if the closed-loop control system includes a negative unity feedback and the open-loop transfer function is as follows:

$$G(s) = \frac{1}{s(s+a)}, a > 0$$

Difficulty level	○ Easy	○ Normal	● Hard
Calculation amount	● Small	○ Normal	○ Large

1) 0
2) ∞
3) a
4) $2a$

7.6 The open-loop transfer function of a control system with a negative unity feedback is as follows:

$$G(s) = \frac{2k}{s^3 + 4s^2 + 5s + 2}$$

Determine its minimum steady-state error to a unit step function.

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ● Normal ○ Large

1) 0.1
2) 0
3) 1
4) ∞

7.7 Consider the control system shown in Fig. 7.4. Determine the steady-state error resulted from the input of $R(s)$ and the noise of $N(s)$ that all are unit step functions.

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ○ Normal ● Large

1) Zero, zero
2) Nonzero constant, infinite
3) Nonzero constant, zero
4) Zero, infinite

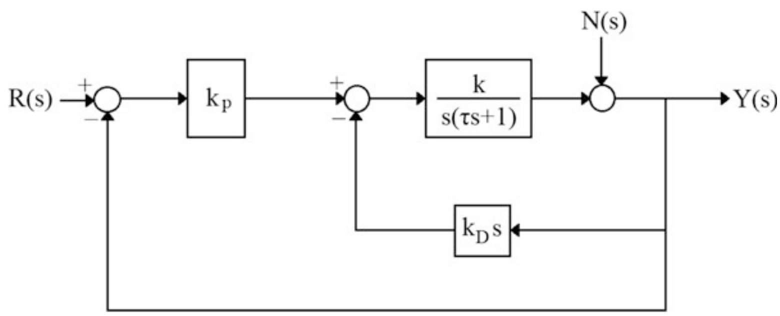

Figure 7.4 The control system of problem 7.7

7.8 Consider the control system shown in Fig. 7.5. Determine the total steady-state error resulted from the input of $R(s)$, which is a unit ramp function, and the disturbance of $D(s)$, which is a step function with the amplitude of d.

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ○ Normal ● Large

1) $\frac{d+kk_h}{k} - \frac{B}{k}$
2) $\frac{B+kk_h}{k_h} - \frac{d}{k_h}$
3) $\frac{B}{k} - \frac{d}{k}$
4) $\frac{B+kk_h}{k} - \frac{d}{k}$

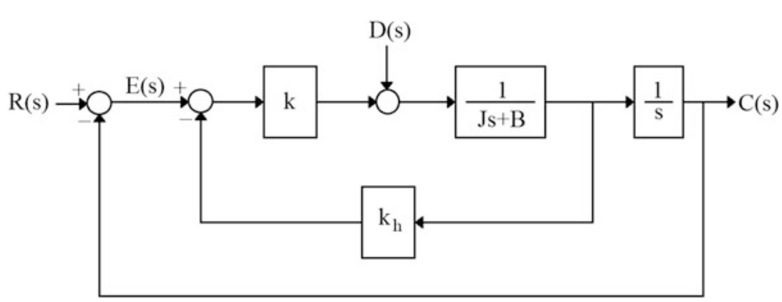

Figure 7.5 The control system of problem 7.8

8 Solutions of Problems: Analysis of Steady-State Response

Abstract

In this chapter, the problems of the seventh chapter are fully solved, in detail, step-by-step, and with different methods.

8.1 Based on the information given in the problem, the closed-loop system is stable. Therefore, checking the stability status of the system is not needed.

Moreover, the open-loop transfer function of the system (see Fig. 8.1) is as follows:

$$G(s) = \frac{k_1}{s(s+k_2)} \tag{1}$$

Based on the definition, type of a control system can be determined from its open-loop transfer function as follows:

$$G(s) = \frac{k \prod_{i=1}^{m} (s+z_i)}{s^T \prod_{j=1}^{n} (s+p_j)} \Rightarrow \begin{cases} T=0 \Rightarrow type\ 0 \\ T=1 \Rightarrow type\ 1 \\ T=2 \Rightarrow type\ 2 \\ \vdots \end{cases} \tag{2}$$

Therefore, the type of the system is one because $T = 1$.

Moreover, the steady-state error of a type-one closed-loop control system to a unit ramp function can be determined as follows (see the table below). Herein, k_v is called velocity error constant.

$$e_{ss} = \frac{1}{k_v} = \frac{1}{\lim\limits_{s\to 0} sG(s)} \tag{3}$$

Solving (1) and (3):

$$e_{ss} = \frac{1}{\lim\limits_{s\to 0} sG(s)} = \frac{1}{\lim\limits_{s\to 0} s \times \frac{k_1}{s(s+k_2)}} = \frac{1}{\frac{k_1}{k_2}} = \frac{k_2}{k_1}$$

Choice (4) is the answer.

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		Type of system			
		Zero	One	Two	Three
Reference input	Step function ($Au(t)$)	$\frac{A}{1+k_p}=\frac{A}{1+\lim_{s\to 0} G(s)}$	0	0	0
	Ramp function ($Atu(t)$)	∞	$\frac{A}{k_v}=\frac{A}{\lim_{s\to 0} sG(s)}$	0	0
	Parabola function ($A\frac{t^2}{2}u(t)$)	∞	∞	$\frac{A}{k_a}=\frac{A}{\lim_{s\to 0} s^2G(s)}$	0

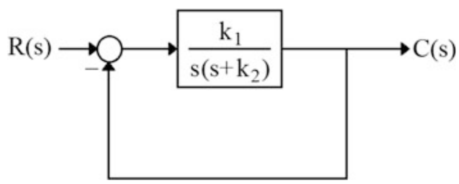

Figure 8.1 The control system of solution of problem 8.1

8.2 First, we need to determine the stability status of the system. The characteristic equation of the closed-loop control system, shown in Fig. 8.2, can be determined as follows:

$$\Delta(s) = 1 + G_c(s)G(s) = 0$$

$$\Rightarrow 1 + \frac{10}{s} \times \frac{1}{(s+1)(s+2)} = 0 \Rightarrow \frac{s^3+3s^2+2s+10}{s(s+1)(s+2)} = 0 \Rightarrow \Delta(s) = s^3+3s^2+2s+10 \tag{1}$$

To determine the stability status of a control system, we can use Routh-Hurwitz table. Suppose that the characteristic equation of a system is as follows:

$$\Delta(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2 s^2 + a_1 s^1 + a_0 s^0 \tag{2}$$

The structure of Routh-Hurwitz table is presented in the following. As can be seen, the coefficients of the characteristic equation are placed on the first two rows of the table with the specific pattern. However, the coefficients of the next rows need to be determined by using (3) and (4), until the last row (s^0) is filled.

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ s^1 & & & & \\ s^0 & & & & \end{array}$$

$$b_{n-1} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}} \tag{3}$$

$$b_{n-3} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}} \tag{4}$$

Based on Routh-Hurwitz table rule, the system is stable if all the elements in the first column of the table are positive.

For this problem, we have:

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & 10 \\ s^1 & -\frac{4}{3} & \\ s^0 & 10 & \end{array}$$

As can be seen, there are two sign changes in the first column of the table. Therefore, the system is unstable, and consequently the steady-state error to a unit step function will be infinite. Choice (1) is the answer.

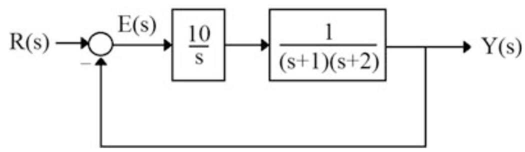

Figure 8.2 The control system of solution of problem 8.2

8.3 Based on the information given in the problem, the transfer function of the closed-loop control system is as follows:

$$T(s) = \frac{s+6}{(s+1)(s+2)(s+3)} \tag{1}$$

However, to calculate the steady-state error or the static error constant, we need to identify the open-loop transfer function ($G(s)$).

Since the system has a negative unity feedback, the relation below exists between its open-loop and closed-loop transfer functions.

$$T(s) = \frac{G(s)}{1+G(s)} \tag{2}$$

Solving (1) and (2):

$$\frac{s+6}{(s+1)(s+2)(s+3)} = \frac{G(s)}{1+G(s)} \Rightarrow G(s) = \frac{\frac{s+6}{(s+1)(s+2)(s+3)}}{1-\frac{s+6}{(s+1)(s+2)(s+3)}}$$

$$\Rightarrow G(s) = \frac{s+6}{s(s^2+6s+10)} \tag{3}$$

Therefore, the type of the system is one since the type of a control system can be determined from its open-loop transfer function as follows:

$$G(s) = \frac{k\prod_{i=1}^{m}(s+z_i)}{s^T\prod_{j=1}^{n}(s+p_j)} \Rightarrow \begin{cases} T=0 \Rightarrow type\ 0 \\ T=1 \Rightarrow type\ 1 \\ T=2 \Rightarrow type\ 2 \\ \vdots \end{cases} \tag{4}$$

The static error constant to a unit ramp function can be determined as follows (see the table below).

$$k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s \times \frac{s+6}{s(s^2+6s+10)} \Rightarrow k_v = \frac{3}{5}$$

This static error constant is called velocity error constant. Choice (1) is the answer.

		Type of system			
		Zero	One	Two	Three
Reference input	Step function ($Au(t)$)	$\frac{A}{1+k_p} = \frac{A}{1+\lim_{s\to 0} G(s)}$	0	0	0
	Ramp function ($Atu(t)$)	∞	$\frac{A}{k_v} = \frac{A}{\lim_{s\to 0} sG(s)}$	0	0
	Parabola function ($A\frac{t^2}{2}u(t)$)	∞	∞	$\frac{A}{k_a} = \frac{A}{\lim_{s\to 0} s^2G(s)}$	0

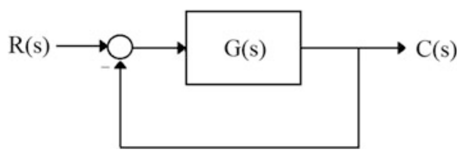

Figure 8.3 The control system of solution of problem 8.3

8.4 Based on the information given in the problem, the open-loop transfer function of the control system that includes a negative unity feedback is as follows:

$$G(s) = \frac{10(s+4)}{s^2(s+1)} \tag{1}$$

First, we need to check the stability status of the system. The characteristic equation of the closed-loop control system can be determined as follows:

$$\Delta(s) = 1 + G(s) = 0$$

$$\Rightarrow 1 + \frac{10(s+4)}{s^2(s+1)} = 0 \Rightarrow \frac{s^3+s^2+10s+40}{s^2(s+1)} = 0 \Rightarrow \Delta(s) = s^3+s^2+10s+40 \tag{2}$$

To determine the stability status of a control system, we can use Routh-Hurwitz table. Suppose that the characteristic equation of a system is as follows:

$$\Delta(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2 s^2 + a_1 s^1 + a_0 s^0 \tag{3}$$

The structure of Routh-Hurwitz table is presented in the following. As can be seen, the coefficients of the characteristic equation are placed on the first two rows of the table with the specific pattern. However, the coefficients of the next rows need to be determined by using (4) and (5), until the last row (s^0) is filled.

$$\begin{array}{c|cccc} s & a & a_{-2} & a_{-4} & \cdots \\ s^{-1} & a_{-1} & a_{-3} & a_{-5} & \cdots \\ s^{-2} & b_{-1} & b_{-3} & b_{-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ s & & & & \\ s & & & & \end{array}$$

$$b_{n-1} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}} \tag{4}$$

$$b_{n-3} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}} \tag{5}$$

Based on Routh-Hurwitz table rule, the system is stable if all the elements in the first column of the table are positive. For this problem, we have:

s^3	1	10
s^2	1	40
s^1	-30	
s^0	40	

As can be seen, there are two sign changes in the first column of the table. Therefore, the system is unstable, and consequently the steady-state error to a unit step function will be infinite. Choice (4) is the answer.

8.5 Based on the information given in the problem, the closed-loop control system includes a negative unity feedback and the open-loop transfer function below.

$$G(s) = \frac{1}{s(s+a)}, a > 0 \tag{1}$$

First, we need to determine the stability status of the system. The characteristic equation of the closed-loop control system can be determined as follows:

$$\Delta(s) = 1 + G_c(s)G(s) = 0$$

$$\Rightarrow 1 + \frac{1}{s(s+a)} = 0 \Rightarrow \frac{s^2 + as + 1}{s(s+a)} = 0 \Rightarrow \Delta(s) = s^2 + as + 1 \tag{2}$$

A second-order system with the characteristic equation of $a_2s^2 + a_1s + a_0$ is stable if and only if all the coefficients are non-zero and have the same sign. In other words:

$$a_2, a_1, a_0 > 0 \tag{3}$$

Solving (2) and (3):

$$a > 0 \tag{4}$$

By considering (1) and (4), it is noticed that the system is stable.

Based on the definition, the type of a control system can be determined from its open-loop transfer function as follows:

$$G(s) = \frac{k\prod_{i=1}^{m}(s+z_i)}{s^T\prod_{j=1}^{n}(s+p_j)} \Rightarrow \begin{cases} T = 0 \Rightarrow type\ 0 \\ T = 1 \Rightarrow type\ 1 \\ T = 2 \Rightarrow type\ 2 \\ \vdots \end{cases} \tag{5}$$

As can be noticed from (1) and (5), the type of the system is one. Hence, based on the table below, the steady-state error to a unit ramp function can be determined as follows:

$$e_{ss} = \frac{1}{k_v} = \frac{1}{\lim_{s\to 0} sG(s)} = \frac{1}{\lim_{s\to 0} s \times \frac{1}{s(s+a)}} = \frac{1}{\frac{1}{a}} \Rightarrow e_{ss} = a$$

Choice (3) is the answer.

		Type of system			
		Zero	One	Two	Three
Reference input	Step function ($Au(t)$)	$\frac{A}{1+k_p} = \frac{A}{1+\lim_{s\to 0} G(s)}$	0	0	0
	Ramp function ($Atu(t)$)	∞	$\frac{A}{k_v} = \frac{A}{\lim_{s\to 0} sG(s)}$	0	0
	Parabola function ($A\frac{t^2}{2}u(t)$)	∞	∞	$\frac{A}{k_a} = \frac{A}{\lim_{s\to 0} s^2G(s)}$	0

8.6 Based on the information given in the problem, the open-loop transfer function of the control system that includes a negative unity function is as follows:

$$G(s) = \frac{2k}{s^3 + 4s^2 + 5s + 2} \tag{1}$$

First, we need to check the stability status of the system. The characteristic equation of the closed-loop control system can be determined as follows:

$$\Delta(s) = 1 + G(s) = 0$$

$$\Rightarrow 1 + \frac{2k}{s^3 + 4s^2 + 5s + 2} = 0 \Rightarrow \frac{s^3 + 4s^2 + 5s + 2 + 2k}{s^3 + 4s^2 + 5s + 2} = 0$$

$$\Rightarrow \Delta(s) = s^3 + 4s^2 + 5s + 2 + 2k \tag{2}$$

To determine the stability status of a control system, we can use Routh-Hurwitz table. Suppose that the characteristic equation of a system is as follows:

$$\Delta(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2s^2 + a_1s^1 + a_0s^0 \tag{3}$$

The structure of Routh-Hurwitz table is presented in the following. As can be seen, the coefficients of the characteristic equation are placed on the first two rows of the table with the specific pattern. However, the coefficients of the next rows need to be determined by using (4) and (5), until the last row (s^0) is filled.

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ s^1 & & & & \\ s^0 & & & & \end{array}$$

$$b_{n-1} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}} \tag{4}$$

$$b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}} \tag{5}$$

Based on Routh-Hurwitz table rule, the system is stable if all the elements in the first column of the table are positive.

For this problem, we have:

$$\begin{array}{c|cc} s^3 & 1 & 5 \\ s^2 & 4 & 2+2k \\ s^1 & \dfrac{18-2k}{4} & \\ s^0 & 2+2k & \end{array}$$

Therefore, the system is stable if:

$$\begin{cases} \dfrac{18-2k}{4} > 0 \\ 2+2k > 0 \end{cases} \Rightarrow \begin{cases} k < 9 \\ k > -1 \end{cases} \overset{\cap}{\Longrightarrow} -1 < k < 9 \tag{6}$$

As can be noticed from (1), the type of the system is zero. Therefore, the steady-state error to a unit step function can be determined as follows (see the table below). Herein, k_p is called position error constant.

$$e_{ss} = \frac{1}{1+k_p} = \frac{1}{1+\lim\limits_{s\to 0} G(s)} = \frac{1}{1+\lim\limits_{s\to 0} \frac{2k}{s^3+4s^2+5s+2}} = \frac{1}{1+k} \tag{7}$$

As can be noticed from (7), the minimum steady-state error to a unit step function will occur for the maximum possible value of k while considering the stability criterion mentioned in (6).

Solving (6) and (7):

$$e_{ss} = \frac{1}{1+9} \Rightarrow e_{ss} = 0.1$$

Choice (1) is the answer.

		Type of system			
		Zero	One	Two	Three
Reference input	Step function ($Au(t)$)	$\frac{A}{1+k_p} = \frac{A}{1+\lim\limits_{s\to 0} G(s)}$	0	0	0
	Ramp function ($Atu(t)$)	∞	$\frac{A}{k_v} = \frac{A}{\lim\limits_{s\to 0} sG(s)}$	0	0
	Parabola function ($A\frac{t^2}{2}u(t)$)	∞	∞	$\frac{A}{k_a} = \frac{A}{\lim\limits_{s\to 0} s^2G(s)}$	0

8.7 Based on the information given in the problem, we know that:

$$r(t) = n(t) = u(t) \xrightarrow{\text{Laplace Trans.}} R(s) = N(s) = \frac{1}{s} \tag{1}$$

The open-loop transfer function of the system with the input and output of $R(s)$ and $Y(s)$, respectively, is as follows:

$$G(s) = k_P \times \frac{\frac{k}{s(\tau s+1)}}{1+\frac{k}{s(\tau s+1)} \times k_D s} = \frac{k_P k}{s(\tau s + 1 + k k_D)} \tag{2}$$

As can be noticed, the type of the system is one. Hence, the steady-state error of the closed-loop control system to the reference input of unit step function ($R(s)$) will be zero, as can be seen in the table below.

$$e_{ss,R} = 0 \tag{3}$$

		Type of system			
		Zero	One	Two	Three
Reference input	Step function ($Au(t)$)	$\frac{A}{1+k_p}=\frac{A}{1+\lim_{s\to 0}G(s)}$	0	0	0
	Ramp function ($Atu(t)$)	∞	$\frac{A}{k_v}=\frac{A}{\lim_{s\to 0}sG(s)}$	0	0
	Parabola function ($A\frac{t^2}{2}u(t)$)	∞	∞	$\frac{A}{k_a}=\frac{A}{\lim_{s\to 0}s^2G(s)}$	0

To calculate the steady-state error resulted from the noise of $N(s)$, we can determine the output ($Y(s)$), while the input of $R(s)$ is turned off. In other words:

$$r(t) = 0 \tag{4}$$

As we know, an error is defined as follows:

$$E(s) = R(s) - Y(s) \xrightarrow{\text{Laplace Inv. Trans.}} e(t) = r(t) - y(t) \tag{5}$$

Solving (4) and (5) for the input of noise in the steady-state condition:

$$e_{ss,N} = -y_{ss,N} \tag{6}$$

Herein, we can use Mason's gain formula to determine the related transfer function, as follows:

$$P = \frac{1}{\Delta}\sum_{k=1}^{N} p_k\Delta_k$$

where:

$$\Delta = 1 - \sum_a L_a + \sum_{a,b} L_aL_b - \sum_{a,b,c} L_aL_bL_c + \ldots$$

P: The total gain from the input point to the output one

Δ: The determinant of the graph which is the same as the characteristic equation of the system

N: The number of forward paths from the input point to the output one

k: The index of forward path from the input point to the output

p_k: The gain of the $k'th$ forward path from the input point to the output one

Δ_k: The determinant of the graph if the $k'th$ forward path is removed

$\sum_a L_a$: The sum of gains of loops

$\sum_{a,b} L_aL_b$: The sum of product of gains of any two non-touching loops (without any common nodes)

$\sum_{a,b,c} L_aL_bL_c$: The sum of product of gains of any three pairwise non-touching loops (without any common nodes)

Now, for the system, shown in Fig. 8.4, we have the following calculations to determine the transfer function of $\frac{Y(s)}{N(s)}$, which is from the input of $N(s)$ to the output of $Y(s)$:

$$N = 1$$

$$p_1 = 1$$

$$\Delta_1 = 1$$

$$\Rightarrow \sum_{k=1}^{N} p_k \Delta_k = p_1 \Delta_1 = 1$$

$$\sum_a L_a = \left(-\frac{k}{s(\tau s+1)} \times k_D s\right) + \left(-k_P \times \frac{k}{s(\tau s+1)}\right) = -\frac{kk_D s}{s(\tau s+1)} - \frac{kk_P}{s(\tau s+1)}$$

$$\sum_{a,b} L_a L_b = 0$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta = 1 - \left(-\frac{kk_D s}{s(\tau s+1)} - \frac{kk_P}{s(\tau s+1)}\right) = 1 + \frac{kk_D s}{s(\tau s+1)} + \frac{kk_P}{s(\tau s+1)}$$

$$\frac{Y(s)}{N(s)} = \frac{1}{1 + \frac{kk_D s}{s(\tau s+1)} + \frac{kk_P}{s(\tau s+1)}} = \frac{s(\tau s+1)}{\tau s^2 + (kk_D+1)s + kk_P}$$

$$\Rightarrow Y(s) = \frac{s(\tau s+1)}{\tau s^2 + (kk_D+1)s + kk_P} N(s) \tag{7}$$

From final value theorem, we know that:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \tag{8}$$

Solving (1), (7), and (8):

$$y_{ss,N} = \lim_{t\to\infty} y_N(t) = \lim_{s\to 0} s \times \frac{s(\tau s+1)}{\tau s^2 + (kk_D+1)s + kk_P} \times \frac{1}{s} = 0 \tag{9}$$

Solving (6) and (9):

$$e_{ss,N} = -y_{ss,N} = 0$$

Choice (1) is the answer.

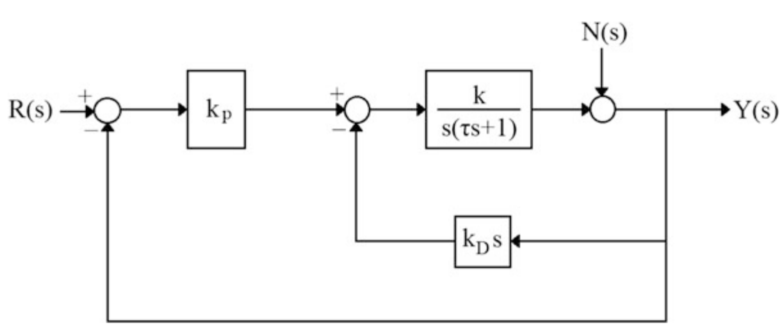

Figure 8.4 The control system of solution of problem 8.7

8.8 Based on the information given in the problem, we have:

$$r(t) = tu(t) \xrightarrow{\text{Laplace Trans.}} R(s) = \frac{1}{s^2} \tag{1}$$

$$d(t) = du(t) \xrightarrow{\text{Laplace Trans.}} D(s) = \frac{d}{s} \tag{2}$$

To calculate the total error ($E(s)$ indicated in Fig. 8.5) resulted from the reference input of $R(s)$ and the disturbance of $D(s)$, we can apply superposition theorem. Therefore, to calculate the error ($E_R(s)$) resulted from the reference input of $R(s)$, we need to turn off $D(s)$. Likewise, to calculate the error ($E_D(s)$) resulted from the disturbance of $D(s)$, we must turn off $R(s)$. Then:

$$E(s) = E_R(s) + E_D(s) \tag{3}$$

After that, the steady-state error can be calculated using final value theorem, as follows:

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s(E_R(s) + E_D(s)) \tag{4}$$

Herein, to calculate the errors, we can use Mason's gain formula, as follows:

$$P = \frac{1}{\Delta}\sum_{k=1}^{N} p_k\Delta_k$$

where:

$$\Delta = 1 - \sum_a L_a + \sum_{a,b} L_aL_b - \sum_{a,b,c} L_aL_bL_c + \ldots$$

P: The total gain from the input point to the output one
Δ: The determinant of the graph which is the same as the characteristic equation of the system
N: The number of forward paths from the input point to the output one
k: The index of forward path from the input point to the output
p_k: The gain of the $k'th$ forward path from the input point to the output one
Δ_k: The determinant of the graph if the $k'th$ forward path is removed
$\sum_a L_a$: The sum of gains of loops
$\sum_{a,b} L_aL_b$: The sum of product of gains of any two non-touching loops (without any common nodes)
$\sum_{a,b,c} L_aL_bL_c$: The sum of product of gains of any three pairwise non-touching loops (without any common nodes)
Now, for the input of $R(s)$ and the output of $E_R(s)$, we have (see Fig. 8.5):

$$N = 1$$

$$p_1 = 1$$

$$\Delta_1 = 1 - \left(-k \times \frac{1}{Js + B} \times k_h\right) = 1 + \frac{kk_h}{Js + B}$$

$$\Rightarrow \sum_{k=1}^{N} p_k \Delta_k = p_1 \Delta_1 = 1 + \frac{kk_h}{Js+B} = \frac{Js+B+kk_h}{Js+B}$$

$$\sum_a L_a = \left(-k \times \frac{1}{Js+B} \times k_h\right) + \left(-k \times \frac{1}{Js+B} \times \frac{1}{s}\right) = -\frac{kk_h}{Js+B} - \frac{k}{s(Js+B)}$$

$$\sum_{a,b} L_a L_b = 0$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta = 1 - \left(-\frac{kk_h}{Js+B} - \frac{k}{s(Js+B)}\right) = 1 + \frac{kk_h}{Js+B} + \frac{k}{s(Js+B)}$$

$$\frac{E_R(s)}{R(s)} = \frac{\frac{Js+B+kk_h}{Js+B}}{1 + \frac{kk_h}{Js+B} + \frac{k}{s(Js+B)}} = \frac{s(Js+B+kk_h)}{Js^2 + (kk_h+B)s + k}$$

$$\Rightarrow E_R(s) = \frac{s(Js+B+kk_h)}{Js^2 + (kk_h+B)s + k} R(s) \tag{5}$$

Likewise, for the input of $D(s)$ and the output of $E_D(s)$, we have (see Fig. 8.5):

$$N = 1$$

$$p_1 = -\frac{1}{Js+B} \times \frac{1}{s} = -\frac{1}{s(Js+B)}$$

$$\Delta_1 = 1$$

$$\Rightarrow \sum_{k=1}^{N} p_k \Delta_k = p_1 \Delta_1 = -\frac{1}{s(Js+B)}$$

$$\sum_a L_a = \left(-k \times \frac{1}{Js+B} \times k_h\right) + \left(-k \times \frac{1}{Js+B} \times \frac{1}{s}\right) = -\frac{kk_h}{Js+B} - \frac{k}{s(Js+B)}$$

$$\sum_{a,b} L_a L_b = 0$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta = 1 - \left(-\frac{kk_h}{Js+B} - \frac{k}{s(Js+B)}\right) = 1 + \frac{kk_h}{Js+B} + \frac{k}{s(Js+B)}$$

$$\frac{E_D(s)}{D(s)} = \frac{-\frac{1}{s(Js+B)}}{1 + \frac{kk_h}{Js+B} + \frac{k}{s(Js+B)}} = -\frac{1}{Js^2 + (kk_h+B)s + k}$$

$$\Rightarrow E_D(s) = -\frac{1}{Js^2 + (kk_h + B)s + k} D(s) \tag{6}$$

Solving (1), (2), (4), (5), and (6):

$$e_{ss} = \lim_{s \to 0} s \left(\frac{s(Js + B + kk_h)}{Js^2 + (kk_h + B)s + k} \times \frac{1}{s^2} - \frac{1}{Js^2 + (kk_h + B)s + k} \times \frac{d}{s} \right)$$

$$\Rightarrow e_{ss} = \lim_{s \to 0} \left(\frac{Js + B + kk_h}{Js^2 + (kk_h + B)s + k} - \frac{d}{Js^2 + (kk_h + B)s + k} \right)$$

$$\Rightarrow e_{ss} = \frac{B + kk_h}{k} - \frac{d}{k}$$

Choice (4) is the answer.

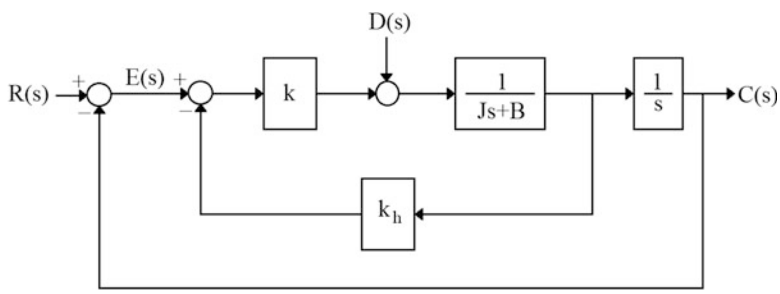

Figure 8.5 The control system of solution of problem 8.8

9 Problems: Graphical Analysis and Design in Time Domain

Abstract
In this chapter, root locus analysis method as a graphical analysis method is applied on the transfer function of an open-loop control system to examine how the poles of the closed-loop system change with the variation of a specific system parameter (loop gain). The analysis includes studying the stability status of the closed-loop control system and evaluating its transient and steady-state responses. In this chapter, the problems are categorized in different levels based on their difficulty levels (easy, normal, and hard) and calculation amounts (small, normal, and large). Additionally, the problems are ordered from the easiest problem with the smallest computations to the most difficult problems with the largest calculations.

9.1 Figure 9.1 illustrates the root locus of a control system with a negative unity feedback (for $k > 0$). Determine the maximum value of loop gain so that the closed-loop system is stable.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) 1
2) 2
3) 3
4) 6

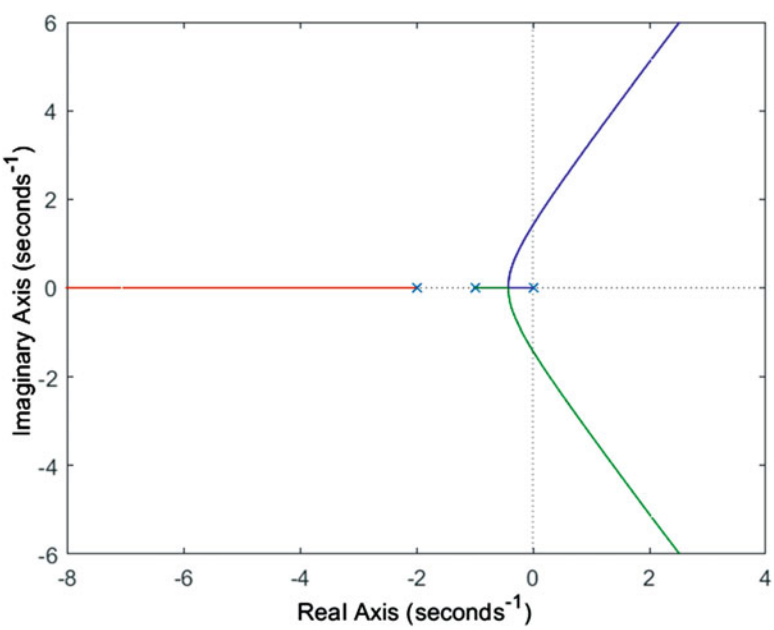

Figure 9.1 The control system of problem 9.1

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9.2 Consider the problem of 9.1 and assume that the system is in the oscillating status. Determine the angular frequency of the oscillations.

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) 1 *rad/sec*
2) $\sqrt{2}\ rad/\ sec$
3) $\sqrt{3}\ rad/\ sec$
4) 4 *rad/sec*

9.3 Figure 9.3 shows the root locus ($k > 0$) of the control system shown in Fig. 9.2. Determine the value of loop gain where the root locus crosses $j\omega - axis$.

Difficulty level	○ Easy	● Normal	○ Hard
Calculation amount	○ Small	● Normal	○ Large

1) 16
2) 160
3) 1.6
4) 0

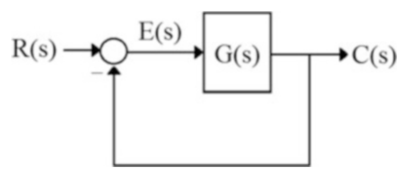

Figure 9.2 The control system of problem 9.3

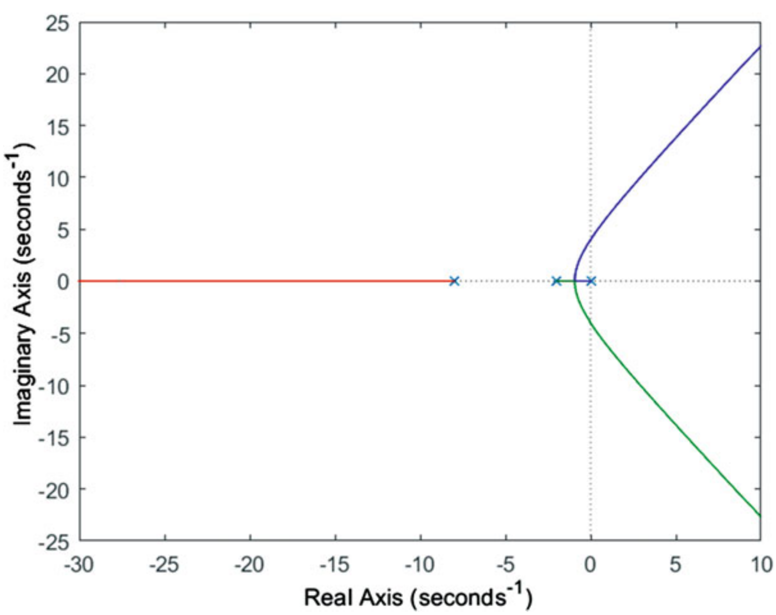

Figure 9.3 The control system of problem 9.3

9.4 If the root locus of the control system of Fig. 9.4 passes from the points of $-1 \pm j$, determine the value of the parameters a and b.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) 5, 4
2) 5, 3
3) 3, 4
4) 3, 3

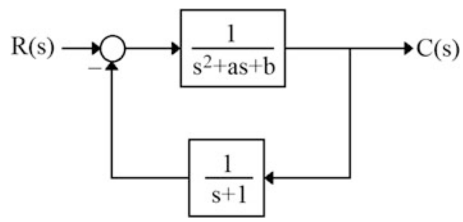

Figure 9.4 The control system of problem 9.4

9.5 In a control system with a negative unity feedback and the open-loop transfer function below, if the parameter of τ increases about 10%, which one of the following choices is correct?

$$G(s) = \frac{1}{s(1+\tau s)}$$

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) The damping ratio will decrease about 5%.
2) The damping ratio will increase about 5%.
3) The damping ratio will decrease about 10%.
4) The damping ratio will increase about 10%.

9.6 Which one of the following choices shows the root locus of the control system of Fig. 9.5 for $k > 0$?

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ○ Normal ● Large

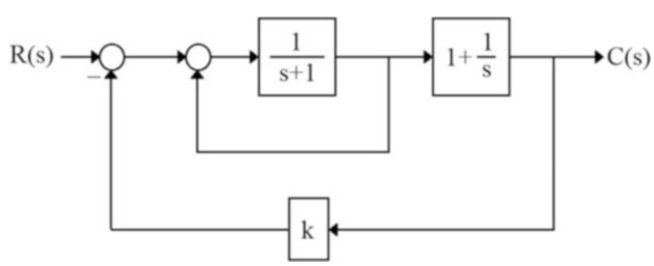

Figure 9.5 The control system of problem 9.6

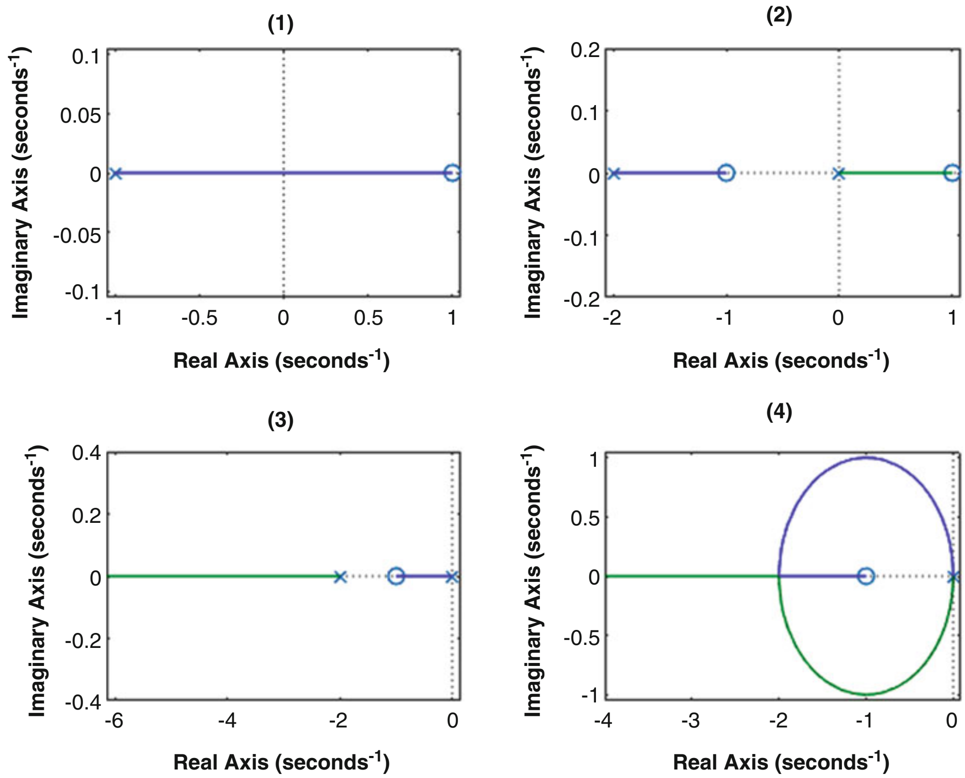

Figure 9.6 The control system of problem 9.6

9.7 Which one of the following choices illustrates the root locus of the control system shown in Fig. 9.7 ($k > 0$)?

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ● Small ○ Normal ○ Large

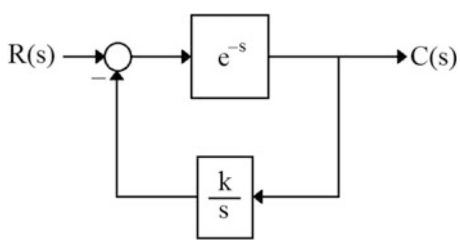

Figure 9.7 The control system of problem 9.7

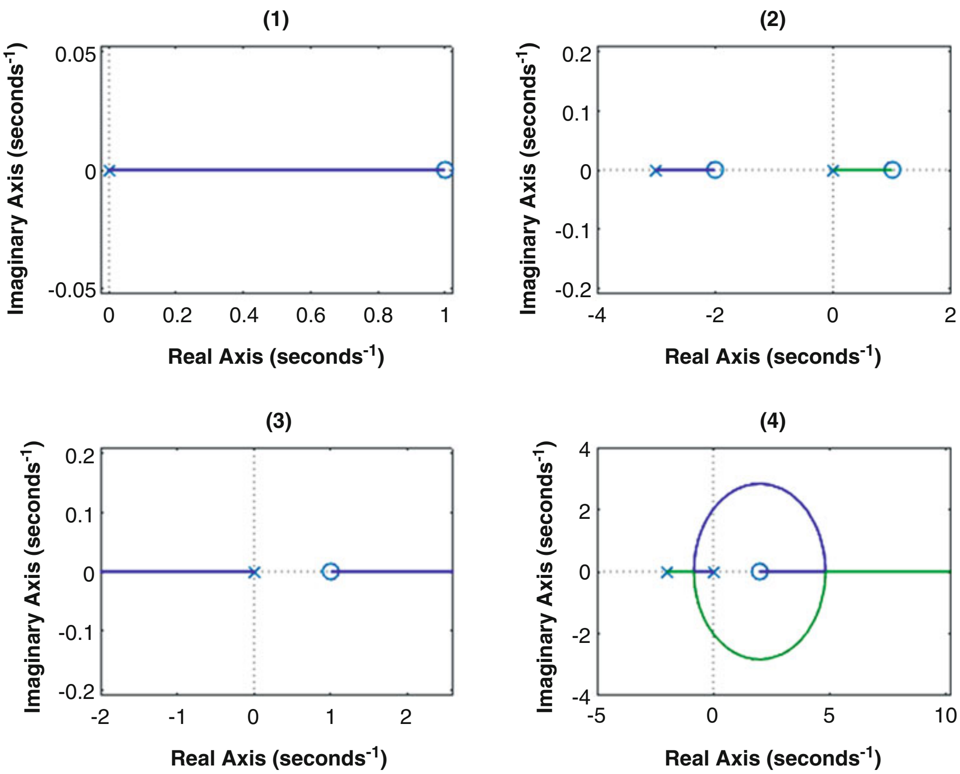

Figure 9.8 The control system of problem 9.7

9.8 The root locus of a control system is shown in Fig. 9.9. Determine the stability status of the system for $k = 40$.

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ● Normal ○ Large

1) Stable.
2) Unstable.
3) Marginally stable.
4) Its stability depends on the other parameters.

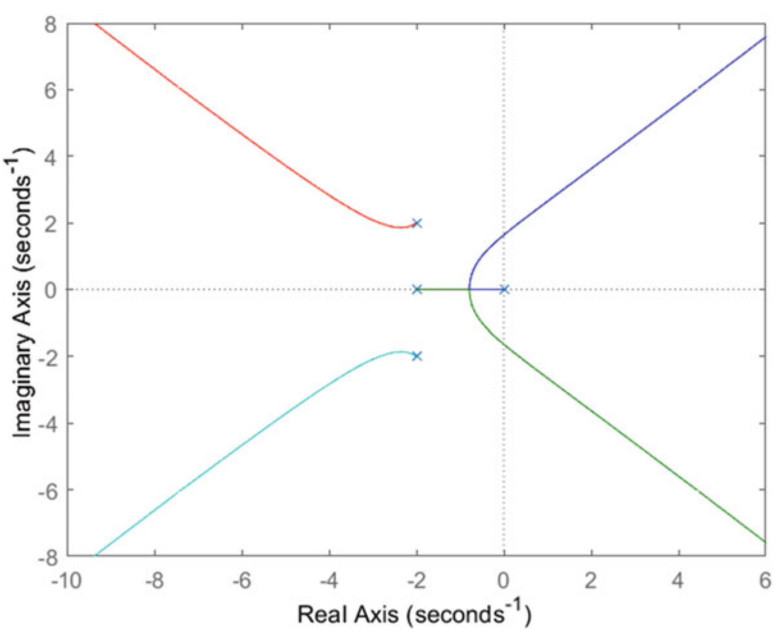

Figure 9.9 The control system of problem 9.8

9.9 Which one of the following options shows the root locus of a closed-loop control system (for $k > 0$) with a negative unity feedback and the open-loop transfer function below?

$$G(s) = \frac{s - k}{s(s+1)}$$

Difficulty level	○ Easy	○ Normal	● Hard
Calculation amount	○ Small	● Normal	○ Large

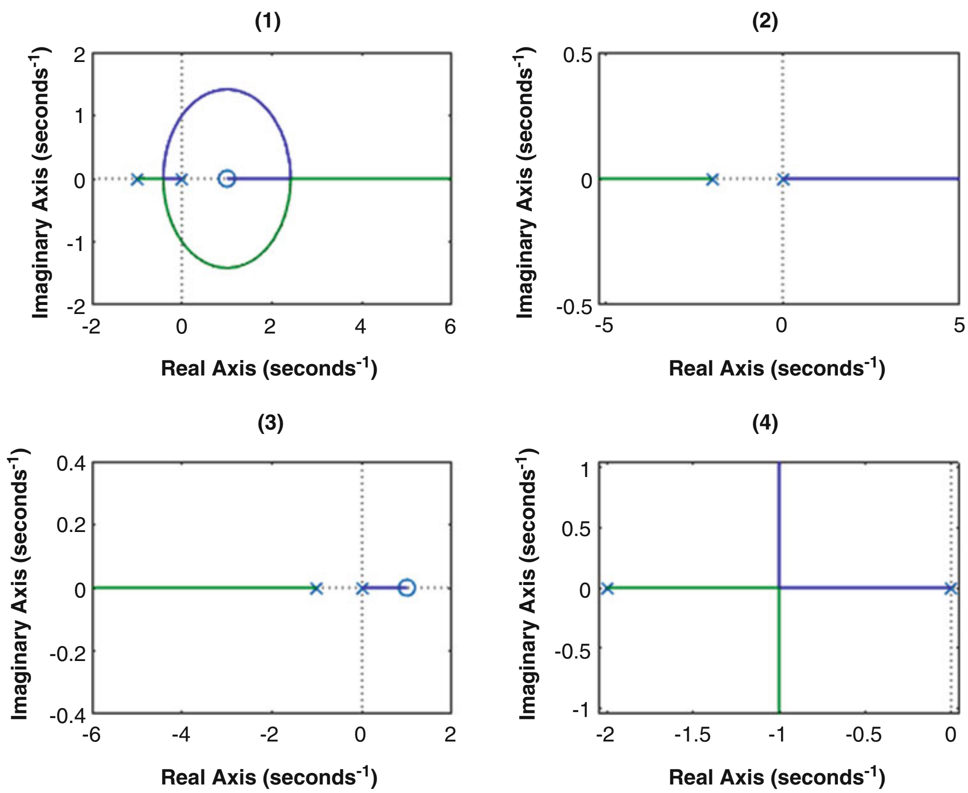

Figure 9.10 The control system of problem 9.9

9.10 Consider a closed-loop control system with a negative unity feedback and the open-loop transfer function below ($k > 0$). If the settling time of the system for the large k is about eight seconds, determine the value of parameter a.

$$G(s) = \frac{k(s+4)}{s(s+2)(s+a)}$$

Difficulty level	○ Easy	○ Normal	● Hard
Calculation amount	○ Small	● Normal	○ Large

1) 1
2) 2
3) 3
4) 4

9.11 Consider a closed-loop control system with a negative unity feedback and the open-loop transfer function below ($k > 0$). Which one of the following statements is correct and complete?

$$G(s) = \frac{k(s+0.5)}{s^2(s^2+4s+8)}$$

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ● Normal ○ Large

1) The points $-1 \pm j$ are on the root locus of the system.
2) The points $-1 \pm j$ are on the root locus of the system, but they are NOT break-away/break-in points.
3) The points $-1 \pm j$ are NOT on the root locus of the system.
4) The points $-1 \pm j$ are on the root locus of the system and they are break-away/break-in points.

9.12 Consider a closed-loop control system with a negative unity feedback and the open-loop transfer function below ($k > 0$). Determine the range of p, so that the transient response of the closed-loop system is always overdamped.

$$G(s) = \frac{k(s+3)}{(s+1)(s+p)}$$

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ● Normal ○ Large

1) $0 < p < 1$
2) $1 < p < 3$
3) $p > 3$
4) $p > 0$

9.13 Which one of the following choices shows the root locus of the control system ($k < 0$) illustrated in Fig. 9.11?

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ● Normal ○ Large

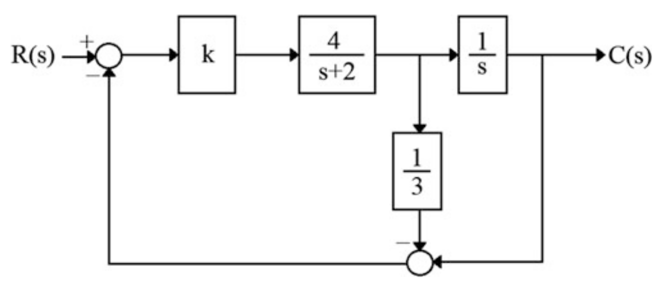

Figure 9.11 The control system of problem 9.13

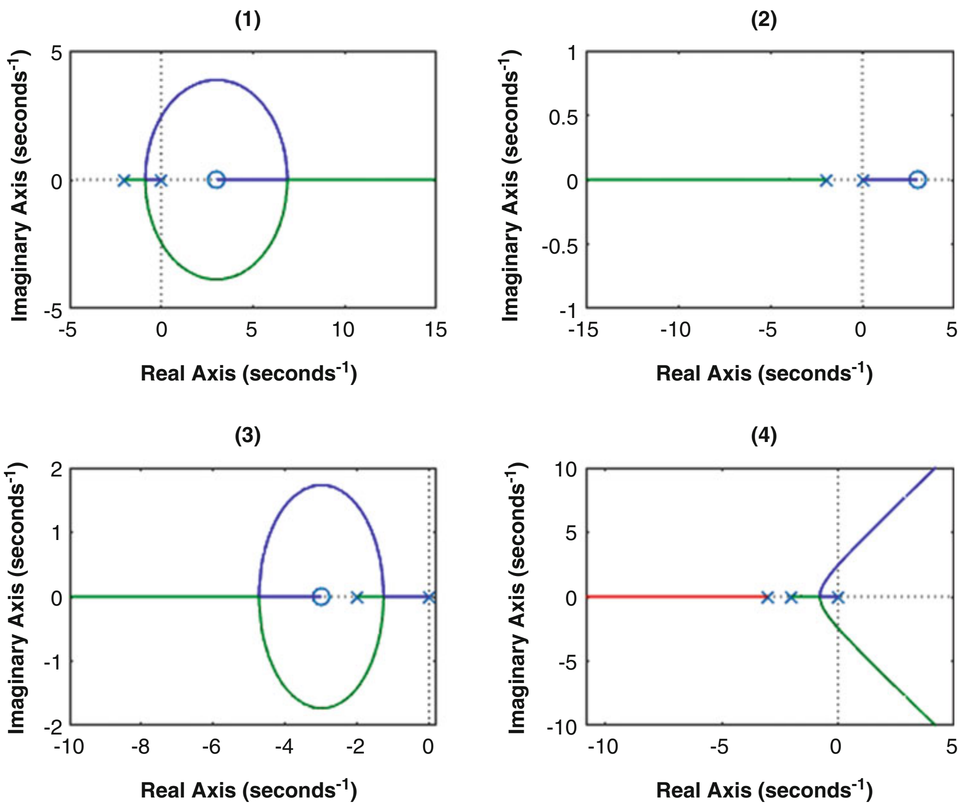

Figure 9.12 The control system of problem 9.13

9.14 In a control system with a negative unity feedback, the open-loop transfer function is as follows:

$$G(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)}$$

Determine the sensitivity of the maximum overshoot percentage of the closed-loop system's response to a unit step input with respect to the damping ratio ($S_{\xi}^{O.S.}$) around the rated damping ratio ($\xi = \frac{\sqrt{2}}{2}$).

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ● Normal ○ Large

1) -2π
2) $-\pi$
3) 2π
4) π

9.15 Consider a closed-loop control system with a negative unity feedback and the open-loop transfer function below ($k > 0$). Determine the characteristic equation of the closed-loop system if its root locus has a break-away/break-in point on the real axis in $-\frac{4}{9}$ and the straight-line asymptotes intersection in $-\frac{11}{9}$.

$$G(s) = \frac{k}{s(s + \alpha)(s + \beta)}$$

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ○ Normal ● Large

1) $s^3 + 11s^2 + 8s + k$
2) $s^3 + \frac{11}{3}s^2 + 8s + k$
3) $s^3 + 11s^2 + \frac{8}{3}s + k$
4) $s^3 + \frac{11}{3}s^2 + \frac{8}{3}s + k$

9.16 The state-transition matrix of a closed-loop control system with a negative unity feedback is as follows:

$$[\varphi(t)] = \begin{bmatrix} 2e^{-t} - 2e^{-2t} + e^{-3t} & e^{-t} - e^{-2t} & 0 \\ 0 & e^{-2t} & -e^{-2t} + e^{-3t} \\ k(e^{-2t} - e^{-3t}) & 0 & e^{-3t} \end{bmatrix}$$

Herein, k is the forward gain of the system. Determine the break-away/break-in point and the asymptotes intersection on the real axis.

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ○ Normal ● Large

1) $\frac{-6+\sqrt{3}}{3}, -6$
2) $\frac{-6+\sqrt{3}}{3}, -2$
3) $\frac{-6-\sqrt{3}}{3}, -2$
4) $\frac{-6-\sqrt{3}}{3}, -6$

10 Solutions of Problems: Graphical Analysis and Design in Time Domain

Abstract
In this chapter, the problems of the ninth chapter are fully solved, in detail, step-by-step, and with different methods.

10.1. Based on the zeros' and poles' locations shown in the root locus of the system in Fig. 10.1, we can determine the open-loop transfer function of the system, as follows:

$$G(s) = \frac{1}{s(s+1)(s+2)} \tag{1}$$

The characteristic equation of the closed-loop system can be determined as follows:

$$1 + kG(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{k}{s(s+1)(s+2)} = 0 \Rightarrow \frac{s^3 + 3s^2 + 2s + k}{s(s+1)(s+2)} = 0 \Rightarrow \Delta(s) = s^3 + 3s^2 + 2s + k \tag{2}$$

To determine the stability status of a control system, we can use Routh-Hurwitz table. Suppose that the characteristic equation of a system is as follows:

$$\Delta(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2 s^2 + a_1 s^1 + a_0 s^0 \tag{3}$$

The structure of Routh-Hurwitz table is presented in the following. As can be seen, the coefficients of the characteristic equation are placed on the first two rows of the table with the specific pattern. However, the coefficients of the next rows need to be determined by using (4) and (5), until the last row (s^0) is filled.

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ s^1 & & & & \\ s^0 & & & & \end{array}$$

$$b_{n-1} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}} \tag{4}$$

M. Rahmani-Andebili, *Feedback Control Systems Analysis and Design*, https://doi.org/10.1007/978-3-030-95277-8_10

$$b_{n-3} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}} \quad (5)$$

Based on Routh-Hurwitz table rule, the system is stable if all the elements in the first column of the table are positive.

For this problem, we have:

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s^1 & \dfrac{6-k}{3} & \\ s^0 & k & \end{array}$$

Based on Routh-Hurwitz rule, the system is stable if:

$$\begin{cases} \dfrac{6-k}{3} > 0 \\ k > 0 \end{cases} \Rightarrow \begin{cases} k < 6 \\ k > 0 \end{cases} \overset{\cap}{\Longrightarrow} 0 < k < 6 \quad (6)$$

As can be seen from (6), the maximum value of k that the system is stable is 6.

$$\Rightarrow k = 6$$

Choice (4) is the answer.

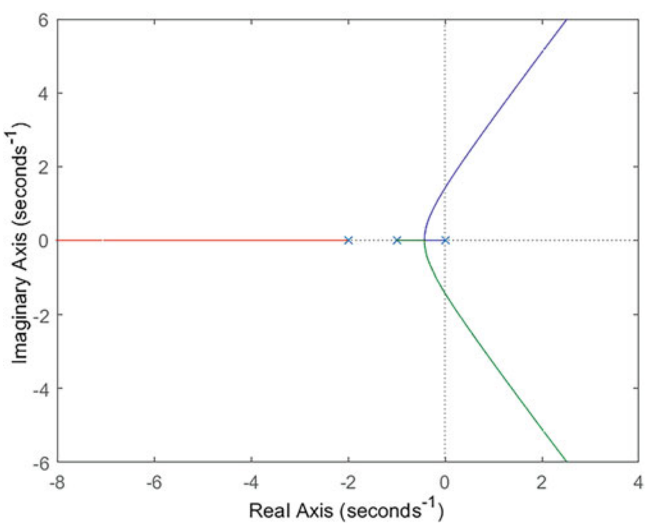

Figure 10.1 The control system of solution of problem 10.1

10.2. To have an oscillating system, all the elements in one of the rows of the Routh-Hurwitz table corresponding to an odd exponent must be zero. As can be seen in the Routh-Hurwitz table, the row corresponding to s^1 is zero if $k = 6$.

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s^1 & \dfrac{6-k}{3} & \\ s^0 & k & \end{array}$$

Moreover, the angular frequency of the oscillations can be determined by using the equation of the previous row ($A(s^2)$, as the auxiliary equation), as follows:

$$A(s^2) = 3s^2 + k = 0 \xrightarrow{k=6} A(s^2) = 3s^2 + 6 = 0 \Rightarrow s = \pm j\sqrt{2} \Rightarrow \omega = \sqrt{2}\ rad/sec$$

Choice (2) is the answer.

10.3. Based on the zeros' and poles' locations shown in the root locus of the system in Fig. 10.3, we can determine the open-loop transfer function of the system, as follows:

$$G(s) = \frac{1}{s(s+2)(s+8)} \tag{1}$$

The characteristic equation of the closed-loop control system can be determined as follows:

$$1 + kG(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{k}{s(s+2)(s+8)} = 0 \Rightarrow \frac{s^3 + 10s^2 + 16s + k}{s(s+2)(s+8)} = 0$$

$$\Rightarrow \Delta(s) = s^3 + 10s^2 + 16s + k \tag{2}$$

Based on the information given in the problem, we know that the root locus crosses $j\omega - axis$. Therefore, by applying Routh-Hurwitz table rule for the system, one of the rows of the table, corresponding to an odd exponent, must be zero since the root locus crosses the $j\omega - axis$ and the system is marginally stable.

Applying Routh-Hurwitz table for this problem:

s^3	1	16
s^2	10	k
s^1	$\frac{160-k}{10}$	
s^0	k	

$$\frac{160-k}{10} = 0 \Rightarrow k = 160$$

Choice (2) is the answer.

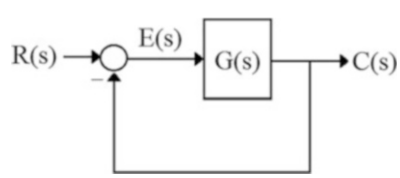

Figure 10.2 The control system of solution of problem 10.3

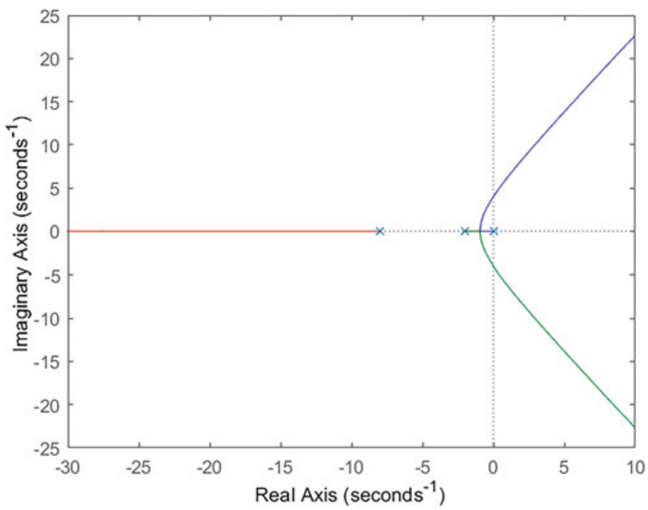

Figure 10.3 The control system of solution of problem 10.3

10.4. Based on the information given in the problem, the root locus passes through the points of $-1 \pm j$. Therefore, these points are on the roots of the system. By assuming the third pole as $(s + p)$, the characteristic equation of the closed-loop system can be in the following form:

$$\Delta(s) = (s + 1 + j)(s + 1 - j)(s + p) = \left(s^2 + 2s + 2\right)(s + p)$$

$$\Rightarrow \Delta(s) = s^3 + (p + 2)s^2 + 2(p + 1)s + 2p \quad (1)$$

The characteristic equation of the closed-loop system can be determined by using another method, as follows:

$$\Delta(s) = 1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{1}{s^2 + as + b} \times \frac{1}{s + 1} = 0 \Rightarrow \frac{s^3 + (a + 1)s^2 + (a + b)s + b + 1}{(s^2 + as + b)(s + 1)} = 0$$

$$\Rightarrow \Delta(s) = s^3 + (a + 1)s^2 + (a + b)s + b + 1 \quad (2)$$

Solving (1) and (2):

$$\begin{cases} p + 2 = a + 1 \\ 2(p + 1) = a + b \Rightarrow a = b = 3 \\ 2p = b + 1 \end{cases}$$

Choice (4) is the answer.

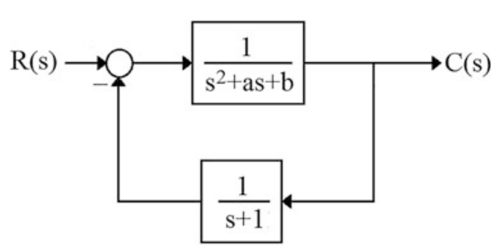

Figure 10.4 The control system of solution of problem 10.4

10.5. Based on the information given in the problem, the control system includes a negative unity feedback, and its open-loop transfer function is as follows:

$$G(s) = \frac{1}{s(1+\tau s)} \tag{1}$$

Based on the choices, it is noticed that the sensitivity of the damping ratio of the system response with respect to the value of parameter of τ is requested.

The characteristic equation of the closed-loop system can be determined as follows:

$$1 + kG(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{1}{s(1+\tau s)} \times 1 = 0 \Rightarrow \frac{\tau s^2 + s + 1}{s(1+\tau s)} = 0 \Rightarrow \tau s^2 + s + 1 = 0 \Rightarrow \Delta(s) = s^2 + \frac{1}{\tau}s + \frac{1}{\tau} \tag{2}$$

By comparing the characteristic equation of the closed-loop control system with the standard second-order characteristic equation, that is, $\Delta(s) = s^2 + 2\xi\omega_n s + {\omega_n}^2$, we can determine the damping ratio, as follows:

$$\begin{cases} {\omega_n}^2 = \frac{1}{\tau} \Rightarrow \omega_n = \frac{1}{\sqrt{\tau}} \\ 2\xi\omega_n = \frac{1}{\tau} \end{cases} \Rightarrow \xi = \frac{\frac{1}{\tau}}{2\frac{1}{\sqrt{\tau}}} = \frac{1}{2\sqrt{\tau}} \tag{3}$$

Sensitivity of a parameter (a) with respect to value of another parameter (b) is defined as follows:

$$S_b^a = \frac{b}{a} \times \frac{da}{db} \tag{4}$$

Solving (3) and (4):

$$S_\tau^\xi = \frac{\tau}{\xi} \times \frac{d\xi}{d\tau} = \frac{\tau}{\frac{1}{2\sqrt{\tau}}} \times \frac{d\left(\frac{1}{2\sqrt{\tau}}\right)}{d\tau} = 2\tau\sqrt{\tau} \times \left(-\frac{1}{4\tau\sqrt{\tau}}\right) = -\frac{1}{2}$$

Therefore, if the parameter of τ increases about 10%, the damping ratio will decrease about 5%. Choice (1) is the answer.

10.6. Based on the information given in the problem, the root locus of the control system for $k > 0$ has been requested.

The characteristic equation of the system can be determined by using Mason's formula, as follows:

$$\Delta = 1 - \sum_a L_a + \sum_{a,b} L_a L_b - \sum_{a,b,c} L_a L_b L_c + \ldots$$

where:

$\sum_a L_a$: The sum of gains of loops

$\sum_{a,b} L_a L_b$: The sum of product of gains of any two non-touching loops (without any common nodes)

$\sum_{a,b,c} L_a L_b L_c$: The sum of product of gains of any three pairwise non-touching loops (without any common nodes)

Now, for the system shown in Fig. 10.5, we have:

$$\sum_a L_a = \left(\frac{1}{s+1}\right) + \left(-\frac{1}{s+1} \times \left(1 + \frac{1}{s}\right) \times k\right) = \frac{1}{s+1} - \frac{k}{s}$$

$$\sum_{a,b} L_a L_b = 0$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta(s) = 1 - \left(\frac{1}{s+1} - \frac{k}{s}\right) = 0 \Rightarrow \Delta(s) = 1 - \frac{1}{s+1} + \frac{k}{s} = \frac{s^2 + s - s + ks + k}{s(s+1)} = \frac{s^2 + k(s+1)}{s(s+1)} = 0$$

$$\Rightarrow s^2 + k(s+1) = 0 \xrightarrow{\times \frac{1}{s^2}} \Delta(s) = 1 + k\frac{s+1}{s^2} \tag{1}$$

By comparing (1) with the standard format of a characteristic equation ($\Delta(s) = 1 + kG(s)H(s)$), we can determine the open-loop transfer function of the system, as follows:

$$L(s) = G(s)H(s) = \frac{s+1}{s^2} \tag{2}$$

To draw the root locus of the system, the rules below must be followed.

$$L(s) = G(s)H(s) = \frac{N(s)}{D(s)} \tag{3}$$

$L(s)$ is the open-loop transfer function excluding k (design gain).
$N(s)$ is the numerator polynomial.
$D(s)$ is the denominator polynomial, or the open-loop characteristic equation.
m is the order of $N(s)$, or the number of open-loop finite zeros.
n is the order of $D(s)$, or the number of open-loop poles, $n \geq m$.
z_i is the i'th open-loop finite zero.
p_j is the j'th open-loop pole.
s_i is a point on the root locus

Rule 1: Number of branches is equal to the number of open-loop poles (n) or the number of closed-loop poles.

Rule 2: On the real axis, the root locus exists to the left of an odd number of open-loop poles and zeros.

Rule 3: Root locus is symmetrical about the real axis. Poles are always in conjugate pairs.

Rule 4: Root loci start at p_j and for $k = 0$ and end at z_i (finite zeros) and infinite zeros of $L(s)$ for $k \to \infty$.

Rule 5: $n - m$ loci approach infinity and converge to straight line asymptotes as $k \to \infty$. Asymptotes are defined by the real-axes intercept σ and the angle θ as follows:

$$\sigma = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n - m} \tag{4}$$

$$\theta = \frac{(2q+1)\pi}{n-m}, q = 0, \pm 1, \pm 2, \ldots \tag{5}$$

Rule 6: Departure angle of a locus from a complex pole is given by the following relation. Herein, $\angle(p_k + z_i)$ and $\angle(p_k + p_j)$ are the angle of the vector from the complex pole of p_k to the zero of z_i and to the pole of p_j, respectively.

$$\theta_k = \sum_{i=1}^{m} \angle(p_k + z_i) - \sum_{\substack{j=1 \\ j \neq k}}^{n} \angle(p_k + p_j) - \pi \tag{6}$$

Rule 7: The breakaway and break-in points from the real axis occur at s, where s can be calculated by using the following equation. Herein, only those s are accepted that are on the root locus.

$$\sum_{i=1}^{m} \frac{1}{s - z_i} = \sum_{j=1}^{n} \frac{1}{s - p_j} \Rightarrow s = s_i \tag{7}$$

Alternatively, s can be determined by using the relation below.

$$1 + kL(s) = 0 \Rightarrow k = \frac{-1}{L(s)} \xrightarrow{\frac{d}{ds}} \frac{d}{ds}(k) = \frac{d}{ds}\left(\frac{-1}{L(s)}\right) = 0 \Rightarrow s = s_i \tag{8}$$

Herein, only those s are accepted that are on the root locus. In other words, those s_i are accepted that their corresponding k_i (use (9)) are real quantities and in the range of $(0, \infty)$.

$$k_i = \left.\frac{-1}{L(s)}\right|_{s=s_i} \tag{9}$$

For this system, the practical rules that we need to apply are as follows:

Rule 2: On the real axis, the left side of the zero is part of root locus.

Rule 6: The departure angles of the loci from the poles can be determined as follows:

$$\theta_{p1=0} = \sum_{i=1}^{m} \angle(p_k + z_i) - \sum_{\substack{j=1 \\ j \neq k}}^{n} \angle(p_k + p_j) - \pi = 0 - \frac{\pi}{2} - \pi \Rightarrow \theta_{p1=0} = -\frac{3\pi}{2} = \frac{\pi}{2} \tag{10}$$

$$\theta_{p2=0} = \sum_{i=1}^{m} \angle(p_k + z_i) - \sum_{\substack{j=1 \\ j \neq k}}^{n} \angle(p_k + p_j) - \pi = 0 - \left(-\frac{\pi}{2}\right) - \pi \Rightarrow \theta_{p2=0} = -\frac{\pi}{2} \tag{11}$$

Rule 7: The breakaway and break-in points from the real axis can be determined as follows:

$$\frac{d}{ds}(k) = \frac{d}{ds}\left(\frac{-1}{\frac{s+1}{s^2}}\right) = \frac{d}{ds}\left(-\frac{s^2}{s+1}\right) = -\frac{2s(s+1) - s^2}{(s+1)^2} = 0 \Rightarrow s^2 + 2s = 0 \Rightarrow s = 0, -2 \tag{12}$$

$$k_1 = -\left.\frac{s^2}{s+1}\right|_{s=0} = 0 \tag{13}$$

$$k_2 = -\left.\frac{s^2}{s+1}\right|_{s=-2} = -\frac{4}{-1} = 4 \tag{14}$$

From (13), it is noticed that the breakaway and break-in points directly start from the poles ($k = 0$). Moreover, $k_2 \in (0, \infty)$; hence, it is a breakaway and break-in point. Therefore, $s = 0, -2$ are the breakaway and break-in points.

Based on the abovementioned calculations, the root locus of the system is shown in Fig. 10.6. Choice (4) is the answer.

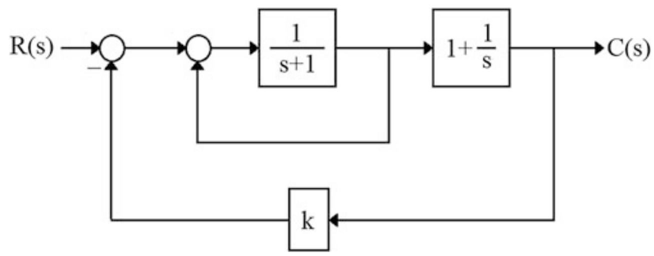

Figure 10.5 The control system of solution of problem 10.6

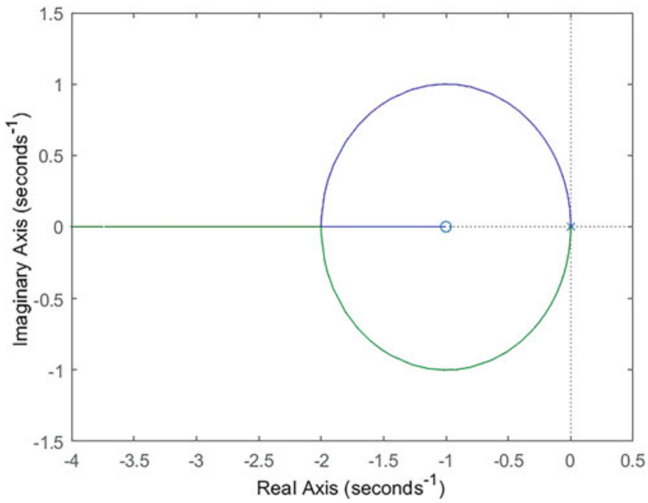

Figure 10.6 The control system of solution of problem 10.6

10.7. The open-loop transfer function of the system is as follows:

$$G(s)H(s) = \frac{k}{s}e^{-s} \tag{1}$$

Herein, we can use the first-order Pade approximation for the term of e^{-s}, as follows:

$$e^{-\theta s} \approx \frac{1 - \frac{\theta}{2}s}{1 + \frac{\theta}{2}s} \tag{2}$$

Solving (1) and (2):

$$\Rightarrow G(s)H(s) \approx \frac{k}{s}\frac{1 - \frac{s}{2}}{1 + \frac{s}{2}} \approx \frac{-k(s-2)}{s(s+2)} \tag{3}$$

Because of the negative sign in the open-loop transfer function, the root locus rules change, and the root locus must be drawn for $k < 0$, or it can be assumed that the system includes a positive unity feedback. The updated rules are as follows:

$L(s)$ is the open-loop transfer function excluding k, where k is the design gain.

$$L(s) = G(s)H(s) = \frac{N(s)}{D(s)} \tag{4}$$

$N(s)$ is the numerator polynomial.
$D(s)$ is the denominator polynomial or the open-loop characteristic equation.

m is the order of $N(s)$ or the number of open-loop finite zeros.
n is the order of $D(s)$ or the number of open-loop poles, $n \geq m$.
z_i is the i'th open-loop finite zero.
p_j is the j'th open-loop pole.
s_i is a point on the root locus.

Rule 1 (like a negative unity feedback system): Number of branches is equal to the number of open-loop poles (n) or the number of closed-loop poles.

Rule 2: On the real axis, the root locus exists to the left of an even number of open-loop poles and zeros.

Rule 3 (like a negative unity feedback system): Root locus is symmetrical about the real axis. Poles are always in conjugate pairs.

Rule 4 (like a negative unity feedback system): Root loci start at p_j and for $k = 0$ and end at z_i (finite zeros) and infinite zeros of $L(s)$ for $k \rightarrow \infty$.

Rule 5: $n - m$ loci approach infinity and converge to straight line asymptotes as $k \rightarrow \infty$. Asymptotes are defined by the real-axes intercept σ and the angle θ as follows:

$$\sigma = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n - m} \tag{5}$$

$$\theta = \frac{2q\pi}{n - m}, q = 0, \pm 1, \pm 2, \ldots \tag{6}$$

Rule 6: Departure angle of a locus from a complex pole is given by the following relation. Herein, $\angle(p_k + z_i)$ and $\angle(p_k + p_j)$ are the angle of the vector from the complex pole of p_k to the zero of z_i and to the pole of p_j, respectively.

$$\theta_k = \sum_{i=1}^{m} \angle(p_k + z_i) - \sum_{\substack{j=1 \\ j \neq k}}^{n} \angle(p_k + p_j) \tag{7}$$

Rule 7: The breakaway and break-in points from the real axis occur at s, where s can be calculated by using the following equation. Herein, only those s are accepted that are on the root locus.

$$\sum_{i=1}^{m} \frac{1}{s - z_i} = \sum_{j=1}^{n} \frac{1}{s - p_j} \Rightarrow s = s_i \tag{8}$$

Alternatively, s can be determined by using the relation below.

$$1 + kL(s) = 0 \Rightarrow k = \frac{-1}{L(s)} \xrightarrow{\frac{d}{ds}} \frac{d}{ds}(k) = \frac{d}{ds}\left(\frac{-1}{L(s)}\right) = 0 \Rightarrow s = s_i \tag{9}$$

Herein, only those s are accepted that are on the root locus. In other words, those s_i are accepted that their corresponding k_i (use (10)) are real quantities and in the range of $(-\infty, 0)$.

$$k_i = \left.\frac{-1}{L(s)}\right|_{s=s_i} \tag{10}$$

In this problem, applying only Rules 2 is enough to identify the root locus. Figure 10.8 approximately shows the root locus of the system. Choice (4) is the answer.

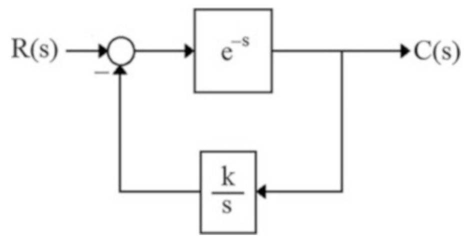

Figure 10.7 The control system of solution of problem 10.7

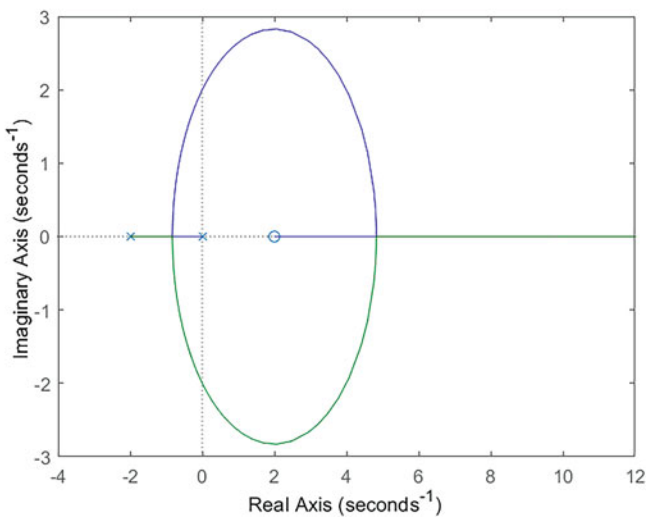

Figure 10.8 The control system of solution of problem 10.7

10.8. If we determine the range of k for the stability of the system, we will be able to determine the stability of the system for $k = 40$.

Based on the zeros' and poles' locations shown in the root locus of the system in Fig. 10.9, we can determine the open-loop transfer function of the system, as follows:

$$G(s)H(s) = \frac{1}{s(s+2)(s+2+j2)(s+2-j2)} = \frac{1}{s(s+2)(s^2+4s+8)} \quad (1)$$

The characteristic equation of the closed-loop control system can be determined as follows:

$$1 + kG(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{k}{s(s+2)(s^2+4s+8)} = 0 \Rightarrow \frac{s^4+6s^3+16s^2+16s+k}{s(s+2)(s^2+4s+8)} = 0$$

$$\Rightarrow \Delta(s) = s^4 + 6s^3 + 16s^2 + 16s + k = 0 \quad (2)$$

Applying Routh-Hurwitz table rule for this problem:

$$\begin{array}{c|ccc} s^4 & 1 & 16 & k \\ s^3 & 6 & 16 & \\ s^2 & \frac{80}{6} & k & \\ s^1 & \frac{80\times 16 - 36k}{80} & & \\ s^0 & k & & \end{array}$$

Based on Routh-Hurwitz table rule, the system is stable if all the elements in the first column of the table are positive. Therefore, the system is stable if:

$$\begin{cases} \dfrac{80 \times 16 - 36k}{80} > 0 \\ k > 0 \end{cases} \Rightarrow \begin{cases} k < \dfrac{80 \times 16}{36} \\ k > 0 \end{cases} \Rightarrow 0 < k < 35.5 \tag{3}$$

From (3), it is seen that the system is unstable for $k = 40$. Choice (2) is the answer.

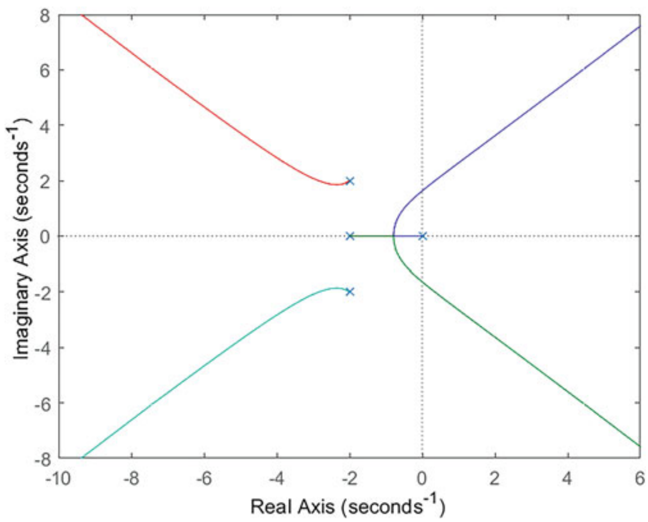

Figure 10.9 The control system of solution of problem 10.8

10.9. Based on the information given in the problem, the system includes a negative unity feedback, and its open-loop transfer function is as follows:

$$G(s) = \frac{s - k}{s(s+1)} \tag{1}$$

As can be noticed from (1), the open-loop transfer function is not in the standard form. The standard format of an open-loop transfer function is as follows:

$$G(s)H(s) = 1 + \frac{k \prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_j)} \tag{2}$$

The open-loop transfer function can be converted to the standard format as follows:

$$1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{s-k}{s(s+1)} \times 1 = 0 \Rightarrow \frac{s^2 + 2s - k}{s(s+1)} = 0 \Rightarrow s(s+2) - k = 0 \xrightarrow{\times \frac{1}{s(s+2)}} 1 + \frac{-k}{s(s+2)} \tag{3}$$

By comparing (2) and (3), it is noticed that:

$$G(s)H(s) = \frac{-1}{s(s+2)} \tag{4}$$

Because of the negative sign in the open-loop transfer function, the root locus rules change, and the root locus must be drawn for $k < 0$, or it can be assumed that the system includes a positive unity feedback. The updated rules are as follows:

$L(s)$ is the open-loop transfer function excluding k, where k is the design gain.

$$L(s) = G(s)H(s) = \frac{N(s)}{D(s)} \tag{5}$$

$N(s)$ is the numerator polynomial.
$D(s)$ is the denominator polynomial or the open-loop characteristic equation.
m is the order of $N(s)$ or the number of open-loop finite zeros.
n is the order of $D(s)$ or the number of open-loop poles, $n \geq m$.
z_i is the i'th open-loop finite zero.
p_j is the j'th open-loop pole.
s_i is a point on the root locus

Rule 1 (like a negative unity feedback system): Number of branches is equal to the number of open-loop poles (n) or the number of closed-loop poles.

Rule 2: On the real axis, the root locus exists to the left of an even number of open-loop poles and zeros.

Rule 3 (like a negative unity feedback system): Root locus is symmetrical about the real axis. Poles are always in conjugate pairs.

Rule 4 (like a negative unity feedback system): Root loci start at p_j and for $k = 0$ and end at z_i (finite zeros) and infinite zeros of $L(s)$ for $k \to \infty$.

Rule 5: $n - m$ loci approach infinity and converge to straight line asymptotes as $k \to \infty$. Asymptotes are defined by the real-axes intercept σ and the angle θ as follows:

$$\sigma = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n - m} \tag{6}$$

$$\theta = \frac{2q\pi}{n - m}, q = 0, \pm 1, \pm 2, \ldots \tag{7}$$

Rule 6: Departure angle of a locus from a complex pole is given by the following relation. Herein, $\angle(p_k + z_i)$ and $\angle(p_k + p_j)$ are the angle of the vector from the complex pole of p_k to the zero of z_i and to the pole of p_j, respectively.

$$\theta_k = \sum_{i=1}^{m} \angle(p_k + z_i) - \sum_{\substack{j = 1 \\ j \neq k}}^{n} \angle\left(p_k + p_j\right) \tag{8}$$

Rule 7: The breakaway and break-in points from the real axis occur at s, where s can be calculated by using the following equation. Herein, only those s are accepted that are on the root locus.

$$\sum_{i=1}^{m} \frac{1}{s - z_i} = \sum_{j=1}^{n} \frac{1}{s - p_j} \Rightarrow s = s_i \tag{9}$$

Alternatively, s can be determined by using the relation below.

$$1 + kL(s) = 0 \Rightarrow k = \frac{-1}{L(s)} \xrightarrow{\frac{d}{ds}} \frac{d}{ds}(k) = \frac{d}{ds}\left(\frac{-1}{L(s)}\right) = 0 \Rightarrow s = s_i \tag{10}$$

Herein, only those s are accepted that are on the root locus. In other words, those s_i are accepted that their corresponding k_i (use (11)) are real quantities and in the range of $(-\infty, 0)$.

$$k_i = \frac{-1}{L(s)}\bigg|_{s=s_i} \tag{11}$$

Figure 10.10 shows the root locus of the system. Herein, applying Rules 2 and 4 are enough to draw the root locus. Choice (2) is the answer.

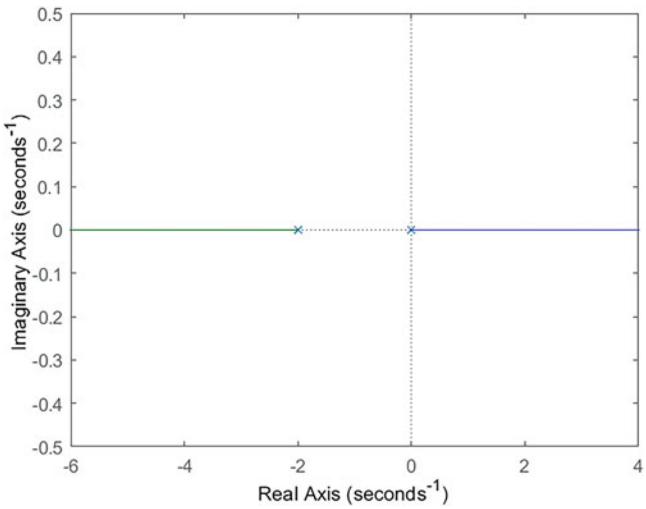

Figure 10.10 The control system of solution of problem 10.9

10.10. Based on the information given in the problem, we know that the control system includes a negative unity feedback, and its open-loop transfer function is as follows:

$$G(s) = \frac{k(s+4)}{s(s+2)(s+a)} \tag{1}$$

Moreover, the settling time of the system is about eight seconds. In other words:

$$t_s = 8 \ sec \tag{2}$$

The settling time of a second-order system in its underdamped status can be determined by the following relation.

$$t_s = \frac{4}{\sigma} \tag{3}$$

Solving (2) and (3):

$$\frac{4}{\sigma} = 8 \Rightarrow \sigma = 0.5 \tag{4}$$

Figure 10.11 shows the root locus of the system for each value of for the parameter of a, presented in the choices. As can be seen in Fig. 10.11.1, the system is unstable for $a = 1$ and large k. Moreover, Fig. 10.11.2 shows that the system is in the undamped status for $a = 2$ and large k.

However, the system has the underdamped response for $a = 3, 4$ and large k, illustrated in Figs. 10.11.3-4. As can be seen, the system has two loci in the infinity for large k, and the value of real part of the dominant poles ($\sigma \pm j\omega_d$) is equal to the real-axes intercept of the asymptotes (σ).

$$\sigma = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n - m} \quad (5)$$

Solving (4) and (5):

$$-0.5 = \frac{(-2-a)-(-4)}{3-1} = \frac{2-a}{2} \Rightarrow 2 - a = -1 \Rightarrow a = 3$$

Choice (3) is the answer.

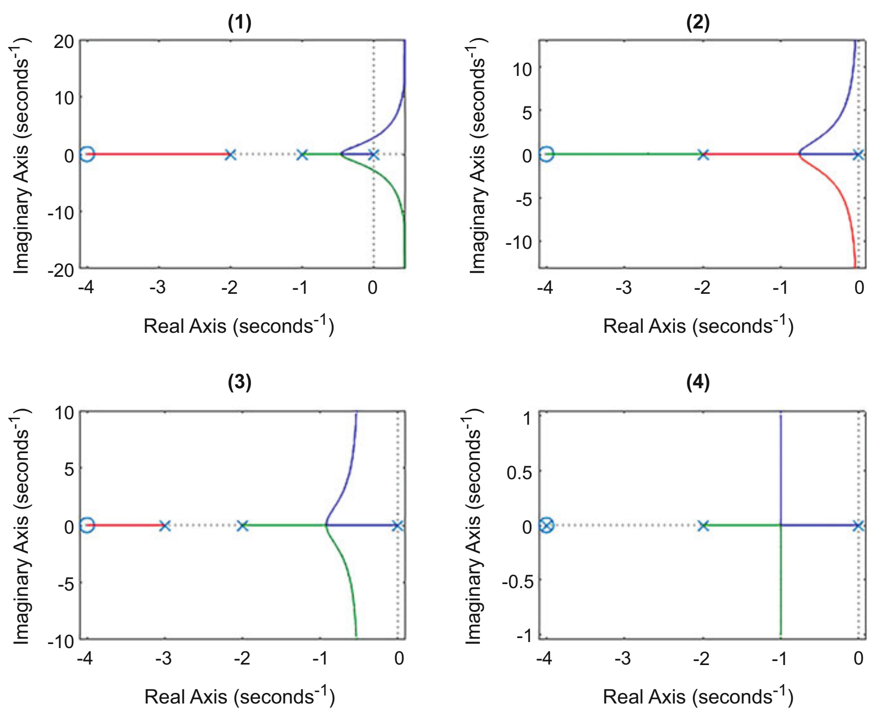

Figure 10.11 The control system of solution of problem 10.10

10.11. Based on the information given in the problem, we know that the control system includes a negative unity feedback, and its open-loop transfer function is as follows:

$$G(s) = \frac{k(s+0.5)}{s^2(s^2+4s+8)} \quad (1)$$

In this problem, we only need to apply Rule 7 as follows.

Rule 7: The breakaway and break-in points from the real axis occur at s, where s can be calculated by using the following equation. Herein, only those s are accepted that are on the root locus.

$$\sum_{i=1}^{m} \frac{1}{s - z_i} = \sum_{j=1}^{n} \frac{1}{s - p_j} \Rightarrow s = s_i \quad (2)$$

Alternatively, s can be determined by using the relation below.

$$1 + kL(s) = 0 \Rightarrow k = \frac{-1}{L(s)} \xrightarrow{\frac{d}{ds}} \frac{d}{ds}(k) = \frac{d}{ds}\left(\frac{-1}{L(s)}\right) = 0 \Rightarrow s = s_i \tag{3}$$

Herein, only those s are accepted that are on the root locus. In other words, those s_i are accepted that their corresponding k_i (use (4)) are real quantities and in the range of $(0, \infty)$.

$$k_i = \frac{-1}{L(s)}\bigg|_{s=s_i} \tag{4}$$

By applying the second method of Rule 7, we have:

$$1 + k\frac{(s+0.5)}{s^2(s^2+4s+8)} = 0 \Rightarrow k = \frac{-1}{\frac{(s+0.5)}{s^2(s^2+4s+8)}} \xrightarrow{\frac{d}{ds}} \frac{d}{ds}(k) = \frac{d}{ds}\left(\frac{-1}{\frac{(s+0.5)}{s^2(s^2+4s+8)}}\right) = 0$$

$$\Rightarrow \frac{d}{ds}\left(-\frac{s^4+4s^3+8s^2}{s+0.5}\right) = 0 \Rightarrow \frac{(4s^3+12s^2+16s)(s+0.5)-(s^4+4s^3+8s^2)}{(s+0.5)^2} = 0$$

$$\Rightarrow 4s^4 + 12s^3 + 16s^2 + 2s^3 + 6s^2 + 8s - s^4 - 4s^3 - 8s^2 = 0$$

$$\Rightarrow 3s^4 + 10s^3 + 14s^2 + 8s = s(3s+4)\left(s^2+2s+2\right) = 0$$

$$\Rightarrow s = -1 \pm j, 0, -\frac{4}{3} \tag{5}$$

Now, we need to check to see if $-1 \pm j$ are on the root locus or not, as follows:

$$k_1 = -\frac{s^2(s^2+4s+8)}{s+0.5}\bigg|_{s=-1+j} = -\frac{s^4+4s^3+8s^2}{s+0.5}\bigg|_{s=-1+j} = -\frac{(-1+j)^4+4(-1+j)^3+8(-1+j)^2}{(-1+j)+0.5} = 8$$

Since $k_1 = 8 \in (0, \infty)$, the point of $-1-j$ and its complex conjugate value, that is, $-1+j$ are the break-away/break-in points, and consequently they are on the root locus of the system. Choice (4) is the answer.

10.12. Based on the information given in the problem, the control system includes a negative unity feedback, and its open-loop transfer function ($k > 0$) is as follows:

$$G(s) = \frac{k(s+3)}{(s+1)(s+p)} \tag{1}$$

Based on the problem, the transient response of the closed-loop system must always be overdamped. Therefore, the root locus of the system must only be on the real axis.

The open-loop system includes one zero and one pole, other than the pole of p. Hence, there are three statuses for the position of the pole of p. Figure 10.12.1-3 show the root locus of the system for $0 < p < 1$, $1 < p < 3$, and $p > 3$, respectively.

As can be seen, $p > 3$ is acceptable for the range of the parameter, as in this condition, the system is stable, the root locus is only on the real axis, and consequently the transient response of the closed-loop system is always overdamped. Choice (3) is the answer.

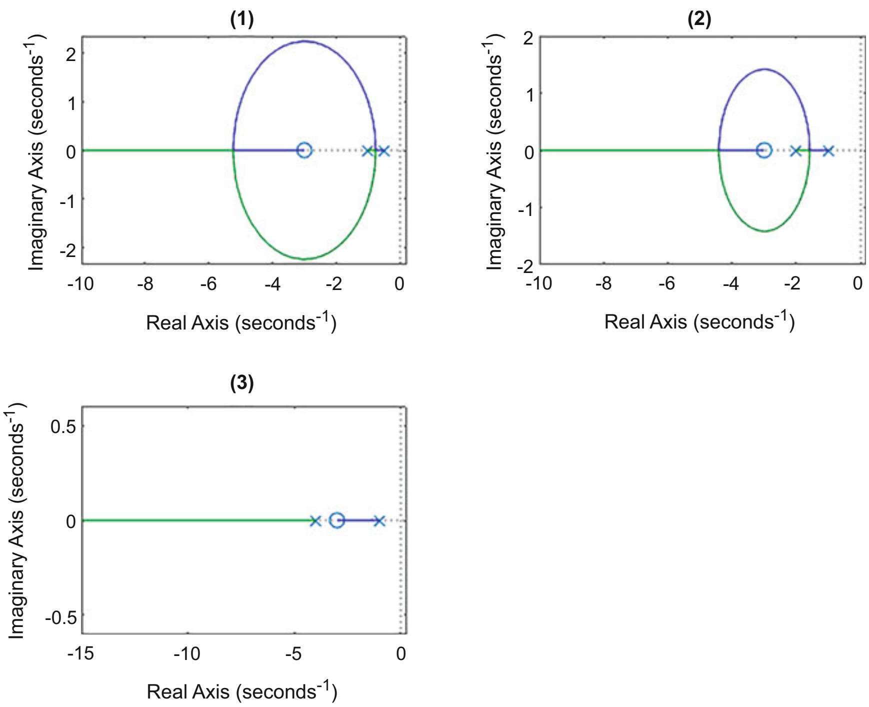

Figure 10.12 The control system of solution of problem 10.12

10.13. Based on the information given in the problem, the root locus of the control system for $k < 0$ has been requested. The characteristic equation of the system can be determined by using Mason's formula, as follows:

$$\Delta = 1 - \sum_{a} L_a + \sum_{a,b} L_a L_b - \sum_{a,b,c} L_a L_b L_c + \ldots$$

where:

$\sum_{a} L_a$: The sum of gains of loops

$\sum_{a,b} L_a L_b$: The sum of product of gains of any two non-touching loops (without any common nodes)

$\sum_{a,b,c} L_a L_b L_c$: The sum of product of gains of any three pairwise non-touching loops (without any common nodes)

Now, for the system shown in Fig. 10.13, we can have:

$$\sum_{a} L_a = \left(k \times \frac{4}{s+2} \times \frac{1}{3}\right) + \left(-k \times \frac{4}{s+2} \times \frac{1}{s}\right) = \frac{\frac{4}{3}k}{s+2} - \frac{4k}{s(s+2)}$$

$$\sum_{a,b} L_a L_b = 0$$

$$\sum_{a,b,c} L_a L_b L_c = 0$$

$$\Rightarrow \Delta = 1 - \left(\frac{\frac{4}{3}k}{s+2} - \frac{4k}{s(s+2)}\right) = 0 \Rightarrow \Delta = 1 + \frac{\left(4 - \frac{4}{3}s\right)k}{s(s+2)} = 1 + \frac{-\frac{4}{3}(s-3)k}{s(s+2)} \quad (1)$$

By comparing (1) with the standard format of a characteristic equation ($\Delta(s) = 1 + kG(s)H(s)$), we have:

$$L(s) = G(s)H(s) = \frac{-\frac{4}{3}(s-3)}{s(s+2)} \tag{2}$$

Because of the negative sign in the open-loop transfer function as well as $k < 0$, the root locus rules do not change; therefore, the root locus must be drawn for $k > 0$, as is shown in Fig. 10.14. Herein, applying only Rules 2 and 4 are enough to draw the root locus. Choice (2) is the answer.

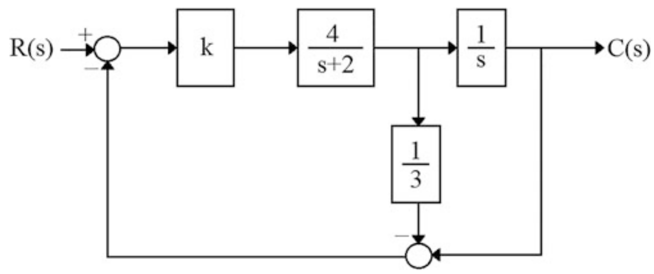

Figure 10.13 The control system of solution of problem 10.13

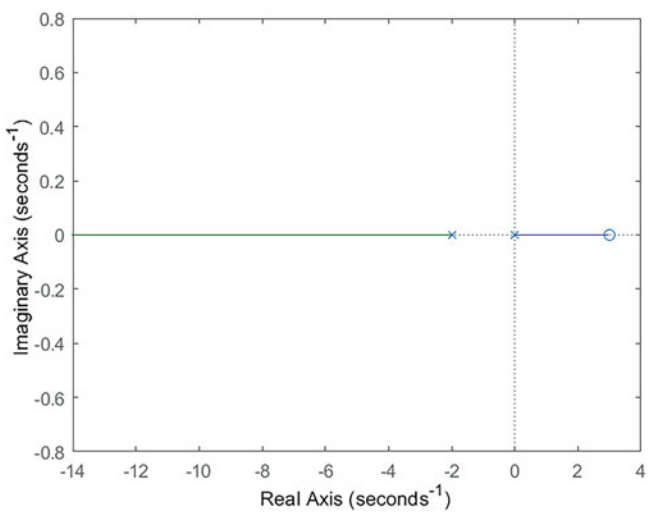

Figure 10.14 The control system of solution of problem 10.13

10.14. Based on the information given in the problem, the control system includes a negative unity feedback, and its open-loop transfer function is as follows:

$$G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)} \tag{1}$$

The characteristic equation of the closed-loop control system can be determined as follows:

$$1 + kG(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{\omega_n^2}{s(s+2\xi\omega_n)} \times 1 = 0 \Rightarrow \frac{s^2 + 2\xi\omega_n s + {\omega_n}^2}{s(s+2\xi\omega_n)} = 0 \Rightarrow \Delta(s) = s^2 + 2\xi\omega_n s + {\omega_n}^2 \tag{2}$$

By comparing the characteristic equation of the system with the standard second-order characteristic equation, that is, $\Delta(s) = s^2 + 2\xi\omega_n s + {\omega_n}^2$, it is noticed they are the same. For a second-order system with the standard format, maximum overshoot percentage is defined as follows:

$$O.S. = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100 \tag{3}$$

Sensitivity of a parameter (a) with respect to value of another parameter (b) is defined as follows:

$$S_b^a = \frac{b}{a} \times \frac{da}{db} \tag{3}$$

Solving (3) and (4):

$$S_\xi^{O.S.} = \frac{\xi}{e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100} \times \frac{d}{d\xi}\left(e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100\right)$$

$$\Rightarrow S_\xi^{O.S.} = \frac{\xi}{e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100} \times \frac{-\pi\sqrt{1-\xi^2} - \frac{-2\xi}{2\sqrt{1-\xi^2}}(-\pi\xi)}{1-\xi^2} e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100$$

$$\Rightarrow S_\xi^{O.S.} = \frac{-\xi\pi}{\left(1-\xi^2\right)\sqrt{1-\xi^2}} \tag{4}$$

Solving (4) for $\xi = \frac{\sqrt{2}}{2}$:

$$S_\xi^{O.S.}\Big|_{\xi=\frac{\sqrt{2}}{2}} = \frac{-\frac{\sqrt{2}}{2}\pi}{\left(1-\frac{1}{2}\right)\sqrt{1-\frac{1}{2}}} \Rightarrow S_\xi^{O.S.}\Big|_{\xi=\frac{\sqrt{2}}{2}} = -2\pi$$

Choice (1) is the answer.

10.15. Based on the information given in the problem, the control system includes a negative unity feedback, and its open-loop transfer function ($k > 0$) is as follows:

$$G(s) = \frac{k}{s(s+\alpha)(s+\beta)} \tag{1}$$

Moreover, the root locus has a break-away/break-in point on the real axis in $-\frac{4}{9}$ and the straight-line asymptotes intersection in $-\frac{11}{9}$. In other words:

$$s_i = -\frac{4}{9} \tag{2}$$

$$\sigma = -\frac{11}{9} \tag{3}$$

Hence, we need to apply Rules 5 and 7 as follows.

Rule 5: $n - m$ loci approach infinity and converge to straight line asymptotes as $k \to \infty$. Asymptotes are defined by the real-axes intercept σ and the angle θ as follows:

$$\sigma = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n-m} \tag{4}$$

Rule 7: The breakaway and break-in points from the real axis occur at s, where s can be calculated by using the following equation. Herein, only those s are accepted that are on the root locus.

$$\sum_{i=1}^{m} \frac{1}{s-z_i} = \sum_{j=1}^{n} \frac{1}{s-p_j} \Rightarrow s = s_i \tag{5}$$

Alternatively, s can be determined by using the relation below.

$$1 + kL(s) = 0 \Rightarrow k = \frac{-1}{L(s)} \xrightarrow{\frac{d}{ds}} \frac{d}{ds}(k) = \frac{d}{ds}\left(\frac{-1}{L(s)}\right) = 0 \Rightarrow s = s_i \tag{6}$$

Herein, only those s are accepted that are on the root locus. In other words, those s_i are accepted that their corresponding k_i (use (7)) are real quantities and in the range of $(0, \infty)$.

$$k_i = \frac{-1}{L(s)}\bigg|_{s=s_i} \tag{7}$$

Solving (3) and (4):

$$-\frac{11}{9} = \frac{(-\alpha-\beta)-(0)}{3-0} \Rightarrow \alpha+\beta = \frac{11}{3} \tag{8}$$

Solving (1) and (6):

$$\frac{d}{ds}(k) = \frac{d}{ds}\left(\frac{-1}{\frac{1}{s(s+\alpha)(s+\beta)}}\right) = \frac{d}{ds}(-s(s+\alpha)(s+\beta)) = 0$$

$$\Rightarrow (s+\alpha)(s+\beta) + s(s+\beta) + s(s+\alpha) = 0 \Rightarrow 3s^2 + 2(\alpha+\beta)s + \alpha\beta = 0 \tag{9}$$

Solving (2) and (9):

$$3\left(-\frac{4}{9}\right)^2 + 2(\alpha+\beta)\left(-\frac{4}{9}\right) + \alpha\beta = 0 \Rightarrow \alpha\beta = \frac{8}{3} \tag{10}$$

The characteristic equation of the closed-loop control system can be determined as follows:

$$1 + kG(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{k}{s(s+\alpha)(s+\beta)} = 0 \Rightarrow \Delta(s) = s^3 + (\alpha+\beta)s^2 + \alpha\beta s + k \tag{11}$$

Solving (8), (10), and (11):

$$\Delta(s) = s^3 + \frac{11}{3}s^2 + \frac{8}{3}s + k$$

Choice (4) is the answer.

10.16. Based on the information given in the problem, the state-transition matrix of the closed-loop control system with a negative unity feedback is as follows:

$$[\varphi(t)] = \begin{bmatrix} 2e^{-t} - 2e^{-2t} + e^{-3t} & e^{-t} - e^{-2t} & 0 \\ 0 & e^{-2t} & -e^{-2t} + e^{-3t} \\ k(e^{-2t} - e^{-3t}) & 0 & e^{-3t} \end{bmatrix} \tag{1}$$

As we know, the relation below exists for any state-transition matrix.

$$[A] = \frac{d}{dt}[\varphi(t)]\Big|_{t=0} \tag{2}$$

Solving (1) and (2):

$$[A] = \left.\begin{bmatrix} -2e^{-t} + 4e^{-2t} - 3e^{-3t} & -e^{-t} + 2e^{-2t} & 0 \\ 0 & -2e^{-2t} & 2e^{-2t} - 3e^{-3t} \\ k(-2e^{-2t} + 3e^{-3t}) & 0 & -3e^{-3t} \end{bmatrix}\right|_{t=0} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ k & 0 & -3 \end{bmatrix} \tag{3}$$

As we know, the characteristic equation of a system can be determined as follows if we know its state matrix ([A]).

$$\Delta(s) = |s[I] - [A]| = 0 \tag{4}$$

Solving (3) and (4):

$$\Delta(s) = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & -1 \\ k & 0 & -3 \end{bmatrix} \right| = \begin{vmatrix} s+1 & -1 & 0 \\ 0 & s+2 & 1 \\ -k & 0 & s+3 \end{vmatrix} = 0$$

$$\Rightarrow \Delta(s) = (s+1)(s+2)(s+3) + (-k)(-1) = 0 \Rightarrow \Delta(s) = 1 + \frac{k}{(s+1)(s+2)(s+3)} \tag{5}$$

By comparing (5) with the standard format of a characteristic equation ($\Delta(s) = 1 + kG(s)H(s)$), we have:

$$L(s) = G(s)H(s) = \frac{1}{(s+1)(s+2)(s+3)} \tag{6}$$

Now, we can use Rules 5 and 7, as follows.

Rule 5: $n - m$ loci approach infinity and converge to straight line asymptotes as $k \to \infty$. Asymptotes are defined by the real-axes intercept σ and the angle θ as follows:

$$\sigma = \frac{\sum_{j=1}^{n} p_j - \sum_{i=1}^{m} z_i}{n - m} \tag{7}$$

Rule 7: The breakaway and break-in points from the real axis occur at s, where s can be calculated by using the following equation. Herein, only those s are accepted that are on the root locus.

$$\sum_{i=1}^{m} \frac{1}{s - z_i} = \sum_{j=1}^{n} \frac{1}{s - p_j} \Rightarrow s = s_i \tag{8}$$

Alternatively, s can be determined by using the relation below.

$$1 + kL(s) = 0 \Rightarrow k = \frac{-1}{L(s)} \xrightarrow{\frac{d}{ds}} \frac{d}{ds}(k) = \frac{d}{ds}\left(\frac{-1}{L(s)}\right) = 0 \Rightarrow s = s_i \tag{9}$$

Herein, only those s are accepted that are on the root locus. In other words, those s_i are accepted that their corresponding k_i (use (10)) are real quantities and in the range of (0, ∞).

$$k_i = \left.\frac{-1}{L(s)}\right|_{s=s_i} \tag{10}$$

Solving (5) and (9):

$$\frac{d}{ds}(k) = \frac{d}{ds}\left(\frac{-1}{\frac{1}{(s+1)(s+2)(s+3)}}\right) = \frac{d}{ds}(-(s+1)(s+2)(s+3)) = 0 \Rightarrow s = \frac{-6 \pm \sqrt{3}}{3} \tag{11}$$

$$k_1 = -(s+1)(s+2)(s+3)|_{s=\frac{-6-\sqrt{3}}{3}} < 0 \tag{12}$$

$$k_2 = -(s+1)(s+2)(s+3)|_{s=\frac{-6+\sqrt{3}}{3}} > 0 \tag{13}$$

Therefore, only the point of $\frac{-6+\sqrt{3}}{3}$ is accepted as a breakaway/break-in point of the system. Alternatively, based on the root locus of the system, shown in Fig. 10.15, it is seen that only the point of $\frac{-6+\sqrt{3}}{3}$ is the breakaway/break-in point. Regarding the asymptote's intersection on the real axis, we can write:

$$\sigma = \frac{(-1-2-3)-(0)}{3-0} \Rightarrow \sigma = -2$$

Choice (2) is the answer.

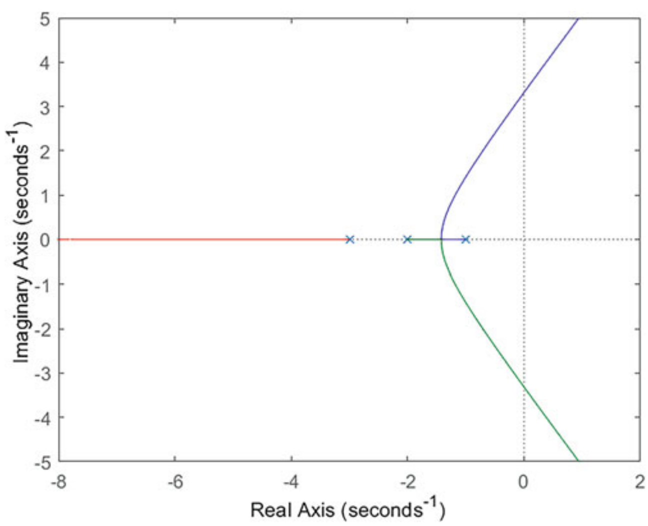

Figure 10.15 The control system of solution of problem 10.16

11 Problems: Controller Design in Time Domain

Abstract

In this chapter, different types of controllers, including proportional controller, proportional-derivative (PD) controller, proportional-integrate (PI) controller, and proportional-integrate-derivative (PID) controller, are designed to achieve the desirable goals in the transient and steady-state responses of the closed-loop control system. In this chapter, the problems are categorized in different levels based on their difficulty levels (easy, normal, and hard) and calculation amounts (small, normal, and large). Additionally, the problems are ordered from the easiest problem with the smallest computations to the most difficult problems with the largest calculations.

11.1. The open-loop transfer function of a control system with a negative unity feedback and a proportional-derivative (PD) controller is as follows:

$$G(s)G_c(s) = \frac{10(k_P + k_D s)}{s^2}$$

Determine the parameters of the controller, so that the closed-loop system is stable and the steady-state error to a unit parabola input is 0.01.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $k_P > 0,\ k_D = 10$
2) $k_P = 10,\ k_D = 100$
3) $k_P = 10,\ k_D > 0$
4) $k_P > 0,\ k_D > 0$

11.2. The open-loop transfer function of the control system, shown in Fig. 11.1, is as follows:

$$G(s) = \frac{2}{(s+3)(s+6)}$$

Design a proportional controller, in the form of $G_c(s) = k_p$, so that the damping ratio of the closed-loop system is 0.7.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $G_c(s) = 11.25$
2) $G_c(s) = 22.5$
3) $G_c(s) = 5.62$
4) $G_c(s) = 45$

M. Rahmani-Andebili, *Feedback Control Systems Analysis and Design*, https://doi.org/10.1007/978-3-030-95277-8_11

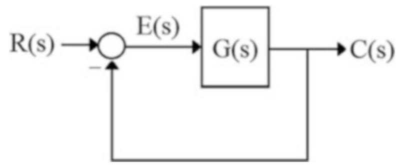

Figure 11.1 The control system of problem 11.2

11.3. The open-loop transfer function of a control system that includes a negative unity feedback is as follows:

$$G(s) = \frac{k}{s(1+T_1 s)(1+T_2 s)}$$

Determine the proportional controller gain (k_P) for a proportional-integral-derivative (PID) controller in Ziegler-Nichols's method.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $\frac{T_1+T_2}{T_1 T_2}$
2) $\frac{5(T_1+T_2)}{3T_1 T_2}$
3) $\frac{3T_1 T_2}{5(T_1+T_2)}$
4) $\frac{3(T_1+T_2)}{5T_1 T_2}$

11.4. The open-loop transfer function of a control system is as follows:

$$G(s) = \frac{k}{s(s+1)^2}$$

Determine the integral time constant (T_I) for a proportional-integral-derivative (PID) controller in Ziegler-Nichols's method.

Difficulty level ○ Easy ● Normal ○ Hard
Calculation amount ○ Small ● Normal ○ Large

1) $\frac{\pi}{2}$
2) π
3) $\frac{3\pi}{2}$
4) 2π

11.5. The open-loop transfer function of a control system is as follows:

$$G(s) = \frac{1}{s^2}$$

Design a controller ($G_c(s)$) in the feedback structure, so that $-1 \pm j2$ are the closed-loop poles of the system.

Difficulty level ○ Easy ○ Normal ● Hard
Calculation amount ○ Small ● Normal ○ Large

1) $25\frac{s+2}{s+12}$
2) $25\frac{s+12}{s+2}$
3) $12\frac{s+2}{s+3}$
4) $12\frac{s+3}{s+2}$

11.6. Which type of the controllers below must be used in a closed-loop control system, with the following open-loop transfer function and a negative unity feedback, to set the undamped natural frequency at $\omega_n = 10\ rad/s$ but without affecting the damping ratio.

$$G(s) = \frac{25}{s(s+8)}$$

Difficulty level	○ Easy	○ Normal	● Hard
Calculation amount	○ Small	● Normal	○ Large

1) Proportional-integral (PI) controller
2) Proportional (P) controller
3) Lead controller with the minimum lead angle of 37°
4) Lead controller with the minimum lead angle of 53°

11.7. The open-loop transfer function of a control system that includes a negative unity feedback is as follows:

$$G(s) = \frac{k}{s(s+4)(s+6)}$$

The uncontrolled closed-loop system response has the overshoot and settling time of 16% and 3.32 seconds, respectively. Design a proportional-derivative (PD) controller (with the format of $G_c(s) = s + z_c$), so that, without changing the overshoot of the system response, a threefold reduction happens in its settling time.

Difficulty level	○ Easy	○ Normal	● Hard
Calculation amount	○ Small	○ Normal	● Large

1) $G_c(s) \approx s + 1$
2) $G_c(s) \approx s + 2$
3) $G_c(s) \approx s + 3$
4) $G_c(s) \approx s + 4$

11.8. The open-loop transfer function of the control system, which is shown in Fig. 11.2, is as follows:

$$G(s) = \frac{2}{(s+3)(s+6)}$$

Design a proportional-integrate (PI) controller, in the form of $G_c(s) = k_p + \frac{k_I}{s}$, so that the damping ratio of the closed-loop system is 0.7.

Difficulty level	○ Easy	○ Normal	● Hard
Calculation amount	○ Small	○ Normal	● Large

1) $G_c(s) = 9 + \frac{27}{s}$
2) $G_c(s) = 9 + \frac{9}{s}$
3) $G_c(s) = 1 + \frac{27}{s}$
4) $G_c(s) = 27 + \frac{9}{s}$

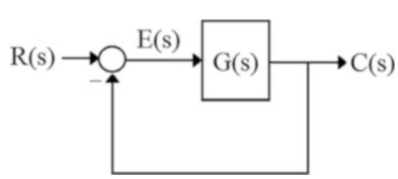

Figure 11.2 The control system of problem 11.8

12 Solutions of Problems: Controller Design in Time Domain

Abstract

In this chapter, the problems of the eleventh chapter are fully solved, in detail, step-by-step, and with different methods.

12.1. Based on the information given in the problem, the closed-loop system that includes a negative unity feedback is stable. In addition, the open-loop transfer function of the system, with a proportional-derivative (PD) controller, is as follows:

$$G(s)G_c(s) = \frac{10(k_P + k_D s)}{s^2} \tag{1}$$

Moreover, the steady-state error of the system to a unit parabola input is 0.01. In other words:

$$e_{ss} = 0.01 \tag{2}$$

The characteristic equation of the closed-loop control system can be determined as follows:

$$1 + G(s)G_c(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{10(k_P + k_D s)}{s^2} \times 1 = 0 \Rightarrow \frac{s^2 + 10k_D s + 10k_P}{s^2} = 0 \Rightarrow \Delta(s) = s^2 + 10k_D s + 10k_P \tag{3}$$

A second-order system with the characteristic equation of $a_2 s^2 + a_1 s + a_0$ is stable if and only if all the coefficients are non-zero and have the same sign. In other words:

$$a_2, a_1, a_0 > 0 \tag{4}$$

Solving (3) and (4):

$$k_D, k_P > 0 \tag{5}$$

The steady-state error of a type-two system to a unit parabola input can be determined as follows:

$$e_{ss} = \frac{1}{\lim\limits_{s \to 0} s^2 G(s)G_c(s)H(s)} \tag{6}$$

Solving (1), (2), and (6):

M. Rahmani-Andebili, *Feedback Control Systems Analysis and Design*, https://doi.org/10.1007/978-3-030-95277-8_12

$$0.01 = \frac{1}{\lim_{s\to 0} s^2 \frac{10(k_P+k_D s)}{s^2} \times 1} = \frac{1}{10k_P} \Rightarrow k_P = 10 \tag{7}$$

From (5) and (7), it is noticed that $k_P = 10$, $k_D > 0$. Choice (3) is the answer.

12.2. Based on the information given in the problem, the open-loop transfer function of the control system, shown in Fig. 12.1, is as follows:

$$G(s) = \frac{2}{(s+3)(s+6)} \tag{1}$$

Moreover, we know that the design objective and the requested controller are as follows:

$$\xi = 0.7 \tag{2}$$

$$G_c(s) = k_p \tag{3}$$

From (1) and (3), we have:

$$G(s)G_c(s) = \frac{2k_p}{(s+3)(s+6)} \tag{4}$$

The root locus of the system is illustrated in Fig. 12.2.

From (2), the angle of $\xi -$ line can be determined as follows:

$$\theta = \cos^{-1}\xi = \cos^{-1}0.7 = 45^\circ \tag{5}$$

By intersecting the $\xi -$ line with the root locus, shown in Fig. 12.3, the design point can be determined. As can be noticed, the $\xi -$ line intersects the vertical branch which is the asymptote of the root locus. Therefore:

$$\sigma = \frac{0-(6+3)}{2-0} = -4.5 \tag{6}$$

From Fig. 12.3, we can write:

$$\tan 45^\circ = \frac{\omega_d}{4.5} \Rightarrow \omega_d = 4.5 \tag{7}$$

From (6) and (7), the design point is:

$$s = -4.5 + j4.5 \tag{8}$$

By applying magnitude criterion, we have:

$$k_p = \left|\frac{1}{L(s)}\right|_{s=-4.5+j4.5} = \left|\frac{(s+3)(s+6)}{2}\right|_{s=-4.5+j4.5} = 11.25 \Rightarrow G_c(s) = 11.25$$

Choice (1) is the answer.

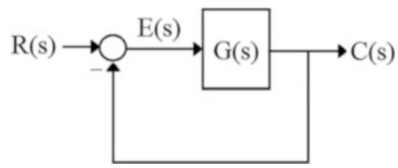

Figure 12.1 The control system of solution of problem 12.2

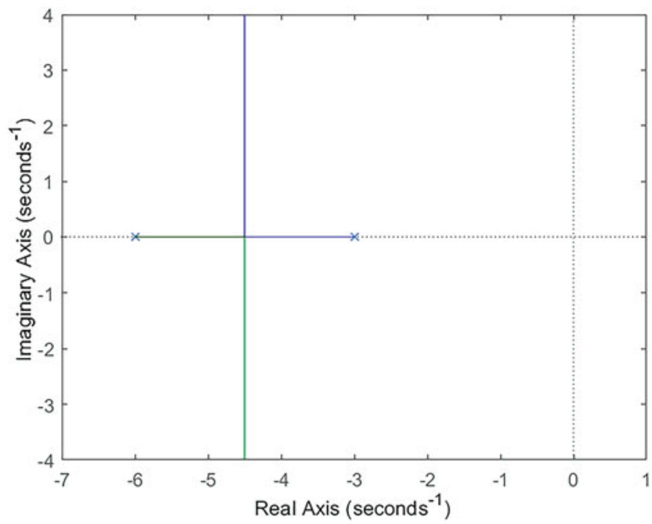

Figure 12.2 The control system of solution of problem 12.2

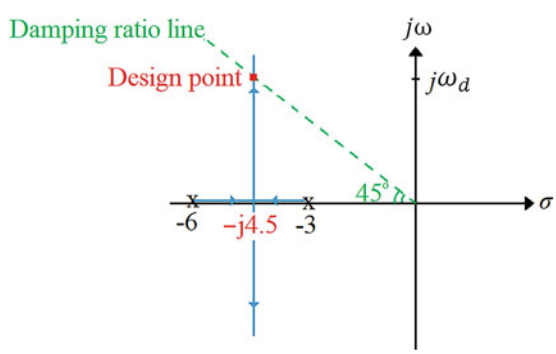

Figure 12.3 The control system of solution of problem 12.2

12.3. Based on the information given in the problem, the open-loop transfer function of the control system that includes a negative unity feedback is as follows:

$$G(s) = \frac{k}{s(1 + T_1 s)(1 + T_2 s)} \quad (1)$$

The characteristic equation of the closed-loop control system can be determined as follows:

$$1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{k}{s(1+T_1s)(1+T_2s)} \times 1 = 0 \Rightarrow \frac{T_1T_2s^3 + (T_1+T_2)s^2 + s + k}{s(1+T_1s)(1+T_2s)} = 0$$

$$\Rightarrow \Delta(s) = T_1T_2s^3 + (T_1+T_2)s^2 + s + k \tag{2}$$

The proportional controller gain (k_P) for a proportional-integral-derivative (PID) controller in Ziegler-Nichols's method can be determined as follows:

$$k_P = 0.6k_u \tag{3}$$

where k_u is the loop gain that puts the system in the oscillating status. In other words, this loop gain causes one of the rows of the Routh-Hurwitz table to be zero.

To apply Routh-Hurwitz rule, suppose that the characteristic equation of a system is as follows.

$$\Delta(s) = a_ns^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_2s^2 + a_1s^1 + a_0s^0 \tag{4}$$

The structure of Routh-Hurwitz table is presented in the following. As can be seen, the coefficients of the characteristic equation are placed on the first two rows of the table with the specific pattern. However, the coefficients of the next rows need to be determined by using (5) and (6), until the last row (s^0) is filled.

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ s^1 & & & & \\ s^0 & & & & \end{array}$$

$$b_{n-1} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} = \frac{a_{n-2}a_{n-1} - a_na_{n-3}}{a_{n-1}} \tag{5}$$

$$b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} = \frac{a_{n-4}a_{n-1} - a_na_{n-5}}{a_{n-1}} \tag{6}$$

For this problem, we have:

$$\begin{array}{c|cc} s^3 & T_1T_2 & 1 \\ s^2 & T_1+T_2 & k \\ s^1 & 1 - \dfrac{kT_1T_2}{T_1+T_2} & \\ s^0 & k & \end{array}$$

The row corresponding to s^1, can be zero as follows:

$$1 - \frac{kT_1T_2}{T_1+T_2} = 0 \Rightarrow k_u = k = \frac{T_1+T_2}{T_1T_2} \tag{7}$$

Solving (3) and (7):

$$k_P = \frac{3(T_1+T_2)}{5T_1T_2}$$

Choice (4) is the answer.

12.4. Based on the information given in the problem, the open-loop transfer function of the control system that includes a negative unity feedback is as follows:

$$G(s) = \frac{k}{s(s+1)^2} \tag{1}$$

The characteristic equation of the closed-loop control system can be determined as follows:

$$1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{k}{s(s+1)^2} \times 1 = 0 \Rightarrow \frac{s^3 + 2s^2 + s + k}{s(s+1)^2} = 0$$

$$\Rightarrow \Delta(s) = s^3 + 2s^2 + s + k \tag{2}$$

The integral time constant (T_I) for a proportional-integral-derivative (PID) controller in Ziegler-Nichols's method can be determined as follows:

$$T_I = 0.5T_u \tag{3}$$

where T_u is the time constant of the oscillations that can be determined from the auxiliary equation ($A(s^2)$). An auxiliary equation is the row of polynomial (corresponding to an even exponent) which is just above the row (corresponding to an odd exponent) containing only zeros in Routh-Hurwitz table.
Applying Routh-Hurwitz rule for this problem:

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 2 & k \\ s^1 & 1 - \frac{k}{2} & \\ s^0 & k & \end{array}$$

As can be seen, for $k = 2$, the row corresponding to s^1 becomes zero. Therefore, the auxiliary equation can be determined as follows:

$$A(s^2) = 2s^2 + 2 = 0 \Rightarrow s = \pm j \Rightarrow \omega_u = 1\ rad/\ sec$$

$$\Rightarrow T_u = \frac{2\pi}{\omega_u} = \frac{2\pi}{1} = 2\pi \tag{4}$$

Solving (3) and (4):

$$T_I = \pi$$

Choice (2) is the answer.

12.5. Based on the information given in the problem, $-1 \pm j2$ are the design points, and the open-loop transfer function of the control system is as follows:

$$G(s) = \frac{1}{s^2} \tag{1}$$

Figure 12.4 shows the root locus of the uncontrolled closed-loop system. As can be seen, the desirable closed-loop poles of $-1 \pm j2$ are in the left-side of the root locus. By using this point and the format of the controllers presented in the choices, it is noticed that the controller must be a lead controller that its format is as follows:

$$G_c(s) = k\frac{s+z}{s+p}, z < p \tag{2}$$

By assigning the zero of the controller at $s = -2$, the open-loop transfer function of the controlled system is as follows:

$$G(s)G_c(s) = k\frac{s+2}{s^2(s+p)} \tag{3}$$

Now, by using angle criterion in Fig. 12.5, we can write:

$$\theta_{z=-2} - 2\theta_{p=0} - \theta_p = \pm(2q+1)\pi$$

$$\Rightarrow \tan^{-1}\left(\frac{2}{1}\right) - 2\left(\pi - \tan^{-1}\left(\frac{2}{1}\right)\right) - \theta_p = -\pi$$

$$\Rightarrow \theta_p \approx \pi - 2\pi + 3\tan^{-1}\left(\frac{2}{1}\right) \approx -\pi + 3 \times 63.4 \approx 10.3^\circ \tag{4}$$

Next, in Fig. 12.5, for $\theta_p = 10.3^\circ$, we can write:

$$tan\,(10.3^\circ) \approx \frac{2}{p-1} \Rightarrow p - 1 \approx 11 \Rightarrow p \approx 12 \tag{5}$$

Solving (3) and (5):

$$G(s)G_c(s) = k\frac{s+2}{s^2(s+12)} \tag{6}$$

In addition, by using magnitude criterion, we have:

$$k = \left|\frac{1}{L(s)}\right|_{s=-1+j2} = \left|\frac{1}{G(s)G_c(s)}\right|_{s=-1+j2} = \left|\frac{s^2(s+12)}{s+2}\right|_{s=-1+j2}$$

$$k = \left|\frac{(-1+j2)^2(-1+j2+12)}{-1+j2+2}\right| \approx 25 \tag{7}$$

Solving (2), (5), and (7):

$$G_c(s) = 25\frac{s+2}{s+12}$$

Choice (1) is the answer.

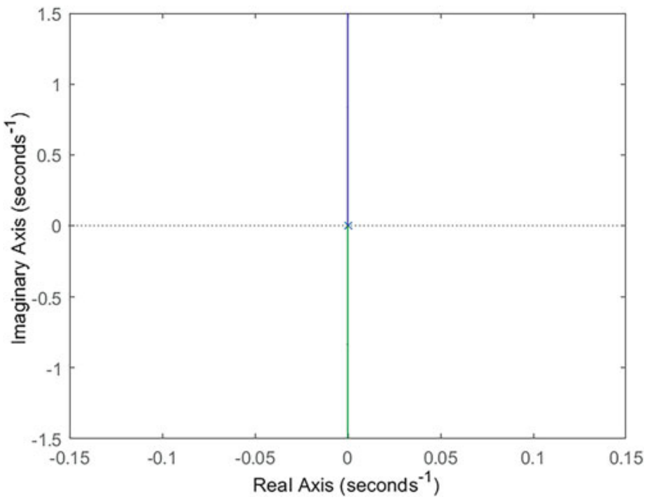

Figure 12.4 The control system of solution of problem 12.5

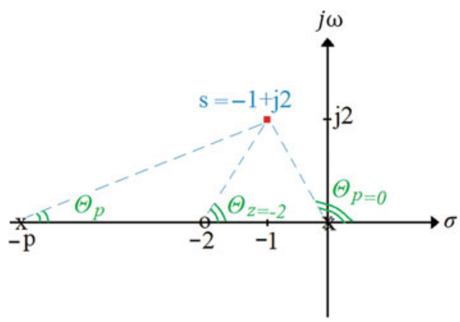

Figure 12.5 The control system of solution of problem 12.5

12.6. Based on the information given in the problem, we have:

$$\omega_{n2} = 10\, rad/s \quad (1)$$

$$\xi_1 = \xi_2 \quad (2)$$

The characteristic equation of the closed-loop control system can be determined as follows:

$$1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + \frac{25}{s(s+8)} = 0 \Rightarrow s^2 + 8s + 25 = 0 \quad (3)$$

The damping ratio of the system response can be determined by comparing the characteristic equation of the system (presented in (3)) with the standard second-order characteristic equation, that is, $\Delta(s) = s^2 + 2\xi\omega_n s + \omega_n{}^2$, as follows:

$$\begin{cases} \omega_n{}^2 = 25 \Rightarrow \omega_n = 5 \\ 2\xi\omega_n = 8 \end{cases} \Rightarrow \xi = \frac{8}{2 \times 5} \Rightarrow \xi_1 = 0.8 \quad (4)$$

Solving (2) and (4):

$$\xi_2 = 0.8 \quad (5)$$

By considering (1) and (5), the design points (desirable points) can be determined as follows:

$$s_2 = -\xi_2\omega_{n2} \pm j\omega_{n2}\sqrt{1-\xi_2{}^2} = -0.8\times 10 \pm j10\sqrt{1-0.8^2} = -8 \pm j6 \quad (6)$$

By using angle criterion for the design point of −8 + *j*6, we can write:

$$\text{Phase angle of } L(s)G_c(s) = \text{Phase angle of } \left(\frac{25}{s(s+8)}G_c(s)\right)\Bigg|_{s=-8+j6} = \pm(2q+1)\pi$$

$$\Rightarrow \textit{Phase angle of } \left(\frac{25}{(-8+j6)(\ j6)}G_c(-8+j6)\right) = -\pi$$

$$\Rightarrow -\theta_1 - \theta_2 + \theta_c = -\pi$$

$$-\left(\pi - \tan^{-1}\left(\frac{6}{8}\right)\right) - \frac{\pi}{2} + \theta_c = -\pi \Rightarrow -143^\circ - \frac{\pi}{2} + \theta_c = -\pi \Rightarrow \theta_c = 53^\circ$$

Therefore, a lead controller with the minimum lead angle of 53° must be used. Choice (4) is the answer.

12.7. Based on the information given in the problem, the system includes a negative unity feedback, and its open-loop transfer function is as follows:

$$G(s) = \frac{k}{s(s+4)(s+6)} \quad (1)$$

Moreover, we know that:

$$G_c(s) = s + z_c \quad (2)$$

$$O.S_1 = O.S_2 = 16\% \quad (3)$$

$$t_{s1} = 3.32 \ \ sec \quad (4)$$

$$t_{s2} = \frac{t_{s1}}{3} = \frac{3.32}{3} = 1.107 \ \ sec \quad (5)$$

As we know:

$$t_s = \frac{4}{\sigma} \quad (6)$$

Solving (5) and (6):

$$\sigma_2 = \frac{4}{1.107} = 3.613 \quad (7)$$

The damping ratio can be determined as follows:

$$\xi = \frac{-\ln\left(\frac{O.S}{100}\right)}{\sqrt{\pi^2 + \left(\ln\left(\frac{O.S}{100}\right)\right)^2}} \quad (8)$$

Solving (3) and (8):

$$\xi_2 = \frac{-\ln\left(\frac{16}{100}\right)}{\sqrt{\pi^2 + \left(\ln\left(\frac{16}{100}\right)\right)^2}} = 0.504 \quad (9)$$

By using (7) and (9) and considering $\omega_n = \frac{\sigma}{\xi}$, we have:

$$\omega_{n2} = \frac{\sigma_2}{\xi_2} = \frac{3.613}{0.504} = 7.168 \quad (10)$$

By considering (7) and (10), the design points (desirable points) can be determined as follows:

$$s_2 = -\sigma_2 \pm j\omega_{n2}\sqrt{1 - \xi_2{}^2} = -3.613 \pm j6.193 \quad (11)$$

In Fig. 12.6, by using angle criterion for the design point of −3.613 + *j*6.193, we can write:

$$\text{Phase angle of } L(s)G_c(s) = \text{Phase angle of } \left(\frac{1}{s(s+4)(s+6)}G_c(s)\right)\bigg|_{s=-3.613+j6.193} = \pm(2q+1)\pi$$

$$\text{Phase angle of } \left(\frac{1}{(-3.613+j6.193)(0.387+j6.193)(2.387+j6.193)}G_c(-3.613+j6.193)\right) = -\pi$$

$$\Rightarrow \theta_c - \theta_1 - \theta_2 - \theta_3 = -\pi$$

$$\Rightarrow \theta_c - \left(\pi - \tan^{-1}\left(\frac{6.193}{3.613}\right)\right) - \tan^{-1}\left(\frac{6.193}{0.387}\right) - \tan^{-1}\left(\frac{6.193}{2.387}\right) = -\pi$$

$$\Rightarrow \theta_c - 120.25 - 86.42 - 68.92 = -\pi$$

$$\Rightarrow \theta_c = -\pi + 120.25 + 86.42 + 68.92 \Rightarrow \theta_c = 95.6^\circ \quad (12)$$

Now, for $\theta_c = 95.6^\circ$, we can write:

$$\tan(\pi - 95.6^\circ) \approx \frac{6.193}{3.613 - z_c} \Rightarrow 3.613 - z_c = 0.61 \Rightarrow z_c \approx 3 \quad (13)$$

Solving (2) and (13):

$$G_c(s) \approx s + 3$$

Choice (3) is the answer.

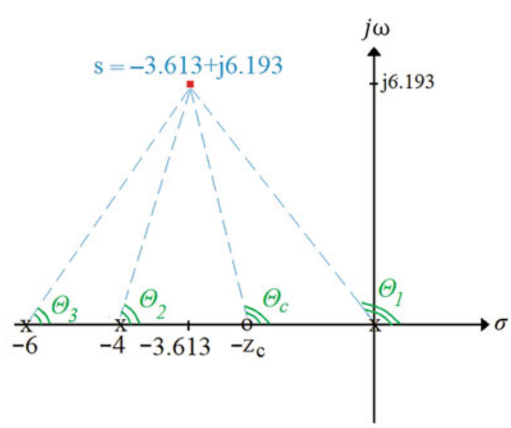

Figure 12.6 The control system of solution of problem 12.7

12.8. Based on the information given in the problem, the open-loop transfer function of the control system (see Fig. 12.7) is as follows:

$$G(s) = \frac{2}{(s+3)(s+6)} \tag{1}$$

Moreover, we know that the design objective and the requested controller are as follows:

$$\xi = 0.7 \tag{2}$$

$$G_c(s) = k_p + \frac{k_I}{s} = \frac{k_P\left(s + \frac{k_I}{k_P}\right)}{s} \tag{3}$$

From (1) and (3), we have:

$$G(s)G_c(s) = \frac{2k_P\left(s + \frac{k_I}{k_P}\right)}{s(s+3)(s+6)} \tag{4}$$

Let us assume:

$$\frac{k_I}{k_P} \triangleq 3 \tag{5}$$

Then we have:

$$G(s)G_c(s) = \frac{2k_P}{s(s+6)} \tag{6}$$

The root locus of the compensated system is shown in Fig. 12.8.

From (2), the angle of $\xi -$ line can be determined as follows:

$$\theta = \cos^{-1}\xi = \cos^{-1}0.7 = 45^\circ \tag{7}$$

By intersecting the $\xi -$ line with the root locus, shown in Fig. 12.9, the design point can be determined. As can be noticed, the $\xi -$ line intersects the vertical branch, which is the asymptote of the root locus. Therefore:

$$\sigma = \frac{0 - (0+6)}{2-0} = -3 \tag{8}$$

From Fig. 12.9, we can write:

$$\tan 45^\circ = \frac{\omega_d}{3} \Rightarrow \omega_d = 3 \tag{9}$$

From (8) and (9), the design point is:

$$s = -3 + j3 \tag{10}$$

By applying magnitude criterion, we have:

$$k_p = \left|\frac{1}{L(s)}\right|_{s=-3+j3} = \left|\frac{s(s+6)}{2}\right|_{s=-3+j3} \Rightarrow k_p = 9 \tag{11}$$

Solving (5) and (11):

$$k_I = 3 \times 9 = 27 \quad (12)$$

Solving (3), (11), and (12):

$$\Rightarrow G_c(s) = 9 + \frac{27}{s}$$

Choice (1) is the answer.

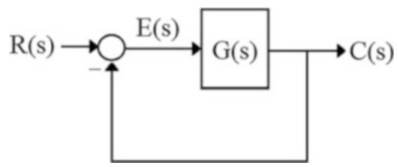

Figure 12.7 The control system of solution of problem 12.8

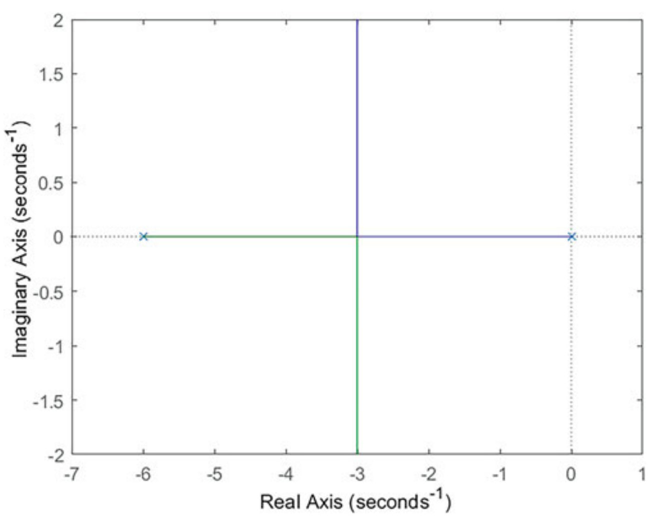

Figure 12.8 The control system of solution of problem 12.8

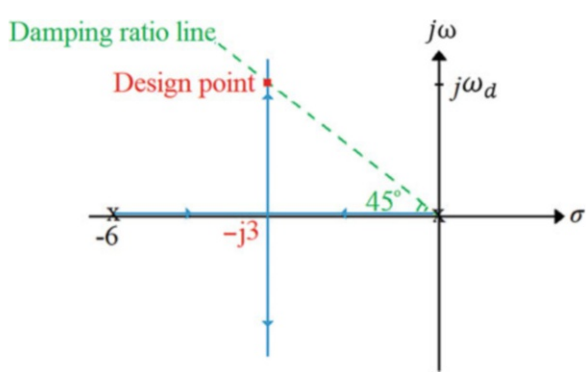

Figure 12.9 The control system of solution of problem 12.8

Index

M. Rahmani-Andebili, *Feedback Control Systems Analysis and Design*, https://doi.org/10.1007/978-3-030-95277-8