ADVANCES IN NUMBER THEORY AND APPLIED ANALYSIS

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Preface

Historically, number theory has been the oldest branch of mathematics which was initially studied for pure beauty of the subject rather than its applications. But with the rapid advancement of science and technology and encouragement toward interdisciplinary research, the demarcation between different branches of mathematics became more and more blurred. Number theory and analysis have been very closely linked with each other. Analytic number theory, theory of curved spaces, and complex analysis are just a few topics to be named where applications of both these branches are heavily exploited.

Presently, the exploration of the applications of different techniques and tools of number theory and mathematical analysis are extensively prevalent in various areas of engineering, mathematical, physical, biological and statistical sciences. This book presents the most recent developments in these two fields through contributions from eminent scientists and mathematicians worldwide.

The book focuses on the current state-of-the-art development in these two areas through original new contributions and surveys. As such, readers will find several useful tools and techniques to develop their skills and expertise in number theory and applied analysis. New research directions are also indicated in each of the chapters. This book is meant for graduate students, faculty and researchers willing to expand their knowledge in number theory and mathematical analysis. The readers of this book will require the minimum prerequisites of analysis, topology, number theory and functional analysis.

Our book is an effort toward presenting the two very important topics in modern mathematics and their applications in diverse areas of science. This book consists of 18 chapters. The first half (Chapters 1–8) of the book deals with applications of number theory, whereas the second half (Chapters 9–18) will elaborate recent applications of mathematical analysis. The first two chapters present surveys on open problems in number theory, properties of partition function and their potential future research directions. The third chapter presents a discussion on the two old standing number theory problems. The fourth chapter is about zeta-functions and allied theta-functions. Sylvester sums on the Frobenius set in arithmetic progression with initial gaps are discussed in Chapter 5. Chapter 6 presents arithmetic properties of minimal excludants of partitions of integers. In Chapter 7, a survey on ℓ-regular partitions is given with recent developments. Chapters 8 elucidates some applications of an algorithm in number theory. From Chapter 9 onwards, the studies in the applications of mathematical analysis have been listed. Chapter 9 presents the component exponential function in scator hypercomplex space. Chapters 10–12 explores the applications of analysis in fixed point results, while Chapter 13 describes sufficient conditions for Mittag-Leffler functions associated with conic regions. Chapters 14–17 respectively investigate controlled optimization problems, integrable function spaces, Borel distribution series and conjugate method for solving nonlinear system of equations. Finally, Chapter 18 presents a closed form of integral transforms in terms of Lauricella function and their numerical simulations.

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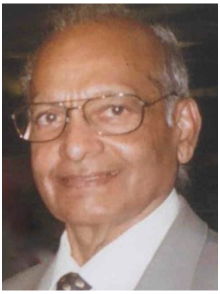
About the Editors

Pradip Debnath is an Assistant Professor in Mathematics at the Department of Applied Science and Humanities, Assam University, Silchar (a central university). He received his PhD in Mathematics from the National Institute of Technology Silchar, India. His research interests include fixed point theory, nonlinear functional analysis, soft computing and mathematical statistics. He has published more than 60 papers in various journals of international repute and is an active reviewer for more than 40 international journals. He is also a reviewer for *Mathematical Reviews* published by the American Mathematical Society. He is the Lead Editor of the books *Metric Fixed Point Theory: Applications in Science, Engineering and Behavioural Sciences* (2021, Springer Nature), *Soft Computing Techniques in Engineering, Health, Mathematical and Social Sciences* (2021, CRC Press), *Fixed Point Theory and Fractional Calculus: Recent Advances and Applications* (2022, Springer Nature) and *Soft Computing: Recent Advances and Applications in Engineering and Mathematical Sciences* (2023, CRC Press). He is a topical advisory panel member of the journal *Axioms* and a Guest Editor of the special issue entitled "Nonlinear Functional Analysis in Natural Sciences" in the same journal. He has successfully guided PhD students in the areas of nonlinear analysis, soft computing and fixed point theory. He has recently completed a major Basic Science Research Project in fixed point theory funded by the UGC, the Government of India. Having been an academic gold medalist during his post graduation studies from Assam University, Silchar, Dr. Debnath has qualified several national-level examinations in mathematics in India.



Hari M. Srivastava is Professor Emeritus in the Department of Mathematics and Statistics at the University of Victoria in Canada since 01 July 2006. He has been Clarivate Analytics [Thomson-Reuters] (Web of Science) Highly Cited Researcher for the years 2015, 2017, 2018, 2019, 2020, 2021 and 2022. He has also been listed and ranked in the sixth place in General Mathematics among the Top 2% scientists in the world. In fact, it is not quite possible to list out his tremendous academic achievements in a few sentences. The details of the same may be found in his regularly-updated website www.math.uvic.ca/~harimsri/.

Kalyan Chakraborty is at present the Director of the Kerala School of Mathematics, India. Previously, he was working as a Professor at Harish-Chandra Research Institute (HRI), Allahabad, India, where he also obtained his PhD in Mathematics. Professor Chakraborty was a postdoctoral fellow at IMSc, Chennai, and at Queen's University, Canada, and a visiting scholar at the University of Paris VI, VII, France; Tokyo Metropolitan University, Japan; Universitá Roma Tre, Italy; the University of Hong Kong, Hong Kong; Northwest University and Shandong University, China; Mahidol University, Thailand; Mandalay University, Myanmar; and many more. His broad area of research is number theory, particularly class groups, Diophantine equations, automorphic forms, arithmetic functions, elliptic curves, and special functions. He has published more than 60 research articles in respected journals and two books on number theory and has been on the editorial boards of various leading journals. Professor Chakraborty is Vice President of the Society for Special Functions and their Applications.

Poom Kumam received the BS, MSc, and PhD degrees in mathematics from Burapha University (BUU), Chiang Mai University (CMU), and Naresuan University (NU), respectively. In 2008, he received a grant from Franco-Thai Cooperation for a short-term visit to the Laboratoire de Mathematiques, Universite de Bretagne Occidentale, France. He was also a Visiting Professor for a short-term research with Professor Anthony To-Ming Lau at the University of Alberta, AB, Canada. He is currently a Full Professor with the Department of Mathematics, King Mongkut's University of Technology Thonburi (KMUTT), where he is also the Head of the KMUTT Fixed Point Theory and Applications Research Group since 2007 and also leading the Theoretical and Computational Science Center (TaCS-Center) in 2014 (now TaCS-Center of Excellence in 2021). He has successfully advised five Master's, and 44 PhD graduates. He had won some of the most important awards for mathematicians. The first one is the TRF-CHE-Scopus Young Researcher Award in 2010 which is the award given by the corporation from three organizations: Thailand Research Fund (TRF), the Commission of Higher Education (CHE), and Elsevier Publisher (Scopus). The second award was in 2012 when he received the TWAS Prize for Young Scientist in Thailand, which is given by the Academy of Sciences for the Developing World TWAS (UNESCO) together with the National Research Council of Thailand. In 2014, the third award is the Fellowship Award for Outstanding Contribution to Mathematics from International Academy of Physical Science, Allahabad, India. In 2015, Dr. Poom Kumam has been awarded Thailand Frontier Author Award 2015, an award for outstanding researcher who has published works and has often been used as a reference or evaluation criteria of the database Web of Science. Moreover, in 2016, Dr. Poom Kumam has been awarded 2016 Thailand Frontier Researcher Awards on Innovation Forum: Discovery, Protection, Commercialization By Intellectual Property & Science, and Thomson Reuters. Dr. Poom Kumam has been a highly cited researcher (HCR 2015, 2016, 2017). Moreover, he has received KMUTT-HALL OF FAME 2017, in honour of the recipients of academic awards, KMUTT Young Researcher Awards, Excellence in Teaching Awards for 2016.

In 2019, he received 2019 CMMSE Prize Winner: The CMMSE prize is given to computational researchers for important contributions in the developments of numerical methods for physics, chemistry, engineering and economics from CMMSE Conference June 30–July 6, 2019, Rota, Cadiz, Spain. He has also been listed and ranked in the 197th Place in General Mathematics among the Top 2% Scientists in the World 2021 (Published by Stanford University in USA).

He served on the editorial boards of various international journals and has also published more than 800 papers in Scopus and Web of Science (WoS) databases the also delivers many invited talks on different international conferences every year all around the world. Furthermore, his research interest focuses on fixed point theory, fractional differential equations and optimization related with optimization problems in both pure science and applied science.

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Chapter 1

Open Problems in Number Theory

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In spite of being one of the oldest branches of mathematics, number theory has an abundance of unsolved and open problems. It will require multiple volumes of a book to list out even a portion of those open problems. Our aim in this introductory chapter is to provide brief commentary about the most celebrated open problems and the progress toward their solutions. New research directions are also indicated accordingly.

1. Introduction

Number theory primarily deals with attributes of the integers and more specifically with the positive integers (i.e. natural numbers). This nomenclature is considered as a misnomer because by the word *number*, the early Greeks meant only the positive integers.

A number greater than 1 is prime if its only positive divisors are 1 and itself. It is also well known that every positive integer greater than 1 can be represented as a product of primes and the representation is unique apart from the order in which the prime factors occur.

The set of positive integers can be partitioned into three classes:

- The unit 1
- The prime numbers $2, 3, 5, 7, 11, 13, 17, \ldots$
- The composite numbers $4, 6, 8, 9, 10, 12, 14, \ldots$

The primes have fascinated mathematicians since the time of Euclid who proved that there is an infinitude of them.

Theorem 1 (Euclid, 350 BC). *There are infinitely many primes.*

Proof. Suppose that there are finite number of primes and $p_1, p_2, p_3, \ldots, p_n$ is the finite list of primes. Consider the product $P = p_1 p_2 p_3 \ldots p_n$ and let $K = P + 1$. Since $K > 1$, it must have a prime divisor p. Hence, p must be one of $p_1, p_2, p_3, \ldots, p_n$. Now, since p divides K and the product P, we must have p divides 1, a contradiction. Therefore, there are infinitely many primes. $\square$

Dirichlet's theorem further extends Euclid's result.

Theorem 2 (Dirichlet, 1837). *Let $a, d \in \mathbb{Z}$ and $gcd(a, d) = 1$. Then there are infinitely many primes in the sequence $a, a + d, a + 2d, \ldots, a + nd. \ldots$ for $n \in \mathbb{N}$.*

Alternately, Dirichlet's theorem states that there are infinitely many primes congruent to a modulo d. Further, the numbers of the form $a + nd$ generate an arithmetic progression and this theorem asserts that this sequence contains infinitely many primes.

Dirichlet's theorem also gives assurance to the existence of primes of particular form. For instance, it ensures that there are infinite number of primes ending with 777 such as $1777, 1000777, \ldots$ because these numbers belong to the arithmetic progression $a + nd$, where $gcd(777, 1000) = 1$.

The most celebrated result concerning the distribution of primes is however the famously known prime number theorem, which was independently proved by Hadamard and Poussin.

Theorem 3 (Hadamard and Poussin, 1896). *Let $\pi(x)$ be the number of primes less than or equal to x. Then $\frac{x}{\log x}$ is a good approximation to $\pi(x)$ in the sense that*

$$\lim_{x \to \infty} \frac{\pi(x)}{\left[\frac{x}{\log x}\right]} = 1.$$

Using asymptotic notation, in 1792, Gauss stated the prime number theorem as

$$\pi(x) \approx \frac{x}{\log x}.$$

Later, Gauss refined his estimate to $\pi(x) \approx Li(x)$, where $Li(x) = \int_2^n \frac{dx}{\ln x}$ is the logarithmic integral.

In spite of being one of the ancient branches of mathematics, number theory has an abundance of unsolved and open problems. It will require multiple volumes of a book to list out even a portion of those open problems. Our aim in this chapter is to provide a brief commentary about the most celebrated open problems and the progress toward their solutions.

From the following section onwards, we present our discussion on some famous unsolved and open problems in number theory.

2. Twin Prime Conjecture

A pair of primes are said to be twin primes if their difference is 2, i.e. the pair is of the form $(p, p+2)$. The first few twin prime pairs are $(3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61), (71,73), (101,103), (107,109), (137,139), \ldots$.

The question of whether or not there are infinitely many twin primes has remained unsolved till date. The **twin prime conjecture** states that *there are infinitely many primes p such that $p+2$ is also a prime.*

In 1849, a more general conjecture was made by de Polignac that for every natural number k, there are infinitely many primes p such that $p+2k$ is a prime. Obviously, when $k=1$, de Polignac's conjecture reduces to the twin prime conjecture.

The following important results related to twin primes are consequences of Wilson's theorem [$(p-1)! \equiv -1 \mod p, p$ is a prime]:

- **Clements, 1949:** $m, m+2$ are twin primes if and only if $4[(m-1)!+1]+m \equiv 0 \mod m(m+2)$.
- **Sergusov, 1971:** $m, m+2$ are twin primes if and only if $\phi(k)\sigma(k) = (k-3)(k+1)$, where $k = m(m+2)$ and ϕ is Euler's totient function (i.e. $\phi(k)$ counts the number of positive integers up to k that are relatively prime to k) and $\sigma(k)$ is the sum of positive divisors of k (including 1 and k).

It is a well known result due to Euler that the infinite series $\sum \frac{1}{p}$ over all primes diverges.

However, in 1919, Norwegian mathematician Viggo Brun proved a remarkable theorem which states that the sum of reciprocals of all the twin primes converges to a finite value known as Brun's constant which is approximately equal to 1.90216054. Formally, this theorem may be stated as follows.

Theorem 4 (Brun's theorem, 1919). *If $\mathbb{P}$ denotes the set of twin primes, then the summation*

$$\sum_{p,p+2\in\mathbb{P}} \left(\frac{1}{p} + \frac{1}{p+2}\right) = \frac{1}{3} + \frac{1}{5} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} \cdots$$

either has finitely or infinitely many terms but converges to Brun's constant.

Chinese mathematician Jing-Run Chen, in 1966 (published in 1978) proved the following which is quite close to proving that there are infinite number of twin primes:

Theorem 5 (Chen, 1978). *There are infinitely many primes p such that $(p+2)$ is either prime or the product of two primes.*

In 2013, Chinese mathematician Yitan Zhang came out with a breakthrough result which is also known as the weak conjecture of twin primes. His theorem roughly states that there are infinitely many pairs of primes that differ by a positive integer which is less than 70 million.

Theorem 6 (Zhang, 2013). *There exists an even integer $M \geq 2$ with the property that there are infinitely many primes of the form $(p, p+M)$. In fact, there exists an M with $M \leq 7 \times 10^7$.*

It was only a matter of time that the upper bound of 70 million in Zhang's theorem got reduced significantly. By July 2013, Australian mathematician Terence Tao, via his Polymath8 project, had a remarkable contribution in reducing this upper bound from 70 million to merely 4680.

In November 2013, British mathematician James Maynard presented an independent proof that pushed down the gap in Zhang's theorem to 600. In 2014, Indo-Canadian mathematician M. R. Pedaprolu Murty at Queen's University also contributed to major developments in this direction. As of April 2014, the gap has been reduced to 246 using the methods of Maynard and Tao. The reduction of the gap up to 2 is still awaited.

3. The Goldbach Conjecture

In 1742, German mathematician Christian Goldbach came up with the conjecture that *every positive even integer greater than 2 is the sum of two prime numbers*. Despite considerable efforts, this conjecture remains unproven till date.

With the help of computer programming, as of 2013, the conjecture has been verified completely to hold true for all integers up to 4×10^{18}.

The weak Goldbach conjecture states that *every odd positive integer greater than 5 can be written as the sum of three primes*. Vinogradov's three primes theorem (1930) is worth mentioning here which states that any sufficiently large odd integer can be represented as the sum of three primes. After enormous efforts, in 2002, the weak Goldbach conjecture was verified for odd numbers greater than 2×10^{1346}. In 2013, the Peruvian mathematician Harald Helfgott came up with his magnificent proof that the weak Goldbach conjecture is true.

4. The Riemann Hypothesis

By $\Re(s)$ and $\Im(s)$, we denote the real and imaginary parts of the complex variable s, respectively. In 1859, Riemann published a groundbreaking article in which he obtained an analytic formula for the number of primes up to a given limit by introducing the Riemann zeta function as a function of the complex variable s, defined in the complex half-plane $\Re(s) > 1$ in terms of the absolutely convergent infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

Riemann showed that $\zeta(s)$ can be extended to $\mathbb{C}$ as a meromorphic function having a simple pole at $s = 1$ with residue 1 and can be derived from the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

The zeros of the zeta function, i.e. the solutions $\omega \in \mathbb{C}$ of the equation $\zeta(\omega) = 0$, play a pivotal role in the representation of this formula. In this article, Riemann established the complex valued function ξ defined in the following manner:

$$\xi(\tau) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

with $s = \frac{1}{2} + i\tau$. He further established that $\xi(\tau)$ is an even entire function of τ and the imaginary parts of those zeros lie between $-i/2$ and $i/2$. He also speculated that in the range between 0 and T, the function $\xi(\tau)$ has approximately $(\frac{T}{2\pi})\log(\frac{T}{2\pi}) - \frac{T}{2\pi}$ zeros. Riemann stated that within that range, all zeros were likely to be real.

It has been proven that the Riemann zeta function has zeros at the negative even integers, i.e. $\zeta(s) = 0$ for $s = -2, -4, -6, \ldots$ and these zeros are known as trivial zeros of the zeta function. The other roots of the zeta function are the complex numbers $\frac{1}{2} + i\alpha$ where α is a root of $\xi(\tau)$. The Riemann hypothesis is all about the locations of the nontrivial zeros.

Riemann hypothesis. *The nontrivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.*

The line $\frac{1}{2} + it, t \in \mathbb{R}$ is called the critical line and the domain $S = \{\rho + it \in \mathbb{C} : \rho \in [0,1], t \in \mathbb{R}\}$ is called the critical strip. It is a proven fact that there are no nontrivial zeros outside the critical strip.

In the year 2000, Clay Mathematics Institute selected seven well-known problems known as The Millennium Prize Problems and pledged US dollar 1 million prize for the correct solution of any of them. Riemann hypothesis is one of those seven problems.

Zeros of the Riemann zeta function are closely linked with the distribution of primes. However, the uses and applications of the Riemann hypothesis is vastly widespread in different branches of science.

There has been a colossal research toward proof of the hypothesis. We list here some of the major developments in this direction:

- **Hardy, 1914:** There are infinitely many zeros of the Riemann zeta function on the critical line.
- **Selberg, 1942:** If $N(\zeta)$ denotes the number of zeros of $\zeta(s)$ on the critical line $\frac{1}{2} + i\tau$ such that $\tau \in (0, T)$, then $N(\zeta) > C\frac{T}{2\pi}\log(\frac{T}{2\pi})$ for some constant $C \in (0, 1)$.
- $\zeta(s)$ has no nontrivial zeros outside the critical strip.
- **Levinson, 1974:** On the critical line, $\zeta(s)$ has at least one-third of its total nontrivial zeros. The bound was further improved to two-fifth by Conrey in 1989.
- **Van de Lune, Riele and Winter** in 1986 computed the first 1,500,000,001 nontrivial zeros of $\zeta(s)$ and speculated that they all were simple zeros.
- **Ramanujan** in 1914 gave an astonishing formula (without proof) for $\zeta(2n+1)$ which states that if $\alpha, \beta > 0, \alpha\beta = \pi^2$, then

$$\alpha^{-n}\left(\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k}-1}\right) = \beta^{-n}\left(\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\right) - 2^{2n}\sum_{k=0}^{n+1}(-1)^k\frac{B_{2k}B_{2n+2-2k}}{(2k)!(2n+2-2k)!}\alpha^{n+1-k}\beta^k,$$

  where B_{2k} denotes the $2k$th Bernoulli number. Berndt gave a complete proof of this formula in 1977.
- Finding the exact order of the error term in the prime number theorem is one of the most famous and important problems in number theory. In 1901, Koch showed that if the Riemann hypothesis were true, then

$$\pi(x) = Li(x) + O(\sqrt{x}\ln x),$$

  where $O(x)$ is the asymptotic notation Big-O.
- There have been numerous claims for the proof of Riemann hypothesis, but none of them happened to be correct. In 2018,

British mathematician Michael Atiyah at the Heidelberg Laureate Forum gave a lecture in which he claimed to have proved the hypothesis. But unfortunately, his proof also turned out to be another failed attempt.

5. The Mersenne Primes

Named after the 17th century French mathematician Martin Mersenne, a Mersenne prime is a prime number which is one less than a power of two. Hence, these are prime numbers of the form $M_n = 2^n - 1$ for some positive integer n. When the primality condition is dropped, a number of this form is called a Mersenne number.

In his book *Cogitata Physica-Mathematica* (1644), Mersenne (incorrectly) conjectured that M_p is prime for $p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$ and composite for all other primes less than 257. More than 300 years later, by 1947, Mersenne's list was corrected to

$$p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127.$$

Thus, Mersenne erroneously conjectured that M_{67} and M_{257} are primes and missed M_{61}, M_{89} and M_{107} from his predicted list. Since 1997, all the newly discovered Mersenne primes have been obtained by the *Great Internet Mersenne Prime Search*, which is a distributed computing project. As of 2020, 51 Mersenne primes have been discovered and the 51st is $2^{82,589,933} - 1$.

There was another conjecture that *if M_n is a prime, then so is M_{M_n}*. If this conjecture were true, it would mean that there are infinitely many Mersenne primes. However, in 1953, a computer search proved that $M_{M_{13}}$ is composite.

Hence, there remain several open problems about Mersenne primes:

- Is the number of Mersenne primes infinite?
- Is it that every Mersenne number happens to be square free?
- Are there infinitely many composite Mersenne numbers?

A positive integer n is said to be perfect if n is equal to the sum of all its positive divisors, excluding n itself.

The numbers $6, 28, 496, 8128, \ldots$ are perfect. However, all the perfect numbers discovered till date are even. A famous open question is as follows:

- Are there odd perfect numbers?

6. The Fermat Numbers

Another class of numbers that provide us with a rich source of open problems and conjectures is the Fermat numbers.

A Fermat number is a number of the form $F_n = 2^{2^n} + 1$ for $n \geq 0$. If F_n is prime, it is called a Fermat prime.

Fermat observed that $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$ are all primes and believed that F_n is a prime for each $n \geq 0$. However, this belief was resolved in the negative by Euler in 1732 as he observed that $F_5 = 429, 496, 7297$ is divisible by 641.

Similar to Mersenne numbers, we have the following open questions:

- Is the number of Fermat primes infinite?
- Is it true that every Fermat number is square free?
- Is the number of composite Fermat numbers infinite?

7. Some Other Famous Open Problems

It is quite clear from the present discussion that one could make a conjecture that there are infinitely many open problems in number theory. In this section, we list out some other famous open problems:

- The Fibonacci numbers may be defined by the following recurrence relation:

  $$F_0 = 1, F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n+2}$$

  for $n > 1$.

  Thus, Fibonacci numbers form a sequence in which each number is the sum of the two preceding ones. A prime number is said to be Fibonacci prime if it appears in the Fibonacci sequence.

An important open question is as follows: *are there infinitely many Fibonacci primes?*

- If an odd prime number p does not divide the numerator of the Bernoulli number B_k, for all even $k \leq p-3$, then it is called regular. An irregular prime is an odd prime that is not regular. The odd numbers $3, 5, 7, \ldots, 31$ are all regular. 37 is the first irregular prime. The infinitude of irregular primes is a well-known fact. However, the following question is open: *is the number of regular primes infinite*?
- The Euler constant γ is defined as

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right).$$

  It is an open problem whether the Euler constant γ is irrational or rational. The question about γ being transcendental is also an open problem.
- The set of Gaussian integers $\mathbb{Z}[i]$ is a unique factorization domain (UFD). However, there exist $m \in \mathbb{Z}$ such that $\mathbb{Z}[m]$ is not a UFD. Exactly nine negative integers have been discovered, known as Heegner numbers, for which $\mathbb{Z}[m]$ is a UFD. These numbers are

$$-1, -2, -3, -7, -11, -19, -43, -67, -163.$$

  It is still not established whether there are infinitely many positive integers m so that $\mathbb{Z}[m]$ is a UFD.
- For the Riemann zeta function $\zeta(s)$, the value of $\zeta(3)$ is well known as the Apéry constant and it is an irrational number. The question about Apéry constant being transcendental is an open problem. Further, the values of $\zeta(2k+1)$, for all $k \in \mathbb{N}$, being irrational or transcendental also remains unsolved.
- Lothar Collatz in 1937 introduced an idea which is popularly known as Collatz conjecture or $3n+1$ problem. Paul Erdös, who offered USD 500 for its correct solution, stated about this conjecture as follows: "Mathematics may not be ready for such problems."

  To state the problem formally, choose a positive integer a_0 and construct the sequence $\{a_n\}$ as

$$a_{n+1} = \begin{cases} \dfrac{a_n}{2}, & \text{if } a_n \text{ is even,} \\ 3a_n + 1, & \text{if } a_n \text{ is odd.} \end{cases}$$

For example, if we choose $a_0 = 1$, we obtain the sequence $1, 4, 2, 1, \ldots$, which repeats in the triplet 1, 4, 2. The Collatz conjecture states that if we initiate with any positive number a_0, the sequence will eventually take the value 1. The conjecture remains unproved till date.

- Lagrange's four-square theorem of 1770 states that every natural number can be expressed as the sum of four integer squares. In the same year, Waring raised the question: how about the minimum number of cubes to do the same? More generally, he wanted to obtain the minimum number of kth powers necessary to express all positive integers. This open problem is known as Waring's problem.

Acknowledgments

For this chapter, much information has been collected from Refs. 1–8 and the references therein.

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Chapter 2

Arithmetic and Congruence Properties of Partition Function

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The purpose of this chapter is to present the systematic development and evolution of the partition function. We particularly discuss the arithmetic and congruence properties. Various types of identities are discussed as well. Finally, we recapitulate different types of Ramanujan congruences and also put forward some new generalizations of Ramanujan congruences on $p(n)$ and $p_j(n)$.

1. Introduction

In mathematics, number theory always occupied a unique position due to its historical importance. It is one of the oldest branches of mathematics. Starting from the Babylonians and then from Pythagoras to Ramanujan, the subject has undergone a huge evolution. The Euclidean Algorithm, developed by Euclid, is an efficient tool toward the application of the Division Algorithm.

In this chapter, we present the systematic development and evolution of the partition function, and discuss the arithmetic and congruence properties and also various types of identities. Finally, different types of Ramanujan congruences are recapitulated and some new generalizations of Ramanujan's congruences on $p(n)$ and $p_j(n)$ are also recommended.

2. Partitions

Partition[1] of a positive integer n is a way by which the number can be expressed as a sum of one or more positive integers.

For example, 3 can be expressed as $3; 1+2; 1+1+1$. Thus, the number of partitions of 3 is 3.

We can denote the number of the partition of n positive number as $p(n)$.

In the following sections, we discuss various types of partitions.

2.1. *Unrestricted partitions*

When in a partition, there is no restriction of any kind on the number or size of the parts, such a partition is called unrestricted partitions.

For example, the positive integer 5 can be expressed as a sum in the following seven ways:

$$5; 1+4; 2+3; 1+2+2; 1+1+3; 1+1+1+2; \text{ and } 1+1+1+1+1.$$

These are all the partitions of 5 and these partitions are unrestricted partitions of 5.

The number of unrestricted partitions of any given positive integer n is denoted by $p(n)$, and for convenience, we consider $p(0) = 1$.

2.2. *Restricted partitions*

If some restrictions are imposed in a_i (like all $a_i^{'}$s are positive), then such types of partitions are known as restricted partitions. Restrictions may be classified as follows:

(1) repetitions of the integers being allowed or not,
(2) partitions into distinct parts,
(3) partitions into odd parts,
(4) partitions into even parts,
(5) partitions into a specified number of parts,
(6) partitions into a specified number of distinct parts.

Besides these, the restrictions can be placed on the size or on the number of parts or both.

Consider an example, where the number $g(n, m, h, k)$ of partitions of n into exactly k summands, each $\leq m$, any h of the positive integers $\leq m$ being used as summands in any partition. Then we have

$$\sum_{n=1}^{\infty} g(n, m, h, k) = \binom{k-1}{h-1}\binom{m}{h}.$$

We denote $p(n, k)$ as the number of partitions of n into exactly k summands, $q(n)$ as the number of partitions of n into distinct parts and $q(n, k)$ as the number of partitions of n into exactly k distinct parts.

2.3. *Perfect partitions*

A partition of n is said to be perfect when it contains just one partition of every number up to n. Hence, one perfect partition of 7 is $1+2+2+2$ because every number up to 7 can be expressed uniquely as a sum by using the summands: $1, 2, 2, 2$. Other such partitions of 7 are as follows:

$$1+1+1+1+1+1+1; 1+1+1+4; 1+2+4.$$

2.4. *Plane and solid partitions*

For a partition of n if the summands are arranged in the form of a matrix so that the elements in each row and in each column are in a descending order, then such a type of arrangement is called a plane partition of n.

Thus,

3	3	2	1
2	2	2	1
1	1		

is a plane partition of 18.

A plane partition is also known as a rowed partition. A k-rowed partition of n includes all partitions of n with k or fewer number of rows.

The number of k-rowed partitions of n is denoted by $t_k(n)$.

In a solid partition, the summands are arranged in a three-dimensional space as they are arranged in a two-dimensional space in the case of a plane partition. The elements are arranged in descending order of magnitude in each of the three principal directions.

2.5. *l-regular partitions*[2]

Let l be a prime that satisfies $p(ln+b) \equiv 0 \pmod{l}$ for $0 < l < b$. Then a partition of n is said to be l-regular if none of its parts are multiples of l. For example, 3-regular partitions of 7 are as follows:

$$7; 5+2; 5+1+1; 4+2+1; 4+1+1+1; 2+2+2+1; 2+2+1+1+1; 2+1+1+1+1+1; 1+1+1+1+1+1+1,$$

l-regular partitions of n by $b_l(n)$.

2.6. *Overpartitions*[3]

An overpartition of the positive integer n is an ordinary partition of n where the first occurrence of parts of each size may be overlined. For example, the overpartitions of the integer 3 are as follows:

$$3; \overline{3}; 2+1; \overline{2}+1; 2+\overline{1}; \overline{2}+\overline{1}; 1+1+1 \text{ and } \overline{1}1+1+1$$

The number of overpartitions of n is denoted by $\overline{p}(n)$, and $\overline{p_0}(n)$ is the number of overpartitions of n in which only odd parts are used.

Hence, $\overline{p}(3) = 8$ and $\overline{p_0}(3) = 4$.

3. Identities

In 1742, Euler developed an expression for the study of partitions which is eagerly known as Euler's identity:

$$\prod_{r=1}^{\infty}(1-x^r) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots$$
$$+ (-1)^k \left\{ x^{\frac{k(3k-1)}{2}} + x^{\frac{k(3k+1)}{2}} \right\} + \cdots$$
$$= \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k+1)}{2}}$$

In 1830, Legendre noted that Euler's identity implies that every number which is not pentagonal can be partitioned into an even number of distinct parts as often as it can be partitioned into an odd number of distinct integers, while the pentagonal number $k(3k+1)/2$ can be partitioned into an even number of distinct parts once oftener or once fewer times than into an odd number of distinct parts based on whether k is even or odd.

In 1843, Cauchy provided an expression as follows:

$$\prod_{r=0}^{n-1}(1+zx^r) = 1 + \frac{1-x^n}{1-x}z - \frac{(1-x^n)(x-x^n)}{(1-x)(1-x^2)}z^2 + \cdots + \frac{(1-x^n)(x-x^n)(x^2-x^n)\dots(x^{n-1}-x^n)}{(1-x)(1-x^2)(1-x^3)\dots(1-x^n)}z^n$$

and

$$\prod_{r=0}^{n-1}(1-zx^r)^{-1} = 1 + \frac{1-x^n}{1-x}z - \frac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)}z^2 + \frac{(1-x^n)(1-x^{n+1})(1-x^{n+2})}{(1-x)(1-x^2)(1-x^3)}z^3 + \cdots$$

This expression was known as Cauchy's identity and one can obtain the Euler's identity from this identity.

Simultaneously, in 1929, Jacobi put forward the applications of elliptic functions to the study of the theory of partitions:

$$\prod_{r=1}^{\infty}(1-x^{2r})(1+zx^{2r-1})(1+z^{-1}x^{2r-1}) = 1 + x(z+z^{-1}) + x^4(z^2+z^{-2}) + x^9(z^3+z^{-3}) + \cdots$$

This new expression was called as Jacobi's identity and it is verified that Euler's identity is a particular case of Jacobi's identity.

In 1966, Sudler[4] provided two enumerative proofs of Jacobi's identity.

In the following section, we discuss some interesting identities known as Ramanujan's Identities.

3.1. *Ramanujan's identities*

Ramanujan established two remarkable identities without proof:

$$p(4) + p(9)x + p(l4)x^2 + \cdots = \frac{5\{f(x)\}^6}{\{f(x)^5\}^5},$$

$$p(5) + p(12)x + p(l9)x^2 + \cdots = \frac{7\{f(x)\}^4}{\{f(x)^7\}^3} + \frac{49x\{f(x)\}^8}{\{f(x)^7\}^7}.$$

Further, these two identities imply the congruences of Ramanujan for the moduli 5, 5^2, 7, 7^2. In 1950, Kruyswijk[5] proved the proof of these two Ramanujan identities using power series.

3.2. *Rogers–Ramanujan identities*

In 1913, Ramanujan rediscovered Rogers's identities:

$$\prod_{r=0}^{\infty}\{(1-x^{5r+1})(1-x^{5r+4})\}^{-1} = 1 + \prod_{r=1}^{\infty}\frac{x^{r^2}}{(1-x)(1-x^2)\ldots(1-x^r)}$$

and

$$\prod_{r=0}^{\infty}\{(1-x^{5r+2})(1-x^{5r+3})\}^{-1} = 1 + \prod_{r=1}^{\infty}\frac{x^{r(r+1)}}{(1-x)(1-x^2)\ldots(1-x^r)}.$$

These two identities are first found by Rogers[6] in 1894, and later on, these identities are known as Rogers–Ramanujan identities. Many mathematicians like Dobbie[7] in 1962 and Alder[8] in 1954 generalized these identities. Simultaneously, in 1961, Gordon[9] and in 1966 Andrews[10,11] generalized these identities in different directions.

4. Generating Function

The function

$$f(x) = \prod_{r=1}^{\infty}(1-x^r)^{-1} \tag{1}$$

is called the generating function for $p(n)$.

The generating function for $q(n)$ is

$$g(x) = \prod_{r=1}^{\infty}(1 + x^r).$$

When a set A consists of all the positive integers, the generating function for $p(n, k)$ is

$$x^k \prod_{r=1}^{k}(1 - x^r)^{-1}.$$

Similarly, the generating function for $q(n, k)$ is

$$x^{\frac{k(k+1)}{2}} \prod_{r=1}^{k}(1 - x^r)^{-1}.$$

4.1. *Generating function for unrestricted partitions*

Euler established the generating function for the number of unrestricted partitions of a number n, denoted by $p(n)$, as

$$P(q) = \sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where $(q;q)_\infty = \prod_{n=1}^{\infty}(1 - q^n)$.

4.2. *Generating function for overpartitions*

The generating function for the overpartition $\overline{p}(n)$ is defined as

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty},$$

where $(a;q)_\infty = \lim_{n\to\infty}(1 - a)(1 - aq)(1 - aq^2)\ldots(1 - aq^{n-1})$.

4.3. *Generalizations of p(n)*

In 1951, Gupta[12] generalized the generating function

$$J(x) = \prod_{r=1}^{\infty}(1 - x^r)^{-r^{j-1}},$$

which is known as the generalization of the generating function $p(n)$.

Another generalization of $p(n)$ is

$$J(x,m) = \prod_{r=1}^{m} (1-x^r)^{-r^{j-1}}.$$

In 1960, Kolberg[13] established another generalization:

$$P(x,j) = \{f(x)\}^{-j} = 1 + \prod_{n=1}^{\infty} P_j(n)x^n.$$

For positive values of j, $p_j(n)$ is a polynomial in j of degree n. Kolberg[14] established another generalization function for $p(n)$:

$$P(x^2,5)P(-x,-2) = \sum_{m=-\infty}^{\infty} (3m+1)x^{m(3m+2)}$$

and

$$P(x,5)P(x^2,-2) = \sum_{m=-\infty}^{\infty} (6m+1)x^{\frac{m(3m+2)}{2}}.$$

In 1953, Carlitz[15] provided one generalization form for $p(n)$ which is equivalent to Newman's formula:

$$P(x,6)P(x^5,-1) = \sum_{m=0}^{\infty} P_5(5m)x^m.$$

5. Congruences

The theory of congruences was first introduced by the German mathematician Carl Friedrich Gauss in his *Disquisitiones Arithmeticae* in 1801. Gauss also introduced the notation of congruence which makes the theory of congruence a powerful technique. According to Gauss, "If a number n measures the difference between two numbers a and b, then a and b are said to be congruent with respect to n; if not incongruent."

Definition 1. Let n be a fixed positive integer. Two integers a and b are said to be congruent modulo n,

$$a \equiv b \pmod{n}$$

if n divides the difference $a - b$, i.e. $a - b = kn$ for some integer k.

For example, consider $n = 5$. Then $29 \equiv 4 \pmod 5$, we say that 29 is congruent to 4 modulo 5.

5.1. *Ramanujan's Conjecture*

In 1919, Ramanujan established the following congruences:

$$p(5m+4) = 0 \pmod 5,\ p(25m+24) = 0 \pmod{25},$$
$$p(7m+5) = 0 \pmod 7,\ p(49m+47) = 0 \pmod{49},$$
$$p(llm+6) = 0 \pmod{11}$$

for any nonnegative m.

These results are known as the Ramanujan conjecture which is stated as

If $p = 5, 7$ or 11 and $(24n-1) \equiv 0 \pmod{p^a}, a \geq 1$, then $p(n) \equiv 0 \pmod{p^a}$.

But after some calculations, Chowla[16] found that the Ramanujan conjecture failed for some $n = 243$. According to the Ramanujan conjecture for the value of $n = 243$,

$$(24n-1) = 5831 \equiv 0 \pmod{7^3}$$

while

$$\begin{aligned} p(243) &\equiv 0 \pmod{7^2} \\ &\not\equiv 0 \pmod{7^3}. \end{aligned}$$

Next, it was seen that the conjecture also failed for $n = 586$.

Finally, Lehmer[17] established a new form of $p(n)$ for the large values of n using the Hardy–Ramanujan and Rademacher series. He provided the following:

$$p(599) \equiv 0 \pmod{54},\ p(721) \equiv 0 \pmod{113},$$
$$p(1224) \equiv 0 \pmod{54},\ p(2052) \equiv 0 \pmod{113},$$
$$p(2474) \equiv 0 \pmod{55},\ p(l4031) \equiv 0 \pmod{114}.$$

In 1948, Lahiri[18] introduced some new types of congruences:

$$p(49m + r) \equiv 0 \pmod{49} \text{ for } r = 19, 33, 40,$$
$$p(125m + r) \equiv 0 \pmod{125} \text{ for } r = 74, 124.$$

In 1952, Rushforth[19] proved another form of congruence:

$$p(121m + 116) \equiv 0 \pmod{121},$$
$$p(49m + r) \equiv 0 \pmod{49} \text{ for } r = 19, 33, 40, 47.$$

In 1960, Newman[20] provided the following for $(m, 30) = 1$:

$$p\left(\frac{l67m^2 + 1}{24}\right) \equiv 0 \pmod{5}.$$

All the above results were the generalized form of the Ramanujan congruences.

Along with the generalization, Watson[21,22] had proved Ramanujan's conjecture completely for powers of 5 and also provided the modification of the conjecture for powers of 7:

If

$$(24n - 1) \equiv 0 \pmod{7^b},$$

then

$$p(n) \equiv 0 \pmod{7^d},$$

where $d = [\frac{(b+2)}{2}]$.

Finally, in 1943 and 1950, Atkin[23] proved the Ramanujan conjecture for powers of 11, 11^2 and 11^3. Accordingly, Atkin and O'Brien[24] studied the Ramanujan conjecture for powers of 13.

5.2. *The Parity of $p(n)$*

In 1920, MacMahon[25] introduced a table for $p(n)$ such that

$$p(n) \equiv \sum_t p(t) \pmod{2},$$

where t is the positive integral value of $8t = 2n - j(j + 1), j \geq 0$.

Garvan,[26] Kolberg,[27] Hirschhorn,[28, 29] Stanton[26] and Subbarao[30] have generalized it for every arithmetic progression with modulus $t \in \{1, 2, 3, 4, 5, 6, 8, 10, 12, 16, 20, 40\}$.

5.3. *Congruences for $p_j(n)$*

In 1956, Newman[31] gave extensive tables of values of $p_j(n)$ for $j \leq 16$.

For $j = 2, 4, 6, 8, 10, 4, 16$ and a prime $p > 3$,

$$j(p + l) \equiv 0 \pmod{24}.$$

Newman[32] established that

$$p_j(np + k) = (-p)^{\frac{(j-2)}{2}} p_j\left(\frac{n}{p}\right),$$

where $k = \frac{j(p^2-1)}{24}$ and $p_j(n) = 0$ if n is not a nonnegative integer.

Newman also provided that

$$p_6(3n + 2) = 9p_6\left(\frac{n}{3}\right).$$

After Newman, Ramanathan[33] provided that

if

$$24n + j \equiv 0 \pmod{5^a}, j \equiv 16, 21, 26 \pmod{30},$$

then

$$p_j(n) \equiv 0 \pmod{5^b},$$

where $b = [\frac{(a+1)}{2}]$.

6. Generalization of the Congruences Modulo for All $p_j(n)$

In 2004, Hammond and Lewis[34] introduced that

$$p_{-2}(5n + l) \equiv 0 \pmod{5},$$

where $l \in 2, 3, 4$.

In 2014, Chen *et al.*[35] established that

$$p_{-2}(25n+23) \equiv 0 \pmod{25}.$$

Recently, in 2018, Tang[36] proved some congruences modulo powers of 5 for $p_j(n)$ with $j \in \{2, 6, 7\}$ such that

$$\begin{aligned} p_{-2}\left(5^{2d-1}n + \frac{7\times 5^{2d-1}+1}{12}\right) &\equiv p_{-6}\left(5^{2d}n + \frac{3\times 5^{2d}+1}{4}\right) \\ &\equiv p_{-7}\left(5^{2d-1}n + \frac{13\times 5^{2d-1}+7}{24}\right) \\ &\equiv 0 \pmod{5^d}. \end{aligned}$$

6.1. *New infinite families of congruences modulo* 11 *for* $p_j(n)$

Some new infinite families of congruences modulo 11 for $p_j(n)$ have been established by using q-identities for any positive integer λ.[37] These are as follows:

If $t = 3, 6, 8, 9, 10,$ then

$$p_{11\lambda+1}(11n+t) \equiv 0 \pmod{11}.$$

Next, if $t = 2, 4, 5, 7, 8, 9,$ then

$$p_{11\lambda+3}(11n+t) \equiv 0 \pmod{11}.$$

One specific case is that

$$p_{11\lambda+6}(11n+8) \equiv 0 \pmod{11}.$$

Next, if $1 \le t \le 10$, then

$$p_{121\lambda+1}(121n+11t+5) \equiv 0 \pmod{11}$$

and if $1 \le t \le 10$, then

$$p_{121\lambda+2}(121n+11t+10) \equiv 0 \pmod{11}.$$

6.2. *Ramanujan-type congruences modulo 8 for the overpartition function $\overline{p_0}(n)$*

Recently, many congruences for the number of overpartitions into odd parts have been discovered. The first Ramanujan-type congruences modulo power of 2 for $\overline{p_0}(n)$ was established in 2006 by Hirschhorn and Sellers.[3] Simultaneously, they also established a number of arithmetic results for Ramanujan-like congruences for the overpartition function $\overline{p_0}(n)$.

$$\overline{p_0}(n) = \begin{cases} 2(\text{mod } 4) \text{ if } n \text{ is a square or } n \text{ is twice a square,} \\ 0(\text{mod } 4) \text{ otherwise.} \end{cases}$$

The prime factorization of $n > 0$ is given by

$$n = 2^{\alpha} \prod p_i^{\alpha_i} q_i^{\beta_i} r_i^{\gamma_i} s_i^{\delta_i},$$

where

$$p_i \equiv 1 \pmod 8, q_i \equiv 3 \pmod 8, r_i \equiv 5 \pmod 8, s_i \equiv 7 \pmod 8.$$

Hirschhorn and Sellers established some more relations on congruence modulo 8 as follows:

(1) $\overline{p_0}(n) \equiv 0 \pmod 8$ if and only if one of the following holds:

- at least one δ_i is odd,
- all δ_i are even and at least one γ_i is odd,
- all δ_i are even, all γ_i are even, at least one β_i is odd, and $\prod(\alpha_i + 1)(\beta_i + 1) \equiv 0 \pmod 4$,
- all δ_i are even, all γ_i are even, all β_i are even, and $\prod(\alpha_i + 1) \equiv 0 \pmod 4$.

(2) $\overline{p_0}(n) \equiv 4 \pmod 8$ if and only if one of the following holds:

- all δ_i are even, all γ_i are even, any β_i is odd, and $\prod(\alpha_i + 1)(\beta_i + 1) \equiv 2 \pmod 4$,
- all δ_i are even, all γ_i are even, all β_i are even, and $\prod(\alpha_i + 1) \equiv 2 \pmod 4$.

(3) $\overline{p_0}(n) \equiv 2 \pmod 8$ if and only if one of the following holds:

- n is an odd square or twice an odd square and $\prod(\alpha_i + \beta_i) \equiv 0 \pmod 4$,

- n is an even square or twice an even square and $\prod(\alpha_i + \beta_i) \equiv 2 \pmod 4$.

(4) $\overline{p_0}(n) \equiv 6 \pmod 8$ if and only if one of the following holds:

- n is an odd square or twice an odd square and $\prod(\alpha_i + \beta_i) \equiv 2 \pmod 4$,
- n is an even square or twice an even square and $\prod(\alpha_i + \beta_i) \equiv 0 \pmod 4$.

In 2021, Mircea Merca[38] established Ramanujan-type congruences modulo 8 for the overpartition function $\overline{p_0}(n)$ considering the divisor function $T_{(odd)}(n)$ that counts the odd positive divisors of n as follows:

$$\overline{p_0}(2^{\alpha}(8n + l)) \equiv r \pmod{16},$$

where $\alpha > 0$ and $l \in \{1, 3, 5, 7\}$.

The partition of the set $\mathbb{N}$ is defined as

$$A_l = \bigcup_{\alpha=0}^{\infty} \{2^{\infty}(8n + l) | n \in \mathbb{N}_{\not{\,}}\}.$$

Theorem 1. *For $n, \alpha \geq 0$:*

(1) *If $l \in \{5, 7\}$, then $\overline{p_0}(2^{\infty}(8n + l)) \equiv 0 \pmod 8$.*
(2) *If $l \in \{1, 3\}$ and $8n + l$ is not a square, then*

$$\overline{p_0}(2^{\infty}(8n + l)) \equiv \begin{cases} 4 \pmod 8, & if\ \dfrac{T_{odd}(8n + l)}{2} \text{ is odd,} \\ 0 \pmod 8, & if\ \dfrac{T_{odd}(8n + l)}{2} \text{ is even.} \end{cases}$$

(3) *If $l \in \{1, 3\}$ and $8n + l$ is a square, then $8n + l$ is one of the form $(8k \pm l)^2$ or $(8k \pm 3)^2$. For $l \in 1, 3$ and $\alpha \in 0, 1$:*

$$\overline{p_0}(2^{\infty}(8n \pm l)^2) \equiv \begin{cases} 6 \pmod 8, & if\ \dfrac{T_{odd}((8n \pm l)^2) - 1}{2} \text{ is odd,} \\ 2 \pmod 8, & if\ \dfrac{T_{odd}((8n \pm l)^2) - 1}{2} \text{ is even.} \end{cases}$$

For $l \in 1, 3$ *and* $\alpha > 1$:

$$\overline{p_0}(2^{\infty}(8n{\pm}l)^2) \equiv \begin{cases} 2 \pmod 8, & if \ \dfrac{T_{odd}((8n \pm l)^2) - 1}{2} \ is\ odd, \\ 6 \pmod 8, & if \ \dfrac{T_{odd}((8n \pm l)^2) - 1}{2} \ is\ even. \end{cases}$$

6.3. *Some new identities for $p(n)$ modulo* 11, 17, 19 *and* 23

By using the theory of modular functions, Paule and Radu[39] obtained a witness identity which is different from Atkin and Lehner in 2017. After that, in 2019, the witness identity in modulo 11 was generalized for the Riemann surface.[40] In 2021, Goswami *et al.*[41] generalized the Ramanujan congruence $p(11n + 6) \equiv 0 \pmod{11}$ and also established some new identities for the generating functions for $p(17n + 5)$, $p(19n + 7)$ and $p(23n + 1)$. Some more works regarding Ramanujan congruence on the Riemann surface can also be found in Ref. 42. In future research, work on this area will be gradually developed and it will help mathematicians to establish more new concepts.

7. Conclusion

In this chapter, we discussed various types of partitions and their arithmetic and congruence properties. We explained congruences and focused on the Ramanujan congruences. Finally, we discussed some generalizations of the Ramanujan congruences. This chapter attempts to relate between the partition function and the Ramanujan congruences along with its generalized forms.

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Chapter 3

Collatz Hypothesis and Kurepa's Conjecture

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We discuss and give some insights on the Collatz conjecture, known as $3N + 1$, and Kurepa's hypothesis on the left factorial. First, the Collatz conjecture is considered and the density of values is compared to Planck's black body radiation in physics, showing a remarkable agreement between the two. We also briefly discuss a generalization of Collatz conjecture for a generic sequence $qN+1$ by means of numerical analysis. Then, we give a brief historical excursus and prove in a simple way some properties of Kurepa's function, also called the left factorial. We introduce Kurepa's hypothesis, propose a new description, and the relation to

Bezout's parameters and the Diophantine equation. A numerical analysis supports Kurepa's hypothesis and the conjecture about distribution for Kurepa's function.

1. Introduction

In this chapter, two different problems of number theory will be discussed. The first one concerns the Collatz function and the aspect of the total stopping time of iterations starting from different starting points, which distribution mimics Planck's black body radiation density for photons.

The second part considers a lesser known problem of Kurepa's function, or left factorial, and its factorizability. A Diophantine equation for this problem has been written, and some numerical results are shown to support that a particular distribution for Kurepa's function approaches a uniform distribution for large numbers.

2. The Collatz Conjecture

A very simple function is the basis of the Collatz conjecture. For $N \in \mathbb{N}$ define the function $C(N)$ as

$$C(N) = \begin{cases} \frac{N}{2} & \text{if } N \text{ even,} \\ 3N+1 & \text{if } N \text{ odd.} \end{cases} \tag{1}$$

A recursive application of this function gives a sequence. Starting from a positive integer N and applying recursively the function $C(N)$ one ends up with a sequence $\{a_i\}_{i\in\mathbb{N}}$ whose generic term a_i could be written as follows:

$$a_i = \begin{cases} N & \text{for } i = 0 \\ C(a_{i-1}) & \text{for } i > 0, \end{cases} \tag{2}$$

so that $a_i = [C(N)]^i$. A couple of examples: from $N = 7$, the resulting sequence is

$$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1,$$

while starting from $N = 1236$ the sequence is

$$1236, 618, 309, 928, 464, 232, 116, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26,$$
$$13, 40, 20, 10, 5, 16, 8, 4, 2, 1.$$

Any sequence is concluded once it reaches the number 1. From the previous examples, the sequence is concluded after 17 steps in the case of $N = 7$, while 27 steps are needed for the case $N = 1236$. It is apparent that for a power of two, $N = 2^k$, $k \in \mathbb{N}$, the value of 1 is reached after k steps. The total stopping time is defined as the number of steps necessary for reaching the value of 1.

The conjecture stated by Collatz[1] in 1937 is the following: using the function (1) and starting from any natural number, the sequence (2) has a finite stopping time. To the present day (2022) his conjecture has neither being proved nor disproved.

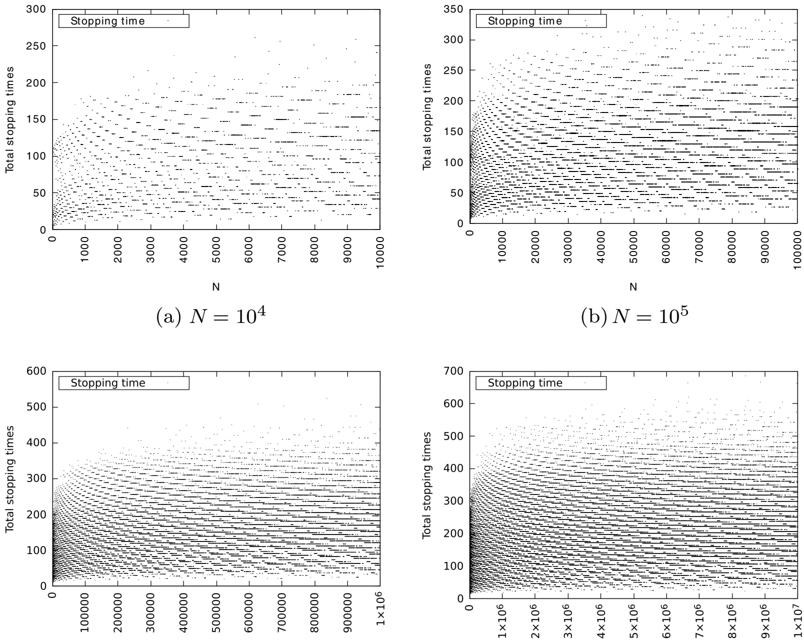
There is plenty of literature on the subject, see for instance Refs. 2–6.

We are here not interested in proposing a possible solution to this conjecture. Rather, we will discuss the problem of total stopping times and its distribution. In Figures 1(a)–(d) we have shown total stopping times for different starting values of N.

A different kind of plot for total stopping times is presented in Figure 2(a) and (b). For fixed N, total stopping times are shown with respect to frequencies. The latter results suggest a close parallel to a well-known problem in quantum physics.

3. Planck's Radiation

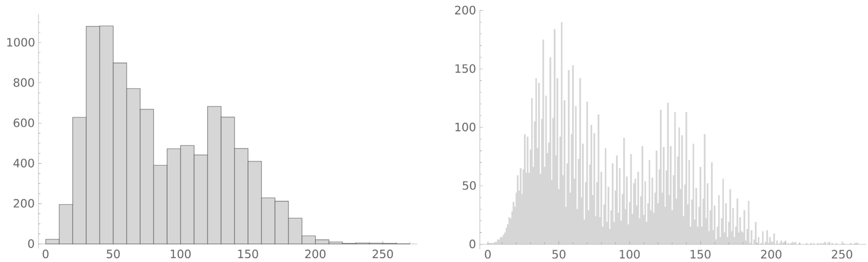
Assume to have a photon gas (i.e. electromagnetic radiation) contained at equilibrium inside a cavity of volume V at a temperature T. Such a system is known as "blackbody cavity." It is well-known that this free electromagnetic field can be described with a sum of harmonic oscillators, each with a fixed frequency ν. Quantum mechanics allows to each photon of frequency ν only discrete energy levels $(n' + \frac{1}{2})2\pi\hbar\nu$, for $n' \in \mathbb{N}$.



(a) $N = 10^4$ (b) $N = 10^5$

(c) $N = 10^6$ (d) $N = 10^7$

Fig. 1. Total stopping times for different starting N.



(a) Bin width = 10 (b) Bin width = 1

Fig. 2. Histograms for $N = 10^4$ of total stopping times with respect to frequencies for different bin widths.

The partition function of such a system at the temperature T, $\beta = 1/k_B T$, can be explicitly calculated as

$$Z = \sum_{n'=0}^{+\infty} \exp(-2\pi\beta\hbar\nu n') = \frac{1}{1 - \exp(-2\pi\beta\hbar\nu)}. \qquad (3)$$

The average energy for a photon of frequency ν is given by

$$\begin{aligned}\langle E\rangle &= -\frac{\partial(\ln Z)}{\partial\beta} = \frac{2\pi\hbar\nu\exp(-2\pi\beta\hbar\nu)}{1-\exp(-2\pi\beta\hbar\nu)} \\ &= \frac{2\pi\hbar\nu}{\exp(2\pi\beta\hbar\nu)-1} = \frac{2\pi\hbar\nu}{\exp[2\pi\hbar\nu/(k_BT)]-1}. \end{aligned} \tag{4}$$

Each photon of frequency ν has a definite momentum $\vec{p} = \hbar\vec{k}$, with wave number $k = |\vec{k}| = (2\pi/c)\nu$. Inside a volume V the number of photon momenta between k and $k + dk$ is given by

$$\frac{V}{(2\pi)^3}8\pi k^2 dk = 8\pi\frac{V}{c^3}\nu^2 d\nu.$$

The internal energy U of the system is therefore obtained integrating the average energy of each photon present inside the cavity

$$U = 8\pi\frac{V}{c^3}\int_0^{+\infty}\langle E\rangle\nu^2 d\nu = 16\pi^2\frac{V}{c^3}\int_0^{+\infty}\frac{\hbar\nu^3}{\exp[2\pi\hbar\nu/(k_BT)]-1}d\nu,$$

so that the internal energy per unit volume is obtained by

$$\frac{U}{V} = \int_0^{+\infty} u(\nu,T)d\nu, \tag{5}$$

where

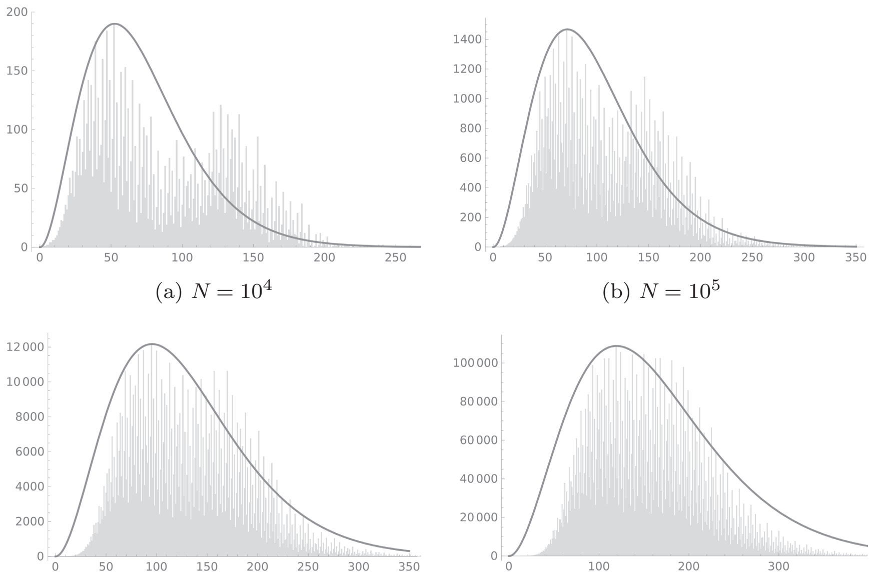
$$u(\nu,T) = 16\pi^2\left(\frac{\hbar}{c^3}\right)\frac{\nu^3}{\exp[2\pi\hbar\nu/(k_BT)]-1}, \tag{6}$$

this is the famous Planck radiation law of photon energy density[7] at frequency ν and fixed temperature T. Calculating the integral (5) one could observe that U/V increases with temperature as T^4, thus obtaining the Stefan–Boltzmann law for the power radiated from a black body. The Stefan–Boltzmann law was discovered even before the advent of quantum mechanics by thermodynamics arguments alone.

Planck's radiation density function (6) has, remarkably, the same qualitative behavior as the total stopping times with respect to frequencies shown in Figures 2(a) and (b).

Table 1. Translation table Collatz–Planck.

Collatz	$\longleftrightarrow$	Planck
Frequency	$\longleftrightarrow$	Photon frequency
Total stopping time	$\longleftrightarrow$	Black body radiation density



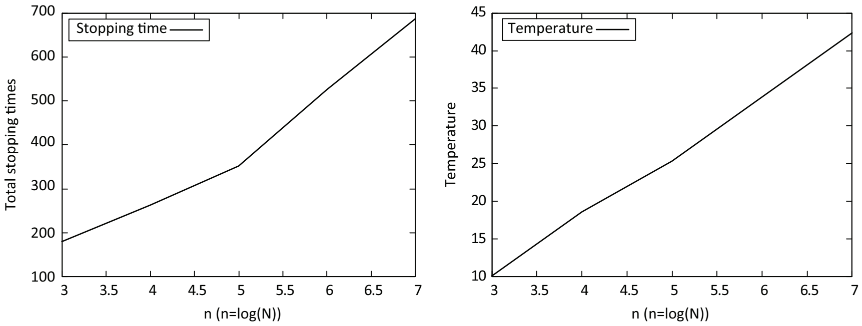
(a) $N = 10^4$ (b) $N = 10^5$ (c) $N = 10^6$ (d) $N = 10^7$

Fig. 3. Histograms: Total stopping times with respect to frequencies. Functions: Planck's black body radiation density with respect to photon frequency.

In Table 1 we have proposed an equivalence language between Collatz distribution of total stopping times and Planck's radiation density.

The results of the comparison of total stopping times and Planck's radiation density are shown in Figures 3(a)–(d) for different values of starting N.

It could be observed that the agreement between the two functions increases with increasing N, and that Planck's radiation law tail overestimates a little the decrease of stopping times with respect to N.



(a) (b)

Fig. 4. (a) Scaling of total stopping times with respect to starting value of n, $n = \log(N)$. (b) Scaling of temperatures with respect to starting value of n, $n = \log(N)$.

Using the results of Ref. 8, Planck's radiation law (6), and the Rosetta Stele of Table 1 we could infer that for large values of frequency ν of photon the total stopping time goes to zero at least as an exponential function

$$\text{Total stopping time} \sim \nu^3 \exp(-\nu). \tag{7}$$

In Figure 4(a) and (b) we show, respectively, the scaling of total stopping times, and the scaling of temperature T of Planck's radiation density with respect to n, the logarithm of starting N, $n = \log(N)(\approx 2.3 \ln(N))$. The temperature T is related to the longest stopping time measured, that is the peak of histogram graphs in Figure 3(a)–(d). Figure 4(a) shows a fairly linear scaling of total stopping times with respect to n, the logarithm of starting N,

$$\text{Total stopping times} \sim \log(N). \tag{8}$$

Figure 4(b) shows an almost perfect linear scaling for temperatures T, or maximum values of total stopping times, with the logarithm of starting N:

$$\text{Temperatures} \sim \log(N). \tag{9}$$

From Stefan–Boltzmann's law discussed before we could also estimate that the sum of total stopping times scales with the fourth power of $\log(N)$:

$$\sum(\text{Total stopping times}) \sim (\log(N))^4. \tag{10}$$

Although they are no rigorous proofs, but rather hints, all these collected results show that for a finite N total stopping times are finite, and their sum is finite as well, thus supporting the Collatz conjecture.

4. Collatz Generalization

Define the generalization of Collatz function (1) $C(N)$, $C(q,N)$, in the following manner:

$$C(q,N) = \begin{cases} \frac{N}{2} & \text{if } N \text{ even,} \\ qN+1 & \text{if } N \text{ odd,} \end{cases} \tag{11}$$

it has fixed points $x = 0$ and $x = -1/(q-1)$, respectively. As even N behaves as the usual Collatz function, we have to focus only on odd N. Let q be even, $q = 2j$, $j \in \mathbb{N}$. For odd $N = 2m+1$, $m \in \mathbb{N}$, one has

$$2j(2m+1)+1 = 2[j(2m+1)]+1$$

which is again an odd number and grows without bounds when applying recursively the function $C(2j, 2m+1)$. Therefore, q has to be odd, $q = 2j+1$.

For $N = 2m+1$, we obtain

$$(2j+1)(2m+1)+1 = 2[(2j+1)m+(j+1)] \tag{12}$$

which is even. Using the notation "$\to$" as "transforms to" we obtain the results presented in Table 2.

Actually, the original Collatz function allegedly loops on the final sequence $4, 2, 1$ for any N, so one could say that the conjecture is verified when any starting number $N \in \mathbb{N}$ enters the aforementioned loop.

Table 2. Transformation table for Collatz generalization.

j	q	$(2m+1) \to$
1	3	$2(3m+2)$
2	5	$2(5m+3)$
3	7	$2(7m+4)$
4	9	$2(9m+5)$

For $q > 3$, the situation is different, probably due to the fact that, considering Table 2, and formula (12), one evinces that the ratio

$$\frac{2j+1}{j+1}$$

is larger than 3/2, so the sequence reaches fewer odd numbers than even numbers.

In the case $q = 5$, we obtain various different loops than the usual sequence $4, 2, 1$. Here are some examples of finite iterations. For $N = 5$:

$$5, 26, 13, 66, 33, 166, 83, 416, 208, 104, 52, 26, 13, \ldots$$

For $N = 13$:

$$13, 66, 33, 166, 83, 416, 208, 104, 52, 26, 13, 66, \ldots$$

For $N = 15$:

$$15, 76, 38, 19, 96, 48, 24, 12, 6, 3, 16, 8, 4, 2, 1, 6, 3, 16, 8, 4, \ldots$$

For $N = 17$:

$$17, 86, 43, 216, 108, 54, 27, 136, 68, 34, 17, 86, \ldots$$

While, for other values of N, $C(5, N)$ diverges, like for $N = 7, 9, 11, 14, 18, \ldots$.

When $q > 5$, the situation drastically worsens, and $C(q, N)$ diverges already for starting $N = 5$, $C(q, 5)$, for the values of $q = 9, 11, 13, 15, 17, 19, 21, \ldots, 211, \ldots$.

Considering all those numerical results, one could conjecture that only the Collatz function case, the one with $q = 3$, has no divergences and ends on a unique loop, $4, 2, 1$, while for $q > 3$ there exist more final loops, all separated by some divergent points.

5. Kurepa's Hypothesis

In 1971, following his seminal works,[9–11] Kurepa introduced[12] the left factorial function, with the symbol $!n$, where $n \in \mathbb{N}$, also known as Kurepa's function,

$$K(n) = !n = \sum_{i=0}^{n-1} i! = \sum_{i=0}^{n-1} \Gamma(i+1), \tag{13}$$

and later[13] extended its definition to arguments z on the complex plane $\Re(z) > 0$

$$K(z) = \int_0^{+\infty} e^{-t} \frac{t^z - 1}{t - 1} dt,$$

$$\Gamma(x+1) = \int_0^{+\infty} e^{-t} t^x dt. \tag{14}$$

For $n \in \mathbb{N}$, $\Gamma(n+1) = n!$. The recurrence relation holds true:

$$\Gamma(x+1) = x\Gamma(x), \tag{15}$$

and its asymptotic behavior (known as Stirling's formula) for $x \to +\infty$ is given by

$$\Gamma(x+1) \sim \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \left[1 + \frac{1}{12x} + \mathcal{O}\left(\frac{1}{x^2}\right)\right]. \tag{16}$$

Substituting (14) in (13), we obtain Kurepa's function on the complex plane as follows:

$$K(n) = \int_0^{+\infty} e^{-t} \sum_{i=0}^{n-1} t^i dt = \int_0^{+\infty} e^{-t} \frac{t^n - 1}{t - 1} dt$$

that is

$$K(x) = \int_0^{+\infty} e^{-t} \frac{t^x - 1}{t - 1} dt. \tag{17}$$

The following theorems for the function $K(x)$ and its relation with function $\Gamma(x)$ were first established in Ref. 13. Here we will show novel simple proofs for them.

Theorem 1. *The following relation holds true for* $z \in \mathbb{C}$:

$$K(z) = K(z+1) - \Gamma(z+1) \tag{18}$$

Proof. Using the expressions of (14) and (17) in (18), we obtain

$$\int_0^{+\infty} e^{-t}\frac{t^z-1}{t-1}dt = \int_0^{+\infty} e^{-t}\left[\frac{t^{z+1}-1}{t-1} - t^z\right]dt$$
$$= \int_0^{+\infty} e^{-t}\left[\frac{t^z-1}{t-1}\right]dt. \tag{19}$$

□

Theorem 2. *Kurepa's function has the following limit for $x \to +\infty$:*

$$\lim_{x\to+\infty}\frac{K(x)}{\Gamma(x)} = 1. \tag{20}$$

Proof. From the definition (13) of $K(x)$, for $n \in \mathbb{N}$ we have

$$K(n) = \sum_{i=0}^{n-1} i! \sim (n-1)! + (n-2)! + \cdots + 1! + 0!. \tag{21}$$

Inserting (21) in (20), we obtain

$$\lim_{n\to+\infty}\frac{K(n)}{\Gamma(n)} = \lim_{n\to+\infty}\frac{(n-1)!+(n-2)!+\cdots}{(n-1)!}$$
$$= \lim_{n\to+\infty}\left[1+\mathcal{O}\left(\frac{1}{n-1}\right)\right] = 1,$$

this result obtained so far is valid for the case $n \in \mathbb{N}$ only. The integrand of (19) is well-defined for any $t \in \mathbb{R}^+$, so $K(z), z \in \mathbb{C}, \Re(z) > 0$ could be rewritten as

$$K(z) = \int_0^{+\infty} e^{-t}t^{z-1}\left[\sum_{n=0}^{+\infty}\left(\frac{1}{t^n} - \frac{1}{t^{n+z}}\right)\right]dt$$
$$\sim \Gamma(z) + \Gamma(z-1) + \Gamma(z-2) + \ldots, \tag{22}$$

the asymptotic behavior is given for $z \to \infty$, and using (15) one has

$$\lim_{z\to\infty}\frac{K(z)}{\Gamma(z)} = \lim_{z\to\infty}\frac{\Gamma(z)+\frac{\Gamma(z)}{z-1}+\mathcal{O}\left[\frac{\Gamma(z)}{(z-1)(z-2)}\right]}{\Gamma(z)} = 1.$$

□

Theorem 3. *Kurepa's function has the following limit for $x \to +\infty$:*

$$\lim_{x\to+\infty} \frac{K(x)}{\Gamma(x+1)} = 0.$$

Proof. Divide (18) by $\Gamma(x+1)$ obtaining

$$\lim_{x\to+\infty} \frac{K(x)}{\Gamma(x+1)} = \lim_{x\to+\infty} \frac{K(x+1)}{\Gamma(x+1)} - 1 = 1 - 1 = 0$$

by virtue of (20). □

6. Diophantine Equation

In this section, we present some equivalent statements for Kurepa's hypothesis, proposing some other new forms, and a suggestion for a new research direction by means of a linear Diophantine equation.

In the following section we present new numerical results supporting the conjecture of uniform distribution for $K(p)$.

Hypothesis 4. Kurepa's hypothesis is that the following, equivalent statements hold true:

$$\begin{aligned} &\gcd(K(n), n!) = 2, && n \geq 2, \\ &\mathrm{mod}(K(n), n) \not\equiv 0 && n > 2, \\ &\mathrm{mod}(K(p), p) \not\equiv 0, && p \geq 3,\ p \text{ prime number.} \end{aligned} \tag{23}$$

Let be $n \in \mathbb{N}, n \geq 2$, then, using the Euclidean algorithm for finding the $\gcd(!n, n!)$, the first formula of (23) could be rewritten as

$$x \cdot !n + y \cdot n! = 2, \tag{24}$$

if such $x, y \in \mathbb{Z}$ exist. This is a Diophantine equation, and (x, y) are Bezout's parameters.[14] From Kurepa's hypothesis, $\gcd(!n, n!) = 2$, therefore, Eq. (24) has an infinite number of solutions, that is Bezout parameters. If (x_0, y_0) is a pair that solves (24), then it follows that

$$x = x_0 + \frac{!n}{2}k,\ y = y_0 - \frac{n!}{2}k$$

with $k \in \mathbb{Z}$ all being Bezout's parameters that solve (24).

For instance, for $n = 4$ we have

$$x \cdot 10 + y \cdot 24 = 2$$

and $(x_0, y_0) = (5, -2)$. Then $x = 5 - 12k$ and $y = -2 + 5k$ are Bezout's parameters of this equation.

Dividing (24) by 2 one could also observe that $!n/2$ and $n!/2$ are coprime numbers (in particular $!n/2$ is odd while $n!/2$ is even). Therefore, $!n/2$ has no factors smaller than n, thus recovering the second formula of (23).

First formula of (23) is also equivalent to

$$\gcd(!!(2n), (2n)!!) = 2,$$
$$\gcd(!!(2n+1), (2n+1)!!) = 2,$$

where we have defined $!!(2n) = \sum_{i=0}^{n-1}(2i)!!$ with $0!! = 0$, $!!(2n+1) = \sum_{i=0}^{n-1}(2i+1)!!$, and have used the relation $\gcd(a, b) \times \mathrm{lcm}(a, b) = a \times b$.

There are many equivalent forms for Kurepa's hypothesis (see Ref. 15 for more details), first one of (23) was stated in his original paper.[12]

In 2004, Ref. 16 presented a proof of Kurepa's hypothesis that was shown in 2011 to be wrong by the same authors,[17] so they retracted the original paper. Until the present time (2022) there does not exist a real proof or a counterexample for (23).

7. Numerical Analysis for Kurepa's Hypothesis

We now present a numerical analysis and some new results of Kurepa's Hypothesis, starting from the last formula of (23). We have analyzed the first 11,000 prime numbers p, from 3 to the 11,000th prime number being 116,447, $!116{,}447 \approx 116{,}446! \approx 1.045 \times 10^{539{,}361}$ (the first approximation is valid because of (20)). According to the hypothesis found in Ref. 18, $K(p)$ modulo p is a random number with random uniform distribution. In order to check this hypothesis, we have shown the distribution of $\mathrm{mod}(K(p), p)$ and have compared it to random numbers, random (p), with uniform distribution in $[0, p)$. In Figure 5, the value of $\mathrm{mod}(K(p), p)$ is shown as a function of the prime p for the first 11,000 prime numbers. There is no value of p for which $\mathrm{mod}(K(p), p) = 0$, thus Kurepa's hypothesis has been

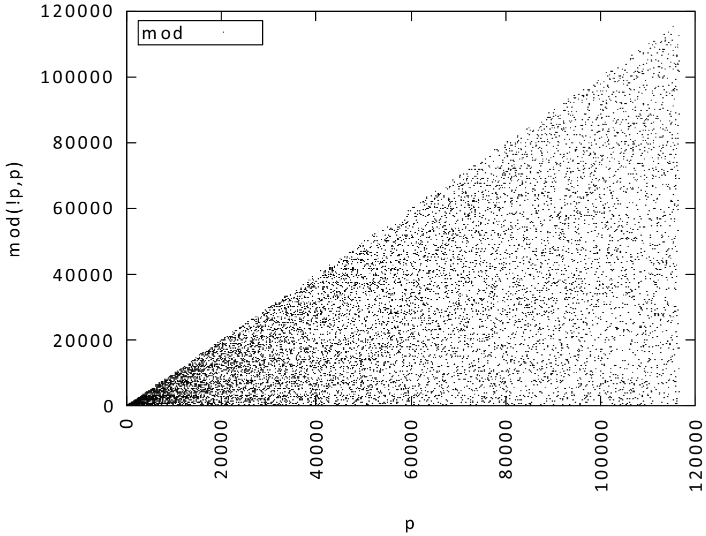


Fig. 5. mod $(!p, p)$ versus p for the first 11,000 prime numbers.

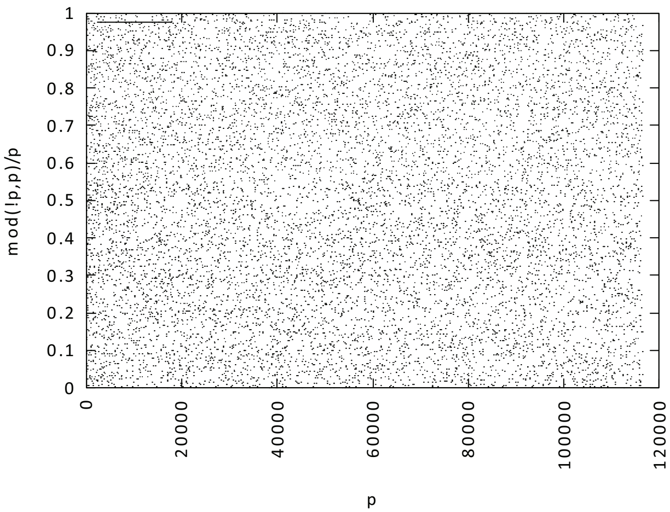


Fig. 6. mod $(!p, p)/p$ versus p for the first 11,000 prime numbers.

verified up to $p = 116{,}447$. Figure 6 shows the normalized plot of mod $(K(p), p)/p$, providing us with the more familiar rectangular uniform distribution of random numbers in $[0, p)$.

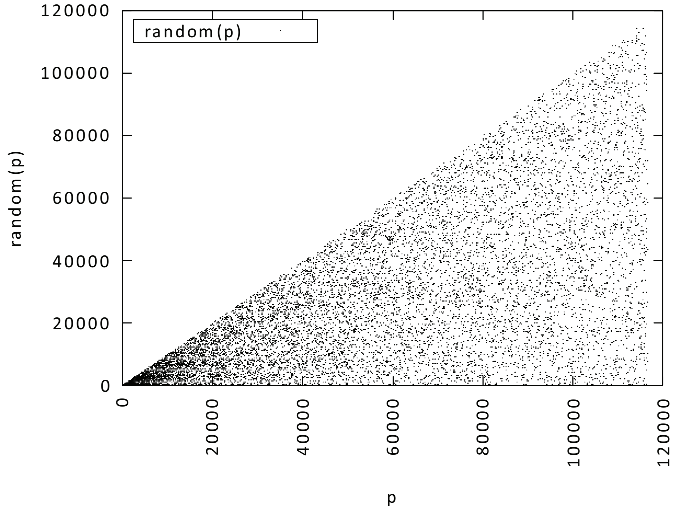


Fig. 7. random (p) versus p for the first 11,000 prime numbers.

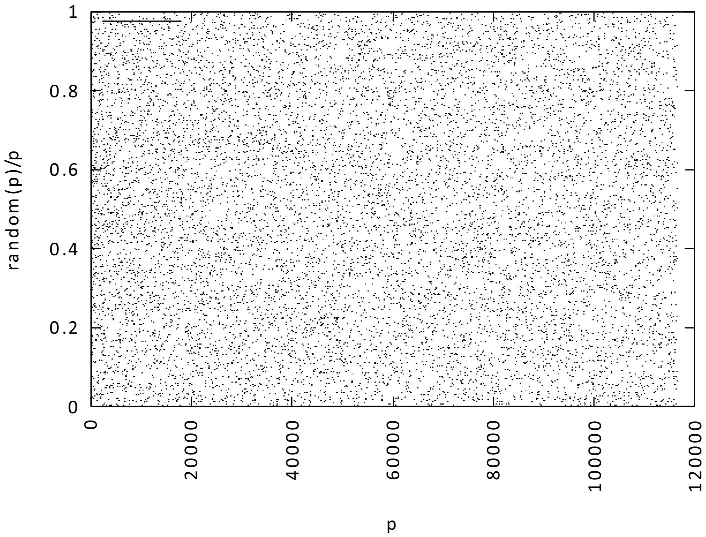


Fig. 8. random $(p)/p$ versus p for the first 11,000 prime numbers.

Figures 7 and 8 show, respectively, the same kind of plots as before, this time for a uniform distribution of random numbers random (p) generated in the range $[0, p)$.

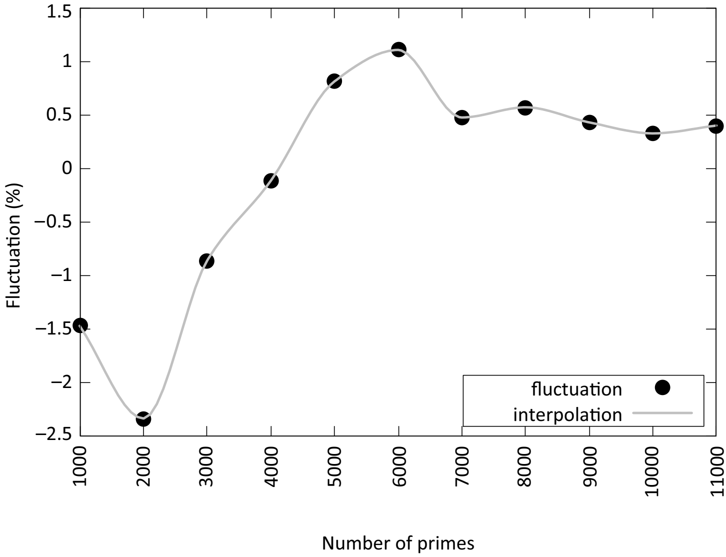


Fig. 9. Difference of $\text{mod}(!p, p)/p$ from a uniform distribution, in percentage, with respect to number of primes.

The comparison of the two groups of results does not show substantial differences in distributions between the results of $\text{mod}(K(p), p)$ and the random number generator random (p). There is also no visible pattern that could possibly spoil the conjectured randomness of $\text{mod}(K(p), p)$ in Figures 5 and 6.

In Figure 9 we compare the results of $\text{mod}(!p, p)/p$, shown in Figure 6, with respect to a uniform distribution between $[0, 1]$, as a function of number of primes taken into account. As expected, with increasing number of samples considered, that is with a larger number of primes, the distribution of $\text{mod}(!p, p)/p$ is getting closer to a uniform distribution, with an error smaller than 0.5%, which further decreases when the number of primes is larger than 7000.

This numerical analysis concludes that Kurepa's hypothesis

$$\text{mod}\ (K(p), p) \not\equiv 0$$

is true for all primes p verified so far, and also that this value is randomly uniformly distributed in the range $[0, p)$.

In Ref. 18, the search for a counterexample of the conjecture was performed, without success, for $p < 2^{34} \approx 1.718 \times 10^{10}$ by means of GPU computing. A fairly exhaustive list of all numerical attempts to solve this problem is also given.

8. Conclusions

Almost 90 years ago, Collatz introduced his simple, often also called $3N + 1$, yet unresolved problem. To use the words of Erdös, "Mathematics may not be ready for such problems." In fact, there is no general method suitable for tackling those kinds of problems. We have shown here heuristically that the stopping of time distributions closely resembles Planck's black body radiation density known in Physics. We have also shown that variations of the Collatz problem, $qN + 1$, lead to different loops separated by divergent points.

About 50 years have passed since the formulation of Kurepa's hypothesis.[12] Many attempts have been made in order to provide a solution to this problem, either by means of a rigorous theorem, with an approximate solution, or with a numerical approach.[18–46] Various tentative solutions increase the importance of Kurepa's hypothesis, and considering its simple formulation it deserves to stay in the realm of other famous unsolved problems in number theory like Collatz conjecture, Goldback's conjecture, de Polignac's conjecture, Legendre's conjecture, Landau's problem, to name a few.

We have brought simplifications to some proofs concerning Kurepa's function. There are some other equivalent statements of Kurepa's hypothesis, and a new study route using a linear Diophantine equation. Quantitative computer simulations support all the aforementioned hypotheses, and show also how the distribution of $\mathrm{mod}(!p, p)/p$ approaches a uniform distribution — a fact not originally envisaged by Kurepa — with increasing prime p.

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Chapter 4

On Zeta Functions and Allied Theta Functions*

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In this chapter, we completely clarify the meaning of a ramified functional equation of Schnee type as the union of the Riemann–Hecke–Bochner (RHB) correspondence and the parity, thus enabling one to describe even and odd parts of an entity whenever it is possible to introduce parity.

We shall uplift the Hurwitz formula between the Hurwitz and Lerch zeta functions to the *ramified Riemann–Hecke–Bochner correspondence*, i.e. to a ramified functional equation whose even, resp. odd, part is in RHB correspondence with the theta function, resp. the conjugate theta function. This is shown not to be coincidental and will be extended to

*Dedicated to Professor Dr. Antanas Laurinčikas with deep respect and friendship.

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a wider class of periodic Dirichlet series. It turns out that the Riemann zeta function $\zeta(s)$ is the even part, however the expression for merely the even part does not lead to $\zeta(s)$ but describes the Clausen function. It is the union of even and odd parts that gives rise to the functional equation for $\zeta(s)$ and entails almost all known formulas in the theory of gamma and zeta functions. The odd part itself vanishes in the functional equation for $\zeta(s)$, but it contains the partial fraction expansion for the cotangent function, an equivalent to it, thereby we establish the following principle: W = E + O, W = Whole, E = Even and O = Odd.

While in the theory of modular relations,[13] equivalent expressions to the functional equation are pursued, we examine here the counterpart of the zeta functions: the theta-function-like objects in the ramified RHB correspondence which govern the intrinsic nature and properties of the corresponding zeta functions. Riemann's famous proof of the functional equation for $\zeta(s)$ by the even part of the theta function ϑ_3 is a typical one and is an example of the Edwald expansion.

Motivated by coincidence of the above ramified functional equation with the one established by Schnee in 1930 as modified by Ishibashi and Kanemitsu in 1999, we turn to the study of Dirichlet series with periodic coefficients in Section 1.1. We complete the evaluation of the Laurent coefficients and give material for Section 2, where we deduce the Chowla–Selberg integral formula as an example of the Fourier–Bessel expansion. It can be proved by the Hurwitz formula, i.e. by RHB correspondence under the beta transform in contrast with the Hecke gamma transform. Then we establish (genesis of) the Lerch–Chowla–Selberg formula as an example of the RHB correspondence as linked by the decomposition theorem.

Thus, we follow the natural flow of ideas: theta function → Hurwitz (and Lerch) zeta-function → Fourier-Bessel expansion with parity.

1. Introduction and Ramified RHB Correspondence

Decidedly one of the most influential and creative source of ideas lies in the Riemann–Hecke–Bochner correspondence, whose naming is in Ref. 1, meaning the correspondence between the transformation formula (of modular form-like objects) and the functional equation (of the corresponding zeta functions). The origin of this correspondence was found by Refs. 2–5. In Ref. 13, equivalent expressions for the functional equation have been developed under the name of modular relations — as a memorial to Bochner.

In this chapter, we clarify the meaning of a ramified functional equation of Schnee type to its root as the union of the Riemann–Hecke–Bochner (RHB) correspondence and the parity, thus enabling one to describe even and odd parts of an entity whenever it is possible to introduce parity. In the specific case of the complex exponential function basis, we may express the ramified functional equation in the form of the generalized Euler identity and as with the Euler identity, we may deduce almost all results from one formula.

As a precursor, we shall describe the RHB correspondence between the theta-like functions and the Hurwitz and the Lerch zeta functions as a natural generalization of the Riemann zeta function. Ehrenpreis (1987) contains another advanced treatment through Appel transform (p. 73). The odd part has been treated in Ref. 6 to some extent and here we make clear the source of the transformation formula and the even and odd parts of the Hurwitz and the Lerch zeta functions. The main results in this section are Theorem 1 and Theorem 2. The odd part alone in the former theorem entails the partial fraction expansion for the cotangent function, an equivalent to the functional equation for the Riemann zeta function while the even part gives slightly restricted information and the union of even and odd parts in the latter theorem describes the whole structure for an even function and includes the functional equation.

We speak of the RHB correspondence or the modular relation to mean the genesis of the functional equation *under the Mellin transform pair* (1) as well as the equivalent expressions to the functional equations.

The ramified functional equation in Definition 1 (18), as well as the Eisenstein formula (41), strongly motivates to shift to the study of Dirichlet series with periodic coefficients in Section 1.1. Already, Schnee,[7] as formulated in Ref. 8, established the same functional equation as the Hurwitz and Lerch zeta functions. Supplementing the results in Ref. 6 etc., we describe the Laurent coefficients of the relevant Dirichlet series including the Dirichlet L-functions, which in particular gives one hand of the decomposition of the Dedekind zeta-function of an imaginary quadratic field.

Then as another direction of the modular relation, we establish the Chowla–Selberg integral formula for the Epstein zeta function with a positive definite quadratic form (the odd case as opposed to indefinite form) as the Fourier–Bessel expansion $H^{1,1}_{1,1} \leftrightarrow H^{2,0}_{0,2}$. This hinges on the beta transform as well as the RHB correspondence between the theta function and the lattice zeta function. This gives the evaluation of the Laurent constant of the Dedekind zeta function, the other part of the decomposition. Equating them, we deduce (genesis of) the Lerch–Chowla–Selberg formula in Section 2.

We write

$$s = \sigma + it, \quad \sigma = \operatorname{Re} s$$

once and for all.

The Mellin transform pair ($\operatorname{Re} x > 0$)

$$\Gamma(s) = \int_0^\infty t^s e^{-t} \frac{\mathrm{d}t}{t}, \sigma > 0, \quad e^{-x} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) x^{-s} \, \mathrm{d}s, \quad c > 0 \tag{1}$$

is the genesis of the RHB correspondence, cf. (48).

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2}$$

for $\sigma > 1$, where the series is absolutely convergent. This is continued analytically to a meromorphic function over the whole plane with a unique simple pole at $s = 1$ with residue 1. The analytic continuation is done by some expressions in $0 < \sigma < 1$ and by the functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \tag{3}$$

for $\sigma < 0$. It turns out that for deeper theory on distribution of prime numbers, one needs the Euler product and Riemann himself defines by the Euler product first, cf. (114).

Reference 9 gives a rather thorough table of functional equations which arise from the theta-transformation formula, Cf. also Kanemitsu and Tsukada (2014), Table 2.1, p. 34. There, after the Riemann zeta function, there comes the Dedekind zeta function and Epstein zeta functions associated to quadratic forms, etc. It is well known that Riemann gave two proofs of the functional equation (3), one depending on a variant of the theta functions ϑ_3 and the other on the contour integral expression. For Riemann's theta-transformation proof, cf. Ref. 10, pp. 61–63, where the Dirichlet L-function case is also treated by the same technique, cf. Remarks 2 and 3.

Let $\ell_s(x)$ be the boundary Lerch zeta function defined by

$$\ell_s(x) = \sum_{n=1}^{\infty} e^{2\pi ixn} n^{-s}, \quad \sigma > 1 \quad \text{or} \quad \sigma > 0, x \notin \mathbb{Z}, \tag{4}$$

which has its counterpart, the Hurwitz zeta function

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \sigma > 1,\ 0 < x \leq 1. \tag{5}$$

This is continued meromorphically over the whole plane with a simple pole at $s = 1$. Both of them reduce to the Riemann zeta function

$$\zeta(s,1) = \ell_s(1) = \zeta(s).$$

These are connected by the *Hurwitz formula* (i.e. the functional equation for the Hurwitz zeta function): for $\sigma > 1, 0 < x \leq 1$,

$$\zeta(1-s,x) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{-\frac{\pi is}{2}} \ell_s(x) + e^{\frac{\pi is}{2}} \ell_s(1-x) \right), \tag{6}$$

while its reciprocal is

$$\ell_{1-s}(x) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{\frac{\pi is}{2}} \zeta(s,x) + e^{-\frac{\pi is}{2}} \zeta(s,1-x) \right), \quad 0 < x < 1. \tag{7}$$

The limiting case with $x \to 1$ of (6) reads

$$\zeta(1-s) = \frac{2\Gamma(s)}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right) \zeta(s), \tag{8}$$

which is seen to be equivalent to (3).

By (7), the boundary Lerch zeta function $\ell_{1-s}(x)$ is continued meromorphically over the whole plane with $s = 0$ a plausible singular point. However, it is a removable singularity if $x \in \mathbb{R} - \mathbb{Z}$ in view of (36). Indeed, comparison of both sides as $s \to 1$ leads to the relation

$$\psi(x) - \psi(1-x) = 2\pi i \ell_0(x) + \pi i, \tag{9}$$

which together with (35) entails (30), cf. the second half of the proof of Corollary 1.

We assemble the identities for $\ell_1(s)$, cf. e.g. Refs. 6 and 11,

$$\sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n} + i\sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n} = \ell_1(x) \tag{10}$$

$$= -\log\left(1 - e^{2\pi ix}\right) = \sum_{n=1}^{\infty} \frac{e^{2\pi inx}}{n} = A_1(x) - \pi i\overline{B}_1(x),$$

$0 < x < 1$, where

$$A_1(x) = -\log 2|\sin \pi x| = \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{n}$$

is its real part (even part), the first Clausen function (or the logsine function) and the imaginary part (odd part) is (27) with $\varkappa = 1$, which reads

$$x - [x] - \frac{1}{2} = \bar{B}_1(x) = -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n} \tag{11}$$

for $x \notin \mathbb{Z}$, where $[x]$ indicates the greatest integer function.

In a totally forgotten paper,[12] Hutchinson considers the zeta function

$$Z_b(s) = Z(a, b, s) = \sum_{\nu=0}^{\infty} (a\nu + b)^{-s}, \quad 0 < b \le a,$$

which is $a^{-s}\zeta(s, x)$, where $x = \frac{b}{a}$, so that $0 < x \le 1$. Hence, this is a slight generalization of the Hurwitz zeta function. He considers the conjugate functions Z_b and Z_{a-b} and states the functional equation satisfied by the even and odd parts, (12), (20), (21):

$$Z_b + Z_{a-b} = \frac{2}{\pi}\left(\frac{2\pi}{a}\right)^s \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\sum_{n=1}^{\infty} \frac{\cos\left(2\pi \frac{b}{a} n\right)}{n^{1-s}}$$

$$= \frac{1}{\pi}\left(\frac{2\pi}{a}\right)^s \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)(\ell_{1-s}(x) + \ell_{1-s}(1-x)) \tag{12}$$

and

$$Z_b - Z_{a-b} = \frac{2}{\pi}\left(\frac{2\pi}{a}\right)^s \cos\left(\frac{\pi s}{2}\right)\Gamma(1-s)\sum_{n=1}^{\infty}\frac{\sin\left(2\pi\frac{b}{a}n\right)}{n^{1-s}}$$
$$= -\frac{i}{\pi}\left(\frac{2\pi}{a}\right)^s \cos\left(\frac{\pi s}{2}\right)\Gamma(1-s)(\ell_{1-s}(x) - \ell_{1-s}(1-x)), \tag{13}$$

where $x = \frac{b}{a}$. He also proves abundance principle[1] for the functions that satisfy the functional equations (12), resp. (13), cf. Ref. 12, Theorems 2 and 3.

In what follows, we use the notation

$$f^e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f^o(x) = \frac{1}{2}(f(x) - f(-x))$$

or

$$f^e(x) = \frac{1}{2}(f(x) + f(1-x)), \quad f^o(x) = \frac{1}{2}(f(x) - f(1-x)) \tag{14}$$

as the case may be, so that

$$f = f^e + f^o.$$

We state the special case of (12) and (13) as a theorem.

Theorem 1. *The even part*

$$\frac{1}{2}(\zeta(s,x) + \zeta(s,1-x)) = \zeta^e(s,x)$$
$$= \frac{1}{\pi}(2\pi)^s \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\ell_{1-s}^e(x)$$
$$= (2\pi)^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)(\ell_{1-s}(x) + \ell_{1-s}(1-x)) \tag{15}$$

is in RHB correspondence with the theta-transformation formula (43)*. But it does not lead to the functional equation since it does not hold for* $x = 1$*.*

On the other hand, the odd part

$$\begin{aligned}\frac{1}{2}(\zeta(s,x)-\zeta(s,1-x)) &= \zeta^o(s,x)\\ &= -\frac{i}{\pi}(2\pi)^s\cos\left(\frac{\pi s}{2}\right)\Gamma(1-s)\ell^o_{1-s}(x)\\ &= -i(2\pi)^{s-1}\cos\left(\frac{\pi s}{2}\right)\Gamma(1-s)(\ell_{1-s}(x)-\ell_{1-s}(1-x)) \qquad (16)\end{aligned}$$

is in RHB correspondence (*in the form of* (56)) *with the transformation formula* (57) *for the conjugate theta function* (55). *In the limit as* $s \to 1$, (16) *in the long run entails the partial fraction expansion for the cotangent function: for nonintegral values of* z,

$$\begin{aligned}\cot \pi z &= \frac{1}{\pi z} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n-z}\right)\\ &= \frac{1}{\pi z} + \frac{2z}{\pi}\sum_{n=1}^{\infty}\frac{1}{z^2-n^2}, \qquad (17)\end{aligned}$$

where the last series is absolutely convergent. It is known that (17) *is equivalent to the functional equation* (3).

Proof of the even part is given in Lemma 1 below while the odd part will be settled in Corollary 1 as well as toward the end of this section.

Definition 1. Specifying the general case of the ramified functional equation (see Ref. 13, Chapter 7), we say that the Dirichlet series $\varphi(s)$ and $\psi_1(s)$, $\psi_2(s)$ satisfy the *ramified functional equation*

$$\begin{aligned}&\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\varphi(s)\\ &\quad = \Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\psi_1(1-s) + \Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\psi_2(1-s). \qquad (18)\end{aligned}$$

The following is the reciprocal of what is enunciated in Ref. 14.

Theorem 2. *The genesis of the functional equation* (3) *for the Riemann zeta function lies in the union of the even part* (15) *and* *the odd part* (16):

$$\zeta(s,x) = \frac{1}{\pi}(2\pi)^s\Gamma(1-s)\left(\sin\left(\frac{\pi s}{2}\right)\ell_{1-s}^e(x) - i\cos\left(\frac{\pi s}{2}\right)\ell_{1-s}^o(x)\right)$$
$$= (2\pi)^{s-1}\Gamma(1-s)\left(\sin\left(\frac{\pi s}{2}\right)(\ell_{1-s}(x)+\ell_{1-s}(1-x))\right.$$
$$\left. - i\cos\left(\frac{\pi s}{2}\right)(\ell_{1-s}(x)-\ell_{1-s}(1-x))\right), \tag{19}$$

which are miniscence of Euler's identity, where the even, resp. odd, part is in RHB correspondence with the theta functions, resp. the conjugate theta function. This leads to the functional equation (8) *in view of vanishing of the odd part, i.e. the odd part is discarded in the functional equation. This amounts to* (18) *with*

$$\varphi(s) = \pi^{-\frac{s}{2}}\zeta(s,x),$$
$$\psi_1(s) = \pi^{-\frac{s}{2}}\ell_s^e(x) = \frac{1}{2}(\pi^{-\frac{s}{2}}\ell_s(x)+\pi^{-\frac{s}{2}}\ell_s(1-x)), \tag{20}$$
$$\psi_2(s) = -i\pi^{-\frac{s}{2}}\ell_s^o(x) = -\frac{i}{2}(\pi^{-\frac{s}{2}}\ell_s(x)-\pi^{-\frac{s}{2}}\ell_s(1-x)).$$

We may say that the zeta functions φ, ψ_1, ψ_2 are in **ramified RHB correspondence** *with the theta function $\vartheta, \tilde{\vartheta}$ in the sense that φ, resp. ψ_i, is in RHB correspondence with ϑ, resp. $\tilde{\vartheta}$, with respective gamma factors.*

Proof. The equation (19) follows from Theorem 1. Equation (19) with $x = 1$ leads to (8). To prove that (20) satisfies (20), we use the well-known results: the duplication formula

$$\Gamma(s) = \sqrt{\pi}^{-1}2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$

and the reciprocal relation

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}.$$

We substitute

$$\Gamma(1-s) = \sqrt{\pi}^{-1} 2^{-s} \Gamma\left(\frac{1-s}{2}\right) \Gamma\left(1-\frac{s}{2}\right),$$
$$\sin\left(\frac{\pi s}{2}\right) = \frac{\pi}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)},$$
$$\cos\left(\frac{\pi s}{2}\right) = \sin\left(\frac{\pi(1-s)}{2}\right) = \frac{\pi}{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}$$

in the right-hand side of (19) and multiplying by $\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)$, we deduce that

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\zeta(s,x) = \frac{1}{\sqrt{\pi}}\pi^s \times \left(\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\ell^e_{1-s}(x) - i\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\ell^o_{1-s}(x)\right).$$

Finally, multiplying by $\pi^{-\frac{s}{2}}$, we conclude (18).

The modification (20) is common and has the effect of taking the basic sequence as $\{\sqrt{\pi}n\}$ in place of $\{n\}$. □

Corollary 1. *The derivative of* (19) *at* $s = 0$ *amounts to*

$$\zeta'(0,x) = -\big(\log 2\pi + \gamma\big)\bar{B}_1(x) + \frac{1}{2}A_1(x) + \frac{i}{\pi}\left(\ell_1'\right)^o(x), \tag{21}$$

where

$$\left(\ell_1'\right)^o(x) = \frac{1}{2}(\ell_1'(x) - \ell_1'(1-x)) = -i\sum_{n=1}^{\infty}\frac{\log n}{n}\sin 2\pi nx. \tag{22}$$

Under the assumption of the Lerch formula

$$\zeta'(0,x) = \log\frac{\Gamma(x)}{\sqrt{2\pi}}, \tag{23}$$

(21) *leads to* **Kummer's Fourier series**

$$\log\frac{\Gamma(x)}{\sqrt{2\pi}} = \sum_{n=1}^{\infty}\left(\frac{1}{2n}\cos 2\pi nx + \frac{\gamma + \log 2\pi n}{\pi n}\sin 2\pi nx\right). \tag{24}$$

For $0 \leq \varkappa \in \mathbb{Z}$ and $0 < x < 1$, we have

$$\ell_\varkappa(x) = \frac{(2\pi i)^{\varkappa-1}}{\varkappa!}\big(A_\varkappa(x) - \pi i \bar{B}_\varkappa(x)\big), \tag{25}$$

where the even part $A_\varkappa(x)$ is the Clausen function of order $\varkappa$

$$\begin{aligned} A_\varkappa(x) &= \frac{\varkappa!}{2(2\pi i)^{\varkappa-1}} \sum_{n=-\infty}^{\infty}{}' \frac{\operatorname{sgn}(n) e^{2\pi i n x}}{n^\varkappa} \\ &= \frac{\varkappa!}{2(2\pi i)^{\varkappa-1}} \Big(\ell_\varkappa(x) + (-1)^{\varkappa-1}\ell_\varkappa(1-x)\Big), \end{aligned} \tag{26}$$

while the odd part $\bar{B}_\varkappa(x)$ is the periodic Bernoulli polynomial of order $\varkappa$:

$$\begin{aligned} \bar{B}_\varkappa(x) &= -\frac{\varkappa!}{(2\pi i)^\varkappa} \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi i n x}}{n^\varkappa} \\ &= -\frac{\varkappa!}{(2\pi i)^\varkappa} \Big(\ell_\varkappa(x) - (-1)^{\varkappa-1}\ell_\varkappa(1-x)\Big) \end{aligned} \tag{27}$$

whereby the formula

$$\zeta(1-\varkappa, \{x\}) = -\frac{1}{\varkappa}\bar{B}_\varkappa(x). \tag{28}$$

These are complementary to each other and satisfy the parity relation

$$A_\varkappa(x) = (-1)^{\varkappa-1} A_\varkappa(1-x), \quad \bar{B}_\varkappa(x) = (-1)^\varkappa \bar{B}_\varkappa(1-x). \tag{29}$$

Regarding the odd part in Theorem 1, (16) *in the limit as $s \to 1$ entails the well-known formula*

$$\psi(x) - \psi(1-x) = -\pi \cot \pi x \tag{30}$$

valid for nonintegral values of x, where

$$\psi(x) = \frac{\Gamma'}{\Gamma}(x) \tag{31}$$

is the Euler digamma function and γ indicates the Euler constant

$$\gamma = -\psi(1). \tag{32}$$

Proof. The derivative of (19) at $s = 0$ amounts to

$$2\pi i\zeta'(0,x) = \left(\log 2\pi - \Gamma'(1)\right)(\ell_1(x) - \ell_1(1-x)) \\ + \frac{\pi}{2} i (\ell_1(x) + \ell_1(1-x)) - (\ell_1'(x) - \ell_1'(1-x)) \quad (33)$$

on correcting (see Ref. 6, (4.38)). Substituting (27), (26) and

$$\ell_1'(x) = -\sum_{n=1}^{\infty} \frac{\log n}{n} e^{2\pi i n x} \quad (34)$$

in (33) and invoking (23), we deduce (21) and (22). Adding (33) with x replaced by $1-x$ to (33), we find the Fourier series for the Clausen function $A_1(x)$ in (26) with $\varkappa = 1$.

(19) with $s = 0$ leads to the $\varkappa = 1$ case of (27) in view of (28). The even part (15) leads only to the parity relation $B_1(x)+B_1(1-x) = 0$. These establish (10), the $\varkappa = 1$ case. Higher order case (25) follows by integration, implying (26) and (27).

On the other hand, regarding the limiting case as $s \to 1$ of (16), we compare the well-known formulas

$$\ell_0(x) = \frac{e^{2\pi i x}}{1 - e^{2\pi i x}} = \frac{1}{2}(-1 + i \cot \pi x) \quad (35)$$

for $x \in \mathbb{R} - \mathbb{Z}$ and the Laurent expansion

$$\zeta(s,x) = \frac{1}{s-1} - \psi(x) + O(s-1) \quad \text{as} \quad s \to 1 \quad (36)$$

to conclude (30).

Higher order case (26) follows by integration.

The Gaussian representation

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(n+x)} \quad (37)$$

and (31) give rise to (17), thereby completing the proof of the odd part assertion in Theorem 1. By (32), the Laurent expansion (36) reduces to

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1) \quad \text{as} \quad s \to 1. \quad (38)$$

□

Remark 1. Milnor (see Ref. 15, Appendix 1) incorporates the parity relation in (6) and Theorem 1 is its ultimate form.

(19) is valid for all x in the form

$$\zeta(s, 1-\{x\}) = \frac{1}{\pi}(2\pi)^s\Gamma(1-s) \times \left(\sin\left(\frac{\pi s}{2}\right)\ell^e_{1-s}(x) - i\cos\left(\frac{\pi s}{2}\right)\ell^o_{1-s}(x)\right), \quad (39)$$

where

$$\{x\} = x - [x] = \bar{B}_1(x) + \frac{1}{2} \quad (40)$$

is the fractional part of x, $[x]$ being the integral part of x defined below (11), cf. the passage after (61).

We note that we need the full force (19) or (5) to derive Kummer's Fourier series. With (the derivative of) the even part (15) alone, we can obtain only the Fourier series for the Clausen function $A_1(x)$ in (26), not Kummer's Fourier series. It can be shown that Kummer's Fourier series is, in the long run, equivalent to the functional equation for the Riemann zeta function (see Refs. 16, p. 108 and 17, pp. 168–175).

Under (37), all equations (9), (17), (30), (35) are equivalent:

$$\psi(x) - \psi(1-x) = 2\pi i\ell_0(x) + \pi i = 2\pi i\frac{e^{2\pi ix}}{1-e^{2\pi ix}} + \pi i$$
$$= -\pi\cot\pi x = -\frac{1}{x} - 2x\sum_{n=1}^{\infty}\frac{1}{x^2-n^2}, \quad 0 < x < 1.$$

Equation (27), resp. (30), being the case $s = 0$ resp. (the limiting case as) $s \to 1$ of (19) are connected by the generalized (inverse) Eisenstein formula

$$\sum_{a=1}^{M-1} e^{-2\pi i\frac{ab}{M}}\bar{B}_1\left(\frac{a}{M}\right) = -\frac{1}{2i}\cot\frac{\pi b}{M}, \quad (41)$$

cf. Corollary 2.

We briefly discuss Ogg's result,[18] stated in Lemma 1, which refers to the RHB correspondence with the theta-transformation formula and establishes the case of the even part of Theorem 1. We return to Ogg's main result, RHB proof of the Jacobi triple product in Section 3.

Let $\vartheta(\tau, x)$ denote $\vartheta_3(\tau, x)$, one of 4 Jacobi theta functions, cf., e.g. Refs. 19, 20:

$$\vartheta(\tau, x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x + \pi i \tau n^2} = \sum_{n=-\infty}^{\infty} z^n q^{n^2}$$

$$= 1 + \sum_{n=1}^{\infty} 2\cos(2\pi n x) q^{n^2}, \tag{42}$$

where $z = e^{2\pi i x}$, $q = e^{\pi i \tau}$ and $\tau \in \mathcal{H}$, $\mathcal{H}$ indicating the upper half-plane. $\vartheta(it, 0) = \sum_{n=-\infty}^{\infty} e^{-\pi t n^2}$ is the theta function used by Riemann to deduce the functional equation for $\zeta(s)$, cf. Remark 2. It is clear that $\vartheta(\tau, x)$ is a doubly periodic function in τ and x with periods 2 and 1, respectively.

Its characteristic property is the *theta-transformation formula*

$$e^{\pi i \tau x^2} \vartheta(\tau, \tau x) = \sqrt{\frac{i}{\tau}} \vartheta\left(-\frac{1}{\tau}, x\right), \tag{43}$$

which we may express as

$$f_j(\tau, x) = \sqrt{\frac{i}{\tau}} f_k\left(-\frac{1}{\tau}, x\right), \quad \{j, k\} = \{1, 2\},$$

on putting

$$f_1(\tau, x) = e^{\pi i \tau x^2} \vartheta(\tau, \tau x), \quad f_2(\tau, x) = \vartheta(\tau, x).$$

We use the case of $\tau = it$, $t > 0$

$$f_j(it, x) = \sqrt{\frac{1}{t}} f_k\left(\frac{i}{t}, x\right), \quad \{j, k\} = \{1, 2\}. \tag{44}$$

Let

$$f_j(\infty) = \lim_{\operatorname{Im} \tau \to \infty} f_j(\tau, x), \quad j = 1, 2.$$

Then

$$f_1(\infty) = \begin{cases} 1 & x \in \mathbb{Z}, \\ 0 & x \notin \mathbb{Z}, \end{cases} \quad f_2(\infty) = 1.$$

For $x \in \mathbb{R}$, consider the Mellin transform

$$\Phi_j(2s, x) = \int_0^\infty t^s (f_j(it, x) - f_j(\infty)) \frac{\mathrm{d}t}{t}, \quad j = 1, 2, \tag{45}$$

where $\sigma > 0$.

Lemma 1 (Ogg). *The functional equation*

$$\Phi_j(s, x) = \Phi_k(1 - s, x), \quad \{j, k\} = \{1, 2\} \tag{46}$$

is in RHB correspondence with (43) *and constitutes the even part in Theorem* 1.

$\Phi_j(s, x)$ *with correction terms is EBV* (*entire and bounded in every vertical strip*, (*see Ref.* 21, *p. xiii*)).

Proof. Note that

$$\begin{aligned} f_1(it, x) &= e^{-\pi t x^2} \vartheta(it, itx) = \sum_{n=-\infty}^{\infty} e^{-\pi t (n+x)^2}, \\ f_2(it, x) &= \vartheta(it, x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x - \pi t n^2}, \end{aligned} \tag{47}$$

As a memorial to Hecke, we call the special case of (1) the *Hecke gamma transform* (for $x > 0$ and $\sigma > 0$):

$$\pi^{-s} \Gamma(s) y^{-s} = \int_0^\infty t^s e^{-\pi y t} \frac{\mathrm{d}t}{t}, \tag{48}$$

which is used with $y = (n + x)^2$, resp. $y = n^2$. By this, we find that

$$\Phi_j(2s, x) = \pi^{-s} \Gamma(s) \phi_j(2s, x), \quad j = 1, 2, \tag{49}$$

where

$$\phi_1(s, x) = \sum_{n=-\infty, n+x \neq 0}^{\infty} |n + x|^{-s}, \quad |n + x|^{-s} = e^{-s \log |n+x|},$$

so that

$$\phi_1(s, x) = \zeta(s, \{x\}) + \zeta(s, 1 - \{x\}) = 2\zeta^e(s, \{x\}) \tag{50}$$

and

$$\begin{aligned}\phi_2(s, x) &= \sum_{n=-\infty, n\neq 0}^{\infty} e^{2\pi i n x}|n|^{-s} = \ell_s(x) + \ell_s(1 - x) \\ &= \sum_{n=1}^{\infty} 2\cos(2\pi n x)n^{-s} = 2\ell_s^e(x).\end{aligned} \tag{51}$$

Equation (49) leads to (46) since the functional equation in RHB correspondence with (44) is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\phi_j(s, x) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1 - s}{2}\right)\phi_k(1 - s, x). \tag{52}$$

By the same argument as in the proof of Theorem 2, we may transform (52) into the even part of Theorem 1. □

Remark 2. $\phi_1(s, x)$ in (50) appears again as the lattice function in (96). This being a perturbed Dirichlet series, which are associated with the Fourier–Bessel expansion (see Ref. 13, Chapter 4), we will see two examples of the Fourier–Bessel expansion in subsequent sections.

In the above argument, the variable $2s$ is used instead of s which mean that one considers the sequence $\{n^2\}$ rather than $\{n\}$ to the effect that one considers the Epstein zeta function in a unary quadratic form.

We analyze Riemann's way of using the theta function (see Ref. 10, pp. 61–62). Let

$$\omega(t) = \frac{1}{2}\vartheta(it, 0) - \frac{1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 t}, \quad \mathrm{Re}\, t > 0.$$

Then (43) reads for $\omega(t)$

$$\omega(t^{-1}) = -\frac{1}{2} + \frac{1}{2}\sqrt{t} + \sqrt{t}\omega(t). \tag{53}$$

Implicit in (45) and (51) with $2s$ replaced by s is

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty t^{\frac{s}{2}}\omega(t)\,\frac{\mathrm{d}t}{t}. \tag{54}$$

Riemann divides the integral in (54) into two parts, $\int_0^1$ and $\int_1^\infty$, makes the change of variable $t \leftrightarrow \frac{1}{t}$ and applies (53) to obtain a symmetric form. Thus, Riemann's method gives rise to the Ewald expansion — expansion with incomplete gamma function coefficients. It is developed by Lavrik and later by Terras, cf. Ref. 13, Chapter 5. The use of the full series amounts to Theorem 2.

Proof of the case of the odd part is given in Theorem 1.

It is natural to introduce the *conjugate* theta function as the theta function-like (modular) function corresponding to (13). This was known to de la Vallée Poussin, cf. Remark 3.

$$\begin{aligned}\tilde{\vartheta}(\tau, x) &= \sum_{n=-\infty}^{\infty} n e^{2\pi i n x + \pi i \tau n^2} = \sum_{n=-\infty}^{\infty} n z^n q^{n^2} = z\frac{\partial}{\partial z}\vartheta(\tau, x)\\ &= \sum_{n=1}^{\infty} 2in\sin(2\pi nx)q^{n^2}.\end{aligned} \tag{55}$$

(18) necessitates the validity of

$$\Gamma\left(\frac{1+s}{2}\right)\varphi(s) = \Gamma\left(1-\frac{s}{2}\right)\psi_2(s). \tag{56}$$

It follows from RHB correspondence (i.e. from the Mellin inversion (1)) that for (56) to hold, it is necessary and sufficient that the transformation formula is to be of the form

$$e^{\pi i \tau x^2}\tilde{\vartheta}(\tau, \tau x) = \sqrt{\frac{i}{\tau^3}}\tilde{\vartheta}\left(-\frac{1}{\tau}, x\right). \tag{57}$$

Summarizing, we have

Theorem 3. *The conjugate theta function* (55) *satisfies the transformation formula* (57) *which is in RHB correspondence with the functional equation* (56). *It is odd w.r.t. x and $\tilde{\vartheta}(\tau, 0) = 0$.*

We quote results of Ref. 22 which are basis of the fact for the odd part in Theorem 1.

Lemma 2.

(i) *The partial fraction expansion for the cotangent function and the functional equation* (3) *for the Riemann zeta function are equivalent.*
(ii) *The infinite product expansion for the sine function and the partial fraction expansion for the cotangent function are equivalent.*

1.1. *Periodic Dirichlet series*

In connection with Remark 1, we summarize the results on periodic Dirichlet series as developed by Refs. 7, 8, 11, 23, etc.

The theory of DFT for arithmetic functions has been developed in Ref. 24 in the case of periodic functions. Cf. also Refs. 16 (Section 8.1) and 25 (Sections 4.1 and 4.3). Let $C(M)$ be the vector space of all periodic arithmetic functions f with period M. As in (14), we let

$$f^e = \frac{1}{2}\left(f(n \bmod M) + f(-n \bmod M)\right),$$
$$f^o = \frac{1}{2}\left(f(n \bmod M) - f(-n \bmod M)\right)$$

be the even, resp. odd, part of f: $f = f_{\text{even}} + f_{\text{odd}}$. Let

$$\varepsilon_j(a) = e^{2\pi i j a/M}, \quad 1 \le j \le M,$$

where a is an integer variable. Then the set $\{\varepsilon_j | 1 \le j \le M\}$ forms a basis of $C(M)$. We define the DFT $\hat{f}$ (or the bth Fourier coefficient) of $f \in C(M)$ by

$$\hat{f}(b) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M} \varepsilon_b(-a) f(a).$$

The parity inherits to the DFT and the Fourier inversion or Fourier expansion formula holds true:

$$f(a) = \frac{1}{\sqrt{M}} \sum_{b=1}^{M} \hat{f}(b)\varepsilon_b(a) = \hat{\hat{f}}(-a). \tag{58}$$

Note that (58) is the expression of f with respect to the basis $\{\varepsilon_j\}$.

$$\hat{f}^e(b) = \frac{1}{2}(\hat{f}(b) + \hat{f}(-b)) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M} \cos\left(2\pi \frac{b}{M} a\right) f(a),$$

$$\hat{f}^o(b) = \frac{1}{2}(\hat{f}(b) - \hat{f}(-b)) = -i \frac{1}{\sqrt{M}} \sum_{a=1}^{M-1} \sin\left(2\pi \frac{b}{M} a\right) f(a).$$

To find another natural basis, let χ_a be the characteristic function χ_a mod M (see Ref. 25, p. 73).

$$\chi_a(n) = \begin{cases} 1 & n \equiv a \bmod M, \\ 0 & n \not\equiv a \bmod M. \end{cases}$$

Then $\{\chi_a | 1 \le a \le M\}$ is a basis of $C(M)$ and

$$\sqrt{M} \hat{\chi}_a(n) = \sum_{j=1}^{M} \varepsilon_n(-j) \chi_a(j) = \varepsilon_a(-n).$$

For $f \in C(M)$, let

$$D(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \tag{59}$$

Since

$$\sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma} \ll \zeta(\sigma),$$

the series in (59) is absolutely convergent for $\sigma > 1$. Let $D(M)$ denote the set of all Dirichlet series of the form (59). Then it forms a vector space of dimension M canonically isomorphic to $C(M)$. One of the bases of $D(M)$ is $\{\ell_s\left(\frac{a}{M}\right) | 1 \le a \le M\}$, where $\ell_s(x)$ is the Lerch zeta function in (4). Hence, we have

$$\begin{aligned} D(s, f) &= \frac{1}{\sqrt{M}} \sum_{a=1}^{M} \hat{f}(a) \ell_s\left(\frac{a}{M}\right) \\ &= \frac{1}{\sqrt{M}} \sum_{a=1}^{M-1} \hat{f}(a) \ell_s\left(\frac{a}{M}\right) + \frac{\hat{f}(M)}{\sqrt{M}} \zeta(s). \end{aligned} \tag{60}$$

It follows that $D(s, f)$ can be continued meromorphically over the whole plane and that it is entire if and only if $\hat{f}(M) = 0$ which is defined by

$$\hat{f}(M) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M} f(a).$$

Another basis of $D(M)$ is $\{D(s, \chi_a) | 1 \le a \le M\}$, where

$$D(s, \chi_a) = \sum_{n=1}^{\infty} \frac{\chi_a(n)}{n^s} = \sum_{\substack{n=1 \\ n \equiv a \bmod M}}^{\infty} \frac{1}{n^s} = \zeta(s, a, M) = M^{-s} \zeta\Big(s, \frac{a}{M}\Big), \tag{61}$$

where $\zeta(s, a, M)$ indicates the partial zeta function. Note that it is $\zeta(s, 1 - \{\frac{a}{M}\})$ that belongs to $D(M)$ rather than $\zeta(s, \{\frac{a}{M}\})$, cf. Remark 1. Hence, parallel to (60), we have another expression

$$D(s, f) = \sum_{a=1}^{M} f(a) \zeta(s, a, M) = \frac{1}{M^s} \sum_{a=1}^{M} f(a) \zeta\left(s, \frac{a}{M}\right).$$

Proposition 1.

$$\begin{aligned} \frac{1}{M^s} \sum_{a=1}^{M} f(a) \zeta\left(s, \frac{a}{M}\right) &= \sum_{a=1}^{M} f(a) \zeta(s, a, M) = D(s, f) \\ &= \frac{1}{\sqrt{M}} \sum_{a=1}^{M} \hat{f}(a) \ell_s\left(\frac{a}{M}\right) \\ &= \frac{1}{\sqrt{M}} \sum_{a=1}^{M-1} \hat{f}(a) \ell_s\left(\frac{a}{M}\right) + \frac{\hat{f}(M)}{\sqrt{M}} \zeta(s), \end{aligned} \tag{62}$$

which entails

$$-\frac{1}{M} \sum_{k=1}^{M} f(k) \psi\left(\frac{k}{M}\right) = D(1, f) = \frac{1}{\sqrt{M}} \sum_{k=1}^{M-1} \hat{f}(k) \ell_1\left(\frac{k}{M}\right) + \frac{\hat{f}(M)}{\sqrt{M}} \gamma \tag{63}$$

as well as

$$D(s, f^o) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M-1} \hat{f}^o(a) \ell_s\left(\frac{a}{M}\right). \tag{64}$$

Proof. Equation (63) is the equality between the Laurent constants of both sides of (62), where (36) and (38) are used. (64) follows since $\hat{f}^o(M) = 0$. □

Corollary 2. (i) *The generalized* (*inverse*) *Eisenstein formula* (41) *is the odd part of* (63) *with* $f(k) = \chi_b(k)$.
(ii) *We have*

$$D'(1, f^o) = \frac{1}{\sqrt{M}} \sum_{a=1}^{M-1} \hat{f}(a) \left(\ell_1'\right)^o \left(\frac{a}{M}\right)$$
$$= -\frac{i}{\sqrt{M}} \sum_{a=1}^{M-1} \hat{f}(a) \sum_{n=1}^{\infty} \frac{\log n}{n} \sin\left(2\pi \frac{a}{M} n\right) \tag{65}$$

which leads, under the assumption of the Lerch formula (23) *and the functional equation* (6), *to*

$$D'(1, f^o) = -\frac{i}{\sqrt{M}} \frac{\pi}{2} \sum_{a=1}^{M-1} \hat{f}(a) \left(\log \Gamma\left(\frac{a}{M}\right) - \log \Gamma\left(1 - \frac{a}{M}\right)\right)$$
$$- \frac{i}{\sqrt{M}} \pi(\gamma + \log 2\pi) \sum_{a=1}^{M-1} \hat{f}(a) \bar{B}_1\left(\frac{a}{M}\right). \tag{66}$$

Proof. The odd part of (63) reads

$$-\frac{1}{M} \sum_{k=1}^{M} f(k) \left(\psi\left(\frac{k}{M}\right) - \psi\left(1 - \frac{k}{M}\right)\right)$$
$$= \frac{1}{\sqrt{M}} \sum_{k=1}^{M-1} \hat{f}(k) \left(\ell_1\left(\frac{k}{M}\right) - \ell_1\left(-\frac{k}{M}\right)\right).$$

By the property of $\{\kappa_a\}$,

$$-\frac{1}{M} \left(\psi\left(\frac{b}{M}\right) - \psi\left(1 - \frac{b}{M}\right)\right)$$
$$= \frac{1}{M} \sum_{k=1}^{M-1} \varepsilon_b(-k) \left(\ell_1\left(\frac{k}{M}\right) - \ell_1\left(-\frac{k}{M}\right)\right).$$

Substituting (27) with $\varkappa = 1$ and (30), we conclude (41).

The equation (65) follows by differentiating (62), whereby we appeal to (34). For (66), the Hurwitz formula (6) and the Kummer Fourier series (24) are used. □

Corollary 3. *Let χ be a nonprincipal Dirichlet character mod M. Then*

$$\frac{1}{M^s}\sum_{a=1}^{M-1}\chi(a)\zeta\left(s,\frac{a}{M}\right)=L(s,\chi)=\frac{1}{\sqrt{M}}\sum_{a=1}^{M-1}\hat{\chi}(a)\ell_s\left(\frac{a}{M}\right).$$

Corresponding to Theorem 2, Schnee's theorem as modified by Ishibashi and Kanemitsu reads as follows.

Theorem 4 (7, 8). *Suppose f is a periodic arithmetic function with period M. Then*

$$D(1-s,f)=\left(\frac{\pi}{M}\right)^{\frac{1}{2}-s}\left(\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}D(s,\hat{f}^e)+i\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}D(s,\hat{f}^o)\right) \tag{67}$$

or

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)&\left(\frac{\pi}{M}\right)^{-\frac{s}{2}}D(s,f)\\ &=\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\left(\frac{\pi}{M}\right)^{-\frac{1-s}{2}}D(1-s,\hat{f}^e)\\ &\quad+i\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)\left(\frac{\pi}{M}\right)^{-\frac{1-s}{2}}D(1-s,\hat{f}^o).\end{aligned} \tag{68}$$

This is equivalent to the generalized Euler identity

$$\begin{aligned}\tilde{D}(1-s,f)&=\sqrt{\pi}^{-1}2^{1-s}\Gamma(s)\\ &\qquad\times\left(\cos\left(\frac{\pi s}{2}\right)\tilde{D}(s,\hat{f}^e)+i\sin\left(\frac{\pi s}{2}\right)\tilde{D}(s,\hat{f}^o)\right),\end{aligned}$$

where

$$\tilde{D}(s,f)=\left(\frac{\pi}{M}\right)^{-\frac{s}{2}}D(s,f).$$

Proof. Equation (67) amounts to (68), i.e. the ramified functional equation (18), on clearing the denominators and multiplying by $\left(\frac{\pi}{M}\right)^{\frac{s-1}{2}}$. □

Definition 2. Define the general Gauss sum

$$\tau(\chi, n) = \sum_{a=1}^{M} \chi(a) e^{\frac{2\pi i n a}{M}} = \sqrt{M}\hat{\chi}(-n) \tag{69}$$

and the (normalized) Gauss sum

$$\tau(\chi) = \tau(\chi, 1) = \sum_{a=1}^{M} \chi(a) e^{\frac{2\pi i a}{M}},$$

recall Ref. 26, Theorem 8.19, p. 171:

Lemma 3. (i) *Suppose χ is a nonprincipal character mod M. Then the general Gauss sum* (69) *is separable, i.e.*

$$\tau(\chi, n) = \bar{\chi}(n)\tau(\chi) \tag{70}$$

if and only if χ is a primitive character mod M.
(ii) *For a Kronecker character, which is real primitive,*

$$\tau(\chi_d) = \begin{cases} i\sqrt{|d|} & d < 0, \\ \sqrt{d} & d > 0. \end{cases} \tag{71}$$

It follows from (69) and (70) that for a primitive χ,

$$\hat{\chi}(n) = \frac{\tau(\chi)}{\sqrt{M}} \chi(-1)\bar{\chi}(n) \tag{72}$$

and

$$\chi(-1)\tau(\chi)\tau(\bar{\chi}) = |\tau(\chi)|^2 = M.$$

Equation (65) reads for a primitive odd character χ:

$$L'(1, \chi) = -\frac{\tau(\chi)}{M} \sum_{a=1}^{M-1} \bar{\chi}(a) \left(\ell_1'\right)^o \left(\frac{a}{M}\right)$$

and under the Lerch formula (23),

$$L'(1, \chi) = \pi i \frac{\tau(\chi)}{M} \left(\sum_{a=1}^{M-1} \bar{\chi}(a) \log \Gamma\left(\frac{a}{M}\right) + (\gamma + \log 2\pi) B_{1,\bar{\chi}} \right), \tag{73}$$

where

$$B_{1,\bar{\chi}} = \sum_{a=1}^{M-1} \bar{\chi}(a)\bar{B}_1\left(\frac{a}{M}\right)$$

is the first generalized Bernoulli number.

Corollary 3 reads for a primitive odd character χ:

$$L(1,\chi) = \pi i \frac{\tau(\chi)}{M} B_{1,\bar{\chi}} \tag{74}$$

by (27).

Substituting (74) in (73), we deduce the following.

Lemma 4. *For a primitive odd character* χ, *we have*

$$\gamma L(1,\chi) + L'(1,\chi) = -\frac{\tau(\chi)}{M} \sum_{a=1}^{M-1} \bar{\chi}(a)\left(\ell_1'\right)^o\left(\frac{a}{M}\right) + \gamma L(1,\chi),$$

which amounts to

$$\gamma L(1,\chi_d) + L'(1,\chi_d) = -\frac{i}{\sqrt{|d|}} \sum_{a=1}^{|d|-1} \bar{\chi}(a)\left(\ell_1'\right)^o\left(\frac{a}{|d|}\right) + \gamma L(1,\chi_d) \tag{75}$$

for the Kronecker character with $d < 0$ *in view of* (71).

Under the Lerch formula (23), *these read*

$$\gamma L(1,\chi) + L'(1,\chi) = \pi i \frac{\tau(\chi)}{M} \sum_{a=1}^{M-1} \bar{\chi}(a) \log\Gamma\left(\frac{a}{M}\right) + (2\gamma + \log 2\pi)L(1,\chi)$$

and

$$\gamma L(1,\chi_d) + L'(1,\chi_d) = -\frac{\pi}{\sqrt{|d|}} \sum_{a=1}^{|d|-1} \bar{\chi}(a) \log\Gamma\left(\frac{a}{|d|}\right) + (2\gamma + \log 2\pi)L(1,\chi_d). \tag{76}$$

Corresponding to Theorem 2, Theorem 4 in the special case reads as follows.

Theorem 5. *For a primitive character* χ, *the functional equation* (67) *reads*

$$\left(\frac{\pi}{M}\right)^{-\frac{1-s}{2}} L(1-s,\chi) = \frac{\tau(\chi)}{\sqrt{M}}\left(\frac{\pi}{M}\right)^{-\frac{s}{2}}$$
$$\left(\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} L^e(s,\bar\chi) - \frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} i L^o(s,\bar\chi)\right)$$

or

$$\left(\frac{\pi}{M}\right)^{-\frac{1-s}{2}} L(1-s,\chi) = \frac{\tau(\chi)}{\sqrt{M}}\left(\frac{\pi}{M}\right)^{-\frac{s}{2}}$$
$$\left(\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \frac{1+\chi(-1)}{2} L(s,\bar\chi) - \frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} \frac{1-\chi(-1)}{2} i L(s,\bar\chi)\right). \tag{77}$$

In the form of a generalized Euler identity,

$$\begin{aligned}
L(s,\chi) &= \frac{\tau(\chi)}{\pi}\left(\frac{2\pi}{M}\right)^s \Gamma(1-s) \\
&\quad \times \left(\sin\left(\frac{\pi s}{2}\right) L^e(1-s,\bar\chi) - i\cos\left(\frac{\pi s}{2}\right) L^o(1-s,\bar\chi)\right) \\
&= \frac{\tau(\chi)}{\pi}\left(\frac{2\pi}{M}\right)^s \Gamma(1-s) \\
&\quad \times \left(\sin\left(\frac{\pi s}{2}\right) \frac{1+\chi(-1)}{2} L(1-s,\bar\chi)\right. \\
&\quad \left. - i\cos\left(\frac{\pi s}{2}\right) \frac{1-\chi(-1)}{2} L(1-s,\bar\chi)\right) \\
&= \frac{\tau(\chi)}{\pi}\left(\frac{2\pi}{M}\right)^s \Gamma(1-s) \\
&\quad \times \left(\sin\left(\frac{\pi s}{2}\right) (1-\mathfrak{a}(\chi)) L(1-s,\bar\chi)\right. \\
&\quad \left. - i\cos\left(\frac{\pi s}{2}\right) \mathfrak{a}(\chi) L(1-s,\bar\chi)\right),
\end{aligned} \tag{78}$$

where $\mathfrak{a}(\chi)$ *is the parity symbol*

$$\mathfrak{a} = \mathfrak{a}(\chi) = \frac{1-\chi(-1)}{2} = \begin{cases} 0 & \chi(-1) = 1, \\ 1 & \chi(-1) = -1. \end{cases}$$

This is the genesis of the functional equation for the Dirichlet L-function, as described by de La Vallée Poussin (as in Davenport).

Proof. The equation (77) is a direct application of Theorem 4 together with (72), the expression of the DFT of χ in view of separability of the general Gauss sum and (78) follows from (77) as in the proof of Theorem 2.

Then the common expression for the functional equation for the Dirichlet L-function

$$\xi(s,\chi) := \left(\frac{\pi}{M}\right)^{-\frac{s}{2}} \Gamma\left(\frac{s+\mathfrak{a}}{2}\right) L(s,\chi) = -\frac{i^{\mathfrak{a}}\tau(\chi)}{\sqrt{M}}\xi(1-s,\bar{\chi})$$

can be read out from (77).

It remains to specify the theta-like functions and their transformation formulas which are in RHB correspondence with $D(s,f^e)$, resp. $D(s,f^o)$, as in Theorem 1:

$$\begin{aligned} \psi(t,f^e) &= \vartheta(it,0,f^e) = \sum_{n=-\infty}^{\infty} f^e(n) e^{-\frac{\pi}{M}tn^2}, \\ \tilde{\psi}(t,f^o) &= \tilde{\vartheta}(it,0,f^o) = \sum_{n=-\infty}^{\infty} n f^o(n) e^{-\frac{\pi}{M}tn^2}, \end{aligned} \tag{79}$$

which satisfies the transformation formula

$$\sqrt{\frac{1}{t}}\psi\left(\frac{1}{t}, f^e\right) = \psi(t,f^e), \quad \sqrt{\frac{1}{t^3}}\psi\left(\frac{1}{t}, f^o\right) = \tilde{\psi}(t,f^o).$$

These follow from the Mellin inversion (1), i.e. RHB correspondence. □

Remark 3. Theorem 5 is a rather special case of Theorem 4 for a primitive χ. It is possible to state more general results for χ not necessarily primitive, cf. Refs. 27 and 28. In the former, separability of Gauss sum in Lemma 3 is deduced from the functional equation. This general case will be studied elsewhere.

In the case of Dirichlet L-functions, (79) has been used by de la Vallée Poussin (see Ref. 29, p. 302) cf., e.g. Ref. 10, pp. 68–71, and similarly, the conjugate theta function (55).

Since our main concern lies in the odd part, only the Lerch zeta expression of Proposition 1 is used and Corollary 3 is to contain the unused part, i.e. (63) is to read for a nonprincipal character χ

$$-\frac{1}{M}\sum_{k=1}^{M}\chi(k)\psi\left(\frac{k}{M}\right) = L(1,\chi),$$

which is in Ref. 16, (8.28), p. 173, and is a basis of the main result of Ref. 30, (also see Ref. 16, Theorem 8.2, pp. 174–175) to the effect that Gauss first formula for ψ is equivalent to finite expressions for $L(1,\chi)$. We return to this elsewhere.

2. Genesis of the Lerch–Chowla–Selberg Formula

In this section, we shall pursue the genesis of the Lerch–Chowla–Selberg formula from the point of view of RHB correspondence. This is a relation between the theta function and the gamma function at rational argument. The latter is already prepared in the preceding sections and the main part is the expression of the constant term of the Laurent expansion of the Dedekind zeta function at $s = 1$ of a quadratic field. It is well known that the Dedekind zeta function may be studied through that of the Epstein zeta function or the Eisenstein series.

We consider $\vartheta(\tau, x) = \vartheta(\tau, z)$ of (42) with $z = 1$. Let $q = e^{\pi i\tau}$, $\tau \in \mathcal{H}$:

$$\theta(\tau) = \vartheta(\tau, 0) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + \sum_{n=1}^{\infty} 2q^{n^2}$$
$$= \prod_{n=1}^{\infty}\left(1 - q^{2n}\right)\left(1 + q^{2n-1}\right)^2, \tag{80}$$

the last equality because of the Jacobi triple product (115). The Dedekind eta function $\eta(\tau)$ is essentially the factor of theta defined by

$$\eta(\tau) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n}) = q^{\frac{1}{12}} \theta^e(\tau)$$

or

$$\log \eta(\tau) = \frac{1}{12} \log q + \sum_{n=1}^{\infty} \log(1 - q^{2n}) = \frac{1}{12} \pi i \tau - \sum_{n=1}^{\infty} \sigma_{-1}(n) q^{2n}.$$

Reference 13 (pp. 12–15) streamlines the historical facts about the Lerch–Chowla–Selberg formula partially following Ref. 31 with reference to Refs. 32 and 43. The main interest lies in the fact that there is an equality of two irrelevant-looking special functions, gamma and theta, at rational arguments. Here, we shall show a plausible genesis of the formula from the viewpoint of RHB correspondence.

It is a combination of two remarkable formulas: Kronecker limit formula and Lerch formula. For the former, there is enormous amount of literature see Refs. 33–37, 53, and references therein. For the latter, there is a detailed account in Ref. 38, especially the references on pp. 67–68 are rather thorough. However, there is no mention of the first statement of the Lerch–Chowla–Selberg formula (see Ref. 39, (26), p. 303) nor earlier works by Berger *et al.*[40] Both Refs. 31 and 38 stress Lerch's formula (23) without resorting to its defining property as the principal solution to the difference equation

$$f(x+1) - f(x) \ = \log^{\alpha} x, \quad x > 0 \tag{81}$$

for each $\alpha \in \mathbb{N}$, cf. Refs. 41 and 42, 61 etc. To obtain the Chowla-Selberg formula for a real quadratic field, the evaluation of the cosine Fourier series is needed (see Ref. 43, (165), p. 180):

$$\Phi(x) = \sum_{n=1}^{\infty} \frac{\log n}{n} \cos 2\pi x n \tag{82}$$

as opposed to the Kummer Fourier series. It had been studied by Lerch[44] before Landau. It was taken up by Gut[45] to derive the Kronecker limit formula for cyclotomic fields. It turns out that the function which Gut introduced is $-(R(x)+R(1-x))$, where $R(x) = R_2(s)$ is the Deninger R-function as the principal solution to (82) with

Table 1. Fourier sine, cosine series and their parity.

Quadr. field	Imaginary	Real
Fourier ser.	$\log\Gamma(x)$	$-\Phi(x)$
Hurwitz zeta	$\zeta'(0,x)$	$R(x)+R(1-x)$
Lerch zeta	$-\frac{i}{2}(\ell_1'(x)-\ell_1'(1-x))$	$-\frac{1}{2}(\ell_1'(x)+\ell_1'(1-x))$

$\alpha = 2$. Deninger's standpoint is that the R-function occupies its space as a proper special function in the same way as $\log\Gamma(x)$ is the principal solution to (82) with $\alpha = 1$ (Bohr–Mollerup theorem). Then from the uniqueness of the principal solution, it follows that

$$R(x) = -\zeta''(0,x),$$

where the prime means the derivative w.r.t. s. As Table 1 shows, the parity of imaginary and real quadratic field and the associated Fourier series are consistent.

Here, $\log\Gamma(x)$ for the Fourier sine series is up to some extra terms. Higher order principal solutions $R_\alpha(x)$ to (81) stimulated the study on higher derivatives of L-functions.[17]

2.1. *Chowla–Selberg integral formula*

In this section, we deduce the Dirichlet class number formula and the Kronecker limit formula for an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$ with discriminant $d < 0$. Let

$$\zeta_K(s) = \sum_{0\neq\mathfrak{a}\in A} (N\mathfrak{a})^{-s},\ \sigma > 1,$$

be the Dedekind zeta function of K, where $N\mathfrak{a} = [\mathfrak{O} : \mathfrak{a}]$ is the norm of the ideal $\mathfrak{a}$ and where $\mathfrak{O} = \mathfrak{O}_K$ is the ring of algebraic integers in K. Then it amounts to finding the residue and the constant term of the Laurent expansion of $\zeta_K(s)$ at $s = 1$. The method is classical and is due to Dirichlet, resp. Kronecker, cf., e.g. Ref. 10, resp. Ref. 35.

Let I/P denote the absolute ideal class group, where I and P signify the group of all fractional ideals and the principal fractional ideals of K, respectively. The order $|I/P|$ is finite and is called the class number of K, denoted h. For an ideal class $A \in I/P$, let

$$\zeta(s, A) = \zeta_K(s, A) = \sum_{0 \neq \mathfrak{a} \in A} (N\mathfrak{a})^{-s}, \ \sigma > 1 \tag{83}$$

be the class zeta function. Then we have the decomposition

$$\zeta_K(s) = \sum_{A \in I/P} \zeta(s, A).$$

Therefore, it suffices to consider the class zeta function and the approach for studying it (for a real or an imaginary quadratic field) remains the same, as described, e.g. in Ref. 37. Let $\mathfrak{O}_K^\times$ denote the unit group of K, which is the finite group of roots of unity of order w for an imaginary quadratic field.

Now, fix $\mathfrak{b} \in A^{-1}$. Then for $\mathfrak{a} \in A$ in (83), $\mathfrak{ab} = (\lambda) \in P$, where $\lambda \in \mathfrak{b}$. Hence, $(N\mathfrak{a})^{-1} = N\mathfrak{b}(N\mathfrak{ab})^{-1} = N\mathfrak{b}|N(\lambda)|^{-1}$. Hence,

$$\zeta(s, A) = \sum_{\lambda \in \mathfrak{b}/\mathfrak{O}_K^\times}{}' \frac{1}{\left(\frac{|N(\lambda)|}{N\mathfrak{b}}\right)^s}, \tag{84}$$

where the prime on the summation sign means that the value 0 is to be omitted.

Then (84) reads

$$\zeta(s, A) = \frac{1}{w} \sum_{\lambda \in \mathfrak{b}}{}' \frac{1}{\left(\frac{N(\lambda)}{N\mathfrak{b}}\right)^s} = \frac{1}{w} \zeta_Q(s), \tag{85}$$

say, the absolute value sign being suppressed in view of $N\lambda = N(\lambda) = |\lambda|^2 > 0$. Here, $Q = Q(m, n) = \frac{|N(\lambda)|}{N\mathfrak{b}}$ in (84) is a positive definite quadratic form with discriminant $d < 0$, i.e. through (85), we view the class zeta function as the Epstein zeta function with a positive definite quadratic form of discriminant $d < 0$ in the setting as in Refs. 46, 47, 53, etc. to state the general Chowla–Selberg integral formula, Theorem 6:

$$Q(m, n) = am^2 + bmn + cn^2 = a|m + n\tau|^2,$$
$$\tau = x + iy = \frac{b + i\sqrt{|d|}}{2a}, \tag{86}$$

where

$$a = \frac{\sqrt{|d|}}{2\,\mathrm{Im}\,\tau}, \tag{87}$$

so that

$$c = a|\tau|^2, \quad b = 2a\,\mathrm{Re}\,\tau, \quad x = \frac{b}{2a}, \quad y = \frac{\sqrt{|d|}}{2a}. \tag{88}$$

Then in conformity with (86), τ is the solution of the quadratic equation with imaginary part positive:

$$a\tau^2 - b\tau + c = 0. \tag{89}$$

Hence, we express (85) as

$$\begin{aligned} \zeta(s, A) &= \frac{1}{w} Z(s), \quad Z(s) = Z(s, Q) = \sum_{m,n}{}' \frac{1}{Q(m,n)^s} = a^{-s}\zeta_{\mathbb{Z}^2}(s), \\ \zeta_{\mathbb{Z}^2}(s) &= \zeta_{\mathbb{Z}^2}(s, \tau) = \sum_{m,n}{}' \frac{1}{|m + n\tau|^{2s}}, \end{aligned} \tag{90}$$

where $\zeta_{\mathbb{Z}^2}(x)$ is the holomorphic Eisenstein series. It has several variations. Equation (90) will be assumed in what follows.

This setting corresponds to

$$\mathfrak{b} = \mathbb{Z} \oplus \mathbb{Z}\tau, \quad \tau \in \mathcal{H}. \tag{91}$$

The choice is possible since (85) is unchanged if we replace $\mathfrak{b}$ by $\alpha\mathfrak{b}$, $K \ni \alpha \neq 0$.

The following lemma is a basis of the beta-transform, which seems to be first used by Hardy[48] (in previous literature, it used to be called as the Mellin–Barnes integral, but this refers to a much wider class of functions), cf. Ref. 49 and Remark 4.

Lemma 5. *The beta-transform reads*

$$(1+x)^{-s} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(z+s)\,\Gamma(-z)}{\Gamma(s)}\, x^z \mathrm{d}z = \frac{1}{\Gamma(s)} G^{1,1}_{1,1}\left(x^{-1} \,\middle|\, \begin{matrix} 1 \\ s \end{matrix}\right)$$

for $x > 0$, $-\mathrm{Re}\, s < c < 0$, *where* (c) *signifies the vertical Bromwich path* $s = c + it, -\infty < t < \infty$.

Theorem 6. (i) *Under the notation* (86)–(90), *in particular,* $\tau = x + iy$, $y = \frac{\sqrt{|d|}}{2a}$, *we have the* **Chowla–Selberg integral formula**

$$\Gamma(s)Z(s) = 2\Gamma(s)\zeta(2s)a^{-s} + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{|d|^{s-\frac{1}{2}}}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1) + Q(s), \tag{92}$$

where

$$Q(s) = \frac{4\pi^s 2^{s+\frac{1}{2}}}{\sqrt{a}|d|^{\frac{s}{2}-\frac{1}{4}}}\sum_{n=1}^{\infty} n^{s-\frac{1}{2}}\sigma_{1-2s}(n)\cos 2\pi nx K_{s-\frac{1}{2}}(2\pi yn), \tag{93}$$

and where

$$\sigma_a(n) = \sum_{d|n} d^a$$

is the sum-of-divisors function.

The Equation (92) *is equivalent to the functional equation*

$$\left(\frac{2\pi}{\sqrt{|d|}}\right)^{-s}\Gamma(s)Z(s) = \left(\frac{2\pi}{\sqrt{|d|}}\right)^{-(1-s)}\Gamma(1-s)Z(1-s).$$

(ii) *The Laurent expansion at* $s = 1$ *holds*:

$$Z(s) = \frac{\frac{2\pi}{\sqrt{|d|}}}{s-1} + \frac{2\pi}{\sqrt{|d|}}\left(2\gamma + \log\frac{a}{|d|}\right) - \frac{4\pi}{\sqrt{|d|}}\log|\eta(\tau)|^2 + O(s-1). \tag{94}$$

Proof. (i) (92) (with (93)) has been proved in many places and elucidated as the Fourier–Bessel expansion (see Ref. 13, Chapter 4), i.e. as the correspondence $G_{1,1}^{1,1} \leftrightarrow G_{0,2}^{2,0}$. However, we reproduce a slightly modified proof of Ref. 46 to exhibit the use of the beta-transform as opposed to the Hecke gamma transform, elucidating the computations in Ref. 53.

We shall prove (92) and (93) in the form

$$Z(s,Q) = a^{-s}\zeta_{\mathbb{Z}^2}(s,\tau) = 2\zeta(2s)a^{-s} + \frac{2\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)y^{-2s+1}a^{-s} + \frac{4a^{-s}}{\Gamma(s)}\sum_{m,n=1}^{\infty}(ny)^{-2s}\frac{\cos(2\pi mnx)}{m}I_{m,n}(s,\tau), \qquad (95)$$

where

$$I_{m,n}(s,\tau) = \frac{1}{\sqrt{\pi}}G^{2,0}_{0,2}\left((\pi mny)^2 \,\middle|\, \begin{matrix} - \\ s, \quad \frac{1}{2}\end{matrix}\right) = 2\pi^s(mny)^{s+\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi mny).$$

We define the lattice zeta function by (50), i.e.

$$\phi_1(s,\{x\}) = \zeta_{\mathbb{Z}}(s,x) = \sum_{n=-\infty}^{\infty}|n+x|^{-s} \qquad (96)$$

for $\{x\} > 0$, where $\{x\}$ is the fractional part of x, (40). Then ϕ_1 is the even part

$$\phi_1(s,x) = 2\zeta^e(s,\{x\}) = \zeta(s,\{x\}) + \zeta(s,1-\{x\}). \qquad (97)$$

In (97), we use the functional equation for the Hurwitz zeta function (6): for $\sigma > 1$, $\{nx\} > 0$, deduce that

$$\phi_1(-2z,\{nx\}) = \zeta(-2z,1-\{nx\}) + \zeta(-2z,\{nx\}) = \frac{2\Gamma(1+2z)}{(2\pi)^{1+2z}}\sin(-\pi z)\,\{\ell_{1+2z}(nx) + \ell_{1+2z}(1-nx)\},$$

i.e. the even part of $\ell_{1+2s}(nx)$, valid for $\{nx\} = 0$, also by interpreting the left-hand side by the right.

Collecting the terms with $n = 0$ in the definition of $\zeta_{\mathbb{Z}^2}(s,\tau)$, (90) gives $2\sum_{m=1}^{\infty} m^{-2s} = 2\zeta(2s)$. In the remaining sum $2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\{(m+nx)^2+n^2y^2\}^{-s}$, we transform the summand into $(ny)^{-2s}\left(1+\left(\frac{m+nx}{ny}\right)^2\right)^{-s}$ and apply the beta transform in Lemma 5. We choose c such that $-\sigma < c < -\frac{1}{2}$ and obtain

$$\zeta_{\mathbb{Z}^2}(s,\tau) = 2\zeta(2s) + \frac{2}{\Gamma(s)}\sum_{n=1}^{\infty}(ny)^{-2s} \times \frac{1}{2\pi i}\int_{(c)} \Gamma(z+s)\Gamma(-z)\phi_1(-2z,\{nx\})(ny)^{-2z}\,\mathrm{d}z. \quad (98)$$

By argument similar to the one in the proof of Theorem 2, we may transform (98) into

$$\zeta_{\mathbb{Z}^2}(s,\tau) = 2\zeta(2s) + \frac{2}{\Gamma(s)}\sum_{n=1}^{\infty}(ny)^{-2s}J_n(c),$$

where

$$\begin{aligned} J_n(c) &= J_n(c,s) \\ &= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(z+s)\Gamma(z+\frac{1}{2})}{\pi^{2z+\frac{1}{2}}} \\ &\quad \times \{\ell_{2z+1}(nx) + \ell_{2z+1}(1-nx)\}(ny)^{-2z}\mathrm{d}z. \end{aligned} \quad (99)$$

Moving the line of integration on the right of (99) to the right up to $\operatorname{Re} z = c_1$, where $-\frac{1}{2} < c_1 < 0$, thereby noting the pole at $z = -\frac{1}{2}$, we deduce that

$$\begin{aligned} \zeta_{\mathbb{Z}^2}(s,\tau) &= 2\zeta(2s) - \frac{2}{\Gamma(s)}\sum_{n=1}^{\infty}(ny)^{-2s+1}\frac{\Gamma(s-\frac{1}{2})}{\pi^{-\frac{1}{2}}}\{\ell_0(nx)+\ell_0(1-nx)\} \\ &\quad + \frac{2}{\Gamma(s)}\sum_{n=1}^{\infty}(ny)^{-2s}J_n(c_1) \\ &= 2\zeta(2s) + \frac{2\sqrt{\pi}\Gamma(s-\frac{1}{2})}{\Gamma(s)}\zeta(2s-1)y^{-2s+1} \\ &\quad + \frac{2}{\Gamma(s)}\sum_{n=1}^{\infty}(ny)^{-2s}J_n(c_1) \end{aligned} \quad (100)$$

on using $\ell_0(nx) + \ell_0(1-nx) = -1$ which follows from (35).

Now,

$$J_n(c_1)$$
$$= \frac{1}{2\pi i\sqrt{\pi}} \sum_{m=1}^{\infty} \frac{2\cos(2\pi mnx)}{m} \int_{(-c_1)} \Gamma(s-z)\Gamma\left(\frac{1}{2}-z\right)(\pi mny)^{2z}\mathrm{d}z$$
$$= \sum_{m=1}^{\infty} \frac{2\cos(2\pi mnx)}{m} I_{m,n}(s,\tau), \tag{101}$$

where

$$I_{m,n}(s,\tau) \;=\; \frac{1}{\sqrt{\pi}} G_{0,2}^{2,0}\left((\pi mny)^2 \,\middle|\, \begin{matrix} - \\ s, \quad \frac{1}{2} \end{matrix}\right). \tag{102}$$

Hence, (100), (101) and (102) after multiplying by a^{-s} lead to (95).

(ii) To deduce (94), we use (92) in the form

$$Z(s) = 2\zeta(2s)a^{-s} + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{|d|^{s-\frac{1}{2}}} \frac{\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)} + \frac{1}{\Gamma(s)}Q(s).$$

Near $s=1$, the second term on the right of (2.1) has the expansion

$$\frac{\frac{2\pi}{\sqrt{|d|}}}{s-1} + \frac{2\pi}{\sqrt{|d|}}\left(2\gamma + \log\frac{a}{|d|}\right)$$

while the first term becomes $2\zeta(2)a^{-1} = \frac{\pi^2}{3a}$, whence it follows that

$$Z(s) = \frac{\frac{2\pi}{\sqrt{|d|}}}{s-1} + \frac{2\pi}{\sqrt{|d|}}\left(2\gamma + \log\frac{a}{|d|}\right) + \frac{\pi^2}{3a} + Q(1) + O(s-1),$$

where

$$Q(1) = \frac{8\sqrt{2}\pi}{\sqrt{a}|d|^{\frac{1}{4}}} \sum_{n=1}^{\infty} n^{\frac{1}{2}}\sigma_{-1}(n)\cos 2\pi xn\sqrt{\frac{1}{4yn}}e^{-2\pi yn}$$
$$= \frac{8\pi}{\sqrt{|d|}} \sum_{n=1}^{\infty} \sigma_{-1}(n)(\cos 2\pi xn)e^{-2\pi yn}$$

in view of

$$K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}. \tag{103}$$

Recalling $q = e^{\pi i\tau}$, we find that

$$q^{2n} + \bar{q}^{2n} = 2e^{-2\pi yn} \cos 2\pi xn,$$

$$\begin{aligned} Q(1) &= \frac{4\pi}{\sqrt{|d|}} \sum_{n=1}^{\infty} \sigma_{-1}(n)(q^{2n} + \bar{q}^{2n}) \\ &= -\frac{4\pi}{\sqrt{|d|}} (\log \eta(\tau) + \log \eta(-\bar{\tau})) + \frac{4\pi}{\sqrt{|d|}} \left(\frac{\pi i}{12} (\tau - \bar{\tau}) \right). \end{aligned}$$

Here, the last term is

$$\frac{4\pi}{\sqrt{|d|}} \frac{\pi i}{12} 2i \operatorname{Im} \tau = -\frac{\frac{2}{3}\pi^2}{\sqrt{|d|}} y,$$

which cancels $\frac{\pi^2}{3a}$, amounting to (94). □

Remark 4. (i) The Chowla–Selberg integral formula is expounded in [Kanemitsu and Tsukada (2014), pp. 136–138]. The case $\boldsymbol{g} = \boldsymbol{h} = \boldsymbol{z}, n = m = 1$ is due to Chowla and Selberg,[47,50] (cf. also Bateman and Grosswald [Bateman and Grosswald (1964)]), the case $m = 1$ is due to Berndt[51] and the general case with $\boldsymbol{g} = \boldsymbol{h} = \boldsymbol{z}$ is due to Terras (see Ref. 52, Example 4, p. 208).

In Ref. 13, the Epstein zeta function is treated as a quadratic form variation of the Fourier–Bessel expansion, $H_{1,1}^{1,1} \leftrightarrow H_{0,2}^{2,0}$, to the effect that the beta transform (or the perturbed Dirichlet series) corresponds to the K-Bessel series (or Fourier series). The K-Bessel function in (93) reduces to the exponential function for $s = 1$, whence the Bessel series is nothing but the Lambert series for the Dedekind eta function and so the Kronecker limit formula arises most naturally in which the theta function or the Dedekind eta function shows up. Note that with $\phi_1(s, x)$ in RHB correspondence is $\vartheta(it, itx)$ (47).

In Ref. 53, this natural deduction of eta function expression is presented suggestive of the modular relation. Substituting $2ay$ for $\sqrt{|\Delta|}$, we see that the coefficient of the second term in (92), resp. in (93), is

$2\sqrt{\pi}a^{-s}y^{1-2s}$, resp. $8\pi^s a^{-s} y^{\frac{1}{2}-s}$, in conformity with Ref. 53, (2.11) and (2.12).
(ii) Note that

$$\Delta = \Delta(q) = (2\pi)^{12} q^2 \prod_{n=1}^{\infty} (1-q^{2n})^{24} = \left((2\pi)\eta(\tau)^2\right)^{12}$$

is a cusp form of weight 12. This leads to the normalization

$$\mathbb{D}(A) = \mathbb{D}(\mathfrak{b}) = \sqrt{|\delta(\mathfrak{b})|}(2\pi)|\eta(\tau)|^2 = \frac{|d|^{\frac{1}{4}}}{\sqrt{a}}(2\pi)|\eta(\tau)|^2, \tag{104}$$

depending only on A, where $\delta(\mathfrak{b})$ is defined by

$$|\delta(\mathfrak{b})| = \left|\det\begin{pmatrix} 1 & 1 \\ \tau & \bar{\tau}\end{pmatrix}\right| = |N\mathfrak{b}|\sqrt{|d|} \quad \text{or} \quad N\mathfrak{b} = \frac{\tau - \bar{\tau}}{i\sqrt{|d|}} = \frac{2\,\mathrm{Im}\,\tau}{\sqrt{|d|}}, \tag{105}$$

with the basis $\{1, \tau\}$ in (91).

Under this, (94) leads to the class invariant form (107) of Meyer.

Theorem 7. (i) *Under the notation* (86)–(90), *we have the* **Chowla–Selberg integral formula**

$$\Gamma(s)\zeta(s, A) = \frac{1}{w}\left(2\Gamma(s)\zeta(2s)a^{-s} + \frac{2^{2s}a^{s-1}\sqrt{\pi}}{|d|^{s-\frac{1}{2}}}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1) + Q_1(s)\right), \tag{106}$$

where

$$Q_1(s) = \frac{4\pi^s 2^{s+\frac{1}{2}}}{\sqrt{a}|d|^{\frac{s}{2}-\frac{1}{4}}} \sum_{n=1}^{\infty} n^{s-\frac{1}{2}} \sigma_{1-2s}(n) \cos 2\pi n x K_{s-\frac{1}{2}}(2\pi y n),$$

which is equivalent to the functional equation

$$\left(\frac{2\pi}{\sqrt{|d|}}\right)^{-s} \Gamma(s)\zeta(s, A) = \left(\frac{2\pi}{\sqrt{|d|}}\right)^{-(1-s)} \Gamma(1-s)\zeta(1-s, A).$$

(ii) *The Laurent expansion at $s = 1$ in Meyer's form reads*

$$\zeta(s,A) = \frac{\frac{2\pi}{w\sqrt{|d|}}}{s-1} + \frac{2\pi}{w\sqrt{|d|}}\left(2\gamma + \log\frac{(2\pi)^2}{\sqrt{|d|}}\right) - \frac{4\pi}{w\sqrt{|d|}}\log\mathbb{D}(A) + O(s-1), \tag{107}$$

where $\mathbb{D}(A)$ is the class invariant defined by (104).

Proof. Only (107) needs explanation. Substitution of (90) in (106) or a direct application of (106) gives

$$\zeta(s,A) = \frac{\frac{2\pi}{w\sqrt{|d|}}}{s-1} + \frac{2\pi}{w\sqrt{|d|}}\left(2\gamma + \log\frac{a}{|d|}\right) - \frac{4\pi}{w\sqrt{|d|}}\log|\eta(\tau)|^2 + O(s-1). \tag{108}$$

where by (105),

$$\mathbb{D}(\mathfrak{b}) = \frac{|d|^{\frac{1}{4}}}{\sqrt{a}}(2\pi)|\eta(\tau)|^2.$$

Hence, (108) amounts to (107). □

Corollary 4. *For an imaginary quadratic field with discriminant d and class number h, we have the Laurent expansion*

$$\zeta_K(s) = \frac{\frac{2\pi h}{w\sqrt{|d|}}}{s-1} + \frac{2\pi h}{w\sqrt{|d|}}\left(2\gamma + \log\frac{(2\pi)^2}{\sqrt{|d|}}\right) - \frac{4\pi}{w\sqrt{|d|}}\sum_{A\in I/P}\log\mathbb{D}(A) + O(s-1). \tag{109}$$

Lemma 6 (Decomposition theorem). *For a quadratic field $K = \mathbb{Q}(\sqrt{D})$ of discriminant d, the decomposition formula holds:*

$$\zeta_K(s) = \zeta(s)L(s,\chi),$$

where $L(s,\chi)$ is the Dirichlet L-function associated to the Kronecker (Dirichlet) character $\chi = \chi_d$ corresponding to K. The Laurent expansion reads

$$\zeta_K(s) = \frac{L(1,\chi)}{s-1} + \gamma L(1,\chi) + L'(1,\chi) + O(s-1).$$

Comparing Corollary 4 and Lemma 6, we state and prove the Dirichlet class number formula and the Lerch–Chowla–Selberg formula for imaginary quadratic fields.

Theorem 8. *Let* $h(d) = |I/P|$ *denote the class number of the quadratic field* $\mathbb{Q}(\sqrt{D})$ *of discriminant* d*. Suppose* $d < 0$.

(i) *The Dirichlet class number formula holds*:

$$h(d) = \frac{w\sqrt{|d|}}{2\pi} L(1, \chi_d). \tag{110}$$

(ii) *The Lerch–Chowla–Selberg formulas hold with class invariants* (*see Refs.* 32, (4.5) *and* 33, *(**), p.* 569):

$$\left(\prod_A \mathbb{D}(A)\right)^2 = \left(\frac{2\pi}{\sqrt{|d|}}\right)^h \prod_{a=1}^{|d|-1} \Gamma\left(\frac{a}{|d|}\right)^{\frac{w}{2}\chi(a)}. \tag{111}$$

In terms of the Lerch zeta function,

$$2\sum_{A\in I/P} \log \mathbb{D}(A) = h\left(\gamma + \log\frac{(2\pi)^2}{\sqrt{|d|}}\right) - \frac{iw}{2\pi}\sum_{a=1}^{|d|-1} \bar{\chi}(a)\left(\ell_1'\right)^o\left(\frac{a}{|d|}\right). \tag{112}$$

Proof. By Corollary 4, the residue of $\zeta_K(s)$ at $s = 1$ is $\frac{2\pi}{w\sqrt{|d|}}h$. Hence, by the decomposition formula, Lemma 6, (110) follows.

By (110), (76) reads

$$\gamma L(1, \chi_d) + L'(1, \chi_d) = -\frac{\pi}{\sqrt{|d|}}\sum_{a=1}^{|d|-1} \bar{\chi}(a) \log\Gamma\left(\frac{a}{|d|}\right) + \frac{2\pi h}{w\sqrt{|d|}}(2\gamma + \log 2\pi).$$

Equating this with the Laurent constant in (109) and dividing both sides by $\frac{2\pi}{w\sqrt{|d|}}$, we conclude that

$$2\sum_{A\in I/P} \log \mathbb{D}(A) = h\log\frac{2\pi}{\sqrt{|d|}} + \frac{w}{2}\sum_{a=1}^{|d|-1} \bar{\chi}(a)\log\Gamma\left(\frac{a}{|d|}\right) \tag{113}$$

(111) is the exponentiated form of this.

Equating (109) and (75), we derive that

$$\frac{2\pi h}{w\sqrt{|d|}}\left(\gamma+\log\frac{(2\pi)^2}{\sqrt{|d|}}\right)-\frac{4\pi}{w\sqrt{|d|}}\sum_{A\in I/P}\log\mathbb{D}(A)$$
$$=-\frac{i}{\sqrt{|d|}}\sum_{a=1}^{|d|-1}\bar{\chi}(a)(\ell_1')^o\left(\frac{a}{|d|}\right),$$

which leads to (112). □

2.2. *Genesis of the formula*

In the light of Tables 2 and 3, Theorem 8 may be described roughly as follows. The class invariant on the left of (113) is the processed (even part of) the theta function with one-time use of the functional equation while the right-hand side of (112) is in terms of the Lerch zeta function which reflects the property of the almost unprocessed conjugate theta function.

In terms of the gamma function, the right side of (113) is processed once by the functional equation and in more orderly form than (112).

Thus, in a sense, this is a relation between the processed theta functions and their counterpart under RHB correspondence linked by the decomposition theorem, Lemma 6, which is a consequence of the Artin reciprocity law (see Ref. 54, pp. 196–199).

Table 2. Lerch–Chowla–Selberg ingredients: Eisenstein ser.

Name	RHB	β trans.	f.e.	Product f.
Eisenstein ser.	$\vartheta\leftrightarrow\zeta^e$	$\zeta_{\mathbb{Z}^2}\to\zeta^e$	$\zeta^e\to\ell^e$	$\eta=\theta^e$

Table 3. Lerch–Chowla–Selberg ingredients: Fourier sine ser.

Name	RHB	Diff. w.r.t. s	Termwise diff.
Der. of Hurwitz	$\vartheta,\tilde{\vartheta}\leftrightarrow\zeta(s,x)$	$\zeta'(0,x)\to(\ell_1^e)'$ (f.e.)	Fourier sine ser.
Der. of Lerch	$\tilde{\vartheta}\leftrightarrow\ell_s^o$	$(\ell_1^e)'$	Fourier sine ser.

3. Product Expansion of Theta-Allied Functions

In previous sections, we worked either in $\sigma > 1$ or $\sigma < 0$ save for the Lerch zeta function which can go into the critical strip $0 < \sigma < 1$. For the Riemann zeta function to go into the critical strip, it needs the Euler product and Riemann defines $\zeta(s)$ by (114) first and then by the additive expression (2). Euler products in general have been one of the most important objects of research from the time of Euler and the most famous Euler product for $\zeta(s)$ reads

$$\zeta(s) = \prod_{p} \left(1 - p^{-s}\right)^{-1} \tag{114}$$

for $\sigma > 1$ and where p runs through all the primes. As is known, this expresses the UFD property of the basic sequence $\{n\}$. Thus, wherever there is the UFD, there is the Euler product.

Here, we speak of the product expressions (80) for theta functions.

Corollary 5 (Infinite product for theta). *For* $|q| < 1$ *and* $z \neq 0$,

$$\vartheta(\tau, x) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + zq^{2n-1})(1 + z^{-1}q^{2n-1}), \tag{115}$$

where $z = e^{2\pi i x}$.

Ogg's[18] proof of Corollary 5 amounts to proving the inverse Mellin transform of the logarithm of the infinite product $\theta_0(\tau, x)$, say, satisfies the functional equation from which the same transformation formula as (43) is deduced. It therefore turns out that the quotient $\frac{\vartheta}{\vartheta_0}$, where ϑ is the additive expression in (42), is an elliptic function in x with periods $1, \tau$ with at most one singularity, so that it is a constant, cf. e.g. Ref. 19, pp. 53–54. Use is made of the second equality in (1) in the form

$$e^{-\pi \frac{\tau}{i} u} = \frac{1}{2\pi i} \int_{(c)} \left(\frac{\tau}{i}\right)^{-s} \Gamma(s)(\pi u)^{-s} \, \mathrm{d}s, \quad c > 0, \quad u > 0.$$

Since the zeta function is $\zeta(s)\zeta(s+1)$ (e.g. Ref. 55) which is in RHB correspondence with $\log \eta$, it may be the case that if we employ an additive expression for the relevant theta-like function, we may argue in the same way as Ogg to derive the product expression. Hitherto only the product expression is used as the definition. This direction will be pursued elsewhere.

We shall elucidate the genesis of Siegel's proof of Hamburger's theorem.[56] Reference 1 (pp. 201–211) contains a rather transparent description of the Siegel–Hecke proof of the Hamburger theorem (especially the translation of introduction of Ref. 57), cf. Refs. 3, 4, 56–58, etc. However, in neither Ref. 1 nor Ref. 59, there is genesis given of Siegel's proof.

We appeal to the following which is (an extract of) Ref. 13, Example 4.2, pp. 120–121.

Lemma 7. *Suppose the Dirichlet series* φ *and* ψ *satisfy the Riemann-type functional equation*

$$\Gamma\left(\frac{1}{2}s\right)\varphi(s) = \Gamma\left(\frac{1}{2}(r-s)\right)\psi(r-s)$$

with a simple pole at $s = r > 0$ *with residue* ρ*. Then the Fourier–Bessel expansion* $H^{1,1}_{1,1} \leftrightarrow H^{2,0}_{0,2}$ *entails*

$$\begin{aligned}
2\Gamma\left(\frac{1}{2}s\right)&\sum_{n=1}^{\infty}\frac{\alpha_n}{(\lambda_n^2+z^2)^{\frac{1}{2}s}}\\
&= 4z^{\frac{1}{2}(r-s)}\sum_{n=1}^{\infty}\frac{\beta_n}{\mu_n^{\frac{1}{2}(r-s)}}K_{\frac{1}{2}(r-s)}(2\mu_n z) + 2\Gamma\left(\frac{1}{2}s\right)\varphi(0)\,z^{-s}\\
&\quad + \rho\Gamma\left(\frac{1}{2}(s-r)\right)\,z^{r-s},
\end{aligned}$$

which is a generalization of Watson's formula.

We specialize Lemma 7 to the case

$$\lambda_n = \mu_n = \sqrt{\pi}n, \quad r = 1, \quad z = \sqrt{\pi}t, \quad s = 1$$

and simplify the resulting expression to derive

$$\sum_{n=1}^{\infty}\alpha_n\left(\frac{1}{t+in}+\frac{1}{t-in}\right) - \pi t H(t) = 2\pi\sum_{n=1}^{\infty}\beta_n e^{-2\pi n t},$$

which is in Ref. 56, (12). After this, Siegel's argument applies. Indeed, both Refs. 56 (5) and 59 (formula before (6)) are the reduction of the K-Bessel function to the exponential function, (103).

Thus, we have seen two occurrences of the Fourier–Bessel expansion. It may be proper to ask if the Hurwitz formula may be another occurrence. But it seems that direct application leads to the K-Bessel function series (see Ref. 13, Example 4.3, pp. 121–122) which suggests a reason why Siegel's proof fails in the case of Hecke (see Ref. 1, p. 207). As is elucidated in Ref. 14, it seems the Ewald expansion leads most naturally to the Hurwitz formula.

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Chapter 5

Sylvester Sums on the Frobenius Set in Arithmetic Progression with Initial Gaps

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Let $a_1, a_2, \ldots, a_k$ be positive integers with $\gcd(a_1, a_2, \ldots, a_k) = 1$. Frobenius number is the largest positive integer that is NOT representable in terms of $a_1, a_2, \ldots, a_k$. When $k \geq 3$, there is no explicit formula in general, but some formulae may exist for special sequences $a_1, a_2, \ldots, a_k$, including those forming arithmetic progressions and their modifications. In this chapter, we give explicit formulae for the sum of nonrepresentable positive integers (Sylvester sum) as well as Frobenius numbers and the number of nonrepresentable positive integers (Sylverster number) for $a_1, a_2, \ldots, a_k$ forming arithmetic progressions with initial gaps.

1. Introduction

Let $a_1, \ldots, a_k$ be positive integers with $\gcd(a_1, \ldots, a_k) = 1$. It is well known that all sufficiently large integers can be represented as a nonnegative integer combination of $a_1, \ldots, a_k$. Then it is important to determine the largest positive integer that is not representable as a nonnegative integer combination of given positive integers that are coprime. Such a problem is known as the *Frobenius Problem* and this largest positive integer is denoted by $g(a_1, \ldots, a_k)$ and called the *Frobenius number* (see Ref. 1 for general references).[1] This problem has also been known as the Coin Exchange Problem, Postage Stamp Problem or Chicken McNugget Problem, and so has a long history. Together with the Frobenius numbers, the number of positive integers with no nonnegative integer representation by $a_1, \ldots, a_k$ has also been studied for a long time. This number is sometimes called the *Sylvester number* (or *genus* in numerical semigroup) and denoted by $n(a_1, \ldots, a_k)$.

According to Sylvester, for positive integers a and b with $\gcd(a, b) = 1$,

$$g(a,b) = (a-1)(b-1) - 1 \quad [4],$$

$$n(a,b) = \frac{1}{2}(a-1)(b-1) \quad [5]$$

There are many kinds of problems related to the Frobenius problem. The problems for the number of solutions (e.g. Ref. 6) and the sum of integer powers of values of the gaps in numerical semigroups (e.g. Refs. 7–9) are popular. In Ref. 10, the various results within the cyclotomic polynomial and numerical semigroup communities are unified. One of the other famous problems is about the so-called *Sylvester sums*:

$$s(a_1, \ldots, a_k) := \sum_{n \in \mathrm{NR}(a_1, \ldots, a_k)} n$$

(see, e.g. Ref. 1, Section 5.5, 11 and references therein), where $\mathrm{NR}(a_1, \ldots, a_k)$ denotes the set of positive integers without nonnegative integer representation by $a_1, \ldots, a_k$. This is exactly the set of gaps in numerical semigroup. It is harder to obtain the Sylvester number than the Frobenius number and even harder to obtain the Sylvester sum. Finally, long time after Sylvester, Brown and Shiue[7]

[1]Some other symbols have also been used by different backgrounds and authors. The symbols used in this paper are mainly based on the literature, such as Refs. 1–3.

found the exact value for positive integers a and b with $\gcd(a,b)=1$,

$$s(a,b)=\frac{1}{12}(a-1)(b-1)(2ab-a-b-1). \tag{1}$$

Rødseth[12] generalized Brown and Shiue's result by giving a closed form for $s_\mu(a,b):=\sum_{n\in\mathrm{NR}(a,b)} n^\mu$, where μ is a positive integer.

When $k=2$, there exist beautiful closed forms for Frobenius numbers, Sylvester numbers and Sylvester sums, but when $k\geq 3$, exact determination of these numbers is extremely difficult. The Frobenius number cannot be given by closed formulas of a certain type,[13] the problem to determine $g(a_1,\ldots,a_k)$ is NP-hard under Turing reduction (see, e.g. Ref. 1). Nevertheless, one convenient formula is found by Johnson.[14] One analytic approach to the Frobenius number can be seen in Refs. 15 and 16.

Though closed forms for the general case are hopeless for $k\geq 3$, several formulae for Frobenius numbers, Sylvester numbers and Sylvester sums have been considered under special cases. For example, one of the best expositions for the Frobenius number in three variables can be seen in Ref. 17. For general $k\geq 3$, the Frobenius number and the Sylvester number for some special cases are calculated, including arithmetic sequences and geometric-like sequences (e.g. Refs. 2, 18–21).

In fact, by introducing the Apéry set, it is possible to determine the functions $g(A)$, $n(A)$ and $s(A)$ for the set of positive integers $A:=\{a_1,a_2,\ldots,a_k\}$ with $\gcd(a_1,a_2,\ldots,a_k)=1$. For $a_1=\min(A)$, we denote by

$$\mathrm{Ape}(A)=\mathrm{Ape}(A,a_1)=\{m_0,m_1,\ldots,m_{a_1-1}\}$$

the *Apéry set* of A, where m_i is the least positive integer that can be represented by a nonnegative integral linear combination of $a_2,\ldots,a_k$, satisfying $m_i\equiv i \pmod{a_1}$ $(1\leq i\leq a_1-1)$. Note that m_0 is defined to be 0. The element 0 is often excluded because it does not affect the calculation.

Lemma 1. *We have*

$$g(a_1,a_2,\ldots,a_k)=\left(\max_{1\leq i\leq a_1-1} m_i\right)-a_1, \quad [22]$$

$$n(a_1, a_2, \dots, a_k) = \frac{1}{a_1} \sum_{i=1}^{a_1 - 1} m_i - \frac{a_1 - 1}{2}, \quad [2]$$

$$s(a_1, a_2, \dots, a_k) = \frac{1}{2a_1} \sum_{i=1}^{a_1 - 1} m_i^2 - \frac{1}{2} \sum_{i=1}^{a_1 - 1} m_i + \frac{a_1^2 - 1}{12}. \quad [3]$$

The third formula appeared with a typo in Ref. 3, and it has been corrected in Refs. 23 and 24. Recently, we study the weighted sums and weighted power sums. When $k = 2$, a general formula can be expressed in terms of the Apostol–Bernoulli numbers.[25] For general k, we can have a formula by using Eulerian numbers.[26] In Ref. 27, a more general formula including $n(A)$ and $s(A)$ is given by using Bernoulli numbers.

As a more general case than the arithmetic sequence, the sequence $a_1 = a$, $a_2 = a + d, \dots, a_k = a + (k - 1)d$, $a_{k+1} = a + Kd$ with $K > k$ has been studied.[2] This is one typical case of the so-called *almost arithmetic sequence*. In this case, there is an additional term after some gap. As special cases, the Frobenius numbers and the Sylvester number are given for the sequences $a, a + 1, a + 2, a + 4$; $a, a + 1, a + 2, a + 5$; $a, a + 1, a + 2, a + 6$ and so on[2,28] i.e. a gap appears in the last. In Ref. 29, the Frobenius numbers of various cases are expressed in which the a_i's lie in an arithmetic progression, but the results are incomplete. In Ref. 30, when the a_i form an almost arithmetic sequence, by considering the Apéry set, algorithms to determine the Sylvester number and sum are given. But, in this paper the computations rely on $\mathrm{Ape}(A, a + (K + 1)d)$ instead of $\mathrm{Ape}(A, a)$. In Ref. 31, the authors gave an alternative description of the Apéry set of the first element in the arithmetic sequence. Their aim was different: they wanted to describe the minimal presentation of the semigroup. The approaches in Refs. 30 and 31 may apply to any almost arithmetic sequence, but they both have an extra burden: they require the pre-computation of a couple of constants depending on the sequence. Some other applications for the case of almost arithmetic sequences can be found in Section 4 of Ref. 32.

In Ref. 21, we give explicit expressions of the Sylvester sum and the weighed sum, where $a_1, a_2, \dots, a_k$ forms arithmetic sequences. In this chapter, we not only study only the Frobenius numbers but also the Sylvester number and the Sylvester sum, where $a_1 = a$,

$a_2 = a + (K+1)d$, $a_3 = a + (K+2)d, \dots, a_{k-K+1} = a + kd$ with $d > 0$, $\gcd(a,d) = 1$ and $k \ge K+1 \ge 2$, i.e. a gap appears in the first. As special cases, we yield these numbers and sums explicitly for the sequences $a, a+2, a+3, a+4$; $a, a+3, a+4, a+5$; $a, a+4, a+5, a+6$; $a, a+5, a+6, a+7$ and so on.

2. Main Result

For positive integers a and d with $a \ge 2$ and $\gcd(a,d) = 1$, consider the sequence $a, a+(K+1)d, a+(K+2)d, \dots, a+kd$. Note that the last $(k-K)$ terms form an arithmetic sequence, and there is a gap in the first part. Determine nonnegative integers q and r as

$$a + K = qk + r, \quad 0 \le r < k, \tag{2}$$

i.e. $q = \lfloor (a+K)/k \rfloor$ and $r = a + K - \lfloor (a+K)/k \rfloor k$.

In this section, we assume that $1 \le K \le (k-1)/2$, i.e. the gaps between the first term and the second term are not so big. The case when $K > (k-1)/2$ is discussed in the following section. In addition to this condition, we shall discuss the two cases separately: $r > K$ or $r \le K$.

Case 1: When $r > K$, all the elements of Apéry set excluding 0 (mod a) can be determined in the following table:

			$a+(K+1)d$	$\dots$	$a+kd$
$2a+(k+1)d$	$\dots$	$2a+(k+K)d$	$2a+(k+K+1)d$	$\dots$	$2a+2kd$
$3a+(2k+1)d$	$\dots$	$3a+(2k+K)d$	$3a+(2k+K+1)d$	$\dots$	$3a+3kd$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$qa+((q-1)k+1)d$	$\dots$	$qa+((q-1)k+K)d$	$qa+((q-1)k+K+1)d$	$\dots$	$qa+qkd$
$(q+1)a+(qk+1)d$	$\dots$	$\dots$	$(q+1)a+(qk+r)d$		
		$(q+\widehat{1)a}+ad$		$\dots$	

The last line consists of $r-1$ terms excluding $(q+1)a+ad = (q+1)a+(qk+r-K)d$ because it is equal to 0 (mod a). Note that by $\gcd(a,d) = 1$, the set of all the elements in this table forms a complete residue system modulo a excluding 0 (mod a):

$$\{(K+1)d, (K+2)d, \dots, (a-1)d, (a+1)d, \dots, (a+K)d\}$$
$$= \{1, 2, \dots, a-1\} \pmod a.$$

Since $K \le (k-1)/2$, $2a+(k+1)d$ can be represented by using the elements of the last $(K-k)$ terms of the given sequence, which appear in the first line, all terms in this table after $2a+(k+1)d$ can be also represented. In addition, none of the elements in the table can be represented by subtracting a. Therefore, the elements in this table form the Apéry set except $0 \pmod a$. Hence,

$$
\begin{aligned}
\sum_{i=1}^{a-1} m_i &= \bigl(1\cdot(k-K)+(2+3+\cdots+q)k+(q+1)(r-1)\bigr)a \\
&\quad + \bigl((K+1)+(K+2)+\cdots+(a-1) \\
&\quad + (a+1)+\cdots+(a+K)\bigr)d \\
&= \left(\frac{q(q+1)}{2}k-K+(q+1)(r-1)\right)a+\left(\frac{a-1}{2}+K\right)ad
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i=1}^{a-1} m_i^2 &= \bigl(1^2\cdot(k-K)+(2^2+3^2+\cdots+q^2)k+(q+1)^2(r-1)\bigr)a^2 \\
&\quad + \bigl((K+1)^2+(K+2)^2+\cdots+(a-1)^2 \\
&\quad + (a+1)^2+\cdots+(a+K)^2\bigr)d^2 \\
&\quad + 2ad\Bigl(1\bigl((K+1)+\cdots+k\bigr)+2\bigl((k+1)+\cdots+(2k)\bigr) \\
&\quad + 3\bigl((2k+1)+\cdots+(3k)\bigr)+\cdots+q\bigl(((q-1)k+1) \\
&\quad + \cdots+(qk)\bigr)+(q+1)\bigl((qk+1)+\cdots+(qk+r)-a\bigr)\Bigr) \\
&= \left(\frac{q(q+1)(2q+1)}{6}k-K+(q+1)^2(r-1)\right)a^2 \\
&\quad + \left(\frac{(a+K)(a+K+1)(2a+2K+1)}{6}\right. \\
&\quad \left. - \frac{K(K+1)(2K+1)}{6}-a^2\right)d^2+2ad\left(\left(\frac{k(k+1)}{2}\right.\right. \\
&\quad \left. -\frac{K(K+1)}{2}\right)+2\left(\frac{2k(2k+1)}{2}-\frac{k(k+1)}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ 3\left(\frac{3k(3k+1)}{2} - \frac{2k(2k+1)}{2}\right) + \cdots \\
&+ q\left(\frac{qk(qk+1)}{2} - \frac{(q-1)k((q-1)k+1)}{2}\right) \\
&+ (q+1)\left(\frac{(qk+r)(qk+r+1)}{2} - \frac{qk(qk+1)}{2} - a\right)\Bigg) \\
= &\left(\frac{q(q+1)(2q+1)}{6}k - K + (q+1)^2(r-1)\right)a^2 \\
&+ \frac{\big(6K(a+K+1)+(a-1)(2a-1)\big)}{6}ad^2 \\
&+ 2ad\Bigg(-\frac{K(K+1)}{2} - \frac{qk(q+1)(2qk+k+3)}{12} \\
&+ \frac{(q+1)(a+K)(a+K+1)}{2} - a(q+1)\Bigg).
\end{aligned}
$$

Therefore, by the third formula in Lemma 1 together with $q = (a+K-r)/k$, we have

$$
\begin{aligned}
&s(a, a+(K+1)d, \dots, a+kd) \\
&= \frac{1}{2a}\sum_{i=1}^{a-1} m_i^2 - \frac{1}{2}\sum_{i=1}^{a-1} m_i + \frac{a^2-1}{12} \\
&= \frac{1}{12k^2}\Bigg(2a^4 + 6(K-1)a^3 + \big(6K(K-2) - k(k+6) \\
&\quad - 6r^2 + 6(k+2)r\big)a^2 + \big(2K^2(K-3) - 2Kk(k+3) \\
&\quad + 4r^3 - 6(K+k+1)r^2 + (6K(k+2) + 2k(k+3))r\big)a \\
&\quad - k^2 + \big(2a^2 + 3(2K-1)a + 6K^2 + 6K + 1\big)k^2d^2 \\
&\quad + \big(4a^3 + 3(4K-3)a^2 \\
&\quad + \big(6K(2K-1) - k(k+6) - 6r^2 + 6(k+2)r\big)a \\
&\quad + 4K^3 - 3K^2(k-1) - Kk(k+3) \\
&\quad + 2r^3 - 3(2K+k+1)r^2 + k(6K+k+3)r\big)kd\Bigg).
\end{aligned}
$$

Case 2: Let $0 \le r \le K$. Since the term $qa + (qk + r - K)d = qa + ad \equiv 0 \pmod{a}$ is excluded, we have

$$\sum_{i=1}^{a-1} m_i = \big(1 \cdot (k-K) + (2+3+\cdots+q)k - q + (q+1)r\big)a + \left(\frac{(a-1)a}{2} + aK\right)d$$
$$= \left(\frac{q(q+1)}{2}k - K - q + (q+1)r\right)a + \left(\frac{a-1}{2} + K\right)ad$$

and

$$\sum_{i=1}^{a-1} m_i^2 = \big(1^2 \cdot (k-K) + (2^2+3^2+\cdots+q^2)k - q^2 + (q+1)^2 r\big)a^2$$
$$+ \big((K+1)^2 + (K+2)^2 + \cdots + (a-1)^2$$
$$+ (a+1)^2 + \cdots + (a+K)^2\big)d^2$$
$$+ 2ad\Big(1\big((K+1) + \cdots + k\big) + 2\big((k+1) + \cdots + (2k)\big)$$
$$+ 3\big((2k+1) + \cdots + (3k)\big) + \cdots + q\big(((q-1)k+1)$$
$$+ \cdots + (qk)\big) - qa\Big) + (q+1)\big((qk+1) + \cdots + (qk+r)\big)\Big)$$
$$= \left(\frac{q(q+1)(2q+1)}{6}k - K - q^2 + (q+1)^2 r\right)a^2$$
$$+ \frac{\big(6K(a+K+1) + (a-1)(2a-1)\big)}{6}ad^2$$
$$+ 2ad\left(-\frac{K(K+1)}{2} - \frac{qk(q+1)(2qk+k+3)}{12}\right.$$
$$\left.+ \frac{(q+1)(a+K)(a+K+1)}{2} - aq\right).$$

Therefore, by the third formula in Lemma 1 together with $q = (a + K - r)/k$, we have

$$
\begin{aligned}
&s(a, a+(K+1)d, \ldots, a+kd)\\
&\quad = \frac{1}{12k^2}(2a^4 + 6(K-1)a^3\\
&\qquad + \big(6K(K-2) - k(k-6) - 6r^2 + 6(k+2)r\big)a^2\\
&\qquad + \big(2K^2(K-3) - 2Kk(k-3)\\
&\qquad + 4r^3 - 6(K+k+1)r^2 + (6K(k+2) - 2k(k-3))r\big)a\\
&\qquad - k^2 + \big(2a^2 + 3(2K-1)a + 6K^2 + 6K + 1\big)k^2d^2\\
&\qquad + \big(4a^3 + 3(4K-3)a^2\\
&\qquad + \big(6K(2K-1) - k(k-6) - 6r^2 + 6(k+2)r\big)a\\
&\qquad + 4K^3 - 3K^2(k-1) - Kk(k+3)\\
&\qquad + 2r^3 - 3(2K+k+1)r^2 + k(6K+k+3)r\big)kd).
\end{aligned}
$$

Theorem 1. *Let a, d, K, k be positive integers with $a \geq 2$, $\gcd(a, d) = 1$ and $K \leq (k-1)/2$. Let $r = a + K - \lfloor (a+K)/k \rfloor k$. If $r > K$, then*

$$
\begin{aligned}
&s(a, a+(K+1)d, \ldots, a+kd)\\
&\quad = \frac{1}{12k^2}(2a^4 + 6(K-1)a^3 + \big(6K(K-2) - k(k+6)\\
&\qquad - 6r^2 + 6(k+2)r\big)a^2 + \big(2K^2(K-3) - 2Kk(k+3)\\
&\qquad + 4r^3 - 6(K+k+1)r^2 + (6K(k+2) + 2k(k+3))r\big)a\\
&\qquad - k^2 + \big(2a^2 + 3(2K-1)a + 6K^2 + 6K + 1\big)k^2d^2\\
&\qquad + \big(4a^3 + 3(4K-3)a^2\\
&\qquad + \big(6K(2K-1) - k(k+6) - 6r^2 + 6(k+2)r\big)a\\
&\qquad + 4K^3 - 3K^2(k-1) - Kk(k+3)\\
&\qquad + 2r^3 - 3(2K+k+1)r^2 + k(6K+k+3)r\big)kd).
\end{aligned}
$$

If $0 \le r \le K$, *then*

$$
\begin{aligned}
&s(a, a+(K+1)d, \dots, a+kd)\\
&\quad = \frac{1}{12k^2}(2a^4 + 6(K-1)a^3\\
&\qquad + \bigl(6K(K-2) - k(k-6) - 6r^2 + 6(k+2)r\bigr)a^2\\
&\qquad + \bigl(2K^2(K-3) - 2Kk(k-3)\\
&\qquad + 4r^3 - 6(K+k+1)r^2 + (6K(k+2) - 2k(k-3))r\bigr)a\\
&\qquad - k^2 + \bigl(2a^2 + 3(2K-1)a + 6K^2 + 6K + 1\bigr)k^2d^2\\
&\qquad + \bigl(4a^3 + 3(4K-3)a^2\\
&\qquad + \bigl(6K(2K-1) - k(k-6) - 6r^2 + 6(k+2)r\bigr)a\\
&\qquad + 4K^3 - 3K^2(k-1) - Kk(k+3)\\
&\qquad + 2r^3 - 3(2K+k+1)r^2 + k(6K+k+3)r\bigr)kd).
\end{aligned}
$$

By applying the first formula in Lemma 1, we can obtain the Frobenius number of the almost arithmetic sequence with initial gaps. Here, integers a, d, K, k, r are determined as in Theorem 1.

Theorem 2. *Under the same conditions as in Theorem* 1, *we have*

$$
\begin{aligned}
&g(a, a+(K+1)d, \dots, a+kd)\\
&\quad = \begin{cases} \dfrac{a(a+K-r)}{k} + (a+K)d & \text{if } r > 0\\ \dfrac{a(a+K-k)}{k} + (a+K)d & \text{if } r = 0 \end{cases}\\
&\quad = \left(\left\lceil \frac{a+K}{k} \right\rceil - 1\right)a + (a+K)d.
\end{aligned}
$$

Proof. If $r > 0$, $(a+K)/k$ is not an integer. Then by $q = (a+K-r)/k$,

$$
\begin{aligned}
g(a, a+(K+1)d, \dots, a+kd) &= (q+1)a + (qk+r)d - a\\
&= \frac{a(a+K-r)}{k} + (a+K)d
\end{aligned}
$$

$$= \left\lfloor \frac{a+K}{k} \right\rfloor a + (a+K)d$$

$$= \left(\left\lceil \frac{a+K}{k} \right\rceil - 1 \right) a + (a+K)d.$$

If $r = 0$, $(a+K)/k = q$ is an integer. Then

$$g(a, a+(K+1)d, \dots, a+kd) = qa + (qk+r)d - a$$

$$= \frac{a(a+K-k)}{k} + (a+K)d. \qquad \square$$

By applying the second formula in Lemma 1, we have the Sylvester number of the almost arithmetic sequence with initial gaps.

Theorem 3. *Under the same conditions as in Theorem* 1, *if* $r > K$, *then*

$$n(a, a+(K+1)d, \dots, a+kd)$$
$$= \frac{a^2 + 2(K-1)a + (a+2K-1)kd + K(K-k-2) - k - r(r-k-2)}{2k}$$

and if $0 \le r \le K$, *then*

$$n(a, a+(K+1)d, \dots, a+kd)$$
$$= \frac{a^2 + 2(K-1)a + (a+2K-1)kd + K(K-k-2) + k - r(r-k-2)}{2k}.$$

Proof. If $r > K$, by the second formula in Lemma 1 together with $q = (a+K-r)/k$, we have

$$n(a, a+(K+1)d, \dots, a+kd)$$
$$= \frac{q(q+1)}{2}k - K + (q+1)(r-1) + \left(\frac{a-1}{2} + K \right) d - \frac{a-1}{2}$$
$$= \frac{a^2 + 2(K-1)a + (a+2K-1)kd + K(K-k-2) - k - r(r-k-2)}{2k}.$$

Other cases are proved similarly and omitted. $\square$

2.1. *Examples*

Consider the sequence $11, 23, 27, 31$. Then, $a = 11$, $d = 4$, $K = 2$, $k = 5$, $q = 2$ and $r = 3$. By Theorem 1, we have $s(11, 23, 27, 31) = 1149$. Indeed,

$$\begin{aligned} s(11,23,27,31) &= 1+2+3+4+5+6+7+8+9+10 \\ &\quad + 12+13+14+15+16+17 \\ &\quad + 18+19+20+21+24+25+26+28 \\ &\quad + 29+30+32+35+36 \\ &\quad + 37+39+40+41+43+47+48+51 \\ &\quad + 52+59+63+70+74 \\ &= 1149. \end{aligned}$$

Consider the sequence $13, 22, 25, 28$. Then, $a = 13$, $d = 3$, $K = 2$, $k = 5$ and $q = 3$. By Theorem 1, we have

$$\begin{aligned} \sum_{i=1}^{a-1} m_i &= 22+25+28+44+47+50+53+56+72+75+81+84 \\ &= 637, \\ \sum_{i=1}^{a-1} m_i^2 &= 22^2+25^2+28^2+44^2+47^2+50^2+53^2+56^2+72^2+75^2 \\ &\quad + 81^2+84^2 \\ &= 38,909 \end{aligned}$$

and $s(13, 22, 25, 28) = 1192$.

Consider the sequence $10, 22, 25, 28, 31, 34, 37, 40$. Then, $a = 10$, $d = 3$, $K = 3$, $k = 10$, $q = 1$ and $r = 3$. By Theorem 1, we have

$$\begin{aligned} \sum_{i=1}^{a-1} m_i &= 22+25+28+31+34+37+53+56+59 \\ &= 345, \end{aligned}$$

$$\sum_{i=1}^{a-1} m_i^2 = 22^2 + 25^2 + 28^2 + 31^2 + 34^2 + 37^2 + 53^2 + 56^2 + 59^2$$
$$= 14,805$$

and $s(10, 22, 25, 28, 31, 34, 37, 40) = 576$.

2.2. *Special patterns*

For an integer $a \geq 2$, let us consider the sequence $a, a+2, a+3, a+4$. So, $K = 1$, $k = 4$ and $d = 1$. Nonnegative integers q and r are determined as

$$a + 1 = 4q + r, \quad 0 \leq r < 4.$$

When $r = 2, 3$, i.e. $a \equiv 1, 2 \pmod 4$, by

$$\sum_{i=1}^{a-1} m_i = \frac{a\bigl(a - 3 + 4q^2 + 2r + 2q(r+1)\bigr)}{2}$$
$$= \frac{a\bigl(a(a+8) - (r-3)^2\bigr)}{8}$$

and

$$\sum_{i=1}^{a-1} m_i^2 = \frac{1}{6}a\bigl(2a^2 + (8q^3 + 6(r+1)q^2 + 4(3r-5)q + 6r - 21)a$$
$$+ 64q^3 + 12(4r+5)q^2 + (6r^2 + 54r - 4)q + 6r^2 + 6r + 1\bigr)$$
$$= \frac{1}{48}a\bigl(a^4 + 14a^3 - 3(r^2 - 6r - 13)a^2$$
$$+ 2(r^3 - 18r^2 + 77r - 64)a + (4r^3 - 42r^2 + 104r + 38)\bigr),$$

we have

$$s(a, a+2, a+3, a+4)$$
$$= \frac{(a - r + 5)\bigl(a^3 + (r+3)a^2 - 2(r^2 - 8r + 8)a - 4r^2 + 22r + 6\bigr)}{96}.$$

When $r = 0$, i.e. $a \equiv 3 \pmod 4$, by

$$\begin{aligned}\sum_{i=1}^{a-1} m_i &= \frac{a(a-1+4q^2+2q)}{2}\\ &= \frac{a(a^2+8a-1)}{8}\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{a-1} m_i^2 &= \frac{1}{6}a\bigl(2a^2+(8q^3+6q^2-8q-3)a+64q^3+60q^2-4q+1\bigr)\\ &= \frac{a(a^4+14a^3+63a^2+40a+38)}{48},\end{aligned}$$

we have

$$s(a,a+2,a+3,a+4) = \frac{(a+1)(a+5)(a^2+2a+6)}{96}.$$

When $r = 1$, i.e. $a \equiv 0 \pmod 4$, by

$$\begin{aligned}\sum_{i=1}^{a-1} m_i &= \frac{a\bigl(a+(2q+1)^2\bigr)}{2}\\ &= \frac{a(a^2+8a+4)}{8}\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{a-1} m_i^2 &= \frac{1}{6}a\bigl(2a^2+(8q^3+12q^2+4q+3)a\\ &\quad + 64q^3+108q^2+56q+13\bigr)\\ &= \frac{a(a^4+14a^3+78a^2+136a+108)}{48},\end{aligned}$$

we have

$$s(a,a+2,a+3,a+4) = \frac{(a+4)(a^3+4a^2+22a+24)}{96}.$$

In conclusion, we have the following.

Corollary 1. *For $a \ge 2$, we have*

$$s(a, a+2, a+3, a+4) = \begin{cases} \dfrac{(a+4)(a^3+4a^2+22a+24)}{96} & \text{if } a \equiv 0 \pmod 4, \\ \dfrac{(a+3)(a^3+5a^2+8a+34)}{96} & \text{if } a \equiv 1 \pmod 4, \\ \dfrac{(a+2)(a^3+6a^2+14a+36)}{96} & \text{if } a \equiv 2 \pmod 4, \\ \dfrac{(a+5)(a^3+3a^2+8a+6)}{96} & \text{if } a \equiv 3 \pmod 4. \end{cases}$$

For example,

$$s(12, 14, 15, 16) = 432,$$
$$s(13, 15, 16, 17) = 530,$$
$$s(14, 16, 17, 18) = 692,$$
$$s(15, 17, 18, 19) = 870.$$

From Theorem 2, we have

$$g(a, a+2, a+3, a+4) = \begin{cases} \dfrac{a^2+4a+4}{4} & \text{if } a \equiv 0 \pmod 4, \\ \dfrac{a^2+3a+4}{4} & \text{if } a \equiv 1 \pmod 4, \\ \dfrac{a^2+2a+4}{4} & \text{if } a \equiv 2 \pmod 4, \\ \dfrac{a^2+a+4}{4} & \text{if } a \equiv 3 \pmod 4. \end{cases}$$

Note that $r = 0, 1, 2, 3$ implies that $a \equiv 3, 0, 1, 2 \pmod 4$, respectively. By using the floor function, we can rewritten as follows.

Corollary 2. *For $a \ge 2$, we have*

$$g(a, a+2, a+3, a+4) = \left(1 + \left\lfloor \frac{a}{4} \right\rfloor\right) a + 1.$$

From Theorem 3, we have the following.

Corollary 3. *For $a \geq 2$, we have*

$$n(a, a+2, a+3, a+4) = \begin{cases} \dfrac{a^2+4a+8}{8} & \text{if } a \equiv 0 \pmod 4, \\ \dfrac{a^2+4a+3}{8} & \text{if } a \equiv 1 \pmod 4, \\ \dfrac{a^2+4a+4}{8} & \text{if } a \equiv 2 \pmod 4, \\ \dfrac{a^2+4a+3}{8} & \text{if } a \equiv 3 \pmod 4. \end{cases}$$

The sequence $a, a+3, a+4, a+5$ also satisfies the condition $1 \leq K \leq (k-1)/2$ as $K = 2$, $k = 5$ and $d = 1$.

Corollary 4. *For $a \geq 3$, we have*

$$\begin{aligned} & s(a, a+3, a+4, a+5) \\ & = \begin{cases} \dfrac{(a+5)(a^3+8a^2+55a+90)}{150} & \text{if } a \equiv 0 \pmod 5, \\ \dfrac{(a+4)(a^3+9a^2+35a+105)}{150} & \text{if } a \equiv 1 \pmod 5, \\ \dfrac{(a+3)(a^3+10a^2+41a+110)}{150} & \text{if } a \equiv 2 \pmod 5, \\ \dfrac{(a+2)(a^3+11a^2+43a+105)}{150} & \text{if } a \equiv 3 \pmod 5, \\ \dfrac{(a+6)(a^3+7a^2+41a+65)}{150} & \text{if } a \equiv 4 \pmod 5. \end{cases} \end{aligned}$$

Corollary 5. *For $a \geq 3$, we have*

$$g(a, a+3, a+4, a+5) = \left(1 + \left\lfloor \frac{a+1}{5} \right\rfloor\right) a + 2.$$

Corollary 6. *For $a \geq 3$, we have*

$$n(a, a+3, a+4, a+5) = \begin{cases} \dfrac{a^2+7a+20}{10} & \text{if } a \equiv 0 \pmod 5, \\ \dfrac{a^2+7a+12}{10} & \text{if } a \equiv 1 \pmod 5, \\ \dfrac{a^2+7a+12}{10} & \text{if } a \equiv 2 \pmod 5, \\ \dfrac{a^2+7a+10}{10} & \text{if } a \equiv 3 \pmod 5, \\ \dfrac{a^2+7a+16}{10} & \text{if } a \equiv 4 \pmod 5. \end{cases}$$

However, the sequence $a, a+4, a+5, a+6$ does not satisfy the condition $1 \leq K \leq (k-1)/2$. In fact, if the gap K becomes bigger compared to k, the situation becomes more complicated. This case is discussed in the following section.

3. Bigger Gaps

When the gap K is bigger, the situation becomes more complicated. Consider the same almost arithmetic sequence $a, a+(K+1)d, a+(K+2)d, \dots, a+kd$ with $a \geq 2$ and $\gcd(a,d) = 1$. In this section, we treat with the case when $K > (k-1)/2$. Nevertheless, this case cannot be treated in a unified manner. Cases still need to be divided.

3.1. *General case*

Assume that $(k-1)/2 < K \leq (2k-2)/3$. Nonnegative integers q and r are determined as in (2). We also assume that $q = \lfloor (a+K)/k \rfloor \geq 2$. In addition to these conditions, we shall discuss four cases separately: $r = 0$, $1 \leq r \leq k-K-1$, $k-K \leq r \leq K$ or $K+1 \leq r < k$.

Case 1: When $r = a + K - \lfloor (a+K)/k \rfloor k = 0$, all the elements of Apéry set of $a, a+(K+1)d, a+(K+2)d, \dots, a+kd$ excluding 0 (mod a) can be determined in the following table:

				$a+(K+1)d$	$\dots$	$a+kd$
		$2a+(2K+2)d$	$\dots$	$2a+(K+k+1)d$	$\dots$	$2a+2kd$
$3a+(2k+1)d$	$\dots$	$3a+(2K+k+2)d$	$\dots$	$3a+(K+2k+1)d$	$\dots$	$3a+3kd$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$qa+((q-1)k+1)d$	$\dots$		$\dots$	$[qa+(qk-K)d]$	$\dots$	$qa+qkd$
		$(q+1)a+(qk+k-K+1)d$	$\dots$	$(q+1)a+(qk+K+1)d$		

Since any of the terms $2a+(k+1)d,\dots,2a+(2K+1)d$ cannot be expressed in terms of $a, a+(K+1)d, a+(K+2)d,\dots,a+kd$, they cannot be elements of the Apéry set, and hence do not exist in this table. In addition, the term $qa+(qk-K)d = qa+ad \equiv 0 \pmod{a}$ cannot exist in this table, and the terms $(q+1)a+(qk+1)d,\dots,$ $(q+1)a+(qk+k-K)d$ are also out. Note that by $\gcd(a,d)=1$,

$$\begin{aligned}
&\{(K+1)d,(K+2)d,\dots,kd,(2K+2)d,\dots,(a-1)d,\\
&\quad (a+1)d,\dots,(a+K)d,(a+k+1)d,\dots,(a+2K+1)d\}\\
&\quad = \{1,2,\dots,a-1\} \pmod{a}.
\end{aligned}$$

It is not difficult to see that all the elements in this table can be represented by $a+(K+1)d, a+(K+2)d,\dots,a+kd$ and that none of the elements can be represented by subtracting a. Hence,

$$\begin{aligned}
\sum_{i=1}^{a-1} m_i &= \big((1+2+\dots+q)k-K-2(2K+1-k)-q\\
&\quad +(q+1)(2K-k+1)\big)a+\big((k+1)+\dots+k+(2K+2)\\
&\quad +\dots+(a-1)+(a+1)+\dots+(a+K)\\
&\quad +(a+k+1)+\dots+(a+2K+1)\big)d\\
&=\left(\frac{q(q+1)}{2}k-5K-2+2k-q+(q+1)(2K-k+1)\right)a\\
&\quad +\left(\frac{a(a-1)}{2}+(3K-k+1)a\right)d\\
&=\frac{a\big(a^2+(6K-k)a+K(5K-7k)+2k(k-1)\big)}{2k}\\
&\quad +\left(\frac{a+1}{2}+3K-k\right)ad
\end{aligned}$$

and

$$
\begin{aligned}
\sum_{i=1}^{a-1} m_i^2 = &\big((1^2+2^2+\cdots+q^2)k - K - 2^2(2K+1-k)\\
&- q^2 + (q+1)^2(2K-k+1)\big)a^2 + \big((k+1)^2+\cdots+k^2\\
&+ (2K+2)^2+\cdots+(a-1)^2+(a+1)^2+\cdots+(a+K)^2\\
&+ (a+k+1)^2+\cdots+(a+2K+1)^2\big)d^2\\
&+ 2ad\Bigg(\left(\frac{k(k+1)}{2} - \frac{K(K+1)}{2}\right) + 2\left(\frac{2k(2k+1)}{2}\right.\\
&\left. - \frac{k(k+1)}{2}\right) - 2\left(\frac{(2K+1)(2K+2)}{2} - \frac{k(k+1)}{2}\right)\\
&+ 3\left(\frac{3k(3k+1)}{2} - \frac{2k(2k+1)}{2}\right)\\
&+ \cdots + q\left(\frac{qk(qk+1)}{2} - \frac{(qk-q)(qk-q+1)}{2}\right)\\
&- qa + (q+1)\left(\frac{(a+2K+1)(a+2K+2)}{2}\right.\\
&\left. - \frac{(a+k)(a+k+1)}{2}\right)\Bigg) = \frac{a^2}{6k^2}\big(2a^3 + 3(6K-k)a^2\\
&+ (6K(5K+3k))a + 14K^3 + (21K-53k+12)Kk\\
&+ 18k^2(k-1)\big)a^2 + \frac{a}{6}\big(2a^2 + 3(6K-2k+1)a\\
&+ 6K(5K+7k) - 6k^2 - 6k + 13\big)d^2\\
&+ \frac{ad}{6k}\big(4a^3 + 3(12K-3k+1)a^2 + (6K(10K+3k+7)\\
&- 19k^2 + 9k + 12)a + K(28K^2+39K+12)\\
&- 3(9K^2+15K+4)k - (7K-6k-6)k^2\big),
\end{aligned}
$$

by the third formula in Lemma 1, we have

$$
\begin{aligned}
&s\big(a, a+(K+1)d, a+(K+2)d, \ldots, a+kd\big)\\
&\quad = \frac{1}{12k^2}\big(2a^4 + 6(3K-k)a^3 + \big(30K^2 - k(7k-12)\big)a^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \bigl(14K^3 + 12k^3 - 4(8K+3)k^2 + 6K(K+2)k\bigr)a - k^2 \\
&+ \bigl(2a^2 + 3(6K - 2k + 1)a + 30K^2 + 42K - 6k^2 \\
&- 6k + 13\bigr)k^2d^2 + \bigl(4a^3 + 3(12K - 4k + 1)a^2 \\
&+ (60K^2 + 42K - 13k^2 + 6k + 12)a + 6k^3 - (7K-6)k^2 \\
&- 3(9K^2 + 15K + 4)k + K(28K^2 + 39K + 12)\bigr)kd\bigr).
\end{aligned}
$$

In addition, by the first and second formulae in Lemma 1, we have

$$
\begin{aligned}
&g\bigl(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\bigr) \\
&\quad = \frac{a(a+K)}{k} + (a+2K+1)d
\end{aligned}
$$

and

$$
\begin{aligned}
&n\bigl(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\bigr) \\
&\quad = \frac{a^2 + 2(3K-k)a + K(5K-7k) + k(2k-1)}{2k} \\
&\qquad + \left(\frac{a+1}{2} + 3K - k\right)d,
\end{aligned}
$$

respectively.

Case 2: Assume that $1 \le r \le k - K - 1$. Since $k - K - 1 < K$, the q-th and the $(q+1)$th lines are replaced by

$$
\begin{aligned}
&\underbrace{qa + ((q-1)k+1)d \dots qa + (a-1)d}_{k-K+r-1}\ \underbrace{[\text{gap}]}_{1} \\
&\underbrace{qa + (a+1)d \dots qa + qkd}_{K-r}
\end{aligned}
$$

and

$$
\begin{aligned}
&\underbrace{(q+1)a + (qk+1)d \dots (q+1)a + (qk+r)d}_{r}\underbrace{[\quad\text{gap}\quad]}_{k-K} \\
&\underbrace{(q+1)a + (a+k+1)d \dots (q+1)a + (a+2K+1)d}_{2K-k+1}\underbrace{[\quad\text{gap}\quad]}_{k-K-r-1}
\end{aligned}
$$

from Case 1, respectively. Then, by

$$\sum_{i=1}^{a-1} m_i = \Bigg(\frac{q(q+1)}{2}k - K - 2(2K+1-k) - q$$

$$+ (q+1)(2K-k+1+r)\Bigg)a + \left(\frac{a(a+1)}{2} + (3K-k)a\right)d$$

$$= \frac{a\big(a^2 + (6K-k)a + K(5K-7k) + 2k(k-1) - r(r+4K-3k)\big)}{2k}$$

$$+ \left(\frac{a+1}{2} + 3K - k\right) ad$$

and

$$\sum_{i=1}^{a-1} m_i^2 = \Bigg(\frac{q(q+1)(2q+1)}{6}k - 9K - 4 + 4k - q^2$$

$$+ (q+1)^2(2K-k+1+r)\Bigg)a^2 + \Bigg(\frac{(a+1)(2a+1)}{6}$$

$$+(3K-k)a + (K+1)(5K+2) - k(k+1)\Bigg) ad^2$$

$$+ \ 2ad\Bigg(-\frac{(K+1)(9K+4)}{2} + k(k+1)$$

$$+ \ \frac{qk(q+1)\big((4q-1)k+3\big)}{12} - qa + (q+1)$$

$$\times \Bigg(\frac{(a+2K+1)(a+2K+2)}{2} - \frac{(a+k)(a+k+1)}{2}$$

$$+ \left(qk + \frac{r+1}{2}\right) r\Bigg)\Bigg),$$

we have

$$s\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big)$$

$$= \frac{1}{12k^2}(2a^4 + 6(3K-k)a^3 + \big(30K^2 - 7k^2 + 12k$$

$$- \ 6(r+4K-3k)r\big)a^2 + \big(K^2(14K+6k)$$

$$
\begin{aligned}
&- 4Kk(8k-3) + 12k^2(k-1) + 4r^3 + 6(K-2k)r^2 \\
&- (24K^2 - 6Kk - 8k^2 + 12k)r\big)a - k^2 \\
&+ \big(2a^2 + 3(6K-2k+1)a + 6K(5K+7) \\
&- (6k^2+6k-13)\big)k^2d^2 + \big(4a^3 + 3(12K-4k+1)a^2 \\
&+ \big(6K(10K+7k) - (13k^2-6k-12) - 6r(r+4K-3k)\big)a \\
&+ 28K^3 - K^2(27k-39) - K(7k^2+45k-12) \\
&+ 6(k-1)k(k+2)\big)kd\big).
\end{aligned}
$$

This also holds for $r=0$.

We also have

$$
\begin{aligned}
&g\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big) \\
&\quad = \frac{a(a+K-r)}{k} + (a+2K+1)d
\end{aligned}
$$

and

$$
\begin{aligned}
&n\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big) \\
&\quad = \frac{a^2 + 2(3K-k)a + K(5K-7k) + k(2k-1) - r(r+4K-3k)}{2k} \\
&\qquad + \left(\frac{a+1}{2} + 3K - k\right)d.
\end{aligned}
$$

Case 3: Assume that $k-K \le r \le K$. Then, the qth, the $(q+1)$th and $(q+2)$th lines are replaced by

$$
\underbrace{qa+((q-1)k+1)d \cdots qa+(a-1)d}_{k-K+r-1}\ \underbrace{[\text{gap}]}_{1}
$$

$$
\underbrace{qa+(a+1)d \cdots qa+qkd}_{K-r}
$$

$$
\underbrace{(q+1)a+(qk+1)d \cdots (q+1)a+(qk+r)d}_{r}\ \underbrace{[\quad \text{gap} \quad]}_{k-K}
$$

$$
\underbrace{(q+1)a+(a+k+1)d \cdots (q+1)a+(a+k+K-r)d}_{K-r}
$$

and

$$\underbrace{(q+2)a+((q+1)k+1)d\cdots(q+2)a+(a+2K+1)d}_{K-k+r+1}\underbrace{[\quad\text{gap}\quad]}_{2k-K-r-1},$$

respectively. Then, by

$$\begin{aligned}\sum_{i=1}^{a-1} m_i &= \left(\frac{q(q+1)}{2}k - K - 2(2K+1-k) - q + (q+1)K\right.\\ &\quad \left. + (q+2)(K-k+r+1)\right)a + \left(\frac{a(a+1)}{2} + (3K-k)a\right)d\\ &= \frac{a\big(a^2+(6K-k)a+(K-r)(5K-5k+r)\big)}{2k}\\ &\quad + \left(\frac{a+1}{2}+3K-k\right)ad\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{a-1} m_i^2 &= \left(\frac{q(q+1)(2q+1)}{6}k - 9K - 4 + 4k - q^2\right.\\ &\quad \left. + (q+1)^2K + (q+2)^2(K-k+r+1)\right)a^2\\ &\quad + \left(\frac{(a+1)(2a+1)}{6} + (3K-k)a + (K+1)(5K+2)\right.\\ &\quad \left. - k(k+1)\right)ad^2 + 2ad\left(-\frac{(K+1)(9K+4)}{2} + k(k+1)\right.\\ &\quad + \frac{qk(q+1)\big((4q-1)k+3\big)}{12} - qa + (q+1)\\ &\quad \times\left(\frac{(a+K+k-r)(a+K+k-r+1)}{2}\right.\\ &\quad \left. - \frac{(a+k)(a+k+1)}{2} + \left(qk+\frac{r+1}{2}\right)r\right)\\ &\quad + (q+2)\,\big((q+1)k(K-k+r+1)\\ &\quad \left. + \frac{(K-k+r+1)(K-k+r+1)}{2}\big)\right),\end{aligned}$$

we have

$$
\begin{aligned}
&s\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big)\\
&\quad = \frac{1}{12k^2}\Bigg(2a^4 + 6(3K-k)a^3 + \big(30K^2 + 12Kk\\
&\qquad - 19k^2 + 24k - 6(r+4K-5k)r\big)a^2\\
&\qquad + \big(K^2(14K+18k) - 8Kk(4k-3) + 4r^3 + 6(K-4k)r^2\\
&\qquad - (24K^2 - 6Kk - 32k^2 + 24k)r\big)a - k^2\\
&\qquad + \big(2a^2 + 3(6K-2k+1)a + 6K(5K+7)\\
&\qquad - 6k^2 - 6k + 13\big)k^2d^2 + \big(4a^3 + 3(12K-4k+1)a^2\\
&\qquad + \big(6K(10K+2k+7) - (25k^2 - 18k - 12)\\
&\qquad - 6r(r+4K-5k)\big)a + 28K^3 - K^2(9k-39)\\
&\qquad - K(19k^2 + 15k - 12) + 2r^3 - 3r^2(2K+3k+1)\\
&\qquad + r\big(-K(24K-18k+36) + (19k^2 + 15k - 12)\big)\big)kd\Bigg).
\end{aligned}
$$

We also have

$$
\begin{aligned}
&g\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big)\\
&\quad = \frac{a(a+K-r+k)}{k} + (a+2K+1)d
\end{aligned}
$$

and

$$
\begin{aligned}
&n\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big)\\
&\quad = \frac{a^2 + 2(3K-k)a + (K-r)(5K-5k+r) + k}{2k}\\
&\qquad + \left(\frac{a+1}{2} + 3K - k\right)d.
\end{aligned}
$$

Case 4: Finally, assume that $K+1 \le r < k$. Then, the qth line consists of no gaps. The $(q+1)$th and $(q+2)$th lines are replaced by

$$\underbrace{(q+1)a+(qk+1)d \dots (q+1)a+(qk+r-K-1)d}_{r-K-1} \underbrace{[\text{gap}]}_{1}$$

$$\underbrace{(q+1)a+(a+1)d \dots (q+1)a+(a+K)d}_{K} \underbrace{[\ \text{gap}\ ]}_{k-r}$$

and

$$\underbrace{[\ \text{gap}\ ]}_{r-K} \underbrace{(q+2)a+(a+k+1)d \dots (q+2)a+(a+2K+1)d}_{2K-k+1}$$

$$\underbrace{[\quad \text{gap} \quad]}_{2k-K-r-1},$$

respectively. Then, by

$$\begin{aligned}\sum_{i=1}^{a-1} m_i &= \left(\frac{q(q+1)}{2}k - K - 2(2K+1-k) + (q+1)(r-1)\right.\\ &\qquad \left. + (q+2)(2K-k+1)\right)a + \left(\frac{a(a+1)}{2} + (3K-k)a\right)d\\ &= \frac{a\big(a^2+(6K-k)a+K(5K-3k)-2k-r(r+4K-3k)\big)}{2k}\\ &\qquad + \left(\frac{a+1}{2} + 3K - k\right)ad\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{a-1} m_i^2 &= \left(\frac{q(q+1)(2q+1)}{6}k - 9K - 4 + 4k + (q+1)^2(r-1)\right.\\ &\qquad \left. +\ (q+2)^2(2K-k+1)\right)a^2 + \left(\frac{(a+1)(2a+1)}{6}\right.\\ &\qquad \left. + (3K-k)a + (K+1)(5K+2) - k(k+1)\right)ad^2\end{aligned}$$

$$
\begin{aligned}
&+\ 2ad\biggl(-\frac{(K+1)(9K+4)}{2}+k(k+1)\\
&+\ \frac{qk(q+1)\bigl((4q-1)k+3\bigr)}{12}+(q+1)\\
&\times\left((r-K-1)\left(qk+\frac{r-K}{2}\right)+K\left(a+\frac{K+1}{2}\right)\right)\\
&+(q+2)\left(\frac{(a+2K+1)(a+2K+2)}{2}\right.\\
&\left.\left.-\ \frac{(a+k)(a+k+1)}{2}\right)\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
&s\bigl(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\bigr)\\
&\quad=\frac{1}{12k^2}(2a^4+6(3K-k)a^3+\bigl(30K^2+24Kk\\
&\qquad-\ 19k^2+12k-6(r+4K-3k)r\bigr)a^2\\
&\qquad+\bigl(K^2(14K+30k)-4Kk(5k-3)\\
&\qquad+\ 12k^2(k-1)+4r^3+6(K-2k)r^2\\
&\qquad-(24K^2+18Kk-20k^2+12k)r\bigr)a-k^2\\
&\qquad+\bigl(2a^2+3(6K-2k+1)a+6K(5K+7)\\
&\qquad-\ 6k^2-6k+13\bigr)k^2d^2+\bigl(4a^3+3(12K-4k+1)a^2\\
&\qquad+\bigl(6K(10K+4k+7)-(25k^2-6k-12)-6r(r+4K-3k)\bigr)a\\
&\qquad+\ 28K^3-3K^2(k-13)-K(7k^2+9k-12)+2r^3\\
&\qquad-\ 3r^2(2K+k+1)+r\bigl(-6K(4K-k+6)\\
&\qquad+\ 7k^2+9k-12\bigr)\bigr)kd).
\end{aligned}
$$

We also have

$$g\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big) = \frac{a(a+K-r+k)}{k} + (a+2K+1)d$$

and

$$n\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big) = \frac{a^2+2(3K-k)a+K(5K-3k)-k-r(r+4K-3k)}{2k} + \left(\frac{a+1}{2}+3K-k\right)d.$$

In conclusion, we have the Sylvester sums.

Theorem 4. *Let* a *and* d *be positive integers with* $a \ge 2$ *and* $\gcd(a,d)=1$ K *and* k *be positive integers with* $(k-1)/2 < K \le (2k-2)/3$, *and let* $r = a+K-\lfloor (a+K)/k \rfloor k$. *Assume that* $\lfloor (a+K)/k \rfloor \ge 2$. *If* $0 \le r \le k-K-1$, *then*

$$\begin{aligned} &s\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big) \\ &= \frac{1}{12k^2}\big(2a^4+6(3K-k)a^3+\big(30K^2-7k^2+12k \\ &\quad - 6(r+4K-3k)r\big)a^2+\big(K^2(14K+6k) \\ &\quad - 4Kk(8k-3)+12k^2(k-1)+4r^3+6(K-2k)r^2 \\ &\quad - (24K^2-6Kk-8k^2+12k)r\big)a-k^2 \\ &\quad + \big(2a^2+3(6K-2k+1)a+6K(5K+7) \\ &\quad - (6k^2+6k-13)\big)k^2d^2+\big(4a^3+3(12K-4k+1)a^2 \\ &\quad + \big(6K(10K+7k)-(13k^2-6k-12)-6r(r+4K-3k)\big)a \\ &\quad + 28K^3-K^2(27k-39)-K(7k^2+45k-12) \\ &\quad + 6(k-1)k(k+2)\big)kd\big). \end{aligned}$$

If $k-K\le r\le K$, *then*

$$
\begin{aligned}
&s\bigl(a,a+(K+1)d,a+(K+2)d,\dots,a+kd\bigr)\\
&\quad=\frac{1}{12k^2}(2a^4+6(3K-k)a^3+\bigl(30K^2+12Kk-19k^2\\
&\qquad+\ 24k-6(r+4K-5k)r\bigr)a^2+\bigl(K^2(14K+18k)\\
&\qquad-\ 8Kk(4k-3)+4r^3+6(K-4k)r^2\\
&\qquad-(24K^2-6Kk-32k^2+24k)r\bigr)a-k^2\\
&\qquad+\bigl(2a^2+3(6K-2k+1)a+6K(5K+7)-6k^2-6k+13\bigr)k^2d^2\\
&\qquad+\bigl(4a^3+3(12K-4k+1)a^2+\bigl(6K(10K+2k+7)\\
&\qquad-(25k^2-18k-12)-6r(r+4K-5k)\bigr)a\\
&\qquad+\ 28K^3-K^2(9k-39)-K(19k^2+15k-12)\\
&\qquad+\ 2r^3-3r^2(2K+3k+1)+r\bigl(-K(24K\\
&\qquad-18k+36)+(19k^2+15k-12)\bigr)\bigr)kd).
\end{aligned}
$$

If $K+1\le r<k$, *then*

$$
\begin{aligned}
&s\bigl(a,a+(K+1)d,a+(K+2)d,\dots,a+kd\bigr)\\
&\quad=\frac{1}{12k^2}(2a^4+6(3K-k)a^3+\bigl(30K^2+24Kk\\
&\quad-\ 19k^2+12k-6(r+4K-3k)r\bigr)a^2\\
&\qquad+\bigl(K^2(14K+30k)-4Kk(5k-3)+12k^2(k-1)+4r^3\\
&\qquad+\ 6(K-2k)r^2-(24K^2+18Kk-20k^2+12k)r\bigr)a-k^2\\
&\qquad+\bigl(2a^2+3(6K-2k+1)a+6K(5K+7)-6k^2\\
&\qquad-\ 6k+13)\bigr)k^2d^2+\bigl(4a^3+3(12K-4k+1)a^2\\
&\qquad+\bigl(6K(10K+4k+7)-(25k^2-6k-12)
\end{aligned}
$$

$$- 6r(r+4K-3k)\big)a + 28K^3 - 3K^2(k-13)$$
$$- K(7k^2+9k-12) + 2r^3 - 3r^2(2K+k+1)$$
$$+ r\big(-6K(4K-k+6)+7k^2+9k-12\big)\big)kd\big).$$

Concerning Frobenius and Sylvester numbers, we have the following. a, d, K, k, r are determined as in Theorem 4.

Theorem 5. *Under the same conditions as in Theorem 4, we have*

$$g\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big)$$
$$= \begin{cases} \dfrac{a(a+K-r)}{k} + (a+2K+1)d & \text{if } 0 \le r \le k-K-1 \\[2ex] \dfrac{a(a+K-r+k)}{k} + (a+2K+1)d & \text{if } k-K \le r < k. \end{cases}$$

Theorem 6. *Under the same conditions as in Theorem 4, if* $0 \le r \le k-K-1$, *then*

$$n\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big)$$
$$= \frac{a^2+2(3K-k)a+K(5K-7k)+k(2k-1)-r(r+4K-3k)}{2k}$$
$$+ \left(\frac{a+1}{2}+3K-k\right)d.$$

If $k-K \le r \le K$, *then*

$$n\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big)$$
$$= \frac{a^2+2(3K-k)a+(K-r)(5K-5k+r)+k}{2k}$$
$$+ \left(\frac{a+1}{2}+3K-k\right)d.$$

If $K+1 \le r < k$, *then*

$$n\big(a, a+(K+1)d, a+(K+2)d, \dots, a+kd\big)$$
$$= \frac{a^2+2(3K-k)a+K(5K-3k)-k-r(r+4K-3k)}{2k}$$
$$+ \left(\frac{a+1}{2}+3K-k\right)d.$$

3.2. *Special patterns*

For an integer $a \ge 2$, let us consider the sequence $a, a+4, a+5, a+6$. Then, we apply the Theorem 4 as $K = 3$, $k = 6$ and $d = 1$, and nonnegative integers q and r are determined by $a + 3 = 6q + r$ with $0 \le r \le 5$. $q \ge 2$ implies that $a \ge 9$. When $r = 0, 1, 2$, i.e. $a \equiv 3, 4, 5 \pmod 6$, we have

$$
\begin{aligned}
s(a, a+4, a+5, a+6) = \frac{1}{216}\bigl(&a^4 + 21a^3 - 3(r^2 - 6r - 66)a^2 \\
&+ (2r^3 - 45r^2 + 162r + 927)a \\
&+ 3(2r^3 - 39r^2 + 78r + 495)\bigr).
\end{aligned}
$$

When $r = 3$, i.e. $a \equiv 0 \pmod 6$, we have

$$
\begin{aligned}
s(a, a+4, a+5, a+6) = \frac{1}{216}\bigl(&a^4 + 21a^3 - 3(r^2 - 18r - 42)a^2 \\
&+ (2r^3 - 81r^2 + 774r - 153)a \\
&+ 3(2r^3 - 75r^2 + 762r - 793)\bigr).
\end{aligned}
$$

When $r = 4, 5$, i.e. $a \equiv 1, 2 \pmod 6$, we have

$$
\begin{aligned}
s(a, a+4, a+5, a+6) = \frac{1}{216}\bigl(&a^4 + 21a^3 - 3(r^2 - 6r - 66)a^2 \\
&+ (2r^3 - 45r^2 + 162r + 1143)a \\
&+ 3(2r^3 - 39r^2 + 78r + 999)\bigr).
\end{aligned}
$$

The final result also holds for $q = 1$, i.e. $a = 8$.

Corollary 7. *For $a \geq 8$, we have*

$$s(a, a+4, a+5, a+6) = \begin{cases} \dfrac{a^4 + 21a^3 + 261a^2 + 1494a + 2808}{216} & \textit{if } a \equiv 0 \pmod 6, \\ \dfrac{a^4 + 21a^3 + 222a^2 + 1199a + 2445}{216} & \textit{if } a \equiv 1 \pmod 6, \\ \dfrac{a^4 + 21a^3 + 213a^2 + 1078a + 1992}{216} & \textit{if } a \equiv 2 \pmod 6, \\ \dfrac{(a+3)(a^3 + 18a^2 + 144a + 495)}{216} & \textit{if } a \equiv 3 \pmod 6, \\ \dfrac{a^4 + 21a^3 + 213a^2 + 1046a + 1608}{216} & \textit{if } a \equiv 4 \pmod 6, \\ \dfrac{a^4 + 21a^3 + 222a^2 + 1087a + 1533}{216} & \textit{if } a \equiv 5 \pmod 6. \end{cases}$$

By applying Theorem 5 as $K = 3$, $k = 6$ and $d = 1$, for $r = 0, 1, 2$, i.e. $a \equiv 3, 4, 5 \pmod 6$, we have

$$g(a, a+4, a+5, a+6) = \frac{a^2 + (9-r)a + 42}{6},$$

and for $r = 3, 4, 5$, i.e. $a \equiv 0, 1, 2 \pmod 6$, we have

$$g(a, a+4, a+5, a+6) = \frac{a^2 + (15-r)a + 42}{6}.$$

The case for $a = 8$ is also valid. We can conclude that

$$g(a, a+4, a+5, a+6) = \begin{cases} \dfrac{a^2 + 12a + 42}{6} & \text{if } a \equiv 0 \pmod 6, \\ \dfrac{a^2 + 11a + 42}{6} & \text{if } a \equiv 1 \pmod 6, \\ \dfrac{a^2 + 10a + 42}{6} & \text{if } a \equiv 2 \pmod 6, \\ \dfrac{a^2 + 9a + 42}{6} & \text{if } a \equiv 3 \pmod 6, \\ \dfrac{a^2 + 8a + 42}{6} & \text{if } a \equiv 4 \pmod 6, \\ \dfrac{a^2 + 7a + 42}{6} & \text{if } a \equiv 5 \pmod 6. \end{cases}$$

The coefficient of a can be unified by using the floor function.

Corollary 8. *For $a \ge 8$, we have*

$$g(a, a+4, a+5, a+6) = \left(2 + \left\lfloor \frac{a}{6} \right\rfloor\right) a + 7.$$

By applying Theorem 6, for $r = 0, 1, 2$, i.e. $a \equiv 3, 4, 5 \pmod 6$, we have

$$n(a, a+4, a+5, a+6) = \frac{a^2 + 12a - (r+3)(r-9)}{12},$$

for $r = 3$, i.e. $a \equiv 0 \pmod 6$, we have

$$n(a, a+4, a+5, a+6) = \frac{a^2 + 12a - r^2 + 18r + 3}{12},$$

for $r = 4, 5$, i.e. $a \equiv 1, 2 \pmod 6$, we have

$$n(a, a+4, a+5, a+6) = \frac{a^2 + 12a - r^2 + 6r + 27}{12}.$$

Finally, we can check manually that the result is also valid for $a = 4, 5, 6$.

Corollary 9. *For $a = 4, 5, 6$ and $a \ge 8$, we have*

$$n(a, a+4, a+5, a+6) = \begin{cases} \dfrac{a^2 + 12a + 48}{12} & \text{if } a \equiv 0 \pmod 6, \\ \dfrac{a^2 + 12a + 35}{12} & \text{if } a \equiv 1 \pmod 6, \\ \dfrac{a^2 + 12a + 32}{12} & \text{if } a \equiv 2 \pmod 6, \\ \dfrac{a^2 + 12a + 27}{12} & \text{if } a \equiv 3 \pmod 6, \\ \dfrac{a^2 + 12a + 32}{12} & \text{if } a \equiv 4 \pmod 6, \\ \dfrac{a^2 + 12a + 35}{12} & \text{if } a \equiv 5 \pmod 6. \end{cases}$$

Remark. The coefficient of the constant term can be unified by using the floor function.

$$\begin{aligned} &n(a, a+4, a+5, a+6) \\ &\quad = \frac{a^2}{12} + a + \frac{1}{12\cdot 15}\bigg(720 + 209a - 14\left\lfloor\frac{a}{6}\right\rfloor - 164\left\lfloor\frac{a+1}{6}\right\rfloor \\ &\qquad - 134\left\lfloor\frac{a+2}{6}\right\rfloor - 284\left\lfloor\frac{a+3}{6}\right\rfloor - 254\left\lfloor\frac{a+4}{6}\right\rfloor - 404\left\lfloor\frac{a+5}{6}\right\rfloor\bigg) \\ &\quad = \frac{a^2}{12} + \frac{389}{180}a + 4 - \frac{1}{90}\bigg(7\left\lfloor\frac{a}{6}\right\rfloor + 82\left\lfloor\frac{a+1}{6}\right\rfloor + 67\left\lfloor\frac{a+2}{6}\right\rfloor \\ &\qquad + 142\left\lfloor\frac{a+3}{6}\right\rfloor + 127\left\lfloor\frac{a+4}{6}\right\rfloor + 202\left\lfloor\frac{a+5}{6}\right\rfloor\bigg). \end{aligned}$$

For an integer $a \ge 2$, let us consider the sequence $a, a+5, a+6, a+7$. Then, $K = 4$, $k = 7$ and $d = 1$ in Theorem 4. Non-negative integers q and r are determined by $a + 4 = 7q + r$ with $0 \le r \le 6$. $q \ge 2$ implies that $a \ge 10$. When $r = 0, 1, 2$, i.e. $a \equiv 3, 4, 5 \pmod 7$, we have

$$\begin{aligned} s(a, a+5, a+6, a+7) = \frac{1}{294}\big(&a^4 + 29a^3 - (3r^2 - 15r - 380)a^2 \\ &+ (2r^3 - 51r^2 + 151r + 2296)r \\ &+ 7(r^3 - 24r^2 + 17r + 672)\big). \end{aligned}$$

When $r = 3, 4$, i.e. $a \equiv 6, 0 \pmod 7$, we have

$$\begin{aligned} s(a, a+5, a+6, a+7) = \frac{1}{294}\big(&a^4 + 29a^3 - (3r^2 - 57r - 296)a^2 \\ &+ (2r^3 - 93r^2 + 991r + 784)r \\ &+ 7(r^3 - 45r^2 + 500r - 210)\big). \end{aligned}$$

When $r = 5, 6$, i.e. $a \equiv 1, 2 \pmod 7$, we have

$$s(a, a+5, a+6, a+7) = \frac{1}{294}\big(a^4 + 29a^3 - (3r^2 - 15r - 422)a^2 + (2r^3 - 51r^2 + 109r + 3346)a + 7(r^3 - 24r^2 + 17r + 1386)\big).$$

We can check manually that the result is also valid for $a = 5, 6, 7$.

Corollary 10. *For $a = 5, 6, 7$ and $a \geq 10$, we have*

$$s(a, a+5, a+6, a+7) = \begin{cases} \dfrac{a^4 + 29a^3 + 476a^2 + 3388a + 7938}{294} & \text{if } a \equiv 0 \pmod 7, \\ \dfrac{a^4 + 29a^3 + 422a^2 + 2866a + 6972}{294} & \text{if } a \equiv 1 \pmod 7, \\ \dfrac{a^4 + 29a^3 + 404a^2 + 2596a + 5880}{294} & \text{if } a \equiv 2 \pmod 7, \\ \dfrac{a^4 + 29a^3 + 380a^2 + 2296a + 4704}{294} & \text{if } a \equiv 3 \pmod 7, \\ \dfrac{a^4 + 29a^3 + 392a^2 + 2398a + 4662}{294} & \text{if } a \equiv 4 \pmod 7, \\ \dfrac{a^4 + 29a^3 + 398a^2 + 2410a + 4326}{294} & \text{if } a \equiv 5 \pmod 7, \\ \dfrac{a^4 + 29a^3 + 440a^2 + 2974a + 6384}{294} & \text{if } a \equiv 6 \pmod 7. \end{cases}$$

By applying Theorem 5 as $K = 4$, $k = 7$ and $d = 1$, for $r = 0, 1, 2$, i.e. $a \equiv 3, 4, 5 \pmod 7$, we have

$$g(a, a+5, a+6, a+7) = \frac{a^2 + (11 - r)a + 63}{7},$$

and for $r = 3, 4, 5, 6$, i.e. $a \equiv 6, 0, 1, 2 \pmod 7$, we have

$$g(a, a+5, a+6, a+7) = \frac{a^2 + (18 - r)a + 63}{7}.$$

The cases for $a = 5, 6, 7, 8$ are also valid. The coefficient of a can be unified by using the floor function.

Corollary 11. *For $a = 5, 6, 7, 8$ and $a \geq 10$, we have*

$$g(a, a+5, a+6, a+7) = \left(2 + \left\lfloor \frac{a+1}{7} \right\rfloor\right) a + 9.$$

By applying Theorem 6, for $r = 0, 1, 2$, i.e. $a \equiv 3, 4, 5 \pmod 7$, we have

$$n(a, a+5, a+6, a+7) = \frac{a^2 + 17a - r^2 + 5r + 52}{14},$$

for $r = 3, 4$, i.e. $a \equiv 6, 0 \pmod 7$, we have

$$n(a, a+5, a+6, a+7) = \frac{a^2 + 17a - r^2 + 19r + 24}{14},$$

for $r = 5, 6$, i.e. $a \equiv 1, 2 \pmod 7$, we have

$$n(a, a+5, a+6, a+7) = \frac{a^2 + 17a - r^2 + 5r + 66}{14}.$$

Finally, we can check manually that the result is also valid for $a = 5, 6, 7$.

Corollary 12. *For $a = 5, 6, 7$ and $a \geq 10$, we have*

$$n(a, a+4, a+5, a+6) = \begin{cases} \dfrac{a^2 + 17a + 84}{14} & \text{if } a \equiv 0 \pmod 7, \\ \dfrac{a^2 + 17a + 66}{14} & \text{if } a \equiv 1 \pmod 7, \\ \dfrac{a^2 + 17a + 60}{14} & \text{if } a \equiv 2 \pmod 7, \\ \dfrac{a^2 + 17a + 52}{14} & \text{if } a \equiv 3 \pmod 7, \\ \dfrac{a^2 + 17a + 56}{14} & \text{if } a \equiv 4 \pmod 7, \\ \dfrac{a^2 + 17a + 58}{14} & \text{if } a \equiv 5 \pmod 7, \\ \dfrac{a^2 + 17a + 72}{14} & \text{if } a \equiv 6 \pmod 7. \end{cases}$$

4. Comments

Similarly, we can consider the sequence $a, a + (K + 1)d, a + (K + 2)d, \ldots, a + kd$ when $(2k - 2)/3 < K \leq (3k - 3)/4$. Then, as a special case, we can get Frobenius number, Sylvester number and sum for $a, a + 6, a + 7, a + 8$ and so on. After that, we may continue to consider the cases $(3k - 3)/4 < K \leq (4k - 4)/5$, $(4k - 4)/5 < K \leq (5k - 5)/6, \ldots$. However, the situation becomes more and more complicated. Is there any more convenient method to find their Sylvester sums?

The approaches in Refs. 30 and 31 may apply to any almost arithmetic sequence, but they both have an extra burden: they require the pre-computation of a couple of constants depending on the sequence. Particularizing their results to ours would be of some interest.

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Chapter 6

Arithmetic Properties of Minimal Excludants of Partitions of Integers

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Andrews and Newman[5] defined the minimal excludant (resp. minimal odd exludant) of the partition ξ of an integer $n \geq 1$ as the smallest positive integer (resp. smallest odd positive integer) which is not a part of the partition ξ and is denoted by $\mathsf{Mex}(\xi)$ (resp. and $\mathsf{Moex}(\xi)$). Some congruence and other divisibility properties of the following two arithmetic functions are discussed:

$$\sigma_{\mathsf{Mex}}(n) = \sum_{\xi \in \mathcal{S}} \mathsf{Mex}(\xi) \quad \text{and} \quad \sigma_{\mathsf{Moex}}(n) = \sum_{\xi \in \mathcal{S}} \mathsf{Moex}(\xi).$$

1. Introduction

Throughout the chapter, define the q-series

$$(\beta; q)_\infty := \prod_{j \geq 0} (1 - \beta q^j), \quad |q| < 1, \tag{1}$$

where β and q are complex numbers. For brevity one often writes

$$(\beta_1; q)_\infty (\beta_2; q)_\infty (\beta_3; q)_\infty \cdots (\beta_m; q)_\infty = (\beta_1, \beta_2, \beta_3, \ldots, \beta_m; q)_\infty. \tag{2}$$

A partition of a positive integer n is a sequence of integers $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_j \geq 1$ such that $\sum_{j=1}^{k} \xi_j = n$. The integers ξ_j are called parts or summands of the partition ξ. If $\lambda(n)$ denotes the number of partitions of the positive integer n, then the generating function of $\lambda(n)$ (due to Euler) is given by

$$\sum_{n\geq 0} \lambda(n)q^n = \frac{1}{(q;q)_\infty} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots . \quad (3)$$

Thus, $\lambda(4) = 5$ with the partitions of 5 given by

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

To satisfy the generating function, it is assumed that $\lambda(0) = 1$. Euler also considered two restricted partition functions $\lambda_d(n)$ (resp. $\lambda_o(n)$) which counts the number of partitions of n into distinct (resp. odd) parts, and proved the famous theorem, $\lambda_d(n) = \lambda_o(n)$. In terms of generating functions or q-series, Euler's theorem is stated as

$$\sum_{n\geq 0} \lambda_d(n)q^n = (-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \sum_{n\geq 0} \lambda_o(n)q^n. \quad (4)$$

Euler also gave the following recursion formula for the partition function $\lambda(n)$: If $\lambda(0) = 1$ and $\lambda(n) = 0$ if $n < 1$, then for $n \geq 1$,

$$\lambda(n) = \lambda(n-1) + \lambda(n-2) - \lambda(n-5) - \lambda(n-7) \cdots . \quad (5)$$

Using Euler's recursion formula (5), MacMahon calculated the values of $\lambda(n)$ upto $n = 200$. By examining the values of $\lambda(n)$ calculated by MacMahon, Ramanujan[1] offered the following congruences for $\lambda(n)$:

$$\lambda(5n+4) \equiv 0 \pmod 5,$$
$$\lambda(7n+5) \equiv 0 \pmod 7,$$

and

$$\lambda(11n+6) \equiv 0 \pmod{11}.$$

Ramanujan published two more papers[2,3] on $\lambda(n)$ to establish his congruences. Ramanujan's congruences of $\lambda(n)$ motivated mathematicians to investigate different restricted partition functions or functions related to $\lambda(n)$ in search of analogous results. One such interesting function is the minimal excludant or the mex function. Fraenkel and Peled[4] defined minimal excludant as the smallest positive integer in a set S of positive integers that is not an element of S. The minimal excludant or the mex function was originally applied to study combinatorial game theory.

The concept of minimal excludants in the partition theory of integer was first introduced by Andrews and Newman[5] in 2019. In partition theory of integer, the minimal excludant of the partition ξ of a positive integer n is defined as the smallest positive integer that is not a part of ξ and is denoted by $\mathsf{Mex}(\xi)$. They also defined minimal odd exludant which is the smallest odd positive integer that is not a part of ξ. The minimal odd exludant of the partition ξ is denoted by $\mathsf{Moex}(\xi)$. For example, for the partition $3+1+1$ of 5, minimal excludant is $\mathsf{Mex}(3+1+1) = 2$ and minimal odd excludant is $\mathsf{Moex}(3+1+1) = 5$.

In Ref. 5, Andrews and Newman considered the following two arithmetic functions connected with the functions $\mathsf{Mex}(\xi)$ and $\mathsf{Moex}(\xi)$: Let S denote the set of all partition of positive integer n. Then

$$\sigma_{\mathsf{Mex}}(n) = \sum_{\xi \in \mathsf{S}} \mathsf{Mex}(\xi) \tag{6}$$

and

$$\sigma_{\mathsf{Moex}}(n) = \sum_{\xi \in \mathsf{S}} \mathsf{Moex}(\xi). \tag{7}$$

Andrews and Newman [5, p. 252, Theorem 4.1] also established that

$$\sum_{n \geq 0} \sigma_{\mathsf{Moex}}(n) q^n = (-q;q)_\infty (-q;q^2)_\infty^2. \tag{8}$$

A generalization of the function $\mathsf{Mex}(\xi)$ is also considered by Andrews and Neuman[6] and gave applications in partition theory of numbers. Andrews and Neuman[6] defined $\rho_{m,\alpha}(n)$ and $\overline{\rho}_{m,\alpha}(n)$ which counts the number of partitions of ξ of n, where $\mathsf{Mex}_{m,\alpha}(\xi) \equiv \alpha \pmod{2m}$ and $\mathsf{Mex}_{m,a}(\xi) \equiv m + a \pmod{2m}$, respectively. They showed that $\lambda(n) = \rho_{m,\alpha}(\xi) + \overline{\rho}_{m,\alpha}(\xi)$. They also gave partition-theoretic interpretation of $\rho_{m,\alpha}(\xi)$ for $(m,\alpha) = (1,1), (3,3), (2,1), (4,2)$ and $(6,3)$. The parity of the function $\rho_{r,r}(n)$ was considered by Silva and Sellers,[7] who discussed the characterization of $\rho_{1,1}(n)$ and $\rho_{3,3}(n)$ completely. Recently, Barman and Singh[8] studied parity of $\rho_{2^t,2^t}(n)$ and $\rho_{3\cdot 2^t,3\cdot 2^t}(n)$ for $t \geq 1$. They also found infinite families of congruences modulo 2 using the theory of modular form.

This chapter is devoted to discuss some divisibility properties of the arithmetic functions $\sigma_{\mathsf{Mex}}(n)$ and $\sigma_{\mathsf{Moex}}(n)$. Specifically, congruences moduli 2 and 4 for $\sigma_{\mathsf{Mex}}(n)$ and $\sigma_{\mathsf{Moex}}(n)$ will be proved and also identities that connect $\sigma_{\mathsf{Mex}}(n)$ and $\sigma_{\mathsf{Moex}}(n)$ with some other partition functions will be established by applying q-operations and some q-series identities. The concept of color partitions of positive integer is also important in proving the results. A part or summand in a partition ξ is said to have ℓ different colors if each part of ξ is considered in ℓ different copies and all of them are taken as distinct. For example, if each part in the partitions of 2 has three colors, say red(r), blue(b) and green(g), then the number of three color partitions of 2 is 9, namely

$$2_r,\ 2_b,\ 2_g,\ 1_r+1_r,\ 1_b+1_b,\ 1_g+1_g,\ 1_g+1_r,\ 1_g+1_b,\ 1_r+1_b.$$

Let $\lambda_\ell(n)$ denote the number of partitions of n with each part having ℓ different colors. Then the generating function of $\lambda_\ell(n)$ is given by

$$\sum_{n\geq 0} \lambda_\ell(n) q^n = \frac{1}{(q;q)_\infty^\ell}. \tag{9}$$

The partition function $\lambda_\ell(n)$ was first introduced by Ramanujan in one of his letters to Hardy.[9] The case $\ell = 1$, that is, $\lambda_1(n)$ is the partition function $\lambda(n)$ given in (3). The case $\ell = -1$ of $\lambda_\ell(n)$ defined in (9) is the celebrated Pentagonal Number Theorem given by

$$\sum_{n\geq 0} \lambda_{-1}(n) q^n = (q;q)_\infty = \begin{cases} (-1)^j & \text{for} \quad n = j(3j \pm 1)/2, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

In fact,

$$\lambda_{-1}(n) = \lambda_{-1}^e(n) - \lambda_{-1}^o(n), \tag{11}$$

where $\lambda^e_{-1}(n)$ (resp. $\lambda^o_{-1}(n)$) counts the number of partitions of integer n with even (resp. odd) number of distinct parts. For example, if $n = 5$, then $\lambda^e_{-1}(5) = 2$ with relevant partitions $4+1$ and $3+2$ and $\lambda^o_{-1}(5) = 1$ with the relevant partition 5. Thus, $\lambda_{-1}(5) = 2 - 1 = 1$. Generalizing the idea, for any integer $\ell \geq 1$, one has

$$\lambda_{-\ell}(n) = \lambda^e_{-\ell}(n) - \lambda^o_{-\ell}(n), \tag{12}$$

where $\lambda^e_{-r}(n)$ (resp. $\lambda^o_{-\ell}(n)$) is the number of partitions of n having even (resp. odd) number of distinct parts such that each part has ℓ distinct colors. One can see Refs. 10 and 11 and references therein for congruences of the $\lambda_{-\ell}(n)$ and $\lambda_{\ell}(n)$.

It is useful to note here that if k, m and ℓ are positive integers, then

$$\frac{1}{(q^k; q^m)^{\ell}_{\infty}} \tag{13}$$

is the generating function of the number of partitions of a positive integer with parts congruent to k modulo m (that is, $\equiv k \pmod{m}$) and each part has ℓ different colors. Similarly,

$$\frac{1}{(q^{k_1}; q^m)^{\ell}_{\infty}\,(q^{k_2}; q^m)^{\ell}_{\infty}} \tag{14}$$

is the generating function of the number of partitions with parts $\equiv k_1$ or $k_2 \pmod{m}$ and each part has ℓ different colors.

It is also noteworthy that using color partition of positive integer, Andrews and Newman [5, p. 250, Theorem 1.1] proved that

$$\sigma_{\mathsf{Mex}}(n) = D_2(n), \tag{15}$$

where $D_2(n)$ counts the number of partitions of n into distinct parts such that each part has two colors. The equality (15) can be stated as

$$\sum_{n\geq 0} \sigma_{\mathsf{Mex}}(n) q^n = (-q;q)^2_{\infty} = \sum_{n\geq 0} D_2(n) q^n. \tag{16}$$

For combinatorial proof of (15), refer paper by Ballantine and Merca.[12]

2. Congruences for $\sigma_{\mathsf{Mex}}(n)$

Theorem 1. *One has*

$$\sigma_{\mathsf{Mex}}(2n+1) \equiv 0 \pmod 2.$$

Proof. From (16), note that

$$\sum_{n\geq 0} \sigma_{\mathsf{Mex}}(n) q^n = (-q;q)_\infty^2. \tag{17}$$

Simplify (17) by employing elementary q-operations to write as

$$\sum_{n\geq 0} \sigma_{\mathsf{Mex}}(n) q^n = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^2}. \tag{18}$$

Using binomial theorem in (1), one can easily show that

$$(q^2;q^2)_\infty \equiv (q;q)_\infty^2 \pmod 2. \tag{19}$$

Employ (19) in (18) to obtain

$$\sum_{n\geq 0} \sigma_{\mathsf{Mex}}(n) q^n \equiv (q^2;q^2)_\infty \pmod 2. \tag{20}$$

It is easily seen that the right-hand side of (20) contains no terms involving odd powers of q. So, extracting the terms involving q^{2n+1} from (20), one can arrive at the desired result. □

To verify Theorem 1, take $n=1$. Then for the partitions 3, $2+1$ and $1+1+1$ of $2n+1 = 2\cdot 1+1 = 3$, $\mathsf{Mex}(3) = 1, \mathsf{Mex}(2+1) = 3$ and $\mathsf{Mex}(1+1+1) = 2$, respectively. So, $\sigma_{\mathsf{Mex}}(3) = 1+3+2 = 6 \equiv 0 \pmod 2$.

Theorem 2. *Let $\delta \geq 0$ be any integer and $\omega \geq 5$ be any prime. Then*

$$\sum_{n\geq 0} \sigma_{\mathsf{Mex}}\left(2\omega^{2\delta} n + \frac{\omega^{2\delta}-1}{12}\right) q^n \equiv (q;q)_\infty \pmod 2. \tag{21}$$

Proof. The method of mathematical induction with δ will be employed to prove the result. Extract the terms containing q^{2n} from (20) and then replace q^2 by q to obtain

$$\sum_{n\geq 0}\sigma_{\mathsf{Mex}}(2n)q^n \equiv (q;q)_\infty \pmod 2. \tag{22}$$

This proves $\delta = 0$ case of (21). Assume that (21) is true for some $\delta \geq 0$. Now, for any prime $\omega \geq 5$, from [13, Theorem 2.2], one has

$$\begin{aligned}(q;q)_\infty = &\sum_{\substack{j=-(\omega-1)/2\\ j\neq(\pm\omega-1)/6}}^{(\omega-1)/2} (-1)^j q^{(3j^2+j)/2} f\\ &\times\left(-q^{(3\omega^2+(6j+1)\omega)/2}, -q^{(3\omega^2-(6j+1)\omega)/2}\right),\\ &+(-1)^{(\pm\omega-1)/6} q^{(\omega^2-1)/24}(q^{\omega^2};q^{\omega^2})_\infty \end{aligned}\tag{23}$$

where $\dfrac{\pm\omega-1}{6} := \begin{cases}(\omega-1)/6, & if \quad \omega\equiv 1 \pmod 6,\\ (-\omega-1)/6, & if \quad \omega\equiv -1 \pmod 6,\end{cases}$ and $f(u,v)$ is the Ramanujan's general theta function defined by

$$f(u,v) = \sum_{t=-\infty}^{\infty} u^{t(t+1)/2} v^{t(t-1)/2}, \qquad |uv|<1.$$

Furthermore, if $-\dfrac{\omega-1}{2} \leq j \leq \dfrac{\omega-1}{2}$ and $j \neq \dfrac{\pm\omega-1}{2}$, then $\dfrac{3j^2+j}{2} \not\equiv \dfrac{\omega^2-1}{24} \pmod{\omega}$.

Employ (23) in (21) to obtain

$$\begin{aligned}&\sum_{n\geq 0}\sigma_{\mathsf{Mex}}\left(2\omega^{2\delta}n + \frac{\omega^{2\delta}-1}{12}\right)\\ &\equiv \sum_{\substack{j=-(\omega-1)/2\\ j\neq(\pm\omega-1)/6}}^{(\omega-1)/2} (-1)^j\ q^{(3j^2+j)/2}\ f\left(-\ q^{(3\omega^2+(6j+1)\omega)/2}, -q^{(3\omega^2-(6j+1)\omega)/2}\right)\\ &\quad +(-1)^{(\pm\omega-1)/6}\ q^{(\omega^2-1)/24}\ (q^{\omega^2};q^{\omega^2})_\infty \pmod 2.\end{aligned}\tag{24}$$

Extract the terms that contain $q^{\omega n+(\omega^2-1)/24}$ from (24) and then divide by $q^{(\omega^2-1)/24}$ and replace q^ω by q to obtain

$$\sum_{n\geq 0}\sigma_{\mathsf{Mex}}\left(2\omega^{2\delta+1}n+\frac{\omega^{2\delta+2}-1}{12}\right)q^n\equiv(q^\omega;q^\omega)_\infty \pmod 2. \tag{25}$$

Again, extract the terms involving $q^{\omega n}$ and replace q^ω by q to obtain

$$\sum_{n\geq 0}\sigma_{\mathsf{Mex}}\left(2\omega^{2(\delta+1)}n+\frac{\omega^{2(\delta+1)}-1}{12}\right)q^n\equiv(q;q)_\infty \pmod 2. \tag{26}$$

Equation (26) is the $\delta+1$ case of (21), and this completes the proof. □

Corollary 1. *For any integer $\delta\geq 0$ and any prime $\omega\geq 5$, one has*

$$\sigma_{\mathsf{Mex}}\left(2\omega^{2\delta+2}n+\frac{\omega^{2\delta+1}(24j+\omega)-1}{12}\right)\equiv 0 \pmod 2,$$

where $j=1,2,3,\ldots,\omega-1$.

Proof. Extract the terms containing $q^{\omega n+j}$ for $j=1,2,3,\ldots,\omega-1$ from (25) to obtain the desired result. □

Corollary 2. *One has*

$$\sigma_{\mathsf{Mex}}(2n)\equiv\left(\lambda^e_{-1}(n)-\lambda^o_{-1}(n)\right) \pmod 2,$$

where $\lambda^e_{-1}(n)$ (resp. $\lambda^o_{-1}(n)$) counts the number of partitions of n with even (resp. odd) number of distinct parts.

Proof. From (10) and (11), it is easily seen that

$$\sum_{n\geq 0}\left(\lambda^e_{-1}(n)-\lambda^o_{-1}(n)\right)q^n=(q;q)_\infty. \tag{27}$$

Employ (22) in (27) to obtain

$$\sum_{n\geq 0}\sigma_{\mathsf{Mex}}(2n)q^n\equiv\sum_{n\geq 0}\left(\lambda^e_{-1}(n)-\lambda^o_{-1}(n)\right)q^n \pmod 2. \tag{28}$$

The required result now follows directly from (28). □

To verify Corollary 2, set $n = 2$. Then, for the partitions 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$ of 4, $\mathsf{Mex}(4) = 1$, $\mathsf{Mex}(3+1) = 2$, $\mathsf{Mex}(2+2) = 1$, $\mathsf{Mex}(2+1+1) = 3$, and $\mathsf{Mex}(1+1+1+1) = 2$, respectively. Therefore, $\sigma_{\mathsf{Mex}}(4) = 1+2+1+3+2 = 9$. Again, $\lambda^e_{-1}(2) = 0$ and $\lambda^o_{-1}(2) = 1$ with the relevant partition 2. Thus, Corollary 2 is true as $9 \equiv (0-1) = -1 \pmod 2$.

Theorem 3. *Let $\lambda^e_{-2}(n)$ (resp. $\lambda^o_{-2}(n)$) be the number of partitions of n with even (resp. odd) number of distinct parts and each part has two colors. Then*

$$\sigma_{\mathsf{Mex}}(n) \equiv \left(\lambda^e_{-2}(n) - \lambda^o_{-2}(n)\right) \pmod 4.$$

Proof. Employ (35) in (18) to obtain

$$\sum_{n\geq 0} \sigma_{\mathsf{Mex}}(n) q^n = \frac{(q^2;q^2)^2_\infty}{(q;q)^2_\infty} \equiv (q;q)^2_\infty \pmod 4. \tag{29}$$

From Pentagonal Number Theorem, one can see that

$$\sum_{n\geq 0} \left(\lambda^e_{-2}(n) - \lambda^o_{-2}(n)\right) q^n = (q;q)^2_\infty. \tag{30}$$

Employ (30) in (29) to obtain

$$\sum_{n\geq 0} \sigma_{\mathsf{Mex}}(n) q^n \equiv \sum_{n\geq 0} \left(\lambda^e_{-2}(n) - \lambda^o_{-2}(n)\right) q^n \pmod 4. \tag{31}$$

Now, extract the coefficients of q^n from both sides of (31) to complete the proof. □

Set $n = 2$ in Theorem 3. One can see that $\sigma_{\mathsf{Mex}}(2) = \mathsf{Mex}(2) + \mathsf{Mex}(1+1) = 1+2 = 3$. Next, consider two colors green(g) and blue(b), then $\lambda^e_{-2}(n) = 1$ with relevant partition $1_b + 1_g$ and $\lambda^o_{-2}(n) = 2$ with relevant partitions 2_b and 2_g. Thus, Theorem 3 is easily verified as $3 \equiv (1-2) = -1 \pmod 4$.

Theorem 4. *Let $\mathcal{C}(n)$ denote the number of partitions of n such that parts $\equiv 0 \pmod 4$ have two colors and parts $\equiv 1, 2,$ or $3 \pmod 4$ have three colors with the generating function defined by*

$$\sum_{n\geq 0} \mathcal{C}(n) q^n = \frac{1}{(q;q)^2_\infty (q,q^2,q^3;q^4)_\infty} = \frac{1}{(q^4;q^4)^2_\infty (q,q^2,q^3;q^4)^3_\infty}.$$

Then

$$\sigma_{\mathsf{Mex}}(2n) \equiv \mathcal{C}(n) \pmod 4.$$

Proof. From [14, Lemma 2.1], one has

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5\,(q^{16};q^{16})_\infty^2} + 2q\,\frac{(q^4;q^4)_\infty^2\,(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5\,(q^8;q^8)_\infty}. \tag{32}$$

Employ (32) in (18) and simplify to obtain

$$\sum_{n\geq 0}\sigma_{\mathsf{Mex}}(n)q^n = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^3\,(q^{16};q^{16})_\infty^2} + 2q\,\frac{(q^4;q^4)_\infty^2\,(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^3\,(q^8;q^8)_\infty}. \tag{33}$$

Extract the terms containing q^{2n} from (33) and then replace q^2 by q to obtain

$$\sum_{n\geq 0}\sigma_{\mathsf{Mex}}(2n)q^n = \frac{(q^4;q^4)_\infty^5}{(q;q)_\infty^3\,(q^8;q^8)_\infty^2}. \tag{34}$$

From (1) and binomial theorem, one has

$$(q^2;q^2)_\infty^2 \equiv (q;q)_\infty^4 \pmod 4. \tag{35}$$

Simplify (34) by employing (35) to obtain

$$\sum_{n\geq 0}\sigma_{\mathsf{Mex}}(2n)q^n \equiv \frac{(q^4;q^4)_\infty}{(q;q)_\infty^3} \pmod 4. \tag{36}$$

Again, simplify (36) by using elementary q-operations to write as

$$\sum_{n\geq 0}\sigma_{\mathsf{Mex}}(2n)q^n \equiv \frac{1}{(q;q)_\infty^2(q,q^2,q^3;q^4)_\infty} = \sum_{n\geq 0}\mathcal{C}(n)q^n \pmod 4. \tag{37}$$

Now, equate the coefficients of q^n on both sides of (37) to complete the proof. □

Setting $n = 2$ in Theorem 4, we see that for the partitions of 4, $\sigma_{\mathsf{Mex}}(4) = 1+2+1+3+2 = 9$, as $\mathsf{Mex}(4) = 1$, $\mathsf{Mex}(3+1) = 2$, $\mathsf{Mex}(2+2) = 1$, $\mathsf{Mex}(2+1+1) = 3$, and $\mathsf{Mex}(1+1+1+1) = 2$. Again, if there are three different colors, say blue(b), green(g) and red(r), then $\mathcal{C}(2) = 9$ with relevant partitions 2_r, 2_g, 2_b, 1_r+1_r, 1_r+1_g, 1_r+1_b, 1_g+1_g, 1_g+1_b, and 1_b+1_b. Thus, Theorem 4 is true as $9 \equiv 9 \pmod 4$.

3. Congruence for $\sigma_{\mathsf{Moex}}(n)$

Theorem 5. *One has*

$$\sigma_{\mathsf{Moex}}(n) \equiv p(n) \pmod 2.$$

Proof. Use elementary q-operations in (8) to obtain

$$\sum_{n\geq 0} \sigma_{\mathsf{Moex}}(n) q^n = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^3 (q^4;q^4)_\infty^2}. \tag{38}$$

Simplify (38) using (19) and then employ (3) to obtain

$$\sum_{n\geq 0} \sigma_{\mathsf{Moex}}(n) q^n \equiv \frac{1}{(q;q)_\infty} = \sum_{n\geq 0} p(n) q^n \pmod 2. \tag{39}$$

Now, extract the coefficients of q^n in (39) to arrive at the desired result. □

Taking $n = 3$ in Theorem 5, we find that $\sigma_{\mathsf{Moex}}(3) = 1+3+3 = 7$ for $\mathsf{Moex}(3) = 1$, $\mathsf{Moex}(2+1) = 3$, and $\mathsf{Moex}(1+1+1) = 3$. Again, $p(3) = 3$ with the relevant partitions 3, $2+1$, and $1+1+1$. Thus, Theorem 5 is readily verified as $7 \equiv 3 \pmod 2$.

Theorem 6. *Let $\mathcal{D}(n)$ be the number of partitions of a positive integer n such that even parts have two colors and odd parts have three colors with the generating function given by*

$$\sum_{n\geq 0} \mathcal{D}(n) q^n = \frac{1}{(q;q)_\infty^2 (q;q^2)_\infty} = \frac{1}{(q^2;q^2)_\infty^2 (q;q^2)_\infty^3}.$$

Then

$$\sigma_{\mathsf{Moex}}(n) \equiv \mathcal{D}(n) \pmod 4.$$

Proof. Simplify (38) by elementary q-operations and then employ (35) to obtain

$$\begin{aligned}\sum_{n\geq 0} \sigma_{\mathsf{Moex}}(n) q^n &\equiv \frac{(q^2;q^2)_\infty}{(q;q)_\infty^3} = \frac{1}{(q;q)_\infty^2 (q;q^2)_\infty} \\ &= \sum_{n\geq 0} \mathcal{D}(n) q^n \pmod 4. \end{aligned} \tag{40}$$

The desired result now easily follows by equating the coefficients of q^n on both sides of (40). □

Setting $n = 2$ in Theorem 6, we see that $\sigma_{moex}(2) = 1 + 3 = 4$ for $\mathsf{Moex}(2) = 1$ and $\mathsf{Moex}(1 + 1) = 3$. Again, if there are three different colors, say red(r), blue(b) and green(g) for odd parts and two colors blue(b) and green(g) for even parts, then $\mathcal{D}(2) = 8$ with relevant partitions 2_b, 2_g, $1_r + 1_r, 1_r + 1_g, 1 + 1_r + 1_b, 1_g + 1_g, 1_b + 1_g$, and $1_b + 1_b$. Thus, Theorem 6 is easily verified as $4 \equiv 8 \pmod 4$ and holds true.

4. Conclusion

In this chapter, some new congruences modulo 2 and 4 for the functions $\sigma_{\mathsf{Mex}}(n)$ and $\sigma_{\mathsf{Moex}}(n)$ are presented. Some identities connecting $\sigma_{\mathsf{Mex}}(n)$ and $\sigma_{\mathsf{Moex}}(n)$ with the color partition functions are also established. The two functions $\sigma_{\mathsf{Mex}}(n)$ and $\sigma_{\mathsf{Moex}}(n)$ can be further investigated for congruences modulo higher primes. One can also note that the proofs of many results related to mex functions in the literature relied on the theory of modular forms, so elementary proofs using q-series are always desired.

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Chapter 7

On Certain ℓ-regular Partitions: A Brief Survey

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In this chapter, we give an introduction to ℓ-regular partitions and ℓ-regular overpartitions. We also give a brief review of the literature on the works done so far on these partition functions.

1. Introduction

A *partition* of a positive integer n is a representation of n as a sum of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that the order of these positive integers occurs being immaterial. We thus do not distinguish between $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ and $n = \lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$, where $\sigma = i_1 i_2 \cdots i_k$ is a permutation of the set $\{1, 2, \ldots, k\}$. The positive integers $\lambda_1, \lambda_2, \ldots, \lambda_k$ occur in a partition called *parts*. If there is no restriction of any kind on the number or size of parts, we call a partition as *unrestricted* partition.

One can restrict a partition in many ways. For example, imposing restrictions on parts has famous partition functions such as partitions into distinct parts and partitions into odd parts. Euler's partition theorem states that the number of partitions of an integer n into odd parts is equal to the number of partitions of an integer n into distinct parts (see Ref. 1).

The number of unrestricted partitions of any given positive integer n is denoted by $p(n)$. Conventionally, we set $p(0) = 1$. For example, 7 can be expressed as a sum in the following 15 ways:

$$7;\ 6+1;\ 5+2;\ 5+1+1;\ 4+3;\ 4+2+1;\ 4+1+1+1;$$
$$3+3+1;\ 3+2+2;\ 3+2+1+1;$$
$$3+1+1+1+1;\ 2+2+2+1;\ 2+2+1+1+1;$$
$$2+1+1+1+1+1;\ 1+1+1+1+1+1+1.$$

These are all the unrestricted partitions of 7. For complex numbers a and q, the symbol $(a;q)_\infty$ stands for the *q-shifted factorial* defined as

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1 - aq^{n-1}), \quad |q| < 1.$$

The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

Ramanujan had found three simple congruences satisfied by $p(n)$:

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$

and

$$p(11n+6) \equiv 0 \pmod{11}.$$

He gave proofs of the first two congruences in Ref. 2 and remarked that there are no simple congruences for $p(n)$ whose moduli involve primes other than $5, 7$ and 11. Ahlgren and Boylan[3] justifies Ramanujan's claim by proving that if ℓ is a prime and $0 \le \beta < \ell$ is an integer for which

$$p(\ell n + \beta) \equiv 0 \pmod{\ell}$$

for every nonnegative integer n, then $(\ell, \beta) \in \{(5,4), (7,5), (11,6)\}$. Ono[4] made a remarkable breakthrough by proving that for every prime $m \geq 5$, there exist infinitely many arithmetic progressions $an+b$ such that for every nonnegative integer n, we have $p(an+b) \equiv 0 \pmod{m}$. Later, Ahlgren[5] has extended this result to include every modulus m coprime to 6. Looking at these congruences, one may wonder exactly how often $p(n)$ is divisible by $m > 1$. The answer to this question is not known for any such m. The well-known parity conjecture of Parkin and Shanks[6] predicts that the values of $p(n)$ are equally distributed modulo 2.

Conjecture 1. *If* $r \in \{0, 1\}$, *then*

$$\lim_{X \to +\infty} \frac{\#\{0 \leq n < X : p(n) \equiv r \pmod{2}\}}{X} = \frac{1}{2}.$$

This conjecture is still open. In fact, very little is known regarding the above conjecture.

For certain ℓ-regular partition functions, the state of knowledge about similar questions as above is better.

A partition of n is ℓ-*regular* if none of its parts are multiples of ℓ. For example, the nine 3-regular partition of 7 are

$$7;\ 5+2;\ 5+1+1;\ 4+2+1;\ 4+1+1+1;\ 2+2+2+1;$$
$$2+2+1+1+1;\ 2+1+1+1+1+1;\ 1+1+1+1+1+1+1.$$

We denote the number of ℓ-regular partitions of n by $b_\ell(n)$ and the generating function[7] is given by

$$\sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{(q^\ell; q^\ell)_\infty}{(q; q)_\infty}.$$

In the classical representation theory, the number of irreducible p-modular representations of the symmetric group S_n is the same as $b_p(n)$ when p is prime (see Ref. 8). Recently, Gordon and Ono[9] proved that the number of p-modular irreducible representations of almost every symmetric group S_n is a multiple of p^j. Many good results are proved for ℓ-regular partitions using q-series identities, modular form, and combinatorics.

In Section 2, we briefly discuss the results for ℓ-regular partitions that are currently available in the literature.

Corteel and Lovejoy[10] introduced overpartitions. An *overpartition* of n is an unrestricted partition of n in which the first occurrence of a number may be overlined. Suppose $\overline{p}(n)$ counts the total number of overpartitions of n. For example, the overpartitions of 4 are

$$4;\ \overline{4};\ 3+1;\ \overline{3}+1;\ 3+\overline{1};\ \overline{3}+\overline{1};\ 2+2;\ \overline{2}+2;\ 2+1+1;$$
$$\overline{2}+1+1;\ 2+\overline{1}+1;\ \overline{2}+\overline{1}+1;\ 1+1+1+1;\ \overline{1}+1+1+1.$$

From the above example, $\overline{p}(4)=14$. The generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n=\frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}.$$

In Ref. 11, Lovejoy investigated the *ℓ-regular overpartition* $\overline{A}_\ell(n)$, which counts the number of overpartitions of n with no parts divisible by ℓ. From the above example, it is easy to see that $\overline{A}_3(4)=10$. The generating function for $\overline{A}_\ell(n)$ is

$$\sum_{n=0}^{\infty}\overline{A}_\ell(n)q^n=\frac{(-q;q)_\infty(q^\ell;q^\ell)_\infty}{(q;q)_\infty(-q^\ell;q^\ell)_\infty}.$$

We give a brief review of the literature on the works done so far for ℓ-regular overpartitions in Section 3.

2. ℓ-Regular Partitions

In Ref. 9, Gordon and Ono consider $\ell=p_1^{a_1}p_2^{a_2}\dots p_m^{a_m}$ as a product of primes and $S_k(X;M)$ is the number of positive integers $n\le X$ for which $b_k(n)\equiv 0\pmod{M}$. They prove the following theorem.

Theorem 1. *Let $\ell=p_1^{a_1}p_2^{a_2}\dots p_m^{a_m}$ be the prime factorization of a positive integer ℓ. If $p_i^{2a_i}\ge\ell$, then for every positive integer j,*

$$\lim_{X\to\infty}\frac{S_k(X;p_i^j)}{X}=1.$$

Although the set of nonnegative integers n for which $b_\ell(n)\equiv 0\pmod{p_i^j}$ has density one when $p_i^{2a_i}\ge\ell$, the authors also show that the set of those m for which $b_\ell(m) \not\equiv 0 \pmod{p_i^j}$ is indeed infinite in Corollary 2.

Lovejoy and Penniston[11] consider the 3-regular partitions, and they find a relation between $b_3(n)$ and the zeta function of a modular $K3$ surface. Particularly, they proved the following theorem.

Theorem 2. *Let X be the $K3$ surface defined by*

$$X : s^2 = x(x+1)y(y+1)(x+8y).$$

If $p \equiv 1 \pmod{12}$ is a prime, then

$$b_3\left(\frac{p-1}{12}\right) = \#X(\mathbb{F}_p) - (p+1)^2 \pmod{9}.$$

Using the Hecke theory and the arithmetic of the Gaussian integers, they also proved the following result.

Theorem 3. *Let n be a positive integer such that $12n+1 = N^2M$ with M square-free, and for a prime divisor p of $12n+1$, define $k_p := ord_p(12n+1)$. If $p \equiv 1 \pmod{12}$, let d_p and e_p be integers such that $3 \mid d_p$ and $p = d_p^2 + e_p^2$.*

- *If there is a prime $p \equiv 5, 7$ or $11 \pmod{12}$ such that $p \mid M$, then*

$$b_3(n) \equiv 0 \pmod{9}.$$

- *If every prime divisor of M satisfies that there is a prime $p \equiv 1$ $\pmod{12}$, then*

$$b_3(n) \equiv (3n+1) \prod_{\substack{p\mid(12n+1)\\ p\equiv 1 \pmod{12}}} (-1)^{k_p d_p}(k_p+1) \times \prod_{\substack{p\mid(12n+1)\\ p\equiv 5 \pmod{12}}} (-1)^{k_{p/2}} \pmod{9}.$$

Remember that $b_2(n)$ also denotes the number of partitions of an integer n into odd parts (which is equal to counting the number of partitions of an integer n into distinct parts). In Refs. 12 and 13, Ono and Penniston address the more general question of the distribution of $b_2(n)$ compared to the previous result by Gordon and Ono.[9] They proved the following results.

Theorem 4. *If j is a positive integer and $1 \le i \le 2^j - 1$, then*

$$\#\{0 \le n \le X : b_2(n) \equiv i \pmod{2^j}\} \gg_{i,j} \sqrt{X}/\log X.$$

It would be difficult to substantially improve Theorem 4 for odd i. However, one can often improve the lower bound in Theorem 4 to $X/\log X$, in case i is even. In the series of articles,[14–16] Penniston studied ℓ-regular partitions intensely and proved many significant results. In Ref. 14, Penniston extended Theorem 4 from $\ell = 2$ to an arbitrary prime power $\ell = p^a$ and proved the following two theorems.

Theorem 5. *Let a be a positive integer, p an odd prime, and let $\ell = p^a$. If j is a positive integer and $1 \le i \le p^j - 1$, then*

$$\#\{0 \le n \le X : b_\ell(n) \equiv i \pmod{p^j}\} \gg_{\ell,i,j} X/\log X.$$

Theorem 6. *Let a be a positive integer, and let $\ell = p^a$. If j is a positive integer and $1 \le i \le 2^j - 1$, then*

$$\#\{0 \le n \le X : b_\ell(n) \equiv i \pmod{2^j}\} \gg_{\ell,i,j} \begin{cases} X/\log X & \text{if } i \text{ is even,} \\ \sqrt{X}/\log X & \text{if } i \text{ is odd.} \end{cases}$$

Now, we discuss the article[15] which is a follow-up of the above article by Penniston.

Theorem 7. *Suppose $3 \le \ell \le 23$ and $p \ge 5$ are distinct primes, and s is a positive integer. Then*

$$\lim_{X\to\infty} \inf \frac{\#\{1 \le n \le X : b_\ell(n) \equiv 0 \pmod{p^s}\}}{X} \ge \begin{cases} \dfrac{p+1}{2p} & \text{if } p \nmid \ell - 1, \\ \dfrac{p-1}{p} & \text{if } p \mid \ell - 1. \end{cases}$$

Theorem 8. *Suppose $3 \le \ell \le 23$ is prime and $m > 1$ is odd. Assume $3 \nmid m$ if $\ell \ne 3$. Further suppose there exists an integer $k' \equiv \ell - 1 \pmod{24}$ such that if we set $n' = \frac{k' - \ell + 1}{24}$, then $n' \in \cap_{p|m} \mathcal{S}_{\ell,p}$ and*

$b_\ell(n')$ is coprime to m. Then for any $1 \le a \le m-1$,

$$\#\{0 \le n \le X : b_\ell(n) \equiv a \pmod m\} \gg X/logX.$$

In Ref. 15, Penniston showed that for any $j \ge 1$, $b_{11}(n)$ is divisible by 5^j for at least 80% of positive integers n and in Ref. 16, the author gave a complete description of 5-divisibility of $b_{11}(n)$ if $5 \nmid n$. Consider the binary quadratic forms

$$F_{a,b,c}(X,Y) = aX^2 + bXY + cY^2,$$

and let p be an odd prime with $\left(\frac{-33}{p}\right) = 1$. One can find $x, y \in \mathbb{Z}$ such that

$$p = F_{a,b,c}(x,y)$$

for $(a,b,c) \in \{(1,0,33),(2,2,17),(6,6,7),(3,0,11)\}$. Now, fix such x and y, and define

$$\delta_p := \begin{cases} 4px^2 & \text{if } p \equiv 1 \pmod{12}, \\ py^2 & \text{if } p \equiv 5 \pmod{12}, \\ p(y-3x)^2 & \text{if } p \equiv 7 \pmod{12}, \\ 4py^2 & \text{if } p \equiv 11 \pmod{12}. \end{cases}$$

Theorem 9. *Suppose n is a positive integer such that $5 \nmid n$. Then $b_{11}(n)$ is divisible by 5 if and only if there exists a prime p such that* (i) $\left(\frac{-33}{p}\right) = -1$ *and* $ord_p(12n+5) \equiv 1 \pmod 2$, *or* (ii) $\left(\frac{-33}{p}\right) = 1$ *and one of the following conditions holds.*

(1) $ord_p(12n+5) \equiv 1 \pmod 2$ *and* $\delta_p \equiv 0 \pmod 2$,

(2) $ord_p(12n+5) \equiv 2 \pmod 2$ *and* $\delta_p \equiv 1 \pmod 2$,

(3) $ord_p(12n+5) \equiv 4 \pmod 2$ *and* $\delta_p \equiv 4 \pmod 2$,

(4) $ord_p(12n+5) \equiv 5 \pmod 2$ *and* $\delta_p \equiv 2 \pmod 2$,

where $ord_p(n)$ denotes the highest power of p that divides n.

Dandurand and Penniston[17] studied the divisibility property for $b_5(n)$, $b_7(n)$, and $b_{11}(n)$. They proved the following results for $b_5(n)$. Consider the prime factorization of $6n+1$ by

$$6n+1=\prod_{i=1}^{s} p_i^{e_i},$$

where $p_1,\ldots,p_s$ are distinct primes and $e_i>0$ for all $1\le i\le s$. For each i with $p_i\equiv 1\pmod 3$, let us write $p_i=x_i^2+3y_i^2$ with $x_i,y_i\in\mathbb{Z}$.

Theorem 10. *Let n be a nonnegative integer. Then $b_5(n)$ is divisible by 5 if and only if for some $1\le i\le s$, one of the following holds*:

(1) $p_i=5$,

(2) $p_i\equiv 2\pmod 3$, $p_i\neq 5$ *and e_i is odd*,

(3) $p_i\equiv 1\pmod 3$, $5\mid x_i$ *and e_i is odd*,

(4) $p_i\equiv 1\pmod 3$, $5\mid y_i$ *and* $e_i\equiv 4\pmod 5$,

(5) $p_i\equiv 1\pmod 3$, $5\mid (x_i^2-y_i^2)$ *and* $e_i\equiv 2\pmod 3$,

(6) $p_i\equiv 1\pmod 3$, $5\nmid x_iy_i(x_i^2-y_i^2)$ *and* $e_i\equiv 5\pmod 6$.

Hirschhorn and Sellers[18] used Jacobi triple product identity and various 2-dissection formulas to prove the following results.

Theorem 11. *For all nonnegative integers n, $b_5(2n)$ is odd if and only if $12n+1$ is a perfect square.*

Theorem 12. *For all nonnegative integers n, $b_5(4n+1)$ is even unless $24n+7 = 2x^2+5y^2$ for some integers x and y.*

Theorem 13. *For all nonnegative integers n, we have*

$$b_5(20n+5)\equiv 0\pmod 2$$

and

$$b_5(20n+13)\equiv 0\pmod 2.$$

Theorem 14. *Suppose that p is any prime greater than 3 such that -10 is a quadratic nonresidue modulo p, u is the reciprocal of 24 modulo p^2, and $r\not\equiv 0\pmod p$. Then, for all m, we have*

$$b_5\left(4p^2m+4upr-28u+1\right)\equiv 0\pmod 2.$$

Lastly, they proved the following density result.

Theorem 15. *$b_5(n)$ is even for at least 75% of the positive integers n.*

Here, we like to mention that Theorems 11 and 13 were also proved by Calkin *et al.*[19] using a different method. In a very recent work, Dai[20] studied the parity results for 25-regular partitions.

Theorem 16. *For all nonnegative integers n, we have*

$$b_{25}(10n+9) \equiv 0 \pmod 2.$$

Theorem 17. *Let n be a nonnegative integer and $p > 5$ be a prime. Then for all $\alpha > 0$ and $1 \le r < 4p$ with $gcd(r,p) = 1$ and $rp \equiv 1 \pmod 4$, we have*

$$b_{25}(20p^{2\alpha}n + 5rp^{2\alpha-1} - 1) \equiv 0 \pmod 2.$$

Xia[21] derived the congruence properties for b_{13}, b_{17} and b_{19} modulo $13, 17$ and 19, respectively, by using some theta-function identities of Ramanujan.

Theorem 18. *For all nonnegative integers n and k, we have*

$$b_{13}\left(3^{6k}n + \frac{3^{6k}-1}{2}\right) \equiv a(6k)b_{13}(n) \pmod{13},$$

where $a(k) = \left(\frac{1}{2} - \frac{\sqrt{17}}{34}\right)\left(\frac{1}{2} + \frac{\sqrt{17}}{2}\right)^k + \left(\frac{1}{2} + \frac{\sqrt{17}}{34}\right)\left(\frac{1}{2} - \frac{\sqrt{17}}{2}\right)^k.$

Theorem 19. *For all nonnegative integers n and k, we have*

$$b_{17}\left(4^{9k}n + \frac{2(4^{9k}-1)}{3}\right) \equiv c(9k)b_{17}(n) \pmod{17},$$

where $c(k) = \left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right)\left(1 + \sqrt{5}\right)^k + \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right)\left(1 - \sqrt{5}\right)^k.$

Theorem 20. *For all nonnegative integers n and k, we have*

$$b_{19}\left(5^{10k}n + \frac{3(5^{10k}-1)}{4}\right) \equiv d(10k)b_{19}(n) \pmod{19},$$

where $d(k) = \left(\frac{1}{2} - \frac{\sqrt{69}}{46}\right)\left(\frac{3}{2} + \frac{\sqrt{69}}{2}\right)^k + \left(\frac{1}{2} + \frac{\sqrt{69}}{46}\right)\left(\frac{3}{2} - \frac{\sqrt{69}}{2}\right)^k.$

Later using Theorems 18–20, the author have the following congruences, respectively:

$$b_{13}\left(\frac{151\times 3^{6k}-1}{2}\right)\equiv 0 \pmod{13},$$

$$b_{17}\left(\frac{89\times 4^{9k}-1}{3}\right)\equiv 0 \pmod{17},$$

and

$$b_{19}\left(\frac{123\times 5^{10k}-1}{4}\right)\equiv 0 \pmod{19}.$$

Ramanujan's general theta-function $f(a,b)$ is defined as

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}.$$

Then, the Jacobi triple product identity [22, Entry 19, p. 36] takes the shape

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The most important two special cases of $f(a,b)$ are

$$\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty} q^{n(n+1)/2},$$

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}.$$

Recently, Cui and Gu[23] developed the following two p-dissection identities for $\psi(q)$ and $f(-q)$.

Theorem 21. *For any odd prime p,*

$$\psi(q)=\sum_{k=0}^{\frac{p-3}{2}} q^{\frac{k^2+k}{2}} f\left(\frac{p^2+(2k+1)p}{2},\frac{p^2-(2k+1)p}{2}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}).$$

Furthermore, $\frac{k^2+k}{2}\not\equiv\frac{p^2-1}{8} \pmod{p}$, *for* $0\le k\le\frac{p-3}{2}$.

Theorem 22. *For any prime* $p \geq 5$,

$$f(-q) = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)$$

$$+ (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^{p^2}),$$

where

$$\frac{\pm p-1}{6} := \begin{cases} \dfrac{p-1}{6} & \text{if } p \equiv 1 \pmod 6; \\ \dfrac{-p-1}{6} & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

Furthermore, for $\frac{-(p-1)}{2} \leq k \leq \frac{p-1}{2}$ *and* $k \neq \frac{\pm p-1}{6}$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

In a series of articles, Cui and Cu[23–25] studied the arithmetic properties of ℓ-regular partitions for several values of ℓ, using p-dissection formulas for Ramanujan's theta functions and several modular equations.

Theorem 23. *For any prime* $p \geq 5$, $\alpha \geq 0$, *and* $n \geq 0$, *we have*

$$b_2\left(p^{2\alpha+1}n + \frac{(24j+1)p^{2\alpha}-1}{24}\right) \equiv 0 \pmod 2,$$

where $0 \leq j \leq p-1$ *and* $\left(\frac{24j+1}{p}\right) = -1$.

Theorem 24. *For any odd prime* p, $\alpha \geq 0$, *and* $n \geq 0$, *we have*

$$b_4\left(p^{2\alpha+1}n + \frac{(8j+1)p^{2\alpha}-1}{8}\right) \equiv 0 \pmod 2,$$

where $0 \leq j \leq p-1$ *and* $\left(\frac{8j+1}{p}\right) = -1$.

Theorem 25. *For any prime* $p \geq 5$, $\left(\frac{-10}{p}\right) = -1$, $\alpha \geq 0$, *and* $n \geq 0$, *we have*

$$b_5\left(4p^{2\alpha}n + \frac{(24i+7p)p^{2\alpha-1}-1}{6}\right) \equiv 0 \pmod{2},$$

where $1 \leq i \leq p-1$.

Theorem 26. *For any prime* $p \not\equiv -1 \pmod{6}$, $\alpha \geq 0$, *and* $n \geq 0$, *we have*

$$b_9\left(2p^{2\alpha}n + \frac{(6i+p)p^{2\alpha-1}-1}{3}\right) \equiv 0 \pmod{3},$$

where $1 \leq i \leq p-1$.

Theorem 27. *For* $k = 0, 2, 3, 4$, $\alpha \geq 1$, *and* $n \geq 0$,

$$b_9\left(5^{2\alpha}n + \frac{(3k+2)5^{2\alpha-1}-1}{3}\right) \equiv 0 \pmod{3}.$$

Theorem 27 was conjectured by Keith[26] and was first proven by Xia and Yao.[27]

3. ℓ-Regular Overpartitions

Andrews[28] defined the *singular overpartition* function $\overline{C}_{k,i}(n)$, which counts the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. For $k \geq 3$ and $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$, the generating function for $\overline{C}_{k,i}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n)q^n = \frac{(q^k;q^k)_\infty(-q^i;q^k)_\infty(-q^{k-i};q^k)_\infty}{(q;q)_\infty}.$$

Note that the number of 3-regular overpartition of n is equal to the number of $(3,1)$-singular overparition of n, i.e. for $n \geq 0$,

$$\overline{A}_3(n) = \overline{C}_{3,1}(n).$$

For example,

$$\overline{A}_3(4) = \overline{C}_{3,1}(4) = 10.$$

In Ref. 28, Andrews established the following congruences.

Theorem 28. *For a nonnegative integer n, we have*

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}.$$

Theorem 29. *Let n be nonnegative integer and $r_2(n)$ be the number of representations of n as the sum of two squares. Then*

$$\overline{C}_{3,1}(9n) \equiv (-1)^n r_2(n) \pmod{3}.$$

Chen *et al.*[29] studied the parity of $\overline{C}_{3,1}(n)$ and showed that $\overline{C}_{3,1}(n)$ is always even. They also prove the following results.

Theorem 30. *Let N be a positive integer which is not expressible as a square or the sum of two positive squares. Then $\overline{C}_{3,1}(N) \equiv 0 \pmod{3}$.*

Theorem 31. *We have*

$$\sum_{n=0}^{\infty}(-1)^n\overline{C}_{3,1}(n)q^n \equiv 1 + 6\sum_{k\geq 1} q^{k^2} + 4\sum_{k\geq 1} q^{2k^2} + 2\sum_{k\geq 1} q^{3k^2} + 4\sum_{k,\ell\geq 1} q^{k^2+3\ell^2} \pmod{8}.$$

Ahmed and Baruah[30] found several congruences for $\overline{C}_{3,1}(n)$ modulo 4, 18 and 36. For example, they proved the following.

Theorem 32. *If $p \geq 5$ is a prime and $1 \leq j \leq p-1$, then for any nonnegative integers α and n, we have*

$$\overline{C}_{3,1}\left(24p^{2\alpha+1}(pn+j) + p^{2\alpha+2}\right) \equiv 0 \pmod{4}.$$

In Ref. 31, Naika and Gireesh proved some infinite families of congruences for $\overline{C}_{3,1}(n)$ modulo 12, 18, 48, and 72.

Theorem 33. *If $p \geq 5$ is a prime, $\left(\frac{-2}{p}\right) = -1$, $\alpha \geq 1$ and $n > 0$, we have*

$$\overline{C}_{3,1}\left(16p^{2\alpha-1}(pn+j) + 6p^{2\alpha}\right) \equiv 0 \pmod{48}.$$

They also conjectured the following.

Conjecture 2. *For each nonnegative integer n,*

$$\overline{C}_{3,1}(12n+11) \equiv 0 \pmod{144}.$$

The above conjecture was confirmed by Barman and the author.[32] Isnaini and Toh[33] extended some of the results mentioned above and proved many new congruences modulo $108, 192, 288$ and 432 for $\overline{C}_{3,1}(n)$.

In a recent work, Barman and Ray[34] studied the arithmetic density of $\overline{C}_{3,1}(n)$ and proved the following.

Theorem 34. *Let k be a fixed positive integer and $p \in \{2, 3\}$. Then $\overline{C}_{3,1}(n)$ is almost always divisible by p^k, namely,*

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{C}_{3,1}(n) \equiv 0 \pmod{p^k}\right\}}{X} = 1.$$

In the same article, the authors also showed that $\overline{C}_{3,1}(n)$ is almost always divisible by $2\cdot 3^k$ for any positive integer k. In Refs. 32 and 34, Barman and the author studied the divisibility properties for $\overline{A}_9(n)$, $\overline{A}_{2k}(n)$, and $\overline{A}_{4k}(n)$ for any positive integer k. Some of the sample results are the following.

Theorem 35. *If $p \geq 5$ is a prime such that $\left(\frac{-2}{p}\right) = -1$. Then for all positive integers n, k and α, we have*

$$\overline{A}_{4k}\left(4p^{2\alpha}n + (4j + 3p)p^{2\alpha-1}\right) \equiv 0 \pmod{16}.$$

Theorem 36. *For any positive integers k and n, we have*

$$\overline{A}_9(n) \equiv \overline{A}_9(3^k n) \pmod{3}.$$

Particularly, if n is odd, then

$$\overline{A}_9(n) \equiv \overline{A}_9(3^k n) \pmod{6}.$$

Theorem 37. *For $i = 1, 2, \ldots, 6$, we have*

$$\overline{A}_8(28n + 4i) \equiv 0 \pmod{7}.$$

In Ref. 35, Chakraborty and the author studied the arithmetic and density properties for $\overline{A}_{p_1^{a_1} p_2^{a_2} \ldots p_m^{a_m}}$, where $p_i \geq 5$'s are primes. More particularly, they proved the following results.

Theorem 38. *Let $\ell = p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}$ where p_i's are primes > 3. If $p_i^{2a_i} \geq \ell$, then for every positive integer j,*

$$\lim_{X\to\infty} \frac{\#\left\{0 < n \leq X : \overline{A}_\ell(n) \equiv 0 \pmod{p_i^j}\right\}}{X} = 1.$$

Theorem 39. *Let k be a positive integer and p be a prime number such that $p \equiv 3 \pmod 4$. Let r be a nonnegative integer such that p divides $4r+3$, then*

$$\overline{A}_5\left(4p^{k+1}n + p(4r+3)\right) \equiv f(p)\overline{A}_5\left(4p^{k-1}n + \frac{4r+3}{p}\right) \pmod 8,$$

where $f(p)$ is defined by

$$f(p) = \begin{cases} -1 & \text{if } p \equiv 3, 7 \pmod{20}, \\ 1 & \text{if } p \equiv 11, 19 \pmod{20}. \end{cases}$$

As a continuation of the article,[35] for any positive integer ℓ and k, the author[36] proved that the distribution for $\overline{A}_{2\ell}$ modulo 2^k is exactly one. In the same article, the author also studied the arithmetic properties of Fourier coefficients of certain integer weight modular form by using Hecke eigenforms and find infinitely many arithmetic progressions where congruences for $\overline{A}_{2\ell}(n)$ hold. One of the sample results is the following.

Theorem 40. *Let k, n be nonnegative integers. For each i with $1 \leq i \leq k+1$, if $p_i \geq 5$ is a prime such that $p_i \equiv 3 \pmod 4$, then for any positive integer ℓ and $j \not\equiv 0 \pmod{p_{k+1}}$, we have*

$$\overline{A}_{2\ell}\left(4p_1^2 \dots p_{k+1}^2 n + (4j + p_{k+1})p_1^2 \dots p_k^2 p_{k+1}\right) \equiv 0 \pmod 4.$$

Suppose $p \geq 5$ be a prime and $p_1 = p_2 = \cdots = p_{k+1} = p$. Then from the above theorem, we have

$$\overline{A}_{2\ell}\left(4p^{2(k+1)}n + 4p^{2k+1}j + p^{2(k+1)}\right) \equiv 0 \pmod 4,$$

where $j \not\equiv 0 \pmod p$. In particular, for all $n \geq 0$ and $j \not\equiv 0 \pmod{11}$, we have

$$\overline{A}_{2\ell}\left(484n + 44j + 121\right) \equiv 0 \pmod 4.$$

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Chapter 8

Some Applications of an Algorithm in Number Theory

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In this chapter, we mainly deal with applications of a known algorithm (called *Algorithm* $\mathcal{M}$) in number theory obtained by the elegant theory of Möbius transformations. An extended version of Fermat's well-known "*two-square theorem*" can be stated as follows: "A positive integer n is the sum of two squares if -1 is a quadratic residue modulo n. Conversely, if $n = x^2 + y^2$ with $(x, y) = 1$ then -1 is a quadratic residue modulo n." The *Algorithm* $\mathcal{M}$ is related to this extended version of Fermat's two-square theorem and computes the integers x and y satisfying the equation $n = x^2 + y^2$ for a given positive integer n such that -1 is a quadratic residue modulo n. First, we consider the positive integers n which can be written in the form $n = x^2 + Ny^2$ with $(x, y) = 1$ (N is a fixed positive integer and n is a given positive integer relatively prime to N). Using the group structure of the Modular group $\Gamma = PSL(2,\mathbb{Z})$, we prove that the *Algorithm* $\mathcal{M}$ can be used to compute the integers x and y satisfying the equation $n = x^2 + Ny^2$ with $(x, y) = 1$ for $N \in \{4, 9, 16, 25\}$. Finally, we consider an open question raised in a recent paper on the existence of a quick way to compute u and v in the form $4A_p = Du^2 + pv^2$ where $p \equiv 3 \pmod 4$ is a prime number and A_n is the well-known generalized Fibonacci sequence defined by

$A_0 = 0, A_1 = 1$ and $A_n = rA_{n-1} + sA_{n-2}$ $(n \geq 2)$ (r and s are fixed integers). We discuss the usage of the *Algorithm* $\mathcal{M}$ for the computation of u and v.

1. Introduction and Preliminaries

Let a, b, c, d be complex constants satisfying $ad - bc \neq 0$. A Möbius transformation (also called fractional linear transformation or bilinear transformation) is a rational function of the form

$$T(z) = \frac{az + b}{cz + d}. \tag{1}$$

The condition $ad - bc \neq 0$ is necessary to ensure a Möbius transformation is a one-to-one mapping since we have

$$T(z_1) - T(z_2) = (ad - bc)\,\frac{z_1 - z_2}{(cz_1 + d)\,(cz_2 + d)},$$

for any complex numbers z_1 and z_2. We may assume that $ad - bc = 1$ by multiplying the numerator and denominator of (1) by a suitable constant. Möbius transformations are directly conformal homeomorphisms of the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (see Refs. 1–5 for the basic knowledge about Möbius transformations).

The set of all Möbius transformations is a group under composition; the identity is the function $I(z) = z$ and the inverse of a Möbius transformation T is

$$T^{-1}(z) = \frac{-dz + b}{cz - a}.$$

The group of Möbius transformations is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$, the projective special linear group of order 2 over $\mathbb{C}$ (see Ref. 4). The Modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ consists of all Möbius transformations with integer coefficients

$$z \to \frac{az + b}{cz + d},\ a,\ b,\ c,\ d \in \mathbb{Z} \text{ and } ad - bc = 1.$$

All elements of Γ can also be considered as projective matrices $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d rational integers and $ad - bc = 1$. It is well known that Γ is generated by the transformations

$$R(z) = -\frac{1}{z} \text{ and } S(z) = -\frac{1}{z+1}$$

and can be presented as

$$\Gamma = \langle R, S; R^2 = S^3 = 1 \rangle. \tag{2}$$

Group theoretically, Γ has the structure of a free product of the cyclic group of order 2 generated by R and the cyclic group of order 3 generated by S, that is, we have $\Gamma \cong C_2 * C_3$ (see Ref. 6). Another set of generators for the Modular group Γ consists of the transformations

$$R(z) = -\frac{1}{z} \text{ and } T(z) = z + 1, \tag{3}$$

where $T(z) = (R \circ S)(z)$.

Besides the applications in mathematics, the theory of Möbius transformations (especially the theory of Modular group) has had wide applications in a range of scientific disciplines, such as physics, engineering, and music. For example, recently Möbius transformations have been used as activation functions in complex valued neural networks. There are many studies on complex valued neural networks using some theoretical properties of Möbius transformations such as fixed points, and cross-ratio (see Refs. 7–9 and the references therein). In Ref. 10, a transformational approach to musical intervals was presented using the theory of principal congruence subgroups of the Modular group Γ.

In this paper, we deal with applications of a known algorithm (called *Algorithm* $\mathcal{M}$) obtained by the group theoretic properties of the Modular group. This algorithm is related to the well-known Fermat's two-square theorem in number theory. Fermat's two-square theorem states that a prime p is expressible as the sum of two squares if and only if -1 is a quadratic residue modulo p (see, for instance, Refs. 11–20 and the references therein). There are many proofs of Fermat's two-square theorem. For example, in Ref. 21, Fine presented a new proof of this theorem using the group structure of the Modular group Γ. Fine proved the following theorem.

Theorem 1.[21] *A positive integer n is the sum of two squares if -1 is a quadratic residue mod n. Conversely, if $n = x^2 + y^2$ with $(x, y) = 1$, then -1 is a quadratic residue mod n.*

Fine's result extends the two-square theorem for an arbitrary positive integer n. Such proofs only deal with the existence of the representation of a prime p (or a positive integer n) without any algorithm to find constructions for the numbers x and y in terms of p (or n). On the other hand, there are many constructions in the literature (see, for example, Refs. 19 and 22).

In Ref. 22, given a positive integer n such that -1 is a quadratic residue modulo n, an algorithm, called *Algorithm* $\mathcal{M}$, that computes the integers x and y in Theorem 1 was presented. To do this, some facts about the group structure of the Modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ together with the generator set (3) were used. The proof of this algorithm is based on the standard algorithm to express any projective matrix $A \in \Gamma$ in terms of the generators R and T (see Ref. 6). The *Algorithm* $\mathcal{M}$ is briefly described as follows (see Ref. 22 for more details).

Algorithm $\mathcal{M}$:

1. Find two coprime integers u and q such that $u^2 = -1 + qn$.
2. Define the following two functions:

$$f : (a,b,c,d) \to (d,-c,-b,a),$$
$$g : (a,b,c,d) \to (a-c, 2a+b-c, c, c+d).$$

3. Start with the quadruple $(-u,n,-q,u)$ and apply f if the first coordinate is positive and apply g if not.
4. Proceed likewise until the quadruple $(0,1,-1,0)$ is obtained.
5. For f, write $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and for r_i times g, write $T^{r_i} = \begin{pmatrix} 1 & r_i \\ 0 & 1 \end{pmatrix}$. Then compute the matrix $B = T^{r_0}RT^{r_1}R\cdots RT^{r_m}$ where only r_0 and r_m may be zero.

If $B = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, then we have $n = x^2 + y^2$ with $(x,y) = 1$. This is an efficient algorithm for the computation of the integers x and y in Theorem 1 since it works easily even for large values of n due to matrix multiplication.

Let N be a fixed positive integer and x, y integers. There are several papers on the solution of the equation $n = x^2 + Ny^2$. Especially, primes written in this form have been studied extensively (see, for example, Refs. 16, 23–30 and the references therein). Recently, Fibonacci numbers written in this form have been investigated (see Refs. 31–34). For the case $N = 1$, besides the *Algorithm* $\mathcal{M}$, there are some other algorithms that compute the integers x and y satisfying the equation $n = x^2 + y^2$ for a given positive integer n such that -1 is a quadratic residue modulo n (see Refs. $16, 18, 19, 24, 35, 36$). For $N > 1$, besides the Smith–Cornacchia algorithm (see Ref. 37), there is another algorithm (called *Algorithm* $\mathcal{H}$) that computes the integers x and y satisfying the equation $n = x^2 + Ny^2$ (for more details, see Ref. 30). For $N = 2$ and $N = 3$, the *Algorithm* $\mathcal{H}$ can be used for all n which can be written as $n = x^2 + Ny^2$. For $N \geq 5$, the *Algorithm* $\mathcal{H}$ was obtained by some facts about the group structure of the Hecke groups $H(\sqrt{N})$ generated by two linear fractional transformations

$$R(z) = -\frac{1}{z} \text{ and } T_{\sqrt{N}}(z) = z + \sqrt{N}.$$

But in these last cases, the *Algorithm* $\mathcal{H}$ can be used as long as some conditions related to the transformations of the corresponding Hecke group $H(\sqrt{N})$ hold (for more details, see Ref. 30 and the references therein). Then a natural problem arises:

Can we use the Algorithm $\mathcal{M}$ *in the cases where the Algorithm* $\mathcal{H}$ *cannot be applied*?

Now, we recall the *Algorithm* $\mathcal{H}$ briefly.

Algorithm $\mathcal{H}$:

1. Find two coprime integers u and q such that $Nu^2 = -1 + qn$.
2. Define the following functions:

$$f : (a, b, c, d) \to (d, -c, -b, a),$$
$$g_{\sqrt{N}} : (a, b, c, d) \to (a - c, 2Na + b - Nc, c, c + d).$$

3. Start with the quadruple $(-u, n, -q, u)$ and apply f if the first coordinate is positive and apply $g_{\sqrt{N}}$ if not.
4. Proceed likewise until the quadruple $(0, 1, -1, 0)$ is obtained.
5. For f, write $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and for r_i times $g_{\sqrt{N}}$, write $T_{\sqrt{N}}^{r_i} = \begin{pmatrix} 1 & r_i\sqrt{N} \\ 0 & 1 \end{pmatrix}$. Then compute the matrix $B = T_{\sqrt{N}}^{r_0} R T_{\sqrt{N}}^{r_1} R \cdots R T_{\sqrt{N}}^{r_m}$ where only r_0 and r_m may be zero.

If $B = \begin{pmatrix} x & y\sqrt{N} \\ z\sqrt{N} & t \end{pmatrix}$, then we have $n = x^2 + Ny^2$ with $(x, y) = 1$; if $B = \begin{pmatrix} x\sqrt{N} & y \\ z & t\sqrt{N} \end{pmatrix}$, then we have $n = Nx^2 + y^2$ with $(x, y) = 1$. It is easy to compute the matrix B since we have

$$T^r_{\sqrt{N}} = \begin{pmatrix} 1 & r\sqrt{N} \\ 0 & 1 \end{pmatrix}, \quad T^r_{\sqrt{N}} R = \begin{pmatrix} -r\sqrt{N} & 1 \\ -1 & 0 \end{pmatrix}$$

$$\text{and } RT^r_{\sqrt{N}} = \begin{pmatrix} 0 & 1 \\ -1 & -r\sqrt{N} \end{pmatrix}$$

for any integer r. This is an efficient algorithm for the computation of the integers x and y for a positive integer n that can be written as $n = x^2 + Ny^2$. We note that Modular group Γ is one of the Hecke groups.

Let us consider the generalized Fibonacci sequence A_n defined by

$$A_0 = 0, A_1 = 1 \text{ and } A_n = rA_{n-1} + sA_{n-2} (n \geq 2),$$

where r and s are fixed integers. Let $p \equiv 3 \pmod 4$ be a prime number. In Ref. 38, it was proved that $4A_p$ is represented by the quadratic form

$$Du^2 + pv^2. \tag{4}$$

Here, $D = r^2 + 4s$ is the discriminant of the sequence. Then an open question was left on the existence of a quick way to compute u and v in representation (4) (see Remark on page 138 in Ref. 38).

In this paper, our aim is to explore some applications of the *Algorithm* $\mathcal{M}$. In Section 2, we prove that the *Algorithm* $\mathcal{M}$ can be used to find the integers x and y satisfying the equation $n = x^2 + Ny^2$ with $(x, y) = 1$ for $N \in \{9, 16, 25\}$. Our arguments work also in the case of $N = 4$. Although there are many simple proofs of the case $N = 4$ in the literature, we include this case in our main theorem. Some remarks and necessary examples are also given. We see that the *Algorithm* $\mathcal{M}$ is an efficient tool to provide a simple solution to the problem when a natural number n (relatively prime to N) can be represented in the form $n = x^2 + Ny^2$ with the advantage of calculating the integers x and y. In Section 3, we discuss the usage of the *Algorithm* $\mathcal{M}$ for the open question on the existence of a quick way to compute u and v in (4). We refer the reader to Refs. 13, 14 and 39 for the basic information on the quadratic reciprocity.

2. An Application of the Algorithm $\mathcal{M}$ to the Computation of the Integers x and y Satisfying the Equation $n = x^2 + Ny^2$

First, we begin with an example that shows us how the *Algorithm* $\mathcal{M}$ works.

Example 1. Let us consider the prime number $p = 1289$. Using the Euler criterion, it is easy to check that -1 is a quadratic residue mod p (see, for example, Refs. 13 and 19 for more details). By Fermat's two-square theorem (also Theorem 1), we know that the prime number $p = 1289$ can be written as the sum of two squares, that is, we can write $1289 = x^2 + y^2$ for some integers x and y. Now, we want to compute the integers x and y using the *Algorithm* $\mathcal{M}$. We can find the integers $810, 509$ such that $(810)^2 = -1 + 1289 \cdot 509$. Then, the matrix

$$A = \begin{pmatrix} -810 & 1289 \\ -509 & 810 \end{pmatrix}$$

is in the Modular group Γ and we can apply the *Algorithm* $\mathcal{M}$. We have

$$\begin{aligned}
&(-810, 1289, -509, 810) \underset{\rightarrow}{g} (-301, 178, -509, 301) \\
&\quad \underset{\rightarrow}{g} (208, 85, -509, -208) \\
&\quad \underset{\rightarrow}{f} (-208, 509, -85, 208) \underset{\rightarrow}{g} (-123, 178, -85, 123) \underset{\rightarrow}{g} (-38, 17, -85, 38) \\
&\quad \underset{\rightarrow}{g} (47, 26, -85, -47) \underset{\rightarrow}{f} (-47, 85, -26, 47) \underset{\rightarrow}{g} (-21, 17, -26, 21) \\
&\quad \underset{\rightarrow}{g} (5, 1, -26, -5) \underset{\rightarrow}{f} (-5, 26, -1, 5) \underset{\rightarrow}{g} (-4, 17, -1, 4) \underset{\rightarrow}{g} (-3, 10, -1, 3) \\
&\quad \underset{\rightarrow}{g} (-2, 5, -1, 2) \underset{\rightarrow}{g} (-1, 2, -1, 1) \underset{\rightarrow}{g} (0, 1, -1, 0).
\end{aligned}$$

Then, we obtain $B = T^2RT^3RT^2RT^5$. If we compute the matrix B, we get

$$\begin{aligned} B &= \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -8 & -35 \\ -5 & -22 \end{pmatrix}. \end{aligned}$$

Hence, we find $1289 = (8)^2 + (35)^2 = 64 + 1225$.

Let N be a fixed positive integer. Now, we investigate whether the *Algorithm* $\mathcal{M}$ can be used to calculate the integers x and y satisfying the equation $n = x^2 + Ny^2$ with $(x, y) = 1$ for a given natural number n relatively prime to N. We see that this is possible for the cases $N = 4, 9, 16$ or 25.

From now on, we will assume that $N = 4, 9, 16$ or 25, $\alpha = \sqrt{N}$, $n > 0$, $n \in \mathbb{N}$ and $(n, N) = 1$.

Theorem 2. *Let $N \in \{4, 9, 16, 25\}$. Suppose that n is a natural number relatively prime to N.*

(i) *Let $N = 4$. If -4 is a quadratic residue mod n, then the Algorithm $\mathcal{M}$ can be used to find an integer solution x, y of the equation $n = x^2 + 4y^2$ with $(x, y) = 1$.*
(ii) *Let $N \in \{9, 25\}$. If $-N$ is a quadratic residue mod n and n is a quadratic residue mod α, then the Algorithm $\mathcal{M}$ can be used to find an integer solution x, y of the equation $n = x^2 + Ny^2$ with $(x, y) = 1$.*
(iii) *Let $N = 16$. If $-N$ is a quadratic residue mod n, n is a quadratic residue mod α and $n \equiv 1, -7 \pmod{16}$, then the Algorithm $\mathcal{M}$ can be used to find an integer solution x, y of the equation $n = x^2 + Ny^2$ with $(x, y) = 1$.*

Proof. Assume that $-N$ is a quadratic residue mod n. We can find some integers k, l such that $kN - ln = 1$ since $(n, N) = 1$. This implies $kN \equiv 1 \bmod n$, and so $-k$ is a quadratic residue mod n. Then $u^2 \equiv -k \bmod n$ for some integer u. We get $u^2N \equiv -kN \bmod n$, $u^2N \equiv -1 \bmod n$, and so we have

$$u^2N = -1 + qn \tag{5}$$

for some integer q. Define the matrix

$$A = \begin{pmatrix} -u\alpha & n \\ -q & u\alpha \end{pmatrix}. \tag{6}$$

The determinant of the matrix A is $-u^2N + qn = 1$, where $\alpha = \sqrt{N}$. Clearly, $A \in \Gamma$ and A has order 2 since $trA = 0$. Now, each element of order 2 in Γ is conjugate to the generator R because of the group structure $\Gamma \cong C_2 * C_3$. That is, we have $A = BRB^{-1}$ for some $B \in \Gamma$. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $a, b, c, d \in \mathbb{Z}, ad - bc = 1$. Then we obtain

$$A = \begin{pmatrix} -(ac + bd) & a^2 + b^2 \\ -(c^2 + d^2) & (ac + bd) \end{pmatrix}. \tag{7}$$

Comparing the entries in (6) and (7), we find

$$n = a^2 + b^2 \tag{8}$$

for some integers a, b. By the determinant condition $ad - bc = 1$, we see that $(a, b) = 1$. Also we have

$$q = c^2 + d^2 \text{ and } u\alpha = ac + bd \tag{9}$$

with $(c, d) = 1$.

Now, if we show that $\alpha \mid a$ or $\alpha \mid b$, then the proof is completed. Indeed, rearranging (8), we have

$$n = (\alpha r)^2 + (b)^2 = Nr^2 + b^2$$

or

$$n = a^2 + (\alpha s)^2 = a^2 + Ns^2,$$

according to $a = \alpha r$ or $b = \alpha s$ $(r, s \in \mathbb{Z})$, respectively. It is evident that $(b, r) = 1$ or $(a, s) = 1$.

For $\alpha = 3$, 4, 5, we use the extra hypothesis that n is a quadratic residue mod α to show that $\alpha \mid a$ or $\alpha \mid b$.

Case 1. Let $N = 4$. By Eq. (5), we have $nq \equiv 1 \pmod 2$. So, we find $n \equiv 1 \pmod 2$ and $q \equiv 1 \pmod 2$. $n = a^2 + b^2 \equiv 1 \pmod 2$ implies $a \equiv 0 \pmod 2$ or $b \equiv 0 \pmod 2$. Similarly, $q = c^2 + d^2 \equiv 1 \pmod 2$ implies $c \equiv 0 \pmod 2$ or $d \equiv 0 \pmod 2$. By the determinant condition $ad - bc = 1$, we find $a \equiv 0 \pmod 2$ and $d \equiv 0 \pmod 2$ or $b \equiv 0 \pmod 2$ and $c \equiv 0 \pmod 2$. That is, $2 \mid a$ or $2 \mid b$.

Case 2. Let $N = 9$. Using the condition n is a quadratic residue modulo 3, we shall prove that $3 \mid a$ or $3 \mid b$. By Eq. (5), we have $nq \equiv 1 (\text{mod } 3)$ and $(3, q) = 1$. If n is a quadratic residue modulo 3, we obtain $n \equiv 1 (\text{mod } 3)$ by Euler's criterion. The conditions $n \equiv 1 (\text{mod } 3)$ and $nq \equiv 1 (\text{mod } 3)$ imply $q \equiv 1 \pmod 3$. Then $n = a^2 + b^2 \equiv 1 \pmod 3$ implies $a \equiv 0 \pmod 3$ or $b \equiv 0 \pmod 3$. Similarly, $q = c^2 + d^2 \equiv 1 \pmod 3$ implies $c \equiv 0 \pmod 3$ or $d \equiv 0 \pmod 3$. By the determinant condition $ad - bc = 1$, we find $a \equiv 0 \pmod 3$ and $d \equiv 0 \pmod 3$ or $b \equiv 0 \pmod 3$ and $c \equiv 0 \pmod 3$. This implies $3 \mid a$ or $3 \mid b$.

Case 3. Let $N = 16$. By Eq. (5), we have $nq \equiv 1 (\text{mod } 4)$ and $(4, q) = 1$. Using the condition n is a quadratic residue modulo 4, we shall prove that $4 \mid a$ or $4 \mid b$. We know that the only quadratic residues modulo 4 are 0 and 1. Hence, we have $n \equiv 1 \pmod 4$ and $q \equiv 1 \pmod 4$. Considering the condition $(n, N) = 1$, it is easy to check that n should be congruent to 1 or -7 modulo 16. Then, taking into account the determinant condition $ad - bc = 1$, it is easy to check that $n = a^2 + b^2 \equiv 1 \pmod 4$ implies $a \equiv 0 \pmod 4$ or $b \equiv 0 \pmod 4$. Similarly, $q = c^2 + d^2 \equiv 1 \pmod 4$ implies $c \equiv 0 \pmod 4$ or $d \equiv 0 \pmod 4$. The determinant condition $ad - bc = 1$ also implies that $a \equiv 0 \pmod 4$ and $d \equiv 0 \pmod 4$ or $b \equiv 0 \pmod 4$ and $c \equiv 0 \pmod 4$.

Case 4. Let $N = 25$. Using the condition n is a quadratic residue modulo 5, we prove that $5 \mid a$ or $5 \mid b$. By Eq. (5), we have $nq \equiv 1 \pmod 5$ and $(5, q) = 1$. If n is a quadratic residue modulo 5, we obtain $n^2 \equiv 1 \pmod 5$ by Euler's criterion. $nq \equiv 1 (\text{mod } 5)$ implies $(nq)^2 \equiv 1 \pmod 5$ and $q^2 \equiv 1 \pmod 5$. Then the possible cases are $n \equiv 1 (\text{mod } 5)$, $q \equiv 1 (\text{mod } 5)$ or $n \equiv 4 (\text{mod } 5)$, $q \equiv 4 \pmod 5$.

$n = a^2 + b^2 \equiv 1 \pmod 5$ implies $a \equiv 0 \pmod 5$ or $b \equiv 0 \pmod 5$ and $q = c^2 + d^2 \equiv 1 \pmod 5$ implies $c \equiv 0 \pmod 5$ or $d \equiv 0 \pmod 5$. By the determinant condition $ad - bc = 1$, we find $a \equiv 0 \pmod 5$ and $d \equiv 0 \pmod 5$ or $b \equiv 0 \pmod 5$ and $c \equiv 0 \pmod 5$. Similarly, $n = a^2 + b^2 \equiv 4 \pmod 5$ implies $a \equiv 0 \pmod 5$ or $b \equiv 0 \pmod 5$ and $q = c^2 + d^2 \equiv 4 \pmod 5$ implies $c \equiv 0 \pmod 5$ or $d \equiv 0 \pmod 5$. Again, we find $a \equiv 0 \pmod 5$ and $d \equiv 0 \pmod 5$ or $b \equiv 0 \pmod 5$ and $c \equiv 0 \pmod 5$ by the determinant condition.

Consequently, the *Algorithm* $\mathcal{M}$ can be used to compute the integers x and y satisfying the equation $n = x^2 + Ny^2$ with $x, y \in Z$, $(x, y) = 1$ for $N = 4, 9, 16, 25$ under the hypothesis of the theorem.

□

Corollary 1. *Let N be as defined in Theorem 2 and $n = p$ be a prime number which does not divide N. Then, the Algorithm $\mathcal{M}$ can be used to find an integer solution (x, y) of the equation $p = x^2 + Ny^2$ with $(x, y) = 1$ if p satisfies the following condition:*

N	p
$N = 4$	$p \equiv 1\ (mod\ 4)$
$N = 9$	$p \equiv 1\ (mod\ 12)$
$N = 16$	$p \equiv -7, 1\ (mod\ 16)$
$N = 25$	$p \equiv -11, 1\ (mod\ 20)$

Note that the *Algorithm* $\mathcal{M}$ enables us to find the matrix B in the proof of Theorem 2. Now, we adapt this algorithm to our cases.

Application of the Algorithm: Let $N \in \{4, 9, 16, 25\}$ be fixed. Suppose that all of the assumptions of Theorem 2 hold for N and a given positive integer n. Let u and $q(> N)$ be the integers satisfying the equation $Nu^2 = -1 + qn$. If we start with the quadruple $(-u\alpha, n, -q, u\alpha)$ and apply the *Algorithm* $\mathcal{M}$, we obtain the matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we find $n = a^2 + b^2$ with $a, b \in \mathbb{Z}$, $(a, b) = 1$, $\alpha \mid a$ or $\alpha \mid b$ for $\alpha = 2, 3, 4, 5$. By rearranging the equality $n = a^2 + b^2$, we obtain the integers x and y satisfying the equation $n = x^2 + Ny^2$.

Remark 1. (1) For any integer r, as we have

$$T^r = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, T^r R = \begin{pmatrix} -r & 1 \\ -1 & 0 \end{pmatrix} \text{ and } RT^r = \begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix},$$

the matrix B can be computed easily using these equalities.

(2) In Example 1, we have easily found $1289 = (8)^2 + (35)^2$ via the *Algorithm* $\mathcal{M}$. Rearranging this equality, we get $1289 = (8)^2 + 25(7)^2$. That is, this last equality is also the solution of the problem that the calculation of the integers x, y satisfying the equation $n = x^2 + 25y^2$ with $(x, y) = 1$ for the given prime number 1289. Note that -25 is

a quadratic residue mod 1289 and 1289 is a quadratic residue mod $\alpha = 5$, then Theorem 2 guarantees that the Algorithm $\mathcal{M}$ can be used to calculate the integers x, y in the equation $n = x^2 + 25y^2$. Note that the equality $1289 = (8)^2 + (35)^2$ can also be rearranged as $1289 = 4(4)^2 + (35)^2$.

(3) Let us consider the matrix A defined in (6). The *Algorithm* $\mathcal{H}$ can also be used to compute the integers x and y satisfying the equation $n = x^2 + Ny^2$ with $(x, y) = 1$ for a given positive integer n. But, in the cases $N \geq 5$, the *Algorithm* $\mathcal{H}$ can be used as long as the following conditions hold:

(i) $-N$ is a quadratic residue mod n,
(ii) n is a quadratic residue mod N,
(iii) $A \in H(\sqrt{N})$ hold (see Ref. 30 for more details). From Ref. 40 we know that $A \in H(\sqrt{N})$ if and only if $\frac{u\alpha}{q}$ is a finite $\sqrt{N}$-fraction. Since we are unable to give the explicit conditions which determine whether $A \in H(\sqrt{N})$ or not, it is not practical to use the *Algorithm* $\mathcal{H}$ for $N \geq 5$. However, in the cases where the *Algorithm* $\mathcal{H}$ can be applied, we see that the *Algorithm* $\mathcal{H}$ is more efficient than the *Algorithm* $\mathcal{M}$ (see Example 3).

In the following example, we see an example of a case that we cannot use the *Algorithm* $\mathcal{H}$ but can apply the *Algorithm* $\mathcal{M}$.

Example 2. Let $N = 9$ and consider the prime number $p = 17257 \equiv 1 \pmod{12}$. We find the integers $342, 61$ such that $9(342)^2 = -1 + 17257 \cdot 61$. Using the *Algorithm* $\mathcal{M}$, we have the following:

$$\begin{aligned}
&(-1026, 17257, -61, 1026) \underset{\rightarrow}{g} (-965, 15266, -61, 965)\\
&\underset{\rightarrow}{g} (-904, 13397, -61, 904) \underset{\rightarrow}{g} (-843, 11650, -61, 843)\\
&\underset{\rightarrow}{g} (-782, 10025, -61, 782) \underset{\rightarrow}{g} (-721, 8522, -61, 721)\\
&\underset{\rightarrow}{g} (-660, 7141, -61, 660) \underset{\rightarrow}{g} (-599, 5882, -61, 599)\\
&\underset{\rightarrow}{g} (-538, 4745, -61, 538) \underset{\rightarrow}{g} (-477, 3730, -61, 477)
\end{aligned}$$

$$\underset{\to}{g}(-416, 2837, -61, 416) \underset{\to}{g}(-355, 2066, -61, 355)$$
$$\underset{\to}{g}(-294, 1417, -61, 294) \underset{\to}{g}(-233, 890, -61, 233)$$
$$\underset{\to}{g}(-172, 485, -61, 172) \underset{\to}{g}(-111, 202, -61, 111)$$
$$\underset{\to}{g}(-50, 41, -61, 50) \underset{\to}{g}(11, 2, -61, -11)$$
$$\underset{\to}{f}(-11, 61, -2, 11) \underset{\to}{g}(-9, 41, -2, 9) \underset{\to}{g}(-7, 25, -2, 7)$$
$$\underset{\to}{g}(-5, 13, -2, 5) \underset{\to}{g}(-3, 5, -2, 3) \underset{\to}{g}(-1, 1, -2, 1)$$
$$\underset{\to}{g}(1, 1, -2, -1) \underset{\to}{f}(-1, 2, -1, 1) \underset{\to}{g}(0, 1, -1, 0).$$

Then $B = T^{17}RT^6RT$. If we compute the matrix B, we obtain

$$\begin{aligned} B &= \begin{pmatrix} -17 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -6 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 101 & 84 \\ 6 & 5 \end{pmatrix}. \end{aligned}$$

Then we find $17257 = (101)^2 + (84)^2 = (101)^2 + 9(28)^2$.

Note that the matrix $A = \begin{pmatrix} -1026 & 17257 \\ -61 & 1026 \end{pmatrix} = \begin{pmatrix} -342\sqrt{9} & 17257 \\ -61 & 342\sqrt{9} \end{pmatrix}$ is in Γ, but we have $A \notin H(\sqrt{9})$ and so we cannot use the *Algorithm* $\mathcal{H}$. Indeed, using the nearest integer algorithm, we find

$$\frac{342\sqrt{9}}{61} = 6\sqrt{9} - \frac{1}{\frac{61\sqrt{9}}{216}}.$$

Hence, $\frac{1026}{61} = \frac{342\sqrt{9}}{61}$ cannot be expanded into a finite $\sqrt{9}$-fraction.

Example 3. Let $N = 4$ and $p = 1109 \equiv 1 \pmod 4$. We can find the integers $177, 113$ such that $4(177)^2 = -1 + 113 \cdot 1109$. Then, the matrix

$$A = \begin{pmatrix} -354 & 1109 \\ -113 & 354 \end{pmatrix} = \begin{pmatrix} -177\sqrt{4} & 1109 \\ -113 & 177\sqrt{4} \end{pmatrix}$$

is in the Hecke group $H(\sqrt{4})$. Hence, by Theorem 2, we can use both the *Algorithm* $\mathcal{M}$ and the *Algorithm* $\mathcal{H}$ to calculate the integers x, y satisfying the equation $n = x^2 + 4y^2$. First, we use the *Algorithm* $\mathcal{M}$:

$$\begin{aligned}
&(-354, 1109, -113, 354) \underset{\rightarrow}{g} (-241, 514, -113, 241) \\
&\quad \underset{\rightarrow}{g} (-128, 145, -113, 128) \underset{\rightarrow}{g} (-15, 2, -113, 15) \\
&\quad \underset{\rightarrow}{g} (98, 85, -113, -98) \underset{\rightarrow}{f} (-98, 113, -85, 98) \\
&\quad \underset{\rightarrow}{g} (-13, 2, -85, 13) \underset{\rightarrow}{g} (72, 61, -85, -72) \underset{\rightarrow}{f} (-72, 85, -61, 72) \\
&\quad \underset{\rightarrow}{g} (-11, 2, -61, 11) \underset{\rightarrow}{g} (50, 41, -61, -50) \underset{\rightarrow}{f} (-50, 61, -41, 50) \\
&\quad \underset{\rightarrow}{g} (-9, 2, -41, 9) \underset{\rightarrow}{g} (32, 25, -41, -32) \underset{\rightarrow}{f} (-32, 41, -25, 32) \\
&\quad \underset{\rightarrow}{g} (-7, 2, -25, 7) \underset{\rightarrow}{g} (18, 13, -25, -18) \underset{\rightarrow}{f} (-18, 25, -13, 18) \\
&\quad \underset{\rightarrow}{g} (-5, 2, -13, 5) \underset{\rightarrow}{g} (8, 5, -13, -8) \underset{\rightarrow}{f} (-8, 13, -5, 8) \\
&\quad \underset{\rightarrow}{g} (-3, 2, -5, 3) \underset{\rightarrow}{g} (2, 1, -5, -2) \underset{\rightarrow}{f} (-2, 5, -1, 2) \\
&\quad \underset{\rightarrow}{g} (-1, 2, -1, 1) \underset{\rightarrow}{g} (0, 1, -1, 0).
\end{aligned}$$

Then $B = T^4 R (T^2 R)^6 T^2$. If we compute the matrix B, we obtain

$$\begin{aligned}
B &= \begin{pmatrix} -4 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}^6 \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -22 & -25 \\ -7 & -8 \end{pmatrix}.
\end{aligned}$$

So, we find $1109 = (22)^2 + (25)^2 = 4(11)^2 + (25)^2$, that is, we have $x = 25$ and $y = 11$.

Now, we use the *Algorithm* $\mathcal{H}$ to make a comparison. Using the *Algorithm* $\mathcal{H}$, we obtain

$$\begin{aligned}
&(-177, 1109, -113, 177) \underset{\longrightarrow}{g_{\sqrt{4}}} (-64, 145, -113, 64) \\
&\quad \underset{\longrightarrow}{g_{\sqrt{4}}} (49, 85, -113, -49) \underset{\rightarrow}{f} (-49, 113, -85, 49) \\
&\quad \underset{\longrightarrow}{g_{\sqrt{4}}} (36, 61, -85, -36) \underset{\rightarrow}{f} (-36, 85, -61, 36)
\end{aligned}$$

$$\xrightarrow{g_{\sqrt{4}}}(25,41,-61,-25)\xrightarrow{f}(-25,61,-41,25)\xrightarrow{g_{\sqrt{4}}}(16,25,-41,-16)$$
$$\xrightarrow{f}(-16,41,-25,16)\xrightarrow{g_{\sqrt{4}}}(9,13,-25,-9)\xrightarrow{f}(-9,25,-13,9)$$
$$\xrightarrow{g_{\sqrt{4}}}(4,5,-13,-4)\xrightarrow{f}(-4,13,-5,4)\xrightarrow{g_{\sqrt{4}}}(1,1,-5,-1)$$
$$\xrightarrow{f}(-1,5,-1,1)\xrightarrow{g_{\sqrt{4}}}(0,1,-1,0).$$

Then, we find

$$B = T_{\sqrt{4}}^2 R(T_{\sqrt{4}}R)^6 T_{\sqrt{4}} = \begin{pmatrix} -2\sqrt{4} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\sqrt{4} & 1 \\ -1 & 0 \end{pmatrix}^6 \begin{pmatrix} 1 & \sqrt{4} \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -11\sqrt{4} & -25 \\ -7 & -4\sqrt{4} \end{pmatrix},$$

and hence, $1109 = 4(11)^2 + (25)^2$.

Remark 2. We note that our arguments do not work for the general cases $N = \alpha^2$, $\alpha \geq 6$, because of the reason that the conditions $-N$ is a quadratic residue mod n and n is a quadratic residue mod α are not sufficient to show that $\alpha \mid a$ or $\alpha \mid b$. For example, let $N = 36$ and $n = 1237$. -36 is a quadratic residue mod 1237 and 1237 is a quadratic residue mod 6. We find the integers $91, 241$ such that $36(91)^2 = -1 + 1237 \cdot 241$. Using the *Algorithm* $\mathcal{M}$, we have

$$(-546,1237,-241,546)\xrightarrow{g}(-305,386,-241,305)$$
$$\xrightarrow{g}(-64,17,-241,64)\xrightarrow{g}(177,130,-241,-177)$$
$$\xrightarrow{f}(-177,241,-130,177)\xrightarrow{g}(-47,17,-130,47)$$
$$\xrightarrow{g}(83,53,-130,-83)\xrightarrow{f}(-83,130,-53,83)\xrightarrow{g}(-30,17,-53,30)$$
$$\xrightarrow{g}(23,10,-53,-23)\xrightarrow{f}(-23,53,-10,23)\xrightarrow{g}(-13,17,-10,13)$$
$$\xrightarrow{g}(-3,1,-10,3)\xrightarrow{g}(7,5,-10,-7)\xrightarrow{f}(-7,10,-5,7)\xrightarrow{g}(-2,1,-5,2)$$
$$\xrightarrow{g}(3,2,-5,-3)\xrightarrow{f}(-3,5,-2,3)\xrightarrow{g}(-1,1,-2,1)\xrightarrow{g}(1,1,-2,-1)$$
$$\xrightarrow{f}(-1,2,-1,1)\xrightarrow{g}(0,1,-1,0).$$

Then, we get $B = T^3R(T^2R)^2T^3R(T^2R)^2T$. If we compute the matrix B, we obtain

$$B = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}^2 \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 34 & 9 \\ 15 & 4 \end{pmatrix}.$$

Consequently, we find $1237 = (34)^2 + (9)^2$. But, neither 34 nor 9 is divisible by 6. Hence, 1237 cannot be written in the form $x^2 + 36y^2$. If we find the sufficient conditions, our arguments used in the proof of Theorem 2 are valid, and hence, we can use the *Algorithm* $\mathcal{M}$ for all $N = \alpha^2$, $\alpha \geq 6$.

Now, before testing the efficiency of the *Algorithm* $\mathcal{M}$ for some large values of n, we give a theorem for the reduction of the steps of the *Algorithm* $\mathcal{M}$.

Theorem 3. *Suppose all of the assumptions of Theorem* 2 *hold. Consider the integers* $u\alpha$ *and* q *obtained from* (5)*. If we have* $u\alpha = mq - \delta$ *where* $0 \leq \delta < q$*, then applying the Algorithm* $\mathcal{M}$ *to the matrix* $T^{-m}AT^m$ *reduces the steps of the Algorithm* $\mathcal{M}$ *by* m *times.*

Proof. The proof is based on the fact that the matrix A, defined in (6), must be conjugate to R in Γ. The purpose of the *Algorithm* $\mathcal{M}$ is to find the matrix B such that $A = BRB^{-1}$, and so, $B^{-1}AB = R$. It is well known that every element of Γ can be expressed as a word in R and T. So, we can write $B = T^{r_0}RT^{r_1}R \cdots RT^{r_n}$ where the r_i, $(0 < i < n)$ are integers and only r_0 and r_n may be zero. Then we have

$$R = B^{-1}AB = (T^{-r_n}R \cdots RT^{-r_1}RT^{-r_0})A(T^{r_0}RT^{r_1}R \cdots RT^{r_n})$$
$$= T^{-r_n}R \cdots RT^{-r_1}R(T^{-r_0}AT^{r_0})RT^{r_1}R \cdots RT^{r_n}.$$

Since f represents the coefficients of the matrix RXR and g represents the coefficients of the matrix $T^{-1}XT$ for any matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, then the proof follows easily. □

Example 4. Let $N = 16$ and consider the prime number $p = 426161 \equiv 1 \pmod{16}$. We find the integers $2996, 337$ such that

$16(2996)^2 = -1 + 426161 \cdot 337$. Since we have $11984 = 36 \cdot 337 - 148$, we shall apply the *Algorithm* $\mathcal{M}$ to the matrix

$$T^{-36}AT^{36} = \begin{pmatrix} 1 & -36 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -11984 & 426161 \\ -337 & 11984 \end{pmatrix} \begin{pmatrix} 1 & 36 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 148 & 65 \\ -337 & -148 \end{pmatrix}.$$

We obtain

$$(148, 65, -337, -148) \underset{\to}{f} (-148, 337, -65, 148) \underset{\to}{g} (-83, 106, -65, 83)$$
$$\underset{\to}{g} (-18, 5, -65, 18) \underset{\to}{g} (47, 34, -65, -47) \underset{\to}{f} (-47, 65, -34, 47)$$
$$\underset{\to}{g} (-13, 5, -34, 13) \underset{\to}{g} (21, 13, -34, -21) \underset{\to}{f} (-21, 34, -13, 21)$$
$$\underset{\to}{g} (-8, 5, -13, 8) \underset{\to}{g} (5, 2, -13, -5) \underset{\to}{f} (-5, 13, -2, 5) \underset{\to}{g} (-3, 5, -2, 3)$$
$$\underset{\to}{g} (-1, 1, -2, 1) \underset{\to}{g} (1, 1, -2, -1) \underset{\to}{f} (-1, 2, -1, 1) \underset{\to}{g} (0, 1, -1, 0).$$

Then, we get $B = T^{36}RT^3R(T^2R)^2T^3RT$. If we compute the matrix B, we obtain

$$B = \begin{pmatrix} -36 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}^2 \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -569 & -320 \\ -16 & -9 \end{pmatrix}.$$

Consequently, we find $426161 = (569)^2 + (320)^2 = (569)^2 + 16(80)^2$, and so, we have $x = 569$ and $y = 80$.

Example 5. Let $N = 25$. We consider the prime number $p = 813089 \equiv -11 \pmod{100}$. We shall use the reduced version of the *Algorithm* $\mathcal{M}$ to find the numbers x, y satisfying the equation $n = x^2 + 25y^2$. We find the integers $4802, 709$ such that $25(4802)^2 = -1 + 813089 \cdot 709$. Since we have $24010 = 34 \cdot 709 - 96$, we shall apply the *Algorithm* $\mathcal{M}$ to the matrix

$$T^{-34}AT^{34} = \begin{pmatrix} 1 & -34 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -24010 & 813089 \\ -709 & 24010 \end{pmatrix} \begin{pmatrix} 1 & 34 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 96 & 13 \\ -709 & -96 \end{pmatrix}.$$

We find

$$(96,13,-709,-96) \underset{\rightarrow}{f} (-96,709,-13,96) \underset{\rightarrow}{g} (-83,530,-13,83)$$
$$\underset{\rightarrow}{g} (-70,377,-13,70) \underset{\rightarrow}{g} (-57,250,-13,57) \underset{\rightarrow}{g} (-44,149,-13,44)$$
$$\underset{\rightarrow}{g} (-31,74,-13,31) \underset{\rightarrow}{g} (-18,25,-13,18) \underset{\rightarrow}{g} (-5,2,-13,5)$$
$$\underset{\rightarrow}{g} (8,5,-13,-8) \underset{\rightarrow}{f} (-8,13,-5,8) \underset{\rightarrow}{g} (-3,2,-5,3)$$
$$\underset{\rightarrow}{g} (2,1,-5,-2) \underset{\rightarrow}{f} (-2,5,-1,2) \underset{\rightarrow}{g} (-1,2,-1,1) \underset{\rightarrow}{g} (0,1,-1,0).$$

Then, we get $B = T^{34}RT^{8}RT^{2}RT^{2}$. If we compute the matrix B, we obtain

$$B = \begin{pmatrix} -34 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -8 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -508 & -745 \\ -15 & -22 \end{pmatrix}.$$

Consequently, we find $813089 = (508)^2 + (745)^2 = (508)^2 + 25(149)^2$, and so, we have $x = 508$ and $y = 149$.

3. Application of the Algorithm $\mathcal{M}$ to an Open Question

Let $p \equiv 3 \pmod 4$ be a prime number. In this section, we consider the open question on the computation of the numbers u and v in the representation $4A_p = Du^2 + pv^2$. This is a hard problem for general values of r and s. It seems that it is difficult to find a fast algorithm for solving it.

The general expression of the first 15 generalized Fibonacci numbers is presented in Table 1.

Table 1. The list of the first 15 generalized Fibonacci numbers.

n	A_n
0	0
1	1
2	r
3	$r^2 + s$
4	$r^3 + 2rs$
5	$r^4 + 3r^2s + s^2$
6	$r^5 + 4r^3s + 3rs^2$
7	$r^6 + 5r^4s + 6r^2s^2 + s^3$
8	$r^7 + 6r^5s + 10r^3s^2 + 4rs^3$
9	$r^8 + 7r^6s + 15r^4s^2 + 10r^2s^3 + s^4$
10	$r^9 + 8r^7s + 21r^5s^2 + 20r^3s^3 + 5rs^4$
11	$r^{10} + 9r^8s + 28r^6s^2 + 35r^4s^3 + 15r^2s^4 + s^5$
12	$r^{11} + 10r^9s + 36r^7s^2 + 56r^5s^3 + 35r^3s^4 + 6rs^5$
13	$r^{12} + 11r^{10}s + 45r^8s^2 + 84r^6s^3 + 70r^4s^4 + 21r^2s^5 + s^6$
14	$r^{13} + 12r^{11}s + 55r^9s^2 + 120r^7s^3 + 126r^5s^4 + 56r^3s^5 + 7rs^6$
15	$r^{14} + 13r^{12}s + 66r^{10}s^2 + 165r^8s^3 + 210r^6s^4 + 126r^4s^5 + 28r^2s^6 + s^7$

The following formula is well known:

$$A_{2n+1} = A_{n+1}^2 + sA_n^2. \tag{10}$$

For the basic facts about the generalized Fibonacci sequence A_n, one can consult Refs. 41 and 42. Replacing n with $2m+1$, we get

$$A_{4m+3} = A_{2m+2}^2 + sA_{2m+1}^2. \tag{11}$$

For $p = 3$, from Ref. 38, we know that

$$4A_3 = D(1)^2 + 3(r)^2, \tag{12}$$

that is, we have $u = 1$ and $v = r$. Since $A_0 = 0$ and $A_1 = 1$, this can also be obtained from (11) as follows:

$$\begin{aligned} 4A_3 &= 4A_2^2 + 4sA_1^2 = 4\left[rA_1 + sA_0\right]^2 + 4sA_1^2 \\ &= \left(r^2 + 4s\right)A_1^2 + 3r^2A_1^2. \end{aligned}$$

For $p = 7$, by (11), observe that we have

$$A_7 = A_4^2 + sA_3^2, \tag{13}$$

and so,

$$4A_7 = 4A_4^2 + 4sA_3^2.$$

It is difficult to rearrange this last equation using the properties of generalized Fibonacci numbers for general r and s.

Now, we discuss the possible applications of the *Algorithm* $\mathcal{M}$ to find the integers u and v in the representation $4A_p = Du^2 + pv^2$ for a given prime number $p \equiv 3 \pmod 4$. If -1 is a quadratic residue modulo $2A_p$, then we can find the integers a, b with $a^2 = -1 + 2bA_p$. Then the matrix

$$A = \begin{pmatrix} -a & 2A_p \\ -b & a \end{pmatrix}$$

is in Γ and has order 2. Then, we can apply the *Algorithm* $\mathcal{M}$ to find the integers α and β such that $2A_p = \alpha^2 + \beta^2$ (see Ref. 22 for more details). Rearranging this last equation, we get the integers u and v such that $4A_p = Du^2 + pv^2$.

Now, we give some illustrative examples.

Example 6. Let $p = 7$, $r = 2$ and $s = 1$. Then $D = 8$. By (11), we have $4A_7 = 4A_4^2 + 4A_3^2$. Observe that 7 (resp. 8) does not divide A_4 or A_3. For the integer $2A_7 = 338$, we obtain the equation $29.338 - 99^2 = 1$. Then, the matrix

$$A = \begin{pmatrix} -99 & 338 \\ -29 & 99 \end{pmatrix}$$

is in Γ and has order 2. Applying the *Algorithm* $\mathcal{M}$ to the matrix

$$\begin{aligned} T^{-4}AT^4 &= \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -99 & 338 \\ -29 & 99 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 17 & 10 \\ -29 & -17 \end{pmatrix}, \end{aligned}$$

we obtain

$$\begin{aligned}&(17,10,-29,-17)\ \underset{\rightarrow}{f}\,(-17,29,-10,17)\ \underset{\rightarrow}{g}\ (-7,5,-10,7)\\ &\underset{\rightarrow}{g}\ (3,1,-10,-3)\ \underset{\rightarrow}{f}\ (-3,10,-1,3)\ \underset{\rightarrow}{g}\ (-2,5,-1,2)\\ &\underset{\rightarrow}{g}\ (-1,2,-1,1)\ \underset{\rightarrow}{g}\ (0,1,-1,0).\end{aligned}$$

We find

$$B = T^4RT^2RT^3 = \begin{pmatrix}-4 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-2 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix} = \begin{pmatrix}7 & 17\\ 2 & 5\end{pmatrix},$$

and so, $2A_7 = (7)^2 + (17)^2$. We have

$$4A_7 = 2(7)^2 + 2(17)^2.$$

Rearranging this equation, we get

$$\begin{aligned}4A_7 &= 7.14 + 578 = 7(2)^2 + 70 + 578\\ &= 7(2)^2 + 648 = 7(2)^2 + 8(9)^2,\end{aligned}$$

that is, $u = 9$ and $v = 2$.

Example 7. Let $p = 11$, $r = 2$ and $s = 1$. We have $2A_{11} = 11482$ and $985.11482 - 3363^2 = 1$. The matrix

$$A = \begin{pmatrix}-3363 & 11482\\ -985 & 3363\end{pmatrix}$$

is in Γ and has order 2. Applying the *Algorithm* $\mathcal{M}$ to the matrix

$$\begin{aligned}T^{-4}AT^4 &= \begin{pmatrix}1 & -4\\ 0 & 1\end{pmatrix}\begin{pmatrix}-3363 & 11482\\ -985 & 3363\end{pmatrix}\begin{pmatrix}1 & 4\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}577 & 338\\ -985 & -577\end{pmatrix},\end{aligned}$$

we obtain

$$
\begin{aligned}
&(577, 338, -985, -577) \underset{\rightarrow}{f} (-577, 985, -338, 577) \\
&\quad \underset{\rightarrow}{g} (-239, 169, -338, 239) \underset{\rightarrow}{g} (99, 29, -338, -99) \\
&\quad \underset{\rightarrow}{f} (-99, 338, -29, 99) \underset{\rightarrow}{g} (-70, 169, -29, 70) \\
&\quad \underset{\rightarrow}{g} (-41, 58, -29, 41) \underset{\rightarrow}{g} (-12, 5, -29, 12) \underset{\rightarrow}{g} (17, 10, -29, -17) \\
&\quad \underset{\rightarrow}{f} (-17, 29, -10, 17) \underset{\rightarrow}{g} (-7, 5, -10, 7) \underset{\rightarrow}{g} (3, 1, -10, -3) \\
&\quad \underset{\rightarrow}{f} (-3, 10, -1, 3) \underset{\rightarrow}{g} (-2, 5, -1, 2) \underset{\rightarrow}{g} (-1, 2, -1, 1) \underset{\rightarrow}{g} (0, 1, -1, 0).
\end{aligned}
$$

Then, we get $B = \left(T^4RT^2R\right)^2 T^3$. If we compute the matrix B, we obtain

$$
\begin{aligned}
B &= \left(\begin{pmatrix} -4 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \right)^2 \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 41 & 99 \\ 12 & 29 \end{pmatrix}.
\end{aligned}
$$

We find $2A_{11} = (41)^2 + (99)^2$, and so, we have

$$
\begin{aligned}
4A_{11} &= 2(99)^2 + 2(41)^2 = 2(11)^2(9)^2 + 3362 \\
&= 11(1782) + 3362 = 11(14)^2 + 8(51)^2.
\end{aligned}
$$

Consequently, we obtain $u = 51$ and $v = 14$.

4. Conclusion and Future Works

In Theorem 2, we have proved that the *Algorithm* $\mathcal{M}$ can be used to compute the integers x and y satisfying the equation $n = x^2 + Ny^2$ with $(x, y) = 1$ for $N \in \{4, 9, 16, 25\}$. This also means that n can be written in the form $n = x^2+Ny^2$ with $(x, y) = 1$ under the hypothesis stated in Theorem 2. So, using the group structure of the Modular group Γ, we have also provided a solution to the problem when a natural number n (relatively prime to N) can be represented in the form $n = x^2 + Ny^2$. As a future work, sufficient conditions can be investigated for further values of N to prove similar results with the help of the modular group Γ. Of course, this method also provides the usage of the *Algorithm* $\mathcal{M}$ in the computation of the integers x and y. It is well known that Cox answered the problem "which primes p can be expressed in the form $p = x^2 + Ny^2$" completely by using the class field theory.[27]

Finally, we note that some Hecke groups and discontinuous groups have been used for this purpose in Refs. 28 and 30 for some special cases of the number N. The interested reader can refer to Refs. 28 and 30 for more details. In Ref. 28, the special cases for $N = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 22, 25, 28, 37, 58$ are considered. As stated there, the importance of these results is that they yield the desired results easily and directly for general numbers n, not only for prime numbers. Furthermore, we do not need the restriction that N is a prime number.

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Chapter 9

The Components Exponential Function in Scator Hypercomplex Space: Planetary Elliptical Motion and Three-Body Choreographies

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Scator algebra is a nondistributive extension of complex algebra and perplex numbers. The rectangular and polar representations are extended to higher dimensions in the additive (rectangular) and multiplicative (polar) representations of components. The complex exponential function is generalized to scator hypercomplex space by the components exponential function. This novel mathematical formalism permits a three-dimensional mapping that can be used to describe elementary curves, such as the conics, as well as other more complex curves. An asset of this approach is that trajectories are naturally parameterized in terms of angle variables. The usual parametrization of the two-body problem is readily obtained using this technique. Three-body choreographies in the trifolium rose, the trisectrix limaçon and a Gerono-type lemniscata are discussed. These two-dimensional projections are ruled out as three-body choreographies. However, these plane curves are encompassed in a scator curve embedded in a three-dimensional space where collisions do not take place. From the constant angular momentum condition, the time dependence of the parameter is readily obtained. Much insight is gained from the examination of these curves. Scator algebra provides a promising route toward the analytical study of two- and three-dimensional choreographies with n bodies in given curves.

1. Introduction

Scator algebras are $1 + n$ dimensional algebras endowed with two operations and a main involution that serves as an order parameter. Scator elements are compound numbers that have one scalar component and an arbitrary number n of director components. Two branches of these algebras, analogous to complex and double (or perplex) numbers, have been acknowledged: (i) Elliptic scators that have been employed in polynomial solutions,[1] powers and roots[2] of scators, and (ii) Hyperbolic scators that have been applied to describe deformed Lorentz spaces[3,4] as well as superluminal motion.[5] A different product rule constitutes the departure between these two ramifications. Here, we shall be mainly concerned with elliptic scators.

In Section 2, the essentials of elliptic scator algebra in $1 + n$ dimensions are presented. The reader may skip this section on a first approach if in haste, since thereafter the dissertation is framed in one scalar and two director components. The scator representations in $1+2$ dimensions and their geometrical interpretation are undertaken in Section 3. The elliptical trajectory and two-body motion in the scator formalism is described in Section 4. The components exponential ($\overset{o}{\mathrm{c}}\mathrm{exp}$) mapping of inclined lines with constant scalar component is introduced in Section 5. The next three sections consider the projections of the $\overset{o}{\mathrm{c}}\mathrm{exp}$ mapping with $m = 3$, namely, the trifolium rose in Section 6, the trisectrix limaçon in Section 7 and the Gerono-type lemniscatae in Section 8.2. The dynamics of the lemniscata are discussed in Section 9. Conclusions are drawn in the last section.

2. Scator Algebra Preamble

Scator elements can be written in two ways, an additive representation that generalizes the rectangular complex representation and a multiplicative representation that generalizes the polar complex representation. In the *additive representation*, scator elements are described by a sum of components,

$$\overset{o}{\varphi} = f_0 + \sum_{j=1}^{n} f_j \check{\mathbf{e}}_j, \tag{1}$$

where the scator additive coefficients, usually written in lowercase Latin letters, are $f_0, f_j \in \mathbb{R}$, the director components are labeled with subindices 1 to $n \in \mathbb{N}$ and $\check{\mathbf{e}}_j \notin \mathbb{R}$. The *multiplicative* or polar *representation* of a scator is

$$\overset{o}{\varphi} = \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j} = \varphi_0 \prod_{j=1}^{n} \exp\left(\varphi_j \check{\mathbf{e}}_j\right). \tag{2}$$

where the scator multiplicative variables, labeled with Greek letters, are $\varphi_0 \in \mathbb{R}^+, \varphi_j \in \mathbb{R}$, and $\check{\mathbf{e}}_j \notin \mathbb{R}$; $\mathbb{R}^+$ represents the interval $[0, \infty)$. The zero subindex component is the scalar component in either representation. For all elements in the $\mathbb{S}^{1+n}$ scator set, the additive scalar component must be different from zero if two or more director components are not zero,

$$\mathbb{S}^{1+n} = \left\{ \overset{o}{\varphi} = f_0 + \sum_{j=1}^{n} f_j \check{\mathbf{e}}_j : f_0 \neq 0 \text{ if } \exists\, f_j f_l \neq 0, \right. \\ \left. \text{for } j \neq l \text{ from 1 to } n \right\}. \tag{3}$$

The *components exponential function*[6] in scator hypercomplex space is a scator function of scator variable $\overset{o}{\text{c}}\text{exp} : \mathbb{R}^{1+n} \to \mathbb{S}^{1+n}$ defined for a scator argument $\overset{o}{\zeta} = z_0 + \sum_{j=1}^{n} z_j \check{\mathbf{e}}_j$, by

$$\overset{o}{\text{c}}\text{exp}\left(\overset{o}{\zeta}\right) \equiv \overset{o}{\text{c}}\text{exp}\left(z_0 + \sum_{j=1}^{n} z_j \check{\mathbf{e}}_j\right) \equiv \exp\left(z_0\right) \prod_{j=1}^{n} \exp\left(z_j \check{\mathbf{e}}_j\right), \tag{4a}$$

where $\exp\left(z_j \check{\mathbf{e}}_j\right) = \cos\left(z_j\right) + \sin\left(z_j\right) \check{\mathbf{e}}_j$. An Euler-type formula is thus satisfied for each hypercomplex component $\check{\mathbf{e}}_j$. In the additive representation, the *components exponential function* is defined by

$$\overset{o}{\text{cexp}}\left(z_0 + \sum_{j=1}^{n} z_j \check{\mathbf{e}}_j\right) \equiv e^{z_0} \prod_{k=1}^{n} \cos\left(z_k\right) + e^{z_0} \sum_{j=1}^{n} \prod_{k \neq j}^{n} \cos\left(z_k\right) \sin\left(z_j\right) \check{\mathbf{e}}_j. \quad (4b)$$

The components exponential function given by (4b) may be viewed as a higher dimensional extension of Euler's formula. The multiplicative scator representation in terms of the $\overset{o}{\text{cexp}}$ function is

$$\overset{o}{\varphi} = \varphi_0 \, \overset{o}{\text{cexp}}\left(\sum_{j=1}^{n} \varphi_j \check{\mathbf{e}}_j\right). \quad (5)$$

The mapping of the multiplicative to additive representations is given by the function $\mathfrak{f}_{mar} : (\mathbb{R}^+; \mathbb{R}^n) \to \mathbb{S}^{1+n}$,

$$\mathfrak{f}_{mar} : \varphi_0 \prod_{j=1}^{n} e^{\varphi_j \check{\mathbf{e}}_j} \longmapsto \underbrace{\varphi_0 \prod_{k=1}^{n} \cos(\varphi_k)}_{f_0} + \underbrace{\varphi_0 \sum_{j=1}^{n} \prod_{k \neq j}^{n} \cos(\varphi_k) \sin(\varphi_j)\, \check{\mathbf{e}}_j}_{f_j}. \quad (6)$$

Scator algebra is endowed with two fundamental operations. The sum of scators $\overset{o}{\alpha} = a_0 + \sum_{j=1}^{n} a_j \check{\mathbf{e}}_j \in \mathbb{R}^{1+n}$ and $\overset{o}{\beta} = b_0 + \sum_{j=1}^{n} b_j \check{\mathbf{e}}_j \in \mathbb{R}^{1+n}$ is

$$\overset{o}{\alpha} + \overset{o}{\beta} \equiv (a_0 + b_0) + \sum_{j=1}^{n} (a_j + b_j)\, \check{\mathbf{e}}_j. \quad (7)$$

Scator addition satisfies Abelian group properties. The product of scators $\overset{o}{\alpha} = \alpha_0 \prod_{j=1}^{n} \exp\left(\alpha_j \check{\mathbf{e}}_j\right)$ and $\overset{o}{\beta} = \beta_0 \prod_{j=1}^{n} \exp\left(\beta_j \check{\mathbf{e}}_j\right)$ is

$$\overset{o}{\alpha}\overset{o}{\beta} \equiv \alpha_0 \beta_0 \prod_{j=1}^{n} \exp\left[\left(\alpha_j + \beta_j\right) \check{\mathbf{e}}_j\right]. \quad (8)$$

In the multiplicative representation, the scator product, provided that zero is excluded $\alpha_0, \beta_0 \neq 0$, satisfies Abelian group properties.

However, in the additive representation this is no longer true. The product $\overset{o}{\alpha}, \overset{o}{\beta} \in \mathbb{S}^{1+n}$ in the additive representation is: if $a_0 b_0 \neq 0$,

$$\overset{o}{\alpha}\overset{o}{\beta} = a_0 b_0 \prod_{k=1}^{n} \left(1 - \frac{a_k b_k}{a_0 b_0}\right) + a_0 b_0 \sum_{j=1}^{n} \left[\prod_{k \neq j}^{n} \left(1 - \frac{a_k b_k}{a_0 b_0}\right) \left(\frac{a_j}{a_0} + \frac{b_j}{b_0}\right)\right] \check{\mathbf{e}}_j; \tag{9a}$$

If $a_0 = 0$ and $b_0 \neq 0$, $\overset{o}{\alpha} = a_l \check{\mathbf{e}}_l$ has a single nonvanishing director component,

$$\overset{o}{\alpha}\overset{o}{\beta} = -a_l b_l + a_l b_0 \check{\mathbf{e}}_l - \sum_{j \neq l}^{n} \left(a_l \frac{b_l b_j}{b_0}\right) \check{\mathbf{e}}_j; \tag{9b}$$

if $a_0 = b_0 = 0$, $\overset{o}{\alpha} = a_l \check{\mathbf{e}}_l$ and $\overset{o}{\beta} = b_j \check{\mathbf{e}}_j$, $\overset{o}{\alpha}\overset{o}{\beta} = -a_l b_j \delta_{lj}$,where δ_{lj} is a Kroneker delta. The scator product in the additive representation does not satisfy group properties because it has zero divisors, i.e. $\overset{o}{\alpha}\overset{o}{\beta} = 0$ for nonzero factors, and is not associative if the scalar component of a product is zero. However, the scator product is always commutative. The main second-order involution is *Conjugation,* defined by the negative of the director components in either representation. The conjugate scator of $\overset{o}{\varphi}$, given by (2), is $\overset{o}{\varphi}^* = \varphi_0 \prod_{j=1}^{n} e^{-\varphi_j \check{\mathbf{e}}_j}$, whereas the conjugate of (1) is $\overset{o}{\varphi}^* \equiv f_0 - \sum_{j=1}^{n} f_j \check{\mathbf{e}}_j$.

The magnitude is given by the positive square root of the scator times its conjugate $\|\overset{o}{\varphi}\| = \sqrt{\overset{o}{\varphi}\overset{o}{\varphi}^*} = \varphi_0$. In the additive representation, for $f_0 \neq 0$,

$$\|\overset{o}{\varphi}\| = \sqrt{\overset{o}{\varphi}\overset{o}{\varphi}^*} = |f_0| \prod_{k=1}^{n} \sqrt{1 + \frac{f_k^2}{f_0^2}}, \tag{10}$$

and if the additive scalar component is zero, there is only one director component, say the lth component, $\|\overset{o}{\varphi}\| = |f_l|$. The multiplicative inverse of a scator $\overset{o}{\varphi}$ is $\overset{o}{\varphi}^{-1} = \overset{o}{\varphi}^* / \|\overset{o}{\varphi}\|^2$.

In the rest of this chapter, we shall restrict to $\mathbb{S}^{1+2}$. A description having two hypercomplex director components will already reveal many of the interesting features and possibilities of scator algebra.

3. Scators in $\mathbb{S}^{1+2}$

In $\mathbb{S}^{1+2}$, the scator $\overset{o}{\varphi}$ in the additive and multiplicative representations is

$$\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y = \varphi_0 \exp\left(\varphi_x \check{\mathbf{e}}_x\right) \exp\left(\varphi_y \check{\mathbf{e}}_y\right). \tag{11a}$$

The additive representation with multiplicative variables is

$$\overset{o}{\varphi} = \varphi_0 \cos\varphi_x \cos\varphi_y + \varphi_0 \sin\varphi_x \cos\varphi_y\,\check{\mathbf{e}}_x + \varphi_0 \cos\varphi_x \sin\varphi_y\,\check{\mathbf{e}}_y. \tag{11b}$$

The relationship between additive and multiplicative variables is

$$s = \varphi_0 \cos\varphi_x \cos\varphi_y, \quad x = \varphi_0 \sin\varphi_x \cos\varphi_y, \quad y = \varphi_0 \cos\varphi_x \sin\varphi_y. \tag{12}$$

This scator can be represented geometrically in our familiar three-dimensional space with three orthogonal axes, letting one axis to represent the scalar axis, labeled with the letter "s" and the remaining two axes representing the $\check{\mathbf{e}}_x$ and $\check{\mathbf{e}}_y$ directions, respectively. An example of such a scator is depicted in Figure 1.

This is, so to speak, an Argand diagram in $\mathbb{S}^{1+2}$-dimensional space. The angle in the $(s, \check{\mathbf{e}}_x)$ projection is $\tan\varphi_x = \frac{x}{s}$ and in the $(s, \check{\mathbf{e}}_y)$ plane $\tan\varphi_y = \frac{y}{s}$.

The angle in the $(\check{\mathbf{e}}_x, \check{\mathbf{e}}_y)$ plane is

$$\frac{y}{x} = \tan\theta = \frac{\varphi_0 \cos\varphi_x \sin\varphi_y}{\varphi_0 \sin\varphi_x \cos\varphi_y} = \frac{\tan\varphi_y}{\tan\varphi_x}. \tag{13}$$

The scator magnitude squared of $\overset{o}{\varphi} = s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$ is

$$\left\|\overset{o}{\varphi}\right\|^2 = s^2 \left(1 + \frac{x^2}{s^2}\right)\left(1 + \frac{y^2}{s^2}\right). \tag{14}$$

The scator $\overset{o}{\varphi}(\varphi_0; \varphi_x, \varphi_y)$ refers to $\overset{o}{\varphi}$ written in terms of its multiplicative (or polar) variables. The magnitude φ_0 in the argument is separated by a semicolon from the director angles φ_x, φ_y. The scator $\overset{o}{\varphi}(s; x, y)$ refers to $\overset{o}{\varphi}$ written in terms of its additive (or rectangular) variables. The additive scalar component is again separated by a semicolon from the director components. However, it does not represent the scator magnitude that in terms of $(s; x, y)$ is given by (14).

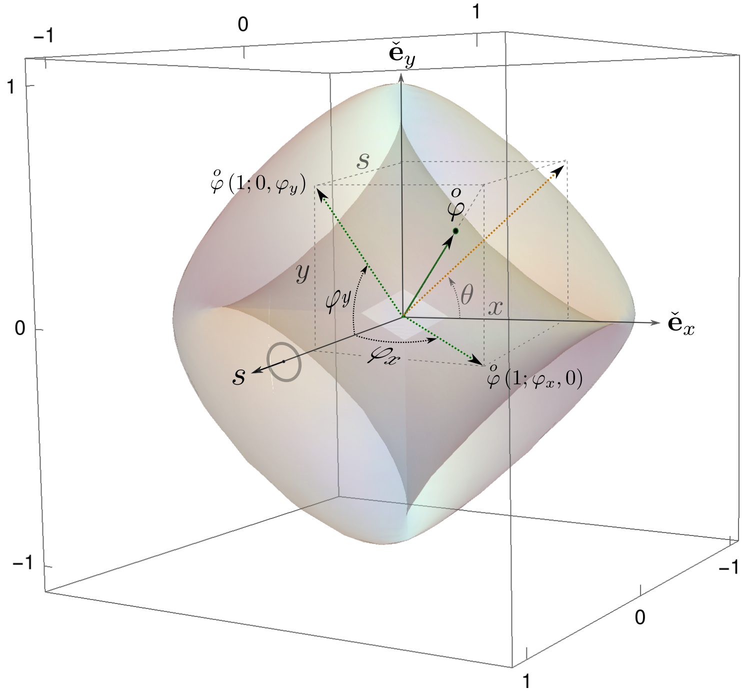


Fig. 1. Geometric representation of a unit magnitude scator $\overset{o}{\varphi}$ (solid green). The tip of the scator lies on the surface of the cusphere (shown in semitransparent pearl). The tips of the projections of $\overset{o}{\varphi}$ (dotted green) in the $(s, \check{\mathbf{e}}_x)$ and $(s, \check{\mathbf{e}}_y)$ planes also lie in the cusphere surface. The scator in the $(\check{\mathbf{e}}_x, \check{\mathbf{e}}_y)$ plane (dotted orange) does not belong to the $\mathbb{S}^{1+2}$ set, its magnitude is not defined.

For a unit magnitude scator $\left\|\overset{o}{\varphi}\right\|^2 = 1$, the equation of the isometric two-dimensional surface embedded in three-dimensional space is

$$s^2 = \left(s^2 + x^2\right)\left(s^2 + y^2\right). \tag{15}$$

This *cusphere* surface is rendered in Figure 1. The scalar and the two director components are represented in three mutually orthogonal axes. The projections of the cusphere surface onto the $s, \check{\mathbf{e}}_x$ and $s, \check{\mathbf{e}}_y$ planes are filled circles while its projection in the $\check{\mathbf{e}}_x, \check{\mathbf{e}}_y$ plane is a filled square (see Ref. 6, Section 6.3); thus, the portmanteau of cube and sphere. The components exponential function produces this mapping for a constant scalar component, $\overset{o}{\text{cexp}}\left(\overset{o}{\zeta}\right) = \overset{o}{\text{cexp}}\left(c_0 + z_x\,\check{\mathbf{e}}_x + z_y\,\check{\mathbf{e}}_y\right) \to$ cusphere.

3.1. *Scator projections*

The unit magnitude scator $\overset{o}{\varphi} = \cos\varphi_x \cos\varphi_y + \sin\varphi_x \cos\varphi_y \,\check{\mathbf{e}}_x + \cos\varphi_x \sin\varphi_y \,\check{\mathbf{e}}_y \in \mathbb{S}^{1+2}$ can be written in terms of unit magnitude projections in the $(s, \check{\mathbf{e}}_x)$ and $(s, \check{\mathbf{e}}_y)$ planes,

$$\begin{aligned}\overset{o}{\varphi}(1; \varphi_x, \varphi_y) &= \exp(\varphi_x \check{\mathbf{e}}_x) \exp(\varphi_y \check{\mathbf{e}}_y) \\ &= (\cos\varphi_x + \sin\varphi_x \,\check{\mathbf{e}}_x)(\cos\varphi_y + \sin\varphi_y \,\check{\mathbf{e}}_y).\end{aligned}$$

Since

$$\overset{o}{\varphi}(1; \varphi_x, 0) = \exp(\varphi_x \check{\mathbf{e}}_x) = (\cos\varphi_x + \sin\varphi_x \,\check{\mathbf{e}}_x).$$
$$\overset{o}{\varphi}(1; 0, \varphi_y) = \exp(\varphi_y \check{\mathbf{e}}_y) = (\cos\varphi_y + \sin\varphi_y \,\check{\mathbf{e}}_y).$$

Thus, $\overset{o}{\varphi}(1; \varphi_x, \varphi_y) = \overset{o}{\varphi}(1; \varphi_x, 0)\, \overset{o}{\varphi}(1; 0, \varphi_y)$.

Important 1. Elements as product of its projections. Scator elements, in contrast with other objects like vectors, can be decomposed as the product of its projections (see Ref. 7, Section 4.)

In the multiplicative representation, to drop out a component (say $\exp(\varphi_y \check{\mathbf{e}}_y)$) is equal to making the corresponding multiplicative coefficient zero i.e. $\varphi_y = 0$.

In general, the scator product does not distribute over addition, as seen from (9a). However, if the director components of the factors are different, the product distributes over addition. In particular, for $\overset{o}{\alpha} = a_0 + a_x \check{\mathbf{e}}_x$ and $\overset{o}{\beta} = b_0 + b_y \check{\mathbf{e}}_y$

$$\overset{o}{\alpha}\overset{o}{\beta} = (a_0 + a_x \check{\mathbf{e}}_x)(b_0 + b_y \check{\mathbf{e}}_y) = a_0 b_0 + b_0 a_x \check{\mathbf{e}}_x + a_0 b_y \check{\mathbf{e}}_y,$$

where $\check{\mathbf{e}}_x \check{\mathbf{e}}_y = 0$. Let $\overset{o}{\varphi}(1; \varphi_x, 0) = \overset{o}{\varphi}(s_1; x_1, 0)$ and $\overset{o}{\varphi}(1; 0, \varphi_y) = \overset{o}{\varphi}(s_2; 0, y_2)$. From the identifications in (12) and $s_1 = \cos\varphi_x$, $x_1 = \sin\varphi_x$, $s_2 = \cos\varphi_y$, $y_2 = \sin\varphi_y$, the unit magnitude factors in terms of additive coefficients are

$$\begin{aligned}\overset{o}{\varphi}(s_1; x_1, 0) = s_1 + x_1 \,\check{\mathbf{e}}_x &= \left(\frac{s}{\sqrt{s^2 + x^2}} + \frac{x}{\sqrt{s^2 + x^2}} \,\check{\mathbf{e}}_x\right) \\ &= (\cos\varphi_x + \sin\varphi_x \,\check{\mathbf{e}}_x), \qquad (16a)\end{aligned}$$

$$\overset{o}{\varphi}(s_2; 0, y_2) = s_2 + y_2\, \check{\mathbf{e}}_y = \left(\frac{s}{\sqrt{s^2+y^2}} + \frac{y}{\sqrt{s^2+y^2}}\, \check{\mathbf{e}}_y \right)$$
$$= (\cos\varphi_y + \sin\varphi_y\, \check{\mathbf{e}}_y). \tag{16b}$$

The scator $\overset{o}{\varphi}(s; x, y) = s + x\, \check{\mathbf{e}}_x + y\, \check{\mathbf{e}}_y$ in terms of its unit magnitude factors in additive variables is then

$$s + x\, \check{\mathbf{e}}_x + y\, \check{\mathbf{e}}_y$$
$$= \left(\frac{s}{\sqrt{s^2+x^2}} + \frac{x}{\sqrt{s^2+x^2}}\, \check{\mathbf{e}}_x \right) \left(\frac{s}{\sqrt{s^2+y^2}} + \frac{y}{\sqrt{s^2+y^2}}\, \check{\mathbf{e}}_y \right).$$

Evaluation of the product

$$s + x\, \check{\mathbf{e}}_x + y\, \check{\mathbf{e}}_y = \frac{s^2}{\sqrt{s^2+x^2}\sqrt{s^2+y^2}} + \frac{sx}{\sqrt{s^2+x^2}\sqrt{s^2+y^2}}\, \check{\mathbf{e}}_x$$
$$+ \frac{sy}{\sqrt{s^2+x^2}\sqrt{s^2+y^2}}\, \check{\mathbf{e}}_y,$$

recalling from (15), that for unit magnitude scators in $\mathbb{S}^{1+2}$, $\sqrt{s^2+x^2}\sqrt{s^2+y^2} = s$ confirms the identity. It is also possible to evaluate the projection of $\overset{o}{\varphi}(s; x, y) = s + x\, \check{\mathbf{e}}_x + y\, \check{\mathbf{e}}_y$ by dropping out one of the director components in the additive representation

$$\overset{o}{\varphi}(s; x) = s + x\, \check{\mathbf{e}}_x = \cos\varphi_x \cos\varphi_y + \sin\varphi_x \cos\varphi_y\, \check{\mathbf{e}}_x, \tag{17}$$

$$\overset{o}{\varphi}(s; y) = s + y\, \check{\mathbf{e}}_y = \cos\varphi_x \cos\varphi_y + \cos\varphi_x \sin\varphi_y\, \check{\mathbf{e}}_y, \tag{18}$$

where the rightmost equalities follow from (12). Therefore, dropping out a component or making it zero in the additive representation gives different results, as can be seen from a comparison of (16a)–(17) and (16b)–(18). If the component is dropped out, say y in (17), it is allowed to acquire all its permitted values, $y = \varphi_0 \cos\varphi_x \sin\varphi_y$. Whereas in the unit magnitude projection, it is restricted to zero, $y = 0$. Hereafter, we refer simply to "projections" when the component is dropped out and explicitly to "unit magnitude projections."

In either projection, the scator projections point in the same direction, since the quotients of components $\frac{x_1}{s_1} = \frac{x}{s} = \tan\varphi_x$ and $\frac{y_1}{s_1} = \frac{y}{s} = \tan\varphi_y$ are the same. However, their magnitudes differ, and this will be an important difference when curves are parameterized. In the $\overset{o}{\varphi}(s;x)$ (dropout) projection, the magnitude is $\left\|\overset{o}{\varphi}(s;x)\right\| = \sqrt{s^2+x^2}$ and for $\overset{o}{\varphi}(s;y)$, the magnitude is $\left\|\overset{o}{\varphi}(s,y)\right\| = \sqrt{s^2+y^2}$. However, if the scalar component is dropped out, the scator magnitude (14) diverges. In the limit when $s \to 0$ with $xy \neq 0$, the scator element $s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y$ is not in the $\mathbb{S}^{1+2}$ set. The scator magnitude is then $\lim_{s\to 0}\|\overset{o}{\varphi}\| = \infty$. Thus, in the limit $s \to 0$, the Euclidean metric is not recovered. It is nonetheless possible to obtain the Euclidean limit for $s^2 \gg x^2, y^2$, following an analogous procedure to the classical limit of a hyperbolic scator in a deformed Lorentz metric.[4] The scalar magnitude of a scator is $\|\overset{o}{\varphi}\|_0 \equiv \left\|\frac{1}{2}(\overset{o}{\varphi} + \overset{o}{\varphi}^*)\right\| = |s|$. The director magnitude is defined in an analogous fashion as it was done for hyperbolic scators

$$\|\overset{o}{\varphi}\|_{\check{\mathbf{e}}} \equiv \sqrt{\|\overset{o}{\varphi}\|^2 - \|\overset{o}{\varphi}\|_0^2} = \left(s^2\left(1+\frac{x^2}{s^2}\right)\left(1+\frac{y^2}{s^2}\right) - s^2\right)^{\frac{1}{2}}. \quad (19)$$

The director magnitude limit when the additive scalar component s becomes much larger than the director components is

$$\lim_{\frac{xy}{s}\to 0}\|\overset{o}{\varphi}\|_{\check{\mathbf{e}}} = \lim_{\frac{xy}{s}\to 0}\left(x^2+y^2+\frac{x^2y^2}{s^2}\right)^{\frac{1}{2}} = \sqrt{x^2+y^2}.$$

In this limit, the two-dimensional Euclidean metric is recovered. This $\sqrt{x^2+y^2}$ circle is depicted on the cusphere surface in Figure 1, for s close to one and small x, y.

There are three variables in the $\mathbb{S}^{1+2}$ scator set but only two in $\mathbb{C}$. So, there are many possibilities in order to embed a curve or trajectory (one-dimensional) in $1+2$-(three)-dimensional space.

4. Elliptical Trajectory-Curve-Motion

The components exponential function $\overset{o}{\text{cexp}}$ in $\mathbb{S}^{1+2}$ with argument $\overset{o}{\zeta} = z_0 + z_x\,\check{\mathbf{e}}_x + z_y\,\check{\mathbf{e}}_y$ is

$$\overset{o}{\text{cexp}}\left(\overset{o}{\zeta}\right) = e^{z_0}\cos z_x \cos z_y + e^{z_0}\sin z_x \cos z_y\,\check{\mathbf{e}}_1 + e^{z_0}\cos z_x \sin z_y\,\check{\mathbf{e}}_2.$$

Consider the scalar and the $\check{\mathbf{e}}_x$ director components to be constant, $z_0 = c_0$ and $z_x = c_1$. The scator $\overset{o}{\zeta} = c_0 + c_1\,\check{\mathbf{e}}_x + z_y\,\check{\mathbf{e}}_y$ is then a line parallel to the $\breve{\mathbf{e}}_y$ axis. The components exponential function maps line segments parallel to one of the director axes into ellipses,[6] as portrayed in Figure 2. Comparison with the additive representation using multiplicative variables (11b) gives the following identifications, the scator magnitude is $\varphi_0 = e^{z_0}$ and $z_x = c_1 = \varphi_x$.

Allow for the constants a, b to be $a = \varphi_0 \sin\varphi_x$ and $b = \varphi_0 \cos\varphi_x$. The scator $\overset{o}{\varphi}_{\text{elli}}$, from (11b), is then

$$\overset{o}{\varphi}_{\text{elli}} = \underbrace{b\cos\varphi_y}_{s} + \underbrace{a\cos\varphi_y}_{x}\,\breve{\mathbf{e}}_x + \underbrace{b\sin\varphi_y}_{y}\,\breve{\mathbf{e}}_y. \tag{20}$$

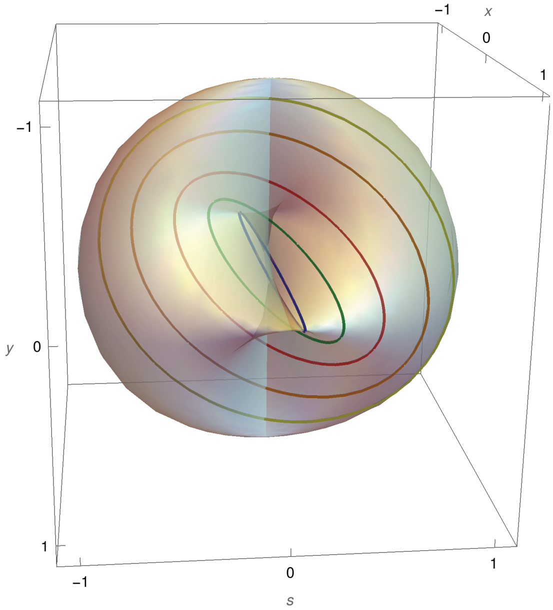
The plane with $\varphi_x = \arctan\frac{a}{b} =$ constant is shown in semitransparent yellow in Figure 3. A semiplane with a given φ_y is drawn in semitransparent green in the same figure. The $\overset{o}{\varphi}_{\text{elli}}$ scator lies on the intersection of these two planes. As the angle φ_y is swept from 0 to 2π, the tip of the $\overset{o}{\varphi}_{\text{elli}}$ scator describes an ellipse in the $\varphi_x =$ constant plane (shown in maroon in Figure 3). To confirm this assertion, perform a clockwise Euclidean rotation by $\chi = \frac{\pi}{2} - \varphi_x$ of the coordinates in the $(s, \breve{\mathbf{e}}_x)$ plane. $\overset{o}{\varphi}{}'_{\text{elli}} = \sqrt{a^2+b^2}\cos\varphi_y\,\breve{\mathbf{e}}_x + b\sin\varphi_y\,\breve{\mathbf{e}}_y$ is indeed an ellipse with semiaxes $\sqrt{a^2+b^2}$ and b in the φ_x= constant plane. Euclidean rotations do not preserve the scator magnitude so they should be treated with great care. Furthermore, in this case, $\overset{o}{\varphi}{}'_{\text{elli}} \notin \mathbb{S}^{1+2}$ is not in the scator set, the sole purpose of the transformation was to show that an ellipse is obtained in the $\varphi_x =$ constant plane.

The projections of the $\overset{o}{\varphi}_{\text{elli}}$ scator are as follows:

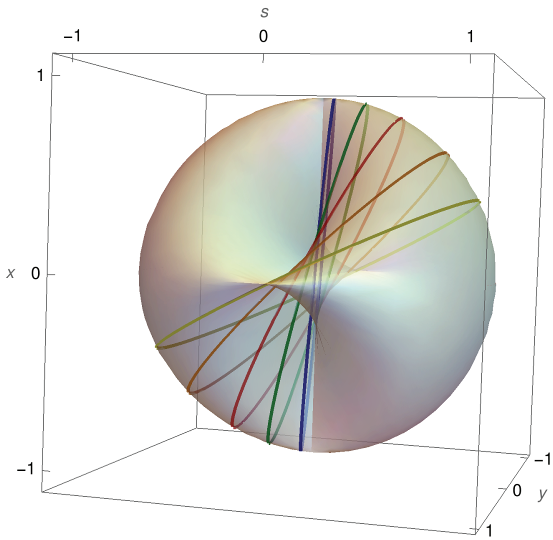
(i) a circle radius b is obtained in the $(s, \breve{\mathbf{e}}_y)$ plane, with parametric representation $\overset{o}{\varphi}_{\text{elli}}\Big|_{(s,\breve{\mathbf{e}}_y)} = \overset{o}{\varphi}_{\text{circle}}(s, \breve{\mathbf{e}}_y) = b\cos\varphi_y + b\sin\varphi_y\,\breve{\mathbf{e}}_y$,

(ii) a line segment with length $2\sqrt{a^2+b^2}$ and slope $\frac{a}{b}$ is obtained in the $(s, \breve{\mathbf{e}}_x)$ plane, with parametric representation $\overset{o}{\varphi}_{\text{elli}}\Big|_{(s,\breve{\mathbf{e}}_x)} = \overset{o}{\varphi}_{\text{line}}(s, \breve{\mathbf{e}}_x) = b\cos\varphi_y + a\cos\varphi_y\,\breve{\mathbf{e}}_x$,

(iii) an ellipse is obtained in the $(\breve{\mathbf{e}}_x, \breve{\mathbf{e}}_y)$ plane with semiaxes a and b, respectively, with parametric representation $\overset{o}{\varphi}_{\text{elli}}\Big|_{(\breve{\mathbf{e}}_x,\breve{\mathbf{e}}_y)} = a\cos\varphi_y\,\breve{\mathbf{e}}_x + b\sin\varphi_y\,\breve{\mathbf{e}}_y$.



(Scator ellipses lie on the cusphere surface)



(Scator ellipses lie on the cusphere surface)

Fig. 2. Cusphere unit magnitude surfaces shown in semitransparency. Ellipses with semiaxes $a = \varphi_0 \sin \varphi_x$ and $b = \varphi_0 \cos \varphi_x$, for $\varphi_x = 0.96\frac{\pi}{2}$ (blue), $\varphi_x = 0.84\frac{\pi}{2}$ (green), $\varphi_x = 0.7\frac{\pi}{2}$ (red), $\varphi_x = 0.5\frac{\pi}{2}$ (orange) and $\varphi_x = 0.3\frac{\pi}{2}$ (yellow).

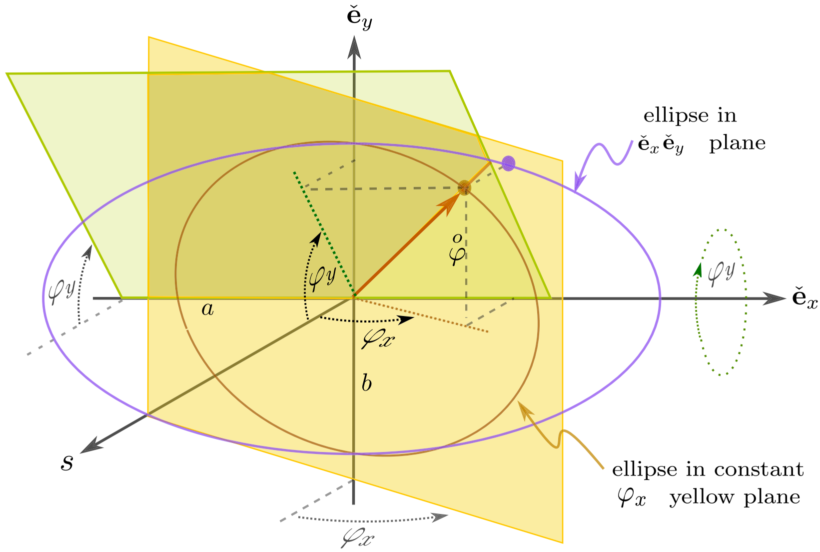
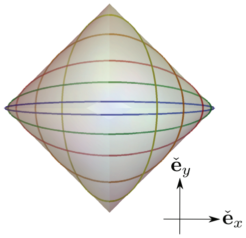


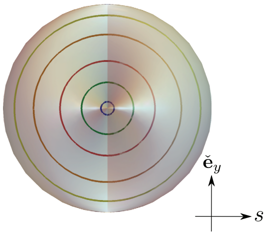
Fig. 3. A scator $\overset{o}{\varphi}$ depicted as an orange arrow in $\mathbb{S}^{1+2}$-dimensional space. The ellipse in maroon lies in the φ_x constant plane (yellow). The φ_y plane (green) intersects the φ_x constant plane at a line (orange). The magnitude is chosen $\varphi_0 = \sqrt{a^2 + b^2}$, where a, b are the semiaxes of the magenta ellipse in the $(\check{\mathbf{e}}_x, \check{\mathbf{e}}_y)$ plane. The angle φ_y is swept from 0 to 2π. The curve thus obtained is an ellipse in the φ_x constant plane (yellow). In scator space the magnitude is constant although the shape of the curve is an ellipse. It lies in the isometric cusphere surface. The projection of this ellipse in the $(\check{\mathbf{e}}_x, \check{\mathbf{e}}_y)$ plane is also an ellipse (magenta). The φ_y angle corresponds to the eccentric anomaly angle E defined in the $(\check{\mathbf{e}}_x, \check{\mathbf{e}}_y)$ plane (see Figure).

These projections are depicted in Figure 4. It is interesting to point out that fixing the scator magnitude and the angle φ_x does not make any of the additive scator coefficients $(s; x, y)$ constant. In contrast, in complex algebra, a constant magnitude and phase define a point in the complex plane. The extra degree of freedom in $\mathbb{S}^{1+2}$ now defines a curve, that is, a one-dimensional object embedded in a three-dimensional space. The scator magnitude squared is

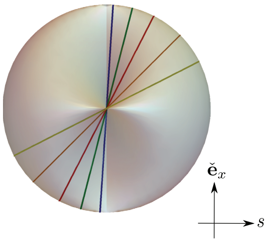
$$\left\|\overset{o}{\varphi}\right\|^2 = b^2 \cos^2 \varphi_y + a^2 \cos^2 \varphi_y + b^2 \sin^2 \varphi_y + \frac{a^2 \cos^2 \varphi_y b^2 \sin^2 \varphi_y}{b^2 \cos^2 \varphi_y} = a^2 + b^2.$$



(Projections in the $(\check{\mathbf{e}}_x, \check{\mathbf{e}}_y)$ plane are ellipses)



(Projections in the $(s, \check{\mathbf{e}}_y)$ plane are circles)



(Projections in the $(s, \check{\mathbf{e}}_x)$ plane are line segments)

Fig. 4. Orthographic projections of scator ellipses.

Thus, $\varphi_0 = \sqrt{a^2 + b^2}$ and the constant angle is $c_1 = \varphi_x = \arctan \frac{a}{b}$. The representation of an ellipse in $\mathbb{S}^{1+2}$ scator space in the multiplicative representation is

$$\overset{o}{\varphi}_{\text{elli}} = \sqrt{a^2 + b^2} \, \exp\left(\arctan \frac{a}{b} \check{\mathbf{e}}_x \right) e^{\varphi_y \check{\mathbf{e}}_y}. \tag{21}$$

The multiplicative representations of the projections in the $(s, \check{\mathbf{e}}_y)$ and $(s, \check{\mathbf{e}}_x)$ planes are $\overset{o}{\varphi}_{\text{circle}} = b e^{\varphi_y \check{\mathbf{e}}_y}$ and $\overset{o}{\varphi}_{\text{line}} = \sqrt{a^2 + b^2} \cos \varphi_y$ $\exp\left(\pm \arctan \frac{a}{b} \check{\mathbf{e}}_x\right)$. However, there is no multiplicative representation of the $(\breve{\mathbf{e}}_x, \breve{\mathbf{e}}_y)$ projection because $\overset{o}{\varphi}_{\text{elli}}\big|_{(\breve{\mathbf{e}}_x,\breve{\mathbf{e}}_y)} \notin \mathbb{S}^{1+2}$. It is therefore preferable to work in scator algebra retaining always the scalar component keeping in mind that the director part represents the trajectory in a two-dimensional plane.

4.1. *Eccentric anomaly*

Consider a shift in the $\breve{\mathbf{e}}_x$ direction to a focus as is usually done in celestial mechanics, $f_x \, \breve{\mathbf{e}}_x \mapsto (f_x - ae) \, \breve{\mathbf{e}}_x$, where the eccentricity is $e = \sqrt{1 - \frac{b^2}{a^2}}$. The $\overset{o}{\varphi}_{\text{elli}}$ scator is shifted to

$$\overset{o}{\varphi}_{\text{elli-ecc}} = \overset{o}{\varphi}_{\text{elli}} - ae \, \breve{\mathbf{e}}_x = b\cos\varphi_y + a\left(\cos\varphi_y - e\right) \, \breve{\mathbf{e}}_x + b\sin\varphi_y \, \breve{\mathbf{e}}_y. \tag{22}$$

Recall that the parametric vector representation of position of a two-body problem with one-body sitting at one focus is[8]

$$\mathbf{r} = a\left(\cos E - e\right)\hat{\mathbf{e}}_x + b\sin E \, \hat{\mathbf{e}}_y, \tag{23}$$

where E is the eccentricity angle. Comparison of (22) and (23) evinces that φ_y is equal to the eccentricity angle E. This is quite an unexpected result since φ_y is the projection angle of a scator in the $(s, \breve{\mathbf{e}}_y)$ plane. The assertion $\varphi_y = E$ implies that φ_y is equal to the eccentricity angle in the $(\breve{\mathbf{e}}_x, \breve{\mathbf{e}}_y)$ plane, as shown in Figure 5.

4.2. *True anomaly*

The φ_y angle can be written in terms of the true anomaly ϕ following the known relationships between eccentricity and true angles. However, it is readily obtained in the scator formalism if the origin is shifted to a focus in the $1+2$-dimensional space. Note that the triangle in the $s, \breve{\mathbf{e}}_x$ plane in Figure 3 $\tan\varphi_x = \frac{a}{b} = \frac{ae}{be}$. Thus, the scalar component has to be shifted by $s \mapsto (s - be)$ and the $\breve{\mathbf{e}}_x$ component by $x \, \breve{\mathbf{e}}_x \mapsto (x - ae) \, \breve{\mathbf{e}}_x$

$$\overset{o}{\varphi}{}' = b\left(\cos\varphi_y - e\right) + a\left(\cos\varphi_y - e\right) \, \breve{\mathbf{e}}_x + b\sin\varphi_y \, \breve{\mathbf{e}}_y. \tag{24}$$

From (13), the angle in the $(\breve{\mathbf{e}}_x, \breve{\mathbf{e}}_y)$ plane is

$$\tan\phi = \frac{\tan\varphi'_y}{\tan\varphi'_x} = \frac{b}{a}\frac{\sin\varphi_y}{\left(\cos\varphi_y - e\right)} = \sqrt{1 - e^2}\frac{\sin\varphi_y}{\left(\cos\varphi_y - e\right)}, \tag{25}$$

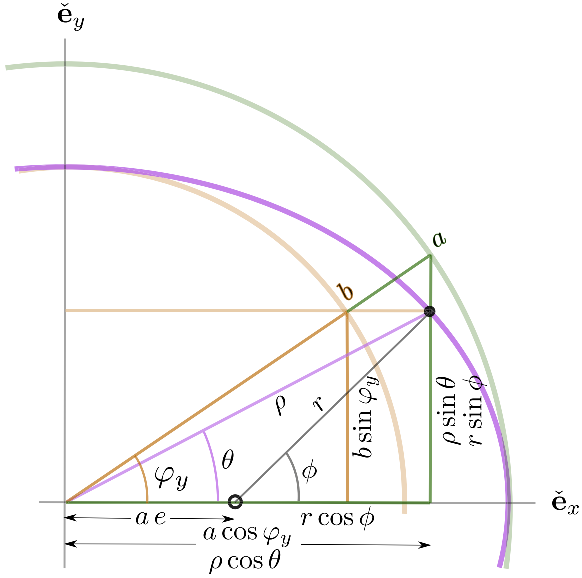


Fig. 5. Ellipse (magenta) in the $(\check{\mathbf{e}}_x, \check{\mathbf{e}}_y)$ plane. The ellipse is circumscribed by the green circle and the orange circle is inscribed in the ellipse. A point in the ellipse is given by (r, θ), where r is clearly not constant but given by the magnitude of (23). Taking a vertical line from the point until it intersects the circumscribed circle or a horizontal line until it intersects the inscribed circle that defines the angle φ_y. The ellipse can then be parametrically represented by $a\cos\varphi_y\,\check{\mathbf{e}}_x + b\sin\varphi_y\,\check{\mathbf{e}}_y$, where a, b are the ellipse semiaxes in $\check{\mathbf{e}}_x$ and $\check{\mathbf{e}}_y$, respectively.

where $\tan\varphi'_y = \frac{b\sin\varphi_y}{b(\cos\varphi_y - e)}$ and $\tan\varphi'_x = \frac{a(\cos\varphi_y - e)}{b(\cos\varphi_y - e)}$. Equation (25) reproduces the usual relationship between eccentric and true anomalies.

4.3. *Dynamics*

To obtain the dynamics, consider the derivative of (23) with respect to time,

$$\mathbf{v} = -a\sin E\,\partial_t E\,\hat{\mathbf{e}}_x + b\cos E\,\partial_t E\,\hat{\mathbf{e}}_y.$$

The angular momentum is then $\mathbf{L} = m\mathbf{r} \times \mathbf{v} = m\,ab\,(1 - e\cos E)\,\partial_t E\,\hat{\mathbf{e}}_z$. Since the system is isolated, the angular momentum should be constant $L = m\,ab\,(1 - e\cos E)\,\partial_t E$, and upon integration Kepler's equation is obtained

$$E - e \sin E = \frac{L}{m\,ab} t.$$

The usual approximation methods are required to solve this transcendental equation for the eccentricity as a function of time $E(t)$.[9]

4.4. *Hyperbolae*

Since the cusphere surface is bounded, only the circle and the ellipse can be obtained within this surface. Hyperbolic or real scators in $\mathbb{S}^{1+2}$ are of the form[10]

$$\begin{aligned}\overset{o}{\varphi} &= s + x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y = \varphi_0 \exp(\varphi_x \hat{\mathbf{e}}_x) \exp(\varphi_y \hat{\mathbf{e}}_y) \\ &= \varphi_0 \cosh\varphi_x \cosh\varphi_y + \varphi_0 \sinh\varphi_x \cosh\varphi_y \hat{\mathbf{e}}_x \\ &\quad + \varphi_0 \cosh\varphi_x \sinh\varphi_y \hat{\mathbf{e}}_y. \end{aligned} \tag{26}$$

Hyperbolic trigonometric functions are now involved in the additive representation with multiplicative variables. The product of real unit directors is $\hat{\mathbf{e}}_j \hat{\mathbf{e}}_j = \delta_{ij}$ (in contrast to $\check{\mathbf{e}}_x \check{\mathbf{e}}_y = -\delta_{ij}$), hats instead of checks are used to distinguish them. The hyperbolic scator magnitude squared of $\overset{o}{\varphi} = s + x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y$ is

$$\left\|\overset{o}{\varphi}\right\|^2 = s^2 \left(1 - \frac{x^2}{s^2}\right)\left(1 - \frac{y^2}{s^2}\right). \tag{27}$$

Consider the constants a, b to be $a = \varphi_0 \sinh\varphi_x$ and $b = \varphi_0 \cosh\varphi_x$, so that the angle φ_x is constant and equal to $\operatorname{arctanh}(\frac{a}{b})$. The constant a is positive or negative depending on the sign of φ_x, while $b \geq 1$ and $b > a$. The scator $\overset{o}{\varphi}_{\text{hyp}}$, from (26), is then

$$\overset{o}{\varphi}_{\text{hyp}} = \underbrace{b\cosh\varphi_y}_{s} + \underbrace{a\cosh\varphi_y}_{x}\,\hat{\mathbf{e}}_x + \underbrace{b\sinh\varphi_y}_{y}\,\hat{\mathbf{e}}_y. \tag{28}$$

A parametric equation of the $x > 0$ branch of a rectangular hyperbola is obtained in the $(s, \hat{\mathbf{e}}_y)$ plane and the positive x branch of a hyperbola with canonical form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \cosh^2\varphi - \sinh^2\varphi = 1$

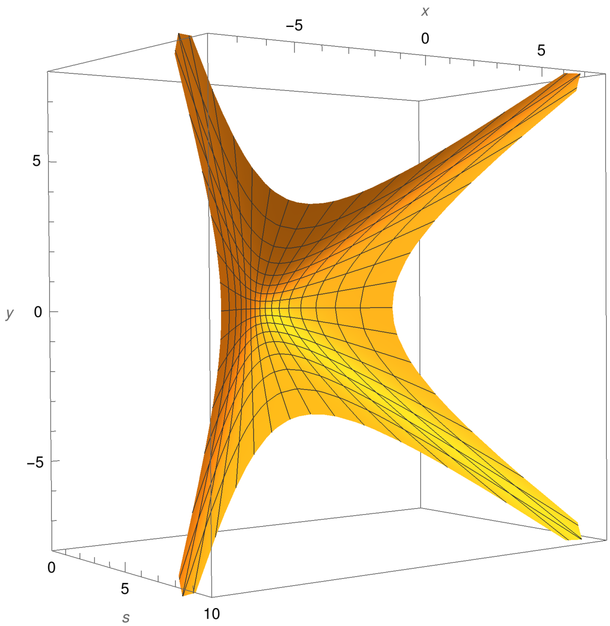


Fig. 6. Families of hyperbolae in the $(s; x, y)$ scator space.

in the $(\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y)$ plane. The multiplicative or polar representation of hyperbolae in scator algebra is

$$\overset{o}{\varphi}_{\text{hyp}} = \sqrt{b^2 - a^2} \exp\left(\operatorname{arctanh}\left(\frac{a}{b}\right)\hat{\mathbf{e}}_x\right) \exp\left(\varphi_y \hat{\mathbf{e}}_y\right).$$

Hyperbolae with major axis in the $\hat{\mathbf{e}}_y$ direction are obtained if the φ_y director component is considered constant. Hyperbolae in $\mathbb{S}^{1+2}$ are illustrated in Figure 6.

5. Three Bodies in the $\mathcal{C}_{13}$ Curve

Consider straight lines in the $z_0 = c_0$ plane with slope m in the $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y$ plane. z_y is then equal to $z_y = m\, z_x + b$, where m, b are real constants. The scator $\overset{o}{\zeta} = c_0 + z_x\, \check{\mathbf{e}}_x + (m\, z_x + b)\, \check{\mathbf{e}}_y$ then describes line segments inclined in the $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y$ plane. The components exponential mapping of

$\overset{o}{\zeta}$ in $\mathbb{S}^{1+2}$ is then

$$\overset{o}{\text{e}}\text{xp}\big(\overset{o}{\zeta}\big) = e^{c_0}\left(\cos\varphi\cos\left(m\varphi\right) + \sin\varphi\cos\left(m\varphi + b\right)\check{\mathbf{e}}_x \right.$$
$$\left. + \cos\varphi\sin\left(m\varphi + b\right)\check{\mathbf{e}}_y\right),$$

where the continuous free parameter is $\varphi = z_x$. If m is rational, the curves are closed. For integer m, odd m coefficients exhibit a π periodicity.[1] For $c_0 = 0$ and $m = 3$,

$$\overset{o}{\varphi} = \cos\varphi_x\cos\left(3\varphi_x\right) + \cos\left(3\varphi_x\right)\sin\varphi_x\check{\mathbf{e}}_x + \cos\varphi_x\sin\left(3\varphi_x\right)\check{\mathbf{e}}_y, \tag{29}$$

This parametric representation of the curve $\mathcal{C}_{13}$ is depicted in Figure 7. $\mathcal{C}_{13}$ is a one-dimensional object embedded in $\mathbb{S}^{1+2}$-(three)-dimensional space, the subindices stand for the φ_x and φ_y coefficients. The components exponential function is one of the few scator holomorphic functions, according to the Cauchy differential quotient criterion.[11, 12] It thus insures that a continuous argument will produce a continuous mapping. The additive and multiplicative variables are related by

$$\tan\varphi_x = \frac{x}{s}, \quad \tan 3\varphi_x = \frac{y}{s}. \tag{30}$$

From the quotient of these two equations, expanding the triple angle,

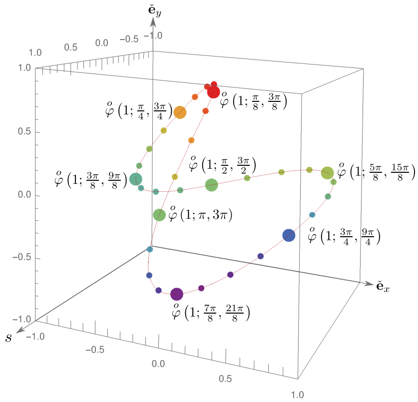
$$\frac{\tan\left(3\varphi_x\right)}{\tan\varphi_x} = \frac{y}{x} = \frac{3 - \tan^2\varphi_x}{1 - 3\tan^2\varphi_x}\frac{3 - \frac{x^2}{s^2}}{1 - 3\frac{x^2}{s^2}}. \tag{31}$$

Substitution of (30) gives y in terms of the other two variables

$$y = x\frac{3s^2 - x^2}{s^2 - 3x^2}. \tag{32a}$$

[1]There is an errata in Ref. 6, First Eq. p. 1031. The $\check{\mathbf{e}}_2$ argument on the left of the equation should be $m\left(z_1 + \pi\right)$. The right side should be multiplied by e^{c_0}. The equation should be

$$\overset{o}{\text{e}}\text{xp}\left(c_0 + \left(z_1 + \pi\right)\check{\mathbf{e}}_1 + m\left(z_1 + \pi\right)\check{\mathbf{e}}_2\right) = -e^{c_0}\cos\left(m\,\pi\right)\cos\left(z_1\right)\cos\left(m\,z_1\right)$$
$$- e^{c_0}\cos\left(m\,\pi\right)\sin\left(z_1\right)\cos\left(m\,z_1\right)\check{\mathbf{e}}_1 - e^{c_0}\cos\left(m\,\pi\right)\cos\left(z_1\right)\sin\left(m\,z_1\right)\check{\mathbf{e}}_2.$$



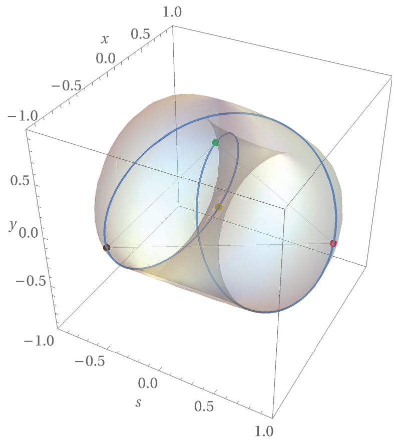


Fig. 7. Curve $\mathcal{C}_{13}$ given by (29) embedded in three-dimensional space.
Notes: Powers of the scator with $\varphi_y = 3\varphi_x$, for $\overset{o}{\varphi}(1; \frac{\pi}{32}, \frac{3\pi}{32})$ in large dots $\overset{o}{\varphi}(1; \frac{\pi}{32}, \frac{3\pi}{32})$ in smaller dots. [Reproduced from Ref. 7]; Curve $\mathcal{C}_{13}$ is shown superimposed on a semitransparent cusphere. Three bodies are positioned at the vertices of an isosceles triangle. A yellow dot at the origin represents the center of mass.

If we write $\frac{\tan(3\varphi_x)}{\tan\varphi_x} = \frac{y}{x}$ in (31) and solve for $\tan^2\varphi_x = \frac{x^2}{s^2}$ from (31), s is obtained

$$s^2 = \frac{x^2(x-3y)}{3x-y}. \tag{32b}$$

In terms of algebraic geometry, the curve $\mathcal{C}_{13}$ in $\mathbb{S}^{1+2}$ is the set of all points $(s; x, y)$ which satisfy the polynomials

$$\left(s^2 + x^2\right)\left(s^2 + y^2\right) - s^2 = 0, \qquad \text{[unit magnitude scator]}$$

$$y\left(s^2 - 3x^2\right) - x\left(3s^2 - x^2\right) = 0. \qquad [\varphi_y = 3\varphi_x \text{ condition}]$$

5.1. *Three bodies in the $\mathcal{C}_{13}$ curve*

Consider three bodies located symmetrically with respect to the parameter φ_x, $\overset{o}{\varphi}_1 = \overset{o}{\varphi}(\varphi)$, $\overset{o}{\varphi}_2 = \overset{o}{\varphi}\left(\varphi + \frac{2\pi}{3}\right)$ and $\overset{o}{\varphi}_3 = \overset{o}{\varphi}\left(\varphi + \frac{4\pi}{3}\right)$, where the subindex in φ_x has been dropped. The parametric representation of the three bodies is

$$\overset{o}{\varphi}_1 = \cos\varphi\cos(3\varphi) + \cos(3\varphi)\sin\varphi\check{\mathbf{e}}_x + \cos\varphi\sin(3\varphi)\,\check{\mathbf{e}}_y, \quad \text{(33a)}$$

$$\overset{o}{\varphi}_2 = \cos\left(\varphi + \frac{2\pi}{3}\right)\cos(3\varphi) + \cos(3\varphi)\sin\left(\varphi + \frac{2\pi}{3}\right)\check{\mathbf{e}}_x + \cos\left(\varphi + \frac{2\pi}{3}\right)\sin(3\varphi)\,\check{\mathbf{e}}_y, \quad \text{(33b)}$$

$$\overset{o}{\varphi}_3 = \cos\left(\varphi - \frac{2\pi}{3}\right)\cos(3\varphi) + \cos(3\varphi)\sin\left(\varphi - \frac{2\pi}{3}\right)\check{\mathbf{e}}_x + \cos\left(\varphi - \frac{2\pi}{3}\right)\sin(3\varphi)\,\check{\mathbf{e}}_y. \quad \text{(33c)}$$

5.2. *Curve projection in two-dimensional planes*

The projections of the three-dimensional curve (29) onto the planes produced by each pair of axes produce the trifolium rhodonea, Pascal's trisectrix limaçon and Gerono's lemniscata are depicted in Figure 8. Let us appraise the possibilities of these curves regarding three-body choreographies. Three elementary conditions imposed are on the motion for an isolated system:

(1) The center of mass should be constant in an adequate inertial frame of reference. No external force acts on the system.
(2) No collisions should take place.

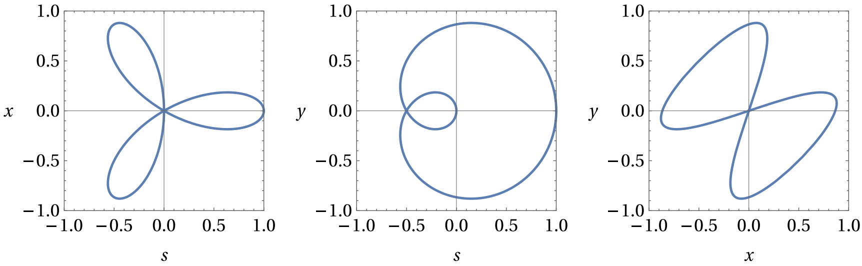


Fig. 8. The projection of the $\varphi_0 = 1, \varphi_y = 3\varphi_x$ scator curve generates: the trifolium rose (left) in the $s, \check{\mathbf{e}}_x$ plane, Pascal's trisectrix (middle) in the $s, \check{\mathbf{e}}_y$ plane and a scaled and rotated Gerono's lemniscata (right) in the $\check{\mathbf{e}}_x, \check{\mathbf{e}}_y$ plane.

(3) The angular momentum should be constant. The direction of the force between bodies lies on the line joining them or is a central force.

6. Trifolium Rose

Substitution of y from (32a) in the unit scator square magnitude (15) gives upon rearrangement,

$$s^2 \left(s^2 - 3x^2\right)^2 = \left(s^2 + x^2\right)^4.$$

From the square root of this expression, the trifolium rose implicit equation is obtained,

$$s \left(s^2 - 3x^2\right) = \left(s^2 + x^2\right)^2.$$

If the equation for the square magnitude is plotted, the trifolium rose and its mirror image are obtained, i.e. an hexa-folium rose. The parametric curve is given by the $s, \check{\mathbf{e}}_x$ components of the 3D curve

$$\overset{o}{\varphi}\Big|_{s,\check{\mathbf{e}}_x} = \cos(\varphi)\cos(3\varphi) + \cos(3\varphi)\sin(\varphi)\,\check{\mathbf{e}}_x,$$

If three bodies are located symmetrically with respect to the angle parameter φ, the center of mass remains at the origin throughout the motion as φ evolves from 0 to 2π. The three bodies lie in the vertices of an equilateral triangle, as shown in Figure 9. As the parameter evolves, the triangle rotates and its length changes. However, the

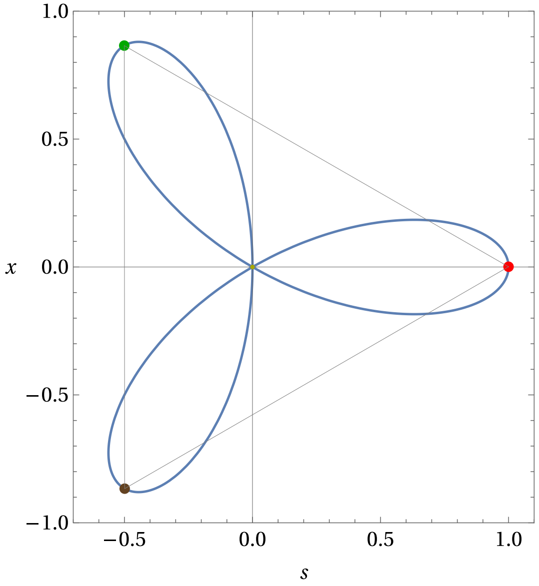


Fig. 9. Three-body choreography in the trifolium rose.

three bodies collide at the origin when $\varphi = \frac{\pi}{2}$, as can be seen from 33(a) to 33(c). So, this trajectory is not a candidate for a three-body choreography. However, it should be noted that four bodies located symmetrically with respect to the angle parameter, $\overset{o}{\varphi}_1 = \overset{o}{\varphi}(\varphi)$, $\overset{o}{\varphi}_2 = \overset{o}{\varphi}\left(\varphi + \frac{\pi}{4}\right)$, $\overset{o}{\varphi}_3 = \overset{o}{\varphi}\left(\varphi + \frac{\pi}{2}\right)$ and $\overset{o}{\varphi}_4 = \overset{o}{\varphi}\left(\varphi + \frac{3\pi}{4}\right)$, still retain the center of mass at the origin but no longer exhibit collisions.

7. Trisectrix Limaçon

From the unit magnitude squared (15), x is equal to

$$x^2 = s^2 \frac{1 - \left(s^2 + y^2\right)}{\left(s^2 + y^2\right)}.$$

If x is substituted in (32a), upon rearrangement, the implicit curve for the trisectrix limaçon of Pascal and its mirror image are obtained. The curve $(2s+1)^2 + 4y^2 = ((2s+1)^2 + 4y^2 - 2(2s+1))^2$, is displaced by 1 in the abscissas and variables scaled by 2 from the usual $(s^2 + y^2) = (s^2 + y^2 - 2s)^2$ limaçon. The parametric curve is given by the $s, \check{\mathbf{e}}_y$ components,

$$\overset{o}{\varphi}\Big|_{s,\check{\mathbf{e}}_y} = \cos(\varphi)\cos(3\varphi) + \cos(\varphi)\sin(3\varphi)\,\check{\mathbf{e}}_y.$$

If three bodies are located symmetrically with respect to the parameter, $\overset{o}{\varphi}_1 = \overset{o}{\varphi}(\varphi)$, $\overset{o}{\varphi}_2 = \overset{o}{\varphi}\left(\varphi + \frac{2\pi}{3}\right)$ and $\overset{o}{\varphi}_3 = \overset{o}{\varphi}\left(\varphi + \frac{4\pi}{3}\right)$, the center of mass remains at the origin throughout the motion as φ evolves from 0 to 2π. It is interesting to note that every position of the three bodies throughout the trajectory is an Euler configuration, that is, the three bodies always lie in a straight line, as shown for $\varphi = 0.252$ in Figure 10. However, two of the bodies, in turn, collide at the crossing point when $\varphi = \frac{\pi}{3} \bmod \frac{\pi}{3}$. If the initial positions are modified (not symmetrical with respect to the parameter) in order to avoid collisions, the center of mass is no longer fixed. So, this trajectory should also be excluded. Nonetheless, it should be noted that in the full three-dimensional curve, there are no collisions, since there are no crossings of the curve $\mathcal{C}_{13}$ in $\mathbb{S}^{1+2}$.

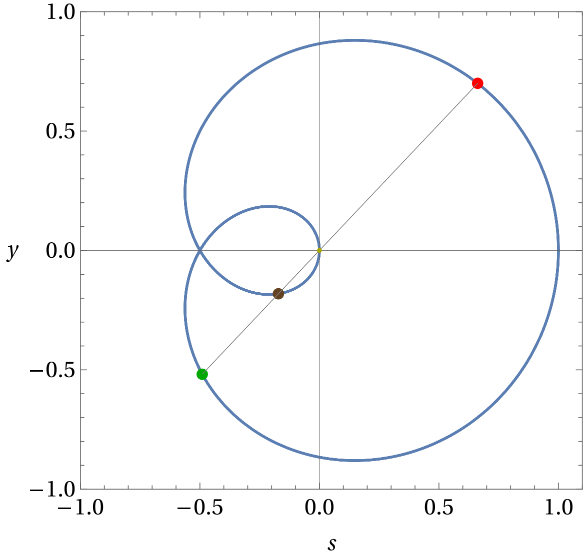


Fig. 10. Three-body configuration in Pascal's limaçon. The center of mass is located at the origin and the three bodies always lie in a line. However, two bodies at the crossing inevitably collide.

8. Lemniscatae

Lemniscatae can be obtained in two ways from the isometric cusphere surface.

8.1. *Lemniscata with constant additive director component*

One possibility is to consider a constant x or y director component, say $y = \cos\varphi_x \sin\varphi_y = c_1$. The magnitude squared is then

$$\left(s^2 + x^2\right)\left(s^2 + c_1^2\right) = s^2. \tag{34}$$

Plots of this curve are shown in Figure 11. For $c_1 = 0$, the curve becomes a circle, for $c_1 \ll 1$, the curves look like wings of butterflies, Papalotl in Nahuatl language. As c_1 approaches one, lemniscatae-like curves are obtained. The scator 11(b), written in terms of the φ_x parameter, is

$$\overset{o}{\varphi} = \cos\varphi_x \sqrt{1 - c_1^2 \sec^2\varphi_x} + \sin\varphi_x \sqrt{1 - c_1^2 \sec^2\varphi_x}\, \check{\mathbf{e}}_x + c_1\, \check{\mathbf{e}}_y. \tag{35}$$

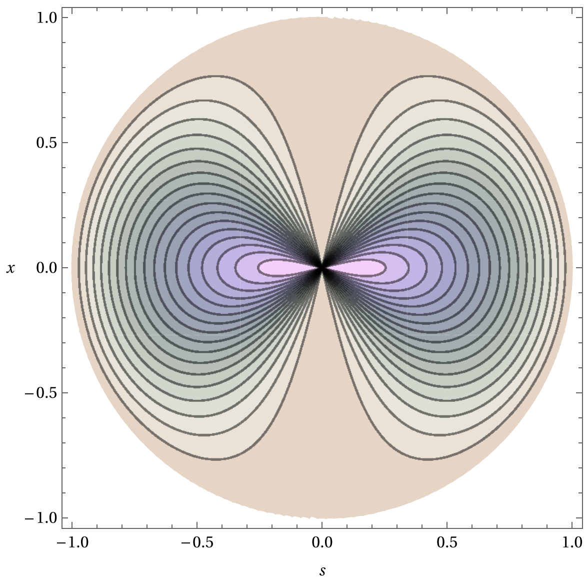


Fig. 11. Papálotl–lemniscata curves for different values of $0 \le c_1 \le 1$.

The implicit equation of the curve is $\left(x^2+y^2\right)\left(x^2+c_1^2\right)=x^2$. The curves can be prevented from becoming ever smaller as $c_1 \to 1$ by an adequate normalization so that its width remains constant, $\left(x^2+y^2\right)\left(x^2\left(1-c_1^2\right)+c_1^2\right)=x^2$. From (35), the coefficients are real for $0 \le |\varphi_x| \le |\arccos c_1|$. As far as generating the lemniscata curve is concerned, this poses no problem. However, if the parameter is used to describe the trajectory of a body, no position is obtained in the intervals where $|\varphi_x| > |\arccos c_1| \mod 2\pi$.

8.2. *Lemniscata from* $\check{\mathbf{e}}_x, \check{\mathbf{e}}_y$ *projection with* $\varphi_y = 3\varphi_x$

The other approach comes from the projection of (7) in the $\check{\mathbf{e}}_x, \check{\mathbf{e}}_y$ plane. Substitution of s from (32b) in the scator magnitude (15) gives

$$1=\frac{x^2\left(x-3y\right)}{3x-y}\left(1+\frac{3x-y}{x-3y}\right)\left(1+\frac{y^2\left(3x-y\right)}{x^2\left(x-3y\right)}\right),$$

that can be rearranged to obtain the implicit scator lemniscata equation

$$\left(3x-y\right)\left(x-3y\right)=4\left(x-y\right)^4. \tag{36}$$

The parametric curve is given by the $\check{\mathbf{e}}_x, \check{\mathbf{e}}_y$ components,

$$\left.\overset{o}{\varphi}\right|_{\check{\mathbf{e}}_x,\check{\mathbf{e}}_y}=\cos\left(\varphi\right)\cos\left(3\varphi\right)+\cos\left(3\varphi\right)\sin\left(\varphi\right)\check{\mathbf{e}}_x+\cos\left(\varphi\right)\sin\left(3\varphi\right)\check{\mathbf{e}}_y.$$

This lemniscata is rotated $\frac{\pi}{4}$ with respect to the coordinate axes (see Figure 8). A Euclidean rotation in the $\check{\mathbf{e}}_x, \check{\mathbf{e}}_y$ plane, $x=\frac{1}{\sqrt{2}}\left(x'+y'\right),\ y=\frac{1}{\sqrt{2}}\left(y'-x'\right)$, involves the substitutions $x-y \to \sqrt{2}x'$, $3x-y \to \sqrt{2}\left(2x'+y'\right)$and $x-3y \to \sqrt{2}\left(2x'-y'\right)$. The implicit equation is then $y'^2=4x'^2\left(1-2x'^2\right)$. Scaling by $y' \to \sqrt{2}y$, $x' \to \frac{x}{\sqrt{2}}$, the polynomial becomes $y^2=x^2\left(1-x^2\right)$, which is the canonical form of the Gerono/Huygens lemniscata. These curves are shown in Figure 12. The Euclidean $\frac{\pi}{4}$ rotation in the parametric variable is

$$x'=\frac{1}{\sqrt{2}}\left(\cos 3\varphi\sin\varphi-\cos\varphi\sin 3\varphi\right)=-\frac{1}{\sqrt{2}}\sin 2\varphi.$$

$$y'=\frac{1}{\sqrt{2}}\left(\cos\varphi\sin 3\varphi+\cos 3\varphi\sin\varphi\right)=\frac{1}{\sqrt{2}}\sin 4\varphi.$$

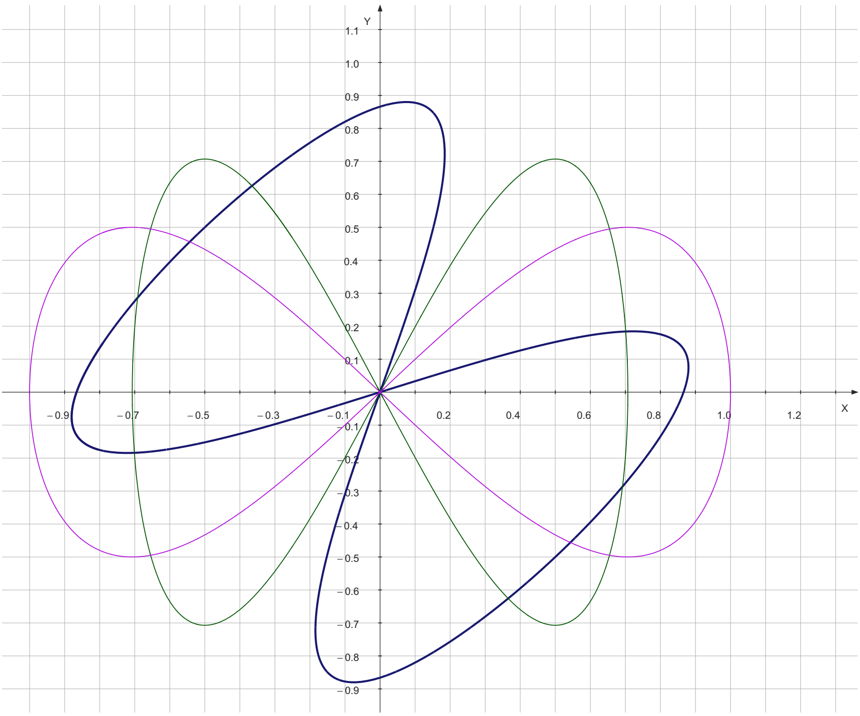


Fig. 12. Lemniscatae in $\check{\mathbf{e}}_x, \check{\mathbf{e}}_y$ plane, scator lemniscata given by Eq. (36) (thick black), $\frac{\pi}{4}$ rotated (green), and scaled by $y' \to \sqrt{2}y$, $x' \to \frac{x}{\sqrt{2}}$, reproducing the Gerono lemniscata (magenta).

The rotated scator is

$$\overset{o}{\varphi}{}' = \cos\varphi \cos 3\varphi \mp \frac{1}{\sqrt{2}} \sin(2\varphi)\ \check{\mathbf{e}}_x + \frac{1}{\sqrt{2}} \sin(4\varphi)\ \check{\mathbf{e}}_y.$$

The minus or plus sign is introduced to allow for the parameter increment to draw the figure counterclockwise or clockwise, respectively. This rotation has to be handled with great care in scator space because it is not scator magnitude invariant. The unrotated scator magnitude equation (15) with rotated variables is

$$s^2 = s^4 + s^2\left(x'^2 + y'^2\right) + \frac{1}{4}\left(x'^2 - y'^2\right)^2. \tag{37}$$

It is clear that $\overset{o}{\varphi}{}' = s + x'\,\check{\mathbf{e}}_x + y'\,\check{\mathbf{e}}_y$ no longer satisfies an expression of the form $s^2 = \left(s^2 + x'^2\right)\left(s^2 + y'^2\right)$ but (37). The $\check{\mathbf{e}}_x, \check{\mathbf{e}}_y$ components of the rotated scator are

$$\overset{o}{\varphi}{}'\Big|_{\check{\mathbf{e}}_x,\check{\mathbf{e}}_y} = \mp\frac{1}{\sqrt{2}}\sin(2\varphi)\ \check{\mathbf{e}}_x + \frac{1}{\sqrt{2}}\sin(4\varphi)\ \check{\mathbf{e}}_y.$$

The π periodicity for $m = 3$, mentioned in Section 5, is explicit in the even arguments of this expression. The parameter is scaled to $\phi = 2\varphi$ to recover the usual 2π periodicity. Recall that the scator products of elements with vanishing scalar component (necessarily with a single director component in order to belong to the $\mathbb{S}^{1+2}$ where the product is defined) is $\check{\mathbf{e}}_x\check{\mathbf{e}}_x = \check{\mathbf{e}}_y\check{\mathbf{e}}_y = -1$ and $\check{\mathbf{e}}_x\check{\mathbf{e}}_y = 0$. In contrast, vectors do not have a scalar component and three different products are defined: product of a scalar with a vector, and inner and outer products between vector elements. The vector set is not a subset of the scator set. However, the real $\mathbb{R} = \mathbb{S}^{1+0}$ and complex $\mathbb{C} = \mathbb{S}^{1+1}$ sets are subsets of the scator set $\mathbb{S}^{1+0} \subset \mathbb{S}^{1+1} \subset \mathbb{S}^{1+2}$. Nonetheless, it is possible to map the scator coefficients onto a vector set, $s + x\,\check{\mathbf{e}}_x + y\,\check{\mathbf{e}}_y \mapsto x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + s\,\hat{\mathbf{e}}_z$.

9. Dynamics of the Gerono-Type Lemniscata

The vector representing the parametric representation of the Gerono type Lemniscata is

$$\mathbf{r} = -\sin\phi\,\hat{\mathbf{e}}_x + \sin(2\phi)\ \hat{\mathbf{e}}_y,$$

where the usual unit vectors (with hats) instead of checks are used. The angle θ, measured counterclockwise from the abscissas, in terms of the parameter ϕ is

$$\tan\theta = \frac{\sin(2\phi)}{-\sin\phi}.$$

The positions of the three bodies are

$$\mathbf{r}_1 = -\sin\phi\,\hat{\mathbf{e}}_x + \sin(2\phi)\ \hat{\mathbf{e}}_y, \tag{38a}$$

$$\mathbf{r}_{2,3} = \left(\frac{1}{2}\sin\phi \mp \frac{\sqrt{3}}{2}\cos\phi\right)\hat{\mathbf{e}}_x + \left(-\frac{1}{2}\sin(2\phi) \mp \frac{\sqrt{3}}{2}\cos(2\phi)\right)\hat{\mathbf{e}}_y, \tag{38b}$$

where the upper sign corresponds to $\mathbf{r}_2$ and the lower sign to $\mathbf{r}_3$. Evaluation of the sum of (38a) and (38b) shows that the center of mass is located at the origin for all times, $\mathbf{r}_{\rm cm} = m\sum_i^3 \mathbf{r}_i = 0$. The moment of inertia is

$$I = m\sum_i^3 \mathbf{r}_i \cdot \mathbf{r}_i = 3\,m. \tag{39}$$

Since the center of mass is zero, the sum of distances between bodies is equal to $\frac{3I}{m}$,

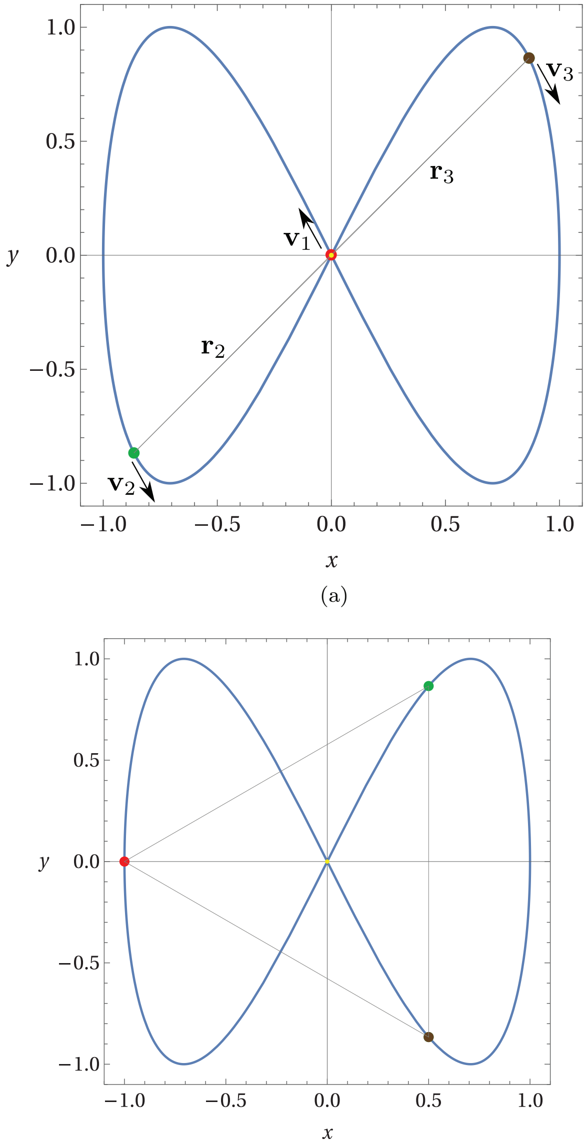
$$\sum_{i<j} |\mathbf{r}_i - \mathbf{r}_j|^2 = \frac{1}{m} 3m \sum \mathbf{r}_i \cdot \mathbf{r}_i = 9. \tag{40}$$

For $\phi = 0$, $\mathbf{r}_1 = 0$ and $\mathbf{r}_{2,3} = \mp\left(\frac{\sqrt{3}}{2}\,\hat{\mathbf{e}}_x + \frac{\sqrt{3}}{2}\,\hat{\mathbf{e}}_y\right)$, $\tan\theta_{2,3} = \mp 1$. The three bodies lie in an Euler configuration, co-linear in a line with $\frac{\pi}{4}$ slope, as shown in Figure 13. The Euler configuration is attained again for $\phi = \frac{\pi}{3}$, but the slope is then $-\frac{\pi}{4}$ and the body 2 sits at the origin. The pattern is repeated after $\frac{\pi}{3}$, with alternating slopes as well as the body located at the origin. For $\phi = \frac{\pi}{2}$, the bodies lie in the vertices of an equilateral triangle, $\mathbf{r}_1 = -1$ and $\mathbf{r}_{2,3} = \frac{1}{2}\,\hat{\mathbf{e}}_x \mp \frac{\sqrt{3}}{2}\,\hat{\mathbf{e}}_y$, $\tan\theta_{2,3} = \mp\frac{\pi}{3}$.

From the three curves, trifolium, limaçon and lemniscata, only the lemniscata is invariant under inversion. The following theorem, due to Ozaki,[13] can then be used to find the position of the three points given one of them:

Theorem (Construction of three points). If a curve γ in $\mathbb{R}^d$ with $d = 2, 3, 4, \ldots$ is invariant under the inversion $q \mapsto -q$, then the set $\{\{q_1, q_2\}\,|q_1, q_2 \in \gamma, q_1 + q_2 + q_3 = 0\}$ for a given $q_3 \in \gamma$ is equal to the set $\left\{\{q, q^*\}\,|q \in \gamma \cap \gamma_{\|}\right\}$ where $\gamma_{\|}$ is the parallel translation $q \mapsto q - q_3$ of the curve γ and $q^* = -q - q_3$.

Consider the point $\mathbf{r}_1 = -1$ (q_3 in Theorem (Construction of three points)), shifting this point to the origin, the intersection of the curves, γ and $\gamma_{\|}$, give the other two positions, as illustrated in Figure 14.



(b)

Fig. 13. Three bodies in a Gerono-type lemniscata. (a) Euler configuration, total angular momentum is zero (b) Equilateral triangle configuration with $\sqrt{3}$ per side.

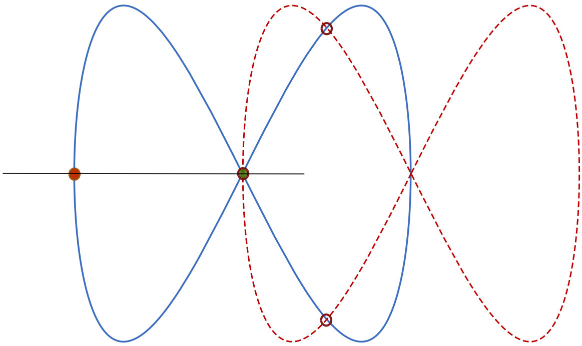


Fig. 14. Construction of three points using Ozaki's theorem.[13] Body 1 (leftmost in red) is displaced to the origin with a parallel translation of the lemniscata curve. The intersections between the two curves give the positions of the other two bodies (equal to configuration in Figure 13(b)).

The velocities of the three bodies, obtained by differentiation of (38a),(38b), are

$$\mathbf{v}_1 = -\cos\phi\,\dot{\phi}\,\hat{\mathbf{e}}_x + 2\cos(2\phi)\ \dot{\phi}\,\hat{\mathbf{e}}_y, \tag{41}$$

and

$$\mathbf{v}_{2,3} = \left(\frac{1}{2}\cos\phi \pm \frac{\sqrt{3}}{2}\sin\phi\right)\dot{\phi}\,\hat{\mathbf{e}}_x + \left(-\cos(2\phi) \pm \sqrt{3}\sin(2\phi)\right)\dot{\phi}\,\hat{\mathbf{e}}_y, \tag{42}$$

The kinetic energy of the system is

$$\sum_i^3 \mathbf{v}_{ix}\cdot\mathbf{v}_{ix} = \frac{3}{2}\dot{\phi}^2, \quad \sum_i^3 \mathbf{v}_{iy}\cdot\mathbf{v}_{iy} = 6\dot{\phi}^2, \quad \sum_i^3 \mathbf{v}_i\cdot\mathbf{v}_i = \frac{15}{2}\dot{\phi}^2. \tag{43}$$

If the parameter ϕ is linear in time, the kinetic energy is conserved, as well as the kinetic energies in each direction. Recall that the sum of velocities in each direction is also constant since the center of mass is constant.

9.1. *Angular momentum*

For the three-body problem, the shape of the orbit, conservation of the center of mass and a constant of motion (the angular momentum or the total energy) determine the motion of the three bodies.[13] Let us determine the angular momentum of the three bodies to describe the motion,

$$(\mathbf{r}_1 \times \mathbf{v}_1)_z = (-2\sin(\phi)\cos(2\phi) + \cos(\phi)\sin(2\phi))\,\dot{\phi}$$

and

$$(\mathbf{r}_{2,3} \times \mathbf{v}_{2,3})_z = \left(\frac{1}{4}\sin\phi\cos(2\phi) - \frac{5}{4}\cos\phi\sin(2\phi)\right)\dot{\phi} \pm \frac{3\sqrt{3}}{4}(\cos\phi\cos(2\phi) + \sin\phi\sin(2\phi))\,\dot{\phi}.$$

The total angular momentum L of the system is then

$$L = \sum_i^3 (\mathbf{r}_i \times \mathbf{v}_i)_z = -\frac{3}{2}\sin(3\phi)\,\dot{\phi}. \tag{44}$$

Therefore, the angular momentum for a linear parameter ϕ is not constant. The same issue is present in the two-body problem (Section 4.3). If L is requested to be constant, upon integration

$$\phi = \frac{1}{3}\arccos(2L\,t), \quad \dot{\phi} = -\frac{1}{3}\frac{2L}{\sqrt{1-4L^2t^2}}.$$

In contrast with the two-body solution, no periodic motion can be achieved with constant angular momentum. This impossibility is due to the fact that $\sin(3\phi)$ is zero for $\frac{\pi}{3}$ mod π. At those points, the body at the origin obviously has zero momentum, and the other two have opposite positions but the same velocity

$$\mathbf{r}_2 = -\mathbf{r}_3 = -\frac{\sqrt{3}}{2}\,\hat{\mathbf{e}}_x - \frac{\sqrt{3}}{2}\,\hat{\mathbf{e}}_y, \qquad \mathbf{v}_2 = \mathbf{v}_3 = \left(\frac{1}{2}\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_y\right)\dot{\phi}.$$

Their momenta are thus opposite $(\mathbf{r}_{2,3} \times \mathbf{v}_{2,3})_z = \pm\frac{3\sqrt{3}}{4}\dot{\phi}$ and their sum is zero, as graphically seen in Figure 13(a). A constant L, (Eq. (44)), then requires a nonphysical infinite parameter velocity $\dot{\phi}$. On the other hand, it is not possible to impose a zero angular momentum condition, $L = 0$, because then ϕ must be constant.

10. Conclusions

Scator algebra provides an adequate framework to describe geometrical objects and trajectories in $1+n$-dimensional spaces. This algebra is endowed with sum and product operations as well as a main second-order involution. It has two representations, additive and multiplicative that correspond to the rectangular and polar representations in complex algebra. These representations provide a natural parametrization of the trajectories when the additive components $(s; x, y)$ are written in terms of the multiplicative angle variables (φ_x, φ_y). Scator elements in $\mathbb{S}^{1+2}$ are represented geometrically in three-dimensional Euclidean space, letting one axis represent the scalar component and the other two axes represent the director components.

Important 2. Cusphere: The cusphere is the scator isometric surface in $\mathbb{S}^{1+2}$ where various curves have been described.

Orthographic projections of the curves lying on the cusphere recreate ellipses, circles and lines when one of the two director angles is considered constant. The projections reproduce the familiar trifolium rose, Pascal's limaçon and Gerono's lemniscata plane curves, when the director angles are in a relationship of 1:3.

Important 3. cexp: The components exponential function is a scator function $\overset{o}{\text{cexp}} \in \mathbb{S}^{1+n}$ of scator variable $\overset{o}{\zeta} \in \mathbb{R}^{1+n}$ that generalizes the complex exponential to higher dimensions.

It maps a constant scalar component argument into a constant magnitude surface, that is, a cusphere. Lines in $\overset{o}{\zeta} \in \mathbb{S}^{1+2}$ parallel to one director axis are mapped into ellipses. In $\mathbb{S}^{1+2}$ geometry, the parameter φ_y represents the angle between the scalar and the $\check{\mathbf{e}}_y$ axes. However, in elliptical trajectories, it also represents the eccentric anomaly angle in the $(\check{\mathbf{e}}_x, \check{\mathbf{e}}_y)$ plane. In contrast with the real and complex formalisms, the ellipse in $\mathbb{S}^{1+2}$ has constant scator magnitude along the trajectory, the two points when the curve crosses the $s = 0$ plane being exceptional points. Being a unitary continuous transformation, the scator formalism provides a simple and straightforward description of the elliptical planetary orbit.

The $\overset{o}{\text{cexp}}$ functions map lines with slope three onto the $\mathcal{C}_{13}$ line that produce the trifolium rose, trisectrix limaçon and Gerono-type lemniscatae in the mutually orthogonal axes planes. Symmetrical placement of three bodies with respect to the φ parameter insures that the center of mass is located at the origin for all times. Three-body choreographies in the trifolium rose and the trisectrix limaçon trajectories have a constant center of mass but exhibit collisions that make them unsuitable in two-dimensional plane trajectories. The Gerono-type lemniscata does not present collisions and has a constant sum of square distances. However, a constant angular momentum is not physically realizable.

The three-body choreography in the Bernoulli lemniscata has been successfully parameterized with elliptic functions by Fujiwara *et al.*[14] A comparison of relevant parameters between these two lemniscatae is shown in Table 1. The main departure comes from the angular momentum, it is null for the Bernoulli lemniscata parameterized with elliptic functions, whereas it is not constant for the Gerono lemniscata parameterized with trigonometric functions. A variable angular momentum is usually compensated by an appropriate dependence of the parameter with respect to time. However, in contrast with the elliptic trajectory, the angular momentum is zero for the Euler configuration and finite for other configurations. Thus, a function without singularities is not realizable in this case. Nonetheless, it should be possible to propose other scator curves where the angular momentum expression is never zero (or always zero) throughout the trajectory. In that case, the time dependence of the parameter

Table 1. Comparison of some relevant quantities in the Bernoulli and Gerono Lemniscatae.

	Bernoulli lemniscata[14]	Gerono-type lemniscata
I	$\sqrt{3}$	3
cm	0	0
$\sum_{i<j} \lvert \mathbf{r}_i - \mathbf{r}_j \rvert^2$	$3\sqrt{3}$	9
$\sum_i^3 \mathbf{v}_i \cdot \mathbf{v}_i$	$\frac{3}{4}$	$\frac{15}{2}\dot{\phi}^2$
$\sum_i^3 (\mathbf{r}_i \times \mathbf{v}_i)_z$	0	$-\frac{3}{2}\sin(3\phi)\,\dot{\phi}$
Parameterize	Jacobian elliptic functions	Trigonometric products

as a function of time could be physically attainable. Rational relationships between the φ_x and φ_y angle parameters produce closed curves. A constant φ_x with two bodies and a 3:1 relation with three bodies has been explored here. A wealth of other trajectories, projections and number of bodies is clearly possible. Some of them look remarkably similar to the numerical trajectories obtained by Simó.[15]

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Chapter 10

Caristi-Type Nonunique Fixed-Point Results and Fixed-Circle Problem on $b_v(s)$-Metric Spaces

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Fixed-point theory has been comprehensively studied with several methods. One of these methods is to generalize the used metric space such as $b_v(s)$-metric spaces. Another method is to analyze the geometric features of the fixed-point set. In the light of these methods, in this chapter, we prove Caristi's fixed-point theorem and new fixed-figure theorems in $b_v(s)$-metric spaces. We present some examples to emphasize the significance of geometrical results. To further strengthen the obtained theoretical results, we establish an application to S-Shaped Rectified Linear Unit ($SReLU$) activation functions.

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1. Introduction and Preliminaries

Fixed-point theory has been a very popular theory in mathematics and sciences. It has a great variety of applications to various kind of problems arise in diverse fields. Existence and uniqueness problems of fixed points are practical in many areas.[1–12]

Since the notion of metric space is a very useful tool for many areas of mathematics, researchers generalize this and then try to find corresponding findings in generalized cases. The notion of b-metric space is the most eminent generalized metric space which was defined in 1989.

Definition 1 (Ref. 13). Let $\mathfrak{E}$ be a nonempty set and $s \geq 1$ a real number. A function $\mathfrak{b} : \mathfrak{E} \times \mathfrak{E} \to [0, \infty)$ is a b-metric if, for all $\mathfrak{e}, \mathfrak{a}, \mathfrak{z} \in \mathfrak{E}$, the following hold:

(b_1) $\mathfrak{b}(\mathfrak{e}, \mathfrak{a}) = 0$ iff $\mathfrak{e} = \mathfrak{a}$,
(b_2) $\mathfrak{b}(\mathfrak{e}, \mathfrak{a}) = \mathfrak{b}(\mathfrak{a}, \mathfrak{e})$,
(b_3) $\mathfrak{b}(\mathfrak{e}, \mathfrak{z}) \leq s[\mathfrak{b}(\mathfrak{e}, \mathfrak{a}) + \mathfrak{b}(\mathfrak{a}, \mathfrak{z})]$.

The pair $(\mathfrak{E}, \mathfrak{b})$ is said to be a b-metric space.

A b-metric space is a generalization of a metric space. With this new concept, mathematicians have obtained numerous fixed-point results on various mappings in this space (see Refs. 14–19).

Definition 2 (Ref. 20). A function $f : \mathfrak{E} \to \mathbb{R}$ is called lower semi-continuous mapping if for any $\{\mathfrak{e}_n\} \subset \mathfrak{E}$ and $\mathfrak{e} \in \mathfrak{E}$

$$\mathfrak{e}_n \to \mathfrak{e} \quad \Rightarrow \quad f(\mathfrak{e}) \leq \lim_{n \to \infty} \inf(f(\mathfrak{e}_n)).$$

After b-metric spaces, some generalizations of this space such as extended b-metric space, partial b-metric space, dislocated b-metric space, and rectangular b-metric space[21] were investigated. The most recent type generalized b-metric space was given in 2017. This metric space which is called $b_v(s)$-metric space is very important because it is the most complicated and generalized version of metric space.

Definition 3 (Ref. 22). Let $\mathfrak{E}$ be a nonempty set, $\mathfrak{b}_{\mathfrak{v}} : \mathfrak{E} \times \mathfrak{E} \to [0, \infty)$ a function and $v \in \mathbb{N}$. d is called a $b_v(s)$-metric space if there is a real number $s \geq 1$ such that for all $\mathfrak{e}, \mathfrak{z} \in \mathfrak{E}$ and for all distinct points $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_v \in \mathfrak{E}$, each of them different from $\mathfrak{e}$ and $\mathfrak{z}$ the following hold:

(b_{v_1}) $\mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \mathfrak{a}) = 0$ iff $\mathfrak{e} = \mathfrak{a}$,
(b_{v_2}) $\mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \mathfrak{a}) = \mathfrak{b}_\mathfrak{v}(\mathfrak{a}, \mathfrak{e})$,
(b_{v_3}) $\mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \mathfrak{z}) \leq s[\mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \mathfrak{a}_1) + \mathfrak{b}_\mathfrak{v}(\mathfrak{a}_1, \mathfrak{a}_2) + \ldots + \mathfrak{b}_\mathfrak{v}(\mathfrak{a}_v, \mathfrak{z})]$.

The pair $(\mathfrak{E}, \mathfrak{b}_\mathfrak{v})$ is said to be a $b_v(s)$-metric space.

Definition 4 (Ref. 22). Let $(\mathfrak{E}, \mathfrak{b}_\mathfrak{v})$ be a $b_v(s)$-metric space, $\{\mathfrak{e}_n\}$ a sequence in $\mathfrak{E}$ and $\mathfrak{e} \in \mathfrak{E}$.

(i) $\{\mathfrak{e}_n\}$ is convergent in $(\mathfrak{E}, \mathfrak{b}_\mathfrak{v})$ and converges to $\mathfrak{e}$, if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{b}_\mathfrak{v}(\mathfrak{e}_n, \mathfrak{e}) < \epsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} \mathfrak{e}_n = \mathfrak{e}$ or $\mathfrak{e}_n \to \mathfrak{e}$ as $n \to \infty$.
(ii) $\{\mathfrak{e}_n\}$ is a Cauchy sequence in $(\mathfrak{E}, \mathfrak{b}_\mathfrak{v})$ if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{b}_\mathfrak{v}(\mathfrak{e}_n, \mathfrak{e}_{n+p}) < \epsilon$ for all $n > n_0$, $p > 0$ or if $\lim_{n\to\infty} \mathfrak{b}_\mathfrak{v}(\mathfrak{e}_n, \mathfrak{e}_{n+p}) = 0$ for all $p > 0$.
(iii) $(\mathfrak{E}, \mathfrak{b}_\mathfrak{v})$ is called a complete $b_v(s)$-metric space if every Cauchy sequence in $\mathfrak{E}$ converges to some $\mathfrak{e} \in \mathfrak{E}$.

Example 1 (Ref. 23). *Define a mapping* $\mathfrak{b}_\mathfrak{v} : \mathfrak{E} \times \mathfrak{E} \to [0, \infty)$ *by*

$$\mathfrak{b}_\mathfrak{v}\left(\frac{1}{k}, \frac{1}{m}\right) = \begin{cases} |k-m|, & |k-m| \neq 1 \\ \dfrac{1}{2}, & |k-m| = 1 \end{cases}$$

on the set $\mathfrak{E} = \left\{\frac{1}{n} \mid n \in \mathbb{N},\ n \geq 2\right\}$. $(\mathfrak{E}, \mathfrak{b}_\mathfrak{v})$ *is a* $b_3(3)$*-metric space.*

$b_v(s)$-metric space generalizes v-generalized metric space, b-metric space, rectangular metric space and rectangular b-metric space. Banach and Reich fixed-point theorems are proved in Ref. 22. Abdullahi and Kumam[24] introduced partial $b_v(s)$-metric spaces. They provided topological features and proved some fixed-point results in this space. Khan and Dass–Gupta type fixed-point theorems are proved.[25] Aydi *et al.*[26] obtained some common fixed-point theorems in partial $b_v(s)$-metric spaces. For more details, see Refs. 27–32.

Next, we recall Caristi's fixed-point theorem.

Theorem 1 (Ref. 20). *Let* $(\mathfrak{E}, d)$ *be a complete metric space. Suppose that* $\xi : \mathfrak{E} \to \mathbb{R}^+$ *is lower semicontinuous and the mapping*

$\Gamma : \mathfrak{E} \to \mathfrak{E}$ *satisfies*

$$d(\mathfrak{e}, \Gamma\mathfrak{e}) \leq \xi(\mathfrak{e}) - \xi(\Gamma\mathfrak{e})$$

for every $\mathfrak{e} \in \mathfrak{E}$*. Then there is* $\mathfrak{e}_0 \in \mathfrak{E}$ *such that* $\Gamma\mathfrak{e}_0 = \mathfrak{e}_0$.

Let $(\mathbb{R}, d)$ be a usual metric space with the metric $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ defined by

$$d(\mathfrak{e}, \mathfrak{a}) = |\mathfrak{e} - \mathfrak{a}|,$$

for all $\mathfrak{e}, \mathfrak{a} \in \mathbb{R}$. Consider the following mapping $\Gamma_i : \mathbb{R} \to \mathbb{R}$ $(i \in \{1, 2, 3, 4\})$ as

$$\Gamma_1\mathfrak{e} = 2\mathfrak{e} + \frac{3}{2},$$
$$\Gamma_2\mathfrak{e} = \mathfrak{e}^3 - 5\mathfrak{e}^2 + 5,$$
$$\Gamma_3\mathfrak{e} = \mathfrak{e} + 100$$

and

$$\Gamma_4\mathfrak{e} = \mathfrak{e},$$

for all $\mathfrak{e} \in \mathbb{R}$. Then we have the fixed-point sets of Γ_i, $Fix(\Gamma_i) = \{\mathfrak{e} \in \mathbb{R} : \mathfrak{e} = \Gamma_i\mathfrak{e}\}$ such that

$$Fix(\Gamma_1) = \left\{\frac{3}{2}\right\}, \; Fix(\Gamma_2) = \{-1, 1, 5\},$$
$$Fix(\Gamma_3) = \emptyset \text{ and } Fix(\Gamma_4) = \mathbb{R}.$$

From the above examples, we say that the number of fixed points of a self-mapping can change. Just in this case, fixed-circle problem has been shown up as a geometric generalization of fixed-point theory.[33] The fundamental point in this problem is to increase the number of fixed points. There are some advantages working this problem. We list a few of them:

- A geometric meaning can be attributed the fixed-point set $Fix(\Gamma)$ when the number of fixed points of Γ is more than one.
- The obtained results for this problem can be applied to some real-life problems.

- Under favor of this problem, some eminent fixed-point results can be generalized.

This problem has been studied from different perspectives. For example, in Ref. 34, some solutions of this problem were given using the Khan-type contractions.[35] In Ref. 36, techniques of Ćirić, Hardy–Rogers, Reich and Chatterjea were used to investigate some solutions on metric spaces (for more of these techniques, see Refs. 37–41). In Ref. 42, Mlaiki *et al.* used Wardowski's approach for fixed-circle problem on S-metric spaces. In Ref. 43, Özgür presented some fixed-disc results via simulation functions. Recently, this problem has been labored on metric and some generalized metric spaces (for more examples, see Refs. 44–47 and the references therein). Also, this problem has been a generalized fixed-figure problem.[48] In this context, fixed-ellipse, fixed-hyperbola, fixed-Cassini curve, fixed-Apollonious circle and fixed-k-ellipse theorems were obtained with diverse methods (see Refs. 48–52).

In this chapter, we get some fixed-point and fixed-figure results using the Caristi-type inequality in $b_v(s)$-metric spaces. The proved results generalize some known results in the literature and they are supported by necessary examples. Finally, we construct an application to *SReLU* activation functions.

2. Main Results

In this section, we first prove Caristi-type fixed-point theorems and investigate some solutions to the fixed-circle problem using the Caristi-type inequality in $b_v(s)$-metric spaces. The next definition is the modification of Definition 1.6 to $b_v(s)$-metric spaces in Ref. 53.

Definition 5. Let $(\mathfrak{E}, \mathfrak{b}_\mathfrak{v})$ be a $b_v(s)$-metric space and Γ a self-mapping on $\mathfrak{E}$. If the set

$$\left\{\mathfrak{a} \in \mathfrak{E} : \lim_i \Gamma^{m_i}\mathfrak{a} = u \Longrightarrow \lim_i \Gamma\Gamma^{m_i}\mathfrak{a} = \Gamma u\right\}$$

is not empty whenever $\{\mathfrak{e} \in \mathfrak{E} : \lim_i \Gamma^{m_i}\mathfrak{e} = u\}$ is nonempty set, then Γ is called weakly orbitally continuous.

In the following four theorems, we suppose that $(\mathfrak{E}, \mathfrak{b}_{\mathfrak{v}})$ is a complete $b_v(s)$-metric space and Γ a self-mapping on $\mathfrak{E}$.

Theorem 2. *Assume that* $\xi : \mathfrak{E} \to [0, \infty)$ *is a function such that for all* $\mathfrak{e}, \mathfrak{a} \in \mathfrak{E}$ *we have*

$$\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e}) \leq \xi(\mathfrak{e}) - \xi(\Gamma\mathfrak{e}). \tag{1}$$

If Γ *is weakly orbitally continuous, then* Γ *has a fixed point.*

Proof. Let $\mathfrak{e}_0 \in \mathfrak{E}$. Consider a sequence $\{\mathfrak{e}_n\}$ by $\mathfrak{e}_1 = \Gamma\mathfrak{e}_0,\ \mathfrak{e}_2 = \Gamma\mathfrak{e}_1, \ldots, \mathfrak{e}_n = \Gamma\mathfrak{e}_{n-1}, \ldots$, that is, $\mathfrak{e}_n = \Gamma^n\mathfrak{e}_0$. By inequality (1), we have

$$\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_0, \mathfrak{e}_1) = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_0, \Gamma\mathfrak{e}_0) \leq \xi(\mathfrak{e}_0) - \xi(\Gamma\mathfrak{e}_0) = \xi(\mathfrak{e}_0) - \xi(\mathfrak{e}_1).$$

In the same manner, we obtain the following inequalities:

$$\begin{aligned}
\mathfrak{b}_{\mathfrak{v}}(u_1, u_2) &\leq \xi(\mathfrak{e}_1) - \xi(\mathfrak{e}_2)\\
\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_2, \mathfrak{e}_3) &\leq \xi(\mathfrak{e}_2) - \xi(\mathfrak{e}_3)\\
\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_3, \mathfrak{e}_4) &\leq \xi(\mathfrak{e}_3) - \xi(\mathfrak{e}_4)\\
\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_4, \mathfrak{e}_5) &\leq \xi(\mathfrak{e}_4) - \xi(\mathfrak{e}_5)\\
&\vdots\\
d(\mathfrak{e}_{n-1}, \mathfrak{e}_n) &\leq \xi(\mathfrak{e}_{n-1}) - \xi(\mathfrak{e}_n)\\
\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1}) &\leq \xi(\mathfrak{e}_n) - \xi(\mathfrak{e}_{n+1}).
\end{aligned}$$

If we add the above inequalities, we obtain

$$\begin{aligned}
&\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_0, \mathfrak{e}_1) + \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_1, \mathfrak{e}_2) + \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_2, \mathfrak{e}_3) + \cdots + \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_{n-1}, \mathfrak{e}_n) + \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1})\\
&\quad \leq \xi(\mathfrak{e}_0) - \xi(\mathfrak{e}_{n+1}) \leq \xi(\mathfrak{e}_0).
\end{aligned}$$

Taking limit as $n \to \infty$, we find

$$\sum_{n=0}^{\infty} \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \leq \xi(\mathfrak{e}_0).$$

From the above inequality, $\{\mathfrak{e}_n\}$ is a Cauchy sequence. From the completeness of $\mathfrak{E}$, there is a point $r \in \mathfrak{E}$ such that

$$\lim_{n\to\infty} \mathfrak{e}_n = r \quad \text{and} \quad \lim_{n\to\infty} \Gamma^{\alpha}\mathfrak{e}_n = r$$

for every $\alpha \geq 1$. Assume that Γ is weakly orbitally continuous mapping. Since $\{\Gamma^n\mathfrak{e}_0\}$ converges for every $\mathfrak{e}_0 \in \mathfrak{E}$, weak orbital continuity implies that there exists $\mathfrak{w}_0 \in \mathfrak{E}$ such that $\Gamma^n \mathfrak{w}_0 \to \mathfrak{z}$ and $\Gamma^{n+1}\mathfrak{w}_0 \to \Gamma\mathfrak{z}$ for some $\mathfrak{z} \in \mathfrak{E}$. Taking limit as $n \to \infty$, we have $\mathfrak{z} = \Gamma\mathfrak{z}$, that is, Γ has the fixed point. Hence, the proof of this theorem is completed. □

Definition 6 (Ref. 54). Let Γ and Θ be self-mappings on a metric space $(\mathfrak{E}, d)$.

(i) A point $\mathfrak{e}_0 \in \mathfrak{E}$ is called a common fixed point of Γ and Θ if $\mathfrak{e}_0 = \Gamma\mathfrak{e}_0 = \Theta\mathfrak{e}_0$.
(ii) A point $\mathfrak{e}_0 \in \mathfrak{E}$ is said to be a coincidence point of Γ and Θ if $\mathfrak{e}_0 = \Gamma\mathfrak{e} = \Theta\mathfrak{e}$.
(iii) The mappings $\Gamma, \Theta : \mathfrak{E} \to \mathfrak{E}$ are called weakly compatible if $\Gamma\Theta\mathfrak{e} = \Theta\Gamma\mathfrak{e}$ whenever $\Gamma\mathfrak{e} = \Theta\mathfrak{e}$.

Theorem 3. *Let Γ, Θ be weakly compatible contractive self-mappings on $\mathfrak{E}$. Let $\xi : \mathfrak{E} \to \mathbb{R}^+$ be a lower semicontinuous function such that*

$$\xi\mathfrak{e} \leq \mathfrak{d}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \Theta\mathfrak{e}) \tag{2}$$

for all $\mathfrak{e} \in \mathfrak{E}$ satisfying

$$\mathfrak{d}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \Gamma\mathfrak{a}) \leq \xi(\Theta\mathfrak{e}) - \xi(\Gamma\mathfrak{e}) + \xi(\Theta\mathfrak{a}) - \xi(\Gamma\mathfrak{a}) \tag{3}$$

for all $\mathfrak{e}, \mathfrak{a} \in \mathfrak{E}$ with $\mathfrak{e} \neq \mathfrak{a}$. Then Γ and Θ have a unique common fixed point.

Proof. The proof will be given in two parts:

(Existence) First, we show the existence of common fixed point. Suppose

$$\Gamma\mathfrak{e}_{n-1} = \Theta\mathfrak{e}_n = \mathfrak{e}_n$$

for each $n \in \mathbb{N}$. Then, by (3) and the above equations, we have

$$\begin{aligned} \mathfrak{d}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1}) = \mathfrak{d}_{\mathfrak{v}}(\Gamma\mathfrak{e}_{n-1}, \Gamma\mathfrak{e}_n) &\leq \xi(\Theta\mathfrak{e}_{n-1}) - \xi(\Gamma\mathfrak{e}_{n-1}) + \xi(\Theta\mathfrak{e}_n) - \xi(\Gamma\mathfrak{e}_n) \\ &= \xi(\mathfrak{e}_{n-1}) - \xi(\mathfrak{e}_n) + \xi(\mathfrak{e}_n) - \xi(\mathfrak{e}_{n+1}) \\ &= \xi(\mathfrak{e}_{n-1}) - \xi(\mathfrak{e}_{n+1}) \\ &\leq \mathfrak{d}_{\mathfrak{v}}(\Gamma\mathfrak{e}_{n-1}, \Theta\mathfrak{e}_{n-1}) - \mathfrak{d}_{\mathfrak{v}}(\Gamma\mathfrak{e}_{n+1}, \Theta\mathfrak{e}_{n+1}). \end{aligned}$$

From the last statements, we find

$$\mathfrak{d}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+2}) \leq \mathfrak{d}_{\mathfrak{v}}(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \tag{4}$$

Γ and Θ are continuous because they are contractive mappings. From the continuity and boundedness from below of ξ, we conclude that $\mathfrak{e}_n$ is decreasing to a point $\mathfrak{e}_0 \in \mathfrak{E}$ and for all $\mathfrak{e} \in \mathfrak{E}$, $\xi(\mathfrak{e}_0) \leq \xi(\mathfrak{e})$.

Since $(\mathfrak{E}, \mathfrak{b}_\mathfrak{v})$ is a complete $b_v(s)$-metric space, then $\mathfrak{e}_n \to \mathfrak{e}_0$. On the other hand, there exists a subsequence of $\{\mathfrak{e}_n\}$ such that $\mathfrak{e}_{n_k} \to \mathfrak{e}_0$ for all $n, k \in \mathbb{N}$. Then we obtain

$$\lim_{n,k\to\infty} \mathfrak{e}_{n_k} = \Gamma\left(\lim_{n,k\to\infty} \mathfrak{e}_{n_k}\right) = \Gamma(\mathfrak{e}_0)$$

and

$$\lim_{n\to\infty} \mathfrak{e}_n = \Theta\left(\lim_{n\to\infty} \mathfrak{e}_n\right) = T(\mathfrak{e}_0).$$

Therefore, $\mathfrak{e}_0$ is a coincidence fixed point of Γ and Θ. Hence, $\mathfrak{e}_0$ is a common fixed point of Γ and Θ.

(Uniqueness) Assume that $\Gamma(\pi) = \Theta(\pi) = \pi$ for all $\pi \in \mathfrak{E}$ such that $\mathfrak{e}_0 \neq \pi$. From (3), we obtain

$$\begin{aligned}
\mathfrak{b}_\mathfrak{v}(\pi, \mathfrak{e}_0) &= d(\Gamma\pi, \Gamma\mathfrak{e}_0) \\
&\leq \xi(\Theta\pi) - \xi(\Gamma\pi) + \xi(\Theta\mathfrak{e}_0) - \xi(\Gamma\mathfrak{e}_0) \\
&\leq \xi(\pi) - \xi(\pi) + \xi(\mathfrak{e}_0) - \xi(\mathfrak{e}_0) \\
&\leq \mathfrak{b}_\mathfrak{v}(\Gamma\pi, \Theta\pi) - \mathfrak{b}_\mathfrak{v}(\Gamma\pi, \Theta\pi) + \mathfrak{b}_\mathfrak{v}(\Gamma\mathfrak{e}_0, \Theta\mathfrak{e}_0) - \mathfrak{b}_\mathfrak{v}(\Gamma\mathfrak{e}_0, \Theta\mathfrak{e}_0) \\
&\leq 0.
\end{aligned}$$

It means that $\pi = \mathfrak{e}_0$, that is, this fixed point is unique. □

In the following theorem, inspired by Ćirić-type contraction[39] and the used technique in Ref. 55, we use the number $C(\mathfrak{e}, \mathfrak{a})$ defined as

$$C(\mathfrak{e}, \mathfrak{a}) = \max\left\{\mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \mathfrak{a}), \mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \Gamma\mathfrak{e}), \mathfrak{b}_\mathfrak{v}(\mathfrak{a}, \Gamma\mathfrak{a}), \mathfrak{b}_\mathfrak{v}(\mathfrak{a}, \Gamma\mathfrak{e}), \mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \Gamma\mathfrak{a})\right\}.$$

Theorem 4. *Let $\varpi : \mathfrak{E} \to [0, \infty)$ be a lower semicontinuous and bounded below function. If the following condition holds for all $\mathfrak{e}, \mathfrak{a} \in \mathfrak{E}$*

$(b_v(s) - \mathfrak{MKCC})$. *Given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \Gamma\mathfrak{e}) > 0$ implies*

$$\varepsilon \leq [\varpi(\mathfrak{e}) - \varpi(\Gamma\mathfrak{e})]\, C(\mathfrak{e}, \mathfrak{a}) < \varepsilon + \delta \Longrightarrow \mathfrak{b}_\mathfrak{v}(\Gamma\mathfrak{e}, \Gamma\mathfrak{a}) < \varepsilon,$$

then given $\mathfrak{e} \in \mathfrak{E}$, *the sequence of iterates* $\{\Gamma^n \mathfrak{e}\}$ *is a Cauchy sequence and* $\lim_{n\to\infty} \Gamma^n \mathfrak{e} = \mathfrak{z}$ *for some* $\mathfrak{z} \in \mathfrak{E}$.

Proof. Using $(b_v(s) - \mathfrak{MKCC})$, we get that if $\mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \Gamma\mathfrak{e}) > 0$ then

$$\mathfrak{b}_\mathfrak{v}(\Gamma\mathfrak{e}, \Gamma\mathfrak{a}) < [\varpi(\mathfrak{e}) - \varpi(\Gamma\mathfrak{e})]\, C(\mathfrak{e}, \mathfrak{a}). \tag{5}$$

Let $\mathfrak{e}_0 \in \mathfrak{E}$ and let us define $\{\mathfrak{e}_n\}$ in $\mathfrak{E}$ by $\mathfrak{e}_n = \Gamma\mathfrak{e}_{n-1}$, that is,

$$\mathfrak{e}_n = \Gamma^n \mathfrak{e}_0.$$

If $\mathfrak{e}_n = \mathfrak{e}_{n+1}$ for some n then $\mathfrak{e}_n = \mathfrak{e}_{n+1} = \mathfrak{e}_{n+2} = \cdots$, that is, $\{\mathfrak{e}_n\} = \{\Gamma^n \mathfrak{e}\}$ is a Cauchy sequence and $\mathfrak{e}_n \in Fix(\Gamma)$. Suppose that $\mathfrak{e}_n \neq \mathfrak{e}_{n+1}$ for each n and $\mathfrak{c}_n = \mathfrak{b}_\mathfrak{v}(\mathfrak{e}_{n-1}, \mathfrak{e}_n)$. Using (5), we have

$$\begin{aligned}
\mathfrak{c}_{n+1} = \mathfrak{b}_\mathfrak{v}(\mathfrak{e}_n, \mathfrak{e}_{n+1}) &= \mathfrak{b}_\mathfrak{v}(\Gamma\mathfrak{e}_{n-1}, \Gamma\mathfrak{e}_n) < [\varpi(\mathfrak{e}_{n-1}) - \varpi(\mathfrak{e}_n)]\, C(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \\
&= [\varpi(\mathfrak{e}_{n-1}) - \varpi(\mathfrak{e}_n)] \max\{\mathfrak{b}_\mathfrak{v}(\mathfrak{e}_{n-1}, \mathfrak{e}_n), \mathfrak{b}_\mathfrak{v}(\mathfrak{e}_n, \mathfrak{e}_{n+1}), \mathfrak{b}_\mathfrak{v}(\mathfrak{e}_{n-1}, \mathfrak{e}_{n+1})\} \\
&= [\varpi(\mathfrak{e}_{n-1}) - \varpi(\mathfrak{e}_n)]\, \beta_N.
\end{aligned} \tag{6}$$

Case 1: If $\beta_N = \mathfrak{b}_\mathfrak{v}(\mathfrak{e}_{n-1}, \mathfrak{e}_n)$, then using (6), we obtain

$$\mathfrak{c}_{n+1} = \mathfrak{b}_\mathfrak{v}(\mathfrak{e}_n, \mathfrak{e}_{n+1}) < [\varpi(\mathfrak{e}_{n-1}) - \varpi(\mathfrak{e}_n)]\, \mathfrak{c}_n$$

and so

$$0 < \frac{\mathfrak{c}_{n+1}}{\mathfrak{c}_n} < \varpi(\mathfrak{e}_{n-1}) - \varpi(\mathfrak{e}_n),$$

for each $n \in \mathbb{N}$. So, $\{\varpi(\mathfrak{e}_n)\}$ is nonincreasing and positive hence it converges to some $t \geq 0$. For all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\sum_{m=1}^{n} \frac{\mathfrak{c}_{m+1}}{\mathfrak{c}_m} &< \sum_{m=1}^{n} [\varpi(\mathfrak{e}_{m-1}) - \varpi(\mathfrak{e}_m)] \\
&= \varpi(\mathfrak{e}_0) - \varpi(\mathfrak{e}_n) \to \varpi(\mathfrak{e}_0) - t < \infty \text{ as } n \to \infty
\end{aligned}$$

and

$$\sum_{m=1}^{n} \frac{\mathfrak{c}_{m+1}}{\mathfrak{c}_m} < \infty \Longrightarrow \lim_{n\to\infty} \frac{\mathfrak{c}_{n+1}}{\mathfrak{c}_n} = 0.$$

Therefore, for $\beta \in (0, 1)$, there is $n_0 \in \mathbb{N}$ such that

$$\frac{\mathfrak{c}_{n+1}}{\mathfrak{c}_n} \leq \beta \text{ for all } n \geq n_0$$

and we get

$$\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \leq \beta \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \text{ for all } n \geq n_0.$$

Case 2: If $\beta_N = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1})$, then using (6), we have

$$\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1}) < [\varpi(\mathfrak{e}_{n-1}) - \varpi(\mathfrak{e}_n)]\, \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1}).$$

By a similar approach used in Case 1, $\{\varpi(\mathfrak{e}_n)\}$ is a positive and nonincreasing sequence and so it converges to some $t \geq 0$. Since $\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1}) > 0$, we get

$$1 < \varpi(\mathfrak{e}_{n-1}) - \varpi(\mathfrak{e}_n) \to 0 \text{ as } n \to \infty,$$

a contradiction.

Case 3: If $\beta_N = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_{n-1}, \mathfrak{e}_{n+1})$, then using (6), we have

$$\begin{aligned}
\mathfrak{c}_{n+1} = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1}) &< [\varpi(\mathfrak{e}_{n-1}) - \varpi(\mathfrak{e}_n)]\, \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_{n-1}, \mathfrak{e}_{n+1}) \\
&\leq [\varpi(\mathfrak{e}_{n-1}) - \varpi(\mathfrak{e}_n)]\, (\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_{n-1}, \mathfrak{e}_n) + \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1})) \\
&= [\varpi(\mathfrak{e}_{n-1}) - \varpi(\mathfrak{e}_n)]\, (\mathfrak{c}_n + \mathfrak{c}_{n+1})
\end{aligned}$$

and so

$$0 < \frac{\mathfrak{c}_{n+1}}{\mathfrak{c}_n + \mathfrak{c}_{n+1}} < \varpi(\mathfrak{e}_{n-1}) - \varpi(\mathfrak{e}_n).$$

$\{\varpi(\mathfrak{e}_n)\}$ converges to some $t \geq 0$ since it is a positive and nonincreasing sequence. For all $n \in \mathbb{N}$, we obtain

$$\begin{aligned}
\sum_{m=1}^{n} \frac{\mathfrak{c}_{m+1}}{\mathfrak{c}_m + \mathfrak{c}_{m+1}} &< \sum_{m=1}^{n} [\varpi(\mathfrak{e}_{m-1}) - \varpi(\mathfrak{e}_m)] \\
&= \varpi(\mathfrak{e}_0) - \varpi(\mathfrak{e}_n) \to \varpi(\mathfrak{e}_0) - t < \infty \text{ as } n \to \infty
\end{aligned}$$

and

$$\sum_{m=1}^{n} \frac{\mathfrak{c}_{m+1}}{\mathfrak{c}_m + \mathfrak{c}_{m+1}} < \infty \Longrightarrow \lim_{n\to\infty} \frac{\mathfrak{c}_{n+1}}{\mathfrak{c}_n + \mathfrak{c}_{n+1}} = 0.$$

Hence, for $\beta' \in \left(0, \frac{1}{2}\right)$, there is $n_0 \in \mathbb{N}$ such that

$$\frac{\mathfrak{c}_{n+1}}{\mathfrak{c}_n + \mathfrak{c}_{n+1}} \leq \beta' \text{ for all } n \geq n_0$$

and we have

$$\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1}) \leq \beta \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_{n-1}, \mathfrak{e}_n) \text{ for all } n \geq n_0,$$

where $\beta = \frac{\beta'}{1-\beta'}$.

Now, we show that $\{\mathfrak{e}_n\}$ is a Cauchy sequence and $\{\mathfrak{e}_n\}$ converges to some $\mathfrak{z} \in \mathfrak{E}$. By the above cases, we say that the sequence $\{\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_n, \mathfrak{e}_{n+1})\}$ is bounded below and nonincreasing. Hence, it converges to some $\mathfrak{e} \geq 0$. Since $\beta < 1$, we easily prove $\mathfrak{e} = 0$. For each $n_1, n_2 \in \mathbb{N}$ $(n_1 > n_2)$, we get

$$\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_{n_1}, \mathfrak{e}_{n_2}) \leq \sum_{m=n_2}^{n_1-1} \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_m, \mathfrak{e}_{m+1}) \leq \frac{\beta^{n_2}}{1-\beta} \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_0, \mathfrak{e}_1),$$

that is,

$$\lim_{n\to\infty} \sup \{\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_{n_1}, \mathfrak{e}_{n_2}) : n_1 > n_2\} = 0.$$

Consequently, $\{\mathfrak{e}_n\}$ is Cauchy and there is $\mathfrak{z} \in \mathfrak{E}$ such that $\{\mathfrak{e}_n\} \to \mathfrak{z}$ since $(\mathfrak{E}, \mathfrak{b}_{\mathfrak{v}})$ is a complete $b_v(s)$-metric space. □

Recently, some fixed-point results have been obtained using the notion of k-continuity with different approaches (for example, see Refs. 56, 57 and the references therein).

A self-mapping $\Gamma : \mathfrak{E} \to \mathfrak{E}$ is called $k_{b_v(s)}$-continuous, $k = 1, 2, 3, \ldots$, if $\Gamma^k \mathfrak{e}_n \to \Gamma \mathfrak{t}$ whenever $\{\mathfrak{e}_n\}$ is a sequence in $\mathfrak{E}$ such that $\Gamma^{k-1} \mathfrak{e}_n \to \mathfrak{t}$.

Theorem 5. *Let $\Gamma : \mathfrak{E} \to \mathfrak{E}$ be a mapping satisfying $(b_v(s) - \mathfrak{MKCC})$ given in Theorem 4. If Γ is $k_{b_v(s)}$-continuous, then Γ has a fixed point $\mathfrak{z}$ in $\mathfrak{E}$, that is, $\mathfrak{z} \in Fix(\Gamma)$.*

Proof. Let $\mathfrak{e}_0 \in \mathfrak{E}$ and let us define a sequence $\{\mathfrak{e}_n\}$ in $\mathfrak{E}$ by $\mathfrak{e}_n = \Gamma \mathfrak{e}_{n-1}$, that is,

$$\mathfrak{e}_n = \Gamma^n \mathfrak{e}_0.$$

Using Theorem 4, we say that $\{\mathfrak{e}_n\}$ is Cauchy. So, there is a point $\mathfrak{z} \in \mathfrak{E}$ such that $\{\mathfrak{e}_n\} \to \mathfrak{z}$ since $(\mathfrak{E}, \mathfrak{b}_{\mathfrak{v}})$ is a complete metric space. Also, we have $\Gamma^p \mathfrak{e}_n \to \mathfrak{z}$ for each $p \geq 1$.

Let Γ be a $k_{b_v(s)}$-continuous self-mapping. $k_{b_v(s)}$-continuity of Γ implies that $\Gamma^k \mathfrak{e}_n \to \Gamma \mathfrak{z}$ since $\Gamma^{k-1} \mathfrak{e}_n \to \mathfrak{z}$ and so we get $\Gamma \mathfrak{z} = \mathfrak{z}$

as $\Gamma^k \mathfrak{e}_n \to \mathfrak{z}$ from the uniqueness of the limit point. Therefore, $\mathfrak{z} \in Fix(\Gamma)$. □

Now, we present new fixed-figure results on a $b_v(s)$-metric space. To do this, we are inspired by the Caristi-type inequality.[20] Let us give the following notions such as a circle and a disc:

Definition 7. Let $(\mathfrak{E}, \mathfrak{b}_\mathfrak{v})$ be a $b_v(s)$-metric space, $r \geq 0$ and $\mathfrak{e}_0 \in \mathfrak{E}$. Then the circle $\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_0,r}$ is defined by

$$\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_0,r} = \{\mathfrak{e} \in \mathfrak{E} : \mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \mathfrak{e}_0) = r\}$$

and the disc $\mathfrak{D}^{b_v(s)}_{\mathfrak{e}_0,r}$ is defined by

$$\mathfrak{D}^{b_v(s)}_{\mathfrak{e}_0,r} = \{\mathfrak{e} \in \mathfrak{E} : \mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \mathfrak{e}_0) \leq r\}.$$

Example 2. Let $\mathfrak{E} = \mathbb{N}$ and the function $\mathfrak{b}_\mathfrak{v} : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ be defined as

$$\mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \mathfrak{a}) = \begin{cases} 0; & \mathfrak{e} = \mathfrak{a} \\ 1; & \mathfrak{e} \text{ or } \mathfrak{a} \notin \{1,2\} \text{ and } \mathfrak{e} \neq \mathfrak{a} \\ 10; & \mathfrak{e}, \mathfrak{a} \in \{1,2\} \text{ and } \mathfrak{e} \neq \mathfrak{a} \end{cases},$$

for all $\mathfrak{e}, \mathfrak{a} \in \mathbb{N}$. Then $(\mathbb{N}, \mathfrak{b}_\mathfrak{v})$ is a $b_v(s)$-metric space with $v = 8$ and $s = \frac{10}{9}$ (see Ref. 58). Then we get

$$\mathfrak{C}^{b_v(s)}_{1,1} = \{\mathfrak{e} \in \mathbb{N} : \mathfrak{b}_\mathfrak{v}(\mathfrak{e}, 1) = 1\} = \mathbb{N} - \{1, 2\},$$

$$\mathfrak{D}^{b_v(s)}_{1,1} = \{\mathfrak{e} \in \mathbb{N} : \mathfrak{b}_\mathfrak{v}(\mathfrak{e}, 1) \leq 1\} = \mathbb{N} - \{2\},$$

$$\mathfrak{C}^{b_v(s)}_{1,10} = \{\mathfrak{e} \in \mathbb{N} : \mathfrak{b}_\mathfrak{v}(\mathfrak{e}, 1) = 10\} = \{2\}$$

and

$$\mathfrak{D}^{b_v(s)}_{1,10} = \{\mathfrak{e} \in \mathbb{N} : \mathfrak{b}_\mathfrak{v}(\mathfrak{e}, 1) \leq 10\} = \mathbb{N}.$$

What is the notion of a fixed figure on a $b_v(s)$-metric space?

Definition 8. Let $(\mathfrak{E}, \mathfrak{b}_\mathfrak{v})$ be a $b_v(s)$-metric space, $\mathcal{F}$ a figure and $\Gamma : \mathfrak{E} \to \mathfrak{E}$ a self-mapping. If $\mathcal{F} \subseteq Fix(\Gamma)$ then $\mathcal{F}$ is called as the fixed figure of Γ.

Especially, if $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$ then $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$ is a fixed circle of Γ and if $\mathfrak{D}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$ then $\mathfrak{D}_{\mathfrak{e}_0,r}^{b_v(s)}$ is a fixed disc of Γ.

Since we obtain some existence results for fixed circle of a self-mapping on $b_v(s)$-metric spaces, we define the function $\varsigma : \mathfrak{E} \to [0,\infty)$ as

$$\varsigma(\mathfrak{e}) = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \mathfrak{e}_0), \tag{7}$$

for all $\mathfrak{e} \in \mathfrak{E}$.

From now on, we assume that $(\mathfrak{E}, \mathfrak{b}_{\mathfrak{v}})$ is a $b_v(s)$-metric space, $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$ is any circle on $\mathfrak{E}$ and the function $\varsigma : \mathfrak{E} \to [0,\infty)$ is defined as in (7).

Theorem 6. *If there is* $\Gamma : \mathfrak{E} \to \mathfrak{E}$ *satisfying*

$$\left(C_{b_v(s)}1\right) \;\; \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e}) \leq \varsigma(\mathfrak{e}) - \varsigma(\Gamma\mathfrak{e})$$

and

$$\left(C_{b_v(s)}2\right) \;\; \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \mathfrak{e}_0) \geq r,$$

for each $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$, *then* $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$.

Proof. Let $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$. Then we prove $\mathfrak{e} \in Fix(\Gamma)$. By $\left(C_{b_v(s)}1\right)$ and $\left(C_{b_v(s)}2\right)$, we get

$$\begin{aligned} \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e}) &\leq \varsigma(\mathfrak{e}) - \varsigma(\Gamma\mathfrak{e}) = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \mathfrak{e}_0) - \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \mathfrak{e}_0) \\ &= r - \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \mathfrak{e}_0) \leq r - r = 0 \end{aligned}$$

and so $\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e}) = 0$, that is, $\mathfrak{e} \in Fix(\Gamma)$. Consequently, we get

$$\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma).$$

□

Example 3. Let $(\mathfrak{E}, d)$ be a $b_v(s)$-metric space and c a constant such that $\mathfrak{b}_{\mathfrak{v}}(\mathfrak{c}, \mathfrak{e}_0) > r$ with any circle $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$. Let us identify $\Gamma : \mathfrak{E} \to \mathfrak{E}$ as

$$\Gamma\mathfrak{e} = \begin{cases} \mathfrak{e}, & \mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \\ \mathfrak{c}, & \text{otherwise}, \end{cases}$$

for all $\mathfrak{e} \in \mathfrak{E}$. Then the conditions $\left(C_{b_v(s)}1\right)$ and $\left(C_{b_v(s)}2\right)$ are satisfied and so $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$. Indeed, for $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$, we get

$$\begin{aligned}\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}) &= \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}) = 0 \leq \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0) - \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0)\\ &= \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0) - \mathfrak{b}_{\mathfrak{v}}(T\mathfrak{e},\mathfrak{e}_0) = \varsigma(\mathfrak{e}) - \varsigma(\Gamma\mathfrak{e})\end{aligned}$$

and

$$\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0) = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0) = r.$$

Theorem 7. *If* $\Gamma : \mathfrak{E} \to \mathfrak{E}$ *satisfies* $\left(C_{b_v(s)}1\right)$ *and*

$$\left(C_{b_v(s)}3\right) \;\; h\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}) + \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0) \geq r,$$

for each $\mathfrak{e} \in \mathfrak{C}_{x_0,r}^{b_v(s)}$ *and some* $h \in [0,1)$, *then* $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$.

Proof. Let $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$. Then we prove $\mathfrak{e} \in Fix(\Gamma)$. On the contrary, suppose $\mathfrak{e} \notin Fix(\Gamma)$. By $\left(C_{b_v(s)}1\right)$ and $\left(C_{b_v(s)}3\right)$, we find

$$\begin{aligned}\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}) &\leq \varsigma(\mathfrak{e}) - \varsigma(\Gamma\mathfrak{e}) = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0) - \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)\\ &= r - \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)\\ &\leq h\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}) + \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0) - \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)\\ &= h\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}) < \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}),\end{aligned}$$

a contradiction with $h \in [0,1)$. Then we get $\mathfrak{e} \in Fix(\Gamma)$ and so

$$\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma).$$

□

Example 4. Let $(\mathfrak{E},\mathfrak{b}_{\mathfrak{v}})$ be a $b_v(s)$-metric space and c a constant such that $\mathfrak{b}_{\mathfrak{v}}(\mathfrak{c},\mathfrak{e}_0) < r$ with any circle $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$. Let us identify $\Gamma : \mathfrak{E} \to \mathfrak{E}$ as

$$\Gamma\mathfrak{e} = \begin{cases}\mathfrak{e}, & \mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}\\ \mathfrak{c}, & \text{otherwise},\end{cases}$$

for all $\mathfrak{e} \in \mathfrak{E}$. Then the conditions $\left(C_{b_v(s)}1\right)$ and $\left(C_{b_v(s)}3\right)$ are satisfied and so $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$. Indeed, for $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$, we get

$$\begin{aligned}\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}) &= \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}) = 0 \leq \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0) - \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0)\\ &= \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0) - \mathfrak{b}_{\mathfrak{v}}(T\mathfrak{e},\mathfrak{e}_0) = \varsigma(\mathfrak{e}) - \varsigma(\Gamma\mathfrak{e})\end{aligned}$$

and

$$h\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e})+\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)=h\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e})+\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0)=r.$$

In the following fixed-circle theorems, we modify the Caristi inequality using the function $\varsigma:\mathfrak{E}\to[0,\infty)$ defined as in (7).

Theorem 8. *If there is* $\Gamma:\mathfrak{E}\to\mathfrak{E}$ *satisfying*

$$\left(C_{b_v(s)}4\right)\ \ \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e})\leq\varsigma(\mathfrak{e})+\varsigma(\Gamma\mathfrak{e})-2r$$

and

$$\left(C_{b_v(s)}5\right)\ \ \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e})+\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)\leq r,$$

for each $\mathfrak{e}\in\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_0,r}$, *then* $\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_0,r}\subseteq Fix(\Gamma)$.

Proof. Let $\mathfrak{e}\in\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_0,r}$. Then we prove $\mathfrak{e}\in Fix(\Gamma)$. On the contrary, suppose $\mathfrak{e}\notin Fix(\Gamma)$. By $\left(C_{b_v(s)}4\right)$ and $\left(C_{b_v(s)}5\right)$, we find

$$\begin{aligned}\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e})&\leq\varsigma(\mathfrak{e})+\varsigma(\Gamma\mathfrak{e})-2r\\&=\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0)+\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)-2r\\&=\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)-r\\&\leq\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)-[\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e})+\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)]\\&=-\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}),\end{aligned}$$

a contradiction. Then we get $\mathfrak{e}\in Fix(\Gamma)$ and so

$$\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_0,r}\subseteq Fix(\Gamma).$$

□

Example 5. Let us consider Example 3. Then the conditions $\left(C_{b_v(s)}4\right)$ and $\left(C_{b_v(s)}5\right)$ are satisfied and so $\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_0,r}\subseteq Fix(\Gamma)$.

Theorem 9. *If there is* $\Gamma:\mathfrak{E}\to\mathfrak{E}$ *satisfying* $\left(C_{b_v(s)}4\right)$ *and*

$$\left(C_{b_v(s)}6\right)\ \ \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)\leq r,$$

for each $\mathfrak{e}\in\mathfrak{C}^{b_v(s)}_{x_0,r}$, *then* $\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_0,r}\subseteq Fix(\Gamma)$.

Proof. Let $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$. Then we prove $\mathfrak{e} \in Fix(\Gamma)$. On the contrary, suppose $\mathfrak{e} \notin Fix(\Gamma)$. By $\left(C_{b_v(s)}4\right)$ and $\left(C_{b_v(s)}5\right)$, we find

$$\begin{aligned}\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}) &\leq \varsigma(\mathfrak{e})+\varsigma(\Gamma\mathfrak{e})-2r\\ &= \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0)+\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)-2r\\ &= \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)-r\\ &\leq r-r=0,\end{aligned}$$

a contradiction. Then we get $\mathfrak{e} \in Fix(\Gamma)$ and so

$$\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma).$$

□

Example 6. Let us consider Example 4. Then the conditions $\left(C_{b_v(s)}4\right)$ and $\left(C_{b_v(s)}6\right)$ are satisfied and so $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$.

Theorem 10. *If there is* $\Gamma : \mathfrak{E} \to \mathfrak{E}$ *satisfying*

$$\left(C_{b_v(s)}7\right)\ \ \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}) \leq \varsigma(\mathfrak{e})-r$$

or

$$\left(C_{b_v(s)}8\right)\ \ \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}) \leq \varsigma(\Gamma\mathfrak{e})-r$$

and

$$\left(C_{b_v(s)}9\right)\ \ \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0) \leq r+h\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}),$$

for each $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$ *and some* $h \in [0,1)$, *then* $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$.

Proof. Let $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$. Then we prove $\mathfrak{e} \in Fix(\Gamma)$. On the contrary, suppose $\mathfrak{e} \notin Fix(\Gamma)$. If $\left(C_{b_v(s)}7\right)$ holds, we get

$$\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}) \leq \varsigma(\mathfrak{e})-r = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0)-r = r-r=0,$$

a contradiction. Then we get $\mathfrak{e} \in Fix(\Gamma)$ and so

$$\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma).$$

On the other hand, if $\left(C_{b_v(s)}7\right)$ holds, using $\left(C_{b_v(s)}9\right)$, we get

$$\begin{aligned}\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}) &\leq \varsigma(\Gamma\mathfrak{e})-r = \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\mathfrak{e}_0)-r\\ &\leq r+h\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e})-r\\ &= h\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}),\end{aligned}$$

a contradiction with $h \in [0,1)$. Then we get $\mathfrak{e} \in Fix(\Gamma)$ and so

$$\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma).$$

□

Example 7. Let us consider Examples 3 and 4. Then the conditions $\left(C_{b_v(s)}7\right)$, $\left(C_{b_v(s)}8\right)$ and $\left(C_{b_v(s)}9\right)$ are satisfied and so $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$.

To obtain new fixed-circle results, the used auxiliary functions can be changed. For example, let us define the mapping $\eta : [0,\infty) \to \mathbb{R}$ as

$$\eta(\mathfrak{e}) = \begin{cases} \mathfrak{e} - r; & \mathfrak{e} > 0 \\ 0; & \mathfrak{e} = 0, \end{cases} \tag{8}$$

for all $\mathfrak{e} \in [0,\infty)$ and any $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$.

Theorem 11. *If there is* $\Gamma : \mathfrak{E} \to \mathfrak{E}$ *satisfying*

(i) $\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \mathfrak{e}_0) = r$ *for each* $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$,
(ii) $\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \Gamma\mathfrak{a}) > r$ *for each* $\mathfrak{e}, \mathfrak{a} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$ *and* $\mathfrak{e} \neq \mathfrak{a}$,
(iii) $\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \Gamma\mathfrak{a}) \leq \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \mathfrak{a}) - \eta(\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e}))$ *for each* $\mathfrak{e}, \mathfrak{a} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$,

then $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$.

Proof. Let $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$. Using (i), we get $\Gamma\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$. Now, we prove $\mathfrak{e} \in Fix(\Gamma)$. On the contrary, let $\mathfrak{e} \notin Fix(\Gamma)$. Using (ii), we have

$$\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \Gamma^2\mathfrak{e}) > r \tag{9}$$

and using (iii), we get

$$\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \Gamma^2\mathfrak{e}) \leq \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e}) - \eta(\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e})) = \Gamma\mathfrak{e} - \Gamma\mathfrak{e} + r = r,$$

a contradiction with (9). Then $\mathfrak{e} \in Fix(\Gamma)$ and so $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$. □

The identity mapping $I_{\mathfrak{E}} : \mathfrak{E} \to \mathfrak{E}$, defined as $I_{\mathfrak{E}}(\mathfrak{e}) = \mathfrak{e}$, $\mathfrak{e} \in \mathfrak{E}$, fixes every circle on a $b_v(s)$-metric space. So, as a natural problem, is there a new theorem which excludes the possibility of identity mapping $I_{\mathfrak{E}}$?

Theorem 12. Γ *satisfies the condition*

$$(I_{b_v(s)})\ \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e})+\sum_{i=1}^{v-1}\mathfrak{b}_{\mathfrak{v}}(u_i,u_{i+1})<\varsigma(\mathfrak{e})-\varsigma(u_v),$$

for all $\mathfrak{e}\in\mathfrak{E}$ *and* $u_i\in\mathfrak{E}$ *such that all* u_i *are different for* $i\in\{1,2,\ldots,v\}$ *if and only if* $\Gamma=I_{\mathfrak{E}}$.

Proof. Let $\mathfrak{e}\in\mathfrak{E}$ and $\mathfrak{e}\notin Fix(\Gamma)$. By $(I_{b_v(s)})$ and $(B3)$, we obtain

$$\begin{aligned}\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e})+\sum_{i=1}^{v-1}\mathfrak{b}_{\mathfrak{v}}(u_i,u_{i+1})<\varsigma(\mathfrak{e})-\varsigma(u_v)&=\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{e}_0)-\mathfrak{b}_{\mathfrak{v}}(u_v,\mathfrak{e}_0)\\&\le\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},u_1)+\mathfrak{b}_{\mathfrak{v}}(u_1,u_2)+\cdots\\&\quad+\mathfrak{b}_{\mathfrak{v}}(u_v,\mathfrak{e}_0)-\mathfrak{b}_{\mathfrak{v}}(u_v,\mathfrak{e}_0)\\&=\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},u_1)+\sum_{i=1}^{v-1}\mathfrak{b}_{\mathfrak{v}}(u_i,u_{i+1})\end{aligned}$$

and so if we take $\Gamma\mathfrak{e}=u_1$, we have

$$\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e})<\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}),$$

a contradiction. Hence, we get $\mathfrak{e}\in Fix(\Gamma)$ for all $\mathfrak{e}\in\mathfrak{E}$, that is, $\Gamma=I_{\mathfrak{E}}$.

The converse statement is clear. □

In the following proposition, we see that the number of fixed circles is not to be one.

Proposition 1. *Let* $\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_1,r_1},\ldots,\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_n,r_n}$ *any given circles on* $\mathfrak{E}$. *Then there is at least one* $\Gamma:\mathfrak{E}\to\mathfrak{E}$ *such that*

$$\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_1,r_1}\subseteq Fix(\Gamma),\ldots,\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_n,r_n}\subseteq Fix(\Gamma).$$

Proof. Assume that $\mathfrak{c}$ is constant such that

$$\mathfrak{b}_{\mathfrak{v}}(\mathfrak{c},\mathfrak{e}_i)\neq r\quad(1\le i\le n).$$

Let us identify $\Gamma : \mathfrak{E} \to \mathfrak{E}$ by

$$\Gamma\mathfrak{e} = \begin{cases} \mathfrak{e}, & \mathfrak{e} \in \bigcup_{i=1}^{n} \mathfrak{C}_{\mathfrak{e}_i, r_i}^{b_v(s)} \\ \mathfrak{c}, & \text{otherwise,} \end{cases}$$

for all $\mathfrak{e} \in \mathfrak{E}$ and the mappings $\xi_i : \mathfrak{E} \to [0, \infty)$ as

$$\xi_i(\mathfrak{e}) = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \mathfrak{e}_i) \ \ (1 \leq i \leq n).$$

The conditions $\left(C_{b_v(s)}1\right)$, $\left(C_{b_v(s)}2\right)$ and $\left(C_{b_v(s)}3\right)$ are satisfied by Γ. Consequently, we get

$$\mathfrak{C}_{\mathfrak{e}_1, r_1}^{b_v(s)} \subseteq Fix(\Gamma), \ldots, \mathfrak{C}_{\mathfrak{e}_n, r_n}^{b_v(s)} \subseteq Fix(\Gamma).$$

□

By Proposition 1, searching for conditions of uniqueness makes sense for a fixed circle. To do this, we use the Banach-type[59] and Ćirić-type[39] contractions.

Theorem 13. *Let $\Gamma : \mathfrak{E} \to \mathfrak{E}$ satisfy $\left(C_{b_v(s)}1\right)$ and $\left(C_{b_v(s)}2\right)$ (or $\left(C_{b_v(s)}3\right)$). If it satisfies the condition*

$$(B) \quad \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \Gamma\mathfrak{a}) \leq h\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \mathfrak{a}),$$

for all $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_0, r}^{b_v(s)}$, $\mathfrak{a} \in \mathfrak{E} - \mathfrak{C}_{\mathfrak{e}_0, r}^{b_v(s)}$ and some $h \in [0, 1)$, then $\mathfrak{C}_{\mathfrak{e}_0, r}^{b_v(s)}$ is unique such that $\mathfrak{C}_{\mathfrak{e}_0, r}^{b_v(s)} \subseteq Fix(\Gamma)$.

Proof. Let $\mathfrak{C}_{\mathfrak{e}_1, r_1}^{b_v(s)}$, $\mathfrak{C}_{\mathfrak{e}_2, r_2}^{b_v(s)}$ be any two circles and $\Gamma : \mathfrak{E} \to \mathfrak{E}$ satisfies $\left(C_{b_v(s)}1\right)$ and $\left(C_{b_v(s)}2\right)$ $\left(\text{or } \left(C_{b_v(s)}3\right)\right)$ for each $\mathfrak{C}_{\mathfrak{e}_1, r_1}^{b_v(s)}$ and $\mathfrak{C}_{\mathfrak{e}_2, r_2}^{b_v(s)}$. Hence, we have

$$\mathfrak{C}_{\mathfrak{e}_1, r_1}^{b_v(s)} \subseteq Fix(\Gamma) \text{ and } \mathfrak{C}_{\mathfrak{e}_2, r_2}^{b_v(s)} \subseteq Fix(\Gamma).$$

Let $\mathfrak{e} \in \mathfrak{C}_{\mathfrak{e}_1, r_1}^{b_v(s)}$ and $\mathfrak{a} \in \mathfrak{C}_{\mathfrak{e}_2, r_2}^{b_v(s)}$ be any arbitrary points such that $\mathfrak{e} \neq \mathfrak{a}$. By (B), we get

$$\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e}, \Gamma\mathfrak{a}) = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \mathfrak{a}) \leq h\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \mathfrak{a}) < \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \mathfrak{a}),$$

a contradiction. Consequently, the fixed circle of Γ is unique. □

In the following theorem, we use the number $C(\mathfrak{e},\mathfrak{a})$.

Theorem 14. *Let* $\Gamma:\mathfrak{E}\to\mathfrak{E}$ *satisfy* $\left(C_{b_v(s)}1\right)$ *and* $\left(C_{b_v(s)}2\right)$ *(or* $\left(C_{b_v(s)}3\right)$*). If it satisfies the condition*

$$(C)\quad \mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\Gamma\mathfrak{a})\leq hC(\mathfrak{e},\mathfrak{a}),$$

for all $\mathfrak{e}\in\mathfrak{C}_{x_0,r}^{b_v(s)}$, $\mathfrak{a}\in\mathfrak{E}-\mathfrak{C}_{x_0,r}^{b_v(s)}$ *and some* $h\in[0,1)$, *then* $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$ *is unique such that* $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}\subseteq Fix(\Gamma)$.

Proof. Let $\mathfrak{C}_{\mathfrak{e}_1,r_1}^{b_v(s)}$, $\mathfrak{C}_{\mathfrak{e}_2,r_2}^{b_v(s)}$ be any two circles and $\Gamma:\mathfrak{E}\to\mathfrak{E}$ satisfies $\left(C_{b_v(s)}1\right)$ and $\left(C_{b_v(s)}2\right)$ $\left(\text{or } \left(C_{b_v(s)}3\right)\right)$ for each $\mathfrak{C}_{\mathfrak{e}_1,r_1}^{b_v(s)}$ and $\mathfrak{C}_{\mathfrak{e}_2,r_2}^{b_v(s)}$. Hence, we have

$$\mathfrak{C}_{\mathfrak{e}_1,r_1}^{b_v(s)}\subseteq Fix(\Gamma)\text{ and }\mathfrak{C}_{\mathfrak{e}_2,r_2}^{b_v(s)}\subseteq Fix(\Gamma).$$

Let $\mathfrak{e}\in\mathfrak{C}_{\mathfrak{e}_1,r_1}^{b_v(s)}$ and $\mathfrak{a}\in\mathfrak{C}_{\mathfrak{e}_2,r_2}^{b_v(s)}$ be any arbitrary points such that $\mathfrak{e}\neq\mathfrak{a}$. By (C), we get

$$\mathfrak{b}_{\mathfrak{v}}(\Gamma\mathfrak{e},\Gamma\mathfrak{a})=\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{a})\leq hC(\mathfrak{e},\mathfrak{a})<C(\mathfrak{e},\mathfrak{a})=\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\mathfrak{a}),$$

a contradiction. Consequently, the fixed circle of Γ is unique. □

Now, we define the following number ρ as

$$\rho=\inf\left\{\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}):\mathfrak{e}\notin Fix(\Gamma),\ \mathfrak{e}\in\mathfrak{E}\right\}\tag{10}$$

and the function $\zeta:\mathfrak{E}\to[0,\infty)$ as

$$\zeta(\mathfrak{e})=\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e},\Gamma\mathfrak{e}),\tag{11}$$

for all $\mathfrak{e}\in\mathfrak{E}$ since we acquire a new fixed-figure theorem. In the following theorems, we use the Meir–Keeler-type technique.

Theorem 15. *If there is* $\mathfrak{e}_0\in\mathfrak{E}$ *such that*

(a) $\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_0,\Gamma\mathfrak{e})\leq\rho$ *and* $0\leq\zeta(\mathfrak{e})\leq 1$ *for all* $\mathfrak{e}\in\mathfrak{D}_{\mathfrak{e}_0,\rho}^{b_v(s)}$,
(b) *For all* $\mathfrak{e}\in\mathfrak{E}$,

$$\zeta(\mathfrak{e})>0\Longrightarrow\zeta(\mathfrak{e})<\left[\zeta(\mathfrak{e})-\zeta(\mathfrak{e}_0)\right]C(\mathfrak{e},\mathfrak{e}_0),$$

then

(1) $\mathfrak{e}_0 \in Fix(\Gamma)$,
(2) $\mathfrak{D}_{\mathfrak{e}_0,\rho}^{b_v(s)} \subseteq Fix(\Gamma)$,
(3) $\mathfrak{C}_{\mathfrak{e}_0,\rho}^{b_v(s)} \subseteq Fix(\Gamma)$.

Proof. (1) Let $\rho = 0$. Then we get $\mathfrak{D}_{\mathfrak{e}_0,\rho}^{b_v(s)} = \mathfrak{C}_{\mathfrak{e}_0,\rho}^{b_v(s)} = \{\mathfrak{e}_0\}$. If $\zeta(\mathfrak{e}_0) > 0$, by (b), we find

$$\zeta(\mathfrak{e}_0) = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_0, \Gamma\mathfrak{e}_0) < [\zeta(\mathfrak{e}_0) - \zeta(\mathfrak{e}_0)]\, C(\mathfrak{e}_0, \mathfrak{e}_0) = 0,$$

a contradiction. So, it should be $\zeta(\mathfrak{e}_0) = 0$, that is,

$$\mathfrak{e}_0 \in Fix(\Gamma).$$

(2) If $\rho = 0$, then using the condition (1), we have

$$\mathfrak{e}_0 \in Fix(\Gamma), \text{ that is, } \mathfrak{D}_{\mathfrak{e}_0,\rho}^{b_v(s)} \subseteq Fix(\Gamma).$$

Let $\rho > 0$ and $\mathfrak{e} \in D_{\mathfrak{e}_0,\rho}^{b_v(s)}$. If $\zeta(\mathfrak{e}) > 0$, by (a), (b), (1) and the definition of ρ, we find

$$\begin{aligned}\zeta(\mathfrak{e}) &= \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e}) < [\zeta(\mathfrak{e}) - \zeta(\mathfrak{e}_0)]\, C(\mathfrak{e}, \mathfrak{e}_0) = \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e}) C(\mathfrak{e}, \mathfrak{e}_0) \\ &= [\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e})]^2,\end{aligned}$$

a contradiction. Hence, it should be $\zeta(\mathfrak{e}) = 0$ and so we have

$$\mathfrak{e} \in Fix(\Gamma), \text{ that is, } \mathfrak{D}_{\mathfrak{e}_0,\rho}^{b_v(s)} \subseteq Fix(\Gamma).$$

(3) This is a natural consequence of the condition (2). □

Example 8. Let $(\mathfrak{E}, d)$ be a $b_v(s)$-metric space and $\Gamma : \mathfrak{E} \to \mathfrak{E}$ defined as

$$\Gamma\mathfrak{e} = \begin{cases} \mathfrak{e}, & \mathfrak{e} \in \mathfrak{D}_{\mathfrak{e}_0,\rho}^{b_v(s)} \\ \mathfrak{e}_0, & \text{otherwise,} \end{cases}$$

for all $\mathfrak{e} \in \mathfrak{E}$ such that $0 \le \mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \mathfrak{e}_0) \le 1$. Then (a) and (b) of Theorem 15 are satisfied and so $\mathfrak{D}_{\mathfrak{e}_0,\rho}^{b_v(s)} \subseteq Fix(\Gamma)$. As a consequence, we have $\mathfrak{e}_0 \in Fix(\Gamma)$ and $\mathfrak{C}_{\mathfrak{e}_0,\rho}^{b_v(s)} \subseteq Fix(\Gamma)$.

To obtain a new fixed-disc result, we need the following definition.

Definition 9 (Ref. 60). Let $\mathbb{F}$ be the family of all functions $F : (0,\infty) \to \mathbb{R}$ such that

(F_1) F is strictly increasing,
(F_2) For every sequence $\{\alpha_n\}$ in $(0,\infty)$ the following holds:

$$\lim_{n\to\infty} \alpha_n = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} F(\alpha_n) = -\infty,$$

(F_3) There is $k \in (0,1)$ such that $\lim\limits_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Some examples of functions that satisfy the above conditions are $F(\mathfrak{e}) = \ln(\mathfrak{e})$, $F(\mathfrak{e}) = \ln(\mathfrak{e}) + \mathfrak{e}$, $F(\mathfrak{e}) = -\frac{1}{\sqrt{\mathfrak{e}}}$ and $F(\mathfrak{e}) = \ln(\mathfrak{e}^2 + \mathfrak{e})$ (see Ref. 60 for more details).

Theorem 16. *If there are* $\mathfrak{e}_0 \in \mathfrak{E}$, $t > 0$ *and* $F \in \mathbb{F}$ *such that*

(a) $\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}_0, \Gamma\mathfrak{e}) \leq \rho$ *and* $0 \leq \zeta(\mathfrak{e}) \leq 1$ *for all* $\mathfrak{e} \in \mathfrak{D}^{b_v(s)}_{\mathfrak{e}_0,\rho}$,
(b) *For all* $\mathfrak{e} \in \mathfrak{E}$,

$$\zeta(\mathfrak{e}) > 0 \Longrightarrow t + F(\zeta(\mathfrak{e})) \leq F([\zeta(\mathfrak{e}) - \zeta(\mathfrak{e}_0)]\, C(\mathfrak{e}, \mathfrak{e}_0)),$$

then

(1) $\mathfrak{e}_0 \in Fix(\Gamma)$,
(2) $\mathfrak{D}^{b_v(s)}_{\mathfrak{e}_0,\rho} \subseteq Fix(\Gamma)$,
(3) $\mathfrak{C}^{b_v(s)}_{\mathfrak{e}_0,\rho} \subseteq Fix(\Gamma)$.

Proof. (1) Let $\rho = 0$. Then it is an immediate result of the condition (b).

(2) If $\rho = 0$, by (1), we have

$$\mathfrak{e}_0 \in Fix(\Gamma), \text{ that is, } \mathfrak{D}^{b_v(s)}_{\mathfrak{e}_0,\rho} \subseteq Fix(\Gamma).$$

Let $\rho > 0$ and $\mathfrak{e} \in D^{b_v(s)}_{\mathfrak{e}_0,\rho}$. If $\zeta(\mathfrak{e}) > 0$, by (a), (b), (1) and the definition of ρ, we find

$$\begin{aligned} t + F(\zeta(\mathfrak{e})) &= t + F(\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e})) \\ &\leq F([\zeta(\mathfrak{e}) - \zeta(\mathfrak{e}_0)]\, C(\mathfrak{e}, \mathfrak{e}_0)) = F\left([\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e})]^2\right) \\ &\leq F(\mathfrak{b}_{\mathfrak{v}}(\mathfrak{e}, \Gamma\mathfrak{e})), \end{aligned}$$

a contradiction with $t > 0$. So, it should be $\zeta(\mathfrak{e}) = 0$ and so we have

$$\mathfrak{e} \in Fix(\Gamma), \text{ that is, } \mathfrak{D}_{\mathfrak{e}_0,\rho}^{b_v(s)} \subseteq Fix(\Gamma).$$

(3) This is obvious. □

Example 9. Let us consider the $b_v(s)$-metric space $(\mathbb{N}, \mathfrak{b}_\mathfrak{v})$ recalled in Example 2. Let us identify $\Gamma : \mathbb{N} \to \mathbb{N}$ as

$$\Gamma\mathfrak{e} = \begin{cases} \mathfrak{e}, & \mathfrak{e} \in \mathbb{N} - \{2\} \\ 1, & \mathfrak{e} = 2, \end{cases}$$

for all $\mathfrak{e} \in \mathbb{N}$. Then we get

$$\rho = \inf\{\mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \Gamma\mathfrak{e}) : \mathfrak{e} \notin Fix(\Gamma),\ \mathfrak{e} \in \mathbb{N}\} = 1$$

and Γ satisfies the conditions (a) and (b) of Theorem 16 with $\mathfrak{e}_0 = 1$, $t = \ln 10$ and $F(\mathfrak{e}) = \ln \mathfrak{e}$. Then we get $1 \in Fix(\Gamma)$, $\mathfrak{D}_{1,1}^{b_v(s)} \subseteq Fix(\Gamma)$ and $\mathfrak{C}_{1,1}^{b_v(s)} \subseteq Fix(\Gamma)$.

Consequently, we attain the following outcomes:

- If we take $v = s = 1$, then Theorem 6 coincides with Theorem 2.1 given in Ref. 33 on a metric space.
- Taking $v = s = 1$, we conclude that Theorem 7 coincides with Theorem 2.3 given in Ref. 33 on a metric space.
- Theorems 15 and 16 generalize Theorem 5, Theorem 8, Corollary 2 and Corollary 4 given in Ref. 61.
- Since $b_v(s)$-metric is the generalization of some metric spaces such as b, rectangular and v-generalized, then the obtained results can be considered in these spaces. Therefore, the obtained fixed-figure results are important in these spaces.
- Theorems 13 and 14 can be considered for Theorems 8–11.
- The contractive conditions used for uniqueness need not be unique. For example, Kannan-type contractions[62] can be used as follows:

  Let $(\mathfrak{E}, \mathfrak{b}_\mathfrak{v})$ be a $b_v(s)$-metric space, $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$, any circle on $\mathfrak{E}$ and $\Gamma : \mathfrak{E} \to \mathfrak{E}$ satisfies $\left(C_{b_v(s)}1\right)$ and $\left(C_{b_v(s)}2\right)$ $\left(\text{or } \left(C_{b_v(s)}3\right)\right)$. If it satisfies the condition

$$(K) \quad \mathfrak{b}_\mathfrak{v}(\Gamma\mathfrak{e}, \Gamma\mathfrak{a}) \leq h\left[\mathfrak{b}_\mathfrak{v}(\mathfrak{e}, \Gamma\mathfrak{e}) + \mathfrak{b}_\mathfrak{v}(\mathfrak{a}, \Gamma\mathfrak{a})\right],$$

for all $\mathfrak{e} \in \mathfrak{C}_{x_0,r}^{b_v(s)}$, $\mathfrak{a} \in \mathfrak{E} - \mathfrak{C}_{x_0,r}^{b_v(s)}$ and some $h \in [0, 1)$, then $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)}$ is unique such that $\mathfrak{C}_{\mathfrak{e}_0,r}^{b_v(s)} \subseteq Fix(\Gamma)$.

- In Theorems 6 and 7, the center of a fixed circle does not have to be fixed. But in Theorems 15 and 16, the center of a fixed circle is a fixed point.

3. An Application to Activation Functions

Activation functions are used in neural network and they are also known transfer functions because they are used to get the output of the node. The main goal of these functions is to determine the output of neural network like yes or no. Activation functions can be divided as follows:

- Linear activation functions,
- Nonlinear activation functions.

There exist a lot of examples of activation functions in the literature (for some examples, see Ref. 63). One of them is Rectified Linear Unit (*ReLU*) activation function (see Ref. 64). *ReLU* has been widely used for deep learning applications. This function is defined by

$$ReLU(\mathfrak{e}) = \max\{0, \mathfrak{e}\} = \begin{cases} \mathfrak{e}, & \mathfrak{e} \geq 0 \\ 0, & \mathfrak{e} < 0, \end{cases}$$

which performs a threshold operation to each input element. Some types of *ReLU* are listed below:

- Leakly ReLU (*LReLU*) is defined by

$$LReLU(\mathfrak{e}) = \alpha\mathfrak{e} + \mathfrak{e} = \begin{cases} \mathfrak{e}, & \mathfrak{e} > 0 \\ \alpha\mathfrak{e}, & \mathfrak{e} \leq 0, \end{cases}$$

  where $\alpha = 0.01$.[65]
- Parametric ReLU (*PReLU*) is defined by

$$PReLU(\mathfrak{e}) = \max\{0, \mathfrak{e}\} + \alpha \min\{0, \mathfrak{e}\} = \begin{cases} \mathfrak{e}, & \mathfrak{e} > 0 \\ \alpha\mathfrak{e}, & \mathfrak{e} \leq 0, \end{cases}$$

  where α is the negative slope controlling parameter.[66]

- Randomized Leaky ReLU ($RReLU$) is defined by

$$RReLU(\mathfrak{e}) = \begin{cases} \mathfrak{e}, & \mathfrak{e} \geq 0 \\ \alpha\mathfrak{e}, & \mathfrak{e} < 0, \end{cases}$$

  where $U(l, u)$ is a random number sampled from a uniform distribution, $\alpha \sim U(l, u)$, $l < u$ and $l, u \in [0, 1]$.[67]
- S-Shaped Rectified Linear Activation Unit ($SReLU$) is defined by

$$SReLU\,(\mathfrak{e}) = \begin{cases} \sqcup_l + \dashv_l (\mathfrak{e} - \sqcup_l), & \mathfrak{e} \leq \sqcup_l \\ \mathfrak{e}, & \sqcup_l < \mathfrak{e} < \sqcup_r \\ \sqcup_r + \dashv_r (\mathfrak{e} - \sqcup_r), & \mathfrak{e} \geq \sqcup_r, \end{cases}$$

  where $\sqcup_l$, $\dashv_l$, $\sqcup_r$, $\dashv_r$ are parameters.[68]

These types of activation functions have been used in the application of fixed-circle problem with different aspects on metric and some generalized metric spaces. For example, to obtain an application, the real valued discontinuous activation functions and $ReLU$ activation functions were used in metric spaces (see Ref. 47 and the references therein for more examples).

To obtain a new application for activation functions, we use $SReLU$ activation functions. For this purpose, let us take $\mathfrak{E} = \mathbb{N}$, $\sqcup_l = 0$, $\sqcup_r = 2$, $\dashv_l = 3$ and $\dashv_r = 4$. Then we get

$$SReLU\,(x) = \begin{cases} 3x, & x = 0 \\ x, & 0 < x < 2 \\ 4x - 6, & x \geq 2 \end{cases}$$

for all $x \in \mathbb{N}$, as seen in Figure 1.

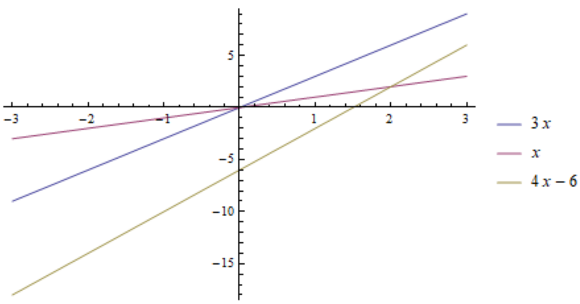

Fig. 1. The constructed activation function: $SReLU\,(x)$.

If we consider the $b_v(s)$-metric space $(\mathbb{N}, \mathfrak{b}_\mathfrak{v})$ given in Example 2, then $SReLU$ satisfies the conditions of Theorem 6 for each $\mathfrak{C}_{2,10}^{b_v(s)} = \{1\}$. Consequently, $\mathfrak{C}_{2,10}^{b_v(s)} \subseteq Fix(SReLU)$. Note that the center of a fixed circle $\mathfrak{C}_{2,10}^{b_v(s)}$ is not fixed by $SReLU$.

4. Conclusion

In this chapter, we prove Caristi-type fixed-point theorems on $b_v(s)$-metric spaces. On the other hand, we investigate some geometric properties of the fixed-point set $Fix(\Gamma)$ of a self-mapping Γ. In this context, we obtain new fixed-circle and fixed-disc results with auxiliary function families. The obtained theoretical results are supported by some illustrative examples. Finally, to show the importance of our results, we give an application to S-shaped rectified linear activation unit. It is hoped that this chapter will inform new nonunique fixed-point theorems with geometric approaches and will shed light on new research areas on generalized metric spaces.

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Chapter 11

Extended Interpolative Hardy–Rogers–Geraghty–Wardowski Contractions and an Application

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The goal of this research is to present the concept of extended interpolative Hardy–Rogers–Geraghty contractions using Wardowski contractions. That is, we combine four ideas: Hardy–Rogers, Geraghty, Wardoowski, and interpolative contractions. We also include an example and an application to confirm the results.

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1. Introduction

Banach's fixed-point theorem[1] is an essential technique in the theory of metric fixed points in mathematics, and it has been widely expanded. Hardy–Rogers contractions,[2] Geraghty contractions,[3] and Wardowski contractions[4] are three prominent Banach contraction extensions. The reader might refer to Refs. 5–11 for more information on the generalizations of this well-known principle.

In 2018, Karapinar[12] introduced the interpolative Kannan Contraction in metric space (MS) and proved its fixed, point theorem. Next, in Ref. 13, Karapinar *et al.* showed that the fixed point of interpolative Kannan contraction mapping may not be unique. To do this, Karapinar introduced the concept of interpolative Ćirić–Reich–Rus (CRR), type mappings to obtain fixed-point theorems in MS.

In 2019, Debonat *et al.*[6] defined a new and modified CRR-type contraction in b-MS and created the corresponding fixed-point result. In the same year, Mohammadi *et al.*[14] introduced an extended interpolative CRR-type F-contraction and produced some related fixed-point results. For more results on interpolation and F-contractions, see Refs. 15 and 16 and Refs. 17–22, respectively.

A new type of contraction mapping on a metric space termed F-contraction was developed in Ref. 4, and a fixed-point theorem for such a map in a complete metric space (CMS) was proved. Wardowski contraction is used to prove some results in fixed-point theory.

Definition 1 (Ref. 4). In an MS $(\Lambda, \aleph)$, a mapping $Ł : \Lambda \to \Lambda$ is called an $\mathcal{Q}$-contraction if

$$\aleph(Ł\jmath, Ł\jmath') > 0 \Rightarrow \tau + \mathcal{Q}(\aleph(Ł\jmath, Ł\jmath')) \leq \mathcal{Q}(\aleph(\jmath, \jmath')),$$

for some $\tau > 0$ and for all $\jmath, \jmath' \in \Lambda$, where $\mathcal{Q} : (0, +\infty) \to (-\infty, +\infty)$ is a mapping so that:

(F1) $\mathcal{Q}$ is strictly increasing,
(F2) for each sequence $\{\alpha_n\}$ of positive real numbers,

$$\lim_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim_{n\to\infty} \mathcal{Q}(\alpha_n) = -\infty,$$

(F3) $\lim_{\alpha\to 0^+} \alpha^\nu \mathcal{Q}(\alpha) = 0$ for some $\nu \in (0, 1)$.

The Kannan contraction appears to be modified in an interpolative Kannan contraction. Karapinar *et al.*[13] improved this notion. Many of the existing contraction-type inequalities have been adjusted to use the pattern of an interpolative Kannan contraction since Karapinar's invention of an interpolative Kannan contraction.[12]

Definition 2 (Ref. 23). A mapping Ł on an MS $(\Lambda, \aleph)$ is called Kannan if there is $\mho \in [0, \frac{1}{2})$ so that

$$\aleph(Łȷ, Łȷ') \leq \mho[\aleph(ȷ, Łȷ) + \aleph(ȷ', Łȷ')]$$

for all $ȷ, ȷ' \in \Lambda$.

Theorem 1 (Ref. 12). *Let* $(\Lambda, \aleph)$ *be a CMS. Let* $Ł : \Lambda \to \Lambda$ *be an interpolative Kannan-type contraction, that is,*

$$\aleph(Łȷ, Łȷ') \leq \mho(\aleph(ȷ, Łȷ))^{\rho}.(\aleph(ȷ', Łȷ'))^{1-\rho}$$

for all $ȷ, ȷ' \in \Lambda$ *with* $ȷ \neq Łȷ$, *where* $\mho \in [0, 1)$ *and* $\rho \in (0, 1)$. *Then there is a unique fixed point in* Λ *for* $Ł$.

In Ref. 13 a contradictory example for Theorem 1 was presented. A modified version of this theorem is the following theorem.

Theorem 2 (Ref. 13). *In a CMS* $(\Lambda, \aleph)$, *let the mapping* $Ł : \Lambda \to \Lambda$ *be given so that*

$$\aleph(Łȷ, Łȷ') \leq \mho(\aleph(ȷ, Łȷ))^{\rho}.(\aleph(ȷ', Łȷ'))^{1-\rho}$$

for all $ȷ, ȷ' \in \Lambda - Fix(Ł)$ *where* $Fix(Ł) = \{z \in \Lambda : Łz = z\}$. *Then* $Ł$ *has a unique fixed point in* Λ.

Hardy and Rogers[2] generalized the principle of Banach contraction as follows:

Theorem 3 (Ref. 2). *In a CMS* $(\Lambda, \aleph)$, *let the mapping* $Ł : \Lambda \to \Lambda$ *be given so that*

$$\aleph(Łȷ, Łȷ') \leq \rho\aleph(ȷ, ȷ') + \varrho\aleph(ȷ, Łȷ) + \sigma\aleph(ȷ', Łȷ') + \eta\left[\frac{\aleph(ȷ, Łȷ') + \aleph(ȷ', Łȷ)}{2}\right]$$

for all $ȷ, ȷ' \in \Lambda$, *where coefficients are nonnegative reals so that their sum is less than one. Then* $Ł$ *possesses a unique fixed point in* Λ.

In Ref. 24, using the interpolation approach, the Hardy–Rogers contraction was generalized as follows:

Definition 3 (Ref. 24). In an MS $(\Lambda, \aleph)$, we say that the mapping $Ł : \Lambda \to \Lambda$ is an interpolative Hardy–Rogers-type contraction if there exists $\mho \in [0, 1)$ and $\rho, \varrho, \sigma \in (0, 1)$ with $\rho + \varrho + \sigma < 1$ so that

$$\aleph(Ł\jmath, Ł\jmath') \leq \mho(\aleph(\jmath, \jmath'))^{\rho}.(\aleph(\jmath, Ł\jmath))^{\varrho}.(\aleph(\jmath', Ł\jmath'))^{\sigma} \cdot \left[\frac{\aleph(\jmath, Ł\jmath') + \aleph(\jmath', Ł\jmath)}{2}\right]^{(1-\rho-\varrho-\sigma)}$$

for all $\jmath, \jmath' \in \Lambda - Fix(Ł)$.

Theorem 4 (Ref. 24). *In a CMS $(\Lambda, \aleph)$, if Ł be an interpolative Hardy–Rogers-type contraction, then Ł possesses a fixed point in Λ.*

In 1972, Reich[25] presented the following theorem as a generalization of the concepts of Kannan and Banach contraction:

Theorem 5 (Ref. 25). *In a CMS $(\Lambda, \aleph)$, let $Ł : \Lambda \to \Lambda$ such that*

$$\aleph(Ł\jmath, Ł\jmath') \leq \rho\aleph(\jmath, \jmath') + \varrho\aleph(\jmath, Ł\jmath) + \sigma\aleph(\jmath', Ł\jmath'),$$

for all $\jmath, \jmath' \in \Lambda$, where $\rho, \varrho, \sigma \geq 0$ so that $\rho + \varrho + \sigma < 1$. Then Ł admits a unique fixed point.

Ćirić proved the following theorem, which is one of the known results in fixed-point theory:

Theorem 6. *In a CMS $(\Lambda, \aleph)$, if $Ł : \Lambda \to \Lambda$ be a self-mapping, such that the inequality*

$$\aleph(Ł\jmath, Ł\jmath') \leq \mho[\aleph(\jmath, \jmath') + \aleph(\jmath, Ł\jmath) + (\aleph(\jmath', Ł\jmath')],$$

for all $\jmath, \jmath' \in \Lambda$, where $\mho \in [0, \frac{1}{3})$, then Ł has a unique fixed point.

In 1973, Geraghty[3] presented the following theorem as a generalization of the Banach contraction principle.

Theorem 7 (Ref. 3). *In a CMS $(\Lambda, \aleph)$, let the mapping $Ł : \Lambda \to \Lambda$ be given so that*

$$\aleph(Ł\jmath, Ł\jmath') \leq \mho(\aleph(\jmath, \jmath'))\aleph(\jmath, \jmath')$$

where $\mho : [0, \infty) \to [0, 1)$ *is a function so that* $\mho(t_n) \to 1$ *yields that* $t_n \to 0$ *as* $n \to \infty$*. Then there is a unique fixed point* $\jmath^* \in \Lambda$ *for* Ł*.*

Definition 4 (Ref. 26). In a b-MS (Λ, b, s), we call the mapping Ł $: \Lambda \to \Lambda$ is an interpolative CRR-type Geraghty contraction if there exists $\rho, \varrho \in (0, 1)$ with $\rho + \varrho < 1$, so that

$$\aleph(Ł\jmath, Ł\jmath') \leq \mho[(\aleph(\jmath, \jmath'))^{\rho}.(\aleph(\jmath, Ł\jmath))^{\varrho}.(\aleph(\jmath', Ł\jmath'))^{(1-\rho-\varrho)}]$$
$$.(\aleph(\jmath, \jmath'))^{\rho}.(\aleph(\jmath, Ł\jmath))^{\varrho}.(\aleph(\jmath', Ł\jmath'))^{(1-\rho-\varrho)}$$

for all $\jmath, \jmath' \in \Lambda - Fix(Ł)$, where $\mho : [0, \infty) \to [0, \frac{1}{s})$ is a function satisfying $\mho(t_n) \to \frac{1}{s}$ which implies $t_n \to 0$ as $n \to \infty$.

In Ref. 14, a contraction of the extended interpolative CRR type was introduced with the Wardowski approach as follows:

Recall that $\mathbb{Q}$ is the set of all functions $\mathcal{Q} : (0, \infty) \to \mathbb{R}$ such that:

(F1) $\mathcal{Q}$ is strictly increasing,
(F2) $\lim_{n\to\infty} t_n = 0$ iff $\lim_{n\to\infty} \mathcal{Q}(t_n) = -\infty$ for each sequence $t_n \in (0, \infty)$,
(F3) $\lim_{t\to 0^+} t^{\nu}\mathcal{Q}(t) = 0$ for some $\nu \in (0, 1)$.

Definition 5 (Ref. 14). In an MS $(\Lambda, \aleph)$, we call the mapping Ł $: \Lambda \to \Lambda$ an extended interpolative CRR-type $\mathcal{Q}$-contraction if there exist $\rho, \varrho \in (0, 1)$ with $\rho + \varrho < 1$, $\tau > 0$ and $\mathcal{Q} \in \mathbb{Q}$ so that

$$\tau + \mathcal{Q}(\aleph(Ł\jmath, Ł\jmath')) \leq \rho\mathcal{Q}(\aleph(\jmath, \jmath')) + \varrho\mathcal{Q}(\aleph(\jmath, Ł\jmath))$$
$$+ (1 - \rho - \varrho)\mathcal{Q}(\aleph(\jmath', Ł\jmath')),$$

for all $\jmath, \jmath' \in \Lambda - Fix(Ł)$ with $\aleph(Ł\jmath, Ł\jmath') > 0$.

Theorem 8 (Ref. 14). *Any extended interpolative CRR-type* $\mathcal{Q}$*-contraction self-mapping on a CMS* $(\Lambda, \aleph)$ *possesses a fixed point in* Λ*.*

To derive novel fixed-point results, we combine and unify the notions of Hardy–Rogers, Geraghty, Wardowski, and interpolative contraction in this chapter. Many of the results in the literature are generalized by our findings.

2. Main Results

In this section, we will first introduce the following definitions and then present the main results.

Let $\mathbb{Q}$ represent the set of all functions $\mathcal{Q} : (0, \infty) \to \mathbb{R}$ such that

($\mathcal{Q}$1) $\mathcal{Q}$ is strictly increasing and continuous.

($\mathcal{Q}$2) $\lim_{n\to\infty} \alpha_n = 1 \Leftrightarrow \lim_{n\to\infty} \mathcal{Q}(\alpha_n) = 0$, for any sequence $\alpha_n \in (0, \infty)$.

Definition 6. In an MS $(\Lambda, \aleph)$, we call the mapping $\text{Ł} : \Lambda \to \Lambda$ an **extended interpolative Hardy–Rogers-type Geraghty $\mathcal{Q}$-contraction** if there are $\rho, \varrho, \sigma \in (0, 1)$ with $\rho+\varrho+\sigma < 1$, $\varrho+\sigma < 1$, $\rho + \varrho < 1$ and $\mathcal{Q} \in \mathbf{Q}$ such that

$$\begin{aligned}\mathcal{Q}(\aleph(\text{Ł}\jmath, \text{Ł}\jmath')) \leq \mathcal{Q}\Bigg(\mho\Bigg(&(\aleph(\jmath, \jmath'))^{\rho}.(\aleph(\jmath, \text{Ł}\jmath))^{\varrho}.(\aleph(\jmath', \text{Ł}\jmath'))^{\sigma}\\ &.\left(\frac{\aleph(\jmath, \text{Ł}\jmath') + \aleph(\jmath', \text{Ł}\jmath)}{2}\right)^{(1-\rho-\varrho-\sigma)}\Bigg)\Bigg)\\ &+ \rho\mathcal{Q}(\aleph(\jmath, \jmath')) + \varrho\mathcal{Q}(\aleph(\jmath, \text{Ł}\jmath)) + \sigma\mathcal{Q}(\aleph(\jmath', \text{Ł}\jmath'))\\ &+ (1 - \rho - \varrho - \sigma)\mathcal{Q}\left(\frac{\aleph(\jmath, \text{Ł}\jmath') + \aleph(\jmath', \text{Ł}\jmath)}{2}\right) \end{aligned} \tag{1}$$

for all $\jmath, \jmath' \in \Lambda - Fix(\text{Ł})$ with $\aleph(\text{Ł}\jmath, \text{Ł}\jmath') > 0$ and $\mho \in \mathcal{S}$ where $\mathcal{S}$ is the class of functions $\mho : [0, \infty) \to [0, 1)$ so that $\mho(\alpha_n) \to 1$ implies $\alpha_n \to 0$ as $n \to \infty$.

Theorem 9. *Any extended interpolative Hardy–Rogers-type Geraghty $\mathcal{Q}$-contraction self-mapping Ł on a CMS $(\Lambda, \aleph)$ possesses a fixed point in Λ.*

Proof. Choose $\jmath_0 \in \Lambda$ and set

$$\jmath_n = \text{Ł}\jmath_{n-1},$$

for every n of positive integers. If there exists m such that $\jmath_m = \jmath_{m+1}$, then $\jmath_m$ is a fixed point of Ł. Suppose that $\jmath_n \neq \jmath_{n+1}$ for all $n \geq 0$.

Then by (1) we have

$$\begin{aligned}
&\mathcal{Q}(\aleph(\jmath_{n+1},\jmath_n))\\
&\quad=\mathcal{Q}(\aleph(\text{Ł}\jmath_n,\text{Ł}\jmath_{n-1}))\\
&\quad\le\mathcal{Q}\Bigg(\mho\Bigg((\aleph(\jmath_n,\jmath_{n-1}))^{\rho}.(\aleph(\jmath_n,\text{Ł}\jmath_n))^{\varrho}.(\aleph(\jmath_{n-1},\text{Ł}\jmath_{n-1}))^{\sigma}\\
&\qquad.\left[\frac{\aleph(\jmath_n,\text{Ł}\jmath_{n-1})+\aleph(\jmath_{n-1},\text{Ł}\jmath_n)}{2}\right]^{(1-\rho-\varrho-\sigma)}\Bigg)\Bigg)\\
&\qquad+\rho\mathcal{Q}(\aleph(\jmath_n,\jmath_{n-1}))+\varrho\mathcal{Q}(\aleph(\jmath_n,\text{Ł}\jmath_n))+\sigma\mathcal{Q}(\aleph(\jmath_{n-1},\text{Ł}\jmath_{n-1}))\\
&\qquad+(1-\rho-\varrho-\sigma)\mathcal{Q}\left(\frac{\aleph(\jmath_n,\text{Ł}\jmath_{n-1})+\aleph(\jmath_{n-1},\text{Ł}\jmath_n)}{2}\right)\\
&\quad\le\mathcal{Q}\Bigg(\mho\Bigg((\aleph(\jmath_n,\jmath_{n-1}))^{\rho}.(\aleph(\jmath_n,\jmath_{n+1}))^{\varrho}.(\aleph(\jmath_{n-1},\jmath_n))^{\sigma}\\
&\qquad.\left[\frac{\aleph(\jmath_{n-1},\jmath_n)+\aleph(\jmath_n,\jmath_{n+1})}{2}\right]^{(1-\rho-\varrho-\sigma)}\Bigg)\Bigg)\\
&\qquad+\rho\mathcal{Q}(\aleph(\jmath_n,\jmath_{n-1}))+\varrho\mathcal{Q}(\aleph(\jmath_n,\jmath_{n+1}))+\sigma\mathcal{Q}(\aleph(\jmath_{n-1},\jmath_n))\\
&\qquad+(1-\rho-\varrho-\sigma)\mathcal{Q}\left(\frac{\aleph(\jmath_{n-1},\jmath_n)+\aleph(\jmath_n,\jmath_{n+1})}{2}\right)\\
&\quad\le\rho\mathcal{Q}(\aleph(\jmath_n,\jmath_{n-1}))+\varrho\mathcal{Q}(\aleph(\jmath_n,\jmath_{n+1}))+\sigma\mathcal{Q}(\aleph(\jmath_{n-1},\jmath_n))\\
&\qquad+(1-\rho-\varrho-\sigma)\mathcal{Q}\left(\frac{\aleph(\jmath_{n-1},\jmath_n)+\aleph(\jmath_n,\jmath_{n+1})}{2}\right). \qquad (2)
\end{aligned}$$

Suppose that $\aleph(\jmath_n,\jmath_{n-1})<\aleph(\jmath_{n+1},\jmath_n)$ for some $n\ge 1$. Thus,

$$\frac{\aleph(\jmath_{n-1},\jmath_n)+\aleph(\jmath_n,\jmath_{n+1})}{2}<d(\jmath_n,\jmath_{n+1}).$$

Consequently, the inequality (2) yields

$$\begin{aligned}
\mathcal{Q}(\aleph(\jmath_{n+1},\jmath_n))&\le\rho\mathcal{Q}(\aleph(\jmath_n,\jmath_{n-1}))+\varrho\mathcal{Q}(\aleph(\jmath_n,\jmath_{n+1}))\\
&\qquad+\sigma\mathcal{Q}(\aleph(\jmath_{n-1},\jmath_n))+(1-\rho-\varrho-\sigma)\mathcal{Q}(\aleph(\jmath_n,\jmath_{n+1})).
\end{aligned}$$

So,

$$(\rho+\sigma)\mathcal{Q}(\aleph(\jmath_{n+1},\jmath_n)) \le (\rho+\sigma)\mathcal{Q}(\aleph(\jmath_n,\jmath_{n-1})),$$

which is a contradiction. Thus, we have $\aleph(\jmath_{n-1},\jmath_n) < \aleph(\jmath_n,\jmath_{n+1})$ for all $n \ge 1$. Hence, $\{\aleph(\jmath_n,\jmath_{n+1})\}$ is a decreasing sequence with positive terms. Therefore, there is $r \ge 0$ so that

$$\lim_{n\to\infty} \aleph(\jmath_{n+1},\jmath_n) = r.$$

We show that $r = 0$.

If $r > 0$, from continuity of $\mathcal{Q}$

$$\begin{aligned}\mathcal{Q}(r) \le \lim_{n\to\infty} \mathcal{Q}\Bigg(\mho\Bigg(&(\aleph(\jmath_n,\jmath_{n-1}))^{\rho}.(\aleph(\jmath_n,\jmath_{n+1}))^{\varrho}.(\aleph(\jmath_{n-1},\jmath_n))^{\sigma}\\ &.\left[\frac{\aleph(\jmath_{n-1},\jmath_n)+\aleph(\jmath_n,\jmath_{n+1})}{2}\right]^{(1-\rho-\varrho-\sigma)}\Bigg)\Bigg)\\ &+\rho\mathcal{Q}(r)+\varrho\mathcal{Q}(r)+\sigma\mathcal{Q}(r)+(1-\rho-\varrho-\sigma)\mathcal{Q}(r)\\ \le{}& \alpha\mathcal{Q}(r)+\varrho\mathcal{Q}(r)+\sigma\mathcal{Q}(r)+(1-\rho-\varrho-\sigma)\mathcal{Q}(r).\end{aligned}$$

So,

$$\begin{aligned}\lim_{n\to\infty} \mathcal{Q}\Bigg(\mho\Bigg(&(\aleph(\jmath_n,\jmath_{n-1}))^{\rho}.(\aleph(\jmath_n,\jmath_{n+1}))^{\varrho}.(\aleph(\jmath_{n-1},\jmath_n))^{\sigma}\\ &.\left[\frac{\aleph(\jmath_{n-1},\jmath_n)+\aleph(\jmath_n,\jmath_{n+1})}{2}\right]^{(1-\rho-\varrho-\sigma)}\Bigg)\Bigg)=0.\end{aligned}$$

which yields that

$$\begin{aligned}\lim_{n\to\infty} \mho\Bigg(&\aleph(\jmath_n,\jmath_{n-1}))^{\rho}.(\aleph(\jmath_n,\jmath_{n+1}))^{\varrho}.(\aleph(\jmath_{n-1},\jmath_n))^{\sigma}\\ &.\left[\frac{\aleph(\jmath_{n-1},\jmath_n)+\aleph(\jmath_n,\jmath_{n+1})}{2}\right]^{(1-\rho-\varrho-\sigma)}\Bigg)=1.\end{aligned}$$

So,

$$\lim_{n\to\infty} (r)^{\rho}.(r)^{\varrho}.(r)^{\sigma}.[r]^{(1-\rho-\varrho-\sigma)} = 0,$$

that is,

$$r = 0. \tag{3}$$

Now, to show that $\{\jmath_n\}$ is a Cauchy sequence, suppose that there is $\varepsilon > 0$ so that for all i there exist m_i and n_i with $i < m_i < n_i$ such that

$$\aleph(\jmath_{m_i}, \jmath_{n_i}) \geq \varepsilon, \tag{4}$$

and

$$\aleph(\jmath_{m_i}, \jmath_{n_i-1}) < \varepsilon. \tag{5}$$

By (4), we have

$$\aleph(\jmath_{m_i-1}, \jmath_{n_i-1}) \leq \aleph(\jmath_{m_i-1}, \jmath_{m_i}) + \aleph(\jmath_{m_i}, \jmath_{n_i-1}). \tag{6}$$

From (3) and (5), we have

$$\limsup_{i\to\infty} \aleph(\jmath_{m_i-1}, \jmath_{n_i-1}) \leq \varepsilon. \tag{7}$$

As a result,

$$\limsup_{i\to\infty} \aleph(\jmath_{m_i-1}, \jmath_{n_i}) \leq \varepsilon.$$

On the other hand, one has

$$\begin{aligned}
&\mathcal{Q}(\aleph(\jmath_{m_i}, \jmath_{n_i})) \\
&\quad = \mathcal{Q}(\aleph(\text{Ł}\iota_{m_i-1}, \text{Ł}\iota_{n_i-1})) \\
&\quad \leq \mathcal{Q}\Bigg(\mho\Bigg(\aleph(\jmath_{m_i-1}, \jmath_{n_i-1}))^{\rho}.(\aleph(\jmath_{m_i-1}, \text{Ł}\jmath_{m_i-1}))^{\varrho} \\
&\qquad .\,(\aleph(\jmath_{n_i-1}, \text{Ł}\jmath_{n_i-1}))^{\sigma} \\
&\qquad .\left[\frac{\aleph(\jmath_{m_i-1}, \text{Ł}\jmath_{n_i-1}) + \aleph(\jmath_{n_i-1}, \text{Ł}\jmath_{m_i-1})}{2}\right]^{(1-\rho-\varrho-\sigma)}\Bigg)\Bigg) \\
&\qquad + \rho\mathcal{Q}(\aleph(\jmath_{m_i-1}, \jmath_{n_i-1})) + \varrho\mathcal{Q}(\aleph(\jmath_{m_i-1}, \text{Ł}\jmath_{m_i-1})) \\
&\qquad + \sigma\mathcal{Q}(\aleph(\jmath_{n_i-1}, \text{Ł}\jmath_{n_i-1})) \\
&\qquad + (1-\rho-\varrho-\sigma)\mathcal{Q}\left(\frac{\aleph(\jmath_{m_i-1}, \text{Ł}\jmath_{n_i-1}) + \aleph(\jmath_{n_i-1}, \text{Ł}\jmath_{m_i-1})}{2}\right)
\end{aligned}$$

$$\le \rho\mathcal{Q}(\aleph(\jmath_{m_i-1}, \jmath_{n_i-1})) + \varrho\mathcal{Q}(\aleph(\jmath_{m_i-1}, \jmath_{m_i})) + \sigma\mathcal{Q}(\aleph(\jmath_{n_i-1}, \jmath_{n_i}))$$
$$+ (1-\rho-\varrho-\sigma)\mathcal{Q}\left(\frac{\aleph(\jmath_{m_i-1}, \jmath_{n_i}) + \aleph(\jmath_{n_i-1}, \jmath_{m_i})}{2}\right). \tag{8}$$

Now, according to (3)–(7), one has

$$\begin{aligned}
\mathcal{Q}(\varepsilon) &\le \mathcal{Q}\left(\limsup_{i\to\infty} \aleph(\jmath_{m_i}, \jmath_{n_i})\right)\\
&\le \rho\mathcal{Q}\left(\limsup_{i\to\infty} \aleph(\jmath_{m_i-1}, \jmath_{n_i-1})\right) + \varrho\mathcal{Q}\left(\limsup_{i\to\infty} \aleph(\jmath_{m_i-1}, \jmath_{m_i})\right)\\
&\quad + \sigma\mathcal{Q}\left(\limsup_{i\to\infty} \aleph(\jmath_{n_i-1}, \jmath_{n_i})\right)\\
&\quad + (1-\rho-\varrho-\sigma)\mathcal{Q}\left(\limsup_{i\to\infty} \frac{\aleph(\jmath_{m_i-1}, \jmath_{n_i}) + \aleph(\jmath_{n_i-1}, \jmath_{m_i})}{2}\right)\\
&\le \rho\mathcal{Q}(\varepsilon) + (1-\rho-\varrho-\sigma)\mathcal{Q}(\varepsilon)\\
&= (1-\varrho-\sigma)\mathcal{Q}(\varepsilon),
\end{aligned} \tag{9}$$

which is a contradiction. So, we have shown that $\{\jmath_n\}$ is a Cauchy sequence in Λ. Since $(\Lambda, \aleph)$ is complete, $\jmath_n \to \jmath$ as $n \to \infty$, for some $\jmath \in \Lambda$, that is, $\lim_{n\to\infty} \jmath_n = \jmath$. Suppose to the contrary that $\jmath \neq \text{Ł}\jmath$.

We now consider the following two cases:

Case 1: There exists a subsequence $\{\jmath_{n_k}\}$ so that $\text{Ł}\jmath_{n_k} = \text{Ł}\jmath$ for all $k \in \mathbb{N}$. Therefore,

$$\aleph(\jmath, \text{Ł}\jmath) = \lim_{k\to\infty} \aleph(\jmath_{n_k+1}, \text{Ł}\jmath) = \lim_{k\to\infty} \aleph(\text{Ł}\jmath_{n_k}, \text{Ł}\jmath) = 0.$$

Case 2: There exists an $N \in \mathbb{N}$ so that $\text{Ł}\jmath_k \neq \text{Ł}\jmath$ for all $k \ge N$. Therefore, applying (1), we have

$$\begin{aligned}
\mathcal{Q}(\aleph(\jmath_{k+1}, \text{Ł}\jmath)) &= \mathcal{Q}(\aleph(\text{Ł}\jmath_k, \text{Ł}\jmath))\\
&\le \mathcal{Q}\Bigg(\mho\Bigg(\aleph(\jmath_k, \jmath))^{\rho}.(\aleph(\jmath_k, \text{Ł}\jmath_k))^{\varrho}.(\aleph(\jmath, \text{Ł}\jmath))^{\sigma}\\
&\quad \cdot\left[\frac{\aleph(\jmath_k, \text{Ł}\jmath) + \aleph(\jmath, \text{Ł}\jmath_k)}{2}\right]^{(1-\rho-\varrho-\sigma)}\Bigg)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ \rho\mathcal{Q}(\aleph(\jmath_k,\jmath)) + \varrho\mathcal{Q}(\aleph(\jmath_k,\L\jmath_k)) + \sigma\mathcal{Q}(\aleph(\jmath,\L\jmath)) \\
&+ (1-\rho-\varrho-\sigma)\mathcal{Q}\left(\frac{\aleph(\jmath_k,\L\jmath)+\aleph(\jmath,\L\jmath_k)}{2}\right) \\
\leq\ & \rho\mathcal{Q}(\aleph(\jmath_k,\jmath)) + \varrho\mathcal{Q}(\aleph(\jmath_k,\jmath_{k+1})) + \sigma\mathcal{Q}(\aleph(\jmath,\L\jmath)) \\
&+ (1-\rho-\varrho-\sigma)\mathcal{Q}\left(\frac{\aleph(\jmath_k,\L\jmath)+\aleph(\jmath,\jmath_{k+1})}{2}\right).
\end{aligned}
\tag{10}
$$

Letting $k\to\infty$ in the inequality (10), one has

$$
\begin{aligned}
\mathcal{Q}(\aleph(\jmath,\L\jmath)) &\leq \sigma\mathcal{Q}(\aleph(\jmath,\L\jmath)) + (1-\rho-\varrho-\sigma)\mathcal{Q}\left(\frac{\aleph(\jmath,\L\jmath)}{2}\right) \\
&\leq \sigma\mathcal{Q}(\aleph(\jmath,\L\jmath)) + (1-\rho-\varrho-\sigma)\mathcal{Q}(\aleph(\jmath,\L\jmath)) \\
&= (1-\rho-\varrho)\mathcal{Q}(\aleph(\jmath,\L\jmath)).
\end{aligned}
$$

We deduce that $\jmath=\L\jmath$, as $\rho+\varrho<1$. □

We obtain the following result if we take $\mathcal{Q}(t) = -\frac{1}{t}+1$ in Theorem 9.

Corollary 1. *For an MS $(\Lambda,\aleph)$, let $\L:\Lambda\to\Lambda$ be a mapping. If there exists $\rho,\varrho,\sigma\in(0,1)$ with $\rho+\varrho+\sigma<1$, such that*

$$
\begin{aligned}
&\aleph(\L\jmath,\L\jmath') \leq \mho\Bigg((\aleph(\jmath,\jmath'))^{\rho}.(\aleph(\jmath,\L\jmath))^{\varrho}.(\aleph(\jmath',\L\jmath'))^{\sigma} \\
&\quad .\left(\frac{\aleph(\jmath,\L\jmath')+\aleph(\jmath',\L\jmath)}{2}\right)^{(1-\rho-\varrho-\sigma)}\Bigg) \\
&\quad + \frac{\aleph(\jmath,\jmath')}{\rho+\varrho\left(\frac{\aleph(\jmath,\jmath')}{\aleph(\jmath,\L\jmath)}\right)+\sigma\left(\frac{\aleph(\jmath,\jmath')}{\aleph(\jmath',\L\jmath')}\right)+(1-\rho-\varrho-\sigma)\left(\frac{2\aleph(\jmath,\jmath')}{\aleph(\jmath,\L\jmath')+\aleph(\jmath',\L\jmath)}\right)},
\end{aligned}
\tag{11}
$$

for all $\jmath,\jmath'\in\Lambda - Fix(\L)$, where $\mho\in\mathcal{S}$, then $\L$ possesses a fixed point in Λ.

Taking $\mho(t)=k$, where $k\in(0,1)$ in Corollary 1, we have the following corollary.

Corollary 2. *In an MS* $(\Lambda, \aleph)$, *if* $Ł : \Lambda \to \Lambda$ *be a mapping and there is* $\rho, \varrho, \sigma \in (0,1)$ *with* $\rho + \varrho + \sigma < 1$ *such that*

$$\aleph(Ł\jmath, Ł\jmath') \leq k + \frac{\aleph(\jmath, \jmath')}{\rho + \varrho\left(\frac{\aleph(\jmath,\jmath')}{\aleph(\jmath, Ł\jmath)}\right) + \sigma\left(\frac{\aleph(\jmath,\jmath')}{\aleph(\jmath', Ł\jmath')}\right) + (1-\rho-\varrho-\sigma)\left(\frac{2\aleph(\jmath,\jmath')}{\aleph(\jmath, Ł\jmath') + \aleph(\jmath', Ł\jmath)}\right)}, \tag{12}$$

for all $\jmath, \jmath' \in \Lambda - Fix(Ł)$, *then* $Ł$ *possesses a fixed point in* Λ.

Example 1. Let $\Lambda = [0,1]$, and let $d : \Lambda^2 \to \mathbb{R}_+$ with

$$\aleph(\jmath, \jmath') = \begin{cases} \max\{\jmath, \jmath'\}, & \text{if } \jmath \neq \jmath'; \\ 0, & otherwise. \end{cases}$$

Then $(\Lambda, \aleph)$ is a CMS. Let $Ł : \Lambda \to \Lambda$ with

$$Ł\jmath = \begin{cases} 0, & \text{if } \jmath \in \left[0, \frac{1}{2}\right); \\ \frac{1}{4}, & \text{if } \jmath \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

for all $\jmath \in \Lambda$, $\mathcal{Q}(t) = \ln t$, $\mho(t) = e^{-t}$ and also $\rho = \varrho = \sigma = \frac{1}{4}$.

Let $\jmath, \jmath' \in \Lambda - Fix(Ł)$ so that $\aleph(Ł\jmath, Ł\jmath') > 0$. Without loss of generality, one has the following case:

$$(\jmath, \jmath') \in \left\{\left[0, \frac{1}{2}\right) \times \left[\frac{1}{2}, 1\right]\right\}.$$

It is easy to conclude that all conditions of Theorem 9 are satisfied and 0 is a fixed point of Ł.

3. An Application to Integral Equations

Let $\Lambda = C([0,T], \mathbb{R})$ be the set of all real valued continuous functions with domain $[0, Ł]$ which is endowed with the metric

$$\aleph(\jmath, \jmath') = \sup_{\iota \in [0,T]} |\jmath(\iota) - \jmath'(\iota)| = \| \jmath - \jmath' \|_\infty .$$

Consider the following Fredholm integral equation:

$$\jmath(\iota) = \int_0^T \Upsilon(\iota, s, \jmath(s))ds + \Gamma(\iota), \tag{13}$$

for all $s, \iota \in [0,T]$, where $\Upsilon : [0,T]^2 \times \mathbb{R} \to \mathbb{R}$ and $\Gamma : [0,T] \to \mathbb{R}$ are continuous.

Now, we consider the following assumption:

for all $\jmath, \jmath' \in \Lambda$, assume that the following condition holds:

$$|\Upsilon(s,\iota,\jmath(s)) - \Upsilon(s,\iota,\jmath'(s))| \leq k/T + \frac{|\jmath(\iota) - \jmath'(\iota)|}{T\rho + T\varrho\left(\frac{\|\jmath-\jmath'\|_\infty}{\|\jmath-Ł\jmath\|_\infty}\right) + T\sigma\left(\frac{\|\jmath-\jmath'\|_\infty}{\|\jmath'-Ł\jmath'\|_\infty}\right) + T(1-\rho-\varrho-\sigma)\left(\frac{2\|\jmath-\jmath'\|_\infty}{\|\jmath-Ł\jmath'\|_\infty + \|\jmath'-Ł\jmath\|_\infty}\right)}.$$

Theorem 10. *Suppose that the above assumptions are true. Then the integral equation* (13) *possesses a unique solution in* Λ*.*

Proof. We define $Ł : \Lambda \to \Lambda$ by

$$Ł\jmath(\iota) = \int_0^T \Upsilon(\iota, \upsilon, \jmath(\upsilon))d\upsilon + \Gamma(\iota), \quad \forall \upsilon, \iota \in [0,T].$$

Then, for every $\jmath, \jmath' \in \Lambda$, we have

$$\begin{aligned}
&|Ł\jmath(\iota) - Ł\jmath'(\iota)| \\
&= \left|\int_0^T \Upsilon(\iota,\upsilon,\jmath(\upsilon)) - \Upsilon(\iota,\upsilon,\jmath'(\upsilon))d\upsilon\right| \\
&\leq \int_0^T |\Upsilon(\iota,\upsilon,\jmath(s)) - \Upsilon(\iota,\upsilon,\jmath'(\upsilon))|d\upsilon \\
&\leq k + \frac{|\jmath(\iota) - \jmath'(\iota)|}{\rho + \varrho\left(\frac{\|\jmath-\jmath'\|_\infty}{\|\jmath-Ł\jmath\|_\infty}\right) + \sigma\left(\frac{\|\jmath-\jmath'\|_\infty}{\|\jmath'-Ł\jmath'\|_\infty}\right) + (1-\rho-\varrho-\sigma)\left(\frac{2\|\jmath-\jmath'\|_\infty}{\|\jmath-Ł\jmath'\|_\infty + \|\jmath'-Ł\jmath\|_\infty}\right)} \\
&\leq k + \frac{\| \jmath - \jmath' \|_\infty}{\rho + \varrho\left(\frac{\|\jmath-\jmath'\|_\infty}{\|\jmath-Ł\jmath\|_\infty}\right) + \sigma\left(\frac{\|\jmath-\jmath'\|_\infty}{\|\jmath'-Ł\jmath'\|_\infty}\right) + (1-\rho-\varrho-\sigma)\left(\frac{2\|\jmath-\jmath'\|_\infty}{\|\jmath-Ł\jmath'\|_\infty + \|\jmath'-Ł\jmath\|_\infty}\right)}.
\end{aligned}$$

Take the supremum on the left hand side to find that

$$\begin{aligned}
&\aleph(Ł\jmath, Ł\jmath') \\
&= \| Ł\jmath - Ł\jmath' \|_\infty \\
&\leq k + \frac{\| \jmath - \jmath' \|_\infty}{\rho + \varrho\left(\frac{\|\jmath-\jmath'\|_\infty}{\|\jmath-Ł\jmath\|_\infty}\right) + \sigma\left(\frac{\|\jmath-\jmath'\|_\infty}{\|\jmath'-Ł\jmath'\|_\infty}\right) + (1-\rho-\varrho-\sigma)\left(\frac{2\|\jmath-\jmath'\|_\infty}{\|\jmath-Ł\jmath'\|_\infty + \|\jmath'-Ł\jmath\|_\infty}\right)} \\
&= k + \frac{\aleph(\jmath,\jmath')}{\rho + \varrho\left(\frac{\aleph(\jmath,\jmath')}{\aleph(\jmath,Ł\jmath)}\right) + \sigma\left(\frac{\aleph(\jmath,\jmath')}{\aleph(\jmath',Ł\jmath')}\right) + (1-\rho-\varrho-\sigma)\left(\frac{2\aleph(\jmath,\jmath')}{\aleph(\jmath,Ł\jmath') + \aleph(\jmath',Ł\jmath)}\right)}.
\end{aligned}$$

From the above inequality, we get

$$\frac{1}{\aleph(Łȷ, Łȷ')} \geq \frac{1}{k} + \rho\left(\frac{1}{\aleph(ȷ, ȷ')}\right) + \varrho\left(\frac{1}{\aleph(ȷ, Łȷ)}\right) + \sigma\left(\frac{1}{\aleph(ȷ', Łȷ')}\right) + (1-\rho-\varrho-\sigma)\left(\frac{2}{\aleph(ȷ, Łȷ') + \aleph(ȷ', Łȷ)}\right).$$

This is equivalent to

$$\frac{-1}{\aleph(Łȷ, Łȷ')} \leq \frac{-1}{k} + \rho\left(\frac{-1}{\aleph(ȷ, ȷ')}\right) + \varrho\left(\frac{-1}{\aleph(ȷ, Łȷ)}\right) + \sigma\left(\frac{-1}{\aleph(ȷ', Łȷ')}\right) + (1-\rho-\varrho-\sigma)\left(\frac{-2}{\aleph(ȷ, Łȷ') + \aleph(ȷ', Łȷ)}\right).$$

Therefore,

$$\begin{aligned}\left(\frac{-1}{\aleph(Łȷ, Łȷ')} + 1\right) \leq \left(\frac{-1}{k} + 1\right) + \rho\left(\frac{-1}{\aleph(ȷ, ȷ')} + 1\right) \\ + \varrho\left(\frac{-1}{\aleph(ȷ, Łȷ)} + 1\right) + \sigma\left(\frac{-1}{\aleph(ȷ', Łȷ')} + 1\right) \\ + (1-\rho-\varrho-\sigma)\left(\frac{-2}{\aleph(ȷ, Łȷ') + \aleph(ȷ', Łȷ)} + 1\right).\end{aligned}$$

Taking $\mathcal{Q}(t) = -\frac{1}{t} + 1$ and $\mho(t) = k$, we get

$$\begin{aligned}\mathcal{Q}(\aleph(Łȷ, Łȷ')) \leq \mathcal{Q}\Bigg(\mho\Bigg((\aleph(ȷ, ȷ'))^{\rho}.(\aleph(ȷ, Łȷ))^{\varrho}.(\aleph(ȷ', Łȷ'))^{\sigma} \\ .\left(\frac{\aleph(ȷ, Łȷ') + \aleph(ȷ', Łȷ)}{2}\right)^{(1-\rho-\varrho-\sigma)}\Bigg)\Bigg) \\ + \rho\mathcal{Q}(\aleph(ȷ, ȷ')) + \varrho\mathcal{Q}(\aleph(ȷ, Łȷ)) + \sigma\mathcal{Q}(\aleph(ȷ', Łȷ')) \\ + (1-\rho-\varrho-\sigma)\mathcal{Q}\left(\frac{\aleph(ȷ, Łȷ') + \aleph(ȷ', Łȷ)}{2}\right).\end{aligned}$$

Therefore, it is obvious that the mapping Ł has all the conditions of Theorem 9. Hence, the Fredholm integral Equation (13) admits a unique solution because Ł possesses a unique fixed point. □

4. Conclusions

In this chapter, we stated the concept of extended interpolative Hardy–Rogers-Geraghty-Wardowski contractions. We have combined the four ideas of Hardy–Rogers, Geraghty, Wardoowski, and interpolative contractions. Using these ideas, we presented a new development of the Banach contraction principle. An example and an application are provided to support the results and distinguish them from other results. Our results extend and generalized the relevant results in Refs. $2, 3, 13, 14, 24$, and 26.

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Chapter 12

(η, ψ)-Rational F-Contractions and Weak-Wardowski Contractions in a Triple-Controlled Modular-Type Metric Space

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In this chapter, we present the concept of triple-controlled modular $\hbar$-metric spaces and (η, ψ)-rational F-contractions and weak-Wardowski

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contractions. We also use these concepts to survey some of the fixed point results in triple-controlled fuzzy $\hbar$-metric spaces. Some examples and an application are provided to confirm the results.

1. Introduction

There are dozens of generalizations of metric spaces, e.g. vector-valued metric spaces,[1,2] double-controlled metric spaces,[3] b-rectangular metric spaces,[4] generalized parametric metric spaces,[5] C^*-algebra-valued metric spaces[6] etc. In these settings, the Banach's fixed point theorem[7] was developed by several authors. For a detailed relevant study on new fixed point results we refer the reader to Refs. 8–11.

In Refs. 5,12,13 the authors used the (α, ψ)-rational type contractive mappings to prove some fixed point theorems. In this chapter, we introduce the concept of triple-controlled modular $\hbar$-metric space and via these concepts obtain some new fixed point results.

Definition 1 (Ref. 14). Let $\mathcal{Q}$ be a nonempty set and $s \geq 1$ be a real number. A mapping $d : \mathcal{Q}^2 \longrightarrow \mathbb{R}$ is called a b-metric on $\mathcal{Q}$, if

(1) $d(\ell, \ell') \geq 0$ for each $\ell, \ell' \in \mathcal{Q}$ and $d(\ell, \ell') = 0$ if and only if $\ell = \ell'$;
(2) $d(\ell, \ell') = d(\ell', \ell)$ for each $\ell, \ell' \in \mathcal{Q}$;
(3) $d(\ell, \ell') \leq s[d(\ell, \ell'') + d(\ell'', \ell')]$ for each $\ell, \ell', \ell'' \in \mathcal{Q}$.

Then $(\mathcal{Q}, d)$ is called a b-metric space.

Definition 2 (Ref. 14). Let $\mathcal{Q}$ be a nonempty set and $\wp : \mathcal{Q} \times \mathcal{Q} \to [1, \infty)$. A function $d_\wp : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ is called an extended b-metric if for all $\ell, \ell', \ell'' \in \mathcal{Q}$ it satisfies

($d_\wp$1) $d_\wp(\ell, \ell') = 0$ if and only if $\ell = \ell'$;
($d_\wp$2) $d_\wp(\ell, \ell') = d_\wp(\ell', \ell)$;
($d_\wp$3) $d_\wp(\ell, \ell'') \leq \wp(\ell, \ell'')[d_\wp(\ell, \ell') + d_\wp(\ell', \ell'')]$.

The pair $(\mathcal{Q}, d_\wp)$ is called an extended b-metric space.

Definition 3 (Ref. 3). Let $\mathcal{Q}$ be a nonempty set and $\wp, \wp' : \mathcal{Q} \times \mathcal{Q} \to [1, \infty)$. A function $\rho : \mathcal{Q} \times \mathcal{Q} \to [0, \infty)$ is called a double-controlled metric if for all $\ell, \ell', \ell'' \in \mathcal{Q}$ it satisfies

($\rho 1$) $\rho(\ell, \ell') = 0$ if and only if $\ell = \ell'$;

($\rho 2$) $\rho(\ell, \ell') = \rho(\ell', \ell)$;

($\rho 3$) $\rho(\ell, \ell'') \leq \wp(\ell, \ell')\rho(\ell, \ell') + \wp'(\ell', \ell'')\rho(\ell', \ell'')$.

The pair $(\mathcal{Q}, \rho)$ is called a double controlled metric space.

Definition 4 (Ref. 15). Let $\mathcal{Q}$ be a nonempty set. The function $\sigma : (0, +\infty) \times \mathcal{Q} \times \mathcal{Q} \to [0, +\infty]$ is said to be a modular metric on $\mathcal{Q}$ if

1. $\ell = \ell'$ if and only if $\sigma_\lambda(\ell, \ell') = 0$ for all $\lambda > 0$;
2. $\sigma_\lambda(\ell, \ell') = \sigma_\lambda(\ell', \ell)$;
3. $\sigma_{\lambda+\mu}(\ell, \ell') \leq \sigma_\lambda(\ell, \ell'') + \sigma_\mu(\ell'', \ell')$ for all $\ell, \ell', \ell'' \in \mathcal{Q}$ and all $\lambda, \mu > 0$, for all $\lambda, \mu > 0$ and for all $\ell, \ell', \ell'' \in \mathcal{Q}$. The pair $(\mathcal{Q}, \sigma)$ is called a modular metric space.

Definition 5 (Ref. 16). Let $\mathcal{Q}$ be a nonempty set and $s \geq 1$ be a real number. A mapping $\widehat{\sigma} : (0, +\infty) \times \mathcal{Q} \times \mathcal{Q} \to [0, +\infty]$ is said to be a modular b-metric if the following statements for all $\ell, \ell', \ell'' \in \mathcal{Q}$ hold:

(a) $\widehat{\sigma}_\lambda(\ell, \ell') = 0$ for all $\lambda > 0$ if and only if $\ell = \ell'$;
(b) $\widehat{\sigma}_\lambda(\ell, \ell') = \widehat{\sigma}_\lambda(\ell', \ell)$;
(c) $\widehat{\sigma}_{\lambda+\mu}(\ell, \ell') \leq s(\widehat{\sigma}_\lambda(\ell, \ell'') + \widehat{\sigma}_\mu(\ell'', \ell'))$. Then $(\mathcal{Q}, \widehat{\sigma})$ is called a modular b-metric space.

Definition 6 (Ref. 17). Let $(\mathcal{Q}, d)$ be a metric space. A mapping $Ł : \mathcal{Q} \to \mathcal{Q}$ is said to be an F-contraction if there is $\tau > 0$ such that for all $\ell, \ell' \in \mathcal{Q}$,

$$d(Ł\ell, Ł\ell') > 0 \Rightarrow \tau + F(d(Ł\ell, Ł\ell')) \leq F(d(\ell, \ell')),$$

where $F : (0, +\infty) \to (-\infty, +\infty)$ is a mapping satisfying the following conditions:

(F1) F is strictly increasing,
(F2)

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} F(a_n) = -\infty$$

for any sequence $\{a_n\}$ of positive real numbers,
(F3) there exists $k \in (0, 1)$ such that $\lim_{a\to 0^+} a^k F(a) = 0$.

Definition 7 (Ref. 18). Let $\mathcal{Q}$ be a nonempty set and let $\alpha : \mathcal{Q}^2 \to [0, \infty)$. A mapping $Ł : \mathcal{Q} \to \mathcal{Q}$ is called an α-admissible mapping if

$$\alpha(\ell, \ell') \geq 1 \Rightarrow \alpha(Ł\ell, Ł\ell') \geq 1 \quad \text{for all } \ell, \ell' \in \mathcal{Q}.$$

In this chapter, we study the new structure of generalized triple-controlled modular metric spaces. In fact, we combine the ideas of controlled metric spaces, modular metric spaces and generalized metric spaces. We present some fixed point results in this new framework.

2. Main Results

In this section, we first introduce the following definitions and then present the main results. Now, we present the concept of a triple-controlled modular $\hbar$-metric space (TCM$\hbar$M space).

Definition 8. Let $\mathcal{Q}$ be a nonempty set and $\alpha_\lambda, \beta_\lambda, \gamma_\lambda : (0, \infty) \times \mathcal{Q}^3 \to [1, \infty)$. Suppose that the mapping $\hbar : (0, \infty) \times \mathcal{Q}^3 \to [0, \infty)$ be a function satisfying the following properties:

($\hbar$1) $\hbar_\lambda(\ell, \ell', \ell'') = 0$ iff $\ell = \ell' = \ell''$,
($\hbar$2)

$$\hbar_{\lambda+\mu+\eta}(\ell, \ell', \ell'') \leq \alpha_\lambda(\ell, a, \ell)\hbar_\lambda(\ell, a, \ell) + \beta_\lambda(\ell', a, \ell')\hbar_\mu(\ell', a, \ell') + \gamma_\lambda(\ell'', a, \ell'')\hbar_\eta(\ell'', a, \ell'')$$

for all $\ell, \ell', \ell'', a \in \mathcal{Q}$ and for all $\lambda, \mu, \eta > 0$ (rectangle inequality).

Then, the function $\hbar$ is called a TCM$\hbar$M on $\mathcal{Q}$ and $(\mathcal{Q}, \hbar)$ is called a TCM$\hbar$M space.

Remark 1. Note that in such extended spaces, we will not have the symmetry property. Otherwise, α_λ, β_λ, and γ_λ must be equal to 1.

Example 1. Let $\mathcal{Q} = C([a, b], (-\infty, +\infty))$ be the set of all continuous real-valued functions on $[a, b]$. Define $\hbar : (0, \infty) \times \mathcal{Q}^3 \to [0, \infty)$ by

$$\hbar_\lambda(\ell, \ell', \ell'') = \sup_{t \in [a,b]} \frac{|\ell(t) - \ell'(t)| + |\ell'(t) - \ell''(t)|}{\lambda}, \quad \lambda > 0,$$

and

$$\alpha_\lambda(\ell(t), \ell'(t), \ell''(t)) = \beta_\lambda(\ell(t), \ell'(t), \ell''(t)) = \gamma_\lambda(\ell(t), \ell'(t), \ell''(t))$$
$$= \sup_{t\in[a,b]} \max\{|\ell(t)|, |\ell'(t)|, |\ell''(t)|\} + \lambda + 1.$$

For all $\ell, \ell', \ell'', a \in \mathcal{Q}$, we have

$$\begin{aligned}
&\hbar_\lambda(\ell(t), \ell'(t), \ell''(t)) \\
&\quad = \sup_{t\in[a,b]} \frac{|\ell(t) - \ell'(t)| + |\ell'(t) - \ell''(t)|}{\lambda} \\
&\quad \leq \sup_{t\in[a,b]} \frac{|\ell(t) - a(t)| + |a(t) - \ell'(t)|}{\lambda} \\
&\qquad + \sup_{t\in[a,b]} \frac{|\ell'(t) - a(t)| + |a(t) - \ell''(t)|}{\lambda} \\
&\qquad + \sup_{t\in[a,b]} \frac{|\ell(t) - a(t)| + |a(t) - \ell''(t)|}{\lambda} \\
&\quad \leq \hbar_\lambda(l(t), a(t), l(t)) + \hbar_\lambda(l'(t), a(t), l'(t)) + \hbar_\lambda(l''(t), a(t), l''(t)) \\
&\quad \leq \alpha_\lambda(l(t), a(t), l(t))\hbar_\lambda(l(t), a(t), l(t)) + \beta_\lambda(l'(t), a(t), l'(t)) \\
&\qquad \hbar_\lambda(l'(t), a(t), l'(t)) + \gamma_\lambda(l''(t), a(t), l''(t))\hbar_\lambda(l''(t), a(t), l''(t)).
\end{aligned}$$

In the last part of the above inequality, we use the fact that for all $\ell, \ell', a \in \mathcal{Q}$, we have $\alpha_\lambda(\ell(t), a(t), \ell(t)) \geq 1$, $\beta_\lambda(\ell'(t), a(t), \ell'(t)) \geq 1$ and $\gamma_\lambda(\ell''(t), a(t), \ell''(t)) \geq 1$. Hence, $\hbar$ is a TCM$\hbar$M space.

Definition 9. Let $(\mathcal{Q}, \hbar)$ be a TCM$\hbar$M space. Then

(i) A sequence $\{\ell_n\}$ is called convergent if and only if there exists $\ell \in \mathcal{Q}$ such that $\hbar_\lambda(\ell_n, \ell, \ell_n)$ or $\hbar_\lambda(\ell, \ell_n, \ell)$ go to 0 as $n \to \infty$. In this case, we write

$$\lim_{n\to+\infty} \ell_n = \ell.$$

(ii) A sequence $\{\ell_n\}$ is called $\hbar$-Cauchy if, for each $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that $\hbar_\lambda(\ell_n, \ell_m, \ell_n) < \epsilon$ for all $m, n \geq N$.

Definition 10. The TCM$\hbar$M space $(\mathcal{Q}, \hbar)$ is called complete if, for each $\hbar$-Cauchy sequence $\{\ell_n\}$, there exists $\ell \in \mathcal{Q}$ such that

$$\lim_{n\to+\infty} \hbar_\lambda(\ell_n, \ell, \ell_n) = 0.$$

Remark 2. Let $(\mathcal{Q}, \hbar)$ be a TCM$\hbar$M space. Then $\hbar_\lambda(\ell_n, \ell, \ell_n) \to 0$ $\Leftrightarrow \hbar_\lambda(\ell, \ell_n, \ell) \to 0$.

Lemma 1. *Let $(\mathcal{Q}, \hbar)$ be a TCM$\hbar$M space with continuous control function β_λ. If there exist sequences $\{\ell_n\}$ and $\{\ell'_n\}$ such that $\lim_{n\to\infty} \ell_n = \ell$ and $\lim_{n\to\infty} \ell'_n = \ell'$, then*

$$\beta_\lambda^{-1}(\ell', \ell, \ell')\beta_\lambda^{-1}(\ell, \ell', \ell)\hbar_\lambda(\ell, \ell', \ell) \le \limsup_{n\to\infty} \hbar_{\frac{\lambda}{9}}(\ell_n, \ell'_n, \ell_n)$$

and

$$\limsup_{n\to\infty} \hbar_\lambda(\ell_n, \ell'_n, \ell_n) \le \beta_\lambda(\ell', \ell, \ell')\beta_\lambda(\ell, \ell', \ell)\hbar_{\frac{\lambda}{9}}(\ell, \ell', \ell).$$

In particular, if $\ell = \ell'$, then we have $\limsup_{n\to\infty} \hbar_\lambda(\ell_n, \ell'_n, \ell_n) = 0$. Moreover, suppose that $\{\ell_n\}$ is convergent to ℓ and $\ell' \in \mathcal{Q}$ is arbitrary. Then, we have

$$\beta_\lambda^{-1}(\ell', \ell, \ell')\beta_\lambda^{-1}(\ell, \ell', \ell)\hbar_\lambda(\ell, \ell', \ell) \le \limsup_{n\to\infty} \hbar_{\frac{\lambda}{9}}(\ell_n, \ell', \ell_n)$$

and

$$\limsup_{n\to\infty} \hbar_\lambda(\ell_n, \ell', \ell_n) \le \beta_\lambda(\ell', \ell, \ell')\beta_\lambda(\ell, \ell', \ell)\hbar_{\frac{\lambda}{9}}(\ell, \ell', \ell).$$

Proof. (a) Using the triangular inequality, one writes

$$\begin{aligned}
\hbar_\lambda(\ell, \ell', \ell) &\le \alpha_\lambda(\ell, \ell_n, \ell)\hbar_{\frac{\lambda}{3}}(\ell, \ell_n, \ell) + \beta_\lambda(\ell', \ell_n, \ell')\hbar_{\frac{\lambda}{3}}(\ell', \ell_n, \ell') \\
&\quad + \gamma_\lambda(\ell, \ell_n, \ell)\hbar_{\frac{\lambda}{3}}(\ell, \ell_n, \ell)) \\
&\le (\alpha_\lambda(\ell, \ell_n, \ell) + \gamma_\lambda(\ell, \ell_n, \ell))\hbar_{\frac{\lambda}{3}}(\ell, \ell_n, \ell) + \beta_\lambda(\ell', \ell_n, \ell') \\
&\quad \times [(\alpha_\lambda(\ell', \ell'_n, \ell') + \gamma_\lambda(\ell', \ell'_n, \ell'))\hbar_{\frac{\lambda}{9}}(\ell', \ell'_n, \ell') \\
&\quad + \beta_\lambda(\ell_n, \ell'_n, \ell_n)\hbar_{\frac{\lambda}{9}}(\ell_n, \ell'_n, \ell_n)]
\end{aligned}$$

and

$$\begin{aligned}\hbar_\lambda(\ell_n,\ell'_n,\ell_n) &\leq \alpha_\lambda(\ell_n,\ell,\ell_n)\hbar_{\frac{\lambda}{3}}(\ell_n,\ell,\ell_n)+\beta_\lambda(\ell'_n,\ell,\ell'_n)\hbar_{\frac{\lambda}{3}}(\ell'_n,\ell,\ell'_n)\\ &\quad+\gamma_\lambda(\ell_n,\ell,\ell_n)\hbar_{\frac{\lambda}{3}}(\ell_n,\ell,\ell_n))\\ &\leq (\alpha_\lambda(\ell_n,\ell,\ell_n)+\gamma_\lambda(\ell_n,\ell,\ell_n))\hbar_{\frac{\lambda}{3}}(\ell_n,\ell,\ell_n)\\ &\quad+\beta_\lambda(\ell'_n,\ell,\ell'_n)[(\alpha_\lambda(\ell'_n,\ell',\ell'_n)+\gamma_\lambda(\ell'_n,\ell',\ell'_n))\hbar_{\frac{\lambda}{9}}\\ &\quad\times(\ell'_n,\ell',\ell'_n)+\beta_\lambda(\ell,\ell',\ell)\hbar_{\frac{\lambda}{9}}(\ell,\ell',\ell)].\end{aligned}$$

Taking the limit as $n\to\infty$, the result is obtained.

(b) Using the triangular inequality,

$$\begin{aligned}\hbar_\lambda(\ell,\ell',\ell) &\leq \alpha_\lambda(\ell,\ell_n,\ell)\hbar_{\frac{\lambda}{3}}(\ell,\ell_n,\ell)+\beta_\lambda(\ell',\ell_n,\ell')\hbar_{\frac{\lambda}{3}}(\ell',\ell_n,\ell')\\ &\quad+\gamma_\lambda(\ell,\ell_n,\ell)\hbar_{\frac{\lambda}{3}}(\ell,\ell_n,\ell))\\ &\leq (\alpha_\lambda(\ell,\ell_n,\ell)+\gamma_\lambda(\ell,\ell_n,\ell))\hbar_{\frac{\lambda}{3}}(\ell,\ell_n,\ell)\\ &\quad+\beta_\lambda(\ell',\ell_n,\ell')[(\alpha_\lambda(\ell',\ell',\ell')+\gamma_\lambda(\ell',\ell',\ell'))\hbar_{\frac{\lambda}{9}}(\ell',\ell',\ell')\\ &\quad+\beta_\lambda(\ell_n,\ell',\ell_n)\hbar_{\frac{\lambda}{9}}(\ell_n,\ell',\ell_n)]\end{aligned}$$

and

$$\begin{aligned}\hbar_\lambda(\ell_n,\ell',\ell_n) &\leq \alpha_\lambda(\ell_n,\ell,\ell_n)\hbar_{\frac{\lambda}{3}}(\ell_n,\ell,\ell_n)+\beta_\lambda(\ell',\ell,\ell')\hbar_{\frac{\lambda}{3}}(\ell',\ell,\ell')\\ &\quad+\gamma_\lambda(\ell_n,\ell,\ell_n)\hbar_{\frac{\lambda}{3}}(\ell_n,\ell,\ell_n))\\ &\leq (\alpha_\lambda(\ell_n,\ell,\ell_n)+\gamma_\lambda(\ell_n,\ell,\ell_n))\hbar_{\frac{\lambda}{3}}(\ell_n,\ell,\ell_n)\\ &\quad+\beta_\lambda(\ell',\ell,\ell')[(\alpha_\lambda(\ell',\ell',\ell')+\gamma_\lambda(\ell',\ell',\ell'))\hbar_{\frac{\lambda}{9}}(\ell',\ell',\ell')\\ &\quad+\beta_\lambda(\ell,\ell',\ell)\hbar_{\frac{\lambda}{9}}(\ell,\ell',\ell)].\end{aligned}$$

□

Let **F** be the set of all functions $F:\mathcal{R}^+\to\mathcal{R}$ such that

($F1$) F is a continuous and strictly increasing mapping,
($F2$) $\lim\mu_n=0$ if and only if $\lim F(\mu_n)=-\infty$ for each sequence $\{\mu_n\}$ in $(0,+\infty)$.

Now, let Ψ be the collection of all mappings $\psi : [0,+\infty) \to [0,+\infty)$ such that

(i) ψ is strictly increasing and upper semicontinuous;
(ii) $\{\psi^n(t)\}_{n\in N}$ converges to 0 as $n \to \infty$, for all $t > 0$;
(iii) $\psi(t) < t$ for all $t > 0$.

Definition 11. Let $\mathcal{Q}$ be a nonempty set and let $\eta : (0,\infty) \times \mathcal{Q}^3 \to [0,\infty]$. A mapping $Ł : \mathcal{Q} \to \mathcal{Q}$ is called an η-admissible mapping if

$$\eta_\lambda(\ell,\ell',\ell) \geq 1 \Rightarrow \eta_\lambda(Ł\ell, Ł\ell', Ł\ell) \geq 1 \quad \textit{for all } \ell,\ell' \in \mathcal{Q} \textit{ and } \lambda > 0.$$

Definition 12. Let $(\mathcal{Q},\hbar)$ be a TCM$\hbar$M space and $\eta : (0,\infty) \times \mathcal{Q}^3 \to [0,\infty]$. We call the mapping $Ł : \mathcal{Q} \to \mathcal{Q}$ an (η,ψ)-rational F-contractive mapping of type-I if

$$\begin{aligned}
&\hbar_\lambda(Ł\ell, Ł\ell', Ł\ell) > 0 \\
&\quad \Rightarrow \tau + F(\beta_\lambda(Ł\ell', Ł\ell, Ł\ell')\beta_\lambda(Ł\ell, Ł\ell', Ł\ell)\eta_\lambda(\ell,\ell',\ell)\hbar_\lambda(Ł\ell, Ł\ell', Ł\ell)) \\
&\quad \leq F(\psi(\mathcal{M}(\ell,\ell',\ell))), \qquad (1)
\end{aligned}$$

for all $\ell,\ell' \in \mathcal{Q}$ where $\psi \in \Psi$, $\tau > 0$ $\lambda > 0$ and

$$\begin{aligned}
\mathcal{M}(\ell,\ell',\ell) = \max\Bigg\{ &\hbar_\lambda(\ell,\ell',\ell), \hbar_\lambda(\ell, Ł\ell, \ell), \hbar_\lambda(\ell', Ł\ell', \ell'), \\
&\frac{\hbar_\lambda(\ell, Ł\ell, \ell)\hbar_\lambda(\ell', Ł\ell', \ell')}{1 + \hbar_\lambda(\ell,\ell',\ell)}, \frac{\hbar_\lambda(\ell, Ł\ell, \ell)\hbar_\lambda(\ell', Ł\ell', \ell')}{1 + \hbar_\lambda(Ł\ell, Ł\ell', Ł\ell)} \Bigg\}.
\end{aligned}$$

Note that if $F(t) = \ln(t)$ for all $t > 0$ and $\beta_\lambda = 1$, then we get

$$\begin{aligned}
\eta_\lambda(\ell,\ell',\ell)\hbar_\lambda(Ł\ell, Ł\ell', Ł\ell)) &\leq e^{-\tau}(\psi(\hbar_\lambda(\ell,\ell',\ell))) \\
&\leq \psi(\hbar_\lambda(\ell,\ell',\ell)), \quad \lambda > 0.
\end{aligned}$$

Therefore, Ł is an (η,ψ)-contraction, which will be a generalization of Samet's work.[18]

We now state the main theorem of this section as follows which says that any (η,ψ)-rational F-contractive mapping of type-I defined in a complete TCM$\hbar$M space has a unique fixed point:

Theorem 1. *Let* $(\mathcal{Q}, \hbar)$ *be a complete TCM*$\hbar$*M space,* $\eta : (0, \infty) \times \mathcal{Q}^3 \to [0, \infty]$*, and* Ł *be an* η*-admissible self-mapping on* $\mathcal{Q}$ *so that*

(i) *there is* $\ell_0 \in \mathcal{Q}$ *such that* $\eta_\lambda(\ell_0, Ł\ell_0, \ell_0) \geq 1$*;*
(ii) Ł *is an* (η, ψ)*-rational* F*-contractive mapping of type-I.*

Then Ł *has a unique fixed point in* $\mathcal{Q}$*.*

Proof. Choose an $\ell_0 \in \mathcal{Q}$ such that it satisfies $\eta_\lambda(\ell_0, Ł\ell_0, \ell_0) \geq 1$, and set

$$\ell_n = Ł(\ell_{n-1}),\ n = 1, 2, \ldots.$$

As Ł is η-admissible,

$$\begin{aligned}\eta_\lambda(\ell_0, Ł\ell_0, \ell_0) = \eta_\lambda(\ell_0, \ell_1, \ell_0) \geq 1 &\Rightarrow \eta_\lambda(Ł\ell_0, Ł\ell_1, Ł\ell_0)\\ &= \eta_\lambda(\ell_1, \ell_2, \ell_1) \geq 1.\end{aligned}$$

By induction, we have $\eta_\lambda(\ell_n, \ell_{n+1}, \ell_n) \geq 1$ for all $n \geq 0$.

By condition (1), we get

$$\begin{aligned}&F(\hbar_\lambda(\ell_{n+1}, \ell_{n+2}, \ell_{n+1}))\\ &\quad = F(\hbar_\lambda(Ł\ell_n, Ł\ell_{n+1}, Ł\ell_n))\\ &\quad \leq F(\beta_\lambda(Ł\ell_{n+1}, Ł\ell_n, Ł\ell_{n+1})\beta_\lambda(Ł\ell_n, Ł\ell_{n+1}, Ł\ell_n)\\ &\qquad \times\ \eta_\lambda(\ell_n, \ell_{n+1}, \ell_n)\hbar_\lambda(Ł\ell_n, Ł\ell_{n+1}, Ł\ell_n))\\ &\quad \leq F(\psi(\mathcal{M}(\ell_n, \ell_{n+1}, \ell_n))) - \tau,\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(\ell_n, \ell_{n+1}, \ell_n) = \max\Bigg\{&\hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n), \hbar_\lambda(\ell_n, Ł\ell_n, \ell_n),\\ &\hbar_\lambda(\ell_{n+1}, Ł\ell_{n+1}, \ell_{n+1}),\\ &\frac{\hbar_\lambda(\ell_n, Ł\ell_n, \ell_n)\hbar_\lambda(\ell_{n+1}, Ł\ell_{n+1}, \ell_{n+1})}{1 + \hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n)},\\ &\frac{\hbar_\lambda(\ell_n, Ł\ell_n, \ell_n)\hbar_\lambda(\ell_{n+1}, Ł\ell_{n+1}, \ell_{n+1})}{1 + \hbar_\lambda(Ł\ell_n, Ł\ell_{n+1}, Ł\ell_n)}\Bigg\}\end{aligned}$$

$$= \max\Big\{\hbar_\lambda(\ell_n,\ell_{n+1},\ell_n), \hbar_\lambda(\ell_n,\ell_{n+1},\ell_n),$$
$$\hbar_\lambda(\ell_{n+1},\ell_{n+2},\ell_{n+1}),$$
$$\frac{\hbar_\lambda(\ell_n,\ell_{n+1},\ell_n)\hbar_\lambda(\ell_{n+1},\ell_{n+2},\ell_{n+1})}{1+\hbar_\lambda(\ell_n,\ell_{n+1},\ell_n)},$$
$$\frac{\hbar_\lambda(\ell_n,\ell_{n+1},\ell_n)\hbar_\lambda(\ell_{n+1},\ell_{n+2},\ell_{n+1})}{1+\hbar_\lambda(\ell_{n+1},\text{Ł}\ell_{n+2},\ell_{n+1})}\Big\}$$
$$= \max\big\{\hbar_\lambda(\ell_n,\ell_{n+1},\ell_n), \hbar_\lambda(\ell_{n+1},\ell_{n+2},\ell_{n+1})\big\}.$$

If $\mathcal{M}(\ell_n,\ell_{n+1},\ell_n) = \hbar_\lambda(\ell_{n+1},\ell_{n+2},\ell_{n+1})$, then we simply see that it is impossible. Therefore, $\mathcal{M}(\ell_n,\ell_{n+1},\ell_n) = \hbar_\lambda(\ell_n,\ell_{n+1},\ell_n)$ for all $n \in \mathbb{N}$, and

$$\begin{aligned} F(\hbar_\lambda(\ell_{n+1},\ell_{n+2},\ell_{n+1})) &\le F(\psi(\mathcal{M}(\ell_n,\ell_{n+1},\ell_n))) - \tau \\ &= F(\psi(\hbar_\lambda(\ell_n,\ell_{n+1},\ell_n))) - \tau \\ &< F(\hbar_\lambda(\ell_n,\ell_{n+1},\ell_n)) - \tau. \end{aligned}$$

This implies that

$$\hbar_\lambda(\ell_{n+1},\ell_{n+2},\ell_{n+1}) < \hbar_\lambda(\ell_n,\ell_{n+1},\ell_n).$$

On the other hand,

$$\begin{aligned} F(\hbar_\lambda(\ell_{n+1},\ell_{n+2},\ell_{n+1})) &\le F(\hbar_\lambda(\ell_n,\ell_{n+1},\ell_n)) - \tau \\ &\le F(\hbar_\lambda(\ell_{n-1},\ell_n,\ell_{n-1})) - 2\tau \\ &\;\;\vdots \\ &\le F(\hbar_\lambda(\ell_0,\ell_1,\ell_0)) - n\tau. \end{aligned}$$

Now, by $(F2)$, if $n \to \infty$, then

$$\hbar_\lambda(\ell_{n+1},\ell_{n+2},\ell_{n+1}) \to 0^+. \tag{2}$$

Now, we show that $\ell_n \neq \ell_m$ for $n \neq m$. Let $\ell_n = \ell_m$ for some $n > m$. So, we have $\ell_{n+1} = L\ell_n = L\ell_m = \ell_{m+1}$. By continuing this method, we conclude that $\ell_{n+k} = \ell_{m+k}$ for all $k \in \mathbb{N}$. Then inequality (1) yields

$$\begin{aligned} F(\hbar_\lambda(\ell_m, \ell_{m+1}, \ell_m)) &= F(\hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n)) \\ &= F(\hbar_\lambda(\text{Ł}\ell_{n-1}, \text{Ł}\ell_n, \text{Ł}\ell_{n-1})) \\ &\leq F(\beta_\lambda(\text{Ł}\ell_n, \text{Ł}\ell_{n-1}, \text{Ł}\ell_n)\beta_\lambda(\text{Ł}\ell_{n-1}, \text{Ł}\ell_n, \text{Ł}\ell_{n-1}) \\ &\quad \times\ \eta_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1})\hbar_\lambda(\text{Ł}\ell_{n-1}, \text{Ł}\ell_n, \text{Ł}\ell_{n-1})) \\ &\leq F(\psi(\mathcal{M}(\ell_{n-1}, \ell_n, \ell_{n-1}))) - \tau, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \mathcal{M}(\ell_{n-1}, \ell_n, \ell_{n-1}) = \max\Bigg\{ & \hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1}), \hbar_\lambda(\ell_{n-1}, \text{Ł}\ell_{n-1}, \ell_{n-1}), \\ & \times\ \hbar_\lambda(\ell_n, \text{Ł}\ell_n, \ell_n), \\ & \times\ \frac{\hbar_\lambda(\ell_{n-1}, \text{Ł}\ell_{n-1}, \ell_{n-1})\hbar_\lambda(\ell_n, \text{Ł}\ell_n, \ell_n)}{1 + \hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1})}, \\ & \times\ \frac{\hbar_\lambda(\ell_{n-1}, \text{Ł}\ell_{n-1}, \ell_{n-1})\hbar_\lambda(\ell_n, \text{Ł}\ell_n, \ell_n)}{1 + \hbar_\lambda(\text{Ł}\ell_{n-1}, \text{Ł}\ell_n, \text{Ł}\ell_{n-1})} \Bigg\} \\ = \max\Bigg\{ & \hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1}), \hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1}), \\ & \times\ \hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n), \\ & \times\ \frac{\hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1})\hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n)}{1 + \hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1})}, \\ & \times\ \frac{\hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1})\hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n)}{1 + \hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n)} \Bigg\} \\ = \max\big\{ & \hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1}), \hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n)\big\}. \end{aligned}$$

If $\mathcal{M}(\ell_{n-1}, \ell_n, \ell_{n-1}) = \hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n)$, then

$$\begin{aligned} F(\hbar_\lambda(\ell_m, \ell_{m+1}, \ell_m)) &\leq F(\psi(\hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n))) - \tau \\ &\leq F(\psi^{n-m+1}(\hbar_\lambda(\ell_m, \ell_{m+1}, \ell_m))) - \tau. \end{aligned}$$

This implies that

$$\hbar_\lambda(\ell_m, \ell_{m+1}, \ell_m) \leq \psi^{n-m+1}(\hbar_\lambda(\ell_m, \ell_{m+1}, \ell_m)). \tag{4}$$

If $\mathcal{M}(\ell_{n-1}, \ell_n, \ell_{n-1}) = \hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1})$, then

$$\begin{aligned} F(\hbar_\lambda(\ell_m, \ell_{m+1}, \ell_m)) &\leq F(\psi(\hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1}))) - \tau \\ &\leq F(\psi^{n-m}(\hbar_\lambda(\ell_m, \ell_{m+1}, \ell_m))) - \tau. \end{aligned}$$

This implies that

$$\hbar_\lambda(\ell_m, \ell_{m+1}, \ell_m) \leq \psi^{n-m}(\hbar_\lambda(\ell_m, \ell_{m+1}, \ell_m)). \tag{5}$$

By (4) and (5), we have

$$\hbar_\lambda(\ell_m, \ell_{m+1}, \ell_m) < \hbar_\lambda(\ell_m, \ell_{m+1}, \ell_m),$$

a contradiction. Hence, $\ell_n \neq \ell_m$ for all $n \neq m$.

Now, we show that $\{\ell_n\}$ is a Cauchy sequence in $(\mathcal{Q}, \hbar)$. Let there be $\varepsilon > 0$ so that for all $i \in \mathbb{N}$, there are m_i, n_i with $i < m_i < n_i$ so that

$$\hbar_\lambda(\ell_{m_i}, \ell_{n_i}, \ell_{m_i}) \geq \varepsilon, \ \textit{for all } \lambda > 0, \tag{6}$$

i.e.

$$\hbar_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i}) < \varepsilon. \tag{7}$$

From (6), we have

$$\begin{aligned} \varepsilon \leq \hbar_{9\lambda}(\ell_{m_i}, \ell_{n_i}, \ell_{m_i}) &\leq \alpha_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i})\hbar_{3\lambda}(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) \\ &\quad + \beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\hbar_{3\lambda}(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})) \\ &\quad + \gamma_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i})\hbar_{3\lambda}(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) \\ &= \Big[\alpha_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) + \gamma_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i})\Big] \\ &\quad \times \hbar_{3\lambda}(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) \\ &\quad + \beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\hbar_{3\lambda}(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i}) \\ &\leq \Big[\alpha_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) + \gamma_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i})\Big] \\ &\quad \times \hbar_{3\lambda}(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) \\ &\quad + \beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\beta_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}}) \\ &\quad \times \hbar_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}}). \end{aligned}$$

Letting $i \to \infty$ and using (2), we get

$$\varepsilon \leq \limsup_{i\to\infty} \beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\beta_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}}) \times \hbar_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}}). \tag{8}$$

On the other hand, we have

$$\begin{aligned}
&F(\beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\beta_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}})\hbar_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}}))\\
&\quad = F(\beta_\lambda(\text{Ł}\ell_{n_{i-1}}, \text{Ł}\ell_{m_i}, \text{Ł}\ell_{n_{i-1}})\beta_\lambda(\text{Ł}\ell_{m_i}, \text{Ł}\ell_{n_{i-1}}, \text{Ł}\ell_{m_i})\\
&\qquad \times\ \hbar_\lambda(\text{Ł}\ell_{m_i}, \text{Ł}\ell_{n_{i-1}}, \text{Ł}\ell_{m_i}))\\
&\quad \leq F(\beta_\lambda(\text{Ł}\ell_{n_{i-1}}, \text{Ł}\ell_{m_i}, \text{Ł}\ell_{n_{i-1}})\beta_\lambda(\text{Ł}\ell_{m_i}, \text{Ł}\ell_{n_{i-1}}, \text{Ł}\ell_{m_i})\\
&\qquad \times\ \eta_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i})\hbar_\lambda(\text{Ł}\ell_{m_i}, \text{Ł}\ell_{n_{i-1}}, \text{Ł}\ell_{m_i}))\\
&\quad \leq F(\psi(\mathcal{M}(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i})) - \tau,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
\mathcal{M}(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i}) &= \max\Bigg\{\hbar_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i}), \hbar_\lambda(\ell_{m_i}, \text{Ł}\ell_{m_i}, \ell_{m_i}),\\
&\qquad \times\ \hbar_\lambda(\ell_{n_{i-1}}, \text{Ł}\ell_{n_{i-1}}, \ell_{n_{i-1}}),\\
&\qquad \times\ \frac{\hbar_\lambda((\ell_{m_i}, \text{Ł}\ell_{m_i}, \ell_{m_i})\hbar_\lambda(\ell_{n_{i-1}}, \text{Ł}\ell_{n_{i-1}}, \ell_{n_{i-1}})}{1 + \hbar_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i})},\\
&\qquad \times\ \frac{\hbar_\lambda((\ell_{m_i}, \text{Ł}\ell_{m_i}, \ell_{m_i})\hbar_\lambda(\ell_{n_{i-1}}, \text{Ł}\ell_{n_{i-1}}, \ell_{n_{i-1}})}{1 + \hbar_\lambda(\text{Ł}\ell_{m_i}, \text{Ł}\ell_{n_{i-1}}, \text{Ł}\ell_{m_i})}\Bigg\}\\
&= \max\Bigg\{\hbar_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i}), \hbar_\lambda(\ell_{m_i}, \ell_{m_{i-1}}, \ell_{m_i}),\\
&\qquad \times\ \hbar_\lambda(\ell_{n_{i-1}}, \ell_{n_i}, \ell_{n_{i-1}}),\\
&\qquad \times\ \frac{\hbar_\lambda((\ell_{m_i}, \ell_{m_{i-1}}, \ell_{m_i})\hbar_\lambda(\ell_{n_{i-1}}, \ell_{n_i}, \ell_{n_{i-1}})}{1 + \hbar_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i})},\\
&\qquad \times\ \frac{\hbar_\lambda((\ell_{m_i}, \ell_{m_{i-1}}, \ell_{m_i})\hbar_\lambda(\ell_{n_{i-1}}, \ell_{n_i}, \ell_{n_{i-1}})}{1 + \hbar_\lambda(\ell_{m_{i-1}}, \ell_{n_i}, \ell_{m_{i-1}})}\Bigg\}.
\end{aligned}$$

Taking the upper limit as $n \to \infty$ and by (2), we have

$$\begin{aligned} F(\varepsilon) &\leq \limsup_{i\to\infty} F(\beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\beta_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}}) \\ &\quad \times\ \hbar_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}})) \\ &\leq \limsup_{i\to\infty} F(\psi(\mathcal{M}(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i}))) - \tau \\ &\leq F(\psi(\varepsilon)) - \tau \\ &< F(\varepsilon), \end{aligned} \tag{10}$$

which is a contradiction. So, $\{\ell_n\}$ is a Cauchy sequence in $\mathcal{Q}$. Since $\mathcal{Q}$ is complete, then there exists $\ell \in \mathcal{Q}$ such that $\ell_n \to \ell$, i.e.

$$\lim_{n\to\infty} \hbar_\lambda(\ell, \ell_n, \ell) = \lim_{n\to\infty} \hbar_\lambda(\ell_n, \ell, \ell_n) = 0.$$

Now, we show that ℓ is a fixed point of Ł.

First, let Ł be continuous. Then we have

$$u = \lim_{n\to\infty} \ell_{n+1} = \lim_{n\to\infty} \text{Ł}\ell_n = \text{Ł}u.$$

Let Ł be not continuous. Now, by Lemma 1, we have

$$\begin{aligned} &F\left(\beta_\lambda(u, \text{Ł}u, u)\beta_\lambda(\text{Ł}u, u, \text{Ł}u)\eta_\lambda(u, u, u)\beta_\lambda^{-1}(u, \text{Ł}u, u)\right. \\ &\qquad \times\ \left.\beta_\lambda^{-1}(\text{Ł}u, u, \text{Ł}u)\hbar_\lambda(u, \text{Ł}u, u)\right) \\ &\quad \leq F\left(\lim\sup_{n\to\infty} \beta_\lambda(\text{Ł}\ell_n, \text{Ł}u, \text{Ł}\ell_n)\beta_\lambda(\text{Ł}u, \text{Ł}\ell_n, \text{Ł}u)\eta_\lambda(\ell_n, u, \ell_n)\right) \\ &\qquad \times \lim\sup_{n\to\infty} \hbar_{\frac{\lambda}{9}}(\text{Ł}\ell_n, \text{Ł}u, \text{Ł}\ell_n)) \\ &\quad \leq F\left(\psi\left(\lim\sup_{n\to\infty} \mathcal{M}(\ell_n, u, \ell_n)\right)\right) - \tau, \end{aligned}$$

where

$$
\begin{aligned}
&\mathcal{M}(\ell_n, u, \ell_n)\\
&\quad = \max\Bigg\{\hbar_{\frac{\lambda}{9}}(\ell_n, u, \ell_n), \hbar_{\frac{\lambda}{9}}(\ell_n, Łℓ_n, \ell_n), \hbar_{\frac{\lambda}{9}}(u, Łu, u),\\
&\qquad \frac{\hbar_{\frac{\lambda}{9}}(\ell_n, Łℓ_n, \ell_n)\hbar_{\frac{\lambda}{9}}(u, Łu, u)}{1+\hbar_{\frac{\lambda}{9}}(\ell_n, u, \ell_n)}, \frac{\hbar_{\frac{\lambda}{9}}(\ell_n, Łℓ_n, \ell_n)\hbar_{\frac{\lambda}{9}}(u, Łu, u)}{1+\hbar_{\frac{\lambda}{9}}(Łℓ_n, Łu, Łℓ_n)}\Bigg\}\\
&\quad = \max\Bigg\{\hbar_{\frac{\lambda}{9}}(\ell_n, u, \ell_n), \hbar_{\frac{\lambda}{9}}(\ell_n, \ell_{n+1}, \ell_n), \hbar_{\frac{\lambda}{9}}(u, Łu, u),\\
&\qquad \frac{\hbar_{\frac{\lambda}{9}}(\ell_n, \ell_{n+1}, \ell_n)\hbar_{\frac{\lambda}{9}}(u, Łu, u)}{1+\hbar_{\frac{\lambda}{9}}(\ell_n, u, \ell_n)}, \frac{\hbar_{\frac{\lambda}{9}}(\ell_n, \ell_{n+1}, \ell_n)\hbar_{\frac{\lambda}{9}}(u, Łu, u)}{1+\hbar_{\frac{\lambda}{9}}(\ell_{n+1}, Łu, \ell_{n+1})}\Bigg\}.
\end{aligned}
$$

This gives, as $n \to \infty$,

$$
\begin{aligned}
&\beta_\lambda(u, Łu, u)\beta_\lambda(Łu, u, Łu)\eta_\lambda(u, u, u)\beta_\lambda^{-1}\\
&\qquad \times (Łu, u, Łu)\beta_\lambda^{-1}(u, Łu, u)\hbar_\lambda(u, Łu, u)\\
&\quad \le \psi\left(\limsup_{n\to\infty} \mathcal{M}(\ell_n, u, \ell_n)\right)\\
&\quad = \psi(\hbar_\lambda(u, Łu, u))\\
&\quad < \hbar_\lambda(u, Łu, u).
\end{aligned}
$$

So, $\psi(\hbar_\lambda(u, Łu, u)) = \hbar_\lambda(u, Łu, u)$. Hence, u is a fixed point of Ł. To show the uniqueness, assume that there exists $v \neq u \in \mathcal{Q}$ such that $Łu = u$ and $Łv = v$. Thus,

$$
\begin{aligned}
F(\hbar_\lambda(u, v, u)) &= F(\hbar_\lambda(Łu, Łv, Łu))\\
&\le F(\beta_\lambda(Łv, Łu, Łv)\beta_\lambda(Łu, Łv, Łu)\\
&\quad \times\ \eta_\lambda(u, v, u)\hbar_\lambda(Łu, Łv, Łu))\\
&\le F(\psi(\mathcal{M}(u, v, u))) - \tau,
\end{aligned}
$$

where

$$\begin{aligned}\mathcal{M}(u,v,u) &= \max\Big\{\hbar_\lambda(u,v,u), \hbar_\lambda(u,Łu,u), \hbar_\lambda(v,Łv,v),\\ &\qquad \frac{\hbar_\lambda(u,Łu,u)\hbar_\lambda(v,Łv,v)}{1+\hbar_\lambda(u,v,u)}, \frac{\hbar_\lambda(u,Łu,u)\hbar_\lambda(v,Łv,v)}{1+\hbar_\lambda(Łu,Łv,Łu)}\Big\}\\ &= \max\Big\{\hbar_\lambda(u,v,u), \hbar_\lambda(u,u,u), \hbar_\lambda(v,v,v),\\ &\qquad \frac{\hbar_\lambda(u,u,u)\hbar_\lambda(v,v,v)}{1+\hbar_\lambda(u,v,u)}, \frac{\hbar_\lambda(u,u,u)\hbar_\lambda(v,v,v)}{1+\hbar_\lambda(u,v,u)}\Big\}\\ &= \hbar_\lambda(u,v,u),\end{aligned}$$

which is a contradiction. Then $u = v$. Therefore, the fixed point of Ł is unique. □

Definition 13. Let $(\mathcal{Q},\hbar)$ be a TCM$\hbar$M space and $\eta : (0,\infty) \times \mathcal{Q}^3 \to [0,\infty]$. We call the mapping $Ł : \mathcal{Q} \to \mathcal{Q}$ an (η,ψ)-rational F-contractive mapping of type-II if for all $\ell,\ell' \in \mathcal{Q}$

$$\begin{aligned}&\hbar_\lambda(Ł\ell,Ł\ell',Ł\ell) > 0\\ &\quad \Rightarrow \tau + F(\beta_\lambda(Ł\ell',Ł\ell,Ł\ell')\beta_\lambda(Ł\ell,Ł\ell',Ł\ell)\eta_\lambda(\ell,\ell',\ell)\hbar_\lambda(Ł\ell,Ł\ell',Ł\ell))\\ &\quad \le F(\psi(\mathcal{M}(\ell,\ell',\ell))),\end{aligned} \tag{11}$$

where $\psi \in \Psi$, $\tau > 0$ and

$$\begin{aligned}\mathcal{M}(\ell,\ell',\ell) = \max\Big\{&\hbar_\lambda(\ell,\ell',\ell), \hbar_\lambda(\ell,Ł\ell,\ell), \hbar_\lambda(\ell',Ł\ell',\ell'),\\ &\frac{\hbar_\lambda(\ell,Ł\ell,\ell)\hbar_\lambda(\ell',Ł\ell',\ell')}{1+\hbar_\lambda(\ell,\ell',\ell)+\hbar_\lambda(\ell,Ł\ell',\ell)+\hbar_\lambda(\ell',Ł\ell,\ell')},\\ &\frac{\hbar_\lambda(\ell,Ł\ell',\ell)\hbar_\lambda(\ell,\ell',\ell)}{1+\hbar_\lambda(\ell,Ł\ell,\ell)+\hbar_\lambda(\ell',Ł\ell,\ell')+\hbar_\lambda(\ell',Ł\ell',\ell')}\Big\}.\end{aligned}$$

We now state another main theorem of this section, which says that any (η,ψ)-rational F-contractive mapping of type-II defined in a complete TCM$\hbar$M space has a unique fixed point.

Theorem 2. *Let $(\mathcal{Q},\hbar)$ be a complete ŁCM$\hbar$M space, $\eta:(0,\infty)\times\mathcal{Q}^3\to[0,\infty]$, and Ł be an η-admissible self-mapping on $\mathcal{Q}$ so that*

(i) *there is $\ell_0\in\mathcal{Q}$ such that $\eta_\lambda(\ell_0, Ł\ell_0, \ell_0)\geq 1$;*
(ii) *Ł is an (η,ψ)-rational F-contractive mapping of type-II.*

Then Ł has a unique fixed point in $\mathcal{Q}$.

Proof. Following the proof of Theorem 1, we can complete the proof. □

Example 2. Let $\mathcal{Q}=C([a,b],[0,1])$ be the set of all continuous real-valued functions on $[a,b]$. Consider the TCM$\hbar$M space and the functions α_λ, β_λ and γ_λ presented in Example 1. Choose

$$Ł\ell(t)=\begin{cases}\dfrac{\ell(t)}{6} & \text{if } \ell(t)\in\left[0,\dfrac{1}{2}\right);\\[2ex] \dfrac{\ell(t)}{12} & \text{if } \ell(t)\in\left[\dfrac{1}{2},1\right].\end{cases}$$

Also, $\psi(t)=\frac{t}{3}$, $\tau=\frac{1}{4}$, $F(\ell)=\ln\ell$ and $\eta_\lambda(\ell(t),\ell'(t),\ell''(t))=1$.

Consider the following cases:

(1) Let $\ell(t),\ell'(t)\in[0,\frac{1}{2})$. Then if $\ell'\leq\ell$, we have

$$\begin{aligned}&\tau+F(\beta_\lambda(Ł\ell'(t),Ł\ell(t),Ł\ell'(t))\beta_\lambda(Ł\ell(t),Ł\ell'(t),Ł\ell(t))\\ &\quad\times\ \eta_\lambda(\ell(t),\ell'(t),\ell(t))\hbar_\lambda(Ł\ell(t),Ł\ell'(t),Ł\ell(t)))\\ &\quad=\tau+\ln\left(\sup_{t\in[a.b]}\max\left\{\left|\frac{\ell(t)}{6}\right|,\left|\frac{\ell'(t)}{6}\right|\right\}+1\right)\\ &\qquad\times\left(\sup_{t\in[a.b]}\max\left\{\left|\frac{\ell(t)}{6}\right|,\left|\frac{\ell'(t)}{6}\right|\right\}+1\right)\left(\sup_{t\in[a.b]}\frac{2|\ell(t)-\ell'(t)|}{6\lambda}\right)\end{aligned}$$

$$\leq \tau + \ln\left(\sup_{t\in[a.b]} |\frac{\ell(t)}{6}| + 1\right)\left(\sup_{t\in[a.b]} |\frac{\ell(t)}{6}| + 1\right)$$

$$\times \left(\sup_{t\in[a.b]} \frac{2|\ell(t)-\ell'(t)|}{6\lambda}\right) \leq \ln\left(\sup_{t\in[a.b]} \frac{|\ell(t)-\ell'(t)|}{3\lambda}\right)$$

$$= F(\psi(\hbar_\lambda(\ell(t),\ell'(t),\ell(t)))) \leq F(\psi(\mathcal{M}(\ell(t),\ell'(t),\ell(t)))).$$

(2) Let $\ell(t), \ell'(t) \in [\frac{1}{2}, 1]$. Then if $\ell' \leq \ell$, we have

$$\tau + F(\beta_\lambda(\text{Ł}\ell'(t), \text{Ł}\ell(t), \text{Ł}\ell'(t))\beta_\lambda(\text{Ł}\ell(t), \text{Ł}\ell'(t), \text{Ł}\ell(t))$$

$$\times\ \eta_\lambda(\ell(t),\ell'(t),\ell(t))\hbar_\lambda(\text{Ł}\ell(t), \text{Ł}\ell'(t), \text{Ł}\ell(t)))$$

$$= \tau + \ln\left(\sup_{t\in[a.b]} \max\left\{\left|\frac{\ell(t)}{12}\right|, \left|\frac{\ell'(t)}{12}\right|\right\} + 1\right)$$

$$\times \left(\sup_{t\in[a.b]} \max\left\{\left|\frac{\ell(t)}{12}\right|, \left|\frac{\ell'(t)}{12}\right|\right\} + 1\right)\left(\sup_{t\in[a.b]} \frac{2|\ell(t)-\ell'(t)|}{12\lambda}\right)$$

$$\leq \tau + \ln\left(\sup_{t\in[a.b]} |\frac{\ell(t)}{12}| + 1\right)\left(\sup_{t\in[a.b]} |\frac{\ell(t)}{12}| + 1\right)$$

$$\times \left(\sup_{t\in[a.b]} \frac{2|\ell(t)-\ell'(t)|}{12\lambda}\right) \leq \ln\left(\sup_{t\in[a.b]} \frac{|\ell(t)-\ell'(t)|}{3\lambda}\right)$$

$$= F(\psi(\hbar_\lambda(\ell(t),\ell'(t),\ell(t)))) \leq F(\psi(\mathcal{M}(\ell(t),\ell'(t),\ell(t)))).$$

(3) Let $\ell'(t) \in \left[0, \frac{1}{2}\right)$ and $\ell(t) \in \left[\frac{1}{2}, 1\right]$. Clearly, $\ell' \leq \ell$ and then we have

$$\tau + F(\beta_\lambda(\text{Ł}\ell'(t), \text{Ł}\ell(t), \text{Ł}\ell'(t))\beta_\lambda(\text{Ł}\ell(t), \text{Ł}\ell'(t), \text{Ł}\ell(t))$$

$$\times\ \eta_\lambda(\ell(t),\ell'(t),\ell(t))\hbar_\lambda(\text{Ł}\ell(t), \text{Ł}\ell'(t), \text{Ł}\ell(t)))$$

$$= \tau + \ln\left(\sup_{t\in[a.b]} \max\left\{\left|\frac{\ell(t)}{12}\right|, \left|\frac{\ell'(t)}{6}\right|\right\} + 1\right)$$

$$\times \left(\sup_{t\in[a.b]} \max\left\{\left|\frac{\ell(t)}{12}\right|, \left|\frac{\ell'(t)}{6}\right|\right\} + 1\right)\left(\sup_{t\in[a.b]} \frac{2|\ell(t)-2\ell'(t)|}{12\lambda}\right)$$

$$\leq \tau + \ln\left(\sup_{t\in[a.b]} |\frac{\ell(t)}{6}| + 1\right)\left(\sup_{t\in[a.b]}\left|\frac{\ell(t)}{6}\right| + 1\right)$$
$$\times\left(\sup_{t\in[a.b]}\frac{2|\ell(t)-\ell'(t)|}{6\lambda}\right) \leq \ln\left(\sup_{t\in[a.b]}\frac{|\ell(t)-\ell'(t)|}{3\lambda}\right)$$
$$= F(\psi(\hbar_\lambda(\ell(t),\ell'(t),\ell(t)))) \leq F(\psi(\mathcal{M}(\ell(t),\ell'(t),\ell(t)))).$$

Therefore, all hypotheses of Theorem 1 are satisfied and Ł has a unique fixed point.

3. Weak-Wardowski Contraction

Let $\mathbf{F}$ be the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ such that

$(F1)$ F is a continuous and strictly increasing mapping,
$(F2)$ $\lim \mu_n = 0$ if and only if $\lim F(\mu_n) = -\infty$ for each sequence $\{\mu_n\}$ in $(0, +\infty)$.

Let Ψ' be the collection of all functions $\psi' : \mathbb{R} \to (0, +\infty)$ such that

(i) ψ' is continuous;
(ii) $\sum_{n=1}^{\infty} \psi'(t_n) = \infty$ and $\liminf \psi'(t_n) > 0$ for all $\{t_n\} \subseteq \mathbb{R}$.

Definition 14. Let $(\mathcal{Q}, \hbar)$ be a TCM$\hbar$M space. We call the mapping Ł $: \mathcal{Q} \to \mathcal{Q}$ a weak-Wardowski contraction if there are $\psi' \in \Psi'$ and $F \in \mathbf{F}$ such that

$$\begin{aligned} &F(\beta_\lambda(Ł\ell, Ł\ell', Ł\ell)\beta_\lambda(Ł\ell', Ł\ell, Ł\ell')\hbar_\lambda(Ł\ell, Ł\ell', Ł\ell)) \\ &\quad \leq F(\hbar_\lambda(\ell,\ell',\ell)) - \psi'(F(\hbar_\lambda(\ell,\ell',\ell))),\ \lambda > 0, \end{aligned} \tag{12}$$

for all $\ell, \ell' \in \mathcal{Q}$ with $Ł\ell \neq Ł\ell'$.

We now state another main theorem of this section, which says that any weak-Wardowski contraction defined in a complete TCM$\hbar$M space has a unique fixed point.

Theorem 3. *Let $(\mathcal{Q}, \hbar)$ be a complete TCM$\hbar$M space. Then, any weak-Wardowski contraction $Ł : \mathcal{Q} \to \mathcal{Q}$ has a unique fixed point.*

Proof. We choose an $\ell_0 \in \mathcal{Q}$, and set

$$\ell_n = Ł(\ell_{n-1}), \ n = 1, 2, \ldots.$$

By condition (12), we get

$$\begin{aligned}
&F(\hbar_\lambda(\ell_{n+1}, \ell_{n+2}, \ell_{n+1}))\\
&\quad = F(\hbar_\lambda(Ł\ell_n, Ł\ell_{n+1}, Ł\ell_n))\\
&\quad \leq F(\beta_\lambda(Ł\ell_n, Ł\ell_{n+1}, Ł\ell_n)\beta_\lambda(Ł\ell_{n+1}, Ł\ell_n, Ł\ell_{n+1})\\
&\qquad \times \hbar_\lambda(Ł\ell_n, Ł\ell_{n+1}, Ł\ell_n))\\
&\quad \leq F(\hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n)) - \psi'(F(\hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n)))\\
&\quad \leq F(\hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1})) - \psi'(F(\hbar_\lambda(\ell_{n-1}, \ell_n, \ell_{n-1})))\\
&\qquad - \psi'(F(\hbar_\lambda(\ell_n, \ell_{n+1}, \ell_n)))\\
&\qquad \vdots\\
&\quad \leq F(\hbar_\lambda(\ell_0, \ell_1, \ell_0)) - \sum_{i=0}^{n-1} \psi'(F(\hbar_\lambda(\ell_{i+1}, \ell_{i+2}, \ell_{i+1})))\\
&\quad \to -\infty, \ (as\ n \to \infty).
\end{aligned}$$

Then from $(F2)$, we have

$$\hbar_\lambda(\ell_{n+1}, \ell_{n+2}, \ell_{n+1}) \to 0^+. \tag{13}$$

Now, we show that $\{\ell_n\}$ is a Cauchy sequence in $(\mathcal{Q}, \hbar_\lambda)$. Let there be $\varepsilon > 0$ so that for all $i \in \mathbb{N}$ there are m_i, n_i with $i < m_i < n_i$ such that

$$\hbar_\lambda(\ell_{m_i}, \ell_{n_i}, \ell_{m_i}) \geq \varepsilon, \ \textit{for all}\ \lambda > 0, \tag{14}$$

i.e.

$$\hbar_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i}) < \varepsilon. \tag{15}$$

From (14), we have

$$\begin{aligned}
\varepsilon &\le \hbar_{9\lambda}(\ell_{m_i}, \ell_{n_i}, \ell_{m_i}) \\
&\le \alpha_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i})\hbar_{3\lambda}(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) \\
&\quad + \beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\hbar_{3\lambda}(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})) \\
&\quad + \gamma_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i})\hbar_{3\lambda}(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) \\
&= \Big[\alpha_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) + \gamma_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i})\Big]\hbar_{3\lambda}(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) \\
&\quad + \beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\hbar_{3\lambda}(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i}) \\
&\le \Big[\alpha_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) + \gamma_\lambda(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i})\Big]\hbar_{3\lambda}(\ell_{m_i}, \ell_{m_{i+1}}, \ell_{m_i}) \\
&\quad + \beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\beta_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}})\hbar_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}}).
\end{aligned}$$

Letting $i \to \infty$ and using (13), we get

$$\begin{aligned}
\varepsilon \le \limsup_{i\to\infty} &\beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\beta_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}}) \\
&\times \hbar_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}}).
\end{aligned} \tag{16}$$

On the other hand, we have

$$\begin{aligned}
&F(\beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\beta_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}})\hbar_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}})) \\
&\quad = F(\beta_\lambda(\text{Ł}\ell_{n_{i-1}}, \text{Ł}\ell_{m_i}, \text{Ł}\ell_{n_{i-1}})\beta_\lambda(\text{Ł}\ell_{m_i}, \text{Ł}\ell_{n_{i-1}}, \text{Ł}\ell_{m_i}) \\
&\qquad \times \hbar_\lambda(\text{Ł}\ell_{m_i}, \text{Ł}\ell_{n_{i-1}}, \text{Ł}\ell_{m_i})) \\
&\quad \le F(\hbar_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i})) - \psi'(F(\hbar_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i})).
\end{aligned} \tag{17}$$

Łaking the upper limit as $n \to \infty$, we have

$$\begin{aligned}
F(\varepsilon) &\le \limsup_{i\to\infty} F(\beta_\lambda(\ell_{n_i}, \ell_{m_{i+1}}, \ell_{n_i})\beta_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}}) \\
&\qquad \times \hbar_\lambda(\ell_{m_{i+1}}, \ell_{n_i}, \ell_{m_{i+1}})) \\
&\le \limsup_{i\to\infty} F(\hbar_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i})) \\
&\qquad - \liminf_{i\to\infty} \psi'(F(\hbar_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i}))) \\
&< F(\varepsilon) - \liminf_{i\to\infty} \psi'(F(\hbar_\lambda(\ell_{m_i}, \ell_{n_{i-1}}, \ell_{m_i}))),
\end{aligned}$$

which is a contradiction. So, $\{\ell_n\}$ is a Cauchy sequence in $\mathcal{Q}$. Since $\mathcal{Q}$ is complete, then there exists $\ell \in \mathcal{Q}$ such that $\ell_n \to \ell$, i.e.

$$\lim_{n\to\infty} \hbar_\lambda(\ell_n, \ell, \ell_n) = 0.$$

Now, we show that ℓ is a fixed point of Ł.

First, let Ł be continuous. Then we have

$$u = \lim_{n\to\infty} \ell_{n+1} = \lim_{n\to\infty} Ł\ell_n = Łu.$$

Let Ł be not continuous. Now, using Lemma 1, we have

$$\begin{aligned}
&F(\beta_\lambda(Łu, u, Łu)\beta_\lambda(u, Łu, u)\beta_\lambda^{-1}(Łu, u, Łu)\beta_\lambda^{-1}(u, Łu, u)\hbar_\lambda(u, Łu, u)) \\
&\quad \le F\left(\limsup_{n\to\infty} \hbar_{\frac{\lambda}{9}}(Ł\ell_n, Łu, Ł\ell_n)\right) \\
&\quad \le \limsup_{n\to\infty} F(\beta_\lambda(Ł\ell_n, Łu, Ł\ell_n)\beta_\lambda(Łu, Ł\ell_n, Łu)\hbar_{\frac{\lambda}{9}}(Ł\ell_n, Łu, Ł\ell_n)) \\
&\quad \le \limsup_{n\to\infty} F(\hbar_{\frac{\lambda}{9}}(\ell_n, u, \ell_n)) - \liminf_{n\to\infty} \psi'(F(\hbar_{\frac{\lambda}{9}}(\ell_n, u, \ell_n)) \\
&\quad < F\left(\limsup_{n\to\infty} \hbar_{\frac{\lambda}{9}}(\ell_n, u, \ell_n)\right) \to -\infty.
\end{aligned}$$

So, this gives that

$$\hbar_\lambda(u, Łu, u) = 0.$$

Hence, u is a fixed point of Ł. To show the uniqueness, assume that there exists $v \neq u \in \mathcal{Q}$ such that $Łu = u$ and $Łv = v$. Thus,

$$\begin{aligned}
F(\hbar_\lambda(u, v, u)) &= F(\hbar_\lambda(Łu, Łv, Łu)) \\
&\le F(\beta_\lambda(Łv, Łu, Łv)\beta_\lambda(Łu, Łv, Łu)\hbar_\lambda(Łu, Łv, Łu)) \\
&\le F(\hbar_\lambda(u, v, u)) - \psi'(F(\hbar_\lambda(u, v, u))) \\
&< F(\hbar_\lambda(u, v, u)),
\end{aligned}$$

which is a contradiction. Then $u = v$. Therefore, the fixed point of Ł is unique. □

Taking $\psi'(t) = \tau$, $\beta_\lambda = 1$, we have the following:

Corollary 1. *Let $(\mathcal{Q}, \hbar)$ be a TCM$\hbar$M space and $Ł : \mathcal{Q} \to \mathcal{Q}$ be a self-mapping such that*

$$\tau + F(\hbar_\lambda(Ł\ell, Ł\ell', Ł\ell)) \leq F(\hbar_\lambda(\ell, \ell', \ell)), \ \lambda > 0, \tag{18}$$

for all $\ell, \ell' \in \mathcal{Q}$ with $Ł\ell \neq Ł\ell'$. Then, $Ł$ has a unique fixed point in $\mathcal{Q}$.

If in the above corollary, we define $F(t) = \ln t$, then we have the following.

Corollary 2. *Let $(\mathcal{Q}, \hbar)$ be a TCM$\hbar$M space and $Ł : \mathcal{Q} \to \mathcal{Q}$ be a self-mapping such that*

$$\hbar_\lambda(Ł\ell, Ł\ell', Ł\ell) \leq k\hbar_\lambda(\ell, \ell', \ell), \ \lambda > 0, \tag{19}$$

for all $\ell, \ell' \in \mathcal{Q}$ with $Ł\ell \neq Ł\ell'$. Then, $Ł$ has a unique fixed point in $\mathcal{Q}$.

Taking $\psi(t) = \tau + \sum_{i=1}^{\infty} t^{2i}$, $\beta_\lambda = 1$, we have the following:

Corollary 3. *Let $(\mathcal{Q}, \hbar)$ be a TCM$\hbar$M space and $Ł : \mathcal{Q} \to \mathcal{Q}$ be a self-mapping such that*

$$\begin{aligned} F(\hbar_\lambda(Ł\ell, Ł\ell', Ł\ell)) &\leq F(\hbar_\lambda(\ell, \ell', \ell)) \\ &\quad - \tau - \sum_{i=1}^{\infty} F(\hbar_\lambda(\ell, \ell', \ell))^{2i}, \ \lambda > 0, \end{aligned} \tag{20}$$

for all $\ell, \ell' \in \mathcal{Q}$ with $Ł\ell \neq Ł\ell'$. Then, $Ł$ has a unique fixed point in $\mathcal{Q}$.

4. Triple-Controlled Fuzzy $\hbar$-Metric Spaces

In this section, we introduce the concept of a triple-controlled fuzzy $\hbar$-metric (TCF$\hbar$M) space. We create a relationship between TCM$\hbar$M and TCF$\hbar$M, and present some new fixed point results in TCF$\hbar$M space.

Definition 15 (Schweizer and Sklar (Ref. 19)). A binary operation $\star : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if it satisfies the following assertions:

(T1) $\star$ is commutative and associative;
(T2) $\star$ is continuous;
(T3) $a \star 1 = a$ for all $a \in [0,1]$;
(T4) $a \star b \leq c \star d$ when $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

Definition 16 (Ref.20). A three-tuple $(\mathcal{Q}, M, *)$ is said to be a fuzzy metric space if $\mathcal{Q}$ is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $\mathcal{Q}^2 \times (0,\infty)$ satisfying the following conditions:

(i) $M(\ell, \ell', t) > 0$;
(ii) $M(\ell, \ell', t) = 1$ for all $t > 0$ if and only if $\ell = \ell'$;
(iii) $M(\ell, \ell', t) = M(\ell', \ell, t)$;
(iv) $M(\ell, \ell', t) * M(\ell', \ell'', s) \leq M(\ell, \ell'', t+s)$;
(v) $M(\ell, \ell', \cdot) : (0,\infty) \to [0,1]$ is continuous; for all $\ell, \ell', \ell'' \in \mathcal{Q}$ and for all $t, s > 0$.

The function $M(\ell, \ell', t)$ denotes the degree of nearness between ℓ and ℓ' with respect to t.

Definition 17. A TCF$\hbar$M is an ordered triple $(\mathcal{Q}, \Lambda, \star)$ such that $\mathcal{Q}$ is a nonempty set, $\star$ is a continuous t-norm and Λ is a fuzzy set on $\mathcal{Q}^3 \times (0,\infty)$ satisfying the following conditions, for all $\ell, \ell', a \in \mathcal{Q}$ and $t, s, r > 0$,

(F1) $\Lambda(\ell, \ell', \ell'', t) > 0$;
(F2) $\Lambda(\ell, \ell', \ell'', t) = 1$ if and only if $\ell = \ell' = z$;
(F3) $\Lambda(\ell, \ell', \ell'', \alpha_t(\ell, a, \ell)t + \beta_s(\ell', a, \ell')s + \gamma_r(\ell'', a, \ell'')r) \geq \Lambda(\ell, a, \ell, t) \star \Lambda(\ell', a, \ell', s) \star \Lambda(\ell'', a, \ell'', r)$;
(F5) $\Lambda(\ell, \ell', \ell, \cdot) : (0,+\infty) \to (0,1]$ is left continuous.

Definition 18. Let $(\mathcal{Q}, \Lambda, \star)$ be a TCF$\hbar$M space and $\{a_n\}$ be a sequence in $\mathcal{Q}$ and $a \in \mathcal{Q}$.

(i) $\{a_n\}$ is said to be convergent and converges to a if $\lim \Lambda(a_n, a, a_n, t) = 1$ or $\lim \Lambda(a, a_n, a, t) = 1$ for all $t > 0$.
(ii) $\{a_n\}$ is said to be a Cauchy sequence if and only if for all $\varepsilon \in (0,1)$ and $t > 0$, there exists n_0 such that $\Lambda(a_n, a_m, a_n, t) > 1-\varepsilon$ for all $m, n \geq n_0$.
(iii) $(\mathcal{Q}, \Lambda, \star)$ is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 19. Let $(\mathcal{Q}, \Lambda, \star)$ be a TCF$\hbar$M space. The TCF$\hbar$M Λ is called rectangular if

$$\begin{aligned}\frac{1}{\Lambda(\ell,\ell',\ell'',\lambda)} - 1 \le \alpha_\lambda(\ell,a,\ell)&\left[\frac{1}{\Lambda(\ell,a,\ell,\lambda)} - 1\right]\\ &+ \beta_\lambda(\ell',a,\ell')\left[\frac{1}{\Lambda(\ell',a,\ell',\lambda)} - 1\right]\\ &+ \gamma_\lambda(\ell'',a,\ell''\left[\frac{1}{\Lambda(\ell'',a,\ell'',\lambda)} - 1\right].\end{aligned}$$

for all $\ell, \ell', a \in \mathcal{Q}$ and $\lambda > 0$.

Remark 3. Note that $\hbar_\lambda(\ell,\ell',\ell'') = \dfrac{1}{\Lambda(\ell,\ell',\ell'',\lambda)} - 1$ is a TCM$\hbar$M space, whenever Λ is a rectangular TCF$\hbar$M.

Definition 20. Let $(\mathcal{Q}, \Lambda, \star)$ be a rectangular TCF$\hbar$M space and $\eta_\lambda : (0,\infty) \times \mathcal{Q}^3 \to [0,\infty]$. We call the mapping $\text{Ł} : \mathcal{Q} \to \mathcal{Q}$ a Fuzzy (η,ψ)-rational F-contractive mapping of type-I if for all $\ell, \ell' \in \mathcal{Q}$, there exists a function $\psi \in \Psi$ and $\tau > 0$ satisfying

$$\begin{aligned}\tau + F(\beta_\lambda(\text{Ł}\ell',\text{Ł}\ell,\text{Ł}\ell')\beta_\lambda(\text{Ł}\ell,\text{Ł}\ell',\text{Ł}\ell)\eta_\lambda(\ell,\ell',\ell))&\frac{1}{\Lambda(\text{Ł}\ell,\text{Ł}\ell',\text{Ł}\ell,\lambda)} - 1)\\ \le F(\psi(\mathcal{M}_\lambda(\ell,\ell',\ell))),\ \lambda > 0,&\end{aligned} \tag{21}$$

where $\eta_\lambda(\ell,\ell',\ell) \ge 1$ and

$$\begin{aligned}\mathcal{M}_\lambda(\ell,\ell',\ell) = \max\Bigg\{&\frac{1}{\Lambda(\ell,\ell',\ell,\lambda)} - 1, \frac{1}{\Lambda(\ell,\text{Ł}\ell,\ell,\lambda)} - 1,\\ &\times\ \frac{1}{\Lambda(\ell',\text{Ł}\ell',\ell',\lambda)} - 1,\\ &\times\ \frac{\left(\frac{1}{\Lambda(\ell,\text{Ł}\ell,\ell,\lambda)} - 1\right)\left(\frac{1}{\Lambda(\ell',\text{Ł}\ell',\ell',\lambda)} - 1\right)}{\frac{1}{\Lambda(\ell,\ell',\ell,\lambda)}},\\ &\times\ \frac{\left(\frac{1}{\Lambda(\ell,\text{Ł}\ell,\ell,\lambda)} - 1\right)\left(\frac{1}{\Lambda(\ell',\text{Ł}\ell',\ell',\lambda)} - 1\right)}{\frac{1}{\Lambda(\text{Ł}\ell,\text{Ł}\ell',\text{Ł}\ell,\lambda)}}\Bigg\}.\end{aligned}$$

We now state another result of this section, which says that any fuzzy (η, ψ)-rational F-contractive mapping of type-I defined in a complete rectangular TCF$\hbar$M space has a unique fixed point.

Theorem 4. *Let $(\mathcal{Q}, \Lambda, \star)$ be a complete rectangular TCF$\hbar$M space, $\eta_\lambda : (0, \infty) \times \mathcal{Q}^3 \to [0, \infty]$ and let Ł be an η_λ-admissible self-mapping on $\mathcal{Q}$, so that*

(i) *there is an $\ell_0 \in \mathcal{Q}$ such that $\eta_\lambda(\ell_0, Ł\ell_0, \ell_0) \geq 1$;*
(ii) *Ł is a Fuzzy (η, ψ)-rational F-contractive mapping of type-I.*

Then Ł has a unique fixed point in $\mathcal{Q}$.

Proof. Using Remark 3 and Theorem 1, we can get the proof. □

Definition 21. Let $(\mathcal{Q}, \Lambda, \star)$ be a rectangular TCF$\hbar$M space. We call the mapping Ł : $\mathcal{Q} \to \mathcal{Q}$ a fuzzy weak-Wardowski contraction if there are $\psi' \in \Psi'$ and $F \in \mathbf{F}$ such that

$$F\left(\beta_\lambda(Ł\ell, Ł\ell', Ł\ell)\beta_\lambda(Ł\ell', Ł\ell, Ł\ell')\frac{1}{\Lambda(Ł\ell, Ł\ell', Ł\ell, \lambda)} - 1\right) \leq F\left(\frac{1}{\Lambda(\ell, \ell', \ell, \lambda)} - 1\right) - \psi'\left(F\left(\frac{1}{\Lambda(\ell, \ell', \ell,_\lambda)} - 1\right)\right), \ \lambda > 0, \tag{22}$$

for all $\ell, \ell' \in \mathcal{Q}$ with $Ł\ell \neq Ł\ell'$.

We now state another result of this section, which says that any weak-Wardowski contraction defined in a complete rectangular TCF$\hbar$M space has a unique fixed point.

Theorem 5. *Let $(\mathcal{Q}, \Lambda, \star)$ be a complete rectangular TCF$\hbar$M space. Then, any fuzzy weak-Wardowski contraction Ł : $\mathcal{Q} \to \mathcal{Q}$ has a unique fixed point.*

Proof. Using Remark 3 and Theorem 3, we can get the proof. □

Corollary 4. *Let $(\mathcal{Q}, \Lambda, \star)$ be a rectangular TCF$\hbar$M space, so that*

$$\tau + F\left(\frac{1}{\Lambda(Ł\ell, Ł\ell', Ł\ell, \lambda)} - 1\right) \leq F\left(\frac{1}{\Lambda(\ell, \ell', \ell, \lambda)} - 1\right), \ \lambda > 0, \tag{23}$$

for all $\ell, \ell' \in \mathcal{Q}$ with $Ł\ell \neq Ł\ell'$. Then, Ł has a unique fixed point in $\mathcal{Q}$.

Corollary 5. *Let $(\mathcal{Q}, \Lambda, \star)$ be a rectangular TCF$\hbar$M space so that*

$$\frac{1}{\Lambda(Ł\ell, Ł\ell', Ł\ell, \lambda)} - 1 \leq k\frac{1}{\Lambda(\ell, \ell', \ell, \lambda)} - 1,\ \lambda > 0, \tag{24}$$

for all $\ell, \ell' \in \mathcal{Q}$ with $Ł\ell \neq Ł\ell'$. Then, $Ł$ has a unique fixed point in $\mathcal{Q}$.

Corollary 6. *Let $(\mathcal{Q}, \Lambda, \star)$ be a rectangular TCF$\hbar$M space so that*

$$\begin{aligned} F\left(\frac{1}{\Lambda(Ł\ell, Ł\ell', Ł\ell, \lambda)} - 1\right) &\leq F\left(\frac{1}{\Lambda(\ell, \ell', \ell, \lambda)} - 1\right) \\ &- \tau - \sum_{i=1}^{\infty} F\left(\frac{1}{\Lambda(\ell, \ell', \ell, \lambda)} - 1\right)^{2i},\ \lambda > 0, \end{aligned} \tag{25}$$

for all $\ell, \ell' \in \mathcal{Q}$ with $Ł\ell \neq Ł\ell'$. Then, $Ł$ has a unique fixed point in $\mathcal{Q}$.

5. Application

Let $\mathcal{Q} = C([a, b], (-\infty, +\infty))$ be the set of real continuous functions defined on $[a, b]$. Consider the following Fredholm integral equation:

$$\ell(t) = \int_a^b f(t, s, \ell(s))ds + g(t) \tag{26}$$

for all $s, t \in [a, b]$, where $f : [a, b]^2 \times (-\infty, +\infty) \to (-\infty, +\infty)$ and $g : [a, b] \to (-\infty, +\infty)$. Define $\hbar : \mathcal{Q}^3 \times (0, +\infty) \to [0, +\infty)$ by

$$\hbar_\lambda(\ell, \ell', \ell'') = \frac{|\ \ell - \ell'\ | + |\ \ell' - z\ |}{\lambda}$$

for all $\ell, \ell', \ell'' \in \mathcal{Q}$ and $\lambda > 0$. Now, we define the mappings $\alpha_\lambda, \beta_\lambda, \gamma_\lambda : (0, \infty) \times \mathcal{Q}^3 \to [1, \infty)$ by

$$\alpha_\lambda(\ell, \ell', \ell'') = \beta_\lambda(\ell, \ell', \ell'') = \gamma_\lambda(\ell, \ell', \ell'') = |\ \ell\ | + |\ \ell'\ | + |\ \ell''\ | + 1,\ t > 0.$$

Then $(\mathcal{Q}, \hbar)$ is a complete TCMhM space. Now, we consider the following assumption:

for all $\ell, \ell' \in \mathcal{Q}$, assume that the following condition holds:

$$\frac{|f(s,t,\ell(s)) - f(s,t,\ell'(s))|}{\lambda}$$
$$\leq \frac{\frac{\frac{1}{\int_a^b f(t,s,\ell(s))ds+g(t)+2\int_a^b f(t,s,\ell'(s))ds+2g(t)+\lambda+1}}{\frac{1}{2\int_a^b f(t,s,\ell(s))ds+2g(t)+\int_a^b f(t,s,\ell'(s))ds+g(t)+\lambda+1}}}{2(b-a)}$$
$$\times F^{-1}\left[F\Big(\psi\left(\frac{|\ell-\ell'|}{\lambda}\right)\Big) - \tau\right].$$

where we have the following theorem.

Theorem 6. *Suppose that above assumptions hold. Then the integral equation* (26) *has a unique solution in* $\mathcal{Q}$.

Proof. We define $Ł : \mathcal{Q} \to \mathcal{Q}$ by

$$Ł(\ell)(t) = \int_a^b f(t,s,\ell(s))ds + g(t), \quad \forall s,t \in [a,b].$$

Then, for every $\ell, \ell' \in \mathcal{Q}$ and $\lambda > 0$, we have

$$\begin{aligned}
\hbar_\lambda(Ł\ell, Ł\ell', Ł\ell) &= \frac{|Ł\ell - Ł\ell'| + |Ł\ell - Ł\ell'|}{\lambda} \\
&= \frac{2}{\lambda}\left|\int_a^b f(t,s,\ell(s)) - f(t,s,\ell'(s))ds\right| \\
&\leq \frac{2}{\lambda}\int_a^b |f(t,s,\ell(s)) - f(t,s,\ell'(s))|ds \\
&\leq \beta_\lambda^{-1}(Ł\ell', Ł\ell, Ł\ell')\beta_\lambda^{-1}(Ł\ell, Ł\ell', Ł\ell)F^{-1} \\
&\quad \times \left[F\left(\psi\left(\frac{2|\ell-\ell'|}{\lambda}\right)\right) - \tau\right]\Bigg) \\
&= \beta_\lambda^{-1}(Ł\ell', Ł\ell, Ł\ell')\beta_\lambda^{-1}(Ł\ell, Ł\ell', Ł\ell)F^{-1} \\
&\quad \times [F(\psi(\hbar_\lambda(\ell,\ell',\ell))) - \tau] \\
&\leq \beta_\lambda^{-1}(Ł\ell', Ł\ell, Ł\ell')\beta_\lambda^{-1}(Ł\ell, Ł\ell', Ł\ell)\eta_\lambda^{-1}(\ell,\ell',\ell)F^{-1} \\
&\quad \times [F(\psi(\mathcal{M}(\ell,\ell',\ell))) - \tau]
\end{aligned}$$

for $\lambda > 0$. This implies that

$$F(\beta_\lambda(Ł\ell', Ł\ell, Ł\ell')\beta_\lambda(Ł\ell, Ł\ell', Ł\ell)\eta_\lambda(\ell, \ell', \ell)\hbar_\lambda(Ł\ell, Ł\ell', Ł\ell)) \\ \leq F(\psi(\mathcal{M}(\ell, \ell', \ell))) - \tau,$$

Therefore, it is obvious that the mapping Ł estimates all the conditions of Theorem 1. Hence, Ł has a unique fixed point, i.e. the Fredholm integral equation (26) has a unique solution. □

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Chapter 13

Sufficient Conditions for Mittag-Leffler Function Associated with Conic Regions

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The main purpose of this chapter is to find sufficient conditions under which the convolution of two functions

$$\mathbb{L}_{p,q}(z) * g(z) = \Gamma(q)\, zE_{p,q}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(q)}{\Gamma(pk+q)} b_k z^{k+1},$$

(where $E_{p,q}(z)$ denotes the well-known Mittag-Leffler function with parameters p and q) belongs to μ-uniformly convex class of type δ, which is denoted by $UCV(\mu, \delta)$, μ-uniformly starlike function of type δ is denoted by $\mathcal{S}_p(\mu, \delta)$ and classes $\mathcal{S}^*_\lambda$ and $\mathcal{C}_\lambda$ defined by Ponnusamy and Rønning.[1]

1. Introduction

Let a class of normalized holomorphic function in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be denoted by $\mathcal{A}$ and have Taylors series expansion of the form

$$g(z) = z + \sum_{k=2}^{\infty} b_k\, z^k. \tag{1}$$

The class $\mathcal{S}$ is the well-known subclass of $\mathcal{A}$ including those functions which are also univalent in $\mathbb{U}$. For each fixed $g \in \mathcal{S}$ and $k \geq 2$, De-Branges[2] has shown that

$$|b_k| \leq k. \tag{2}$$

A function $g \in \mathcal{A}$ is said to be starlike if $g(\mathbb{U})$ is starlike w.r.t. origin, and if $g(\mathbb{U})$ is convex domain, it is said to be convex. These classes are subsets of $\mathcal{A}$ and are denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. The classes $\mathcal{S}^*(\mu)$ and $\mathcal{C}(\mu)$ consist of those functions which are starlike and convex of order μ, $0 \leq \mu < 1$. These generalized classes are defined by holomorphic characterization $\Re\left(z\xi'(z)/\xi(z)\right) > \mu$ and $\Re\left(z\xi''(z)/\xi'(z) + 1\right) > \mu$ for $z \in \mathbb{U}$ and $\xi(z) \in \mathcal{A}$. It can be easily seen that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}^* = \mathcal{C}$. The detailed study of classes $\mathcal{C}(\mu)$ and $\mathcal{S}^*(\mu)$ are done by Robertson[3] (see also Ref. 4).

A function $\zeta(z) \in \mathcal{A}$ is said to be uniformly convex in $\mathbb{U}$ if it has the condition that the image arc $\zeta(\gamma)$ is a convex arc for every circular arc γ included in the unit disc $\mathbb{U}$, with center λ (also in $\mathbb{U}$). Goodman[5] first introduced this class. He introduced the uniform starlike functions in another study.[6]

Bharti *et al.*[7] introduced the subclasses $UCV(\mu, \delta)$ with $\mu \geq 0$ and $0 \leq \delta < 1$ as

$$UCV\,(\mu, \delta) := \left\{\zeta \in \mathcal{A} : \Re\left\{\frac{z\zeta''(z)}{\zeta'(z)} + 1\right\} \geq \mu\left|\frac{z\zeta''(z)}{\zeta'(z)}\right| + \delta, z \in \mathbb{U}\right\}. \tag{3}$$

The class $\mathcal{S}_p(\mu,\delta)$ was also discussed in Ref. 7, as

$$\mathcal{S}_p(\mu,\delta) := \left\{\zeta \in \mathcal{A} : \Re\left\{\frac{z\zeta'(z)}{\zeta(z)} + 1\right\} \geq \mu\left|\frac{z\zeta'(z)}{\zeta(z)}\right| + \delta, z \in \mathbb{U}\right\}. \tag{4}$$

It is clear to observe that

$$\zeta(z) \in UCV(\mu,\delta) \Leftrightarrow z\zeta'(z) \in \mathcal{S}_p(\mu,\delta).$$

Further, $UCV(\mu,0) = \mu - UCV$ and $\mathcal{S}_p(\mu,0) = \mu - \mathcal{S}_p$, where $\mu - UCV$ is called class of μ-uniformly convex function and $\mu - \mathcal{S}_p$ is called class of μ-starlike function in $\mathbb{U}$. The class $\mu - UCV$ was introduced by Kanas and Wisniowska[8] and the class $\mu - \mathcal{S}_p$ was investigated in Ref. 9. In particular, when $\mu = 1$, we obtain $1 - UCV \equiv UCV$ and $1 - \mathcal{S}_p \equiv \mathcal{S}_p$, where UCV and $\mathcal{S}_p$ are well-known classes of uniformly convex functions and parabolic starlike functions in the unit disc $\mathbb{U}$, respectively (see works of Goodman,[5,6] Ma and Minda,[10] and Rønning,[11] and also Refs.[12–19]).

A function $g \in \mathcal{A}$ is said to be in the class $P_m^\tau(n)$ if it satisfies the inequality

$$\left|\frac{(1-m)\frac{g(z)}{z} + mg'(z) - 1}{2\tau(1-n) + (1-m)\frac{g(z)}{z} + mg'(z) - 1}\right| < 1, \tag{5}$$

where $0 \leq m \leq 1$, $n < 1$ and $\tau \in \mathbb{C} \setminus \{0\}$. The class $P_m^\tau(n)$ was introduced by Swaminathan.[20] The importance of this class lies in the fact that for $m = 1$, the class $P_1^\tau(n)$ reduces in the class studied by Gangadharan *et al.*.[21] Further, for $\tau = e^{i\xi}cos(\xi)$ where $\xi \in (-\pi/2, \pi/2)$, class $P_m^\tau(n)$ reduces in the class studied earlier by Kim and Srivastava.[22] *Throughout this Chapter, we will take* $\tau = e^{i\xi}cos(\xi)$.

Next, we consider the classes

$$\mathcal{S}_\lambda^* = \left\{g \in \mathcal{A} : \left|\frac{zg'(z)}{g(z)} - 1\right| < \lambda, (z \in \mathbb{U}, \lambda > 0)\right\} \tag{6}$$

and

$$\mathcal{C}_\lambda = \left\{g \in \mathcal{A} : \left|\frac{zg''(z)}{g'(z)}\right| < \lambda, (z \in \mathbb{U}, \lambda > 0)\right\}, \tag{7}$$

obviously,

$$f(z) \in \mathcal{C}_\lambda \Longleftrightarrow zf'(z) \in \mathcal{S}^*_\lambda, \quad \lambda > 0.$$

The classes $\mathcal{S}^*_\lambda$ and C_λ were introduced by Ponnusamy and Rønning.[1]

The function $E_p(z)$ defined by

$$E_p(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kp+1)} \quad (z \in \mathbb{C},\, \Re(p) > 0) \tag{8}$$

was introduced by Mittag-Leffler.[23] A more general function $E_{p,q}(z)$ is defined by

$$E_{p,q}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kp+q)} \quad (z, p, q \in \mathbb{C};\, \Re(p) > 0). \tag{9}$$

This function, sometimes called a Mittag-Leffler type function, first appeared in a paper by Wiman.[24,25]

The importance of the Mittag-Leffler function was rediscovered when its connection to fractional calculus was fully understood. The Mittag-Leffler function appears in the solution of fractional order integral and differential equations, especially in the study of fractional generalization of kinetic equation, modeling fractional order viscoelastic materials, study of certain deterministic models, stochastic models, super-diffusive transport and in the study of complex systems. The most essential properties of these entire functions can be found in Refs. 26–31.

Note that Mittag-Leffler function $E_{p,q}$ does not belong to the family $\mathcal{A}$. Thus, it is obvious to consider the following normalization of Mittag-Leffler function:

$$\begin{aligned} \mathbb{L}_{p,q}(z) &= \Gamma(q)\, zE_{p,q}(z) \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(q)}{\Gamma(pk+q)} z^{k+1} \;\; (z, p, q \in \mathbb{C};\, Re(p) > 0, q \neq 0, -1, \ldots). \end{aligned} \tag{10}$$

Formula (10) holds for p, $q, z \in \mathbb{C}$, but in this chapter, we restrict ourselves to the case where p, $q \in \mathbb{R}$ and $z \in \mathbb{U}$. It is easy to note that $\mathbb{L}_{p,q}(z)$ contains the following well-known functions as its special case, e.g.

$$\begin{cases} \mathbb{L}_{0,1}(z) = \dfrac{z}{1-z}, \ \mathbb{L}_{1,1}(z) = z\ \exp(z), \ \mathbb{L}_{2,1}(z) = z\cosh(\sqrt{z}), \\ \mathbb{L}_{1,2}(z) = \exp(z) - 1, \quad \mathbb{L}_{1,3}(z) = \dfrac{2(\exp(z) - z - 1)}{z}, \\ \mathbb{L}_{1,4}(z) = \dfrac{6(\exp(z) - 1 - z) - 3z^2}{z^2}, \quad \mathbb{L}_{2,2}(z) = \sqrt{z}\ \sinh(\sqrt{z}). \end{cases} \tag{11}$$

If $g, h \in \mathcal{A}$, where $g(z)$ is given by (1) and $h(z)$ is defined by

$$h(z) = z + \sum_{k=2}^{\infty} c_k\ z^k,$$

then their convolution (Hadamard product) $g * h$ is defined by

$$(g * h)(z) = z + \sum_{k=2}^{\infty} b_k c_k\ z^k \ \ (z \in \mathbb{U}).$$

In view of the above definition, we have

$$\begin{aligned} \mathbb{L}_{p,q}(z) * g(z) &= z + \sum_{k=2}^{\infty} \frac{\Gamma(q)}{\Gamma(p(k-1)+q)} b_k z^k \\ &= z + \sum_{k=2}^{\infty} t_k b_k z^k \left(\text{where } t_k = \frac{\Gamma(q)}{\Gamma(p(k-1)+q)}\right). \end{aligned} \tag{12}$$

Recently, several mathematicians have studied some well-known special functions (from geometric function theory point of view) and obtained sufficient conditions such that these special functions have certain geometric properties such as close-to-convexity, convexity, starlikeness, etc. in $\mathbb{U}$. In this context, many results are available in the literature for different special functions. One can refer to works regarding the hypergeometric functions in Refs. 32–34, Bessel function in Refs. 35 and 36, Wright function in Ref. 37, Mittag-Leffler function in Ref. 38 and Mathieu-type power series in Ref. 39.

Finding the relation between various classes of holomorphic functions is an interesting research problem and has contributed many results in the past. We are particularly interested in the following problem.

Problem 1. *For a class of holomorphic function* $f \in P_m^{\tau}(n)$, *find sufficient conditions such that the* $\mathbb{L}_{p,q}(z) * g(z)$ *belongs to the classes* $UCV(\mu,\delta)$, $\mathcal{S}_p(\mu,\delta)$, $\mathcal{S}_\lambda^*$ *and* $\mathcal{C}_\lambda$. *Here,* $\mathbb{L}_{p,q}(z)$ *is the Mittag-Leffler operator of the form* (10).

One can refer to the works of Dixit and Porwal,[13] Gangadharan *et al.*,[21] Din and Yalcin,[40] Porwal and Moin[41] and references therein, which have earlier solved the similar Problem 1 for different linear operator (other than the Mittag-Leffler function).

2. Preliminaries

The following lemmas will be required to prove our key findings.

Lemma 1 (see Ref. 7). *Let* $g \in \mathcal{A}$. *If the inequality*

$$\sum_{k=2}^{\infty} [(1+\mu)k - (\mu+\delta)]\, k\, |b_k| \leq 1-\delta \tag{1}$$

holds, then $g \in UCV(\mu,\delta)$.

Lemma 2 (see Ref. 7). *Let* $g \in \mathcal{A}$. *If the inequality*

$$\sum_{k=2}^{\infty} [k(1+\mu) - (\mu+\delta)]\, |b_k| \leq 1-\delta \tag{2}$$

holds, then $g \in \mathcal{S}_p(\mu,\delta)$.

Lemma 3 (see Ref. 20). *Let* $g(z) \in \mathcal{S}$ *and it belongs to class given in* (1). *If* $g \in P_m^{\tau}(n)$, *then*

$$|b_k| \leq \frac{2\,|\tau|\,(1-n)}{1+m(k-1)}. \tag{3}$$

Lemma 4 (see Ref. 21). *Let* $g \in \mathcal{A}$. *If the inequality*

$$\sum_{k=2}^{\infty} (\lambda+k-1)\, |b_k| \leq \lambda,\ \ \lambda > 0, \tag{4}$$

holds, then $g \in \mathcal{S}_\lambda^*$.

Lemma 5 (see Ref. 21). *Let $g \in \mathcal{A}$. If the inequality*

$$\sum_{k=2}^{\infty} k\,(\lambda + k - 1)\,|b_k| \leq \lambda, \;\; \lambda > 0, \tag{5}$$

holds, then $g \in \mathcal{C}_\lambda$.

3. Main Results

The first theorem provides sufficient condition, so that $\mathbb{L}_{p,q}(z) * g(z) \in UCV(\mu, \delta)$, whenever $g(z)$ of Eq. (1) belongs to the class $P_m^\tau(n)$.

Theorem 1. *Let $g \in P_m^\tau(n) \cap \mathcal{S}$ be of the form (1) with $\tau = e^{i\xi} cos(\xi)$ $(\xi \in (-\pi/2, \pi/2)), 0 < m \leq 1, n < 1$ and $p, q > 0$. If*

$$\frac{2(1-n)\cos\xi}{m}\left(\frac{(q+1)\,((1+\mu)(q+1)+(1-\delta)q)}{q^3}\right) \leq (1-\delta),$$

then $\mathbb{L}_{p,q}(z) * g(z) \in UCV(\mu, \delta)$.

Proof. Using Lemma 1 on (12), the convolution operator $\mathbb{L}_{p,q}(z) * g(z) \in UCV(\mu, \delta)$, provided that

$$\sum_{k=2}^{\infty} k\,[k(1+\mu) - (\mu+\delta)]\,|t_k| \leq 1 - \delta. \tag{6}$$

Using Lemma 3 with $\tau = e^{i\xi} cos(\xi)$, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} k\,[k(1+\mu) - (\mu+\delta)]\,\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}\,|b_k| \\
&\quad \leq 2(1-n)\cos\xi \sum_{k=2}^{\infty} k\,[k(1+\mu) - (\mu+\delta)]\,\frac{\Gamma(q)}{\Gamma(p(k-1)+q)} \\
&\qquad \cdot \frac{1}{1+m(k-1)} \\
&\quad \leq \frac{2(1-n)\cos\xi}{m}\left\{(1+\mu)\sum_{k=2}^{\infty}\frac{k\Gamma(q)}{\Gamma(p(k-1)+q)}\right. \\
&\qquad \left. - (\mu+\delta)\sum_{k=2}^{\infty}\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2(1-n)\cos\xi}{m}\left\{(1+\mu)\sum_{k=2}^{\infty}\left(\frac{(k-1+1)\Gamma(q)}{\Gamma(p(k-1)+q)}\right)\right. \\
&\quad \left. -(\mu+\delta)\sum_{k=2}^{\infty}\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}\right\} \\
&= \frac{2(1-n)\cos\xi}{m}\left\{(1+\mu)\sum_{k=2}^{\infty}\left(\frac{(k-1)\Gamma(q)}{\Gamma(p(k-1)+q)}\right)\right. \\
&\quad \left. +(1-\delta)\sum_{k=2}^{\infty}\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}\right\} \\
&\leq \frac{2(1-n)\cos\xi}{m}\left\{(1+\mu)\sum_{k=2}^{\infty}\frac{(k-1)}{(q)_{k-1}}+(1-\delta)\sum_{k=2}^{\infty}\frac{1}{(q)_{k-1}}\right\} \\
&\leq \frac{2(1-n)\cos\xi}{m}\left\{\frac{(1+\mu)}{q}\left(1-\frac{1}{(q+1)}\right)^{-2}\right. \\
&\quad \left. +\frac{(1-\delta)}{q}\left(1-\frac{1}{(q+1)}\right)^{-1}\right\} \\
&= \frac{2(1-n)\cos\xi}{m}\,\frac{(q+1)\left((1+\mu)(q+1)+(1-\delta)q\right)}{q^3} \\
&\leq (1-\delta),
\end{aligned}
$$

using the inequality

$$
\frac{\Gamma(q)}{\Gamma(p(k-1)+q)} \leq \frac{1}{(q)_{k-1}} \leq \frac{1}{q(q+1)^{k-2}} \; (\forall\, k \geq 2) \tag{7}
$$

and by the given hypothesis. This completes the proof of Theorem 1. □

The second theorem provides sufficient condition, so that $\mathbb{L}_{p,q}(z) * g(z) \in \mathcal{S}_p(\mu,\delta)$, whenever $g(z)$ of the form (1) belongs to the class $P_m^{\tau}(n)$.

Theorem 2. *Let $g \in P_m^{\tau}(n) \cap \mathcal{S}$ be of the form* (1) *with* $0 < m \leq 1, n < 1$ *and* $p, q > 0$. *If*

$$\frac{2(1-n)\cos\xi}{m}\left\{\frac{(q+1)}{2q^2}(3+2\mu-\delta)\right\} \leq (1-\delta),$$

then $\mathbb{L}_{p,q}(z) * g(z) \in \mathcal{S}_p(\mu,\delta)$.

Proof. Using Lemma 2 on (12), the convolution operator $\mathbb{L}_{p,q}(z) * g(z) \in \mathcal{S}_p(\mu,\delta)$, provided that

$$\sum_{k=2}^{\infty}\left[k(1+\mu)-(\mu+\delta)\right]|t_k| \leq 1-\delta.$$

Using Lemma 3 with $\tau = e^{i\xi}cos(\xi)$, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left[k(1+\mu)-(\mu+\delta)\right]\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}|b_k| \\
&\quad \leq 2(1-n)\cos\xi\sum_{k=2}^{\infty}\left[k(1+\mu)-(\mu+\delta)\right]\frac{\Gamma(q)}{\Gamma(p(k-1)+q)} \\
&\qquad \cdot\frac{1}{1+m\,(k-1)} \\
&\quad \leq \frac{2(1-n)\cos\xi}{m}\left\{(1+\mu)\sum_{k=2}^{\infty}\left(\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}\right)\right. \\
&\qquad \left. -(\mu+\delta)\sum_{k=2}^{\infty}\frac{\Gamma q}{k\Gamma(p(k-1)+q)}\right\} \\
&\quad = \frac{2(1-n)\cos\xi}{m}\left\{(1+\mu)\sum_{k=2}^{\infty}\left(\frac{(k-1+1)\Gamma(q)}{k\Gamma(p(k-1)+q)}\right)\right. \\
&\qquad \left. -(\mu+\delta)\sum_{k=2}^{\infty}\frac{\Gamma(q)}{k\Gamma(p(k-1)+q)}\right\}
\end{aligned}$$

$$= \frac{2(1-n)\cos\xi}{m}\left\{(1+\mu)\sum_{k=2}^{\infty}\left(\frac{(k-1)\Gamma(q)}{k\Gamma(p(k-1)+q)}\right)\right.$$
$$\left. + (1-\delta)\sum_{k=2}^{\infty}\frac{\Gamma(q)}{k\Gamma(p(k-1)+q)}\right\}$$
$$\leq \frac{2(1-n)\cos\xi}{m}\left\{(1+\mu)\sum_{k=2}^{\infty}\left(\frac{k-1}{k}\right)\frac{1}{(q)_{k-1}}\right.$$
$$\left. + (1-\delta)\sum_{k=2}^{\infty}\left(\frac{1}{k}\right)\frac{1}{(q)_{k-1}}\right\}$$
$$\leq \frac{2(1-n)\cos\xi}{m}\left\{\frac{1+\mu}{q}+\frac{(1-\delta)}{2q}\right\}\left(1-\frac{1}{q+1}\right)^{-1}$$
$$= \frac{2(1-n)\cos\xi}{m}\left\{\frac{(q+1)}{2q^2}(3+2\mu-\delta)\right\} \leq (1-\delta),$$

using the inequality (7) and by the given hypothesis. This completes the proof of Theorem 2. □

The third theorem provides sufficient condition, so that $\mathbb{L}_{p,q}(z) * g(z) \in S_\lambda^*$, whenever $g(z)$ of the form (1) belongs to the class $P_m^\tau(n)$.

Theorem 3. *Let $g \in P_m^\tau(n) \cap \mathcal{S}$ be of the form (1) with $0 < m \leq 1$, $n < 1$ and $p, q > 0$. If*

$$\frac{(1-n)\cos\xi}{m}\left\{\frac{(q+1)(\lambda+1)}{q^2}\right\} \leq \lambda,$$

then $\mathbb{L}_{p,q}(z) * g(z) \in \mathcal{S}_\lambda^*$.

Proof. In view of Lemma 4 and (12), it is sufficient to show that

$$\sum_{k=2}^{\infty}(\lambda+k-1)\,|t_k| \leq \lambda.$$

Now, using Lemma 3 with $\tau = e^{i\xi}cos(\xi)$, we have

$$\sum_{k=2}^{\infty}(\lambda+k-1)\,|t_k|$$

$$= \sum_{k=2}^{\infty}(\lambda+k-1)\,\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}\,|b_k|$$

$$\leq \frac{2(1-n)\cos\xi}{m}\left\{\sum_{k=2}^{\infty}\left(\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}\right)\right.$$

$$\left. + \,(\lambda-1)\sum_{k=2}^{\infty}\left(\frac{\Gamma(q)}{k\Gamma p(k-1)+q}\right)\right\}$$

$$\leq \frac{2(1-n)\cos\xi}{m}\left\{\sum_{k=2}^{\infty}\frac{1}{(q)_{k-1}}+(\lambda-1)\sum_{k=2}^{\infty}\frac{1}{k(q)_{k-1}}\right\}$$

$$\leq \frac{(1-n)\cos\xi}{m}\left\{\frac{(q+1)\,(\lambda+1)}{q^2}\right\} \leq \lambda,$$

using the inequality (7) and by the given hypothesis. This completes the proof of Theorem 3. □

The fourth theorem provides sufficient condition, so that $\mathbb{L}_{p,q}(z) * g(z) \in C_\lambda$, whenever $g(z)$ of the form (1) belongs to the class $P_m^\tau(n)$.

Theorem 4. *Let* $g \in P_m^\tau(n) \cap \mathcal{S}$ *be of the form* (1) *with* $0 < m \leq 1, n < 1$ *and* $p, q > 0$. *If*

$$\frac{2(1-n)\cos\xi}{m}\left\{\frac{(q+1)(q(1+\lambda)+1)}{q^3}\right\} \leq \lambda,$$

then $\mathbb{L}_{p,q}(z) * g(z) \in \mathcal{C}_\lambda$.

Proof. In view of Lemma 5 and (12), it is sufficient to show that

$$\sum_{k=2}^{\infty} k\,(\lambda+k-1)\,|t_k| \leq \lambda.$$

Now, using Lemma 3 with $\tau = e^{i\xi}cos(\xi)$, we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} k\,(\lambda+k-1)\,|t_k| \\
&= \sum_{k=2}^{\infty} k\,(\lambda+k-1)\,\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}\,|b_k| \\
&\leq \frac{2(1-n)\cos\xi}{m}\left\{\sum_{k=2}^{\infty}\left(\frac{k\Gamma(q)}{\Gamma(p(k-1)+q)}\right)\right. \\
&\quad\left. + (\lambda-1)\sum_{k=2}^{\infty}\left(\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}\right)\right\} \\
&\leq \frac{2(1-n)\cos\xi}{m}\left\{\sum_{k=2}^{\infty}\frac{(k-1)}{(q)_{k-1}} + \lambda\sum_{k=2}^{\infty}\frac{1}{(q)_{k-1}}\right\} \\
&\leq \frac{2(1-n)\cos\xi}{m}\left\{\frac{(q+1)(q(1+\lambda)+1)}{q^3}\right\} \leq \lambda
\end{aligned}$$

by the given hypothesis. This completes proof of Theorem 4. □

The fifth theorem provides sufficient condition, so that $\mathbb{L}_{p,q}(z)*g(z) \in \mathcal{S}_p(\mu,\delta)$, whenever $g(z)$ of the form (1) belongs to the class $\mathcal{S}$.

Theorem 5. *Let $p > 0,\ q > 1$ satisfy the inequality*

$$\left[\frac{4(1+\mu)(q+1)}{q(q-1)} + \frac{(2+\mu-\delta)(q+1)^2}{q^3} + \frac{(1-\mu)(q+1)}{q^2}\right] \leq (1-\delta).$$

Then the convolution $\mathbb{L}_{p,q}(z) * g(z)$ maps $g(z) \in \mathcal{S}$ into $\mathcal{S}_p(\mu,\delta)$.

Proof. Let $g(z) \in \mathcal{S}$. Using Lemma 1, it is sufficient to show that

$$\sum_{k=2}^{\infty}\left[k(1+\mu) - (\mu+\delta)\right]|t_k| \leq 1-\delta.$$

Using (2), we have

$$= \sum_{k=2}^{\infty} [k(1+\mu)-(\mu+\delta)] \frac{\Gamma(q)}{\Gamma(p(k-1)+q)} |b_k|$$

$$= (1+\mu)\sum_{k=2}^{\infty}\left(\frac{k\Gamma(q)}{\Gamma(p(k-1)+q)}\right)|b_k|$$

$$-(\mu+\delta)\sum_{k=2}^{\infty}\left(\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}\right)|b_k|$$

$$\leq (1+\mu)\sum_{k=2}^{\infty}\left(\frac{k^2\Gamma(q)}{\Gamma(p(k-1)+q)}\right)$$

$$-(\mu+\delta)\sum_{k=2}^{\infty}\left(\frac{k\Gamma(q)}{\Gamma(p(k-1)+q)}\right)$$

$$= (1+\mu)\sum_{k=2}^{\infty}\frac{((k-1)^2+2(k-1)+1)\Gamma(q)}{\Gamma(p(k-1)+q)}$$

$$-(\mu+\delta)\sum_{k=2}^{\infty}\frac{(k-1+1)\Gamma(q)}{\Gamma(p(k-1)+q)}$$

$$= (1+\mu)\sum_{k=2}^{\infty}\frac{(k-1)^2\Gamma(q)}{\Gamma(p(k-1)+q)} + (2+\mu-\delta)\sum_{k=2}^{\infty}\frac{(k-1)\Gamma(q)}{\Gamma(p(k-1)+q)}$$

$$+(1-\delta)\sum_{k=2}^{\infty}\frac{\Gamma(q)}{\Gamma(p(k-1)+q)}.$$

Now, using the inequality $(k-1)^2 \leq 2^k \ \ \forall k \geq 2$ and inequality (7), we have

$$\sum_{k=2}^{\infty} [k(1+\mu)-(\mu+\delta)] \frac{\Gamma(q)}{\Gamma(p(k-1)+q)} |b_k|$$

$$\leq \frac{(1+\mu)2^2}{q}\sum_{k=0}^{\infty}\left(\frac{2}{q+1}\right)^k + \frac{(2+\mu-\delta)}{q}\sum_{k=0}^{\infty}\left(\frac{k+1}{(q+1)^k}\right)$$

$$+\frac{(1-\delta)}{q}\left(\sum_{k=0}^{\infty}\frac{1}{q+1}\right)^k$$

$$= \frac{4(1+\mu)(q+1)}{q(q-1)} + \frac{(2+\mu-\delta)(q+1)^2}{q^3} + \frac{(1-\mu)(q+1)}{q^2}$$
$$\leq 1-\delta$$

by the given hypothesis. This completes the proof of Theorem 5. □

The sixth theorem provides sufficient condition, so that $\mathbb{L}_{p,q}(z) * g(z) \in \mathcal{S}^*_\lambda$, whenever $g(z)$ of the form (1) belongs to the class $\mathcal{S}$.

Theorem 6. *Let $p > 0, q > 1$ satisfy the inequality*

$$\frac{(q+1)}{q^3(q-1)} \left(q^2(2\lambda+5) - q\lambda - (\lambda+1)\right) \leq \lambda.$$

Then $\mathbb{L}_{p,q}(z) * g(z)$ maps $g(z) \in \mathcal{S}$ into $\mathcal{S}^*_\lambda$.

Proof. The proof of Theorem 6 is similar to the proof of Theorem 5, therefore we omit the details. □

The seventh theorem provides sufficient condition, so that $\mathbb{L}_{p,q}(z) * g(z) \in \mathcal{C}_\lambda$, whenever $g(z)$ of the form (1) belongs to the class $\mathcal{S}$.

Theorem 7. *Let $p > 0, q > 2$ satisfy the inequality*

$$\frac{(q+1)}{q^3(q-1)(q-2)} \left(q^3(7\lambda+2) - 3q^2(5\lambda-2) + 4(\lambda-1)\right) \leq \lambda.$$

Then $\mathbb{L}_{p,q}(z) * g(z)$ maps $g(z) \in \mathcal{S}$ into $\mathcal{C}_\lambda$.

Proof. Using the inequality $k^3 \leq 3^k \ \forall k \geq 2$ and proceeding similar to the proof of Theorem 5, we obtain the required result. □

4. Conclusion

The importance of the results discussed in Section 3 lies in the fact that when we give particular values to various parameters in (12), we obtain the operators which transform functions belonging to class $P^\tau_m(n)$ into the functions belonging to class $UCV(\mu, \delta)$, $\mathcal{S}_p(\mu, \delta)$, $\mathcal{S}_\lambda$ and $\mathcal{C}_\lambda$.

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Chapter 14

Results on the Existence of Solutions for Some Controlled Optimization Problems

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In this chapter, by using some generalized convexity and Fréchet differentiability hypotheses of the involved functionals, we establish several connections between the solutions of some new (weak) vector-controlled variational inequalities and (proper, weak) efficient solutions associated with certain multiobjective-controlled variational problems. Also, the notion of invex set with respect to some given functions has an important role for proving the main results derived in the chapter.

1. Introduction

The importance of multiobjective/vector optimization problems in the mathematical modelling of natural phenomena or engineering is well known. For the study of these types of problems, it was necessary to define some concepts of *efficient solutions*. Geoffrion[1] introduced a pretty narrow definition of efficiency called *proper efficient solutions*. On the other side, Klinger[2] analyzed *improper solutions* for the vector maximum problem. Later, Kazmi[3] investigated, by using vector variational-like inequalities, the existence of a *weak minimum* for constrained optimization problems. Moreover, Ghaznavighosoni and Khorram[4] stated conditions of efficiency for approximating (weakly, properly) efficient points in general multiobjective optimization problems.

The mathematical concept of *convexity* is almost inevitable in optimization theory to formulate the optimality or efficiency conditions. Moreover, its generalization was necessary in the study of some concrete problems in applied sciences, engineering, or natural phenomena. In consequence, Hanson[5] defined the notion of *invexity* and, over time, many other extensions have been introduced (see, for instance, Antczak,[6] Arana-Jiménez *et al.*,[7] Mishra *et al.*,[8] Ahmad[9]), such as pseudoinvexity, univexity, quasiinvexity, preinvexity, approximate convexity, and so on. Also, some of these generalized convexities have been transposed into the multidimensional context determined by multiple or curvilinear integrals (see, for instance, Mititelu and Treanţă,[10] Treanţă[11]).

On the other hand, variational inequalities have been defined and studied to model concrete problems in mechanics, physics, engineering, traffic analysis, or natural phenomena. In this regard, Giannessi[12] analyzed, obtaining some remarkable results, the vector variational inequalities. As we all know, an application of scalar/vector variational inequalities is the study of existence of solutions for scalar/multiobjective optimization problems under some suitable hypotheses. Thus, many papers investigated the connections between the solutions of these inequalities and optimal or efficient solutions associated to the corresponding optimization problems (see, for instance, Ruiz-Garzón *et al.*,[13] Marinoschi,[14] Jayswal *et al.*[15]).

We divide variational problems into (1) classical variational problems and (2) continuous-time variational problems. Some interesting results on multiobjective continuous-time program and vector variational inequality have been formulated by Kim.[16] Also, since optimal control problems can be regarded as continuous-time variational problems, their analysis from this point of view has generated various important results in engineering problems, economics, or processes coming from game theory, operations research. Thus, quite recently, Treanţă[17] and Jha *et al.*[18] have established some necessary and sufficient optimality or efficiency conditions, saddle-point criterion, and well-posedness results for several multidimensional control problems driven by functionals of curvilinear or multiple integral type.

In this chapter, we introduce a class of (weak) vector-controlled variational inequalities and multiobjective variational control problems defined by functionals of multiple integral type. We state certain connections between the solutions of the multidimensional variational problems under examination. The main novelty element in this mathematical framework is the presence of the control variables. Also, the notion of invex set with respect to some given functions has a decisive role in proving the main results included in the chapter.

The chapter is continued with some preliminaries. In Section 3, we state some characterization results for the solutions associated with the considered variational problems. Finally, Section 4 includes the conclusions of this chapter and further research development.

2. Problem Description

In this chapter, we begin with P as a set in $\mathbb{R}^m$, that is supposed to be compact, and $P \ni \varsigma = (\varsigma^\gamma)$, $\gamma = \overline{1, m}$, as a multi-variable of evolution. Also, we introduce Q as the space of all piecewise differentiable *state* functions $x : P \to \mathbb{R}^n$ and T as the space of all *control* functions $\eta : P \to \mathbb{R}^k$, which are supposed to be piecewise continuous. In addition, on $\mathsf{Q} \times \mathsf{T}$, we define the scalar product

$$\begin{aligned}\langle (x, \eta), (\pi, v) \rangle &= \int_P \Big[x(\varsigma) \cdot \pi(\varsigma) + \eta(\varsigma) \cdot v(\varsigma)\Big] d\varsigma \\ &= \int_P \left[\sum_{i=1}^{n} x^i(\varsigma)\pi^i(\varsigma) + \sum_{j=1}^{k} \eta^j(\varsigma) v^j(\varsigma)\right] d\varsigma, \\ &\quad (\forall)(x, \eta), (\pi, v) \in \mathsf{Q} \times \mathsf{T},\end{aligned}$$

together with the norm induced by it.

By using the vector-valued functions $\phi = (\phi^l) : P \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^p$, $l = \overline{1, p}$, we introduce the following vector functional defined by multiple integrals:

$$\Phi : \mathsf{Q} \times \mathsf{T} \to \mathbb{R}^p, \quad \Phi(x,\eta) = \int_P \phi\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma$$
$$= \left(\int_P \phi^1\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma, \ldots, \int_P \phi^p\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma\right).$$

In the following, $D_\alpha,\ \alpha \in \{1, \ldots, m\}$, denotes the operator of total derivative, and we assume that the Lagrange-type densities

$$\phi = \left(\phi^1, \ldots, \phi^p\right) : P \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^p$$

are of C^1-class. Also, throughout the chapter, we will use the following rules associated with equalities and inequalities:

$$a = b \Leftrightarrow a^l = b^l,\ a \leq b \Leftrightarrow a^l \leq b^l,\ a < b \Leftrightarrow a^l < b^l,$$
$$a \preceq b \Leftrightarrow a \leq b,\ a \neq b,\ l = \overline{1,p}$$

for any p-tuples $a = \left(a^1, \ldots, a^p\right),\ b = \left(b^1, \ldots, b^p\right)$ in $\mathbb{R}^p$.

Further, we introduce the following partial differential equation constrained *vector variational control problem*:

$$(P) \quad \min_{(x,\eta)} \left\{\Phi(x,\eta) = \int_P \phi\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma\right\} \text{ subject to } (x,\eta) \in S,$$

where

$$\Phi(x,\eta) = \int_P \phi\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma$$
$$= \left(\int_P \phi^1\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma, \ldots, \int_P \phi^p\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma\right)$$
$$= \left(\Phi^1(x,\eta), ..., \Phi^p(x,\eta)\right)$$

and

$$S = \Big\{(x,\eta) \in \mathsf{Q} \times \mathsf{T} \mid x^i_\alpha(\varsigma) = \frac{\partial x^i}{\partial \varsigma^\alpha}(\varsigma) = H^i_\alpha\left(\varsigma, x(\varsigma), \eta(\varsigma)\right),$$
$$Y\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) \leq 0,\ (x,\eta)|_{\partial P} = \text{given}\Big\}.$$

In the definition of S, we have considered that the C^1-class functions $H_\alpha = \left(H^i_\alpha\right) : P \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n,\ i = \overline{1,n},\ \alpha = \overline{1,m}$, define the following partial differential equations of evolution:

$$x^i_\alpha(\varsigma) = H^i_\alpha\left(\varsigma, x(\varsigma), \eta(\varsigma)\right), \quad i = \overline{1,n},\ \alpha = \overline{1,m}$$

and verify the closeness relations $D_\gamma H^i_\alpha = D_\alpha H^i_\gamma,\ \alpha,\gamma = \overline{1,m},\ \alpha \neq \gamma,\ i = \overline{1,n}$. Also, $Y = (Y^r) : P \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^q,\ r = \overline{1,q}$, are assumed to be C^1-class functions.

Definition 1 (Mititelu and Treanţă[10]). A point $(x^0, \eta^0) \in S$ is called an *efficient solution* in (P) if there exists no other $(x, \eta) \in S$ such that $\Phi(x,\eta) \preceq \Phi(x^0,\eta^0)$ or, equivalently, $\Phi^l(x,\eta) - \Phi^l(x^0,\eta^0) \leq 0,\ (\forall) l = \overline{1,p}$, with strict inequality for at least one l.

Definition 2 (Geoffrion[1]). A point $(x^0,\eta^0) \in S$ is called a *proper efficient solution* in (P) if $(x^0,\eta^0) \in S$ is an efficient solution in (P) and there exists a positive real number M such that, for all $l = \overline{1,p}$, we have

$$\Phi^l(x^0,\eta^0) - \Phi^l(x,\eta) \leq M\left(\Phi^s(x,\eta) - \Phi^s(x^0,\eta^0)\right)$$

for some $s \in \{1,\ldots,p\}$ such that

$$\Phi^s(x,\eta) > \Phi^s(x^0,\eta^0),$$

whenever $(x,\eta) \in S$ and

$$\Phi^l(x,\eta) < \Phi^l(x^0,\eta^0).$$

Definition 3. A point $(x^0,\eta^0) \in S$ is called a *weak efficient solution* in (P) if there exists no other $(x,\eta) \in S$ such that $\Phi(x,\eta) < \Phi(x^0,\eta^0)$ or, equivalently, $\Phi^l(x,\eta) - \Phi^l(x^0,\eta^0) < 0,\ (\forall) l = \overline{1,p}$.

According to Treanţă,[11] for $x \in \mathsf{Q}$ and $\eta \in \mathsf{T}$, we consider the vector functional of multiple integral type

$$L : \mathsf{Q} \times \mathsf{T} \to \mathbb{R}^p, \quad L\left(x,\eta\right) = \int_P \kappa\left(\varsigma, x(\varsigma), x_\alpha(\varsigma), \eta(\varsigma)\right) d\varsigma$$

and introduce the concepts of invexity and pseudoinvexity associated with L.

Definition 4. If there exist

$$\rho : P \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n,$$
$$\rho = \rho\left(\varsigma, x(\varsigma), \eta(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right) = \left(\rho^i\left(\varsigma, x(\varsigma), \eta(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right)\right),$$
$$i = \overline{1, n},$$

of C^1-class with $\rho\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right) = 0,\ (\forall)\varsigma \in P,\ \rho|_{\partial P} = 0$, and

$$\sigma : P \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k,$$
$$\sigma = \sigma\left(\varsigma, x(\varsigma), \eta(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right)$$
$$= \left(\sigma^j\left(\varsigma, x(\varsigma), \eta(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right)\right),\ j = \overline{1, k},$$

of C^0-class with $\sigma\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right) = 0,\ (\forall)\varsigma \in P$, $\sigma|_{\partial P} = 0$, such that

$$L(x, \eta) - L\left(x^0, \eta^0\right) \geq \int_P \left[\frac{\partial \kappa}{\partial x}\left(\varsigma, x^0(\varsigma), x^0_\alpha(\varsigma), \eta^0(\varsigma)\right)\rho \right.$$
$$\left. + \frac{\partial \kappa}{\partial x_\alpha}\left(\varsigma, x^0(\varsigma), x^0_\alpha(\varsigma), \eta^0(\varsigma)\right) D_\alpha \rho\right] d\varsigma$$
$$+ \int_P \left[\frac{\partial \kappa}{\partial \eta}\left(\varsigma, x^0(\varsigma), x^0_\alpha(\varsigma), \eta^0(\varsigma)\right)\sigma\right] d\varsigma,$$

for any $(x, \eta) \in \mathsf{Q} \times \mathsf{T}$, then L is said to be *invex at* $\left(x^0, \eta^0\right) \in \mathsf{Q} \times \mathsf{T}$ *with respect to* ρ *and* σ.

Definition 5. In the above definition, with $(x, \eta) \neq \left(x^0, \eta^0\right)$, if we replace $\geq$ with $>$, we say that L is *strictly invex at* $\left(x^0, \eta^0\right) \in \mathsf{Q} \times \mathsf{T}$ *with respect to* ρ *and* σ.

Definition 6. If there exist

$$\rho : P \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n,$$
$$\rho = \rho\left(\varsigma, x(\varsigma), \eta(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right)$$
$$= \left(\rho^i\left(\varsigma, x(\varsigma), \eta(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right)\right),\ i = \overline{1, n},$$

of C^1-class with $\rho\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right) = 0,\ (\forall)\varsigma \in P,\ \rho|_{\partial P} = 0$, and

$$\begin{aligned}&\sigma : P \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k,\\ &\sigma = \sigma\left(\varsigma, x(\varsigma), \eta(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right)\\ &\quad = \left(\sigma^j\left(\varsigma, x(\varsigma), \eta(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right)\right),\ j = \overline{1,k},\end{aligned}$$

of C^0-class with $\sigma\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma), x^0(\varsigma), \eta^0(\varsigma)\right) = 0,\ (\forall)\varsigma \in P$, $\sigma|_{\partial P} = 0$, such that

$$\begin{aligned}&L(x,\eta) - L\left(x^0,\eta^0\right) < 0\\ &\quad \Rightarrow \int_P \left[\frac{\partial \kappa}{\partial x}\left(\varsigma, x^0(\varsigma), x^0_\alpha(\varsigma), \eta^0(\varsigma)\right)\rho\right.\\ &\qquad \left. + \frac{\partial \kappa}{\partial x_\alpha}\left(\varsigma, x^0(\varsigma), x^0_\alpha(\varsigma), \eta^0(\varsigma)\right) D_\alpha \rho\right] d\varsigma\\ &\qquad + \int_P \left[\frac{\partial \kappa}{\partial \eta}\left(\varsigma, x^0(\varsigma), x^0_\alpha(\varsigma), \eta^0(\varsigma)\right)\sigma\right] d\varsigma < 0,\end{aligned}$$

or, equivalently,

$$\begin{aligned}&\int_P \left[\frac{\partial \kappa}{\partial x}\left(\varsigma, x^0(\varsigma), x^0_\alpha(\varsigma), \eta^0(\varsigma)\right)\rho\right.\\ &\quad \left. + \frac{\partial \kappa}{\partial x_\alpha}\left(\varsigma, x^0(\varsigma), x^0_\alpha(\varsigma), \eta^0(\varsigma)\right) D_\alpha \rho\right] d\varsigma\\ &\quad + \int_P \left[\frac{\partial \kappa}{\partial \eta}\left(\varsigma, x^0(\varsigma), x^0_\alpha(\varsigma), \eta^0(\varsigma)\right)\sigma\right] d\varsigma \geq 0 \Rightarrow L(x,\eta)\\ &\quad - L\left(x^0,\eta^0\right) \geq 0,\end{aligned}$$

for any $(x,\eta) \in \mathsf{Q} \times \mathsf{T}$, then L is said to be *pseudoinvex at* $\left(x^0,\eta^0\right) \in \mathsf{Q} \times \mathsf{T}$ *with respect to* ρ *and* σ.

Some examples of invex and pseudoinvex multiple integral functionals can be consulted in Treanţă[11] (see Examples 1 and 2).

Definition 7. The nonempty subset $\mathsf{X} \times \mathsf{U} \subset \mathsf{Q} \times \mathsf{T}$ is said to be *invex with respect to* ρ *and* σ if

$$(x^0,\eta^0) + \lambda\left(\rho\left(\varsigma, x, \eta, x^0, \eta^0\right), \sigma\left(\varsigma, x, \eta, x^0, \eta^0\right)\right) \in \mathsf{X} \times \mathsf{U},$$

for all $(x,\eta),\ (x^0,\eta^0) \in \mathsf{X} \times \mathsf{U}$ and $\lambda \in [0,1]$.

Now, in order to formulate and prove some results on the existence of solutions for problem (P), we introduce the following *(weak) vector-controlled variational inequalities*:

I. Find $(x^0, \eta^0) \in S$ such that there exists no $(x, \eta) \in S$ satisfying

$$\text{(VI)} \quad \left(\int_P \left[\frac{\partial \phi^1}{\partial x} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^1}{\partial \eta} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma \right] d\varsigma, \ldots, \right.$$
$$\left. \int_P \left[\frac{\partial \phi^p}{\partial x} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^p}{\partial \eta} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma \right] d\varsigma \right) \leq 0.$$

II. Find $(x^0, \eta^0) \in S$ such that there exists no $(x, \eta) \in S$ satisfying

$$\text{(WVI)} \quad \left(\int_P \left[\frac{\partial \phi^1}{\partial x} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^1}{\partial \eta} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma \right] d\varsigma, \ldots, \right.$$
$$\left. \int_P \left[\frac{\partial \phi^p}{\partial x} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^p}{\partial \eta} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma \right] d\varsigma) \right) < 0.$$

Next, we present an illustrative example to verify that the above-mentioned class of vector-controlled variational inequalities is solvable at a given point.

Example 1. Let us consider $m = p = 2, n = k = 1, P = [0, 1] \times [0, 1]$. Also, we assume that $x, \eta : P \to \mathbb{R}$ are piecewise differentiable functions and $\rho, \sigma : P \times \mathbb{R}^4 \to \mathbb{R}$ are given by: $\rho = 0$, and $\sigma = e^{x^0(\varsigma)} - e^{x(\varsigma)}$, $(\forall)\varsigma \in P \setminus \partial P$ and $\sigma = 0$ for $\varsigma \in \partial P$. Define the following Lagrange-type densities

$$\phi = \left(\phi^1, \phi^2\right) : P \times \mathbb{R}^2 \to \mathbb{R}^2,$$

as

$$\phi^1 \left(\varsigma, x(\varsigma), \eta(\varsigma)\right) = -\eta(\varsigma) - \frac{1}{2} - x(\varsigma),$$
$$\phi^2 \left(\varsigma, x(\varsigma), \eta(\varsigma)\right) = e^{\eta(\varsigma)} + \frac{1}{2}.$$

Further, we can easily note that $(x^0, \eta^0) = (0, 0)$ is a solution for the associated vector-controlled variational inequality $(VCVI)$.

Indeed, we have

$$\left(\int_P \left[\frac{\partial \phi^1}{\partial x} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^1}{\partial \eta} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma \right] d\varsigma, \right.$$

$$\left. \int_P \left[\frac{\partial \phi^2}{\partial x} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^2}{\partial \eta} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma \right] d\varsigma \right)$$

$$= \left(\int_P \left(e^{x(\varsigma)} - 1 \right) d\varsigma, \ \int_P e^{\eta(\varsigma)} \left(1 - e^{x(\varsigma)} \right) d\varsigma \right) \nleq (0,0)$$

for all piecewise differentiable functions $x, \eta : P \to \mathbb{R}$.

3. Main Results

In this section, we will state some characterization results and connections between the solutions of the considered (weak) vector-controlled variational inequalities and (proper, weak) efficient solutions of the introduced vector variational control problem (P).

Theorem 1. *Let $S \subset \mathsf{Q} \times \mathsf{T}$ be an invex set with respect to ρ and σ, and let $(x^0, \eta^0) \in S$ be a proper efficient solution of (P). If each multiple integral $\int_P \phi^l\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma$, $l = \overline{1,p}$, is Fréchet differentiable at $(x^0, \eta^0) \in S$, then the pair (x^0, η^0) solves (VI).*

Proof. By *reductio ad absurdum*, consider that $(x^0, \eta^0) \in S$ is a proper efficient solution of (P), but it does not satisfy (VI). In consequence, there exists $(x, \eta) \in S$ such that, for all $l = \overline{1,p}$, we have

$$\int_P \left[\frac{\partial \phi^l}{\partial x} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^l}{\partial \eta} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma \right] d\varsigma < 0 \qquad (1)$$

and, for $s \neq l$,

$$\int_P \left[\frac{\partial \phi^s}{\partial x} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^s}{\partial \eta} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma \right] d\varsigma \leq 0. \qquad (2)$$

By hypothesis, we have that $S \subset \mathsf{Q} \times \mathsf{T}$ is an invex set with respect to ρ and σ. Thus, we can consider the pair

$$(z, w) = (x^0, \eta^0) + \lambda_n \left(\rho\left(\varsigma, x, \eta, x^0, \eta^0\right), \sigma\left(\varsigma, x, \eta, x^0, \eta^0\right)\right) \in S, \ (\forall) n,$$

for some sequence $\{\lambda_n\}$ of positive real numbers, satisfying $\lambda_n \to 0$ as $n \to \infty$.

Further, we apply that each multiple integral $\int_P \phi^l(\varsigma, x(\varsigma), \eta(\varsigma)) d\varsigma$, $l = \overline{1, p}$ is Fréchet differentiable at $(x^0, \eta^0) \in S$ and obtain the following equality:

$$\begin{aligned} &\Phi^l(z, w) - \Phi^l(x^0, \eta^0) \\ &\quad = \int_P \lambda_n \left[\frac{\partial \phi^l}{\partial x}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^l}{\partial \eta}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma\right] d\varsigma \\ &\qquad + \parallel \lambda_n \left(\rho\left(\varsigma, x, \eta, x^0, \eta^0\right), \sigma\left(\varsigma, x, \eta, x^0, \eta^0\right)\right) \parallel \cdot G^l(z, w), \quad (3) \end{aligned}$$

where $G^l : V_{(x^0, \eta^0)} \to \mathbb{R}$ is a continuous function defined on a neighborhood of (x^0, η^0), denoted by $V_{(x^0, \eta^0)}$, with $\lim_{n \to \infty} G^l(z, w) = 0$. By dividing (3) with λ_n and taking the limit, we get

$$\begin{aligned} &\lim_{n \to \infty} \frac{1}{\lambda_n} \left[\Phi^l(z, w) - \Phi^l(x^0, \eta^0)\right] \\ &\quad = \int_P \left[\frac{\partial \phi^l}{\partial x}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^l}{\partial \eta}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma\right] d\varsigma. \quad (4) \end{aligned}$$

Combining relations (1) and (4), it results that

$$\Phi^l(z, w) - \Phi^l(x^0, \eta^0) < 0$$

for some $n \geq N$, with N a natural number.

Next, since $(x^0, \eta^0) \in S$ is a proper efficient solution of (P), we consider the nonempty set

$$\mathcal{M} = \left\{s \in \{1, \ldots, p\} \mid \Phi^s(x^0, \eta^0) - \Phi^s(z, w) \leq 0, \ (\forall) n \geq N\right\}.$$

For $s \in \mathcal{M}$, by considering the Fréchet differentiability of $\int_P \phi^s\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma$ at $(x^0, \eta^0) \in S$, we obtain

$$\begin{aligned}
&\Phi^s(z,w) - \Phi^s(x^0,\eta^0) \\
&\quad = \int_P \lambda_n \left[\frac{\partial \phi^s}{\partial x}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\rho + \frac{\partial \phi^s}{\partial \eta}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\sigma\right] d\varsigma \\
&\qquad + \parallel \lambda_n \left(\rho\left(\varsigma, x, \eta, x^0, \eta^0\right), \sigma\left(\varsigma, x, \eta, x^0, \eta^0\right)\right) \parallel \cdot G^s(z,w), \qquad (5)
\end{aligned}$$

where $G^s : V_{(x^0,\eta^0)} \to \mathbb{R}$ is a continuous function defined on a neighborhood of (x^0, η^0), denoted by $V_{(x^0,\eta^0)}$, with $\lim_{n\to\infty} G^s(z,w) = 0$. By dividing (5) with λ_n and taking the limit, we get

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{\lambda_n}\left[\Phi^s(z,w) - \Phi^s(x^0,\eta^0)\right] \\
&\quad = \int_P \left[\frac{\partial \phi^s}{\partial x}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\rho + \frac{\partial \phi^s}{\partial \eta}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\sigma\right] d\varsigma.
\end{aligned}$$

By using the property of the set $\mathcal{M}$, for $n \geq N$, we get

$$\int_P \left[\frac{\partial \phi^s}{\partial x}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\rho + \frac{\partial \phi^s}{\partial \eta}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\sigma\right] d\varsigma \geq 0. \qquad (6)$$

Combining relations (2) and (6), we get

$$\int_P \left[\frac{\partial \phi^s}{\partial x}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\rho + \frac{\partial \phi^s}{\partial \eta}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\sigma\right] d\varsigma = 0$$

for some $n \geq N$, with N a natural number, and $s \neq l,\ s \in \mathcal{M}$.

Finally, for $s \neq l,\ s \in \mathcal{M}$, by computing the limit

$$\frac{\frac{1}{\lambda_n}\left[\Phi^l(x^0,\eta^0) - \Phi^l(z,w)\right]}{\frac{1}{\lambda_n}\left[\Phi^s(z,w) - \Phi^s(x^0,\eta^0)\right]},$$

we find that it is ∞ as $n \to \infty$, which contradicts the proper efficiency of (x^0, η^0) for (P). The proof is now complete. □

The next theorem provides a charaterization of the efficient solutions for (P) by using the vector-controlled variational inequality (VI).

Theorem 2. *Let $(x^0, \eta^0) \in S$ be a solution of (VI). If each multiple integral*

$$\int_P \phi^l \left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma, \quad l = \overline{1, p}$$

is Fréchet differentiable and invex at $(x^0, \eta^0) \in S$ with respect to ρ and σ, then the pair (x^0, η^0) is an efficient solution of (P).

Proof. By *reductio ad absurdum*, consider that $(x^0, \eta^0) \in S$ is a solution of (VI), but it is not an efficient solution of (P). In consequence, there exists $(x, \eta) \in S$ such that, for all $l = \overline{1, p}$,

$$\Phi^l(x, \eta) - \Phi^l(x^0, \eta^0) \leq 0, \tag{7}$$

with strict inequality for at least one l.

By hypothesis, each multiple integral $\int_P \phi^l \left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma$, $l = \overline{1, p}$ is Fréchet differentiable and invex at $(x^0, \eta^0) \in S$ with respect to ρ and σ. In consequence, we have

$$\begin{aligned} &\Phi^l(x, \eta) - \Phi^l(x^0, \eta^0) \\ &\quad \geq \int_P \left[\frac{\partial \phi^l}{\partial x} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^l}{\partial \eta} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma \right] d\varsigma \end{aligned} \tag{8}$$

for any $(x, \eta) \in S$ and $l = \overline{1, p}$.

On combining inequalities (7) and (8), we find that, for all $l = \overline{1, p}$, there exists $(x, \eta) \in S$ such that

$$\int_P \left[\frac{\partial \phi^l}{\partial x} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \rho + \frac{\partial \phi^l}{\partial \eta} \left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right) \sigma \right] d\varsigma \leq 0,$$

with strict inequality for at least one l, which contradicts that $(x^0, \eta^0) \in S$ is solution of (VI). The proof is now complete. □

The next result formulates a sufficient condition for a pair $(x^0, \eta^0) \in S$ to be a solution of (WVI).

Theorem 3. *Let $S \subset \mathsf{Q} \times \mathsf{T}$ be an invex set with respect to ρ and σ, and let $(x^0, \eta^0) \in S$ be a weak efficient solution of (P). If each multiple integral $\int_P \phi^l\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma$, $l = \overline{1,p}$, is Fréchet differentiable at $(x^0, \eta^0) \in S$, then the pair (x^0, η^0) solves (WVI).*

Proof. Since $(x^0, \eta^0) \in S$ is a weak efficient solution of (P), it results that there exists no other feasible solution $(x, \eta) \in S$ such that $\Phi(x, \eta) < \Phi(x^0, \eta^0)$, or, equivalently,

$$\Phi^l(x, \eta) - \Phi^l(x^0, \eta^0) < 0, \quad (\forall) l = \overline{1,p}. \tag{9}$$

By hypothesis, we have that $S \subset \mathsf{Q} \times \mathsf{T}$ is an invex set with respect to ρ and σ. Thus, for all $\lambda \in [0, 1]$, we have

$$(z, w) = (x^0, \eta^0) + \lambda \left(\rho\left(\varsigma, x, \eta, x^0, \eta^0\right), \sigma\left(\varsigma, x, \eta, x^0, \eta^0\right)\right) \in S.$$

Thus, by using (9), we can see that there exists no other feasible solution $(x, \eta) \in S$ such that $\Phi(z, w) < \Phi(x^0, \eta^0)$, or, equivalently,

$$\Phi^l(z, w) - \Phi^l(x^0, \eta^0) < 0, \quad (\forall) l = \overline{1,p}. \tag{10}$$

Further, we apply that each multiple integral $\int_P \phi^l(\varsigma, x(\varsigma), \eta(\varsigma)) d\varsigma$, $l = \overline{1,p}$, is Fréchet differentiable at $(x^0, \eta^0) \in S$ and, proceeding as in the proof of Theorem 1, by (10), we obtain that there exists no other feasible solution $(x, \eta) \in S$ such that

$$\int_P \left[\frac{\partial \phi^l}{\partial x}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\rho + \frac{\partial \phi^l}{\partial \eta}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\sigma\right] d\varsigma < 0$$

for all $l = \overline{1,p}$. This ends the proof. □

The following theorem provides a charaterization of weak efficient solutions for (P) by using the weak vector-controlled variational inequality (WVI).

Theorem 4. *Let $(x^0, \eta^0) \in S$ be a solution of (WVI). If each multiple integral $\int_P \phi^l\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma$, $l = \overline{1,p}$, is Fréchet differentiable and pseudoinvex at $(x^0, \eta^0) \in S$ with respect to ρ and σ, then the pair (x^0, η^0) is a weak efficient solution of (P).*

Proof. By *reductio ad absurdum*, consider that $(x^0, \eta^0) \in S$ is a solution of (WVI) but it is not a weak efficient solution of (P). In consequence, there exists $(x, \eta) \in S$ such that, for all $l = \overline{1,p}$,

$$\Phi^l(x, \eta) - \Phi^l(x^0, \eta^0) < 0.$$

By hypothesis, each multiple integral $\int_P \phi^l\left(\varsigma, x(\varsigma), \eta(\varsigma)\right) d\varsigma,\ l = \overline{1,p}$, is Fréchet differentiable and pseudoinvex at $(x^0, \eta^0) \in S$ with respect to ρ and σ. In consequence, we have

$$\int_P \left[\frac{\partial \phi^l}{\partial x}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\rho + \frac{\partial \phi^l}{\partial \eta}\left(\varsigma, x^0(\varsigma), \eta^0(\varsigma)\right)\sigma\right] d\varsigma < 0$$

for any $(x, \eta) \in S$ and $l = \overline{1,p}$. This contradicts that $(x^0, \eta^0) \in S$ is a solution of (WVI). The proof is complete. □

4. Conclusions

In this chapter, we have established certain connections between the solutions of the considered (weak) vector-controlled-variational inequalities and (proper, weak) efficient solutions associated with a class of multiobjective controlled variational problems defined by functionals of multiple integral type. A crucial role for proving the main results formulated in this chapter was played by the concept of invex set with respect to some given functions, the generalized convexity and Fréchet differentiability of the functionals which were involved. As further development, taking into account the concept of *variational derivative* (see, for instance, Treanţă[17]) for functionals of multiple integral type, new aspects can be formulated based on the results presented in this chapter.

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Chapter 15

The Theory of Nonabsolute Integrable Function Spaces over R$^\infty$ and Its Various Applications

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We develop a Banach space $K^p(\mathbf{R}^\infty)$, $1 \leq p \leq \infty$. We discuss the Lebesgue spaces L^p over $\mathbf{R}^\infty$ are subsets of $K^p(\mathbf{R}^\infty)$ as dense continuous compact embedding. The spaces of finitely additive measure $\mathcal{M}$ over $\mathbf{R}^\infty$ are subsets of $K^p(\mathbf{R}^\infty)$ as dense continuous compact embedding. Also, we discuss that $K^p(\mathbf{R}^\infty)$ contains Henstock–Kurzweil integrable functions over $\mathbf{R}^\infty$. Further, we obtain Hilbert space $GD^2(\mathbf{R}^\infty_\mathbf{I})$. Finally, we demonstrate that Fourier transformation and convolution operators can be extended on the Hilbert spaces $K^2(\mathbf{R}^\infty)$ as linear operators.

1. Introduction

The formation of Banach spaces with Luxemburg norm of Henstock–Kurzweil-integrable function is not possible (see Refs. 1–9). Yeong[10] addressed this drawback, which can be solved if a canonical approach is developed. In Kuelbs–Steadman spaces denoted by $KS^p(S)$, $\forall\, 1 \leq r \leq \infty$, where $S \subset \mathbf{R^m}$ was developed by Gill and Zachary (see Ref. 11). An interesting fact about Kuelbs–Steadman spaces is that they are canonical in nature (see Refs. 11, 12). These spaces are directly related to the Feynman operator calculus and path integral. Kuelbs–Steadman spaces are separable, also Kuelbs–Steadman spaces containing the Lebesgue space L^p are continuously dense compact embedding. These spaces contains L^p as dense, continuous compact embedding and they are also separable. Gill and Zachary mainly focused in a Banach space which contains Henstock–Kurzweil integrable functions. Henstock–Kurzweil integrable functions are equivalent to the Denjoy integrable function in a restricted sense (see Refs. 3, 5, 13, 14). The motivation behind the construction of KS^p spaces was to find a Banach space with additive measures as well as containing Henstock–Kurzweil integrable functions. Simultaneously, Gill and Myers introduced an equivalent approach of Lebesgue measure theory over $\mathbf{R}^\infty$ (see Ref. 15). These ideas seem useful to construct separable Banach spaces containing Henstock–Kurzweil integrable functions. Gill *et al.*[16] discussed the approach of the Kuelbs–Steadman spaces over $\mathbf{R}^\infty$. Kalita *et al.*,[17] expanded the theory of the spaces KS^p on separable Banach spaces. In Ref. 16, we indicated that all theorems of $K^p(\mathbf{R_I^m})$ are extendable to $K^p(\mathbf{R_I^\infty})$. In this chapter, we discuss the construction of $K^p(\mathbf{R_I^\infty})$ and we will discuss all results that we mention in Ref. 16.

2. Preliminaries

In this section, we recall few definitions, which are important in our main section. We denote the set of real numbers by $\mathbf{R}$.

Definition 1. A σ-algebra on a set X is a nonempty collection Σ of subsets of X closed under complement and closed under countable unions and countable intersections. The pair (X, Σ) is called a measurable space.

Definition 2. Let (X, Σ_1, μ_1) and (Y, Σ_1, μ_2) be two measurable spaces. A product measure $\mu_1 \times \mu_2$ is defined to be a measure on the measurable space $(X \times Y, \Sigma_1 \times \Sigma_2)$, if it satisfies the property

$$(\mu_1 \times \mu_2)(A \times B) = \mu_1(A)\mu_2(B),$$

for $A \in \Sigma_1, B \in \Sigma_2$.

Definition 3. The Borel measures on the Euclidean space $\mathbf{R}^n$ can be obtained as the product of n copies of Borel measures on the real line $\mathbf{R}$.

Definition 4. Let μ be a measure on $\mathbf{R}^n$ equipped with the Borel σ-algebra $B(\mathbf{R}^n)$. Then μ is said to be translation-invariant or invariant under translations if $\mu(x+A) = \mu(A)$ for all $x \in \mathbf{R}^n$ and $A \in B(\mathbf{R}^n)$.

Definition 5. Let (X, Σ, μ) be a measurable space. A transformation $T : X \to X$ is said to be measurable if $T^{-1}(A) \in \Sigma$ for all $A \in \Sigma$. We say that T is a measure-preserving transformation or equivalently μ is said to be T-invariant measure if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \Sigma$.

Definition 6 ([3, Definition 9.3]). A function $\mathbf{f} : [a, b] \to \mathbf{R}$ is Henstock–Kurzweil integrable on $[a, b]$ if for each $\epsilon > 0$, there exists a positive function δ on $[a, b]$ and a real number A such that

$$\left| \sum_{i=1}^{n} \mathbf{f}(x_i)(d_i - c_i) - A \right| < \epsilon.$$

Let us denote $HK(\mathbf{R})$ as the space of Henstock–Kurzweil integrable functions. Recall the norm for $\mathbf{f} \in HK(\mathbf{R})$ as

$$||\mathbf{f}||_{HK(\mathbf{R})} = \sup_s \left| \int_{-\infty}^{s} \mathbf{f}(r)dx(r) \right|. \tag{1}$$

The norm defined by Eq. (1) was introduced by Alexiewicz (see Ref. 9). One can follow Refs. 3,9 for the details of Henstock–Kurzweil integrals. Throughout the chapter, we consider $\mathbf{I} = [-\frac{1}{2}, \frac{1}{2}]$.

Definition 7 ([11, Definition 2.5]). If $\mathbf{A}_m = A \times \mathbf{I}_m$; $\mathbf{B}_m = B \times \mathbf{I}_m$ (mth order box sets in $\mathbf{R}^\infty$). We recall

(i) $\mathbf{A}_m \cup \mathbf{B}_m = (A \cup B) \times \mathbf{I}_m$;
(ii) $\mathbf{A}_m \cap \mathbf{B}_m = (A \cap B) \times \mathbf{I}_m$;
(iii) $\mathbf{B}_m^c = B^c \times \mathbf{I}_m$.

Definition 8 ([15, Definition 1.11]). Let us consider $\mathbf{R}^m_{\mathbf{I}} = \mathbf{R}^m \times \mathbf{I}_m$. Also, let $\mathbf{T}$ be a linear transformation on $\mathbf{R}^m$. The linear transformation on $\mathbf{R}^m_{\mathbf{I}}$ can be considered as $\mathbf{T}_I(\mathbf{A}_m) = \mathbf{T}(A)$, where $\mathbf{A}_m = A \times \mathbf{I}_m$.

Let us denote the Borel σ-algebra for $\mathbf{R}^m_{\mathbf{I}}$ by $B(\mathbf{R}^m_{\mathbf{I}})$. We can define the topology on $\mathbf{R}^m_{\mathbf{I}}$ via the class of open sets $\mathbf{D}_m = \{U \times \mathbf{I}_m\}$, where $U \subset \mathbf{R}^m$ is an open set.

For given two measurable spaces, one can obtain a product measurable space and a product measure on that space with the similar concept of the Cartesian product of sets and the product of sets and the product topology of two topological spaces. When A in $B(\mathbf{R}^m)$, and $\nu_\infty(\mathbf{A}_m)$ on $\mathbf{R}^m_{\mathbf{I}}$, the properties of product measure gives $\nu_\infty(\mathbf{A}_m) = \nu_m(A) \times \pi^{\infty}_{i=m+1}\nu_{\mathbf{I}}(\mathbf{I}) = \nu_m(A)$, where $\nu_m(A)$ is Lebesgue measure of A. Recalling (see Ref. 11 Theorem 2.7), if $\nu_\infty(.)$ is a measure on $B(\mathbf{R}^m_{\mathbf{I}})$ then $\nu_\infty(.)$ coincides with the m-dimensional Lebesgue measure on $\mathbf{R}^m$.

Corollary 1 ([11, Corollary 2.8]). *For each $m \in \mathbf{N}$, the measure $\nu_\infty(.)$ is translation as well as rotational invariant on $\big(\mathbf{R}^m_{\mathbf{I}}, B(\mathbf{R}^m_{\mathbf{I}})\big)$.*

The theory on $\mathbf{R}^m_{\mathbf{I}}$ is completely parallel to that on $\mathbf{R}^m$. Since $\mathbf{R}^m_{\mathbf{I}} \subset \mathbf{R}^{m+1}_{\mathbf{I}}$, one can define $\widehat{\mathbf{R}}^\infty_{\mathbf{I}} = \lim\limits_{n\to\infty} \mathbf{R}^m_{\mathbf{I}} = \bigcup_{k=1}^\infty \mathbf{R}^k_{\mathbf{I}}$. Let $\mathbf{X}_1 = \widehat{\mathbf{R}}^\infty_{\mathbf{I}}$. We consider the topology τ_1 induced by the class of open sets $D \subset \mathbf{X}_1$ so that $D = \bigcup\limits_{n=1}^{\infty} \mathbf{D}_m = \bigcup\limits_{n=1}^{\infty} \{U \times \mathbf{I}_m : U \text{ is open in } \mathbf{R}^m\}$. Let $\mathbf{X}_2 = \mathbf{R}^\infty \setminus \widehat{\mathbf{R}}^\infty_{\mathbf{I}}$, and we assume a discrete topology τ_2 on $\mathbf{X}_2$ induced by the discrete metric so that for $x, y \in \mathbf{X}_2,\ x \neq y,\ \mathbf{d}_2(x,y) = 1$ and for $x = y,\ \mathbf{d}_2(x,y) = 0$. We denote the co-product of $\big(\mathbf{X}_1, \tau_1\big)$ and $\big(\mathbf{X}_2, \tau_2\big)$ on $\big(\mathbf{R}^\infty_{\mathbf{I}}, \tau\big)$ by $\big(\mathbf{X}_1, \tau_1\big) \otimes \big(\mathbf{X}_2, \tau_2\big)$. In this case, every open set in $\big(\mathbf{R}^\infty_{\mathbf{I}}, \tau\big)$ is the disjoint union of two open sets $\mathbf{G}_1 \cup \mathbf{G}_2$ with $\mathbf{G}_1$ in $\big(\mathbf{X}_1, \tau_1\big)$ and $\mathbf{G}_2$ in $\big(\mathbf{X}_2, \tau_2\big)$.

Being sets, $\mathbf{R}^\infty_{\mathbf{I}} = \mathbf{R}^\infty$. But, since each point in $\mathbf{X}_2$ is open and closed in $\mathbf{R}^\infty_{\mathbf{I}}$ and points in $\mathbf{R}^\infty$ are neither open nor closed, hence, as a topological space, $\mathbf{R}^\infty_{\mathbf{I}} \neq \mathbf{R}^\infty$.

Likewise, if $B(\mathbf{R}^m_{\mathbf{I}})$ is the Borel σ-algebra of $\mathbf{R}^m_{\mathbf{I}}$, at that time, $B\big(\mathbf{R}^m_{\mathbf{I}}\big) \subset B\big(\mathbf{R}^{m+1}_{\mathbf{I}}\big)$ can be defined as

$$\widehat{B}\big(\mathbf{R}^\infty_{\mathbf{I}}\big) = \lim_{m\to\infty} B(\mathbf{R}^m_{\mathbf{I}}) = \bigcup_{k=1}^{\infty} B(\mathbf{R}^k_{\mathbf{I}}).$$

If $B(\mathbf{R}_\mathbf{I}^\infty)$ is the tiniest σ-algebra carrying $\widehat{B}(\mathbf{R}_\mathbf{I}^\infty) \cup P\left(\mathbf{R}^\infty \setminus \bigcup_{k=1}^{\infty} (\mathbf{R}_\mathbf{I}^k)\right)$, where $P(.)$ is the power set, then clearly the class $B(\mathbf{R}_\mathbf{I}^\infty)$ coincides with the Borel σ-algebra produced by the τ-topology on $\mathbf{R}_\mathbf{I}^\infty$. We can find $\widehat{B}(\mathbf{R}_\mathbf{I}^\infty) \subset B(\mathbf{R}_\mathbf{I}^\infty)$ (see Ref. 15 Lemma 1.15).

3. Measurable Functions

In Ref. 15, Gill and Myres have shown that the Lebesgue measure ν_m of $\mathbf{R}^m$ can be extended to the measure $\nu_\infty(.)$ of $\mathbf{R}^\infty$. The theory of measurable functions on $\mathbf{R}_\mathbf{I}^\infty$ was discussed in Refs. 11, 16, 18. We recall the measurable function on $\mathbf{R}_\mathbf{I}^\infty$. Assume $\mathbf{x} = (x_1, x_2, \dots) \in \mathbf{R}_\mathbf{I}^\infty$, $\mathbf{I}_m = \Pi_{k=m+1}^{\infty}[\frac{-1}{2}, \frac{1}{2}]$ and let $\mathbf{h}_m(\widehat{x}) = \chi_{\mathbf{I_m}}(\widehat{\mathbf{x}})$, where $\widehat{\mathbf{x}} = (x_i)_{i=m+1}^{\infty}$. Assume the class of measurable functions on $\mathbf{R}^m$ is represented by M^m. Let $\mathbf{x} \in \mathbf{R}_\mathbf{I}^\infty$ and $f^m \in M^m$. Let an essentially tame measurable function of order m (or e_m-tame) on $\mathbf{R}_\mathbf{I}^\infty$ be

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}^m(\overline{\mathbf{x}}) \otimes \mathbf{h}_m(\widehat{\mathbf{x}}),$$

where $\overline{\mathbf{x}} = (x_i)_{i=1}^{m}$. Then $M_\mathbf{I}^m = \left\{\mathbf{f}(x) : \mathbf{f}(x) = \mathbf{f}^m(\overline{\mathbf{x}}) \otimes \mathbf{h}_m(\widehat{\mathbf{x}}), \mathbf{x} \in \mathbf{R}_\mathbf{I}^\infty\right\}$ is the class of all e_m-tame functions.

Definition 9 ([11, Definition 2.47]). We call a mapping $\mathbf{f} : \mathbf{R}_\mathbf{I}^\infty \to \mathbf{R}$ to be measurable with $\mathbf{f} \in M_\mathbf{I}$, if there exists a sequence $\left\{\mathbf{f}_m \in M_\mathbf{I}^m\right\}$ of e_m-tame functions, so that

$$\lim_{m\to\infty} \mathbf{f}_m(x) \to \mathbf{f}(x) \quad \textit{with } \nu_\infty - (a.e.).$$

Recalling the existence of functions fulfilling Definition 9 is not obvious.

Theorem 1 ([11, Theorem 2.48] (Existence)). *Let us consider* $\mathbf{f} : \mathbf{R}_\mathbf{I}^\infty \to (-\infty, \infty)$ *in addition to* $\mathbf{f}^{-1}(A) \in B(\mathbf{R}_\mathbf{I}^\infty)$ *for all* $A \in B(\mathbf{R})$, *then for a family of functions* $\{\mathbf{f}_m\}$, *such that* $\mathbf{f}_m(x) \to \mathbf{f}(x)$, ν_∞ $(-a.e.)$, *where* $\mathbf{f}_m \in M_\mathbf{I}^m$.

Remark 1. Recognizing that any measurable set A, with nonzero measure, is concentrated in $\mathbf{X}_1$ that is to say, $\nu_\infty(A) = \nu_\infty(A \cap \mathbf{X}_1)$. In addition it follows that the essential support of the limit function

$\mathbf{f}(x)$ in Definition 9, i.e. $\{x: \mathbf{f}(x) \neq 0\}$ is concentrated in $\mathbf{R}_{\mathbf{I}}^{N}$, for some N.

4. $\mathbf{R}_{\mathbf{I}}^{\infty}$ and Its Integration Theory

The construction of the integration theory on $\mathbf{R}_{\mathbf{I}}^{\infty}$ is similar to the integration theory of $\mathbf{R}^{m}$. All theorems for Lebesgue measure are also applicable in the theory of $\mathbf{R}_{\mathbf{I}}^{m}$. Let us consider the class of integrable functions on $\mathbf{R}_{\mathbf{I}}^{m}$ be $L^1(\mathbf{R}_{\mathbf{I}}^{m})$. As $L^1(\mathbf{R}_{\mathbf{I}}^{m}) \subset L^1(\mathbf{R}_{\mathbf{I}}^{m+1})$, recall $L^1(\widehat{\mathbf{R}}_{\mathbf{I}}^{\infty}) = \bigcup_{m=1}^{\infty} L^1(\mathbf{R}_{\mathbf{I}}^{m})$.

A measurable function $\mathbf{f}$ is in $L^1(\mathbf{R}_{\mathbf{I}}^{\infty})$ if the following condition satisfies $\mathbf{f} \in L^1(\mathbf{R}_{\mathbf{I}}^{\infty})$, if for a Cauchy-sequence $\{\mathbf{f}_m\} \subset L^1\big(\widehat{\mathbf{R}}_{\mathbf{I}}^{\infty}\big)$ in the company of $\mathbf{f}_m \in L^1(\mathbf{R}_{\mathbf{I}}^{m})$, along with

$$\lim_{m\to\infty} \mathbf{f}_m(x) = \mathbf{f}(x),\ \nu_\infty - (a.e.).$$

Theorem 2 ([16, Theorem 1.8]). $L^1(\mathbf{R}_{\mathbf{I}}^{\infty}) = L^1(\widehat{\mathbf{R}}_{\mathbf{I}}^{\infty})$.

For $f \in L^1(\mathbf{R}_{\mathbf{I}}^{\infty})$, the integral of $\mathbf{f}$ is defined as follows:

$$\int_{\mathbf{R}_{\mathbf{I}}^{\infty}} \mathbf{f}(x) d\nu_\infty(x) = \lim_{m\to\infty} \int_{\mathbf{R}_{\mathbf{I}}^{\infty}} \mathbf{f}_m(x) d\nu_\infty(x), \tag{2}$$

for any Cauchy-sequence $\{f_m\} \subset L^1(\mathbf{R}_{\mathbf{I}}^{\infty})$ converges to $\mathbf{f}(x)$-a.e. (see Ref. 16 Definition 1.9). We can find from Ref. 15, Theorem 3.15, that when $\mathbf{f} \in L^1({\mathbf{R}_{\mathbf{I}}}^{\infty})$, then the integral (2) exists and all theorems that are true for $\mathbf{f} \in L^1(\mathbf{R}_{\mathbf{I}}^{m})$, also hold for $\mathbf{f} \in L^1(\mathbf{R}_{\mathbf{I}}^{\infty})$.

5. Kuelbs–Steadman Spaces $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ $1 \leq p \leq \infty$

Recalling Henstock–Kurzweil integrals generalize the Riemann and Lebesgue integrals. In the theory of Henstock[6] and Kurzweil,[14] finitely additive measure can be generated by the Henstock–Kurzweil integral (HK integral). In the restricted sense, Denjoy integrals are equivalent to the Henstock–Kurzweil integrals and also the Perron integrals (see Ref. 3 and references therein). The theory of Henstock–Kurzweil integral is easier in comparison with the theory of Lebesgue integral. Henstock–Kurzweil integrals give useful variants of the theorems of Lebesgue integral. The widespread use of the Henstock–Kurzweil integral in mathematics, engineering, physics and chemistry is unexplored due to the lack of a Banach space structure of Henstock–Kurzweil integral. The possibility of development of this condition began indirectly in 1965, when Gross[19] proved that every separable Banach space contains a separable Hilbert space as a continuous dense embedding. This was the extension work of Wieners theory, which used the (densely embedded Hilbert) Sobolev space $\mathcal{H}_0^1[0,1] \subset C_0[0,1]$. Kuelbs generalized Gross's theorem (see Ref. 12) in the year 1970.

Theorem 3 ([16, Theorem 2.1] Gross–Kuelbs). *For a separable Banach space $\overline{B}$ there exist two other Banach spaces, H_1, H_2, such that $H_1 \subset \overline{B} \subset H_2$ all as continuous dense embedding with the followings*

$$\left\langle T_{12}^{\frac{1}{2}}u, T_{12}^{\frac{1}{2}}v \right\rangle_1 = \left\langle u, v \right\rangle_2 \text{ and}$$
$$\left\langle T_{12}^{\frac{-1}{2}}u, T_{12}^{\frac{-1}{2}}v \right\rangle_2 = \left\langle u, v \right\rangle_1,$$

where T_{12} is a positive trace class operator defined on H_2.

This work appeared for the first time in Ref. 20 in association to the Henstock–Kurzweil integral.

Lemma 1 ([12, Kuelbs Lemma]). *Let $\overline{B}$ be a separable Banach space then $H \supset \overline{B}$ is a continuous dense embedding, where H is a separable Hilbert space.*

It is very straight forward to see that some Henstock-Kurzweil integrable-type functions are in H. It is very complicated to predict whether these nonabsolute integrals are Henstock–Kurzweil or not. Steadman gave an effective hint for this (see Ref. 20). Before the construction of $K^p(\mathbf{R}_\mathbf{I}^\infty)$, we need to understand the construction of $K^p(\mathbf{R}_\mathbf{I}^m)$.

Formation of $K^p(\mathbf{R}_\mathbf{I}^m)$: Let us consider $m \in \mathbf{N}$ and let $\mathbf{Q}_\mathbf{I}^\mathbf{m}$ be the set $\{\mathbf{x} \in \mathbf{R}_\mathbf{I}^m\}$, so that $(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m)$ are initial co-ordinates. These co-ordinates are rational. Clearly, this is a countable dense set of $\mathbf{R}_{\mathbf{I}}^{m}$. For every k and i, we can arrange $\mathbf{Q}_{\mathbf{I}}^{\mathbf{m}} = \{\mathbf{x_1}, \mathbf{x_2}, ...\}$. Let us assume a closed cube $\mathcal{B}_k(x_i)$ centered at x_i along with edge $e_k = \frac{1}{2^{l-1}\sqrt{n}}$, $l \in \mathbf{N}$. We can choose a map $\mathbf{N} \times \mathbf{N}$ bijectively to $\mathbf{N}$ of natural order and suppose the resulting set of all closed cubes

$$\{\mathcal{B}_k(x_i)|\ (l, i) \in \mathbf{N} \times \mathbf{N}\}$$

centered at a point in $\mathbf{Q}_{\mathbf{I}}^{\mathbf{m}}$ be $\{\mathcal{B}_k :\ k \in \mathbf{N}\}$. Let the characteristic function of $\mathcal{B}_k$ be $\mathbf{E_k}(\mathbf{x})$ such that for $1 \leq p < \infty$, $\mathbf{E_k}(\mathbf{x}) \in \mathbf{L^p}(\mathbf{R}_{\mathbf{I}}^{\mathbf{m}}) \cap \mathbf{L}^{\infty}(\mathbf{R}_{\mathbf{I}}^{\mathbf{m}})$. We can define $F_k(.)$ on $L^1(\mathbf{R}_{\mathbf{I}}^{m})$ by

$$F_k(\mathbf{f}) = \int_{\mathbf{R}_{\mathbf{I}}^{m}} \mathbf{E_k}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_{\infty}(\mathbf{x}).$$

Since each $\mathcal{B}_k$ is a cube with sides parallel to the co-ordinate axes, $F_k(.)$ is well defined for all Henstock–Kurzweil integrable functions and is a bounded linear functional on $L^p(\mathbf{R}_{\mathbf{I}}^{m})$ for $1 \leq p \leq \infty$. Let $\mathbf{t}_k > 0$ so that $\sum_{k=1}^{\infty} \mathbf{t}_k = 1$. We can define an inner product $< \ . \ >$ on $L^1(\mathbf{R}_{\mathbf{I}}^{m})$ as follows:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{k=1}^{\infty} \mathbf{t}_k \left(\int_{\mathbf{R}_{\mathbf{I}}^{m}} \mathbf{E_k}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_{\infty}(\mathbf{x}) \right) \times \left(\int_{\mathbf{R}_{\mathbf{I}}^{n}} \mathbf{E_k}(\mathbf{y})\mathbf{g}(\mathbf{y})\mathbf{d}\nu_{\infty}(\mathbf{y}) \right)^{\mathbf{c}}.$$

With this inner product, the completion of $L^1(\mathbf{R}_{\mathbf{I}}^{m})$ is the p-summable space $K^2(\mathbf{R}_{\mathbf{I}}^{m})$. This space contains Henstock–Kurzweil integrable functions. We can find this as follows:

$$||\mathbf{f}||_{K^2}^2 = \sum_{k=1}^{\infty} \mathbf{t}_k \left| \int_{\mathbf{R}_{\mathbf{I}}^{n}} \mathbf{E_k}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_{\infty}(\mathbf{x}) \right|^{\mathbf{2}}.$$

If we replace $\mathbf{R}$ by $\mathbf{R}_{\mathbf{I}}^{m}$, then the Alexiewicz norm defined in Eq. (1) becomes

$$||\mathbf{f}||_{HK(\mathbf{R}_{\mathbf{I}}^{m})} = \sup_{r>0} \left| \int_{\mathcal{B}_r} \mathbf{f}(x) d\nu_{\infty}(x) \right| < \infty, \tag{3}$$

and $||\mathbf{f}||_{K^2}^2 \leq \sup_k |\int_{\mathcal{B}_k} \mathbf{f}(x) d\nu_{\infty}(x)|^2 < \infty$. So, $\mathbf{f} \in K^2(\mathbf{R}_{\mathbf{I}}^{m})$.

Theorem 4 ([16, Theorem 2.3]). *For every* p, $1 \leq p \leq \infty$, $L^p(\mathbf{R}_\mathbf{I}^m)$ *is a densely continuous subspace of* $K^2(\mathbf{R}_\mathbf{I}^m)$.

For $1 \leq p \leq \infty$, the norm of $K^p(\mathbf{R}_\mathbf{I}^m)$ can be defined as

$$||\mathbf{f}||_{K^p(\mathbf{R}_\mathbf{I}^m)} = \begin{cases} \left(\sum_{k=1}^{\infty} \mathbf{t}_k \left| \int_{\mathbf{R}_\mathbf{I}^m} \mathbf{E}_\mathbf{k}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_\infty(\mathbf{x}) \right|^p \right)^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty; \\ \sup_{k \geq 1} \left| \int_{\mathbf{R}_\mathbf{I}^m} \mathbf{E}_\mathbf{k}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_\infty(\mathbf{x}) \right|, & \text{for } p = \infty \end{cases}$$

with $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots) \in \mathbf{R}_\mathbf{I}^\infty$ and $\mathbf{R}_\mathbf{I}^\infty = \lim_{m \to \infty} \mathbf{R}_\mathbf{I}^m = \bigcup_{m=1}^{\infty} \mathbf{R}_\mathbf{I}^k$ and $\mathbf{R}_\mathbf{I}^m = \mathbf{R}^m \times \mathbf{I}_m$, where $\mathbf{I} = [\frac{-1}{2}, \frac{1}{2}]$.

Obviously, $||f||_{K^p}$ defines on L^p and with respect to this norm L^p is the completion of K^p. As L^p is the completion of K^p, then we have the following theorem:

Theorem 5 ([16, Theorem 2.4]). *For every* q, $1 \leq q \leq \infty$, $L^q(\mathbf{R}_\mathbf{I}^m)$ *is a densely continuous subspace of* $K^p(\mathbf{R}_\mathbf{I}^m)$. *Also, the inclusion is continuous embedding.*

Remark 2. For each p, $1 \leq p < \infty$, the Hölder and Minkowski inequalities hold for $L^p(\mathbf{R}_\mathbf{I}^m)$. The norm of $K^p(\mathbf{R}_\mathbf{I}^m)$ is derived from $L^p(\mathbf{R}_\mathbf{I}^m)$ and the completion of $L^p(\mathbf{R}_\mathbf{I}^m)$ is $K^p(\mathbf{R}_\mathbf{I}^m)$. So, for $1 \leq p < \infty$, one can easily show that Hölder and Minkowski inequalities hold in $K^p(\mathbf{R}_\mathbf{I}^m)$.

Theorem 6 ([16, Theorem 2.5]). *For* $p, 1 \leq p \leq \infty$, *the following are true:*

(1) *When* $\mathbf{f}_m \to \mathbf{f}$ *weakly in* $L^p(\mathbf{R}_\mathbf{I}^m)$, *then* $\mathbf{f}_m \to \mathbf{f}$ *strongly in* $K^p(\mathbf{R}_\mathbf{I}^m)$.

(2) $K^p(\mathbf{R}_\mathbf{I}^m)$ *is uniformly convex for* $1 < p < \infty$.

(3) $K^q(\mathbf{R}_\mathbf{I}^m)$ *the dual space of* $K^p(\mathbf{R}_\mathbf{I}^m)$ *for* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

(4) $K^\infty(\mathbf{R}_\mathbf{I}^m) \subset K^p(\mathbf{R}_\mathbf{I}^m)$ *for* $1 \leq p < \infty$.

(5) $C_c(\mathbf{R}_\mathbf{I}^m)$ *is dense subset of* $K^p(\mathbf{R}_\mathbf{I}^m)$.

Theorem 7 ([16, Theorem 2.6]). *For every* p, $1 \leq p \leq \infty$, *then the test function spaces* $D(\mathbf{R}_\mathbf{I}^m) \subset K^p(\mathbf{R}_\mathbf{I}^m)$ *are continuously embedding.*

6. GD^2 Gross Steadman–Hilbert Spaces

Hilbert spaces play an important role in the extension of the Feynman operator Calculus to the nonreflexive Banach spaces. In this section, we will construct a Hilbert space on $\mathbf{R}_{\mathbf{I}}^m$, which is a natural compliment to $K^2(\mathbf{R}_{\mathbf{I}}^m)$ in a certain general sense. For the construction of this Hilbert space, we fix $\overline{B}$ and define $GD^2_{\overline{B}}(\mathbf{R}_{\mathbf{I}}^m)$ by

$$GD^2_{\overline{B}}(\mathbf{R}_{\mathbf{I}}^m) = \left\{ \mathbf{f} \in \overline{B} : \ \sum_{n=1}^{\infty} \mathbf{t}_m^{-1} | < \mathbf{f}, \ \mathbf{E_m} >_{\mathbf{2}} |^{\mathbf{2}} < \infty \right\},$$

and $\left\langle \mathbf{f}, \ \mathbf{g} \right\rangle_1 = \sum_{m=1}^{\infty} \mathbf{t}_m^{-1} \left\langle \mathbf{f}, \mathbf{E_m} \right\rangle_{\mathbf{2}} \left\langle \mathbf{E_m}, \mathbf{g} \right\rangle_{\mathbf{2}}$ where $\mathbf{g} \in \overline{B}$. We can extend the Gross–Kuelbs theorem in this sense as follows:

Theorem 8. *The operator T_{12} is a positive trace class operator on $\overline{B}$ with a bounded extension to $K^2(\mathbf{R}_{\mathbf{I}}^m)$. Additionally, $GD^2_{\overline{B}}(\mathbf{R}_{\mathbf{I}}^m) \subset \overline{B} \subset K^2(\mathbf{R}_{\mathbf{I}}^m)$ (as continuous dense embedding), $\left\langle T_{12}^{\frac{1}{2}} u, T_{12}^{\frac{1}{2}} v \right\rangle_1 = \left\langle u, v \right\rangle_2$ and $\left\langle T_{12}^{\frac{-1}{2}} u, T_{12}^{\frac{-1}{2}} v \right\rangle_2 = \left\langle u, v \right\rangle_1$.*

Proof. We can prove the theorem with the similar technique of Ref. 11 Theorem 3.30. □

Changing $\overline{B}$ by $K^p(\mathbf{R}_{\mathbf{I}}^m)$, one can find the following relation:

$$GS^2_{\overline{B}}(\mathbf{R}_{\mathbf{I}}^m) \to K^p(\mathbf{R}_{\mathbf{I}}^m) \to K^2(\mathbf{R}_{\mathbf{I}}^m).$$

7. The Family $K^p(\mathbf{R}_{\mathbf{I}}^\infty)$

In this section, we will construct the spaces $K^p(\mathbf{R}_{\mathbf{I}}^\infty), \ 1 \leq p \leq \infty$. We are motivated by the approach of the construction of $L^1(\mathbf{R}_{\mathbf{I}}^\infty)$. Using the same approach that led to $L^1[\mathbf{R}_{\mathbf{I}}^\infty]$, since $\mathbf{R}_{\mathbf{I}}^m \subset \mathbf{R}_{\mathbf{I}}^{m+1}$, then $K^p(\mathbf{R}_{\mathbf{I}}^m) \subset K^p(\mathbf{R}_{\mathbf{I}}^{m+1})$. We can have an increasing sequence so that $K^p(\widehat{\mathbf{R}}_{\mathbf{I}}^\infty) = \bigcup_{m=1}^{\infty} K^p(\mathbf{R}_{\mathbf{I}}^m)$.

Definition 10 ([16, Definition 2.7]). For $1 \leq p \leq \infty$, we say that a measurable function $\mathbf{f} \in K^p(\mathbf{R}_{\mathbf{I}}^\infty)$ if we can find a Cauchy sequence

$\{\mathbf{f}_m\} \subset K^p(\widehat{\mathbf{R}}_{\mathbf{I}}^{\infty}]$ along with $\mathbf{f}_m \in K^p(\mathbf{R}_{\mathbf{I}}^m)$ such that $\lim_{m\to\infty} \mathbf{f}_m(x) = \mathbf{f}(x)$ ν_∞-a.e.

The functions of $K^p(\widehat{\mathbf{R}}_{\mathbf{I}}^{\infty})$ are different from the functions in its closure of $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ by those sets of measure zero.

Theorem 9 ([16, Theorem 2.8]). $K^p(\widehat{\mathbf{R}}_{\mathbf{I}}^{\infty}) = K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$.

Definition 11. For $f \in K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$, the integral of $\mathbf{f}$ is defined as follows:

$$\int_{\mathbf{R}_{\mathbf{I}}^{\infty}} \mathbf{f}(x) d\nu_\infty(x) = \lim_{m\to\infty} \int_{\mathbf{R}_{\mathbf{I}}^{\infty}} \mathbf{f}_m(x) d\nu_\infty(x), \tag{4}$$

where $\{f_m\} \subset K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ is any Cauchy sequence converging to $\mathbf{f}(x)$-a.e.

Theorem 10. *Suppose* $\mathbf{f} \in K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$. *The integral of* $\mathbf{f}$ *defined in Definition 11 exists and its uniqueness holds for every* $\mathbf{f} \in K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$.

Proof. Since the family of functions $\{\mathbf{f_m}\}$ is Cauchy, the uniqueness holds if the integral exists. We can prove the existence with standard argument by assuming $\mathbf{f}(x) \geq 0$. In addition, we consider the sequence to to be increasing such that the integral should exist. Finally, for the general case, we follow the standard decomposition. $\square$

To form $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$, let us consider a countable dense set of functions $\{\mathbf{E_m}(\mathbf{x})\}_{\mathbf{m=1}}^{\infty}$ on the unit ball of $L^1(\mathbf{R}_{\mathbf{I}}^{\infty})$ and considering $\{\mathbf{E}_{\mathbf{m}}^*\}_{\mathbf{m=1}}^{\infty}$ be any analogous set of duality mapping in $L^\infty(\mathbf{R}_{\mathbf{I}}^{\infty})$ along with if $\overline{B}$ is $L^1(\mathbf{R}_{\mathbf{I}}^{\infty})$. We can find from the Kuelbs lemma that a certain class of Henstock–Kurzweil-type integrable functions are in the Hilbert space H. We are not confident enough that the Henstock–Kurzweil-type integrable functions in H are Henstock–Kurzweil integrable. The known fact is that $\widehat{\mathbf{Q}}_{\mathbf{I}}^{\infty} = \lim_{m\to\infty} \mathbf{Q}_{\mathbf{I}}^{\mathbf{m}} = \bigcup_{\mathbf{k=1}}^{\infty} \mathbf{Q}_{\mathbf{I}}^{\mathbf{k}}$, where $\mathbf{Q}_{\mathbf{I}}^{\mathbf{n}}$ is the set $\{\mathbf{x} \in \mathbf{R}_{\mathbf{I}}^n\}$ so that the initial (first) co-ordinates $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ are rational. Clearly, this is a countable dense set of $\mathbf{R}_{\mathbf{I}}^{\infty}$. For every k and i, we can arrange $\mathbf{Q}_{\mathbf{I}}^{\infty} = \{\mathbf{x_1}, \mathbf{x_2}, ...\}$. Let us assume a closed cube $\mathcal{B}_k(x_i)$ centered at x_i along with edge $e_k = \frac{1}{2^{l-1}\sqrt{m}}$, $l \in \mathbf{N}$.

We can choose a map $\mathbf{N} \times \mathbf{N}$ bijectively to $\mathbf{N}$ of natural order and suppose the resulting set of all closed cubes

$$\{\mathcal{B}_k(x_i)|\ (l,i) \in \mathbf{N} \times \mathbf{N}\}$$

centered at a point in $\mathbf{Q}_{\mathbf{I}}^{\infty}$ be $\{\mathcal{B}_k:\ k \in \mathbf{N}\}$. Let the characteristic function of $\mathcal{B}_k$ be $\mathbf{E_k}(\mathbf{x})$ such that for $1 \leq p < \infty$, $\mathbf{E_k}(\mathbf{x}) \in \mathbf{L^p}(\mathbf{R}_{\mathbf{I}}^{\infty}) \cap \mathbf{L}^{\infty}(\mathbf{R}_{\mathbf{I}}^{\infty})$. We can define $F_k(.)$ on $L^1(\mathbf{R}_{\mathbf{I}}^{\infty})$ by

$$F_k(\mathbf{f}) = \int_{\mathbf{R}_{\mathbf{I}}^{\infty}} \mathbf{E_k}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_{\infty}(\mathbf{x}).$$

Now, $\mathcal{B}_k$ are cubes with sides parallel to the co-ordinate axes. So, $F_k(.)$ is well defined for all Henstock–Kurzweil integrable functions and is a bounded linear functional on $L^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ for $1 \leq p \leq \infty$. Let us fix $\mathbf{t}_k > 0$ so that $\sum_{k=1}^{\infty} \mathbf{t}_k = 1$. We can define an inner product $\langle . \rangle$ on $L^1(\mathbf{R}_{\mathbf{I}}^{\infty})$ as follows:

$$\begin{aligned}\big\langle \mathbf{f}, \mathbf{g} \big\rangle = \sum_{k=1}^{\infty} \mathbf{t}_k \bigg(\int_{\mathbf{R}_{\mathbf{I}}^{\infty}} \mathbf{E_k}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_{\infty}(\mathbf{x}) \bigg) \\ \times \bigg(\int_{\mathbf{R}_{\mathbf{I}}^{m}} \mathbf{E_k}(\mathbf{y})\mathbf{g}(\mathbf{y})\mathbf{d}\nu_{\infty}(\mathbf{y}) \bigg)^{\mathbf{c}}.\end{aligned}$$

With this inner product, the completion of $L^1(\mathbf{R}_{\mathbf{I}}^{\infty})$ is the p-summable space $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$. This space contains Henstock–Kurzweil integrable functions. We can find this as follows:

$$||\mathbf{f}||_{K^2}^2 = \sum_{k=1}^{\infty} \mathbf{t}_k \bigg| \int_{\mathbf{R}_{\mathbf{I}}^{\infty}} \mathbf{E_k}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_{\infty}(\mathbf{x}) \bigg|^{\mathbf{2}}.$$

If we replace $\mathbf{R}_{\mathbf{I}}^{n}$ by $\mathbf{R}_{\mathbf{I}}^{\infty}$, then the Alexiewicz norm defined in Eq. (3) becomes

$$||\mathbf{f}||_{HK(\mathbf{R}_{\mathbf{I}}^{\infty})} = \sup_{r>0} \bigg| \int_{\mathcal{B}_r} \mathbf{f}(x) d\nu_{\infty}(x) \bigg| < \infty,$$

and $||\mathbf{f}||_{K^2}^2 \leq \sup_k \bigg| \int_{\mathcal{B}_k} \mathbf{f}(x) d\nu_{\infty}(x) \bigg|^2 < \infty$. So, $\mathbf{f} \in K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$.

Theorem 11. *For every p, $1 \leq p \leq \infty$, $L^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ is a densely continuous subspace of $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$.*

Proof. Since for every p, $1 \leq p \leq \infty$, $L^p(\mathbf{R}_\mathbf{I}^m)$ is a densely continuous subspace of $K^2(\mathbf{R}_\mathbf{I}^n)$. Moreover, the closure of $\bigcup_{n=1}^{\infty}, K^2(\mathbf{R}_\mathbf{I}^n)$ is $K^2(\mathbf{R}_\mathbf{I}^\infty)$. In this way, $K^2(\mathbf{R}_\mathbf{I}^\infty)$ contains $\bigcup_{n=1}^{\infty} L^p(\mathbf{R}_\mathbf{I}^m)$ which is dense in $L^p(\mathbf{R}_\mathbf{I}^\infty)$, as it is the closure. □

For $1 \leq p \leq \infty$, the norm of $K^p(\mathbf{R}_\mathbf{I}^\infty)$ spaces can be defined as

$$||\mathbf{f}||_{K^p(\mathbf{R}_\mathbf{I}^\infty)} = \left(\begin{array}{ll} \left(\sum_{k=1}^{\infty} \mathbf{t}_k \left| \int_{\mathbf{R}_\mathbf{I}^\infty} \mathbf{E}_\mathbf{k}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_\infty(\mathbf{x}) \right|^p \right)^{\frac{1}{p}}, & \text{for } 1 \leq p < \infty; \\ \sup_{k \geq 1} \left| \int_{\mathbf{R}_\mathbf{I}^\infty} \mathbf{E}_\mathbf{k}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_\infty(\mathbf{x}) \right|, & \text{for } p = \infty \end{array} \right.$$

with $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots) \in \mathbf{R}_\mathbf{I}^\infty$. and $\mathbf{R}_\mathbf{I}^\infty = \lim_{m \to \infty} \mathbf{R}_\mathbf{I}^m = \bigcup_{m=1}^{\infty} \mathbf{R}_\mathbf{I}^k$ and $\mathbf{R}_\mathbf{I}^m = \mathbf{R}^m \times \mathbf{I}_m$, where $\mathbf{I} = [\frac{-1}{2}, \frac{1}{2}]$.

Obviously, $||f||_{K^p}$ defines on L^p and with respect to this norm L^p is the completion of K^p. As L^p is the completion of K^p, then we have the following theorem:

Theorem 12. *For every q, $1 \leq q \leq \infty$, $L^q(\mathbf{R}_\mathbf{I}^\infty)$ is a densely continuous subspace of $K^p(\mathbf{R}_\mathbf{I}^\infty)$. Also, the inclusion is continuous embedding.*

Proof. Proof is similar to Ref. 16 Theorem 2.3. □

Remark 3. For each p, $1 \leq p < \infty$, the Hölder and Minkowski inequalities hold for $L^p(\mathbf{R}_\mathbf{I}^\infty)$. The norm of $K^p(\mathbf{R}_\mathbf{I}^\infty)$ is derived from $L^p(\mathbf{R}_\mathbf{I}^\infty)$ and the completion of $L^p(\mathbf{R}_\mathbf{I}^\infty)$ is $K^p(\mathbf{R}_\mathbf{I}^\infty)$. So, for $1 \leq p < \infty$, one can easily show that Hölder and Minkowski inequalities hold in $K^p(\mathbf{R}_\mathbf{I}^\infty)$.

Theorem 13. *For each p, $1 \leq p \leq \infty$, the following are true for $K^p(\mathbf{R}_\mathbf{I}^\infty)$.*

(a) *If $\mathbf{f}_m \to \mathbf{f}$ weakly in $L^p(\mathbf{R}_\mathbf{I}^\infty)$. Then $\mathbf{f}_m \to \mathbf{f}$ strongly in $K^p(\mathbf{R}_\mathbf{I}^\infty)$.*

(b) *If $1 < p < \infty$, then $K^p(\mathbf{R}_\mathbf{I}^\infty)$ is uniformly convex.*

(c) *If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the dual space of $K^p(\mathbf{R}_\mathbf{I}^\infty)$ is $K^q(\mathbf{R}_\mathbf{I}^\infty)$.*

(d) *$K^\infty(\mathbf{R}_\mathbf{I}^\infty) \subset K^p(\mathbf{R}_\mathbf{I}^\infty)$ for $1 \leq p < \infty$.*

(e) *$C_c(\mathbf{R}_\mathbf{I}^\infty) \subset K^p(\mathbf{R}_\mathbf{I}^\infty)$ as dense.*

Proof. (a) Let $\{\mathbf{f}_m\}$ weakly converge with its limit in $L^p(\mathbf{R}_\mathbf{I}^\infty)$. At this point, for each k, we have

$$\int_{\mathbf{R}_\mathbf{I}^m} \mathbf{E_k}(\mathbf{x})|\mathbf{f_m}(\mathbf{x}) - \mathbf{f}(\mathbf{x})|\mathbf{d}\nu_\infty(\mathbf{x}) \to \mathbf{0}.$$

Now, for every $\mathbf{f}_m \in K^p(\mathbf{R}_\mathbf{I}^m)$ for every m, we can have

$$\lim_{m\to\infty} \int_{\mathbf{R}_\mathbf{I}^m} \mathbf{E_k}(\mathbf{x})|\mathbf{f_m}(\mathbf{x}) - \mathbf{f}(\mathbf{x})|\mathbf{d}\nu_\infty(\mathbf{x}) \to \mathbf{0}.$$

(b) For each m, $L^p(\mathbf{R}_\mathbf{I}^m)$ is uniformly convex. We know for $1 \leq q \leq \infty$, $L^p(\mathbf{R}_\mathbf{I}^m)$ is dense in $K^q(\mathbf{R}_\mathbf{I}^m)$ with compact embedding. So, for each n, and for $1 \leq q \leq \infty$, $\bigcup\limits_{m=1}^{\infty} L^p(\mathbf{R}_\mathbf{I}^m)$ is uniformly convex in $\bigcup\limits_{m=1}^{\infty} K^q(\mathbf{R}_\mathbf{I}^m)$ as a dense and compact embedding. But $L^p(\widehat{\mathbf{R}}_\mathbf{I}^\infty] = \bigcup\limits_{m=1}^{\infty} L^p(\mathbf{R}_\mathbf{I}^m)$. So, for $1 \leq q \leq \infty$, $L^p(\widehat{\mathbf{R}}_\mathbf{I}^\infty)$ is uniformly convex in $K^q(\widehat{\mathbf{R}}_\mathbf{I}^\infty)$ as dense and compact embedding.

Since the closure of $K^q(\widehat{\mathbf{R}}_\mathbf{I}^\infty)$ is $K^q(\mathbf{R}_\mathbf{I}^\infty)$, $K^q(\mathbf{R}_\mathbf{I}^\infty)$ is uniformly convex.

(c) Using (2), we have for $1 < p < \infty$, $K^p(\mathbf{R}_\mathbf{I}^\infty)$ is reflexive. At the same time, for all k and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \{K^p(\mathbf{R}_\mathbf{I}^k)\}^* &= K^q(\mathbf{R}_\mathbf{I}^k) \\ K^p(\mathbf{R}_\mathbf{I}^k) \subset K^p(\mathbf{R}_\mathbf{I}^{k+1}), \Rightarrow & \bigcup_{k=1}^{\infty} \left\{K^p(\mathbf{R}_\mathbf{I}^k)\right\}^* \\ &= \bigcup_{k=1}^{\infty} K^q(\mathbf{R}_I^k). \end{aligned}$$

As each $\mathbf{f} \in K^p(\mathbf{R}_\mathbf{I}^\infty)$ is the limit of a sequence $\{\mathbf{f}_m\} \subset \bigcup\limits_{k=1}^{\infty} K^p(\mathbf{R}_\mathbf{I}^k)$, hence for $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, the dual space of $K^p(\mathbf{R}_\mathbf{I}^\infty)$ is $K^q(\mathbf{R}_\mathbf{I}^\infty)$.

(d) Suppose $\mathbf{f} \in K^\infty(\mathbf{R}_\mathbf{I}^\infty)$. Then for all k, $\left|\int_{\mathbf{R}_\mathbf{I}^\infty} \mathbf{E_k}(\mathbf{x})\mathbf{f}(\mathbf{x}) \mathbf{d}\nu_\infty(\mathbf{x})\right|$ is uniformly bounded. Consequently, for $1 \leq p < \infty$,

$\left|\int_{\mathbf{R}_{\mathbf{I}}^{\infty}} \mathbf{E}_{\mathbf{k}}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_{\infty}(\mathbf{x})\right|^p$ is uniformly bounded. Finally, the definition of $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ gives

$$\left[\sum \left|\int_{\mathbf{R}_{\mathbf{I}}^{\infty}} \mathbf{E}_{\mathbf{k}}(\mathbf{x})\mathbf{f}(\mathbf{x})\mathbf{d}\nu_{\infty}(\mathbf{x})\right|^p\right]^{\frac{1}{p}} \leq M||f||_{K^p(\mathbf{R}_{\mathbf{I}}^{\infty})} < \infty.$$

Hence, $\mathbf{f} \in K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$.

(e) Since $C_c(\mathbf{R}_{\mathbf{I}}^n) \subset K^p(\mathbf{R}_{\mathbf{I}}^n)$ is dense, so easily we can find $C_c^{\infty}(\mathbf{R}_{\mathbf{I}}^{\infty})$ is dense in $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$, $\forall\, p$. □

Corollary 2. $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ *is strictly convex.*

Proposition 1. $C_c^{\infty}(\mathbf{R}_{\mathbf{I}}^{\infty})$ *is subset of* $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$ *as densely continuous.*

Proof. Since $\forall p$, $C_c^{\infty}(\mathbf{R}_{\mathbf{I}}^{m})$ is in $L^p(\mathbf{R}_{\mathbf{I}}^{m})$ as dense. Also, $L^p(\mathbf{R}_{\mathbf{I}}^{\infty}) \subset K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$. Hence, $C_c^{\infty}(\mathbf{R}_{\mathbf{I}}^{\infty})$ is a subset of $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$ as densely continuous. □

Remark 4. $C_0^{\infty}(\mathbf{R}_{\mathbf{I}}^{\infty})$ is a dense subset of $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$.

Theorem 14. $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ *is a subset of* $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$ *as dense and continuous embedding.*

Proof. Since $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ is separable, assume any fixed set of corresponding duality mappings $\{\mathbf{u}_m\}$ $\mathbf{f}_m \in K^q(\mathbf{R}_{\mathbf{I}}^{\infty})$. Then,

$$\begin{aligned}\mathbf{f}_m(\mathbf{u}_m) &= \langle \mathbf{u}_m, \mathbf{f}_m\rangle \\ &= ||\mathbf{u}_m||^2_{K^p(\mathbf{R}_{\mathbf{I}}^{\infty})} \\ &= ||\mathbf{f}_m||^2_{K^q(\mathbf{R}_{\mathbf{I}}^{\infty})}.\end{aligned}$$

Assume a set of positive numbers $\{r_m\}_{m=1}^{\infty}$ such that $\sum\limits_{m=1}^{\infty} r_m = 1$, also assume

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \sum_{m=1}^{\infty} r_m \mathbf{f}_m(\mathbf{u})\overline{\mathbf{f}_m}(\mathbf{v}).$$

Since $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$ is Hilbert space, the completion of $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ generated with respect to this inner product is

$$\begin{aligned}||\mathbf{u}||^2 &= \sum_{n=1}^{\infty} r_m |\mathbf{f}_m(\mathbf{u})|^2 \\ &\leq \sup_m |\mathbf{f}_m(\mathbf{u})|^2 \\ &= ||\mathbf{u}||^2_{K^p(\mathbf{R}_{\mathbf{I}}^{\infty})}.\end{aligned}$$

Clearly, it is an inner product on $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$. Hence, $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ is dense in $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$ and it is continuous embedding. □

Remark 5. We know that $L^1(\mathbf{R}_{\mathbf{I}}^{\infty}) \subset K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$. For $1 < p < \infty$, $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ is reflexive. Let the space of bounded finitely additive set functions be $\mathcal{M}(\mathbf{R}_{\mathbf{I}}^{\infty})$ which is defined on the Borel sets $B(\mathbf{R}_{\mathbf{I}}^{\infty})$. Then, $\{L^1(\mathbf{R}_{\mathbf{I}}^{\infty})\}^{**} = \mathcal{M}(\mathbf{R}_{\mathbf{I}}^{\infty}) \subset K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$. In the area of Feynman integral, the Dirac delta measure and the free-particle Green's functions are contained by these spaces.

8. The Space $\mathcal{M}[\mathbf{R}_{\mathbf{I}}^{\infty}]$

We recall that a uniformly bounded sequence $\{\nu_n\} \subset \mathcal{M}(\mathbb{R}^m)$ is weakly convergent to $\nu(\nu_n \to \nu)$ if for each bounded and uniformly continuous function $\mathbf{h}(x)$, the following holds:

$$\int_{\mathbf{R}^m} \mathbf{h}(x) d\nu_m \to \int_{\mathbf{R}^m} \mathbf{h}(x) d\nu.$$

Definition 12. We called a uniformly bounded sequence $\{\nu_m\} \subset \mathcal{M}(\mathbf{R}_{\mathbf{I}}^{\infty})$ as weakly convergent to ν $(\nu_m \to \nu)$ if the following holds: if for each bounded uniformly continuous function $\mathbf{h}(x)$,

$$\lim_{m\to\infty} \int_{\mathbf{R}_{\mathbf{I}}^{\infty}} \mathbf{h}(x) d\nu_m \to \int_{\mathbf{R}_{\mathbf{I}}^{\infty}} \mathbf{h}(x) d\nu.$$

Proposition 2. *$\nu_m \to \nu$ strongly in $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$ when $\nu_m \to \nu$ weakly in $\mathcal{M}(\mathbf{R}_{\mathbf{I}}^{\infty})$.*

Proof. As $\nu_n \to^w \nu$ weakly in $\mathcal{M}(\mathbf{R}_\mathbf{I}^\infty]$, it gives

$$\lim_{m\to\infty} \int_{\mathbf{R}_\mathbf{I}^\infty} \mathbf{h}(x) d\nu_m \to \int_{\mathbf{R}_\mathbf{I}^\infty} \mathbf{h}(x) d\nu \text{ for every } n.$$

Now,

$$\lim_{m\to\infty} ||\nu_m - \nu||_{K^p(\mathbf{R}_\mathbf{I}^\infty)} = 0.$$

Hence, $\nu_m \to \nu$ in $K^p(\mathbf{R}_\mathbf{I}^\infty)$ strongly. □

9. $K^p(\mathbf{R}_\mathbf{I}^\infty)$ and Fourier and Convolution Operators

We will extend the theory of operators in $K^2(\mathbf{R}_\mathbf{I}^\infty)$ from $K^p(\mathbf{R}_\mathbf{I}^\infty)$. To execute this, we need $GD^2(\mathbf{R}_\mathbf{I}^\infty)$. In the similar approach of the construction of $L^1(\mathbf{R}_\mathbf{I}^\infty)$ and $K^p(\mathbf{R}_\mathbf{I}^\infty)$, we can construct $GD^2(\mathbf{R}_\mathbf{I}^\infty)$. We will mention the construction in brief as follows:

Since $GD^2(\mathbf{R}_\mathbf{I}^m) \subset GD^2(\mathbf{R}_\mathbf{I}^{m+1})$, $\forall m$, therefore, $GD^2(\widehat{\mathbf{R}}_\mathbf{I}^\infty) = \bigcup_{m=1}^\infty GD^2(\mathbf{R}_\mathbf{I}^m)$. Like in other spaces in the setting above, we can say $\mathbf{f} \in GD^2(\mathbf{R}_\mathbf{I}^\infty)$ if we can find a Cauchy sequence $\{\mathbf{f}_m\} \subset GD^2(\widehat{\mathbf{R}}_\mathbf{I}^\infty)$ along with $\mathbf{f}_m \in GD^2(\mathbf{R}_\mathbf{I}^\infty)$ and $\lim_{m\to\infty} \mathbf{f}_m(x) = \mathbf{f}(x)$, ν_∞-a.e.

Theorem 15. $GD^2(\mathbf{R}_\mathbf{I}^\infty)$ *is a dense subset of* $K^2(\mathbf{R}_\mathbf{I}^\infty)$ *with continuous embedding.*

Proof. The closure of $\bigcup_{m=1}^\infty K^2(\mathbf{R}_\mathbf{I}^m)$ is $K^2(\mathbf{R}_\mathbf{I}^\infty)$. Moreover, $\bigcup_{m=1}^\infty K^2(\mathbf{R}_\mathbf{I}^m)$ is in $K^2(\mathbf{R}_\mathbf{I}^\infty)$, a dense subset as its closure is $GD^2(\mathbf{R}_\mathbf{I}^\infty)$. $GD^2(\mathbf{R}_\mathbf{I}^\infty)$ is a dense subset of $K^2(\mathbf{R}_\mathbf{I}^\infty)$ with continuous embedding. □

Theorem 16. $GD^2(\mathbf{R}_\mathbf{I}^\infty)$ *is a dense subset of* $K^p(\mathbf{R}_\mathbf{I}^\infty)$ *with continuous embedding*

Proof. Proof is similar to the proof of Theorem 15. □

We can find $K^p(\mathbf{R}_\mathbf{I}^\infty)$ is in $K^2(\mathbf{R}_\mathbf{I}^\infty)$ as dense. We can find the inclusion relation from Theorems 13 and 16 as follows:

$$GD^2(\mathbf{R}_\mathbf{I}^\infty) \subset K^p(\mathbf{R}_\mathbf{I}^\infty) \subset K^2(\mathbf{R}_\mathbf{I}^\infty).$$

The separable Banach space $K^p(\mathbf{R}_\mathbf{I}^\infty)$ is in $K^2(\mathbf{R}_\mathbf{I}^\infty)$ as a dense subset. Moreover, $K^p(\mathbf{R}_\mathbf{I}^\infty)$ is in $K^2(\mathbf{R}_\mathbf{I}^\infty)$ as a continuous dense embedding. We find that the densely closed linear operator of $K^p(\mathbf{R}_\mathbf{I}^\infty)$ has a densely closed linear operator in $K^2(\mathbf{R}_\mathbf{I}^\infty)$. Let $L\big[K^p(\mathbf{R}_\mathbf{I}^\infty)\big]$, $L\big[K^2(\mathbf{R}_\mathbf{I}^\infty)\big]$ be the class of linear operators of the spaces $K^p(\mathbf{R}_\mathbf{I}^\infty)$ and $K^2(\mathbf{R}_\mathbf{I}^\infty)$, respectively.

Theorem 17. *Let the separable Banach spaces $K^p(\mathbf{R}_\mathbf{I}^\infty)$ be continuous dense embedding in $K^2(\mathbf{R}_\mathbf{I}^\infty)$ with a bounded linear operator A_L on $K^p(\mathbf{R}_\mathbf{I}^\infty)$. Let A_L be self-adjoint in nature on $K^2(\mathbf{R}_\mathbf{I}^\infty)$ implying A_L is bounded in nature on $K^2(\mathbf{R}_\mathbf{I}^\infty)$ with $||A_L||_{K^2(\mathbf{R}_\mathbf{I}^\infty)} \leq K_s||A_L||_{K^p(\mathbf{R}_\mathbf{I}^\infty)}$, for a positive constant K_s.*

Proof. Let $\mathbf{u} \in \mathbf{K^p}(\mathbf{R}_\mathbf{I}^\infty)$. Let us assume $k=1$ and $||\mathbf{u}||_{\mathbf{K^2}(\mathbf{R}_\mathbf{I}^\infty)} = \mathbf{1}$. Since A_L is self-adjoint, we get

$$\begin{aligned} ||A_L\mathbf{u}||^{\mathbf{2}}_{\mathbf{K^2}(\mathbf{R}_\mathbf{I}^\infty)} &= \big\langle A_L\mathbf{u}, \mathbf{A_L u}\big\rangle \\ &= \big\langle \mathbf{u}, \mathbf{A_L^2 u}\big\rangle \\ &\leq ||\mathbf{u}||_{\mathbf{K^2}(\mathbf{R}_\mathbf{I}^\infty)}||\mathbf{A_L^2 u}||_{\mathbf{K^2}(\mathbf{R}_\mathbf{I}^\infty)} \\ &= ||A_L^2\mathbf{u}||_{\mathbf{K^2}(\mathbf{R}_\mathbf{I}^\infty)}. \end{aligned}$$

Thus, $||A_L\mathbf{u}||^{\mathbf{2m}}_{\mathbf{K^2}(\mathbf{R}_\mathbf{I}^\infty)} \leq ||\mathbf{A_L^{2m} u}||_{\mathbf{K^2}(\mathbf{R}_\mathbf{I}^\infty)}$ for all m. Therefore,

$$\begin{aligned} ||A_L\mathbf{u}||_{\mathbf{K^2}(\mathbf{R}_\mathbf{I}^\infty)} &\leq \big(||A_L^{2m}\mathbf{u}||_{\mathbf{K^2}(\mathbf{R}_\mathbf{I}^\infty)}\big)^{\frac{\mathbf{1}}{\mathbf{2m}}} \\ &\leq \big(||A_L^{2m}\mathbf{u}||_{\mathbf{K^p}(\mathbf{R}_\mathbf{I}^\infty)}\big)^{\frac{\mathbf{1}}{\mathbf{2m}}} \\ &\leq \big(||A_L^{2m}||_{K^p(\mathbf{R}_\mathbf{I}^\infty)}\big)^{\frac{1}{2m}}\big(||\mathbf{u}||_{\mathbf{K^p}(\mathbf{R}_\mathbf{I}^\infty)}\big)^{\frac{\mathbf{1}}{\mathbf{2m}}} \\ &\leq ||A_L||_{K^p(\mathbf{R}_\mathbf{I}^\infty)}\big(||\mathbf{u}||_{\mathbf{K^p}(\mathbf{R}_\mathbf{I}^\infty)}\big)^{\frac{\mathbf{1}}{\mathbf{2m}}}. \end{aligned}$$

Since $m \to \infty$, $||A_L\mathbf{u}||_{\mathbf{K^2}(\mathbf{R}_\mathbf{I}^\infty)} \leq ||\mathbf{A_L}||_{\mathbf{K^p}(\mathbf{R}_\mathbf{I}^\infty)}$ for $\mathbf{u}$ in a dense set of the unit ball of $K^2(\mathbf{R}_\mathbf{I}^\infty)$. □

Theorem 18. *Suppose $K^p(\mathbf{R}_\mathbf{I}^\infty)$ is a separable Banach space. If $K^p(\mathbf{R}_\mathbf{I}^\infty) \subset K^2(\mathbf{R}_\mathbf{I}^\infty)$, then a bounded linear operator A_L on $K^p(\mathbf{R}_\mathbf{I}^\infty)$*

can be extended to $Li\left[K^2(\mathbf{R}_{\mathbf{I}}^{\infty})\right]$ *with the following*:

$$||A_L||_{K^2(\mathbf{R}_{\mathbf{I}}^{\infty})} \leq k||A_L||_{K^p(\mathbf{R}_{\mathbf{I}}^{\infty})},$$

with $k = \inf\left\{M||A_L^* A_L||_{K^p(\mathbf{R}_{\mathbf{I}}^{\infty})} \leq M||A_L||^2_{K^p(\mathbf{R}_{\mathbf{I}}^{\infty})}\right\}$ *for* $M > 0$.

Proof. Let us assume a bounded self-adjoint linear operator A_L on $K^p(\mathbf{R}_{\mathbf{I}}^{\infty})$. Using Lax's theorem (see Ref. 11 Theorem 3.29), A_L is bounded on $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$. Also,

$$\begin{aligned} ||A_L^* A_L||_{K^2(\mathbf{R}_{\mathbf{I}}^{\infty})} &\leq ||A_L||^2_{K^2(\mathbf{R}_{\mathbf{I}}^{\infty})} \\ &\leq ||A_L^* A_L||_{K^p(\mathbf{R}_{\mathbf{I}}^{\infty})} \\ &\leq k||A_L||^2_{K^p(\mathbf{R}_{\mathbf{I}}^{\infty})}. \end{aligned}$$

□

Recalling that $K^2(\mathbf{R}^m)$ is an effective alternate to formulate the extension of the Feynman operator, it is found that the Fourier and convolution have an extension to the Hilbert space $K^2(\mathbf{R}^m)$. The Schrödinger and Heisenberg theories have faithful representations on $K^2(\mathbf{R}^m)$. We demonstrate that the Fourier and convolution operators can be extended on $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$. Let us consider F_r as the Fourier transform on $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$. Let C_r be the convolution operator on $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$.

Theorem 19. *The Fourier transform* F_r *can be extended on* $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$ *as a bounded linear operator.*

Proof. We know that $C_0(\mathbf{R}_{\mathbf{I}}^{\infty})$ is in $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$. At the same time $F_r : L^1(\mathbf{R}_{\mathbf{I}}^{\infty}) \to C_0(\mathbf{R}_{\mathbf{I}}^{\infty})$ is a bounded linear operator. Also, we know that $L^{\infty}(\mathbf{R}_{\mathbf{I}}^{\infty}) \subset K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$. Again, $L^1(\mathbf{R}_{\mathbf{I}}^{\infty})$ is dense in $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$. Hence, we can claim with Theorem 18 that F_r can be extended on $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$ to a bounded linear operator. □

Theorem 20. *The convolution operator* (*transformation*) C_r *can be extended on* $K^2(\mathbf{R}_{\mathbf{I}}^{\infty})$ *as a bounded linear operator.*

Proof. Proof is similar to Theorem 19. □

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Chapter 16

Subclass of Analytic Functions Involving Mittag-Leffler Type Borel Distribution Series

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In this chapter, we discuss a subclass $TS_{\wp,\varsigma}^{\xi}(\vartheta, \hbar, \ell)$ of univalent mappings with negative coefficients related to Borel distribution series in the unit disk $U = \{\omega \in C : |\omega| < 1\}$. For mappings in our class, we obtain fundamental properties, such as coefficient inequality, distortion and covering theorems, radii of starlikeness, convexity and close-to-convexity, extreme points, Hadamard product, and closure theorems.

1. Introduction

Let $\mathcal{A}$ indicate the class of all mappings $\aleph(\omega)$ of the type

$$\aleph(\omega) = \omega + \sum_{\kappa=2}^{\infty} a_{\kappa}\omega^{\kappa} \tag{1}$$

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in the open, the unit disk $\mathbb{U} = \{\omega : |\omega| < 1\}$. Let S be the subclass of $\mathcal{A}$ consisting of univalent mappings satisfying the following usual normalization condition $\aleph(0) = 0$ and $\aleph'(0) = 1$. We denote by S the subclass of $\mathcal{A}$ consisting of $\aleph(\omega)$ which are all univalent in $\mathbb{U}$. A mapping $\aleph \in \mathcal{A}$ is a starlike mapping of the order υ, $\upsilon(0 \leq \upsilon < 1)$ if it satisfies

$$\Re\left\{\frac{\omega\aleph'(\omega)}{\aleph(\omega)}\right\} > \upsilon, \quad (\omega \in \mathbb{U}), \tag{2}$$

we denote this class by $S^*(\upsilon)$.

A mapping $\aleph \in \mathcal{A}$ is a convex mapping of the order $\upsilon, \upsilon(0 \leq \upsilon < 1)$ if it satisfies

$$\Re\left\{1 + \frac{\omega\aleph''(\omega)}{\aleph'(\omega)}\right\} > \upsilon, \quad (\omega \in \mathbb{U}), \tag{3}$$

we denote this class with $K(\upsilon)$.

Denote by T the subclass of $\mathcal{A}$ consisting of mappings $\aleph$ of the form

$$\aleph(\omega) = \omega - \sum_{\kappa=2}^{\infty} a_\kappa \omega^\kappa, \qquad (a_\kappa \geq 0). \tag{4}$$

Silverman pioneered and extensively researched this subclass.[1]

For $\aleph \in \mathcal{A}$ given by (1) and $g(\omega)$ given by

$$g(\omega) = \omega + \sum_{\kappa=2}^{\infty} b_\kappa \omega^\kappa \tag{5}$$

their convolution (or Hadamard product), indicated by $(\aleph * g)$, is defined as

$$(\aleph * g)(\omega) = \omega + \sum_{\kappa=2}^{\infty} a_\kappa b_\kappa \omega^\kappa = (g * \aleph)(\omega), \quad (\omega \in \mathbb{U}). \tag{6}$$

Note that $\aleph * g \in \mathcal{A}$.

1.1. *Mittag-Leffler type Borel distribution*

Geometric mapping theory, complex analysis, and other related fields rely heavily on operator theory. Convolution of certain analytic mappings can express a wide range of derivative and integral operators. Let $E_\varsigma(\omega)$ and $E_{\varsigma,\xi}(\omega)$ be mappings defined by

$$E_\varsigma(\omega) = \sum_{\kappa=0}^{\infty} \frac{\omega^\kappa}{\Gamma(\varsigma\kappa+1)}, \quad (\omega \in \mathbb{C},\ \Re(\varsigma) > 0)$$

and

$$E_{\varsigma,\xi}(\omega) = \frac{1}{\Gamma(\xi)} + \sum_{\kappa=1}^{\infty} \frac{\omega^\kappa}{\Gamma(\varsigma\kappa+\xi)} \quad (\varsigma, \xi \in \mathbb{C}, \Re(\varsigma) > 0, \Re(\xi) > 0).$$

It can also be written in another way:

$$E_{\varsigma,\xi}(\omega) = \frac{1}{\Gamma(\xi)} + \sum_{\kappa=2}^{\infty} \frac{\omega^{\kappa-1}}{\Gamma(\varsigma(\kappa-1)+\xi)},$$
$$\times(\varsigma, \xi \in \mathbb{C}, \Re(\varsigma) > 0, \Re(\xi) > 0).$$

Mittag-Leffler[2] acquainted the mapping $E_\varsigma(\omega)$ and it is thus renowned as the Mittag-Leffler mapping.

A more general mapping $E_{\varsigma,\xi}$ generalizing $E_\varsigma(\omega)$ was acquainted by Wiman[3] and specified by

$$E_{\varsigma,\xi}(\omega) = \sum_{\kappa=0}^{\infty} \frac{\omega^\kappa}{\Gamma(\varsigma\kappa+\xi)}, \quad (\omega, \varsigma, \xi \in \mathrm{C}, \Re(\varsigma) > 0, \Re(\xi) > 0).$$

As a special case, the mapping $E_{\varsigma,\xi}$ possesses many well-known mappings, for example,

$$E_{1,1}(\omega) = \mathrm{e}^\omega, \quad E_{1,2}(\omega) = \frac{\mathrm{e}^\omega - 1}{\omega},$$

$$E_{2,1}\left(\omega^2\right) = \cosh\omega, \quad E_{2,1}\left(-\omega^2\right) = \cos\omega, \quad E_{2,2}\left(\omega^2\right) = \frac{\sinh\omega}{\omega},$$

$$E_{2,2}\left(-\omega^2\right) = \frac{\sin\omega}{\omega}c,$$

$$E_3(\omega) = \frac{1}{2}\left[e^{\omega^{1/3}} + 2e^{-\frac{1}{2}\omega^{1/3}}\cos\left(\frac{\sqrt{3}}{2}\omega^{1/3}\right)\right],$$

$$E_4(\omega) = \frac{1}{2}\left[\cos\omega^{1/4} + \cosh\omega^{1/4}\right].$$

The Mittag-Leffler mapping appears naturally when solving fractional order differential and integral equations. There are many other properties of the Mittag-Leffler mapping and the generalized Mittag-Leffler mapping that can be realized, e.g. in Refs. 4–10. Observe that Mittag-Leffler mapping, $E_{\varsigma,\xi}(\omega)$, does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of Mittag-Leffler function as below:

$$E_{\varsigma,\xi}(\omega) = \omega\Gamma(\xi)E_{\varsigma,\xi}(\omega) = \omega + \sum_{\kappa=2}^{\infty} \frac{\Gamma(\xi)}{\Gamma(\varsigma(\kappa-1)+\xi)}\omega^{\kappa}, \qquad (7)$$

is holds for complex parameters ς, ξ and $\omega \in \mathbb{C}$. In this chapter, we focus on the case of real-valued mappings ς, ξ and $\omega \in \mathbb{U}$.

Elementary distributions such as Pascal, Poisson, logarithmic, binomial, and beta negative binomial have been examined theoretically in geometric function theory. We suggest readers to see Refs.11 and 12 for a more complete research.

Wanas and Khuttar[13] recently added the Borel distribution (BD), whose probability mass mapping is

$$P(x=\rho) = \frac{(\rho\wp)^{\rho-1}e^{-\wp\rho}}{\rho!}, \quad \rho = 1, 2, 3, \dots$$

and they added a series $\mathcal{M}_{\wp}(\omega)$ whose coefficients are probabilities of the Borel distribution (BD):

$$\mathcal{M}_{\wp}(\omega) = \omega + \sum_{\kappa=2}^{\infty} \frac{[\wp(\kappa-1)]^{\kappa-2}e^{-\wp(\kappa-1)}}{(\kappa-1)!}\omega^{\kappa}, (0 < \wp \leq 1). \qquad (8)$$

The above series is convergent with the domain of convergence of the entire complex plane, according to a well-known ratio test.

The Mittag-Leffler-type Borel distribution was described as follows:

$$\mathcal{P}_{\wp}(\varsigma, \xi; \rho) = \frac{(\wp\rho)^{\rho-1}}{E_{\varsigma,\xi}(\wp\rho)\Gamma(\varsigma\rho+\xi)}, \quad \rho = 0, 1, 2, \dots,$$

where

$$E_{\varsigma,\xi}(\omega) = \sum_{\kappa=0}^{\infty} \frac{\omega^{\kappa}}{\Gamma(\varsigma\kappa+\xi)} \quad (\varsigma,\ \xi \in \mathbb{C},\ \Re(\varsigma) > 0,\ \Re(\xi) > 0).$$

Thus, by using (7) and (8) and the convolution operator, the Mittag-Leffler-type Borel distribution series is specified as

$$\mathscr{B}_{\wp}(\varsigma,\xi)(\omega) = \omega + \sum_{\kappa=2}^{\infty} \frac{[\wp(\kappa-1)]![\wp(\kappa-1)]^{\kappa-2}e^{-\wp(\kappa-1)}}{(\kappa-1)!E_{\varsigma,\xi}(\wp(\kappa-1))\Gamma(\varsigma(\kappa-1)+\xi)} \times\omega^{\kappa}(0<\wp\leq 1).$$

Further, by the convolution operator, we specify

$$\begin{aligned}\mathscr{B}_{\wp}(\varsigma,\xi)\aleph(\omega) &= \mathscr{B}_{\wp}(\varsigma,\xi)(\omega) * \aleph(\omega)\\ &= \omega + \sum_{\kappa-2}^{\infty} \frac{[\wp(\kappa-1)]![\wp(\kappa-1)]^{\kappa-2}e^{-\wp(\kappa-1)}}{(\kappa-1)!E_{\varsigma,\xi}(\wp(\kappa-1))\Gamma(\varsigma(\kappa-1)+\xi)} a_{\kappa}\omega^{\kappa}\\ &= \omega + \sum_{\kappa=2}^{\infty} \Lambda(\kappa)a_{\kappa}\omega^{\kappa}, \quad (\varsigma,\xi\in\mathbb{C},\Re(\varsigma)>0,\\ &\quad \Re(\xi)>0, 0<\wp\leq 1),\end{aligned} \tag{9}$$

where

$$\Lambda(\kappa) = \frac{[\wp(\kappa-1)]![\wp(\kappa-1)]^{\kappa-2}\mathrm{e}^{-\wp(\kappa-1)}}{(\kappa-1)!E_{\varsigma,\xi}(\wp(\kappa-1))\Gamma(\varsigma(\kappa-1)+\xi)}. \tag{10}$$

If $\aleph \in T$ is indicated by (1), then we have

$$\mathscr{B}_{\wp}(\varsigma,\xi)\aleph(\omega) = \omega - \sum_{\kappa=2}^{\infty} \Lambda(\kappa)a_{\kappa}\omega^{\kappa}, \tag{11}$$

where $\Lambda(\kappa)$ is indicated by (9).

The study of operators is essential in geometric function theory. Many differential and integral operators can be expressed as convolutions of analytic functions. This approach is found to facilitate further mathematical research and to increase comprehension of the geometric and symmetric features of such operators. The significance of convolution in operator theory may be clearly appreciated from the work in Ref. 14. Furthermore, probability is applied in a wide range of real-life applications from insurance to forecasting and politics to economic forecasting. We recommend Ref. 15 to the reader for other applications. For more details, see Refs. 16–21.

Now, we establish a new subclass of mappings in the class $\mathcal{A}$ by using the Mittag-Leffler-type Borel distribution series $\mathscr{B}\wp(\varsigma,\xi)$,

Definition 1. For $0 \leq \vartheta < 1, 0 \leq \hbar < 1, 0 < \ell < 1, \delta \geq 0, \sigma \in [0,1]$ and $\varrho \in \mathbb{N}$, let $TS^{\xi}_{\wp,\varsigma}(\vartheta,\hbar,\ell)$ be the subclass of $\aleph$ consisting of mappings of the form (4) and its geometrical condition satisfies

$$\left| \frac{\vartheta\left((\mathscr{B}^{\xi}_{\wp,\varsigma}\aleph(\omega))' - \frac{\mathscr{B}^{\xi}_{\wp,\varsigma}\aleph(\omega)}{\omega} \right)}{\hbar\left(\mathscr{B}^{\xi}_{\wp,\varsigma}\aleph(\omega)\right)' + (1-\vartheta)\frac{\mathscr{B}^{\xi}_{\wp,\varsigma}\aleph(\omega)}{\omega}} \right| < \ell, \quad \omega \in \mathbb{U},$$

where $\mathscr{B}^{\xi}_{\wp,\varsigma}\aleph(\omega)$ is given by (9).

2. Coefficient Inequality

Theorem 1. *Let* $\aleph \in TS^{\xi}_{\wp,\varsigma}(\vartheta,\hbar,\ell)$ *if and only if*

$$\sum_{\kappa=2}^{\infty}[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)a_\kappa \leq \ell(\hbar + (1-\vartheta)), \tag{12}$$

where $0 < \ell < 1, 0 \leq \vartheta < 1, 0 \leq \hbar < 1, \delta \geq 0, \sigma \in [0,1]$ *and* $\varrho \in \mathbb{N}$. *The result* (12) *is sharp for the mapping*

$$\aleph(\omega) = \omega - \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)}\omega^\kappa, \quad \kappa \geq 2.$$

Proof. Suppose (12) holds true and $|\omega| = 1$. Then we obtain

$$\begin{aligned}
&\left|\vartheta\left((\mathscr{B}^{\xi}_{\wp,\varsigma}\aleph(\omega))' - \frac{\mathscr{B}^{\xi}_{\wp,\varsigma}\aleph(\omega)}{\omega}\right)\right| \\
&\qquad - \ell\left|\hbar\left(\mathscr{B}^{\xi}_{\wp,\varsigma}\aleph(\omega))' + (1-\vartheta)\frac{\mathscr{B}^{\xi}_{\wp,\varsigma}\aleph(\omega)}{\omega}\right)\right| \\
&\quad = \left|-\vartheta\sum_{\kappa=2}^{\infty}(\kappa-1)\Lambda(\kappa)a_\kappa\omega^{\kappa-1}\right| \\
&\qquad - \ell\left|\hbar + (1-\vartheta) - \sum_{\kappa=2}^{\infty}(\kappa\hbar + 1 - \vartheta)\Lambda(\kappa)a_\kappa\omega^{\kappa-1}\right|
\end{aligned}$$

$$\leq \sum_{\kappa=2}^{\infty}[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]\Lambda(\kappa)a_\kappa-\ell(\hbar+(1-\vartheta))$$
$$\leq 0.$$

Hence, by maximum modulus principle, $\aleph \in TS_{\wp,\varsigma}^{\xi}(\vartheta,\hbar,\ell)$.

Now, assume that $\aleph \in TS_{\wp,\varsigma}^{\xi}(\vartheta,\hbar,\ell)$ so that

$$\left|\frac{\vartheta\left((\mathscr{B}_{\wp,\varsigma}^{\xi}\aleph(\omega))'-\frac{\mathscr{B}_{\wp,\varsigma}^{\xi}\aleph(\omega)}{\omega}\right)}{\hbar(\mathscr{B}_{\wp,\varsigma}^{\xi}\aleph(\omega))'+(1-\vartheta)\frac{\mathscr{B}_{\wp,\varsigma}^{\xi}\aleph(\omega)}{\omega}}\right|<\ell, \quad \omega\in\mathbb{U}.$$

Hence,

$$\left|\vartheta\left((\mathscr{B}_{\wp,\varsigma}^{\xi}\aleph(\omega))'-\frac{\mathscr{B}_{\wp,\varsigma}^{\xi}\aleph(\omega)}{\omega}\right)\right|$$
$$<\ell\left|\hbar\left(\mathscr{B}_{\wp,\varsigma}^{\xi}\aleph(\omega))'+(1-\vartheta)\frac{\mathscr{B}_{\wp,\varsigma}^{\xi}\aleph(\omega)}{\omega}\right)\right|.$$

Therefore, we get

$$\left|-\sum_{\kappa=2}^{\infty}\vartheta(\kappa-1)\Lambda(\kappa)a_\kappa\omega^{\kappa-1}\right|$$
$$<\ell\left|\hbar+(1-\vartheta)-\sum_{\kappa=2}^{\infty}(\kappa\hbar+1-\vartheta)\Lambda(\kappa)a_\kappa\omega^{\kappa-1}\right|.$$

Thus,

$$\sum_{\kappa=2}^{\infty}[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]\Lambda(\kappa)a_\kappa\leq\ell(\hbar+(1-\vartheta)),$$

and this completes the proof. □

Corollary 1. *Let* $\aleph \in TS_{\wp,\varsigma}^{\xi}(\vartheta,\hbar,\ell)$. *Then*

$$a_\kappa\leq\frac{\ell(\hbar+(1-\vartheta))}{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]\Lambda(\kappa)}\omega^\kappa, \quad \kappa\geq 2.$$

3. Distortion and Covering Theorem

Theorem 2. *Let the mapping* $\aleph \in TS_{\wp,\varsigma}^{\xi}(\vartheta,\hbar,\ell)$. *Then*

$$\begin{aligned}|\omega| - &\frac{\ell(\hbar+(1-\vartheta))}{\Lambda(2)[\vartheta+\ell(2\hbar+1-\vartheta)]}|\omega|^2\\ &\leq |\aleph(\omega)|\\ &\leq |\omega| + \frac{\ell(\hbar+(1-\vartheta))}{\Lambda(2)[\vartheta+\ell(2\hbar+1-\vartheta)]}|\omega|^2.\end{aligned}$$

The result is sharp and attained as

$$\aleph(\omega) = \omega - \frac{\ell(\hbar+(1-\vartheta))}{\Lambda(2)[\vartheta+\ell(2\hbar+1-\vartheta)]}\omega^2.$$

Proof.

$$\begin{aligned}|\aleph(\omega)| = \left|\omega - \sum_{\kappa=2}^{\infty} a_\kappa \omega^\kappa\right| &\leq |\omega| + \sum_{\kappa=2}^{\infty} a_\kappa |\omega|^\kappa\\ &\leq |\omega| + |\omega|^2 \sum_{\kappa=2}^{\infty} a_\kappa.\end{aligned}$$

By Theorem 1, we get

$$\sum_{\kappa=2}^{\infty} a_\kappa \leq \frac{\ell(\hbar+(1-\vartheta))}{[\vartheta+\ell(2\hbar+1-\vartheta)]\Lambda(\kappa)}. \tag{13}$$

Thus,

$$|\aleph(\omega)| \leq |\omega| + \frac{\ell(\hbar+(1-\vartheta))}{\Lambda(2)[\vartheta+\ell(2\hbar+1-\vartheta)]}|\omega|^2.$$

Also,

$$\begin{aligned}|\aleph(\omega)| &\geq |\omega| - \sum_{\kappa=2}^{\infty} a_\kappa |\omega|^\kappa\\ &\geq |\omega| - |\omega|^2 \sum_{\kappa=2}^{\infty} a_\kappa\\ &\geq |\omega| - \frac{\ell(\hbar+(1-\vartheta))}{\Lambda(2)[\vartheta+\ell(2\hbar+1-\vartheta)]}|\omega|^2.\end{aligned}$$

Then the proof of the theorem follows. □

Theorem 3. *Let* $\aleph \in TS_{\wp,\varsigma}^{\xi}(\vartheta, \hbar, \ell)$. *Then*

$$1 - \frac{2\ell(\hbar + (1-\vartheta))}{\Lambda(2)[\vartheta + \ell(2\hbar + 1 - \vartheta)]}|\omega| \leq |\aleph'(\omega)|$$
$$\leq 1 + \frac{2\ell(\hbar + (1-\vartheta))}{\Lambda(2)[\vartheta + \ell(2\hbar + 1 - \vartheta)]}|\omega|$$

with equality for

$$\aleph(\omega) = \omega - \frac{2\ell(\hbar + (1-\vartheta))}{\Lambda(2)[\vartheta + \ell(2\hbar + 1 - \vartheta)]}\omega^2.$$

Proof. Note that

$$\begin{aligned}\Lambda(2, \varrho, \sigma, \delta)[\vartheta + \ell(2\hbar + 1 - \vartheta)] \sum_{\kappa=2}^{\infty} \kappa a_\kappa \\ \leq \sum_{\kappa=2}^{\infty} \kappa[\vartheta(\kappa - 1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)a_\kappa \\ \leq \ell(\hbar + (1-\vartheta)) \qquad (14)\end{aligned}$$

from Theorem 1. Thus,

$$\begin{aligned}|\aleph'(\omega)| &= \left|1 - \sum_{\kappa=2}^{\infty} \kappa a_\kappa \omega^{\kappa-1}\right| \\ &\leq 1 + \sum_{\kappa=2}^{\infty} \kappa a_\kappa |\omega|^{\kappa-1} \\ &\leq 1 + |\omega| \sum_{\kappa=2}^{\infty} \kappa a_\kappa \\ &\leq 1 + |\omega| \frac{2\ell(\hbar + (1-\vartheta))}{\Lambda(2)[\vartheta + \ell(2\hbar + 1 - \vartheta)]}. \qquad (15)\end{aligned}$$

On the other hand,

$$\begin{aligned}|\aleph'(\omega)| &= \left|1-\sum_{\kappa=2}^{\infty}\kappa a_\kappa \omega^{\kappa-1}\right|\\ &\geq 1-\sum_{\kappa=2}^{\infty}\kappa a_\kappa |\omega|^{\kappa-1}\\ &\geq 1-|\omega|\sum_{\kappa=2}^{\infty}\kappa a_\kappa\\ &\geq 1-|\omega|\frac{2\ell(\hbar+(1-\vartheta))}{\Lambda(2)[\vartheta+\ell(2\hbar+1-\vartheta)]}. \qquad (16)\end{aligned}$$

Combining (15) and (16), we get the result. □

4. Radii of Starlikeness, Convexity and Close-to-Convexity

Theorem 4. *Let* $\aleph \in TS_{\wp,\varsigma}^{\xi}(\vartheta,\hbar,\ell)$. *Then* $\aleph$ *is starlike in* $|\omega| < R_1$ *of order* ϑ, $0 \leq \vartheta < 1$, *where*

$$R_1 = \inf_{\kappa}\left\{\frac{(1-\vartheta)(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa,\wp,\vartheta)}{(\kappa-\vartheta)\ell(\hbar+(1-\vartheta))}\right\}^{\frac{1}{\kappa-1}}, \quad \kappa \geq 2. \qquad (17)$$

Proof. $\aleph$ is starlike of order $\vartheta, 0 \leq \vartheta < 1$ if

$$\Re\left\{\frac{\omega\aleph'(\omega)}{\aleph(\omega)}\right\} > \vartheta.$$

Thus, it is enough to show that

$$\left|\frac{\omega\aleph'(\omega)}{\aleph(\omega)}-1\right| = \left|\frac{-\sum\limits_{\kappa=2}^{\infty}(\kappa-1)a_\kappa\omega^{\kappa-1}}{1-\sum\limits_{\kappa=2}^{\infty}a_\kappa\omega^{\kappa-1}}\right| \leq \frac{\sum\limits_{\kappa=2}^{\infty}(\kappa-1)a_\kappa|\omega|^{\kappa-1}}{1-\sum\limits_{\kappa=2}^{\infty}a_\kappa|\omega|^{\kappa-1}}.$$

Thus,

$$\left|\frac{\omega\aleph'(\omega)}{\aleph(\omega)}-1\right| \leq 1-\vartheta \quad \text{if} \quad \sum_{\kappa=2}^{\infty}\frac{(\kappa-\vartheta)}{(1-\vartheta)}a_\kappa|\omega|^{\kappa-1} \leq 1. \qquad (18)$$

Hence, by Theorem 1, (18) will be true if

$$\frac{\kappa-\vartheta}{1-\vartheta}|\omega|^{\kappa-1} \leq \frac{(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa)}{\ell(\hbar+(1-\vartheta)}$$

or if

$$|\omega| \leq \left[\frac{(1-\vartheta)(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa)}{(\kappa-\vartheta)\ell(\hbar+(1-\vartheta))}\right]^{\frac{1}{\kappa-1}}, \kappa \geq 2. \quad (19)$$

The theorem follows easily from (19). □

Theorem 5. *Let* $\aleph \in TS^{\xi}_{\wp,\varsigma}(\vartheta,\hbar,\ell)$*. Then* $\aleph$ *is convex in* $|\omega| < R_2$ *of order* $\vartheta, 0 \leq \vartheta < 1$*, where*

$$R_2 = \inf_{\kappa}\left\{\frac{(1-\vartheta)(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa)}{\kappa(\kappa-\vartheta)\ell(\hbar+(1-\vartheta))}\right\}^{\frac{1}{\kappa-1}}, \ \kappa \geq 2. \quad (20)$$

Proof. $\aleph$ is convex of order $\vartheta, 0 \leq \vartheta < 1$ if

$$\Re\left\{1+\frac{\omega\aleph''(\omega)}{\aleph'(\omega)}\right\} > \vartheta.$$

As a result, demonstrating that

$$\left|\frac{\omega\aleph''(\omega)}{\aleph'(\omega)}\right| = \left|\frac{-\sum\limits_{\kappa=2}^{\infty}\kappa(\kappa-1)a_\kappa\omega^{\kappa-1}}{1-\sum\limits_{\kappa=2}^{\infty}\kappa a_\kappa\omega^{\kappa-1}}\right| \leq \frac{\sum\limits_{\kappa=2}^{\infty}\kappa(\kappa-1)a_\kappa|\omega|^{\kappa-1}}{1-\sum\limits_{\kappa=2}^{\infty}\kappa a_\kappa|\omega|^{\kappa-1}}.$$

Thus,

$$\left|\frac{\omega\aleph''(\omega)}{\aleph'(\omega)}\right| \leq 1-\vartheta \ \text{ if } \ \sum_{\kappa=2}^{\infty}\frac{\kappa(\kappa-\vartheta)}{(1-\vartheta)}a_\kappa|\omega|^{\kappa-1} \leq 1. \quad (21)$$

Hence, by Theorem 1, (21) will be true if

$$\frac{\kappa(\kappa-\vartheta)}{1-\vartheta}|\omega|^{\kappa-1} \leq \frac{(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa)}{\ell(\hbar+(1-\vartheta)}$$

or if

$$|\omega| \leq \left[\frac{(1-\vartheta)(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa)}{\kappa(\kappa-\vartheta)\ell(\hbar+(1-\vartheta))}\right]^{\frac{1}{\kappa-1}}, \kappa \geq 2. \qquad (22)$$

The theorem follows easily from (22). □

Theorem 6. *Let* $\aleph \in TS_{\wp,\varsigma}^{\xi}(\vartheta,\hbar,\ell)$. *Then* $\aleph$ *is close-to-convex in* $|\omega| < R_3$ *of order* ϑ, $0 \leq \vartheta < 1$, *where*

$$R_3 = \inf_{\kappa}\left\{\frac{(1-\vartheta)(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa,\wp,\vartheta)}{\kappa\ell(\hbar+(1-\vartheta))}\right\}^{\frac{1}{\kappa-1}}, \quad \kappa \geq 2. \qquad (23)$$

Proof. $\aleph$ is close-to-convex of order $\vartheta, 0 \leq \vartheta < 1$ if

$$\Re\left\{\aleph'(\omega)\right\} > \vartheta.$$

Thus, it is enough to show that

$$|\aleph'(\omega)-1| = \left|-\sum_{\kappa=2}^{\infty}\kappa a_\kappa \omega^{\kappa-1}\right| \leq \sum_{\kappa=2}^{\infty}\kappa a_\kappa |\omega|^{\kappa-1}.$$

Thus,

$$|\aleph'(\omega)-1| \leq 1-\vartheta \;\text{ if }\; \sum_{\kappa=2}^{\infty}\frac{\kappa}{(1-\vartheta)}a_\kappa|\omega|^{\kappa-1} \leq 1. \qquad (24)$$

Hence, by Theorem 1, (24) will be true if

$$\frac{\kappa}{1-\vartheta}|\omega|^{\kappa-1} \leq \frac{(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa)}{\ell(\hbar+(1-\vartheta))}$$

or if

$$|\omega| \leq \left[\frac{(1-\vartheta)(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa)}{\kappa\ell(\hbar+(1-\vartheta))}\right]^{\frac{1}{\kappa-1}}, \kappa \geq 2. \qquad (25)$$

The theorem follows easily from (25). □

5. Extreme Points

Theorem 7. *Let* $\aleph_1(\omega) = \omega$ *and*

$$\aleph_\kappa(\omega) = \omega - \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)}\omega^\kappa, \quad \text{for} \quad \kappa = 2, 3, \ldots.$$

Then $\aleph \in TS^{\xi}_{\wp,\varsigma}(\vartheta, \hbar, \ell)$ *if and only if it can be expressed in the form*

$$\aleph(\omega) = \sum_{\kappa=1}^{\infty} \theta_\kappa \aleph_\kappa(\omega), \quad \text{where} \quad \theta_\kappa \geq 0 \quad \text{and} \quad \sum_{\kappa=1}^{\infty} \theta_\kappa = 1.$$

Proof. Assume that $\aleph(\omega) = \sum\limits_{\kappa=1}^{\infty} \theta_\kappa \aleph_\kappa(\omega)$, hence we get

$$\aleph(\omega) = \omega - \sum_{\kappa=2}^{\infty} \frac{\ell(\hbar + (1-\vartheta))\theta_\kappa}{[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)}\omega^\kappa.$$

Now, $\aleph \in TS^{\xi}_{\wp,\varsigma}(\vartheta, \hbar, \ell)$, since

$$\begin{aligned} &\sum_{\kappa=2}^{\infty} \frac{[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)}{\ell(\hbar + (1-\vartheta))} \\ &\qquad \times \frac{\ell(\hbar + (1-\vartheta))\theta_\kappa}{[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)} \\ &= \sum_{\kappa=2}^{\infty} \theta_\kappa = 1 - \theta_1 \leq 1. \end{aligned}$$

Conversely, suppose $\aleph \in TS^{\xi}_{\wp,\varsigma}(\vartheta, \hbar, \ell)$. Then we show that $\aleph$ can be written in the form $\sum_{\kappa=1}^{\infty} \theta_\kappa \aleph_\kappa(\omega)$.
Now, $\aleph \in TS^{\xi}_{\wp,\varsigma}(\vartheta, \hbar, \ell)$ implies, from Theorem 1,

$$a_\kappa \leq \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)}.$$

Setting $\theta_\kappa = \frac{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]\Lambda(\kappa)}{\ell(\hbar+(1-\vartheta))} a_\kappa, \kappa = 2, 3, \ldots$ and $\theta_1 = 1 - \sum_{\kappa=2}^{\infty} \theta_\kappa$, we obtain $\aleph(\omega) = \sum_{\kappa=1}^{\infty} \theta_\kappa \aleph_\kappa(\omega)$. □

6. Hadamard Product

Theorem 8. *Let* $\aleph, g \in TS^{\xi}_{\wp,\varsigma}(\vartheta,\hbar,\ell)$*. Then* $\aleph * g \in TS^{\xi}_{\wp,\varsigma}(\vartheta,\hbar,\ell)$ *for*

$$\aleph(\omega) = \omega - \sum_{\kappa=2}^{\infty} a_\kappa \omega^\kappa, g(\omega) = \omega - \sum_{\kappa=2}^{\infty} b_\kappa \omega^\kappa \quad \text{and } (\aleph * g)(\omega)$$
$$= \omega - \sum_{\kappa=2}^{\infty} a_\kappa b_\kappa \omega^\kappa,$$

where

$$\zeta \geq \frac{\ell^2(\hbar + (1-\vartheta))\vartheta(\kappa-1)}{[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]^2\Lambda(\kappa) - \ell^2(\hbar + (1-\vartheta))(\kappa\hbar + 1 - \vartheta)}.$$

Proof. $\aleph \in TS^{\xi}_{\wp,\varsigma}(\vartheta,\hbar,\ell)$ and so

$$\sum_{\kappa=2}^{\infty} \frac{[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)}{\ell(\hbar + (1-\vartheta))} a_\kappa \leq 1 \tag{26}$$

and

$$\sum_{\kappa=2}^{\infty} \frac{[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)}{\ell(\hbar + (1-\vartheta))} b_\kappa \leq 1. \tag{27}$$

We have to find the smallest number ζ such that

$$\sum_{\kappa=2}^{\infty} \frac{[\vartheta(\kappa-1) + \zeta(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)}{\zeta(\hbar + (1-\vartheta))} a_\kappa b_\kappa \leq 1. \tag{28}$$

By Cauchy–Schwarz inequality

$$\sum_{\kappa=2}^{\infty} \frac{[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)}{\ell(\hbar + (1-\vartheta))} \sqrt{a_\kappa b_\kappa} \leq 1. \tag{29}$$

Therefore, it is enough to show that

$$\frac{[\vartheta(\kappa-1) + \zeta(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)}{\zeta(\hbar + (1-\vartheta))} a_\kappa b_\kappa$$
$$\leq \frac{[\vartheta(\kappa-1) + \ell(\kappa\hbar + 1 - \vartheta)]\Lambda(\kappa)}{\ell(\hbar + (1-\vartheta))} \sqrt{a_\kappa b_\kappa},$$

i.e.

$$\sqrt{a_\kappa b_\kappa} \le \frac{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]\zeta}{[\vartheta(\kappa-1)+\zeta(\kappa\hbar+1-\vartheta)]\ell}. \tag{30}$$

From (29),

$$\sqrt{a_\kappa b_\kappa} \le \frac{\ell(\hbar+(1-\vartheta))}{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]\Lambda(\kappa)}.$$

Thus, it is enough to show that

$$\frac{\ell(\hbar+(1-\vartheta))}{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]\Lambda(\kappa)} \le \frac{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]\zeta}{[\vartheta(\kappa-1)+\zeta(\kappa\hbar+1-\vartheta)]\ell},$$

which simplifies to

$$\zeta \ge \frac{\ell^2(\hbar+(1-\vartheta))\vartheta(\kappa-1)}{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]^2\Lambda(\kappa)-\ell^2(\hbar+(1-\vartheta))(\kappa\hbar+1-\vartheta)}.$$

□

7. Closure Theorems

Theorem 9. *Let* $\aleph_j \in TS^{\xi}_{\wp,\varsigma}(\vartheta,\hbar,\ell), j=1,2,\ldots,s$. *Then*

$$\aleph_g(\omega)=\sum_{j=1}^{s} c_j \aleph_j(\omega) \in T\ S^{\xi}_{\wp,\varsigma}(\vartheta,\hbar,\ell).$$

For $\aleph_j(\omega)=\omega-\sum\limits_{\kappa=2}^{\infty} a_{\kappa,j}\omega^\kappa$, *where* $\sum\limits_{j=1}^{s} c_j = 1$.

Proof. Now,

$$\begin{aligned} g(\omega) &= \sum_{j=1}^{s} c_j \aleph_j(\omega) \\ &= \omega - \sum_{\kappa=2}^{\infty}\sum_{j=1}^{s} c_j a_{\kappa,j}\omega^\kappa \\ &= \omega - \sum_{\kappa=2}^{\infty} e_\kappa \omega^\kappa, \end{aligned}$$

where $e_\kappa = \sum_{j=1}^{s} c_j a_{\kappa,j}$. Thus, $g(\omega) \in TS_{\wp,\varsigma}^{\xi}(\vartheta, \hbar, \ell)$ if

$$\sum_{\kappa=2}^{\infty} \frac{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]\Lambda(\kappa)}{\ell(\hbar+(1-\vartheta))} e_\kappa \leq 1,$$

i.e. if

$$\begin{aligned}
&\sum_{\kappa=2}^{\infty}\sum_{j=1}^{s} \frac{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]\Lambda(\kappa)}{\ell(\hbar+(1-\vartheta))} c_j a_{\kappa,j} \\
&\quad = \sum_{j=1}^{s} c_j \sum_{\kappa=2}^{\infty} \frac{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]\Lambda(\kappa)}{\ell(\hbar+(1-\vartheta))} a_{\kappa,j} \\
&\quad \leq \sum_{j=1}^{s} c_j = 1.
\end{aligned}$$

□

Theorem 10. *Let* $\aleph, g \in TS_{\wp,\varsigma}^{\xi}(\vartheta, \hbar, \ell)$. *Then*

$$h(\omega) = \omega - \sum_{\kappa=2}^{\infty}(a_\kappa^2 + b_\kappa^2)\omega^\kappa \in TS_{\wp,\varsigma}^{\xi}(\vartheta, \hbar, \ell), where$$

$$\zeta \geq \frac{2\vartheta(\kappa-1)\ell^2(\hbar+(1-\vartheta))}{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]^2\Lambda(\kappa) - 2\ell^2(\hbar+(1-\vartheta))(\kappa\hbar+1-\vartheta)}.$$

Proof. Since $\aleph, g \in TS_{\wp,\varsigma}^{\xi}(\vartheta, \hbar, \ell)$, Theorem 1 yields

$$\sum_{\kappa=2}^{\infty}\left[\frac{(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa)}{\ell(\hbar+(1-\vartheta))} a_\kappa\right]^2 \leq 1$$

and

$$\sum_{\kappa=2}^{\infty}\left[\frac{(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa)}{\ell(\hbar+(1-\vartheta))} b_\kappa\right]^2 \leq 1.$$

We obtain from the last two inequalities

$$\sum_{\kappa=2}^{\infty}\frac{1}{2}\left[\frac{(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa)}{\ell(\hbar+(1-\vartheta))}\right]^2(a_\kappa^2+b_\kappa^2)\leq 1. \qquad (31)$$

But $h(\omega)\in TS_{\wp,\varsigma}^{\xi}(\vartheta,\hbar,\ell,\zeta)$ if and only if

$$\sum_{\kappa=2}^{\infty}\frac{[\vartheta(\kappa-1)+\zeta(\kappa\hbar+1-\vartheta)]\Lambda(\kappa)}{\zeta(\hbar+(1-\vartheta))}(a_\kappa^2+b_\kappa^2)\leq 1, \qquad (32)$$

where $0<\zeta<1$, however, (31) implies (32) if

$$\frac{[\vartheta(\kappa-1)+\zeta(\kappa\hbar+1-\vartheta)]\Lambda(\kappa)}{\zeta(\hbar+(1-\vartheta))} \leq \frac{1}{2}\left[\frac{(\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta))\Lambda(\kappa)}{\ell(\hbar+(1-\vartheta))}\right]^2.$$

Simplifying, we get

$$\zeta\geq\frac{2\vartheta(\kappa-1)\ell^2(\hbar+(1-\vartheta))}{[\vartheta(\kappa-1)+\ell(\kappa\hbar+1-\vartheta)]^2\Lambda(\kappa)-2\ell^2(\hbar+(1-\vartheta))(\kappa\hbar+1-\vartheta)}.$$

□

8. Concluding Remarks

This research has introduced the study of the Mittag-Leffler-type Borel distribution series related to analytic function and some basic properties of geometric function theory. Accordingly, some results to coefficient estimates, growth and distortion theorem, radii of starlikeness, convexity, close-to-convexity and convolution properties have also been considered, inviting future research for this field of study. We hope that this distribution series play a significant role in several branches of mathematics, science and technology.

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Chapter 17

Hybrid Accelerated Conjugate Gradient Method for Solving Nonlinear System of Equations

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Conjugate gradient methods have proved to be the most appropriate iterative techniques for managing large-scale nonlinear equation systems due to their low storage requirement and global convergence properties.

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This chapter presents a hybrid method by incorporating the approach proposed by Halilu and Waziri[14] with the conjugate gradient method to obtain a new conjugate gradient parameter. Under suitable conditions, the global convergence of our method is achieved through a derivative-free line search. The numerical results shown in this paper illustrate the effectiveness of the proposed method.

1. Introduction

The typical nonlinear system of equations is defined by

$$\Phi(z) = 0, \tag{1}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear map. Throughout this article, the space $\mathbb{R}^n$ denotes the $n-$dimensional real space and $\Phi_k = \Phi(z_k)$.

Problem (1) can be applied in control theory and compressive sensing problems.[1–3] Some iterative methods used to solve these problems include Newton and quasi-Newton methods,[4–7] Gauss–Newton methods,[8,9] Levenberg–Marquardt methods[10–12] and derivative-free methods.[13–16]

Moreover, problem (1) is analogous to the following problem of unconstrained optimization[8]:

$$\min \phi(z), \quad z \in \mathbb{R}^n. \tag{2}$$

Let ϕ be a merit function defined as

$$\phi(z) = \frac{1}{2}\|\Phi(z)\|^2, \tag{3}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\| \cdot \|$ is the Euclidean norm.

The Newton method, which can obtain its search direction based on solving the following linear equation, is the most common scheme for solving (1):

$$\Phi_k + \Phi'_k d_k = 0, \tag{4}$$

where Φ'_k is the Jacobian matrix of Φ_k at x_k or its approximation and d_k is the search direction. Although the Newton method can easily be implemented with quadratic rate of convergence,[4] it is however not applicable for solving large-scale problem because it requires the Jacobian matrix computation at each iteration.[4,5]

Due to these shortcomings of the Newton method, conjugate gradient (CG) approaches are appropriate.[3,16–18,20] The methods are very promising for solving the large-scale problems due to their simplicity and low storage requirements. A typical CG method is implemented using the iterative procedure

$$z_{k+1} = z_k + s_k, \quad s_k = \alpha_k d_k, \quad k = 0, 1, \ldots, \tag{5}$$

where α_k, $s_k = x_{k+1} - x_k$ is a step length and d_k is calculated by

$$d_k = \begin{cases} -\Phi_k, & \text{if } k = 0, \\ -\Phi_k + \beta_k d_{k-1} & \text{if } k \geq 1, \end{cases} \tag{6}$$

where β_k is a called CG parameter. The motivation after any CG method is to have a helpful CG parameter. As a result, each CG approach corresponds to a specific CG parameter. Among the well-known CG parameters are

$$\beta_k^{FR} = \frac{\|\Phi_k\|^2}{\|\Phi_{k-1}\|^2}, \quad \beta_k^{PRP} = \frac{\Phi_k^T y_{k-1}}{\|\Phi_{k-1}\|^2},$$
$$\beta_k^{HS} = \frac{\Phi_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}, \quad \text{and} \quad \beta_k^{DY} = \frac{\|\Phi_k\|^2}{d_{k-1}^T y_{k-1}},$$

which are called Fletcher–Reeves (FR),[17] Polak–Ribire–Polyak (PRP),[18] Hestenes–Stiefel (HS),[19] and Dai–Yuan (DY),[20] respectively. Note that $y_{k-1} = \Phi_k - \Phi_{k-1}$.

Numerous scientists have built various formulas for the CG parameters to make it easier to apply these methods to multiple areas that produce global convergence properties with good numerical outcomes. As a result, hybrid methods have been developed, which combine various approaches to present a practical approach with improved numerical performance and convergence properties.

In recent decades, adaptive modifications of the classical CG methods discussed above have been incorporated into the solution of problem (1). Cheng in Ref. 21 demonstrated the global convergence of a nonlinear monotone PRP-type algorithm by employing the approach of PRP for unconstrained optimization (2). In addition, Ming *et al.*[22] introduced a family of CG methods in Ref. 23 for finding (2). The numerical results demonstrated the efficacy of the technique in Ref. 22 when compared to the existing method. Motivated by Ref. 23, Waziri *et al.*[24] introduced a method that is globally converged. Subsequently, a Hager–Zhang family of CG methods for monotone nonlinear equations has been presented by Waziri *et al.*,[9] who were motivated by the global optimization problems in Ref. 25 and the hyperplane approach in Ref. 26.

Hybrid methods and parametric families have been introduced to address the shortcomings of some existing methods. As the name implies, they are based on combining different techniques to present a better approach with improved numerical performance and convergence properties. Ahmad and Storey[27] proposed the following hybrid approach for unconstrained optimization problems:

$$\beta_k = \begin{cases} \beta_k^{PRP} & \text{if} \quad 0 \leq \beta_k^{PRP} \leq \beta_k^{FR}, \\ \beta_k^{FR} & \text{otherwise}. \end{cases} \tag{7}$$

When an iteration jam occurs again, the PRP update parameter is used. Consequently, Hu and Storey[28] proposed that β_k be defined as

$$\beta_k = \max\{0, \min\{\beta_k^{PRP}, \beta_k^{FR}\}\}. \tag{8}$$

According to Nocedal and Gilbert,[29] β_k^{PRP} can be negative. Consequently, they proposed the following CG parameter:

$$\beta_k = \max\{-\beta_k^{FR}, \min\{\beta_k^{PRP}, \beta_k^{FR}\}\} \tag{9}$$

in an attempt to broaden the available options for the PRP update parameter. In Ref. 30, Dai and Yuan used the DY parameter in conjunction with HS parameter to present the following CG parameter:

$$\beta_k = \max\{0, \min\{\beta_k^{DY}, \beta_k^{HS}\}\} \tag{10}$$

and

$$\beta_k = \max\{-c\beta_k^{DY}, \min\{\beta_k^{DY}, \beta_k^{HS}\}\}, \quad c = \frac{1-\sigma}{1+\sigma} > 0, \tag{11}$$

where $0 < \sigma < 1$.

Dai and Ni investigated various CG methods for large-scale unconstrained optimization problems in Ref. 31 and found that the hybrid CG approach (10) produced the best results. By combining the LS and DY parameters as a convex combination, Liu and Li introduced a new CG parameter[32] in the following way:

$$\beta_k = (1 - \theta_k)\beta_k^{LS} + \theta_k \beta_k^{DY}, \tag{12}$$

where $\theta_k \in [0, 1]$ is the hybridization parameter. It can be seen that when $\theta_k = 0$, $\beta_k = \beta_k^{LS}$ and when $\theta_k = 1$, $\beta_k = \beta_k^{DY}$.

Another type of hybrid conjugate gradient methods is developed by combining the gradient methods with the Picard–Mann hybrid iterative process[33] defined by

$$\begin{cases} x_1 = x \in \Omega, \\ q_k = (1 - \varrho_k)x_k + \varrho_k T x_k, \\ x_{k+1} = T q_k, \quad k \in \mathbb{N}, \end{cases} \tag{13}$$

where $\{\varrho_k\}$ is in $(0, 1)$. Let Ω be a nonempty convex subset of a normed space $\mathbf{E}$, and let $T : \Omega \longrightarrow \Omega$ be a mapping. In Ref. 34, Petrovic *et al.* combined the gradient descent (SM) method[35] with the iterative process in Ref. 33. Numerical results reveal that the hybrid method[34] converges faster than SM method in Ref. 35. Furthermore, Petrovic *et al.*[36] presented a modified hybrid method to improve the numerical version of the double direction method in Ref. 37. As a result, Halilu *et al.*[38] combined the hybrid process in Ref. 33 with the method presented in Ref. 14 and proposed the CG method for solving convex constrained nonlinear monotone equations. Recently, a hybrid double direction method for solving nonlinear monotone equations with convex constraint has been introduced by Halilu *et al.*[39] via the hybrid iterative process presented in Ref. 33.

The above literature clearly shows that the hybridization process enhances the effectiveness of some current methods. The appealing properties of hybridized approaches and the scarcity of hybridization methods for solving (1) in the literature motivated this study.

Following is how this chapter is organized. The proposed method's algorithm is presented in Section 2. Section 3 demonstrates the proposed algorithm's convergence analysis. Section 4 contains a report on some numerical experiments. Section 5 concludes the chapter.

2. Hybrid Accelerated Conjugate Gradient Algorithm

This section contains details of the proposed method. Let us begin with the technique proposed in Ref. 14. The method approximated the Jacobian via

$$\Phi'_k \approx \gamma_k I, \tag{14}$$

where $\gamma_k > 0$ is a nonnegative parameter and I is an identity matrix. The technique in Ref. 14 develops a sequence of iterates $\{z_k\}$ with $z_{k+1} = z_k + (\alpha_k + \alpha_k^2 \gamma_k) d_k$ and the direction d_k is given as

$$d_k = -\gamma_k^{-1} \Phi_k. \tag{15}$$

The parameter $\gamma_k > 0$ is obtained as

$$\gamma_{k+1} = \frac{y_k^T y_k}{y_k^T s_k}, \tag{16}$$

where $y_k = \Phi_{k+1} - \Phi_k$ and $s_k = (\alpha_k + \alpha_k^2 \gamma_k) d_k$. Even though the convergence properties of the method[14] are strong, its numerical performance is defined as weaker. Consequently, we are inspired to enhance its numerical convergence by introducing a hybrid method with efficacious numerical results. This is made possible by combining (6) and (15) as follows:

$$\gamma_k^{-1} \Phi_k = (\Phi_k - \beta_k s_{k-1}). \tag{17}$$

By substituting (16) into (17), we have

$$(y_{k-1}^T s_{k-1}) \Phi_k = y_{k-1}^T y_{k-1} (\Phi_k - \beta_k s_{k-1}). \tag{18}$$

By multiplying (18) by y_{k-1}^T, it can be written as

$$\begin{aligned} \beta_k (y_{k-1}^T y_{k-1})(y_{k-1}^T s_{k-1}) = {} & (y_{k-1}^T y_{k-1})(y_{k-1}^T \Phi_k) \\ & - (y_{k-1}^T \Phi_k)(y_{k-1}^T s_{k-1}). \end{aligned} \tag{19}$$

Now, we write the CG parameter as follows:

$$\beta_k = \frac{(y_{k-1} - s_{k-1})^T y_{k-1} (y_{k-1}^T \Phi_k)}{(y_{k-1}^T y_{k-1})(y_{k-1}^T s_{k-1})}, \tag{20}$$

and the search direction is given by

$$d_k = \begin{cases} -\Phi_k & \text{if} \quad k = 0, \\ -\Phi_k + \dfrac{(y_{k-1} - s_{k-1})^T y_{k-1}(y_{k-1}^T \Phi_k)}{(y_{k-1}^T y_{k-1})(y_{k-1}^T s_{k-1})} s_{k-1} & \text{if} \quad k \geq 1. \end{cases} \tag{21}$$

To achieve a better direction toward the solution, we proposed the following CG search direction:

$$d_k = -\Phi + \beta_k^{HACG} s_{k-1}, \tag{22}$$

with

$$\beta_k = \frac{(y_{k-1} - s_{k-1})^T y_{k-1}(y_{k-1}^T \Phi_k)}{\psi_k}, \tag{23}$$

where

$$\psi_k = \max\{(y_{k-1}^T y_{k-1})(y_{k-1}^T s_{k-1}), \rho \|s_{k-1}\|^3\}.$$

The following line search proposed in Ref. 8 can be used to compute the step length α_k.

Let $\Omega_1 > 0$, $\Omega_2 > 0$ and $r \in (0, 1)$ be a constant, and let $\{\sigma_k\}$ be a positive sequence such that

$$\sum_{k=0}^{\infty} \sigma_k < \sigma < \infty, \tag{24}$$

with

$$\phi(z_k + \alpha_k d_k) - \phi(z_k) \leq -\Omega_1 \|\alpha_k \Phi_k\|^2 - \Omega_2 \|\alpha_k d_k\|^2 + \sigma_k \phi(z_k). \tag{25}$$

Let a_k be the smallest nonnegative integer a such that (25) holds for $\alpha = r^a$. Let $\alpha_k = r^{a_k}$.

Remark 1. It is clear that the line search (25) is well defined. Otherwise, for any integer $i > 0$,

$$\phi(z_k + r^a d_k) - \phi(z_k) > -\Omega_1 \|r^a \Phi_k\|^2 - \Omega_2 \|r^a d_k\|^2 + \sigma_k \phi(z_k).$$

Let $a \to \infty$, then $0 \geq \sigma_k f(z_k)$. This leads to a contradiction since $\sigma_k \phi(z_k)$ is positive.

Algorithm 1 (HACG).
Step 1: Given x_0, $d_0 = -\Phi_0$, $\epsilon = 10^{-4}$, set $k = 0$.
Step 1: Compute Φ_k.
Step 2: If $\|\Phi_k\| \leq \epsilon$, then stop, else go to Step 2.
Step 3: Compute α_k (using (25)).
Step 4: Set $z_{k+1} = z_k + \alpha_k d_k$.
Step 5: Compute d_{k+1} using (21).
Step 6: Set $k = k + 1$ and go to Step 2.

3. Convergence Analysis

In this section, the convergence result of Algorithm 1 (HACG) is presented. Now, let us defined the level set

$$\S = \{z | \|\Phi(z)\| \leq \|\Phi_0\|\}. \tag{26}$$

Assumption 1. To analyze the convergence properties of Algorithm 1, the following assumptions are required:

Assumption 1: There exists $z^* \in \mathbb{R}^n$ such that $\Phi(z^*) = 0$.
Assumption 2: Φ is continuously differentiable in some neighborhood, say N of z^* containing $\S$.
Assumption 3: The Jacobian of Φ is bounded and positive definite on Q, i.e. there exists a positive constants $H > h > 0$ such that

$$\|\Phi'(z)\| \leq H \quad \forall z \in Q \tag{27}$$

and

$$h\|d\|^2 \leq d^T \Phi'(z) d \quad \forall z \in Q, d \in \mathbb{R}^n. \tag{28}$$

Remark 2. Assumption 1 implies that there exists a constant $H > h > 0$ such that

$$h\|d\| \leq \|\Phi'(z)d\| \leq H\|d\| \quad \forall z \in Q, d \in \mathbb{R}^n, \tag{29}$$

$$h\|z - y\| \leq \|\Phi(z) - \Phi(y)\| \leq H\|z - y\| \quad \forall y, z \in N. \tag{30}$$

Lemma 1. *Suppose that Assumptions* 1–3 *hold and Algorithm* 1 *generates* $\{z_k\}$. *Then there exists a constant* $h > 0$ *such that for all* k,

$$y_k^T s_k \geq h\|s_k\|^2. \tag{31}$$

Proof. By mean-value theorem, we have $y_k^T s_k = s_k^T(\Phi_{k+1} - \Phi_k) = s_k^T \Phi'(\xi) s_k \geq h\|s_k\|^2$, where $\xi = z_k + \zeta(z_{k+1} - z_k)$, $\zeta \in (0,1)$; the last inequality follows from (28). □

From Lemma 1 and (30), we have

$$\frac{y_k^T s_k}{\|s_k\|} \geq h, \qquad \frac{\|y_k\|^2}{y_k^T s_k} \leq \frac{H^2}{h}. \tag{32}$$

Lemma 2. *Suppose that Assumptions* 1–3 *hold and Algorithm* 1 *generates* $\{z_k\}$. *Then we have*

$$\lim_{k\to\infty} \|\alpha_k d_k\| = \lim_{k\to\infty} \|s_k\| = 0 \tag{33}$$

and

$$\lim_{k\to\infty} \|\alpha_k \Phi_k\| = 0. \tag{34}$$

Proof. By (25), we have, for all $k > 0$,

$$\begin{aligned} \Omega_2\|\alpha_k d_k\|^2 &\leq \Omega_1\|\alpha_k \Phi_k\|^2 + \Omega_2\|\alpha_k d_k\|^2 \\ &\leq \|\Phi_k\|^2 - \|\Phi_{k+1}\|^2 + \sigma_k\|\Phi_k\|^2. \end{aligned} \tag{35}$$

By summing the above inequality, we have

$$\begin{aligned} \Omega_2 \sum_{i=0}^{k} \|\alpha_i d_i\|^2 &\leq \sum_{i=0}^{k} \left(\|\Phi_i\|^2 - \|\Phi_{i+1}\|^2\right) + \sum_{i=0}^{k} \sigma_i\|\Phi_i\|^2 \\ &= \|\Phi_0\|^2 - \|\Phi_{k+1}\|^2 + \sum_{i=0}^{k} \sigma_i \|\Phi_i\|^2 \\ &\leq \|\Phi_0\|^2 + \|\Phi_0\|^2 \sum_{i=0}^{k} \sigma_i \\ &\leq \|\Phi_0\|^2 + \|\Phi_0\|^2 \sum_{i=0}^{\infty} \sigma_i. \end{aligned} \tag{36}$$

From (26) and (24), the series $\sum_{i=0}^{\infty} \|\alpha_i d_i\|^2$ is convergent. This implies (33). Using similar arguments as above, (34) holds. □

Lemma 3. *Suppose that Assumptions* 1–3 *hold and Algorithm* 1 *generates* $\{z_k\}$*. Then there exist a constant* $m_1 > 0$ *such that for all* $k > 0$*,*

$$\|d_k\| \leq m_1. \tag{37}$$

Proof. From the level set, (23) and (30), we have

$$\begin{aligned}
|\beta_k^{HACG}| &= \left|\frac{(y_{k-1} - s_{k-1})^T y_{k-1}(y_{k-1}^T \Phi_k)}{\psi_k}\right| \\
&= \left|\frac{\|y_{k-1}\|^2 y_{k-1}^T \Phi_k - (y_{k-1}^T s_{k-1})(y_{k-1}^T \Phi_k)}{\psi_k}\right| \\
&\leq \frac{|\|y_{k-1}\|^2 y_{k-1}^T \Phi_k + (y_{k-1}^T s_{k-1})(y_{k-1}^T \Phi_k)|}{\psi_k} \\
&\leq \frac{|\|y_{k-1}\|^2 y_{k-1}^T \Phi_k + (y_{k-1}^T s_{k-1})(y_{k-1}^T \Phi_k)|}{\rho\|s_{k-1}\|^3} \\
&\leq \frac{\|y_{k-1}\|^3\|\Phi_k\|}{\rho\|s_{k-1}\|^3} + \frac{\|y_{k-1}\|^2\|s_{k-1}\|\|\Phi_k\|}{\rho\|s_{k-1}\|^3} \\
&\leq \frac{H^3\|s_{k-1}\|^3\|\Phi_k\|}{\rho\|s_{k-1}\|^3} + \frac{H^2\|s_{k-1}\|^3\Phi_k\|}{\rho\|s_{k-1}\|^3} \\
&\leq \frac{H^3\Phi_0\|}{\rho} + \frac{H^2\Phi_0\|}{\rho} \\
&= \frac{(H+1)H^2\|\Phi_0\|}{\rho}.
\end{aligned} \tag{38}$$

Taking $\lambda = \frac{(H+1)H^2\|\Phi_0\|}{\rho}$, we have

$$|\beta_k^{HACG}| \leq \lambda. \tag{39}$$

Therefore, from (22) and (39), we have

$$\begin{aligned}
\|d_k\| &= \| - \Phi_k + \beta_k^{HACG} s_{k-1}\| \\
&\leq \|\Phi_k\||\beta_k^{HACG}|\|s_{k-1}\| \\
&\leq \|\Phi_k\|\lambda\|s_{k-1}\| \\
&\leq \|\Phi_0\|\lambda P,
\end{aligned} \tag{40}$$

Since $\lim_{k\to\infty} \|s_k\| = 0$, then there exists a positive constant P such that $\|s_{k-1}\| < P$. Taking $m_1 = \|\Phi_0\|\lambda P$, we have (37). We can deduce that for all k, (37) holds. □

Lemma 4. *Suppose that Assumptions* 1–3 *hold and Algorithm* 1 *generates* $\{z_k\}$. *Then* $\{z_k\} \subset \S$.

Proof. From the norm decent line search in (25), we have $\|\Phi_{k+1}\| \leq \|\Phi_k\|$. Moreover, we have, for all k,

$$\|\Phi_{k+1}\| \leq \|\Phi_k\| \leq \|\Phi_{k-1}\| \leq \cdots \leq \|\Phi_0\|.$$

This implies that $\{z_k\} \subset \S$. □

Theorem 1. *Suppose that Assumptions* 1–3 *hold and Algorithm* 1 *generates* $\{z_k\}$. *Assume further for all* $k > 0$,

$$\alpha_k \geq c\frac{|\Phi_k d_k|}{\|d_k\|^2}, \tag{41}$$

where $c > 0$. *Then*

$$\lim_{k\to\infty} \|\Phi_k\| = 0. \tag{42}$$

Proof. From (37), (33) and the boundedness of $\{\|d_k\|\}$, we have

$$\lim_{k\to\infty} \alpha_k \|d_k\|^2 = 0. \tag{43}$$

From (41) and (43), it follows that

$$\lim_{k\to\infty} |\Phi_k^T d_k| = 0. \tag{44}$$

On the other hand, (22) and (39) lead to

$$\begin{aligned}\|\Phi_k\|^2 &= -\Phi_k^T d_k + \beta_k^{HACG}\|\Phi_k\|\|s_{k-1}\|,\\ &\leq |\Phi_k^T d_k| + |\beta_k^{HACG}|\|\Phi_k\|\|s_{k-1}\|,\\ &\leq |\Phi_k^T d_k| + \lambda\|\Phi_0\|\|s_{k-1}\|.\end{aligned}$$

Thus,

$$0 \leq \|\Phi_k\|^2 \leq |\Phi_k^T d_k| + \lambda\|\Phi_0\|\|s_{k-1}\| \longrightarrow 0. \tag{45}$$

Therefore,

$$\lim_{k\to\infty} \|\Phi_k\| = 0. \tag{46}$$

□

The proof is completed.

4. Numerical Results

This section compares the proposed method (HACG) to the following existing methods to demonstrate its efficiency and robustness:

(1) IDFDD algorithm proposed in Ref. 14,
(2) ADLCG algorithm proposed in Ref. 24.

All the three algorithms have been implemented with the same line search technique and the following parameters are set: $\Omega_1 = \Omega_2 = 10^{-4}$, $r = 0.2$ and $\sigma_k = \dfrac{1}{(k+1)^2}$. The computational codes used in this study were written in Matlab 9.4.0 (R2018a) and operated on a computer with a 1.80 GHz CPU processor and 4 GB RAM. The program is terminated when the total number of iterations overreaches 1000 or when $\|\Phi_k\| \leq 10^{-4}$. We employ the symbol "–" to represent a failure due to the following conditions:

(i) memory requirements;
(ii) when the iterations reach 1000; however, no z_k satisfies the stopping criterion.

Using some Benchmark test problems with various initial points, the methods were tested. Problems 1 and 2 are from Ref. 9 and Problems 3 is from Ref. 14. We experimented with the three methods with the dimension (n values), 1000, 10,000, and 100,000 with different initial points:

$$z1 = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right)^T,$$

$$z2 = \left(1-1, 1-\frac{1}{2}, 1-\frac{1}{3}, \ldots, 1-\frac{1}{n}\right)^T,$$

$$z3 = (2, 4, 6, \ldots, 2n)^T,$$

$$z4 = (1, 3, 5, \dots, 2n-1)^T,$$
$$z5 = (0.5, 0.5, \dots, 0.5)^T.$$

Problem 1:

$$\Phi_i(z) = 2\left(n + i(1-\cos z_i) - \sin z_i - \sum_{j=1}^{n} \cos z_j\right)(2\sin z_i - \cos z_i)$$
$$i = 1, 2, 3, \dots, n.$$

Problem 2:

$$\Phi_i(z) = z_i - \left(1 - \frac{c}{2n}\sum_{j=1}^{n} \frac{\mu_i z_j}{\mu_i + \mu_j}\right)^{-1}$$
$$i = 1, 2, 3, \dots, n, \quad j = 1, 2, 3, \dots, n,$$

with $c \in [0, 1)$ and $\mu = \frac{i-0.5}{n}$ (we take $c = 0.1$ in this experiment).

Problem 3:

$$\Phi(z) = \begin{pmatrix} 2 & -1 & & & \\ 0 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix} z + (\sin z_1 - 1, \sin z_2 - 1, \dots, \sin z_n - 1)^T.$$

Table 1. Numerical results for Problem 2

		HACG			**ADLCG**			**IDFDD**		
Dimension	Guess	ITRN	TM	$\|\Phi_k\|$	ITRN	TM	$\|\Phi_k\|$	ITRN	TM	$\|\Phi_k\|$
1000	z1	6	0.00146	2.30E-05	8	0.00176	2.82E-05	95	0.01652	8.94E-05
	z2	5	0.00138	7.03E-07	—	—	—	49	0.01145	8.82E-05
	z3	8	0.00241	7.04E-07	9	0.00226	4.30E-05	88	0.01505	8.94E-05
	z4	6	0.00133	1.71E-06	—	—	—	17	0.00403	7.38E-05
10,000	z1	4	0.00153	6.30E-11	—	—	—	37	0.00906	9.71E-05
	z2	5	0.00123	1.73E-06	5	0.00131	6.11E-05	36	0.00492	9.63E-05
	z3	6	0.00164	1.81E-06	9	0.00159	6.51E-05	20	4.37E-03	9.52E-05
	z4	6	0.00147	2.47E-06	8	0.00196	2.83E-05	39	0.00584	8.86E-05
100,000	z1	4	0.00137	2.90E-05	4	0.00174	2.49E-05	25	0.00639	7.36E-05
	z2	3	0.00104	7.72E-08	—	—	—	25	0.00403	8.29E-05
	z3	7	0.00155	3.34E-07	11	0.00274	3.26E-05	38	0.00924	8.74E-05
	z4	5	0.00174	4.81E-06	6	0.00185	3.15E-05	24	0.00451	7.80E-05

Table 2. Numerical results for Problem 2

		HACG			ADLCG			IDFDD		
Dimension	Guess	ITRN	TM	$\|\Phi_k\|$	ITRN	TM	$\|\Phi_k\|$	ITRN	TM	$\|\Phi_k\|$
1000	z1	5	0.01489	2.56E-06	26	0.04494	6.62E-05	31	0.05437	8.90E-05
	z2	4	0.01312	5.68E-05	29	0.06025	9.51E-05	45	0.08011	9.58E-05
	z3	5	0.01882	6.38E-07	32	0.04791	8.32E-05	67	0.11789	7.21E-05
	z4	4	0.01257	1.52E-06	32	0.04575	8.28E-05	67	0.11571	7.17E-05
10,000	z1	5	0.01592	9.76E-05	25	0.03704	8.80E-05	31	0.05514	9.48E-05
	z2	4	0.01512	1.59E-04	31	0.04863	9.48E-05	53	7.99E-02	6.89E-05
	z3	5	0.01331	2.70E-05	32	0.05302	6.54E-05	81	0.14985	7.41E-05
	z4	5	0.01312	9.52E-06	38	0.0515	6.53E-05	81	0.13035	7.41E-05
100,000	z1	4	0.01638	1.02E-04	26	0.03682	7.46E-05	32	0.0603	8.72E-05
	z2	5	0.01756	1.81E-04	33	0.04575	7.71E-05	60	0.09579	7.04E-05
	z3	5	0.01299	2.87E-05	41	0.05545	9.57E-05	95	0.14284	7.60E-05
	z4	5	0.01167	4.09E-07	41	0.06811	9.57E-05	95	0.14711	7.60E-05

Table 3. Numerical results for Problem 3

		HACG			ADLCG			IDFDD		
Dimension	Guess	ITRN	TM	$\|\Phi_k\|$	ITRN	TM	$\|\Phi_k\|$	ITRN	TM	$\|\Phi_k\|$
1000	z1	8	0.01852	1.27E-05	11	0.0029	5.01E-05	27	0.01349	9.24E-05
	z2	18	0.00692	3.78E-05	20	0.00478	5.25E-05	—	—	—
	z3	33	0.00854	5.68E-07	32	0.00843	3.95E-05	—	—	—
	z4	33	0.00767	5.59E-07	32	0.00753	3.98E-05	—	—	—
	z5	23	0.01325	5.23E-05	21	0.03219	2.57E-05	—	—	—
10,000	z1	9	0.00251	6.33E-07	11	0.00293	6.19E-05	30	0.01161	7.94E-05
	z2	23	0.02048	1.13E-05	23	0.0056	9.75E-05	—	—	—
	z3	43	0.00591	1.63E-07	41	0.06081	5.25E-05	—	—	—
	z4	43	0.00815	1.62E-07	41	0.00892	3.72E-05	—	—	—
	z5	28	0.09247	7.64E-05	25	0.09236	9.00E-05	—	—	—
100,000	z1	9	0.00246	1.38E-05	11	0.00317	3.19E-05	22	0.00694	7.85E-05
	z2	28	0.00502	2.72E-06	29	0.00699	3.10E-05	—	—	—
	z3	61	0.00958	4.76E-07	56	0.01182	5.99E-05	—	—	—
	z4	61	0.03234	4.76E-07	59	0.0136	9.18E-05	—	—	—
	z5	32	0.80454	6.91E-05	28	0.76903	8.54E-01	—	—	—

Table 4. Overview of the results recorded in the Tables 1–3

Methods	**ITRN**	**Percentage**	**TM**	**Percentage**
HACG	27	69.23	29	74.36
ADLCG	9	23.08	10	25.64
IDFDD	0	0	0	0
Undecided	3	7.69	0	0

The implemented numerical results of the HACG, ADLCG, and IDFDD methods are reported in Tables 1–3, where the following can be noted:

- "Guess" denotes the initial point,
- "ITRN" denotes the total number of iterations,
- "TM' denotes the CPU time (seconds),
- "$\|\Phi_k\|$" is the norm of residual at the termination point.

Although all three methods solved the numerical experiment's test problems, the HACG method converges to the solution of (1) more quickly because it has fewer iterations than the ADLCG and IDFDD methods, except for Problem 3, where the ADLCG beats the HACG in some iterations. Moreover, the ADLCG and IDFDD methods failed during the iteration process by clear indication from Tables 1 and 3. This shows that the proposed method is very effective solving problem (1). Moreover, as shown in Tables 1 and 3, the HACG method solved the three problems with less CPU time. Our method beats the ADLCG and IDFDD methods tested for nearly all problems because it has the fewest iterations and processor time of the methods compared.

To illustrate which approach is the best, Tables 4 recapitulates the results recorded in Tables 1–3. From Table 4, the HACG approach solved 69.23% (27 out of 39) of iterations. At the same time, the ADLCG and IDFDD methods have solved 23.08% (9 out of 39) and 0% (0 out of 39), respectively. The summary table indicates that seven problems have the same number of iterations in the HACG and ADLCG methods, which is 7.69% and is undecided. For the processor time, the methods ADLCG and IDFDD are respectively resolved 25.64% (10 out of 39) and 0 percent (0 out of 39), and the HACG method solved 74.36% (29 out of 39) of the problems with less CPU time.

Dolan and Moré[40] evaluation tool displayed figures of each of the three methods used in the experimentations to analyze the results described in Tables 1–3. Figures 1 and 2 depict the performance profiles of the HACG, ADLCG, and IDFDD methods in terms of iteration numbers and CPU time, using data from Tables 1–3 and the strategy proposed in Ref. 40. A fraction p(τ) of the problems is plotted from two graphs where a method is within τ of the best time. The top curve in both figures corresponds to the HACG scheme.

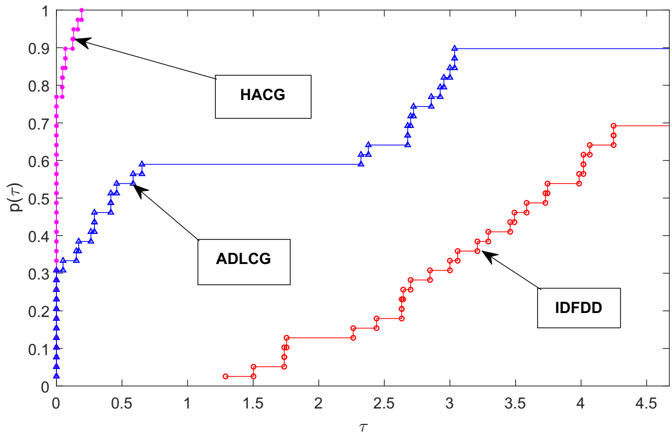

Fig. 1. Performance profile for the function evaluation.

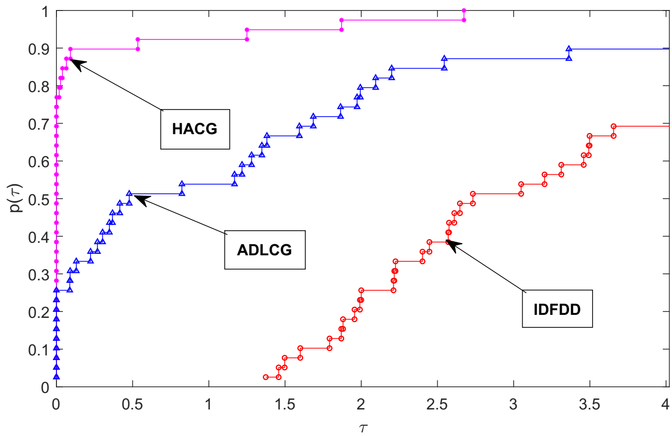

Fig. 2. Performance profile for the CPU time (seconds).

As a result, it outperforms the other approaches in terms of minimum iteration numbers and CPU time, hence we deduce that HACG method is effective for solving problem (1).

5. Conclusion

In this work, a hybrid accelerated conjugate gradient method for solving systems of nonlinear equations is presented. The proposed method (HACG) is a derivative-free, it has no computation and storage of the Jacobian matrix at each iteration, therefore it has low memory requirements and computational cost. In addition, it has improved the convergence of the double direction method.[14] As a result, it was compared with the ADLCG[24] and IDFDD methods.[14] In addition, the numerical results of the three (3) methods are reported in Tables 1–3. It can be noted that the HACG method is very effective because it has the smallest number of iterations and CPU time compared to the ADLCG and IDFDD methods. Table 4 summarized Tables 1–3, highlighting that the HACG method is a winner in terms of CPU time and the number of iterations. In Figures 1 and 2, the performance profile of Dolan and Moré[40] was used to interpret the results in Tables 1–3. The figures depict the effectiveness of the HACG method. Therefore, we deduce that the proposed method is promising and appropriate for solving problem (1). Future work includes modifying the proposed method using the inertial-based approach for nonlinear system equations with a motion control application.

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Chapter 18

A Closed Form of Integral Transforms in terms of Lauricella Function $F_A^{(n)}$ and Their Numerical Simulations

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The main object of this chapter is to investigate integral transforms involving the product of two Bessel functions and the Whittaker function. These integral transforms are given in terms of the Lauricella function $F_A^{(3)}$ of three variables and $F_A^{(4)}$ of four variables. Interesting special cases of our main results are deduced by taking suitable values of the index of the Whittaker function. We also perform some numerical simulations using the Laguerre–Gauss quadrature method and it is found that there is a good agreement between the numerical and theoretical

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evaluations. These results can be used in the Fresnel diffraction by a helical axicon of many laser fields, such as Laguerre–Bessel–Gaussian beams.

1. Introduction and Preliminaries

In this chapter, we evaluate integral transforms using special functions, with a focus on Bessel functions. The use of Bessel functions is motivated by their ability to handle complex and abstract situations (see, Ref. 13). These functions have been widely applied in mathematical physics and have been used to solve problems in areas such as mathematics, physics, optimization, cybernetic technology, and biology by reducing the complexity of the problem to the evaluation of integral transforms involving special functions such as Bessel, Whittaker, and Kummer functions (see Refs. 2–5). Earlier, we have published a series of papers in laser physics (see Refs. 6–9) and we have elaborated some applications of these theories in this field. Our results will be useful in a variety of domains of physics, such as plasma, radio and laser physics.

The following definitions are essential for the present investigation: We recall some definitions as the hypergeometric ${}_pF_q$ given by (see Refs. 10 and 11)

$$ {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}. \tag{1} $$

In this definition, we assume that $\beta_j \neq 0,-1,-2,\ldots; j=1,\ldots,q$.

The shifted factorial (the Pochhammer symbol) is defined in terms of the familiar Gamma function (see Ref. 11)

$$ (\lambda)_\nu=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}. \tag{2} $$

We give the definition of Humbert's confluent hypergeometric series of two variables (see Ref. 11)

$$ \Psi_2\left[\alpha;\gamma,\gamma';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{3} $$

with $|x|<\infty$ and $|y|<\infty$, and the series representation for the Bessel function of the first kind of order ν for each $x \in C\ \ (-\infty,0)$

is given by (see Ref. 12)

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{\nu+2k}}{k!\Gamma(\nu+k+1)}. \tag{4}$$

The denominator parameters of the above functions are neither zero nor negative integers.

Indeed, recall the triple hypergeometric function $F_A^{(3)}$ given by

$$\begin{aligned} &F_A^{(3)}[a, b_1, b_2, b_3; c_1, c_2, c_3; x, y, z] \\ &= \sum_{l,m,n=0}^{\infty} \frac{(a)_{l+m+n}(b_1)_l(b_2)_m(b_3)_n}{(c_1)_l(c_2)_m(c_3)_n} \frac{x^l}{l!}\frac{y^m}{m!}\frac{z^n}{n!}, \end{aligned} \tag{5}$$

with $|x| + |y| + |z| < 1$, and the hypergeometric function $F_A^{(4)}$ of four variables defined in the form

$$\begin{aligned} &F_A^{(4)}[a, b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; x, y, z, t] \\ &= \sum_{j,l,m,n=0}^{\infty} \frac{(a)_{j+l+m+n}(b_1)_j(b_2)_l(b_3)_m(b_4)_n}{(c_1)_j(c_2)_l(c_3)_m(c_4)_n} \frac{x^j}{j!}\frac{y^l}{l!}\frac{z^m}{m!}\frac{t^n}{n!}, \end{aligned} \tag{6}$$

with $|x| + |y| + |z| + |t| < 1$.

The Whittaker function $M_{s,\xi}$ is defined by (see Ref. 11)

$$M_{s,\xi}(z) = z^{\xi+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) {}_1F_1\left(\frac{1}{2} + \xi - s; 2\xi + 1; z\right). \tag{7}$$

We also recall some simple special cases of the Whittaker function by following the notation of Buchholz,[10] which are deduced from (7) as follows:

$$M_{\mp r, -r-\frac{1}{2}}(z) = z^{-r} \exp\left(\mp\frac{z}{2}\right), \tag{8}$$

$$M_{0,\frac{1}{2}}(z) = 2\sinh\left(\frac{z}{2}\right), \tag{9}$$

$$M_{a+\frac{1}{4}, -\frac{1}{4}}\left(z^2\right) = (-1)^a \frac{a!}{(2a)!} \exp\left(-\frac{z^2}{2}\right) \sqrt{z} H_{2a}(z), \tag{10}$$

$$M_{a+\frac{3}{4},\frac{1}{4}}\left(z^2\right)=\frac{(-1)^a}{2}\frac{a!}{(2a+1)!}\exp\left(-\frac{z^2}{2}\right)\sqrt{z}H_{2a+1}\left(z\right), \quad (11)$$

$$M_{\pm\left(b+\frac{1+a}{2}\right)\frac{a}{2}}\left(z\right)=\frac{b!}{(a+1)_b}z^{\frac{1+a}{2}}\exp\left(\pm\frac{z}{2}\right)L_b^{(a)}\left(\mp z\right), \quad (12)$$

$$M_{0,r}\left(z\right)=2^{2r}\Gamma\left(r+1\right)\sqrt{z}I_r\left(\frac{z}{2}\right), \quad (13)$$

$$M_{r-\frac{1}{2},r}\left(z\right)=2r\exp\left(\frac{z}{2}\right)z^{\frac{1}{2}-r}\gamma\left(2r,z\right), \quad (14)$$

$$M_{-\frac{1}{4},\frac{1}{4}}\left(z^2\right)=\frac{\exp\left(z^2/2\right)}{2}\sqrt{\pi z}erf\left(z\right), \quad (15)$$

and

$$M_{\pm\left(\frac{1}{2}-r\right)r}\left(z\right)=z^{r+\frac{1}{2}}\exp\left(\pm\frac{z}{2}\right){}_1F_1\left(2r;2r+1;\mp z\right), \quad (16)$$

with $L_b^{(a)}$ and H_a as the generalized Laguerre polynomial and the Hermite polynomial, respectively. The functions γ and erf are known as incomplete Gamma and error functions.

2. Main Results

In this section, we evaluate closed forms of two integral transforms involving the product of Bessel functions and Whittaker functions. The used weight in the integrand of our transformation is $e^{-p\rho^2+2q\rho}$ with $\Re(p)>0$. The considered integrals will be expressed in terms of Lauricella functions $F_A^{(3)}$ of three variables and $F_A^{(4)}$ of four variables.

Theorem 1. *Let the conditions* $\Re(p)>0, \Re(\mu)>-1$ *and* $\frac{|\chi|^2}{16}+|\lambda|^2+|q|^2<|p|$ *be verified, then the undermentioned transformation holds true:*

$$\int_0^\infty \rho^\mu J_\nu(\chi\rho)J_\varepsilon(\lambda\rho)e^{-p\rho^2+2q\rho}d\rho$$

$$=R_0\left\{\frac{\Gamma\left(\alpha_J\right)}{2\sqrt{p}}F_A^{(3)}\left[\alpha_J,\_,\_,\_;\varepsilon+1,\nu+1,\frac{1}{2};x,y,z\right]\right.$$

$$\left.+\frac{q}{p}\Gamma\left(\beta_J\right)F_A^{(3)}\left[\beta_J,\_,\_,\_;\varepsilon+1,\nu+1,\frac{3}{2};x,y,z\right]\right\},$$

where

$$R_0 = \frac{\left(\frac{\lambda}{2\sqrt{p}}\right)^{\varepsilon}}{\left(\sqrt{p}\right)^{\mu}\varepsilon!}\frac{\left(\frac{\chi}{2\sqrt{p}}\right)^{\nu}}{\nu!},$$

$$x = -\frac{\lambda^2}{4p}, \quad y = -\frac{\chi^2}{4p}, \quad z = \frac{q^2}{p}, \tag{17}$$

$$\alpha_J = \frac{\mu+\nu+\varepsilon+1}{2}, \quad \beta_J = \frac{\mu+\nu+\varepsilon}{2} + 1, \tag{18}$$

and $F_A^{(3)}$ *is the Lauricella series in three variables given by* (5).

Proof. Substituting the expansion of Bessel function J_ε, we use Lemma 1 (see Appendix) and denoting the left-hand side of (17) by R, we get, after some simplifications, the following expression:

$$R = \frac{\left({}^{\lambda}/_{2\sqrt{p}}\right)^{\varepsilon}}{2p^{\frac{\mu+1}{2}}}\frac{\left({}^{\chi}/_{2\sqrt{p}}\right)^{\nu}}{\nu!}\left[\sum_{l,m,n=0}^{\infty} A_{l\ m\ n}\frac{x^l}{n!}\frac{y^m}{m!}\frac{z^k}{k!}\right.$$

$$\left. + \frac{2q}{\sqrt{p}}\sum_{l,m,n=0}^{\infty} B_{l\ m\ n}\frac{x^l}{n!}\frac{y^m}{m!}\frac{z^k}{k!}\right], \tag{19}$$

where

$$A_{l\ m\ n} = \frac{\Gamma\left(\frac{\mu+\nu+\varepsilon+1}{2}+l\right)\left(\frac{\mu+\nu+\varepsilon+1}{2}+l\right)_{m+n}}{\Gamma\left(\varepsilon+l+1\right)\left(\nu+1\right)_m\left(\frac{1}{2}\right)_n} \tag{20}$$

and

$$B_{l\ m\ n} = \frac{\Gamma\left(\frac{\mu+\nu+\varepsilon}{2}+1+l\right)\left(\frac{\mu+\nu+\varepsilon}{2}+1+l\right)_{m+n}}{\Gamma\left(\varepsilon+l+1\right)\left(\nu+1\right)_m\left(\frac{3}{2}\right)_n}, \tag{21}$$

with x, y and z given by (17).

By using the identity (see Ref. 11),

$$(\lambda+m)_n = \frac{(\lambda)_{m+n}}{(\lambda)_m} \tag{22}$$

the expressions of $A_{l\ m\ n}$ and $B_{l\ m\ n}$ become

$$A_{l\ m\ n} = \frac{1}{\varepsilon!}\Gamma\left(\alpha_J\right)\frac{\left(\alpha_J\right)_{l+m+n}}{(\varepsilon+1)_l(\nu+1)_m\left(\frac{1}{2}\right)_n} \tag{23}$$

and

$$B_{l\ m\ n} = \frac{1}{\varepsilon!}\Gamma\left(\beta_J\right)\frac{\left(\beta_J\right)_{l+m+n}}{(\varepsilon+1)_l(\nu+1)_m\left(\frac{3}{2}\right)_n}, \tag{24}$$

where α_J and β_J are expressed in (18).

Finally, with the help of these last equations, (23) and (24), (19) yields to (17) by noting that the triple summation of (19) is equal to the Lauricella series $F_A^{(3)}$. This completes the proof. □

Theorem 2. *Let the conditions* $\Re(p) > 0, \Re(\mu) > -1$ *and* $\frac{|\chi|^2}{16} + |\lambda|^2 + |q|^2 < |p|^2$ *be verified, then the undermentioned integral transform holds true:*

$$\begin{aligned}
&\int_0^\infty \rho^\mu J_\nu(\chi\rho)J_\varepsilon(\lambda\rho)M_{s,\zeta}(2\gamma\rho^2)e^{-p\rho^2+2q\rho}d\rho \\
&= L_0\Bigg\{\frac{\Gamma\left(\alpha_L\right)}{2\sqrt{p+\gamma}}F_A^{(4)}\Bigg[\alpha_L, \frac{1}{2}+\zeta-s, _, _, _; 2\zeta \\
&\quad +1, \varepsilon+1, \nu+1, \frac{1}{2}; x, y, z, t\Bigg] \\
&\quad + \frac{q}{p+\gamma}\Gamma\left(\beta_L\right)F_A^{(4)}\Bigg[\beta_L, \frac{1}{2}+\zeta-s, _, _, _; 2\zeta+1, \varepsilon \\
&\quad +1, \nu+1, \frac{3}{2}; x, y, z, t\Bigg]\Bigg\},
\end{aligned} \tag{25}$$

where

$$L_0 = \frac{(2\gamma)^{\zeta+\frac{1}{2}}}{(p+\gamma)^{\frac{\mu+\nu+\varepsilon+1}{2}+\zeta}}\frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}\frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{\varepsilon!}, \tag{26}$$

$$x = \frac{2\gamma}{(p+\gamma)},\ y = -\frac{\lambda^2}{(p+\gamma)},\ z = -\frac{\chi^2}{4\left(p+\gamma\right)},\ t = \frac{q^2}{(p+\gamma)}, \tag{27}$$

$$\alpha_L = \frac{\mu+\nu+\varepsilon}{2}+\zeta+1 \ \textit{and}\ \beta_L = \frac{\mu+\nu+\varepsilon}{2}+\zeta+\frac{3}{2}. \tag{28}$$

Proof. The expression of the Whittaker function is given by (see Ref. 10)

$$M_{s,\,\zeta}(2\gamma\rho^2) = (2\gamma)^{\zeta+\frac{1}{2}} \sum_{l=0}^{\infty} \frac{\left(\frac{1}{2}+\zeta-s\right)_l}{(2\zeta+1)_l} \frac{(2\gamma)^l}{l!} e^{-\gamma\rho^2} \rho^{2l+2\zeta+1}. \quad (29)$$

Substituting (29) in (32) and denoting its left-hand side by L, (32) becomes

$$L = (2\gamma)^{\zeta+\frac{1}{2}} \sum_{l=0}^{\infty} \frac{\left(\frac{1}{2}+\zeta-s\right)_l}{(2\zeta+1)_l} \frac{(2\gamma)^l}{l!} L_l, \quad (30)$$

where

$$\begin{aligned} L_l = & \frac{\left(\frac{\lambda}{2\sqrt{p+\gamma}}\right)^{\varepsilon}}{\varepsilon!(p+\gamma)^{\frac{\mu+1}{2}+l+\zeta}} \frac{\left(\frac{\chi}{2\sqrt{p+\gamma}}\right)^{\nu}}{\nu!} \\ & \times \left\{ \frac{\Gamma\left(\alpha_L^l\right)}{2\sqrt{p+\gamma}} \sum_{m,\,n,\,k=0}^{\infty} \frac{\left(\alpha_L^l\right)_{m+n+k}}{(\varepsilon+1)_m(\nu+1)_n\left(\frac{1}{2}\right)_k} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^k}{k!} \right. \\ & \left. + \frac{q}{(p+\gamma)} \Gamma\left(\beta_L^l\right) \sum_{m,\,n,\,k=0}^{\infty} \frac{\left(\beta_L^l\right)_{m+n+k}}{(\varepsilon+1)_m(\nu+1)_n\left(\frac{3}{2}\right)_k} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^k}{k!} \right\}, \end{aligned}$$

with

$$\alpha_L^l = \alpha_L + l \;\; and \;\; \beta_L^l = \beta_L + l. \quad (31)$$

Making the use of the identity,

$$\Gamma\left(\lambda+l\right)\left(\lambda+l\right)_{m+n+k} = \Gamma\left(\lambda\right)\left(\lambda\right)_{l+m+n+k}, \quad (32)$$

(30) can be rearranged as

$$\begin{aligned} L = & (2\gamma)^{\zeta+\frac{1}{2}} \frac{\left(\frac{\lambda}{2\sqrt{p+\gamma}}\right)^{\varepsilon}}{\varepsilon!(p+\gamma)^{\frac{\mu+1}{2}+\zeta}} \frac{\left(\frac{\chi}{2\sqrt{p+\gamma}}\right)^{\nu}}{\nu!} \left\{ \frac{\Gamma\left(\alpha_L\right)}{2\sqrt{p+\gamma}} \right. \\ & \times \sum_{l,m,n,k=0}^{\infty} \frac{(\alpha_L)_{l+m+n+k}\left(\frac{1}{2}+\zeta-s\right)_l}{(2\zeta+1)_l(\varepsilon+1)_m(\nu+1)_n\left(\frac{1}{2}\right)_k} \frac{x^l}{l!} \frac{y^m}{m!} \frac{z^n}{n!} \frac{t^k}{k!} \end{aligned} \quad (33)$$

$$+\frac{q}{(p+\gamma)}\Gamma\left(\beta_L\right)\sum_{l,m,n,k=0}^{\infty}$$

$$\times\frac{(\beta_L)_{l+m+n+k}\left(\frac{1}{2}+\zeta-s\right)_l}{(2\zeta+1)_l(\varepsilon+1)_m(\nu+1)_n\left(\frac{3}{2}\right)_k}\frac{x^l}{l!}\frac{y^m}{m!}\frac{z^n}{n!}\frac{t^k}{k!}\Bigg\}. \tag{34}$$

Finally, with the resulting expression deduced from (34) in terms of the Lauricella function of four variables $F_A^{(4)}$, we get the required result. This completes the proof of Theorem 2. □

3. Special Cases

In this section, we recall the identities (8)–(18) of the Whittaker function given in terms of some other special functions. By taking some particular values of the index of the Whittaker function, we establish the following corollaries.

3.1. *Case of* $\mu=-\nu=\frac{1}{2}$

In this case, we use the well-known identity (see Ref. 12):

$$J_{-\frac{1}{2}}\left(z\right)=\sqrt{\frac{2}{\pi z}}\cos\left(z\right). \tag{35}$$

Corollary 1. *The following integral transform holds true*:

$$\int_0^{\infty}\cos(\chi\rho)J_{\varepsilon}(\lambda\rho)e^{-p\rho^2+2q\rho}d\rho$$

$$=\frac{\left(\frac{\lambda}{2\sqrt{p}}\right)^{\varepsilon}}{\varepsilon!}\left\{\frac{\Gamma\left(\frac{\varepsilon+1}{2}\right)}{2\sqrt{p}}F_A^{(3)}\left[\frac{\varepsilon+1}{2},\_,\_,\_;\varepsilon+1,\frac{1}{2},\frac{1}{2};x,y,z\right]\right.$$

$$\left.+\frac{q}{p}\Gamma\left(\frac{\varepsilon}{2}+1\right)F_A^{(3)}\left[\frac{\varepsilon}{2}+1,\_,\_,\_;\varepsilon+1,\frac{1}{2},\frac{3}{2};x,y,z\right]\right\}, \tag{36}$$

where x, y, and z are given by (17), *with the conditions*, $\Re(p>0)$ *and* $\frac{|\chi|^2}{16}+|\lambda|^2+|q|^2<|p|$.

Proof. Taking $\mu = -\nu = \frac{1}{2}$ and applying Theorem 1, (48) is proved. □

Corollary 2. *The following integral transform holds true*:

$$\int_0^\infty \cos(\chi\rho) J_\varepsilon(\lambda\rho) M_{s,\zeta}(2\gamma\rho^2) e^{-p\rho^2+2q\rho} d\rho$$
$$= \frac{(2\gamma)^{\zeta+\frac{1}{2}}}{(p+\gamma)^{\frac{\varepsilon+1}{2}+\zeta}} \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{\varepsilon!}$$
$$\times \left\{ \frac{\Gamma\left(\frac{\varepsilon}{2}+\zeta+1\right)}{2\sqrt{p+\gamma}} F_A^{(4)}\left[\frac{\varepsilon}{2}+\zeta+1, \frac{1}{2}+\zeta-s, \_, \_, \_; 2\zeta+1, \varepsilon\right.\right.$$
$$\left. +1, \frac{1}{2}, \frac{1}{2}; x, y, z, t\right]$$
$$+ \frac{q}{p+\gamma}\Gamma\left(\frac{\varepsilon+3}{2}+\zeta\right)$$
$$\times F_A^{(4)}\left[\frac{\varepsilon+3}{2}+\zeta, \frac{1}{2}+\zeta-s, \_, \_, \_; 2\zeta+1, \varepsilon\right.$$
$$\left.\left. +1, \frac{1}{2}, \frac{3}{2}; x, y, z, t\right]\right\}, \tag{37}$$

where x, y, z *and* t *are given by* (2.13) *with the condition* $\Re(p) > 0$, $\Re(\mu) > -1$ *and* $\frac{|\chi|^2}{16} + |\lambda|^2 + |q|^2 < |p|^2$.

Proof. Let $\mu = -\nu = \frac{1}{2}$ and with the help of Theorem 2, one obtains (51). □

3.2. *Case of* $\mu = \nu = \frac{1}{2}$

The Bessel function is expressed as (see Ref. 12):

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin(z). \tag{38}$$

Corollary 3. *Let* $\Re(p) > 0$, $\Re(\mu) > -1$ *and* $|x|^2 + |y|^2 + |z|^2 < 1$. *Then*

$$\int_0^\infty \sin(\chi\rho) J_\varepsilon(\lambda\rho) e^{-p\rho^2+2q\rho} d\rho$$
$$= \frac{\left(\frac{\lambda}{2\sqrt{p}}\right)^\varepsilon}{\varepsilon!} \frac{\chi}{\sqrt{p}} \left\{ \frac{\Gamma\left(\frac{\varepsilon}{2}+1\right)}{2\sqrt{p}} F_A^{(3)}\left[\frac{\varepsilon}{2}+1, \_, \_, \_; \varepsilon+1, \frac{3}{2}, \frac{1}{2}; x, y, z\right]\right.$$
$$\left. + \frac{q}{p}\Gamma\left(\frac{\varepsilon+3}{2}\right) F_A^{(3)}\left[\frac{\varepsilon+3}{2}, \_, \_, \_; \varepsilon+1, \frac{3}{2}, \frac{3}{2}; x, y, z\right]\right\}, \quad (39)$$

where x, y *and* z *are given in* (17).

Proof. Making use of $\mu = \nu = \frac{1}{2}$ and (56) and with the aid of Theorem 1, (57) is proved. □

Corollary 4. *The following integral transform holds true*:

$$\int_0^\infty \sin(\chi\rho) J_\varepsilon(\lambda\rho) M_{s,\zeta}(2\gamma\rho^2) e^{-p\rho^2+2q\rho} d\rho$$
$$= \frac{(2\gamma)^{\zeta+\frac{1}{2}}\chi}{(p+\gamma)^{\frac{\varepsilon}{2}+\zeta+1}} \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{\varepsilon!} \left\{ \frac{\Gamma\left(\frac{\varepsilon+3}{2}+\zeta\right)}{2\sqrt{p+\gamma}}\right.$$
$$\times F_A^{(4)}\left[\frac{\varepsilon+3}{2}+\zeta, \frac{1}{2}+\zeta-s, \_, \_, \_; 2\zeta+1, \varepsilon+1, \frac{3}{2}, \frac{1}{2}; x, y, z, t\right]$$
$$+ \frac{q}{p+\gamma}\Gamma\left(\frac{\varepsilon}{2}+\zeta+2\right)$$
$$\left. \times F_A^{(4)}\left[\frac{\varepsilon}{2}+\zeta+2, \frac{1}{2}+\zeta-s, \_, \_, \_; 2\zeta+1, \varepsilon+1, \frac{3}{2}, \frac{1}{2}; x, y, z, t\right]\right\}, \quad (40)$$

where x, y, z *and* t *are given by* (27).

Proof. Taking $\mu = \nu = \frac{1}{2}$ and applying Theorem 2, one obtains (60). □

3.3. *Case of $s = \mp r$ and $\xi = -r - \frac{1}{2}$*

The Whittaker function is given as (see Ref. 12)

$$M_{\mp r, -r-\frac{1}{2}}(z) = z^{-r} \exp\left(\mp \frac{z}{2}\right). \tag{41}$$

Corollary 5. *The following integral transform holds true*:

$$\begin{aligned}
&\int_0^\infty \rho^{\mu-2r} J_\nu(\chi\rho) J_\varepsilon(\lambda\rho) e^{-(p\pm\gamma)\rho^2+2q\rho} d\rho \\
&\quad = L_5 \Bigg\{ \frac{\Gamma(\alpha_K)}{2\sqrt{p+\gamma}} F_A^{(4)} \Bigg[\alpha_L, -r \pm r, _, _, _; -2r, \varepsilon+1, \nu \\
&\qquad + 1, \frac{1}{2}; x, y, z, t \Bigg] \\
&\qquad + \frac{q}{(p+\gamma)} \Gamma(\beta_L) F_A^{(4)} \Bigg[\beta_L, -r \pm r, _, _, _; -2r, \varepsilon+1, \nu \\
&\qquad + 1, \frac{3}{2}; x, y, z, t \Bigg] \Bigg\},
\end{aligned}$$

where

$$L_5 = \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\mu+\nu+\varepsilon}{2}-r} \varepsilon!} \frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{42}$$

$$\alpha_L = \frac{\mu+\nu+\varepsilon}{2} + \frac{1}{2} - r \text{ and } \beta_L = \frac{\mu+\nu+\varepsilon}{2} - r + 1. \tag{43}$$

x, y, z and t are given by (27).

Proof. With the help of Theorem 2 and the use of (41), we arrive at (42). □

Corollary 6. *The following integral transform holds true*:

$$\int_0^\infty \rho^{-2r}\cos(\chi\rho)J_\varepsilon(\lambda\rho)e^{-(p\pm\gamma)\rho^2+2q\rho}d\rho$$
$$= \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\varepsilon}{2}-r}\varepsilon!}\Bigg\{\frac{\Gamma\left(\frac{\varepsilon+1}{2}-r\right)}{2\sqrt{p+\gamma}}$$
$$\times F_A^{(4)}\left[\frac{\varepsilon+1}{2}-r,-r\pm r,\_,\_,\_;-2r,\varepsilon+1,\frac{1}{2},\frac{1}{2};x,y,z,t\right]$$
$$+\frac{q}{(p+\gamma)}\Gamma\left(\frac{\varepsilon}{2}-r+1\right) \tag{44}$$
$$\times F_A^{(4)}\left[\frac{\varepsilon}{2}-r+1,-r\pm r,\_,\_,\_;-2r,\varepsilon+1,\frac{1}{2},\frac{3}{2};x,y,z,t\right]\Bigg\},$$

where x, y, z and t are given by (27).

Proof. Using (41) and Theorem 2 with $\mu=-\nu=\frac{1}{2}$, (71) is proved. □

Corollary 7. *The following integral transform holds true*:

$$\int_0^\infty \rho^{-2r}\sin(\chi\rho)J_\varepsilon(\lambda\rho)e^{-(p\pm\gamma)\rho^2+2q\rho}d\rho$$
$$= \frac{\left(\frac{\lambda}{2}\right)^\varepsilon\chi}{\varepsilon!(p+\gamma)^{\frac{\varepsilon+1}{2}-r}}\Bigg\{\frac{\Gamma\left(\frac{\varepsilon}{2}-r+1\right)}{2\sqrt{p+\gamma}}$$
$$\times F_A^{(4)}\left[\frac{\varepsilon}{2}-r+1,-r\pm r,\_,\_,\_;-2r,\varepsilon+1,\frac{3}{2},\frac{1}{2};x,y,z,t\right]$$
$$+\frac{q}{(p+\gamma)}\Gamma\left(\frac{\varepsilon+3}{2}-r\right) \tag{45}$$
$$\times F_A^{(4)}\left[\frac{\varepsilon+3}{2}-r,-r\pm r,\_,\_,\_;-2r,\varepsilon+1,\frac{3}{2},\frac{3}{2};x,y,z,t\right]\Bigg\},$$

where x, y, z and t are given by (27).

Proof. Using Theorem 2 and (41) with $\mu=\nu=\frac{1}{2}$, one easily proves (76). □

3.4. *Case of* $s = \pm\left(\frac{1}{2} - r\right)$ *and* $\xi = r$

The Whittaker function becomes (see Ref. 12)

$$M_{\pm\left(\frac{1}{2}-r\right)r}(z) = z^{r+\frac{1}{2}} \exp\left(\pm\frac{z}{2}\right) {}_1F_1(2r; 2r+1; \pm z). \tag{46}$$

Corollary 8. *The following integral transform holds true*:

$$\begin{aligned}
&\int_0^\infty \rho^{\mu+2r+1} J_\nu(\chi\rho)\, J_\varepsilon(\lambda\rho)\, {}_1F_1(2r; 2r+1; \pm 2\gamma\rho^2) e^{-(\pm\gamma+p)\rho^2+2q\rho} d\rho \\
&= L_8 \Bigg\{ \frac{\Gamma(\alpha_L)}{2\sqrt{p+\gamma}} \\
&\quad \times F_A^{(4)}\left[\alpha_L, \frac{1}{2} + r \pm \left(\frac{1}{2} - r\right), _, _, _; 2r+1, \varepsilon+1, \nu \right. \\
&\quad \left. +1, \frac{1}{2}; x, y, z, t\right] \\
&\quad + \frac{q}{(p+\gamma)} \Gamma(\beta_L) \\
&\quad \times F_A^{(4)}\left[\beta_L, \frac{1}{2} + r \pm \left(\frac{1}{2} - r\right), _, _, _; 2r+1, \varepsilon+1, \nu \right. \\
&\quad \left. +1, \frac{3}{2}; x, y, z, t\right] \Bigg\},
\end{aligned} \tag{47}$$

where

$$L_8 = \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\mu+\nu+\varepsilon+1}{2}+r} \varepsilon!} \frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{48}$$

$$\alpha_L = \frac{\mu+\varepsilon+\nu}{2} + r + 1 \quad \text{and} \quad \beta_L = \frac{\mu+\varepsilon+\nu}{2} + r + \frac{3}{2}. \tag{49}$$

x, y, z and t are given by (27)*.*

Proof. Taking $s = \pm\left(\frac{1}{2} - r\right)$ and $\xi = r$, and applying Theorem 2 and (46), we arrive at (47). □

Corollary 9. *The following integral transform holds true*:

$$\int_0^\infty \rho^{2r+1}\cos(\chi\rho)\, J_\varepsilon(\lambda\rho)\,{}_1F_1(2r;2r+1;\pm 2\gamma\rho^2)e^{-(\pm\gamma+p)\rho^2+2q\rho}d\rho$$

$$= \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\varepsilon+1}{2}+r}\varepsilon!}\left\{\frac{\Gamma\left(\frac{\varepsilon}{2}+r+1\right)}{2\sqrt{p+\gamma}}\right.$$

$$\times F_A^{(4)}\left[\frac{\varepsilon}{2}+r+1,\frac{1}{2}+r\pm\left(\frac{1}{2}-r\right),\_,\_,\_;2r+1,\varepsilon\right.$$

$$\left.+1,\frac{1}{2},\frac{1}{2};x,y,z,t\right]$$

$$+\frac{q}{(p+\gamma)}\Gamma\left(\frac{\varepsilon+3}{2}+r\right)$$

$$\times F_A^{(4)}\left[\frac{\varepsilon+3}{2}+r,\frac{1}{2}+r\pm\left(\frac{1}{2}-r\right),\_,\_,\_;2r+1,\varepsilon\right.$$

$$\left.\left.+1,\frac{1}{2},\frac{3}{2};x,y,z,t\right]\right\}, \tag{50}$$

where x, y, z and t are given by (27).

Proof. Using $\mu=-\nu=\frac{1}{2}$ and applying (46) and Theorem 2, we prove (89). □

Corollary 10. *The following integral transform holds true*:

$$\int_0^\infty \rho^{2r+1}\sin(\chi\rho)\, J_\varepsilon(\lambda\rho)\,{}_1F_1(2r;2r+1;\pm 2\gamma\rho^2)e^{-(\pm\gamma+p)\rho^2+2q\rho}d\rho$$

$$= \frac{\left(\frac{\lambda}{2}\right)^\varepsilon\chi}{(p+\gamma)^{\frac{\varepsilon}{2}+r+1}\varepsilon!}\left\{\frac{\Gamma\left(\frac{\varepsilon+3}{2}+r\right)}{2\sqrt{p+\gamma}}\right.$$

$$\times F_A^{(4)}\left[\frac{\varepsilon+3}{2}+r,\frac{1}{2}+r\pm\left(\frac{1}{2}-r\right),\_,\_,\_;2r+1,\varepsilon\right.$$

$$\left.+1,\frac{3}{2},\frac{1}{2};x,y,z,t\right]$$

$$+\frac{q}{(p+\gamma)}\Gamma\left(\frac{\varepsilon}{2}+r+2\right)$$

$$\times F_A^{(4)}\left[\frac{\varepsilon}{2}+r+2,\frac{1}{2}+r\pm\left(\frac{1}{2}-r\right),\_,\_,\_;2r+1,\varepsilon\right.$$
$$\left.+1,\frac{3}{2},\frac{3}{2};x,y,z,t\right]\Bigg\},$$

where x, y, z and t are given by (27).

Proof. Taking $\mu=\nu=\frac{1}{2}$ and the identity (46), and applying Theorem 2, we prove (94). □

3.5. *Case of* $s=\pm\left(\frac{a+1}{2}+b\right)$ *and* $\xi=\frac{a}{2}$

The Whittaker function becomes (see Ref. 12)

$$M_{\pm\left(b+\frac{1+a}{2}\right),\frac{a}{2}}(z)=\frac{L_b^{(a)}(\pm z)}{L}z^{\frac{a+1}{2}}\exp\left(\mp\frac{z}{2}\right),\tag{51}$$

where

$$L=\frac{(a+1)_b}{b!}.\tag{52}$$

Corollary 11. *The following integral transform holds true*:

$$\int_0^\infty \rho^{\mu+a+1}J_\nu(\chi\rho)\,J_\varepsilon(\lambda\rho)L_b^{(a)}\left(\pm2\gamma\rho^2\right)e^{-(\pm\gamma+p)\rho^2+2q\rho}d\rho$$
$$=\frac{(1+a)_b}{b!}L_{11}\Bigg\{\Gamma\left(\alpha_L\right)$$
$$\times F_A^{(4)}\left[\alpha_L,\frac{1}{2}+\frac{a}{2}\pm\left(\frac{a+1}{2}+b\right),\_,\_,\_;a+1,\varepsilon+1,\nu\right.$$
$$\left.+1,\frac{1}{2};x,y,z,t\right]$$
$$+\frac{q}{(p+\gamma)}\Gamma\left(\beta_L\right)$$
$$\times F_A^{(4)}\left[\beta_L,\frac{1}{2}+\frac{a}{2}\pm\left(\frac{a+1}{2}+b\right),\_,\_,\_;a+1,\varepsilon+1,\nu\right.$$
$$\left.+1,\frac{3}{2};x,y,z,t\right]\Bigg\},\tag{53}$$

where

$$L_{11} = \frac{\left(\frac{\lambda}{2}\right)^{\varepsilon}}{(p+\gamma)^{\frac{\mu+\nu+\varepsilon+a+1}{2}}\varepsilon!}\frac{\left(\frac{\chi}{2}\right)^{\nu}}{\nu!}, \tag{54}$$

$$\alpha_L = \frac{\mu+\nu+\varepsilon+a}{2}+1 \text{ and } \beta_L = \frac{\mu+\nu+\varepsilon+a+3}{2}. \tag{55}$$

x, y, z and t are given by (27).

Proof. Leting the condition $s = \pm\left(\frac{a+1}{2}+b\right)$ and $\xi = \frac{a}{2}$, and applying Theorem 2, one easily finds (101). □

Corollary 12. *The following integral transform holds true*:

$$\begin{aligned}
&\int_0^\infty \rho^{a+1}\cos(\chi\rho)\,J_\varepsilon(\lambda\rho)L_b^{(a)}\left(\pm 2\gamma\rho^2\right)e^{-(\pm\gamma+p)\rho^2+2q\rho}d\rho \\
&= L_{12}\Bigg\{\Gamma\left(\frac{\varepsilon+a}{2}+1\right) \\
&\quad\times F_A^{(4)}\left[\frac{\varepsilon+a}{2}+1, \frac{1}{2}+\frac{a}{2}\pm\left(\frac{a+1}{2}+b\right), _, _, _; a+1, \varepsilon \right. \\
&\qquad \left. +1, \frac{1}{2}, \frac{1}{2}; x, y, z, t\right] \\
&\quad + \frac{2q}{\sqrt{p+\gamma}}\Gamma\left(\frac{\varepsilon+a+3}{2}\right) \\
&\quad\times F_A^{(4)}\left[\frac{\varepsilon+a+3}{2}, \frac{1}{2}+\frac{a}{2}\pm\left(\frac{a+1}{2}+b\right), _, _, _; a+1, \varepsilon \right. \\
&\qquad \left. +1, \frac{1}{2}, \frac{3}{2}; x, y, z, t\right]\Bigg\},
\end{aligned} \tag{56}$$

where

$$L_{12} = \frac{(1+a)_b}{b!}\frac{\left(\frac{\lambda}{2}\right)^{\varepsilon}}{(p+\gamma)^{\frac{a+\varepsilon+1}{2}}\varepsilon!}. \tag{57}$$

x, y, z and t are given by (27).

Proof. Using $\mu = -\nu = \frac{1}{2}$ and under the conditions of Theorem 2 and the use of (51), one obtains (108). □

Corollary 13. *The following integral transform holds true*:

$$\int_0^\infty \rho^{a+1}\sin(\chi\rho)\,J_\varepsilon(\lambda\rho)L_b^{(a)}\left(\pm 2\gamma\rho^2\right)e^{-(\pm\gamma+p)\rho^2+2q\rho}d\rho$$

$$= L_{13}\Bigg\{\Gamma\left(\frac{\varepsilon+a+3}{2}\right)$$

$$\times F_A^{(4)}\left[\frac{\varepsilon+a+3}{2},\frac{1}{2}+\frac{a}{2}\pm\left(\frac{a+1}{2}+b\right),\_,\_,\_;a+1,\varepsilon\right.$$

$$\left.+1,\frac{3}{2},\frac{1}{2};x,y,z,t\right]$$

$$+\frac{2q}{\sqrt{p+\gamma}}\Gamma\left(\frac{\varepsilon+a}{2}+2\right)$$

$$\times F_A^{(4)}\left[\frac{\varepsilon+a}{2}+2,\frac{1}{2}+\frac{a}{2}\pm\left(\frac{a+1}{2}+b\right),\_,\_,\_;a+1,\varepsilon\right.$$

$$\left.+1,\frac{3}{2},\frac{3}{2};x,y,z,t\right]\Bigg\}, \tag{58}$$

where

$$L_{13}=\frac{(1+a)_b}{b!}\frac{\left(\frac{\lambda}{2}\right)^\varepsilon\chi}{(p+\gamma)^{\frac{a+\varepsilon}{2}+1}\varepsilon!}. \tag{59}$$

x, y, z and t are given by (27)*.*

Proof. Let $\mu=\nu=\frac{1}{2}$ and with (51), and by using Theorem 2, we find (114). □

3.6. *Case of* $s = 0$*,* $\xi = r$

The Whittaker function is given as (see Ref. 12)

$$M_{0,r}(z)=\frac{1}{C\sqrt{2\gamma}}\sqrt{z}I_r\left(\frac{z}{2}\right), \tag{60}$$

where

$$C=\frac{1}{2^{2r}\sqrt{2\gamma}r!} \tag{61}$$

and I_r is the modified Bessel function of order r.

Corollary 14. *The following integral transform holds true*:

$$\int_0^\infty \rho^{\mu+1} J_\nu(\chi\rho)\, J_\varepsilon(\lambda\rho) I_r\left(\gamma\rho^2\right) e^{-p\rho^2+2q\rho} d\rho$$

$$= \frac{(2\gamma)^r}{2^{2r} r!} L_{14} \Bigg\{ \Gamma\left(\alpha_L\right) F_A^{(4)}\left[\alpha_L, \frac{1}{2}+r, \_, \_; 2r+1, \varepsilon+1, \nu + 1, \frac{1}{2}; x, y, z, t\right] + \frac{q}{(p+\gamma)} \Gamma\left(\beta_L\right) F_A^{(4)}\left[\beta_L, \frac{1}{2}+r, \_, \_; 2r+1, \varepsilon+1, \nu + 1, \frac{3}{2}; x, y, z, t\right] \Bigg\}, \tag{62}$$

where

$$L_{14} = \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\mu+\nu+\varepsilon+1}{2}+r} \varepsilon!} \frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{63}$$

$$\alpha_L = \frac{\mu+\nu+\varepsilon}{2} + r + 1 \textit{ and } \beta_L = \frac{\mu+\nu+\varepsilon}{2} + r + \frac{3}{2}. \tag{64}$$

x, y, z and t are given by (27).

Proof. By taking $s = 0$, $\xi = r$ and using Theorem 2, one obtains (122). □

Corollary 15. *The following integral transform holds true*:

$$\int_0^\infty \rho \cos(\chi\rho)\, J_\varepsilon(\lambda\rho) I_r\left(\gamma\rho^2\right) e^{-p\rho^2+2q\rho} d\rho$$

$$= L_{15} \Bigg\{ \Gamma\left(\frac{\varepsilon}{2}+r+1\right) F_A^{(4)}\left[\frac{\varepsilon}{2}+r+1, \frac{1}{2}+r, \_, \_; 2r+1, \varepsilon + 1, \frac{1}{2}, \frac{1}{2}; x, y, z, t\right] + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{\varepsilon}{2}+r+\frac{3}{2}\right) \times F_A^{(4)}\left[\frac{\varepsilon}{2}+r+\frac{3}{2}, \frac{1}{2}+r, \_, \_; 2r+1, \varepsilon+1, \frac{1}{2}, \frac{3}{2}; x, y, z, t\right] \Bigg\}, \tag{65}$$

where

$$L_{15} = \frac{(2\gamma)^r}{2^{2r} r!} \frac{\left(\frac{\lambda}{2}\right)^{\varepsilon}}{(p+\gamma)^{\frac{\varepsilon+1}{2}+r} \varepsilon!}. \tag{66}$$

x, y, z and t are given by (27).

Proof. With $\mu = -\nu = \frac{1}{2}$ and the identity (60), we arrive at (127) by applying Theorem 2. □

Corollary 16. *The following integral transform holds true*:

$$\int_0^{\infty} \rho \sin(\chi\rho)\, J_{\varepsilon}(\lambda\rho) I_r\left(\gamma\rho^2\right) e^{-p\rho^2+2q\rho} d\rho$$
$$= L_{16} \left\{ \Gamma\left(\frac{\varepsilon}{2} + r + \frac{3}{2}\right) F_A^{(4)}\left[\frac{\varepsilon}{2} + r + \frac{3}{2}, \frac{1}{2} + r, \_, \_; 2r+1, \varepsilon + 1, \frac{3}{2}, \frac{1}{2}; x, y, z, t\right] + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{\varepsilon}{2} + r + 2\right) \times F_A^{(4)}\left[\frac{\varepsilon}{2} + r + 2, \frac{1}{2} + r, \_, \_; 2r+1, \varepsilon+1, \frac{3}{2}, \frac{3}{2}; x, y, z, t\right] \right\},$$

where

$$L_{16} = \frac{(2\gamma)^r}{2^{2r} r!} \frac{\chi\left(\frac{\lambda}{2}\right)^{\varepsilon}}{(p+\gamma)^{\frac{\varepsilon}{2}+r+1} \varepsilon!}. \tag{67}$$

x, y, z and t are given by (27).

Proof. As the above corollary, we take $\mu = \nu = \frac{1}{2}$, (60) and under the conditions of Theorem 2, one proves (132). □

3.7. *Case of* $s = 0$ *and* $\xi = \frac{1}{2}$

In this case, the Whittaker function is expressed as (see Ref. 12)

$$M_{0,\frac{1}{2}}(z) = 2 \sinh\left(\frac{z}{2}\right). \tag{68}$$

Corollary 17. *The following integral transform holds true*:

$$\int_0^\infty \rho^{\mu+1} J_\nu(\chi\rho)\, J_\varepsilon(\lambda\rho) \sinh(\gamma\rho^2) e^{-p\rho^2+2q\rho} d\rho$$
$$= \gamma L_{17} \Bigg\{ \Gamma\left(\alpha_L\right) F_A^{(4)} \left[\alpha_L, 1, \_, \_; 2, \varepsilon+1, \nu+1, \frac{1}{2}; x, y, z, t\right]$$
$$+ \frac{q}{(p+\gamma)} \Gamma\left(\beta_L\right) F_A^{(4)} \left[\beta_L, 1, \_, \_; 2, \varepsilon+1, \nu+1, \frac{3}{2}; x, y, z, t\right] \Bigg\}, \tag{69}$$

where

$$L_{17} = \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\mu+\nu+\varepsilon}{2}+1} \varepsilon!} \frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{70}$$

$$\alpha_L = \frac{\mu+\nu+\varepsilon}{2} + \frac{3}{2} \text{ and } \beta_L = \frac{\mu+\nu+\varepsilon}{2} + 2. \tag{71}$$

x, y, z and t are given by (27).

Proof. With $s = 0$, $\xi = \frac{1}{2}$ and (137), one proves (138) by the use of Theorem 2. □

Corollary 18. *The following integral transform holds true*:

$$\int_0^\infty \rho \cos(\chi\rho)\, J_\varepsilon(\lambda\rho) \sinh(\gamma\rho^2) e^{-p\rho^2+2q\rho} d\rho$$
$$= \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\varepsilon}{2}+1} \varepsilon!} \gamma \Bigg\{ \Gamma\left(\frac{\varepsilon+3}{2}\right) F_A^{(4)} \left[\frac{\varepsilon+3}{2}, 1, \_, \_, \_; 2, \varepsilon\right.$$
$$\left. +1, \frac{1}{2}, \frac{1}{2}; x, y, z, t\right] + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{\varepsilon}{2}+2\right) F_A^{(4)}$$
$$\times \left[\frac{\varepsilon}{2}+2, 1, \_, \_, \_; 2, \varepsilon+1, \frac{1}{2}, \frac{3}{2}; x, y, z, t\right] \Bigg\}, \tag{72}$$

where x, y, z and t are given by (27).

Proof. Let $\mu = -\nu = \frac{1}{2}$ and (137), one can deduce (143) with the aid of Theorem 2. □

Corollary 19. *The following integral transform holds true*:

$$\int_0^\infty \rho \sin(\chi\rho)\, J_\varepsilon(\lambda\rho) \sinh(\gamma\rho^2) e^{-p\rho^2+2q\rho} d\rho$$

$$= \frac{\left(\frac{\lambda}{2}\right)^\varepsilon \chi}{(p+\gamma)^{\frac{\varepsilon+3}{2}} \varepsilon!} \gamma \left\{ \Gamma\left(\frac{\varepsilon}{2}+2\right) F_A^{(4)} \left[\frac{\varepsilon}{2}+2, 1, \_, \_; 2, \varepsilon \right.\right.$$

$$\left. +1, \frac{3}{2}, \frac{1}{2}; x, y, z, t\right] + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{\varepsilon+5}{2}\right) F_A^{(4)}$$

$$\left. \times \left[\frac{\varepsilon+5}{2}, 1, \_, \_; 2, \varepsilon+1, \frac{3}{2}, \frac{3}{2}; x, y, z, t\right] \right\}, \tag{73}$$

where x, y, and z are given by (27).

Proof. With $\mu = \nu = \frac{1}{2}$, $s = 0$ and $\xi = \frac{1}{2}$, (146) can be proved easily by using Theorem 2. □

3.8. *Case of $s = a + \frac{1}{4}$ and $\xi = -\frac{1}{4}$*

In this case, the Whittaker function is given by (see Ref. 12)

$$M_{a+\frac{1}{4}, -\frac{1}{4}}\left(z^2\right) = \frac{S}{(2\gamma)^{1/4}} \exp\left(-\frac{z^2}{2}\right) \sqrt{z} H_{2a}(z), \tag{74}$$

where

$$S = (-1)^a \frac{a!}{(2a)!} (2\gamma)^{1/4}, \tag{75}$$

with $\gamma \neq 0$.

Corollary 20. *The following integral transform holds true*:

$$\int_0^\infty \rho^{\mu+\frac{1}{2}} J_\nu(\chi\rho)\, J_\varepsilon(\lambda\rho) H_{2a}(\sqrt{2\gamma}\rho) e^{-p\rho^2+2q\rho} d\rho$$

$$= L_{20} \frac{(2a)!}{(-1)^a a!} \Bigg\{ \Gamma(\alpha_L)\, F_A^{(4)} \left[\alpha_L, -a, \_, \_, \_; \frac{1}{2}, \varepsilon + 1, \nu + 1, \frac{1}{2}; x, y, z, t \right] + \frac{q}{(p+\gamma)} \Gamma(\beta_L)\, F_A^{(4)} \left[\beta_L, 1, \_, \_, \_; \frac{1}{2}, \varepsilon + 1, \nu + 1, \frac{3}{2}; x, y, z, t \right] \Bigg\}, \tag{76}$$

where

$$L_{20} = \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\mu+\nu+\varepsilon}{2}+\frac{1}{4}} \varepsilon!} \frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{77}$$

$$\alpha_L = \frac{\mu+\nu+\varepsilon}{2} + \frac{3}{4} \text{ and } \beta_L = \frac{\mu+\nu+\varepsilon}{2} + \frac{5}{4}. \tag{78}$$

x, y, z and t are given by (27).

Proof. Setting $s = a + \frac{1}{4}$ and $\xi = -\frac{1}{4}$ and using (74), it's easy to deduce (76) with the help of Theorem 2. □

Corollary 21. *The following integral transform holds true*:

$$\int_0^\infty \rho^{\frac{1}{2}} \cos(\chi\rho)\, J_\varepsilon(\lambda\rho) H_{2a}(\sqrt{2\gamma}\rho) e^{-p\rho^2+2q\rho} d\rho$$

$$= L_{21} \Bigg\{ \Gamma\left(\frac{\varepsilon}{2}+\frac{3}{4}\right) F_A^{(4)} \left[\frac{\varepsilon}{2}+\frac{3}{4}, -a, \_, \_, \_; \frac{1}{2}, \varepsilon+1, \frac{1}{2}, \frac{1}{2}; x, y, z, t \right] + \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{\varepsilon}{2}+\frac{5}{4}\right) F_A^{(4)} \left[\frac{\varepsilon}{2}+\frac{5}{4}, 1, \_, \_, \_; \frac{1}{2}, \varepsilon+1, \frac{1}{2}, \frac{3}{2}; x, y, z, t \right] \Bigg\}, \tag{79}$$

where

$$L_{21} = \frac{(2a)!}{(-1)^a a!} \frac{\left(\frac{\lambda}{2}\right)^{\varepsilon}}{(p+\gamma)^{\frac{\varepsilon}{2}+\frac{1}{4}} \varepsilon!}. \tag{80}$$

x, y, z and t are given by (27).

Proof. With $\mu = -\nu = \frac{1}{2}$ and under the conditions of the above corollary, one obtains (156). □

Corollary 22. *The following integral transform holds true*:

$$\int_0^\infty \rho^{\frac{1}{2}} \sin(\chi\rho)\, J_\varepsilon(\lambda\rho) H_{2a}(\sqrt{2\gamma}\rho) e^{-p\rho^2+2q\rho} d\rho$$
$$= L_{22}\Bigg\{\Gamma\left(\frac{\varepsilon}{2}+\frac{5}{4}\right) F_A^{(4)}\left[\frac{\varepsilon}{2}+\frac{5}{4}, -a, \_, \_, \_; \frac{1}{2}, \varepsilon + 1, \frac{3}{2}, \frac{1}{2}; x, y, z, t\right]$$
$$+ \frac{2q}{\sqrt{p+\gamma}} \Gamma\left(\frac{\varepsilon}{2}+\frac{7}{4}\right) F_A^{(4)}\left[\frac{\varepsilon}{2}+\frac{7}{4}, 1, \_, \_, \_; \frac{1}{2}, \varepsilon + 1, \frac{3}{2}, \frac{3}{2}; x, y, z, t\right]\Bigg\}, \tag{81}$$

where

$$L_{22} = \frac{(2a)!}{(-1)^a a!} \frac{\chi\left(\frac{\lambda}{2\sqrt{p+\gamma}}\right)^{\varepsilon}}{(p+\gamma)^{\frac{\varepsilon}{2}+\frac{3}{4}} \varepsilon!}. \tag{82}$$

x, y, z and t are given by (27).

Proof. If we take $\mu = \nu = \frac{1}{2}$, and under the conditions of Corollary 20, we arrive at (160). □

3.9. *Case of* $s = a + \frac{3}{4}$ *and* $\xi = \frac{1}{4}$

The Whittaker function is expressed as (see Ref. 12)

$$M_{a+\frac{3}{4},\frac{1}{4}}\left(z^2\right) = \frac{S'}{(2\gamma)^{1/4}} \exp\left(-\frac{z^2}{2}\right) \sqrt{z} H_{2a+1}(z), \tag{83}$$

where

$$S' = \frac{(-1)^a}{2} \frac{a!(2\gamma)^{1/4}}{(2a+1)!}, \tag{84}$$

with $\gamma \neq 0$.

Corollary 23. *The following integral transform holds true*:

$$\begin{aligned}
&\int_0^\infty \rho^{\mu+\frac{1}{2}} J_\nu(\chi\rho)\, J_\varepsilon(\lambda\rho) H_{2a+1}(\sqrt{2\gamma}\rho) e^{-p\rho^2+2q\rho} d\rho \\
&= L_{23} \frac{2\,(a+1)!}{(-1)^a a!} \sqrt{2\gamma} \Bigg\{ \Gamma\left(\alpha_L\right) F_A^{(4)} \Bigg[\alpha_L, -a, _, _, _; \frac{3}{2}, \varepsilon \\
&\quad +1, \nu+1, \frac{1}{2}; x, y, z, t \Bigg] \\
&\quad + \frac{q}{(p+\gamma)} \Gamma\left(\beta_L\right) F_A^{(4)} \Bigg[\beta_L, -a, _, _, _; \frac{3}{2}, \varepsilon+1, \nu \\
&\quad +1, \frac{3}{2}; x, y, z, t \Bigg] \Bigg\},
\end{aligned} \tag{85}$$

where

$$L_{23} = \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\mu+\nu+\varepsilon}{2}+\frac{3}{4}} \varepsilon!} \frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{86}$$

$$\alpha_L = \frac{\mu+\nu+\varepsilon}{2} + \frac{5}{4} \text{ and } \beta_L = \frac{\mu+\nu+\varepsilon}{2} + \frac{7}{4}. \tag{87}$$

x, y, z and t are given by (27).

Proof. This transformation is obtained if one uses s and ξ of the present case and if we apply Theorem 2. □

Corollary 24. *The following integral transform holds true*:

$$\int_0^\infty \rho^{\frac{1}{2}} \cos(\chi\rho)\, J_\varepsilon(\lambda\rho) H_{2a+1}(\sqrt{2\gamma}\rho) e^{-p\rho^2+2q\rho} d\rho$$

$$= L_{24}\Bigg\{\Gamma\left(\frac{\varepsilon}{2}+\frac{5}{4}\right) F_A^{(4)}\left[\frac{\varepsilon}{2}+\frac{5}{4}, -a, \_, \_, \_; \frac{3}{2}, \varepsilon + 1, \frac{1}{2}, \frac{1}{2}; x, y, z, t\right]$$

$$+ \frac{2q}{\sqrt{p+\gamma}}\Gamma\left(\frac{\varepsilon}{2}+\frac{7}{4}\right) F_A^{(4)}\left[\frac{\varepsilon}{2}+\frac{7}{4}, -a, \_, \_, \_; \frac{3}{2}, \varepsilon + 1, \frac{1}{2}, \frac{3}{2}; x, y, z, t\right]\Bigg\}, \quad (88)$$

where

$$L_{24} = \frac{2\,(a+1)!}{(-1)^a a!} \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\varepsilon}{2}+\frac{3}{4}}\varepsilon!}\sqrt{2\gamma}. \quad (89)$$

x, y, z and t are given by (27).

Proof. Under the conditions $\mu = -\nu = \frac{1}{2}$ and the conditions of Corollary 23, we arrive at (171). □

Corollary 25. *The following integral transform holds true*:

$$\int_0^\infty \rho^{\frac{1}{2}} \sin(\chi\rho)\, J_\varepsilon(\lambda\rho) H_{2a+1}(\sqrt{2\gamma}\rho) e^{-p\rho^2+2q\rho} d\rho$$

$$= L_{25}\Bigg\{\Gamma\left(\frac{\varepsilon}{2}+\frac{7}{4}\right) F_A^{(4)}\left[\frac{\varepsilon}{2}+\frac{7}{4}, -a, \_, \_, \_; \frac{3}{2}, \varepsilon + 1, \frac{3}{2}, \frac{1}{2}; x, y, z, t\right]$$

$$+ \frac{2q}{\sqrt{p+\gamma}}\Gamma\left(\frac{\varepsilon}{2}+\frac{9}{4}\right) F_A^{(4)}\left[\frac{\varepsilon}{2}+\frac{9}{4}, -a, \_, \_, \_; \frac{3}{2}, \varepsilon + 1, \frac{3}{2}, \frac{3}{2}; x, y, z, t\right]\Bigg\}, \quad (90)$$

where

$$L_{25} = \frac{2\,(a+1)!}{(-1)^a a!}\frac{\left(\frac{\lambda}{2}\right)^{\varepsilon}\chi}{(p+\gamma)^{\frac{\varepsilon}{2}+\frac{5}{4}}\varepsilon!}\sqrt{2\gamma}. \tag{91}$$

x, y, z and t are given by (27).

Proof. Let the condition $\mu = \nu = \frac{1}{2}$, and under the conditions of Corollary 23, one obtains (175). □

3.10. *Case of $s = r - \frac{1}{2}$ and $\xi = r$*

The Whittaker function becomes (see Ref. 12)

$$M_{r-\frac{1}{2},r}(z) = 2r\exp\left(\frac{z}{2}\right)z^{\frac{1}{2}-r}\gamma\,(2r,z), \tag{92}$$

with $\gamma\,(a,x)$ as the incomplete gamma function.

Corollary 26. *The following integral transform holds true*:

$$\begin{aligned}
&\int_0^{\infty}\rho^{-2r+1}cos(\chi\rho)J_{\varepsilon}(\lambda\rho)\,\gamma(2r,2\gamma'\rho^2)e^{-(p-\gamma')\rho^2+2q\rho}d\rho\\
&= L_{26}\left\{\Gamma\left(\frac{\varepsilon}{2}+r+1\right)F_A^{(4)}\left[\frac{\varepsilon}{2}+r+1,1,_,_,_;2r+1,\varepsilon\right.\right.\\
&\quad\left.+1,\frac{1}{2},\frac{1}{2};x,y,z,t\right]\\
&\quad+\frac{2q}{\sqrt{p+\gamma'}}\Gamma\left(\frac{\varepsilon}{2}r+\frac{3}{2}\right)F_A^{(4)}\left[\frac{\varepsilon}{2}+r+\frac{3}{2},1,_,_,_;2r+1,\varepsilon\right.\\
&\quad\left.\left.+1,\frac{1}{2},\frac{3}{2};x,y,z,t\right]\right\},
\end{aligned} \tag{93}$$

where

$$L_{26} = \frac{\left(\frac{\lambda}{2}\right)^{\varepsilon}}{(p+\gamma)^{\frac{\varepsilon+1}{2}+r}\varepsilon!}. \tag{94}$$

x, y, z and t are given by (27).

Proof. With the help of $\mu = -\nu = \frac{1}{2}$ and (92), and applying Theorem 2, one obtains (180). □

Corollary 27. *The following integral transform holds true*:

$$\int_0^\infty \rho^{-2r+1}\sin(\chi\rho)J_\varepsilon(\lambda\rho)\,\gamma(2r,2\gamma'\rho^2)e^{-(p-\gamma')\rho^2+2q\rho}d\rho$$

$$= L_{27}\left\{\Gamma\left(\frac{\varepsilon}{2}+r+\frac{3}{2}\right)F_A^{(4)}\left[\frac{\varepsilon}{2}+r+\frac{3}{2},1,\_,\_,\_;2r+1,\varepsilon\right.\right.$$

$$\left.+1,\frac{3}{2},\frac{1}{2};x,y,z,t\right]$$

$$+\frac{2q}{\sqrt{p+\gamma'}}\Gamma\left(\frac{\varepsilon}{2}+r+2\right)F_A^{(4)}\left[\frac{\varepsilon}{2}+r+2,1,\_,\_,\_;2r+1,\varepsilon\right.$$

$$\left.\left.+1,\frac{3}{2},\frac{3}{2};x,y,z,t\right]\right\}, \tag{95}$$

where

$$L_{27} = \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{\varepsilon!(p+\gamma)^{\frac{\varepsilon}{2}+r+1}}. \tag{96}$$

x, y, z and t are given by (27).

Proof. By the use of $\mu = \nu = \frac{1}{2}$, and (92), and with the help of Theorem 2, it is easy to deduce (184). □

3.11. *Case of $s = -\frac{1}{4}$ and $\xi = \frac{1}{4}$*

In this case, the Whittaker function can be expressed in terms of the error function erf as (see Ref. 12)

$$M_{-\frac{1}{4},\frac{1}{4}}\left(z^2\right) = \frac{\exp\left(z^2/2\right)}{2}\sqrt{\pi z}erf(z). \tag{97}$$

Corollary 28. *The following integral transforms holds true*:

$$\int_0^\infty \rho^{\mu-2r+1} J_\nu(\chi\rho) J_\varepsilon(\lambda\rho)\, erf(\sqrt{2\gamma}\rho) e^{-(p-\gamma)\rho^2+2q\rho} d\rho$$

$$= 2\sqrt{\frac{2\gamma}{\pi}} L_{28} \Bigg\{ \Gamma(\alpha_L) F_A^{(4)} \left[\alpha_L, \frac{1}{2}, \_, \_, \_; \frac{3}{2}, \varepsilon+1, \nu \right.$$

$$\left. +1, \frac{1}{2}; x, y, z, t \right]$$

$$+ \frac{q}{(p+\gamma')} \Gamma(\beta_L) F_A^{(4)} \left[\beta_L, \frac{1}{2}, \_, \_, \_; \frac{3}{2}, \varepsilon+1, \nu \right.$$

$$\left. +1, \frac{3}{2}; x, y, z, t \right] \Bigg\}, \tag{98}$$

where

$$L_{28} = \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\mu+\nu+\varepsilon}{2}+\frac{3}{4}} \varepsilon!} \frac{\left(\frac{\chi}{2}\right)^\nu}{\nu!}, \tag{99}$$

$$\alpha_L = \frac{\mu+\nu+\varepsilon}{2} + \frac{5}{4} \ \textit{and} \ \ \beta_L = \frac{\mu+\nu+\varepsilon}{2} + \frac{7}{4}. \tag{100}$$

x, y, z and t are given by (27).

Proof. With $s = -\frac{1}{4}$, $\xi = \frac{1}{4}$ and the use of (97), Theorem 2 yields (188). □

Corollary 29. *The following integral transform holds true*:

$$\int_0^\infty \rho^{-2r+1} cos(\chi\rho) J_\varepsilon(\lambda\rho)\, erf(\sqrt{2\gamma}\rho) e^{-(p-\gamma)\rho^2+2q\rho} d\rho$$

$$= 2\sqrt{\frac{2\gamma}{\pi}} \frac{\left(\frac{\lambda}{2}\right)^\varepsilon}{(p+\gamma)^{\frac{\varepsilon}{2}+\frac{3}{4}} \varepsilon!} \Bigg\{ \Gamma\left(\frac{\varepsilon}{2}+\frac{5}{4}\right) F_A^{(4)} \left[\frac{\varepsilon}{2}+\frac{5}{4}, \frac{1}{2}, \_, \_, \_; \frac{3}{2}, \varepsilon \right.$$

$$\left. +1, \frac{1}{2}, \frac{1}{2}; x, y, z, t \right]$$

$$+\frac{2q}{\sqrt{p+\gamma'}}\Gamma\left(\frac{\varepsilon}{2}+\frac{7}{4}\right)F_A^{(4)}\left[\frac{\varepsilon}{2}+\frac{7}{4},\frac{1}{2},-,-,-;\frac{3}{2},\varepsilon\right.$$
$$\left.+1,\frac{1}{2},\frac{3}{2};x,y,z,t\right]\Bigg\}, \tag{101}$$

where x, y, z and t are given by (27).

Proof. Taking the conditions $\mu=-\nu=\frac{1}{2}$ and $s=-\xi=-\frac{1}{4}$, and applying (97) and Theorem 2, one arrives at (194). □

Corollary 30. *The following integral transform holds true*:

$$\int_0^\infty \rho^{-2r+1}\sin(\chi\rho)J_\varepsilon(\lambda\rho)\,erf(\sqrt{2\gamma}\rho)e^{-(p-\gamma)\rho^2+2q\rho}d\rho$$
$$=2\sqrt{\frac{2\gamma}{\pi}}\frac{\left(\frac{\lambda}{2}\right)^\varepsilon\chi}{\varepsilon!(p+\gamma)^{\frac{\varepsilon}{2}+\frac{5}{4}}}\Bigg\{\Gamma\left(\frac{\varepsilon}{2}+\frac{7}{4}\right)F_A^{(4)}\left[\frac{\varepsilon}{2}+\frac{7}{4},\frac{1}{2},-,-,-;\frac{3}{2},\varepsilon\right.$$
$$\left.+1,\frac{3}{2},\frac{1}{2};x,y,z,t\right]$$
$$+\frac{2q}{\sqrt{p+\gamma'}}\Gamma\left(\frac{\varepsilon}{2}+\frac{9}{4}\right)F_A^{(4)}\left[\frac{\varepsilon}{2}+\frac{9}{4},\frac{1}{2},-,-,-;\frac{3}{2},\varepsilon\right.$$
$$\left.+1,\frac{3}{2},\frac{3}{2};x,y,z,t\right]\Bigg\}, \tag{102}$$

where x, y, z and t are given by (27).

Proof. As the above corollary, we arrive at (197) if we take $\mu=\nu=\frac{1}{2}$. □

4. Numerical Simulations

In this section, we compare our theoretical results of (17) and (32) given by its second number and its first number, which is the integral evaluated by using the Laguerre–Gauss quadrature. This integral,

Table 1. Abscissas and weight factors for Laguerre integration.

i	x_i	$w_i e^{x_i}$
1	0.093307812017	0.239578170311
2	0.492691740302	0.560100842793
3	1.215595412071	0.887008262919
4	2.269949526204	1.22366440215
5	3.667622721751	1.57444872163
6	5.425336627414	1.94475197653
7	7.565916226613	2.34150205664
8	10.120228568019	2.77404192683
9	13.130282482176	3.25564334640
10	16.654407708330	3.80631171423
11	20.776478899449	4.45847775384
12	25.623894226729	5.27001778443
13	31.407519169754	6.35956346973
14	38.530683306486	8.03178763212
15	48.026085572686	11.5277721009

denoted by R, can be expressed as

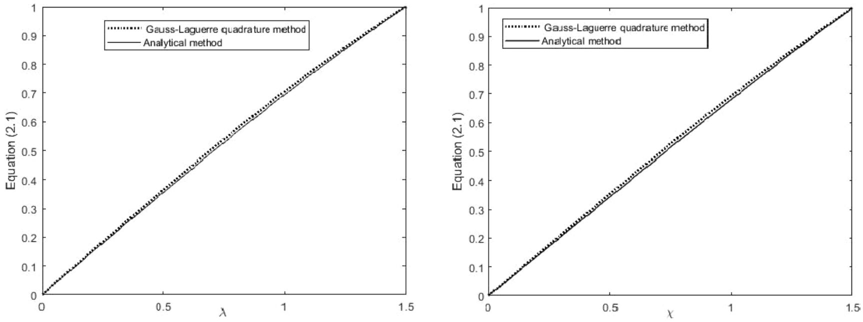
$$R = \sum_{i=1}^{15} \omega_i e^{x_i} g(x_i), \tag{103}$$

where x_i and $\omega_i e^{x_i}$ are given in Table 1 (see Ref. 13) and $g(x_i) = {x_i}^{\mu} J_\nu(\chi x_i) J_\varepsilon(\lambda x_i) e^{-p{x_i}^2+2qx_i}$, which relates to (17) and $g(x_i) = {x_i}^{\mu} J_\nu(\chi x_i) J_\varepsilon(\lambda x_i) M_{s,\zeta}\left(2\gamma\rho^2\right) e^{-p{x_i}^2+2qx_i}$, which corresponds to (32).

To compare our numerical simulations and the second member of (17), we illustrate in Figure 1 both the expressions with the following calculation parameters: $\nu = 1, \mu = 1, \epsilon = 1$, where the condition of Theorem 1 is verified. One can deduce from this figure that there is an excellent agreement between the two considered evaluations.

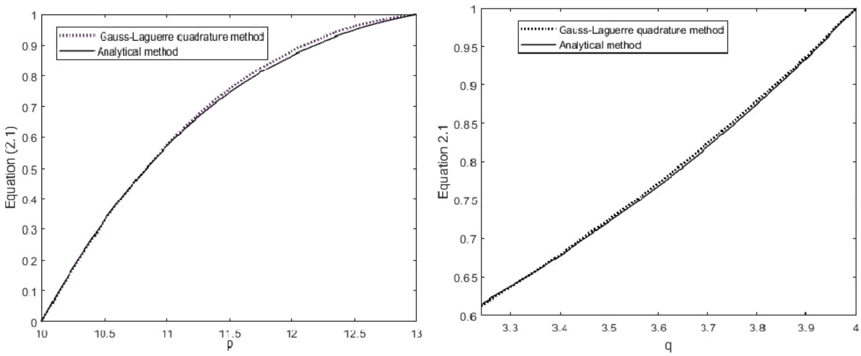
Also, we show in Figure 2, a comparison between our theoretical and numerical results with the following set of parameters: $\nu = 1, \mu = 1, \epsilon = 1, \chi = 2$ and $\lambda = 2$. It is clear from this illustration that there is an excellent agreement between (103) and the second member of (17).

To show the equivalence between our numerical simulations and the second member of (36), we display the two formulas in Figure 1 with the following calculation parameters: $\nu = 1, \mu = 1, \zeta = 1,$



(a) (b)

Fig. 1. Illustration of (17) as a function of: (a) λ with $\chi = 2, p = 14$ and $q = 2$ and (b) χ with $\lambda = 6, p = 14$ and $q = 5$.

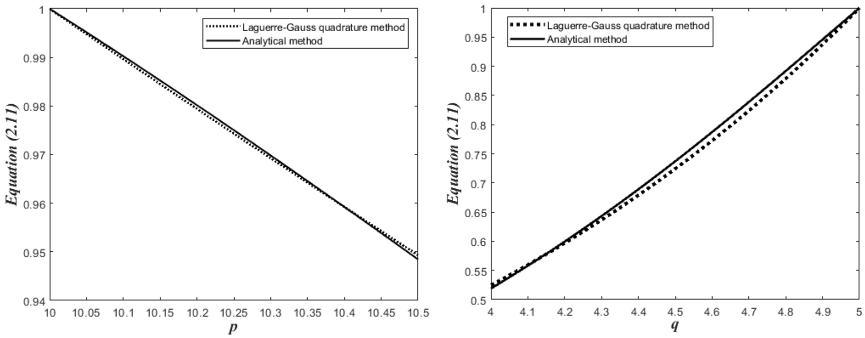


(a) (b)

Fig. 2. Illustration of (17) as a function of: (a) p with $q = 3, \chi = 2$ and $\lambda = 2$ and (b) q with $p = 20, \chi = 2$ and $\lambda = 8$.

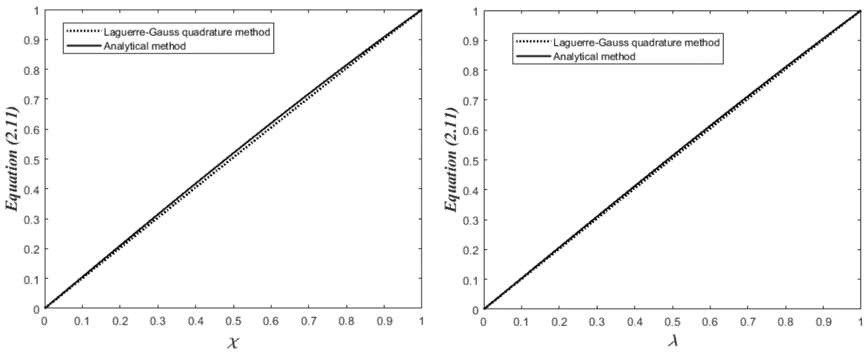
$s = 1, \epsilon = 1$, and $\gamma = 1$, where the condition of Theorem 2 is verified. The obtained results show good compatibility of the numerical solution obtained using the Laguerre–Gauss quadrature and the closed-form solution.

Moreover, we illustrate a comparison of our theoretical and numerical results in Figure 3 and the other calculation parameters are the same as those taken in Figure 4. From these figures, we clearly observe that our result obtained theoretically is identical to that obtained numerically.



(a) (b)

Fig. 3. Illustration of (36) as a function of: (a) p with $q = 1, \chi = 2$ and $\lambda = 3$ and (b) q with $p = 14, \chi = 3$ and $\lambda = 3$.



(a) (b)

Fig. 4. Illustration of (36) as a function of: (a) χ with $\lambda = 6$, $p = 12$ and $q = 7$ and (b) λ with $\chi = 8$, $p = 20$ and $q = 1$.

5. Conclusion

In this investigation, we have derived two integral transforms in terms of the Lauricella function of three and four variables, involving the product of two Bessel and Whittaker functions. Some corollaries are derived from the main theorems as particular cases. To compare our theoretical and numerical results, some numerical simulations have been done. The obtained results show that there is an excellent agreement between the numerical solution obtained using the Laguerre–Gauss quadrature and our theoretical results. These results can be used in the Fresnel diffraction by a helical axicon of many laser fields, such as Laguerre–Bessel–Gaussian beams.

6. Appendix

We use the following lemma in this study.

Lemma 1. *For* $\Re(p) > 0$, $\Re(\mu) > -1$, *and* $|x|^2 + |y|^2 < 1$, *the following integral transform holds true*:

$$\int_0^\infty \rho^\mu J_\nu(\chi\rho) e^{-p\rho^2+2q\rho} d\rho$$
$$= K_0 \left\{ \Gamma(\alpha)\, \Psi_2 \left[\alpha; \nu+1, \frac{1}{2}; x, y \right] + \frac{2q}{\sqrt{p}} \Gamma(\beta) \times \Psi_2 \left[\beta; \nu+1, \frac{3}{2}; x, y \right] \right\},$$

where

$$K_0 = \frac{1}{2p^{\frac{\mu+1}{2}}} \frac{\left(\frac{\chi}{2\sqrt{p}}\right)^\nu}{\nu!}, \tag{104}$$

$$x = -\frac{\chi^2}{4p}, \quad y = \frac{q^2}{p} \tag{105}$$

$$\alpha = \frac{\mu+\nu+1}{2} \textit{ and } \beta = \frac{\mu+\nu}{2} + 1. \tag{106}$$

Proof. By denoting the left-hand side of (104) by K and by using the identity $e^x = chx + shx$ and the series representation of the Bessel function J_ν of the first kind of order ν given by (4), we get

$$K = (\chi/2)^\nu \sum_{k=0}^\infty \frac{(-\chi^2/4)^k}{k!\Gamma(\nu+k+1)} K_k, \tag{107}$$

where

$$K_k = \int_0^\infty \rho^{\mu+\nu+2k} e^{-p\rho^2} ch(2q\rho) d\rho + \int_0^\infty \rho^{\mu+\nu+2k} e^{-p\rho^2} sh(2q\rho) d\rho. \tag{108}$$

Making use of the following identities (Ref. 12)

$$\int_0^\infty x^{2\alpha-1}e^{-\beta x^2}sh(\gamma x)dx$$
$$= \frac{\Gamma(2\alpha)}{2(2\beta)^\alpha}e^{\gamma^2/8\beta}\left[D_{-2\alpha}\left(-\frac{\gamma}{\sqrt{2\beta}}\right) - D_{-2\alpha}\left(\frac{\gamma}{\sqrt{2\beta}}\right)\right], \qquad (109)$$

with $\Re(\alpha) > -\frac{1}{2},\ \Re(\beta) > 0$,

and

$$\int_0^\infty x^{2\alpha-1}e^{-\beta x^2}ch(\gamma x)dx$$
$$= \frac{\Gamma(2\alpha)}{2(2\beta)^\alpha}e^{\gamma^2/8\beta}\left[D_{-2\alpha}\left(-\frac{\gamma}{\sqrt{2\beta}}\right) + D_{-2\alpha}\left(\frac{\gamma}{\sqrt{2\beta}}\right)\right], \qquad (110)$$

with $\Re(\alpha) > 0,\ \Re(\beta) > 0$,

(107) becomes

$$K = \frac{(\chi/2)^\nu}{(2p)^{\frac{\mu+\nu+1}{2}}}e^{\frac{q^2}{2p}}\sum_{k=0}^{\infty}\frac{\left(-\chi^2/8p\right)^k}{k!}\frac{\Gamma(\mu+\nu+1+2k)}{\Gamma(\nu+1+k)}$$
$$\times D_{-(\mu+\nu+1+2k)}\left(-\sqrt{\frac{2}{p}}q\right). \qquad (111)$$

In these last equations, $D_{-2\delta}$ is the parabolic cylinder function given in terms of the Kummer's function as

$$D_{-2\delta}(z) = \frac{\sqrt{\pi}}{2^\delta}e^{-\frac{z^2}{4}}\left[\frac{1}{\Gamma\left(\frac{1}{2}+\delta\right)}\,{}_1F_1\left(\delta;\frac{1}{2};\frac{z^2}{2}\right)\right.$$
$$\left. -\frac{\sqrt{2}z}{\Gamma(\delta)}\,{}_1F_1\left(\frac{1}{2}+\delta;\frac{3}{2};\frac{z^2}{2}\right)\right]. \qquad (112)$$

The use of this last identity yields the following expression:

$$K = \frac{\sqrt{\pi}}{(4p)^{\frac{\mu+1}{2}}}\left(\chi/_{4\sqrt{p}}\right)^{\nu}\left\{\sum_{k=0}^{\infty}\frac{\left(-\chi^2/_{16p}\right)^k}{k!\Gamma(\nu+1+k)}\varepsilon_{1\,1}^{k}\right.$$
$$\times F_1\left(\frac{\mu+\nu+1}{2}+k;\frac{1}{2};\frac{q^2}{2}\right)$$
$$\left.+\frac{2q}{\sqrt{p}}\sum_{k=0}^{\infty}\frac{\left(-\chi^2/_{16p}\right)^k}{k!\Gamma(\nu+1+k)}\varepsilon_{2\,1}^{k}F_1\left(\frac{\mu+\nu}{2}+1+k;\frac{3}{2};\frac{q^2}{2}\right)\right\}, \tag{113}$$

where

$$\varepsilon_1^k = \frac{\Gamma(\mu+\nu+1+2k)}{\Gamma\left(\frac{\mu+\nu}{2}+1+k\right)} \tag{114}$$

and

$$\varepsilon_2^k = \frac{\Gamma(\mu+\nu+1+2k)}{\Gamma\left(\frac{\mu+\nu+1}{2}+k\right)}. \tag{115}$$

To evaluate the expression of ε_1^k and ε_2^k, we use the identities (see Ref. 11)

$$(\lambda+m)_n = \frac{(\lambda)_{m+n}}{(\lambda)_m} \tag{116}$$

and

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \quad z \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots \tag{117}$$

Now, in view of (114), (115), (116), and (117), K can be expressed in terms of the Humbert function of two variables ψ_2 and the required result (104) easily follows. This completes the proof. □

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