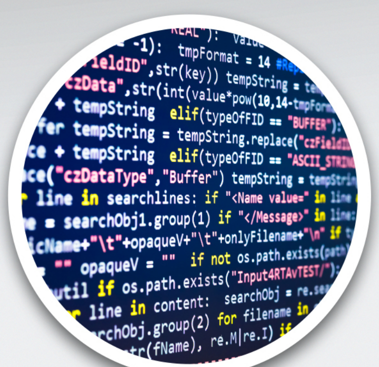
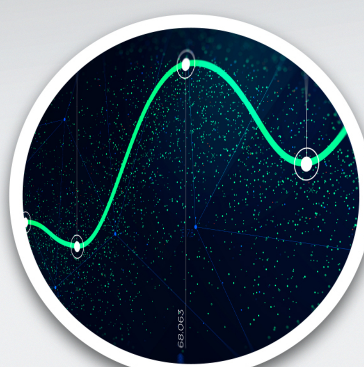
MATHEMATICS RESEARCH DEVELOPMENTS

CONTEMPORARY ALGORITHMS

THEORY AND APPLICATIONS

CHRISTOPHER I. ARGYROS
SAMUNDRA REGMI
IOANNIS K. ARGYROS, PHD
SANTHOSH GEORGE, PHD

VOLUME III

NOVA

Mathematics Research Developments

nova
science publishers

www.novapublishers.com

Mathematics Research Developments

Contemporary Algorithms: Theory and Applications. Volume II
Christopher I. Argyros, Samundra Regmi,
Ioannis K. Argyros and Santhosh George (Authors)
2022. ISBN: 979-8-88697-109-5 (Hardcover)
2022. ISBN: 979-8-88697-479-9 (eBook)

Contemporary Algorithms: Theory and Applications. Volume I
Christopher I. Argyros, Samundra Regmi,
Ioannis K. Argyros and Santhosh George (Authors)
2022. ISBN: 978-1-68507-994-9 (Hardcover)
2022. ISBN: 979-8-88697-425-6 (eBook)

Non-Euclidean Geometry in Materials of Living and Non-Living Matter in the Space of the Highest Dimension
Gennadiy Zhizhin (Author)
2022. ISBN: 978-1-68507-885-0 (Hardcover)
2022. ISBN: 979-8-88697-064-7 (eBook)

Frontiers in Mathematical Modelling Research
M. Haider Ali Biswas and M. Humayun Kabir (Editors)
2022. ISBN: 978-1-68507-430-2 (Hardcover)
2022. ISBN: 978-1-68507-845-4 (eBook)

Mathematical Modeling of the Learning Curve and Its Practical Applications
Charles Ira Abramson and Igor Stepanov (Authors)
2022. ISBN: 978-1-68507-737-2 (Hardcover)
2022. ISBN: 978-1-68507-851-5 (eBook)

Partial Differential Equations: Theory, Numerical Methods and Ill-Posed Problems
Michael V. Klibanov and Jingzhi Li (Authors)
2022. ISBN: 978-1-68507-592-7 (Hardcover)
2022. ISBN: 978-1-68507-727-3 (eBook)

More information about this series can be found at https://novapublishers.com/product-category/series/mathematics-research-developments/

Christopher I. Argyros
Samundra Regmi
Ioannis K. Argyros
and Santhosh George

Contemporary Algorithms

Theory and Applications

Volume III

nova
science publishers
www.novapublishers.com

https://doi.org/10.52305/BYUE0534

Library of Congress Cataloging-in-Publication Data
Names: Argyros, Christopher I., editor. | Regmi, Samundra, editor. | Argyros, Ioannis K., editor. | George, Santhosh, editor.
Title: Contemporary algorithms: theory and applications / Christopher I. Argyros, Samundra Regmi, Ioannis K. Argyros, and Santhosh George, editors.
Identifiers: LCCN 2022051224 (print) | LCCN 2022051225 (ebook) | ISBN 9781685079949 (volume 1; hardcover) | ISBN 9798886971095 (volume 2; hardcover)| ISBN 9798886979060 (volume 3; hardcover)| ISBN 9798886974256 (volume 1; adobe pdf) | ISBN 9798886974614 (volume 2; adobe pdf) | ISBN 9798886979855 (volume 3; hardcover)
Subjects: LCSH: Computer algorithms.
Classification: LCC QA76.9.A43 A73 2023 (print) | LCC QA76.9.A43 (ebook) DDC 005.13--dc23/eng/20221202
LC record available at https://lccn.loc.gov/2022051224
LC ebook record available at https://lccn.loc.gov/2022051225

Published by Nova Science Publishers, Inc. † New York

The first author dedicates this book to his Beloved grandparents
Jolanda, Mihallaq, Anastasia and Konstantinos.

The second author dedicates this book to his mother Madhu Kumari Regmi
and Father Moti Ram Regmi.

The third author dedicates this book to his Wonderful children
Christopher, Gus, Michael, and lovely wife Diana.

The fourth author dedicates this book to Prof. M. T. Nair
and Prof. V. Krishnakumar.

Contents

Preface **xiii**

1 Local Convergence for a Two-Step Third Order Secant-Like Method without Bilinear Operators **1**

1. Introduction 1
2. Convergence 1

References 4

2 Semi-Local Convergence for a Derivative Free Method with Error Controlled Controlled Iterates for Solving Nonlinear Equations **7**

1. Introduction 7
2. Convergence of Majorizing Sequences 8
3. Convergence of Method (2.2) 12

References 15

3 On the Semi-Local Convergence of a Sharma-Gupta Fifth Order Method for Solving Nonlinear Equations **19**

1. Introduction 19
2. Convergence 20

References 24

4 On the Semi-Local Convergence of a Seventh Order Method for Nonlinear Equations Convergence for a Seventh Order Method for Equations **29**

1. Introduction 29
2. Semi-Local Convergence 30

References 34

5 On the Semi-Local Convergence of a Fifth Order Method for Solving Nonlinear Equations **39**

1. Introduction 39
2. Sequences Majorizing Method (5.2) 40
3. Analysis for Method (5.2) 47

References 50

6 On the Semi-Local Convergence of a Fifth Order Efficient Method for Solving Nonlinear Equations **55**
1. Introduction 55
2. Semi-Local Analysis 56
References 60

7 Improved Convergence of Derivative-Free Halley's Scheme with Two Parameters **63**
1. Introduction 63
2. Semi-Local Convergence I 65
3. Semi-Local Convergence Part 2 68
References 74

8 Extended Convergence for a Third Order Traub-Like Method with Parameter for Solving Equations **77**
1. Introduction 77
2. Majorizing Sequence 78
3. Convergence of Method (8.3) 82
4. Numerical Experiments 85
References 86

9 Ball Convergence Comparison for Methods Whose First Step is Given by Newton's for Solving Nonlinear Equations **89**
1. Introduction 89
2. Ball Convergence 90
References 98

10 Extending the Convergence of a Steffensen-Like Method without Derivatives for Solving Nonlinear Equations **101**
1. Introduction 101
2. Majorizing Sequences 102
References 105

11 On the Semi-Local Convergence of a Fourth Order Derivative Free Two-Step Steffensen Method for Solving Equations **109**
1. Introduction 109
2. Majorizing Sequences 110
3. Convergence for Method (11.2) 113
References 115

12 A Semi-Local Convergence for a Class of Fourth Order Method for Solving Equations **119**
1. Introduction 119
2. Semi-Local Convergence 120
References 123

13 Local Convergence Comparison between Two Competing Fifth Order Iterations 127
1. Introduction . . . 127
2. Real Majorizing Function . . . 128
3. Local Analysis . . . 130
References . . . 133

14 Semi-Local Convergence Analysis of High Convergence Order Iterations under the Same Set of Conditions 137
1. Introduction . . . 137
2. Semi-Local Analysis . . . 139
3. Numerical Example . . . 151
References . . . 151

15 A Collection of Iterative Methods with Order Five, Six and Seven and Their Semi-Local Convergence 155
1. Introduction . . . 155
2. Convergence Criteria . . . 161
References . . . 162

16 Extended Semi-Local Convergence Analysis for the Two-Step Newton Method under Generalized Lipschitz Conditions 165
1. Introduction . . . 165
2. Semi-Local Convergence . . . 166
References . . . 170

17 Semi-Local Convergence for Jarratt-Like Methods under Generalized Conditions for Solving Equations 173
1. Introduction . . . 173
2. SL of Method (17.3) and Method (17.2) . . . 175
References . . . 180

18 On the Semi-Local Convergence of a Derivative Free Fourth Order Method for Solving Equations 183
1. Introduction . . . 183
2. Majorizing Sequence . . . 184
3. Semi-Local Convergence . . . 184
References . . . 187

19 Extended and Unified Kantorovich Theory for Solving Generalized Nonlinear Equations 191
1. Introduction . . . 191
2. Mathematical Background . . . 192
3. Majorizing Sequence . . . 193
4. Main Result . . . 194
References . . . 197

20 A Derivative Free Two-Step Fourth Order Method for Solving Equations in Banach Space 201
1. Introduction 201
2. Local Convergence Analysis 201
References 205

21 Local Convergence for an Efficient Derivative Free Fourth Order Method for Solving Equations in Banach Space 209
1. Introduction 209
2. Convergence 209
References 212

22 Extended Local Convergence of Steffensen-Type Methods for Solving Nonlinear Equations in Banach Space 215
1. Introduction 215
2. Local Analysis of STM 216
References 219

23 Extended Convergence Analysis of Optimal Eighth Order Methods for Solving Nonlinear Equations in Banach Space 223
1. Introduction 223
2. Local Convergence 223
3. Special Cases 226
References 230

24 Efficient Fifth Convergence Order Methods for Solving Equations in Banach Space 233
1. Introduction 233
2. Local Convergence 234
3. Semi-Local Convergence 238
References 241

25 Efficient Derivative Free Seventh Order Methods for Solving Equations in Banach Space 245
1. Introduction 245
2. Convergence for Method (25.4) 246
3. Special Cases 250
4. Convergence of Method (25.7) 252
References 253

26 Necessary and Sufficient Conditions for the Q-Order of Convergence of Iterative Methods 257
1. Introduction 257
2. Local Convergence 257
References 261

27 Necessary and Sufficient Conditions for the Convergence of Sixth Order or Higher Derivative Free Methods 265
1. Introduction 265
2. Local Convergence 265
References 269

28 The Convergence Analysis of Some Fourth and Fifth Order Methods 273
1. Introduction 273
2. Local Convergence 274
3. Semi-Local Convergence 277
References 280

29 High Convergence Order Derivative Free Methods-I 283
1. Introduction 283
2. Local Convergence Analysis 284
3. Semi-Local Convergence 288
References 293

30 High Convergence Order Derivative Free Methods-II 297
1. Introduction 297
2. Local Convergence 297
3. Semi-Local Convergence 300
References 303

31 High Convergence Order Derivative Free Methods-III 307
1. Introduction 307
2. Local Convergence 308
3. Semi-Local Convergence 311
References 313

32 Fourth Convergence Order Derivative Free Methods with or without Memory 317
1. Introduction 317
2. Local Convergence 318
3. Semi-Local Convergence 319
References 321

33 Convergence Radius of an Efficient Iterative Method with Frozen Derivatives 325
1. Introduction 325
2. Convergence for Method (33.2) 326
3. Numerical Examples 330
References 331

34 A Comparison between Some Derivative Free Methods 335
1. Introduction 335
2. Local Convergence 336
References 343

35 High Convergence Order Methods **347**
1. Introduction . . . 347
2. Semi-Local Convergence . . . 347
References . . . 368

36 On a Family of Optimal Eighth Order Methods for Equations in Banach Space **371**
1. Introduction . . . 371
2. Local Analysis . . . 371
References . . . 375

37 Extended Ball Convergence of a Xiao-Yin Fifth Order Scheme for Equations **379**
1. Introduction . . . 379
2. Ball Convergence . . . 380
3. Numerical Experiments . . . 383
References . . . 384

38 On the Semi-Local Convergence of King's Fourth Order Method for Solving Equations **387**
1. Introduction . . . 387
2. Scalar Majorizing Sequences . . . 388
3. Analysis . . . 392
4. Convergence Result for Method 38.3 . . . 393
5. Applications . . . 396
References . . . 396

39 On the Convergence of Two Eighth Order Methods with Divided Differences and Derivatives **401**
1. Introduction . . . 401
2. Local Convergence . . . 402
3. Semi-Local Convergence . . . 405
References . . . 408

40 Bilinear Operator Free Method for Solving Nonlinear Equations **411**
1. Introduction . . . 411
2. History of the Procedure . . . 412
3. Convergence Analysis . . . 412
4. Numerical Experiments . . . 418
References . . . 419

Index **423**

About the Authors **425**

Preface

The book provides different avenues to study algorithms. It also brings new techniques and methodologies to problem solving in computational Sciences, Engineering, Scientific Computing and Medicine (imaging, radiation therapy) to mention a few.

A plethora of algorithms which are universally applicable is presented on a sound analytical way.

The chapters are written independently of each other, so they can be understood without reading earlier Chapters. But some knowledge of Analysis, Linear Algebra and some Computing experience is required.

The organization and content of the book cater to senior undergraduate, graduate students, researchers, practitioners, professionals and academicians in the aforementioned disciplines. It can also be used as a reference book and includes numerous references and open problems.

In order to avoid repetitions whenever the "ω", "φ", "ψ", "γ", "δ", "h" symboles are used as functions connected to the local or semi-local convergence analysis of iterative algorithms, then it is assume to be continuous, nondecreasing and defined on a domain with nonnegative valued and with range in the real number system. Moreover, they are related to the operator appearing on the equation to be solved, its derivative or its divided difference of order one as follows for x^* denoting a simple solution of equation $F(x) = 0$:

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq \omega_0(\|x - x^*\|)$$

for all $x \in \Omega$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \omega(\|x - y\|)$$

for all $x, y \in \Omega_0 \subset \Omega$,

$$\|F'(x^*)^{-1}F'(x)\| \leq \omega_1(\|x - x^*\|)$$

for all $x \in \Omega_0$,

$$\|I - a[x, x^*; F]\| \leq \gamma_0(\|x - x^*\|),$$

$$\|I + b[x, x^*; F]\| \leq \gamma(\|x - x^*\|),$$

$$\|I + c[x, x^*; F]\| \leq \gamma_1(\|x - x^*\|),$$

$$\|I + d[x, x^*; F]\| \leq \gamma_2(\|x - x^*\|),$$

$$\|I + L[x, x^*; F]\| \leq \gamma_3(\|x - x^*\|),$$

where $a,b,c,d \in \mathbb{R}$, L a linear operator, and x^* can be also replaced by x_* in the preceding conditions involving the "γ" functions, and $x \in \Omega$ or $x \in \Omega_0$.

$$\|F'(x^*)^{-1}[z,x;F]\| \leq \varphi_0(\|z-x^*\|,\|x-x^*\|),$$

for all $x,z \in \Omega$,

$$\|F'(x^*)^{-1}([z,x;F]-[x,x^*;F])\| \leq \varphi(\|z-x^*\|,\|x-x^*\|),$$

$$\|F'(x^*)^{-1}([z,x;F]-[y,x;F])\| \leq \varphi_1(\|z-x^*\|,\|x-x^*\|),$$

$$\|F'(x^*)^{-1}([z,x;F]-[y,z;F])\| \leq \varphi_2(\|z-x^*\|,\|x-x^*\|,\|y-x^*\|),$$

or

$$\|F'(x^*)^{-1}([z,x;F]-[y,v;F])\| \leq \varphi(\|z-x^*\|,\|y-x^*\|,\|x-x^*\|,\|v-x*\|),$$

for all $x,y,z,v \in \Omega_0$,

$$\|F'(x^*)^{-1}[x,x^*;F]\| \leq \delta(\|x-x^*\|),$$

for all $x \in \Omega_0$,

$$\|F'(x^*)^{-1}([x,x^*;F]-F'(x^*))\| \leq \varphi_3(\|x-x^*\|),$$

for all $x \in \Omega$,

$$\|F'(x_0)^{-1}[x,x_0;F]\| \leq \delta(\|x-x_0\|),$$

for all $x \in \Omega_0$,

$$\|F'(x_0)^{-1}([x,x^*;F]-F'(x_0))\| \leq \psi_0(\|z-x_0\|,\|x-x_0\|),$$

for all $x \in \Omega$,

$$\|F'(x_0)^{-1}([x,x^*;F]-[y,x;F])\| \leq \psi_1(\|z-x_0\|,\|x-x_0\|,\|y-x_0\|),$$

$$\|F'(x_0)^{-1}([z,x;F]-[y,z;F])\| \leq \psi_2(\|z-x_0\|,\|x-x_0\|,\|y-x_0\|),$$

for all $x,y,z \in \Omega_0$.

Moreover, the "φ" and the "ψ" functions are assumed to be symmentric in each variable. Furthermore, we use the same notation to denote the "φ" or the "ψ" functions in case on variable other than x^* or x_0 is missing in the corresponding condition.

Chapter 1

Local Convergence for a Two-Step Third Order Secant-Like Method without Bilinear Operators

1. Introduction

We are concerned with the convergence of the two-step Secant-like method of order three for solving the equation

$$F(x) = 0. \tag{1.1}$$

Here, $F : \Omega \subset B_1 \longrightarrow B_2$ is a nonlinear operator, B_1 and B_2 are Banach spaces and $\Omega \neq \emptyset$ open set. We denote the solution of (1.1) by x^*. The convergence of the following iterative method was studied in [15],

$$\begin{aligned} y_n &= x_n + A_n^{-1} F(x_n) \\ \text{and} \quad & \\ x_{n+1} &= y_n - A_n^{-1} F(y_n), \end{aligned} \tag{1.2}$$

where $A_n = A(x_n) = [x_n - \gamma_n F(x_n), x_n + \gamma_n F(x_n); F]$, $[.,.;F] : \Omega \times \Omega \longrightarrow L(B_1, B_2)$ is a divided difference of order one $[1,2,3,4,5,6,7]$ and γ_n are given linear operators chosen to force convergence.

Throughout the chapter $U(x_0, R) = \{x \in X : \|x - x_0\| < R\}$ and $U[x_0, R] = \{x \in X : \|x - x_0\| \leq R\}$ for some $R > 0$.

2. Convergence

It is convenient to introduce parameters and nonnegative functions. Let $T = [0, \infty)$, and α, β be nonnegative parameters.

Suppose there exist functions:

(i) $\varphi_0 : T \times T \longrightarrow T$ continuous and nondecreasing such that equation $\varphi_0(\alpha t, \beta t) - 1 = 0$ has a smallest solution $\rho \in T - \{0\}$. Set $T_1 = [0, \rho)$. Define function $\psi_1 : T_1 \longrightarrow T$ by

$$\psi_1(t) = \frac{\varphi(\alpha t, \beta t) + 2\varphi_1(t)}{1 - \varphi_0(\alpha t, \beta t)}$$

for some continuous and nondecreasing functions $\varphi : T_1 \times T_1 \longrightarrow T$ and $\varphi_1 : T_1 \longrightarrow T$.

(ii) Equation $\psi_2(t) - 1 = 0$ has a smallest solution $r \in (0, \rho)$, where

$$\psi_2(t) = \frac{\varphi(\alpha t + \psi_1(t)t, \beta t)\psi_1(t)}{1 - \varphi_0(\alpha t, \beta t)}.$$

The parameter r is shown to be a convergence radius for method (1.2). Set $T_2 = [0, r)$. It follows by these definitions that for each $t \in T_2$

$$0 \le \varphi_0(\alpha t, \beta t) < 1, \tag{1.3}$$

$$0 \le \psi_1(t) < 1 \tag{1.4}$$

and

$$0 \le \psi_1(t) < 1. \tag{1.5}$$

The conditions (H) shall be used.

Suppose:

(H1) There exists a simple solution $x^* \in \Omega$ of equation $F(x) = 0$, and parameters $\alpha \ge 0$, $\beta \ge 0$ such that

$$\|I - \gamma(x)[x, x^*; F]\|\|\| \le \alpha \text{ and } \|I - \gamma(x)[x, x^*; F]\|\|\| \le \beta.$$

(H2) $\|F'(x^*)^{-1}(A(x - \gamma(x)F(x), x + \gamma(x)F(x)) - F'(x^*))\| \le \varphi_0(\|x - \gamma(x)F(x) - x^*\|, \|x + \gamma(x)F(x) - x^*\|)$.

Set $\Omega_1 = U(x^*, \rho) \cap \Omega$.

(H3) $\|F'(x^*)^{-1}(A(x - \gamma(x)F(x), x + \gamma(x)F(x)) - [x, x^*; F])\| \le \varphi(\|\gamma(x)\|, \|x + \gamma(x)F(x) - x^*\|)\| \; \|F'(x_0)^{-1}[x, x^*; F]\| \le \varphi_1(\|x - x^*\|)$ for each $x \in \Omega_1$ and

(H4) $U[x_0, R] \subset \Omega$, where $R = \max\{r, \alpha r, \beta r, \psi_1(r)r\}$.

The local convergence analysis of method (1.2) follows.

Theorem 1. Suppose conditions (H) hold. Then, iteration $\{x_n\}$ generated by method (1.2) converges to x^* provided that $x_0 \in U(x_0, r) - \{x^*\}$.

Proof. By hypothesis $x_0 \in U(x_0, r) - \{x^*\}$. Then, by (H1), (H2), the definition of r and (1.3), we obtain in turn

$$\begin{aligned}
\|F'(x^*)^{-1}(A_0 - F'(x^*))\| &= \|F'(x^*)^{-1}([x_0 - \gamma_0 F(x_0), x_0 + \gamma_0 F(x_0); F] - F'(x^*))\| \\
&\leq \varphi_0(\|(1 - \gamma_0[x_0, x^*; F])(x_0 - x^*)\|, \\
&\qquad \|(1 + \gamma_0[x_0, x^*; F])(x_0 - x^*)\|) \\
&\leq \varphi_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|) \\
&\leq \varphi_0(\alpha r, \beta r) < 1.
\end{aligned}$$

So, $A_0^{-1} \in L(B_2, B_1)$ and

$$\|A_0^{-1}F'(x^*)\| \leq \frac{1}{1 - \varphi_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|)} \tag{1.6}$$

followed by the Banach Lemma on linear operators with inverses. Moreover, iterates y_0 and x_1 are defined by method (1.2) (first step). We can also write

$$\begin{aligned}
y_0 - x^* &= x_0 - x^* - A_0^{-1}F(x_0) + 2A_0^{-1}F(x_0) \\
&= A_0^{-1}(A_0 - [x_0, x^*; F])(x_0 - x^*) + 2A_0^{-1}F(x_0).
\end{aligned} \tag{1.7}$$

It follows by (H3), (1.6) and (1.7) that

$$\begin{aligned}
\|y_0 - x^*\| &\leq \frac{(\varphi(\|x_0 - \gamma_0 F(x_0) - x^*\|, \|x_0 + \gamma_0 F(x_0) - x^*\|) + 2\varphi_1(\|x_0 - x^*\|))\|x_0 - x^*\|}{1 - \varphi_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|)} \\
&\leq \frac{(\varphi(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|) + 2\varphi_1(\|x_0 - x^*\|))\|x_0 - x^*\|}{1 - \varphi_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|)} \\
&\leq \psi_1(\|x_0 - x^*\|)\|x_0 - x^*\|.
\end{aligned} \tag{1.8}$$

Similarly, by the second substep of method (1.2), we can write in turn that

$$\begin{aligned}
x_1 - x^* &= y_0 - x^* - A_0^{-1}F(y_0) \\
&= A_0^{-1}(A_0 - [y_0, x^*; F])(y_0 - x^*).
\end{aligned} \tag{1.9}$$

Hence, by (1.5)- (1.9) and (H3), we get

$$\begin{aligned}
\|x_1 - x^*\| &\leq \frac{\varphi(\|x_0 - \gamma_0 F(x_0) - y_0 - x^* + x^*\|, \|x_0 + \gamma_0 F(x_0) - x^*\|)\|y_0 - x^*\|}{1 - \varphi_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|)} \\
&\leq \frac{(\varphi(\alpha\|x_0 - x^*\| + \psi_1(\|x_0 - x^*\|)\|x_0 - x^*\|, \beta\|x_0 - x^*\|))\|y_0 - x^*\|}{1 - \varphi_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|)} \\
&\leq \psi_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r,
\end{aligned}$$

so $x_1 \in U(x^*, r)$. If we simply replace x_0, y_0, γ_0, x_1 by $x_m, y_m, \gamma_m, x_{m+1}$, we get

$$\|x_{m+1} - x^*\| \leq c\|x_m - x^*\| < r, \tag{1.10}$$

where $c = \dfrac{(\varphi(\alpha\|x_0 - x^*\| + \psi_1(\|x_0 - x^*\|)\|x_0 - x^*\|, \beta\|x_0 - x^*\|))\psi_1(\|x_0 - x^*\|)}{1 - \varphi_0(\alpha\|x_0 - x^*\|, \beta\|x_0 - x^*\|)} \in [0, 1)$.
Hence, we conclude by (1.10) that $lim_{m \longrightarrow \infty} x_m = x^*$. □

Concerning the uniqueness of the solution, we provide such information. But the following condition is used.

(H6)

$$\|F'(x^*)^{-1}([x^*,y;F]-F'(x^*))\| \leq \varphi_2(\|y-x^*\|) \tag{1.11}$$

for all $y \in \Omega_3 = U(x^*,\tau) \subset \Omega$ for some $\tau > 0$.

In particular,

Proposition 1. *Suppose*

(i) The point x^* is a simple solution of equation $F(x) = 0$ in Ω_3.

(ii) Condition (1.11) holds.

(iii) There exists $\tau_1 \geq \tau$ such that

$$\varphi_2(\tau_1) < 1. \tag{1.12}$$

Set $\Omega_4 = U[x^*,\tau_1] \cap \Omega$. Then, the point x^* is the only solution of equation $F(x) = 0$ in the set Ω_4.

Proof. Let $q \in \Omega_4$ with $F(q) = 0$. Then, using (1.11) and (1.12), we get for $M = [x^*,q;F]$:

$$\|F'(x^*)^{-1}(M-F'(x^*))\| \leq \varphi_2(\|q-x^*\|) \leq \varphi_2(\tau_1) < 1,$$

thus, $x^* = q$ by invertibility of M and $M(x^*-q) = F(x^*)-F(q) = 0$. □

Conditions (H) have not been used. If we suppose they hold, then we can set $\tau = r$. These ideas can also be used on methods studied in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

Bibliography

[1] Argyros, I.K., *Computational theory of iterative methods*. Series: Studies in Computational Mathematics, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[2] Argyros, I.K. *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[3] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[4] Argyros, I.K., A Newton-Kantorvich theorem for equations involving m Frchet differentiable operators and applications in radiative transfer, *J. Comput. Appl. Math.*, 131(1-2), (2001), 149-159.

[5] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)* 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[6] Gutiérrez, J.M., A new semilocal convergence theorem for Newton's method, *J. Comput. Appl. Math.*, 79, (1997), 131-145.

[7] Hernández, M.A., Romero, N., On a characterization of some Newton-like methods of R-order at least three, *J. Comput. Appl. Math.*, 183(1), (2005) 53-66.

[8] Hernández, M.A., Romero, N., General Study of Iterative Processes of R-Order at Least Three under Weak Convergence Conditions, *J. Optim. Th. Appl.*, 133(2), (2007), 163-177.

[9] Hernández, M.A., Romero, N., A uniparametric family of iterative processes for solving nondifferentiable equations, *J. Math. Anal. Appl.*, 275(2), (2002), 821-834.

[10] Hernández, M.A., Romero, N., Semilocal convergence of the secant method under mild convergence conditions of differentiability, *Comput, Math. Appli.*, 44(3-4), (2002),277-285.

[11] Ezquerro, J.A., Hernández, M.A., Halley's method for operators with unbounded second derivative, *Appl. Numer. Math.*, 57(3), (2007), 354-360.

[12] Kantorovich, L.V., Akilov, G.P., *Functional Analysis*, Pergamon Press, Oxford, (1982).

[13] Kou, J., Li, Y., Wang, X., A modification of Newton method with third-order convergence, *Appl. Math. Comput.*, 182, (2007), 1106-1111.

[14] Kou, J., Li, Y., A family of modified super-Halley methods with fourth-order convergence, *Appl. Math. Comput.*, 189, (2007), 366-370.

[15] Kou, J., Wang, X., Some variant of Chebyshev-Halley methods for solving nonlinear equations, *Appl. Math. Comput.*, 189, (2007), 1839-1843.

Chapter 2

Semi-Local Convergence for a Derivative Free Method with Error Controlled Controlled Iterates for Solving Nonlinear Equations

1. Introduction

We study the convergence of a Secant-like method of order three for solving the equation

$$F(x) = 0. \tag{2.1}$$

Here, $F : \Omega \subset X \longrightarrow Y$ is a nonlinear operator, X and Y are Banach spaces and $\Omega \neq \emptyset$ open set. We denote the solution of (2.1) by x^*. The convergence of the following iterative method was studied in [15],

$$\begin{aligned} y_n &= x_n + B_n^{-1} F(x_n) \\ \text{and} \quad x_{n+1} &= y_n - B_n^{-1} F(y_n), \end{aligned} \tag{2.2}$$

where $B_n = B(x_n) = [x_n - c_n F(x_n), x_n + c_n F(x_n); F]$, $[.,.;F] : \Omega \times \Omega \longrightarrow L(X,Y)$ is a divided difference of order one [1,2,3,4,5,6,7] and c_n are given linear operators chosen to force convergence. Relevant work can be found in [6,7,8,9,10,11,12,13,14,15].

2. Convergence of Majorizing Sequences

Real sequences are developed to majorize the method (2.2). Let $\alpha,\beta,\gamma,A>0$, and $\eta,\delta\ \geq 0$ be given parameters. Define sequences $\{t_n\},\{s_n\}$ by $t_0=0,s_0=\eta$,

$$t_{n+1}=s_n+\frac{q(s_n-t_n)^2+2A(s_n-t_n)}{1-L_0(\alpha+\beta)t_n+2\gamma\delta)} \tag{2.3}$$

and

$$s_{n+1}=t_{n+1}+\frac{L(t_{n+1}-t_n+s_n-t_n+2\gamma A(s_n-t_n))(t_{n+1}-s_n)}{1-L_0((\alpha+\beta)t_{n+1}+2\gamma\delta)},$$

where $q=(1+2\gamma A)L$.
Next, the result of the convergence of these sequences is provided.

Lemma 1. Suppose

$$2L_0\gamma\delta<1 \tag{2.4}$$

and

$$t_n<t^{**}=\frac{1-2L_0\gamma\delta}{\alpha+\beta} \tag{2.5}$$

for each $n=0,1,2,\ldots$
Then, the following hold

$$0\leq t_n\leq s_n\leq t_{n+1}<t^{**} \tag{2.6}$$

and

$$t^*=lim_{n\to\infty}t_n\leq t^{**}. \tag{2.7}$$

Proof. By (2.3)-(2.5), estimates (2.6) and (2.7) hold, for t^* being the least upper(unique) of sequence t_n. □

The next result provides stronger convergence criteria but which are easier to check than (2.4) and (2.5).
Let us develop auxiliary functions on the interval $[0,1)$ and parameters:

$$g_1(t)=(q+L_0(\alpha+\beta))t-q$$

and

$$g_2(t)=Lt^2+Lat-Lt-La+L_0(\alpha+\beta)t^2, a=2(1+\gamma A).$$

Notice that

$$p_1=\frac{q}{q+L_0(\alpha+\beta)}\in[0,1)$$

is the only solution of equation $g_1(t) = 0$. By the definition of g_2, we get $g_2(0) = -La$, and $g_2(1) = 2L_0(\alpha+\beta)$. Denote by p_2 the least solution of an equation $g_2(t) = 0$ assured to exist by the intermediate value theorem. Moreover, define the function f_1 on $[0,1)$ by

$$f_1(t) = t^2 - (1+\lambda)t + \lambda + L_0(\alpha+\beta)\eta.$$

Suppose

$$4L_0(\alpha+\beta)\eta < (1-\lambda)^2 \tag{2.8}$$

and

$$2(A + L_0\gamma\delta) < 1, \tag{2.9}$$

where $\lambda = 2(A + L_0\gamma\delta)$.
Then, we get $f_1(0) = \lambda + L_0(\alpha+\beta)\eta$ and $f_1(1) = (1-\lambda)^2 - 4L_0(\alpha+\beta)\eta < 0$. Denote by μ_1 the smallest solution in [0,1). We also have

$$f_1(t) \leq 0 \tag{2.10}$$

for each $t \in [\mu_1, 1)$.
Define the function f_2 on [0,1) by

$$f_2(t) = \frac{L_0(\alpha+\beta)\eta}{1-t} + 2L_0\gamma\delta - 1. \tag{2.11}$$

Set

$$\mu = 1 - \frac{L_0(\alpha+\beta)\eta}{1 - 2L_0\gamma\delta}$$

and suppose

$$2L_0\gamma\delta + L_0(\alpha+\beta)\eta < 1. \tag{2.12}$$

Then, we get $\mu \in [0,1)$, and

$$f_2(t) \leq 0 \tag{2.13}$$

for each $t \in [0, \mu_2)$.
Set

$$a_1 = \frac{qn + 2A}{1 - L_0\gamma\delta}, \quad a_2 = \frac{L(t_1 + n + 2\gamma An)(t_1 - n)}{n(1 - L_0((\alpha+\beta)t_1 + 2\gamma\delta)},$$

$c = max\{a_1, a_2, \mu_1\}, p_0 = min\{p_1, p_2\}$, and $p = max\{p_1, p_2\}$.

Lemma 2. Under conditions (2.8),(2.9),(2.12), further suppose.

$$c \leq p_0 \leq p \leq \mu. \tag{2.14}$$

Then, the conclusion of Lemma 1 hold for sequence $\{t_n\}$ for $t^{**} = \frac{\eta}{1-p}$.
Moreover, the following estimates hold

$$0 \leq s_n - t_n \leq \delta(s_{n-1} - t_{n-1}) \leq \delta^n \eta, \tag{2.15}$$
$$0 \leq (t_{n+1} - s_n) \leq \delta(s_{n-1} - t_{n-1}) \leq \delta^n \eta \tag{2.16}$$

and

$$t_n \leq \frac{1-\delta^{n+1}}{1-\delta}\eta \tag{2.17}$$

Proof. Mathematical induction is used to show

$$0 \leq \frac{q(s_k - t_k) + 2A}{1 - L_0((\alpha+\beta)t_k + 2\gamma\delta)} \leq p \tag{2.18}$$

and

$$o \leq \frac{L[(t_{k+1} - t_k) + (s_k - t_k) + 2\gamma A(s_k - t_k)](t_{k+1} - s_k)}{1 - L_0((\alpha+\beta)t_{k+1} + 2\gamma\delta)} \leq p(s_k - t_k). \tag{2.19}$$

These estimates hold true for $k = 0$ by (2.14), so

$$t_0 \leq s_0 \leq t_1,$$
$$0 \leq s_1 - t_1 \leq p(s_0 - t_0) = p\eta,$$
$$0 \leq t_0 - s_0 \leq p\eta$$

and

$$t_0 \leq \eta + \eta p = \frac{1-p^2}{1-p}\eta.$$

Suppose (2.18) and (2.19) hold for all integers smaller or equal to $k-1$. Suppose

$$0 \leq s_k - t_k \leq \delta^n \eta,$$
$$0 \leq t_{k+1} - s_k \leq \delta^{k+1}$$

and

$$t_k \leq \frac{1-p^{k+1}}{1-p}\eta.$$

Then, evidently (2.18) is true if

$$\frac{qp^n\eta + 2A}{1 - L_0((\alpha+\beta)t_k + 2\gamma\delta} \leq p$$

or

$$qp^n\eta+2A-p+2L_0\gamma\delta+L_0(\alpha+\beta)\frac{1-p^{n+1}}{1-p}\eta\leq 0. \tag{2.20}$$

Estimate (2.20) motivates us to define on the interval $[0,1)$ recurrent polynomial $f_n^{(1)}(t)$ by

$$f_n^{(1)}(t)=qt^n\eta+L_0(\alpha+\beta)(1+t+...+t^n)\eta+\lambda-t. \tag{2.21}$$

Then, a relationship can be found between two consecutive functions as follows:

$$\begin{aligned}f_{n+1}^{(1)}(t)&=f_{n+1}^{(1)}(t)-f_n^{(1)}(t)+f_n^{(1)}(t)\\&=qt^{n+1}\eta+L_0(\alpha+\beta)(1+t+...+t^{n+1})\eta+\lambda-t\\&+f_n^{(1)}(t)-qt^{n+1}\eta-L_0(\alpha+\beta)(1+t+...+t^{n+1})\eta-\lambda+t\\&=f_n^{(1)}(t)+g_1(t)t^n\eta.\end{aligned}$$

In particular, by the definition of p_1, estimate (2.20) holds if

$$f_n^{(1)}(t)\leq 0 \;\; at \;\; t=p_1. \tag{2.22}$$

Define function

$$f_\infty^{(1)}(t)=lim_{n\to\infty}f_n^{(1)}(t). \tag{2.23}$$

Then, by (2.21) and (2.23), we obtain

$$f_\infty^{(1)}(t)=\frac{L_0(\alpha+\beta)\eta}{1-t}+\lambda-t.$$

Hence, (2.22) holds if

$$f_\infty^{(1)}(t)\leq 0 \;\; or \;\; f_1^{(1)}(t)\leq 0 \;\; at \;\; t=p_1,$$

which is true by (2.10) and (2.14).
Similarly, estimate (2.19) certainly holds if

$$\frac{L[p(s_n-t_n)+2(s_n-t_n)+2\gamma A(s_n-t_n)]p(s_n-t_n)}{1-L_0((\alpha+\beta)t_{n+1}+2\gamma\delta}\leq p(s_n-t_n)$$

or

$$Lp^{n+1}\eta+Lap^n\eta+L_0(\alpha+\beta)\frac{1-p^{n+2}}{1-p}\eta+2L_0\gamma\delta-1\leq 0$$

or

$$f_n^{(2)}(t)\leq 0 \;\; at \;\; t=p_2, \tag{2.24}$$

where

$$f_n^{(2)}(t)=Lt^{n+1}\eta+Lat^n\eta+L_0(\alpha+\beta)(1+t+t^2+...+t^{n+1})\eta+2L_0\gamma\delta-1. \tag{2.25}$$

This time, we have

$$\begin{aligned}
f_{n+1}^{(2)}(t) =& f_{n+1}^{(2)}(t) - f_n^{(2)}(t) + f_n^{(2)}(t) \\
&= f_n^{(2)}(t) + Lt^{n+2}\eta + Lat^{n+1}\eta - L_0(\alpha+\beta)(1+t+\ldots+t^{n+2})\eta \\
&+ 2L_0\gamma\delta - 1 - Lt^{n+1}\eta - Lat^n\eta - L_0(\alpha+\beta)(1+t+\ldots+t^{n+1})\eta - 2L_0\gamma\delta + 1 \\
&= f_n^{(2)}(t) + g_2(t)t^n\eta.
\end{aligned}$$

In particular, we have

$$f_{n+1}^{(2)}(t) = f_n^{(2)}(t) \;\; at \;\; t = p_2.$$

Define the function $f_\infty^{(2)}(t)$ on the interval [0,1) by

$$f_\infty^{(2)}(t) = lim_{n\to\infty} f_n^{(2)}(t). \tag{2.26}$$

Then, by (2.24) holds, since $f_\infty^{(2)}(t) \leq 0$ by choice of μ. The induction for (2.18) and (2.19) is completed. Hence, estimates (2.15)-(2.17) hold. The rest is given in Lemma 2. □

3. Convergence of Method (2.2)

The conditions (H) are used in the convergence analysis. Suppose:

(H_1) There exist $x_0 \in D, \eta \geq 0, \alpha \geq 0, \beta \geq 0, \delta \geq 0, \gamma \geq 0, L_0 \geq 0, A \in (0, \frac{1}{2})$ such that $F'(x_0)^{-1}, B_0^{-1} \in \delta(B,B)$,
$\|B_0^{-1}F(x_0)\| \leq \eta$, $\|F(x_0)\| \leq \delta$, $\|I - \gamma(x)[x,x_0;F]\| \leq \alpha$,
$\|I + \gamma(x)[x,x_0;F] \leq \beta$, $\|\gamma(x)\| \leq \gamma$, $\|A(x)\| \leq A$
and

$$\begin{aligned}
&\|F'(x_0)^{-1}([x-\gamma(x)F(x), x+\gamma(x)F(x);F] - F'(x_0))\| \\
&\qquad \leq L_0(\|x-\gamma(x)F(x)-x_0\| + \|x+\gamma(x)F(x)-x_0\|).
\end{aligned}$$

Suppose $2L_0\gamma\delta < 1$. Define

$$D_1 = U(x_0, \frac{1-2\gamma\delta L_0}{\alpha+\beta}) \cap D.$$

(H_2) There exist $L > 0$ such that

$$\|F'(x_0)^{-1}([z,w;F] - A(x))\| \leq L(\|z-x+\gamma(x)F(x)\| + \|w-x-\gamma(x)F(x)\|)$$

for all $x, z, w \in D_1$.

(H_3) Hypothesis of Lemma 1 or Lemma 2 hold
and

(H_4) $U(x_0,\rho) \subseteq D$, where $\rho = max\{t^*, \alpha t^* + \gamma\delta, \beta t^* + \gamma\delta\}$.

The main semi-local result for method (2.2) follows using conditions (H) and the preceding terminology.

Theorem 2. Suppose conditions (H) hold. Then, iteration $\{x_n\}$ generated by method (2.2) is well defined in $U[x_0,t^*]$, remains in $U[x_0,t^*]$ for each $n = 0,1,3,...$ and converges to some $x^* \in U[x_0,t^*]$ solving equation $F(x) = 0$. Moreover, the following estimates hold

$$\|y_n - x_n\| \leq s_n - t_n, \tag{2.27}$$

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n \tag{2.28}$$

and

$$\|x^* - x_n\| \leq t^* - t_n. \tag{2.29}$$

Proof. It follows by (H_1) and (2.4) that

$$\|y_0 - x_0\| \leq \eta = s_0 - t_0,$$

so (2.27) holds for $n = 0$ and $y_0 \in U[x_0,t^*]$. Let $v \in U[x_0,t^*]$. Then, by (H_1) and $(H_2,)$ we get in turn

$$\begin{aligned}
\|F'(x_0)^{-1}(B_n - F'(x_0)\| &\leq \|F'(x_0)^{-1}([x_n - \gamma_n F(x_n), x_n + \gamma_n F(x_n); F] - F'(x_0))\| \\
&\leq L_0(\|x_n - \gamma_n F(x_n) - x_0\| + \|x_n + \gamma_n F(x_n) - x_0\|) \\
&\leq L_0(\alpha\|x_n - x_0\| + \gamma\|F(x_0)\| + \beta\|x_n - x_0\| + \gamma\|F(x_0)\|) \\
&\leq L_0((\alpha+\beta)\|x_n - x_0\| + 2\gamma\delta) \\
&\leq L_0(\alpha+\beta)t_n + 2\gamma\delta) < 1.
\end{aligned} \tag{2.30}$$

Thus, $B_0^{-1} \in \delta(X,X)$

$$\|B_n^{-1}F'(x_0)\| \leq \frac{1}{1 - L_0((\alpha+\beta)t_n + 2\gamma\delta)} \tag{2.31}$$

follows by a lemma due to Banach on linear operators with inverses [12],
where we also used

$$\begin{aligned}
&\|x_n - x_0 - \gamma_n(F(x_n) - F(x_0)) - \gamma_n F(x_0)\| \\
&\leq \|(I - \gamma_n[x_n, x_0; F])(x_n - x_0) - \gamma_n F(x_0)\| \\
&\leq \alpha t^* + \gamma\delta,
\end{aligned}$$

and

$$\|x_n - x_0 + \gamma_n(F(x_n) - F(x_0) + \gamma_n F(x_0)\| \leq \beta t^* + \gamma\delta.$$

Therefore, the iterate $x_n + \gamma_n F(x_n), x_n - \gamma_n F(x_n) \in U[x_0,\rho]$. Then, we get by method (2.2),(2.31) and (H_3) that

$$\begin{aligned}
\|x_{n+1} - y_n\| &\leq \|B_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_n)\| \\
&\leq \frac{L(s_n - t_n + 2\gamma A(s_n - t_n))(s_n - t_n) + 2A(s_n - t_n)}{1 - L_0((\alpha+\beta)t_n + 2\gamma\delta)} \\
&= t_{n+1} - s_n,
\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-x_0\| &\leq \|x_{n+1}-y_n\|+\|y_n-x_0\| \\ &\leq t_{n+1}-s_n+s_n-t_0=t_{n+1},\end{aligned}$$

where we also used

$$\begin{aligned}F(y_n) &= F(y_n)-F(x_n)+F(x_n) \\ &= F(y_n)-F(x_n)+B_n(y_n-x_n) \\ &= ([y_n,x_n;F]-B_n)(y_n-x_n)+2B_n(y_n-x_n), \qquad (2.32)\end{aligned}$$

$$\|y_n-x_n-\gamma_n F(x_n)\| \leq s_n-t_n+\gamma A(s_n-t_n)$$

and

$$\|x_n-x_n-\gamma_n F(x_n)\| \leq \gamma A(s_n-t_n).$$

By method (2.2) and the identity

$$\begin{aligned}F(x_{n+1}) &= F(x_{n+1})-F(y_n)+F(y_n) \\ &= ([x_{n+1},y_n;F]-B_n)(x_{n+1}-y_n),\end{aligned}$$

we obtain

$$\|F'(x_0)^{-1}F(x_{n+1}\| \leq L(t_{n+1}-t_n+s_n-t_n+2\gamma A(s_n-t_n)) \qquad (2.33)$$

and

$$\begin{aligned}\|y_{n+1}-x_{n+1}\| &\leq \|B_{n+1}^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\| \\ &\leq \frac{L(t_{n+1}-t_n+s_n-t_n+2\gamma A(s_n-t_n))(t_{n+1}-s_n)}{1-L_0((\alpha+\beta)t_{n+1}+2\gamma\delta)} \\ &= s_{n+1}-t_{n+1}. \qquad (2.34)\end{aligned}$$

The induction for (2.27) and (2.28) is completed. Sequence $\{t_n\}$ is fundamental since it converges. These for a sequence $\{x_n\}$ is also fundamental in Banach space and as such it converges to some $x^* \in U[x_0,t^*]$. By letting $n \to \infty$ in (2.33), we deduce that $F(x^*) = 0$. Then, for $j \geq 0$, we get from the

$$\|x_{n+j}-x_n\| \leq t_{n+j}-t_n. \qquad (2.35)$$

Therefore, (2.29) follows by letting $j \to \infty$ in (2.35) □

Next, is the uniqueness of the solution result follows, where conditions (H) are not all necessarily needed.

Proposition 2. *Suppose:*

(i) There exist a simple solution $x^* \in U(x_0,R_0) \subset D$ for equation $F(x) = 0$ for some $R_0 > 0$.

(ii) There exist $R \geq R_0$ such that

$$L_0(R_0 + R) < 2. \tag{2.36}$$

Set $D_2 = U[x_0, R] \cap D$.
Then, the element x^* is the only solution of equation $F(x) = 0$ in the set D_2.

Proof. Let $v^* \in D_2$ with $F(v^*) = 0$. Define $T = [v^*, x^*]$. Then, in view of (2.2) and (2.36) we have in turn that

$$\begin{aligned} \|F'(x_0)^{-1}(T - F'(x_0))\| &\leq L_0(\|v^* - x_0\| + \|x^* - x_0\|) \\ &\leq L_0(R_0 + R) < 1. \end{aligned}$$

Hence, $v^* = x^*$ is implied by the invertibility of T and the identity
$T(v^* - x^*) = F(v^*) - F(x^*) = 0$ □

Remark. (i) Under conditions (H), we can set $\rho = R_0$.

(ii) The parameter $\dfrac{1 - 2L_0\gamma\delta}{\alpha + \beta}$ under Lemma 1 or $\dfrac{\eta}{1 - p}$ under Lemma 2 given in closed form can replace t^* in Theorem 2.

Bibliography

[1] Argyros, I.K., *Computational theory of iterative methods. Series: Studies in Computational Mathematics*, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[2] Argyros, I.K. *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[3] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[4] Argyros, I.K., A Newton-Kantorvich theorem for equations involving m Frchet differentiable operators and applications in radiative transfer, *J. Comput. Appl. Math.*, 131(1-2), (2001), 149-159.

[5] Argyros, I. K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)* 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[6] Gutiérrez, J. M., A new semilocal convergence theorem for Newton's method, *J. Comput. Appl. Math.*, 79, (1997), 131-145.

[7] Hernández, M. A., Romero,N., On a characterization of some Newton-like methods of R-order at least three, *J. Comput. Appl. Math.*, 183(1), (2005) 53-66.

[8] Hernández, M. A., Romero,N., General Study of Iterative Processes of R-Order at Least Three under Weak Convergence Conditions, *J. Optim. Th. Appl.*, 133(2), (2007), 163-177.

[9] Hernández, M. A., Romero, N., A uniparametric family of iterative processes for solving nondifferentiable equations, *J. Math. Anal. Appl.*, 275(2), (2002), 821-834.

[10] Hernández, M. A., Romero, N., Semilocal convergence of the secant method under mild convergence conditions of differentiability, *Comput, Math. Appli.*, 44(3-4), (2002), 277-285.

[11] Ezquerro, J. A., Hernández, M. A., Halley's method for operators with unbounded second derivative, *Appl. Numer. Math.*, 57(3), (2007), 354-360.

[12] Kantorovich, L.V., Akilov, G.P., *Functional Analysis*, Pergamon Press, Oxford, (1982).

[13] Kou, J., Li, Y., Wang, X., A modification of Newton method with third-order convergence, *Appl. Math. Comput.*, 182, (2007), 1106-1111.

[14] Kou, J., Li, Y., A family of modified super-Halley methods with fourth-order convergence, *Appl. Math. Comput.*, 189, (2007), 366-370.

[15] Kou, J., Wang, X., Some variant of Chebyshev-Halley methods for solving nonlinear equations, *Appl. Math. Comput.*, 189, (2007), 1839-1843.

Chapter 3

On the Semi-Local Convergence of a Sharma-Gupta Fifth Order Method for Solving Nonlinear Equations

The semi-local convergence for a Sharma-Gupta method (not given before) of order five is studied using assumptions only on the first derivative of the operator involved. The convergence of this method was shown by assuming that the sixth order derivative of the operator not on the method exists and hence it is limiting its applicability. Moreover, no computational error bounds or uniqueness of the solution are given. We address all these problems using only the first derivative that appears on the method. Hence, we extend the applicability of the method. Our techniques can be used to obtain the convergence of other similar higher-order methods using assumptions on the first derivative of the operator involved.

1. Introduction

Let $F : D \subset E_1 \longrightarrow E_2$ be a nonlinear operator acting between Banach spaces E_1 and E_2. Consider the problem of solving the nonlinear equation

$$F(x) = 0. \tag{3.1}$$

Iterative methods are used to approximate a solution x^* of the equation (3.1). The following iterative method was studied in [26],

$$\begin{aligned} y_n &= x_n - \gamma F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - F'(y_n)^{-1}F(x_n) \\ \text{and} \quad x_{n+1} &= z_n - (2F'(y_n)^{-1} - F'(x_n)^{-1})F(z_n), \end{aligned} \tag{3.2}$$

where $\gamma \in \mathbb{R}$. If $\gamma = \frac{1}{2}$, (3.2) reduces to the method in [26]. It was shown to be of order five using hypotheses on the sixth derivative.

In this study, we study the convergence of method (3.2) using assumptions only on the first derivative of F, unlike earlier studies [26] where the convergence analysis required assumptions on the derivatives of F up to the order six. These methods can be used on other methods and relevant topics along the same lines [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29].

For example: Let $X = Y = \mathbb{R}$, $D = [-\frac{1}{2}, \frac{3}{2}]$. Define f on D by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we have $f(1) = 0$,

$$f'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously, $f'''(t)$ is not bounded by D. So, the convergence of the method (3.2) is not guaranteed by the analysis in [26].

Throughout the article $U(x_0, R) = \{x \in X : \|x - x_0\| < R\}$ and $U[x_0, R] = \{x \in X : \|x - x_0\| \leq R\}$ for some $R > 0$.

The chapter contains local convergence analysis in Section 2, and the numerical examples are given in Section 3.

2. Convergence

Let L_0, L, L_1 be positive parameters and $\eta \geq 0$. Define sequence $\{t_n\}$ by

$$\begin{aligned} t_0 &= 0,\ s_0 = |\gamma|\eta, \\ u_n &= s_n + \frac{1}{|\gamma|}(|\gamma - 1| + \frac{L(s_n - t_n)}{1 - L_0 s_n})(s_n - t_n), \\ t_{n+1} &= u_n + \frac{L}{2}(u_n - t_n)^2 + L_1(u_n - s_n) + \left|\frac{\gamma - 1}{\gamma}\right| L_1(s_n - t_n) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} s_{n+1} &= t_{n+1} + \frac{|\gamma|}{1 - L_0 t_{n+1}} \left[\frac{L L_1^6 (s_n - t_n)}{(1 - (L_0 t_n + L(s_n - t_n)))^2} \right. \\ &\quad \left. + \frac{L L_1^3 (u_n - t_n + \frac{1}{2}(t_{n+1} - u_n))}{(1 - L_0 t_n + L(s_n - t_n))} \right] (t_{n+1} - u_n). \end{aligned}$$

Next, sufficient convergence criteria are given for sequence $\{t_n\}$.

Lemma 3. Suppose that for each $n = 0, 1, 2, \ldots$

$$L_0 t_n < 1, L_0 s_n < 1 \text{ and } L_0 t_n + L(s_n - t_n) < 1. \tag{3.4}$$

Then, sequence $\{t_n\}$ is non-decreasing, bounded from above by $\frac{1}{L_0}$ and converges to its unique least upper bound $t^* \in [0, \frac{1}{L_0}]$.

Proof. It follows by the definition of sequence $\{t_n\}$ that $0 \leq t_n \leq s_n \leq u_n \leq t_{n+1} < \frac{1}{L_0}$, so $\lim_{n\longrightarrow\infty} t_n = t^* \in [0, \frac{1}{L_0}]$. □

The conditions (H) are used in the semi-local convergence of method (3.2). Suppose:

(h1) $\exists\, x_0 \in D, \eta \geq 0, \delta \geq 0$ such that $F'(x_0)^{-1} \in L(Y,X)$, $\|F'(x_0)^{-1}F(x_0)\| \leq \eta$ and $\|F'(x_0)^{-1}\| \leq \delta$.

(h2) $\exists\, L_0 > 0$ such that $\forall x \in D$

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_0\|x - x_)\|.$$

Set $D_1 = D \cap U(x_0, \frac{1}{L_0})$.

(h3) $\exists L > 0, L_1 > 0$ such that for all $x, y \in D_1$

$$\|F'(x_0)^{-1}(F'(y) - F'(x)\| \leq L\|y - x\|$$

and

$$\|F'(x)\| \leq \frac{L_1}{\delta}.$$

(h4) Conditions of Lemma 3 hold
and

(h5) $U[x_0, t^*] \subset D$.

Next, the semi-local convergence of method (3.2) is presented using conditions (H).

Theorem 3. Suppose conditions (H) hold. Then, sequence $\{x_n\}$ produced by method (3.2) is well defined in $U[x_0,t^*]$, remains in $U[x_0,t^*]$ and converges to a solution $x^* \in U[x_0,t^*]$ of equation $F(x) = 0$. Moreover, the following estimates hold

$$\|y_n - x_n\| \leq s_n - t_n, \tag{3.5}$$

$$\|z_n - y_n\| \leq u_n - s_n \tag{3.6}$$

and

$$\|x_{n+1} - z_n\| \leq t_{n+1} - u_n. \tag{3.7}$$

Proof. Let $a \in U[x_0,t^*]$. Then, by (h2)

$$\|F'(x_0)^{-1}(F'(a) - F'(x_0))\| \leq L_0\|a - x_0\| \leq L_0 t^* < 1. \tag{3.8}$$

In view of estimate (3.8) and the Banach lemma on linear invertible operators [15] $F'(a)^{-1} \in L(Y,X)$ and

$$\|F'(a)^{-1}F'(x_0)\| \leq \frac{1}{1 - L_0\|a - x_0\|}. \tag{3.9}$$

Iterate y_0 is well defined by the first substep of method (3.2) and

$$\|y_0 - x_0\| = |\gamma| \|F'(x_0)^{-1} F(x_0)\| \leq |\gamma|\eta = s_0 - t_0 \leq t^*.$$

Hence, the iterate $y_0 \in U[x_0, t^*]$ and (3.5) holds for $n = 0$. We can write in turn by the second substep of method (3.2) for $n = 0$ (since $F'(y_0)^{-1}$ exists by (3.9) for $a = y_0$)

$$\begin{aligned} z_0 &= y_0 + \gamma F'(x_0)^{-1} F(x_0) - F'(y_0)^{-1} F(x_0) \\ &= y_0 + (\gamma - 1) F'(x_0)^{-1} F(x_0) \\ &\quad + F'(x_0)^{-1} (F'(y_0) - F'(x_0)) F'(y_0)^{-1} F(x_0). \end{aligned} \tag{3.10}$$

It follows by (3.3), (h3), (3.9) and (3.10) that

$$\begin{aligned} \|z_0 - y_0\| &\leq \left| \frac{\gamma - 1}{\gamma} \right| \|y_0 - x_0\| + \|F'(y_0)^{-1} F'(x_0)\| \\ &\quad \|F'(x_0)^{-1} (F'(x_0) - F'(y_0))\| \|F'(x_0)^{-1} F(x_0)\| \\ &\leq \left| \frac{\gamma - 1}{\gamma} \right| \|y_0 - x_0\| + \frac{1}{1 - L_0 \|y_0 - x_0\|} L \frac{\|y_0 - x_0\|^2}{|\gamma|} \\ &\leq \frac{1}{|\gamma|} \left(|\gamma - 1| + \frac{L(s_0 - t_0)}{1 - L_0 s_0} \right) (s_0 - t_0) = u_0 - s_0, \end{aligned} \tag{3.11}$$

and

$$\|z_0 - x_0\| \leq \|z_0 - y_0\| + \|y_0 - x_0\| \leq u_0 - s_0 + s_0 - t_0 = u_0 \leq t^*,$$

so (3.6) holds and $z_0 \in U[x_0, t^*]$. Iterate x_1 is well defined by the third substep of method (3.2) for $n = 0$, since $F'(x_0)^{-1}$ and $F'(y_0)^{-1}$ exist. We can write

$$\begin{aligned} F(z_0) &= F(z_0) - F(x_0) + F(x_0) = F(z_0) - F(x_0) - \frac{1}{\gamma} F'(x_0)(y_0 - x_0) \\ &= F(z_0) - F(x_0) - F'(x_0)(z_0 - x_0) + F'(x_0)(z_0 - x_0) \\ &\quad - F'(x_0)(y_0 - x_0) + F'(x_0)(y_0 - x_0) - \frac{1}{\gamma} F'(x_0)(y_0 - x_0) \\ &= \int_0^1 (F'(x_0 + \theta(z_0 - x_0)) d\theta - F'(x_0)(z_0 - x_0) \\ &\quad + (1 - \frac{1}{\gamma}) F'(x_0)(y_0 - x_0). \end{aligned} \tag{3.12}$$

In view of (3.9) (for $a = x_0, y_0$), (h3), (3.11) and (3.12), we get

$$\begin{aligned} \|F'(x_0)^{-1} F(z_0)\| &\leq \frac{L}{2} \|z_0 - x_0\|^2 + L_1 \|z_0 - y_0\| + \frac{|\gamma - 1|}{|\gamma|} L_1 \|y_0 - x_0\| \\ &\leq \frac{L}{2} (u_0 - t_0)^2 + L_1 (u_0 - s_0) \\ &\quad + \left| \frac{\gamma - 1}{\gamma} \right| L_1 (s_0 - t_0) = v_0. \end{aligned} \tag{3.13}$$

Then, we can write

$$x_1 - z_0 = -(F'(y_0)^{-1} - F'(x_0)^{-1}) F(z_0) - F'(y_0)^{-1} F(z_0), \tag{3.14}$$

so

$$\|x_1 - x_0\| \leq \left[\frac{L(s_0 - t_0)}{(1 - L_0 t_0)(1 - L_0 s_0)} + \frac{1}{1 - L_0 s_0}\right] v_0 = t_1 - u_0,$$

and

$$\begin{aligned}\|x_1 - x_0\| &\leq \|x_1 - z_0\| + \|z_0 - y_0\| + \|y_0 - x_0\| \\ &\leq t_1 - u_0 + u_0 - s_0 + s_0 - t_0 = t_1 \leq t^*.\end{aligned}$$

Therefore, the iterate $x_1 \in U[x_0, t^*]$ and (3.7) holds for $n = 0$. We can write

$$\begin{aligned}F(x_1) &= F(x_1) - F(z_0) + F(z_0) \\ &= \int_0^1 (F'(z_0 + \theta(x_1 - x_0))d\theta - B_0)(x_1 - x_0), \end{aligned} \tag{3.15}$$

where $B_0^{-1} = (2F'(y_0)^{-1} - F'(x_0)^{-1}$, so $B_0 = F'(x_0)(2F'(x_0) - F'(y_0)^{-1}F'(y_0)$. Notice that

$$\|F'(x_0)^{-1}(2F'(x_0) - F'(y_0) - F'(x_0))\| \leq L_0(\|x_0 - x_0\| + \|y_0 - x_0\|) \leq L_(s_0 - t_0) < 1.$$

It follows that,

$$\|(2F'(x_0) - F'(y_0)^{-1}F'(x_0)\| \leq \frac{1}{1 - L(s_0 - t_0)}.$$

Set $\int_0^1 F'(z_0 + \theta(x_1 - x_0))d\theta = b_0$. Then, we get

$$\begin{aligned}b_0 - B_0 &= b_0 - F'(x_0)B_0^{-1}F'(y_0) - b_0 B_0^{-1}F'(y_0) + b_0 B_0^{-1}F'(y_0) \\ &= b_0(I - B_0^{-1}F'(y_0)) + (b_0 - F'(x_0))B_0^{-1}F'(y_0) \\ &= b_0 B_0^{-1}(B_0 - F'(y_0)) + (b_0 - F'(x_0))B_0^{-1}F'(y_0) \\ &= b_0 B_0^{-1}(F'(y_0) - F'(x_0))B_0^{-1}F'(y_0) \\ &\quad + (b_0 - F'(x_0))B_0^{-1}F'(y_0). \end{aligned} \tag{3.16}$$

It follows by (h3) that

$$\|F'(x_0)^{-1}F(x_1)\| \leq \frac{LL_1^6(s_0 - t_0)}{(1 - (L_0 t_n + L(s_n - t_n)))^2} + \frac{LL_1^3(u_0 - t_0 + \frac{1}{2}(t - 1 - u_0))}{1 - (L_0 t_0 + L(s_0 - t_0))} \tag{3.17}$$

leading to

$$\begin{aligned}\|y_1 - x_1\| &\leq |\gamma| \|F'(x_1)^{-1}F'(x_0)\| \|F'(x_0)^{-1}F(x_1)\| \\ &\leq |\gamma| \left(\frac{LL_1^6(s_0 - t_0)}{(1 - (L_0 t_n + L(s_n - t_n)))^2}\right. \\ &\quad \left. + \frac{LL_1^3(u_0 - t_0 + \frac{1}{2}(t - 1 - u_0))}{1 - (L_0 t_0 + L(s_0 - t_0))}\right)(t_1 - u_0) \\ &= s_1 - t_1, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\|y_1 - x_0\| &\leq \|y_1 - x_1\| + \|x_1 - z_0\| + \|z_0 - y_0\| + \|y_0 - x_0\| \\ &\leq s_1 - t_1 + t_1 - u_0 + u_0 - s_0 + s_0 - t_0 = s_1 \leq t^*.\end{aligned}$$

Thus, the iterate $y_1 \in U[x_0,t^*]$ and (3.5) holds. Simply replace x_0,y_0,z_0,x_1 by $x_m.y_m,z_m,x_{m+1}$ in the preceding calculations to terminate the induction for items (3.5)-(3.7). Hence, sequences $\{x_m\},\{y_m\},\{z_m\}$ are fundamental in Banach space X and as such they converge to some $x^* \in U[x_0,t^*]$. Then, by the estimate (see (3.13)

$$\|F'(x_0)^{-1}F(z_m)\| \leq \frac{L}{2}(u_m-t_m)^2+L_1(u_m-s_m)+\left|\frac{\gamma-1}{\gamma}\right|L_1(s_m-t_m) \tag{3.19}$$

and the continuity of F we get $F(x^*)=0$ provided that $m \longrightarrow \infty$. □

The uniqueness of the solution results under conditions (H) is found in earlier Chapters.

Bibliography

[1] Argyros, I.K., On the Newton - Kantorovich hypothesis for solving equations, *J. Comput. Math.*, 169,(2004), 315-332.

[2] Argyros, I.K., *Computational theory of iterative methods*. Series: Studies in Computational Mathematics, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[3] Argyros, I.K. *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[4] Argyros, I.K., Hilout, S., Weaker conditions for the convergence of Newton's method. *J. Complexity*, 28, (2012), 364–387.

[5] Argyros, I.K., Hilout, S., On an improved convergence analysis of Newton's methods, *Applied Mathematics and Computation*, 225, (2013), 372-386.

[6] Argyros, I.K., Magréñan, A.A., *Iterative methods and their dynamics with applications*, CRC Press, New York, USA, 2017.

[7] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative methods*, Elsevier (Academic Press), New York, 2018.

[8] Arroyo, V., Cordero, A., Torregrosa, J.R., Approximation of artificial satellites preliminary orbits: The efficiency challenge. *Mathematical and Computer modeling*, 54, (7-8), (2011), 1802-1807.

[9] Behl, R., Maroju, P., Martinez, E., Singh, S., A study of the local convergence of a fifth order iterative method , *Indian J. Pure Appl. Math.*, 51,2, (2020), 439-455.

[10] Cătinaş, E., The inexact, inexact perturbed, and quasi-Newton methods are equivalent models, *Math. Comp.*, 74, (2005), 291-301.

[11] Ezquerro, J.A., Gutiérrez, J.M., Hernández, M.A., Romero, N., Rubio, M.J., The Newton method: from Newton to Kantorovich(Spanish), *Gac. R. Soc. Mat. Esp.*, 13,(2010), 53-76.

[12] Ezquerro, J. A., Hernandez, M. A., *Newton's method: An updated approach of Kantorovich's theory*, Cham Switzerland, (2018).

[13] Grau-Sánchez, Miquel and Grau, Àngela and Noguera, Miquel, Ostrowski type methods for solving systems of nonlinear equations. *Appl. Math. Comput.*, 281,(2011), 2377-2385.

[14] Homeier, H.H.H., A modified Newton method with cubic convergence: the multivariate case, *Journal of Computational and Applied Mathematics*, 169 (2004), 161-169.

[15] Kantorovich, L.V., Akilov, G.P., *Functional Analysis*, Pergamon Press, Oxford, (1982).

[16] Magréñan, A.A., Argyros, I.K., Rainer, J.J., Sicilia, J.A., Ball convergence of a sixth-order Newton-like method based on means under weak conditions, *J. Math. Chem.*, (2018) 56:2117-2131, https://doi.org/10.1007/ s10910-018-0856-y.

[17] Magréñan, A.A., Gutiérrez, J.M., Real dynamics for damped Newton's method applied to cubic polynomials, *J. Comput. Appl. Math.*, 275, (2015), 527–538.

[18] Ortega, L.M., Rheinboldt, W.C., *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, (1970).

[19] Ostrowski, A.M., *Solution of equations in Euclidean and Banach spaces*, Elsevier, 1973.

[20] Potra, F.A. and Pták, V., Nondiscrete induction and iterative processes. *Research Notes in Mathematics*, 103. Pitman (Advanced Publishing Program), Boston,MA. (1984).

[21] Proinov P.D., General local convergence theory for a class of iterative processes and its applications to Newton's method, *J. Complexity*, 25, (2009) , 38-62.

[22] Proinov P.D., New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems, *J. Complexity*, 26, (2010), 3-42.

[23] Rheinboldt, W.C., *An adaptive continuation process of solving systems of nonlinear equations*, Polish Academy of Science, Banach Ctr. Publ. 3,(1978), 129-142.

[24] Shakhno, S.M., Gnatyshyn, O.P., On an iterative algorithm of order 1.839... for solving nonlinear least squares problems, *Appl. Math. Applic.*, 161, (2005), 253-264.

[25] Shakhno, S. M., Iakymchuk, R.P., Yarmola, H.P., Convergence analysis of a two step method for the nonlinear squares problem with decomposition of operator, *J. Numer. Appl. Math.*, 128, (2018), 82-95.

[26] Sharma, J.R. and Gupta, P., An efficient fifth order method for solving systems of nonlinear equations, *Comput. and Math. Appl.*, 4, (2014), 591-601.

[27] Sharma, J.R, Kumar, S., A class of accurate Newton-Jarratt like methods with applications to nonlinear models, *Comput. Appl. Math*, (2022).

[28] Traub, J.F., *Iterative methods for the solution of equations* Prentice Hall, New Jersey, U.S.A, (1964).

[29] Verma, R., *New Trends in Fractional Programming*, Nova Science Publishers, New York, USA, (2019).

Chapter 4

On the Semi-Local Convergence of a Seventh Order Method for Nonlinear Equations Convergence for a Seventh Order Method for Equations

1. Introduction

The local convergence for a Xiao-Yin method of order seven is studied using assumptions only on the first derivative of the operator involved. The convergence of this method was shown by assuming that the eighth order derivative of the operator not on the method exists and hence it is limiting its applicability. Moreover, no computational error bounds or uniqueness of the solution are given. We address all these problems using only the first derivative that appears on the method. Hence, we extend the applicability of the method. Our techniques can be used to obtain the convergence of other similar higher-order methods using assumptions only on the first derivative of the operator involved. The semi-local convergence not given in [29] is also included.

Let $F : \Omega \subset B \longrightarrow B_1$ be a nonlinear operator acting between Banach spaces B and B_1. Consider the problem of solving the nonlinear equation

$$F(x) = 0. \tag{4.1}$$

Iterative methods are used to approximate a solution x^* of the equation (4.1). Define iterative method by,

$$\begin{aligned} y_n &= x_n - \delta F'(x_n)^{-1} F(x_n), \\ z_n &= x_n - F'(y_n)^{-1} F(x_n), \\ w_n &= z_n - (2F'(y_n)^{-1} - F'(x_n)^{-1}) F(z_n) \end{aligned}$$

and

$$x_{n+1} = w_n - (2F'(y_n)^{-1} - F'(x_n)^{-1}) F(w_n).$$

$\delta \in \mathbb{R}$, if $\delta = \dfrac{1}{2}$ then, the method reduces to the one in [26].

We develop the semi-local convergence of method (4.2) using assumptions only on the first derivative of F, unlike earlier studies [26] where the convergence analysis required assumptions on the derivatives of F up to the order eight. This technique can be used on other methods and relevant topics along the same lines [1,2,3,4,5,6,7,8,9,10,11,12,13, 14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29].

For example: Let $X = Y = \mathbb{R}$, $D = [-\frac{1}{2}, \frac{3}{2}]$. Define f on D by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we have $f(1) = 0$,

$$f'''(t) = 6\log t^2 + 60t^2 - 24t + 22.$$

Obviously, $f'''(t)$ is not bounded by D. So, the convergence of the method (4.2) is not guaranteed by the analysis in [26].

Throughout this Chapter $U(x_0, R) = \{x \in X : \|x - x_0\| < R\}$ and $U[x_0, R] = \{x \in X : \|x - x_0\| \leq R\}$ for some $R > 0$.

2. Semi-Local Convergence

Let L_0, L, L_1 and $\eta \geq 0$ be given parameters. Define sequence $\{t_n\}$ by $t_0 = 0$,$s_0 = |r|\eta$,

$$\begin{aligned}
u_n &= s_n + \frac{1}{|\gamma|}(|\gamma - 1| + \frac{L(s_n - t_n)}{1 - L_0 s_n})(s_n - t_n), \qquad (4.2)\\
p_n &= u_n + \frac{1}{1 - L_0 s_n}(1 + \frac{L(s_n - t_n)}{1 - L_0 s_n})\gamma_n,\\
v_n &= \frac{L}{2}(u_n - t_n)^2 + L_1(u_n - s_n) + |\frac{\gamma - 1}{\gamma}|L_1(s_n - t_n),\\
t_{n+1} &= p_n + \frac{1}{1 - L_0 s_n}(1 + \frac{L(s_n - t_n)}{1 - L_0 s_n})\bar{v}_n)(p_n - t_n),\\
\bar{v}_n &= \frac{L{L_1}^6(s_n - t_n)}{(1 - (L_0 t_n + L(s_n - t_n)))^2} + \frac{L{L_1}^3(u_n - t_n + \frac{1}{2}(p_n - u_n))}{1 - (L_0 t_n + L_1(s_n - t_n))},\\
s_{n+1} &= t_{n+1} + \frac{|\gamma|}{1 - L_0 t_{n+1}}\bar{\bar{v}}_n)(t_{n+1} - p_n)
\end{aligned}$$

and

$$\bar{\bar{v}}_n = \frac{L{L_1}^6(s_n - t_n)}{(1 - (L_0 t_n + L(s_n - t_n)))^2} + \frac{L{L_1}^3(p_n - t_n + \frac{1}{2}(t_{n+1} - p_n))}{1 - (L_0 t_n + L_1(s_n - t_n))}.$$

Convergence criteria for sequences $\{t_n\}, \{s_n\}, \{u_n\}$ are provided in the auxiliary result that follows.

Lemma 4. Suppose that for each $n = 0, 1, 2, \ldots, L_0 t_n < 1, L_0 s_n < 1$ and $L_0 t_n + L(s_n - t_n) < 1$,(4.2). Then, sequence $\{t_n\}$ is non-decreasing, bounded from above by $\frac{1}{L_0}$ and converges to its unique least upper bound $t^* \in [0, \frac{1}{L_0}]$.

Proof. It follows by the definition of sequence $\{t_n\}$ and (4.2) that

$$0 \le t_n \le s_n \le u_n \le p_n \le t_{n+1} < \frac{1}{L_0}, \tag{4.3}$$

so

$$lim_{n\to\infty} t_n = t^* \in [0, \frac{1}{L_0}].$$

□

The conditions (H) are used in the semi-local convergence of method (4.2). Suppose:

(h1) $\exists x_0 \in D, \eta \ge 0, \delta \ge 0$ such that $F'(x_0)^{-1} \in L(Y,X), \|F'(x_0)^{-1}F(x_0)\| \le \eta$ and $\|F'(x_0)^{-1}\| \le \delta$.

(h2) $\exists L_0 > 0$ such that $\forall x \in D$

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \le L_0\|x - x_)\|.$$

Set $D_1 = D \cap U(x_0, \frac{1}{L_0})$.

(h3) $\exists L > 0, L_1 > 0$ such that for all $x, y \in D_1$

$$\|F'(x_0)^{-1}(F'(y) - F'(x)\| \le L\|y - x\|$$

and

$$\|F'(x)\| \le \frac{L_1}{\delta}.$$

(h4) Conditions of Lemma 4 hold
and

(h5) $U[x_0, t^*] \subset D$.

Next, the semi-local convergence of method (4.2) is presented using conditions (H).

Theorem 4. Suppose conditions (H) hold. Then, sequence $\{x_n\}$ produced by method (4.2) is well defined in $U[x_0,t^*]$, remains in $U[x_0,t^*]$ and converges to a solution $x^* \in U[x_0,t^*]$ of equation $F(x) = 0$. Moreover, the following estimates hold

$$\|y_n - x_n\| \le s_n - t_n, \tag{4.4}$$

$$\|z_n - y_n\| \le u_n - s_n, \tag{4.5}$$

$$\|w_n - z_n\| \le p_n - u_n \tag{4.6}$$

and

$$\|x_{n+1} - z_n\| \le t_{n+1} - u_n. \tag{4.7}$$

Proof. Let $a \in U[x_0, t^*]$. Then, by (h2)

$$\|F'(x_m)^{-1}(F'(a) - F'(x_m))\| \leq L_0 \|a - x_m\| \leq L_0 t^* < 1. \tag{4.8}$$

In view of estimate (4.14) and the Banach lemma on linear invertible operators [15] $F'(a)^{-1} \in L(Y, X)$ and

$$\|F'(a)^{-1} F'(x_m)\| \leq \frac{1}{1 - L_0 \|a - x_m\|}. \tag{4.9}$$

Iterate y_m is well defined by the first substep of method (4.2) and

$$\|y_m - x_m\| = |\gamma| \|F'(x_m)^{-1} F(x_m)\| \leq |\gamma| \eta = s_m - t_m \leq t^*.$$

Hence, the iterate $y_m \in U[x_0, t^*]$ and (4.4) holds for $n = m$. We can write in turn by the second substep of method (4.2) for $n = m$ (since $F'(y_m)^{-1}$ exists by (4.9) for $a = y_m$)

$$\begin{aligned} z_m &= y_m + \gamma F'(x_m)^{-1} F(x_m) - F'(y_m)^{-1} F(x_m) \\ &= y_m + (\gamma - 1) F'(x_m)^{-1} F(x_m) \\ &\quad + F'(x_m)^{-1}(F'(y_m) - F'(x_m)) F'(y_m)^{-1} F(x_m). \end{aligned} \tag{4.10}$$

It follows by (4.2), (h3), (4.9) and (4.10) that

$$\begin{aligned} \|z_m - y_m\| &\leq \left| \frac{\gamma - 1}{\gamma} \right| \|y_m - x_m\| + \|F'(y_m)^{-1} F'(x_m)\| \\ &\quad \|F'(x_m)^{-1}(F'(x_m) - F'(y_m))\| \|F'(x_m)^{-1} F(x_m)\| \\ &\leq \left| \frac{\gamma - 1}{\gamma} \right| \|y_m - x_m\| + \frac{1}{1 - L_0 \|y_m - x_m\|} L \frac{\|y_m - x_m\|^2}{|\gamma|} \\ &\leq \frac{1}{|\gamma|} \left(|\gamma - 1| + \frac{L(s_m - t_m)}{1 - L_0 s_m} \right) (s_m - t_m) = u_m - s_m \end{aligned} \tag{4.11}$$

and

$$\|z_m - x_m\| \leq \|z_m - y_m\| + \|y_m - x_m\| \leq u_m - s_m + s_m - t_m = u_m \leq t^*,$$

so (4.5) holds and $z_m \in U[x_0, t^*]$. Iterate x_{m+1} is well defined by the third substep of method (4.2) for $n = m$, since $F'(x_m)^{-1}$ and $F'(y_m)^{-1}$ exist. We can write

$$\begin{aligned} F(z_m) &= F(z_m) - F(x_m) + F(x_m) = F(z_m) - F(x_m) - \frac{1}{\gamma} F'(x_m)(y_m - x_m) \\ &= F(z_m) - F(x_m) - F'(x_m)(z_m - x_m) + F'(x_m)(z_m - x_m) \\ &\quad - F'(x_m)(y_m - x_m) + F'(x_m)(y_m - x_m) - \frac{1}{\gamma} F'(x_m)(y_m - x_m) \\ &= \int_0^1 (F'(x_m + \theta(z_m - x_m)) d\theta - F'(x_m)(z_m - x_m) \\ &\quad + (1 - \frac{1}{\gamma}) F'(x_m)(y_m - x_m). \end{aligned} \tag{4.12}$$

In view of (4.9) (for $a = x_m, y_m$), (h3), (4.11) and (4.12), we get

$$\begin{aligned}\|F'(x_m)^{-1}F(z_m)\| &\leq \frac{L}{2}\|z_m - x_m\|^2 + L_1\|z_m - y_m\| + \frac{|\gamma-1|}{|\gamma|}L_1\|y_m - x_m\| \\ &\leq \frac{L}{2}(u_m - t_m)^2 + L_1(u_m - s_m) \\ &\quad + \left|\frac{\gamma-1}{\gamma}\right| L_1(s_m - t_m) = v_m. \end{aligned} \tag{4.13}$$

Then, we can also write

$$w_m - z_m = -(F'(y_m)^{-1} - F'(x_m)^{-1})F(z_m) - F'(y_m)^{-1}F(z_m),$$

so

$$\|w_m - z_m\| \leq \left(\frac{L(s_m - t_m)}{(1 - L_0t_m)(1 - L_0s_m)} + \frac{1}{1 - L_0s_m}\right)v_m = p_m - u_m$$

and

$$\begin{aligned}&\|w_m - x_m\|\|w_m - z_m\| + \|z_m - y_m\| + \|y_m - x_m\| \\ &\qquad \leq p_m - u_m + u_m - s_m + s_m + s_m - t_m = p_m \leq t^*,\end{aligned}$$

so (4.6) holds for $n = m$ and $w_m \in U[x_0, t^*]$.
We can write

$$\begin{aligned}F(w_m) &= F(w_m) - F(z_m) + F(z_m) = F(w_m) - F(z_m) - B_m(w_m - z_m) \\ &= \int_0^1 (F'(z_m + \theta(w_m - z_m))d\theta - B_m)(w_m - z_m) = (a_m - B_m)(w_m - z_m),\end{aligned}$$

where

$$a_m = \int_0^1 F'(z_m + \theta(w_m - z_m))d\theta,\ and\ B_m = F'(x_m)(zF'(x_m) - F'(y_m))F'(y_m).$$

Moreover, we obtain in turn Furthermore, we can write

$$\begin{aligned} a_m - B_m &= a_m B_m^{-1}(F'(y_m) - F'(x_m))B_m^{-1}F'(y_m) \\ &\quad + (a_m - F'(x_m))B_m^{-1}F'(y_m), \\ \|F'(x_m)^{-1}(2F'(x_m) - F'(y_m) - F'(x_m))\| &\leq \|F'(x_m)^{-1}(F'(x_m) - F'(x_m))\| \\ &\quad + \|F'(x_m)^{-1}(F'(x_m) - F'(y_m))\| \\ &\leq L_0\|x_m - x_m\| + \|y_m - x_m\| \\ &\leq L_0t_m + L(s_m - t_m) < 1,\end{aligned}$$

so,

$$\begin{aligned}\|B_m^{-1} &\leq \|F'(y_m)^{-1}\|\|(zF'(x_m) - F'(y_m))^{-1}F'(x_m)\|\|F'(x_m)^{-1}F'(x_m)\| \\ &\leq \frac{L_1}{\delta}\frac{L_1.\delta}{1 - (L_0t_m + L(s_m - t_m))}\end{aligned}$$

so

$$\|F'(x_m)^{-1}F(w_m)\| \le [\frac{LL_1^6(s_m-t_m)}{(1-(L_0t_m+L(s_m-t_m)))^2} + \frac{LL_1^3(u_m-t_m+\frac{1}{2}(p_m-u_m))}{1-(L_0t_m+L(s_m-t_m))}](p_m-u_m) = \bar{v}_m(p_m-u_m). \tag{4.14}$$

Consequently, from the third substep of method (4.2), we also get

$$\begin{aligned}\|x_{m+1}-w_m\| &= \|-(F'(y_m)^{-1}-F'(x_m)^{-1})F(w_m)-F'(y_m)^{-1}F'(w_m)\| \\ &\le (\frac{L(s_m-t_m)}{(1-L_0t_m)(1-L_0s_m)}+\frac{1}{1-L_0s_m})\bar{v}_m(p_m-u_m) \\ &= t_{m+1}-p_m\end{aligned}$$

and

$$\begin{aligned}\|x_{m+1}-x_0\| &\le \|x_{m+1}-w_m\|+\|w_m-z_m\|+\|z_m-y_m\|+\|y_m-x_0\| \\ &\le t_{m+1}-p_m+p_m+-u_m+u_m-s_m+s_m-t_0 \\ &= t_{m+1}\le t^*,\end{aligned}$$

which show (4.7) for $n=m+1$ and $x_{m+1}\in U[x_0,t^*]$. So iterate y_{m+1} is well defined and we have the estimate

$$\|y_{m+1}-x_{m+1}\| \le |\gamma|\|F'(x_{m+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{m+1})\|.$$

By exchanging the role of w_m by x_m in (4.14), we get

$$\|F'(x_0)^{-1}F(x_{m+1})\| \le \bar{\bar{v}}_m$$

Hence, we get eventually

$$\|y_{m+1}-x_{m+1}\| \le \frac{|\gamma|}{1-L_0t_{m+1}}\bar{\bar{v}}_m = s_{m+1}-t_{m+1}$$

and

$$\begin{aligned}\|y_{m+1}-x_0\| &\le \|y_{m+1}-x_{m+1}\|+\|x_{m+1}-x_0\| \\ &\le s_{m+1}-t_{m+1}+t_{m+1}-t_0 = s_{m+1}\le t^*,\end{aligned}$$

which completes the induction. Hence, the sequence $\{x_m\}$ is fundamental in Banach space X and as such it converges to some $x^*\in U[x_0,t^*]$. By the continuity of F and if we let $m\to\infty$ in (4.14), we conclude $F(x^*)=0$. □

Bibliography

[1] Argyros, I.K., On the Newton - Kantorovich hypothesis for solving equations, *J. Comput. Math.*, 169, (2004), 315-332.

[2] Argyros, I.K., *Computational theory of iterative methods*. Series: Studies in Computational Mathematics, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[3] Argyros, I.K., *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[4] Argyros, I.K., Hilout, S., Weaker conditions for the convergence of Newton's method. *J. Complexity*, 28, (2012), 364–387.

[5] Argyros, I.K., Hilout, S., On an improved convergence analysis of Newton's methods, *Applied Mathematics and Computation*, 225, (2013), 372-386.

[6] Argyros, I.K., Magréñan, A.A., *Iterative methods and their dynamics with applications*, CRC Press, New York, USA, 2017.

[7] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative methods*, Elsevier (Academic Press), New York, 2018.

[8] Arroyo, V., Cordero, A., Torregrosa, J.R., Approximation of artificial satellites preliminary orbits: The efficiency challenge. *Mathematical and Computer Modeling*, 54, (7-8), (2011), 1802-1807.

[9] Behl, R., Maroju, P., Martinez, E., Singh, S., A study of the local convergence of a fifth order iterative method , *Indian J. Pure Appl. Math.*, 51,2, (2020), 439-455.

[10] Cătinaş, How many steps still left to "x^*", *Siam Review*, 63, 3, (2021).

[11] Ezquerro, J.A., Gutiérrez, J.M., Hernández, M.A., Romero, N., Rubio, M.J., The Newton method: from Newton to Kantorovich (Spanish), *Gac. R. Soc. Mat. Esp.*, 13, (2010), 53-76.

[12] Ezquerro, J.A., Hernandez, M.A., *Newton's method: An updated approach of Kantorovich's theory*, Cham Switzerland, (2018).

[13] Grau-Sánchez, Miquel and Grau, Àngela and Noguera, Miquel, Ostrowski type methods for solving systems of nonlinear equations. *Appl. Math. Comput.*, 281, (2011), 2377-2385.

[14] Homeier, H.H.H., A modified Newton method with cubic convergence: the multivariate case, *Journal of Computational and Applied Mathematics*, 169, (2004), 161-169.

[15] Kantorovich, L.V., Akilov, G.P., *Functional Analysis*, Pergamon Press, Oxford, (1982).

[16] Magréñan, A.A., Argyros, I.K., Rainer, J.J., Sicilia, J.A., Ball convergence of a sixth-order Newton-like method based on means under weak conditions, *J. Math. Chem.*, (2018), 56, 2117-2131, https://doi.org/10.1007/ s10910-018-0856-y

[17] Magréñan, A.A., Gutiérrez, J.M., Real dynamics for damped Newton's method applied to cubic polynomials, *J. Comput. Appl. Math.*, 275, (2015), 527–538.

[18] Ortega, L.M., Rheinboldt, W.C., *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, (1970).

[19] Ostrowski, A.M., *Solution of equations in Euclidean and Banach spaces*, Elsevier, 1973.

[20] Potra, F.A. and Pták, V., Nondiscrete induction and iterative processes. *Research Notes in Mathematics*, 103. Pitman(Advanced Publishing Program), Boston, MA. (1984).

[21] Proinov P.D., General local convergence theory for a class of iterative processes and its applications to Newton's method, *J. Complexity*, 25, (2009), 38-62.

[22] Proinov, P.D., New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems, *J. Complexity*, 26, (2010), 3-42.

[23] Rheinboldt, W.C. *An adaptive continuation process of solving systems of nonlinear equations*. Polish Academy of Science, Banach Ctr. Publ. 3, (1978), 129-142.

[24] Shakhno, S.M., Gnatyshyn, O.P., On an iterative algorithm of order 1.839... for solving nonlinear least squares problems, *Appl. Math. Applic* 161, (2005), 253-264.

[25] Shakhno, S.M., Iakymchuk, R.P., Yarmola, H. P., Convergence analysis of a two step method for the nonlinear squares problem with decomposition of operator, J. Numer. Appl. Math., 128, (2018), 82-95.

[26] Sharma, J.R., Kumar, S., A class of accurate Newton-Jarratt like methods with applications to nonlinear models, *Comput. Appl. Math*, (2022).

[27] Traub, J.F., *Iterative methods for the solution of equations*, Prentice Hall, New Jersey, U.S.A, (1964).

[28] Verma, R., *New Trends in Fractional Programming*, Nova Science Publishers, New York, USA, (2019).

[29] Xiao, X., Yin, H., A new class of methods with high order of convergence for solving systems of nonlinear equations, *Appl. Math. Comput.*, 264, (2015), 300-309.

Chapter 5

On the Semi-Local Convergence of a Fifth Order Method for Solving Nonlinear Equations

The semi-local convergence of a fifth-order method is studied using assumptions only on the first derivative of the operator involved. The convergence of this method was shown by assuming that the sixth-order derivative of the operator does not exist and hence it limiting its applicability. Moreover, no computational error bounds or uniqueness of the solution are given. We address all these problems using only the first derivative that appears on the. Hence, we extend the applicability of the. Our techniques can be used to obtain the convergence of other similar higher-order schemes using assumptions on the first derivative of the operator involved.

1. Introduction

Let $F : D \subset X \longrightarrow Y$ be a nonlinear operator acting between Banach spaces X and Y. Consider the problem of solving the nonlinear equation

$$F(x) = 0. \tag{5.1}$$

Iterative methods are used to approximate a solution x^* of the equation (5.1). The following iterative method was studied in [21],

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - \frac{1}{12}(13I - 9A_n)F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - \frac{1}{2}(5I - 3A_n)F'(x_n)^{-1}F(z_n), \end{aligned} \tag{5.2}$$

where $\alpha \in \mathbb{R}, A_n = F'(x_n)^{-1}F'(y_n)$. If $\alpha = \frac{2}{3}$, (5.2) reduces to the method in [21]. This method was shown to be of order five using hypotheses on the sixth derivative. The local convergence was given in [21]. We present the semi-local convergence of method (5.2) using assumptions only on the first derivative of F, unlike earlier studies [21] where the convergence analysis required assumptions on the derivatives of F up to the order six. This technique can be used on other methods and relevant topics along the same lines [1,2,3,4, 5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29].

For example: Let $X = Y = \mathbb{R}$, $D = [-\frac{1}{2}, \frac{3}{2}]$. Define f on D by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we have $f(1) = 0$,

$$f'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously, $f'''(t)$ is not bounded by D. So, the convergence of the method (5.2) is not guaranteed by the analysis in [21].

Throughout the article $U(x_0, R) = \{x \in X : \|x - x_0\| < R\}$ and $U[x_0, R] = \{x \in X : \|x - x_0\| \leq R\}$ for some $R > 0$.

2. Sequences Majorizing Method (5.2)

Let L_0, L, L_1 be positive and η be a nonnegative parameter. Define sequence $\{t_n\}$ for $n = 0, 1, 2, \ldots$ by $t_0 = 0, s_0 = \eta$

$$\begin{aligned} u_n &= s_n + \frac{1}{|\alpha|}\left(\frac{3L(s_n - t_n)}{4(1 - L_0 t_n)} + \frac{1}{3}\right)(s_n - t_n), \\ t_{n+1} &= u_n + \left[\frac{3L(s_n - t_n)}{4(1 - L_0 t_n)^2} + \frac{2}{3(1 - L_0 t_n)}\right] u_n, \\ v_n &= \frac{L}{2}(u_n - t_n)^2 + L_1(u_n - s_n) + |1 - \frac{1}{\alpha}| L_1 (s_n - t_n), \\ s_{n+1} &= t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2L_1(t_{n+1} - s_n) + 2|1 - \frac{1}{\alpha}| L_1 (s_n - t_n)}{2(1 - L_0 t_{n+1})}. \end{aligned} \tag{5.3}$$

Next, some convergence criteria are given for this sequence.

Lemma 5. Suppose that for all $n = 0, 1, 2, \ldots$

$$t_n < \frac{1}{L_0} \tag{5.4}$$

Then, sequence $\{t_n\}$ is non-decreasing, bounded from above by $\frac{1}{L_0}$ and converges to its unique least upper bound $t^* \in [0, \frac{1}{L_0}]$.

Proof. It is implied by (5.3) and (5.4) that $0 \leq t_n \leq s_n \leq t_{n+1} < \frac{1}{L_0}$. Hence, we conclude that $t^* = lim_{n \to \infty} t_n \in [0, \frac{1}{L_0}]$. □

We can provide a second result on a majorizing sequence where the convergence criteria are stronger but easier to test than (5.3). But first, some polynomials and parameters are needed.
Define recurrent polynomials on $[0,1)$ by

$$\begin{aligned}
f_n^{(1)}(t) &= \frac{3L}{4|\alpha|}t^n\eta + 2L_0t(1+t+\dots+t^n)\eta \\
&- \frac{2L_0}{3|\alpha|}(1+t+\dots+t^n)\eta - t + \frac{1}{3|\alpha|}, \\
f_n^{(2)}(t) &= \frac{3L^2}{2}(1+t)^2t^{2n}\eta^2 + 3LL_1t^{n+1}\eta \\
&+ 3LL_1|1-\frac{1}{\alpha}|t^n\eta + \frac{L}{3}(1+t)^2t^n\eta + \frac{2}{3}L_1t \\
&+ \frac{2L_1}{3}|1-\frac{1}{\alpha}| + 2L_0t(1+t+\dots+t^n)\eta - t, \\
f_n^{(3)}(t) &= L(1+2t)t^n\eta + 4L_0t(1+t+\dots+t^{n+1})\eta \\
&+ 2L_1(1+t) + 2|1-\frac{1}{\alpha}|L_1 - 2t, \\
g_1(t) &= \frac{3Lt}{4|\alpha|} - \frac{3L}{4|\alpha|} + 2L_0t^2 - \frac{2L_0}{3|\alpha|}t, \\
f_\infty^{(1)}(t) &= f_1(t) = t^2 - (1+\frac{1}{3|\alpha|})(1-2L_0\eta)t + \frac{1}{3|\alpha|}(1-2L_0\eta), \\
g_n^{(2)}(t) &= \frac{3L^2}{2}(1+t)^2t^{n+2}\eta - \frac{3L^2}{2}(1+t)^2t^n\eta \\
&+ \frac{3LL_1}{2}t^2 - \frac{3}{2}LL_1t + 3LL_1|1-\frac{1}{\alpha}|t \\
&- 3LL_1|1-\frac{1}{\alpha}| + \frac{L}{3}(1+t)^2t - \frac{L}{3}(1+t)^2 + 2L_0t^3, \\
f_3(t) &= (L_1-1)t^2 + 2((1-2L_0\eta) + 2|1-\frac{1}{\alpha}|L_1)t \\
&- (L_1 + |1-\frac{1}{\alpha}|L_1 + |1-\frac{1}{\alpha}|),
\end{aligned}$$

and

$$g_3(t) = L(1+2t)t - L(1+2t) + 4L_0t^3.$$

By these definitions, we get

$$\begin{aligned}
g_1(0) &= -\frac{3L}{4|\alpha|} < 0, g_1(1) = 2L_0(1-\frac{1}{3|\alpha|}) > 0, \\
g_3(0) &= -L, \textit{ and } g_3(1) = 4L_0 > 0.
\end{aligned}$$

Hence, polynomials g_1 and g_3 have roots in the interval (0,1) by the intermediate value theorem. Denote the smallest such roots by δ_1 and δ_3 respectively. Equation $f_3(t) = 0$ has a unique positive root denoted by p_3 by Descartes' rule of signs(assuming $L_1 \geq 0$ and $2L_0\eta \leq 1$).

Set

$$\eta_0 = \frac{1-\frac{1}{3|\alpha|}}{\frac{3L}{4|\alpha|}+4L_0(1-\frac{1}{3|\alpha|})}.$$

Then, we have

$$f_1(0) = \frac{1}{3|\alpha|}(1-2L_0\eta) > 0$$

and

$$f_1(1) = (\frac{3L}{4|\alpha|}+4L_0(1-\frac{1}{3|\alpha|})\eta-(1-\frac{1}{3|\alpha|}).$$

Suppose

$$\eta \leq \eta_0.$$

Then, $f_1(1) < 0$, and polynomial f_1 has roots in $(0,1)$. Denote by p_1 the smallest such root. Moreover, define the polynomial

$$f_2(t) = (\frac{2}{3}L_1-1)t^2+[(1-2L_0\eta)+\frac{2}{3}L_1(|1-\frac{1}{\alpha}|-1)t-\frac{2L_1}{3}|1-\frac{1}{\alpha}|].$$

Let

$$\begin{aligned} g_2(t) = g_1^{(2)}(t) &= \frac{3L^2}{2}(1+t)^2t^3\eta-\frac{3L^2}{2}(1+t)^2t\eta \\ &+\frac{3LL_1}{2}t^2-\frac{3LL_1}{2}t+3LL_1|1-\frac{1}{\alpha}|t \\ &-3LL_1|1-\frac{1}{\alpha}|+\frac{L}{3}(1+t)^2t-\frac{L}{3}(1+t)^2+2L_0t^3. \end{aligned}$$

By this definition

$$g_2(0) = -3LL_1-\frac{L}{3} < 0 \textit{ and } g_2(1) = 2L_0 > 0.$$

Denote by δ_2 the smallest root of g_2 in $(0,1)$.
The preceding polynomials are connected.

Lemma 6. The following assertions hold:

(1) $f_{n+1}^{(1)}(t) = f_n^{(1)}(t)+g_1(t)t^n\eta.$

(2) $f_{n+1}^{(2)}(t) = f_n^{(2)}(t)+g_n^{(2)}(t)t^n\eta.$

(3) $g_{n+1}^{(2)}(t) \geq g_n^{(2)}(t).$

(4) $f_{n+1}^{(3)}(t) = f_n^{(3)}(t)+g_3(t)t^n\eta.$

Proof. By the definition of these polynomials we get in turn:

(1)

$$\begin{aligned}
f_{n+1}^{(1)}(t)-f_n^{(1)}(t)=&\frac{3L}{4|\alpha|}t^{n+1}\eta-\frac{3Lt^n\eta}{4|\alpha|}\\
&+2L_0t(1+t+\ldots+t^{n+1})\eta-2L_0t(1+t+\ldots+t^n)\eta\\
&-\frac{2L_0}{3|\alpha|}(1+t+\ldots+t^{n+1})\eta+\frac{2L_0}{3|\alpha|}(1+t+\ldots+t^{n+1})\\
&=\frac{3L}{4|\alpha|}t^{n+1}\eta-\frac{3Lt^n\eta}{4|\alpha|}+2L_0t^{n+2}\eta-\frac{2L_0t^{n+1}\eta}{3|\alpha|}\\
&=g_1(t)t^n\eta.
\end{aligned}$$

(2)

$$\begin{aligned}
f_{n+1}^{(2)}(t)-f_n^{(2)}(t)=&\frac{3L^2}{2}(1+t)^2t^{2n+2}\eta^2-\frac{3L^2}{2}(1+t)^2t^{2n}\eta^2\\
&+3LL_1t^{n+2}\eta-3LL_1t^{n+1}\eta+3LL_1|1-\frac{1}{\alpha}|t^{n+1}\eta\\
&-3LL_1|1-\frac{1}{\alpha}|t^n\eta+\frac{L}{3}(1+t)^2t^{n+1}\eta-\frac{L}{2}(1+t)^2t^n\eta\\
&+2L_0t^{n+2}\eta=g_n^{(2)}(t)t^n\eta.
\end{aligned}$$

(3)

$$\begin{aligned}
g_{n+1}^{(2)}(t)-g_n^{(2)}(t)=&\frac{3L^2}{2}(1+t)^2t^{n+3}\eta-\frac{3L^2}{2}(1+t)^2t^{n+2}\eta\\
&-\frac{3L^2}{2}(1+t)^2t^{n+1}\eta+\frac{3L^2}{2}(1+t)^2t^n\eta\\
&=\frac{3L^2}{2}(1+t)^2t^{n+2}\eta(t-1)-\frac{3L^2}{2}(1+t)^2t^n\eta(t-1)\\
&=\frac{3L^2}{2}(1+t)^3(t-1)^2t^n\eta\geq 0.
\end{aligned}$$

(4)

$$\begin{aligned}
f_{n+1}^{(3)}(t)-f_n^{(3)}(t)=&L(1+2t)t^{n+1}\eta-L(1+2t)t^n\eta\\
&+4L_0t(1+t+\ldots+t^{n+2})\eta-4L_0t(1+t+\ldots+t^{n+1})\eta\\
&=L(1+2t)t^{n+1}\eta-L(1+2t)t^n\eta+4L_0t^{n+3}\eta\\
&=g_3(t)t^n\eta.
\end{aligned}$$

□

Set

$$a=\frac{3L\eta}{4|\alpha|}+\frac{1}{3|\alpha|}, b=(\frac{3L\eta}{4}+\frac{2}{3}[\frac{L}{2}u_0^2+L_1(u_0-s_0)+|1-\frac{1}{\alpha}|L_1\eta]\frac{1}{\eta}, \eta\neq 0$$

$$c=\frac{Lt_1^2+2L_1(t_1-\eta)}{1\eta(1-L_0t_1)},$$

$$d=max\{a,b,c\}, \delta_0=min\{\delta_1,\delta_2,\delta_3\},$$

$$\delta=max\{\delta_1,\delta_2,\delta_3\}, p=min\{p_1,p_3,1-4L_0\eta\}$$

and

$$\eta_1=min\{\eta_0,\frac{1}{4L_0}\}.$$

Then, we can show the second convergence result for sequence (5.3).

Lemma 7. Suppose

$$\eta\leq\eta_1, L_0t_1<1, \tag{5.5}$$

$$0\leq d\leq\delta_0 \text{ and } \delta\leq p<1. \tag{5.6}$$

Then, sequence $\{t_n\}$ is non-decreasing, bounded from above by $t^{**}=\frac{2\eta}{1-\delta}$ and converges to its unique least upper bound $t^*\in[0,\frac{2\eta}{1-\delta}]$.

Notice that (5.5) and (5.6) determine the smallness of η for convergence. Moreover, the following assertions hold

$$0\leq s_n-t_n\leq\delta(s_{n-1}-t_{n-1})\leq\delta^n\eta, \tag{5.7}$$

$$0\leq u_n-s_n\leq\delta(s_n-t_n)\leq\delta^{n+1}\eta, \tag{5.8}$$

$$0\leq t_{n+1}-u_n\leq\delta(s_n-t_n)\leq\delta^{n+1}\eta \tag{5.9}$$

and

$$0\leq t_n\leq s_n\leq u_n\leq t_{n+1}. \tag{5.10}$$

Proof. Mathematical induction is utilized to show

$$0 \leq \frac{3L(s_n-t_n)}{4|\alpha|(1-L_0t_n)}+\frac{1}{3|\alpha|}\leq\delta, \tag{5.11}$$

$$2t_n \leq 1, \tag{5.12}$$

$$0 \leq (\frac{3L(s_n-t_n)}{4(1-L_0t_n)^2}+\frac{2}{3(1-L_0t_n)})v_n\leq\delta(s_n-t_n) \tag{5.13}$$

and

$$\frac{L(t_{n+1}-t_n)^2+2L_1(t_{n+1}-s_n)+2L_1|1-\frac{1}{\alpha}|(s_n-t_n)}{2(1-L_0t_{n+1})}\leq\delta(s_n-t_n). \tag{5.14}$$

These estimates hold for for $n=0$ by the initial conditions, (5.5) and (5.6). It follows that $0\leq s_0-t_0=\eta$, $0\leq u_0-s_0\leq\delta(s_0-t_0)$, $0\leq t_1-u_0\leq\delta(s_0-t_0)$, $0\leq s_1-t_1\leq\delta(s_0-t_0)$.

So, (5.7)-(5.10) hold at the beginning. Suppose that hold for all integer values smaller than n. Then, notice that we can get in turn

$$\begin{aligned} t_{k+1} &\leq u_k+\delta^{k+1}\eta \leq s_k+\delta^{k+1}\eta+\delta^{k+1}\eta \leq t_k+\delta^k\eta+\delta^{k+1}\eta+\delta^{k+1}\eta \\ &\leq u_{k-1}+2(\delta^k\eta+\delta^{k+1}\eta) \leq ... \leq t_{k-1}+\delta^{k-1}\eta+2(\delta^k\eta+\delta^{k+1}\eta) \\ &\leq 2\frac{1-\delta^{k+2}}{1-\delta}\eta < \frac{2\eta}{1-\delta} = t^{**}. \end{aligned} \tag{5.15}$$

Evidently, (5.11) certainly holds if

$$\frac{3L\delta^k\eta}{4|\alpha|} \leq (\delta-\frac{1}{3|\alpha|})(1-2L_0(1+\delta+...+\delta^k)\eta)$$

or

$$\frac{3L\delta^k\eta}{4|\alpha|}+2L_0(1+\delta+...+\delta^k)\eta-\delta+\frac{1}{3|\alpha|}-\frac{2L_0}{3|\alpha|}+(1+\delta+...+\delta^k)\eta \leq 0$$

or

$$f_n^{(1)}(t) \leq 0 \; at \; t=\delta_1. \tag{5.16}$$

In particular, by Lemma 6 (1) and the definition of δ_1, we have

$$f_{n+1}^{(1)}(t) = f_n^{(1)}(t) \; at \; t=\delta_1.$$

Define the function $f_\infty^{(1)}(t) = lim_{k\to\infty} f_k^{(1)}(t)$.
It follows by definition of polynomial $f_k^{(1)}$ that

$$f_\infty^{(1)}(t) = \frac{2L_0 t\eta}{1-t}-t+\frac{1}{3|\alpha|}-\frac{2L_0\eta}{3|\alpha|}.$$

Thus, (5.16) holds if

$$f_\infty^{(1)}(t) \leq 0 \; at \; t=\delta_1$$

or

$$f_1(t) \leq 0 \; at \; t=\delta_1$$

which is true by (5.6).
Estimate (5.12) holds since
$2t_n = 2\cdot 2\frac{\eta}{1-\delta} \leq 1$ by choice of δ in (5.6). Then notice that $\frac{1}{1-L_0 t_n} \leq 2$. Hence, estimate (5.13) holds if

$$(3L\delta^n\eta+\frac{2}{3})(\frac{L}{2}(1+\delta)^2\delta^n\eta+L_1\delta+|1-\frac{1}{\alpha}|L_1)+2L_0(1+\delta+\ldots+\delta^n)\eta-\delta \leq 0$$

or

$$f_n^{(2)}(t) \leq 0 \; at \; t=\delta_2. \tag{5.17}$$

By Lemma 6 (2) and (3), we have

$$\begin{aligned} f_{n+1}^{(2)}(t) &= f_n^{(2)}(t) + g_n^{(2)}(t)t^n\eta \\ &\geq f_n^{(2)}(t) + g_1^{(2)}(t)t^n\eta \\ &= f_n^{(2)}(t) + g_2(t)t^n\eta \end{aligned}$$

so

$$f_{n+1}^{(2)}(t) \geq f_n^{(2)}(t) \; at \; t = p_2.$$

by the definition of p_2.
Define function $f_\infty^{(2)}(t) = lim_{k\to\infty} f_k^{(2)}(t)$.
It follows by the definition of polynomials $f_k^{(2)}$ that

$$f_\infty^{(2)}(t) = \frac{2}{3}L_1 t + \frac{2L_1}{3}|1 - \frac{1}{\alpha}| + \frac{2L_0 t}{1-t}\eta - t.$$

Hence, (5.17) holds if $f_\infty^{(2)}(t) \leq 0$ at $t = p_2$ or

$$f_2(t) \leq 0 \; at \; t = p_2$$

which is true by (5.6). Moreover, to show (5.15) it suffices to have

$$L(1+2\delta)\delta^n\eta + 4L_0\delta(1+\delta+\ldots+\delta^{n+2})\eta + 2L_1(1+\delta) + 2|1-\frac{1}{\alpha}|L_1 - 2\delta \leq 0$$

or

$$f_n^{(3)}(t) \leq 0 \; at \; t = \delta_3. \tag{5.18}$$

By Lemma 6 (4), and the definition of δ_3, we have

$$f_{n+1}^{(3)}(t) = f_n^{(3)}(t) \; at \; t = \delta_3.$$

Define function $f_\infty^{(3)} = lim_{k\to\infty} f_k^{(3)}(t)$. Then, we obtain

$$\begin{aligned} f_\infty^{(3)}(t) &= 4L_0\eta t + 2L_1(1-t^2) + 2L_1|1-\frac{1}{\alpha}| \\ &+ 2|1-\frac{1}{\alpha}| - 2|1-\frac{1}{\alpha}|L_1 t - 2t + 2t^2. \end{aligned}$$

Hence, (5.18) holds for $f_\infty^{(3)}(t) \leq 0$ or

$$(L_1-1)t^2 + 2[(1-2L_0\eta) + 2L_1|1-\frac{1}{\alpha}|]t - (L_1 + L_1|1-\frac{1}{\alpha}| + |1-\frac{1}{\alpha}|) \leq 0$$

or $f_3(t) \leq 0$, which is true by (5.6). The induction for items (5.11)-(5.15) is furnished. Hence, assertions (5.7)-(5.10) hold true too. Therefore, we deduce $lim_{k\to\infty} t_k = t^* \in [0, t^{**}]$.

□

3. Analysis for Method (5.2)

Consider conditions (T).

Suppose:

(T_1) There exist $x_0 \in D, \eta \geq 0$ so that $F^{'}(x_0)^{-1}$ exist and

$$\|F^{'}(x_0)^{-1}F(x_0)\| \leq \eta.$$

(T_2) There exist $L_0 > 0$ so that for all $w \in D$

$$\|F^{'}(x_0)^{-1}(F^{'}(w) - F^{'}(x_0))\| \leq L_0\|w - x_0\|.$$

Define $D_1 = U(x_0, \frac{1}{L_0}) \cap D$

(T_3) There exist $L > 0, L_1 \geq 1$ so that for all $w_1, w_2 \in D_1$

$$\|F^{'}(x_0)^{-1}(F^{'}(w_2) - F^{'}(w_1))\| \leq L\|w_2 - w_1\|$$

and

$$\|F^{'}(x_0)^{-1}F^{'}(w_2)\| \leq L_1.$$

(T_4) Conditions of Lemma 5 or Lemma 7 hold
and

(T_5) $U[x_0, t^*] \subset D$

Next, conditions T are used to show convergence for (5.2).

Theorem 5. Under conditions T the following assertions hold

$$\{x_n\} \subset U[x_0, t^*], \tag{5.19}$$
$$\|y_n - x_n\| \leq s_n - t_n, \tag{5.20}$$
$$\|z_n - y_n\| \leq u_n - s_n \tag{5.21}$$

and

$$\|x_{n+1} - z_n\| \leq t_{n+1} - u_n. \tag{5.22}$$

Moreover, $lim_{n\to\infty}x_n = x^* \in U[x_0, t^*]$ and so that x^* solves (5.1).

Proof. Conditions (T_1), method (5.2) and (5.3) give

$$\|y_0 - x_0\| = \|f^{'}(x_0)^{-1}F(x_0)\| \leq \eta = s_0 - t_0 \leq t^*,$$

so $y_0 \leq U[x_0, t^*]$ and (5.20) is true for $n = 0$. Pick $w \leq U[x_0, t^*]$. Then, by using (T_1) and (T_2), we obtain

$$\|F(x_0)^{-1}(F^{'}(w) - F(w_0))\| \leq L_0\|w - x_0\| \leq L_0 t^* < 1. \tag{5.23}$$

Thus, $F^{'}(w)^{-1}$ exists by (5.23) and a Lemma due to Banach on linear operators with inverses [15] and

$$\|F^{'}(w)^{-1}F^{'}(x_0)\| \leq \frac{1}{1-L_0\|w-x_0\|}. \tag{5.24}$$

Then, iterates y_0, z_0 and x_1 exist.
Suppose assertion (5.19)-(5.22) hold for all $k \leq n$. We need in turn the estimates taken from method (5.2), using the induction hypothesis, (5.24)(for $w = x_k$), and (T_3)

$$\begin{aligned}
z_k - y_k &= -\frac{1}{12}(9I-9A_k)F^{'}(x_k)^{-1}F(x_k) - \frac{1}{3}F^{'}(x_k)^{-1}F(x_k)\\
&= -\frac{3}{4}(I-A_k)F^{'}(x_k)^{-1}F(x_k) - \frac{1}{3}F^{'}(x_k)^{-1}F(x_k)\\
&= -\frac{3}{4}(I-F^{'}(x_k)^{-1}F^{'}(y_k))F^{'}(x_k)^{-1}F(x_k) - \frac{1}{3}F^{'}(x_k)^{-1}F(x_k)\\
&= -\frac{3}{4}F^{'}(x_k)^{-1}(F^{'}(x_k)-F^{'}(y_n))F^{'}(x_k)^{-1}F(x_k) - \frac{1}{3}F^{'}(x_k)^{-1}F(x_k),
\end{aligned}$$

so

$$\begin{aligned}
\|z_k - y_k\| &\leq \frac{3}{4}\frac{L\|y_k-x_k\|\|\frac{y_k-x_k}{\alpha}\|}{(1-L_0\|x_k-x_0\|)} + \frac{1}{3|\alpha|}\|y_k-x_k\|\\
&\leq (\frac{3L(s_k-t_k)}{4|\alpha|(1-L_0t_k)} + \frac{1}{3|\alpha|})(s_k-t_k) = u_k - s_k
\end{aligned}$$

and

$$\|z_k - x_0\| \leq \|z_k-y_k\| + \|y_k-x_0\| \leq u_k - s_k + s_k - t_0 = u_k \leq t^*.$$

Therefore, (5.21) holds and $z_k \in U[x_0,t^*]$. Similarly, by the second substep of method (5.2), we get in turn

$$\begin{aligned}
x_{k+1} - z_k &= \frac{3}{4}F^{'}(x_k)^{-1}(F^{'}(x_k)-F^{'}(y_k))F^{'}(x_k)^{-1}F(z_k)\\
&\quad -\frac{2}{3}F^{'}(x_k)^{-1}F(z_k) = B_kF(z_k)
\end{aligned}$$

so

$$\begin{aligned}
\|x_{k+1} - z_k\| &\leq [\frac{3}{4}\frac{L\|y_k-x_k\|}{(1-L_0\|x_k-x_0\|)^2} + \frac{2}{3(1-L_0\|x_k-x_0)}]v_k,\\
&\leq [\frac{3}{4}\frac{L(s_k-t_k)}{(1-L_0t_k)^2} + \frac{2}{3(1-L_0t_k)}]v_k = t_{k+1} - u_k
\end{aligned}$$

and

$$\|x_{k+1} - x_0\| \leq \|x_{k+1}-z_k\| + \|z_k-x_0\| \leq t_{k+1} - u_k + u_k - t_0 = t_{k+1} \leq t^*.$$

Thus, (5.22) holds and $x_{k+1} \in U[x_0,t^*]$.
We also used

$$\begin{aligned} F(z_k) &= F(z_k) - F(x_k) + F(x_k) \\ &= F(z_k) - F(x_k) - F'(x_k)(z_k - x_k) + F'(x_k)(z_k - y_k) \\ &+ (1 - \frac{1}{\alpha})F'(x_k)(y_k - x_k), \end{aligned}$$

so

$$\begin{aligned} \|F'(x_0)^{-1}F(z_k)\| &\leq \frac{L}{2}\|z_k - x_k\|^2 + L_1\|z_k - y_k\| \\ &+ L_1|1 - \frac{1}{\alpha}|\|y_n - x_k\| \leq v_k. \end{aligned}$$

Set $b_k = \int_0^1 F(x_k + \theta(x_{k+1} - x_k))d\theta(x_{k+1} - x_k)$.
Moreover, by method (5.2), we can write instead

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) + F(x_k) \\ &= b_k(x_{k+1} - x_k) - \frac{F'(x_k)}{\alpha}(y_k - x_k) \\ &= b_k(x_{k+1} - x_k) - F'(x_k)(x_{k+1} - x_k) + F'(x_k)(x_{k+1} - x_k) \\ &- F'(x_k)(y_k - x_k) + F'(x_k)(y_n - x_k) - \frac{F'(x_k)}{\alpha}(y_k - x_k) \\ &= (b_k - F'(x_k))(x_{k+1} - x_k) \\ &+ F'(x_k)(x_{k+1} - y_k) + |1 - \frac{1}{\alpha}|F'(x_k)(y_k - x_k), \end{aligned}$$

so

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{k+1})\| &\leq \frac{L}{2}\|x_{k+1} - x_k\|^2 + L_1\|x_{k+1} - y_k\| \\ &+ |1 - \frac{1}{\alpha}|L_1\|y_k - x_k\| \\ &\leq \frac{L}{2}(t_{k+1} - t_k)^2 + L_1(t_{k+1} - s_k) + L_1|1 - \frac{1}{\alpha}|(s_k - t_k). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \|y_{k+1} - x_{k+1}\| &\leq \|F'(x_{k+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{k+1}\| \\ &\frac{\leq \frac{L}{2}(t_{k+1} - t_k)^2 + L_1(t_{k+1} - s_k) + L_1|1 - \frac{1}{\alpha}|(s_k - t_k)}{1 - L_0\|x_{k+1} - x_0\|} \\ &\leq s_{k+1} - t_{k+1} \end{aligned}$$

and

$$\begin{aligned} \|y_{k+1} - x_0\| &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \\ &\leq s_{k+1} - t_{k+1} + t_{k+1} - t_0 = s_{k+1} \leq t^*, \end{aligned}$$

so (5.20) holds and $y_{k+1} \in U[0,t^*]$.

Remark. The parameter $\frac{1}{L_0}, \frac{2\eta}{1-\delta}$ given in closed form can replace t^* in (T_5) assuming that conditions of Lemma 5 or Lemma 7 hold, respectively.

The uniqueness of the solution is provided.

Proposition 3. *Assume*

(1) The point $x^* \in U[x_0, \delta] \subset D$ *is such that* $F(x^*) = 0$ *for some* $\delta > 0$ *and* $F'(x^*)^{-1}$ *exists.*

(2) Condition (T_2) *holds.*

(3) There exist $r \geq \delta$ *such that*

$$\frac{L_0}{2}(r+\delta) < 1. \tag{5.25}$$

Define $D_1 = U[x^*, r] \cap D$,
Set $Q = \int_0^1 F'(x^* + \theta(\lambda - x^*))d\theta$ *for some* $\lambda \in D_1$ *with* $F(\lambda) = 0$. *Then, in view of* (T_2) *and (5.25), we have*

$$\begin{aligned}\|F'(x_0)^{-1}(Q - F'(x_0))\| &\leq L_0 \int_0^1 [(1-\theta)\|x_0 - x^*\| + \theta\|\lambda - x^*\|]d\theta \\ &\leq \frac{L_0}{2}(r+\delta) < 1.\end{aligned}$$

Hence, we conclude $\lambda = x^*$.

□

Remark. Notice that not all conditions (H) are used in Proposition 3. But if they were used, then we can certainly set $\rho_0 = t^*$.

Bibliography

[1] Argyros, I.K., On the Newton - Kantorovich hypothesis for solving equations, *J. Comput. Math.*, 169, (2004), 315-332.

[2] Argyros, I.K., *Computational theory of iterative methods. Series: Studies in Computational Mathematics*, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[3] Argyros, I.K., *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[4] Argyros, I.K., Hilout, S., Weaker conditions for the convergence of Newton's method. *J. Complexity*, 28, (2012), 364–387.

[5] Argyros, I.K., Hilout, S., On an improved convergence analysis of Newton's methods, *Applied Mathematics and Computation*, 225, (2013), 372-386.

[6] Argyros, I.K., Magréñan, A.A., *Iterative methods and their dynamics with applications*, CRC Press, New York, USA, 2017.

[7] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative methods*, Elsevier (Academic Press), New York, 2018.

[8] Arroyo, V., Cordero, A., Torregrosa, J.R., Approximation of artificial satellites preliminary orbits: The efficiency challenge. *Mathematical and Computer Modeling*, 54, (7-8), (2011), 1802-1807.

[9] Behl, R., Maroju, P., Martinez, E., Singh, S., A study of the local convergence of a fifth order iterative method, *Indian J. Pure Appl. Math.*, 51, 2, (2020), 439-455.

[10] Cătinaş, Emil The inexact, inexact perturbed, and quasi-Newton methods are equivalent models, *Math. Comp.*, 74, (2005), 291-301.

[11] Ezquerro, J.A., Gutiérrez, J.M., Hernández, M.A., Romero, N., Rubio, M.J., The Newton method : from Newton to Kantorovich(Spanish), *Gac. R. Soc. Mat. Esp.*, 13, (2010), 53-76.

[12] Ezquerro, J.A., Hernandez, M.A., *Newton's method: An updated approach of Kantorovich's theory*, Cham Switzerland, (2018).

[13] Grau-Sánchez, Miquel and Grau, Àngela and Noguera, Miquel, Ostrowski type methods for solving systems of nonlinear equations. *Appl. Math. Comput* 281, (2011), 2377-2385.

[14] Homeier, H.H.H., A modified Newton method with cubic convergence: the multivariate case, *Journal of Computational and Applied Mathematics*, 169, (2004), 161-169.

[15] Kantorovich, L.V., Akilov, G.P., *Functional Analysis*, Pergamon Press, Oxford, (1982).

[16] Magréñan, A. A., Argyros, I. K., Rainer, J.J., Sicilia, J.A., Ball convergence of a sixth-order Newton-like method based on means under weak conditions, *J. Math. Chem* (2018), 56, 2117-2131, https://doi.org/10.1007/ s10910-018-0856-y.

[17] Magréñan, A.A., Gutiérrez, J.M., Real dynamics for damped Newton's method applied to cubic polynomials, *J. Comput. Appl. Math.*, 275, (2015), 527–538.

[18] Ortega, L.M., Rheinboldt, W.C., *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, (1970).

[19] Ostrowski, A. M., *Solution of equations in Euclidean and Banach spaces*, Elsevier, 1973.

[20] Potra, F.A. and Pták, V., Nondiscrete induction and iterative processes. *Research Notes in Mathematics*, 103. Pitman(Advanced Publishing Program), Boston,MA. (1984).

[21] Proinov, P.D., General local convergence theory for a class of iterative processes and its applications to Newton's method, *J. Complexity*, 25, (2009), 38-62.

[22] Proinov, P.D., New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems, *J. Complexity*, 26, (2010), 3-42.

[23] Rheinboldt, W.C. *An adaptive continuation process of solving systems of nonlinear equations*. Polish Academy of Science, Banach Ctr. Publ. 3,(1978), 129-142.

[24] Shakhno, S.M., Gnatyshyn, O.P., On an iterative algorithm of order 1.839... for solving nonlinear least squares problems, *Appl. Math. Applic.*, 161, (2005), 253-264.

[25] Shakhno, S.M., Iakymchuk, R.P., Yarmola, H.P., Convergence analysis of a two step method for the nonlinear squares problem with decomposition of operator, *J. Numer. Appl. Math.*, 128, (2018), 82-95.

[26] Sharma, J.R., Guha, R.K. and Sharma, R., An efficient fourth order weighted - Newton method for systems of nonlinear equations. *Numer. Algorithms*, 62, (2013), 307-323.

[27] Sharma, J.R, Kumar, S., A class of accurate Newton-Jarratt like methods with applications to nonlinear models, *Comput. Appl. Math*, (2022).

[28] Traub, J.F., *Iterative methods for the solution of equations*, Prentice Hall, New Jersey, U.S.A, (1964).

[29] Verma, R., *New Trends in Fractional Programming*, Nova Science Publishers, New York, USA, (2019).

Chapter 6

On the Semi-Local Convergence of a Fifth Order Efficient Method for Solving Nonlinear Equations

1. Introduction

The semi-local convergence for a Xiao-Yin method of order five is studied using assumptions only on the first derivative of the operator involved. The convergence of this method was shown by assuming that the sixth order derivative of the operator not on the method exists and hence it is limiting its applicability. Moreover, no computational error bounds or uniqueness of the solution are given. We address all these problems using only the first derivative that appears on the method. Hence, we extend the applicability of the method. Our techniques can be used to obtain the convergence of other similar higher-order methods using assumptions on the first derivative of the operator involved.

Let $F : D \subset E_1 \longrightarrow E_2$ be a nonlinear operator acting between Banach spaces E_1 and E_2 and $D \neq \emptyset$ be an open set. Consider the problem of solving the nonlinear equation

$$F(x) = 0. \tag{6.1}$$

Iterative methods are used to approximate a solution x^* of the equation (6.1). The following iterative method was studied in [26],

$$\begin{aligned} y_n &= x_n - \beta F'(x_n)^{-1} F(x_n), \\ z_n &= x_n - \frac{1}{4}(3F'(y_n)^{-1} + F'(x_n)^{-1})F(x_n), \end{aligned}$$

and

$$x_{n+1} = z_n - \frac{1}{2}(3F'(y_n)^{-1} - F'(x_n)^{-1})F(z_n).$$

In this chapter, we study the semi-local convergence of method (6.2) using assumptions only on the first derivative of F, unlike earlier studies [26] where the convergence analysis required assumptions on the derivatives of F up to the order six. This method can be used on other methods and relevant topics along the same lines [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29].

For example: Let $X = Y = \mathbb{R}$, $D = [-\frac{1}{2}, \frac{3}{2}]$. Define f on D by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we have $f(1) = 0$,

$$f'''(t) = 6\log t^2 + 60t^2 - 24t + 22.$$

Obviously, $f'''(t)$ is not bounded by D. So, the convergence of the method (6.2) is not guaranteed by the analysis in [26].

Throughout the chapter $U(x_0, R) = \{x \in X : \|x - x_0\| < R\}$ and $U[x_0, R] = \{x \in X : \|x - x_0\| \leq R\}$ for some $R > 0$.

The chapter contains a semi-local convergence analysis in Section 2.

2. Semi-Local Analysis

The convergence analysis is based on the majorizing sequence. Let $B > 0, L_0 > 0, L_1 > 0, L > 0$ and $R \geq 0$ be given parameters. Define scalar sequence $\{t_n\}$ by $t_0 = 0$, $s_0 = R$,

$$\begin{aligned} u_n &= s_n + \frac{L|1 - \frac{3}{4B}|(s_n - t_n)^2 + L_1(s_n - t_n)}{1 - L_0 s_n}, \\ v_n &= \frac{L}{2}(u_n - t_n)^2 + L_1(u_n - s_n) + L_1|1 - \frac{1}{B}|(s_n - t_n), \\ t_{n+1} &= u_n + \frac{1}{1 - L_0 s_n}\left(1 + \frac{L(s_n - t_n)}{2(1 - L_0 t_n)}\right) v_n \end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2L_1(t_{n+1} - s_n) + 2|1 - \frac{1}{B}|L_1(s_n - t_n)}{2(1 - L_0 t_{n+1})}.$$

Next, a convergence result for a sequence (6.2) is provided.

Lemma 8. Suppose that for each $n = 0, 1, 2, \cdots$

$$t_n < \frac{1}{L_0} \quad and \quad s_n < \frac{1}{L_0}. \tag{6.2}$$

Then, sequence $\{t_n\}$ is non-decreasing, bounded from above by $\frac{1}{L_0}$ and converges to its unique least upper bound $t^\star \in [0, \frac{1}{L_0}]$.

Proof. It follows from (6.2) and (6.2) that $0 \leq t_n \leq s_n \leq u_n \leq t_{n+1} < \frac{1}{L_0}$, so $\lim_{n\to\infty} t_n = t^\star \in [0, \frac{1}{L_0}]$. □

Sequence $\{t_n\}$ shall be shown to be majorizing for method (1.2). But first we introduce the needed conditions (A).

Suppose:

(A_1) There exists $x_0 \in \Omega$, $R \geq 0$ such that $F'(x_0)^{-1} \in \mathcal{L}(Y,X)$ and $\|F'(x_0)^{-1}F(x_0)\| \leq R$.

(A_2) There exists $L_0 > 0$ such that for all $x \in \Omega$, $\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_0\|x - x_0\|$.

Define $\Omega_1 = U(x_0, \frac{1}{L_0}) \cap \Omega$.

(A_3) There exist $L > 0$, $L_1 > 0$ such that for each $x, y \in \Omega_1$, $\|F'(x_0)^{-1}(F'(y) - F'(x))\| \leq L\|y - x\|$ and $\|F'(x_0)^{-1}F'(x)\| \leq L_1$.

(A_4) Conditions of Lemma 8 hold.

and

(A_5) $U[x_0, t^\star] \subset \Omega$.

Next, the semi-local convergence of method (1.2) is presented based on conditions A.

Theorem 6. Suppose conditions A hold. Then, sequence $\{x_n\}$ generated by method (1.2) is well defined in $U[x_0, t^\star]$, remains in $U[x_0, t^\star]$ and converges to a solution $x^\star \in U[x_0, t^\star]$ of equation $F(x) = 0$. Moreover, the following items hold

$$\begin{aligned} \|y_n - x_n\| &\leq s_n - t_n, \\ \|z_n - y_n\| &\leq \|u_n - s_n\|, \\ \|x_{n+1} - z_n\| &\leq t_{n+1} - u_n \end{aligned}$$

and

$$\|x^\star - x_n\| \leq t^\star - t_n. \tag{6.3}$$

Proof. Estimates $(6.3) - (6.3)$ are shown using induction. By (6.2) and (A_1) we have $\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq R = s_0 - t_0 = s_0 < t^{\star\star}$, so $y_0 \in U[x_0, t^\star]$ and (6.3) holds for $n = 0$. Let $x \in U[x_0, t^\star]$. Then, using (A_1) and (A_2), we get $\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_0\|x - x_0\| \leq L_0 t^\star < 1$, so $F'(x)^{-1} \in \mathcal{L}(Y,X)$ and

$$\|F'(x)^{-1}F'(x_0)\| \leq \frac{1}{1 - L_0\|x - x_0\|} \tag{6.4}$$

by the Banach lemma on invertible linear operators [1, 2, 3, 4, 5, 6, 7, 15, 19]. Moreover, iterates z_0 and x_k exist by (6.4) and since x_0 and y_0 belong in $U[x_0, t^\star]$. We can write by the first two substeps of method (1.2) and supposing estimates (6.3) hold for all values k smaller than n.

$$\begin{aligned} z_k &= y_k + [BF'(x_k)^{-1} - \frac{1}{4}(3F'(y_k)^{-1} + F'(x_k)^{-1})]F(x_k) \\ &= y_k + [\left(B - \frac{3}{4}\right)F'(x_k)^{-1} - \frac{3}{4}F'(y_k)^{-1}]F(x_k) \\ &= y_k - [\left(B - \frac{3}{4}\right)(F'(y_k)^{-1} - F'(x_k)^{-1}) - BF'(y_k)^{-1}]F(x_k) \\ &= y_k - (B - \frac{3}{4})F'(y_k)^{-1}(F'(x_k) - F'(y_k))F'(x_k)^{-1}F(x_k) \\ &\quad + BF'(y_k)^{-1}F(x_k). \end{aligned}$$

Using $(6.2), (A_3), (6.4)$ (for $x = x_k, y_k$), (6.5) we obtain in turn that

$$\begin{aligned} \|z_k - y_k\| &\leq \frac{|B - \frac{3}{4}|L\|y_k - x_k\| \frac{1}{|B|}\|y_k - x_k\| + L_1\|y_k - x_k\|}{1 - L_0\|y_k - x_0\|} \\ &\leq \frac{L|1 - \frac{3}{4B}|\|s_k - t_k\|^2 + L_1\|s_k - t_k\|}{1 - L_0 s_k} = u_k - s_k \end{aligned}$$

and

$$\|z_k - x_0\| \leq \|z_k - y_k\| + \|y_k - x_0\| \leq u_k - s_k + s_k - t_0 = u_k < t^\star,$$

so (6.3) holds and the iterate $z_k \in U[x_0, t^\star]$.

We need an estimate:

$$\begin{aligned} F(z_k) &= F(z_k) - F(x_k) + F(x_k) \\ &= F(z_k) - F(x_k) - \frac{F'(x_k)}{B}(y_k - x_k) \\ &= F(z_k) - F(x_k) - F'(x_k)(z_k - x_k) + F'(x_k)(z_k - x_k) \\ &\quad - \frac{F'(x_k)}{B}(y_k - x_k) \\ &= F(z_k) - F(x_k) - F'(x_k)(z_k - x_k) + F'(x_k)(z_k - x_k) \\ &\quad + F'(x_k)(y_k - x_k) - F'(x_k)(y_k - x_k) - \frac{F'(x_k)}{B}(y_k - x_k) \\ &= F(z_k) - F(x_k) - F'(x_k)(z_k - x_k) + F'(x_k)(z_k - y_k) \\ &\quad + (1 - \frac{1}{B})F'(x_k)(y_k - x_k), \end{aligned}$$

so

$$\begin{aligned} \|F'(x_0)^{-1}F(z_k)\| &\leq \frac{L}{2}\|z_k - x_k\|^2 + L_1\|z_k - y_k\| + |1 - \frac{1}{B}|L_1\|y_k - x_k\| \\ &\leq \frac{L}{2}(u_k - t_k)^2 + L_1(u_k - s_k) + |1 - \frac{1}{B}|L_1(s_k - t_k) \\ &= v_k. \end{aligned}$$

Then, by the third substep of method (1.2), we can write in turn that

$$\begin{aligned} x_{k+1} - z_k &= -\frac{1}{2}(F'(y_k)^{-1} - F'(x_k)^{-1})F(z_k) - F'(y_k)^{-1}F(z_k) \\ &= -\frac{1}{2}F'(y_k)^{-1}(F'(x_k) - F'(y_k))F'(x_k)^{-1}F(z_k) - F'(y_k)^{-1}F(z_k). \end{aligned}$$

Using $(6.2), 6.4$ (for $x = x_k, y_k$,) $(A_3), (6.5)$ and (6.5), we get in turn that

$$\begin{aligned} \|x_{k+1} - z_k\| &\leq \left[\frac{L\|y_k - x_k\|}{2(1 - L_0\|y_k - x_0\|)(1 - L_0\|x_k - x_0\|)} + \frac{1}{1 - L_0\|y_k - x_0\|}\right]v_k \\ &\leq \frac{1}{1 - L_0 s_k}\left(1 + \frac{L(s_k - t_k)}{2(1 - L_0 t_k)}\right)v_k = t_{k+1} - u_k \end{aligned}$$

and

$$\begin{aligned}\|x_{k+1}-x_0\| &\leq \|x_{k+1}-z_k\|+\|z_k-x_0\| \leq t_{k+1}-u_k+u_k-t_0\\ &= t_{k+1}<t^\star,\end{aligned}$$

so (6.3) holds and the iterate $x_{k+1} \in U[x_0,t^\star]$.

Set $d_k = \int_0^1 F'(x_k+\theta(x_{k+1}-x_k))d\theta$. Then, we can write in turn by the first substep of method (1.2)

$$\begin{aligned}F(x_{k+1}) &= F(x_{k+1})-F(x_k)+F(x_k)\\ &= d_k(x_{k+1}-x_k)-\frac{F'(x_k)}{B}(y_k-x_k)\\ &= d_k(x_{k+1}-x_k)-F'(x_k)(x_{k+1}-x_k)+F'(x_k)(x_{k+1}-x_k)-F'(x_k)(y_k-x_k)\\ &\quad +F'(x_k)(y_k-x_k)-\frac{F'(x_k)}{B}(y_k-x_k)\\ &= (d_k-F'(x_k))(x_{k+1}-x_k)+F'(x_k)(x_{k+1}-y_k)+\left(1-\frac{1}{B}\right)F'(x_k)(y_k-x_k),\end{aligned}$$

$$\begin{aligned}\|F'(x_0)^{-1}F(x_{k+1})\| &\leq \frac{L}{2}\|x_{k+1}-x_k\|^2+L_1\|x_{k+1}-y_k\|+|1-\frac{1}{B}|L_1\|y_k-x_k\|\\ &\leq \frac{L}{2}(t_{k+1}-t_k)^2+L_1(t_{k+1}-s_k)+|1-\frac{1}{B}|L_1(s_k-t_k),\end{aligned}$$

so

$$\begin{aligned}\|y_{k+1}-x_{k+1}\| &\leq \|F'(x_{k+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{k+1})\|\\ &\leq \frac{L(t_{k+1}-t_k)^2+2L_1(t_{k+1}-s_k)+2|1-\frac{1}{B}|L_1(s_k-t_k)}{2(1-L_0t_{k+1})}\\ &= s_{k+1}-t_{k+1}\end{aligned}$$

and

$$\begin{aligned}\|y_{k+1}-x_0\| &\leq \|y_{k+1}-x_{k+1}\|+\|x_{k+1}-x_0\| \leq s_{k+1}-t_{k+1}+t_{k+1}-t_0\\ &= s_{k+1}<t^\star,\end{aligned}$$

so (6.3) holds and $y_{k+1} \in U[x_0,t^\star]$.

The induction for items (6.3) – (6.3) is completed. Hence, sequence $\{x_k\}$ is fundamental in Banach space X, so it converges to some $x^\star \in U[x_0,t^\star]$. By letting $k \to \infty$ in (6.5) and using the continuity of F we conclude $F(x^\star) = 0$. □

Concerning the uniqueness of the solution $x^\star$ we have:

Proposition 4. *Suppose*

(1) *There exists a solution* $x^\star \in U[x_0,r] \subset \Omega$ *for some* $r>0$ *and* $F'(x^\star)^{-1} \in \mathcal{L}(Y,X)$.

(2) *Condition* (A_2) *holds and*

(3) *There exists* $r_1 \geq r$ *such that*

$$L_0(r+r_1) < 2. \tag{6.5}$$

Define $\Omega_2 = U[x_0, r_1] \cap \Omega$. *Then, the solution* $x^\star$ *is unique in the set* Ω_2.

Proof. Define $T = \int_0^1 F'(x^\star + \tau(b - x^\star))d\tau$ for some $b \in \Omega_2$ with $F(b) = 0$. Using (A_2) and (6.5), we get in turn

$$\begin{aligned} \|F'(x_0)^{-1}(T - F'(x_0))\| &\leq \int_0^1 L_0((1-\tau)\|x_0 - x^\star\| + \tau\|b - x_0\|)d\tau \\ &\leq \frac{L_0}{2}(r+r_1) < 1, \end{aligned}$$

so $b = x^\star$ by the invertibility of T and the identity $T(x^\star - b) = F(x^\star) - F(b) = 0$. □

Remark. (1) The parameter $\frac{1}{L_0}$ given in the closed form can be replace $t^\star$ in A_5.

(2) Conditions A with the exception of (A_2) are not assume in Proposition 4. But if they are, then, we can certainly set $r = t^\star$ and drop (1) in Proposition 4

Bibliography

[1] Argyros, I.K., On the Newton - Kantorovich hypothesis for solving equations, *J. Comput. Math.*, 169,(2004), 315-332.

[2] Argyros, I.K., *Computational theory of iterative methods*. Series: Studies in Computational Mathematics, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[3] Argyros, I.K., *Convergence, and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[4] Argyros, I.K. Hilout, S., Weaker conditions for the convergence of Newton's method. *J. Complexity*, 28, (2012), 364–387.

[5] Argyros, I.K. Hilout, S., On an improved convergence analysis of Newton's methods, *Applied Mathematics and Computation*, 225, (2013), 372-386.

[6] Argyros, I.K., Magréñan, A.A., *Iterative methods and their dynamics with applications*, CRC Press, New York, USA, 2017.

[7] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative methods*, Elsevier (Academic Press), New York, 2018.

[8] Arroyo, V., Cordero, A., Torregrosa, J.R., Approximation of artificial satellites preliminary orbits: The efficiency challenge. *Mathematical and Computer modeling*, 54, (7-8), (2011), 1802-1807.

[9] Behl, R., Maroju, P., Martinez, E., Singh, S. A study of the local convergence of a fifth order iterative method, *Indian J. Pure Appl. Math.*, 51,2, (2020), 439-455.

[10] Cătinaş, E., The inexact, inexact perturbed, and quasi-Newton methods are equivalent models, *Math. Comp.*, 74, (2005), 291-301.

[11] Ezquerro, J.A., Gutiérrez, J.M., Hernández, M.A., Romero, N., Rubio, M.J., The Newton method: from Newton to Kantorovich (Spanish), *Gac. R. Soc. Mat. Esp.*, 13,(2010), 53-76.

[12] Ezquerro, J.A., Hernandez, M.A., *Newton's method: An updated approach of Kantorovich's theory*, Cham Switzerland, (2018).

[13] Grau-Sánchez, M., Grau, A., Noguera, M., Ostrowski type methods for solving systems of nonlinear equations. *Appl. Math. Comput.*, 281,(2011), 2377-2385.

[14] Homeier, H.H.H., A modified Newton method with cubic convergence: the multivariate case, *Journal of Computational and Applied Mathematics* 169 (2004) 161-169.

[15] Kantorovich, L.V., Akilov, G.P., *Functional Analysis*, Pergamon Press, Oxford, (1982).

[16] Magréñan, A.A., Argyros, I.K., Rainer, J.J., Sicilia, J.A., Ball convergence of a sixth-order Newton-like method based on means under weak conditions, *J. Math. Chem.*, (2018), 56, 2117-2131, https://doi.org/10.1007/ s10910-018-0856-y.

[17] Magréñan, A.A., Gutiérrez, J.M., Real dynamics for damped Newton's method applied to cubic polynomials, *J. Comput. Appl. Math.*, 275, (2015), 527–538.

[18] Ortega, L.M., Rheinboldt, W.C., *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, (1970).

[19] Ostrowski, A.M., *Solution of equations in Euclidean and Banach spaces*, Elsevier, 1973.

[20] Potra, F.A., Pták, V., Nondiscrete induction and iterative processes. *Research Notes in Mathematics*, 103. Pitman (Advanced Publishing Program), Boston, MA. (1984).

[21] Proinov, P.D., General local convergence theory for a class of iterative processes and its applications to Newton's method, *J. Complexity*, 25 (2009), 38-62.

[22] Proinov, P.D., New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems, *J. Complexity*, 26, (2010), 3-42.

[23] Rheinboldt, W.C., *An adaptive continuation process of solving systems of nonlinear equations*. Polish Academy of Science, Banach Ctr. Publ. 3, (1978), 129-142.

[24] Shakhno, S.M., Gnatyshyn, O.P., On an iterative algorithm of order 1.839... for solving nonlinear least squares problems, *Appl. Math. Applic.*, 161, (2005), 253-264.

[25] Shakhno, S.M. Iakymchuk, R.P., Yarmola, H.P., Convergence analysis of a two step method for the nonlinear squares problem with decomposition of operator, *J. Numer. Appl. Math.*, 128, (2018), 82-95.

[26] Sharma, J.R., Kumar, S., A class of accurate Newton-Jarratt like methods with applications to nonlinear models, *Comput. Appl. Math*, (2022).

[27] Traub, J.F., *Iterative methods for the solution of equations*, Prentice Hall, New Jersey, U.S.A, (1964).

[28] Verma, R., *New Trends in Fractional Programming*, Nova Science Publishers, New York, USA, (2019).

[29] Xiao, X., Yin, H., Accelerating the convergence speed of iterative methods for solving equations, *Appl. Math. Comput.*, 333, (2018), 8-19.

Chapter 7

Improved Convergence of Derivative-Free Halley's Scheme with Two Parameters

Halley's scheme for nonlinear equations is extended with no additional conditions. Iterates are shown to belong to a smaller domain resulting in tighter Lipschitz constants and a finer convergence analysis than in earlier works.

1. Introduction

Many applications in computational Sciences require finding a solution x^* of the nonlinear equation

$$\mathcal{G}(x) = 0, \tag{7.1}$$

where $\mathcal{G} : \Omega \subset E_1 \longrightarrow E_2$ acting between Banach spaces E_1 and E_2. Higher convergence order schemes have been used extensively to generate a sequence approximating x^* under certain conditions [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25]. In particular, the third order scheme [25] has been used to define for each $i = 0, 1, 2, \ldots$ by

$$x_{i+1} = x_i - T_i \mathcal{G}'(x_i)^{-1}(\mathcal{G}(x_i), \tag{7.2}$$

where $a \in \mathbb{R}$, $K_i = \mathcal{G}'(x_i)^{-1}\mathcal{G}''(x_i)\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i)$, and $T_l = \frac{1}{2}K_i(I - aK_i)^{-1}$. If $a = 0, \frac{1}{2}, 1$, then (7.2) reduces to Chebyshev, Halley, and Super-Halley Schemes, respectively.

The convergence conditions used are:

(R1) $\|\mathcal{G}'(x_0)^{-1}\| \leq \beta$.

(R2) $\|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)\| \leq \eta$.

(R3) $\|\mathcal{G}''(x)\| \leq K_1\|$ for each $x \in D$.

(R4) $\|\mathcal{G}''(x) - \mathcal{G}''(y)\| \leq K_1\|x - y\|$ for each $x, y \in D$.

But there are examples [25] where condition (R4) is not satisfied. That is why in references [25] the following conditions are used

(S3) $\|\mathcal{G}''(x)\| \le K_1$ for each $x \in D$.

(S4) $\|\mathcal{G}''(x) - \mathcal{G}''(y)\| \le w_1(\|x-y\|)$ for each $x, y \in D$, where $w_1(0) \ge 0$, and for $t > 0$, function w_1 is continuous and nondecreasing.

(S5) There exists $\bar{w}_1(ts) \le \bar{w}_1(t) w_1(s)$ for $t \in [0,1]$ and $s \in (0, +\infty)$.

Using (R1), (R2), (S3)-(S5) the Halley scheme was shown to be of $R-$order at least two [25]. In particular, if $w_1(t) = \sum_{i=1}^{j} K_i t^{q_i}$, the Halley scheme is of $R-$ order at least $2+q$, where $\gamma = \min\{\gamma_1, \gamma_2, \ldots \gamma_j\}$, $\gamma_i \in [0,1].i = 1, 2, \ldots, j$.

If $a \neq 0$, scheme (7.2) requires the evaluation of the inverse of the linear operator $I - aK_i$ at each step. That is why to reduce the computational cost of this inversion and increase the $R-$order scheme

$$\begin{aligned} y_i &= x_i - \mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i), \\ z_i &= x_i - A_i \mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i) \\ \text{and} \\ x_{k+1} &= z_i - B_i \mathcal{G}'(x_i)^{-1}\mathcal{G}(y_i). \end{aligned} \tag{7.3}$$

was studied in [25], where $D_i = 3\mathcal{G}'(x_i)^{-1}(\mathcal{G}'(x_i) - \mathcal{G}'(x_i - \frac{1}{3}\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i)),$

$A_i = I + \frac{1}{2}D_i + \frac{1}{2}D_i^2(I - aD_i)^{-1}$, $B_i = I + D_i + bD_i^2$, $a \in [0,1]$ and $b \in [-1,1]$. Consider condition

(T4) $\|\mathcal{G}''(x) - \mathcal{G}''(y)\| \le w_2(\|x-y\|)$ for each $x, y \in D_1 \subseteq D$,

where D_1 is a non-empty convex set, $w_2(t)$ is continuous and nondecreasing scalar function with $w_2(0) \ge 0$, and there exists non-negative real function $w_3 \in C[0,1]$ satisfying $w_3(t) \le 1$ and $w_2(ts) \le w_2(t) w_2(s)$, $t \in [0,1]$, $s \in (0, \infty)$.

Using conditions (R1)-(R3) and (T4) the $R-$order was increased. In particular, if the second derivative satisfies (R4) the $R-$order of the scheme (7.3) is at least five which is higher than Chebyshev's, Halley's, and Super-Halley's.

In this chapter, we are concerned with optimization considerations. We raise the following questions. Can we:

(Q1) Increase the convergence domain?

(Q2) Weaken the sufficient semi-local convergence criteria?

(Q3) Improve the estimate on error bounds on the distances $\|x_{k+1} - x_i\|$, $\|x_i - x^*\|$?

(Q4) Can we improve the uniqueness information on the location of x^*?

(Q5) Can we use weaker conditions ?
and

(Q6) Provide the radius in affine invariant form.

The advantages of (Q6) are well known [25]. Denote this set of questions by (Q). We would like question Q to be answered positively without additional or even weaker conditions. This can be achieved by finding at least as small $K_1, K_1, w_1, \bar{w}_1, w_2$ and w_3.

Note that the conditions on the higher-order derivatives, reduce the applicability of the method.

For example: Let $E_1 = E_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define f on Ω by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we have $t_* = 1$, and

$$f'''(t) = 6\log t^2 + 60t^2 - 24t + 22.$$

Obviously, $f'''(t)$ is not bounded on Ω. So, the convergence of the above schemes is not guaranteed by the analysis in earlier papers.

In Section 2 we achieve this goal. Another concern involves conditions (R4) or (S4) or (Q4). Denote the set of nonlinear equations where the operator $\mathcal{G}$ satisfies say (T4) by S1. Moreover, denote by S2 the set of nonlinear equations where the operator $\mathcal{G}'$ does not satisfy (Q4). Then, clearly, S1 is a strict subset of S2. Therefore, working on S2 instead of S1 is interesting, since the applicability of scheme (7.3) is extended. We show how to do this by dropping condition (T4) in Section 3.

2. Semi-Local Convergence I

The results are presented in the affine invariant form. So condition (R1) is dropped. Conditions (U) are used:

(U1) $\|\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(x) - \mathcal{G}'(x_0))\| \leq K_0\|x - x_0\|$ for each $x \in D$.
Set $D_0 = B[x_0, \frac{1}{K_0}] \cap D$.

(U2) $\|\mathcal{G}'(x_0)^{-1}\mathcal{G}''(x)\| \leq K$ for each $x \in D_0$.

(U3) $\|\mathcal{G}'(x_0)^{-1}(\mathcal{G}''(x) - \mathcal{G}''(y))\| \leq w(\|x - y\|)\|$ for each $x, y \in D_0$,

where w is a continuous and nondecreasing function with $w(0) \geq 0$, and there exists non-negative function $w_0 \in C[0,1]$ such that $w_0(t) \leq 1$ and $w(ts) \leq w_0(t)w(s), t \in [0,1], s \in (0, \infty)$.

Remark. It follows by the definition of the set D_0 that

$$D_0 \subseteq D, \tag{7.4}$$

so

$$K_0 \leq \beta K_1, \tag{7.5}$$

$$K \leq \beta K_1 \tag{7.6}$$

and

$$w(t) \leq \beta w_1(t). \tag{7.7}$$

Notice also that using (R3) the following estimate was used in the earlier studies [25]:

$$\|\mathcal{G}'(x)^{-1}\| \leq \frac{\beta}{1-\beta K_1\|x-x_0\|}. \tag{7.8}$$

But using weaker and actually needed (U1) we obtained an instead tighter estimate

$$\|\mathcal{G}'(x)^{-1}\mathcal{G}'(x_0)\| \leq \frac{1}{1-K_0\|x-x_0\|}, \tag{7.9}$$

Moreover, suppose

$$K_0 \leq K. \tag{7.10}$$

Otherwise, the results that follow hold with K_0 replacing M.

Next, we state the semi-local convergence result [25], but first, we define some scalar functions. Consider real functions given by

$$g(\theta_1) = p(\theta_1) + \frac{\theta_1}{2}(1+\theta_1+|b|\theta_1^2)[1+\frac{\theta_1}{1-a\theta_1}+p(\theta_1)^2],$$

$$h(\theta_1) = \frac{1}{1-\theta_1 g(\theta_1)},$$

$$\begin{aligned} f_1(\theta_1,\theta_2) &= \Big[\frac{\theta_2}{Q+1)3^Q}+\theta_1^2(1+|b|+|b|\theta_1)+\frac{1}{Q+1}(1+\theta_1+|b|\theta_1^2)\theta_2]f_2(\theta_1,\theta_2) \\ &\quad +\frac{\theta_1^2}{2}(1+\frac{\theta_1}{1-a\theta_1})(1+\theta_1+|b|\theta_1^2)f_2(\theta_1,\theta_2) \\ &\quad +\frac{\theta_1}{2}(1+\theta_1+|b|\theta_1^2)f_2(\theta_1,\theta_2)^2, \end{aligned}$$

where

$$p(\theta_1) = 1+\frac{1}{2}\theta_1+\frac{\theta_1^2}{2(1-a\theta_1)},$$

$$\begin{aligned} f_2(\theta_1,\theta_2) &= \frac{\theta_1^2}{2}(1+\frac{1}{1-a\theta_1}+\frac{\theta_1}{1-a\theta_1})+\frac{\theta_1^3}{8}(1+\frac{\theta_1}{1-a\theta_1})^2 \\ &\quad +\frac{\theta_2}{2(Q+1)3^q}+\frac{\theta_2}{(Q+1)(Q+2)}. \end{aligned}$$

Let $f_3(\theta_1)=\theta_1 g(\theta_1)-1$, since, $f_3(0)=-1<0$, $f_3(\frac{1}{2})>\frac{27}{256}>0$, we know that $f_3(\theta_1)=0$ has a root in $(0,\frac{1}{2})$. Define s^* as the smallest positive root of equation $\theta_1 g(\theta_1)-1=0$, then $s^*<\frac{1}{2}$.

Theorem 7. Suppose: $\mathcal{G} : \Omega \subseteq E_1 \longrightarrow E_2$ is twice Fréchet differentiable and conditions (R1)-(R3) and (T4) hold,

$$B[x_0, \rho\eta] \subset \Omega,\ \rho = \frac{p(a_0)}{1-d_0},$$

where $a_0 = K_1\beta\eta$, $b_0 = \beta\eta w_2(\eta)$, $d_0 = h(a_0)f_2(a_0,b_0)$ satisfy $a_0 < s^*$ and $h(a_0)d_0 < 1$. Then, the following items hold

$$\{x_i\} \subset B[x_0, \rho\eta],$$

and there exists $\lim_{k\longrightarrow\infty} x_i = x^* \in B[x_0, \rho\eta]$ with $\mathcal{G}(x^*) = 0$,

$$\|x^* - x_i\| \leq r_i = p(a_0)\eta\lambda^k\gamma^{\frac{2^k-1}{2}}\frac{1}{1-\lambda\gamma^{3k}},$$

where $\gamma = h(a_0)d_0$ and $\lambda = \dfrac{1}{h(a_0)}$ only solution of equation $\mathcal{G}(x) = 0$ in the region $B(x_0, \rho_1\eta) \cap \Omega$, where $R_1 = \dfrac{2}{a_0} - \rho$.

But in the new case:

Theorem 8. Suppose $\mathcal{G} : \Omega \subseteq E_1 \longrightarrow E_2$ is twice Fréchet differentiable and conditions (U) hold.

$$B[x_0, \rho_0\eta] \subset \Omega,\ \rho_0 = \frac{p(\bar{a}_0)}{1-\bar{d}_0},$$

where $\bar{a}_0 = M\beta\eta$, $\bar{b}_0 = \eta w(\eta)$, $\bar{d}_= h(\bar{a}_0)f_2(\bar{a}_0,\bar{b}_0)$ satisfy $\bar{a}_0 < s^*$ and $h(\bar{a}_0)\bar{d}_0 < 1$. Then, the following items hold

$$\{x_i\} \subset B[x_0, \rho_0\eta],$$

and there exists $\lim_{k\longrightarrow\infty} x_i = x^* \in B[x_0, \rho_0\eta]$ so that $\mathcal{G}(x^*) = 0$,

$$\|x^* - x_i\| \leq \bar{r}_i = p(\bar{a}_0)\eta\lambda_0^k\gamma_0^{\frac{2^k-1}{2}}\frac{1}{1-\lambda_0\gamma_0^{3k}},$$

where $\gamma_0 = h(\bar{a}_0)\bar{d}_0$ and $\lambda_0 = \dfrac{1}{h(\bar{a}_0)}$. Moreover, the point x^* is the only solution of equation $\mathcal{G}(x) = 0$ in the region $B(x_0, \bar{\rho}_1\eta) \cap \Omega$, where $\bar{\rho}_1 = \dfrac{2}{K_0} - \rho_0$.

Proof. Simply use $K_1, w_2, a_0, b_0, d_0, \rho, \rho_1, \lambda, \gamma$ used in Theorem 7 with $M, w, \bar{a}_0, \bar{b}_0, \bar{d}_0, \rho_0\bar{\rho}_1, \lambda_0, \gamma_0$ respectively. □

Remark. In view of (7.4)-(7.7), we have

$$a_0 < s^* \Rightarrow \bar{a}_0 < s^*,$$

$$\bar{a}_0 < a_0,$$

$$h(a_0)d_0 < 1 \Rightarrow h(\bar{a}_0)\bar{d}_0 < 1$$

$$\bar{r}_i \leq r_i$$

and

$$\rho_1 \leq \bar{\rho}_1.$$

These estimates show that questions (Q) have been answered positively under our technique.

3. Semi-Local Convergence Part 2

The results are also presented in an affine invariant form and the restrictive condition (T4) is dropped.
Suppose

(U4) $\|\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(x)-\mathcal{G}'(y))\| \leq v(\|x-y\|)$ for all $x, y \in D_0$,

where v is a real continuous and nondecreasing function defined on the interval $[0,\infty)$. Denote conditions (R2), (U1), (U2) and weaker (U3) or (T4) or (S4) or (R4). The semi-local convergence is based on conditions (U)':

Consider nonnegative scalar sequences $\{t_i\}, \{s_i\}$ and $\{u_i\}$ for $i = 0, 1, 2, \ldots$ by

$$\begin{aligned} t_0 &= 0, s_0 = \eta, \\ u_i &= s_i + \gamma_i(s_i - t_i), \\ t_{i+1} &= u_i + \frac{\beta_i \alpha_i}{1 - K_0 t_i} \end{aligned} \tag{7.11}$$

and

$$s_{i+1} = t_{i+1} + \frac{\delta_{i+1}}{1 - K_0 t_{i+1}}, \tag{7.12}$$

where

$$\begin{aligned} \alpha_i &= \frac{1}{2}K(s_i - t_i)^2 + \frac{1}{2}\frac{K^2 q_i^2 (s_i - t_i)^2}{1 - K_0 t_i} \\ &\quad + K\int_0^1 v(\theta(u_i - t_i))d\theta(u_i - t_i), \\ \beta_i &= 1 + \frac{K(s_i - t_i)}{1 - K_0 t_i} + |b|\left(\frac{K(s_i - t_i)}{1 - K_0 t_i}\right)^2, \\ q_i &= \frac{1}{1 - p_i}, \\ p_i &= \frac{|a|K(s_i - t_i)}{1 - K_0 t_i} + |b|\frac{K^2(s_i - t_i)^2}{(1 - K_0 t_i)^2}, \\ \gamma_i &= \frac{K(s_i - t_i)}{1 - K_0 t_i}[1 + \frac{K(s_i - t_i)}{1 - K_0 t_i} q_i] \end{aligned}$$

and

$$\begin{aligned}\delta_{i+1} &= \frac{2K\alpha_i(s_i-t_i)}{1-K_0t_i}+\frac{|b|K(s_i-t_i)}{1-K_0t_i}\\ &+K(s_i-t_i)\left(\frac{K(s_i-t_i)}{1-K_0t_i}+\frac{|b|K^2(s_i-t_i)^2}{(1-K_0t_i)}\right)\frac{\alpha_i}{1-K_0t_i}\\ &+2K(s_i-t_i)(t_{i+1}-u_i)+K(u_i-s_i)(t_{i+1}-u_i)\\ &+\int_0^1 v(\theta(t_{i+1}-u_i))d\theta(t_{i+1}-u_i).\end{aligned}$$

Sequence $\{t_i\}$ shall be shown to be majorizing for the scheme (7.3). But first, we need some auxiliary convergence results for it.

Lemma 9. Suppose

$$K_0t_i < 1 \text{ for each } i=0,1,2,\ldots. \tag{7.13}$$

Then, the following hold

$$t_i \le s_i \le u_i \le t_{i+1} < \frac{1}{K_0} \tag{7.14}$$

and

$$\lim_{i\longrightarrow\infty} x_i = t^* \le t^{**} = \frac{1}{K_0}, \tag{7.15}$$

where the point t^* is the upper bound of sequence $\{t_i\}$.

Proof. By using (7.11) and (7.13), we see that (7.14) holds and so (7.15) follows too. □

Stronger convergence criteria than (7.13) are provided in the second convergence result. But they are considered easier to verify.

Lemma 10. Suppose for each $i=0,1,2,\ldots$

$$0 \le s_i - t_i \le \eta, \tag{7.16}$$

$$2K_0t_{i+1} \le 1 \tag{7.17}$$

and

$$2p_i \le 1. \tag{7.18}$$

Then, the following estimates hold:

$$\begin{aligned}\gamma_i &\le K\eta(1+4K\eta)=\mu_3,\\ \alpha_i &\le \left(\frac{K}{2}\eta+2K^2\eta+K\int_0^1 v(\theta(1+\mu_3)\eta)d\theta(1+\mu_3)\right)\\ &= \mu_0(s_i-t_i),\\ \frac{\beta_i\alpha_i}{1-K_0t_i} &\le 2(1+2K\eta+4|b|K^2\eta^2)\mu_0(s_i-t_i)\\ &= \mu_2(s_i-t_i)\end{aligned}$$

and

$$\begin{aligned}\frac{\delta_{i+1}}{1-K_0t_{i+1}} &\leq 2[4K\mu_0\eta+2|b|K\\ &+2K(2K\eta+2|b|\eta^2)\mu_0\eta+2K\mu_2\eta\\ &+K\mu_3\mu_2\eta+\int_0^1 v(\theta\mu_2\eta)d\theta\mu_2](s_i-t_i)\\ &= \mu_3(s_i-t_i).\end{aligned}$$

Lemma 11. Under conditions (7.14) and (7.15) further suppose $m=\max\{\mu_1,\mu_2,\mu_3\}$ for $\ell_0=\max\{\eta,\, u_0-s_0,\, t_1-u_0\}$, $\ell_1=\min\{\mu_1,\mu_2,\mu_3\}$ and

$$2|a|K\eta<1,\ 0\leq\ell_0\leq\ell_1\leq m<1-2K_0\eta. \tag{7.19}$$

Then, the following assertions hold

$$0\leq u_k-s_k\leq m(s_k-t_k)\leq m^{k-1}\eta, \tag{7.20}$$

$$0\leq s_k-t_k\leq m(s_{k-1}-t_{k-1})\leq m^k\eta, \tag{7.21}$$

$$0\leq t_{k+1}-u_k\leq m(s_k-t_k)\leq m^{k+1}\eta, \tag{7.22}$$

$$0\leq t_{k+i}-t_k\leq B\eta m^{k-1}\frac{1-m^{i-1}}{1-m}\leq\frac{B\eta}{1-m}m^{k-1} \tag{7.23}$$

and there exists $t^*=\lim_{k\longrightarrow\infty}t_k$ such that

$$0\leq t^*-t_k\leq\frac{B\eta}{1-m}m^{k-1} \tag{7.24}$$

and

$$2K_0t_k\leq 1, \tag{7.25}$$

where $B=1+m+m^2$.

Proof. It follows from (7.11), (7.13), (7.15) and (7.16) that estimates (7.17)-(7.19) hold. Let $i\geq 0$ be an integer. Then, we can write in turn that

$$\begin{aligned}0 &\leq t_{k+i}-t_k=(t_{k+i}-t_{k+i-1})+(t_{k+i-1}-t_{k+i-2})\\ &\quad +\ldots+(t_{k+1}-t_k)\\ &\leq B\eta(m^{k+i-2}+\ldots+t^{k-i})\\ &= B\eta m^{k-1}\frac{1-m^{i-1}}{1-m}\leq\frac{B\eta m^{k-1}}{1-m},\end{aligned} \tag{7.26}$$

so (7.20) holds. Hence, the sequence $\{t_k\}$ is complete, and as such it converges to some t^*. By letting $k\longrightarrow\infty$ in (7.26), we obtain (7.21). Notice also that

$$2K_0t_k\leq 2K_0\frac{1-m^{k+1}}{1-m}\eta\leq\frac{2K_0\eta}{1-m}\leq 1,$$

by the right hand side of (7.16), so

$$\frac{1}{1-K_0t_k} \leq 2.$$

Similarly, we have

$$2p_i \leq 2|a|K(s_i-t_i) \leq 2|a|K\eta < 1,$$

which holds by condition (7.19). □

Remark. Condition (7.16) is the sufficient convergence criterion for a sequence $\{t_i\}$. Such a criterion is standard in this type of study. It shows how close x_0 should be to the solution (i.e. how small η should be) to obtain convergence.

Notice also that each $u_i < 1, i = 1,2,3$ can be solved for η, which depends on K_0, M, b, c, and v, i.e., the initial data.

The following Ostrowski-like representations are needed.

Lemma 12. [25] Suppose iterates $\{x_i\}$ exists for each $i = 0,1,2,\ldots$. Then, the following items hold.

$$\begin{aligned}
\mathcal{G}(x_{k+1}) &= 3\int_0^1 \mathcal{G}''(x_i+\theta(-\frac{1}{3}\mathcal{G}'(x_i)^{-1}\mathcal{G}(z_i))d\theta \\
&\quad (-\frac{1}{3}\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i))\mathcal{G}'(x_i)^{-1}\mathcal{G}(z_i) \\
&\quad +3\mathcal{G}''(x_i)(\frac{1}{3}\mathcal{G}'(x_i)^{-1}\mathcal{G}(z_i) \\
&\quad +3b(\mathcal{G}'(x_i-\frac{1}{3}\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i))-\mathcal{G}'(x_i))D_i\mathcal{G}'(x_i)^{-1}\mathcal{G}(z_i) \\
&\quad -\mathcal{G}''(x_i)(y_i-x_i)(D_i+bD_i^2)\mathcal{G}'(x_i)^{-1}\mathcal{G}(z_i) \qquad (7.27) \\
&\quad +\int_0^1 \mathcal{G}''(x_i+\theta(y_i-x_i))d\theta(y_i-x_i)(x_{i+1}-z_i) \\
&\quad -\mathcal{G}''(x_i)(y_i-x_i)(x_{i+1}-z_i) \\
&\quad +\int_0^1 \mathcal{G}''(y_i+\theta(z_i-y_i))d\theta(x_{i+1}-z_i) \\
&\quad +\int_0^1 (\mathcal{G}'(z_i+\theta(x_{i+1}-z_i))-\mathcal{G}'(z_i))(x_{i+1}-z_i)d\theta \qquad (7.28)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{G}(z_i) &= -\frac{3}{2}(\mathcal{G}'(x_i)-\mathcal{G}'(x_i-\frac{1}{3}\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i)))\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i) \\
&\quad -\frac{3}{2}(\mathcal{G}'(x_i)-\mathcal{G}'(x_i-\frac{1}{3}\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i)))D_i(I-aA_i)^{-1}\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i) \\
&\quad +\int_0^1 (\mathcal{G}'(x_i+\theta(z_i-x_i))-\mathcal{G}'(x_i))d\theta(z_i-x_i). \qquad (7.29)
\end{aligned}$$

Next, we present the second semi-local convergence result for the scheme (7.3).

Theorem 9. Under the conditions (U)' further suppose $B[x_0,t^*] \subset D$, if conditions of Lemma 9 or Lemma 10 hold. Then, the following assertions hold $\{x_i\} \subset B[x_0,t^*]$ and there exists $\lim_{k\longrightarrow\infty} x_i = x^* \in B[x_0,t^*]$ so that

$$\|x^* - x_i\| \leq t^* - t_i. \tag{7.30}$$

Proof. Estimates

$$\|y_i - x_i\| \leq s_i - t_i, \tag{7.31}$$

$$\|z_i - y_i\| \leq \theta_i - s_i \tag{7.32}$$

and

$$\|x_{i+1} - z_i\| \leq t_{i+1} - \theta_i, \tag{7.33}$$

shall be shown using induction. By (7.11) and the first substep of method (7.3), we have

$$\|y_0 - x_0\| = \|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)\| \leq \eta = s_0 - t_0 < t^*,$$

so $y_0 \in B[x_0,t^*]$ and (7.31) holds for $n = 0$.

Consider $u \in B[x_0,t^*]$. Using (U1), we have

$$\|\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(u) - \mathcal{G}'(x_0))\| \leq K_0\|u - x_0\| \leq K_0 t^* < 1,$$

leading to $\mathcal{G}'(u)^{-1} \in L(E_2,E_1)$ with

$$\|\mathcal{G}'(u)^{-1}\mathcal{G}'(x_0)\| \leq \frac{1}{1 - K_0\|u - x_0\|} \tag{7.34}$$

by the celebrated lemma due to Banach on linear operators that are invertible [2]. Numerous estimates are needed to be derived by conditions (U)' and Lemma 12. We get in turn

$$\begin{aligned}
\|D_i\| &\leq \frac{3K\frac{1}{3}\|y_i - x_i\|}{1 - K_0\|x_i - x_0\|} \\
&\leq \frac{K\|y_i - x_i\|}{1 - K_0\|x_i - x_0\|} \leq \frac{K(s_i - t_i)}{1 - K_0 t_i}, \\
\|aD_i\| &\leq |a|\|D_i\| \leq \frac{|a|K(s_i - t_i)}{1 - K_0 t_i} < 1,
\end{aligned}$$

so

$$\|(I - aD_i)^{-1}\| \leq q_i,$$

$$\begin{aligned}
\|\mathcal{G}'(x_0)^{-1}\mathcal{G}(z_i)\| &\leq \frac{3}{2}K\frac{1}{3}\|y_i - x_i\|^2 \\
&\quad + \frac{3}{2}\frac{1}{3}\frac{K^2\|y_i - x_i\|q_i}{1 - K_0\|x_i - x_0\|}\|y_i - x_i\| \\
&\quad + K\int_0^1 v(\theta\|z_i - x_i\|)d\theta\|z_i - x_i\| \leq \alpha_i,
\end{aligned}$$

$$\begin{aligned}
z_i &= x_i - \mathcal{G}'(z_i)^{-1}\mathcal{G}(x_i) + (I - A_i)\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i) \\
&\quad y_i + (I - A_i)\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i), \\
\|z_i - y_i\| &\leq \|\frac{1}{2}D_i + \frac{1}{2}D_i^2(I - aD_i)^{-1}\|\|y_i - x_i\| \\
&\leq \frac{1}{2}\left(\frac{K\|y_i - x_i\|}{1 - K_0\|x_i - x_0\|} + \left(\frac{K\|y_i - x_i\|}{1 - K_0\|x_i - x_0\|}\right)^2 q_i\right)\|y_i - x_i\| \\
&\leq \gamma_n(s_i - t_i) \leq u_i - s_i,
\end{aligned}$$

$$\begin{aligned}
\|x_{i+1} - z_i\| &\leq \left(1 + \frac{K\|y_i - x_i\|}{1 - K_0\|x_i - x_0\|}\right. \\
&\quad \left. + \|b|\left(\frac{K\|y_i - x_i\|}{1 - K_0\|x_i - x_0\|)}\right)^2\right)\frac{\alpha_i}{1 - K_0\|x_i - x_0\|} \\
&\leq \left(1 + \frac{K(s_i - t_i)}{1 - K_0 t_i} + |b|\left(\frac{K(s_i - t_i)}{1 - K_0 t_i}\right)^2\right)\frac{\alpha_i}{1 - K_0 t_i} \\
&\leq \frac{\beta_i\alpha_i}{1 - K_0 t_i} \leq t_{i+1} - u_i,
\end{aligned}$$

$$\begin{aligned}
\|y_{i+1} - x_{i+1}\| &\leq \|\mathcal{G}'(x_{i+1})^{-1}\mathcal{G}'(x_0)\|\|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_{i+1})\| \\
&\leq \frac{1}{1 - K_0\|x_{i+1} - x_0\|}\left[\frac{K\|y_i - x_i\|\alpha_i}{1 - K_0\|x_i - x_0\|}\right. \\
&\quad + \frac{K\|y_i - x_i\|\alpha_i}{1 - K_0\|x_i - x_0\|} + \frac{|b|K\|y_i - x_i\|^2}{1 - K_0\|x_i - x_0\|} \\
&\quad + K\|y_i - x_i\|\left(\frac{K\|y_i - x_i\|}{1 - K_0\|x_{i+1} - x_0\|} + |b|\left(\frac{K\|y_i - x_i\|}{1 - K_0\|x_{i+1} - x_0\|}\right)^2\right) \\
&\quad + 2K\|y_i - x_i\|\|x_{i+1} - z_i\| + K\|z_i - y_i\|\|x_{i+1} - z_i\| \\
&\quad \left. + \int_0^1 v(\theta\|x_{i+1} - z_i\|)d\theta\|x_{i+1} - z_i\|\right] \\
&\leq \frac{\delta_{i+1}}{1 - K_0\|x_{i+1} - x_0\|} \\
&\leq \frac{\delta_{i+1}}{1 - K_0 t_{i+1}} \leq s_{i+1} - t_{i+1},
\end{aligned}$$

where we also used

$$\begin{aligned}
\|z_i - y_i\| &\leq u_i - s_i, \\
\|x_{i+1} - z_i\| &\leq t_{i+1} - u_i, \\
\|z_i - x_0\| &\leq \|z_i - y_i\| + \|y_i - x_0\| \leq u_i - s_i + s_i - t_0 = u_i < t^*, \\
\|x_{i+1} - x_0\| &\leq \|x_{i+1} - z_i\| + \|z_i - x_0\| \leq t_{i+1} - u_i + u_i - t_0 = t_{i+1} < t^*, \\
\|x_i + \theta(y_i - x_i) - x_0\| &\leq (1 - \theta)\|x_i - x_0\| + \theta\|y_i - x_0\| \\
&< (1 - \theta)t^* + \theta t^* = t^*
\end{aligned}$$

and for $\bar{u}_i = x_i - \frac{1}{3}\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i)$,

$$\begin{aligned}\|\bar{u}_i - x_0\| &\leq \|\bar{u}_i - x_i\| + \|x_i - x_0\| \\ &\leq \frac{1}{3}\|\mathcal{G}'(x_i)^{-1}\mathcal{G}(x_i)\| + \|x_i - x_0\| \\ &\leq \frac{1}{2}(s_i - t_i) + t_i = \frac{1}{3}(s_i + 2t_i) \leq t^*,\end{aligned}$$

$$\begin{aligned}\|z_i + \theta(x_{i+1} - z_i) - x_0\| &\leq (1-\theta)\|z_i - x_0\| + \theta\|x_{i+1} - x_0\| \\ &< (1-\theta)t^* + \theta t^* = t^*,\end{aligned}$$

and

$$\begin{aligned}\|y_{i+1} - x_0\| &\leq \|y_{i+1} - x_{i+1}\| + \|x_{i+1} - x_0\| \\ &\leq s_{i+1} - t_{i+1} + t_{i+1} - t_0 = s_{i+1} < t^*,\end{aligned}$$

so, $z_i, x_{i+1}, x_i + \theta(y_i - x_i)$, $z_i + \theta(x_{i+1} - z_i)$, $y_{i+1} \in B(x_0, t^*)$ and the induction for estimates is completed.

It follows that sequence $\{t_i\}$ is complete in E_1 and as such it converges to some $x^* \in B[x_0, t^*]$. By letting $n \longrightarrow \infty$ in the estimation (see (7.33))

$$\|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_{i+1})\| \leq \delta_{i+1}$$

and using the continuity of $\mathcal{G}$ we obtain $\mathcal{G}(x^*) = 0$. Moreover, see (7.21) for the proof of (7.28). □

Remark. The condition $B[x_0, t^*] \subset D$ can be replaced by $B[x_0, \frac{1}{K_0}] \subset D$ if conditions of Lemma 9 hold or $B[x_0, \frac{\eta}{1-m}] \subset D$ under conditions of Lemma 11 where $\frac{1}{K_0}$ and $\frac{\eta}{1-m}$ are given in closed form in contrast to t^*.

The uniqueness of the solution x^* result follows without necessarily using conditions of Theorem 7 or Theorem 8 or Theorem 9.

Proposition 5. *Suppose $x^* \in B(x_0, \xi_0) \subset D$ is a simple solution of equation $\mathcal{G}(x) = 0$; Condition (U2) holds and there exists $\xi \geq \xi_0$ such that*

$$K_0(\xi_0 + \xi) < 2. \tag{7.35}$$

Set $G = B[x_0, \xi] \cap D$. Then, the point x^* is the only solution of equation $\mathcal{G}(x) = 0$ in the set G.

Proof. Let $y^* \in G$ with $\mathcal{G}(y^*) = 0$. Define linear operator $Q = \int_0^1 \mathcal{G}'(x^* + \theta(y^* - x^*))d\theta$. By using (U2) and (7.35) we get in turn

$$\begin{aligned}\|\mathcal{G}'(x_0)^{-1}(Q - \mathcal{G}'(x_0))\| &\leq K_0 \int_0^1 ((1-\theta)\|x^* - x_0\| + \theta\|y^* - x_0\|)d\theta \\ &\leq \frac{K_0}{2}(\xi_0 + \xi) < 1,\end{aligned}$$

so $x^* = y^*$ is implied since $Q^{-1} \in L(E_2, E_1)$ and $Q(y^* - x^*) = \mathcal{G}(y^*) - \mathcal{G}(x^*) = 0$. □

Bibliography

[1] Argyros, I.K. Hilout, S., Inexact Newton-type procedures. *J Complex*, 26(6), (2010), 577-590.

[2] Argyros, I.K., *The theory and applications of iteration* 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[3] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative procedures*, Elsevier (Academic Press), New York, 2018.

[4] Argyros, I.K. George, S., *Mathematical modeling for the solution of equations and systems of equations with applications*, Volume-IV, Nova Publishers, NY, 2021.

[5] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics*, 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[6] Behl, R. Maroju, P., Martinez, E., Singh, S., A study of the local convergence of a fifth order iterative method, *Indian J. Pure Appl. Math.*, 51, 2, (2020), 439-455.

[7] Ortega, J.M., Rheinboldt, W.C., *Iterative Solution of Nonlinear Equation in Several Variables*, Academic Press, New York, 1970.

[8] Gutiérrez, J.M., Hernández, M.A., A family of ChebyshevHalley type methods in Banach spaces, *Bull. Aust. Math. Soc.*, 55, (1997), 113130.

[9] Gutiérrez, J.M., Hernández, M.A., Recurrence relations for the super-Halley method, *Comput. Math. Appl.*, 36, (1998) 18.

[10] Argyros, I.K., Chen, D., Results on the Chebyshev method in Banach spaces, *Proyecciones*, 12 (2), (1993), 119128.

[11] Babajee, D.K.R., Dauhoo, M.Z., Darvishi, M.T., Karami, A., Barati, A.: Analysis of two Chebyshev-like third order methods free from second derivatives for solving systems of nonlinear equations. *J. Comput. Appl. Math.*, 233, (2010), 20022012.

[12] Candela, V. Marquina, A., Recurrence relations for rational cubic methods i, *The Halley Method Comput.*, 44, (1990), 169184.

[13] Chen, D., Argyros, I.K., Qian, Q.S., A note on the Halley method in Banach spaces, *Appl. Math. Comput.*, 58, (1993), 215224.

[14] Candela, V. Marquina, A., Recurrence relations for rational cubic methods II, *The Chebyshev Method Comput.*, 45, (1990), 355367.

[15] Ezquerro, J.A., Hernández, M.A., On the r-order of the Halley method, *J. Math. Anal. Appl.*, 303, (2005), 591601.

[16] Ezquerro, J.A., Hernández, M.A., Recurrence relations for Chebyshev-type methods. *Appl. Math. Optim*, 41(2), (2000), 227236.

[17] Ezquerro, J.A., Hernández, M.A., On the R-order of the Halley method. *J. Math. Anal. Appl* 303, 591601, (2005).

[18] Ezquerro, J.A., Hernández, M.A., New iterations of R-order four with reduced computational cost, *BIT Numer Math*, 49, 325342, (2009).

[19] Gutiérrez, J.M., Hernández, M.A., A family of Chebyshev-Halley type methods in banach spaces. *Bull. Aust. Math. Soc.*, 55, 113130, (1997).

[20] Gutiérrez, J.M., Hernández, M.A., Recurrence relations for the super-Halley method, *Comput. Math. Appl.*, 36, 18, (1998).

[21] Ganesh, M., Joshi, M.C., Numerical solvability of Hammerstein integral equations of mixed type, *IMA J. Numer. Anal.*, 11, (1991), 21-31.

[22] Hernández, M.A., Reduced recurrence relations for the Chebyshev method, *J. Optim. Theory Appl.*, 98, (1998), 385397.

[23] Hernández, M.A., Chebyshev's approximation algorithms and applications, *Comput. Math. Appl.*, 41, (2001), 433445.

[24] Kou, J., Li, Y.: A family of modified super-Halley methods with fourth-order convergence. *Appl Math. Comput.*, 189, 366370, (2007).

[25] Wang X., Kou, J., Semilocal convergence of multipoint improved super-Halley-type methods without the second derivative under generalized conditions, *Numer. Algor.*, 71, (2016), 567-584.

Chapter 8

Extended Convergence for a Third Order Traub-Like Method with Parameter for Solving Equations

The local convergence for a Traub-like method of order three is studied using assumptions only on the first derivative of the operator involved. The convergence of this method was shown by assuming that the fourth order derivative of the operator not on the method exists and hence it is limiting its applicability. Moreover, no computational error bounds or uniqueness of the solution are given. We address all these problems using only the first derivative that appears on the method. Hence, we extend the applicability of the method. Our techniques can be used to obtain the convergence of other similar higher-order methods using assumptions on the first derivative of the operator involved.

1. Introduction

In this Chapter, we are concerned with the semi-local convergence of the Traub-like method of order three for solving the equation

$$F(x) = 0. \tag{8.1}$$

Here $F : D \subset B_1 \longrightarrow B_2$ be a nonlinear operator, B_1 and B_2 are Banach spaces and $D \neq \emptyset$ open set. We denote the solution of (8.1) by x^*. The local convergence of the following iterative method was studied in [7],

$$\begin{aligned} z_n &= x_n + \gamma F'(x_n)^{-1} F(x_n) \\ \text{and} & \\ x_{n+1} &= x_n - [(I - \frac{1}{2\gamma}) + \frac{1}{2\gamma} F'(x_n)^{-1} F'(z_n)] F'(x_n)^{-1} F(x_n). \end{aligned} \tag{8.2}$$

But, the more interesting semi-local convergence case was not given in [7]. In this Chapter, we study the semi-local convergence of the simplified form of (8.2) defined by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ &\text{and} \\ x_{n+1} &= y_n + \frac{1}{2\gamma}F'(x_n)^{-1}(F'(x_n)-F'(y_n)(y_n-x_n). \end{aligned} \tag{8.3}$$

Our convergence analysis uses assumptions only on the first derivative of F, unlike earlier studies [7] where the convergence analysis required assumptions on the derivatives of F up to the order four. This method can be used on other methods and relevant topics along the same lines [6,8,9,10,12,13,15,16,17,18,19,20,21,22]. The assumptions on the fourth-order derivative reduce the applicability of the method (8.3).

For example: Let $B = B_1 = \mathbb{R}, D = [-\frac{1}{2}, \frac{3}{2}]$. Define f on D by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we have $f(1) = 0$,

$$f'''(t) = 6\log t^2 + 60t^2 - 24t + 22.$$

Obviously, $f'''(t)$ is not bounded by D. So, the convergence of the method (8.3) is not guaranteed by the analysis in [7]. Throughout the chapter $U(x_0,R) = \{x \in X : \|x - x_0\| < R\}$ and $U[x_0,R] = \{x \in X : \|x - x_0\| \leq R\}$ for some $R > 0$.

The chapter contains the semi-local convergence analysis in Section 2, and the numerical examples are given in Section 3.

2. Majorizing Sequence

Let L_0, L, L_1 and η be positive parameters. Define sequence $\{t_n\}$ for each $n = 0,1,2,\ldots$ by

$$\begin{aligned} t_0 &= 0,\ s_0 = \eta, \\ t_{n+1} &= s_n + \frac{L(s_n - t_n)^2}{2(1 - L_0 t_n)} \\ &\text{and} \\ s_{n+1} &= t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2L_1(t_{n+1} - s_n)}{2(1 - L_0 t_{n+1})}. \end{aligned} \tag{8.4}$$

This sequence is shown to be majorizing in Section 3. But first, we present two convergence results for it.

Lemma 13. Suppose

$$L_0 t_n < 1\ \forall\ n = 0,1,2,\ldots. \tag{8.5}$$

Then, the following assertions hold

$$0 \leq t_n \leq s_n \leq t_{n+1} < t^{**} = \frac{1}{L_0}$$

and there exists $t^* \in [0, t^{**}]$ such that

$$\lim_{n \longrightarrow \infty} t_n = t^*.$$

Proof. By (8.4) and (8.5) sequence $\{t_n\}$ is nondecreasing, bounded from above by $\frac{1}{L_0}$ so it converges to its unique least upper bound t^*. □

The second result uses stronger convergence conditions but which are easier to verify (8.5). It is convenient for us to define recurrent polynomials defined on the interval $[0,1)$ by

$$h_n^{(1)}(t) = Lt^{n-1}\eta + 2L_0(1+t+\ldots+t^n)\eta - 2,$$

$$h_n^{(2)}(t) = L(1+t)^2 t^{n-1}\eta + 2L_1 L t^{n-1}\eta + 2L_0(1+t+\ldots+t^{n+1})\eta - 2,$$

polynomials on the same interval

$$p_1(t) = 2L_0 t^2 + Lt - L$$

and

$$p_2(t) = L(1+t)^2 t - L(1+t)^2 + 2L_1 L t - 2L_1 L + 2L_0 t^3.$$

By these definitions we have $p_1(0) = -L, p_1(1) = 2L_0, p_2(0) = -L(1+2L_1)$ and $p_2(1) = 2L_0$. The intermediate value theorem assures the existence of zeros for polynomials p_1 and p_2 on $(0,1)$. Denote by δ_1 and δ_2 the smallest such zeros for p_1 and p_2, respectively. Moreover, define parameters

$$a_1 = \frac{L\eta}{2}, a_2 = \frac{L(t_1-t_0)^2 + 2L_1(t_1-s_0)}{2\eta(1-L_0 t_1)}, \text{ for } \eta \neq 0$$

$$a = \max\{a_1, a_2\}, b = \min\{\delta_1, \delta_2\} \text{ and } \delta = \max\{\delta_1, \delta_2\}.$$

Then, we can show the second convergence result for a sequence $\{t_n\}$.

Lemma 14. Suppose

$$L_0 t_1 < 1 \tag{8.6}$$

and

$$a \leq b \leq \delta \leq 1 - 2L_0\eta. \tag{8.7}$$

Then, the conclusions of Lemma 13 hold for sequence $\{t_n\}$ with $t^{**} = \frac{\eta}{1-\delta}$. Moreover, the following assertions hold

$$0 \leq s_n - t_n \leq \delta(s_{n-1} - t_{n-1}) \leq \delta^n \eta. \tag{8.8}$$

and

$$0 \leq t_{n+1} - s_n \leq \delta(s_n - t_n) \leq \delta^{n+1}\eta. \tag{8.9}$$

Proof. Assertions (8.8) and (8.9) can be shown if

$$0 \leq \frac{L(s_k - t_k)}{2(1 - L_0 t_k)} \leq \delta \tag{8.10}$$

and

$$0 \leq \frac{L(t_{k+1} - t_k)^2 + 2L_1(t_{k+1} - s_k)}{2(1 - L_0 t_{k+1})} \leq \delta(s_k - t_k). \tag{8.11}$$

These shall be shown using induction on k. Assertions (8.10) and (8.11) hold true for $k = 0$ by the choice of a, (8.6) and (8.7). It follows that

$$0 \leq t_1 - s_0 \leq \delta(s_0 - t_0),\ 0 \leq s_1 - t_1 \leq \delta(s_0 - t_0),$$

so (8.8) and (8.9) hold for $n = 0$. We also get

$$t_1 \leq s_0 + \delta\eta = (1 + \delta)\eta = \frac{1 - \delta^2}{1 - \delta}\eta < t^*.$$

Suppose

$$0 \leq s_k - t_k \leq \delta^k \eta,\ 0 \leq t_{k+1} - s_k \leq \delta^{k+1}\eta$$

and

$$t_k \leq \frac{1 - \delta^{k+1}}{1 - \delta}\eta.$$

Then, evidently (8.10) certainly holds if

$$L\delta^k\eta + 2\delta L_0(1 + \delta + \ldots + \delta^k)\eta - 2\delta \leq 0$$

or

$$h_k^{(1)}(t) \leq 0 \text{ at } t = \delta_1. \tag{8.12}$$

Two consecutive polynomials $h_k^{(1)}$ are related as follows:

$$\begin{aligned} h_{k+1}^{(1)}(t) &= h_{k+1}^{(1)}(t) + h_k^{(1)}(t) - h_k^{(1)}(t) \\ &= Lt^k\eta + 2L_0(1 + t + \ldots + t^{k+1})\eta - 2 \\ &\quad -Lt^{k-1}\eta - 2L_0(1 + t + \ldots + t^k)\eta + 2 \\ &= h_k^{(1)}(t) + Lt^k\eta - Lt^{k-1}\eta + 2L_0 t^{k+1}\eta \\ &= h_k^{(1)}(t) + p_1(t)t^{k-1}\eta. \end{aligned} \tag{8.13}$$

In particular $h_{k+1}^{(1)}(t) = h_k^{(1)}(t)$ at $t = \delta_1$. Define function $h_\infty^{(1)}$ on $[0,1)$ by

$$h_\infty^{(1)}(t) = \lim_{k \longrightarrow \infty} h_k^{(1)}(t). \tag{8.14}$$

It follows by the definition of polynomial $h_k^{(1)}$ and (8.4) that

$$h_\infty^{(1)}(t) = 2\left(\frac{L_0\eta}{1 - t} - 1\right). \tag{8.15}$$

Hence, (8.12) holds if

$$\frac{L_0\eta}{1-t} - 1 \le 0 \text{ at } t = \delta_1,$$

which is true by (8.7). Similarly, by the first substep of sequence (8.4) and (8.10), estimate (8.11) holds for

$$t_{n+1} - t_n = t_{n+1} - s_n + s_n - t_n \le (1+\delta)(s_n - t_n)$$

if

$$\frac{L(1+\delta)^2(s_n-t_n)^2 + \frac{L_1L(s_n-t_n)^2}{1-L_0t_n}}{2(1-L_0t_{n+1})} \le \delta(s_n - t_n). \tag{8.16}$$

By the right hand side of (8.7), the induction hypotheses and the definition of t^{**}, we have $2t_nL_0 \le 2L_0\frac{\eta}{1-\delta} \le 1$, so $\frac{1}{1-L_0t_n} \le 2$. Then, (8.16) holds if

$$\frac{L(1+\delta)^2(s_n-t_n) + 2L_1L(s_n-t_n)}{2(1-L_0t_{n+1})} \le \delta$$

or

$$L(1+\delta)^2\delta^n\eta + 2L_1L\delta^n\eta + 2L_0\delta\frac{1-\delta^{n+2}}{1-\delta}\eta - 2\delta \le 0$$

or

$$h_n^{(2)}(t) \le 0 \text{ at } t = \delta_2. \tag{8.17}$$

We also have

$$\begin{aligned}
h_{k+1}^{(2)}(t) &= h_{k+1}^{(2)}(t) + h_k^{(2)}(t) - h_k^{(2)}(t)\\
&= L(1+t)^2t^k\eta + 2L_1Lt^k\eta + 2L_0(1+t+\ldots+t^{k+2})\eta - 2\\
&\quad -L(1+t)^2t^{k-1}\eta - 2L_1Lt^{k-1}\eta - 2L_0(1+t+\ldots+t^{k+1})\eta + 2\\
h_k^{(2)}(t) &= h_k^{(2)}(t) + L(1+t)^2t^k\eta - L(1+t)^2t^{k-1}\eta\\
&\quad +2L_1Lt^k\eta - 2L-1Lt^{k-1}\eta + 2L_0t^{k+2}\eta\\
&= h_k^{(2)}(t) + p_2(t)t^{k-1}\eta.
\end{aligned}$$

In particular $h_{k+1}^{(2)}(t) = h_k^{(2)}(t)$ at $t = \delta_2$. Define function $h_\infty^{(2)}$ on $[0,1)$ by

$$h_\infty^{(2)}(t) = \lim_{k\longrightarrow\infty} h_k^{(2)}(t). \tag{8.18}$$

In view of (8.18) and the definition of polynomial $h_k^{(2)}$, we also deduce

$$h_\infty^{(2)}(t) = 2\left(\frac{L_0\eta}{1-t} - 1\right),$$

so (8.17) holds, since $h_\infty^{(2)}(t) \le 0$ at $t = \delta_2$. Sequence $\{t_n\}$ is nondecreasing by (8.4), (8.10) and (8.11), and bounded from above t^{**}. Hence, it converges to t^*. □

3. Convergence of Method (8.3)

We introduce conditions (H) to be used in the semilocal convergence of method (8.3).

Suppose:

(H1) There exists $x_0 \in D, \eta \geq 0$ such that $F'(x_0)^{-1} \in L(B_2, B_1)$ and

$$\|F'(x_0)^{-1}F'(x_0)\| \leq \eta.$$

(H2) $\|F'(x_0)^{-1}(F'(u) - F'(x_0))\| \leq L_0\|u - x_0\|$ for all $u \in D$.
Set $D_1 = U(x_0, \frac{1}{L_0}) \cap D$.

(H3)

$$\|F'(x_0)^{-1}(F'(u) - F'(v))\| \leq L\|u - v\|$$

and

$$\|F'(x_0)^{-1}F'(u)\| \leq L_1.$$

for all $u, v \in D_1$ or $u \in D_1$ and $v = u - F'(u)^{-1}F(u)$.

(H4) Conditions of Lemma 13 or Lemma 14 hold.
and

(H5) $U[x_0, t^*] \subset D$.

Notice that under the second choice of u and v parameter L can be smaller. Denote by $\tilde{L}$ the corresponding parameter.

Next, the semilocal convergence of method (8.3) is developed.

Theorem 10. Under the conditions (H) the following items hold: $\{x_n\} \subset U(x_0, t^*)$, and there exists $x^* \in U[x_0, t^*]$ solving equation $F(x) = 0$ such that $x^* = \lim_{n \longrightarrow \infty} x_n$ and

$$\|x^* - x_n\| \leq t^* - t_n. \tag{8.19}$$

Proof. Estimates

$$\|y_k - x_k\| \leq s_k - t_k \tag{8.20}$$

and

$$\|x_{k+} - x_k\| \leq t_{k+1} - s_k \tag{8.21}$$

are shown using induction on k. By (8.4), (H1) and the first substep of method (8.3), we have

$$\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = s_0 - t_0 < t^*,$$

so $y_0 \in U(x_0, t^*)$ and (8.20) holds for $k = 0$. Let $v \in U(x_0, t^*)$. In view of (H1), (H2) and the definition of t^*, we get

$$\|F'(x_0)^{-1}(F'(v) - F'(x_0))\| \leq L_0\|v - x_0\| \leq L_0 t^* < 1,$$

$F'(v)^{-1} \in L(B_2, B_1)$ and

$$\|F'(v)^{-1}F'(x_0)\| \leq \frac{1}{1 - L_0\|v - x_0\|} \tag{8.22}$$

by a lemma due to Banach on linear operators with inverses [12]. We have

$$\begin{aligned}
\|z_0 - x_0\| &= \|\gamma F'(x_0)^{-1}F(x_0)\| \\
&\leq |\gamma|\|F'(x_0)^{-1}F(x_0)\| \\
&\leq \|F'(x_0)^{-1}F(x_0)\| \leq \eta < t^*,
\end{aligned}$$

so $z_0 \in U(x_0, t^*)$. Suppose $x_k, y_k, z_k \in U(x_0, t^*)$. Then, by the second substep of method (8.3), (H3), (8.22) (for $v = x_k$), and (8.4), we get

$$\begin{aligned}
\|x_{k+1} - y_k\| &\leq \frac{1}{2|\gamma|}\|F'(x_k)^{-1}F'(x_0)\| \\
&\quad \times \|F'(x_0)^{-1}(F'(x_k) - F'(z_k))\|\|F'(x_k)^{-1}F(x_k)\| \\
&\leq \frac{1}{2|\gamma|}\frac{|\gamma|\|y_k - x_k\|^2}{1 - L_0\|x_k - x_0\|} \leq t_{k+1} - s_k,
\end{aligned}$$

and

$$\begin{aligned}
\|x_{k+1} - x_k\| &\leq \|x_{k+1} - y_k\| + \|y_k - x_0\| \\
&\leq t_{k+1} - s_k + s_k - t_0 = t_{k+1} < t^*,
\end{aligned}$$

so (8.21) holds and $x_{k+1} \in U(x_0, t^*)$.

Next, using the first substep of method (8.3) we can write

$$\begin{aligned}
F(x_{k+1}) &= F(x_{k+1}) - F(x_k) + F(x_k) \\
&= F(x_{k+1}) - F(x_k) - F'(x_k)(y_k - x_k) \\
&\quad -F'(x_k)(x_{k+1} - x_k) + F'(x_k)(x_{k+1} - x_k) \\
&= F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k) \\
&\quad +F'(x_k)(x_{k+1} - y_k).
\end{aligned} \tag{8.23}$$

It folloows by (8.4), (H3), (8.23) and the induction hypotheses

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{k+1})\| &\leq \frac{L}{2}\|x_{k+1} - x_k\|^2 + L_1\|x_{k+1} - y_k\| \\
&\leq \frac{L}{2}(t_{k+1} - t_k)^2 + L_1(t_{k+1} - s_k).
\end{aligned} \tag{8.24}$$

Then, by (8.4), the first substep of method (8.3), (8.22) (for $v = x_{k+1}$) and (8.24) we obtain

$$\begin{aligned}
\|y_{k+1} - x_{k+1}\| &\leq \|F'(x_{k+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{k+1})\| \\
&\leq \frac{L(t_{k+1} - t_k)^2 + 2L_1(t_{k+1} - s_k)}{2(1 - L_0\|x_{k+1} - x_0\|)} \leq s_{k+1} - t_{k+1}
\end{aligned}$$

and

$$\begin{aligned}\|y_{k+1}-x_0\| &\leq \|y_{k+1}-x_{k+1}\|+\|x_{k+1}-x_0\| \\ &\leq s_{k+1}-t_{k+1}+t_{k+1}-t_0=s_{k+1}<t^*.\end{aligned}$$

Thus, $y_{k+1} \in U(x_0,t^*)$ and the induction for (8.20) is completed too. Sequence $\{t_k\}$ is fundamental as convergent. In view of (8.20) and (8.21), sequence $\{x_k\}$ is fundamental too in Banach space B_1. Hence, it converges to some $x^* \in U[x_0,t^*]$. By letting $k \longrightarrow \infty$ in (8.24), we conclude $F(x^*)=0$. □

The uniqueness of the solution x^* result follows without necessarily using conditions of Theorem 10.

Proposition 6. *Suppose $x^* \in U(x_0,\xi_0) \subset D$ for some $\xi_0>0$ is a simple solution of equation $F(x)=0$; Condition (H2) holds and there exists $\xi \geq \xi_0$ such that*

$$M_0(\xi_0+\xi)<2. \tag{8.25}$$

Set $G=U[x_0,\xi]\cap D$. Then, the point x^* is the only solution of equation $F(x)=0$ in the set G.

Proof. Let $y^* \in G$ with $F(y^*)=0$. Define linear operator $Q=\int_0^1 F'(x^*+\theta(y^*-x^*))d\theta$. By using (H2) and (8.25) we get in turn

$$\begin{aligned}\|F'(x_0)^{-1}(Q-F'(x_0))\| &\leq M_0\int_0^1((1-\theta)\|x^*-x_0\|+\theta\|y^*-x_0\|)d\theta \\ &\leq \frac{M_0}{2}(\xi_0+\xi)<1.\end{aligned}$$

Thus, $x^*=y^*$ is implied since $Q^{-1}\in L(B_2,B_1)$ and $Q(y^*-x^*)=F(y^*)-F(x^*)=0$. □

Remark. (1) The conclusions of Theorem 10 hold if (H5) is replaced by
(H5)' $U[x_0,\frac{1}{L_0}]\subset D$ if conditions of Lemma 13 hold
or
(H5)" $U[x_0,\frac{\eta}{1-\delta}]\subset D$ if conditions of Lemma 14 hold. Notice that $\frac{1}{L_0}$ and $\frac{\eta}{1-\delta}$ are given in closed form.

(2) Condition (H3) can be replaced by stronger
(H3)' $\|F'(x_0)^{-1}(F'(u)-F'(v))\|\leq L_2\|u-v\|$

$$\|F'(x_0)^{-1}F'(u)\|\leq L_3$$

for all $u,v\in D_0$ used by us in [1,2,3,4,5]
or
(H3)"$\|F'(x_0)^{-1}(F'(u)-F'(v))\|\leq L_4\|u-v\|$

$$\|F'(x_0)^{-1}F'(u)\|\leq L_5$$

for all $u,v\in D$ used by us others.

Notice that $L_0\leq L_4$ and $L\leq L_2\leq L_4$.

(3) Tighter Lipschitz constant will further improve the convergence criteria. As an example. Define $S = U(x_1, \frac{1}{L_0} - \eta) \cap D$ provided that $L_0\eta < 1$ and suppose $S \subset D$. Then, we have $S \subset D_0$. Hence, the tighter Lipschitz constants on S can be used. It is worth noticing that in all these cases we use the initial information, since $x_1 = x_0 - F'(x_0)^{-1}F(x_0)$.

(4) The second condition in (H3) can be dropped if we replace $L-1$ by $1+L_0t$, since

$$\begin{aligned} \|F'(x_0)^{-1}F(u)\| &\leq \|F'(x_0)^{-1}(F'(u) - F'(x_0) + F'(x_0))\| \\ &\leq 1 + L_0\|u - x_0\|. \end{aligned}$$

Then, $1+L_0t$ can replace L_1. This is important when $1+L_0t \leq L_1$.

(5) Under all the conditions (H), set $\xi_0 = t^*$.

4. Numerical Experiments

We compute the radius of convergence in this section.

Example 1. Let $B_1 = B_2 = \mathbb{R}$. Let us consider a scalar function F defined on the set $\Omega = U[x_0, 1-q]$ for $q \in (0,1)$ by

$$F(x) = x^3 - q.$$

Choose $x_0 = 1$. Then, we obtain the estimates $\eta = \frac{1-q}{3}$,

$$\begin{aligned} |F'(x_0)^{-1}(F'(x) - F'(x_0))| &= |x^2 - x_0^2| \\ &\leq |x+x_0||x-x_0| \leq (|x-x_0| + 2|x_0|)|x-x_0| \\ &= (1-q+2)|x-x_0| = (3-q)|x-x_0|, \end{aligned}$$

for all $x \in \Omega$, so $L_0 = 3-q$, $\Omega_0 = U(x_0, \frac{1}{L_0}) \cap \Omega = U(x_0, \frac{1}{L_0})$,

$$\begin{aligned} |F'(x_0)^{-1}(F'(y) - F'(x)| &= |y^2 - x^2| \\ &\leq |y+x||y-x| \leq (|y-x_0+x-x_0+2x_0)||y-x| \\ &= (|y-x_0| + |x-x_0| + 2|x_0|)|y-x| \\ &\leq (\frac{1}{L_0} + \frac{1}{L_0} + 2)|y-x| = 2(1+\frac{1}{L_0})|y-x|, \end{aligned}$$

for all $x, y \in \Omega$ and so $L = 2(1+\frac{1}{L_0})$.

$$\begin{aligned} |F'(x_0)^{-1}(F'(y) - F'(x)| &= (|y-x_0| + |x-x_0| + 2|x_0|)|y-x| \\ &\leq (1-q+1-q+2)|y-x| = 2(2-q)|y-x|, \end{aligned}$$

for all $x, y \in D$ and $L_2 = 2(2-q)$.

Notice that for all $q \in (0,1), L_1 = (1+\frac{1}{L_0})^2$,

$$L_0 < L < L_2.$$

Next, set $y = x - F'(x)^{-1}F(x), x \in D$. Then, we have

$$y + x = x - F'(x)^{-1}F(x) + x = \frac{5x^3+q}{3x^2}.$$

Define function $\bar{F}$ on the interval $D = [q, 2-q]$ by

$$\bar{F}(x) = \frac{5x^3+q}{3x^2}.$$

Then, we get by this definition that

$$\begin{aligned} \bar{F}'(x) &= \frac{15x^4 - 6xq}{9x^4} \\ &= \frac{5(x-q)(x^2+xq+q^2)}{3x^3}, \end{aligned}$$

where $p = \sqrt[3]{\frac{2q}{5}}$ is the critical point of function $\bar{F}$. Notice that $q < p < 2-q$. It follows that this function is decreasing on the interval (q,p) and increasing on the interval $(q, 2-q)$, since $x^2 + xq + q^2 > 0$ and $x^3 > 0$. Hence, we can set

$$K_2 = \frac{5(2-q)^2+q}{9(2-q)^2}$$

and

$$K_2 < L_0.$$

But if $x \in D_0 = [1 - \frac{1}{L_0}, 1 + \frac{1}{L_0}]$, then

$$\tilde{L} = \frac{5\rho^3+q}{9\rho^2},$$

where $\rho = \frac{4-q}{3-q}$ and $K < K_1$ for all $q \in (0,1)$.

Bibliography

[1] Argyros, I.K., *Computational theory of iterative methods*. Series: Studies in Computational Mathematics, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[2] Argyros, I.K. *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[3] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[4] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)*, 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[5] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative methods*, Elsevier (Academic Press), New York, 2018.

[6] Arroyo, V., Cordero, A., Torregrosa, J. R., Approximation of artificial satellites preliminary orbits: The efficiency challenge. *Mathematical and Computer modeling*, 54, (7-8), (2011), 1802-1807.

[7] BaBajee, D.K.R, Dauhoo, M.Z, Durvishi, M.T., Karami, A., Barati, A., Analysis of two Chebyshev-like third order methods free from second derivative for solving systems of nonlinear equations,*J. Comput. Appl. Math.*, 233, (2010), 2002-2012.

[8] Behl, R., Maroju, P., Martinez, E., Singh, S., A study of the local convergence of a fifth order iterative method, *Indian J. Pure Appl. Math.*, 51, 2, (2020), 439-455.

[9] Ezquerro, J.A., Hernandez, M.A., *Newton's method: An updated approach of Kantorovich's theory*, Cham Switzerland, (2018).

[10] Grau-Sánchez, Miquel and Grau, Àngela and Noguera, Miquel, Ostrowski type methods for solving systems of nonlinear equations. *Appl. Math. Comput.*, 281, (2011), 2377-2385.

[11] Homeier, H.H.H., A modified Newton method with cubic convergence: the multivariate case, *Journal of Computational and Applied Mathematics*, 169, (2004), 161-169.

[12] Kantorovich, L.V., Akilov, G.P., *Functional Analysis*, Pergamon Press, Oxford, (1982).

[13] Magréñan, A.A., Argyros, I.K., Rainer, J.J., Sicilia, J.A., Ball convergence of a sixth-order Newton-like method based on means under weak conditions, *J. Math. Chem.*, (2018), 56, 2117-2131, https://doi.org/10.1007/ s10910-018-0856-y.

[14] Potra, F.A. and Pták, V., Nondiscrete induction and iterative processes, *Research Notes in Mathematics*, 103, Pitman (Advanced Publishing Program), Boston, MA. (1984).

[15] Proinov, P.D., General local convergence theory for a class of iterative processes and its applications to Newton's method, *J. Complexity*, 25, (2009), 38-62.

[16] Proinov, P.D., New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems, *J. Complexity*, 26, (2010), 3-42.

[17] Rheinboldt, W.C., *An adaptive continuation process of solving systems of nonlinear equations*, Polish Academy of Science, Banach Ctr. Publ. 3, (1978), 129-142.

[18] Shakhno, S.M., Gnatyshyn, O.P., On an iterative algorithm of order 1.839... for solving nonlinear least squares problems, *Appl. Math. Applic.*, 161, (2005), 253-264.

[19] Shakhno, S.M., Iakymchuk, R.P., Yarmola, H.P., Convergence analysis of a two step method for the nonlinear squares problem with decomposition of operator, *J. Numer. Appl. Math.*, 128, (2018), 82-95.

[20] Sharma, J.R, Kumar, S., A class of accurate Newton-Jarratt like methods with applications to nonlinear models, *Comput. Appl. Math*, (2022).

[21] Traub, J.F., *Iterative methods for the solution of equations*, Prentice Hall, New Jersey, U.S.A, (1964).

[22] Verma, R., *New Trends in Fractional Programming*, Nova Science Publishers, New York, USA, (2019).

Chapter 9

Ball Convergence Comparison for Methods Whose First Step is Given by Newton's for Solving Nonlinear Equations

1. Introduction

A plethora of applications from diverse areas can be brought in the form

$$F(x) = 0, \tag{9.1}$$

where $F : \Omega \subset X \longrightarrow Y$ is a continuous operator between Banach spaces X, Y and $\Omega \neq \phi$ is an open or closed set. The solution x^* if it exists of the non-linear (9.1) is needed in closed form. But this is possible only in special cases. That is why researchers and practitioners resort to the conclusion of iterative methods of approximating x^*.

Newton's method defined for $x_0 \in \Omega$, $n = 0, 1, 2, ...$ as

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \tag{9.2}$$

is by no doubt the most popular quadratically convergent method. But to increase the order of convergence too numerous methods have been proposed with (9.2) as a first step. Below we provide an incomplete list of popular methods of order higher than two:

$$x_{n+1} = y_n - F'(x_n)^{-1}F(x_n), \tag{9.3}$$

$$x_{n+1} = y_n + \frac{1}{2}L_n(I-\gamma L_n)^{-1}F'(x_n)^{-1}F(x_n), \tag{9.4}$$

$$L_n = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n),$$

$$x_{n+1} = y_n + \frac{1}{2}L_n(I+L_n(I-\gamma L_n)^{-1})F'(x_n)^{-1}F(x_n), \tag{9.5}$$

$$x_{n+1} = y_n + \frac{1}{2}K_n(I+K_n(I-\gamma K_n)^{-1})F'(x_n)^{-1}F(x_n), \tag{9.6}$$

$$K_n = F'(x_n)^{-1}F''(x_n - \frac{1}{3}F'(x_n)^{-1}F(x_n))F'(x_n)^{-1}F(x_n),$$

$$x_{n+1} = y_n - F'(x_n)^{-1}(F'(x_n + \frac{1}{2}(y_n-x_n)) - F'(x_n))(y_n-x_n), \tag{9.7}$$

$$z_n = x_n - [I + \frac{1}{2}Q_n + \frac{1}{2}Q_n{}^2(I-\alpha_1 Q_n)^{-1}]F'(x_n)^{-1}F(x_n), \tag{9.8}$$

$$x_{n+1} = z_n - [I + Q_n + \alpha_2 Q_n{}^2)]F'(x_n)^{-1}F(z_n), \tag{9.9}$$

$$Q_n = F'(x_n)^{-1}(F'(x_n) - F'(V_n)),\ V_n = x_n - \frac{1}{3}F'(x_n)^{-1}F(x_n),$$

$$z_n = x_n - [I + \frac{1}{2}L_n + \frac{\alpha}{2}L_n{}^2 + \frac{\alpha^2}{2}L_n{}^3]F'(x_n)^{-1}F(x_n), \tag{9.10}$$

$$x_{n+1} = z_n - [I + L_n + \beta\Delta_n]F'(x_n)^{-1}F(z_n) \tag{9.11}$$

and

$$\Delta_n = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(z_n).$$

Here α, β, γ are real parameters. Methods (9.2),(9.3); (9.2),(9.4); (9.2),(9.5); (9.2),(9.6); (9.2),(9.7); (9.2),(9.2),(9.8),(9.9); and (9.2),(9.10),(9.11), are of order three or higher.

In this chapter, we determine the ball of convergence for these methods, so we can compute them under the same set of conditions. As far as we know such a study has not been done. Our technique is very general so it can be used even if we exchange (9.2) by

$$y_n = x_n + F'(x_n)^{-1}F(x_n) \tag{9.12}$$

and use as a second step say

$$x_{n+1} = y_n - F'(x_n)^{-1}F(y_n) \tag{9.13}$$

or

$$x_{n+1} = x_n - F'(x_n)^{-1}[x_n, y_n; F]F'(x_n)^{-1}F(x_n). \tag{9.14}$$

2. Ball Convergence

Let $\varphi_0, \varphi, \varphi_1$ and φ_2 be continuous and non-decreasing function defined on the interval $T = [0, +\infty)$. The common conditions used are:

Suppose

(C_1) $F : \Omega \longrightarrow Y$ is twice continuously differentiable, and there exists a simple solution $x^* \in \Omega$ of equation (9.1).

(C_2) Function $\varphi_0(t) - 1$ has the smallest zero $\rho \in T - \{0\}$.

Set $\Omega_1 = U(x^*, \rho) \cap \Omega$.

(C_3)
$$\|F'(x^*)^{-1}(F'(w) - F'(x^*)\| \leq \varphi_0(\|w - x^*\|$$

for each $w \in \Omega$.

(C_4)
$$\|F'(x^*)^{-1}(F'(u) - F'(v)\| \leq \varphi(\|u - v\|),$$

$$\|F'(x^*)^{-1}F'(u)\| \leq \varphi_1(\|u - x^*\|)$$

and
$$\|F'(x^*)^{-1}F''(u)\| \leq \varphi_2(\|u - x^*\|)$$

for each $u, v \in \Omega_1$.

(C_5) $U[x^*, s] \subset \Omega$, where s is determined later depending on the method.

In view of conditions (C), we have in turn the estimates :

Method (9.2) [1,2,3,4,5,6,7]

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - F'(x_n{}^{-1}F(x_n) \\ &= [F'(x_n{}^{-1}F(x_n)][\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_n - x^*)) - F'(x_n)d\theta(x_n - x^*)], \end{aligned}$$

so,

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \frac{\int_0^1 \varphi((1-\theta)\|x_n - x^*\|)d\theta\|x_n - x^*\|}{1 - \varphi_0(\|x_n - x^*\|)} \\ &\leq \varphi_1(\|x_{n+1} - x^*\|)\|x_{n+1} - x^*\| \leq \|x_{n+1} - x^*\| < r, \end{aligned}$$

where

$$\varphi_1(t) = \frac{\int_0^1((1-\theta)t)d\theta}{1 - \varphi_0(t)}$$

and r is (if it exists) the smallest solution of an equation $\varphi_1(t) - 1 = 0$ in (0,1).
Then, $\{x_n\} \subset U[x^*, r]$ and

$$\lim_{n \longrightarrow \infty} x_n = x^*. \tag{9.15}$$

Method (9.2), (9.3) [13,14,15]
As in the previous case we deduce $\|y_n - x^*\| \leq \varphi_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r$.
It then follows in turn from the second sub-step.

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|y_n - x^* - F'(y_n)^{-1}F(y_n) + (F'(y_n)^{-1} - F'(x_n)^{-1})F(y_n)\| \\
&\leq \frac{\int_0^1 \varphi((1-\theta)\|y_n - x^*\|)d\theta\|y_n - x^*\|}{1 - \varphi(\|y_n - x^*\|)} + \frac{m_n \int_0^1 \varphi_1(\theta\|y_n - x^*\|)d\theta\|y_n - x^*\|}{1 - \varphi_0(\|y_n - x^*\|)(1 - \varphi_0(\|x_n - x^*\|)} \\
&\leq \frac{\int_0^1 \varphi((1-\phi)\varphi_1(\|x_n - x^*\|)\|x_n - x^*\|)d\theta\varphi_1(\|x_n - x^*\|)\|x_n - x^*\|}{1 - \varphi_0(\varphi_1(\|x_n - x^*\|)\|x_n - x^*\|)} \\
&+ \frac{m_n \int_0^1 \varphi(\phi\varphi_1(\|x_n - x^*\|)\|x_n - x^*\|)d\theta\varphi_1(\|x_n - x^*\|)\|x_n - x^*\|}{(1 - \varphi_0(\varphi_1(\|x_n - x^*\|)\|x_n - x^*\|))(1 - \varphi_0(\|x_n - x^*))} \\
&\leq \varphi_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|,
\end{aligned}$$

where

$$m_n = \varphi_0(\|y_n - x^*\|) + \varphi_0(\|x_n - x^*\|)$$

or

$$m_n = \varphi(\|y_n - x_n\|),$$

so

$$\begin{aligned}
\varphi_2(t) &= \frac{\int_0^1 \varphi((1-\theta)\varphi_1(t)t)d\theta\varphi_1(t)}{1 - \varphi_0(\varphi_1(t)t)} + \frac{m_n \int_0^1 \varphi(\theta\varphi_1(t)t)d\theta\varphi_1(t)}{(1 - \varphi_0(\varphi_1(t)t)(1 - \varphi_0(t)} \\
m(t) &= \varphi_0(t) + \varphi_0(\varphi_1(t)t)
\end{aligned}$$

or

$$m = \varphi((1 + \varphi_1(t)t)$$

and r is the smallest solution (if it exists in $(0, \rho)$) of the solutions $\varphi_1(t) - 1 = 0, \varphi_2(t) - 1 = 0$.
Then, (9.15) holds for this method.

Method (9.2),(9.4) [17,18,19]

It follows from the second sub-step (9.4)

$$\|x_{n+1} - x^*\| = x_n - x^* - F'(x_n)^{-1}F(x_n) - \frac{1}{2}L_n(I - \gamma L_n)^{-1}F'(x_n)^{-1}F(x_n),$$

so

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \frac{\int_0^1 \varphi((1-\theta)\|x_n - x^*\|)d\theta\|x_n - x^*\|}{1 - \varphi_0(\|x_n - x^*\|)} + \frac{1}{2}p_n \frac{\int_0^1 \varphi_1(\theta\|x_n - x^*\|)d\theta\|x_n - x^*\|}{(1 - |\gamma|p_n\|x_n - x^*\|)(1 - \phi_0(\|x_n - x^*\|)} \\
&\leq \varphi_2(\|x_n - x^*\|)
\end{aligned}$$

where we used

$$\|\gamma L_n\| \le |\gamma| \frac{\varphi_1(\|x_n - x^*\|)\varphi_2(\|x_n - x^*\|)\|x_n - x^*\|}{1 - \varphi_0(\|x_n - x^*\|)} = |\gamma| p_n \|x_n - x^*\|,$$

$$p(t) = \frac{\varphi_1(t)\varphi_2(t)}{1 - \varphi_0(t)},$$

and

$$\varphi_2(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1 - \varphi_0(t)} + \frac{1}{2}p(t)\frac{\int_0^1 \varphi_1(\theta t)d\theta t}{(1 - |\gamma| p(t))(1 - \varphi_0(t))}$$

Then r is determined as in the previous case and (9.15) holds for this method too.

Method(9.2),(9.5) [19]
As in the previous case, we get

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|y_n - x^* + \frac{1}{2}L_n F'(x_n)^{-1}F(x_n) + \frac{1}{2}L_n^2(I - \gamma L_n)^{-1}F'(x_n)^{-1}F(x_n)\| \\
&\le \|y_n - x^*\| + \frac{1}{2}p_n \int_0^1 \varphi_1(\theta\|x_n - x^*\|)d\theta\|x_n - x^*\|^2 \\
&+ \frac{1}{2}p_n^2 \frac{\int_0^1 \varphi_1(\theta\|x_n - x^*\|)d\theta\|x_n - x^*\|^3}{(1 - |\gamma| p_n\|x_n - x^*\|)(1 - \varphi_0(\|x_n - x^*\|)} \\
&\le \varphi_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r, \qquad (9.16)
\end{aligned}$$

where

$$\varphi_2(t) = \varphi_1(t) + \frac{1}{2}p(t)\frac{\int_0^1 \varphi_1(\theta t)d\theta t}{1 - \varphi_0(t)} + \varphi_1(t) + \frac{1}{2}p(t)^2\frac{\int_0^1 \varphi_1(\theta t)d\theta t^2}{(1 - |\gamma| p(t))(1 - \varphi_0(t))}.$$

Method(9.2),(9.6) [20]
As in the previous case for φ_1 and φ_2. But we also have to show $v_n = x_n - \frac{1}{3}F'(x_n)^{-1}F(x_n) \in U[x^*, r)$.
But we have

$$\begin{aligned}
\|v_n - x^*\| &= \|x_n - x^* - F'(x_n)^{-1}F(x_n) + \frac{2}{3}F'(x_n)^{-1}F(x_n)\| \\
&\le \|x_n - x^* - F'(x_n)^{-1}F(x_n)\| + \frac{2}{3}\|F'(x_n)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_n)\| \\
&\le \frac{[\int_0^1 \varphi((1-\theta)\|x_n - x^*\|)d\theta + \frac{2}{3}\int_0^1 \varphi_1(\theta\|x_n - x^*\|)d\theta]\|x_n - x^*\|}{1 - \varphi_0(\|x_n - x^*\|)} \\
&\le \varphi_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r.
\end{aligned}$$

where,

$$\varphi_3(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta + \frac{2}{3}\int_0^1 \varphi_1(\theta t)d\theta}{1 - \varphi_0(t)}.$$

Then, r is taken as the smallest solution in $(0,\rho)$ of equation $\varphi_1(t)=0, \varphi_2(t)-1=0$ and $\varphi_3(t)-1=0$

Method(9.2),(9.7) [18]
By the second sub-step, we get in turn that

$$\begin{aligned}
\|x_{n+1}-x^*\| &\le \|y_n-x^*\| + \frac{\lambda_n\int_0^1\varphi_1(\theta\|x_n-x^*\|)d\theta\|x_n-x^*\|}{(1-\varphi_0(\|x_n-x^*\|)^2}\\
&\le [\varphi_1(\|x_n-x^*\|) + \frac{\lambda_n\int_0^1\varphi_1(\theta\|x_n-x^*\|)d\theta}{(1-\varphi_0(\|x_n-x^*\|)}]\|x_n-x^*\|\\
&\le \varphi_2(\|x_n-x^*\|)\|x_n-x^*\| \le \|x_n-x^*\| < r,\\
where, \lambda_n &= \varphi(\frac{1}{2}\|y_n-x_n\|)\\
or, \lambda_n &= \varphi_0(\frac{1}{2}(1+\varphi_1(\|x_n-x^*\|))\|x_n-x^*\|) + \varphi_0(\|x_n-x^*\|),\\
\lambda &= \lambda(t) = \varphi(\frac{1}{2}(1+\varphi_1(t))t)
\end{aligned}$$

and

$$\varphi_2(t) = \varphi_1(t) + \frac{\int_0^1\varphi_1(\theta t)d\theta}{(1-\varphi_0(t))^2}.$$

Method (9.2),(9.8),(9.9) [16]
By (9.8), we have in turn that

$$\begin{aligned}
\|z_n-x^*\| &\le \|y_n-x^*\| + \frac{1}{2}q_n(1+\frac{q_n}{1-|\alpha_1|q_n})\frac{\int_0^1\varphi_1(\theta\|x_n-x^*\|)d\theta\|x_n-x^*\|}{1-\varphi_0(\|x_n-x^*\|)}\\
&\le \varphi_2(\|x_n-x^*\|)\|x_n-x^*\| \le \|x_n-x^*\| < r,
\end{aligned}$$

where

$$\begin{aligned}
\varphi_2(t) &= \varphi_1(t) + \frac{1}{2}q(1+\frac{q}{1-|\alpha_1|q})\frac{\int_0^1\varphi_1(d\theta)d\theta}{1-\varphi_0(t)},\\
q = q(t) &= 3\frac{\varphi(\frac{\int_0^1\varphi_1(\theta t)d\theta t}{1-\varphi_0(t)}}{1-\varphi_0(t)}
\end{aligned}$$

and we also used the estimate

$$\begin{aligned}
\|\alpha_1 Q_n\| = |\alpha_1|\|Q_n\| &\le 3|\alpha_1|\frac{\varphi(\frac{1}{3}F'(x_n)^{-1}F(x_n))}{1-\varphi_0(\|x_n-x^*\|)}\\
&\le 3|\alpha_1|\frac{\int_0^1\varphi_1(\frac{\theta\|x_n-x^*\|)d\theta\|x_n-x^*\|}{1-\varphi_0(\|x_n-x^*\|)})}{1-\varphi_0(\|x_n-x^*\|)} \le |\alpha_1|q_n.
\end{aligned}$$

We also consider equation $\varphi_3(t)-1=0$ given before required to show $v_n \in U[x^*,r)$. Moreover, in view of the third sub-step, we obtain in turn that

$$\begin{aligned}
\|x_n-x^*\| &= \|z_n-x^*-F'(z_n)^{-1}F(z_n)+(F'(z_n)^{-1}-F'(x_n)^{-1})F(z_n)\| \\
&\leq \|z_n-x^*-F'(z_n)^{-1}F(z_n)\|+\frac{\|F'(z_n)^{-1}F(x^*)\|\|F'(x_*)^{-1}(F'(x_n)-F'(z_n))\|}{\|F'(x_n)^{-1})F'(x^*)\|\|F'(x^*)^{-1})F(z_n)} \\
&\leq \frac{\int_0^1 \varphi((1-\theta)\|z_n-x^*\|)d\theta\|z_n-x^*\|}{1-\varphi_0(\|z_n-x^*\|)} \\
&+\frac{(\varphi_0(\|z_n-x^*\|)+\varphi_0(\|x_n-x^*\|))\int_0^1 \varphi_1(\theta\|z_n-x^*\|)d\theta\|z_n-x^*\|}{(1-\varphi_0(\|z_n-x^*\|)(1-\varphi_0(\|x_n-x^*\|))} \\
&+q_n(1+|\alpha_2|q_n)\frac{\int_0^1 \varphi_1(\theta\|z_n-x^*\|)d\theta\|z_n-x^*\|}{1-\varphi_0(\|x_n-x^*\|} \\
&\leq \varphi_4(\|x_n-x^*\|)\|x_n-x^*\| \leq \|x_n-x^*\|,
\end{aligned}$$

where

$$\begin{aligned}
\varphi_4(t) = &\frac{\int_0^1 \varphi((1-\theta)\varphi_2(t)t)\varphi(t)}{1-\varphi_0(\varphi_2(t)t)} \\
&+\frac{(\varphi_0(\varphi_2(t)t)+\varphi_0(t))\int_0^1 \varphi_1(\theta\varphi_2(t)t)d\theta\varphi_2(t)}{1-\varphi_0(\varphi_2(t)t))(1-\varphi_0(t))} \\
&+\frac{q(1+|\alpha_2|q)\int_0^1 \varphi_1(\theta\varphi_2(t)t)d\theta\varphi_2(t)}{1-\varphi_0(t)}.
\end{aligned}$$

Then, the smallest of the solution of equations(if it exists)
$\varphi_i(t)-1=0, i=1,2,3,4$ shall be r.

<u>**Method (9.2),(9.10),(9.11)**</u> [15]
By (9.10),we in turn that

$$\begin{aligned}
\|z_n-x^*\| &\leq \|y_n-x^*\|+\frac{1}{2}\frac{p_n(1+|\alpha|p_n+|\alpha|^2p_n^2)\int_0^1 \varphi_1(\theta\|x_n-x^*\|)d\theta\|x_n-x^*\|}{1-\varphi_0(\|x_n-x^*\|)} \\
&\leq \varphi_2(\|x_n-x^*\|)\|x_n-x^*\| \leq \|x_n-x^*\| < r,
\end{aligned} \tag{9.17}$$

where,

$$\varphi_2(t) = \varphi_1(t)+\frac{1}{2}\frac{p(t)(1+|\alpha|p(t)+|\alpha|^2p(t)^2)\int_0^1 \varphi_1(\theta t)d\theta}{1-\varphi_0(t)}.$$

Moreover, from (9.11), we get in turn that

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|z_n-x^*-(F'(z_n)^{-1}F(z_n))+(F'(z_n)^{-1}(F'(x_n)-F'(z_n))F'(x_n)^{-1}F(z_n)\\ &\quad -L_n(F'(x_n)^{-1}F(z_n)-\beta V_n(F'(x_n)^{-1}F(z_n)\|\\ &\leq \frac{\int_0^1 \varphi(\theta\|z_n-x^*\|)d\theta\|z_n-x^*\|}{1-\varphi_0(\|z_n-x^*\|)}\\ &\quad+\frac{(\varphi_0(\|z_n-x^*\|)+\varphi_0(\|x_n-x^*\|))\int_0^1\varphi_1(\theta\|z_n-x^*\|)d\theta\|z_n-x^*\|}{(1-\varphi_0(\|z_n-x^*\|))(1-\varphi_0(\|x_n-x^*\|))}\\ &\quad+\frac{p_n\int_0^1\varphi_1(\theta\|z_n-x^*\|)d\theta}{1-\varphi_0(\|x_n-x^*\|)}+|\beta|\|\Delta_n\|\frac{\int_0^1\varphi_1(\theta\|z_n-x^*\|)}{(1-\varphi_0(\|x_n-x^*\|))^2}\\ &\leq \varphi_3(\|x_n-x^*\|)\|x_n-x^*\|\leq\|x_n-x^*\|,\end{aligned}$$

where we used

$$\|\Delta_n\|\leq\frac{\varphi_2(\|x_n-x^*\|)\int_0^1\varphi_1(\theta\|z_n-x^*\|)d\theta\|z_n-x^*\|}{1-\varphi_0(\|x_n-x^*\|))^2}\leq\delta$$

and

$$\begin{aligned}\varphi_3(t) &= \frac{\int_0^1\varphi(\theta\varphi_2(t)t)d\theta\varphi_2(t)}{1-\varphi_0(\varphi_2(t)t)}+(\varphi_0(\varphi_2(t)t)+\varphi_0(t))\int_0^1\varphi_1(\theta\varphi_2(t)t)d\theta\varphi_2(t)\\ &\quad+P\frac{\int_0^1\varphi_1(\theta\varphi_2(t)t)d\theta\varphi_2(t)}{1-\varphi_0(t)}+\frac{|\beta|\delta\int_0^1\varphi_1(\theta\varphi_2(t)t)d\theta\varphi_2(t)}{1-\varphi_0(t)}\end{aligned}$$

Method (9.12),(9.13) [21]
By (9.12),we can write

$$\begin{aligned}y_n-x^* &= x_n-x^*-F'(x_n)^{-1}F(x_n)+2F'(x_n)^{-1}F(x_n),\\ so,\|y_n-x^*\| &\leq \frac{[\int_0^1\varphi((1-\theta)\|x_n-x^*\|d\theta+2\int_0^1\varphi_1(\theta\|x_n-x^*\|)d\theta]\|x_n-x^*\|}{1-\varphi_0(\|x_n-x^*\|)}\\ &\leq \bar{\varphi}_1(t)\|x_n-x^*\|=a\|x_n-x^*\|\leq ar,\end{aligned}$$

where

$$\bar{\varphi}_1(t)=\frac{\int_0^1\varphi((1-\theta)t)d\theta+2\int_0^1\varphi_1(\theta t)d\theta}{1-\varphi_0(t)}$$

and

$$\begin{aligned}\|\|x_{n+1}-x^*\| &= \|y_n-x^*-F'(y_n)^{-1}F(y_n)+(F'(y_n)^{-1}-F'(x_n)^{-1})F(y_n)\|\\ &\leq \frac{\int_0^1\varphi((1-\theta)\|y_n-x^*\|)d\theta\|y_n-x^*\|}{1-\varphi_0(\|y_n-x^*\|)}\\ &\quad+\frac{\int_0^1\varphi_1(\theta\|y_n-x^*\|)d\theta\|y_n-x^*\|(\varphi_0\|x_n-x^*\|+y_0(\|y_n-x^*\|))}{(1-\varphi_0(\|x_n-x^*\|))(1-\varphi_0(\|y_n-x^*\|)}\\ &\leq \varphi_2(\|x_n-x^*\|)\|x_n-x^*\|\leq\|x_n-x^*\|<r,\end{aligned}$$

where

$$\varphi_2(t) = \frac{\int_0^1 \varphi((1-\theta)at)d\theta a}{1-\varphi_0(at)} + \frac{\int_0^1 \varphi_1(\theta at)d\theta a(\varphi_0(t)+\varphi_0(at))}{1-\varphi_0(t))(1-\varphi_0(at)},$$

and $a = \varphi(\bar{t})$.
Then the radius r is chosen as the smallest solution of an equation $\varphi_2(t) - 1 = 0$, in $(0,\rho)$ and provided that (C_5) is replaced by $(C_5)' U[x^*, ar] \subset \Omega$.

In order to study method (9.12),(9.14) we suppose instead of the last condition in (C_4)

$$\|F'(x^*)^{-1}([x,y;F] - F'(x))\| \le \bar{\varphi}_2(\|x_n - x^*\|, \|y_n - x^*\|),$$

where $\bar{\varphi}_2 : [0,\rho) \times [0,\rho) \longrightarrow T$ is continuous and non-decreasing in both arguments.

Method (9.12),(9.14) [22]
As in the previous case

$$\|y_n - x^*\| \le a\|x_n - x^*\|$$

and

$$\begin{aligned}
\|x_n - x^*\| &= \|x_n - x^* - F'(x_n)^{-1}F(x_n) + F'(x_n)^{-1}F(x_n) - F'(x_n)^{-1}[x_n, y_n; F]F'(x_n)^{-1}F(x_n)\| \\
&\le \|x_n - x^* - F'(x_n)^{-1}F(x_n)\| + \|F'(x_n)^{-1}(F(x_n) - [x_n, y_n; F]F'(x_n)^{-1}F(x_n)\| \\
&\le \frac{\int_0^1 \varphi((1-\theta)\|x_n - x^*\|)d\theta\|x_n - x^*\|}{1-\varphi_0(\|x_n - x^*\|)} \\
&+ \frac{\bar{\varphi}_2(\|x_n - x^*\|, \|y_n - x^*\| \int_0^1 \varphi_1(\theta\|x_n - x^*\|)d\theta\|x_n - x^*\|}{(1-\varphi_0(\|x_n - x^*\|))^2} \\
&\le \varphi_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < r,
\end{aligned}$$

where

$$\varphi_2(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1-\varphi_0(t)} + \frac{\bar{\varphi}_2(t, at)\int_0^1 \varphi_1(\theta t)d\theta}{(1-\varphi_0(t))^2}.$$

Here, the radius r is determined by solving $\varphi_2(t) - 1 = 0$, and choosing the smallest solution $r \in (0,\rho)$. Moreover ,(C_5) is replaced by $C_5)'$.

Remark. Notice that in the first condition in (C_4), we can suppose instead
$\|F'(x^*)^{-1}(F'(u) - F'(v)\| \le \bar{\varphi}(\|u - v\|$
for all $u \in \Omega_1$and $v = u - F'(u)^{-1}F(u)$,
so $\bar{\varphi}(t) = \varphi(t)$.
Then, the tighter function $\bar{\varphi}$ can replace φ in all previous results to obtain tighter error bounds and larger radius r.

Next, the uniqueness of the solution x^* result follows which is the same for all methods.

Proposition 7. *Suppose:*

(1) The element $x^* \in U(x^*, R_0) \subset \Omega$ *for some* $s^* > 0$ *is a simple solution of equation* $F(x) = 0$.

(2) Condition (C_3) *holds.*

(3) There exists $R \geq R_0$ *so that*

$$\int_0^1 \varphi_0((1-\theta)R + \theta R_0)d\theta < 1. \tag{9.18}$$

Set $\Omega_2 = \Omega \cap U[x^*, R]$. *Then,* x^* *is the unique solution of equation (9.1) in the domain* Ω_2.

Proof. Let $\lambda \in \Omega_1$ with $F(\lambda) = 0$. Define $S = \int_0^1 F'(\lambda + \theta(x^* - \lambda))d\theta$. Using (C_3) and (9.18) one obtains

$$\begin{aligned} \|F'(x_0)^{-1}(S - F'(x_0))\| &\leq \int_0^1 \varphi_0((1-\theta)\|\lambda - x_0\| + \theta\|x^* - x_0\|)d\theta \\ &\leq \int_0^1 \varphi_0((1-\theta)R + \theta R_0)d\theta < 1, \end{aligned}$$

so $\lambda = x^*$, follows from the invertibility of S and the identity $S(\lambda - x^*) = F(\lambda) - F(x^*) = 0 - 0 = 0$. $\square$

Bibliography

[1] Argyros, I.K., On the Newton - Kantorovich hypothesis for solving equations, *J. Comput. Math.*, 169, (2004), 315-332.

[2] Argyros, I.K., *Computational theory of iterative methods*. Series: Studies in Computational Mathematics, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[3] Argyros, I.K. *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[4] Argyros, I.K., Hilout, S., Weaker conditions for the convergence of Newton's method. *J. Complexity*, 28, (2012), 364–387.

[5] Argyros, I.K., Hilout, S., On an improved convergence analysis of Newton's methods, *Applied Mathematics and Computation*, 225, (2013), 372-386.

[6] Argyros, I.K., Magréñan,A.A., *Iterative methods and their dynamics with applications*, CRC Press, New York, USA, 2017.

[7] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative methods*, Elsevier (Academic Press), New York, 2018.

[8] Arroyo, V., Cordero, A., Torregrosa, J.R., Approximation of artificial satellites preliminary orbits: The efficiency challenge. *Mathematical and Computer modeling*, 54, (7-8), (2011), 1802-1807.

[9] Bruns, D.D., Bailey, J.E., Nonlinear feedback control for operating a nonisothermal CSTR near an unstable steady state. *Chem. Eng. Sci* 32, (1977), 257-264.

[10] Babajee, D.K.R., Dauhoo, M.Z., Darvishi, M.T., Karami, A., Barati, A.: Analysis of two Chebyshev-like third order methods free from second derivatives for solving systems of nonlinear equations, *J. Comput. Appl. Math.*, 233, (2010), 2002-2012.

[11] Candela, V., Marquina, A., Recurrence relations for rational cubic methods I: the Halley method, *Computing*, 44, 169-184 (1990).

[12] Chen, D., Argyros, I.K., Qian, Q.S., A note on the Halley method in Banach spaces. *App. Math. Comput.*, 58, (1993), 215-224.

[13] Ezquerro, J.A., Hernndez, M.A.: Recurrence relations for Chebyshev-type methods. *Appl. Math. Optim.*, 41, (2), (2000), 227-236.

[14] Ezquerro, J.A., Hernández, M.A., On the R-order of the Halley method. *J. Math. Anal. Appl.*, 303, (2005), 591-601.

[15] Ezquerro, J.A., Hernández, M.A., New iterations of R-order four with reduced computational cost, *BIT Numer Math.*, 49, (2009), 325-342.

[16] Ganesh, M., Joshi, M.C., Numerical solvability of Hammerstein integral equations of mixed type, IMA. *J. Numer. Anal.*, 11, (1991), 21-31.

[17] Gutiérrez, J.M., Hernández, M.A.: A family of Chebyshev-Halley type methods in banach spaces, *Bull. Aust. Math. Soc.*, 55, (1997), 113-130.

[18] Gutiérrez, J.M., Hernández, M.A., Recurrence relations for the super-Halley method, *Comput. Math. Appl.*, 36, (1998), 1-8.

[19] Hernández, M.A., Second-derivative-free variant of the Chebyshev method for nonlinear equations, *J. Optim. Theory Appl*, 104, (3), (2000), 501-515.

[20] Kou, J., Li, Y., A family of modified super-Halley methods with fourth-order convergence, *Appl Math. Comput.*, 189, (2007), 366-370.

[21] Ostrowski, A.M., *Solution of Equations in Euclidean and Banach Spaces*, 3rd edn., Academic Press, New York, (1973).

[22] Wang, X., Kou, J., Semilocal convergence of multi-point improved supper-Halley-type methods with the second derivative under generalized weak conditions, *Numer. Algor.*, 71, (2016), 567-584.

Chapter 10

Extending the Convergence of a Steffensen-Like Method without Derivatives for Solving Nonlinear Equations

1. Introduction

By eliminating the Taylor series tool from the existing convergence theorem, an extended local convergence of a seventh-order method without derivatives is developed. The first derivative is all that is required for our convergence result, unlike the preceding concept. In addition, the error estimates, convergence radius, and region of uniqueness for the solution are calculated. As a result, the usefulness of this effective algorithm is enhanced. The convergence zones of this algorithm for solving polynomial equations with complex coefficients are also shown. This aids in the selection of beginning points with the purpose of obtaining solutions to nonlinear equations. Our convergence result is validated by numerical testing. Let $F : \Omega \subset X \to Y$ such that

$$F(x) = 0 \tag{10.1}$$

Sharma, Arora fifth order method [7]

$$\begin{aligned} y_n &= x_n - A_n^{-1} F(x_n), \\ x_{n+1} &= y_n - B_n A_n^{-1} F(y_n), \\ A_n &= [z_n, x_n; F], z_n = x_n + F(x_n) \end{aligned} \tag{10.2}$$

and

$$B_n = 3I - A_n^{-1}([y_n, x_n; F] - [y_n, z_n; F]),$$

where $[\cdot,\cdot] : \Omega \times \Omega \to d(x,y)$ is a divided difference of order one $[1,2,3]$.

2. Majorizing Sequences

Let $b, L, L_0, L_1, L_2 > 0$ and $a, R \geq 0$ be parameters. Define sequences $\{t_n\}, \{s_n\}$ by $t_0, s_0 = R$,

$$\begin{aligned} v_n &= L(s_n - t_n) + L_2(t_n + a)(s_n - t_n), \\ t_{n+1} &= s_n + \left[\frac{L(s_n - t_n) + L_2 t_n + a + L_1}{(1 - L_0(2+b)t_n)^2} + \frac{2}{1 - L_0(2+b)t_n}\right] v_n, \end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - s_n) + L(s_n - t_n + L_2 t_n + a)(s_n - t_n)}{1 - L_0(2+b)t_{n+1}}. \tag{10.3}$$

These scalar sequences shall be shown to be majorizing for the method (10.2). But first, we need some convergence results for them.

Lemma 15. Suppose that for all $n = 0, 1, 2, \cdots$

$$(2+b)t_n < \frac{1}{L_0}. \tag{10.4}$$

Then, sequence $\{t_n\}$ is non-decreasing, bounded from above by $\dfrac{1}{(2+b)L_0}$ and converges to its unique least upper bound $t^\star \in [0, \dfrac{1}{L_0}]$.

Proof. It follows from (10.3) and (10.4) that $t_n \leq s_n \leq t_{n+1} < \dfrac{1}{L_0}$, so $t^\star = \lim_{n\to\infty} t_n \in [0, \dfrac{1}{L_0}]$. □

The semilocal convergence of method (10.2) is based on conditions (C).
Suppose:

(C_1) There exists $x_0 \in \Omega, a \geq 0, b \geq 0, R \geq 0$ such that $A_0^{-1} \in \mathcal{L}(Y,X)$, $\|F(x_0)\| \leq a$, $[x, x_0; F] \leq b$ and $\|A_0^{-1}F(x_0)\| \leq R$.

(C_2) $\|A_0^{-1}([z,x;F] - A_0)\| \leq L_0(\|z - z_0\| + \|x - x_0\|)$ for some $L_0 > 0$ and all $z, x \in \Omega$.

Set $\Omega_1 = U(x_0, \dfrac{1}{(2+b)L_0}) \cap \Omega$.

(C_3) $\|A_0^{-1}([z,w,F] - [y,x;F])\| \leq L(\|z - y\| + \|w - x\|)$, $\|A_0^{-1}[y,z;F]\| \leq L_1$ and $\|F(x) - F(x_0)\| \leq L_2\|x - x_0\|$, for some $L, L_1, L_2 \geq 0$ and all $x, y, z, w \in \Omega_1$,

(C_4) Conditions of Lemma 15 hold

and

(C_5) $U[x_0, r] \subset \Omega$, where $r = (1 + L_2)t^\star + a$.

Next, we present the semilocal convergence of method (10.2) using conditions (C) and the aforementioned terminology.

Theorem 11. Suppose conditions C hold. Then, sequence $\{x_n\}$ generated by method (10.2) is well defined in $U[x_0,t^\star]$, remains in $U[x_0,t^\star]$ for all $n=0,1,2,\cdots$ and converges to a solution $x^\star \in U[x_0,t^\star]$ of equation $F(x)=0$. Moreover, the following error estimates hold

$$\begin{aligned} \|y_n - x_n\| &\leq s_n - t_n, \\ \|x_{n+1} - y_n\| &\leq t_{n+1} - s_n \end{aligned}$$

and

$$\|x^\star - x_n\| \leq t^\star - t_n. \tag{10.5}$$

Proof. Mathematical induction is used to show (10.5) and (10.5). Using (10.3) and (C_1) we get $\|y_0 - x_0\| = \|A_0^{-1}F(x_0)\| \leq R = s_0 - t_0 \leq t^\star$, so $y_0 \in U[x_0,t^\star]$ and (10.5) holds for $n=0$. Let $z_k, x_k \in U[x_0,t^\star]$. Then, by $(C_1)-(C_3)$ we have

$$\begin{aligned} \|A_0^{-1}(A_k - A_0)\| &\leq L_0(\|z_k - z_0\| + \|x_k - x_0\|) \\ &\leq L_0(t_k + bt_k + t_k) = L_0(2+b)t_k < 1, \end{aligned}$$

where we also used

$$\begin{aligned} \|z_k - z_0\| &= \|x_k + F(x_k) - x_0 - F(x_0)\| \leq \|x_k - x_0\| + \|F(x_k) - F(x_0)\| \\ &\leq t_k + bt_k = (1+b)t_k. \end{aligned}$$

It follows by the Banach lemma on invertible linear operators [1,2,6] and (10.6) that $A_k^{-1} \in \mathcal{L}(Y,X)$ and

$$\|A_k^{-1}A_0\| \leq \frac{1}{1 - L_0(2+b)t_k}. \tag{10.6}$$

By the second substep of method (10.2) we can write

$$x_{k+1} - y_k = -A_k^{-1}([z_k,x_k;F] - [y_k,x_k;F] + [y_k,z_k;F])A_k^{-1}F(y_k) + 2A_k^{-1}F(y_k). \tag{10.7}$$

By (C_3), we obtain in turn using (10.2), (10.3) and (C_3) the estimates:

$$\begin{aligned} F(y_k) &= F(y_k) - F(x_k) + F(x_k) \\ &= F(y_k) - F(x_k) - A_k(y_k - x_k) \\ &= ([y_k,x_k;F] - [z_k,x_k;F])(y_k - x_k), \end{aligned}$$

$$\begin{aligned} \|A_0^{-1}F(y_k)\| &\leq L\|z_k - y_k\|\|y_k - x_k\| \\ &\leq L(s_k - t_k + L_2 t_k + a)(s_k - t_k) = v_k, \end{aligned}$$

where we also used

$$\begin{aligned} \|z_k - y_k\| &= \|x_k + F(x_k) - y_k\| \leq \|y_k - x_k\| + \|F(x_k)\| \\ &\leq s_k - t_k + \|F(x_k) - F(x_0)\| + \|F(x_0)\| \\ &\leq s_k - t_k + L_2 t_k + a. \end{aligned}$$

Moreover, we have

$$\begin{aligned} 3I - A_k^{-1}([y_k,x_k;F] - [y_k,z_k;F]) &= A_k^{-1}(3A_k - [y_k,x_k;F] - [y_k,z_k;F]) \\ &= A_k^{-1}(A_k - [y_k,x_k;F] - [y_k,z_k;F]) + 2I \\ &= A_k^{-1}([z_k,x_k;F] - [y_k,x_k;F] - [y_k,z_k;F]) + 2I, \end{aligned}$$

so by (10.3) and (10.7) summing up we obtain

$$\begin{aligned} \|x_{k+1} - y_k\| &\leq \left[\frac{L(s_k - t_k) + L_2 t_k + a + L_1}{(1 - L_0(2+b)t_k)^2} + \frac{2}{1 - L_0(2+b)t_k}\right] v_k \\ &= t_{k+1} - s_k \end{aligned}$$

and

$$\|x_{k+1} - x_0\| \leq \|x_{k+1} - y_k\| + \|y_k - x_0\| \leq t_{k+1} - s_k + s_k - t_0 = t_{k+1} \leq t^\star.$$

Thus, (10.5) holds and the iterate $x_{n+1} \in U[x_0, t^\star]$. Then, in view of method (10.2), we can write in turn

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(y_k) + F(y_k) - F(x_k) + F(x_k) \\ &= [x_{k+1}, y_k; F](x_{k+1} - y_k) + [y_k, x_k; F](y_k - x_k) - A_k(y_k - x_k) \\ &= [x_{k+1}, y_k; F](x_{k+1} - y_k) + ([y_k, x_k; F] - [z_k, x_k; F])(y_k - x_k). \end{aligned}$$

By taking norms in (10.8), we get in turn

$$\begin{aligned} \|A_0^{-1}F(x_{k+1})\| &\leq L_1\|x_{k+1} - y_k\| + L\|y_k - z_k\|\|y_k - x_k\| \\ &\leq L_1(t_{k+1} - s_k) + L(s_k - t_k + L_2 t_k + a). \end{aligned}$$

Then, by (10.2), (10.3), (10.6), we obtain

$$\begin{aligned} \|y_{k+1} - x_{k+1}\| &\leq \|A_{k+1}^{-1}A_0\|\|A_0^{-1}F(x_{k+1})\| \\ &\leq \frac{L_1(t_{k+1} - s_k) + L(s_k - t_k + L_2 t_k + a)(s_k - t_k)}{1 - L_0(2+b)t_{k+1}} = s_{k+1} - t_{k+1} \end{aligned}$$

and

$$\begin{aligned} \|y_{k+1} - x_0\| &\leq \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \leq s_{k+1} - t_{k+1} + t_{k+1} - t_0 \\ &= s_{k+1} \leq t^\star. \end{aligned}$$

Hence, the iterate $y_{k+1} \in U[x_0, t^\star]$ and the induction for assertions (10.5) and (10.5) is completed. Sequence $\{t_k\}$ is fundamental as convergent. Consequently, $\{x_k\}$ is fundamental too by (10.5) and (10.5) in Banach space X. Hence, it converges to some $x^\star \in U[x_0, t^\star]$. By letting $k \to \infty$ in (10.8) and using the continuity of F we conclude $F(x^\star) = 0$. □

The uniqueness of the solution result follows.

Proposition 8. *Suppose*

(1) *There exists a simple solution* $x^\star \in U[x_0, \rho] \subset D$ *for some* $\rho > 0$.

(2) *Condition* (C_2) *holds.*

(3) *There exists* $\rho_1 \geq \rho$ *such that*

$$L_0(\rho + \rho_1 + a) < 1. \tag{10.8}$$

Set $\Omega_2 = U[x_0, \rho_1] \cap \Omega$. *Then, the only solution of the equation* $F(x) = 0$ *in the region* Ω_2 *is* $x^\star$.

Proof. Let $d \in \Omega_2$ with $F(d) = 0$. Set $[x^\star, d; F] = T$. Then, by C_2, (1), (2) and (10.8) we have in turn

$$\begin{aligned} \|A_0^{-1}(T - A_0)\| &\leq L_0(\|x^\star - z_0\| + \|y^\star - x_0\|) \\ &\leq L_0(\rho + a + \rho_1) < 1, \end{aligned}$$

where we also used

$$\|x^\star - z_0\| \leq \|x^\star - x_0\| + \|F(x_0)\| \leq \rho + a.$$

It follows by (10.9) and the identity $T(x^\star - d) = F(x^\star) - F(d) = 0$ that $d = x^\star$. □

Remark. (1) In the view of (10.4), we can replace r by r_1 in (C_5), where

$$r_1 = \frac{(1 + L_2)}{L_0(2 + b)} + a,$$

which is given in closed form.

(2) The only Condition from the C conditions used in Proposition 8 is (C_2). But if we use all conditions C, then we can certainly set $\rho = t^\star$ in this Proposition.

Bibliography

[1] Argyros, I.K. and Magreñán, A.A., *Iterative Methods and Their Dynamics with Applications: A Contemporary Study*, CRC Press, 2017.

[2] Argyros, I.K., Sharma, D., Argyros, C.I., Parhi, S.K. and Sunanda, S.K., Extended iterative schemes based on decomposition for nonlinear models. *Journal of Applied Mathematics and Computing*, 68(3):14851504, June 2022.

[3] Behl, R., Bhalla, S., Magreñán, A.A. and Moysi, A. An Optimal Derivative Free Family of ChebyshevHalleys Method for Multiple Zeros, *Mathematics*, 9(5):546, Jan. 2021. Number: 5 Publisher: Multidisciplinary Digital Publishing Institute.

[4] Ezquerro, J.A., Hernández, M.A., Romero, N. and Velasco, A. I., On Steffensens method on Banach spaces, *Journal of Computational and Applied Mathematics*, 249:923, Sept. 2013.

[5] Kantorovich, L.V. and Akilov, G.P., *Functional analysis*. Pergamon Press, Oxford, New York, 2d ed edition, 1982.

[6] Magreñán, A.A. and Argyros, I.K., *A contemporary study of iterative methods: convergence, dynamics and applications*, Academic Press, London, 2018.

[7] Sharma, J.R. and Arora, H., An efficient derivative free iterative method for solving systems of nonlinear equations, *Applicable Analysis and Discrete Mathematics*, 7(2):390403, 2013.

Chapter 11

On the Semi-Local Convergence of a Fourth Order Derivative Free Two-Step Steffensen Method for Solving Equations

1. Introduction

A fourth order two-step Steffensen method for solving equations is studied in this Chapter.

Let X and Y be Banach spaces and $\Omega \subset X$ be an open set. We are concerned with the problem of approximating a locally unique solution $x^* \in \Omega$ of the nonlinear equation

$$F(x) = 0, \tag{11.1}$$

where $F : D \longrightarrow Y$ is a nonlinear operator. A plethora of applications reduces to solving equation (11.1) [1, 2, 3, 4, 5, 6, 7, 8]. The solution x^* is needed in a closed form. But this is attainable only in special cases. That explains why most solution methods are iterative. In particular, we study the semi-local convergence of the Steffensenmethod (SM) defined $\forall n = 0, 1, 2, \ldots$ by

$$y_n = x_n - A_n^{-1} F(x_n)$$

and

$$x_{n+1} = y_n - B_n^{-1} F(y_n), \tag{11.2}$$

where $A_n = [u_n, v_n; F], B_n = 2[y_n, x_n; F] - [u_n, v_n; F], [., .; F] : D \times D \longrightarrow L(B, B)$ is a divided difference of order one for operator $F, u_n = x_n - F(x_n)$ and $n_n = x_n - F(x_n)$. It was found to be of a fourth convergence order. In particular, the radius of convergence was established. We show that this radius can be enlarged without new conditions. Other benefits include tighter error bounds on distances $\|x_n - x^*\|$ and better information on the uniqueness of the solution. The technique is independent of method (11.2). Thus, it can be used to extend the applicability of other methods. This process provides the location of the SM iterates and computable error distances$\|x_{n+1} - x_n\|$ and $\|x_n - x^*\|$.

2. Majorizing Sequences

Let $L_0, L, L_1 > 0$ and $a \geq 0, \eta \geq 0$ be parameters. Define scalar sequence $\{t_n\}$ by $t_0 = 0, s_0 = \eta$,

$$t_{n+1} = s_n + \frac{L(s_n - t_n + 2L_1 t_n + 2a)(s_n - t_n)}{1 - [L_0(s_n + t_n + 2a) + L(s_n - t_n + 2L_1 t_n + 2a)]} \tag{11.3}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - s_n + 2(s_n - t_n) + 2L_1 t_n + 2a)(t_{n+1} - s_n)}{1 - 2L_0(1 + L_1)t_{n+1}}.$$

Sequence $\{t_n\}$ shall be shown to be majorizing for method (11.2) in the next section. But first, we develop two convergence results for it.

Lemma 16. Suppose that for all $n = 0, 1, 2, \ldots$

$$L_0(s_n + t_n + 2a) + L(s_n - t_n + 2L_1 t_n + 2a) < 1 \tag{11.4}$$

and

$$2L_0(1 + L_1)t_{n+1} < 1. \tag{11.5}$$

Then, the following assertions hold

$$0 \leq t_n \leq s_n \leq t_{n+1} < \frac{1}{2L_0(1 + L_1)} \tag{11.6}$$

and

$$t^* = \lim_{n \longrightarrow \infty} t_n \in \left[0, \frac{1}{2L_0(1 + L_1)}\right]. \tag{11.7}$$

Proof. It follows from (11.3)-(11.5) that sequence $\{t_n\}$ is non-decreasing and bounded from above by $\frac{1}{2L_0(1 + L_1)} = t_1^{**}$ and as such it converges to its unique least upper bound $t^* \in [0, t_1^{**}]$. □

In order to develop the second convergence result for sequence $\{t_n\}$ we need to define recurrent polynomials on the interval $[0, 1)$ by

$$\begin{aligned} h_n^{(1)}(t) &= Lt^n\eta + 2LL1(1 + t + \cdots + t^n)\eta + 2La + 2L_0 t(1 + t + \cdots + t^n)\eta) + 2aL_0 t \\ &\quad + Lt^{n+1}\eta + 2LL_1 t(1 + t + \cdots + t^n)\eta + 2Lat - t, \\ h_n^{(2)}(t) &= Lt^{n+1}\eta + 2Lt^n\eta + 2LL_1(1 + t + \cdots + t^n)\eta + 2La \\ &\quad + 2L_0(1 + L_1)(1 + t + \cdots + t^n)\eta - t, \end{aligned}$$

polynomials on $[0, 1)$

$$\begin{aligned} p_1(t) &= Lt - 1 + 2LL_1 t + Lt^2 - Lt + 2LL_1 t^2, \\ p_2(t) &= Lt^2 - Lt + 2Lt - 2L + 2LL_1 t + 2L_0(1 + L_1)t^3 \end{aligned}$$

and parameters

$$b_1 = \frac{L(\eta+2a)}{1-[L_0(\eta+2a)+L(\eta+2a)]},$$
$$b_2 = \frac{L(t_1+\eta+2a)}{1-L_0(1+L_1)t_1},$$
$$b_3 = \frac{2La}{1-2a(L_0+L)}$$

and $b = \max\{b_1, b_2, b_3\}$. By the definition of polynomials p_1 and p_2, we have $p_1(0) = -L < 0, p_1(1) = 4LL_1 > 0, p_2(0) = -2L < 0$ and $p_2(1) = 2LL_1 + 2L_0(HL_1) > 0$. Then, the intermediate value theorem assures that the polynomials have roots in $(0,1)$. Denote by S_1 and S_2 the least such roots of polynomials p_1 and p_2, respectively. Define $r = \min\{S_1, S_2\}$ and $S = \max\{S_1, S_2\}$.

The convergence of sequence $\{t_n\}$ is shown under the conditions (A) :

$$(L_0+L)(\eta+2a) < 1, \tag{11.8}$$
$$L_0(1+L_1)t_1 < 1, \tag{11.9}$$
$$2a(L_0+L) < 1, \tag{11.10}$$
$$h_1^{(1)}(t) \leq 0, \quad \text{at } t = S \tag{11.11}$$
$$h_2^{(1)}(t) \leq 0 \quad \text{at } t = S \tag{11.12}$$

and

$$b \leq r \leq S. \tag{11.13}$$

Conditions (11.8)-(11.10) assure that parameters b_1, b_2, b_3, b are well-defined. Moreover conditions (11.11) and (11.12) can be written respectively as

$$[Lt + 2LL_1(1+t) + 2L_0t(1+t) + Lt^3 + 2LL_1t(1+t)]\eta + t[2La + 2aL_0 - 1] + 2La \leq 0$$

and

$$[Lt^2 + 2Lt + 2L_1(1+t) + 2L_0(1+L_1)t(1+t+t^2)]\eta + 2La - t \leq 0$$

which can certainly hold for η sufficiently small and the choice of b_3. The second convergence result on sequence $\{t_n\}$ is based on conditions (A) and the preceding terminology.

Lemma 17. Under conditions A the following assertions hold:

$$0 \leq t_{n+1} - s_n \leq S(s_n - t_n) \leq S^{2n+1}\eta, \tag{11.14}$$
$$0 \leq s_n - t_n \leq S(t_n - s_{n-1}) \leq S^{2n}\eta \tag{11.15}$$

and

$$t^* = \lim_{n\longrightarrow\infty} t_n \in [0, t^{**}], t^* = \frac{\eta}{1-S}. \tag{11.16}$$

Proof. Mathematical induction is used to show

$$0 \leq \frac{L(s_n - t_n + 2L_1t_n + 2a)}{1-[L_0(s_n+t_n+2a)+L(s_n-t_n+2L_1t_n+2a)]} \leq S \tag{11.17}$$
$$0 \leq \frac{L(t_{n+1} - s_n + 2(s_n-t_n) + 2L_1t_n + a)}{1-2L_0(1+L_1)t_{n+1}} \leq S \tag{11.18}$$
$$0 \leq t_n \leq s_n \leq t_{n+1}. \tag{11.19}$$

Estimates (11.17)-(11.19) hold true for $n = 0$ by (11.3),the definition of b_1 and b_2 and (11.13). Suppose (11.17)-(11.19) hold true for all $k = 1,2,\ldots,n$. In view of (11.14),(11.15) and the induction hypothesis we have in turn that

$$s_k \leq t_k + S^{2k}\eta \leq s_{k-1} + S^{2k-1}\eta + S^{2k}\eta \leq \eta + S\eta + \cdots + S^{2k}\eta = \frac{1-S^{2k+1}}{1-S}\eta < t^{**} \quad (11.20)$$

and

$$t_{k+1} \leq s_k + S^{2k+1}\eta \leq t_k + S^{2k}\eta + S^{2k+1}\eta \leq s_0 + S\eta + \cdots + S^{2k+1}\eta = \frac{1-S^{2k+2}}{1-S}\eta < t^* * . \quad (11.21)$$

Notice that $S^{2k} \leq S^k$. Hence, (11.17) holds if

$$LS^k\eta + 2LL_1\frac{1-S^{k+1}}{1-S}\eta + 2La - S + 2L_0S\frac{1-S^{n+1}}{1-S}\eta + 2aL_0S + S^{n+1}L\eta$$
$$+2LL_1S\frac{1-S^{n+1}}{1-S}\eta + 2LaS \leq 0$$

or

$$h_n^{(1)}(t) \leq 0 \quad \text{at } t = S_1. \quad (11.22)$$

Recurrent polynomials $h_n^{(1)}(t)$ are connected . Indeed, we have in turn

$$\begin{aligned} h_{n+1}^{(1)}(t) - h_n^{(1)} &= Lt^{n+1}\eta - Lt^n\eta + 2LL_1t^{n+1}\eta + 2L_0t^{n+2}\eta + Lt^{n+2}\eta - Lt^{n+1}\eta + 2LL_1t^{n+2}\eta \\ &= p_1(t)t^n\eta. \end{aligned}$$

In particular, we have

$$h_{n+1}^{(1)}(t) = h_n^{(1)}(t) \quad \text{at } \ t = S_1. \quad (11.23)$$

So, (11.22) holds by (11.11). Similarly, estimate (11.18) holds if

$$LS^{k+1}\eta + 2LS^k\eta + 2LL_1\frac{1-S^{n+1}}{1-S}\eta + 2La + 2L_0S(1+L_1)\frac{1-S^{k+2}}{1-S}\eta - S \leq 0$$

or

$$h_k^{(2)}(t) \leq 0 \quad \text{at } t = S_2. \quad (11.24)$$

This time we have

$$h_{k+1}^{(2)}(t) - h_k^{(2)}(t) = Lt^{k+2}\eta - Lt^{k+1}\eta - 2Lt^k\eta + 2LL_1t^{k+1}\eta + 2L_0(1+L_1)t^{k+3}\eta = p_2(t)t^k\eta.$$

In particular, we have

$$h_{k+1}^{(1)}(t) = h_k^{(1)}(t) \quad \text{at } t = S_2.$$

Hence, (11.24) holds by (11.12). It then follows from (11.17),(11.18) and (11.3) that (11.19) holds. Using (11.19) and (11.21) we deduce that

$$\lim_{n\longrightarrow\infty} t_n = t^* \in [0, t^{**}].$$

□

3. Convergence for Method (11.2)

The conditions (C) are needed: Suppose

(C1) There exist $x_0 \in \Omega$ such that $A_0^{-1} \in \mathcal{L}(Y,X)$

$$\|F(x_0)\| \leq a \text{ and } \|A_0^{-1}F(x_0)\| \leq \eta.$$

(C2) There exists $L_0 > 0$ such that for all $x, y \in \Omega$

$$\|A_0^{-1}([x,y;F] - A_0)\| \leq L_0(\|x-u_0\| + \|y-v_0\|).$$

Define $\Omega_1 = U(x_0, \frac{1}{2L_0(1+L_1)}) \cap \Omega$ and $\|F(x) - F(x_0)\| \leq L_1\|x-x_0\|$.

(C3) There exist $L > 0, L_1 > 0$ such that for all $x, y, z, w \in \Omega_1$

$$\|A_0^{-1}([x,y;F] - [z,w;F])\| \leq L(\|x-z\| + \|y-w\|).$$

(C4) Conditions of Lemma 16 or Lemma 17 hold.

(C5) $U[x_0, T] \subset \Omega, T = t^* + a$.

The semi-local convergence of method(11.2) uses conditions (C).

Theorem 12. Suppose conditions (C) hold. Then, sequence $\{x_n\}$ generated by method(11.2) is well-defined in $U[x_0,t^*]$, remains in $U[x_0,t^*]$ for all $n = 1,2,\ldots$ and converges to a solution $x^* \in U[x_0,t^*]$ of equation $F(x) = 0$, so that

$$\|y_n - x_n\| \leq s_n - t_n, \tag{11.25}$$

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n \tag{11.26}$$

and

$$\|x^* - x_n\| \leq t^* - t_n. \tag{11.27}$$

Proof. Using (C1) and (11.3), we get

$$\|y_0 - x_0\| = \|A_0^{-1}F(x_0)\| \leq \eta = s_0 - t_0 \leq s_0 < t^*,$$

so (11.25) holds for $n = 0$ and $y_0 \in U[x_0,t^*]$. We have

$$\|v_n - v_0\| = \|x_n - F(x_n) - x_0 - F(x_0)\| \leq \|x_n - x_0\| + \|F(x_n) - F(x_0)\| \leq (1+L_1)\|x_n - x_0\|$$

and

$$\|u_n - u_0\| = \|x_n + F(x_n) - x_0 - F(x_0)\| \leq (1+L_1)\|x_n - x_0\|$$

In view of (C_1) and (C_2)

$$\begin{aligned}\|A_0^{-1}(A_n - A_0)\| &= \|A_0^{-1}([u_n,v_n;F] - [u_0,v_0;F])\| \leq L_0(\|u_n - u_0\| + \|v_n - v_0\|)\\ &\leq 2L_0(1+L_1)\|x_n - x_0\| \leq 2L_0(1+L_1)\|x_n - x_0\| \leq 2L_0(1+L_1)t^* < 1.\end{aligned}$$

Thus, $A_n^{-1} \in \mathcal{L}(Y,X)$ by the Banach lemma on invertible operators [1] and

$$\|A_n^{-1}A_0\| \leq \frac{1}{1-2L_0(1+L_1)\|x_n-x_0\|}. \tag{11.28}$$

We need the estimates

$$\|x_n-v_n\| = \|x_n-v_n+F(x_n)\| \leq \|F(x_n)\| \leq \|F(x_n)-F(x_0)\|+\|F(x_0)\| \leq L_1\|x_n-x_0\|+a,$$

$$\begin{aligned}\|y_n-u_n\| &= \|y_n-x_n-F(x_n)-F(x_0)+F(x_0)\| \\ &\leq \|y_n-x_n\|+\|F(x_n)-F(x_0)\|+\|F(x_0)\| \leq s_n-t_n+L_1t_n+a,\end{aligned}$$

$$\begin{aligned}\|x_n-v_0\| &= \|x_n-x_0+F(x_0)\| \leq \|x_n-x_0\|+\|F(x_0)\| \leq t_n+a,\\ \|y_n-u_0\| &= \|y_n-x_0-F(x_0)\| \leq \|y_n-x_0\|+\|F(x_0)\| \leq s_n+a,\\ \|u_n-x_0\| &= \|x_n-x_0-F(x_0)\| \leq t^*+a,\\ &\text{and}\\ \|v_n-x_0\| &= \|x_n-x_0+F(x_0)\| \leq t^*+a\end{aligned}$$

Then, by (C_2) and (C_3)

$$\begin{aligned}\|A_0^{-1}(B_n-A_0)\| &\leq \|A_0^{-1}([y_n,x_n;F]-[u_0,v_0;F])\|+\|A_0^{-1}([y_n,x_0;F]-[u_n,v_n;F])\|\\ &\leq L_0(\|y_n-u_0\|+\|x_n-v_0\|)+L(\|y_n-u_n\|+\|x_n-v_n\|)\\ &\leq L_0(s_n+t_n+2a)+L(s_n-t_n+2L_1t_n+2a)<1,\end{aligned}$$

so

$$\|B_n^{-1}A_0\| \leq \frac{1}{1-[L_0(s_n+t_n+2a)+L(s_n-t_n+2L_1t_n+2a)]}. \tag{11.29}$$

Moreover, iterate x_1 is well-defined. We can write

$$F(y_n) = F(y_n)-F(x_n)+F(x_n) = ([y_n,x_n;F]-[u_n,v_n;F])(y_n-x_n)$$

$$\begin{aligned}\|A_0^{-1}F(y_n)\| &\leq L(\|y_n-u_n\|+\|x_n-v_n\|)\|y_n-x_n\|\\ &\leq L(s_n-t_n+L_1t_n+a+L_1t_n+a)(s_n-t_n)\end{aligned}$$

and

$$\|x_{n+1}-y_n\| \leq \|B_n^{-1}A_0\|\|A_0^{-1}F(y_n)\| \leq t_{n+1}-s_n,$$
$$\|x_{n+1}-x_0\| \leq \|x_{n+1}-y_n\|+\|y_n-x_0\| \leq t_{n+1}-s_n+s_n-t_0 = t_{n+1}-s_n+s_n-t_0 = t_{n+1} \leq t^*,$$

so (11.26) holds and $x_{n+1} \in U[x_0,t^*]$.

Furthermore, we can write

$$F(x_{n+1}) = F(x_{n+1})-F(y_n)+F(x_n) = ([x_{n+1},y_n;F]-B_n)(x_{n+1}-y_n),$$

so

$$\begin{aligned}\|A_0^{-1}F(x_{n+1})\| &\leq L(\|x_{n+1}-y_n\|+\|y_n-x_n\|)\\ &\leq L(t_{n+1}-s_n+s_n-t_n+s_n-t_n+L_1t_n+a+L_1t_n+a)(t_{n+1}-s_n)\end{aligned}$$

and

$$\begin{aligned}\|y_{n+1}-x_{n+1}\| &\leq \|A_{n+1}^{-1}A_0\|\|A_0^{-1}F(x_{n+1})\|\\ &\leq s_{n+1}-t_{n+1}\\ \|y_{n+1}-x_0\| &\leq \|y_{n+1}-x_{n+1}\|+\|x_{n+1}-x_0\|\\ &\leq s_{n+1}-t_{n+1}+t_{n+1}-t_0=s_{n+1}\leq t^*.\end{aligned}$$

Hence, the iterate $y_{n+1}\in U[x_0,t^*]$. The induction for assertions (11.25) and (11.26) is terminated. It follows that sequence $\{x_n\}$ is complete in a Banach Space X and as such it converges some $x^*\in U[x_0,t^*]$. Finally using the continuity of F and letting $n\longrightarrow\infty$ we conclude $F(x^*)=0$. □

Next, the uniqueness of the solution result follows.

Proposition 9. *Assume*

(1) There exist a simple solution $x^*\in U(x_0,\rho)\subset\Omega$ of equation $F(x)=0$.

(2) Condition (C_2) holds and $\|F(x_0)\|\leq a$.

(3) There exist $\rho_1\geq\rho$ such that

$$L_0(\rho+\rho_1+2a)<1 \tag{11.30}$$

Define $\Omega_2=U[x_0,\rho_1]\cap\Omega$. Then,$x^*$ is the only solution of equation $F(x)=0$ in the set Ω_2.

Proof. Let $v^*\in\Omega_2$ be such that $F(v^*)=0$. Define $M=[v^*,x^*;F]$. Then, in view of (C_1), and (1)-(3), we get

$$\begin{aligned}\|A_0^{-1}(M-A_0)\| &\leq L_0(\|v^-u_0\|+\|x-v_0\|)\\ &\leq L_0(\|v^*-x_0\|+\|F(x_0)\|+\|x^*-x_0\|+\|F(x_0)\|)\\ &\leq L_0(\rho_1+a+\rho+a)<1.\end{aligned}$$

Therefore, we deduce $v^*=x^*$ by the invertability of M and the identity

$$M(v^*-x^*)=F(v^*)-F(x^*)=0.$$

□

Remark. (1) The parameters $t_1^* *$ and $\dfrac{\eta}{1-\eta}$ given in closed form can replace t^* in conditions (C_4) and (C_5).

(2) Notice that in proposition 3.2 we did not use all conditions C. But if they were used then we can set $\rho=t^*$.

Bibliography

[1] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[2] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)*, 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[3] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative methods*, Elsevier (Academic Press), New York, 2018.

[4] Argyros, I.K., Cordero, A., Magreńán, A.A., Torregrosa, J.R., Third-degree anomalies of Traub's method, *J. Comput. Appl. Math.*, 309, (2017), 511-521.

[5] Ezquerro, J.A., Grau-Sánchez, M., Hernández, M. A., Noguera, M., Semilocal convergence of secant-like methods for differentiable and nondifferentiable operator equations, *J. Math. Anal. Appl.*, 398, (1), (2013), 100-112.

[6] Moccari, M, Lofti, T., On a two step optimal Steffensen-type method: Relaxed local and semi-local convergence analysis and dynamical stability, *J. Math. Anal. Appl.*, 468, (2018), 240-269.

[7] Magréñan, A.A., Argyros, I.K.,New improved convergence analysis for the secant method, *Math. Comput. Simulation*, 119, (2016), 161-170.

[8] Traub, J.F., Iterative methods for the solution of equations, *American Mathematical Soc.*, 1982.

Chapter 12

A Semi-Local Convergence for a Class of Fourth Order Method for Solving Equations

1. Introduction

Let X and Y be Banach spaces and $D \subset X$ be an open set. We are concerned with the problem of approximating a locally unique solution $x^* \in D$ of the nonlinear equation

$$F(x) = 0, \tag{12.1}$$

where $F : D \longrightarrow Y$ is a nonlinear operator. A plethora of applications reduces to solving equation (12.1). The solution x^* is needed in a closed form. But this is attainable only in special cases. That explains why most solution methods are iterative. In particular, we study the semi-local convergence of the method defined $\forall n = 0, 1, 2, \ldots$ by

$$\begin{aligned} x_0 \in D,\, z_n &= x_n + F(x_n), \\ y_n &= x_n - A_n^{-1}F(x_n) \end{aligned}$$

and

$$x_{n+1} = y_n - M_n^{-1}B_nM_n^{-1}F(y_n), \tag{12.2}$$

where $A_n = [z_n, x_n; F], M_n = [y_n, x_n; F], B_n = [y_n, x_n; F] - [z_n, x_n; F], [., .; F] : D \times D \longrightarrow L(B, B)$ is a divided difference of order one for operator F [1,2,3,6]. This method was studied in [12]. It was found to be of a fourth convergence order. In particular, the radius of convergence was established. We show that this radius can be enlarged without new conditions. Other benefits include tighter error bounds on distances $\|x_n - x^*\|$ and better information on the uniqueness of the solution. The technique is independent of method (12.2). Thus, it can be used to extend the applicability of other methods. This process specifies a more precise location of the method iterates leading to an at least as tight Lipschitz parameters which are specializations of the ones in [12]. Hence, no additional computational effort is required for these benefits either. Relevant work can be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

2. Semi-Local Convergence

Let $L_0>0, L>0, L_1\geq 1, a\geq 0, b\geq 0$ and $\eta\geq 0$ be parameters. The convergence of method (12.2) is based on these parameters and scalar sequence $\{t_n\}$ defined for all $n=0,1,2,\ldots$ by

$$\begin{aligned} t_0 &= 0, s_0=\eta, \\ q_n &= L_0(s_n+t_n+b), \\ t_{n+1} &= s_n+\left(1+\frac{L(s_n-t_n)}{1-q_n}\right)\frac{L(s_n-t_n+at_n+b)(s_n-t_n)}{1-q_n}, \\ P_n &= L_0(2+a)t_n, \\ V_n &= L_0(s_n+t_n+b)+L(s_n-t_n) \end{aligned} \tag{12.3}$$

and

$$s_{n+1} = t_{n+1}+\frac{L_1(1-v_n+L_1)(t_{n+1}-s_n)}{(1-v_n)(1-P_{n+1})}.$$

Sequence $\{t_n\}$ shall be shown to be majorizing for method (12.2). But first, we show a convergence result for it.

Lemma 18. Suppose that for each $n=0,1,2,\ldots$

$$0\leq P_n<1, 0\leq q_n<1 \text{ and } 0\leq v_n<1. \tag{12.4}$$

Then, the following assertions hold

$$0\leq t_n\leq s_n\leq t_{n+1} < \frac{P}{L_0(2+a)} \tag{12.5}$$

and

$$\lim_{n\to\infty} t_n = t_* \leq \frac{P}{L_0(2+a)}. \tag{12.6}$$

Proof. By the definition of sequence $\{t_n\}$ and condition (12.4), it follows that sequence $\{t_n\}$ is non-decreasing, bounded from above by $\dfrac{P}{L_0(2+a)}$ so it converges to t^* satisfying (12.6). □

The semi-local convergence relies on conditions (H).

Suppose:

(h_1) There exist $x_0\in D, L_0>0, \eta\geq 0, a\geq 0, b\geq 0$ such that for all $x,y\in D$,

$$\begin{aligned} A_0^{-1} &\in L(B_2,B_1), \\ \|A_0^{-1}([z,x;F]-[z,y;F])\| &\leq L\|x-y\|, \\ \|A_0^{-1}F(x_0) &\leq \eta, \\ \|F(x)-F(x_0)\| &\leq a\|x-x_0\|, \\ &\text{and} \\ \|F(x_0)\| &\leq b. \end{aligned}$$

Define $D_1=U\left(x_0,\dfrac{1}{L_0(2+a)}\right)\cap D$.

(h_2) There exist $L > 0$ and $L_1 \geq 1$ such that for all $x, y, z \in D_1$

$$\|A_0^{-1}([z, x; F] - [z, y; F])\| \leq L\|x - y\|$$

and

$$\|A_0^{-1}[x, y; F] \leq L_1.$$

(h_3) Conditions of Lemma 18 hold.
and

(h_4) $U[x_0, \rho] \subset D$, where $\rho = (1 + a)t_* + b$.

Theorem 13. Suppose conditions (H) hold. Then, sequence $\{x_n\}$ generated by method (12.2) is well defined in $U[x_0, t_*]$, remains in $U[x_0, t_*]$ for each $h = 0, 1, 2, \ldots$ and converges to a solution $x_* \in U[x_0, t_*]$ of equation $F(x) = 0$. Moreover, the following assertion holds

$$\|x_* - x_n\| \leq t_* - t_n. \tag{12.7}$$

Proof. Mathematical Induction is employed to show

$$\|y_n - x_n\| \leq s_n - t_n \tag{12.8}$$

and

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n. \tag{12.9}$$

By condition (h_1) and sequence $\{t_n\}$ it follows

$$\|y_0 - x_0\| = \|A_0^{-1}F(x_0)\| \leq \eta = s_0 - t_0 \leq t_*.$$

Thus, $y_0 \in U[x_0, t^*]$ and (12.8) holds for $n = 0$.
Using $(h_1), (h_2)$ for $z_n, x_n \in U[x_0, t_*]$

$$\|A_0^{-1}(A_n - A_0)\| \leq L_0(\|z_n - z_0\| + \|x_n - x_0\|) \leq L_0(t_n + at_n + t_n) = L_0(2 + a)t_n = P_n < 1,$$

so $A_n^{-1} \in L(B_2, B_1)$ and

$$\|A_n^{-1}A_0\| \leq \frac{1}{1 - P_n} \tag{12.10}$$

by the Banach Lemma for linear invertible operators [2, 3, 4, 5, 6, 7, 8, 9, 10], where

$$\begin{aligned}\|z_n - z_0\| &= \|x_n + F(x_n) - x_0 - F(x_0) \leq \|x_n - x_0\| + \|F(x_n - F(x_0))\| \\ &\leq t_n + a\|x_n - x_0\| \leq (1 + a)t_n\end{aligned}$$

and

$$\|z_n - x_0\| \leq \|z_n - z_0\| + \|z_0 - x_0\| \leq (1 + a)t_n + b \leq (1 + a)t_* + b = s.$$

Similarly,

$$\|A_0^{-1}([y_n, x_n; F] - [z_0, x_0; F])\| \leq L_0(\|y_n - z_0)\| + \|x_n - x_0\|) \leq L_0(s_n + b + t_n) = q_n < 1,$$

so,

$$\|[y_n,x_n;F]^{-1}A_0\| \leq \frac{P}{1-q_n},$$

where

$$\|y_n-z_0\| = \|y_n-x_0-F(x_0)\| \leq \|y_n-x_0\|+\|F(x_0)\| \leq s_n+b.$$

is well-defined. It also follows that x_{n+1} is well-defined.
Then, by the second substep of method (12.2),(h_3) and (12.3), one gets in turn

$$\begin{aligned}\|x_{n+1}-y_n\| &\leq \|I+[y_n,x_n]^{-1}([z_n,x_n;F]-[z_n,y_n;F])\| \\ &\times \|[y_n,x_n;F]^{-1}A_0\|\|A_0^{-1}F(y_n)\|\end{aligned}$$

$$\left(1+\frac{L\|y_n-x_n}{1-q_n}\right)\frac{L\|y_n-z_n\|\|y_n-x_n\|}{1-q_n} \leq t_{n+1}-s_n$$

and

$$\begin{aligned}\|x_{n+1}-x_0\| &\leq \|x_{n+1}-y_n\|+\|y_n-x_0\| \\ &\leq t_{n+1}-s_n+s_n-t_0=t_{n+1}\leq t_*.\end{aligned}$$

Hence, $x_{n+1}\in U[x_0,t^*]$ and (12.9) hold. The following estimate was also used.

$$F(y_n)=F(y_n)-F(x_n)+F(x_n)=([y_n,x_n;F]-A_n)(y_n-x_n),$$

so

$$\begin{aligned}\|A_0^{-1}F(y_n)\| &\leq L\|y_n-z_n\|\|y_n-x_n\| \leq L(\|y_n-x_n\|+\|F(x_n)\|)\|y_n-x_n\| \\ &\leq L(s_n-t_n+\|F(x_n)-F(x_0))\|+\|F(x_0)\|)(s_n-t_n) \\ &\leq L(s_n-t_n+at_n+b)(s_n-t_n).\end{aligned}$$

Then, by method (12.2) it follows that

$$\begin{aligned}F(x_{n+1}) &= F(x_{n+1})-F(y_n)+F(y_n) \\ &= F(x_{n+1})-F(y_n)-C_n(x_{n+1}-y_n) \\ &= ([x_{n+1},y_n;F]-C_n)(x_{n+1}-y_n),\end{aligned}$$

where $C_n=[y_n,x_n;F]B_n^{-1}[y_n,x_n;F]$.

Notice that B_n^{-1} exists, since

$$\begin{aligned}\|A_0^{-1}(B_n-A_0)\| &\leq \|A_0^{-1}([y_n,x_n;F]-[z_0,x_0;F])\|+\|A_0^{-1}([z_n,x_n;F]-[z_n,y_n;F])\| \\ &\leq L_0(\|y_n-z_0\|+\|x_n-x_0\|)+L(y_n-x_n) \\ &\leq L_0(s_n+t_n+b)+L(s_n-t_n)=v_n<1,\end{aligned}$$

so

$$\|B_n^{-1}A_0\| \leq \frac{1}{1-v_n}.$$

It then follows from (12.11) and (12.11) that

$$\|A_0^{-1}F(x_{n+1})\| \leq L_1\left(1+\frac{L_1}{1-v_n}\right)(t_{n+1}-s_n),$$

$$\begin{aligned}\|y_{n+1}-x_{n+1}\| &\leq \|A_{n+1}^{-1}A_0\|\|A_0^{-1}F(x_{n+1})\| \\ &\leq L_1\left(1+\frac{L_1}{1-v_n}\right)\frac{1}{1-P_{n+1}}(t_{n+1}-s_n) = s_{n+1}-t_{n+1}\end{aligned}$$

and

$$\|y_{n+1}-x_0\| \leq \|y_{n+1}-x_{n+1}\|+\|x_{n+1}-x_0\| \leq s_{n+1}-t_{n+1}+t_{n+1}-t_0 = s_{n+1} \leq t_*.$$

Thus, $y_{n+1} \in U[x_0,t_*]$ and the induction for items (12.8) and (12.9) is completed.
It follows that sequence $\{x_n\}$ is Cauchy in Banach space B_1, so it converges to some $x_* \in U[x_0,t_*]$. By using the continuity of F and letting $n \to \infty$ in estimate (12.11) we conclude $F(x_*) = 0$. □

Concerning the uniqueness of the solution.

Proposition 10. *Suppose:*

1. *The point $x_* \in U[x_0,\rho_1] \subset D$ is a simple solution of equation $F(x) = 0$ for some $\rho_1 > 0$.*
2. *The fourth estimate in (h_1) and (h_2) hold.*
3. *There exists $\rho_2 \geq \rho_1$ such that*

$$L_0(\rho_1+\rho_2+b) < 1. \qquad (12.11)$$

Define $D_2 = U[x_0,\rho_2] \cap D$.
Then, the point x_* is the only solution of equation $F(x) = 0$ in the region D_2.

Proof. Consider $d \in D_2$ such that $F(d) = 0$. Define $S = [x_*,d,F]$. Then, it follows from $1-3$ that

$$\begin{aligned}\|A_0^{-1}(S-A_0)\| &\leq L_0(\|x_*-z_0\|+\|d-x_0\|) \\ &\leq L_0(\rho_1+b+\rho_2) < 1.\end{aligned}$$

Hence, $d = x_*$ is implied by the invertibility of S and the identity $S(x_*-d) = F(x_*)-F(d) = 0$. □

Remark. (1) The parameter $\bar{\rho}$ given in closed form can replace ρ in condition (h_4), where $\bar{\rho} = \dfrac{(1+a)}{L_0(2+a)}+b.$

(2) Notice that not all conditions of Theorem 13 are used in Proposition 10. Otherwise, set $s_1 = t_*$.

Bibliography

[1] Argyros, I.K., *Computational theory of iterative methods*. Series: Studies in Computational Mathematics, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[2] Argyros, I.K. *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[3] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[4] Argyros, I.K., A Newton-Kantorvich theorem for equations involving m Frchet differentiable operators and applications in radiative transfer, *J. Comput. Appl. Math.*, 131(1-2), (2001), 149-159.

[5] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)*, 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[6] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative methods*, Elsevier (Academic Press), New York, 2018.

[7] Argyros, I.K, Behl, R., Motsa, S.S., Unifying semilocal and local convergence of Newton's method on Banach space with a convergence structure, *Appl. Numer. Math.*, 115 (2017), 225-234.

[8] Argyros, I. K., Cordero, A., Magreñán, A. A., Torregrosa, J. R., Third-degree anomalies of Traub's method, *J. Comput. Appl. Math.*, 309 (2017), 511-521.

[9] Argyros, I.K., Cordero, A., Magreñán, A.A., Torregrosa, J. R., On the convergence of a higher order family of methods and its dynamics, *J. Comput. Appl. Math.*, 309, (2017), 542-562.

[10] Cordero, A., Fazlollah Soleymani, Juan R. Torregrosa, Stanford Shateyi Basins of attraction for various Steffensen-type methods, *J. Appl. Math.*, 2014.

[11] Ezquerro, J.A., Grau-Sánchez, M., Hernández, M. A., Noguera, M., Semilocal convergence of secant-like methods for differentiable and nondifferentiable operator equations, *J. Math. Anal. Appl.*, 398, (1), (2013), 100-112.

[12] Liu, Z., Zheng, Q., Zhao, P., A variant of Steffensen's method of fourth order convergence and its applications, *Appl. Math. Comput.*, 2146, (2010), 1978-1983.

[13] Moccari, M., Lofti, T., On a two step optimal Steffensen-type method: Relaxed local and semi-local convergence analysis and dynamical stability, *J. Math. Anal. Appl.*, 468, (2018), 240-269.

[14] Magréñan, A.A., Argyros, I.K., New improved convergence analysis for the secant method, *Math. Comput. Simulation*, 119 (2016), 161-170.

Chapter 13

Local Convergence Comparison between Two Competing Fifth Order Iterations

1. Introduction

The radius of convergence between two fifth convergence order iterations is determined for solving equations in Banach spaces. Let $F : D \subset X \longrightarrow Y$ be a nonlinear operator acting between Banach spaces X and Y. Consider the problem of solving the nonlinear equation

$$F(x) = 0. \tag{13.1}$$

Iterative methods are used to approximate a solution x^* of the equation (13.1). The following iterative method was studied by Sharma and Gupta in [2],

$$\begin{aligned} y_n &= x_n - \frac{1}{2}F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - \frac{1}{2}(2F'(y_n)^{-1} - F'(x_n)^{-1})F(x_n) \end{aligned} \tag{13.2}$$

and

$$x_{n+1} = z_n - (2F'(y_n)^{-1} - F'(x_n)^{-1})F(z_n)$$

and Cordero-Torregrosa in [1] considered the following iterative method

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - (F'(y_n) + F'(x_n))^{-1})F(x_n) \end{aligned} \tag{13.3}$$

and

$$x_{n+1} = z_n - F'(y_n)^{-1}F(z_n).$$

These methods were shown to be of order five using hypotheses on the sixth derivative. We present the semi-local convergence of method (13.2) using assumptions only on the first derivative of F, unlike earlier studies [1, 2] where the convergence analysis required assumptions on the derivatives of F up to the order six. This technique can be used on other methods and relevant topics along the same lines.

For example: Let $X = Y = \mathbb{R}$, $D = [-\frac{1}{2}, \frac{3}{2}]$. Define f on D by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we have $f(1) = 0$,

$$f'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously, $f'''(t)$ is not bounded by D. So, the convergence of the method (13.2) is not guaranteed by the analysis in [1,2].

Throughout the chapter, $U(x_0, R) = \{x \in X : \|x - x_0\| < R\}$ and $U[x_0, R] = \{x \in X : \|x - x_0\| \leq R\}$ for some $R > 0$.

2. Real Majorizing Function

The convergence of method (13.2) is presented first. Set $S = [0, \infty)$.
Suppose:

(1) $\exists$ function $w_0 : S \longrightarrow S$ continuous and nondecreasing such that equation

$$w_0(t) - 1 = 0$$

has a smallest positive solution ρ_0. Set $S_1 = [0, \rho_0)$.

(2) $\exists$ functions $w : S_1 \longrightarrow S_1$, $W_1 : S_1 \longrightarrow S$ such that equation

$$\varphi_1(t) - 1 = 0$$

has a smallest solution $r_1 \in S_1 - \{0\}$, where function $\varphi_1 : S_1 \longrightarrow S$ is defined by

$$\varphi_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta + \frac{1}{2}\int_0^1 w_1(\theta t)d\theta}{1 - w_0(t)}.$$

(3) Equation

$$w_0(\varphi_1(t)t) - 1 = 0$$

has a smallest solution $\rho_1 \in S_1 - \{0\}$. Set $\rho = \min\{\rho_0, \rho_1\}$ and $S_2 = [0, \rho)$.

(4) Define r_2 by $r_2 = \varphi_2(r_1)r_1$, where

$$\varphi_2(t) = 1 + \frac{1}{2}\left(\frac{2}{1 - w_0(\varphi_1(t)t)} + \frac{1}{1 - w_0(t)}\right)\int_0^1 w_1(\theta t)d\theta.$$

(5) Equation

$$\varphi_3(t) - 1 = 0$$

has a smallest solution $r_3 \in S_2 - \{0\}$, where

$$\varphi_3(t) = \left(1 + \left(\frac{2}{1 - w_0(\varphi_1(t)t)} + \frac{1}{1 - w_0 t)}\right) \int_0^1 w_1(\theta\varphi_2(t)t)d\theta\right) \varphi_2(t).$$

The parameter

$$r = \min\{r_m\}, m = 1, 2, 3 \tag{13.4}$$

is shown to be a radius of convergence for iteration (13.2). Let $S_3 = [0, r)$. Then, it follows by these definitions that for $t \in S_3$

$$\begin{aligned} 0 &\leq w_0(t) < 1, & (13.5)\\ 0 &\leq w_0(\varphi_1(t)t) < 1, & (13.6)\\ 0 &\leq \varphi_1(t) < 1, & (13.7)\\ 0 &\leq \varphi_2(t)t \leq r_2 & (13.8) \end{aligned}$$

and

$$0 \leq \varphi_3(t) < 1. \tag{13.9}$$

In a similar way radii, R_i and function ψ_i are defined. Suppose

(6) Equation

$$\psi_1(t) - 1 = 0$$

has a smallest solution $R_1 \in S_1 - \{0\}$, where

$$\psi_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)}.$$

(7) Equation

$$p(t) - 1 = 0$$

has a smallest solution $\rho_p \in S - \{0\}$. Let $\rho_2 = \min\{\rho_0, \rho_p\}$, and $S_4 = [0, \rho_2)$, where

$$p(t) = \frac{1}{2}(w_0(t) + w_0(\psi_1(t)t)).$$

(8) Equation

$$\psi_2(t) - 1 = 0$$

has a smallest solution $R_2 \in S_4 - \{0\}$, where

$$\psi_2(t) = \psi_1(t) + \frac{w_0(\psi_1(t)t) + w_0(t)) \int_0^1 w_1(\theta t)d\theta}{2(1 - w_0(t))(1 - p(t))}.$$

(9) Equation

$$w_0(\psi_1(t)t)-1=0$$

and

$$w_0(\psi_2(t)t)-1=0$$

have smallest solutions $\rho_3,\rho_4 \in S-\{0\}$, respectively. Let $\rho_5=\min\{\rho_2,\rho_3,\rho_4\}$ and $S_5=[0,\rho_5]$.

(10) Equation

$$\psi_3(t)-1=0$$

has a smallest solution $R_3 \in S_5-\{0\}$, where

$$\psi_3(t) = \left[\psi_1(\psi_2(t)t) + \frac{(w_0(\psi_1(t)t)+w_0(\psi_2(t)t))\int_0^1 w_1(\theta\psi_2(t)t)d\theta}{(1-w_0(\psi_1(t)t))91-w_0(\psi_2(t)))}\right]\psi_2(t).$$

The parameter

$$R=\min\{R_i\},\ i=1,2,3 \tag{13.10}$$

is shown to be a radius of convergence for iteration (13.3). Let $S_6=[0,R)$. It follows by these definitions that for each $t \in S_6$

$$0 \leq w_0(t)<1, \tag{13.11}$$
$$0 \leq w_0(\varphi_1(t)t)<1, \tag{13.12}$$
$$0 \leq w_0(\varphi_2(t)t)<1 \tag{13.13}$$

and

$$0 \leq \varphi_i(t)<1. \tag{13.14}$$

Let $U(x^*,d)$ and $U[x^*,d]$ stand for the open and closed ball in X with center $x^* \in X$ and radius $d>0$.

3. Local Analysis

The common set of conditions for both iterations connecting the scalar functions of the previous section to operator D,x^*, and F' are conditions (A). Suppose:

(A1) $\exists$ a simple solution x^* of equation $F(x)=0$.

(A2) $\|F'(x^*)^{-1}(F'(x)-F'(x_0))\| \leq w_0(\|x-x_0\|)\ \forall x \in D$.

Set $D_0=D\cap U(x^*,\rho_0)$.

(A3)

$$\|F'(x^*)^{-1}(F'(y)-F'(x))\| \leq w(\|x-y\|)$$

and

$$\|F'(x^*)^{-1}F'(x)\| \leq w_1(\|x-x^*\|)\ \forall x,y \in D_0.$$

(A4) $U[x^*,\tau] \subseteq D$, where $\tau = \max\{r, r_2\}$ is case of iteration (13.2) and $\tau = R$ in case of iteration (13.3). Denote by (I), (II) aforementioned conditions for iterations (13.2) and (13.3), respectively.

Theorem 14. **Under condition (I)**, sequence $\{x_n\}$ generated by iteration (13.2) is well defined, remains in $U(x^*,r)\,\forall n = 0,1,2,\ldots$ and converges to x^* according to

$$\|y_n - x^*\| \leq \varphi_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{13.15}$$

$$\|z_n - x^*\| \leq \varphi_2(r_1)r_1 \leq r_2 \tag{13.16}$$

and

$$\|x_{n+1} - x^*\| \leq \varphi_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r. \tag{13.17}$$

Proof. Estimates (13.15)-(13.17) and the conclusions follow from the calculations

$$\|F'(x^*)^{-1}(F'(u) - F'(x^*))\| \leq w_0(\|u - x^*\|) < 1\,\forall u \in U(x^*,r);$$

$$y_n - x^* = x_n - x^* - F'(x_n)^{-1}F(x_n) + \frac{1}{2}F'(x_n)^{-1}F9x_n),$$

$$\begin{aligned}\|y_n - x^*\| &\leq \frac{(\int_0^1 w((1-\theta)\|x_n - x^*\|)d\theta + \frac{1}{2}\int_0^1 w_1(\theta\|x_n - x^*\|)d\theta)\|x_n - x^*\|}{1 - w_0(\|x_n - x^*\|)} \\ &\leq \varphi_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r;\end{aligned}$$

$$\begin{aligned}\|z_n - x^*\| &\leq \|y_n - x^*\| + \frac{1}{2}[2\|F'(y_n)^{-1}F'(x^*)\| \\ &\quad + \|F'(x_n)^{-1}F'(x^*)\|]\|F'(x^*)^{-1}F(x_n)\| \\ &\leq [\varphi_1(\|x_n - x^*\|) + \frac{1}{2}\left(\frac{2}{1 - w_0(\|y_n - x^*\|)}\right. \\ &\quad \left. + \frac{1}{1 - w_0(\|x_n - x^*\|)}\right)\int_0^1 w_1(\theta\|x_n - x^*\|)d\theta]\|x_n - x^*\| \\ &\leq \varphi_2(\|x_n - x^*\|)\|x_n - x^*\| \leq r_2; \\ \|x_{n+1} - x^*\| &\leq \|z_n - x^*\| + \left(\frac{2}{1 - w_0(\varphi_1(\|x_n - x^*\|)\|x_n - x^*\|)}\right. \\ &\quad \left. + \frac{1}{1 - w_0(\|x_n - x^*\|)}\right)\int_0^1 w_1(\theta\|x_n - x^*\|)d\theta\|x_n - x^*\| \\ &\leq \varphi_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r,\end{aligned}$$

and

$$\|x_{n+1} - x^*\| \leq c\|x_n - x^*\| < r,$$

where $c = \varphi_3(\|x_n - x^*\|) \in [0,1)$. □

Theorem 15. **Under condition (II)**, sequence $\{x_n\}$ generated by iteration (13.3) is well defined, remains in $U(x^*,R)\,\forall n=0,1,2,\ldots$ and converges to x^* according to

$$\|y_n - x^*\| \le \psi_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < R, \tag{13.18}$$

$$\|z_n - x^*\| \le \psi_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < R \tag{13.19}$$

and

$$\|x_{n+1} - x^*\| \le \psi_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < R. \tag{13.20}$$

Proof. As in the previous calculations

$$\begin{aligned}
\|y_n - x^*\| &\le \frac{\int_0^1 w((1-\theta)\|x_n - x^*\|)d\theta\|x_n - x^*\|}{1 - w_0(\|x_n - x^*\|)} \\
&\le \psi_1(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < R_1; \\
\|z_n - x^*\| &\le y_n - x^* + F'(x_n)^{-1}(F'(y_n) - F'(x_n))(F'(x_n) \\
&\quad + F'(y_n))^{-1}F(x_n),
\end{aligned}$$

$$\begin{aligned}
\|z_n - x^*\| &\le \left[\psi_1(\|x_n - x^*\|) + \frac{(w_0(\|x_n - x^*\|) + w_0(\|y_n - x^*\|))}{2(1 - w_0(\|x_n - x^*\|))(1 - p_n)}\right. \\
&\quad \left.\int_0^1 w_1(\theta\|x_n - x^*\|)d\theta\right]\|x_n - x^*\| \\
&\le \psi_2(\|x_n - x^*\|)\|x_n - x^*\| \le R_2; \\
\|x_{n+1} - x^*\| &\le \|z_n - x^* - F'(z_n)^{-1}F(z_n) \\
&\quad + F'(z_n)^{-1}(F'(y_n) - F'(z_n))F'(y_n)^{-1}F(z_n)\| \\
&\le \left[\psi(\|z_n - x^*\|) + \frac{(w_0(\|y_n - x^*\|) + w_0(\|z_n - x^*\|))}{(1 - w_0(\|z_n - x^*\|))(1 - w_0(\|y_n - x^*\|))}\right]\|x_n - x^*\| \\
&\le \psi_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\| < R_3,
\end{aligned}$$

$$\|x_{n+1} - x^*\| \le d\|x_n - x^*\| < R_3,$$

$d = \psi_3(\|x_n - x^*\|) \in [0,1)$, where we also used

$$\begin{aligned}
&\|2F'(x^*)^{-1}(F'(x_n) + F'(y_n) - 2F'(x^*))\| \\
\le\ &\frac{1}{2}(\|F'(x^*)^{-1}(F'(x_n) - F'(x^*))\| + \|F'(x^*)^{-1}(F'(y_n) - F'(x^*))\| \\
\le\ &\frac{1}{2}(w_0(\|x_n - x^*\|) + w_0(\|y_n - x^*\|)) \le p_n < 1,
\end{aligned}$$

so

$$\|(F'(x_n) + F'(y_n))^{-1}F'(x^*)\| \le \frac{1}{2(1 - p_n)}.$$

□

Concerning the uniqueness of the solution ball:

Proposition 11. *Suppose:*

(1) *x^* is a simple solution of equation (13.1) in $U(x^*, \rho_0)$ for some* $\rho_0 > 0$.

(2) *Condition (A2) holds.*

(3) $\exists\, \rho \geq \rho_0$ *such that*

$$\int_0^1 w_0((1-\theta)\rho_0 + \theta\rho)d\theta < 1.$$

Set $U_1 = U[x^*, \rho] \cap D$. Then, x^* is the only solution of equation $F(x) = 0$ in the domain U_1.

Proof. Let $q \in U_1$ with $F(q) = 0$. Set $M = \displaystyle\int_0^1 F'(x^* + \theta(q - x^*))d\theta$. Then, by (1)-(3) and the following calculation

$$\begin{aligned} \|F'(x^*)^{-1}(M - F'(x^*))\| &\leq \int_0^1 w_0((1-\theta)\|x^* - x^*\| + \theta\|q - x^*\|)d\theta \\ &\leq \int_0^1 w_0((1-\theta)\rho_0 + \theta\rho)d\theta < 1, \end{aligned}$$

$q = x^*$, since $M^{-1} \in L(Y, X)$ and $M(q - x^*) = F(q) - F(x^*) = 0 - 0 = 0$. □

Numerical examples where we choose functions w_0, w and w_1 and find r, R in concrete applications can be found in earlier Chapters using different iterations.

Bibliography

[1] Cordero, A., Torregrosa, J.R., Variants of Newton's method using fifth order quadrature formulas, *Appl. Math. Comput.*, 190, (2007), 686-698.

[2] Sharma, R.J., Gupta, P., An efficient fifth order method for solving systems of nonlinear equations, *Computers and Mathematics with applications*, 67, (2014), 591-601.

Chapter 14

Semi-Local Convergence Analysis of High Convergence Order Iterations under the Same Set of Conditions

The semi-local convergence analysis of efficient third, fourth and fifth convergence order iterations is presented for solving equations in a Banach space setting. The convergence analysis uses conditions only on the first derivative.

1. Introduction

Let B_1 and B_2 be Banach spaces and $D \subset B_1$ be an open and convex set. The determination of a solution $x^* \in D$ of the nonlinear equation

$$F(x) = 0, \tag{14.1}$$

where $F : D \subset B_1 \longrightarrow B_1$ is a continuous operator, is one of the most challenging problems in computational Mathematics. Most solution procedures are iterative since the analytic representation of x^* is rarely attainable. A plethora of problems from diverse disciplines reduces to solving equations (14.1) [1,2,10].

The local convergence analysis has been provided for the following efficient and popular iterations:

Fifth order [3]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - 2(F'(y_n) + F'(x_n))^{-1}F(x_n) \end{aligned} \tag{14.2}$$

and

$$x_{n+1} = z_n - F'(y_n)^{-1}F(z_n).$$

Fifth order [9]

$$\begin{aligned} y_n &= x_n - \frac{1}{2}F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - F'(y_n)^{-1}F(x_n) \end{aligned} \tag{14.3}$$

and

$$x_{n+1} = z_n - (2F'(y_n)^{-1} - F'(x_n)^{-1})F(z_n).$$

Fifth order [8]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n),\ u_n = \frac{x_n + y_n}{2}, v_n = \frac{y_n - x_n}{2\sqrt{3}}, \\ z_n &= x_n - 2A_n^{-1}F(x_n), \\ A_n &= F'(u_n - v_n) + F'(u_n + v_n) \end{aligned} \tag{14.4}$$

and

$$x_{n+1} = z_n - F'(y_n)^{-1}F(z_n).$$

Fourth order [5]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - A_n^{-1}F(x_n) \end{aligned} \tag{14.5}$$

and

$$A_n = \frac{1}{6}F'(x_n) + \frac{2}{3}F'(\frac{x_n + z_n}{2}) + \frac{1}{6}F'(z_n).$$

Third order [4]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - 6A_n^{-1}F(x_n) \end{aligned} \tag{14.6}$$

and

$$A_n = F'(x_n) + 4F'(\frac{x_n + y_n}{2}) + F'(y_n).$$

Third order [4]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - 3A_n^{-1}F(x_n) \end{aligned} \tag{14.7}$$

and

$$A_n = 2F'(\frac{3x_n + y_n}{4}) - F'(\frac{x_n + y_n}{2}) + 2F'(\frac{x_n + 3y_n}{4}).$$

Third order [7]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - 8A_n^{-1}F(x_n) \end{aligned} \tag{14.8}$$

and

$$A_n = F'(x_n) + 3F'(\frac{2x_n + y_n}{3}) + 3F'(\frac{x_n + 2y_n}{3}) + F'(y_n).$$

and

Third order [6]

$$y_n = x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \tag{14.9}$$

and

$$x_{n+1} = x_n - \frac{1}{2}(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n).$$

The local convergence of the preceding iterations was given using Taylor series equations and conditions reaching derivatives one higher than the convergence order of the iteration at least. Hence, for the convergence of a fifth-order iteration $F^{(6)}$ should exist. But this limits the applicability to equations involving operators that are that many times differentiable.

Example 2. Let $B_1 = B_2 = \mathbb{R}$, $D = [-\frac{3}{2}, \frac{1}{2}]$. Define F on D by

$$F(x) = x^3 \log x^2 + x^5 - x^4$$

Then

$$F'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$F''(x) = 6x \log x^2 + 20x^3 - 12x^2 + 10x$$

and

$$F'''(x) = 6 \log x^2 + 60x^2 - 24x + 22.$$

Obviously $F'''(x)$ is not bounded on S. Hence, the applicability of the methods is limited.

Thus, these results cannot guarantee convergence. However, the iteration may converge. On the other hand, these iterations do not use derivatives of orders higher than one. In order to extend the applicability of these iterations we use only conditions on the derivative appearing on them and in the more interesting semi-local case.

2. Semi-Local Analysis

The same set of conditions (C) is used for all iterations:

(C1) $\exists\, x_0 \in D, d > 0$ such that $F'(x_0) \in L(B_2, B_1)$ and $\|F'(x_0)^{-1}F(x_0)\| \leq d$.

(C2) $\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_0\|x - x_0\|, L_0 > 0\ \forall x \in D$.

Let $D_1 = D \cap U[x_0, \frac{1}{L_0}]$.

(C3) $\|F'(x_0)^{-1}(F'(y) - F'(x))\| \leq L\|y - x\|, L > 0\ \forall x, y \in D_1$.

(C4) $U[x_0, t^*] \subset D$, where t^* is the limit point of corresponding majorizing sequences for iterations (14.2)-(14.9).

(C5) $\exists t_1^* \geq t^*$ such that

$$L_0(t^* + t_1^*) < 2.$$

Let $D_2 = D \cap U[x_0, t_1^*]$.

Theorem 16. Under conditions (C1)-(C5) iterations (14.2)-(14.9) are well defined, remains in $U[x_0, t^*]$ and converges to a unique solution x^* of equation $F(x) = 0$ in the region D_2.

The proof is based on majorizing sequences.

Majorizing sequence for iteration (14.2)

$$\begin{aligned}
t_0 &= 0,\ s_0 = r,\\
u_n &= s_n + \frac{(1+L_0t_n)L(s_n-t_n)^2}{2(1-L_0t_n)(1-q_n)},\ q_n = \frac{1}{2}L_0(s_n+t_n),\\
t_{n+1} &= u_n + \frac{(1+L_0t_n)(s_n-t_n)+(1+\frac{L_0}{2}(u_n+t_n))(u_n-t_n)}{1-L_0s_n}
\end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1}-t_n)^2 + 2(1+L_0t_n)(t_{n+1}-s_n)}{2(1-L_0t_{n+1})}.$$

Next, conditions are presented for the convergence of the majorizing sequence.

Lemma 19. Suppose

$$L_0t_n < 1 \quad \text{and} \quad L_0s_n < 1\ \forall n = 0,1,2,\ldots.$$

Then, the sequence $\{t_n\}$ is strictly increasing and converges to its unique least upper bound $t^* \in [d, \frac{1}{L_0}]$.

Proof. If follows immediately by the condition of the Lemma and the definition of the sequence. □

Proof. **Proof of Theorem 16 using Lemma 19**: If follows by the first two substeps

$$\begin{aligned}
z_n - y_n &= F'(x_n)^{-1}F(x_n) - 2(F'(y_n)+F'(x_n))^{-1}F(x_n)\\
&= F'(x_n)^{-1}(F'(y_n)+F'(x_n)-2F'(x_n))(F'(y_n)+F'(x_n))^{-1}F(x_n)\\
&= F'(x_n)^{-1}(F'(y_n)-F'(x_n))(F'(y_n)+F'(x_n))^{-1}F(x_n).
\end{aligned}$$

But

$$\begin{aligned}
&\|(2F'(x_0))^{-1}(F'(y_n)+F'(x_n)-2F'(x_0))\|\\
\leq\ & \frac{1}{2}[\|F'(x_0)^{-1}(F'(y_n)-F'(x_0))\| + \|F'(x_0)^{-1}(F'(x_n)-F'(x_0))\|]\\
\leq\ & \frac{1}{2}L_0[\|y_n-x_0\| + \|x_n-x_0\|]\\
\leq\ & \frac{L_0}{2}(s_n+t_n) = q_n < 1.
\end{aligned}$$

Hence,

$$\|\|(F'(y_n)-F'(x_0))^{-1}F'(x_0)\| \leq \frac{1}{2(1-q_n)}$$

and

$$\|z_n - y_n\| \leq \frac{L(1+L_0t_n)(s_n-t_n)^2}{291-L_0t_n)(1-q_n)} = u_n - s_n,$$

where it is also used

$$F(x_n) = F'(x_n)(y_n-x_n) = ((F'(x_n)-F'(x_0))+F'(x_0))(y_n-x_n),$$

so

$$\|F'(x_0)^{-1}F(x_n)\| \leq (1+L_0t_n)(s_n-t_n).$$

Moreover, the third substep can be written as

$$x_{n+1} - z_n = -F'(y_n)^{-1}(F(z_n)-F(x_n)+F(x_n)).$$

Therefore, by

$$\begin{aligned} F(z_n)-F(x_n) &= \int_0^1 F'(x_n+\theta(z_n-x_n))d\theta(z_n-x_n) \\ &= [\int_0^1 (F'(x_n+\theta(z_n-x_n))d\theta - F'(x_0)) + F'(x_0)](z_n-x_n) \end{aligned}$$

and

$$\begin{aligned} & \|F'(x_0)^{-1}[\int_0^1 (F'(x_n+\theta(z_n-x_n))-F'(x_0))d\theta + F'(x_0)](z_n-x_n)\| \\ \leq\ & (\frac{L_0}{2}(\|x_n-x_0\|+\|z_n-x_0\|)+1)\|z_n-x_n\| \\ \leq\ & (\frac{L_0}{2}(t_n+u_n)+1)(u_n-t_n). \end{aligned}$$

Hence, it follows

$$\begin{aligned} \|x_{n+1}-z_n\| &\leq \frac{(1+L_0t_n)(s_n-t_n)+(1+\frac{L_0}{2}(u_n+t_n)))(u_n-t_n)}{1-L_0s_n} \\ &= t_{n+1}-u_n. \end{aligned}$$

By the first substep

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1})-F(x_n)-F'(x_n)(y_n-x_n) \\ &= F(_{n+1})-F(x_n)-F'(x_n)(x_{n+1}-x_n)+F'(x_n)(x_{n+1}-y_n), \end{aligned}$$

so

$$\begin{aligned} \|y_{n+1}-x_{n+1}\| &\leq \frac{\frac{L}{2}(t_{n+1}-t_n)^2+(1+L_0t_n)(t_{n+1}-s_n)}{1-L_0t_{n+1}} \\ &= s_{n+1}-t_{n+1}. \end{aligned}$$

Therefore, the sequence $\{t_n\}$ majorizes sequence $\{x_n\}$. Thus, $\{x_n\}$ is complete in a Banach space B_1 and as such it converges to some $x^* \in U[x_0, t^*]$. By letting $n \longrightarrow \infty$ in the estimate

$$\|F'(x_0)^{-1}F(x_{n+1})\| \leq \frac{L}{2}(t_{n+1}-t_n)62+(1+L_0t_n)(t_{n+1}-s_n).$$

Moreover, using the continuity of $F, F(x^*) = 0$. That is limit point x^* solves equation $F(x) = 0$. Finally, in order to show the uniqueness part, let $M = \int_0^1 F'(x^*+\theta(p-x^*))d\theta$ for some $p \in D_2$ with $F(p) = 0$. It then follows from (C2) and (C5)

$$\begin{aligned}\|F'(x_0)^{-1}(M-F'(x_0))\| &\leq \frac{1}{2}L_0(\|x^*-x_0\|+\|p-x_0\|)\\ &\leq \frac{1}{2}(t^*+t_1^*) < 1.\end{aligned}$$

Hence, it follows $p = x^*$ by the invertibility of M guaranteed by the Banach lemma on invertible operators and the identity $M(p-x^*) = F(p)-F(x^*) = 0-0 = 0$. □

Similarly, convergence is established for the rest of the iterations.

Majorizing sequence for iteration (14.3)

$$\begin{aligned}t_0 &= 0,\, s_0 = d,\\ u_n &= s_n+\frac{1}{1-L_0s_n}\left(1+\frac{L(s_n-t_n)}{1-L_0t_n}\right)(1+L_0t_n)(s_n-t_n),\\ t_{n+1} &= u_n+\frac{1}{1-L_0s_n}\left(1+\frac{L(s_n-t_n)}{1-L_0t_n}\right)\\ &\quad\times[(1+\frac{L_0}{2}(u_n+t_n)(u_n-t_n)+2(1+L_0t_n)(s_n-t_n)]\end{aligned}$$

and

$$s_{n+1} = t_{n+1}+\frac{L(t_{n+1}-t_n)^2+2(1+L_0t_n)(t_{n+1}-s_n)+2(1+L_0t_n)(s_n-t_n)}{2(1-L_0t_{n+1})}.$$

Lemma 20. Suppose

$$L_0t_n < 1 \text{ and } L_0s_n < 1\ \forall n = 0,1,2,\ldots.$$

Then, sequence $\{t_n\}$ is increasing and convergent to its unique least upper bound $t^* \in [d, \frac{1}{L_0}]$.

Proof. It is given in Lemma 19. □

Proof. **Proof of Theorem 16 using Lemma 20**: It follows from the first two substeps of the method (14.3)

$$\begin{aligned}z_n-y_n &= \frac{1}{2}F'(x_n)^{-1}F(x_n)-F'(y_n)^{-1}F(x_n)\\ &= \frac{1}{2}(F'(x_n)^{-1}-F'(y_n)^{-1})F(x_n)-\frac{1}{2}F'(y_n)^{-1}F(x_n)\\ &= \frac{1}{2}F'(x_n)^{-1}(F'(y_n)-F'(x_n))F'(y_n)^{-1}F(x_n)-\frac{1}{2}F'(y_n)^{-1}F(x_n),\end{aligned}$$

so

$$\|z_n - y_n\| \le \frac{1}{2}\left(\frac{L(s_n - t_n)}{(1-L-0t_n)(1-L_0 s_n)} + \frac{1}{1-L_0 s_n}\right) 2(1+L_0 t_n)(s_n - t_n) = u_n - s_n.$$

Similarly,

$$\begin{aligned}
\|x_{n+1} - z_n\| &\le \|[(F'(y_n)^{-1} - F'(x_n)^{-1}) + F'(y_n)^{-1}]F(z_n)\| \\
&\le \left(\frac{L(s_n - t_n)}{(1-L_0 t_n)(1-L-0s_n)} + \frac{1}{1-L_0 s_n}\right) \\
&\quad (1+\frac{L_0}{2}(u_n + t_n))(u_n - t_n) + 2(1+L_0 t_n)(s_n - t_n) \\
&= t_{n+1} - u_n; \\
F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - 2F'(x_n)(y_n - x_n) \\
&= F(x_{n+1}) - F(x_n) - F'(x_n)(x_{n+1} - x_n) \\
&\quad + F'(x_n)(x_{n+1} - x_n) - F'(x_n)(y_n - x_n) - F'(x_n)(y_n - x_n).
\end{aligned}$$

Thus,

$$\|y_{n+1} - x_{n+1}\| \le \frac{\frac{1}{2}(t_{n+1} - t_n)^2 + (1+L_0 t_n)(t_{n+1} - s_n) + (1+L_0 t_n)(s_n - t_n)}{1 - L_0 t_{n+1}} = s_{n+1} - t_{n+1},$$

where it is also used

$$F(x_n) = 2F'(x_n)(y_n - x_n) \Rightarrow \|F'(x_0)^{-1}F(x_n)\| \le 2(1+L_0 t_n)(s_n - t_n)$$

and

$$\begin{aligned}
F(z_n) - F(x_n) &= \int_0^1 F'(x_n + \theta(z_n - x_n))d\theta(z_n - x_n) \\
&= \int_0^1 [F'(x_n + \theta(z_n - x_n)) - F'(x_0) + F'(x_0)]d\theta(z_n - x_n),
\end{aligned}$$

so

$$\|F'(x_0)^{-1}(F(x_n) - F(x_n))\| \le (1+\frac{L_0}{2}(u_n + t_n))(u_n - t_n).$$

The rest follows as in the previous proof. □

Majorizing sequence for iteration (14.4)

$$\begin{aligned}
t_0 &= 0, s_0 = d, \\
w_n &= s_n + \frac{L(1+L_0 t_n)(s_n - t_n)^2}{2(1-L_0 t_n)[1 - \frac{L_0}{\sqrt{3}}((\sqrt{3}-1)t_n + (\sqrt{3}+1)s_n}, \\
t_{n+1} &= w_n + \frac{(1+\frac{L_0}{3}(t_n + w_n))(w_n - t_n) + (1+L_0 t_n)(s_n - t_n)}{1 - L_0 s_n}
\end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2(1+L_0 t_n)(t_{n+1} - s_n)}{2(1-L_0 t_{n+1})}.$$

Lemma 21. Suppose

$$L_0 t_n < 1,\ L_0 s_n < 1 \text{ and } \frac{L_0}{\sqrt{3}}((\sqrt{3}-1)t_n+(\sqrt{3}+1)s_n) < 1\ \forall n = 0,1,2,\ldots.$$

Then, sequence $\{t_n\}$ is increasing and convergent to its unique least upper bound $t^* \in [d, \frac{1}{L_0}]$.

Proof. It is given in Lemma 19. □

Proof. **Proof of Theorem 16 using Lemma 21** It follows from the first two substeps of the method (14.4)

$$\begin{aligned} z_n &= y_n + (F'(x_n)^{-1} - 2A_n^{-1})F(x_n) \\ &= y_n + (F'(x_n)^{-1}(A_n - 2F'(x_n))A_n^{-1}F(x_n), \end{aligned}$$

so

$$\|z_n - y_n\| \leq \frac{L(1+L_0t_n)(s_n-t_n)^2}{2(1-L-0t_n)[1-\frac{L_0}{\sqrt{3}}((\sqrt{3}-1)t_n+(\sqrt{3}+1)s_n)]} = w_n - s_n,$$

where the estimates were also used

$$\begin{aligned} \|u_n+v_n-x_0\| &\leq \frac{1}{2\sqrt{3}}((\sqrt{3}-1)t_n+(\sqrt{3}=1)s_n), \\ \|u_n-v_n-x_0\| &= \frac{1}{2\sqrt{3}}[(\sqrt{3}+1)x_n+(\sqrt{3}-1)y_n-2\sqrt{3}x_0]\| \\ &\leq \frac{1}{2\sqrt{3}}((\sqrt{3}-1)t_n+\sqrt{3}+1)s_n), \\ \|u_n-v_n-x_n\| &= \frac{1}{2\sqrt{3}}(\sqrt{3}-1)\|y_n-x_n\| \\ &\leq \frac{\sqrt{3}-1}{2\sqrt{3}}(s_n-t_n), \end{aligned}$$

$$\begin{aligned} \|F'(x_0)^{-1}(A_n-2F'(x_n))\| &\leq \|F'(x_0)^{-1}(F'(u_n-v_n)-F'(x_n))\| \\ &\quad +\|F'(x_0)^{-1}(F'(u_n+v_n)-F'(x_n))\| \\ &\leq L(\|u_n-v_n-x_n\|+\|u_n+v_n-x_n\|), \\ \|(2F'(x_0))^{-1}(A_n-2F'(x_0))\| &\leq L_0(\|u_n-v_n-x_0\|+\|u_n+v_n-x_0\|) < 1, \end{aligned}$$

so,

$$\|A_n^{-1}F'(x_0)\| \leq \frac{1}{2(1-\frac{L_0}{\sqrt{3}}((\sqrt{3}-1)t_n+(\sqrt{3}+1)s_n)]},$$

$$\begin{aligned} \|x_{n+1}-z_n\| &\leq \|-F'(y_n)^{-1}(F(z_n)-F(x_n)+F(x_n))\| \\ &\leq \frac{(1+\frac{L_0}{2}(t_n+w_n))(w_n-t_n)+(1+L_0t_n)(s_n-t_n)}{1-L_0s_n} \\ &= t_{n+1}-w_n; \end{aligned}$$

and as in the proof using iteration (14.2)

$$\|y_{n+1}-x_{n+1}\| \leq s_{n+1}-t_{n+1}.$$

The rest follows as in the proof using iteration (14.2). ☐

Majorizing sequence for iteration (14.5)

$$\begin{aligned}
t_0 &= 0, s_0 = d,\\
u_n &= s_n + \frac{[(1+\frac{L_0}{2}(t_n+s_n))+(1+L_0t_n)](s_n-t_n)}{1-L_0t_n},\\
t_{n+1} &= u_n + \frac{L(u_n-t_n)(1+L_0t_n)(s_n-t_n)}{2(1-L_0t_n)(1-\frac{L_0}{2}(t_n+s_n))}\\
&\quad + \frac{[(1+\frac{L_0}{2}(s_n+t_n))+(1+L_0t_n)](s_n-t_n)}{2(1-L_0t_n)(1-\frac{L_0}{2}(t_n+s_n))}
\end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1}-t_n)^2+2(1+L_0t_n)(t_{n+1}-s_n)}{2(1-L_0t_{n+1})}.$$

Lemma 22. Suppose

$$L_0t_n < 1 \text{ and } \frac{L_0}{2}(t_n+s_n) < 1 \ \forall n = 0,1,2,\ldots.$$

Then, sequence $\{t_n\}$ is increasing and convergent to its unique least upper bound $t^* \in [d, \frac{1}{L_0}]$.

Proof. It is given in Lemma 19. ☐

Proof. **Proof of Theorem 16 using Lemma 22**: It follows from the first two substeps of the method (14.5)

$$\begin{aligned}
z_n &= y_n + (F'(x_n)^{-1}F(x_n) - F'(x_n)^{-1})(F(x_n)+F(y_n))\\
&= y_n - (F'(x_n)^{-1}((F(y_n)-F(x_n)+F(x_n)),
\end{aligned}$$

so

$$\|z_n-y_n\| \leq \frac{[1+\frac{L_0}{2}(t_n+s_n)+(1+L_0t_n)](s_n-t_n)}{1-L-0t_n} = u_n-s_n.$$

Similarly,

$$\begin{aligned}
\|x_{n+1}-z_n\| &\leq \|F'(x_n)^{-1}(A_n-F'(x_n))A_n^{-1}F(x_n)\|\\
&\quad +\|F'(x_n)^{-1}F(y_n)\\
&\leq \frac{\frac{L}{2}(u_n-t_n)(1+L_0t_n)(s_n-t_n)+(1+\frac{L_0}{2}(t_n+s_n))(s_n-t_n)+(1+L_0t_n)(s_n-t_n)}{(1-L_0s_n)((1-\frac{L_0}{2}(t_n+s_n))}\\
&= t_{n+1}-u_n,
\end{aligned}$$

where the estimates are also used

$$F(y_n)-F(x_n)=\int_0^1((F'(x_n+\theta(y_n-x_n))-F'(x_0))d\theta+F'(x_0))(y_n-x_n).$$

Hence,

$$\|F'(x_0)^{-1}(F(y_n)-F(x_n))\|\leq(1+\frac{L_0}{2}(t_n+s_n))(s_n-t_n),$$

$$\begin{aligned}\|F'(x_0)^{-1}(A_n-F'(x_n))\| &\leq \frac{2}{3}\|F'(x_0)^{-1}(F'(\frac{x_n+z_n}{2})-F'(x_n))\|\\ &\quad \frac{1}{6}\|F'(x_0)^{-1}(F'(z_n)-F'(x_n))\|\\ &\leq \frac{2L}{3}\left(\frac{u_n-t_n}{2}\right)+\frac{1}{6}(u_n-t_n)\\ &= \frac{L}{2}(u_n-t_n),\end{aligned}$$

$$\begin{aligned}&\|F'(x_0)^{-1}(\frac{1}{6}(F'(x_n)-F'(x_0))+\frac{4}{6}(F'(\frac{x_n+y_n}{2})-F'(x_0))\\ &+\frac{1}{6}(F'(z_n)-F'(x_0))\|\\ \leq\ &\frac{L_0}{2}(t_n+2(t_n+u_n)+u_n)=\frac{L_0}{2}(t_n+u_n)<1.\end{aligned}$$

Thus,

$$\|A_n^{-1}F'(x_0)\|\leq\frac{1}{1-\frac{L_0}{2}(t_n+u_n)}.$$

Moreover,

$$\|y_{n+1}-x_{n+1}\|\leq s_{n+1}-t_{n+1}$$

is obtained as previously. The rest follows as in the proof of Theorem 16 using iteration (14.2). □

Majorizing sequence for iteration (14.6)

$$\begin{aligned}t_0 &= 0, s_0=d,\\ t_{n+1} &= s_n+\frac{L(1+L_0t_n)(s_n-t_n)^2}{2(1-L_0t_n)(1-\frac{L_0}{2}(t_n+s_n)}\end{aligned}$$

and

$$s_{n+1} = t_{n+1}+\frac{L(t_{n+1}-t_n)^2+2(1+L_0t_n)(t_{n+1}-s_n)}{2(1-L_0t_{n+1})}.$$

Lemma 23. Suppose

$$L_0t_n<1 \text{ and } \frac{L_0}{2}(t_n+s_n)<1\ \forall n=0,1,2,\ldots.$$

Then, sequence $\{t_n\}$ is increasing and convergent to its unique least upper bound $t^*\in[d,\frac{1}{L_0}]$.

Proof. It is given in Lemma 19. □

Proof. **Proof of Theorem 16 using Lemma 23**: It follows from the first two substeps of the method (14.6)

$$x_{n+1}-y_n=F'(x_n)^{-1}(A_n-6F'(x_n))A_n^{-1}F(x_n),$$

so

$$\|x_{n+1}-y_n\|\leq\frac{L(1+L_0t_n)(s_n-t_n)^2}{2(1-L_0t_n)(1-\frac{L_0}{2}(t_n+s_n))}=t_{n+1}-s_n,$$

where the following estimates are also used

$$\begin{aligned}
A_n-6F'(x_n) &= 4(F'(\frac{x_n+y_n}{2})-F'(x_n))+(F'(y_n)-F'(x_n)),\\
\|F'(x_0)^{-1}(A_n-F'(x_n))\| &\leq L(2\|y_n-x_n\|+\|y_n-x_n\|)\\
&\leq 3L\|y_n-x_n\|\leq 3L(s_n-t_n),\\
\|(6F'(x_0))^{-1}(A_n-6F'(x_0))\| &\leq \frac{L_0}{6}(\|x_n-x_0\|+\|y_n-x_0\|\\
&\quad +2(\|x_n-x_0\|+\|y_n-x_0\|))\\
&= \frac{L_0}{2}(t_n+s_n)<1.
\end{aligned}$$

Hence,

$$\|A_n^{-1}F(x_0)\|\leq\frac{1}{6(1-\frac{L_0}{2}(t_n+s_n))}.$$

Moreover,

$$\|y_{n+1}-x_{n+1}\|\leq s_{n+1}-t_{n+1}$$

is obtained as previously. The rest follows as in the proof of Theorem 16 using iteration (14.2). □

Majorizing sequence for iteration (14.7)

$$\begin{aligned}
t_0 &= 0,\ s_0=d,\\
t_{n+1} &= s_n+\frac{2L(1+L_0t_n)(s_n-t_n)^2}{3(1-L_0t_n)(1-\frac{5L_0}{6}(t_n+s_n)}
\end{aligned}$$

and

$$s_{n+1} = t_{n+1}+\frac{L(t_{n+1}-t_n)^2+2(1+L_0t_n)(t_{n+1}-s_n)}{2(1-L_0t_{n+1})}.$$

Lemma 24. Suppose

$$L_0t_n<1 \text{ and } \frac{5L_0}{6}(t_n+s_n)<1\ \forall n=0,1,2,\ldots.$$

Then, sequence $\{t_n\}$ is increasing and convergent to its unique least upper bound $t^*\in[d,\frac{1}{L_0}]$.

Proof. It follows as the proof of Lemma 19 for iteration (14.2). □

Proof. **Proof of Theorem 16 using Lemma 24**: It follows from the first two substeps of the method (14.7)

$$x_{n+1}-y_n = F'(x_n)^{-1}(A_n-3F'(x_n))A_n^{-1}F(x_n),$$

so

$$\|x_{n+1}-y_n\| \leq \frac{2L(1+L_0t_n)(s_n-t_n)^2}{3(1-L_0t_n)(1-\frac{5L_0}{6}(t_n+s_n))} = t_{n+1}-s_n,$$

where the following estimates are also used.

$$\begin{aligned} A_n-3F'(x_n) &= 4(F'(\frac{3x_n+y_n}{4})-F'(\frac{x_n+y_n}{2}))+(F'(\frac{3x_n+y_n}{4})-F'(x_n)) \\ &\quad +2(F'(\frac{x_n+3y_n}{4})-F'(x_n)), \end{aligned}$$

so,

$$\begin{aligned} \|F'(x_0)^{-1}(A_n-3F'(x_n))\| &\leq L\Bigg(\frac{\|3x_n-2x_n+y_n-2y_n\|}{4}+\frac{\|y_n-x_n\|}{4}) \\ &\quad +\frac{3}{2}\|y_n-x_n\|) \\ &\leq 2L\|y_n-x_n\| \leq 2L(s_n-t_n) \end{aligned}$$

and

$$\begin{aligned} \|(3F'(x_0))^{-1}(A_n-3F'(x_0))\| &\leq \frac{L_0}{3}\Bigg[\frac{\|3x_n+y_n-4x_0\|}{2}+\frac{\|2x_0-x_n-y_n\|}{2} \\ &\quad +\frac{\|x_n+3y_n-4x_0\|}{2}] \\ &\leq \frac{L_0}{6}(3t_n+s_n+s_n+t_n+t_n+3s_n) \\ &= \frac{5L_0}{6}(t_n+s_n) < 1, \end{aligned}$$

so

$$\|A_n^{-1}F(x_0)\| \leq \frac{1}{3(1-\frac{5L_0}{6}(t_n+s_n))}.$$

Moreover, estimate

$$\|y_{n+1}-x_{n+1}\| \leq s_{n+1}-t_{n+1}$$

is shown as before.

Finally, the rest is given as in the proof of Theorem 16 using iteration (14.2). □

Majorizing sequence for iteration (14.8)

$$\begin{aligned} t_0 &= 0, s_0 = d, \\ t_{n+1} &= s_n + \frac{3L(1+L_0t_n)(s_n-t_n)^2}{8(1-L_0t_n)(1-\frac{L_0}{2}(t_n+s_n)} \end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1}-t_n)^2+2(1+L_0t_n)(t_{n+1}-s_n)}{2(1-L_0t_{n+1})}.$$

Lemma 25. Suppose

$$L_0 t_n < 1 \text{ and } \frac{L_0}{2}(t_n + s_n) < 1 \ \forall n = 0, 1, 2, \ldots.$$

Then, sequence $\{t_n\}$ is increasing and convergent to its unique least upper bound $t^* \in [d, \frac{1}{L_0}]$.

Proof. It is similar to the proof of Lemma 19 for iteration (14.2). □

Proof. **Proof of Theorem 16 using Lemma 25**: It follows from the first two substeps of the method (14.8)

$$x_{n+1} - y_n = F'(x_n)^{-1}(A_n - 8F'(x_n))A_n^{-1}F(x_n).$$

Thus

$$\|x_{n+1} - y_n\| \leq \frac{3L(1 + L_0 t_n)(s_n - t_n)^2}{8(1 - L_0 t_n)(1 - \frac{L_0}{2}(t_n + s_n))} = t_{n+1} - s_n,$$

where the following estimates are also used.

$$\begin{aligned} A_n - 8F'(x_n) &= 3(F'(\frac{2x_n + y_n}{3}) - F'(x_n)) + 3(F'(\frac{x_n + 2y_n}{3}) - F'(x_n)) \\ &\quad + (F'(y_n) - F'(x_n)), \end{aligned}$$

so,

$$\begin{aligned} \|F'(x_0)^{-1}(A_n - 8F'(x_n))\| &\leq 3L\|y_n - x_n\| \\ &\leq 3L(s_n - t_n) \end{aligned}$$

and

$$\begin{aligned} \|(8F'(x_0))^{-1}(A_n - 8F'(x_0))\| &\leq \frac{1}{8}(\|F'(x_0)^{-1}(F'(x_n) - F'(x_0))\| \\ &\quad + 3\|F'(x_0)^{-1}(F'(\frac{2x_n + y_n}{3}) - F'(x_0))\| \\ &\quad + 3\|F'(x_0)^{-1}(F'(\frac{x_n + 2y_n}{3}) - F'(x_0))\| \\ &\quad + \|F'(x_0)^{-1}(F'(y_n) - F'(x_0))\|) \\ &\leq \frac{L_0}{2}(t_n + s_n) < 1, \end{aligned}$$

so

$$\|A_n^{-1}F'(x_0)\| \leq \frac{1}{8(1 - \frac{L_0}{2}(t_n + s_n))}.$$

The estimate

$$\|y_{n+1} - x_{n+1}\| \leq s_{n+1} - t_{n+1}$$

is given previously.

The rest follows from as in the proof of Theorem 16 using iteration (14.2). □

Majorizing sequence for iteration (14.9)

$$\begin{aligned} t_0 &= 0, s_0 = d, \\ t_{n+1} &= s_n + \frac{3L(s_n - t_n)^2}{8(1 - \frac{L_0}{2}(t_n + s_n)} \end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2(1 + L_0 t_n)(t_{n+1} - s_n) + (1 + L_0 t_n)(s_n - t_n)}{2(1 - L_0 t_{n+1})}.$$

Lemma 26. Suppose

$$L_0 t_n < 1 \text{ and } \frac{L_0}{2}(t_n + 3s_n) < 1 \ \forall n = 0, 1, 2, \ldots.$$

Then, sequence $\{t_n\}$ is increasing and convergent to its unique least upper bound $t^* \in [d, \frac{1}{L_0}]$.

Proof. It is similar to the proof of Lemma 19 for iteration (14.2). □

Proof. **Proof of Theorem 16 using Lemma 26**: It follows from the first two substeps of the method (14.9)

$$\begin{aligned} x_{n+1} - y_n &= \frac{2}{3} F'(x_n)^{-1} F(x_n) \\ &\quad - \frac{1}{2}(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n) \\ &= -\frac{1}{2}(3F'(y_n) - F'(x_n))^{-1}(F'(y_n) - F'(x_n))\frac{3}{2}(y_n - x_n). \end{aligned}$$

Hence,

$$\|x_{n+1} - y_n\| \leq \frac{3L(s_n - t_n)^2}{8(1 - \frac{L_0}{2}(t_n + 3s_n))} = t_{n+1} - s_n,$$

where the following estimates are also used.

$$\|(2F'(x_0))^{-1}(3F'(y_n) - F'(x_n) - 3F'(x_0) + F'(x_0))\| \leq \frac{L_0}{2}(t_n + 3s_n) < 1,$$

so

$$\|(3F'(y_n) - F'(x_n))^{-1} F'(x_0)\| \leq \frac{1}{2(1 - \frac{L_0}{2}(t_n + 3s_n))}.$$

Moreover,

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - F'(x_n)(x_{n+1} - x_n) \\ &\quad + F'(x_n)(x_{n+1} - x_n) - \frac{3}{2}F'(x_n)(y_n - x_n), \end{aligned}$$

so

$$\begin{aligned} |y_{n+1} - x_{n+1}\| &\leq \frac{L(t_{n+1} - t_n)^2 + 2(1 + L_0 t_n)(t_{n+1} - s_n) + (1 + L_0 t_n)(s_n - t_n)}{2(1 - L_0 t_{n+1})} \\ &= s_{n+1} - t_{n+1}. \end{aligned}$$

The rest follows from as in the proof of Theorem 16 using iteration (14.2). □

3. Numerical Example

We verify convergence criteria using method (14.2).

Example 3. Let $B_1 = B_2 = \mathbb{R}$. Let us consider a scalar function F defined on the set $D = U[x_0, 1-q]$ for $q \in (0, \frac{1}{2})$ by

$$F(x) = x^3 - q.$$

Choose $x_0 = 1$. Then, the conditions (C1)-(C3) are verified for $d = \frac{1-q}{3}$, $L_0 = 3-q$, $D_1 = U(x_0, \frac{1}{L_0}) \cap D = U(x_0, \frac{1}{L_0})$, and $L = 2(1 + \frac{1}{L_0})$.

The conditions of Lemma 23 are satisfied.

Table 14.1. Sequence for (14.6)

n	1	2	3	4	5	6
s_n	0.0333	0.0388	0.0389	0.0389	0.0389	0.0389
t_{n+1}	0.0350	0.0388	0.0389	0.0389	0.0389	0.0389
$L_0 s_n$	0.0736	0.0816	0.0817	0.0817	0.0817	0.0817
$L_0(t_n + s_n)$	0.0350	0.0775	0.0816	0.0817	0.0817	0.0817

Bibliography

[1] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[2] Argyros, M.I., Argyros, I.K., Regmi, S. George, S., Generalized Three Step Numerical Methods for Solving Equations in Banach Spaces, *Mathematics* 2022, 10, 2621. https:// doi.org/10.3390/math10152621

[3] Cordero, A. Torregrosa, J.R., Variants of Newtons method using fifth- order quadrature formulas, *Appl. Math. Comput.*, 190, (2007), 686698.

[4] Cordero, A. Torregrosa, J.R., Variants of newtons method using fifth-order quadrature formulas, *Appl. Math. Comput.*, 190, (2007), 686698.

[5] Darvishi, M.T., Barati, A., A fourth-order method from quadrature formulae to solve systems of nonlinear equations, *Appl. Math. Comput.*, 188, (2007), 257261.

[6] Jarratt, P., Some fourth order multipoint iterative methods for solving equations, *Math. Comp.*, 20, (1966), 434437.

[7] Khirallah, M.Q., Hafiz, M.A., Novel three order methods for solving a system of nonlinear equations, *Bull. Math. Sci. Appl.*, 2, (2012), 0114.

[8] Liu, Z., Zheng, Q., Huang, C.E., Third and fifth-order Newton-Gauss method for solving nonlinear equations with n variables, *Appl. Math. Comput.*, 290, (2016), 250-257.

[9] Sharma, J.R., Gupta, P., An efficient fifth order method for solving systems of nonlinear equations, *Comput. Math. Appl.*, 67, (2014), 591-601.

[10] Traub, J.F., *Iterative methods for the solution of equations*, American Mathematical Soc. 1982.

Chapter 15

A Collection of Iterative Methods with Order Five, Six and Seven and Their Semi-Local Convergence

1. Introduction

This Chapter uses conditions and proving techniques from the preceding one. The conclusions are the same but for different iterative methods. That is why only the method and its majorizing sequence are reported. The notation (I_q) stands for iteration of convergence order q. The studied methods are listed below:

(I_6) [3]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - \frac{1}{2}(F'(y_n)^{-1} + F'(x_n)^{-1})F(x_n) \end{aligned} \tag{15.1}$$

and

$$x_{n+1} = z_n - \frac{1}{2}(F'(x_n)^{-1} + F'(y_n)^{-1}F'(x_n)F'(y_n)^{-1})F(z_n).$$

The corresponding majorizing sequence is given for $t_0 = 0, s_0 > 0$ by

$$\begin{aligned} u_n &= s_n + \frac{L(s_n - t_n)^2}{2(1 - L_0 s_n)}, \\ t_{n+1} &= u_n + \frac{1}{2}\left[s_n - t_n + \frac{(1 + \frac{L_0}{2}(t_n + u_n))(u_n - t_n)}{1 - L_0 t_n} \right. \\ &\quad + \frac{(1 + \frac{L_0}{2}(t_n + u_n)(1 + L_0 t_n)(u_n - t_n)}{(1 - L_0 s_n)^2} \\ &\quad \left. + \frac{(1 + L_0 t_n)^2 (s_n - t_n)}{(1 - L_0 s_n)^2} \right] \end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2(1 + L_0 t_n)(t_{n+1} - s_n)}{2(1 - L_0 t_{n+1})}.$$

(I_5) [10]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - \frac{1}{2}(F'(y_n)^{-1} + F'(x_n)^{-1})F(x_n) \end{aligned} \tag{15.2}$$

and

$$x_{n+1} = z_n - F'(y_n)^{-1})F(z_n).$$

The corresponding majorizing sequence is given for $t_0 = 0, s_0 > 0$ by

$$\begin{aligned} u_n &= s_n + \frac{L(s_n - t_n)^2}{2(1 - L_0 s_n)}, \\ t_{n+1} &= u_n + \frac{(1 + \frac{1}{2}(t_n + u_n))(u_n - t_n) + (1 + L_0 t_n)(s_n - t_n)}{1 - L_0 t_n} \end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2(1 + L_0 t_n)(t_{n+1} - s_n)}{2(1 - L_0 t_{n+1})}.$$

(I_6) [12]:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - \frac{1}{2}(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n) \end{aligned} \tag{15.3}$$

and

$$x_{n+1} = z_n - [(\frac{1}{2}(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))]F'(x_n)^{-1})F(z_n).$$

The corresponding majorizing sequence is given for $t_0 = 0, s_0 > 0$ by

$$\begin{aligned} u_n &= s_n + \frac{(3L(s_n - t_n) + 4(1 + L_0 t_n))(s_n - t_n)}{8(1 - p_n)}, \\ t_{n+1} &= u_n + \frac{1}{16(1 - p_n)^2}(4 + L_0(t_n + 3s_n))^2 \\ & [\frac{1}{1 - L_0 t_n}(1 + \frac{L_0}{2}(u_n + t_n))(u_n - t_n) + \frac{3}{2}(s_n - t_n)] \end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2(1 + L_0 t_n)(t_{n+1} - s_n) + (1 + L_0 t_n)(s_n - t_n)}{2(1 - L_0 t_{n+1})},$$

where $p_n = \frac{L_0}{2}(t_n + 3s_n)$.

(I_6) [13]:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - \frac{1}{2}(3F'(y_n) - F'(x_n))^{-1}(3F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n) \end{aligned}$$

and (15.4)

$$x_{n+1} = z_n - (\frac{3}{2}F'(y_n)^{-1} - \frac{1}{2}F'(x_n)^{-1})F(z_n).$$

The corresponding majorizing sequence is given for $t_0 = 0, s_0 > 0$ by

$$\begin{aligned}
u_n &= s_n + \frac{(3L(s_n - t_n) + 4(1 + L_0 t_n))(s_n - t_n)}{8(1 - p_n)}, \\
t_{n+1} &= u_n + \frac{1}{1 - L_0 s_n}(1 + \frac{L(s_n - t_n)}{2(1 - L_0 t_n)} \\
&\quad \left[1 + \frac{L_0}{2}(u_n + t_n)(u_n - t_n) + (1 + L_0 t_n)(s_n - t_n)\right] \\
&\quad + \frac{(1 + \frac{L_0}{2}(t_n + u_n)(1 + L_0 t_n)(u_n - t_n)}{(1 - L_0 s_n)^2}
\end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2(1 + L_0 t_n)(t_{n+1} - s_n) + (1 + L_0 t_n)(s_n - t_n)}{2(1 - L_0 t_{n+1})},$$

where $p_n = \frac{L_0}{2}(t_n + s_n)$.

(I_6) [4]

$$\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n), \\
z_n &= x_n - 2(F'(x_n) + F'(y_n))^{-1}F(x_n) \qquad (15.5)
\end{aligned}$$

and

$$x_{n+1} = z_n - (\frac{7}{2}I - 4F'(x_n)^{-1}F'(y_n) + \frac{3}{2}(F'(x_n)F'(y_n))^2]F'(x_n)^{-1}F(z_n).$$

The corresponding majorizing sequence is given for $t_0 = 0, s_0 > 0$ by

$$\begin{aligned}
u_n &= s_n + \frac{L(s_n - t_n)^2}{2(1 - L_0(s_n + t_n))}, \\
t_{n+1} &= u_n + [s_n - t_n + \frac{1}{1 - L_0 s_n}(u_n + t_n)(u_n - t_n)] \\
&\quad \left(3\left(\frac{L(s_n - t_n)}{2(1 - L_0 t_n)}\right)^2 + 2\left(\frac{L(s_n - t_n)}{1 - L_0 t_n}\right)\right)
\end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2(1 + L_0 t_n)(t_{n+1} - s_n)}{2(1 - L_0 t_{n+1})}.$$

(I_6) [2]

$$\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n), \\
z_n &= y_n + \frac{1}{3}(F'(x_n)^{-1} + 2(F'(x_n) - 3F'(y_n))^{-1}F(x_n) \qquad (15.6)
\end{aligned}$$

and

$$x_{n+1} = x_n + \frac{1}{3}(-F'(x_n)^{-1} + 4(F'(x_n) - 3F'(y_n))^{-1})F'(z_n).$$

The corresponding majorizing sequence is given for $t_0 = 0, s_0 > 0$ by

$$
\begin{aligned}
u_n &= s_n + \frac{L(s_n - t_n)^2}{2(1 - \frac{L_0}{2}(3s_n + t_n))},\\
t_{n+1} &= u_n + \frac{1}{3}[s_n - t_n + \frac{(1 + \frac{L_0}{2}(u_n + t_n))(u_n - t_n)}{1 - L_0 s_n}]\\
&\quad \left((1 + \frac{L_0}{2})(u_n - t_n) + (1 + L_0 t_n)(s_n - t_n)\right)
\end{aligned}
$$

and

$$
s_{n+1} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2 + 2(1 + L_0 t_n)(t_{n+1} - s_n)}{2(1 - L_0 t_{n+1})}.
$$

(I_7) [11]

$$
\begin{aligned}
y_n &= x_n + F'(x_n)^{-1}F(x_n),\\
z_n &= y_n - F'(x_n)^{-1}F(y_n),\\
w_n &= z_n - F'(x_n)^{-1}F'(y_n)F'(x_n)^{-1}F(z_n)
\end{aligned} \tag{15.7}
$$

and

$$
x_{n+1} = w_n - F'(x_n)^{-1}F'(y_n)F'(x_n)^{-1}F(w_n).
$$

The corresponding majorizing sequence is given for $a_0 = 0, b_0 > 0$ by

$$
\begin{aligned}
c_n &= b_n + \frac{(1 + \frac{L_0}{2}(a_n + b_n)) + (1 + L_0 a_n))(b_n - a_n)}{1 - L_0 a_n},\\
d_n &= c_n + \frac{(1 + L_0 b_n)D}{(1 - L_0 a_n)^2},\\
a_{n+1} &= d_n + \frac{(1 + L_0 b_n)[(1 + \frac{L_0}{2}(b_n + c_n)(d_n - c_n) + D]}{(1 - L_0 a_n)^2}
\end{aligned}
$$

and

$$
b_{n+1} = a_{n+1} + \frac{(1 + \frac{L_0}{2}(a_{n+1} + a_n))(a_{n+1} - a_n) + (1 + L_0 a_n)(b_n - a_n)}{1 - L_0 a_{n+1}},
$$

where $D = (1 + \frac{L_0}{2}(b_n + c_n))(c_n - b_n) + (1 + \frac{L_0}{2}(a_n + b_n)) + (1 + L_0 a_n)(b_n - a_n)$.

(I_5) [4]

$$
\begin{aligned}
y_n &= x_n + F'(x_n)^{-1}F(x_n),\\
z_n &= y_n - F'(x_n)^{-1}F(y_n)
\end{aligned} \tag{15.8}
$$

and

$$
x_{n+1} = z_n + (F'(y_n)^{-1} - 2F'(x_n)^{-1})F(z_n).
$$

The corresponding majorizing sequence is given for $t_0 = 0, s_0 > 0$ by

$$\begin{aligned} w_n &= s_n + \frac{(2+\frac{L_0}{2}(3t_n+s_n))(s_n-t_n)}{1-L_0t_n}, \\ t_{n+1} &= w_n + \frac{1}{1-L_0t_n}(1+\frac{L(s_n-t_n)}{1-L_0s_n})(2+\frac{L_0}{2}(2t_n+s_n+w_n))(w_n-s_n) \end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{(1+\frac{L_0}{2}(t_{n+1}+t_n))(t_{n+1}-t_n)+(1+L_0t_n)(s_n-t_n)}{1-L_0t_{n+1}}.$$

(I_5) [1]

$$y_n = x_n - F'(x_n)^{-1}F(x_n) \tag{15.9}$$

and

$$x_{n+1} = y_n - \frac{1}{4}(5I - 2F'(y_n)^{-1}F'(x_n) + (F'(y_n)^{-1}F'(x_n)^2)F'(y_n)^{-1}F(y_n).$$

The corresponding majorizing sequence is given for $t_0 = 0, s_0 > 0$ by

$$\begin{aligned} t_{n+1} &= s_n + \frac{1}{4(1-L_0s_n)}(3+\frac{2L(s_n-t_n)}{1-L_0s_n}) + \left(\frac{1+L_0t_n}{1-L_0s_n}\right)^2) \\ &\times(2+\frac{L_0}{2}(3t_n+s_n))(s_n-t_n) \end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1}-t_n)^2+2(1+L_0t_n)(t_{n+1}-s_n)}{2(1-L_0t_{n+1})}.$$

(I_5) [5]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \qquad (15.10) \\ z_n &= x_n - 2A_n^{-1}F(x_n) \end{aligned}$$

and

$$x_{n+1} = z_n - F'(y_n)^{-1}F(z_n),$$

where $A_n = F'(\frac{x_n+y_n}{2} + \frac{y_n-x_n}{2\sqrt{3}}) + F'(\frac{x_n+y_n}{2} - \frac{y_n-x_n}{2\sqrt{3}})$. The corresponding majorizing sequence is given for $t_0 = 0, s_0 > 0$ by

$$\begin{aligned} u_n &= s_n + \frac{L(1+\frac{1}{\sqrt{3}})(s_n-t_n)^2}{2(1-L_0t_n)(1-p_n)}, \\ t_{n+1} &= u_n + \frac{(1+\frac{L_0}{2}(u_n+t_n))(u_n-t_n)+(1+L_0t_n)(s_n-t_n)}{1-L_0s_n} \end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1}-t_n)^2+2(1+L_0t_n)(t_{n+1}-s_n)}{2(1-L_0t_{n+1})}.$$

(I_5) [11]

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n), \qquad (15.11)\\
z_n &= x_n - A_n^{-1}F(x_n)
\end{aligned}
$$

and

$$x_{n+1} = z_n - F'(y_n)^{-1}F(z_n),$$

where A_n is as method (15.10). The corresponding majorizing sequence is given for $t_0 = 0, s_0 > 0$ by

$$
\begin{aligned}
u_n &= s_n + \frac{(1+L_0t_n+3L(s_n-t_n))(s_n-t_n)}{16(1-\frac{L_0}{2}(t_n+s_n))(1-L_0t_n)},\\
t_{n+1} &= u_n + \frac{(1+\frac{L_0}{2}(u_n+t_n))(u_n-t_n)+(1+L_0t_n)(s_n-t_n)}{1-L_0s_n}
\end{aligned}
$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1}-t_n)^2+2(1+L_0t_n)(t_{n+1}-s_n)}{2(1-L_0t_{n+1})}.$$

(I_5) [12]

$$
\begin{aligned}
y_n &= x_n - \frac{1}{2}F'(x_n)^{-1}F(x_n), \qquad (15.12)\\
z_n &= x_n - F'(y_n)^{-1}F(x_n)
\end{aligned}
$$

and

$$x_{n+1} = z_n - (2F'(y_n)^{-1} - F'(x_n)^{-1})F(z_n).$$

The corresponding majorizing sequence is given for $t_0 = 0, s_0 > 0$ by

$$
\begin{aligned}
u_n &= s_n + \frac{2}{1-L_0s_n}(\frac{1}{2}+L_0(s_n-t_n)^2)(1+L_0t_n)(s_n-t_n),\\
t_{n+1} &= u_n + \frac{1}{1-L_0s_n}(1+\frac{L(s_n-t_n)}{1-L_0t_n})\\
&\quad \times[(1+\frac{L_0}{2}(t_n+u_n))(u_n-t_n)+2(1+L_0t_n)(s_n-t_n)]
\end{aligned}
$$

and

$$s_{n+1} = t_{n+1} + \frac{L(t_{n+1}-t_n)^2+2(1+L_0t_n)(t_{n+1}-t_n)}{2(1-L_0t_{n+1})}.$$

(I_5) [15]

$$
\begin{aligned}
y_n &= x_n - \frac{1}{2}F'(x_n)^{-1}F(x_n), \qquad (15.13)\\
z_n &= x_n - \frac{1}{2}F'(y_n)^{-1}F(x_n),\\
w_n &= z_n - A_n^{-1}F(z_n)
\end{aligned}
$$

and

$$x_{n+1} = w_n - A_n^{-1}F(w_n),$$

where $A_n = 2F'(y_n) - F'(x_n)$. The corresponding majorizing sequence is given for $a_0 = 0, a_0 > 0$ by

$$\begin{aligned}
c_n &= b_n + \frac{2}{1-L_0 b_n}(\frac{1}{2} + L_0(b_n - a_n)^2)(1+L_0 a_n)(b_n - a_n),\\
d_n &= c_n + \frac{(1+\frac{L_0}{2}(a_n+c_n)(c_n-a_n) + (1+L_0 a_n)(b_n - a_n)}{1-(L(b_n-a_n)+L_0(b_n-a_n))},\\
a_{n+1} &= d_n + \frac{(1+\frac{L_0}{2}(a_n+d_n))(d_n-a_n) + 2(1+L_0 a_n)(b_n-a_n)}{1-(L(b_n-a_n)+L_0(b_n-a_n))}
\end{aligned}$$

and

$$b_{n+1} = a_{n+1} + \frac{L(a_{n+1}-a_n)^2 + 2(1+L_0 a_n)(a_{n+1}-a_n)}{1-L_0 a_{n+1}}.$$

(I_7) [14]

$$\begin{aligned}
y_n &= x_n - \frac{1}{2}F'(x_n)^{-1}F(x_n), \qquad (15.14)\\
z_n &= x_n - F'(y_n)^{-1}F(x_n),\\
w_n &= z_n - B_n F(z_n)
\end{aligned}$$

and

$$x_{n+1} = w_n - B_n F(w_n),$$

where $B_n = 2F'(y_n)^{-1} - F'(x_n)^{-1}$. The corresponding majorizing sequence is given for $a_0 = 0, a_0 > 0$ by

$$\begin{aligned}
c_n &= b_n + \frac{2}{1-L_0 b_n}(\frac{1}{2} + L(b_n-a_n)^2)(1+L_0 a_n)(b_n-a_n),\\
d_n &= c_n + \frac{1}{1-L_0 b_n}(1+\frac{L(b_n-a_n)}{1-L_0 a_n})((1+\frac{L_0}{2}(a_n+c_n)(c_n-a_n)+2(1+L_0 a_n)(b_n-a_n)),\\
a_{n+1} &= d_n + \frac{1}{1-L_0 b_n}(1+\frac{L(b_n-a_n)}{1-L_0 a_n})((1+\frac{L_0}{2}(a_n+c_n)(c_n-a_n)+2(1+L_0 a_n)(b_n-a_n))
\end{aligned}$$

and

$$b_{n+1} = a_{n+1} + \frac{L(a_{n+1}-a_n)^2 + 2(1+L_0 a_n)(a_{n+1}-a_n)}{2(1-L_0 a_{n+1})}.$$

2. Convergence Criteria

Recall that according to the previous Chapter the convergence criteria for the fifteen majorizing sequences are respectively. Suppose $\forall n = 0, 1, 2, \ldots$

$$L_0 t_n < 1, L_0 s_n < 1, \qquad (15.15)$$

$$L_0 t_n < 1, L_0 s_n < 1, \qquad (15.16)$$

$$L_0 t_n < 1, \frac{L_0}{2}(t_n + 3s_n) < 1, \qquad (15.17)$$

$$L_0 t_n < 1, \frac{L_0}{2}(t_n + s_n) < 1, \qquad (15.18)$$

$$L_0 t_n < 1, L_0(s_n + t_n) < 1, \tag{15.19}$$

$$L_0 t_n < 1, \frac{L_0}{2}(t_n + 3s_n) < 1, \tag{15.20}$$

$$L_0 a_n < 1, \tag{15.21}$$

$$L_0 t_n < 1, L_0 s_n < 1, \tag{15.22}$$

$$L_0 t_n < 1, L_0 s_n < 1, \tag{15.23}$$

$$L_0 t_n < 1, L_0 s_n < 1, L_0(t_n + s_n + \frac{s_n - t_n}{\sqrt{3}}) < 1, \tag{15.24}$$

$$L_0 t_n < 1, L_0 s_n < 1, \tag{15.25}$$

$$L_0 t_n < 1, L_0 s_n < 1, \tag{15.26}$$

$$L_0 t_n < 1, L_0 s_n < 1, L_0 s_n + L(s_n - t_n) < 1, \tag{15.27}$$

and

$$L_0 a_n < 1, L_0 b_n < 1. \tag{15.28}$$

Numerical examples involving majorizing sequences can be found in the previous Chapters.

Bibliography

[1] Cordero, A., Gómez, E., and Torregrosa, J. R., Efficient high-order iterative methods for solving nonlinear systems and their application on heat conduction problems, *Complexity*, 2017, 6457532. https://doi.org/10.1155/2017/6457532.

[2] Esmaeili, H. and Ahmadi, M., An efficient three-step method to solve system of nonlinear equations, *Appl. Math. Comput.*, 266, 2015, 10931101.

[3] Grau-Sanchez, M., Grau, A. and Noguera, M., On the computational efficiency index and some iterative methods for solving systems of nonlinear equations, *J. Comput. Appl. Math.*, 236, 2011, 12591266.

[4] Lotfi, T., Bakhtiari, P., Cordero, A., Mahdiani, K. and Torregrosa, J. R., Some new efficient multipoint iterative methods for solving nonlinear systems of equations, *Int. J. Comput. Math.* 92, 2014, 19211934.

[5] Liu, Z., Zheng, Q., and Huang, C. E, Third-and fifth-order Newton-Gauss methods for solving nonlinear equations with n variables, *Applied Mathematics and Computation*, 290, 2016, 250-257.

[6] Michael, I.A., Argyros, I.K, Regmi, S. and George, S., Generalized three-step numerical methods for solving equations in Banach spaces, *Mathematics*, 2022, 10, 2621. https:// doi.org/10.3390/math10152621

[7] Sharma, J.R., and Gupta, P., An efficient fifth order method for solving systems of nonlinear equations, *Computers & Mathematics with Applications*, 67, 3, 2014, 591-601.

[8] Sharma, R., Sharma, J.R., Kalra, N., A modified Newton-*Ö*zban composition for solving nonlinear systems, *International Journal of Computational Methods*, 17, 2020 1-17.

[9] Sharma, J.R. and Kumar, D., A fast and efficient composite Newton-Chebyshev method for system of nonlinear equations, *Journal of Complexity*, 49, 2018, 56-73.

[10] Sharma, R., Sharma, R.J, Kalra, N., A modified Newton-Ozban composition for solving nonlinear systems, *International J. Comput. Math.*, (2019), 1950047.

[11] Singh, H., Sharma, J.R., Simply yet highly efficient techniques for systems of nonlinear equations, *Comput. Appl. Math.*, (2022).

[12] Soleymani, F., Lotfi, T. and Bakhtiari, P., A multi-step class of iterative methods for nonlinear systems, *Optim Lett.*, 2013, doi:10.1007/s11590-013-0617-6.

[13] Wang, X., Kou, J. and Gu, C., Semilocal convergence of a sixth-order Jarrat method in Banach spaces, *Numer. Algo.*, 57, 2011, 441456.

[14] Xiao, X., and Yin, H., Increasing the order of convergence for iterative methods to solve nonlinear systems, *Calcolo*, 53, 3, 2016, 285-300.

[15] Xu, Z., and Jieqing, T., The fifth order of three-step iterative methods for solving systems of nonlinear equations, *Mathematica Numerica Sinica*, 35, 3, 2013, 297-304.

Chapter 16

Extended Semi-Local Convergence Analysis for the Two-Step Newton Method under Generalized Lipschitz Conditions

1. Introduction

Let B_1, B_2 denote Banach spaces and Ω be a nonempty and open convex subset of B_1. Let also $L(B_1, B_2)$ stand for the space of bounded linear operators from B_1 into B_2. A plethora of applications from diverse disciplines can be brought in the form of the nonlinear equation

$$F(x) = 0, \tag{16.1}$$

using mathematical modeling [1,2,3,4,5], where $F : \Omega \longrightarrow B_2$ is a Fréchet differentiable operator. A solution x^* of equation (16.1) is needed in closed form. However, this is attainable only in special cases. That explains most solution methods for equation $F(x) = 0$ are iterative. The most popular iterative method for solving equation (16.1) is Newton's (NM) which is defined by

$$x_0 \in \Omega, x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \; \forall n = 0, 1, 2, \ldots. \tag{16.2}$$

Here, $F'(x)$ denotes the Fréchet-derivative of F. The local, as well as the semi-local convergence of NM, has been studied under various Lipschitz, Hölder and generalized conditions of F' [1,2,3,4,5,6,7,8]. The same has been done for the two-step Newton method (TSNM) defined by

$$y_n = x_n - F'(x_n)^{-1}F(x_n)$$

and

$$x_{n+1} = y_n - F'(x_n)^{-1}F(y_n).$$

In particular, the semi-local convergence of TSNM was given in [8] under generalized Lipschitz conditions. The convergence domain is small in general. That is why in the present Chapter we extended the convergence domain without additional conditions. More benefits are weaker sufficient convergence criteria; tighter error estimates on the distances $\|x_{n+1}-x_n\|$, $\|x_n-x^*\|$, and a piece of more precise information on the location of the solution. From now on $U(x,r)$ denotes an open ball with center $x\in B_1$ and of radius $\rho>0$. Then, by $U[x_0,r]$ we denote the closure of the open ball $U(x_0,r)$.

2. Semi-Local Convergence

Some generalized Lipschitz conditions are introduced and compared. Suppose that there exists $x_0\in\Omega$ such that $F'(x_0)^{-1}\in L(B_2,B_1)$. We assume from now on

$$S=\sup\{t\geq 0: U(x_0,t)\subset\Omega\}.$$

We also set $D=U(x_0,\rho)$.

Definition 1. Operator F' is said to satisfy the center $-L_0-$ average Lipschitz condition on D if $\forall x\in\Omega$

$$\|F'(x_0)^{-1}(F'(x)-F'(x_0))\|\leq\int_0^{\|x-x_0\|}L_0(u)du, \tag{16.3}$$

where L_0 is a positive nondecreasing function on the interval $[0,\rho)$.

Suppose that equation

$$\int_0^r L_0(u)du-1=0 \tag{16.4}$$

has a smallest solution $\rho_0\in(0,r)$. Define the set $D_0=U(x_0,\rho_0)$.

Definition 2. Operator F' is said to satisfy the restricted center $-L-$ average Lipschitz condition on the set D_0 if $\forall x,y\in D_0$

$$\|F'(x_0)^{-1}(F'(y)-F'(x))\|\leq\int_{\|x-x_0\|}^{\|x-x_0\|+\|y-x\|}L(u)du, \tag{16.5}$$

where L is a positive nondecreasing function on the interval $[0,\rho_0)$.

Definition 3. Operator F' is said to satisfy the $-L_1$ average Lipschitz condition on the set D if $\forall x,y\in D$ with $\|x-x_0\|+\|y-x\|<\rho$

$$\|F'(x_0)^{-1}(F'(y)-F'(x))\|\leq\int_{\|x-x_0\|}^{\|x-x_0\|+\|y-x\|}L_1(u)du, \tag{16.6}$$

where L_1 is a positive nondecreasing function on the interval $[0,\rho)$.

Remark. By the definition of sets D_0 and D, we get

$$D_0\subset D. \tag{16.7}$$

It follows that

$$L_0(u)\leq L_1(u) \tag{16.8}$$

and

$$L(u)\leq L_1(u) \tag{16.9}$$

$\forall u \in [0,\rho_0)$. It is assumed from now on that

$$L_0(u) \leq L(u)\ \forall u \in [0,\rho_0). \tag{16.10}$$

If not, replaced L by $\bar{L}$, where the latter function is the largest of functions L_0 and L on the interval $[0,\rho_0)$. The crucial modification in our analysis is the fact that L_0 can replace L in the computation of the upper bounds on $\|F'(x)^{-1}F'(x_0)\|$. Indeed, let us define functions h_0, h and h_1 on the interval $[0,\rho_0)$ for some $b \geq 0$ by

$$h_0(t) = b - t + \int_0^1 L_0(u)(t-u)du,$$

$$h(t) = b - t + \int_0^1 L(u)(t-u)du$$

and

$$h_1(t) = b - t + \int_0^1 L_1(u)(t-u)du.$$

It follows from these definitions that

$$h_0(t) \leq h(t) \leq h_1(t)\ \forall t \in [0,\rho_0). \tag{16.11}$$

Set

$$b := \int_0^{\rho_0} L(u)udu \text{ and } \beta = \|F'(x_0)^{-1}F(x_0)\|. \tag{16.12}$$

Moreover, define the scalar sequence $\{t_n\}$, $\{s_n\}$ $\forall n = 0,1,2,\ldots$ by

$$\begin{aligned} t_0 &= 0, \\ s_n &= t_n - h'(t_n)^{-1}h(t_n) \\ \text{and} \\ t_{n+1} &= s_n - h'(t_n)^{-1}h(s_n). \end{aligned} \tag{16.13}$$

These sequences shall be shown to be majorizing for TSNM at the end of this Section. A convergence criterion for these iterations has been given in [8].

Lemma 27. If $0 < \beta < b$, then h is decreasing on $[0,\rho_0]$ and increasing on $[\rho_0,R]$, and

$$h(\beta) > 0,\ h(\rho_0) = \beta - b < 0,\ h(R) = \beta > 0.$$

Moreover, h has a unique zero in each interval, denoted by t^* and t^{**}. They satisfy

$$\beta < t^* < \frac{\rho_0}{b},\ \beta < \rho_0 < t^{**} < R.$$

The following Lemmas relate operators to functions "h" and scalar majorizing sequences $\{t_n\}$ and $\{s_n\}$.

Lemma 28. Assume that $\|x - x_0\| \leq t < t^*$. If the first derivative F' satisfies the center L_0- average Lipschitz condition (16.3) in $U(x^*,t)$, then $F'(x)$ is nonsingular and

$$\|F'(x)^{-1}F'(x_0)\| \leq -\frac{1}{h'(\|x - x_0\|)} \leq -\frac{1}{h'(t)}.$$

In particular, F' is nonsingular in $U(x_0,t^*)$.

Proof. Take $x \in U[x^*, t], 0 \leq t < t^*$. By using the center L_0- average Lipschitz condition (16.3) we have

$$\|F'(x_0)^{-1}F'(x) - I\| \leq \int_0^{\|x-x_0\|} L(u)du = h'(\|x-x_0\|) - h'(0).$$

But $h'(0) = -1$ and h' is strictly increasing in $(0, t^*)$. Therefore, the Banach lemma is applicable to conclude the result. □

Lemma 29. Let $\{s_k\}$ and $\{t_k\}$ be generated by (16.13). Assume that F' satisfies the restricted $L-$ average Lipschitz condition (16.4) i $U(x_0, t^*)$. If $0 < \beta \leq b$, then the sequence $\{x_k\}$ and $\{y_k\}$ generated by the TSNM with initial guess x_0 are well defined and contained in $U(x_0, t^*)$. Moreover, $\forall k = 0, 1, 2, \ldots$, we have

(i) $F'(x_k)^{-1}$ exists and $\|F'(x_k)^{-1}F'(x_0)\| \leq -\dfrac{1}{h'(\|x-x_0\|)} \leq -\dfrac{1}{h'(t_k)}$.

(ii) $\|F'(x_0)^{-1}F(x_k)\| \leq h(t_k)$,

(iii) $\|y_k - x_k\| \leq s_k - t_k$.

(iv) $\|x_{k+1} - y_k\| \leq (t_{k+1} - s_k)\left(\dfrac{\|y_k - x_k\|}{s_k - t_k}\right)^2 \leq t_{k=1} - s_k$.

(v) $\|x_{k+1} - x_k\| \leq t_{k+1} - t_k$.

Proof. Replace h_1 by h in the proof of Lemma 4 in [5]. □

Lemma 30. Under the same assumptions of Lemma 29. Then, the sequence $\{x_k\}$ converges to a point $x^* \in U[x_0, t^*]$ with $F(x^*) = 0$. Moreover, we have

$$\|x^* - x_k\| \leq t^* - t_k, \ k \geq 0,$$

and

$$\|x^* - y_k\| \leq (t^* - s_k)\left(\frac{\|x^* - x_k\|}{t^* - t_k}\right)^2, \ k \geq 0.$$

Proof. Replace h_1 by h in the proof of Lemma 5 in [5]. □

Lemma 31. Under the same assumptions of Lemma 30 and the assumption $2 + t^*\dfrac{h''(t^*)}{h'(t^*)} > 0$, we have

$$\frac{\|y_k - x_k\|}{s_k - t_k} \leq \frac{1 - \frac{h''(t^*)}{2h'(t^*)}(t^* - t_k)}{1 + \frac{h''(t^*)}{2h'(t^*)}(t^* - t_k)} \frac{\|x^* - x_k\|}{t^* - t_k}, \ k \geq 0.$$

Proof. Replace h_1 by h in the proof of Lemma 6 in [5]. □

Remark. The corresponding estimate of Lemma 28 in [6] gives using (16.6) the less precise estimate

$$\|F'(x)^{-1}F'(x_0)\| \leq -\frac{1}{h_1'(\|x-x_0\|)}.$$

But by (16.11) the following holds

$$\|F'(x)^{-1}F'(x_0)\| \leq -\frac{1}{h_0'(\|x-x_0\|)} \leq -\frac{1}{h'(\|x-x_0\|)} \leq -\frac{1}{h_1'(\|x-x_0\|)}.$$

Hence, the sequence $\{\bar{t}_n\}, \{\bar{s}_n\}$ used in [5] and defined $\forall n = 0,1,2,\ldots$ by

$$\begin{aligned} \bar{t}_0 &= 0, \\ \bar{s}_n &= \bar{t}_n - h_1'(\bar{t}_n)^{-1}h(\bar{t}_n) \end{aligned} \tag{16.14}$$

and

$$\bar{t}_{n+1} = \bar{s}_n - h_1'(\bar{t}_n)^{-1}h(\bar{s}_n). \tag{16.15}$$

are less precise than the new majorizing sequence $\{t_n\}$ and $\{s_n\}$.

Next, the main semi-local result for TSNM is presented.

Theorem 17. Let $F : D \subset X \longrightarrow Y$ be a continuously Fréchet differentiable nonlinear operator in open convex subset D. Assume that there exists an initial guess $x_0 \in D$ such that $F'(x_0)^{-1}$ exists and that F' satisfies the $L-$ average Lipschitz condition (16.5) in $U(x_0,t^*)$. Let $\{x_k\}$ be the iterates generated by the TSNM with initial guess x_0. If $0 < \beta \leq b$, then $\{x_k\}$ is well defined and converges $Q-$ super quadratically to a solution $x^* \in U[x_0,t^*]$ of (16.1) and this solution x^* is unique on $U[x_0,r]$, where $t^* \leq r < t^{**}$. Moreover, if

$$2+\frac{t^*h''(t^*)}{h'(t^*)} > 0 \Leftrightarrow 2-\frac{t^*L(t^*)}{1-\int_0^{t^*}L(u)du} > 0,$$

then the order of convergence is cubic at least and we have the following error bounds

$$\|x^*-x_{n+1}\| \leq \frac{1}{2}H_*62\frac{2-t^*H_*}{2+t^*H_*}\|x^*-x_k\|^3,\ k \geq 0,$$

where $H_* \backsimeq \dfrac{h''(t^*)}{h'(t^*)}$.

Proof. Simply replace function $h-1$ by h in the proof of Theorem 1 in [5]. □

Remark. Popular choices for the "L" functions are
Kantorovich-type Case [1,4,5,6]:

$$h_0(t) = \frac{L_0}{2}t^2 - t + \beta,$$

$$h(t) = \frac{L}{2}t^2 - t + \beta$$

and

$$h_1(t) = \frac{L_1}{2}t^2 - t + \beta,$$

where L_0, L and L_1 are constant functions and $b = \frac{1}{2L}$. Notice that

$$L_0 \leq L \leq L_1. \tag{16.16}$$

Smale-Wang-type Case [7,8]:

$$h_0(t) = \frac{\gamma_0 t^2}{1-\gamma_0 t} - t + \beta \ \forall t \in [0, \frac{1}{\gamma_0}),$$

$$h(t) = \frac{\gamma t^2}{1-\gamma t} - t + \beta \ \forall t \in [0, \frac{1}{\gamma})$$

and

$$h_1(t) = \frac{\gamma_1 t^2}{1-\gamma_1 t} - t + \beta \ \forall t \in [0, \frac{1}{\gamma_1}),$$

where $b = \frac{1}{\gamma}(3 - 2\sqrt{2})$.

Notice that

$$\gamma_0 \leq \gamma \leq \gamma_1. \tag{16.17}$$

Examples where (16.16) and (16.17) (therefore the aforementioned advantages hold) can be found in [1,4,6].

Bibliography

[1] Argyros, I.K., A new semilocal convergence theorem for Newtons method in Banach space using hypotheses on the second frechet-derivativ., *J. Comput. Appl. Math.*, 130, (1-2), 369373, (2001).

[2] Argyros, I.K., Hilout, S., Extending the applicability of the GaussNewton method under average Lipschitztype conditions, *Numer. Algorithms*, 58, 1, 2352, (2011).

[3] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[4] Argyros, I.K., George, S., *Mathematical modeling for the solution of equations and systems of equations with applications, Volume-IV*, Nova Science Publishers, NY, 2019.

[5] Ling, Y., Liang, J. and Lin, W., On the semilocal convergence analysis for two-step Newton method under generalized conditions in Banach spaces, *Numer. Algor.*, 90, (2022), 577-606.

[6] Regmi, S., Argyros, C.I., Argyros, I.K.,George, S., Convergence Criteria of Three Step Schemes for Solving Equations, *Mathematics*, 2021, 9, 3106.

[7] Smale, S., Complexity theory and numerical analysis, *Acta Numer.*, 6, 523552, (1997).

[8] Wang, X., Convergence of Newtons method and inverse functions theorem in Banach space, *Math. Comput.*, 68, 169186, (1999).

Chapter 17

Semi-Local Convergence for Jarratt-Like Methods under Generalized Conditions for Solving Equations

The semi-local convergence is presented for sixth convergence order Jarratt-like methods for solving a nonlinear equation. The convergence analysis is based on the first Fréchet derivative that only appears in the method. Numerical examples are provided where the theoretical results are tested.

1. Introduction

The semi-local convergence is developed for two sixth convergence order Jarratt-like methods for solving the nonlinear equation

$$F(x) = 0, \tag{17.1}$$

where $F : \Omega \subset \mathcal{X} \longrightarrow \mathcal{Y}$ is continuously Fréchet differentiable, $\mathcal{X}, \mathcal{Y}$ are Banach spaces, and Ω is a nonempty convex set.

The methods under consideration in this chapter are:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - (\frac{23}{8}I - \frac{3}{2}F'(x_n)^{-1}F'(y_n)(3I - \frac{9}{8}F'(x_n)^{-1}F'(y_n))] \\ &\quad \times F'(x_n)^{-1}F(x_n) \end{aligned} \tag{17.2}$$

and

$$x_{n+1} = z_n - (\frac{5}{2}I - \frac{3}{2}F'(x_n)^{-1}F'(y_n))F'(x_n)^{-1}F(z_n).$$

and

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n), \\
z_n &= x_n - (I + \frac{21}{8}T_n - \frac{9}{2}T_n^2 + \frac{15}{8}T_n^3)F'(x_n)^{-1}F(x_n) \qquad (17.3) \\
\text{and} \\
x_{n+1} &= z_n - (3I - \frac{5}{2}T_n + \frac{1}{2}T_n^2)F'(x_n)^{-1})F(z_n),
\end{aligned}
$$

where $T_n = F'(x_n)^{-1}F'(y_n)$.

The sixth local convergence order of these methods was shown in [1,9], respectively when $\mathcal{X} = \mathcal{Y} = \mathbb{R}^m$ using Taylor expansions and conditions up to the seven order derivative which does not appear on these methods. These conditions restrict the applicability of these methods.

For example: Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define f on Ω by

$$
f(t) = \begin{cases} 0 & if\ t = 0 \\ t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0. \end{cases}
$$

Then, the third derivative is given as

$$
f'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.
$$

Obviously, $f'''(t)$ is not bounded on Ω. Thus the convergence of these methods is not guaranteed by the analysis in these papers. But these methods may converge. Our convergence analysis is based on the first Fréchet derivative that only appears in the method. The analysis includes computable error estimates on $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ and the uniqueness ball of the solution. The results significantly extend the applicability of these methods. This new process provides a new way of looking at iterative methods. Notice also that the semi-local convergence is more challenging than the local one.

The semi-local convergence analysis is given in Section 2 and the numerical examples are in Section 3.

2. SL of Method (17.3) and Method (17.2)

Let L_0, L and η denote positive parameters. Define the scalar sequences $\{s_n\}, \{u_n\}, \{t_n\}, \forall n = 0, 1, \ldots$ by

$$\begin{aligned} t_0 &= = 0,\, s_0 = \eta, \\ u_n &= s_n + \frac{3}{2}\left(\frac{1}{3} + \frac{15L^2(1+L_0 s_n)(s_n-t_n)^2}{8(1-L_0 t_n)^3}\right. \\ &\quad \left. + \frac{3L(1+L_0 s_n)(s_n-t_n)}{4(1-L_0 t_n)^2}\right)(s_n-t_n), \\ t_{n+1} &= u_n + \frac{1}{1-L_0 t_n}\left(2 + \frac{3L(s_n-t_n)}{1-L_0 t_n} + \frac{L^2(s_n-t_n)^2}{(1-L_0 t_n)^2}\right) \\ &\quad [(1+\frac{L_0}{2}(t_n+u_n))(u_n-t_n) + \frac{3}{2}(s_n-t_n)] \end{aligned} \tag{17.1}$$

and

$$s_{n+1} = t_{n+1} + \frac{3L(t_{n+1}-t_n)^2 + 4(1+L_0 t_n)(t_{n+1}-s_n) + 2(1-L_0 t_n)(t_{n+1}-t_n)}{6(1-L_0 t_{n+1})}.$$

These sequences shall be shown to be majorizing for method (17.3) in Theorem 18.

Next, a convergence result is presented for these methods.

Lemma 32. Suppose:

$$t_n < \frac{1}{L_0} \;\forall n = 0, 1, 2, \ldots. \tag{17.2}$$

Then, sequence given by formula (17.1) are strictly increasing and convergent to $t_* \in (0, \frac{1}{L_0}]$. The limit t_* is the unique least upper bound of these sequences.

Proof. The assertions follow from the definition (17.1) of these sequences and condition (17.2). □

The aforementioned parameters L_0, L and η are connected to the initial data (Ω, x_0, F, F') as follows. Suppose

(A1) There exist $x_0 \in D, \eta > 0$ such that $F'(x_0)^{-1} \in L(\mathcal{Y}, \mathcal{X})$ with $\|F'(x_0)^{-1}F(x_0)\| \leq \eta$.

(A2) There exists $L_0 > 0$ such that $\forall x \in \Omega$

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_0\|x - x_0\|.$$

Define the set $M_0 = U(x_0, \frac{1}{L_0}) \cap \Omega$.

(A3) There exists $L > 0$ such that $\forall x, y \in M_0$

$$\|F'(x_0)^{-1}(F'(y) - F'(x))\| \leq L\|y - x\|.$$

(A4) Conditions of Lemma 32 hold.
and

(A5) $U[x_0,t_*] \subset \Omega$.

The SL of method (17.3) uses the conditions (A1)-(A5).

Theorem 18. Suppose that the conditions (A1)-(A5) hold. Then, sequences $\{x_n\},\{y_n\},\{z_n\}$ are well defined in the ball $U(x_0,t_*)$ remain in the ball $U[x_0,t_*]\,\forall n=0,1,2,\ldots$ and converge to a solution $x_* \in U[x_0,t_*]$ of equation $F(x)=0$. Moreover, the following error estimates hold $\forall n=0,1,2,\ldots$

$$\|x_*-x_n\| \leq t_*-t_n. \tag{17.3}$$

Proof. The following items shall be shown $\forall m=0,1,2,\ldots$ by mathematical induction

$$\|y_m-x_m\| \leq s_m-t_m, \tag{17.4}$$

$$\|z_m-y_m\| \leq u_m-s_m \tag{17.5}$$

and

$$\|x_{m+1}-z_m\| \leq t_{m+1}-u_m. \tag{17.6}$$

Condition (A1) and the formula (17.1) imply

$$\|y_0-x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = s_0-t_0 < t_*.$$

Thus the item (17.4) holds for $m=0$ and the iterate $y_0 \in U(x_0,t_*)$. It also follows by the formula (17.1) that iterates z_0 and x_1 are well defined. Let $w \in U(x_0,t_*)$. By applying condition (A2) and Lemma 32

$$\|F'(x_0)^{-1}(F'(w)-F'(x_0))\| \leq L_0\|w-x_0\| \leq L_0t_* < 1.$$

Hence, $F'(w)^{-1} \in L(\mathcal{Y},\mathcal{X})$ and the estimate

$$\|F'(w)^{-1}F'(x_0)\| \leq \frac{1}{1-L_0\|w-x_0\|} \tag{17.7}$$

follows by the Banach perturbation lemma on invertible operators [3, 6]. Suppose items (17.4)-(17.6) hold $\forall m=0,1,2,\ldots,n$. By the first two substeps of method (17.3)

$$z_n-y_n = \frac{3}{2}(\frac{1}{3}I+B_n)(y_n-x_n), \tag{17.8}$$

where $B_n = \frac{21}{8}T_n - \frac{9}{2}T_n^2 + \frac{15}{8}T_n^3 = T_n(\frac{15}{8}(I-T_n)+\frac{3}{4}I)(I-T_n)$. It then follows by (17.7) for $w=x_n$ and the conditions (A2) and (A3) that

$$\|B_n\| \leq \frac{15L^2(1+L_0s_n)(s_n-t_n)^2}{8(1-L_0t_n)^3} + \frac{3L(1+L_0s_n)(s_n-t_n)}{4(1-L_0t_n)^2}, \tag{17.9}$$

where we also used

$$\begin{aligned} \|F'(x_0)^{-1}F'(y_n)\| &= \|F'(x_0)^{-1}((F'(y_n)-F'(x_0))+F'(x_0))\| \\ &\leq 1+L_0\|y_n-x_0\| \leq 1+L_0s_n. \end{aligned}$$

Therefore, the identity (17.1), formula (17.1) and (17.9) give

$$\begin{aligned}\|z_n - y_n\| &\leq \left(\frac{1}{3} + \frac{15L^2(1+L_0 s_n)(s_n - t_n)^2}{8(1-L_0 t_n)^3}\right. \\ &\quad \left. + \frac{3L(1+L_0 s_n)(s_n - t_n)}{4(1-L_0 t_n)^2}\right) \\ &\quad \times[(1+\frac{L_0}{2}(t_n+u_n))(u_n-t_n)+\frac{3}{2}(s_n-t_n)] \\ &= u_n - s_n\end{aligned}$$

and

$$\|z_n - x_0\| \leq \|z_n - y_n\| + \|y_n - x_0\| \leq u_n - s_n + s_n - t_0 = u_n < t_*,$$

so item (17.5) holds and the iterate $z_n \in U(x_0, t_*)$. Moreover, by the third substep of method (17.3)

$$\|x_{n+1} - z_n\| = \|C_n F'(x_n)^{-1} F(z_n)\|, \tag{17.10}$$

where $C_n = 3I - \frac{5}{2}T_n + \frac{1}{2}T_n^2 = (T_n - I)^2 - 3(T_n - I) + 2I$, so

$$\|C_n\| \leq \frac{L^2(s_n - t_n)^2}{(1-L_0 t_n)^2} + \frac{3L(s_n - t_n)}{1-L_0 t_n} + 2. \tag{17.11}$$

Consequently, (17.1), (17.10) and (17.11) give

$$\begin{aligned}\|x_{n+1} - z_n\| &\leq \frac{1}{1-L_0 t_n}(2 + \frac{L^2(s_n - t_n)^2}{(1-L_0 t_n)^2} + \frac{3L(s_n - t_n)}{1-L-0t_n}) \\ &\quad [(1+\frac{L_0}{2}(t_n+u_n))(u_n-t_n)+\frac{3}{2}(s_n-t_n)] \\ &= t_{n+1} - u_n,\end{aligned}$$

and

$$\|x_{n+1} - x_0\| \leq \|x_{n+1} - z_n\| + \|z_n - x_0\| \leq t_{n+1} - u_n + u_n - t_0 = t_{n+1} < t_*.$$

Thus, item (17.6) holds and the iterate $x_{n+1} \in U(x_0, t_*)$, where the identity

$$\begin{aligned}F(z_n) &= F(z_n) - F(x_n) + F(x_n) \\ &= \int_0^1 F'(x_n + \theta(z_n - x_n))d\theta(z_n - x_n) - \frac{3}{2}F'(x_n)(y_n - x_n)\end{aligned}$$

is used to obtain

$$\begin{aligned}\|F'(x_0)^{-1}F(z_n)\| &\leq \|\int_0^1 F'(x_0)^{-1}(F'(x_n + \theta(z_n - x_n))d\theta - F'(x_0)) \\ &\quad + I\|\|z_n - x_n\| \\ &\quad + \frac{3}{2}(\|I\| + \|F'(x_0)^{-1}(F'(x_n) - F'(x_0))\|)\|y_n - x_n\| \\ &\leq (1 + \frac{L_0}{2}(u_n + t_n)(u_n - t_n) + \frac{3}{2}(1 + L_0 t_n)(s_n - t_n).\end{aligned}$$

It also follows that iterate y_{n+1} is well defined, since $x_{n=1} \in U(x_0,t_*)$. Then, the first substep of method (17.3) gives

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1})-F(x_n)-F'(x_n)(x_{n+1}-x_n) \\ &\quad +F'(x_n)(x_{n+1}-x_n)-\frac{2}{3}F'(x_n)(y_n-x_n) \\ &= (F(x_{n+1})-F(x_n)-F'(x_n)(x_{n+1}-x_n)) \\ &\quad +\frac{2}{3}F'(x_n)(x_{n+1}-y_n)+\frac{1}{3}F'(x_n)(x_{n+1}-x_n), \end{aligned}$$

leading to

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{n+1})\| &\leq \frac{L}{2}(t_{n+1}-t_n)^2+\frac{2}{3}(1+L_0t_n)(t_{n+1}-s_n) \\ &\quad +\frac{1}{3}(1+L_0t_n)(t_{n+1}-t_n). \end{aligned} \tag{17.12}$$

Consequently, it follows by formula (17.1), estimate (17.7) (for $w=x_{n+1}$) and (17.12) that

$$\begin{aligned} \|y_{n+1}-x_{n+1}\| &\leq \|F'(x_{n+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\| \\ &\leq \frac{1}{1-L_0t_{n+1}}(\frac{L}{2}(t_{n+1}-t_n)^2 \\ &\quad +\frac{2}{3}(1+L_0t_n)(t_{n+1}-s_n)+\frac{1}{3}(1+L_0t_n)(t_{n+1}-t_n)) \\ &= s_{n+1}-t_{n+1} \end{aligned} \tag{17.13}$$

and

$$\|y_{n+1}-x_0\| \leq \|y_{n+1}-x_{n+1}\|+\|x_{n+1}-x_0\| \leq s_{n+1}-t_{n+1}+t_{n+1}-t_0 = s_{n+1} < t_*.$$

Thus item (17.4) holds and the iterate $y_{n+1} \in U(x_0,t_*)$. The induction for items (17.4)-(17.6) is completed. The scalar sequence $\{t_n\}$ is complete by condition (A4) (as convergent). Hence, sequence $\{x_n\}$ is convergent too. So, there exists $x_* \in U[x_0,t_*]$ such that $F(x_*)=0$. By letting $n \longrightarrow \infty$ in (17.12) and using the continuity of operator F it follows that $F(x_*)=0$. Finally, estimate (17.3) follows by letting $i \longrightarrow \infty$ in the estimate

$$\|x_{i+n}-x_n\| \leq t_{i+n}-t_n.$$

□

Next, the uniqueness of the solution result is provided but where not all conditions (A1)-(A5) are used.

Proposition 12. *Suppose:*

(a) There exists a solution $p \in U(x_0,\rho)$ for some $\rho > 0$ of equation $F(x)=0$.

(b) Condition (A2) holds such that $U(x_0,\rho) \subset \Omega$.

(c) There exists $\rho_1 \geq \rho$ *such that*

$$L_0(\rho+\rho_1) < 2. \tag{17.14}$$

Define the set $M_1 = U(x_0,\rho_1)\cap\Omega$*. Then, the only solution of equation* $F(x) = 0$ *in the set* M_1 *is the point* p*.*

Proof. Let $q \in M_1$ be such that $F(q) = 0$. Define the linear operator $S = \int_0^1 F'(q+\theta(p-q))d\theta$. It follows by the conditions (A2) and (17.14) that

$$\begin{aligned}\|F'(x_0)^{-1}(S-F'(x_0))\| &\leq L_0\int_0^1((1-\theta)\|p-x_*\|+\theta\|q-x_*\|)d\theta\\ &\leq \frac{L_0}{2}(\rho+\rho_1) < 1.\end{aligned}$$

Therefore, the estimate $p = q$ follows by the invertibility of linear operator S and the identity

$$S(p-q) = F(p)-F(q) = 0-0 = 0.$$

□

If all conditions (A1)-(A5) hold then, set $\rho = t_*$ in the Proposition 12.

Method (17.2) is studied in a similar way. Define the sequences $\{t_n\},\{s_n\},\{u_n\}$ by

$$\begin{aligned}t_0 &= = 0,\ s_0 = \eta,\\ u_n &= s_n+\frac{1}{24}\left(1+27\left(\frac{L(s_n-t_n)}{1-L_0t_n}\right)^2\right.\\ &\quad\left.+18\left(\frac{L(s_n-t_n)}{1-L_0t_n}\right)\right)(s_n-t_n),\\ t_{n+1} &= u_n+\frac{1}{2}(2+\frac{3L(s_n-t_n)}{1-L_0t_n})\\ &\quad[(1+\frac{L_0}{2}(t_n+u_n))(u_n-t_n)+\frac{3}{2}(s_n-t_n)]\end{aligned} \tag{17.15}$$

and

$$s_{n+1} = t_{n+1}+\frac{3L(t_{n+1}-t_n)^2+4(1+L_0t_n)(t_{n+1}-s_n)+2(1-L0t_n)(t_{n+1}-t_n)}{6(1-L_0t_{n+1})}.$$

Then, these sequences are also increasing and convergent to their least upper bound and the conditions of Lemma 32. Moreover, notice that under conditions (A1)-(A5) the following estimates are obtained:

$$\begin{aligned}z_n-y_n &= (\frac{23}{8}I-\frac{3}{2}T_n(3I-\frac{9}{8}T_n))\frac{3}{2}(y_n-x_n)\\ &= \frac{1}{16}(I+27(T_n-I)^2-18(T_n-I)),\end{aligned}$$

implying

$$\begin{aligned}\|z_n-y_n\| &\leq \frac{1}{16}\left(27\left(\frac{L(s_n-t_n)}{1-L_0t_n}\right)^2+18\left(\frac{L(s_n-t_n)}{1-L_0t_n}\right)+1\right)\\ &\quad\frac{2}{3}(s_n-t_n) = u_n-s_n.\end{aligned}$$

Moreover,

$$x_{n+1} - z_n = -\frac{1}{2}(2I + 3(I - T_n)F'(x_n)^{-1}F(z_n),$$

so

$$\begin{aligned} \|x_{n+1} - z_n\| &\leq \frac{1}{2}\left(2 + 3\left(\frac{L(s_n - t_n)}{1 - L_0 t_n}\right)\right) \\ & [(1 + \frac{L_0}{2}(u_n + t_n))(u_n - t_n) + \frac{3}{2}(s_n - t_n)] \\ &= t_{n+1} - u_n. \end{aligned}$$

The rest follows as in the proof of Theorem 18. Hence, the conclusions of Theorem 18 hold for method (17.2). Clearly, the same is true for the conclusions of Proposition 12.

Bibliography

[1] Abbasbandy, S., Bakhtiari, P., Cordero,A., Torregrosa, J. R., Lofti, T., New efficient methods for solving nonlinear systems of equations with arbitrary even order, *Appl. Math. Comput.*, 287, 288, (2016), 94-103.

[2] Argyros, I.K., A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach spaces, *J. Math. Anal. Appl.*, 298, (2004), 374-397.

[3] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[4] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics*, 2021, 9, (16), 1942; https://doi.org/10.3390/math9161942.

[5] Argyros, M.I., Argyros, I.K., Regmi, S., George, S., Generalized Three-Step Numerical Methods for Solving Equations in Banach Spaces, *Mathematics*, 2022, 10, 2621. https://doi.org/10.3390/math10152621.

[6] Argyros, I.K., George, S., Magreñán, A.A., Local convergence for multi-point-parametric Chebyshev-Halley-type method of higher convergence order, *J. Comput. Appl. Math.*, 282, 2015, 215-224.

[7] Argyros, I.K., George, S., On the complexity of extending the convergence region for Traubs method, *Journal of Complexity*, 56, 101423.

[8] Jaiswal, J.P., Semilocal convergence of an eighth-order method in Banach spaces and its computational efficiency, *Numer. Algorithms*, 71, (2016), 933-951.

[9] Sharma, J.R, Arora, H., Efficient Jarratt-like method for solving systems of nonlinear equations, *Calcolo*, 51, 1, (2014), 193-210.

Chapter 18

On the Semi-Local Convergence of a Derivative Free Fourth Order Method for Solving Equations

1. Introduction

The semi-local convergence is developed for fourth convergence order methods for solving the nonlinear equation

$$F(x) = 0, \tag{18.1}$$

where $F : D \subset B \longrightarrow B$ is continuously Fréchet differentiable, B is a Banach spaces, and D is a nonempty convex set.

The methods under consideration in this chapter are defined by

$$\begin{aligned} x_0 \in D,\, y_n &= x_n - [u_n, x_n; F]^{-1} F(x_n), \\ u_n &= x_n + bF(x_n) \\ \text{and} & \\ x_{n+1} &= y_n - (3I - G_n(3I - G_n))[u_n, x_n; F]^{-1} F(y_n), \end{aligned} \tag{18.2}$$

where $a, b \in \mathbb{R}, z_n = y_n + cF(y_n)$, $G_n = [u_n, x_n; F]^{-1}[z_n, y_n; F]$ and $[., .; F] : B \times B \longrightarrow L(B, B)$ is a divided difference of order one [11, 12, 13]. The fourth order of method (18.2) is established in [11, 12, 13] for $B = \mathbb{R}^i$ using the fifth order derivative, not on these methods. The analysis involved local convergence. But we examine the most interesting semi-local convergence analysis by utilizing only the first divided difference appearing on the method and in the more general setting of a Banach space. Hence, the applicability of the method is extended. The technique can be used in other methods [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The notation $U(x, R0$ is used for the open ball in B of center $x \in B$ and of radius $R > 0$.

2. Majorizing Sequence

Let $\eta_0, \eta, \lambda, \beta, \gamma$ and δ be nonnegative parameters. Let also w, w_k, $k = 0, 1, 2$ be continuous and nondecreasing functions. Define sequences

$$\begin{aligned} p_n^1 &= \beta t_n + |b|\eta_0, \\ p_n^2 &= s_n + \gamma t_n + |b|\eta_0, \\ p_n^3 &= \delta s_n + |c|\eta_0, \\ p_n^4 &= p_n^1 + p_n^3, \\ p_n^5 &= |b|\lambda t_n + |b|\eta_0 \end{aligned}$$

and

$$q_n = 1 - w_0(p_n^1, t_n).$$

Moreover, define scalar sequences $\{t_n\}$ and $\{s_n\}$ by

$$\begin{aligned} t_0 &= = 0, s_0 = \eta, \\ t_{n+1} &= s_n + \frac{1}{q_n^2}(3p_n^4 + w_2(p_n^3, s_n))w(p_n^5, t_n, s_n, p_n^1)(s_n - t_n) \end{aligned} \tag{18.3}$$

and

$$s_{n+1} = t_{n+1} + \frac{w(p_n^5, t_n, t_{n+1}, p_n^1)(t_{n+1} - t_n) + w_2(p_n^1, t_n)(t_{n+1} - s_n)}{q_{n+1}}.$$

The sequence defined by formula (18.3) shall be shown to be majorizing for method (18.2).

But first, we presented a convergence result for it.

Lemma 33. Suppose:

$$w_0(p_n^1, t_n) < 1 \text{ and } t_n \le \mu \text{ for some } \mu \ge 0 \quad \forall n = 0, 1, 2, \ldots. \tag{18.4}$$

Then, the sequence $\{t_n\}$is non-decreasing and convergent to its unique least upper bound $t^* \in [0, \mu]$.

Proof. It follow by (18.3) and (18.4) that sequences $\{t_n\}$ is nondecreasing and bounded from above by μ and as such it converges to t^*. □

3. Semi-Local Convergence

The analysis is based on the following conditions:

(h1) There exist $x_0 \in D, \eta \ge 0$ and $\eta \ge 0$ such that $[u_0, x_0; F]^{-1}, F'(x_0)^{-1} \in L(B, B)$ with $\|F'(x_0)\| \le \eta_0$ and $[u_0, x_0; F]^{-1}F(x_0)\| \le \eta$.

(h2)

$$\begin{aligned} \|F'(x_0)^{-1}([u, x : F] - F'(x_0))\| &\le w_0(\|u - x_0\|, \|x - x_0\|)\ \forall x, u \in D, \\ \|[x, x_0; F]\| &\le \lambda, \\ \|I + b[x, x_0; F]\| &\le \beta, \\ \|I - b[x, x_0; F]\| &\le \gamma \end{aligned}$$

and

$$\|I+c[y,x_0;F]\| \leq \delta\, \forall x,y,u,z \in U(x_0,\rho).$$

(h3) Suppose that there exists a smallest solution $\rho \in (0,\infty)$ of equation

$$w_0(\beta t+|b|\eta_0,t)-1=0.$$

Set $T=[0,\rho)$.

(h4)

$$\|F'(x_0)^{-1}([y,x:F]-[u,x;F])\| \leq w(\|y-u\|,\|x-x_0\|,\|y-x_0\|,\|u-x_0\|),$$
$$\|F'(x_0)^{-1}([u,x:F]-[z,y;F])\| \leq w_1(\|u-z\|,\|x-y\|)$$

and

$$\|F'(x_0)^{-1}[z,y:F]\| \leq w_2(\|z-x_0\|,\|y-x_0\|),$$

(h5) Conditions of Lemma 33 hold.
and

(h6) $U[x_0,R] \subset D$, where $R=\max\{t^*,R_1,R_2\}$, with $R_1=\beta t^*+|b|\eta_0$ and $R_2=\delta t^*+|c|\eta_0$.

Next, the semi-local convergence analysis of the method (18.2) is presented using the developed conditions and terminology.

Theorem 19. Suppose that the conditions (h1)-(h6) hold. Then, sequences $\{x_n\}$, is well defined, remain in $U[x_0,R]\, \forall n=0,1,2,\ldots$ and converge to a solution $x^* \in U[x_0,R]$ of equation $F(x)=0$. Moreover, the following error estimates hold $\forall n=0,1,2,\ldots$

$$\|y_n-x_n\| \leq s_n-t_n. \tag{18.5}$$

and

$$\|x_{n+1}-y_n\| \leq t_{n+1}-s_n. \tag{18.6}$$

Proof. It follows by (h1), (h2) and the definition of ρ that

$$\begin{aligned}\|F'(x_0)^{-1}([u_n,x_n;F]-F'(x_0))\| &\leq w_0(\|u_n-x_0\|,\|x_n-x_0\|)\\ &\leq q_n<1 \text{ (by Lemma 33).}\end{aligned}$$

Thus, by the Banach Lemma on invertible operators $[1,2,3]$ $[u_n,x_n;F]^{-1} \in L(B,B)$ and

$$\|[u_n,x_n;F]^{-1}F'(x_0)\| \leq \frac{1}{q_n}, \tag{18.7}$$

where we also used

$$u_n-x_0=x_n-x_0+bF(x_n)=(I+b[x_n,x_0;F])(x_n-x_0)+bF(x_0)$$

so

$$\|u_n-x_0\| \leq p_n^1 \leq R_1 \leq R.$$

Using the first substep of method (18.2), we have

$$\begin{aligned} F(y_n) &= F(y_n) - F(x_n) - [u_n, x_n; F](y_n - x_n) \\ &= ([y_n, x_n; F] - [u_n, x_n; F])(y_n - x_n) \\ &\leq w(p_n^5, t_n, s_n, p_n^1)(s_n - t_n) \end{aligned}$$

and by (h3)

$$\|F'(x_0)^{-1}F(y_n)\| \leq w(\|y_n - x_n\|, \|x_n - x_0\|, \|y_n - x_0\|, \|u_n - x_0\|)\|y_n - x_n\|. \tag{18.8}$$

Moreover, by the second substep of method (18.2) we get

$$x_{n+1} - y_n = (-3(I - G_n)[u_n, x_n; F]^{-1} - G_n^2[u_n, x_n; F]^{-1})F(y_n).$$

The following estimates are needed

$$\begin{aligned} I - G_n &= I - [u_n, x_n; F]^{-1}[z_n, y_n; F] \\ &= [u_n, x_n; F]^{-1}([u_n, x_n; F] - [z_n, y_n; F]). \end{aligned} \tag{18.9}$$

It follows by formula (18.3), (18.7)-(18.9) and (h4) in turn that

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq \left(\frac{3p_n^4}{q_n^2} + \frac{w_2(p_n^3, s_n)}{q_n^3}\right) w(p_n^5, t_n, s_n, p_n^1)(s_n - t_n) \\ &= t_{n+1} - s_n, \end{aligned} \tag{18.10}$$

where we also used $\|y_n - x_n\| \leq s_n - t_n$,

$$\begin{aligned} \|y_n - u_n\| &= \|y_n - x_n + bF(x_n)\| \\ &= \|y_n - x_0 + x_0 - x_n + b(F(x_n) - F(x_0)) + bF(x_0)\| \\ &\leq \|y_n - x_0\| + \|(I - b[x_n, x_0; F])(x_n - x_0)\| + |b|\|F(x_0)\| \\ &\leq s_n + \gamma t_n + |b|\eta_0 = p_n^2, \end{aligned}$$

$$\begin{aligned} \|z_n - x_0\| &= \|y_n - x_0 + c(F(y_n) - F(x_0)) + cF(x_0)\| \\ &\leq \|(I + c[y_n, x_0; F])(y_n - x_0)\| + |c|\|F(x_0)\| \\ &\leq p_n^3 \leq R - 2 \leq R \end{aligned}$$

and

$$\begin{aligned} \|u_n - z_n\| &= \|u_n - x_0 + x_0 - z_n\| \\ &\leq \|u_n - x_0\| + \|z_n - x_0\| \\ &\leq p_n^1 + p_n^3 = p_n^4. \end{aligned}$$

Furthermore, by the first substep of method (18.2) it follows that

$$\begin{aligned} F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - [u_n, x_n; F](y_n - x_n) \\ &= ([x_{n+1}, x_n; F] - [u_n, x_n; F])(x_{n+1} - x_n)) \\ &\quad + [u_n, x_n; F](x_{n+1} - y_n), \end{aligned}$$

so

$$\begin{aligned}\|F'(x_0)^{-1}F(x_{n+1})\| &\leq w(p_n^5, t_n, t_{n+1}, p_n^1)(t_{n+1}-t_n)\\ &\quad + w_2(p_n 61, t_n)(t_{n+1}-t_n). \end{aligned} \tag{18.11}$$

Consequently, we obtain

$$\begin{aligned}\|y_{n+1}-x_{n+1}\| &\leq \|[u_{n+1}, x_{n+1}; F]^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\|\\ &= s_{n+1}-t_{n+1}. \end{aligned} \tag{18.12}$$

Hence, sequence $\{x_n\}$ is fundamental and sequence $\{t_n\}$ is fundamental too as convergent. That is there exists $x_* \in U[x_0, t_*]$ such that $\lim_{n\longrightarrow\infty} x_n = x^*$. By letting $n \longrightarrow \infty$ in (18.11) and using the continuity of operator F we deduce $F(x_*) = 0$. □

Next, we present a uniqueness result for the solution of equation $F(x) = 0$.

Proposition 13. *Suppose:*

(1) There exists a solution $x^* \in U(x_0, \rho_1)$ *for some* $\rho_1 > 0$ *of equation* $F(x) = 0$.

(2) Condition (h2) holds.

(3) There exists $\rho_2 \geq \rho_1$ *such that*

$$w_0(\rho_2, \rho_1) < 1. \tag{18.13}$$

Set $D_2 = U[x_0, \rho_2] \cap D$. *Then, the only solution of equation* $F(x) = 0$ *in the region* D_2 *is the point* x^*.

Proof. Let $y^* \in D_2$ be such that $F(y^*) = 0$. Define the linear operator $S = [y^*, x^*; F]$. It follows by the conditions (h2) and (18.13) that

$$\begin{aligned}\|F'(x_0)^{-1}(S - F'(x_0))\| &\leq w_0(\|y^*-x_0\|, \|x_*-x_0\|)\\ &\leq w_0(\rho_2, \rho_1) < 1.\end{aligned}$$

Hence, we conclude that $x^* = y^*$ by the invertibility of the linear operator S and the identity

$$S(y^*-x^*) = F(y^*) - F(x^*) = 0 - 0 = 0.$$

□

Remark. Notice that not all conditions of Theorem 19 hold. But if we assume all conditions of Theorem 19, then we can set $\rho_1 = t^*$.

Bibliography

[1] Abbasbandy, S., Bakhtiari, P., Cordero, A., Torregrosa, J.R., Lofti, T., New efficient methods for solving nonlinear systems of equations with arbitrary even order, *Appl. Math. Comput.*, 287, 288, (2016), 94-103.

[2] Argyros, I.K., A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach spaces, *J. Math. Anal. Appl.*, 298, (2004), 374-397.

[3] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[4] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics*, 2021, 9, (16), 1942; https://doi.org/10.3390/math9161942.

[5] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[6] Argyros, I.K., *Advances in the Efficiency of Computational Methods and Applications*, World Scientific Publishing Oxon, United Kingdom (2000).

[7] Argyros, I.K., George, S., *Mathematical modeling for the solution of equations and systems of equations with applications*, Volume-II, Nova Science Publishers, NY, 2018.

[8] Argyros, M.I., Argyros, I.K., Regmi, S., George, S., Generalized Three-Step Numerical Methods for Solving Equations in Banach Spaces, *Mathematics*, 2022, 10, 2621. https://doi.org/10.3390/math10152621.

[9] Argyros, I.K., George, S., Magreñán, A.A., Local convergence for multi-point-parametric Chebyshev-Halley-type method of higher convergence order, *J. Comput. Appl. Math.*, 282, 2015, 215-224.

[10] Argyros, I.K., George, S., On the complexity of extending the convergence region for Traubs method, *Journal of Complexity*, 56, 101423.

[11] Grau-Sánchez, M., Grau, A., Noguera, M., Ostrowski type methods for solving systems of nonlinear equations, *Appl. Math. Comput.*, 218, 2011, 23772385.

[12] Grau-Sánchez, M., Noguera, M., A technique to choose the most efficient method between secant method and some variants, *Appl Math. Comput.*, 218, 2012, 64156426.

[13] Grau-Sánchez M., Grau, A., Noguera, M., Frozen divided difference scheme for solving systems of nonlinear equations, *J. Comput. Appl. Math.*, 235, 2011, 17391743.

[14] Jaiswal, J.P., Semilocal convergence of an eighth-order method in Banach spaces and its computational efficiency, *Numer. Algorithms*, 71, (2016), 933-951.

[15] Sharma, J.R., Arora, H., Efficient Jarratt-like method for solving systems of nonlinear equations, *Calcolo*, 51, 1, (2014), 193-210.

Chapter 19

Extended and Unified Kantorovich Theory for Solving Generalized Nonlinear Equations

1. Introduction

Let B_1 and B_2 stand for Banach spaces; $L(B_1,B_2)$ be the space of bounded linear operators mapping B_1 into B_2; $F : B_1 \rightrightarrows B_2$ be a multi-operator and $G : B_1 \rightrightarrows B_2$ be a differentiable operator in the Frechet sense. Numerous problems from diverse disciplines such as optimization, Mathematical Biology, Chemistry, Economics, Physics, Programming, Scientific Computing and also Engineering to mention a few can be written in the form of generalized nonlinear equation (GNE)

$$0 \in F(x)+G(x). \tag{19.1}$$

Equation (19.1) specializes to two popular cases:

(a) $F \equiv 0$ leading to the generalized nonlinear equation [1,2,3,4,5,6,7,8,9,10].

$$G(x) = 0.$$

(b) F is N_K. That is the normal cone of convex and closed subset $K \subset B_1$. In this case GNE is called a variational inequality problem (VIP) [11,12,13,14,15,16,17,18].

A solution x^* of GNE is desired in closed form. But this is possible only in special cases. That is why most solution methods of GNE are iterative when a sequence is generated converging to x^* under certain conditions [1,2,3,4].

A generalized iterative scheme (GIS) is

$$0 \in G(x_n)+T_n(x_{n+1}-x_n)+F(x_{n+1})\ \forall n = 0,1,2,\ldots, \tag{19.2}$$

where $A_n = A(x_n), A : B_1 \longrightarrow L(B_1,B_2)$. Linear operator A_n is an approximation to the Fréchet derivative F' of operator F.

If $A(x) = F'(x)$ GIS reduces to a scheme studied by many authors [1,2,4,18] under Lipschitz-type conditions on F' and various techniques. GIS was also studied in [2] under

Hölder-type conditions. But there are even simple examples [3,4] where operators F' or A do not satisfy Lipschitz-type or Hölder-type conditions. That is why a semi-local convergence is presented using more general conditions on F' and A as well as a majorizing sequence. This way the applicability of GIS is extended.

The background is given in Section 2 followed by the properties of majorizing sequences in Section 3, and the semi-local convergence of GIS in Section 4.

2. Mathematical Background

Denote by $U(x,\gamma)$ the open ball with center $x \in B_1$ and of radius $\gamma > 0$. Let also $G : D \subset B_1 \longrightarrow B_2$, where D is an open set. The following standard definitions are needed:

$$gph F = \{(u_1,u_2) \in B_1 \times B_2 : u_2 \in F(u_1)\};$$

$$dom F = \{u \in B_1 : F(u) \neq 0\};$$

$$rge F = \{v \in B_2 : \exists v = F(u) \text{ for some } u \in B_1\};$$

$F^{-1} : B_2 \longrightarrow B_1$ is such that

$$F^{-1}(u_2) = \{u_1 \in B_1 : u_2 = F(u_1)\};$$

for $C_1 \subset B_1$ and $C_2 \subset B_1$,

$$d(x,C_2) = \inf_{u_2 \in C_2} \|u_1 - u_2\|$$

and

$$e(C_1,C_2) = \sup_{u_1 \in C_1} d(u_1,C_2).$$

The following conventions are used:

$$d(x,C_2) = +\infty \text{ for } C_2 = \emptyset,$$

$$e(\emptyset,C_2) = 0 \text{ for } C_2 \neq 0$$

and

$$e(\emptyset,\emptyset) = +\infty.$$

Definition 4. [2] A multifunction $S : B_2 \rightrightarrows B_1$ is Aubin continuous at $(u,v) \in gph(S)$ with parameter $\beta > 0$ if $\exists$ parameters $\delta_1 > 0$ and $\delta_2 > 0$ such that neighborhoods M_1 and M_2 of v satisfy $\forall w_1, w_2 \in U(v,\delta_2)$

$$e(S(w_1) \cap U(u,\delta_1)) \leq \beta \|u - v\|.$$

Theorem 20. [2] Let $M : B_1 \rightrightarrows B_2$ be a multifunction and point $u \in B_1$. Suppose that $\exists$ parameter $r > 0$ and $\tau \in (0,1)$ such that the set $gph M \cap (U[u,r] \times U[u,r])$ is closed and the following conditions hold:

(a) $d(u,M(u)) \leq R(1-\tau)$
and

(b) $\forall p_1, p_2 \in U[u,r]$

$$e(M(p_1) \cap U[u,r], M(p_2)) \leq \tau \|p_2 - p_1\|.$$

Then, operator M has a fixed point in $U[u,R]$, i.e., $\exists u_0 \in U[u,R]$ such that $u_0 \in M(u_0)$.

3. Majorizing Sequence

Let us define the scalar sequence $\{t_n\}$ for some continuous functions $\varphi : [0,\infty) \longrightarrow (-\infty,\infty)$, $\psi_0 : [0,\infty) \longrightarrow (-\infty,\infty)$ and $\psi : [0,\infty) \longrightarrow (-\infty,\infty)$ and some $\lambda > 0$ by

$$\begin{aligned} t_0 &= 0,\ t_1 = \lambda \\ \text{and} \qquad\qquad & \\ t_{n+2} &= t_{n+1} + \frac{\beta(\int_0^1 \varphi((1-\theta)(t_{n+1}-t_n))d\theta + \psi(t_n))(t_{n+1}-t_n)}{1-\beta(\psi(0)+\psi_0(t_{n+1}))}, \end{aligned} \tag{19.3}$$

$\forall n = 0,1,2,\ldots$. Sequence $\{t_n\}$ shall be shown to be majorizing for sequence $\{x_n\}$ in the next section. But first, some convergence results for it are needed.

Lemma 34. Suppose that condition

$$\beta(\psi(0)+\psi(t_{n+1})) < 1\ \forall n = 0,1,2\ldots \tag{19.4}$$

holds. Then, sequence $\{t_n\}$ defined by formula (19.3) is non-decreasing. Moreover, suppose

$$t_n \le \mu \text{ for some } \mu > 0. \tag{19.5}$$

Then, sequence $\{t_n\}$ converges to its unique least upper bound $t_* \in [0,\mu]$.

Proof. It follows by formula (19.3) and conditions (19.4), (19.5) that sequence $\{t_n\}$ is non-decreasing and bounded from above by μ, and as such it converges to t^*. □

Next, a second and strong convergence result is provided.

Lemma 35. Suppose that there exists parameter $\alpha \in (0,1)$ so that

$$0 \le \frac{\beta(\int_0^1 \varphi((1-\theta)\lambda)d\theta + \psi(0))}{1-\beta(\psi(0)+\psi_0(\lambda))} \le \alpha \tag{19.6}$$

and

$$0 \le \frac{\beta(\int_0^1 \varphi((1-\theta)\lambda)d\theta + \psi(\frac{\lambda}{1-\alpha}))}{1-\beta(\psi(0)+\psi_0(\frac{\lambda}{1-\alpha}))} \le \alpha. \tag{19.7}$$

Then, sequence $\{t_n\}$ is non-decreasing, bounded from above by $\dfrac{\lambda}{1-\alpha}$ and converges to its unique least upper bound t_* such that $t_* \in [0, \dfrac{\lambda}{1-\alpha}]$.

Proof. Mathematical induction is used to show

$$0 \le t_{n+1}-t_n \le \alpha(t_n - t_{n-1})\ \forall n = 1,2,\ldots. \tag{19.8}$$

Assertion (19.8) holds for $n = 1$ by formula (19.3) and condition (19.6). Assume

$$0 \le t_{k+1}-t_k \le \alpha(t_k-t_{k-1}) \le \alpha^k(t_1-t_0) = \alpha^k\lambda \tag{19.9}$$

holds for all integer values k smaller than n. then, it follows from (19.9)

$$\begin{aligned} t_{k+1} &= t_k+\alpha^k\lambda \leq t_{k-1}+\alpha^{k-1}\lambda+\alpha^k\lambda \\ &\leq \ldots \leq (1+\alpha\lambda+\ldots+\alpha^k)\lambda = \frac{1-\alpha^{k+1}}{1-\alpha}\lambda < \frac{\lambda}{1-\alpha}. \end{aligned}$$

Evidently (19.8) holds by condition (19.7). Hence, the induction for (19.8) is completed. It follows that sequence $\{t_k\}$ is non-decreasing and bounded from above by $\frac{\lambda}{1-\alpha}$ and as such it converges to its unique least upper bound $t_* \in [0, \frac{\lambda}{1-\alpha}]$. □

4. Main Result

The semi-local convergence of GIS is provided. Let $x_0 \in B_1$ and $\lambda \geq 0$ be given, $G : \Omega_0 \subseteq B_1 \longrightarrow B_2$ is Frechet-differentiable and F has a closed graph.

Theorem 21. Suppose:

(a) $\exists$ continuous functions $\varphi : [0,\infty) \longrightarrow (-\infty,\infty)$, $\psi_0 : [0,\infty) \longrightarrow (-\infty,\infty)$, $\psi : [0,\infty) \longrightarrow (-\infty,\infty)$ such that $\forall x, y \in \Omega_0$

$$\|G'(y)-G'(x)\| \leq \varphi(\|y-x\|), \tag{19.10}$$

$$\|T(x)-T(x_0)\| \leq \psi_0(\|x-x_0\|) \tag{19.11}$$

and

$$\|G'(x)-T(x)\| \leq \psi(\|x-x_0\|). \tag{19.12}$$

(b) $\exists x_1 \in \Omega_0$ defined by GIS such that $\|x_1-x_0\| \leq \lambda$ and the multifunction $(G(x_0)+G'(x_0)(.-x_0)+F(.))^{-1}$ is Aubin continuous at $(0,x_1)$ with corresponding radii δ_1 and δ_2 and modulus $\beta > 0$.

(c) Conditions of Lemma 35 hold.

(d)

$$2s-\lambda \leq \delta_1, \tag{19.13}$$

$$(2\int_0^1 \varphi(2(1-\theta)s)d\theta + \int_0^1 \varphi((1-\theta)s)d\theta + \psi(s))s \leq \delta_2, \tag{19.14}$$

$$\beta(\psi(0)+\psi_0(s)) < 1 \tag{19.15}$$

and

$$s(\alpha-1)+\lambda \leq 0. \tag{19.16}$$

Then, the sequence $\{x_n\}$ generated by GIS is well defined in $U(x_0,t_*)$ remains in $U(x_0,t_*)$ $\forall n = 0,1,2,\ldots$ and converges to a point $x_* \in U[x_0,t_*]$ such that $0 \in G(x_*)+F(x_*)$.

The following auxiliary result is needed.

Lemma 36. Suppose that conditions of Theorem 21 hold. Then, if $\{x_m\}$ and $\{t_m\}$ are the sequences generated by GIS and (19.3), respectively. Then, the following assertion holds.

$$\|x_{m=1} - x_m\| \leq t_{m+1} - t_n \ \forall m = 0, 1, 2, \ldots. \tag{19.17}$$

Proof. We prove (19.17) by induction. Since $t_0 = 0$ and $t_1 = \lambda$, by condition (b), we have $\|x_1 - x_0\| = \lambda \leq t_1 - t_0$. Suppose $\exists x_1, x_2, \ldots, x_m$ obtained by GIS such that

$$\|x_n - x_{n-1}\| \leq t_n - t_{n-1} \ \forall n = 0, 1, \ldots m-1. \tag{19.18}$$

Hence, if $n = 0, 1, 2, \ldots m$ then

$$\|x_n - x_0\| \leq \sum_{j=0}^{n-1} \|x_{j+1} - x_j\| \leq \sum_{j=0}^{n-1} t_{j=1} - t_j = t_n \leq t_*, \tag{19.19}$$

so

$$\|x_n - x_1\| \leq \sum_{j=1}^{n-1} t_{j+1} - t_j \leq t_n - t_1 \leq t_* - \lambda. \tag{19.20}$$

Therefore, $\forall x \in U(x_m, \|x_m - x_0\|)$, we have

$$\|x - x_1\| \leq \|x - x_m\| + \|x_m - x_1\| \leq 2t_* - \lambda \leq \delta_1. \tag{19.21}$$

Let $p_0(x) = f(x_0) + f'(x_0)(x - x_0) + F(x)$. Define the multifunction

$$\Phi_m(x) = p_0^{-1}[f(x_0) + f'(x_0)(x - x_0) - f(x_m) - T_m(x - x_m)].$$

Next, we check all the conditions in Theorem 21. First, we have from the definition of the method that

$$\begin{aligned}
& \|f(x_0) + f'(x_0)(x - x_0) - f(x_m) - T_m(x - x_m)\| \\
\leq\ & \|f(x) - f(x_0) - f'(x_0)(x - x_0)\| \\
& + \|f(x) - f(x_m) - f'(x_m)(x - x_m)\| \\
& + \|(f'(x_m) - T_m)(x - x_m)\| \\
\leq\ & \int_0^1 \varphi((1-\theta)\|x - x_0\|)d\theta\|x - x_0\| \\
& + \int_0^1 \varphi((1-\theta)\|x - x_m\|)d\theta\|x - x_m\| \\
& + \psi(\|x_m - x_0\|)\|x - x_m\|.
\end{aligned}$$

It follows that $\forall x \in U(x_m, \|x_m - x_0\|)$

$$\begin{aligned}
& \|f(x_0) + f'(x_0)(x - x_0) - f(x_m) - T_m(x - x_m)\| \\
\leq\ & 2\int_0^1 \varphi(2(1-\theta)\|x_m - x_0\|)d\theta\|x_m - x_0\| \\
& + \int_0^1 \varphi((1-\theta)\|x_m - x_0\|)d\theta\|x_m - x_0\| \\
& + \psi(\|x_m - x_0\|)\|x_m - x_0\|.
\end{aligned}$$

Hence, by (19.19) for $n = m$ it follows

$$\begin{aligned}\|f(x_0)+f'(x_0)(x-x_0)-f(x_m)-T_m(x-x_m)\| &\leq (2\int_0^1 \varphi(2(1-\theta)t_*)d\theta \\ &+\int_0^1 \varphi((1-\theta)t_*)d\theta+\psi(t_*))t_*\end{aligned}$$

for $\|x_m - x_0\| \leq t_*$. Therefore, since $t-* \leq s$, we get

$$\begin{aligned}\|f(x_0)+f'(x_0)(x-x_0)-f(x_m)-T_m(x-x_m)\| &\leq (2\int_0^1 \varphi(2(1-\theta)s)d\theta \\ &+\int_0^1 \varphi((1-\theta)s)d\theta\psi(s))s \leq \delta_2.\end{aligned}$$

Second, we note that $x_m \in p_0^{-1}[f(x_0)+f'(x_0)(x_m-x_0)-f(x_{m-1})-T_{m-1}(x_m-x_{m-1})]$. Using the Aubin property of $p_0^{-1}(.)$ at $(0,x_1)$ with modulus κ and constants δ_1 and δ_2, we obtain

$$\begin{aligned}d(x_m,\Phi_m(x_m)) &\leq e\{p_0^{-1}[f(x_0)+f'(x_0)(x_m-x_0)-f(x_{m-1})-T_{m-1}(x_m-x_{m-1})] \\ &\quad \cap U(x_1,\delta_1),\Phi_m(x_m)\} \\ &\leq \kappa\|f(x_m)-f(x_{m-1})-T_{m-1}(x_m-x_{m-1})] \\ &\leq \kappa\|(f(x_{m-1})-T_{m-1})(x_m-x_{m-1}))\| \\ &\leq \beta(\int_0^1 \varphi((1-\theta)(t_m-t_{m-1}))d\theta+\psi(t_{m-1}))\|x_m-x_{m-1}\| \\ &= R(1-\beta(\psi(0)+\psi_0(t_m))),\end{aligned}$$

where $R = \dfrac{\beta(\int_0^1 \varphi((1-\theta)(t_m-t_{m-1}))d\theta+\psi(t_{m-1}))}{1-\beta(\psi(0)+\psi_0(t_m)))}\|x_m - x_{m-1}\|$. But, if $p,q \in U(x_m,\|x_m-x_0\|)$ we can write

$$\begin{aligned}e\{\Phi_m(p)\cap U(x_m,\|x_m-x_0\|),\Phi_m(q)\} &\leq e\{\Phi_m(p)\cap U(x_m,\delta_1),\Phi_m(q)\} \\ &\leq \beta\|(f'(x_0)-T_m)(p-q)\| \\ &\leq \beta(\|f'(x_0)-T_0\|+\|T_0-T_m\|)\|p-q\| \\ &\leq \beta(\psi(0)+\psi_0(\|x_m-x_0\|))\|p-q\|.\end{aligned}$$

In view of $\|x_m-x_0\| \leq t_* \leq s$ and $\beta(\psi(0)+\psi_0(\|x_m-x_0\|)) < 1$, the application of Theorem 21 with $\Phi = \Phi_m$, $\bar{x} = x_*$ and $\tau = \beta(\psi(0)+\psi_0(\|x_m-x_0\|))$ to deduce that $\exists x_{m+1} \in U[x_*,R]$ such that

$$x_{m+1} \in p_0^{-1}[f(x_0)+f'(x_0)(x_{m+1}-x_0)-f(x_m)-T_m(x_{m+1}-x_m)].$$

Thus, x_{m+1} is a Newton iterative. It also follows by the induction hypotheses

$$\begin{aligned}\|x_{m+1}-x_m\| &\leq \frac{\beta(\int_0^1 \varphi((1-\theta)\|x_m-x_{m-1}\|)d\theta+\psi(\|x_{m-1}-x_0\|)}{1-(\psi(0)+\psi_0(\|x_m-x_0\|))}\|x_m-x_{m-1}\| \\ &\leq \frac{\beta(\int_0^1 \varphi((1-\theta)(t_m-t_{m-1}))d\theta+\psi(t_{m-1}))}{1-\beta(\psi(0)+\psi_0(t_m)))}(t_m-t_{m-1}).\end{aligned}$$

Thus, we have

$$\begin{aligned} \|x_{m+1}-x_m\| &\leq \\ &\leq \\ &= t_{m+1}-t_m. \end{aligned}$$

Proof of Theorem 21 Using Lemma 36, sequence $\{t_m\}$ converges to t_*. Moreovere, it follows by Lemma 36

$$\sum_{m=m_0}^{\infty} \|x_{m+1}-x_m\| \leq \sum_{m=m_0}^{\infty} (t_{m+1}-t_m) = t_* - t_m < +\infty,$$

for any $m_0 \in \mathbb{N}$. Hence, $\{x_m\}$ is a complete sequence in $U(x_0,t_*)$ and as such it converges to some $x_* \in U[x_0,t_*]$. Consequently,

$$x_* - x_m\| \leq t_* - t_m.$$

By the definition of $\{x_m\}$ in GIS we get

$$0 \in f(x_m) + A(x_m)(x_{m+1}-x_m) + F(x_{m=1}) \; \forall m = 0,1,\ldots.$$

By letting $m \longrightarrow \infty$ we can conclude that

$$0 \in f(x_*) + F(x_*), \; x_* \in U(x_0,t_*).$$

□

Proposition 14. *Suppose conditions of Theorem 21 hold. Then, the following assertions hold*

$$\|x_n - x_*\| \leq \frac{\alpha}{1-\alpha}(t_n - t_{n-1}) \; \forall n = 0,1,\ldots$$

and

$$\|x_n - x_*\| \leq \frac{\lambda}{1-\alpha}\alpha^n \; \forall n = 0,1,\ldots.$$

Proof. By the proof of Lemma 36, we have

$$\|x_{m+1}-x_m\| \leq t_{m+1}-t_m \leq \alpha(t_m - t_{m-1} \leq \alpha^m \lambda,$$

we show that

$$\|x_{k+n}-x_n\| \leq \alpha(1+\alpha+\ldots+\alpha^{k-1})(t_n - t_{n-1}).$$

It follows that since $\alpha < 1$

$$\|x_{k+n}-x_n\| \leq \frac{\alpha}{1-\alpha}(t_n - t_{n-1}).$$

By letting $k \longrightarrow +\infty$, the proof is complete. □

Bibliography

[1] Adly. S., Van Ngai, H. et al., Newton's method for solving generalized equations: Kantorovich's and smale's approaches, *Journal of Mathematical Analysis and Applications*, 439, (1), (2016), 396-418.

[2] Amaral, V.S., Santos, P.S.M., Silva, E.N, Souza, S.S., A Newton-Kantorovich method for solving generalized equations, *J. Comput. Appl. Math.*, (2021).

[3] Argyros, I.K., A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach spaces, *J. Math. Anal. Appl.*, 298, (2004), 374-397.

[4] Argyros, I.K., *The theory and applications of iteration methods, 2nd Edition, Engineering Series*, CRC Press, Taylor and Francis Group, 2022.

[5] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics*, 2021, 9, (16), 1942; https://doi.org/10.3390/math9161942.

[6] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[7] Argyros, I.K., Hilout. S, On the convergence of Newton-type methods under mild differentiability conditions, *Numerical Algorithms*, 52, (4), (2009), 701-726.

[8] Daniel, J.W., Newton's method for nonlinear inequalities, *Numerische Mathematik*, 21, (5), (1973), 381-387.

[9] Dontchev, A.L, Rockafellar. R.T., *Implicit functions and solution mappings*, Vol. 543, Springer, 2009.

[10] Dontchev, A.L., Local analysis of a newton-type method based on partial linearization, *Lectures in Applied Mathematics - American Mathematical Society*, 32, (1996), 295-306.

[11] Josephy, N.H, *Newton's method for generalized equations and the pies energy model*, (1980).

[12] Kantorovich. L, Akilov. G, Functional analysis in normed spaces, *International series of monographs in pure and applied mathematics*, Vol. 46, xiii, 773, (1964).

[13] Marcotte, P., Dussault. J.P., A note on a globally convergent newton method for solving monotone variational inequalities, *Operations Research Letters*, 6, (1), (1987), 35-42.

[14] Ortega, J.M., Rheinboldt. W.C., Iterative solution of nonlinear equations in several variables, *SIAM*, 2000.

[15] Robinson, S.M., Strongly regular generalized equations, Mathematics of Operations Research, 5, (1), (1980), 43-62.

[16] Regmi, S., *Optimized Iterative Methods with Applications in Diverse Disciplines*, Nova Science Publishers, New York, 2021.

[17] Rokne, J., Newton's method under mild differentiability conditions with error analysis, *Numerische Mathematik*, 18, (5), (1971), 401-412.

[18] Robinson, S.M., Extension of newton's method to nonlinear functions with values in a cone, *Numerische Mathematik*, 19, (4), (1972), 341-347.

Chapter 20

A Derivative Free Two-Step Fourth Order Method for Solving Equations in Banach Space

1. Introduction

Let B be a Banach space and $D \subset B$ be an open set. We are concerned with the problem of approximating a locally unique solution $x^* \in D$ of equation

$$F(x) = 0, \tag{20.1}$$

where $F : D \subset B \longrightarrow B$ is a Fréchet differentiable, operator. The solution x^* is being approximated by the method

$$\begin{aligned} y_n &= x_n - [u_n, x_n; F]^{-1} F(x_n) \\ \text{and} & \\ x_{n+1} &= y_n + (aI + M_n(3-2a)I + (a-2)M_n))[u_n, x_n; F]^{-1} F(y_n), \end{aligned} \tag{20.2}$$

where $a, b, c \in \mathbb{R}, u_n = x_n + bF(x_n), z_n = y_n + cF(y_n), M_n = [u_n, x_n; F]^{-1}[z_n, y_n; F]$ and $[., .; F] : D \times D \longrightarrow L(B, B)$ is a divided difference of order one [1,2,3,4,5,6,7,8,9,10]. The fourth convergence order of method (20.2) was shown in [15] under hypotheses on the fifth derivative of operator F when $B = \mathbb{R}^k$. But we extend the applicability of method (20.2) in a Banach space setting and using any conditions on the divided difference of order one that only appears on the method (20.2).

2. Local Convergence Analysis

Let a, b, c and δ be parameters. It is convenient to define some scalar functions and parameters. Set $T = [0, \infty)$.

Suppose that there exists:

(1) The function $w_0 : T \times T \longrightarrow \mathbb{R}$ is continuous and nondecreasing such that equation $q(t) = 0$, where

$$q(t) = w_0((1+|b|\delta(1+w_0(t),0))t,t) - 1$$

has a smallest solution $\rho \in T - \{0\}$. Set $T_1 = [0,\rho)$.

(2) The function $w : T_1 \times T_1 \longrightarrow \mathbb{R}$ is continuous and decreasing such that equation

$$g_1(t) - 1 = 0$$

has a smallest solution $r_1 \in T - 1 - \{0\}$, where

$$g_1(t) = \frac{w(|b|\delta(1+w_0(t,0))t,t)}{q(t)}.$$

(3) The functions $w_1 : T_1 \times T_1 \longrightarrow \mathbb{R}$, $w_2 : T_1 \times T_1 \longrightarrow \mathbb{R}$ are continuous and nondecreasing such that equation

$$g_2(t) - 1 = 0$$

has a smallest solution $r_2 \in T_1 - \{0\}$, where

$$g_2(t) = \left(1 + \frac{\gamma(t)w_2(g_2(t)t,0)}{q(t)}\right)g_1(t),$$

$$\begin{aligned}\gamma(t) \quad = \quad & \frac{|a|w_1(\beta_1(t),\beta_2(t))}{q(t)} \\ & + \frac{w_2(\beta_3(t),(1+g_1(t)t)}{q(t)} \\ & + \frac{|a-2|w_2(\beta_3(t),(1+g_1(t)t)w_1(\beta_1(t),\beta_2(t))}{q^2(t)},\end{aligned}$$

$$\begin{aligned}\beta_1(t) \quad &= \quad |b|\delta w_2(t,(1+g_1(t)t))(1+g_1(t))t \\ & \quad\quad |b-c|\delta w_2((1+g_1(t))t,0)g_2(t)t, \\ \beta_2(t) \quad &= \quad (1+g_1(t))t\end{aligned}$$

and

$$\beta_3(t) = (1+|c|\delta w_2((1+g_1(t)t,0))g_2(t)t.$$

The parameter r defined by

$$r = \min\{r_i\},\ i = 1,2 \tag{20.3}$$

shall be shown to be a radius of convergence for method (20.2). Set $T_2 = [0,r)$. It follows from these definitions that

$$q(t) < 0 \tag{20.4}$$

and

$$g_i(t) < 1 \tag{20.5}$$

for each $t \in T_2$.

Let $U(x,\xi)$ stand for the open ball with center $x \in B$ and of radius $\xi > 0$. Then, the set $U[x,\xi]$ denotes the corresponding closed ball.

The following conditions relate operators to scalar functions. Suppose:

(H1) There exist a solution $x^* \in D$, $\delta > 0$ such that $F'(x^*)^{-1} \in L(B,B)$ and $\|F'(x^*)\| \le \delta$.

(H2) $\|F'(x^*)^{-1}([u,x;F] - F'(x^*))\| \le w_0(\|u-x^*\|, \|x-x^*\|)$ for each $x,u \in D$.
Set $U_1 = U(x^*,\rho) \cap D$.

(H3)

$$\|F'(x^*)^{-1}([u,x;F] - [x,x^*;F])\| \le w(\|u-x\|, \|x-x^*\|),$$

$$\|F'(x^*)^{-1}([x,y;F] - [z,w;F])\| \le w_1(\|x-z\|, \|y-w\|)$$

and

$$\|F'(x^*)^{-1}[x,y;F]\| \le w_2(\|x-x^*\|, \|y-x^*\|).$$

and

(H4) $U[x^*,R] \subset D$, where $R_1 = (c+|b|\delta(1+w_0(r,0)))r$, $R_2 = (c+|b|\delta w_2(g_1(r)r,0))g_1(r)r$ and $R = \max\{R_1,R_2,r\}$.

But first, it is convenient to define items (if they exist)

$$\begin{aligned}
\alpha_n^1 &= |b|\delta w_2(\|x_n-x^*\|, \|y_n-x^*\|)(1+g-1(\|x_n-x^*\|)\|x_n-x^*\| \\
&\quad +|b-c|\delta w_2(\|y_n-x^*\|,0)g_2(\|x_n-x^*\|)\|x_n-x^*\|, \\
\alpha_n^2 &= (1+g_1(\|x_n-x^*\|)\|x_n-x^*\|, \\
\alpha_n^3 &= (1+|c|\delta w_2(\|y_n-x^*\|,0))\|y_n-x^*\|
\end{aligned}$$

and

$$\begin{aligned}
\gamma_n &= \frac{|a|w_1(\alpha_n^1,\alpha_n^2)}{1-w_0(\|u_n-x^*\|, \|x_n-x^*\|)} \\
&\quad + \frac{w_2(\alpha_n^3,(1+g_1(\|x_n-x^*\|))\|x_n-x^*\|}{1-w_0(\|u_n-x^*\|, \|x_n-x^*\|)} \\
&\quad + \frac{|a-2|w_2(\alpha_n^3,(1+g_1(\|x_n-x^*\|))\|x_n-x^*\|)w_1(\alpha_n^1,\alpha_n 62)}{1-w_0(\|u_n-x^*\|, \|x_n-x^*\|)}.
\end{aligned}$$

Moreover, notice that after some algebraic manipulation linear operator A_n can be rewritten as

$$\begin{aligned}
A_n &= a[u_n,x_n;F]^{-1}([u_n,x_n;F] - [z_n,y_n;F]) \\
&\quad +[u_n,x_n;F]^{-1}[z_n,y_n;F] + (a-2)[u_n,x_n;F]^{-1} \\
&\quad \times [z_n,y_n;F][u_n,x_n;F]^{-1}([z_n,y_n;F] - [u_n,x_n;F]), \\
u_n - x^* &= x_n - x^* + bF(x_n), \\
u_n - x_n &= bF(x_n) = b[x_n,x^*;F](x_n-x^*) \\
&= bF'(x^*)F'(x^*)^{-1}([x_n,x^*;F] - F'(x^*) + F'(x^*))(x_n-x^*)
\end{aligned}$$

and

$$\begin{aligned} u_n - z_n &= b(F(x_n) - F(y_n)) + (b-c)F(y_n) \\ &= bF'(x^*)F'(x^*)^{-1}[x_n, y_n; F](x_n - y_n) \\ &\quad +(b-c)F'(x^*)F'(x^*)^{-1}[y_n, x^*; F](y_n - x^*). \end{aligned}$$

Theorem 22. Under conditions (H1)-(H4), sequence $\{x_n\}$ generated by method (20.2) is well defined in $U(x^*, r)$ and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r \tag{20.6}$$

and

$$\|x_{n+1} - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{20.7}$$

where the functions g_i are defined previously and the radius r is given by formula (20.3).

Proof. It follows by (H1), (H2), (20.3) and (20.4) that

$$\begin{aligned} & \|F'(x^*)^{-1}([u_n, x_n; F] - F'(x^*))\| \\ \leq\ & w_0(\|u_n - x^*\|, \|x_n - x^*\|) \\ \leq\ & w_0((1+|b|\delta(1+w_0(\|x_n - x^*\|, 0))\|x_n - x^*\|, \|x_n - x^*\|) \\ \leq\ & w_0((1+|b|\delta(1+w_0(r,0))r, r) < 1 \end{aligned}$$

so

$$\|[u_n, x_n; F]^{-1}F'(x^*)\| \leq \frac{1}{q_n} \tag{20.8}$$

(by the Banach Lemma on invertible operators [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]), where we also used that $\|u_n - x^*\| \leq (1+|b|\delta(1+w_0(r,0))r \leq R$, so $u_n \in U[x^*, R]$. Hence, iterate y_n is well defined. Then, we can write by the first substep of method (20.2)

$$\begin{aligned} y_n - x^* &= x_n - x^* - [u_n, x_n; F]^{-1}F(x_n) \\ &= [u_n, x_n; F]^{-1}([u_n, x_n; F] - [x_n, x^*; F])(x_n - x^*). \end{aligned} \tag{20.9}$$

By (20.3), (20.5), (H3), (20.7) and (20.8), we get in turn that

$$\begin{aligned} \|y_n - x^*\| &\leq \frac{w(|b|\delta(1+w_0(\|x_n - x^*\|, 0))\|x_n - x^*\|, \|x_n - x^*\|)\|x_n - x^*\|}{q_n} \\ &\leq \|x_n - x^*\| < r. \end{aligned} \tag{20.10}$$

Thus, the iterate $y_n \in U(x^*, r)$ and estimate (20.6) holds. We need an upper bound on $\|A_n\|$. It follows by (20.7) and (H3) that

$$\begin{aligned} \|A_n\| &\leq \frac{|a|w_1(\|u_n - z_n\|, \|y_n - x_n\|)}{q_n} + \frac{w_2(\|z_n - x^*\|, \|y_n - x^*\|)}{q_n} \\ &\quad + \frac{|a-2|w_2(\|z_n - x^*\|, \|y_n - x^*\|)w_1(\|z_n - u_n\|, \|y_n - x_n\|)}{q_n^2} \\ &\leq \frac{|a|w_1(\alpha_n^1, \alpha_n^2)}{q_n} + \frac{w_2(\alpha_n^2, (1+g_1(\|x_n - x^*\|))\|x_n - x^*\|}{q_n} \\ &\quad + \frac{|a-2|w_2(\alpha_n^3, (1+g_1(\|x_n - x^*\|))\|x_n - x^*\|)w_1(\alpha_n^1, \alpha_n^3)}{q_n^2}, \end{aligned} \tag{20.11}$$

where we also used

$$\begin{aligned}\|u_n - z_n\| &\leq |b|\delta w_2(\|x_n - x^*\|, \|y_n - x^*\|)(1 + g_1(\|x_n - x^*\|)\|x_n - x^*\| \\ &\quad + |c - b|\delta w_2(\|y_n - x^*\|, 0)g_2(\|x_n - x^*\|) = \alpha_n^1 \\ \|y_n - x_n\| &\leq (1 + g_1(\|x_n - x^*\|))\|x_n - x^*\| = \alpha_n^2\end{aligned}$$

and

$$\begin{aligned}\|z_n - x^*\| &= \|y_n - x^* + c[y_n, x^*; F](y_n - x^*)\| \\ &\leq (1 + |c|\delta w_2(\|y_n - x^*\|, 0))\|y_n - x^*\| = \alpha_n^3.\end{aligned}$$

It then follows from the second substep of method (20.2) and the preceding estimates that

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|y_n - x^* + A_n[u_n, x_n; F]^{-1}[y_n, x^*; F](y_n - x^*)\| \\ &\leq (1 + \frac{\|A_n\| w_2(\|y_n - x^*\|, 0)}{q_n})\|y_n - x^*\| \\ &\leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r. \end{aligned} \tag{20.12}$$

Hence, the iterate $x_{n+1} \in U(x^*, r)$ and (20.7) holds, where we also used

$$\|u_n - x^*\| \leq (1 + |b|\delta(1 + w_0(r, 0)))r = R_1 \leq R$$

and

$$\|z_n - x^*\| \leq (1 + |c|\delta w_2(g_1(r)r, 0))g_1(r)r = R_2 \leq R.$$

□

Next, a uniqueness result follows for the solution of an equation $F(x) = 0$.

Proposition 15. *Suppose:*

(i) The point $x^* \in D$ is a simple solution of equation $F(x) = 0$.

(ii) Condition (H2) holds.

(iii) There exists $\lambda \geq r$ such that

$$w_0(\lambda, 0) < 1. \tag{20.13}$$

Set $D_2 = D \cap U[x^*, \lambda]$. Then, the point x^* is the only solution of equation $F(x) = 0$ in the set D_2.

Proof. Let $p \in D_2$ be a solution of equation $F(x) = 0$. Define the linear operator $S = [x^*, p; F]$. Then, by (H2) and (20.13)

$$\begin{aligned}\|F'(x^*)^{-1}(S - F'(x^*))\| &\leq w_0(\|p - x^*\|, 0) \\ &\leq w_0(\lambda, 0) < 1,\end{aligned}$$

so linear operator is S invertible and the identity $S(p - x^*) = F(p) - F(x^*) = 0$ implies $p = x^*$. □

Bibliography

[1] Abbasbandy, S., Bakhtiari, P., Cordero, A., Torregrosa, J.R., Lofti, T., New efficient methods for solving nonlinear systems of equations with arbitrary even order, *Appl. Math. Comput.*, 287, 288, (2016), 94-103.

[2] Argyros, I.K., A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach spaces, *J. Math. Anal. Appl.*, 298, (2004), 374-397.

[3] I.K., Argyros, *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[4] I.K., Argyros, Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics*, 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[5] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[6] Argyros, I.K., *Advances in the Efficiency of Computational Methods and Applications*, World Scientific Publishing Oxon, United Kingdom (2000).

[7] I.K. Argyros, S. George, *Mathematical modeling for the solution of equations and systems of equations with applications*, Volume-II, Nova Science Publishers, NY, 2018.

[8] Argyros, M.I., Argyros, I.K., Regmi, S., George, S, Generalized Three-Step Numerical Methods for Solving Equations in Banach Spaces, *Mathematics* 2022, 10, 2621. https://doi.org/10.3390/math10152621.

[9] Argyros, I.K., George, S., Magreñán, A.A., Local convergence for multi-point-parametric Chebyshev-Halley-type method of higher convergence order, *J. Comput. Appl. Math.*, 282, 215-224, (2015).

[10] Argyros, I.K., George, S., On the complexity of extending the convergence region for Traubs method, *Journal of Complexity*, 56, 101423.

[11] Grau-Sánchez, M., Grau, A., Noguera, M., Ostrowski type methods for solving systems of nonlinear equations, *Appl. Math. Comput.*, 218, 2011, 23772385.

[12] Grau-Sánchez, M., Noguera, M., A technique to choose the most efficient method between secant method and some variants, *Appl Math. Comput.*, 218, 2012, 64156426.

[13] Grau-Sánchez M., Grau, A., Noguera, M.: Frozen divided difference scheme for solving systems of nonlinear equations, *J. Comput. Appl. Math.*, 235, 2011, 17391743.

[14] Jaiswal, J.P., Semilocal convergence of an eighth-order method in Banach spaces and its computational efficiency, *Numer. Algorithms*, 71, (2016), 933-951.

[15] Sharma, J.R., Arora, H., Petkovic, M.S., An efficient derivative free family of fourth order methods for solving systems of equations, *Appl. Math. Comput.*, 235, (2014), 383-393.

Chapter 21

Local Convergence for an Efficient Derivative Free Fourth Order Method for Solving Equations in Banach Space

1. Introduction

Let B be a Banach space and $D \subset B$ be an open set. We are concerned with the problem of approximating a locally unique solution $x^* \in D$ of equation

$$F(x) = 0, \tag{21.1}$$

where $F : D \subset B \longrightarrow B$ is a Fréchet differentiable, operator. The solution x^* is being approximated by the fourth-order method

$$\begin{aligned} y_n &= x_n - A_n^{-1} F(x_n) \\ \text{and} & \\ x_{n+1} &= y_n - B_n F(y_n), \end{aligned} \tag{21.2}$$

where $A_n = [x_n + F(x_n), x_n; F]$, $B_n = [y_n, x_n; F]^{-1} C_n [y_n, y_n; F]^{-1}$, $C_n = [y_n, x_n; F] - [y_n, x_n + F(x_n); F] + [x_n + F(x_n), x_n; F]$ and $[.,.; F] : D \times D \longrightarrow L(B, B)$ is a divided difference of order one [1, 2, 3]. The fourth convergence order of method (21.2) was shown in [11] under hypotheses on the fifth derivative of operator F when $B = \mathbb{R}^k$. But we extend the applicability of method (21.2) in a Banach space setting and using only conditions on the divided difference of order one that only appears on the method (21.2). Relevant work can be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

2. Convergence

The local convergence analysis uses real functions and parameters. Set $M = [0, \infty)$.

Suppose that there exists:

(a1) Functions $w_2 : M \longrightarrow \mathbb{R}$, $w_0 : M \times M \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation $p(t) = 0$, where $p : M \longrightarrow M$ is given by

$$p(t) = 1 - w_0(t + w_2(t), t)$$

has a smallest solution denoted by $r \in M - \{0\}$. Set $M_0 = [0, r)$.

(a2) Functions $w_5, w_1 : M_0 \times M_0 \longrightarrow \mathbb{R}$, $w_3, w_4 : M_0 \longrightarrow \mathbb{R}$ continuous and decreasing such that the equations

$$h_1(t) - 1 = 0$$

and

$$h_2(t) = 0$$

have smallest solutions $\rho_1, \rho_2 \in M_0 - \{0\}$, respectively, where

$$h_1(t) = \frac{w_1(w_2(t), t)}{p(t)}$$

and

$$h_2(t) = h_1(t) + w_3(h_1(t)t(1 + \frac{w_4(\beta + w_2(t)) + w_5(t + w_2(t))}{p^2(t)}).$$

The parameter ρ given by

$$\rho = \min\{\rho_i\},\ i = 1, 2 \tag{21.3}$$

shall be shown to be a radius of convergence for method (21.2). Set $M_1 = [0, \rho)$. It follows by these definitions that for all $t \in M_1$

$$p(t) > 0 \tag{21.4}$$

and

$$h_i(t) < 1. \tag{21.5}$$

By $U(x, \alpha)$ we denote the open ball with center $x \in D$ and of radius $\alpha > 0$. The ball $U[x, \alpha]$ denotes its closure.
Next, the "w" functions and parameter β are related to the conditions:

(a3) There exists a solution $x^* \in D$ of equation $F(x) = 0$ such that $F'(x^*)^{-1} \in L(B_2, B_1)$.

(a4) $\|F'(x^*)^{-1}([y, x; F] - F'(x^*))\| \leq w_0(\|y - x^*\|, \|x - x^*\|)$,

$$\|F(x)\| \leq w_2(\|x - x^*\|) \text{ for all } x, y \in D.$$

Set $D_1 = D \cap U(x^*, r)$.

(a5) $\|F'(x^*)^{-1}([x + F(x), x; F] - [x, x^*; F])\| \leq w_1(\|F(x)\|, \|x - x^*\|)$,

$$\|[y, x^*; F]\| \leq w_3(\|y - x^*\|),$$

$$\|F'(x^*)^{-1}([y, x; F] - [y, x + F(x); F])\| \leq w_4(\|F(x)\|)$$

and

$$\|F'(x^*)^{-1}[y, x; F]\| \leq w_5(\|y - x^*\|, \|x - x^*\|)$$

for all $x, y \in D_1$.
and

(a6) $U[x^*,R] \subset D,$. where $R = \rho + w_2(\rho)$.

Next, the local convergence of method (21.2) is presented using the preceding terminology.

Theorem 23. Under conditions (a1)-(a6) hold. Then, sequence $\{x_n\}$ generated by method (21.2) for $x_0 \in U(x^*,\rho)$ is well defined in the ball $U[x^*,\rho]$ and converges to x^* so that

$$\|y_n - x^*\| \leq h_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho \tag{21.6}$$

and

$$\|x_{n+1} - x^*\| \leq h_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{21.7}$$

where the functions h_i are defined previously and the radius r is given by formula (21.3).

Proof. The first substep of method (21.2) gives

$$\begin{aligned} y_n - x^* &= x_n - x^* - A_n^{-1}F9x_n) \\ &= A_n^{-1}([x_n + F(x_n), x_n; F] - [x_n, x^*; F])(x_n - x^*) \end{aligned}$$

leading by (a3)-(a5) to

$$\begin{aligned} \|y_n - x^*\| &\leq \frac{w_1(w_2(\|x_n - x^*\|, \|x_n - x^*\|)\|x_n - x^*\|}{1 - w_0(\|x_n - x^*\| + w_2(\|x_n - x^*\|), \|x_n - x^*\|)} \\ &\leq h_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho, \end{aligned} \tag{21.8}$$

where we also used

$$\|A_n^{-1}F'(x^*)\| \leq \frac{1}{p_n} \tag{21.9}$$

implied by the Banach lemma on invertible operators [1,2,3] and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}([x_n + F(x_n), x_n; F] - F'(x^*))\| &\leq w_0(\|x_n - x^* + F(x_n)\|, \|x_n - x^*\|) \\ &\leq w_0(\|x_n - x^*\| + \|F(x_n)\|, \|x_n - x^*\|) \\ &\leq w_0(\rho + w_2(\rho), \rho) < 1. \end{aligned}$$

Notice that we also used

$$\|x_n - x^* + F(x_n)\| \leq \rho + w_2(\rho) = R.$$

It follows by (21.3), (21.4) and (21.8) that (21.6) holds (for $n = 0$) and $y_n \in U(x^*,\rho)$. Similarly, from the second substep of method (21.2) we can write

$$\begin{aligned} x_{n+1} - x^* &= ([y_n, x^*; F] - [y_n, x_n; F])^{-1}([y_n, x_n; F] - [y_n, x_n + F(x_n); F] \\ &\quad + [x_n + F(x_n), x_n; F])[y_n, x_n; F]^{-1}[y_n, x^*; F](y_n - x^*) \end{aligned}$$

leading to

$$\begin{aligned} \|x_{n+1} - x^*\| &= \{w_3(\|y_n - x^*\| + \frac{w_3(\|y_n - x^*\|)}{p_n^2} \\ &\quad (w_4(w_2(\|x_n - x^*\|)) + w_5(\|x_n - x^*\| + w_2(\|x_n - x^*\|))\|y_n - x^*\| \\ &\leq h_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|. \end{aligned}$$

Therefore, (21.7) holds and $x_{n+1} \in U(x^*, \rho)$ (for $n = 1$). Using induction the same calculations complete the induction for (21.6) and (21.7). Then, the estimate

$$\|x_{n+1} - x^*\| \leq \gamma \|x_n - x^*\| \leq \gamma^{n+1} \|x_0 - x^*\| < \rho$$

leads to $\lim_{n \longrightarrow \infty} x_n = x^*$. □

The uniqueness of the solution of results as similar to [1, 2, 3] is omitted.

Bibliography

[1] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[2] Argyros, I.K., *Advances in the Efficiency of Computational Methods and Applications*. World Scientific Publishing Oxon, United Kingdom (2000).

[3] Argyros, I.K., George, S., *Mathematical modeling for the solution of equations and systems of equations with applications*, Volume-II, Nova Publishers, NY, 2018.

[4] Cordero, A., Hueso, J.L., Martínez, E., and Torregrosa, J.R., A modified Newton-Jarratts composition, *Numer. Algor.*, 2010, 55, 87–99.

[5] Cordero, A., Hueso, J.L., Martínez, E., Torregrosa, J.R., Efficient high-order methods based on golden ratio for nonlinear systems, *Appl. Math. Comput.*, 2011, 217, 4548–4556.

[6] Ezquerro, J.A.; Hernández, M.A.; Romero, N. Solving nonlinear integral equations of Fredholm type with high order iterative methods, *J. Comput. Appl. Math.*, 2011, 236, 1449–1463.

[7] Ezquerro, J.A.; Grau,A.; Grau-Sánchez, M.; Hernández, M.A.; Noguera, M. Analysing the efficiency of some modifications of the secant method, *Comput. Math. Appl.*, 2012, 64, 2066–2073.

[8] Grau-Sánchez, M.; Grau, A.; Noguera, M. Ostrowski type methods for solving systems of nonlinear equations, *Appl. Math. Comput.*, 2011, 218, 2377–2385.

[9] Grau-Sánchez, M.; Noguera, M.; Amat, S. On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative methods, *J. Comput. Appl. Math.*, 2013, 237, 363–372.

[10] Grau-Sánchez, M.; Grau, .; Noguera, M. Frozen divided difference scheme for solving systems of nonlinear equations, *J. Comput. Appl. Math.*, 2011, 235, 1739–1743.

[11] Liu, Z., Zheng, Q., Zhao, P.A., variant of Steffensens method of fourth-order convergence and its applications, *Appl. Math. Comput.*, 2010, 216, 1978–1983.

[12] Ostrowski, A.M., *Solutions of Equations and Systems of Equations*; Academic Press: New York, NY, USA, London, UK, 1966.

[13] Potra, F.A., Pták, V., *Nondiscrete Induction and Iterative Processes*, Pitman Publishing: Boston, MA, USA, 1984.

[14] Wang, X., Zhang, T., Qian, W., Teng, M., Seventh-order derivative-free iterative method for solving nonlinear systems, *Numer. Algor.*, 2015, 70, 545–558.

[15] Wang, X., Zhang, T.A., family of steffensen type methods with seventh-order convergence, *Numer. Algor.*, 2013, 62, 429–444.

Chapter 22

Extended Local Convergence of Steffensen-Type Methods for Solving Nonlinear Equations in Banach Space

1. Introduction

Let B be a Banach space and $D \subset B$ be an open set. We are concerned with the problem of approximating a locally unique solution $x^* \in D$ of the nonlinear equation

$$F(x) = 0, \tag{22.1}$$

where $F : D \longrightarrow B$ is a Fréchet-differentiable operator. A plethora of applications reduces to solving equation (22.1). The solution x^* is needed in a closed form. But this is attainable only in special cases. That explains why most solution methods are iterative. In particular, we study the local convergence of the Steffensen-type method (STM) defined $\forall n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - A_n^{-1} F(x_n) \\ \text{and} & \\ x_{n+1} &= y_n - B_n^{-1} F(y_n), \end{aligned} \tag{22.2}$$

where $A_n = [x_n, w_n; F]$, $[.,.;F] : D \times D \longrightarrow L(B,B)$ is a divided difference of order one for operator $F, w_n = x_n + F(x_n)$ and $B_n = [y_n, w_n; F] + [y_n, x_n; F] - [x_n, w_n; F]$. STM was studied in [13]. It was found to be of a fourth convergence order. In particular, the radius of convergence was established. We show that this radius can be enlarged without new conditions. Other benefits include tighter error bounds on distances $\|x_n - x^*\|$ and better information on the uniqueness of the solution. The technique is independent of method (22.2). Thus, it can be used to extend the applicability of other methods. This process specifies a more precise location of the STM iterates leading to at least as tight Lipschitz parameters which are specializations of the ones in [13]. Hence, no additional computational effort is required for these benefits either.

2. Local Analysis of STM

The analysis is based on certain parameters and real functions. Let L_0, L, and α be positive parameters. Set $T_1 = [0, \frac{1}{(2+\alpha)L_0}]$ provided that $(2+\alpha)L_0 < 1$.

Define function $h_1 : T_1 \longrightarrow \mathbb{R}$ by

$$h_1(t) = \frac{(1+\alpha)Lt}{1-(2+\alpha)L_0 t}.$$

Notice that parameter ρ

$$\rho = \frac{1}{(1+\alpha)L + (2+\alpha)L_0}$$

is the only solution to the equation

$$h_1(t) - 1 = 0$$

in the set T_1. Define the parameter ρ_0 by

$$\rho_0 = \frac{1}{(2+\alpha)(L_0+L)}.$$

Notice that $\rho_0 < \rho$. Set $T_0 = [0, \rho_0]$.

Define function $h_2 : T_0 \longrightarrow \mathbb{R}$ by

$$h_2(t) = \frac{(2+2\alpha+h_1(t))Lh_1(t)t}{1-(2+\alpha)(L_0+L)t}.$$

The equation

$$h_2(t) - 1 = 0$$

has a smallest solution $R \in T_0 - \{0\}$ by the intermediate value theorem, since $h_2(0) - 1 = -1$ and $h_2(t) \longrightarrow \infty$ as $y \longrightarrow \rho_0^-$. It shall be shown that R is a radius of convergence for method (22.2). It follows by these definitions that $\forall t \in T_0$

$$0 \leq (L_0+L)(2+\alpha)t < 1, \tag{22.3}$$

$$0 \leq h_1(t) < 1 \tag{22.4}$$

and

$$0 \leq h_2(t) < 1. \tag{22.5}$$

Let $U(x_0, \lambda) = \{x \in B : \|x - x_0\| < \lambda\}$ and $U[x_0, \lambda] = \{x \in B : \|x - x_0\| \leq \lambda\}$ for some $\lambda > 0$. The following conditions are used:

(C1) There exists a solution $x^* \in D$ of equation $F(x) = 0$ such that $F'(x^*)^{-1} \in L(B,B)$.

(C2) There exist positive parameters L_0 and α such that $\forall v, z \in D$

$$\|F'(x^*)^{-1}([v,z;F] - F'(x^*))\| \leq L_0(\|v - x^*\| + \|z - x^*\|)$$

and

$$\|F(x)\| \leq \alpha \|x - x^*\|.$$

Set $D_1 = U(x^*, \rho) \cap D$.

(C3) There exists a positive constant $L > 0$ such that $\forall x, y, v, z \in D_1$

$$\|F'(x^*)^{-1}([x,y;F] - [v,z;F])\| \leq L(\|x - v\| + \|y - z\|)$$

and

(C4) $U[x_0, R] \subset D$.

Next, the local convergence of method (22.2) is presented using the preceding terminology and conditions.

Theorem 24. Under conditions (C1)-(C4) further suppose that $x_0 \in U(x^*, R)$. Then, sequence STM generated by method (22.2) is well defined in $U(x^*, R)$, Stays in $U(x^*, R)$ $\forall n = 0, 1, 2, \ldots$ and is convergent to x^* so that

$$\|y_n - x^*\| \leq h_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < R \tag{22.6}$$

and

$$\|x_{n+1} - x^*\| \leq h_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{22.7}$$

where functions h_1, h_2 and radius R are defined previously.

Proof. It follows by method (22.2), (C1), (C2) and $x_0 \in U(x^*, R)$ in turn that

$$\begin{aligned} \|F'(x^*)^{-1}(A_0 - F'(x^*))\| &= \|F'(x^*)^{-1}([x_0, x_0 + F(x_0); F] - F'(x^*))\| \\ &\leq L_0(2\|x_0 - x^*\| + \|F(x_0) - F(x^*)\| \\ &\leq L_0(2 + \alpha)\|x_0 - x^*\| \\ &< L_0(2 + \alpha)R. \end{aligned} \tag{22.8}$$

It follows by (22.8) and the Banach lemma on invertible operators [2] that $A_0^{-1} \in L(B, B)$ and

$$\|A_0^{-1}F'(x^*)\| \leq \frac{1}{1 - (2 + \alpha)L_0\|x_0 - x^*\|}. \tag{22.9}$$

Hence, iterate y_0 exists by the first substep of method (22.2) for $n = 0$. It follows from the first substep of method (22.2), (C2) and (C3) that

$$\begin{aligned} \|y_0 - x^*\| &\leq \|x_0 - x^* - A_0^{-1}F(x_0) \\ &\quad \|A_0^{-1}F'(x^*)F'(x^*)^{-1}(A_0 - (F(x_0) - F(x^*)))(x_0 - x^*)\| \\ &\leq \|A_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}(A_0 - (F(x_0) - F(x^*)))\|\|x_0 - x^*\| \\ &\leq \frac{L(\|x_0 - x^*\| + \|F(x_0) - F(x^*))}{1 - L_0(2 + \alpha)\|x_0 - x^*\|} \\ &\leq h_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < R. \end{aligned} \tag{22.10}$$

Thus, $y_0 \in U(x^*,R)$ and (22.6) holds for $n=0$. Similarly, by the second substep of method (22.2), we have

$$\begin{aligned}
\|F'(x^*)^{-1}(B_0 - F'(x^*))\| &= \|F'(x^*)^{-1}([y_0,w_0;F]-[y_0,x_0;F]-[x_0,w_0;F]-[x^*,x^*;F])\| \\
&\leq L\|y_0-w_0\| + L_0(\|y_0-x^*\|+\|w_0-x^*\|) \\
&\leq L(\|y_0-x^*\|+\|w_0-x^*\|)+L_0(\|y_0-x^*\|+\|w_0-x^*\|) \\
&\leq (L+L_0)(2+\alpha)R < \frac{L+L_0}{L+L_0} = 1, \qquad (22.11)
\end{aligned}$$

so, $B_0^{-1} \in L(B,B)$ and

$$\|B_0^{-1}F'(x^*)\| \leq \frac{1}{1-(L+L_0)(2+\alpha)\|x_0-x^*\|}. \qquad (22.12)$$

Hence, iterate x_1 exists by the second sub-step of method (22.2). Then, as in (22.14) we get in turn that

$$\begin{aligned}
\|x_1-x^*\| &\leq \|y_0-x^*-B_0^{-1}F(y_0)\| \\
&\leq \|B_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}(B_0-(F(y_0)-F(x^*)))\|\|y_0-x^*\| \\
&\leq \frac{\|F'(x^*)^{-1}([y_0,w_0;F]+[y_0,x_0;F]-[x_0,w_0;F]-[y_0,x^*:F])\|}{1-(L+L_0)(2+\alpha)\|x_0-x^*\|}\|y_0-x^*\| \\
&\leq \frac{L(2+2\alpha+h_2(\|x_0-x^*\|))\|x_0-x^*\|}{1-(L+L_0)(2+\alpha)\|x_0-x^*\|}h_1(\|x_0-x^*\|)\|x_0-x^*\| \qquad (22.13) \\
&\leq h_2(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\| < R.
\end{aligned}$$

Therefore, $x_1 \in U(x^*,R)$ and (22.7) holds for $n=0$.

Simply replace x_0,y_0,x_1 by x_m,y_m,x_{m+1}, $\forall m = 0,1,2\ldots$ in the preceding calculations to complete the induction for (22.6) and (22.7). It then follows from the estimate

$$\|x_{m+1}-x^*\| \leq b\|x_m-x^*\| < R, \qquad (22.14)$$

4 where, $b = h_2(\|x_0-x^*\|) \in [0,1)$ that $x_{m+1} \in U(x^*,R)$ and $lim_{m\longrightarrow\infty}x_m = x^*$. □

Concerning the uniqueness of the solution x^* (not given n [13]), we provide the result.

Proposition 16. *Suppose:*

(i) The point x^* is a simple solution $x^* \in U(x^*,r) \subset D$ for some $r>0$ of equation $F(x)=0$.

(ii) There exists positive parameter L_1 such that $\forall y \in D$

$$\|F'(x^*)^{-1}([x^*,y;F]-F'(x^*))\| \leq L_1\|y-x^*\| \qquad (22.15)$$

(iii) There exists $r_1 \geq r$ such that

$$L_1 r_1 < 1. \qquad (22.16)$$

Set $D_2 = U[x^*,r_1] \cap D$. Then, x^* is the only solution of equation $F(x)=0$ in the set D_2.

Proof. Set $S = [x^*, y^*; F]$ for some $y^* \in D_2$ with $F(y^*) = 0$. It follows by (i), (22.15) and (22.16) that

$$\|F'(x^*)^{-1}(S - F'(x^*))\| \leq L_1 \|y^* - x^*\|) < 1,$$

so, $x^* = y^*$ by invertibility of S and identity $S(x^* - y^*) = F(x^*) - F(y^*) = 0$. □

Remark. (i) Notice that not all conditions of Theorem 24 are used in Proposition 16. But if they were, then we can set $r_1 = R$.

(ii) By the definition of set D_1 we have

$$D_1 \subset D. \tag{22.17}$$

Therefore, the parameter

$$L \leq M, \tag{22.18}$$

where M is the corresponding Lipschitz constant in [1, 11, 12, 13] appearing in the condition $\forall x, y, z \in D$

$$\|F'(x^*)^{-1}([x, y; F] - [v, z; F])\| \leq M(\|x - v\| + \|y - z\|). \tag{22.19}$$

So, the radius of convergence R_0 in [1, 11, 12] uses M instead of L. That is by (22.18)

$$R_0 \leq R. \tag{22.20}$$

Examples where (22.17), (22.18) and (22.20) holds can be found in [2, 3, 4, 5, 6, 7, 8, 9, 10, 14]. Hence, the claims made in the introduction have been justified.

Bibliography

[1] Amat, S., Bermudez, C., Busquier, S., and Plaza, S., On a third order Newton-type method free of linear operators, *Numer. Lin. Alg. Appl.*, 17, (2010), 639-653.

[2] Argyros, I.K., *Computational theory of iterative methods. Series: Studies in Computational Mathematics*, 15, Editors: C.K.Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[3] Argyros, I.K. *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[4] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[5] Argyros, I.K., A Newton-Kantorvich theorem for equations involving m Frchet differentiable operators and applications in radiative transfer, *J. Comput. Appl. Math.*, 131, (1-2), (2001), 149-159.

[6] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)*, 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[7] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative methods*, Elsevier (Academic Press), New York, 2018.

[8] Argyros, I.K, Behl, R., Motsa, S.S., Unifying semilocal and local convergence of Newton's method on Banach space with a convergence structure, *Appl. Numer. Math.*, 115, (2017), 225-234.

[9] Argyros, I.K., Cordero, A., Magreñán, A.A., Torregrosa, J.R., Third-degree anomalies of Traub's method, *J. Comput. Appl. Math.*, 309 (2017), 511-521.

[10] Argyros, I.K., Cordero, A., Magreñán, A.A., Torregrosa, J. R., On the convergence of a higher order family of methods and its dynamics, *J. Comput. Appl. Math.*, 309, (2017), 542-562.

[11] Cordero, A., Fazlollah Soleymani, Juan R. Torregrosa, Stanford Shateyi, Basins of attraction for various Steffensen-type methods, *J. Appl. Math.*, (2014).

[12] Ezquerro, J.A., Grau-Sánchez, M., Hernández, M.A., Noguera, M., Semilocal convergence of secant-like methods for differentiable and nondifferentiable operator equations, *J. Math. Anal. Appl.*, 398, (1) (2013), 100-112.

[13] Moccari, M., Lofti, T., On a two step optimal Steffensen-type method: Relaxed local and semi-local convergence analysis and dynamical stability, *J. Math. Anal. Appl.*, 468, (2018), 240-269.

[14] Magréñan, A.A., Argyros, I.K., New improved convergence analysis for the secant method, *Math. Comput. Simulation*, 119 (2016), 161-170.

Chapter 23

Extended Convergence Analysis of Optimal Eighth Order Methods for Solving Nonlinear Equations in Banach Space

1. Introduction

Let X and Y be a Banach space and $\Omega \subset X$ be an open set. We are concerned with the problem of approximating a locally unique solution $x^* \in \Omega$ of the nonlinear equation

$$F(x) = 0, \tag{23.1}$$

where $F : \Omega \longrightarrow Y$ is a Fréchet-differentiable operator. We consider the iterative method defined $\forall n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1} F(x_n), \\ z_n &= d_4(x_n, y_n) \\ \text{and} \\ x_{n+1} &= d_8(x_n, y_n, z_n), \end{aligned} \tag{23.2}$$

where α is a real parameter and $d_n : \Omega \times \Omega \longrightarrow X$, $d_8 : \Omega \times \Omega \times \Omega :\longrightarrow X$ are continuous operators. If $X = Y = \mathbb{R}$, $\alpha = 1$, and d_4, d_8 are iteration functions of order four and eight respectively, then method (23.2) was shown to be of order eight in [5].

2. Local Convergence

The local convergence analysis uses some scalar functions and positive parameters. Let $S = [0, \infty)$.

Suppose there exist:

(i) function $w_0 : S \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$w_0(t) - 1 = 0$$

has a smallest solution $\rho_0 \in S - \{0\}$. Let $S_0 = [0, \rho_0)$.

(ii) function $w : S_0 \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$g_1(t) - 1 = 0$$

has a smallest solution $r_1 \in S_0 - \{0\}$, where function $g_1 : S_0 \longrightarrow \mathbb{R}$ is defined by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta + |1-\alpha|(1+\int_0^1 w_0(\theta t)d\theta)}{1-w_0(t)}.$$

(iii) functions $g_2, g_3 : S_0 \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equations

$$g_2(t) - 1 = 0$$

and

$$g_3(t) - 1 = 0$$

has smallest solutions r_2, r_3 in $S_0 - \{0\}$, respectively. The parameter r defined by

$$r = \min\{r_m\}, \ m = 1,2,3 \tag{23.3}$$

shall be shown to be a radius of convergence for method (23.2). Set $S_1 = [0, r)$. it follows by this definition and (23.3) that $\forall t \in S_1$

$$0 \leq w_0(t) < 1 \tag{23.4}$$

and

$$0 \leq g_n(t) < 1. \tag{23.5}$$

Let $U(x,\mu) = \{x \in B : \|x - x_0\| < \mu\}$ and $U[x,\mu] = \{x \in B : \|x - x_0\| \leq \mu\}$ for some $\mu > 0$.

The aforementioned functions and parameters are connected to the following conditions:

(h1) $\exists$ a simple solution $x^* \in \Omega$ of equation $F(x) = 0$.

(h2)

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|) \text{ for each } x \in \Omega.$$

Set $U_0 = U(x^*, \rho_0) \cap \Omega$.

(h3)

$$\|F'(x^*)^{-1}(F'(y) - F'(x))\| \leq w(\|y - x\|) \text{ for each } x, y \in U_0.$$

(h4)

$$\|d_n(x,y) - x^*\| \leq g_2(\|x - x^*\|)\|x - x^*\|$$

for each $x \in \Omega_0$ and $y = x - \alpha F'(x)^{-1}F(x)$.

(h5)

$$\|d_8(x,y,z) - x^*\| \leq g_3(\|x - x^*\|)\|x - x^*\|$$

for each $x \in U_0, y = x - \alpha F'(x)^{-1}F(x), d_4(x, x - \alpha F'(x)^{-1}F(x))$.

(h6) $U[x^*, r] \subset \Omega$.

Next, the main local convergence result is presented using the developed terminology and the "h" conditions.

Theorem 25. Under conditions (h1)-(h6) hold, choose $x_0 \in U(x^*, r) - \{x^*\}$. Then, sequence $\{x_n\}$ generated by method (23.2) is well defined in $U(x^*, r)$, and is convergent to x^*. Moreover, the following estimates hold for all $n = 0, 1, 2, \ldots$

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{23.6}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{23.7}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{23.8}$$

where functions g_m are previously defined and parameter r is given by formula (23.3).

Proof. Let $v \in U(x^*, r)$. Then, it follows by applying (h1), (h2) and using (23.3) that

$$\|F'(x^*)^{-1}(F'(v) - F'(x^*))\| \leq w_0(\|v - x^*\|) < 1,$$

thus $F'(v)^{-1} \in L(Y, X)$ and

$$\|F'(v)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|v - x^*\|)} \tag{23.9}$$

by the Banach Lemma on linear operators with inverses [4]. If $v = x_0$, then (23.9) implies $F'(x_0)^{-1}$ is invertible. Hence, iterate y_0 is well defined by method (23.2) for $n = 0$. We can also write

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0) + (1 - \alpha)F'(x_0)^{-1}F(x_0). \tag{23.10}$$

By using (23.3), (23.5) (for $m = 1$), (23.9) (for $v = x_0$) and (h1)-(h3) we get

$$\begin{aligned} \|y_0 - x^*\| &\leq \frac{1}{1 - w_0(\|x_0 - x^*\|)}\Big[\int_0^1 w((1-\theta)\|x_0 - x^*\|)d\theta \\ &\quad + |1 - \alpha|(1 + \int_0^1 w_0(\theta\|x_0 - x^*\|)d\theta)\Big]\|x_0 - x^*\| \\ &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{23.11}$$

proving (23.6) for $n = 0$ and $y_0 \in U(x^*, r)$. Similarly, using (23.2), (h4) and (h5)

$$\|z_0 - x^*\| \leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| \tag{23.12}$$

and

$$|x_0 - x^*\| \leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|, \tag{23.13}$$

showing (23.7), (23.8), respectively and $z_0, x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, x_1 by x_i, y_i, z_i, x_{i+1}, in the preceding calculations to complete the induction for estimates (23.6)-(23.8). Then, by the estimate

$$\|x_{i+1} - x^*\| \leq b\|x_i - x^*\| < r, \tag{23.14}$$

where, $b = g_3(\|x_0 - x^*\|) \in [0, 1)$, we conclude $lim_{i\longrightarrow\infty} x_i = x^*$ and $x_{i+1} \in U(x^*, r)$. □

Next, a uniqueness of the solution result for equation $F(x) = 0$ is presented.

Proposition 17. *Assume:*

(i) There exists a simple solution $x^* \in U(x^*, \rho)$ of equation $F(x) = 0$ for some $\rho > 0$.

(ii) The condition (h2) holds.

(iii) There exists $\rho_1 \geq \rho$ such that

$$\int_0^1 w_0(\theta\rho_1)d\theta < 1. \tag{23.15}$$

Set $U_1 = U[x^*, \rho_1] \cap \Omega$. Then, the point x^* is the only solution of equation $F(x) = 0$ in the set U_1.

Proof. Let $y^* \in U_1$ with $F(y^*) = 0$. Define the linear operator $T = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$. Then, by using (h2) and (23.15) we obtain

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \leq \int_0^1 w_0(\theta\|y^* - x^*\|)d\theta \leq \int_0^1 w_0(\theta\rho_1)d\theta < 1.$$

Therefore, $x^* = y^*$ by invertibility of T and identity $T(x^* - y^*) = F(x^*) - F(y^*) = 0$. □

Remark. The uniqueness of the solution x^* was shown in Proposition by using only condition (h2). However, if all "h" conditions are used, then set $r = \rho$.

3. Special Cases

In this Section we specialize method (23.2), to determine functions g_2 and g_3.
Case 1 Choose

$$d_4(x_n, y_n) = y_n - A_n F(y_n)$$

and

$$d_8 = z_n - B_n F(z_n),$$

where $A_n = (2[y_n, x_n; F] - F'(x_n))^{-1}$, $B_n = [z_n, x_n; F]^{-1}[z_n, y_n; F]C_n^{-1}$ and $C_n = 2[z_n, y_n; F] - [z_n, x_n; F]$. Let functions $w_1, w_2, w_3, w_5 : S_0 \times S_0 \longrightarrow \mathbb{R}$, $w_4 : S_0 \times S_0 \times S_0 \longrightarrow \mathbb{R}$ be continuous and nondecreasing. Moreover, define scalar sequences and functions provided that iterates x_n, y_n, z_n exist and $\|x_n - x^*\| \leq t$.

$$\begin{aligned}
p_n &= w_1(\|z_n - x^*\|, \|y_n - x^*\|) + w_4(\|x_n - x^*\|, \|y_n - x^*\|) \\
&\leq w_1(g_2(t)t, g_1(t)t) + w_4(g_2(t)t, g_1(t)t) = p(t), \\
q_n &= \frac{1}{1 - P_n} \leq \frac{1}{1 - p(t)} = q(t), \\
s_n &= w_4(\|x_n - x^*\|, \|y_n - x^*\|, \|z_n - x^*\|) + w_3(\|x_n - x^*\|, \|y_n - x^*\|) \\
&\leq w_4(t, g_1(t)t, g_2(t)t) + w_3(t, g_1(t)t) = s(t).
\end{aligned}$$

Define functions g_2 and g_3 by

$$g_2(t) = \frac{(w_2(t, g_1(t)t) + w_3(t, g_1(t)t)g_1(t)}{1 - (w_1(t, g_1(t)t) + w_2(t, g_1(t)t))}$$

and

$$g_3(t) = \frac{(w_4(t, g_1(t)t, g_2(t)t) + q(t)s(t)w_5(g_2(t)t, g_1(t)t))g_2(t)}{1 - w_1(g_2(t)t, t)}.$$

Notice that function g_2 and g_3 are well defined on $S_1 = [0, \gamma)$ provided that equation

$$w_1(t, g_1(t)t) + w_2(t, g_1(t)t) - 1 = 0$$

and

$$w_1(g_1(t)t, t) - 1 = 0$$

have smallest solutions γ_1 and γ_2, respectively in $S_0 - \{0\}$ and

$$\gamma = \min\{\rho, \gamma_1, \gamma_2\}.$$

Then, consider conditions (h7) replacing (h4) and (h5):

(h7)

$$\begin{aligned}
\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| &\leq w_1(\|x - x^*\|, \|y - x^*\|), \\
\|F'(x^*)^{-1}([x, y; F] - F'(y))\| &\leq w_2(\|x - x^*\|, \|y - x^*\|), \\
\|F'(x^*)^{-1}([y, x; F] - [y, x^*; F])\| &\leq w_3(\|y - x^*\|, \|x - x^*\|), \\
\|F'(x^*)^{-1}([z, y; F] - [z, y; F])\| &\leq w_4(\|x - x^*\|, \|y - x^*\|, \|z - x^*\|), \\
\|F'(x^*)^{-1}[z, y; F]\| &\leq w_5(\|z - x^*\|, \|y - x^*\|), \\
\|F'(x^*)^{-1}F'(x)\| &\leq w_6(\|x - x^*\|), \\
\|F'(x^*)^{-1}([x, x^*; F] - F'(y))\| &\leq w_7(\|x - x^*\|, \|y - x^*\|)
\end{aligned}$$

and

$$\|F'(x^*)^{-1}[x, x^*; F]\| \leq w_8(\|x - x^*\|).$$

Then, we use the estimates

$$\begin{aligned} \|F'(x^*)^{-1}(A_n^{-1}-F'(x^*))\| &\leq w_1(\|x_n-x^*\|,\|y_n-x^*\|) \\ &\quad +w_2(\|x_n-x^*\|,\|y_n-x^*\|)=p_n<1, \end{aligned}$$

so

$$\|A_nF'(x^*)\|\leq \frac{1}{1-p_n},$$

$$\begin{aligned} z_n-x^* &= y_n-x^*-(2[y_n,x_n;F]-F'(x_n))^{-1}F(y_n) \\ &= A_n(2[y_n,x_n;F]-F'(x_n)-[y_n,x^*;F])(y_n-x^*), \end{aligned}$$

thus

$$\begin{aligned} \|z_n-x^*\| &\leq \|A_nF'(x^*)\|\|F'(x^*)^{-1}([y_n,x_n;F]-F'(x_n)) \\ &\quad +([y_n,x_n;F]-[z_n,x^*;F])\|\|y_n-x^*\| \\ &\leq g_2(\|x_n-x^*\|)\|x_n-x^*\|\leq\|x_n-x^*\|<r, \\ \|F'(x^*)^{-1}(C_n-F'(x^*))\| &\leq w_1(\|z_0-x^*\|,\|y_n-x^*\|) \\ &\quad +w_4(\|x_n-x^*\|,\|y_n-x^*\|,\|z_n-x^*\|) \\ &\leq p_n. \end{aligned}$$

Hence

$$\|C_n^{-1}F'(x^*)\|\leq\frac{1}{1-p_n}=q_n.$$

Moreover, we can write in turn that

$$\begin{aligned} & z_n-x^*-[z_n,x_n;F]^{-1}[z_n,y_n;F]C_n^{-1}[z_n,x^*;F](z_n-x^*) \\ = & [z_n,y_n;F]^{-1}\{([z_n,x_n;F]-[z_n,x_n;F]) \\ & +[z_n,y_n;F]C_n^{-1}(2[z_n,y_n;F]-[z_n,x_n;F]-[z_n,x^*;F])\}(z_n-x^*), \end{aligned}$$

so

$$\begin{aligned} \|x_{n+1}-x^*\| &= \|z_n-x^*-B_nF(z_n)\| \\ &\leq \frac{1}{1-w_1(\|z_n-x^*\|,\|x_n-x^*\|)} \\ &\quad [w_4(\|x_n-x^*\|,\|y_n-x^*\|,\|z_n-x^*\|) \\ &\quad +q_ns_nw_5(\|z_n-x^*\|,\|y_n-x^*\|)]\|z_n-x^*\| \\ &\leq g_3(\|x_n-x^*\|)\|x_n-x^*\|\leq\|x-n-x^*\|. \end{aligned}$$

Thus, we arrived:

Theorem 26. Suppose that conditions (h1), (h2), (h3), (h6), and (h7) hold. Then, the conclusions of Theorem 25 hold. Concerning the uniqueness of the solution, we can also use function w_1 (see (h7)) instead of w_0 (see (h2)).

Proposition 18. *Suppose:*

(i) *There exists a simple solution* $x^* \in U(x^*, \rho)$ *of equation* $F(x) = 0$ *for some* $\rho > 0$.

(ii) *The first condition in (h7) holds.*

(iii) *There exists* $\rho^* \geq \rho$ *such that*

$$w_1(0, \rho^*) < 1. \tag{23.16}$$

Set $U_2 = U[x^*, \rho^*] \cap \Omega$. *Then, the point* x^* *is the only solution of equation* $F(x) = 0$ *in the set* U_2.

Proof. Let $y^* \in U_2$ with $F(y^*) = 0$. Define the linear operator $T_1 = [x^*, y^*; F]$. Then, by using the first condition in (h7) and (23.16), we get

$$\|F'(x^*)^{-1}(T_1 - F'(x^*))\| \leq w_1(0, \|y^* - x^*\|) \leq w_1(0, \rho^*) < 1,$$

implying $x^* = y^*$. □

Comments similar to Remark 2. can follow.

Case 2 Choose:

$$d_4(x_n, y_n) = y_n - A_n F(y_n),$$

where

$$A_n = [y_n, x_n; F]^{-1} F'(x_n) [y_n, x_n; F]^{-1}.$$

Then, assuming the iterates x_n, y_n exist, we have the estimate

$$\begin{aligned} z_n - x^* &= y_n - x^* - [y_n, x_n; F]^{-1} F'(x_n) [y_n, x_n; F]^{-1} [y_n, x^*; F](y_n - x^*) \\ &= \{I - [y_n, x_n; F]^{-1} F'(x_n) [y_n, x_n; F]^{-1} [y_n, x^*; F]\}(y_n - x^*) \end{aligned}$$

The expression inside the braces can be written as

$$[y_n, x_n; F]^{-1}([y_n, x_n; F] - F'(x_n)) + F'(x_n)[y_n, x_n; F]^{-1}([y_n, x_n; F] - [y_n, x^*; F]).$$

Composing by $F'(x^*)^{-1}$ and using the condition (h7) we get

$$\begin{aligned} & \|F'(x^*)^{-1}([y_n, x_n; F] - F'(x_n))\| + \|F'(x^*)^{-1} F'(x_n)\| \\ & \|[y_n, x_n; F]^{-1} F'(x^*)\| \|F'(x^*)^{-1}([y_n, x_n; F] - [y_n, x^*; F])\| \\ \leq\ & w_2(\|x_n - x^*\|, \|y_n - x^*\|) + \frac{w_6(\|x_n - x^*\| w_3(\|x_n - x^*\|, \|y_n - x^*\|)}{1 - w_1(\|x_n - x^*\|, \|y_n - x^*\|)} \\ =\ & e_n, \end{aligned}$$

so

$$\|z_n - x^*\| \leq \frac{1}{1 - w_1(\|x_n - x^*\|, \|y_n - x^*\|)} e_n \|y_n - x^*\|.$$

It follows that function g_2 can be defined by

$$g_2(t) = \frac{1}{1 - w_1(t, g_1(t)t)} \left(w_2(t, g_1(t)t) + \frac{w_6(t) w_3(t, g_1(t)t)}{1 - w_1(t, g_1(t)t)}\right).$$

Case 3 Choose:

$$d_4(x_n, y_n) = y_n - A_n F(y_n),$$

where

$$A_n = 2[y_n, x_n; F]^{-1} - F'(x_n)^{-1}.$$

This time we get

$$\begin{aligned} z_n - x^* &= y_n - x^* - A_n[y_n, x^*; F](y_n - x^*) \\ &= \{I - A_n[y_n, x^*; F]\}(y_n - x^*). \end{aligned}$$

The expression inside the braces can be written as

$$\begin{aligned} &F'(x_n)^{-1}(F'(x_n) - [y_n, x^*; F]) - 2[y_n, x_n; F]^{-1} \\ &(F'(x_n) - [y_n, x_n; F])F'(x_n)^{-1}[y_n, x^*; F]. \end{aligned}$$

Using the conditions (h7) we obtain

$$\begin{aligned} & \|F'(x_n)^{-1}F'(x^*)\| \|F'(x^*)^{-1}(F'(x_n) - [y_n, x^*; F])\| \\ & +2\|[y_n, x_n; F]^{-1}F'(x^*)\| \|F'(x^*)^{-1}(F'(x_n) - [y_n, x_n; F])\| \\ & \|F'(x_n)^{-1}F'(x^*)\| \|F'(x^*)^{-1}[y_n, x^*; F]\| \\ \leq \; & \frac{w_7(\|x_n - x^*\|, \|y_n - x^*\|)}{1 - w_0(\|x_n - x^*\|)} \\ & +2\frac{w_2(\|x_n - x^*\|, \|y_n - x^*\|)w_8(\|y_n - x^*\|)}{(1 - w_1(\|x_n - x^*\|, \|y_n - x^*\|))(1 - w_0(\|x_n - x^*\|))} = \lambda_n, \end{aligned}$$

thus

$$\|z_n - x^*\| \leq \lambda_n \|y_n - x^*\|.$$

Hence, the function g_2 can be defined by

$$g_2(t) = \frac{w_7(t, g_1(t)t)}{1 - w_0(t)} + 2\frac{w_2(t, g_1(t)t)w_8(g_1(t)t)}{(1 - w_0(t))(1 - w_1(t, g_1(t)t))}.$$

Bibliography

[1] Chun, C., Neta, B., A new sixth-order scheme for nonlinear equations, *Appl. Math. Lett.*, 215, (2012), 185-189.

[2] Kou, J., Li,Y., Wang, X., Some variants of Ostrowskis method with seventh-order convergence, *J. Comput. Appl. Math.*, 209, (2007), 153-159.

[3] Jarratt, P., Some fourth order multipoint methods for solving equations, *Math. Comput.*, 20, (1966), 434-437.

[4] Magréñan, A.A., A new tool to study real dynamics: the convergence plane, *Appl. Math. Comput.*, 248, (2014), 215-224.

[5] Sharma J.R. and Arora, H., A new family of optimal eighth order methods with dynamics for nonlinear equations, *Appl. Math. Comput.*, 273, (2016), 924-933.

[6] Traub, J.F., *Iterative Methods for the Solution of Equations*, Chelsea Publishing Company, New York (1982).

Chapter 24

Efficient Fifth Convergence Order Methods for Solving Equations in Banach Space

1. Introduction

Sharma and Guha (2014) in [14], studied the following iterative method of order five defined $\forall n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - 5F'(x_n)^{-1}F(y_n) \\ \text{and} \\ x_{n+1} &= y_n - \frac{9}{5}F'(x_n)^{-1}F(y_n) - \frac{1}{5}F'(x_n)^{-1}F(z_n), \end{aligned} \tag{24.1}$$

for approximating a locally unique solution x^* of the nonlinear equation

$$F(x) = 0. \tag{24.2}$$

Here $F : D \subset B_1 \longrightarrow B_2$ is a Fréchet-differentiable operator between Banach spaces B_1, B_2, and D is an open convex set in B_1. A plethora of applications reduce to solving equation (24.2). The solution x^* is needed in a closed form. But this is attainable only in special cases. That explains why most solution methods are iterative.

We show that the radius can be enlarged without new conditions than the conditions used in [14]. Other benefits include tighter error bounds on distances $\|x_n - x^*\|$ and better information on the uniqueness of the solution. The technique is independent of method (24.1). Thus, it can be used to extend the applicability of other methods [1,2,3,4,5,6,7,8, 9,10,11,12,13,15,16,17,18,19]. Moreover, the semi-local convergence of method (24.1) not given before in [14] is presented.

2. Local Convergence

The local convergence analysis of method (24.1) utilizes some function parameters. Let $S = [0,\infty)$. Suppose:

(i) $\exists$ function $w_0 : S \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$w_0(t) - 1 = 0$$

has a smallest solution $R_0 \in S - \{0\}$. Let $S_0 = [0,R_0)$.

(ii) $\exists$ function $w : S_0 \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$g_1(t) - 1 = 0$$

has a smallest solution $r_1 \in S_0 - \{0\}$, where the function $g_1 : S_0 \longrightarrow \mathbb{R}$ defined by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)}.$$

(iii) Equation

$$w_0(g_1(t)t) - 1 = 0$$

has a smallest solution $R_1 \in S_0 - \{0\}$. Let

$$R = \min\{R_0, R_1\}$$

and $S_1 = [0,R)$.

(iv) Equation

$$g_2(t) - 1 = 0$$

has a smallest solution $r_2 \in S_1 - \{0\}$, where the function $g_2 : S_1 \longrightarrow \mathbb{R}$ is defined as

$$\begin{aligned} g_2(t) &= \left[\frac{\int_0^1 w((1-\theta)g_1(t)t)d\theta}{1-w_0(g_1(t)t)}\right. \\ &\quad + \frac{w((1+g_1(t))t)(1+\int_0^1 w_0(\theta g_1(t)t)d\theta)}{(1-w_0(t))(1-w_0(g_1(t)t))} \\ &\quad \left. + \frac{4(1+\int_0^1 w_0(\theta g_1(t)t)d\theta}{1-w_0(t)}\right] g_1(t). \end{aligned}$$

(v) Equation

$$g_3(t) - 1 = 0$$

has a smallest solution $r_3 \in S_1 - \{0\}$, where the function $g_3 : S_1 \longrightarrow \mathbb{R}$ is defined by

$$\begin{aligned} g_3(t) &= g_1(t) + \frac{9(1+\int_0^1 w_0(\theta g_1(t)t)d\theta)g_1(t)}{5(1-w_0(t))} \\ &\quad \times (1+\int_0^1 w_0(\theta t)d\theta)g_2(t). \end{aligned}$$

The parameter r defined by

$$r = \min\{r_j\} \;\; j = 1,2,3 \tag{24.3}$$

shall be shown to be a radius of convergence for method (24.1) in Theorem 27. Let $S_2 = [0,r)$. Then, it follows by these definitions that for each $t \in S_2$

$$0 \leq w_0(t) < 1, \tag{24.4}$$

$$0 \leq w_0(g_1(t)t) < 1 \tag{24.5}$$

and

$$0 \leq g_i(t) < 1. \tag{24.6}$$

Let $U(x_0,\lambda) = \{x \in B_1 : \|x - x_0\| < \lambda\}$ and $U[x_0,\lambda] = \{x \in B_1 : \|x - x_0\| \leq \lambda\}$ for some $\lambda > 0$.

The conditions required are:

(C1) Equation $F(x) = 0$ has a simple solution $x^* \in D$.

(C2) $\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|)$ for all $x \in D$. Set $D_1 = U(x^*,R_0) \cap D$.

(C3) $\|F'(x^*)^{-1}(F'(y) - F'(x))\| \leq w(\|y - x\|)$ for all $x, y \in D_1$.
and

(C4) $U[x_0,r] \subset D$.

Next, the main local convergence result follows for method (24.1).

Theorem 27. Suppose that conditions (C1)-(C4) hold and $x_0 \in U(x^*,r) - \{x^*\}$. Then, sequence $\{x_n\}$ generated by method (24.1) is well defined in $U(x^*,R)$, remains in $U(x^*,R)$ $\forall n = 0,1,2,\ldots$ and is convergent to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{24.7}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{24.8}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{24.9}$$

where functions g_i are defined previously and the radius r is given by (24.3).

Proof. Let $u \in U(x^*,r) - \{x^*\}$. By using conditions (C1), (C2) and (24.3) we have that

$$\|F'(x^*)^{-1}(F'(u) - F'(x^*))\| \leq w_0(\|x_0 - x^*\|) \leq w_0(r) < 1. \tag{24.10}$$

It follows by (24.10) and the Banach lemma on invertible operators [2] that $F'(u)^{-1} \in L(B_2,B_1)$ and

$$\|F'(u)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x_0 - x^*\|)}. \tag{24.11}$$

If $u = x_0$, then iterate y_0 is well defined by the first substep of method (24.1) and we can write

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= F'(x_0)^{-1}\int_0^1 (F'(x^* + \theta(x_0 - x^*))d\theta - F'(x_0))(x_0 - x^*). \end{aligned} \tag{24.12}$$

In view of (C1) - (C3), (24.11) (for $u = x_0$), (24.6) (for $j = 1$) and (24.12), we get in turn that

$$\begin{aligned} \|y_0 - x^*\| &\leq \frac{\int_0^1 w((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{1 - w_0(\|x_0 - x^*\|)} \\ &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned} \tag{24.13}$$

so, iterate $y_0 \in U(x^*, r)$ and (24.7) holds for $n = 0$. Iterate z_0 is well defined by the second substep of the method and we can write

$$\begin{aligned} z_0 - x^* &= y_0 - x_0 - 5F'(x_0)^{-1}F(y_0) \\ &= y_0 - x^* - F'(y_0)^{-1}F(y_0) \\ &\quad + F'(y_0)^{-1}(F(x_0) - F'(y_0))F'(x_0)^{-1}F(y_0) \\ &\quad - 4F'(x_0)^{-1}F(y_0). \end{aligned} \tag{24.14}$$

Notice that linear operator $F'(y_0)^{-1}$ exists by (24.11) (for $u = y_0$). It follows by (24.3), (24.6) (for $j = 1$), (C3), (24.11) (for $u = x_0, y_0$), in turn that

$$\begin{aligned} \|z_0 - x^*\| &\leq \left[\frac{\int_0^1 w((1-\theta)\|y_0 - x^*\|)d\theta}{1 - w_0(\|y_0 - x^*\|)} \right. \\ &\quad + \frac{w(\|y_0 - x_0\|)(1 + \int_0 61 w_0(\theta\|y_0 - x^*\|)d\theta)}{(1 - w_0(\|x_0 - x^*\|))(1 - w_0(\|y_0 - x^*\|))} \\ &\quad \left. + \frac{4(1 + \int_0^1 w_0(\theta\|y_0 - x^*\|)d\theta}{1 - w_0(\|x_0 - x^*\|)} \right] \|y_0 - x^*\| \\ &\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|. \end{aligned} \tag{24.15}$$

Thus, iterate $z_0 \in U(x^*, r)$ and (24.8) holds for $n = 0$, where we also used (C1) and (C2) to obtain the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F(y_0)\| &= \|F'(x^*)^{-1}[\int_0^1 F'(x^* + \theta(y_0 - x^*))d\theta - F'(x^*) + F'(x^*)](y_0 - x^*)\| \\ &\leq (1 + \int_0^1 w_0(\theta\|y_0 - x^*\|)d\theta)\|y_0 - x^*\|. \end{aligned}$$

Moreover, iterate x_1 is well defined by the third substep of method (24.1), and we can write

$$x_1 - x^* = y_0 - x^* - \frac{1}{5}F'(x_0)^{-1}(9F(y_0) + F(z_0)),$$

leading to

$$\begin{aligned}\|x_1-x^*\| &\leq \|y_0-x^*\|+\frac{1}{5}\left(\frac{9(1+\int_0^1 w_0(\theta\|y_0-x^*\|)d\theta)\|y_0-x^*\|}{1-w_0(\|y_0-x^*\|)}\right.\\ &\quad \left.+(1+\int_0^1 w_0(\theta\|z_0-x^*\|)d\theta)\|z_0-x^*\|\right)\\ &\leq g_3(\|x_0-x^*\|)\|x_0-x^*\|\leq\|x_0-x^*\|<r, \qquad (24.16)\end{aligned}$$

so, iterate $x_1\in U(x^*,r)$ and (24.9) holds for $n=0$.

Simply replace x_0,y_0,z_0,x_1 by x_m,y_m,y_m,x_{m+1}, $\forall m=0,1,2\ldots$ in the preceding calculations to complete the induction for (24.7)- (24.9). Then, by the estimate

$$\|x_{m+1}-x^*\|\leq d\|x_m-x^*\|<r, \qquad (24.17)$$

where, $d=g_3(\|x_0-x^*\|)\in[0,1)$ that $x_{m+1}\in U(x^*,r)$ and $lim_{m\longrightarrow\infty}x_m=x^*$. □

The uniqueness of the solution result for method (24.1) follows.

Proposition 19. *Suppose:*

(i) Equation $F(x)=0$ has a simple solution $x^*\in U(x^*,\rho)\subset D$ for some $\rho>0$.

(ii) Condition (C2) holds.

(iii) There exists $\rho_1\geq\rho$ such that

$$\int_0^1 w_0(\theta\rho_1)d\theta<1. \qquad (24.18)$$

Set $D_2=U[x^*,\rho_1]\cap D$. Then, the only solution of equation $F(x)=0$ in the set D_2 is x^*.

Proof. Let $y^*\in D_2$ be such that $F(y^*)=0$. Define a linear operator $T=\int_0^1 F'(x^*+\theta(y^*-x^*))d\theta$. It then follows by (ii) and (24.18) that

$$\begin{aligned}\|F'(x^*)^{-1}(T-F'(x^*))\| &\leq \int_0^1 w_0(\theta\|y^*-x^*\|)d\theta\\ &\leq \int_0^1 w_0(\theta\rho_1)d\theta<1,\end{aligned}$$

so, we deduce $x^*=y^*$ by invertibility of T and the estimate $T(x^*-y^*)=F(x^*)-F(y^*)=0$. □

Remark. Under all conditions of Theorem 27, we can set $\rho=r$.

3. Semi-Local Convergence

As in the local case, we use some functions and parameters. Suppose:
There exists function $v_0 : S \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$v_0(t) - 1 = 0$$

has a smallest solution $\tau_0 \in S - \{0\}$. COnsider function $v : S_0 \longrightarrow \mathbb{R}$ continuous and non-decreasing. Define scalar sequences for $\eta \geq 0$ and all $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} t_0 &= 0,\ s_0 = \eta, \\ u_n &= s_n + \frac{5\int_0^1 v(\theta(s_n - t_n))d\theta(s_n - t_n)}{1 - v_0(t_n)}, \\ t_{n+1} &= u_n + \frac{1}{1 - v_0(t_n)}[(1 + \int_0^1 v_0(u_n + \theta(u_n - s_n))d\theta(u_n - s_n) \\ &\quad +3\int_0^1 v(\theta(s_n - t_n))d\theta(s_n - t_n)] \end{aligned} \tag{24.19}$$

and

$$\begin{aligned} s_{n+1} &= t_{n+1} + \frac{1}{1 - v_0(t_{n+1})}[\int_0^1 v(\theta(t_{n+1} - t_n))d\theta(t_{n+1} - t_n) \\ &\quad +(1 + \int_0^1 v_0(\theta t_n)d\theta(t_{n+1} - s_n)]. \end{aligned}$$

This sequence shall be shown to be majorizing for method (24.1) in Theorem 28. But first, we provide a general convergence result for sequence (24.19).

Lemma 37. Suppose that for all $n = 0, 1, 2, \ldots 9$

$$v_0(t_n) < 1 \tag{24.20}$$

and there exists $\tau \in [0, \tau_0)$ such that

$$t_n \leq \tau. \tag{24.21}$$

Then, sequence $\{t_n\}$ converges to some $t^* \in [0, \tau]$.

Proof. It follows by (24.19)-(24.21) that sequence $\{t_n\}$ is non-decreasing and bounded from above by τ and as such it converges to its unique least upper bound t^*. □

Next, the operator F is connected to the scalar functions. Suppose:

(h1) There exists $x_0 \in D, \eta \geq 0$ such that $F'(x_0)^{-1}L(B_2, B_1)$ and $\|F'(x_0)^{-1}F(x_0)\| \leq \eta$.

(h2) $\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq v_0(\|x - x_0\|)$ for all $x \in D$.
Set $D_3 = D \cap U(x_0, \tau_0)$.

(h3) $\|F'(x_0)^{-1}(F'(y) - F'(x))\| \leq v(\|y - x\|)$ for all $x, y \in D_3$.

(h4) Conditions of Lemma 37 hold.
and

(h5) $U[x_0, t^*] \subset D$.

Next, we present the semi-local convergence result for method (24.1).

Theorem 28. Suppose that conditions (h1)-(h5) hold. Then, sequence $\{x_n\}$ is well defined, remains in $U[x_0, t^*]$ and converges to a solution $x^* \in U[x_0, t^*]$ of equation $F(x) = 0$. Moreover, the following estimates hold

$$\|y_n - x_n\| \leq s_n - t_n, \tag{24.22}$$

$$\|z_n - y_n\| \leq u_n - s_n \tag{24.23}$$

and

$$\|x_{n+1} - z_n\| \leq t_{n+1} - u_n. \tag{24.24}$$

Proof. Mathematical induction is utilized to show estimates (24.22)-(24.24). Using (h1) and method (24.1) for $n = 0$

$$\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = s_0 - t_0 \leq t^*$$

so, iterate $y_0 \in U[x_0, t^*]$ and (24.22) holds for $n = 0$.

Let $u \in U[x_0, t^*]$. Then, as in Theorem 27, we get

$$\|F'(u)^{-1}F'(x_0)\| \leq \frac{1}{1 - v_0(\|u - x_0\|}. \tag{24.25}$$

Thus, if $u = x_0$ iterates y_0, z_0 and x_1 are well defined by method (24.1) for $n = 0$. Suppose iterates x_k, y_k, z_k, x_{k+1} also exist for all integer values k smaller than n. Then, we have the estimates

$$\begin{aligned} \|z_n - y_n\| &= 5\|F'(x_n)^{-1}F(y_n)\| \\ &\leq \frac{5\int_0^1 v(\theta\|y_n - x_n\|)d\theta\|y_n - x_n\|}{1 - v_0(\|x_n - x_0\|)} \\ &\leq \frac{5\int_0^1 v(\theta\|s_n - t_n))d\theta(s_n - t_n)}{1 - v_0(t_n)} = u_n - s_n, \end{aligned}$$

$$\begin{aligned} \|x_{n+1} - z_n\| &= \|\frac{1}{5}F'(x_n)^{-1}(F(y_n) - F(z_n)) + 3F'(x_n)^{-1}F(y_n)\| \\ &\leq \frac{1}{1 - v_0(\|x_n - x_0\|)}[(1 + \frac{1}{5}\int_0^1 v_0(\|z_n - x_0\| + \theta\|z_n - y_n)d)\|y_n - x_n\| \\ &\quad +3\int_0^1 v(\theta\|y_n - x_n\|d\theta\|y_n - x_n\|] \\ &\leq t_{n+1} - u_n \end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1}-x_{n+1}\| &= \|F'(x_{n+1})^{-1}F(x_{n+1})\| \\
&\leq \|F'(x_{n+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\| \\
&\leq \frac{1}{1-v_0(\|x_{n+1}-x_0\|)}[\int_0^1 v(\theta\|x_{n+1}-x_n\|)d\theta\|x_{n+1}-x_n\| \\
&\quad +(1+\int_0^1 v_0(\theta\|x_n-x_0\|)d\theta)\|x_{n+1}-y_n\|] \\
&\leq s_{n+1}-t_{n+1},
\end{aligned}$$

where we also used

$$\begin{aligned}
F(y_n) &= F(y_n)-F(x_n)-F'(x_n)(y_n-x_n) \\
&= \int_0^1 [F'(x_n+\theta(y_n-x_n))d\theta-F'(x_n)](y_n-x_n),
\end{aligned}$$

so

$$\|F'(x_0)^{-1}F(y_n)\| \leq \int_0^1 v(\theta\|y_n-x_n\|)d\theta\|y_n-x_n\|$$

and

$$\begin{aligned}
F(x_{n+1}) &= F(x_{n+1})-F(x_n)-F'(x_n)(y_n-x_n) \\
&\quad -F'(x_n)(x_{n+1}-x_n)+F'(x_n)(x_{n+1}-x_n) \\
&= F(x_{n+1})-F(x_n)-F'(x_n)(x_{n+1}-x_n)+F'(x_n)(x_{n+1}-y_n),
\end{aligned}$$

so

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq \int_0^1 v(\theta\|x_{n+1}-x_n\|)d\theta\|x_{n+1}-x_n\| \\
&\quad +(1+v_0(\|x_n-x_0\|))\|x_{n+1}-y_n\| \\
&\leq \int_0^1 v(\theta(t_{n+1}-t_n))d\theta(t_{n+1}-t_n) \\
&\quad +(1+v_0(t_n))(t_{n+1}-s_n), \qquad (24.26) \\
\|z_n-x_0\| &\leq \|z_n-y_n\|+\|y_n-x_0\| \qquad (24.27) \\
&\leq u_n-s_n+s_n-t_0 \leq t^*
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-x_0\| &\leq \|x_{n+1}-z_n\|+\|z_n-x_0\| \\
&\leq t_{n+1}-u_n+u_n-t_0 \leq t^*.
\end{aligned}$$

Hence, sequence $\{t_n\}$ is majorizing for method (24.1) and iterates $\{x_n\},\{y_n\},\{z_n\}$ belong in $U[x_0,t^*]$. The sequence $\{x_n\}$ is complete in Banach space B_1 and as such it converges to some $x^* \in U[x_0,t^*]$. By using the continuity of F and letting $n \longrightarrow \infty$ in (24.26) we conclude $F(x^*)=0$. □

Proposition 20. *Suppose:*

(i) There exists a solution $x^* \in U(x_0, \rho)$ *of equation* $F(x) = 0$ *for some* $\rho > 0$.

(ii) Condition (h2) holds.

(iii) There exists $\rho_1 \geq \rho$ *such that*

$$\int_0^1 v_0((1-\theta)\rho + \theta\rho_1)d\theta < 1. \tag{24.28}$$

Set $D_4 = D \cap U[x_0, \rho_1]$. *Then,* x^* *is the only solution of equation* $F(x) = 0$ *in the region* D_4.

Proof. Let $y^* \in D_4$ with $F(y^*) = 0$. Define the linear operator $Q = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$. Then, by (h2) and (24.28), we obtain in turn that

$$\begin{aligned} \|F'(x_0)^{-1}(Q - F'(x_0))\| &\leq \int_0^1 v_0((1-\theta)\|x_0 - y^*\| + \theta\|x_0 - x^*\|)d\theta \\ &\leq \int_0^1 v_0((1-\theta)\rho_1 + \theta\rho)d\rho < 1, \end{aligned}$$

so, $x^* = y^*$. □

Bibliography

[1] Amat, S., Bermudez, C., Busquier, S., Plaza, S., On a third order Newton-type method free of linear operators, *Numer. Lin. Alg. Appl.*, 17, (2010), 639-653.

[2] Argyros, I.K., *Computational theory of iterative methods. Series: Studies in Computational Mathematics*, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co., New York, U.S.A, 2007.

[3] Argyros, I.K. *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[4] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[5] Argyros, I.K., A Newton-Kantorvich theorem for equations involving m Frchet differentiable operators and applications in radiative transfer, *J. Comput. Appl. Math.*, 131, (1-2), (2001), 149-159.

[6] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)*, 2021, 9(16), 1942. https://doi.org/10.3390/math9161942.

[7] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative methods*, Elsevier (Academic Press), New York, 2018.

[8] Argyros, I.K, Behl, R., Motsa, S.S., Unifying semilocal and local convergence of Newton's method on Banach space with a convergence structure, *Appl. Numer. Math.*, 115, (2017), 225-234.

[9] Argyros, I.K., Cordero, A., Magreńán, A.A., Torregrosa, J.R., Third-degree anomalies of Traub's method, *J. Comput. Appl. Math.*, 309, (2017), 511-521.

[10] Cordero, A., Torregrosa, J.R., Variants of Newtons method for functions of several variables, *Appl. Math. Comput.*, 183, (2006), 199-208.

[11] Cordero, A., Hueso, J.L., Martińez, E., Torregrosa, J.R., A modified Newton-Jarratts composition, *Numer. Algor.*, 55, (2010), 87-99.

[12] Potra, F.A., Pták, V., *Nondiscrete Induction and Iterative Processes*, Pitman Publishing, Boston, (1984).

[13] Magréñan, A.A., Argyros, I.K., New improved convergence analysis for the secant method, *Math. Comput. Simulation*, 119, (2016), 161-170.

[14] Sharma, J.R., Guha, R.K., Simple yet efficient Newton-like method for systems of nonlinear equations, *Calcolo*, 53, (2016), 451-473.

[15] Sharma, J.R., Guha, R.K., Sharma, R., An efficient fourth order weighted-Newton method for systems of nonlinear equations, *Numer. Algor.*, 62, (2013), 307-323.

[16] Xiao, X., Yin. H., A new class of methods with higher order of convergence for solving systems of nonlinear equations, *Appl. Math. Comput.*, 264, (2015), 300-309.

[17] Xiao, X., Yin. H., A simple and efficient method with higher order convergence for solving systems of nonlinear equations, *Comput. Appl. Math.*, 69, (2015), 1220-1231.

[18] Xiao, X.Y., Yin, H.W., Increasing the order of convergence for iterative methods to solve nonlinear systems, *Calcolo*, 53, (2016), 285-300.

[19] Wang, X., An Ostrowski-type method with memory using a novel self-accelerating parameters, *J. Comput. Appl. Math.*, 330, (2018), 710-720.

Chapter 25

Efficient Derivative Free Seventh Order Methods for Solving Equations in Banach Space

1. Introduction

Let X be a Banach space and $\Omega \subset X$ be an open set. We are concerned with the problem of approximating a locally unique solution $x^* \in \Omega$ of the nonlinear equation

$$F(x) = 0, \tag{25.1}$$

where $F : \Omega \longrightarrow X$ is a Fréchet-differentiable operator. Iterative methods are used to approximate x^*. The well-known iterative methods are:
Newton Method (Second order [1,2,10]:

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \forall n = 0,1,2,\ldots, \tag{25.2}$$

where F' denote the Fréchet derivative of operator F [7,8].

In order to avoid the expensive in the general computation of F', Traub suggested:
Traub Method (second order) [12]:

$$\begin{aligned} x_n &= x_n + \alpha F(x_n) \\ \text{and} & \\ x_{n+1} &= x_n - [w_n, x_n; F]^{-1}F(x_n), \end{aligned} \tag{25.3}$$

where $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$ is a given parameter, operator $[.,.;F] : \Omega \times \Omega \longrightarrow L(X,X)$ is a divided difference of order one. In order to increase the convergence order some seventh-order methods are developed.

Wang-Zang Method (seventh order) [13]:

$$\begin{aligned} w_n &= x_n + \alpha F(x_n), \\ y_n &= x_n - [w_n, x_n; F]^{-1} F(x_n), \\ z_n &= d_4(x_n, y_n) \end{aligned} \tag{25.4}$$

and

$$x_{n+1} = z_n - A_n^{-1} F(z_n),$$

where, $\alpha \in \mathbb{R}$ or $\alpha \in \mathbb{C}$, $d_4(x_n, y_n)$ is any fourth-order Steffensen-type iteration method and $A_n = [z_n, x_n; F] + [z_n, y_n; F] - [y_n, x_n; F]$. Some interesting special cases of this class are defined by

$$\begin{aligned} w_n &= x_n + \alpha F(x_n), \\ y_n &= x_n - [w_n, x_n; F]^{-1} F(x_n), \\ z_n &= d_4^1(x_n, y_n) = y_n - B_n^{-1} F(y_n) \end{aligned} \tag{25.5}$$

and

$$x_{n+1} = z_n - A_n^{-1} F(z_n),$$

where $B_n = [y_n, x_n; F] + [y_n, w_n; F] - [w_n, x_n; F]$ and

$$\begin{aligned} w_n &= x_n + \alpha F(x_n), \\ y_n &= x_n - [w_n, x_n; F]^{-1} F(x_n), \\ z_n &= d_4^2(x_n, y_n) = y_n - C_n F(y_n) \end{aligned} \tag{25.6}$$

and

$$x_{n+1} = z_n - A_n^{-1} F(z_n),$$

where $C_n = [y_n, x_n; F]^{-1}([y_n, x_n; F] - [y_n, w_n; F] + [w_n, x_n; F])[y_n, x_n; F]$.

Sharma-Arora Method (seven order) [13]:

$$\begin{aligned} w_n &= x_n + \alpha F(x_n), \\ y_n &= x_n - [w_n, x_n; F]^{-1} F(x_n), \\ z_n &= y_n - (3I - [w_n, x_n; F]^{-1}([y_n, x_n; F + [y_n, w_n; F]))[w_n, x_n; F]^{-1} F(y_n) \end{aligned} \tag{25.7}$$

and

$$x_{n+1} = z_n - [z_n, y_n; F]^{-1}([w_n, x_n; F + [y_n, x_n; F]) - [z_n, x_n; F])[w_n, x_n; F]^{-1} F(z_n).$$

Method (25.5) and method (25.6) use four operators, five divided differences, and three linear operator inversions. But method (25.7) utilizes four operators, five divided differences, and two linear operator inversions.

2. Convergence for Method (25.4)

The convergence for all methods of the introduction requires the introduction of real functions and parameters. Set $M = [0, \infty)$.

Suppose:

(i) $\exists$ functions $v_0 : M \times M \longrightarrow \mathbb{R}$, $v_1 : M \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$v_0(v_1(t)t,t) - 1 = 0$$

has a smallest solution denoted by $\rho_0 \in M - \{0\}$. Let $M_0 = [0, \rho_0)$.

(ii) $\exists$ function $v : M_0 \times M_0 \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$g_1(t) - 1 = 0$$

has a smallest solution $r_1 \in M_0 - \{0\}$, where function $g_1 : M_0 \longrightarrow \mathbb{R}$ is defined by

$$g_1(t) = \frac{v(v_1(t)t,t)}{1 - v_0(v_1(t)t,t)}.$$

(iii) $\exists$ functions $g_2, g_3 : M_0 \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equations

$$g_2(t) - 1 = 0$$

has smallest solutions $\rho_2 \in M_0 - \{0\}$.

(iv) $\exists$ function $v_2 : M_0 \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$v_0(g_2(t)t, g_1(t)t) + v_2(t, g_1(t)t, g_2(t)t) - 1 = 0$$

has a smallest solution $\bar{\rho} \in M_0 - \{0\}$. Set $\bar{\bar{\rho}} = \min\{\rho_0, \rho\}$ and $M_1 = [0, \bar{\bar{\rho}})$.

(v) Equation

$$g_3(t) - 1 = 0$$

has a smallest solution $\rho_3 \in M_0 - \{0\}$, where

$$g_3(t) = \frac{(v(g_2(t)t,t) + v_2(t, g_1(t)t, g_2(t)t))g_2(t)}{1 - s(t)},$$

where

$$s(t) = v_0(g_2(t)t, g_1(t)t) + v_2(t, g_1(t)t, g_2(t)t).$$

The parameter ρ defined by

$$\rho = \min\{\rho_k\},\ k = 1,2,3 \tag{25.8}$$

shall be shown to be a radius of convergence for method (25.2). Set $M_2 = [0, \rho)$. Then, it follows that $\forall t \in M_2$

$$0 \le v_0(v_1(t)t,t) < 1, \tag{25.9}$$

$$0 \le s(t) < 1 \tag{25.10}$$

and

$$0 \le g_k(t) < 1. \tag{25.11}$$

The sets $U(x,\tau), U[x,\tau]$ denote open and closed balls respectively of center $x \in X$ and of radius $\tau > 0$.

The following conditions (H) shall be used although not all of them for each method.

Suppose:

(H1) There exists a simple solution $x^* \in \Omega$ of equation $F(x) = 0$.

(H2)
$$\|F'(x^*)^{-1}([p,x;F] - F'(x^*))\| \leq v_0(\|p - x^*\|, \|x - x^*\|)$$

and
$$\|I + \alpha[x,x^*;F]\| \leq v_1(\|x - x^*\|)$$

$\forall p, x \in \Omega$. Set $\Omega_1 = U(x^*, \rho_0) \cap \Omega$.

(H3)
$$\|F'(x^*)^{-1}([q,x;F] - [x,x^*;F])\| \leq v(\|q - x^*\|, \|x - x^*\|),$$
$$\|F'(x^*)^{-1}([z,y;F] - [y,x;F])\| \leq v_2(\|x - x^*\|, \|y - x^*\|, \|z - x^*\|),$$
$$\|F'(x^*)^{-1}[x,x^*;F]\| \leq v_3(\|x - x^*\|)$$

and
$$\|F'(x^*)^{-1}([p,x;F] - [y,x^*;F])\| \leq v_4(\|x - x^*\|, \|y - x^*\|, \|p - x^*\|)$$

$\forall x, y, z, p, q \in \Omega_1$.

(H4)
$$\|d_4(x,y) - x^*\| \leq g_2(\|x - x^*\|)\|x - x^*\|$$

$\forall x, y \in \Omega_1$
and

(H5) $U[x^*, r] \subset \Omega$, where $r = \max\{\rho, v_1(\rho)\rho\}$.

Notice that in the case of method (25.4) we shall use only the first two conditions from (H3).

Next, the following convergence result follows for method (25.4).

Theorem 29. Under conditions (H) hold and $x_0 \in U(x^*, \rho) - \{x^*\}$. Then, sequence $\{x_n\}$ generated by method (25.4) with starter x_0 is well defined and converges to the solution x^* of equation $F(x) = 0$.

Proof. By applying conditions (H1), (H2), (H3) using (25.8) and (25.9) we have in turn that

$$\begin{aligned}
\|w_0 - x^*\| &= \|x_0 - x^* + \alpha[F(x) - F(x^*)]\| \\
&= \|(I + \alpha[x,x^*;F])(x - x^*)\| \\
&\leq \|I + \alpha[x_0,x^*;F]\|\|x_0 - x^*\| \leq v_1(\rho)\rho \leq R
\end{aligned}$$

and

$$\|F'(x^*)^{-1}([w_0,x_0;F] - F'(x^*))\| \leq v_0(v_1(\|x_0 - x^*\|\|x_0 - x^*\|, \|x_0 - x^*\|) < 1.$$

It follows by the last estimate and a lemma due to Banach on linear invertible operators [1,2,7,10] that $[w_0,x_0;F]^{-1} \in L(X,X)$

$$\|[w_0,x_0;F]^{-1}F'(x^*)\| \leq \frac{1}{1-v_0(v_1(\|x_0-x^*\|)\|x_0-x^*\|,\|x_0-x^*\|)} \tag{25.12}$$

and iterate y_0 is well defined by the first substep of method (25.4) for $n=0$. Then, we can also write that

$$\begin{aligned} y_0-x^* &= x_0-x^*-[w_0,x_0;F]^{-1}F(x_0) \\ &= [w_0,x_0;F]^{-1}([w_0,x_0;F]-[x_0,x^*;F])(x_0-x^*). \end{aligned} \tag{25.13}$$

Using the first condition in (H3), (25.8), (25.11) (for $k=1$), (25.13) we get

$$\begin{aligned} \|y_0-x^*\| &\leq \frac{v(v_1(\|x_0-x^*\|)\|x_0-x^*\|,\|x_0-x^*\|)\|x_0-x^*\|}{1-v_0(v_1(\|x_0-x^*\|\|x_0-x^*\|,\|x_0-x^*\|)} \\ &\leq g_1(\|x_0-x^*\|)\|x_0-x^*\| < \|x_0-x^*\| < \rho. \end{aligned} \tag{25.14}$$

It follows by (H4), (25.11) (for $k=0$) and the definition of the second substep of method (25.4) that

$$\|z_0-x^*\| \leq g_2(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\|. \tag{25.15}$$

As in the derivation of (25.12) we obtain in turn

$$\begin{aligned} \|F'(x^*)^{-1}(A_0-F'(x^*))\| &\leq \|F'(x^*)^{-1}([z_0,y_0;F]-F'(x^*))\| \\ &\quad +\|F'(x^*)^{-1}([z_0,x_0;F]-[y_0,x_0;F])\| \\ &\leq v_0(\|z_0-x^*\|,\|y_0-x^*\|) \\ &\quad +v_2(\|x_0-x^*\|,\|y_0-x^*\|,\|z_0-x^*\|) \\ &\leq s(\|x_0-x^*\|) < 1, \end{aligned}$$

so

$$\|A_0^{-1}F'(x^*)\| \leq \frac{1}{1-s(\|x_0-x^*\|)} \tag{25.16}$$

and iterate x_1 is well defined by the third substep of method (25.4). We can also write

$$x_1-x^* = z_0-x^*-A_0^{-1}F9z_0) = A_0^{-1}(A_0-[z_0,x^*;F])(z_0-x^*)$$

leading to

$$\begin{aligned} \||x_0-x^*\| &\leq \|A_0^{-1}F'(x^*)\|\|F'(x^*)^{-1}(A_0-[z_0,x^*;F])\|\|z_0-x^*\| \\ &\leq \frac{[v(\|z_0-x^*\|,\|x_0-x^*\|)+v_2(\|x_0-x^*\|,\|y_0-x^*\|,\|z_0-x^*\|)]}{1-s(\|x_0-x^*\|)}\|z_0-x^*\| \\ &\leq g_3(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\|, \end{aligned} \tag{25.17}$$

where we also used (25.11) (for $k=3$), (H3) and (25.16), to obtain

$$\|F'(x^*)^{-1}(A_0-[z_0,x^*;F])\| \leq v(\|z_0-x^*\|,\|x_0-x^*\|)+v_2(\|x_0-x^*\|,\|y_0-x^*\|,\|z_0-x^*\|).$$

By simply revisiting these calculations for x_0, w_0, y_0, z_0, x_1 by $x_i, w_i, y_i, z_i, x_{i+1}$, we get

$$\|y_i - x^*\| \leq g_1(\|x_0 - x^*\|)\|x_i - x^*\| \leq \|x_i - x^*\|,$$

$$\|z_i - x^*\| \leq g_2(\|x_0 - x^*\|)\|x_i - x^*\| \leq \|x_i - x^*\|,$$

and

$$\|x_{i+1} - x^*\| \leq \mu\|x_i - x^*\| < \|x_i - x^*\|,$$

where, $\mu = g_3(\|x_0 - x^*\|) \in [0,1)$, concluding that $lim_{i \longrightarrow \infty} x_i = x^*$ and $x_{i+1} \in U(x^*, \rho)$. □

Next, we present a result on the uniqueness of the solution.

Proposition 21. *Suppose:*

(i) There exists a simple solution $x^* \in U(x^*, \rho_4)$ *of equation* $F(x) = 0$ *for some* $\rho_4 > 0$.

(ii) The first condition in (H2) holds and there exists $\rho_5 \geq \rho_4$ *such that*

$$v_0(0, \rho_5) d\theta < 1. \tag{25.18}$$

Set $\Omega_2 = U[x^*, \rho_5] \cap \Omega$. *Then, the is the only solution of equation* $F(x) = 0$ *in the region* Ω_2 *is* x^*.

Proof. Let $y^* \in \Omega_2$ with $F(y^*) = 0$. Define the operator $T = [x^8, y^*; F]$. By applying the first condition in (H2) and (25.18) we obtain

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \leq v_0(\|x^* - x^*\|, \|y^* - x^*\|) \leq v_0(0, \rho_5) < 1.$$

Hence $x^* = y^*$ follows from the invertibility of T and the identity $T(x^* - y^*) = F(x^*) - F(y^*) = 0$. □

Remark. If all conditions H are used, then certainly we can set $\rho_5 = \rho$.

3. Special Cases

The functions g_2 is determined for method (25.5) and method (25.6), respectively.
Case of method (25.5) The function g_2 is defined by

$$g_2(t) = \frac{(v(t, g_1(t)t) + v_2(t, g_1(t)t, v_1(t)t))g_1(t)}{1 - (v_0(t, g_1(t)t) + v_2(t, g_1(t)t, v_1(t)t))}. \tag{25.19}$$

Indeed, following the proof of Theorem 29, we can write by the second substep of the method

$$z_n - x^* = y_n - x^* - B_n^{-1} F(y_n) = B_n^{-1}(B_n - [y_n, x^*; F])(y_n - x^*). \tag{25.20}$$

We need the estimates

$$\begin{aligned}\|F'(x^*)^{-1}(B_n-[y_n,x^*;F])\| &= \|F'(x^*)^{-1}[([y_n,x_n;F]\\ &\quad -[y_n,x^*;F])+([y_n,w_n;F]-[w_n,x_n;F])]\|\\ &\leq \|F'(x^*)^{-1}([y_n,x_n;F]-[y_n,x^*;F])\|\\ &\quad +\|F'(x^*)^{-1}([y_n,w_n;F]-[w_n,x_n;F])\|\\ &\leq v(\|x_n-x^*\|,\|y_n-x^*\|)\\ &\quad v_2(\|x_n-x^*\|,\|y_n-x^*\|,v_1(\|x_n-x^*\|)\|x_n-x^*\|\end{aligned} \tag{25.21}$$

and

$$\begin{aligned}\|F'(x^*)^{-1}(B_n-F'(x^*)) &\leq v_0(\|x_n-x^*\|,\|y_n-x^*\|)\\ &\quad +v_2(\|x_n-x^*\|,\|y_n-x^*\|,v_1(\|x_n-x^*\|)\|x_n-x^*\|).\end{aligned} \tag{25.22}$$

Then, the choice of function g_2 follows from (25.20)-(25.22).
Case of method (25.6) The function g_2 is defined by

$$g_2(t)=\left(\frac{v(t,g_1(t)t)}{1-v_0(t,g_1(t)t)}+\frac{v_3(g_1(t)t)v_2(t,g_1(t)t,v_1(t)t)}{(1-v_0(t,g_1(t)t))^2}\right)g_1(t). \tag{25.23}$$

It follows by the second substep of method (25.6) that

$$z_n-x^*=D_n(y_n-x^*), \tag{25.24}$$

where

$$\begin{aligned}D_n &= I-[y_n,x_n;F]^{-1}([y_n,x_n;F]-[y_n,w_n;F]+[w_n,x_n;F])\\ &\quad [y_n,x_n;F]^{-1}[y_n,x^*;F]\\ &= [y_n,x_n;F]^{-1}([y_n,x_n;F]-[y_n,x^*;F])-[y_n,x_n;F]^{-1}\\ &\quad ([w_m,x_n;F]-[y_n,w_n;F])[y_n,x_n;F]^{-1}[y_n,x^*;F]\end{aligned}$$

and

$$\begin{aligned}\|D_n\| &\leq \frac{v(\|y_n-x^*\|,\|x_n-x^*\|)}{1-v_0(\|y_n-x^*\|,\|x_n-x^*\|)}\\ &\quad +\frac{v_3(\|y_n-x^*\|)v_2(\|x_n-x^*\|,\|y_n-x^*\|,\|w_n-x^*\|)}{(1-v_0(\|y_n-x^*\|,\|x_n-x^*\|))^2}.\end{aligned} \tag{25.25}$$

Estimates (25.24) and (25.25) justify the choice of function g_2. Then, with these choices of function g_2 the conclusions of Theorem 29 hold for method (25.5) and method (25.6).

4. Convergence of Method (25.7)

In this Section, the proof of Theorem 29 is also followed.

The second substep of method (25.7) can be written as

$$z_n - x^* = E_n(y_n - x^*), \tag{25.26}$$

where

$$\begin{aligned} E_n &= I - (3I - [w_n, x_n; F]^{-1}([y_n, x_n; F] + [y_n, w_n; F])) \\ &\quad [w_n, x_n; F]^{-1}[y_n, x^*; F] \\ &= [w_n, x_n; F]^{-1}([w_n, x_n; F] - [y_n, x^*; F]) \\ &\quad + [w_n, x_n; F]^{-1} G_n \end{aligned} \tag{25.27}$$

and

$$\begin{aligned} G_n &= [y_n, x_n; F][w_n, x_n; F][y_n, x^*; F] - [y_n, x^*; F] \\ &\quad + [[y_n, w_n; F][w_n, x_n; F]^{-1}[y_n, x*; F] - [y_n, x*; F] \\ &= ([y_n, x_n; F] - [w_n, x_n; F])[w_n, x_n; F]^{-1}[y_n, x^*; F] \\ &\quad + ([y_n, w_n; F] - [w_n, x_n; F])[w_n, x_n; F]^{-1}[y_n, x^*; F], \end{aligned}$$

so

$$\begin{aligned} \|E_n\| &\leq \|[w_n, x_n; F]^{-1}([w_n, x_n; F] - [y_n, x^*; F])\| \\ &\quad + \|[w_n, x_n; F]^{-1} F'(x^*)\| \|F'(x^*)^{-1} G_n\| \\ &\leq \frac{v_4(\|x_n - x^*\|, \|y_n - x^*\|, \|w_n - x^*\|)}{1 - v_0(\|x_n - x^*\|, \|w_n - x^*\|)} \\ &\quad + 2\frac{v_2(\|x_n - x^*\|, \|y_n - x^*\|, \|w_n - x^*\|) v_3(\|y_n - x^*\|}{1 - v_0(\|x_n - x^*\|, \|w_n - x^*\|)}. \end{aligned}$$

Hence, we can choose

$$\begin{aligned} g_2(t) &= \frac{1}{1 - v_0(t, v_1(t)t)}[v_4(t, g_1(t)t), v_1(t)t) \\ &\quad + 2\frac{v_2(t, g_1(t)t, v_1(t)t) v_3(g_1(t)t)}{1 - v_0(t, g_1(t)t)}\Bigg] g_1(t). \end{aligned}$$

The third substep of method (25.7) can be written as

$$x_{n+1} - x^* = H_n(z_n - x^*),$$

where

$$\begin{aligned} H_n &= I - [z_n, y_n; F]^{-1}([w_n, x_n; F] + [y_n, x_n; F] - [z_n, x_n; F]) \\ &\quad [w_n, x_n; F]^{-1}[z_n, x^*; F] \\ &= [z_n, y_n; F]^{-1}([z_n, y_n; F] - [z_n, x^*; F]) + [z_n, y_n; F]^{-1} \\ &\quad ([y - n, x_n; F] - [z_n, x_n; F])[w_n, x_n; F]^{-1}[z_n, x^*; F], \end{aligned}$$

so

$$\begin{aligned}\|H_n\| &\leq \frac{v(\|y_n-x^*\|,\|z_n-x^*\|)}{1-v_0(\|z_n-x^*\|,\|y_n-x^*\|)}\\ &+\frac{v_2(\|x_n-x^*\|,\|y_n-x^*\|,\|z_n-x^*\|)v_3(\|z_n-x^*\|)}{(1-v_0(\|z_n-x^*\|,\|y_n-x^*\|))(1-v_0(\|w_n-x^*\|,\|x_n-x^*\|))}.\end{aligned}$$

Thus, the choice of the function g_3 is

$$\begin{aligned}g_3(t) &= [\frac{v(g_1(t)t,g_2(t)t)}{1-v_0(g_2(t)t,g_1(t)t)}\\ &+\frac{v_2(t,g_1(t)t,g_2(t)t)v_3(g_2(t)t)}{(1-v_0(g_2(t)t,g_1(t)t))(1-v_0(v_1(t)t,t))}]g_2(t).\end{aligned}$$

Then, the conclusions of Theorem 29 with g_1 and g_2 hold for method (25.7).

Bibliography

[1] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[2] Argyros, I.K., George, S., Thapa, N., *Mathematical Modeling For The Solution Of Equations and Systems Of Equations With Applications*, Volume-II, Nova Publishers, NY, 2018.

[3] Cordero, A., Hueso, J.L., Martínez, E., Torregrosa, J.R., A modified Newton-Jarratts composition, *Numer. Algor.*, 55, 8799, (2010).

[4] Grau-Sánchez, M., Grau, Á., Noguera, M., Frozen divided difference scheme for solving systems of nonlinear equations, *J. Comput. Appl. Math.*, 235, 17391743, (2011).

[5] Grau-Sánchez, M., Peris, J.M., Gutiérrez, J.M., Accelerated iterative methods for finding solutions of a system of nonlinear equations, *Appl. Math. Comput.*, 190, 18151823, (2007).

[6] Liu, Z., Zheng, Q., Zhao, P., A variant of Steffensens method of fourth-order convergence and its applications, *Appl. Math. Comput.*, 216, 19781983, (2010).

[7] Ortega, J.M., Rheinboldt, W.C., *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, New York, (1970).

[8] Ostrowski, A.M., *Solution of Equations and Systems of Equations*. Academic Press, New York, (1966).

[9] Petković, M.S., Remarks on On a general class of multipoint root-finding methods of high computational efficiency, *SIAM J. Numer. Anal*, 49, 13171319, (2011).

[10] Potra, F.A., Pták, V., *Nondiscrete Induction and Iterative Processes*, Pitman Publishing, Boston, (1984).

[11] Sharma J.R and Arora, H., A novel derivative free algorithm with seventh order convergence for solving systems of nonlinear equations, *Numer. Algor.*, 67, (2014), 917-933.

[12] Traub, J.F., *Iterative Methods for the Solution of Equations*, Chelsea Publishing Company, New York, (1982).

[13] Wang, X., Zhang, T., A family of Steffensen type methods with seventh-order convergence, *Numer. Algor.*, 62, 429444, (2013).

Chapter 26

Necessary and Sufficient Conditions for the Q-Order of Convergence of Iterative Methods

1. Introduction

Let X and Y be a Banach space and $D \subset X$ be an open set. We are concerned with the problem of approximating a locally unique solution $x^* \in D$ of the nonlinear equation

$$F(x) = 0, \tag{26.1}$$

where $F : D \longrightarrow Y$ is a Fréchet-differentiable operator. Iterative methods are used to approximate x^*. In this Chapter, we consider the necessary and sufficient conditions for the convergence of the following method

$$\begin{aligned} y_n &= x_n - F'(x_n)F(x_n), \\ z_n &= y_n - a_n F'(x_n)^{-1}F(y_n) \\ \text{and} \quad & \\ x_{n+1} &= z_n - b_n F'(x_n)^{-1}F(z_n), \end{aligned} \tag{26.2}$$

where, $a_n = a(x_n - y_n)$, $a : D \times D \longrightarrow L(X,Y)$, $b_n = b(x_n, y_n, z_n)$, $b : D \times D \times D \longrightarrow L(X,Y)$. Relevant work but for special choices of the sequences $\{a_n\}$ and $\{b_n\}$ can be found in [4].

2. Local Convergence

The convergence uses some real functions and parameters. Set $T = [0, \infty)$.

Suppose:

(i) There exists function $\psi_0 : T \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$\psi_0(t) - 1 = 0$$

has a smallest solution $\rho_0 \in T - \{0\}$. Set $T_0 = [0, \rho_0)$.

(ii) There exists function $\psi : T_0 \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$h_1(t) - 1 = 0$$

has a smallest solution $r_1 \in T_0 - \{0\}$, where function $h_1 : T_0 \longrightarrow \mathbb{R}$ is defined by

$$h_1(t) = \frac{\int_0^1 \psi((1-\theta)t)d\theta}{1-\psi_0(t)}.$$

(iii) Equations

$$\psi_0(h_1(t)t) - 1 = 0$$

has smallest solutions $\rho_1 \in T_0 - \{0\}$.
Set $\rho_2 = \min\{\rho_0, \rho_1\}$ and $T_1 = [0, \rho_2)$.

(iv) There exist functions $\psi_1 : T_1 \times T_1 \longrightarrow \mathbb{R}$, $\psi_3 : T_1 \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$h_2(t) - 1 = 0$$

has a smallest solution $r_2 \in T_1 - \{0\}$, where the function $h_2 : T_1 \longrightarrow \mathbb{R}$ is defined by

$$\begin{aligned} h_2(t) &= \left[\frac{\int_0^1 \psi(\theta h_1(t)t)d\theta}{1-\psi_0(h_1(t)t)}\right. \\ &\left. + \frac{(\psi(1+h_1(t)t) + \psi_1(t, h_1(t)t)\psi_3(h_1(t)t)(1+\int_0^1 \psi_0(\theta h_1(t)t)d\theta)}{(1-\psi_0(t))(1-\psi_0(h_1(t)t))}\right] h_1(t). \end{aligned}$$

(v) Equation

$$\psi_0(h_2(t)t) - 1 = 0$$

has a smallest solution $\rho_3 \in T_0 - \{0\}$. Set $\rho = \min\{\rho_2, \rho_3\}$ and $T_2 = [0, \rho)$.

(vi) There exists function $\psi_2 : T_2 \times T_2 \times T_2 \longrightarrow \mathbb{R}$ continuous and nondecreasing such that equation

$$h_3(t) - 1 = 0$$

has a smallest solution $r_3 \in T_2 - \{0\}$, where the function $h_3 : T_2 \longrightarrow \mathbb{R}$ is defined by

$$\begin{aligned} h_3(t) &= \left[\frac{\int_0^1 \psi(\theta h_2(t)t)d\theta}{1-\psi_0(h_2(t)t)}\right. \\ &\left. + \frac{(\psi(1+h_2(t)t) + \psi_2(t, h_1(t)t, h_2(t)t)\psi_3(h_2(t)t)(1+\int_0^1 \psi_0(\theta h_2(t)t)d\theta}{(1-\psi_0(t))(1-\psi_0(h_2(t)t))}\right] h_2(t). \end{aligned}$$

The parameter r defined by

$$r = \min\{r_i\}, i = 1, 2, 3 \tag{26.3}$$

shall be shown to be a radius of convergence in Theorem 30 for method (26.2). Set $T_3 = [0, r)$.

It follows by (26.3) that $\forall t \in T_3$

$$0 \leq \psi_0(t) < 1, \tag{26.4}$$

$$0 \leq \psi_0(h_1(t)t) < 1, \tag{26.5}$$

$$0 \leq \psi_0(h_2(t)t) < 1 \tag{26.6}$$

and

$$0 \leq h_i(t) < 1. \tag{26.7}$$

By $U(x,R), U[x,R]$ we denote the open and closed balls in X of center $x \in X$ and of radius $R > 0$, respectively.

The following conditions are needed.
Suppose:

(c1) There exists a simple solution $x^* \in D$ of equation $F(x) = 0$.

(c2)

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq \psi_0(\|x - x^*\|) \ \forall x \in D.$$

Set $U_1 = U(x^*, \rho_0) \cap D$.

(c3)

$$\|(I - a(x))\| \leq \psi_1(\|x - x^*\|, \|y - x^*\|)$$

and

$$\|F'(x^*)^{-1}F'(x)\| \leq \psi_3(\|x - x^*\|)$$

for each $x, y \in U_1$. Set $U_2 = D \cap U(x^*, \rho)$.

(c4)

$$\|(I - b(x))\| \leq \psi_2(\|x - x^*\|, \|y - x^*\|, \|z - x^*\|)$$

for each $x, y, z \in U_2$.

(c5) $U[x^*, r] \subset D$.

Next, the main local convergence analysis of method (26.2) follows using the conditions (c1)-(c5).

Theorem 30. Suppose conditions (c1)-(c5) hold. Then, sequence $\{x_n\}$ generated by method (26.2) is well defined, remains in $U(x^*, r)$, and converges to x^* provided that $x_0 \in U(x^*, r) - \{0\}$. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$\|y_n - x^*\| \leq h_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{26.8}$$

$$\|z_n - x^*\| \leq h_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{26.9}$$

and

$$\|x_{n+1} - x^*\| \leq h_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{26.10}$$

where functions h_i are previously defined and the radius r is given by formula (26.3).

Proof. Assertions (26.8)-(26.10) are shown using mathematical induction. Let $u \in U(x^*,r) - \{x^*\}$. By applying (c1), (c2), (26.3) and (26.4) we obtain

$$\|F'(x^*)^{-1}(F'(u) - F'(x^*))\| \leq \psi_0(\|u - x^*\|) \leq \psi_0(r) < 1,$$

implying $F'(u)^{-1} \in L(Y,X)$ and

$$\|F'(u)^{-1}F'(x^*)\| \leq \frac{1}{1 - \psi_0\|u - x^*\|} \tag{26.11}$$

in view of the Banach Lemma on linear operators with inverses [1, 2, 3]. If follows by method (26.2) and (26.9) that iterate y_0, z_0 and x_1 are well defined. Then, we can write by the first substep of method (26.2)

$$y_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0),$$

so by (26.11) (for $u = x_0$), (c2), (26.3)and (26.7) (for $i = 1$)

$$\begin{aligned}\|y_0 - x^*\| &\leq \frac{\int_0^1 \psi((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\|}{1 - \psi_0(\|x_0 - x^*\|)} \\ &\leq h_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r. \end{aligned} \tag{26.12}$$

Thus, the iterate $y_0 \in U(x^*,r)$ and (26.8) holds for $n = 0$.

Then, by the second substep of method (26.2) we can write

$$\begin{aligned} z_0 - x^* &= y_0 - x^* - F'(y_0)^{-1}F(y_0) + F'(y_0)^{-1}F(y_0) - a_0F'(x_0)^{-1}F(y_0) \\ &= (y_0 - x^* - F'(y_0)^{-1}F(y_0) \\ &\quad + F'(y_0)^{-1}[(F'(x_0) - F'(y_0)) + F'(y_0)(I - a_0)]F'(x_0)^{-1}F(y_0). \end{aligned} \tag{26.13}$$

It follows by (26.3), (26.5), (26.7) (for $i = 2$), (26.11) (for $u = y_0$), (c1)-(c3) and (26.13) that

$$\begin{aligned} \|z_0 - x^*\| &\leq \left[\frac{\int_0^1 \psi(\theta\|y_0 - x^*\|)d\theta}{1 - \psi_0(\|y_0 - x^*\|)} \right. \\ &\quad \left. + \frac{(\psi(\|y_0 - x^*\|) + \|(I - a_0)\|\|F'(x^*)^{-1}F'(y_0)\|}{(1 - \psi_0(\|y_0 - x^*\|))(1 - \psi_0(\|x_0 - x^*\|))} \right] F'(x^*)^{-1}F'(y_0) \\ &\leq h_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|, \end{aligned} \tag{26.14}$$

implying $z_0 \in U(x^*,r)$ and (26.9) holds for $n = 0$, where we also used

$$F(y_0) = F(y_0) - F(x^*) = \int_0^1 F'(x^* + \theta(y_0 - x^*))d\theta(y_0 - x^*).$$

Hence,

$$\begin{aligned} \|F'(x^*)^{-1}F(y_0)\| &\leq \|F'(x^*)^{-1}[\int_0^1 F'(x^* + \theta(y_0 - x^*))d\theta - F'(x^*) \\ &\quad + F'(x^*)](y_0 - x^*)\| \\ &\leq (1 + \int_0^1 \psi_0(\theta\|y_n - x^*\|)d\theta)\|y_n - x^*\|. \end{aligned}$$

Similarly with the derivation of (26.14) but using the third substep of method (26.2) and the estimate

$$\begin{aligned} x_1 - x^* &= (z_0 - x^* - F'(z_0)^{-1}F(z_0) \\ &\quad + F'(z_0)^{-1}[(F'(x_0) - F'(z_0)) \\ &\quad + F'(z_0)(I - b_0)]F'(x_0)^{-1}F(z_0). \end{aligned} \tag{26.15}$$

Then, by exchanging y_0, z_0 by z_0, b_0 in (26.14), we get

$$\|x_1 - x^*\| \leq h_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|, \tag{26.16}$$

showing $x_1 \in U(x^*, r)$ and (26.10) holds for $n = 0$. By simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1}, in the preceding calculations to complete the induction for estimates (26.8)-(26.10). Then, by the estimate

$$\|x_{k+1} - x^*\| \leq \lambda\|x_k - x^*\| < r, \tag{26.17}$$

where, $\lambda = h_3(\|x_0 - x^*\|) \in [0, 1)$, we conclude $lim_{k \longrightarrow \infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. □

The proof of uniqueness result that follows is omitted since it is given in an earlier Chapter.

Proposition 22. *Suppose:*

(i) There exists a simple solution $x^* \in U(x^*, \mu)$ of equation $F(x) = 0$ for some $\mu > 0$.

(ii) The condition (c1) and (c2) holds

(iii) There exists $\mu_1 \geq \mu$ such that

$$\int_0^1 \psi_0(\theta\mu_1)d\theta < 1. \tag{26.18}$$

Set $M = U[x^*, \mu_1] \cap D$. Then, the point x^* is the only solution of equation $F(x) = 0$ in the set M.

The convergence of method (26.2) has been given under weak conditions. Stronger conditions are needed to show the order of convergence. That is why we state the following results but for $X = Y = \mathbb{R}^j$.

Theorem 31. [4] Suppose $F : D \subset \mathbb{R}^j \longrightarrow \mathbb{R}^j$ is sufficiently many times differentiable and x^* is a simple solution of equation $F(x) = 0$. Then, if x_0 is closed enough to the solution x^*, then the sequence

$$x_{n+1} = x_n - a_n F'(x_n)^{-1}F(x_n)$$

has order four if and only if $a_n = I + G_n + 2G_n^2 + O(h^3)$, where $h = \|F(x_n)\|$ and $G_n = \frac{1}{2}F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)$.

Theorem 32. [4] Suppose conditions of Theorem 31 hold. Then, method (26.2) has an order of convergence q if and only if operators a_n and b_n are given by the formulas in Table 1.

q	a_n	b_n
5	$I+O(G_n)$ $I+2G_n+\beta G_n^2$	$I+2G_n+\beta G_n^2$ $I+O(G_n)$
6	$I+2G_n+O(G_n^2)$	$I+2G_n+O(G_n)^2)$
7	$I+2G_n+6G_n^2+3d_n$	$I+2G_n+O(G_n^2)$

Bibliography

[1] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[2] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)*, 2021, 9(16), 1942, https://doi.org/10.3390/math9161942.

[3] Argyros, I.K., George, S., Thapa, N., *Mathematical Modeling For The Solution Of Equations And Systems Of Equations With Applications*, Volume-II, Nova Science Publishers, NY, 2018.

[4] Zhanlav, T., Chun, C., Otgondorj, Kh., Ulziibayar, V., High-Order iteration for systems of nonlinear equations, *International Journal of Computer Mathematics*, 97, 8, 1704-1724, DOI:10.1080/00207160.2019.1652739.

Chapter 27

Necessary and Sufficient Conditions for the Convergence of Sixth Order or Higher Derivative Free Methods

1. Introduction

Let $F : \Omega \subset X \longrightarrow Y$, be a nonlinear operator between the Banach spaces X and Y and Ω be a non empty convex subset of X. Consider the nonlinear equation

$$F(x) = 0. \tag{27.1}$$

Iterative methods are used to approximate the solution x^* of (27.1) since a closed-form solution is not easy to find. In this Chapter, we consider the iterative method defined for $n = 0, 1, 2, \dots$ by

$$\begin{aligned} w_n &= x_n + \delta F(x_n),\ v_n = x_n - \delta F(x_n), \\ y_n &= x_n - [w_n, v_n; F]^{-1} F(x_n), \\ z_n &= y_n - \alpha_n [w_n, v_n; F]^{-1} F(y_n) \end{aligned} \tag{27.2}$$

and

$$x_{n+1} = z_n - \beta_n [w_n, v_n; F]^{-1} F(z_n),$$

$\delta \in S$ where $S \in \mathbb{R}$ or $S \in \mathbb{C}$ is a given parameter, $[.,.;F] : \Omega \times \Omega \longrightarrow \mathcal{L}(X,Y)$ is a divided difference of order one [1,2,3] and $\alpha, \beta : \Omega \longrightarrow \Omega$ are linear operator with $\alpha_n = \alpha(x_n)$ and $\beta_n = \beta(x_n)$.

2. Local Convergence

It is convenient to define some functions and parameters. Set $T = [0, \infty)$.

Suppose:

1. There exist functions $a : T \longrightarrow \mathbb{R}, b : T \longrightarrow \mathbb{R}$ and $\varphi_0 : T \times T \longrightarrow \mathbb{R}$ continuous and nondecreasing such that the equation

$$\varphi_0(b(t)t, a(t)t) - 1 = 0$$

has a smallest solution $R_0 \in T - \{0\}$.
Set $T_0 = [0, R_0)$.

2. There exist function $\varphi : T_0 \times T_0 \longrightarrow \mathbb{R}$ such that the equation

$$h_1(t) - 1 = 0$$

has a smallest solution $r_1 \in T_0 - \{0\}$, where the function $h_1 : T_0 \longrightarrow \mathbb{R}$ is defined by

$$h_1(t) = \frac{\varphi((1 + b(t))t, a(t)t)}{1 - \varphi_0(b(t)t, a(t)t)}.$$

3. There exist functions $\varphi_1 : T_0 \longrightarrow \mathbb{R}, \varphi_2 : T_0 \times T_0 \longrightarrow \mathbb{R}, \varphi_3 : T_0 \times T_0 \times T_0 \longrightarrow \mathbb{R}$ continuous and nondecreasing such that the equations

$$\varphi_0(h_1(t)t, 0) - 1 = 0$$
and
$$\varphi_0(h_2(t)t, 0) - 1 = 0$$

have smallest solutions $R_1, R_2 \in T_0 - \{0\}$, respectively.
Set $T_1 = [0, \min\{R_0, R_1, R_2\})$. The equations

$$h_2(t) - 1 = 0$$

and

$$h_3(t) - 1 = 0$$

have smallest solutions $r_2, r_3 \in T_1 - \{0\}$, respectively, where

$$h_2(t) = \frac{[\varphi(h_1(t) + b(t))t, a(t)t) + \varphi_2(t, h_1(t)t)\varphi_1(h_1(t)t]h_1(t)}{(1 - \varphi_0(h_1(t)t, 0))(1 - \varphi_0(b(t)t, a(t)t))}$$
and
$$h_3(t) = \frac{[\varphi(h_2(t) + b(t))t, a(t)t) + \varphi_3(t, h_1(t)t), h_2(t)t\varphi_1(h_2(t)t]h_2(t)}{(1 - \varphi_0(h_2(t)t, 0))(1 - \varphi_0(b(t)t, a(t)t))}.$$

The parameter r is defined by

$$r = min\{r_j\},\ j = 1, 2, 3 \tag{27.3}$$

shall be shown to be a radius of convergence for method (27.2). Set $T_2 = [0, r)$.
Then, it follows by these definitions that for each $t \in T_2$

$$0 \leq \varphi_0(b(t)t, a(t)t) < 1, \tag{27.4}$$

$$0 \leq \varphi_0(h_2(t)t, 0) < 1, \tag{27.5}$$

$$0 \leq \varphi_0(h_2(t)t, 0) < 1 \tag{27.6}$$

and

$$0 \leq h_j(t) < 1. \tag{27.7}$$

Denote by $U(x, \lambda)$, $U[x, \lambda]$ the open and closed balls in X with center $x \in X$ and of radius $\lambda > 0$. The following conditions are utilized:

(C_1) There exist a simple solution $x^* \in \Omega$ of equation $F(x) = 0$.

(C_2)

$$\|I - \delta[x, x^*; F]\| \leq a(\|x - x^*\|),$$
$$\|I + \delta[x, x^*; F]\| \leq b(\|x - x^*\|)$$

and

$$\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \leq \varphi_0(\|x - x^*\|, \|y - x^*\|)$$

for all $x, y \in \Omega$.

(C_3)

$$\|F'(x^*)^{-1}([x, x^*; F]\| \leq \varphi_1(\|x - x^*\|) \;\; for\ all\ x \in T_1,$$
$$\|I - \alpha(x)\| \leq \varphi_2(\|x - x^*\|, \|y - x^*\|) \;\; for\ all\ x, y \in T_1,$$

and

$$\|I - b(x)\| \leq \varphi_3(\|x - x^*\|, \|y - x^*\|, \|z - x^*\|) \;\; for\ all\ x, y, z \in T_1.$$

(C_4) $U[x^*, r^*] \subset \Omega$, where $x^* = max\{r, b(r)r, a(r)r\}$.

Next, the local convergence analysis of method (27.2) is presented based on the developed terminology and the conditions $(C_1) - (C_4)$

Theorem 33. Suppose that the conditions $(C_1) - (C_4)$ hold. Then, the sequence $\{x_n\}$ generated by the method (27.2) is well defined, remains in $U[x^*, r^*)$ for all $n = 0, 1, 2, ...$ and converges to x^*, provided that the initial point $x_0 \in U[x^*, r) - \{x^*\}$. Moreover, the following assertions hold for all $n = 0, 1, 2, ...$

$$\|y_n - x^*\| \leq h_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{27.8}$$

$$\|y_n - x^*\| \leq h_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{27.9}$$

and

$$\|x_{n+1} - x^*\| \leq h_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{27.10}$$

where the functions h_j are defined previously and the radius of convergence r is given by formula (27.3).

Proof. Mathematical induction is used to show assertions (27.8)-(27.10), and the estimates(as in the previous chapter)

$$\|[w_n,v_n;F]^{-1}F'(x^*)\| \leq \frac{1}{1-\varphi_0(\|w_n-x^*\|,\|v_n-x^*\|)}, \tag{27.11}$$

$$\begin{aligned}
y_n-x^* &= x_n-x^*-[w_n,v_n;F]^{-1}F(x_n)\\
&= [w_n,v_n;F]^{-1}([w_n,v_n;F]-[x_n,x^*;F])(x_n-x^*),\\
\|y_n-x^*\| &\leq \frac{\varphi(\|w_n-x_n\|,\|v_n-x^*\|)\|x_n-x^*\|}{1-\varphi_0(\|w_n-x^*\|,\|v_n-x^*\|)}\\
&\leq \frac{\varphi((1+b(\|x_n-x^*\|)\|x_n-x^*\|,a(\|x_n-x^*\|)\|x_n-x^*\|)\|x_n-x^*\|}{1-\varphi_0(b(\|x_n-x^*\|)\|x_n-x^*\|,a(\|x_n-x^*\|)\|x_n-x^*\|)}\\
&\leq h_1(\|x_n-x^*\|)\|x_n-x^*\| \leq \|x_n-x^*\| < r,\\
\|w_n-x^*\| &= (I+\delta[x_n,x^*;F])(x_n-x^*)\| \leq a(\|x_n-x^*\|)\|x_n-x^*\|\\
&\leq b(r)r \leq r^*,\\
\|[y_n,x^*;F]^{-1}F'(x^*)\| &\leq \frac{1}{1-\varphi_0(h_1(\|x_n-x^*\|)\|x_n-x^*\|,0)},\\
\|[z_n,x^*;F]^{-1}F'(x^*)\| &\leq \frac{1}{1-\varphi_0(h_2(\|x_n-x^*\|)\|x_n-x^*\|,0)},\\
z_n-x^* &= y_n-x^*-[y_n,x^*;F]F(y_n)+[y_n,x^*;F]^{-1}F(y_n)\\
&\quad -a_n[w_n,v_n;F]^{-1}F(y_n)\\
&= [y_n,x^*;F]^{-1}([w_n,v_n;F]-[y_n,x^*;F]+[y_n,x^*;F]\\
&\quad \times[w_n,v_n;F]^{-1}(I-a_n))(y_n-x^*),
\end{aligned}$$

and

$$y_n-x^*-[y_n,x^*;F]F(y_n) = y_n-x^*-[y_n,x^*;F][y_n,x^*;F](y_n-x^*) = 0.$$

In the special case when $X=Y=\mathbb{R}^i$, the following results were given in [4]. □

Theorem 34. [4]. Suppose that operator F is sufficiently many times differentiable and $x^*\in\Omega$ is a simple solution of equation $F(x)=0$.
Then, the sequence $\{x_n\}$ generated by

$$y_n = x_n - L^{-1}F(x_n) \tag{27.12}$$

and

$$x_{n+1} = y_n - a_nL^{-1}F(y_n)$$

ha order of convergence equal to four if and only if

$$a_n = F'(y_n)^{-1}L+O(h^2),\ \ h=\|F(x_n)\|,$$

where

$$L_n = [w_n,x_n;F].$$

Theorem 35. [4] Suppose that conditions of Theorem 34 hold: Then, the convergence order of method(27.2) is equal to six if and only if

$$a_n = I + 2p_n + q_n + O(h^2),$$

where

$$p_n = \frac{1}{2}F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n),$$

and

$$q_n = \frac{1}{2}F'(x_n)^{-1}F''(x_n)\delta F(x_n),$$

Notice that the results in the last two theorems hold provided that F is sufficiently many times differentiable limiting the applicability of method (27.12) and method (27.2) in the special case although these methods may converge.

Similar results have been provided for methods with convergence orders higher than six in [4].

Bibliography

[1] Argyros,I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[2] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)*, 2021, 9, (16), 1942. https://doi.org/10.3390/math9161942.

[3] Argyros, I.K., George, S., Thapa, N., *Mathematical Modeling For The Solution of Equations And Systems of Equations With Applications*, Volume-II, Nova Science Publishers, NY, 2018.

[4] Zhanlav, T., Otgondorj, Kh., Saruul, L., A unified approach to the construction of higher order derivative free iterative methods for solving nonlinear system of equations, *Int. J. Computer. Math*, (2021).

Chapter 28

The Convergence Analysis of Some Fourth and Fifth Order Methods

1. Introduction

The local as well as the semi-local convergence analysis of certain fourth and fifth-order methods under the same conditions as in the previous chapter.

Cordero et al. method (Fourth order) [3]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ \text{and} \\ x_{n+1} &= y_n - (2I - F'(x_n))^{-1}F(y_n). \end{aligned} \tag{28.1}$$

Sharma et al. method (Fourth order) [5]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ \text{and} \\ x_{n+1} &= y_n - (3I - F'(x_n)^{-1}[y_n, x_n; F])F'(x_n)^{-1}F(y_n). \end{aligned} \tag{28.2}$$

Grau et al. method (Fifth order) [4]

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - \frac{1}{2}(I + F'(y_n)^{-1}F'(x_n))F'(x_n)^{-1}F(y_n) \\ \text{and} \\ x_{n+1} &= z_n - F'(y_n)^{-1}F(z_n). \end{aligned} \tag{28.3}$$

Xiao et al. method (Fifth order) [6]

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1}F(x_n),\ \alpha \in \mathbb{R} - \{0\} \\ z_n &= y_n - A_n^{-1}F'(x_n)^{-1}F(y_n), \\ A_n &= (1 - \frac{1}{2\alpha})I + \frac{1}{2\alpha}F'(x_n)^{-1}F'(y_n)) \end{aligned} \tag{28.4}$$

and

$$x_{n+1} = z_n - B_nF'(x_n)^{-1}F(z_n),$$

where

$$B_n = I + 2(\frac{1}{2\alpha}F'(y_n) + (1 - \frac{1}{2\alpha})F'(x_n)^{-1}F'(x_n).$$

2. Local Convergence

As in the previous Chapters the convergence theorems rely on some estimates and the corresponding functions.

Method (28.1)
It follows in turn from the two substeps of method (28.1), respectively

$$\|y_n - x^*\| \leq \frac{\int_0^1 \varphi((1-\theta)\|x_n - x^*\|)d\theta\|x_n - x^*\|}{1 - \varphi_0(\|x_n - x^*\|)} \leq g_1(\|x_n - x^*\|)\|x_nb - x^*\|$$

and

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|y_n - x^* - F'(y_n)^{-1}F(y_n) + F'(x_n)^{-1}(F'(x_n) - F'(y_n))F(y_n) \\ &\quad + F'(y_n)^{-1}(F'(x_n) - F'(y_n))F'(x_n)^{-1}F(y_n)\| \\ &\leq \left[\frac{\int_0^1 \varphi((1-\theta)\|y_n - x^*\|)d\theta}{1 - \varphi_0(\|y_n - x^*\|)} + \left(\frac{\varphi_0(\|x_n - x^*\|) + \varphi_0(\|y_n - x^*\|)}{1 - \varphi_0(\|x_n - x^*\|)}\right.\right. \\ &\quad \left. + \frac{\varphi_0(\|x_n - x^*\|) + \varphi_0(\|y_n - x^*\|)}{(1 - \varphi_0(\|x_n - x^*\|))(1 - \varphi_0(\|y_n - x^*\|))}\right) \\ &\quad \left. \times (1 + \int_0^1 \varphi_0(\theta\|y_n - x^*\|)d\theta)\right]\|y_n - x^*\| \\ &\leq g_2(\|x_n - x^*\|)\|x_n - x^*\|. \end{aligned}$$

Hence, we can define

$$g_1(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1 - \varphi_0(t)}$$

and

$$\begin{aligned} g_2(t) &= \left[\frac{\int_0^1 \varphi((1-\theta)g_1(t)t)d\theta}{1 - \varphi_0(g_1(t)t)} + \left(\frac{\varphi_0(t) + \varphi_0(g_1(t)t)}{1 - \varphi_0(t)}\right.\right. \\ &\quad \left.\left. + \frac{\varphi_0(t) + \varphi_0(g_1(t)t)}{(1 - \varphi_0(t))(1 - \varphi_0(g_1(t)t))}\right)(1 + \int_0^1 \varphi_0(\theta g_1(t)t)d\theta)\right]g_1(t). \end{aligned}$$

Method (28.2)

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|y_n-x^*-F'(y_n)^{-1}F(y_n)+F'(y_n)^{-1}(F'(x_n)-F'(y_n))^{-1}F'(x_n)^{-1}F(y_n)\\
&\quad +2F'(x_n)^{-1}(F'(x_n)-[y_n,x_n;F])F(y_n)\|\\
&\leq \left[\frac{\int_0^1\varphi((1-\theta)\|y_n-x^*\|)d\theta}{1-\varphi_0(\|y_n-x^*\|)}+\left(\frac{\varphi_0(\|x_n-x^*\|)+\varphi_0(\|y_n-x^*\|)}{(1-\varphi_0(\|x_n-x^*\|))(1-\varphi_0(\|y_n-x^*\|))}\right.\right.\\
&\quad \left.+\,2\frac{\varphi_1(\|x_n-x^*\|,\|y_n-x^*\|)}{1-\varphi_0(\|x_n-x^*\|)}\right)\\
&\quad \left.\times\,(1+\int_0^1\varphi_0(\theta\|y_n-x^*\|)d\theta)\right]\|y_n-x^*\|.
\end{aligned}$$

Thus, the function g_1 is the same as in method (28.1), where the function g_2 is defined by

$$\begin{aligned}
g_2(t) &= \left[\frac{\int_0^1\varphi((1-\theta)g_1(t)t)d\theta}{1-\varphi_0(g_1(t)t)}+\left(\frac{\varphi_0(t)+\varphi_0(g_1(t)t)}{(1-\varphi_0(t))(1-\varphi_0(g_1(t)t))}\right.\right.\\
&\quad \left.+\,2\frac{\varphi_1(t,g_1(t)t)}{1-\varphi_0(t)}\right)\\
&\quad \left.\times\,(1+\int_0^1\varphi_0(\theta g_1(t)t)d\theta)\right]g_1(t)t.
\end{aligned}$$

Method (28.3)

$$\begin{aligned}
\|z_n-x^*\| &= \|y_n-x^*-F'(y_n)^{-1}F(y_n)\\
&\quad +(F'(y_n)^{-1}-\frac{1}{2}(F'(x_n)^{-1}+F'(y_n)^{-1})F(y_n)\|\\
&\leq \left[\frac{\int_0^1\varphi((1-\theta)\|y_n-x^*\|)d\theta}{1-\varphi_0(\|y_n-x^*\|)}+\frac{1}{2}\left(\frac{\varphi_0(\|x_n-x^*\|)+\varphi_0(\|y_n-x^*\|)}{(1-\varphi_0(\|x_n-x^*\|))(1-\varphi_0(\|y_n-x^*\|))}\right.\right.\\
&\quad \left.\times\,(1+\int_0^1\varphi_0(\theta\|y_n-x^*\|)d\theta)\right]\|y_n-x^*\|
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|z_n-x^*-F'(z_n)^{-1}F(z_n)\\
&\quad +F'(z_n)^{-1}(F'(y_n)-F'(z_n))F'(y_n)^{-1})F(z_n)\|\\
&\leq \left[\frac{\int_0^1\varphi((1-\theta)\|z_n-x^*\|)d\theta}{1-\varphi_0(\|z_n-x^*\|)}+\left(\frac{\varphi_0(\|y_n-x^*\|)+\varphi_0(\|z_n-x^*\|)}{(1-\varphi_0(\|y_n-x^*\|))(1-\varphi_0(\|z_n-x^*\|))}\right.\right.\\
&\quad \left.\times\,(1+\int_0^1\varphi_0(\theta\|z_n-x^*\|)d\theta)\right]\|z_n-x^*\|.
\end{aligned}$$

Hence, the function g_1 is the same as in method (28.2) and

$$\begin{aligned} g_2(t) &\leq \left[\frac{\int_0^1 \varphi((1-\theta)g_1(t)t)d\theta}{1-\varphi_0(g_1(t)t)} + \left(\frac{\varphi_0(t)+\varphi_0(g_1(t)t)}{2(1-\varphi_0(t))(1-\varphi_0(g_1(t)t))}\right.\right. \\ &\quad \left.\times (1+\int_0^1 \varphi_0(\theta g_1(t)t)d\theta)\right] g_1(t) \end{aligned}$$

and

$$\begin{aligned} g_3(t) &\leq \left[\frac{\int_0^1 \varphi((1-\theta)g_2(t)t)d\theta}{1-\varphi_0(g_2(t)t)} + \frac{\varphi_0(g_1(t)t)+\varphi_0(g_2(t)t)}{(1-\varphi_0(g_1(t()t))(1-\varphi_0(g_2(t)t))}\right. \\ &\quad \left.\times (1+\int_0^1 \varphi_0(\theta g_2(t)t)d\theta)\right] g_2(t). \end{aligned}$$

Method (28.4)

$$\begin{aligned} \|y_n - x^*\| &= \|x_n - x^* - F'(x_n)^{-1}F(x_n) + (1-\alpha)F'(x_n)^{-1})F(x_n)\| \\ &\leq [\frac{\int_0^1 \varphi((1-\theta)\|x_n - x^*\|)d\theta + |1-\alpha|(1+\int_0^1 \varphi_0(\theta\|x_n - x^*\|)d\theta)}{1-\varphi_0(\|x_n - x^*\|)}] \\ &\quad \|x_n - x^*\|, \end{aligned}$$

$$\begin{aligned} \|z_n - x^*\| &= \|y_n - x^* - F'(y_n)^{-1}F(y_n) + (F'(y_n)^{-1} - C_n^{-1})F(y_n)\| \\ &\leq \left[\frac{\int_0^1 \varphi((1-\theta)\|y_n - x^*\|)d\theta}{1-\varphi_0(\|y_n - x^*\|)}\right. \\ &\quad \left. + \left(1+\frac{1}{2|\alpha|}\right)\frac{(\varphi_0(\|x_n - x^*\|)+\varphi_0(\|y_n - x^*\|))(1+\int_0^1 \varphi_0(\theta\|y_n - x^*\|)d\theta)}{(1-\varphi_0(\|x_n - x^*\|))(1-p(\|x_n - x^*\|))}\right] \\ &\quad \|y_n - x^*\|, \end{aligned}$$

where we also used

$$C_n = (1-\frac{1}{2\alpha})F'(x_n) + \frac{1}{2\alpha}F'(y_n) + F'(y_n),$$

so

$$\begin{aligned} \|F'(x^*)^{-1}(C_n - F'(x^*))\| &\leq \|F'(x^*)^{-1}(F'(x_n) - F'(x^*))\| \\ &\quad + \frac{1}{2|\alpha|}\|F'(x^*)^{-1}(F'(x_n) - F'(y_n))\| \\ &\leq (1+\frac{1}{2|\alpha|})\varphi_0(\|x_n - x^*\|) + \frac{1}{2|\alpha|}\varphi_0(\|y_n - x^*\|) \\ &\leq p(\|x_n - x^*\|), \end{aligned}$$

where

$$p(t) = (1+\frac{1}{2|\alpha|})\varphi_0(t) + \frac{1}{2|\alpha|}\varphi_0(g_1(t)t)$$

and

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|z_n - x^* - F'(z_n)^{-1}F(z_n)\\ &\quad + F'(z_n)^{-1}(F'(x_n)-F'(z_n))F'(x_n)^{-1})F(z_n) - 2C_n^{-1}F(z_n)\|\\ &\leq \left[\frac{\int_0^1 \varphi((1-\theta)\|z_n-x^*\|)d\theta}{1-\varphi_0(\|z_n-x^*\|)} + \left(\frac{\varphi_0(\|y_n-x^*\|)+\varphi_0(\|z_n-x^*\|)}{(1-\varphi_0(\|y_n-x^*\|))(1-\varphi_0(\|z_n-x^*\|))}\right.\right.\\ &\quad \left.\left. + \frac{2}{1-p(\|x_0-x^*\|}\right)(1+\int_0^1 \varphi_0(\theta\|z_n-x^*\|)d\theta)\right]\|z_n-x^*\|.\end{aligned}$$

Therefore, the corresponding functions are

$$g_1(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta + |1-\alpha|(1+\int_0^1 \varphi_0(\theta t)d\theta)}{1-\varphi_0(t)},$$

$$\begin{aligned}g_2(t) &= \left[\frac{\int_0^1 \varphi((1-\theta)g_1(t)t)d\theta}{1-\varphi_0(g_1(t)t)}\right.\\ &\quad \left. + \left(1+\frac{1}{2|\alpha|}\right)\frac{(\varphi_0(t)+\varphi_0(g_1(t)t))(1+\int_0^1 \varphi_0(\theta g_1(t)t)d\theta)}{(1-\varphi_0(t))(1-p(t))}\right] g_1(t)\end{aligned}$$

and

$$\begin{aligned}g_3(t) &= \left[\frac{\int_0^1 \varphi((1-\theta)g_2(t)t)d\theta}{1-\varphi_0(g_2(t)t)} + \left(\frac{\varphi_0(t)+\varphi_0(g_2(t)t)}{(1-\varphi_0(t))(1-\varphi_0(g_2(t)t))}\right.\right.\\ &\quad \left.\left. + \frac{2}{1-p(t)}\right)(1+\int_0^1 \varphi_0(\theta g_2(t)t)d\theta)\right] g_2(t).\end{aligned}$$

3. Semi-Local Convergence

We have in turn the estimates

$$\begin{aligned}\|x_{n+1}-z_n\| &= \|(I-F'(x_n)^{-1}F'(y_n))F'(x_n)^{-1}F(y_n) - F'(x_n)^{-1}F(y_n)\|\\ &\leq \left[\frac{\psi_0(\|x_n-x^*\|)+\psi_0(\|y_n-x_0\|)}{(1-\psi_0(\|x_n-x_0\|))^2} + \frac{1}{1-\psi_0(\|x_n-x_0\|)}\right]\\ &\quad \int_0^1 \psi((1-\theta)\|y_n-x_n\|)d\theta\|y_n-x_n\|\\ &\leq t_{n+1}-s_n\end{aligned}$$

and as before

$$\|y_{n+1}-x_{n+1}\| \leq s_{n+1}-t_{n+1}.$$

Hence, we define

$$\begin{aligned} t_{n+1} &= s_n + \left(\frac{\psi_0(t_n)+\psi_0(s_n)}{(1-\psi_0(t_n))^2} + \frac{1}{1-\psi_0(t_n)} \right) \\ &\quad \times \int_0^1 \psi((1-\theta)(s_n-t_n))d\theta(s_n-t_n) \end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{\int_0^1 \psi((1-\theta)(t_{n+1}-t_n))d\theta(t_{n+1}-t_n) + (1+\psi_0(t_n))(t_{n+1}-s_n)}{1-\psi_0(t_{n+1})}.$$

Method (28.2)

$$\begin{aligned} \|x_{n+1}-y_n\| &= \|2(I-F'(x_n)^{-1}[y_n,x_n;F])F'(x_n)^{-1}F(y_n)+F'(x_n)^{-1}F(y_n)\| \\ &\leq \left(\frac{2\psi_1(\|x_n-x^*\|,\|y_n-x_0\|)}{(1-\psi_0(\|x_n-x_0\|))^2} + \frac{1}{1-\psi_0(\|x_n-x_0\|)} \right) \\ &\quad \times \int_0^1 \psi((1-\theta)\|y_n-x_n\|)d\theta\|y_n-x_n\| \end{aligned}$$

and

$$\|y_{n+1}-x_{n+1}\| \leq s_{n+1}-t_{n+1},$$

where

$$t_{n+1} = s_n + \left(\frac{2\psi_1(t_n,s_n)}{(1-\psi_0(t_n))^2} + \frac{1}{1-\psi_0(t_n)} \right) \int_0^1 \psi((1-\theta)(s_n-t_n))d\theta(s_n-t_n)$$

and s_{n+1} is given as in method (28.1).

Method (28.3)

$$\begin{aligned} \|z_n-y_n\| &= \frac{1}{2}\|F'(x_n)^{-1}+F'(y_n)^{-1})F(y_n)\| \\ &\leq \frac{1}{2}\left(\frac{1}{1-\psi_0(\|x_n-x_0\|)} + \frac{1}{1-\psi_0(\|y_n-x_0\|)} \right) \\ &\quad \times \int_0^1 \psi((1-\theta)\|y_n-x_n\|)d\theta\|y_n-x_n\| \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1}-z_n\| &= \|F'(z_n)^{-1}F(z_n) \\ &\leq \frac{1}{1-\psi_0(\|y_n-x_0\|)} \\ &\quad [(1+\int_0^1 \psi((1-\theta)\|z_n-y_n\|)d\theta)\|z_n-y_n\| \\ &\quad + \int_0^1 \psi((1-\theta)\|y_n-x_n\|)d\theta\|y_n-x_n\|)], \end{aligned}$$

where we also used

$$\begin{aligned} F(z_n) &= F(z_n) - F(y_n) + F(y_n) \\ &= \int_0^1 F'(y_n + \theta(z_n - y_n))d\theta(z_n - y_n) \\ &\quad + \int_0^1 [F'(x_n + \theta(y_n - x_n))d\theta - F'(x_n)](y_n - x_n). \end{aligned}$$

Thus, the sequences are chosen as

$$u_n = s_n + \frac{1}{2}\left(\frac{1}{1-\psi_0(t_n)} + \frac{1}{1-\psi_0(s_n)}\right)\int_0^1 \psi(\theta(s_n - t_n))d\theta(s_n - t_n)$$

and

$$\begin{aligned} t_{n+1} &= u_n + \frac{1}{1-\psi_0(t_n)}\left(1 + \int_0^1 \psi_0(\theta(u_n - s_n))d\theta)(u_n - s_n)\right. \\ &\quad \left. + \int_0^1 \psi(\theta(s_n - t_n))d\theta(s_n - t_n)\right) \end{aligned}$$

where the iterate s_{n+1} is given as in the previous method.

Method (28.4)

$$\begin{aligned} \|z_n - y_n\| &= \|C_n^{-1}F(y_n)\| \\ &\leq \frac{(\int_0^1 \psi(\theta\|y_n - x_n\|)d\theta + \frac{1}{2|\alpha|}(1 + \psi_0(\|x_n - x_0\|)))\|y_n - x_n\|}{1 - q_n(\|x_n - x_0\|)}, \end{aligned}$$

where we also used

$$\begin{aligned} C_n - F'(x_0) &= F'(x_n) - F'(x_0) + \frac{1}{2\alpha}(F'(y_n) - F'(x_n)), \\ \|F'(x_0)^{-1}(C_n - F'(x_0))\| &\leq \psi_0(\|x_n - x_0\|) + \frac{1}{2|\alpha|}\psi(\|y_n - x_n\|) \\ &\leq q_n(\|x_n - x_0\|), \end{aligned}$$

$$q_n = \psi_0(\|x_n - x_0\|) + \frac{1}{2|\alpha|}\psi(\|y_n - x_n\|),$$

$$\begin{aligned} F(y_n) &= F(y_n) - F(x_n) - \frac{1}{\alpha}F'(x_n)(y_n - x_n) - F'(x_n)(y_n - x_n), \\ \|F'(x_0)^{-1}F(y_n)\| &\leq \left(\int_0^1 \psi(\theta\|y_n - x_n\|)d\theta + \frac{1}{2|\alpha|}(1 + \psi_0(\|x_n - x_0\|))\right)\|y_n - x_n\|, \end{aligned}$$

$$\begin{aligned}
\|x_{n+1}-z_n\| &= \|-F'(x_n)^{-1}F(z_n)+2C_n^{-1}F(z_n)\| \\
&= \|(C_n^{-1}-F'(x_n)^{-1})F(z_n)+C_n^{-1}F(z_n)\| \\
&= C_n^{-1}(F'(x_n)-D_n)F'(x_n)^{-}F(z_n)+C_n^{-1}F(z_n)\| \\
&\leq \left(\frac{\psi(\|y_n-x_n\|)}{2|\alpha|(1-\psi_0(\|x_n-x_0\|))(1-q_n)}+\frac{1}{1-q_n}\right) \\
&\quad \Big[(1+\int_0^1 \psi_0(\theta\|z_n-y_n\|)d\theta)\|z_n-y_n\| \\
&\quad +(\int_0^1 \psi(\theta\|y_n-x_n\|)d\theta+\frac{1}{2|\alpha|}(1+\psi_0(\|x_n-x_0\|))\|y_n-x_n\|\Big],
\end{aligned}$$

$$\begin{aligned}
\|y_{n+1}-x_{n+1}\| &\leq \frac{1}{1-\psi_0(\|x_{n+1}-x_0\|)}\Big[\int_0^1 \psi((1-\theta)\|x_{n+1}-x_n\|)d\theta\|x_{n+1}-x_n\| \\
&\quad +(1+\psi_0(\|x_n-x_0\|))(\|x_{n+1}-x_n\|+\frac{1}{|\alpha|}\|y_n-x_n\|\Big],
\end{aligned}$$

where we also used

$$\begin{aligned}
F(x_{n+1}) &= F(x_{n+1})-F(x_n)+F'(x_n)(x_{n+1}-x_n) \\
&\quad -\frac{1}{\alpha}F'(x_n)(y_n-x_n)+F'(x_n)(x_{n+1}-x_n),
\end{aligned}$$

leading to the numerator of the previous expression if composed in norm by $F'(x_0)^{-1}F(x_{n+1})$. Thus, we define the sequences by

$$\begin{aligned}
u_n &= s_n+\frac{(\int_0^1 \psi(\theta(s_n-t_n))d\theta+\frac{1}{2|\alpha|}(1+\psi_0(t_n)))(s_n-t_n)}{1-q_n}, \\
q_n &= \psi_0(t_n)+\frac{1}{2|\alpha|}\psi(s_n-t_n), \\
t_{n+1} &= u_n+\left(\frac{\psi(s_n-t_n)}{2|\alpha|(1-\psi_0(t_n))(1-q_n)}+\frac{1}{1-q_n}\right) \\
&\quad \Big[(1+\int_0^1 \psi(\theta(u_n-s_n))d\theta)(u_n-s_n) \\
&\quad +(\int_0^1 \psi(\theta(s_n-t_n))d\theta+\frac{1}{2|\alpha|}(1+\psi_0(t_n)))(s_n-t_n)\Big]
\end{aligned}$$

and

$$\begin{aligned}
s_{n+1} &= t_{n+1}+\frac{1}{1-\psi_0(t_{n+1})} \\
&\quad [\int_0^1 \psi((1-\theta)(t_{n+1}-t_n)d\theta(t_{n+1}-t_n)+(1+\psi_0(t_n))(t_{n+1}-t_n+\frac{1}{|\alpha|}(s_n-t_n)].
\end{aligned}$$

Bibliography

[1] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[2] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)*, 2021, 9, (16), 1942. https://doi.org/10.3390/math9161942.

[3] Cordero, A., Martnez, E., Torregrossa, J.R., Iterative methods of order four and five for systems of nonlinear equations, *J. Comput. Appl. Math.*, 231, 541551, (2012).

[4] Grau-Sanchez, M., Grau, A., Noguera, M., Ostrowski type method for solving systems of nonlinear equations, *Appl. Math. Comput.*, 218 (2011), 2377-2385.

[5] Sharma, J.R., Arora, H., On efficient weighted-Newton methods for solving systems of nonlinear equations, *Applied Mathematics and Computation*, 222, 1, (2013), 497-506.

[6] Xiao, X.Y., Yin, H.W., Increasing the order of convergence for iterative methods to solve nonlinear systems, *Calcolo*, 53, (2016), 285-300.

Chapter 29

High Convergence Order Derivative Free Methods-I

1. Introduction

The solution of the nonlinear equation

$$F(x) = 0 \tag{29.1}$$

are approximated using the following fourth-order derivative-free methods defined for all $n = 0, 1, 2, \ldots$, starting point $x_0 \in D$ and for some parameter $\alpha \in \mathbb{R} - \{0\}$ by

$$y_n = x_n - [v_n, x_n; F]^{-1} F(x_n), \quad v_n = x_n + \alpha F(x_n) \tag{29.2}$$

and

$$x_{n+1} = y_n - (3I - [v_n, x_n; F]^{-1}([y_n, x_n; F] + [v_n, v_n; F]))[v_n, x_n; F]^{-1} F(y_n).$$

$$y_n = x_n - [v_n, x_n; F]^{-1} F(x_n) \tag{29.3}$$

and

$$x_{n+1} = y_n - [y_n, x_n; F]^{-1} A_n [y_n, x_n; F]^{-1} F(x_n).$$

where

$$A_n = [y_n, x_n; F] - [y_n, v_n; F] + [v_n, x_n; F],$$

$$y_n = x_n - [v_n, x_n; F]^{-1} F(x_n)$$

and

$$x_{n+1} = y_n - B_n^{-1} F(y_n) \tag{29.4}$$

for

$$B_n = [y_n, x_n; F] + [y_n, v_n; F] - [v_n, x_n; F].$$

The local convergence order is shown using the condition on the fifth derivative when $B = \mathbb{R}^k$ [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11].

The semi-local convergence is not given. In this chapter, both the local as well as the semi-local convergence are presented in the more general setting of a Banach space. Moreover, conditions only on the operators appearing in these methods are used in the analysis. Hence, the applicability of these methods is extended.

2. Local Convergence Analysis

The analysis uses the same conditions for both methods. Let $M = [0,\infty)$. Suppose:

(i) There exist functions $\varphi_0 : M \longrightarrow \mathbb{R}, \gamma : M \longrightarrow \mathbb{R}$ such that the equation

$$\varphi_0(\gamma(t)t,t) - 1 = 0$$

has the smallest solution $\rho \in M - \{0\}$.
Let $M_0 : [0,\rho)$.

(ii) There exist functions $\delta : M_0 \longrightarrow \mathbb{R}, \varphi : M_0 \times M_0 \longrightarrow \mathbb{R}$,
$\varphi_1 : M_0 \times M_0 \times M_0 \longrightarrow \mathbb{R}, \varphi_2 : M_0 \times M_0 \times M_0 \longrightarrow \mathbb{R}$
such that the equations

$$g_i(t) - 1 = 0, \quad i = 1,2$$

have the smallest solutions $\mu_i \in M_0 - \{0\}$, where

$$g_1(t) = \frac{\varphi(\gamma(t)t,t)}{1-\varphi_0(\gamma(t)t,t)}$$

and

$$g_2(t) = [\frac{\varphi_1(\gamma(t)t,t,g_1(t)t)}{1-\varphi_0(\gamma(t)t,t)} + \frac{(\varphi_1(\gamma(t)t,t,g_1(t)t)+\varphi_2(\gamma(t)t,t,g_1(t)t))\delta(t)t}{(1-\varphi_0(\gamma(t)t,t))^2}]g_1(t).$$

We assume that the functions φ, φ_1, and φ_2 are symmetric.
The parameter r defined by

$$r = min\{r_1, r_2\} \tag{29.5}$$

shall be shown to be a radius of convergence for method (29.2) in Theorem 2.1.
Let $M_1 = [0,\delta)$. Then, it follows by these definitions that for all $t \in M_1$

$$0 \le \varphi_0(\gamma(t)t,t) < 1 \tag{29.6}$$

and

$$0 \le g_i(t) \le 1. \tag{29.7}$$

By $U(x^*,\lambda), U[x^*,\lambda]$ we denote the open and closed balls in B with center x^* and of radius $\lambda > 0$.
The scalar functions are connected to the operators appearing on the methods as follows, Suppose:

(a_1) There exist a simple solution $x^* \in D$ of the equation $F(x) = 0$ such that $F^{'}(x^*)^{-1} \in \delta(B_1, B)$.

(a_2)

$$\|I+\alpha[x,x^*;F]\| \leq \gamma(\|x-x^*\|)$$

and

$$\|F^{'}(x^*)^{-1}([z,x;F]-F^{'}(x^*)\| \leq \varphi_0(\|z-x^*\|,\|x-x^*\|)$$

for all $x,z \in D$.
Set $D_0 = U[x^*,\delta) \cap D$.

(a_3)

$$\|F^{'}(x^*)^{-1}[y,x^*;F]\| \leq \delta(\|y-x^*\|,$$
$$\|F^{'}(x^*)^{-1}([z,x;F]-[x,x^*;F])\| \leq \varphi(\|z-x^*\|,\|x-x^*\|),$$
$$\|F^{'}(x^*)^{-1}([z,x;F]-[y,x;F])\| \leq \varphi_1(\|z-x^*\|,\|y-x^*\|,\|x-x^*\|)$$

and

$$\|F^{'}(x^*)^{-1}([z,x;F]-[y,z;F])\| \leq \varphi_2(\|z-x^*\|,\|x-x^*\|,\|y-x^*\|)$$

for all $x,y,z \in D_0$.
and

(a_4) $U[x^*,R] \subset D, R = max\{r,\gamma(r)r\}$

Next, the local convergence analysis for method (29.2) is presented which is based on the conditions $(a_1)-(a_4)$ and the developed terminology.

Theorem 36. Suppose that the conditions $(a_1)-(a_4)$ hold. Then, if $x_0 \in U(x^*,r)-\{x^*\}$ the sequence $\{x_n\}$ generated by method (29.2) is well defined, remains in $U(x^*,r)$ for all $n = 0,1,2,...$ and converges to x^*. Moreover, the following assertion hold for $e_n = \|x_n - x^*\|$:

$$\|y_n - x^*\| \leq g_1(e_n)e_n \leq e_n < r \tag{29.8}$$

and

$$e_{n+1} \leq g_2(e_n)e_n \leq e_n, \tag{29.9}$$

where the functions y_i are defined previously and the convergence radius r is given by formula (29.5).

Proof. Estimates (29.8) and (29.9) are shown using mathematical induction. Let $\mu = x + \alpha F(x) \in U(x^*,r)-\{x^*\}$. By applying (a_2) and using (a_1),(a_2) and (29.5) we obtain in turn

$$\begin{aligned}\|F^{'}(x^*)^{-1}([\mu,x;F]-F^{'}(x^*))\| &\leq \varphi_0(\|\mu-x^*\|,\|x-x^*\|)\\ &\leq \varphi_0(\gamma(\|x-x^*\|,\|x-x^*\|),\|x-x^*\|)\\ &\leq \varphi_0(\gamma(r)r,r) < 1,\end{aligned} \tag{29.10}$$

since $\|\mu - x^*\| = \|(I+[x,x^*;F])(x-x^*)\| \leq \gamma(\|x-x^*\|)\|x-x^*\|$.
It follows that $[\mu,x;F]^{-1} \in \delta^{-1}(B,B)$ according to the Banach lemma on invertible operators $[1,2,3,4]$, and

$$\|[x+\alpha F(x),x;F]^{-1}F^{'}(x^*)\| \leq \frac{1}{1-\varphi_0(\gamma(\|x-x^*\|)\|x-x^*\|,\|x-x^*\|)}. \tag{29.11}$$

In particular, if $x = x_0$, the iterate y_0 is well defined by the first sub-step of the method (29.2). Moreover, we can write

$$y_0 - x^* = ([v_0, x_)); F]^1 F'(x^*)^{-1}([v_0, x_0; F] - [x_0, x^*; F])(v_0 - x^*). \tag{29.12}$$

Using (29.5),(C_3),(29.7) (for $i = 1$),(29.11)(for $x = x_0$) and (29.12)

$$\begin{aligned} \|y_0 - x^*\| &\leq \frac{\varphi(|\alpha|\beta(\|x_0 - x^*\|)\|x_0 - x^*\|, \|x_0 - x^*\|)\|x_0 - x^*\|}{1 - \varphi_0(\gamma(\|x_0 - x^*\|)\|x_0 - x^*\|, \|x_0 - x^*\|)} \\ &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r. \end{aligned} \tag{29.13}$$

Hence, the iterate $x_0 \in U(x^*, r)$ and assertion (29.8) hold for $n = 0$. Notice that the iterate x_1 is also well defined by the second sub-step of method (29.2) for $n = 0$. Furthermore, we can write in turn that

$$\begin{aligned} x_1 - x^* &= y_0 - x^* - [v_0, x_0; F]^{-1}F(y_0) \\ &+ (2I - [v_0, x_0; F]^{-1}([y_0, x_0; F] - [y_0, v_0; F])[v_0, x_0; F]^{-1}F(y_0) \\ &= y_0 - x^* - [v_0, x_0; F]^{-1}F(y_0) \\ &+ \{[v_0, x_0; F]^{-1}([v_0, x_0; F] - [y_0, x_0; F]) + [v_0, x_0; F]^{-1} \\ &([v_0, x_0; F] - [y_0, v_0; F])\}[v_0, y_0; F]^{-1}F(y_0). \end{aligned} \tag{29.14}$$

Then, as in (29.13) but using (29.7) for $i = 2$ and (29.14) we get in turn that

$$\begin{aligned} \|x_1 - x^*\| \leq [&\frac{\varphi_1(\|v_0 - x^*\|, \|x_0 - x^*\|, \|y_0 - x^*\|)}{1 - \varphi_0(\gamma(\|x_0 - x^*\|)\|x_0 - x^*\|, \|x_0 - x^*\|)} \\ &+ \frac{1}{(1 - \varphi_0(\gamma(\|x_0 - x^*\|)\|x_0 - x^*\|, \|x_0 - x^*\|))^2} \\ &(\varphi_1(\|v_0 - x^*\|, \|x_0 - x^*\|, \|y_0 - x^*\|) \\ &+ \varphi_2(\|v_0 - x^*\|, \|x_0 - x^*\|, \|y_0 - x^*\|))\delta(\|y_0 - x^*\|))]\|y_0 - x^*\| \\ &\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|. \end{aligned} \tag{29.15}$$

Thus, the iterate $x_1 \in U(x^*, r)$ and the assertion (29.9) holds for $n = 0$. Suppose these assertions hold for all integer values of the index smaller than n. Then, simply switch x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding calculations to terminate the induction for the assertions (29.8) and (29.9). Then, in view of the estimate

$$e_{k+1} \leq ae_k < r, \tag{29.16}$$

where $a = g_2(\|x_0 - x^*\|)$, we conclude that the iterate $x_{k+1} \in U(x^*, r)$ and $lim_{n \longrightarrow \infty} x_k = x^*$. □

Next, The uniqueness of the solution result is presented.

Proposition 23. *Suppose:*

(*i*) *There exist a simple solution* $x^* \in U(x^*, \rho_1)$ *of the equation* $F(x) = 0$.

(ii) $\|F'(x^*)^{-1}([x,x^*;F]-F'(x^*))\| \leq \varphi_3(\|x-x^*\|)$ *for all* $x \in D$ *and some function* $\varphi_3 : M \longrightarrow \mathbb{R}$ *which is continuous and nondecreasing.*
and

(iii) *There exist* $\rho_2 \geq \rho_1$ *such that*

$$\varphi_3(\rho_2) < 1. \tag{29.17}$$

Let $D_1 = U[x^*, \rho_2] \cap D$.
Then, the point x^* *is the only solution of the equation* $F(x) = 0$ *in the region* D_1.

Proof. Define the linear operator $T = [x^*, y^*; F]$, where $y^* \in D_1$ with $F(y^*) = 0$. Using $(i), (ii)$ and (29.17), we have

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \leq \varphi_3(\|y^* - x^*\|) \leq \varphi_3(\rho_2) < 1,$$

tending to $x^* = t^*$ by the invertibility of the operator T and the identity $T(y^* - x^*) = F(y^*) - F(x^*) = 0$. □

Remark. It is worth noticing that not all hypotheses of Theorem 36 are used to establish the uniqueness of the solution in Proposition 23.

Next, we similarly present the local convergence of method (29.3) under the conditions $(a_1) - (a_4)$ but we define the function g_2 by

$$g_2(t) = \left[\frac{\varphi(t, g_1(t)t)}{1 - \varphi_0(t, g_1(t)t)} + \frac{(\varphi_2(\gamma(t)tt, t, g_1(t)t)\delta(g_1(t)t)}{(1 - \varphi_0(t, g_1(t)t))^2}\right] g_1(t).$$

The definition of the function g_2 is a consequence of the estimate obtained from the second sub-step of the method (29.3) as follows

$$\begin{aligned} x_{n+1} - x^* &= y_n - x^* - [y_n, x_n; F]^{-1} F(y_n) \\ &\quad + [y_n, x_n; F]^{-1}([v_n, x_n; F] - [y_n, v_n; F])[y_n, x_n; F]^{-1} F(y_n), \end{aligned}$$

so

$$\|x_{n+1} - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r,$$

where we also used the estimate

$$\|[y_n, x_n; F]^{-1} F'(x^*)\| \leq \frac{1}{1 - \varphi_0(\|y_n - x^*\|, \|x_n - x^*\|)}.$$

Then, the result in Theorem 36 and Proposition 23 can be rewritten with the above changes but for method (29.3) replacing method (29.2).

3. Semi-Local Convergence

The analysis is given based on majorizing sequence
The following sequence $\{t_n\}$ shall be shown to be majorizing in Theorem 37. This sequence is defined for all $n = 0, 1, 2, ..., t_0 = 0, s_0 = \Omega$ by

$$t_{n+1} = s_n + [\frac{\varphi_1(c(t_n)t_n + b, t_n, s_n)}{1 - \varphi_0(c(t_n)t_n + b, t)} + \frac{1}{(1 - \varphi_0(c(t_n)t_n + b, s_n))^2}$$
$$(\varphi_1(c(t_n)t_n + b, t_n, s_n) + \varphi_2(c(t_n)t_n + b, t_n, s_n)) \tag{29.18}$$
$$\varphi_1(c(t_n)t_n + b, t_n, s_n)](s_n - t_n)$$

and

$$s_{n+1} = t_{n+1} + \frac{\sigma_{n+1}}{1 - \varphi_0(c(t_{n+1})t_{n+1} + b, t_{n+1})}$$

where

$$\sigma_{n+1} = (1 + \varphi_0(t_{n+1}, t_n))(t_{n+1} - t_n) + (1 + \varphi_0(c(t_n)t_n + b, t_n))(s_n - t_n),$$

where the functions "φ" are as the "φ" functions of the previous section. But first we present a convergence result for sequence $\{t_n\}$.
The conditions for the semi-local convergence for both methods are:

(h_1) There exist $x_0 \in D, \Omega \geq 0$ such that

$$F'(x_0)^{-1} \in \delta(B, B), \quad |\alpha| \|F(x_0)\| \leq b \text{ and } \|[v_0, x_0; F]^{-1} F(x_0)\| \leq \Omega.$$

(h_2)

$$\|I + \alpha[x, x_0; F]\| \leq c(\|x - x_0\|$$

and

$$\|F'(x_0)^{-1}([z, x_0; F] - F(x_0)))\| \leq \varphi_0(\|z - x_0\|, \|x - x_0\|)$$

for all $x, z \in D$. Set $D_2 = U(x_0, b_0) \cap D$.

(h_3)

$$\|F'(x_0)^{-1}([z, x; F] - [y, x; F])\| \leq \varphi_1(\|v - x_0\|, \|x - x_0\|, \|y - x_0\|)$$

and

$$\|F'(x_0)^{-1}([z, x; F] - [y, z; F])\| \leq \varphi_2(\|v - x_0\|, \|x - x_0\|, \|y - x_0\|).$$

(h_4) Conditions of Lemma 38 hold
and

(h_5) $U[x_0, R^*] \subset D$, where $R^* = c(t^*)t^* + b$.

Lemma 38. Suppose

$$\varphi_0(c(t_n)t_n + |\alpha|\Omega_0, t_n) < 1, t_n < \tau \quad for \ \ some\, \tau > 0 \tag{29.19}$$

for all $n = 0, 1, 2, \ldots$. Then, the sequence $\{t_n\}$ is non-decreasing and converges to its unique least upper bound $t^* \in [0, \tau]$.

Proof. It follows by (29.18) and (29.19) that sequence $\{t_n\}$ is nondecreasing and bounded from above by τ and as such it converges to t^*. □

We left the choice of τ in (29.19) as uncluttered as possible. A possible choice is given as follows. Suppose there exist a smallest positive solution of the equation

$$\varphi_0(\varphi(t)t + b, t) - 1 = 0 \tag{29.20}$$

denoted by τ_0. Then, we can set $\tau = \tau_0$ in (29.19)
Next, the semi-local convergence analysis of the method (29.2) is presented.

Theorem 37. Suppose that the conditions $(h_1) - (h_5)$ hold. Then, the sequence $\{x_n\}$ generated by method (29.2) is well defined and converges to a solution $x^* \in U[x_0, t^*]$ of the equation $F(x) = 0$. Moreover, the following assertions hold for all $n = 0, 1, 2, \ldots$

$$\|y_n - x_n\| \le s_n - t_n \tag{29.21}$$

and

$$\|x_{n+1} - y_n\| \le t_{n+1} - s_n \tag{29.22}$$

Proof. Mathematical induction is used to show assertions (29.21) and (29.22) as well as the methodology of Theorem 36.
The following estimates are needed:

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \|[v_n, x_n; F]^{-1} F(y_n) \\
&\quad + [v_n, x_n; F]^{-1}([v_n, x_n; F] - [y_n, x_n; F])[v_n, x_n; F]^{-1} F(y_n) \\
&\quad + [v_n, x_n; F]^{-1}([v_n, x_n; F] - [y_n, x_n; F])[v_n, x_n; F]^{-1} F(y_n)\| \\
&\le \frac{\varphi_1(\varphi(\|x_n - x_0\|)\|x_n - x_0\| + b, \|x_n - x_0\|, \|y_n - x_0\|)}{1 - \varphi_0(\varphi(\|x_n - x_0\|)\|x_n - x_0\| + b, \|x_n - x_0\|)} \\
&\quad + \frac{1}{(1 - \varphi_0(\varphi(\|x_n - x_0\|)\|x_n - x_0\| + b, \|x_n - x_0\|))^2} \\
&\quad (\varphi_1(\varphi(\|x_n - x_0\|)\|x_n - x_0\| + b, \|x_n - x_0\|, \|y_n - x_0\|) \\
&\quad (\varphi_2(\varphi(\|x_n - x_0\|)\|x_n - x_0\| + b, \|x_n - x_0\|, \|y_n - x_0\|)) \\
&\le t_{n+1} - s_n
\end{aligned}$$

where we also used

$$\begin{aligned}
\|v_n - x_0\| &= \|x_n + \alpha F(x_n) - x_0\| \\
&= \|(I + \alpha[x_n, x_0; F])(x_n - x_0) + \alpha F(x_0)\| \\
&\leq \|(I + \alpha[x_n, x_0; F])\| \|x_n - x_0\| + |\alpha| \|F(x_0)\| \\
&\leq \varphi(\|x_n - x_0\|)\|x_n - x_0\| + b \leq \varphi(t_n)t_n + b \leq R^*, \\
\|y_n - x_0\| &\leq s_n, \ \|x_n - x_0\| \leq t_n
\end{aligned}$$

and

$$\begin{aligned}
F(y_n) &= F(y_n) - F(x_n) - [v_n, x_n; F](y_n - x_n) \\
&= ([y_n, x_n; F] - [v_n, x_n; F])(y_n - x_n),
\end{aligned}$$

so

$$\begin{aligned}
\|F'(x_0)^{-1}F(y_n)\| &\leq \varphi_1(\varphi(\|x_n - x_0\|)\|x_n - x_0\| + b, \|x_n - x_0\|, \|y_n - x_0\|)\|y_n - x_0\| \\
&\leq \varphi_1(\varphi(t_n)t_n + b, t_n, s_n)(s_n - t_n),
\end{aligned}$$

and

$$\begin{aligned}
\|[v_n, x_n; F]^{-1}F'(x_0)\| &\leq \frac{1}{1 - \varphi_0(\|v_n - x_0\|, \|x_n - x_0\|)} \\
&\leq \frac{1}{1 - \varphi_0(\varphi(t_n)t_n + b, t_n)}.
\end{aligned}$$

Moreover, by the second substep of method (29.2), we can write

$$\begin{aligned}
F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - [v_n, x_n; F](y_n - x_n) \\
&\quad - [x_{n+1}, x_n; F](x_{n+1} - x_n) \\
&\quad + [x_{n+1}, x_n; F](x_{n+1} - x_n) \\
&= [x_{n+1}, x_n; F](x_{n+1} - x_n) - [v_n, x_n; F](y_n - x_n) \\
&= ([x_{n+1}, x_n; F] - F'(x_0) + F'(x_0))(x_{n+1} - x_n) \\
&= ([v_n, x_n; F] - F'(x_0) + F'(x_0))(y_n - x_n)
\end{aligned}$$

so

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq (1 + \varphi_0(\|x_{n+1} - x_0\|, \|x_n - x_0\|))\|x_{n+1} - x_n\| \\
&\quad + (1 + \varphi_0(\|v_n - x_0\|, \|x_n - x_0\|))\|y_n - x_n\| \qquad (29.23) \\
&\leq (1 + \varphi_0(t_{n+1}, t_n))(t_{n+1} - t_n) \\
&\quad + (1 + \varphi_0(\varphi(t_n)t_n + b, t_n)(s_n - t_n) \\
&= \sigma_{n+1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_{n+1} - x_{n+1}\| &\leq \|[v_{n+1}, x_{n+1}; F]^{-1}F'(x_0)\| \|F'(x_0)^{-1}F(x_{n+1})\| \\
&\leq \frac{\sigma_{n+1}}{1 - \varphi_0(\varphi(t_{n+1}t_{n+1} + b, t_{n+1})} = s_{n+1} - t_{n+1}.
\end{aligned}$$

The induction is terminated. Hence, sequence $\{x_n\}$ is complete in a Banach space B, and as such it converges to some $x^*U[x^*, t^*]$. By letting $n \longrightarrow \infty$ in (29.23), we conclude $lim_{n \longrightarrow \infty} x_n = x^*$. □

The uniqueness of the solution is given in the following result.

Proposition 24. *Suppose*

(i) *There exists a solution* $x^* \in U(x_0, \lambda_0)$ *for some* $\lambda_0 > 0$.

(ii) *The second condition in* (h_2) *holds in* $U(x_0, \lambda_0)$ *for* $x = x^*$.

(iii) *There exist* $\lambda \geq \lambda_0$ *such that*

$$\bar{\varphi_0}(0, \lambda) < 1 \tag{29.24}$$

Set $D_3 = U[x_0, \lambda] \cap D$.

Then, the only solution of the equation $F(x) = 0$ *in the region* D_3 *is* x^*.

Proof. Let $y^* \in D_3$ with $F(y^*) = 0$. Set $Q = [x^*, y^*; F]$. It then follows from (ii) and (29.24) that

$$\|F'(x_0)^{-1}(Q - F'(x_0))\| \leq \bar{\varphi_0}(\|x^* - x\|, \|y^* - x^*\|) \leq \bar{\varphi_0}(0, \lambda_1) < 1.$$

Hence, we deduce $x^* = y^*$, since the linear operator Q is invertible and
$Q(x^* - y^*) = F(x^*) - F(y^*) = 0$. □

Notice that the function $\bar{\varphi_0}$ is at least as small as φ. Hence, the radius of uniqueness is at least as large as if function φ is used.
The majorizing sequence for method (29.3) is similarly defined by $t_0 = 0, s_0 = \Omega$

$$\begin{aligned} t_{n+1} &= s_n + (1 + \frac{\varphi_2(c(t_n)t_n + |\alpha|\Omega_0, t_n, s_n)}{1 - \varphi_0(s_n, t_n)} \\ &\quad (\frac{\varphi_1(c(t_n)t_n + |\alpha|\Omega_0, t_n, s_n)}{1 - \varphi_0(s_n, t_n)} \\ s_{n+1} &= t_{n+1} + \frac{\sigma_{n+1}}{1 - \varphi_0(c(t_{n+1})t_{n+1} + |\alpha|\Omega_0, t_{n+1})}. \end{aligned} \tag{29.25}$$

Lemma 39. Suppose that for all $n = 0, 1, 2, \ldots$

$$\varphi_0(\varphi(t_n)t_n + b, t_n) < 1, \ \varphi_0(t_n, s_n) < 1 \textit{ and } t_n \leq \tau \textit{ for some } \tau > 0 \tag{29.26}$$

Then, the sequence $\{t_n\}$ generated by (29.25) is nondecreasing and convergent to its unique least upper bound t^*.

Proof. It follows from (29.25) and (29.26) that sequence $\{t_n\}$ is nondecreasing and bounded above by τ and as such it converges to t^*. □

Theorem 38. Suppose that the conditions $(h_1) - (h_5)$ but with Lemma 39 replacing Lemma 38. Then, the conclusions of Theorem 37 hold for the method (29.3).

Proof. It follows as in the proof of Theorem 37 but we have used instead

$$\begin{aligned}
\|x_{n+1}-y_n\| &= \|[y_n,x_n;F]^{-1}F(y_n)\\
&\quad [y_n,x_n;F]^{-1}([y_n,v_n;F]-[v_n,x_n;F])[y_n,x_n;F]^{-1}F(y_n)\|\\
&\leq (1+\frac{\varphi_2(\|v_n-x_0\|,\|x_n-x_0\|,\|v_n-x_0\|}{1-\varphi_0(\|y_n-x_0\|,\|x_n-x_0\|)})\\
&\quad \frac{\|F^{'}(x_0)^{-1}F(y_n)\|}{1-\varphi_0(\|y_n-x_0\|,\|x_n-x_0\|)}\\
&\leq t_{n+1}-s_n,
\end{aligned}$$

where we also used

$$\|[y_n,x_n;F]^{-1}F^{'}(x_0)\| \leq \frac{1}{1-\varphi_0(s_n,t_n)}.$$

□

Concerning the local convergence analysis of method (29.4) we use the second substep of it to get

$$\begin{aligned}
x_{n+1}-x^* &= y_n-x^*-[y_n,x^*;F]^{-1}F(y_n)\\
&\quad +([y_n,x^*;F]^{-1}-B_n^{-1})F(y_n)=0+([y_n,x^*;F]^{-1}-B_n^{-1})F(y_n)\\
&= [y_n,x^*;F]^{-1}(B_n-[y_n,x^*;F])B_n^{-1}[y_n,x^*;F](y_n-x^*)
\end{aligned}$$

leading to

$$\begin{aligned}
\|x_{n+1}-x^*\| &\leq \frac{1}{(1-p_n)(1-\varphi_0(\|y_n-x^*\|,0))}\\
&\quad (\varphi_1(\|v_n-x^*\|,\|x_n-x^*\|,\|y_n-x^*\|)\\
&\quad +\varphi(\|v_n-x^*\|,\|y_n-x^*\|))\delta(\|y_n-x^*\|)\|y_n-x^*\|), \qquad (29.27)
\end{aligned}$$

where we also used

$$\begin{aligned}
\|F^{'}(x^*)^{-1}(B_n-F^{'}(x^*))\| &\leq \|F^{'}(x^*)^{-1}([y_n,x^n;F]-[v_n,x^n;F]\|\\
&\quad +\|F^{'}(x^*)^{-1}([y_n,v^n;F]-F^{'}(x^*))\|\\
&\leq \varphi_1(\|v_n-x^*\|,\|x_n-x^*\|,\|y_n-x^*\|\\
&\quad +\varphi(\|v_n-x^*\|,\|y_n-x^*\|)=p_n.
\end{aligned}$$

Hence, the function g_2 is defined by

$$g_2(t)=\frac{\delta(g_1(t)t)(\varphi_1(\gamma(t),t,g_1(t)t)+\varphi(\gamma(t)t,g_1(t)t))g_1(t)}{(1-p(t))(1-\varphi_0(g_1(t)t,0)},$$

where

$$p(t)=\varphi_1(\gamma(t)t,t,g_1(t)t)+\varphi(\gamma(t)t,g_1(t)t).$$

Under this modification of function g_2 the conclusions of Theorem 36 hold for the method (29.4).
The semilocal convergence of method (29.4) requires the estimate

$$\|x_{n+1}-y_n\| = \|B_n^{-1}F(y_n)\| \leq \|B_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_n)\|$$

But

$$\begin{aligned}
\|F'(x_0)^{-1}(B_n - F'(x_0))\| &\leq \|F'(x_0)^{-1}([y_n,x_n;F]-[v_n,x_n;F])\| \\
&\quad + \|F'(x_0)^{-1}([y_n,v_n;F]-F'(x_0))\| \\
&\leq \varphi_1(\|v_n-x_0\|,\|x_n-x_0\|,\|y_n-x_0\|) \\
&\quad + \varphi_0(\|y_n-x_0\|,\|v_n-x_0\|) \\
&\leq \varphi_1(\varphi(t_n)t_n+b-t_n-s_n) \\
&\quad + \varphi_0(s_n,\varphi(t_n)t_n+b) = q_n,
\end{aligned}$$

so

$$\|x_{n+1}-y_n\| \leq \frac{\varphi_1(\varphi(t_n)t_n,t_n,s_n)(s_n-t_n)}{1-q_n}.$$

Hence, we define

$$t_{n+1} = s_n + \frac{\varphi_1(\varphi(t_n)t_n,t_n,s_n)(s_n-t_n)}{1-q_n}. \tag{29.28}$$

The iterate s_{n+1} is defined as in (29.25), i.e.

$$s_{n+1} = t_{n+1} + \frac{\sigma_{n+1}}{1-\varphi_0(\varphi(t_{n+1})t_{n+1}+b,t_{n+1})}.$$

The corresponding Lemma 38 to Lemma 39 for method (29.4) uses the condition

$$\varphi_0(\varphi(t_n)t_n+b,t_n) < 1, q_n < 1 \textit{ and } t_n < \tau \tag{29.29}$$

instead of (29.26).
Under these modifications, the conclusions of Theorem 37 hold for method (29.4).

Bibliography

[1] Grau-Sánchez, M., Grau, A., Noguera, M., Frozen divided difference solving systems of nonlinear equations, *J. Comput. Appl. Math.*, 235 (2011), 1739-1743.

[2] Grau-Sánchez, M., Noguera, V., A technique to choose the most efficient method between secant method and some variants, *Appl. Math. Comput.*, 218, (2012), 6415-6426.

[3] Kelley, C.T., Solving nonlinear equations with Newton's Method, *SIAM*, Philadelphia, 2003.

[4] Potra, F.A., Pták, V., *Nondiscrete Induction and Iterative Processes*, Pitman Publishing, Boston, 1984.

[5] Ren, H., Wu, A., Bi, W., A class of two-step Steffensen type methods with fourth-order convergence, *Appl. Math. Comput.*, 209, (2009), 206-210.

[6] Sharma, J.R., Guha, R.K., Gupta, P., Some efficient derivative free methods with memory for solving nonlinear equations, *Appl. Math. Comput.*, 219, (2012), 699-707.

[7] Steffensen, J.F., Remarks on iteration, *Skand. Aktuar Tidskr.*, 16, (1933), 64-72.

[8] Traub, J.F., *Iterative Methods for the Solution of Equations*, Prentice-Hall, New Jersey, 1964.

[9] Wang, X., Zhang, T., A family of Steffensen type methods with seventh-order convergence, *Numer. Algor.*, 62, (2013), 229-444.

[10] Wolfram, S., *The Mathematica Book*, fifth ed. Wolfram Media, 2003.

[11] Sharma, J.R., Arora, H., *An efficient derivative free iterative method for solving systems of nonlinear equations*, 390-403.

Chapter 30

High Convergence Order Derivative Free Methods-II

1. Introduction

The study of the local and semi-local convergence of iterative methods continues with the following sixth convergence order derivative-free methods [6,7,8,9,10].

$$\begin{aligned} y_n &= x_n - [w_n, v_n; F]^{-1} F(x_n), w_n = x_n + F(x_n), v_n = x_n - F(x_n), \\ z_n &= y_n - A_n^{-1} F(y_n) \\ \text{and} & \\ x_{n+1} &= z_n - A_n^{-1} F(z_n), \end{aligned} \tag{30.1}$$

where

$$\begin{aligned} A_n &= 2[x_n, y_n; F] - [w_n, v_n; F], \\ y_n &= x_n - [w_n, y_n; F]^{-1} F(x_n), \\ z_n &= y_n - (3I - 2[w_n, v_n; F]^{-1}[y_n, x_n; F])[w_n, v_n, F]^{-1} F(y_n) \\ \text{and} & \\ x_{n+1} &= z_n - (3I - 2[w_n, v_n; F]^{-1}[y_n, x_n; F])[w_n, v_n; F]^{-1} E(z_n). \end{aligned} \tag{30.2}$$

2. Local Convergence

The conditions for both methods are:

(c1) There exists a simple solution $x^* \in D$ of the equation $F(x) = 0$.

(c2)

$$\|I - [x, x^*; F]\| \leq \gamma_0(\|x - x^*\|),$$
$$\|I + [x, x^*; F]\| \leq \gamma(\|x - x^*\|)$$

and

$$\|F'(x^*)^{-1}([w, v, F] - F'(x^*))\| \leq \varphi_0(\|x - x^*\| \| \| \|y - x^*\|,$$

for all $w, v \in D$.

(c3) There exists a smallest positive solution ρ of the equation

$$\varphi_0(\gamma(t)t, \gamma_0(t)t) - 1 = 0.$$

Set $D_0 = U(x^*, \rho) \cap D$.

(c4)

$$\|F'(x^*)^{-1}([w, v; F] - [x, x^*; F])\| \leq \varphi(\|w - x^*\|, \|v - x^*\|, \|x - x^*\|),$$
$$\|F'(x^*)^{-1}([x, y; F] - [y, x^*; F])\| \leq \varphi_1(\|x - x^*\|, \|y - x^*\|)$$

and

$$\|F'(x^*)^{-1}([y, x; F] - [w, v; F])\| \leq \varphi_2(\|w - x^*\|, \|v - x^*\|, \|y - x^*\|, \|x - x^*\|),$$

for all $x, y, v, w \in D_0$.
and

(c4) $U[x^*, R] \subset D$, where $R = \max\{r, \gamma_0(r)r, \gamma(r)r\}$, with the radius r to be defined later.

Let $M = [0, \rho)$. Define the functions $g_i : M \longrightarrow \mathbb{R}, i = 1, 2, 3$ by

$$g_1(t) = \frac{\varphi(\gamma(t)t, \gamma_0(t)t, t)}{1 - \varphi_0(\gamma(t)t, \gamma_0(t)t)},$$

$$g_2(t) = \frac{(\varphi_2(\gamma(t)t, \gamma_0(t)t, g_1(t)t, t) + \varphi_1(t, g_1(t)t))g_1(t)}{1 - p(t)},$$
$$p(t) = 2\varphi_0(t, g_1(t)t) + \varphi_0(\gamma(t)t, \gamma_0(t)t)$$

and

$$g_3(t) = \frac{(\varphi_2(\gamma(t)t, \gamma_0(t)t, t, g_1(t)t) + \varphi_0(t, g_1(t)t, g_2(t)t)g_2(t)}{1 - p(t)}.$$

Suppose that there exist smallest positive solutions $\rho_i \in (0, \rho)$ of the equations

$$g_i(t) - 1 = 0.$$

Then, a radius of convergence for method (30.1) is defined by

$$r = \min\{r_i\}. \tag{30.3}$$

The motivation for the introduction of the function g_i follows as previously from the series of estimates:

$$\|w_n - x^*\| = \|(I + [x_n, x^*; F])(x_n - x^*)\| \leq \gamma(\|x_n - x^*\|)\|x_n - x^*\|,$$

$$\|v_n - x^*\| = \|(I_[x_n, x^*; F])(x_n - x^*)\| \leq \gamma_0(\|x_n - x^*\|)\|x_n - x^*\|,$$

$$\|[w_n, v_n; F]^{-1}F'(x^*)\| \leq \frac{1}{1 - \varphi_0(\|w_n - x^*\|, \|v_n - x^*\|)},$$

$$\begin{aligned}
\|y_n - x^*\| &= \|[w_n, v_n; f]^{-1}([w_n, v_n; F] - [x_n, x^*; F])(x_n - x^*)\| \\
&\leq \frac{\varphi(\|w_n - x^*\|, \|v_n - x^*\|, \|x_n - x^*\|)\|x_n - x^*\|}{1 - \varphi_0(\|w_n - x\|\|, \|v_n - x^*\|)} \\
&\leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \\
&\leq \|x_n - x^* < r, \\
\|F'(x^*)^{-1}(A_n - F'(x^*))\| &\leq 2\|F'(x^*)^{-1}([x_n, y_n; F] - F'(x^*))\| \\
&\quad + F'(x^*)^{-1}([w_n, v_n; F] - F'(x^*))\| \\
&\leq 2\varphi_0(\|x_n - x^*\|, \|y_n - x^*\|) \\
&\quad + \varphi_0(\|w - x^*\|, \|v_n - x^*\|) = p_n,
\end{aligned}$$

$$\begin{aligned}
\|z_n - x^*\| &= \|y_n - x^* - A_n^{-1}F(y_n)\| \\
&= \|A_n^{-1}(A_n - [y_n, x^*; F])(y_n - x^*)\| \\
&\leq \|A_n^{-1}F'(x^*)\|[\|\|F'(x^*)^{-1}([x_n, y_n; F] - [w_n, v_n; F]) \\
&\quad + \|F'(x^*)^{-1}([x_n, y_n; F] - [y_n, x^*; F])\|\|\|y_n - x^*\| \\
&\leq \frac{1}{1 - p_n}(\varphi_2(\|w_n - x^*\|, \|v_n - x^*\|, \|x_n - x^*\|, \|y_n - x^*\|) \\
&\quad + \varphi_1(\|x_n - x^*\|, \|y_n - x^*\|))\|y_n - x^*\| \\
&\leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|z_n - x^* - A_n^{-1}F(z_n)\| \\
&= \|A_n^{-1}(A_n - [z_n, x^*; F])(z_n - x^*)\| \\
&\leq \|A_n^{-1}F'(x^*)\|[\|\|F'(x^*)^{-1}([x_n, y_n; F] - [w_n, v_n; F]) \\
&\quad + \|F'(x^*)^{-1}([x_n, y_n; F] - [z_n, x^*; F])\|\|\|z_n - x^*\| \\
&\leq \frac{1}{1 - p_n}(\varphi_2(\|w_n - x^*\|, \|v_n - x^*\|, \|x_n - x^*\|, \|y_n - x^*\|) \\
&\quad + \varphi(\|x_n - x^*\|, \|y_n - x^*\|, \|z_n - x^*\|))\|z_n - x^*\| \\
&\leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|.
\end{aligned}$$

Concerning, the convergence of method (30.2) we add the condition

(c4') $\|F'(x^*)^{-1}[x, x^*; F]\| \leq \delta(\|x - x^*\|)$ for all $x \in D_0$.

Then, the function g_1 is the same as for the method (30.1), whereas the functions g_2 and g_3 are defined by.

$$\begin{aligned}
g_2(t) &= \left[\frac{\varphi_1(t, \gamma(t)t, \gamma_0(t)t)}{1 - \varphi_0(\gamma(t)t, \gamma_0(t)t)}\right. \\
&\quad \left. + 2\frac{\varphi_2(\gamma_1(t)t, \gamma_0(t)t, g_1(t)t, t)\,\delta(g_1(t)t)}{(1 - \varphi_0(\gamma(t)t, \gamma_0(t)t))^2}\right] g_1(t)
\end{aligned}$$

$$\begin{aligned} g_3(t) &= \left[\frac{\varphi(g_2(t)t,\gamma(t)t,\gamma_0(t)t)}{1-\varphi_0(\gamma(t)t,\gamma_0(t)t)}\right. \\ &\left. +\frac{2\varphi_2(\gamma(t)t,\gamma_0(t)t,g_1(t)t,t)\delta(g_2(t)t)}{(1-\varphi_0(\gamma(t)t,\gamma_0(t)t))^2}\right] g_2(t). \end{aligned}$$

The motivation for the introduction of the function g_1 is already given, whereas for the functions g_2 and g_3, we have the estimates

$$\begin{aligned} \|z_n - x^*\| &= \|y_n - x^* - [w_n, v_n; F]^{-1} F(y_n) \\ &\quad +2[w_n, v_n; F]^{-1}([w_n, v_n; F] - [y_n, x_n; F])[w_n, v_n; F]^{-1}[y_n, x^*; F](y_n - x^*)\| \\ &\leq \left[\frac{\varphi_1(\|y_n - x^*\|, \|w_n - x^*\|, \|v_n - x^*)}{(1-\varphi_0(\|w_n - x^*\|, \|w_n - x^*))^2}\right. \\ &\leq g_2(\|xn - x^*\|)(\|x_n - x^*\| \leq \|x_n - x^*\| \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \left[\frac{\varphi(\|z_n - x^*\|, \|w_n - x^*\|, \|v_n - x^*\|)}{(1-\varphi_0(\|w_n - x^*\|, \|v_n - x^*\|))^2}\right] \\ &\leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|. \end{aligned}$$

Under these modifications of the functions g_2 and g_3, the radius of convergence is given again by formula (30.3).

Hence, we arrived at the common local convergence result of method (30.1) and method (30.2).

Theorem 39. Under the conditions (c1)-(c5) for method (30.1) or the conditions (c1)-(c5) and (c4') for method (30.2) these methods converge to x^* provided that $x_0 \in U(x^*, r) - \{x^*\}$. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r,$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|.$$

The uniqueness of the solution results as identical to the ones given before in earlier Chapters are omitted.

3. Semi-Local Convergence

The common conditions are:

(h1) There exists a point $x_0 \in D$ and parameters $b \geq 0, \eta \geq 0$ such that $F'(x_0)^{-1}, A_0^{-1} \in \mathcal{L}(Y, X)$, $\|F(x_0)\| \leq b$ and $\|A_0^{-1}F(x_0)\| \leq \eta$.

(h2)

$$\|I-[x,x_0;F]\| \leq \gamma(\|x-x_0\|),$$
$$\|I+[x,x_0;F]\| \leq \gamma_0(\|x-x_0\|)$$

and

$$\left\|F'(x_0)^{-1}\left([w,v;F]-F'(x_0)\right)\right\| \leq \psi_0(\|w-x_0\|,\|v-x_0\|)$$

for all $x,w,v \in D$.

(h3) The equation $\psi_0(\gamma(t)t+b,\gamma_0(t)t+b)-1=0$ has a smallest positive solution ρ. Set $D_1=[0,\rho)$.

(h4)

$$\left\|F'(x_0)^{-1}([x,y;F]-[w,v;F])\right\| \leq \psi(\|x-x_0\|,\|y-x_0\|,\|w-x_0\|,\|v-x_0\|)$$

for all $x,y,w,v \in D_1$.

(h5) The conditions of Lemma 40 hold for method (30.1) or the conditions of Lemma 41 hold for method (30.2)(see below (30.5)) and

(h6) $U[x_0,R^*]\subset D$, where $R^*=\min\{t^*,\gamma(t^*)t^*+b,\gamma_0(t^\star)t^*+b\}$ and the point t^* is given in the Lemmas that follow.

Define the scalar sequences $\{t_n\}$, respectively for method (30.1) and method (30.2)for $t_0=0,s_0=\eta$ by

$$\begin{aligned}
u_n &= s_n+\frac{\psi(t_n,s_n,\gamma(t_n)t_n,\gamma_0(t_n)t_n)(s_n-t_n)}{1-q_n},\\
q_n &= \psi(t_n,s_n,\gamma(t_n)t_n,\gamma_0(t_n)t_n)+\psi_0(t_n,s_n), \qquad (30.4)\\
t_{n+1} &= u_n+\frac{(1+\psi_0(u_n,s_n))(u_n-s_n)+\psi(t_n,s_n,\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)(s_n-t_n)}{1-q_n},\\
s_{n+1} &= t_{n+1}+\frac{\psi(t_n,s_n,\gamma(t_n)t_n,\gamma_0(t_n)t_n))(t_{n+1}-t_n)}{1-\psi_0(\gamma(t_{n+1})t_{n+1}+b,\gamma_0(t_{n+1})t_{n+1}+b)}\\
&\quad+\frac{(1+\psi_0(\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b))(t_{n+1}-s_n)}{1-\psi_0(\gamma(t_{n+1})t_{n+1}+b,\gamma_0(t_{n+1})t_{n+1}+b)},
\end{aligned}$$

$$\begin{aligned}
u_n &= s_n+\left[1+2\frac{\psi(t_n,s_n,\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)}{1-\psi_0(\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)}\right]\\
&\quad\times\frac{\psi(t_n,s_n,\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)}{1-\psi_0(\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)}, \qquad (30.5)\\
t_{n+1} &= u_n+[1+\psi(t_n,s_n;\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)]\\
&\quad\frac{(1+\psi_0(u_n,s_n))(u_n-s_n)+\psi(t_n,s_n;\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)(s_n-t_n)}{1-\psi_0(\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)},
\end{aligned}$$

and s_{n+1} is defined as in sequence (30.4).

The next two Lemmas provide sufficient convergence conditions for majorizing sequences (30.4) and (30.5).

Lemma 40. suppose that for all $n = 0,1,2,\ldots$

$$\psi_0(\gamma(t_n)t_n + b, \gamma_0(t_n)t_n + b) < 1,\ q_n < 1, \text{ and } t_n \leq \tau \tag{30.6}$$

for some $\tau \in [0,\rho)$. Then, the sequence $\{t_n\}$ is non-decreasing and converges to its unique least upper bound $t^* \in [0,\tau]$.

Proof. The sequence $\{t_n\}$ is nondecreasing and bounded form above by τ. Hence, it converges to t^*. □

Lemma 41. Suppose that for all $n = 0,1,2,\ldots$

$$\psi_0\left(\gamma(t_n)t_n + b, \gamma_0(t_n)t_n + b\right) < 1..$$

Then, the conclusions of Lemma 40 hold for the sequence $\{t_n\}$ given by (30.6).

The motivation for the introduction of these sequences is due as before to the following estimates:

$$\begin{aligned}
\|w_n - x_0\| &= \|(I + [x_n, x_0; F](x_n - x_0) + F(x_0))\| \\
&\leq \|I + [x_n, x_0; F]\| \|x_n - x_0\| + \|F(x_0)\| \\
&\leq \gamma(\|x_n - x_0\|)\|x_n - x_0\| + b \leq \gamma(r)r + b \leq R^*,
\end{aligned}$$

$$\begin{aligned}
\|v_n - x_0\| &= \|(I - [x_n, x_0; F](x_n - x_0) + F(x_0))\| \\
&\leq \|I - [x_n, x_0; F]\| \|x_n - x_0\| + \|F(x_0)\| \\
&\leq \gamma(\|x_n - x_0\|)\|x_n - x_0\| + b \leq \gamma(r)r + b \leq R^*,
\end{aligned}$$

$$\|[w_n, x_n; F]^{-1}F'(x_0)\| \leq \frac{1}{1 - \psi(\|w_n - x_0\|, \|v_n - x_0\|)},$$

$$\begin{aligned}
\|F'(x_0)^{-1}(A_n - F'(x_0))\| &\leq \|F'(x_0)^{-1}([x_n, y_n; F] - F'(x_0))\| \\
&\quad \|F'(x_0)^{-1}([x_n, y_n; F] - [w_n, v_n; F])\| \\
&\leq \psi_0(\|x_n - x_0\|, \|y_n - x_0\|) \\
&\quad + \psi(\|x_n - x_0\|, \|y_n - x_0\|, \|v_n - x_0\|, \|w_n - x_0\|) \\
&= q_n,
\end{aligned}$$

$$A_n^{-1}F'(x_0)\| \leq \frac{1}{1 - q_n},$$

$$\begin{aligned}
\|z_n - y_n\| &\leq \|A_n^{-1}F'(x_0)\| \|F'(x_0)^{-1}F(y_n\| \\
&\leq \frac{\psi(\|x_n - x_0\|, \|y_n - x_0\|, \|w_n - x_0\|, \|v_n - x_0\|)\|y_n - x_0\|}{1 - q_n} \\
&\leq u_n - s_n, \\
\|x_{n+1} - z_n\| &\leq \|A_n^{-1}F'(x_0)\| \|F'(x_0)^{-1}(F(z_n) - F(y_n) + F(y_n))\| \\
&\leq [1 + \frac{\psi((t_n, s_n, \gamma(t_n)t_n + b, \gamma_0(t_n)t_n + b)}{1 - \psi_0(\gamma(t_n)t_n + b, \gamma_0(t_n)t_n + b)}] \\
&\quad [\frac{(1 + \psi_0(u_n, s_n))(u_n - s_n) + \psi(t_n, s_n, \gamma(t_n)t_n + b, \gamma_0(t_n)t_n + b)(s_n - t_n)}{1 - \psi_0(\gamma(t_n)t_n + b, \gamma_0(t_n)t_n + b)}] \\
&\leq t_{n=1} - u_n,
\end{aligned}$$

$$\begin{aligned} \|F'(x_0)^{-1}F(x_n)\| &= \|F'(x_0)^{-1}(F(x_n)-F(x_n)-[w_n,x_n;F](y_n-x_n))\| \\ &\leq \|F'(x_0)^{-1}([x_{n+1},x_n;F]-[w_n,v_n;F])\| \\ &\quad +\|[w_n,v_n;F](x_{n+1}-y_n)\| \\ &\leq \psi(\|x_{n+1}-x_0\|,\|w_n-x_0\|,\|v_n-x_0\|,\|x_n-x_0\|) \\ &\quad +(1+\psi_0(\|w_n-x_0\|,\|v_n-x_0\|))\|x_{n+1}-y_n\|, \end{aligned}$$

so

$$\begin{aligned} \|y_{n+1}-x_0\| &\leq \|[w_{n+1},v_{n+1};F]^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\| \\ &\leq s_{n+1}-t_{n+1}. \end{aligned}$$

Concerning the estimates for the method (30.2), we get similarly

$$\begin{aligned} \|z_n-y_n\| &\leq \left[1+2\frac{\psi(t_n,s_n;\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)}{1-\psi_0(\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)}\right] \\ &\quad \frac{\psi(t_n,s_n;\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)}{1-\psi_0(\gamma(t_n)t_n+b,\gamma_0(t_n)t_n+b)} \\ &\leq u_n-s_n \end{aligned}$$

and

$$\|x_{n+1}-z_n\| \leq t_{n+1}-u_n.$$

The iterate s_{n+1} is defined as in the sequence (30.4). Hence, we arrived at the semi-local convergence of both methods (30.1) and method (30.2).

Theorem 40. suppose that the conditions (h1)-(h6) hold. Then, the sequences generated by method (30.1) or the method(30.2) converge to a solution $x^* \in U[x_0,t*]$ of the equation $F(x)=0$. Moreover, the following estimates hold

$$\begin{aligned} \|y_n-x_n\| &\leq s_n-t_n, \\ \|z_n-y_n\| &\leq u_n-s_n \end{aligned}$$

and

$$\|x_{n+1}-z_n\| \leq t_{n+1}-u_n.$$

The uniqueness of the solution results can be found in previous Chapters.

Bibliography

[1] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[2] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)* 2021, 9(16), 1942. https://doi.org/10.3390/math9161942.

[3] Regmi, S., Argyros, I.K., George, S., Argyros, C.I., Extended Convergence of Three Step Iterative Methods for Solving Equations in Banach Space with Applications, *Symmetry*, 14 (7), 1484.

[4] Regmi, S., Argyros, I.K., George, S., Argyros, C.I., Numerical Processes for Approximating Solutions of Nonlinear Equations, *Axioms*, 11, (7), 307.

[5] Grau-Sánchez, M., Grau Á., Noguera, M., On the computational efficiency index and some iterative methods for solving systems of nonlinear equations, *J Comput. Appl. Math.*, 236, (2011), 1259-1266.

[6] Grau-Sánchez M., Grau Á., Noguera M., Ostrowski type methods for solving systems of nonlinear equations, *Appl. Math. Comput.*, 218, (2011), 2377-2385.

[7] Sharma, J.R., Arora, H., On efficient weighted-Newton methods for solving systems of nonlinear equations, *Applied Mathematics and Computation*, 222, 1(2013), 497-506.

[8] Liu Z., Zheng Q., Zhao P., A variant of Steffensen's method of fourth-order convergence and its applications, *Appl. Math. Comput.*, 216, (2010), 1978-1983.

[9] Sharma, J.R., Arora, R., Efficient Jarratt-like methods for solving systems of nonlinear equations, *Calcolo*, 51, (2014), 193-210. doi:10.1007/s10092-013-0097-1.

[10] Sharma, J.R., Guha, R.K., Sharma, R., An efficient fourth order weighted-Newton method for systems of nonlinear equations, *Numer Algorithms*, 62, (2013), 307-323.

[11] Steffensen, J.F., *Remarks on iteration, Skand Aktuar Tidskr*, 16, (1933), 64-72.

[12] Traub, J.F., *Iterative methods for the solution of equations*, Prentice-Hall, New Jersey, 1964.

[13] Wang, X., Zhang, T., A family of Steffensen type methods with seventh-order convergence, *Numer. Algorithms*, 62, (2013), 229-444.

[14] Xiao, X.Y., Yin, H.W., Increasing the order of convergence for iterative methods to solve nonlinear systems, *Calcolo*, 53, (2016), 285-300.

Chapter 31

High Convergence Order Derivative Free Methods-III

1. Introduction

The investigation of high convergence order derivative-free methods continues. In particular, we present the local and semi-local convergence of the following methods:

Zheng et al. [15] (fourth order):

$$\begin{aligned} y_n &= x_n - [w_n, x_n; F]^{-1} F(x_n), w_n = x_n + F(x_n) \\ \text{and} & \\ x_{n+1} &= y_n - A_n^{-1} [y_n, x_n; F]^{-1} F(y_n), \\ \text{where} & \\ A_n &= [y_n, x_n; F] + 2[y_n, w_n; F] - [w_n, x_n; F]. \end{aligned} \tag{31.1}$$

Narang et al. [7] (seventh order):

$$\begin{aligned} y_n &= x_n - Q_n^{-1} F(x_n), \, Q_n = [w_n, v_n; F], \\ w_n &= x_n + aF(x_n), \, v_n = x_n - aF(x_n), \\ z_n &= y_n - Q_n^{-1} F(y_n), \\ x_{n+1} &= z_n - (\frac{17}{4} I + Q_n^{-1} B_n \\ & (-\frac{27}{4} I + Q_n^{-1} B_n (\frac{19}{4} I - \frac{5}{4} Q_n^{-1} B_n))) Q_n^{1} F(z_n), \end{aligned} \tag{31.2}$$

where

$$B_n = [p_n, q_n; F], \; p_n = z_n + bF(z_n), q_n = z_n - bF(z_n),$$

$$\begin{aligned} y_n &= x_n - [w_n, v_n; F]^{-1} F(x_n), \\ w_n &= x_n + F(x_n),\ v_n = x_n - F(x_n), \\ z_n &= y_n - (3I - 2[w_n, v_n; F]^{-1}[y_n, x_n; F])[w_n, v_n; F]^{-1} F(y_n) \end{aligned}$$

and (31.3)

$$\begin{aligned} x_{n+1} &= z_n - (\frac{13}{4}I + [w_n, v_n; F]^{-1}[z_n, y_n; F] \\ &\quad \times \left(\frac{7}{4}I - \frac{5}{4}[w_n, v_n; F]^{-1}[z_n, y_n; F]\right)) \\ &\quad \times [w_n, v_n; F]^{-1} F(z_n). \end{aligned}$$

Related work can be found in previous chapters and [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]

2. Local Convergence

As before we define the function "g" under the same conditions.

Method (31.1)

The estimates are in turn:

$$\|[w_n, x_n; F]^{-1} F'(x^*)\| \leq \frac{1}{1 - \varphi_0(\|w_n - x^*\|, \|x_n - x^*\|)},$$

$$\begin{aligned} y_n - x^* &= x_n - x^* - [w_n, x_n; F]^{-1}[x_n, x^*; F](x_n - x^*) \\ &= [w_n, x_n; F]^{-1}([w_n, x_n; F] - [x_n, x^*; F])(x_n - x^*), \end{aligned}$$

$$\begin{aligned} \|y_n - x^*\| &\leq \frac{\varphi(\|w_n - x^*\|, \|x_n - x^*\|)\|x_n - x^*\|}{1 - \varphi_0(\|w_n - x^*\|, \|x_n - x^*\|)}, \\ x_{n+1} - x^* &= y_n - x^* + A_n^{-1}(I + [y_n, x_n; F]^{-1} \\ &\quad ([y_n, w_n; F] - [w_n, x_n; F])F(y_n), \end{aligned}$$

$$\begin{aligned} \|[y_n, x_n; F]^{-1} F'(x^*)\| &\leq \frac{1}{1 - \varphi_0(\|y_n - x^*\|, \|x_n - x^*\|)}, \\ \|F'(x^*)^{-1}(A_n - F'(x^*))\| &\leq \|F'(x^*)^{-1}([y_n, x_n; F] - F'(x^*))\| \\ &\quad + 2\|F'(x^*)^{-1}([y_n, x_n; F] - [w_n, x_n; F])\| \\ &\leq \varphi_0(\|y_n - x^*\|, \|x_n - x^*\|) \\ &\quad + 2\varphi_1(\|x_n - x^*\|, \|y_n - x^*\|, \|w_n - x^*\|) \\ &= p_n < 1, \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq [1 + \frac{1}{1 - p_n}(1 + \frac{1}{1 - \varphi_0(\|y_n - x^*\|, \|x_n - x^*\|)}) \\ &\quad \times \delta(\|y_n - x^*\|)]\|y_n - x^*\|. \end{aligned}$$

Hence, we can define

$$g_1(t) = \frac{\varphi(\gamma(t)t, t)}{1 - \varphi_0(\gamma(t)t, t)}$$

and

$$g_2(t) = [1 + \frac{1}{1-p(t)}(1 + \frac{1}{1-\varphi_0(t, g_1(t)t)})\delta(g_1(t)t)]g_1(t),$$

$$r = \min\{r_1, r_2\}, R = \max\{r, \gamma(r)r\}.$$

Method (31.2)

$$\|Q_n^{-1}F'(x^*)\| \le \frac{1}{1-\varphi_0(\gamma(\|x_n-x^*\|)\|x_n-x^*\|, \gamma_0(\|x_n-x^*\|)\|x_n-x^*\|)},$$

$$y_n - x^* = x_n - x^* - Q_n^{-1}F(x_n) = Q_n^{-1}([w_n, v_n; F] - [x_n, x^*; F])(x_n - x^*),$$

$$\|y_n - x^*\| \le \frac{\varphi(\|x_n-x^*\|, \|v_n-x^*\|, \|w_n-x^*\|)\|x_n-x^*\|}{1-\varphi_0(\gamma(\|x_n-x^*\|)\|x_n-x^*\|, \gamma_0(\|x_n-x^*\|)\|x_n-x^*\|)},$$

$$z_n - x^* = Q_n^{-1}([w_n, v_n; F] - [y_n, x^*; F])(y_n - x^*),$$

$$\|z_n - x^*\| \le \frac{\varphi(\|y_n-x^*\|, \|v_n-x^*\|, \|w_n-x^*\|)\|y_n-x^*\|}{1-\varphi_0(\gamma(\|x_n-x^*\|)\|x_n-x^*\|, \gamma_0(\|x_n-x^*\|)\|x_n-x^*\|)},$$

$$\begin{aligned} x_{n+1} - x^* &= z_n - x^* - Q_n^{-1}F(z_n) - \frac{1}{4}(I - Q_n^{-1}B_n) \\ &\quad \times [5(I - Q_n^{-1}B_n) - 9(I - Q_n^{-1}B_n) + 4]Q_n^{-1}F(z_n) \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \left[\frac{\varphi(\|z_n-x^*\|, \|w_n-x^*\|, \|v_n-x^*\|)}{1-\varphi_0(\gamma(\|x_n-x^*\|)\|x_n-x^*\|, \gamma_0(\|x_n-x^*\|)\|x_n-x^*\|)} \right. \\ &\quad \left. + \frac{1}{4}\beta_n\alpha_n\right] \|z_n - x^*\|, \end{aligned}$$

where

$$\alpha_n = \frac{(5\beta_n^2 + 9\beta_n + 4)\delta(\|z_n-x^*\|)}{1-\varphi_0(\gamma(\|x_n-x^*\|)\|x_n-x^*\|, \gamma_0(\|x_n-x^*\|)\|x_n-x^*\|)}$$

and

$$\beta_n = \frac{\varphi_1(\|w_n-x^*\|, \|v_n-x^*\|, \|p_n-x^*\|, \|q_n-x^*\|)}{1-\varphi_0(\gamma(\|x_n-x^*\|)\|x_n-x^*\|, \gamma_0(\|x_n-x^*\|)\|x_n-x^*\|)}.$$

Thus, we can choose

$$g_1(t) = \frac{\varphi(t, \gamma_0(t)t, \gamma(t)t)}{1-\varphi_0(\gamma(t)t, \gamma_0(t)t)},$$

$$g_2(t) = \frac{\varphi(g_1(t)t, \gamma_0(t)t, \gamma(t)t)g_1(t)}{1-\varphi_0(\gamma(t)t, \gamma_0(t)t)},$$

$$g_3(t) = [\frac{\varphi(g_2(t)t, \gamma_0(t)t, \gamma(t)t)}{1-\varphi_0(\gamma(t)t, \gamma_0(t)t)} + \frac{1}{4}\beta(t)\alpha(t)]g_2(t),$$

where

$$\beta(t) = \frac{\varphi_1(\gamma(t)t, \gamma_0(t)t, \gamma_1(t)t, g_1(t)t)}{1 - \varphi_0(\gamma(t)t, \gamma_0(t)t)}$$

and

$$\alpha(t) = \frac{(5\beta(t)^2 + 9\beta(t) + 4)\delta(g_2(t)t)}{1 - \varphi_0(\gamma(t)t, \gamma_0(t)t)},$$

where we also added the conditions $\|I - b[x, x^*; F]\| \leq \gamma_1(\|x - x^*\|)$ and $\|I + b[x, x^*; F]\| \leq \gamma_2(\|x - x^*\|)$, and γ_1, γ_2 are continuous and nondecreasing functions with the same domain as γ_0 and γ. Then, we also define $r = \min\{r_1, r_2, r_3\}$ and $R = \max\{r, \gamma_0(r)r, \gamma(r)r, \gamma_1(r)r, \gamma_2(r)r\}$.

Method (31.3)
The estimate on $\|y_n - x^*\|$ is as in method (31.2) but $a = 1$. Then,

$$\begin{aligned}
z_n - x^* &= y_n - x^* - [w_n, v_n; F]^{-1}F(y_n) \\
&\quad + 2[w_n, v_n; F]^{-1}([w_n, v_n; F] - [y_n, x_n; F])[w_n, v_n; F]^{-1}F(z_n), \\
\|z_n - x^*\| &\leq \Bigg[\frac{\varphi(\|y_n - x^*\|, \|w_n - x^*\|, \|v_n - x^*\|)}{1 - \varphi_0(\gamma(\|x_n - x^*\|)\|x_n - x^*\|, \gamma_0(\|x_n - x^*\|)\|x_n - x^*\|)} \\
&\quad \frac{2\varphi_1(\|x_n - x^*\|, \|y_n - x^*\|, \|v_n - x^*\|, \|w_n - x^*\|)}{(1 - \varphi_0(\gamma(\|x_n - x^*\|)\|x_n - x^*\|, \gamma_0(\|x_n - x^*\|)\|x_n - x^*\|))^2}\Bigg] \\
&\quad \times \|y_n - x^*\|,
\end{aligned}$$

$p_n = [w_n, v_n; F]$, $T_n = [z_n, y_n; F]$,

$$\begin{aligned}
x_{n+1} - x^* &= z_n - x^* - p_n^{-1}F(z_n) - \frac{1}{4}(I - P_n^{-1}T_n) \\
&\quad \times (5(I - p_n^{-1}T_n)^2 - 9(I - p_n^{-1}T_n) + 4I)p_n^{-1}F(z_n), \\
\|x_{n+1} - x^*\| &\leq \Bigg[\frac{\varphi(\|z_n - x^*\|, \|w_n - x^*\|, \|v_n - x^*\|)}{1 - \varphi_0(\gamma(\|x_n - x^*\|)\|x_n - x^*\|, \gamma_0(\|x_n - x^*\|)\|x_n - x^*\|)} \\
&\quad + \frac{1}{4}\lambda_n\mu_n\Bigg]\|z_n - x^*\|,
\end{aligned}$$

where

$$\lambda_n = \frac{\varphi_1(\|x_n - x^*\|, \|y_n - x^*\|, \|v_n - x^*\|, \|w_n - x^*\|)}{1 - \varphi_0(\gamma(\|x_n - x^*\|)\|x_n - x^*\|, \gamma_0(\|x_n - x^*\|)\|x_n - x^*\|)}$$

and

$$\mu_n = \frac{(5\lambda_n^2 + 9\lambda_n + 4)\delta(\|z_n - x^*\|)}{1 - \varphi_0(\gamma(\|x_n - x^*\|)\|x_n - x^*\|, \gamma_0(\|x_n - x^*\|)\|x_n - x^*\|)}.$$

Therefore, we choose the function g_1 as in method (31.2) but for $a = 1$,

$$\begin{aligned}
g_2(t) &= \Bigg[\frac{\varphi(g_1(t)t, \gamma(t)t, \gamma_0(t)t)}{1 - \varphi_0(\gamma(t)t, \gamma_0(t)t)} \\
&\quad + 2\frac{\varphi_1(t, g_1(t)t, \gamma(t)t, \gamma_0(t)t)}{(1 - \varphi_0(\gamma(t)t, \gamma_0(t)t))^2}\Bigg] g_1(t),
\end{aligned}$$

$$g_3(t) = [\frac{\varphi(g_2(t)t, \gamma(t)t, \gamma_0(t)t)}{1-\varphi_0(\gamma(t)t, \gamma_0(t)t)} + \frac{1}{4}\lambda(t)\mu(t)]g_2(t),$$

where

$$\lambda(t) = \frac{\varphi_1(t, g_1(t)t, \gamma_0(t)t, \gamma_0(t)t)}{1-\varphi_0(\gamma(t)t, \gamma_0(t)t)},$$

$$\mu(t) = \frac{(5\lambda(t)^2+9\gamma(t)+4)\delta(g_2(t)t)}{1-\varphi_0(\gamma(t)t, \gamma_0(t)t)},$$

$r = \min\{r_1, r_2, r_3\}$ and $R = \max\{r, \gamma_0(r)r, \gamma(r)r\}$.

3. Semi-Local Convergence

The majorizing sequences are defined for each method using the estimates depending on the function "ψ".

Method (31.1)

$$\begin{aligned}\|F'(x_0)^{-1}(A_n - F'(x_0))\| &\leq \|F'(x_0)^{-1}([y_n, x_n; F] - F'(x_0))\| \\ &\quad +2\|F'(x_0)^{-1}([y_n, w_n; F] - [w_n, x_n; F])\| \\ &\leq \psi_0(\|y_n - x_0\|, \|x_n - x_0\|) \\ &\quad +\psi_1(\|y_n - x_0\|, \|x_n - x_0\|, \|w_n - x_0\|) \\ &= q_n < 1,\end{aligned}$$

$$\|A_n^{-1}F'(x_0)\| \leq \frac{1}{1-q_n},$$

$$\begin{aligned}F(y_n) &= F(y_n) - F(x_n) - [w_n, x_n; F](y_n - x_n) \\ &= ([y_n, x_n; F] - [w_n, v_n; F])(y_n - x_n),\end{aligned}$$

$$\|F'(x_0)^{-1}F(y_n)\| \leq \psi_1(\|x_n - x_0\|, \|y_n - x_0\|, \|w_n - x_0\|, \|v_n - x_0\|)\|y_n - x_0\|,$$

$$\begin{aligned}\|x_{n+1} - y_n\| &\leq \frac{1}{q_n}(1 + \frac{\psi_1(\|x_n - x_0\|, \|y_n - x_0\|, \|v_n - x_0\|, \|w_n - x_0\|)}{1-\psi_0(\|x_n - x_0\|, \|y_n - x_0\|)}) \\ &\quad \times\psi_1(\|x_n - x_0\|, \|y_n - x_0\|, \|v_n - x_0\|, \|w_n - x_0\|)\|y_n -_n\| \leq t_{n+1} - s_n,\end{aligned}$$

where

$$\begin{aligned}F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - [w_n, x_n; F](y_n - x_n) \\ &\quad -[w_n, x_n; F](x_{n+1} - x_n) + [w_n, x_n; F](x_{n+1} - x_n) \\ &= ([x_{n+1}, x_n; F] - [w_n, x_n; F])(x_{n+1} - x_n) \\ &\quad +[w_n, x_n; F](x_{n+1} - y_n), \\ \|F'(x_0)^{-1}F(x_{n+1})\| &\leq \psi(\|x_n - x_0\|, \|w_n - x_0\|, \|x_{n+1} - x_0\|)\|x_{n_1} - x_n\|) \\ &\quad +(1 + \psi_0(\|w_n - x_0\|, \|x_n - x_0\|)\|x_{n+1} - y_n\|, \\ \|y_{n+1} - x_{n+1}\| &\leq \|[w_{n+1}, v_{n+1}; F]^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\| \\ &\leq s_{n+1} - t_{n+1}.\end{aligned}$$

Hence, the sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = \eta$ by

$$\begin{aligned}
\tilde{q}_n &= \psi_0(t_n, s_n) + \psi_1(t_n, s_n, \gamma(t_n)t_n, \gamma_0(t_n)t_n), \\
t_{n+1} &= s_n + \frac{1}{1-\tilde{q}_n}(1 + \frac{\psi_1(t_n, s_n, \gamma(t_n)t_n, \gamma_0(t_n)t_n)}{1-\psi_0(t_n, s_n)}) \\
&\quad \times \psi_1(t_n, s_n, \gamma(t_n)t_n, \gamma_0(t_n)t_n)(s_n - t_n)
\end{aligned}$$

and

$$\begin{aligned}
s_{n+1} &= t_{n+1} + \frac{1}{1-\psi_0(\gamma(t_{n+1})t_{n+1}, \gamma_0(t_{n+1})t_{n+1})} \\
&\quad \times (\psi(t_n, \gamma(t_n)t_n, t_{n+1})(t_{n+1} - t_n) + (1 + \psi_0(t_n, \gamma(t_n)t_n))(t_{n+1} - s_n).
\end{aligned}$$

Method (31.2)

$$\begin{aligned}
\|z_n - y_n\| &\leq \|Q_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_n)\| \\
&\leq \frac{\psi_1(t_n, s_n, \gamma(t_n)t_n, \gamma_0(t_n)t_n)}{1-q_n}(s_n - t_n) \\
&\leq u_n - s_n,
\end{aligned}$$

$$F(z_n) = F(z_n) - F(y_n) + F(y_n),$$

$$\begin{aligned}
F'(x_0)^{-1}F(z_n)\| &\leq (1 + \psi_0(\|z_n - x_0\|, \|y_n - x_0\|))\|z_n - y_n\| \\
&\quad + \psi_1(\|x_n - x_0\|, \|y_n - x_0\|, \|w_n - x_0\|, \|v_n - x_0\|) \\
&\quad \times \|y_n - x_n\| = \xi_n,
\end{aligned}$$

$$\|x_{n+1} - z_n\| \leq \frac{1}{4(1-q_n)} d_n \xi_n \leq t_{n+1} - u_n$$

$$c_n = \frac{\psi_1(\|x_n - x_0\|, \|y_n - x_0\|, \|w_n - x_0\|, \|v_n - x_0\|)}{1 - \psi_0(\|x_n - x_0\|, \|y_n - x_0\|)},$$

where we also used

$$\begin{aligned}
&\frac{17}{4}I + Q_n^{-1}B_n(-\frac{27}{4}I + Q_n^{-1}B_n(\frac{19}{4}I - \frac{5}{4}Q_n^{-1}B_n)) \\
&= \frac{1}{4}(5(I - Q_n^{-1}B_n)^3 + 4(I - Q_n^{-1}B_n)^2 + 4(I - Q_n^{-1}B_n) + 4),
\end{aligned}$$

$$\begin{aligned}
F(x_{n+1}) &= ([x_{n+1}, x_n; F] - [w_n, v_n; F])(x_{n+1} - x_n) \\
&\quad + [w_n, v_n; F](x_{n+1} - y_n),
\end{aligned}$$

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq \psi_1(\|x_{n+1} - x_0\|, \|x_n - x_0\|, \|w_n - x_0\|, \|v_n - x_0\|)\|x_{n+1} - x_n\| \\
&\quad + (1 + \psi_0(\|w_n - x_0\|, \|v_n - x_0\|)) \\
&\quad \times \|x_{n+1} - y_n\| = \sigma_{n+1},
\end{aligned}$$

$$\|y_{n+1} - x_{n+1}\| \leq \frac{\sigma_{n=1}}{1 - q_{n+1}} \leq s_{n+1} - t_{n+1}.$$

Hence, we can define

$$u_n = s_n + \frac{\psi_1(t_n, s_n, \gamma(t_n)t_n, \gamma_0(t_n)t_n, \gamma(t_n)t_n)(s_n - t_n)}{\tilde{q}_n},$$

$$\tilde{q}_n = \psi_0(t_n, s_n) + 2\psi_1(t_n, s_n, \gamma(t_n)t_n, \gamma_0(t_n)t_n),$$

$$t_{n+1} = u_n + \frac{1}{4(1-\tilde{q}_n}\tilde{d}_n\tilde{s}_n,$$

$$\tilde{c}_n + \frac{\psi_1(t_n, s_n, \gamma(t_n)t_n, \gamma_0(t_n)t_n)}{1-\psi_0(t_n, s_n)},$$

$$\tilde{d}_n = 5\tilde{c}_n^3 + 4\tilde{c}_n^2 + 4\tilde{c}_n + 4,$$

and the iterate s_{n+1} is as defined in the method (31.1).

Method (31.3)
We have in turn

$$\begin{aligned}
\|z_n - y_n\| &= \|[I - 2[w_n, v_n; F]^{-1} \\
&\quad ([w_n, v_n; F] - [y_n, x_n; F])[w_n, v_n; F]^{-1}F(y_n)\| \\
&\leq (1 + 2\frac{\psi_1(\|x_n - x_0\|, \|y_n - x_0\|, \|v_n - x_0\|, \|w_n - x_0\|)}{1-\psi_0(\gamma(t_n)t_n, \gamma_0(t_n)t_n)} \\
&\quad \times \frac{\psi_1(\|x_n - x_0\|, \|y_n - x_0\|, \|v_n - x_0\|, \|w_n - x_0\|)}{1-\psi_0(\gamma(t_n)t_n, \gamma_0(t_n)t_n)})\|y_n - x_n\| \\
&\leq u_n - s_n.
\end{aligned}$$

Thus, we define

$$\begin{aligned}
u_n &= s_n + (1 + \frac{\psi_1(t_n, s_n, \gamma(t_n)t_n, \gamma_0(t_n)t_n)}{1-\psi_0(\gamma(t_n)t_n, \gamma_0(t_n)t_n))} \\
&\quad \times \frac{\psi_1(t_n, s_n, \gamma(t_n)t_n, \gamma_0(t_n)t_n)}{1-\psi_0(\gamma(t_n)t_n, \gamma_0(t_n)t_n))}(s_n - t_n),
\end{aligned}$$

whereas t_{n+1} and s_{n+1} as define in the method (31.2).

Concerning the convergence of the majorizing sequences, the conditions are imposed as in the previous lemmas. As an example for the majorizing sequence corresponding to the method (31.1) the conditions are:

$$\tilde{q}_n < 1, \psi_0(t_n, s_n) < 1 \text{ and } \psi_0(\gamma(t_n)t_n, \gamma_0(t_n)t_n) < 1$$

for all $n = 0, 1, 2, \ldots$.

Bibliography

[1] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[2] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)*, 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[3] Argyros, I.K., *Computational Theory of Iterative Methods Methods*. In: Chui, C.K., Wuytack, L. (eds.) Series: Studies in Computational Mathematics, vol. 15. Elsevier Publ. Co., New York, (2007).

[4] Regmi, S., Argyros,I . K., George, S., Argyros, C.I., Extended Convergence of Three Step Iterative Methods for Solving Equations in Banach Space with Applications, *Symmetry*,14, (7), 1484.

[5] Regmi, S., Argyros, I. K., George, S., Argyros, C.I., Numerical Processes for Approximating Solutions of Nonlinear Equations, *Axioms*, 11, (7), 307.

[6] Grau-Sánchez, M., Grau, Á., Noguera, M., Frozen divided difference scheme for solving systems of nonlinear equations, *J. Comput. Appl. Math.*, 235, 2011, 17391743.

[7] Narang, M., Bhatia, S., Kanwar, V., New efficient derivative free family of seventh-order methods for solving systems of nonlinear equations, *Numerical Algorithms*, 76, (2017), 283307.

[8] Sharma, J.R., Arora, H., An efficient derivative free iterative method for solving systems of nonlinear equations, *Appl. Anal. Discrete Math.*, 7, 390403, (2013).

[9] Sharma, J.R., Arora, H., A novel derivative free algorithm with seventh order convergence for solving systems of nonlinear equations, *Numer. Algor.*, 4, 917933, (2014).

[10] Sharma, J.R., Arora, H., Efficient derivative-free numerical methods for solving systems of nonlinear equations, *Comp. Appl. Math.*, 35, 269284, (2016). doi:10.1007/s40314-014-0193-0.

[11] Steffensen, J.F., Remarks on iteration, *Skand Aktuar Tidsr.*, 16, 6472, (1933).

[12] Traub, J.F., *Iterative Methods for the Solution of Equations*, Prentice-Hall, New Jersey, (1964).

[13] Wang, X., Zhang, T., A family of Steffensen type methods with seventh-order convergence, *Numer. Algor.*, 62, 429444, (2013).

[14] Wang, X., Zhang, T., Qian, W., Teng, M., Seventh-order derivative-free iterative method for solving nonlinear systems, *Numer. Algor.*, 70, 545558, (2015).

[15] Zheng, Q., Zhao, P., Huang, F., A family of fourth-order Steffensen-type methods with the applications on solving nonlinear ODEs, *Appl. Math. Comput.*, 217, 81968203, (2011).

Chapter 32

Fourth Convergence Order Derivative Free Methods with or without Memory

1. Introduction

The local convergence of the following methods using high-order derivatives in the finite-dimensional Euclidean space given in [5,6,7] is defined by

$$\begin{aligned} y_n &= x_n - [v_n, x_n; F]^{-1} F(x_n), v_n = x_n + aF(x_n) \\ \text{and} & \\ x_{n+1} &= y_n - (\alpha I + A_n((3-2\alpha)I + (\alpha-2)A_n))[v_n, x_n; F]^{-1} F(y_n) \\ \text{where} & \\ A_n &= [v_n, x_n; F]^{-1}[z_n, y_n; F], z_n = y_n + bF(y_n) \end{aligned} \tag{32.1}$$

and

$$\begin{aligned} y_n &= x_n - [v_n, x_n; F]^{-1} F(x_n), w_n = x_n + B_n F(x_n), \\ B_n &= -[v_{n-1}, x_{n-1}; F]^{-1} \\ \text{and} & \\ x_{n+1} &= y_n - (\alpha I + A_n((3-2\alpha)I + (\alpha-2)A_n))[w_n, x_n; F]^{-1} F(y_n). \end{aligned} \tag{32.2}$$

Both methods are variations of the Steffensen method. The first one is without memory with order $2+\sqrt{5}$ and the second one is with memory with order $2+\sqrt{6}$.

The benefits of using these methods have been well explained in [5,6,7] We present a local convergence analysis based only on the divided difference in these methods. Hence, we extend their applicability in a Banach space setting. We also provide the semi-local convergence (not given in [5,6,7]) under the conditions of the previous Chapters.

2. Local Convergence

Method (32.1)

We obtain in turn the estimates

$$
\begin{aligned}
y_n - x^* &= x_n - x^* - [v_n, x_n; F]^{-1} F(x_n) \\
&= [v_n, x_n; F]^{-1}([v_n, x_n; F] - [y_n, x^*; F])(x_n - x^*),
\end{aligned}
$$

$$
\|[v_n, x_n; F]^{-1} F'(x^*)\| \le \frac{1}{1 - \varphi_0(\|v_n - x^*\|, \|x_n - x^*\|)},
$$

$$
\begin{aligned}
\|y_n - x^*\| &\le \frac{\varphi_0(\|v_n - x^*\|, \|x_n - x^*\|)\|x_n - x^*\|}{1 - \varphi_0(\|v_n - x^*\|, \|x_n - x^*\|)}, \\
\|v_n - x^*\| &= \|[I + a[x_n, x^*; F])(x_n - x^*)\| \\
&\le \gamma(\|x_n - x^*\|)\|x_n - x^*\|, \\
x_{n+1} - x^* &= y_n - x^* - [v_n, x_n; F]^{-1} F(y_n) \\
&\quad -((\alpha - 2)I + A_n((3 - 2\alpha)I \\
&\quad +(\alpha - 2)A_n))(v_n, x_n; F]^{-1} F(y_n), \\
\|x_{n+1} - x^*\| &\le [g_1(\|y_n - x^*\|) + \sigma_n(1 + |\alpha - 2|\sigma_n) \\
&\quad \left.\frac{\delta\|y_n - x^*\|}{1 - \varphi_0(\|v_n - x^*\|, \|x_n - x^*\|)}\right] \|y_n - x^*\|.
\end{aligned}
$$

Notice that $\|z_n - x^*\| \le \bar{\gamma}(\|x_n - x^*\|)\|x_n - x^*\|$, where we also imposed $\|I - b[x, x^*; F]\| \le \bar{\gamma}(\|x - x^*\|)$ for all $x \in \Omega$ and the function $\bar{\gamma}$ is as the function γ.

$$
g_2(t) = [g_1(g_1(t)t) + \sigma(t)(1 + |\alpha - 2|)\frac{\delta g_1(t)t}{1 - \varphi_0(\gamma(t)t, t)}]g_1(t),
$$

where

$$
\sigma_n = \frac{\varphi_1(\|v_n - x^*\|, \|x_n - x^*\|, \|z_n - x^*\|}{1 - \varphi_0(\|v_n - x^*\|, \|x_n - x^*\|)}
$$

and

$$
\sigma(t) = \frac{\varphi_1(\gamma(t)t, t, \bar{\gamma}(t)t)}{1 - \varphi_0(\gamma(t)t, t)}.
$$

Method (32.2) Similarly, we have in turn

$$
\begin{aligned}
\|y_n - x^*\| &= \|[w_n, x_n; F]^{-1}([w_n, x_n; F] - [x_n, x^*; F])(w_n - x^*)\| \\
&\le \frac{\varphi(\|w_n - x^*\|, \|x_n - x^*\|)\|x_n - x^*\|}{1 - \varphi_0(\|w_n - x^*\|, \|x_n - x^*\|)}, \\
\|w_n - x^*\| &= \|(I - [v_{n-1}, x_{n-1}; F]^{-1})(x_n - x^*)\| \\
&\le h(\|v_{n-1} - x^*\|, \|x_{n-1} - x^*\|,
\end{aligned}
$$

where we also impose the condition

$$
\|I - [v, x; F]^{-1}\| \le h(\|v - x^*\|, \|x - x^*\|)
$$

for all $v, x \in \Omega$,

$$\begin{aligned} x_{n+1} - x^* &= y_n - x^* - [w_n, x_n; F]^{-1} F(y_n), \\ &\quad -(I - A_n)(I + (\alpha - 2)(I - A_n))[w_n, x_n; F]^{-1} F(Y_n)) \\ \|x_{n+1} - x^*\| &\leq \Big[\frac{\varphi(\|w_n - x^*\|, \|x_n - x^*\|)}{1 - \varphi_0(\|w_n - x^*\|, \|x_n - x^*\|)} \\ &\quad + \lambda_n (1 + |\alpha - 2| \lambda_n) \frac{\delta \|y_n - x^*\|}{1 - \varphi_0(\|w_n - x^*\|, \|x_n - x^*\|)} \Big] \|y_n - x^*\|, \end{aligned}$$

where

$$\lambda_n = \frac{\varphi(\|w_n - x^*\|, \|x_n - x^*\|, \|y_n - x^*\|)}{1 - \varphi_0(\|w_n - x^*\|, \|x_n - x^*\|)}.$$

Hence, we define

$$g_1(t) = \frac{\varphi(h(\gamma(t)t, t), t)}{1 - \varphi_0(h(\gamma(t)t, t), t)}$$

and

$$g_2(t) 7 g_1(t) + \lambda(t)(1 + |\alpha - 2| \lambda(t)) \frac{\delta(h_1(t)t)}{1 - \varphi_0(h(\gamma(t)t, t)}] g_1(t),$$

where

$$\lambda(t) = \frac{\varphi_1(h(\gamma(t)t, t), t, g_1(t)t)}{1 - \varphi_0(h(\gamma(t)t, t), t)}.$$

Notice that $v_{-1}, x_{-1} \in U(x^*, r)$, $r = \min\{r_1, r_2\}$ and

$$R = \max\{r, \gamma(r)r, \bar{\gamma}(r)r, h(\gamma(r)r, r)\}.$$

3. Semi-Local Convergence

The majorizing sequences for these methods are respectively

$$t_{n+1} = s_n + (1 + d_n(1 + |\alpha - 2| d_n)) \frac{\psi(\bar{\gamma}(t_n)t_n, t_n, s_n)}{1 - \psi_0(\gamma(t_n)t_n, t_n)} (s_n - t_n)$$

and

$$\begin{aligned} s_{n+1} &= t_{n+1} + \frac{1}{1 - \psi_0(\gamma(t_{n+1})t_{n+1}, t_{n+1})} \\ &\quad ((1 + \psi_0(t_{n+1}, t_n))(t_{n+1} - t_n) + (1 + \psi_0(t_{n+1}, t_n))(t_{n+1} - t_n) \\ &\quad + (1 + \psi_0)(\gamma(t_n)t_n, t_n)(s_n - t_n)), \\ t_{n+1} &= s_n + (1 + q_n(1 + |\alpha - 2| q_n)) \frac{1}{1 - \psi_0(\gamma(t_n), t_n)} \\ &\quad (\psi(h(\gamma(t_n)t_n, t_n), t_n, s_n)(s_n - t_n) \end{aligned}$$

and

$$\begin{aligned} s_{n+1} &= t_{n+1} + \frac{1}{1 - \psi_0(\gamma(t_{n+1})t_{n+1}, t_{n+1})} \\ &\quad ((1 + \psi_0(t_{n+1}, t_n))(t_{n+1} - t_n) + (1 + \psi_0(h(\gamma(t_n)t_n, t_n, t_n))(s_n, t_n), \end{aligned}$$

where

$$d_n = \frac{\psi_1(\gamma(t_n)t_n, \bar{\gamma}(t_n)t_n, s_n, t_n)}{1 - \psi_0(\gamma(t_n)t_n, t_n)}$$

and

$$q_n = \frac{\psi_1(h(\gamma(t_n)t_n), \bar{\gamma}(t_n)t_n, s_n, t_n)}{1 - \psi_0(\gamma(t_n)t_n, t_n)}.$$

The convergence criteria of the corresponding Lemma for these methods

$$\psi_0(\gamma(t_n)t_n, t_n) < 1$$

and

$$\psi_0(h(\gamma(t_n)t_n, t_n) < 1.$$

The motivation for the introduction of these real sequences follows from some estimates.

Method (32.1)
We have in turn

$$\begin{aligned} \|x_{n+1} - y_n\| &= \|(-I + (I - A_n)(I + (\alpha - 2)A_n))[v_n, x_n; F]^{-1}F(y_n)\| \\ &\leq (1 + c_n(1 + |\alpha - 2|c_n)) \\ &\quad \times \frac{\psi(\|v_n - x_0\|, \|x_n - x_0\|, \|y_n - x_0\|)\|y_n - x_n\|}{1 - \psi_0(\gamma(\|x_n - x_0\|)\|x_n - x_0\|, \|x_n - x_0\|)} \\ &\leq t_{n+1} - s_n, \end{aligned}$$

where we used

$$\begin{aligned} c_n &= \frac{\psi_1(\|v_n - x_0\|, \|z_n - x_0\|, \|y_n - x_0\|, \|x_n - x_0\|)}{1 - \psi_0(\gamma(\|x_n - x_0\|)\|x_n - x_0\|, \|x_n - x_0\|)} \\ &\leq \delta_n \\ \|F'(x_0)^{-1}F(y_n)\| &= \|F'(x_0)^{-1}(F(y_n) - F(x_n) - [v_n, x_n; F](y_n - x_n))\| \\ &= \|F'(x_0)^{-1}([\varepsilon_n, x_n; F] - [v_n, x_n; F])(y_n - x_n)\| \\ &\leq \frac{\psi(\|v_n - x_0\|, \|x_n - x_0\|, \|y_n - x_0\|)}{1 - \psi_0(\gamma(\|x_n - x_0\|)\|x_n - x_0\|)}, \\ F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - [x_{n+1}, x_n; F](x_{n+1} - x_n) \\ &\quad + [x_{n+1}, x_n; F](x_{n+1} - x_n) - [v_n, x_n; F](y_n - x_n), \end{aligned}$$

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{n+1})\| &\leq (1 + \psi_0(\|x_{n+1} - x_0\|, \|x_n - x_0\|))\|x_{n+1} - x_n\| \\ &\quad + (1 + \psi_0(\|v_n - x_0\|, \|x_n - x_0\|))\|y_n - x_n\|, \end{aligned}$$

so

$$\begin{aligned} \|y_{n+1} - x_{n+1}\| &\leq \|F'(x_0)^{-1}F(x_{n+1})\|\|F'(x_0)^{-1}F(x_{n+1})\| \\ &\leq s_{n+1} - t_{n+1}. \end{aligned}$$

Method (32.2)
As in the method (32.1) but replace v_n, γ by w_n, h to obtain

$$\|x_{n+1} - y_n\| \leq t_{n+1} - s_n.$$

Similarly, replace v_n, γ by w_n, h to get

$$\|y_{n+1} - x_{n+1}\| \leq s_{n+1} - t_{n+1}.$$

Notice that in the case of the method (32.2), $x_{-1}, v_{-1} \in U(x_0, t^*)$.

Bibliography

[1] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[2] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)*, 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[3] Regmi, S., Argyros, I.K., George, S., Argyros, C.I., Extended Convergence of Three Step Iterative Methods for Solving Equations in Banach Space with Applications, *Symmetry*, 14, (7), 1484.

[4] Regmi, S., Argyros, I. K., George, S., Argyros, C.I., Numerical Processes for Approximating Solutions of Nonlinear Equations, *Axioms*, 11, (7), 307.

[5] Grau-Sánchez, M., Noguera, M., A technique to choose the most efficient method between secant method and some variants, *Appl Math. Comput.*, 218, 6415-6426, (2012).

[6] Grau-Sánchez, M., Peris, J.M., Gutiérrez, J.M., Accelerated iterative methods for finding solutions of a system of nonlinear equations, *Appl. Math. Comput.*, 190, 1815-1823, (2007).

[7] Sharma, J.R., Arora, H., Petković, M.S., An efficient derivative free family of fourth order methods for solving systems of nonlinear equations, *Appl. Math. Comput.*, 235, 383-393, (2014).

[8] Steffensen, J.F., Remarks on iteration, *Skand. Aktuar Tidskr*, 16, 64-72, (1933).

Chapter 33

Convergence Radius of an Efficient Iterative Method with Frozen Derivatives

We determine a radius of convergence for an efficient iterative method with frozen derivatives to solve Banach space-defined equations. Our convergence analysis used $\omega-$ continuity conditions only on the first derivative. Earlier studies have used hypotheses up to the seventh derivative, limiting the applicability of the method. Numerical examples complete the article.

1. Introduction

We consider solving an equation

$$F(x) = 0, \tag{33.1}$$

where $F : D \subset X \longrightarrow Y$ is continuously Fréchet differentiable, X, Y are Banach spaces and D is a nonempty convex set.

Iterative methods are used to generate a sequence converging to a solution x_* of equation (33.1) under certain conditions [1,2,3,4,5,6,7,8,9,10,11,12]. Recently a surge has been noticed in the development of efficient iterative methods with frozen derivatives. The convergence order is obtained using Taylor expansions and conditions on high-order derivatives not appearing in the method. These conditions limit the applicability of the methods.

For example: Let $X = Y = \mathbb{R}$, $D = [-\frac{1}{2}, \frac{3}{2}]$. Define f on D by

$$f(s) = \begin{cases} s^3 \log s^2 + s^5 - s^4 & if\ s \neq 0 \\ 0 & if\ s = 0. \end{cases}$$

Then, we have $x_* = 1$, and

$$f'(s) = 3s^2 \log s^2 + 5s^4 - 4s^3 + 2s^2,$$

$$f''(s) = 6x \log s^2 + 20s^3 - 12s^2 + 10s$$

and

$$f'''(s) = 6\log s^2 + 60s^2 - 24s + 22.$$

Obviously, $f'''(s)$ is not bounded on D. So, the convergence of these methods is not guaranteed by the analysis in these papers.

Moreover, no comparable error estimates are given on the distances involved or the uniqueness of the solution results. That is why we develop a technique so general that it can be used on iterative methods and address these problems by using only the first derivative which only appears in these methods.

We demonstrate this technique on the $3(i+1)$, $(i = 1,2,\ldots)$ convergence order method defined for all $n = 0,1,2,\ldots$, $y_n^{(-1)} = y_n$ and $h_n = h(x_n, y_n) = \frac{7}{2}I + A_n(-4I + \frac{3}{2}A_n)$, $A_n = F'(x_n)^{-1}F'(y_n)$ by

$$\begin{aligned}
y_n &= x_n - F'(x_n)^{-}F(x_n),\\
y_n^{(0)} &= y_n - \frac{1}{2}(I - A_n)F'(x_n)^{-1}F(x_n),\\
y_n^{(1)} &= y_n^{(0)} - h(x_n,y_n)F'(x_n)^{-1}F(y_n^{(0)}),\\
y_n^{(2)} &= y_n^{(1)} - h(x_n,y_n)F'(x_n)^{-1}F(y_n^{(1)}),\\
y_n^{(3)} &= y_n^{(2)} - h(x_n,y_n)F'(x_n)^{-1}F(y_n^{(2)}),\\
&\vdots\\
y_n^{(i-1)} &= y_n^{(i-2)} - h(x_n,y_n)F'(x_n)^{-1}F(y_n^{(i-2)})\\
\text{and}\quad y_n^{(i)} &= x_{n+1} = y_n^{(i-1)} - h(x_n,y_n)F'(x_n)^{-1}F(y_n^{(i-1)}).
\end{aligned} \tag{33.2}$$

The efficiency, convergence order, and comparisons with other methods using similar information were given in [10] when $X = Y = \mathbb{R}^k$. The convergence was shown using the seventh derivative. We include error bounds on $\|x_n - x_*\|$ and uniqueness results not given in [10]. Our technique is so general that it can be used to extend the usage of other methods [1,2,3,4,5,6,7,8,9,10,11,12]

The chapter contains local convergence analysis in Section 2 and numerical examples in Section 3.

2. Convergence for Method (33.2)

Set $S = [0,\infty)$. Let $w_0 : S \longrightarrow S$ be a continuous and nondecreasing function. Suppose that equation

$$\omega_0(t) - 1 = 0 \tag{33.3}$$

has a least positive solution ρ_0. Set $S_0 = [0,\rho_0)$. Let $\omega : S_0 \longrightarrow S$ and $\omega_1 : S_0 \longrightarrow S$ be continuous and nondecreasing functions.

Suppose that equations

$$\varphi_{-1}(t) - 1 = 0, \tag{33.4}$$

$$\varphi_0(t) - 1 = 0 \tag{33.5}$$

and

$$\psi^m(t)\varphi_0(t) - 1 = 0, m = 1, 2, \ldots, i \tag{33.6}$$

have least solutions $r_{-1}, r_0, r_m \in (0, \rho_0)$, respectively, where

$$\varphi_{-1}(t) = \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)},$$

$$\varphi_0(t) = \varphi_{-1}(t)t + \frac{(\omega_0(t)+\omega_0(\varphi_{-1}(t)t))\int_0^1 \omega_1(\theta t)d\theta}{2(1-\omega_0(t))^2},$$

and

$$\begin{aligned}\psi(t) &= \varphi_{-1}(\varphi_0(t)t) + \frac{(\omega_0(t)+\omega_0(\varphi_{-1}(t)t))\int_0^1 \omega_1(\theta t)d\theta}{(1-\omega_0(t))^2} \\ &+ \frac{1}{2}\left(3\left(\frac{\omega_0(t)+\omega_0(\varphi_{-1}(t)t)}{1-\omega_0(t)}\right)^2 + 2\left(\frac{\omega_0(t)+\omega_0(\varphi_{-1}(t)t)}{1-\omega_0(t)}\right)\right) \\ &\times \int_0^1 \frac{\omega_1(\theta t)d\theta}{1-\omega_0(t)}.\end{aligned}$$

Define

$$r = \min\{r_j\},\ j = -1, 0, 1, \ldots, m. \tag{33.7}$$

It follows by the definition of r that for each $t \in [0, r)$

$$0 \le \omega_0(t) < 1 \tag{33.8}$$

and

$$0 \le \varphi_j(t) < 1. \tag{33.9}$$

We shall show that r is a radius of convergence for method (33.2). Let $B(x,\alpha), \bar{B}(x,\alpha)$ denote the open and closed balls respectively in X with center $x \in X$ and of radius $\alpha > 0$.

The following set of conditions (A) shall be used in the local convergence analysis of the method (33.2).

(A1) $F : D \subset X \longrightarrow Y$ is Fréchet continuously differentiable and there exists $x_* \in D$ such that $F(x_*) = 0$ and $F'(x_*)^{-1} \in L(Y, X)$.

(A2) There exists function $\omega_0 : S \longrightarrow S$ continuous and nondecreasing such that for each $x \in D$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \le \omega_0(\|x - x_*\|).$$

Set $D_0 = D \cap B(x_*, \rho_0)$

(A3) There exists functions $\omega : S_0 \longrightarrow S, \omega_1 : S_0 \longrightarrow S$ continuous and nondecreasing such that for each $x, y \in D_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \le \omega(\|y - x\|)$$

and

$$\|F'(x_*)^{-1}F'(x)\| \le \omega_1(\|x - x_*\|).$$

(A4) $\bar{B}(x_*, r) \subset D$, where r is defined by (33.7).

(A5) There exists $r_* \geq r$ such that

$$\int_0^1 \omega_0(\theta r_*)d\theta < 1.$$

Set $D_1 = D \cap \bar{B}(x_*, r_*)$.

Next, the local convergence of method (33.2) is given using the conditions (A) and the aforementioned notation.

Theorem 41. Suppose that condition (A) holds. Then, sequence $\{x_n\}$ generated by method (33.2) for any starting point $x_0 \in B(x_*, r) - \{x_*\}$ is well defined in $B(x_*, r)$, remains in $B(x_*, r)$ and converges to x_* so that for all $n = 0, 1, 2, \ldots, m = 1, 2, \ldots, i$,

$$\|y_n - x_*\| \leq \varphi_{-1}(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| < r, \tag{33.10}$$

$$\|y_n^{(0)} - x_*\| \leq \varphi_0(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|, \tag{33.11}$$

$$\|y_n^{(m)} - x_*\| \leq \psi^m(\|x_n - x_*\|)\varphi_0(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| \tag{33.12}$$

and

$$\|x_{n+1} - x_*\| \leq \psi^i(\|x_n - x_*\|)\varphi_0(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|. \tag{33.13}$$

Proof. Let $v \in B(x_*, r) - \{x_*\}$. Using (33.7), (33.8), (A1) and (A2), we get in turn

$$\|F'(x_*)^{-1}(F'(v) - F'(x_*))\| \leq \omega_0(\|v - x_*\|) \leq \omega_0(r) < 1. \tag{33.14}$$

It follows by (33.14) and a perturbation Lemma by Banach [2,8] that $F'(v)^{-1} \in L(Y, X)$ and

$$\|F'(v)^{-1}F'(x_*)\| \leq \frac{1}{1 - \omega_0(\|v - x_*\|)}. \tag{33.15}$$

It also follows that $y_0, y_0^{(0)}, \ldots y_0^{(m)}$ exist by method (33.2). By the first substep of method (33.2), (33.9) for $j = -1$, (A3) and (33.15), we have in turn

$$\begin{aligned} \|y_0 - x_*\| &= \|x_0 - x_* - F'(x_0)^{-1}F(x_0)\| \\ &= \|F'(x_0)^{-1}\int_0^1 (F'(x_* + \theta(x_0 - x_*)) - F'(x_0))d\theta(x_0 - x_*)\| \\ &\leq \frac{\int_0^1 \omega((1-\theta)\|x_0 - x_*\|)d\theta\|x_0 - x_*\|}{1 - \omega_0(\|x_0 - x_*\|)} \\ &\leq \varphi_{-1}(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \end{aligned} \tag{33.16}$$

so $y_0 \in B(x_*, r)$. Then, by the second substep of method (33.2), (33.9) for $j = 0$, (A3) and (33.15), we obtain in turn

$$\begin{aligned} \|y_0^{(0)} - x_*\| &= \|y_0 - x_* + \frac{1}{2}F'(x_0)^{-1}F'(x_*)F'(x_*)^{-1}(F'(x_0) - F'(y_0)) \\ &\quad \times F'(x_0)^{-1}F'(x_*)F'(x_*)^{-1}F(x_0)\| \qquad (33.17) \\ &\leq [\varphi_{-1}(\|x_0 - x_*\|)\|x_0 - x_*\| \\ &\quad + \frac{1}{2}\frac{(\omega_0(\|x_0 - x_*\|) + \omega_0(\|y_0 - x_*\|))\int_0^1 \omega_1(\theta\|x_0 - x_*\|)d\theta}{(1 - \omega_0(\|x_0 - x_*\|))^2}\Bigg]\|x_0 - x_*\| \\ &\leq \varphi_0(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \end{aligned} \tag{33.18}$$

so $y_0^{(0)} \in B(x_*, r)$. Next, by the rest of the substeps of method (33.2), (33.9) for $j = 1, 2, \ldots, m$ and (33.15), we have in turn

$$\begin{aligned} y_0^{(1)} - x_* &= y_0^{(0)} - x_* - (\frac{7}{2}I + A_0(-4I + \frac{3}{2}A_0))F'(x_0)^{-1}F(y_0^{(0)}) \\ &= y_0^{(0)} - x_* - F'(x_0)^{-1}F(y_0^{(0)}) \\ &\quad -\frac{1}{2}(3(A_0 - I)^2 - 2(A_0 - I))F'(x_0)^{-1}F(y_0^{(0)}) \\ &= y_0^{(0)} - x_* - F'(y_0^{(0)})^{-1}F(y_0^{(0)} \\ &\quad +(F'(y_0^{(0)})^{-1} - F'(x_0)^{-1})F(y_0^{(0)}) \\ &\quad -\frac{1}{2}(3(A_0 - I)^2 - 2(A_0 - I))F'(x_0)^{-1}F(y_0^{(0)}), \end{aligned} \tag{33.19}$$

which by the triangle inequality leads to

$$\begin{aligned} \|y_0^{(1)} - x_*\| &\leq [\varphi_{-1}(\|y_0^{(0)} - x_*\|) \\ &\quad + \frac{(\omega_0(\|x_0 - x_*\|) + \omega_0(\|y_0^{(0)} - x_*\|)) \int_0^1 \omega_1(\theta\|y_0^{(0)} - x_*\|)d\theta}{(1 - \omega_0(\|y_0^{(0)} - x_*\|))(1 - \omega_0(\|x_0 - x_*\|))} \|y_0^{(0)} - x_*\| \\ &\quad + \frac{1}{2}\left(3\left(\frac{\omega_0(\|x_0 - x_*\|) + \omega_0(\|y_0^{(0)} - x_*\|)}{1 - \omega_0(\|x_0 - x_*\|)}\right)^2\right. \\ &\quad \left.+2\left(\frac{\omega_0(\|x_0 - x_*\|) + \omega_0(\|y_0^{(0)} - x_*\|)}{1 - \omega_0(\|x_0 - x_*\|)}\right)\right) \frac{\int_0^1 \omega_1(\theta\|y_0^{(0)} - x_*\|)d\theta}{1 - \omega_0(\|x_0 - x_*\|)} \\ &\leq \varphi_1(\|x_0 - x_*\|)\varphi_)(\|x_0 - x_*\|)\|x_0 - x_*\| \\ &\leq \|x_0 - x_*\| < r. \end{aligned} \tag{33.20}$$

Thus, $y_0^{(01)} \in B(x_*, r)$.

Similarly,

$$\begin{aligned} \|y_0^{(m)} - x_*\| &\leq \underbrace{\varphi_1(\|x_0 - x_*\|) \ldots \varphi_1(\|x_0 - x_*\|)}_{m-\text{times}} \varphi_0(\|x_0 - x_*\|)\|x_0 - x_*\| \\ &= \psi^m(\|x_0 - x_*\|)\varphi_0(\|x_0 - x_*\|)\|x_0 - x_*\| \\ &\leq \|x_0 - x_*\|. \end{aligned}$$

Hence, $y_0^{(m)} \in B(x_*, r)$, and

$$\|x_1 - x_*\| \leq \psi^i(\|x_0 - x_*\|)\varphi_0(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|. \tag{33.21}$$

Therefore, $x_1 \in B(x_*, r)$. Hence, estimates (33.10)-(33.13) are shown for $n = 0$. By replacing $x_0, y_0, y_0^{(1)}, \ldots y_0^{(m)}, x_1$ by $x_k, y_k, y_k^{(1)}, \ldots y_k^{(m)}, x_{k+1}, k = 0, 1, \ldots n$, we show (33.10)-(33.13) hold for each $n = 0, 1, 2, \ldots, j = -1, 0, 1, \ldots i$. Thus, we get

$$\|x_{k+1} - x_*\| \leq c\|x_k - x_*\|, \tag{33.22}$$

where $c = \psi^i(\|x_0 - x_*\|)\varphi_0(\|x_0 - x_*\|) \in [0,1)$, concluding that $\lim_{k \longrightarrow \infty} x_k = x_*$, and $x_{k+1} \in B(x_*, r)$.

Finally, let $x_{**} \in D_1$ with $F(x_{**}) = 0$. Set $Q = \int_0^1 F'(x_{**} + \theta(x_* - x_{**}))d\theta$. Using (A1), (A2), (A5) and (33.14), we get

$$\|F'(x_*)^{-1}(Q - F'(x_*))\| \leq \int_0^1 \omega_0(\theta\|x_* - x_{**})\|d\theta \leq \int_0^1 \omega_0(\theta r_{**})d\theta < 1,$$

so $Q^{-1} \in L(Y,X)$. Consequently, from $0 = F(x_{**}) - F(x_*) = Q(x_{**} - x_*)$, we obtain $x_{**} = x_*$. □

Remark. If $\{x_n\}$ is an iterative sequence converging to x_*, then the COC is defined as

$$\xi = \ln\left(\frac{\|x_{n+1} - x_*\|}{\|x_n - x_*\|}\right) / \ln\left(\frac{\|x_n - x_*\|}{\|x_{n-1} - x_*\|}\right)$$

where the ACOC is

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

The calculation of these parameters not needing high order derivatives.

3. Numerical Examples

Example 4. Consider the kinematic system

$$F_1'(x) = e^x,\ F_2'(y) = (e-1)y + 1,\ F_3'(z) = 1$$

with $F_1(0) = F_2(0) = F_3(0) = 0$. Let $F = (F_1, F_2, F_3)$. Let $B_1 = B_2 = \mathbb{R}^3, D = \bar{B}(0,1), p = (0,0,0)^t$. Define function F on D for $w = (x,y,z)^t$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^t.$$

Then, we get

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so $\omega_0(t) = (e-1)t, \omega(t) = e^{\frac{1}{e-1}}t, \omega_1(t) = e^{\frac{1}{e-1}}$. Then, the radii are

$$r_{-1} = 0.382692, r_0 = 0.234496, r_1 = 0.11851.$$

Example 5. Consider $B_1 = B_2 = C[0,1], D = \overline{B}(0,1)$ and $F : D \longrightarrow B_2$ defined by

$$F(\phi)(x) = \varphi(x) - 5\int_0^1 x\theta\phi(\theta)^3 d\theta. \tag{33.23}$$

We have that

$$F'(\phi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\phi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, we get that $x^* = 0$, so $\omega_0(t) = 7.5t, \omega(t) = 15t$ and $\omega_1(t) = 2$. Then, the radii are

$$r_{-1} = 0.06667, r_0 = 0.0420116, r_1 = 0.0182586.$$

Example 6. By the academic example of the introduction, we have $\omega_0(t) = \omega(t) = 96.6629073t$ and $\omega_1(t) = 2$. Then, the radii are

$$r_{-1} = 0.00689682, r_0 = 0.003543946, r_1 = 0.00150425.$$

Bibliography

[1] Amat, S., Hernández, M.A. Romero, N., Semilocal convergence of a sixth order iterative method for quadratic equations, *Applied Numerical Mathematics*, 62, (2012), 833-841.

[2] Argyros, I.K., *Computational Theory of Iterative Methods*, Series: Studies in Computational Mathematics, 15, Editors: Chui C.K. and Wuytack L. Elsevier Publ. Company, New York, (2007).

[3] Argyros, I.K., Magreñán, A.A., *Iterative method and their dynamics with applications*, CRC Press, New York, USA, 2017.

[4] Behl, R., Cordero, A., Motsa,S.S., Torregrosa, J.R., Stable high order iterative methods for solving nonlinear models, *Appl. Math. Comput.*, 303, 70-88, (2017).

[5] Cordero, A., Hueso, J.L., Martínez, E., Torregrosa, J.R., A modified Newton-Jarratt's composition, *Numer. Algor.*, 55, 87-99, (2010).

[6] Magreñán, A.A., Different anomalies in a Jarratt family of iterative root finding methods, *Appl. Math. Comput.*, 233, (2014), 29-38.

[7] Noor, M.A., Wassem, M., Some iterative methods for solving a system of nonlinear equations, *Appl. Math. Comput.*, 57, 101–106, (2009).

[8] Rheinboldt, W.C., An adaptive continuation process for solving systems of nonlinear equations, In: *Mathematical models and numerical methods* (A.N. Tikhonov et al. eds.) pub.3, (1977), 129-142, Banach Center, Warsaw Poland.

[9] Sharma, J.R., Kumar S., A class of computationally efficient Newton-like methods with frozen operator for nonlinear systems, *Intern. J. Non. Sc. Numer. Simulations*.

[10] Traub, J.F., *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, (1964).

[11] Sharma, J.R, Arora, H., Improved Newton-like methods for solving systems of nonlinear equations, *SeMA*, 74, 147-163, (2017).

[12] Weerakoon, S., Fernando, T.G.I., A variant of Newton's method with accelerated third-order convergence, *Appl. Math. Lett.*, 13, 87–93 (2000).

Chapter 34

A Comparison between Some Derivative Free Methods

1. Introduction

In this Chapter, we study derivative-free iterative methods for approximately solving the nonlinear equation

$$F(x) = 0,\ G : \Omega \subset X \longrightarrow X, \tag{34.1}$$

where $F : \Omega \subset X \longrightarrow X$ are nonlinear operators, X is a Banach space and Ω is an open convex subset of X.

The methods are defined by

$$\begin{aligned} y_n &= x_n - [x_n, v_n; F]^{-1} F(x_n), v_n = x_n + F(x_n), \\ D_n &= [x_n, v_n; F] + [y_n, \bar{v}_n; F], \\ z_n &= x_n - 2D_n^{-1} F(y_n), \bar{v}_n = y_n + F(y_n) \end{aligned} \tag{34.2}$$

and

$$x_{n+1} = z_n - [y_n, \bar{v}_n; F]^{-1} F(z_n),$$

$$y_n = x_n - \frac{1}{2} A_n^{-1} F(x_n), A_n = [x_n, G(x_n); F]$$

and

$$\begin{aligned} z_n &= x_n - B_n^{-1} F(x_n), \\ x_{n+1} &= z_n - (2B_n^{-1} - A_n^{-1}) F(z_n), \end{aligned} \tag{34.3}$$

where

$$B_n = [y_n, G(x_n); F],$$

$$\begin{aligned} y_n &= x_n - [v_n, x_n; F]^{-1} F(x_n), \\ v_n &= x_n + F(x_n),\ w_n = x_n - F(x_n), \\ z_n &= y_n - M_n^{-1} F(y_n) \\ \text{and} & \\ x_{n+1} &= z_n - M_n^{-1} F(z_n), \\ \text{where} & \\ M_n &= 2[y_n, x_n; F] - [v_n, w_n; F] \end{aligned} \tag{34.4}$$

and

$$\begin{aligned} y_n &= x_n - [v_n, x_n; F]^{-1} F(x_n), \\ v_n &= x_n + dF(x_n), \\ z_n &= y_n - [y_n, x_n; F]^{-1} T_n [y_n, x_n; F]^{-1} F(y_n) \\ \text{and} & \\ x_{n+1} &= z_n - Q_n^{-1} F(z_n), \end{aligned} \tag{34.5}$$

where $d \in \mathbb{R}, T_n = [y_n, x_n; F] - [y_n, v_n; F] + [v_n, x_n; F]$ and $Q_n = [z_n, x_n; F] + [z_n, y_n; F] - [y_n, x_n; F]$. The local convergence of these methods using similar information was given in [6] and [8], respectively. The convergence orders are five, five, seven, and seven, respectively. We extend the local as well as the semi-local convergence (not given in [6, 7, 8]) under weaker conditions. The functions "ω", "φ" and "ψ" are as given in the preface. The proofs and conditions similar to the ones given before are omitted.

Related work can be found in the previous chapters and [1, 2, 3, 4, 5].

2. Local Convergence

Method (34.2)

The estimates are:

$$\begin{aligned} y_n - x^* &= [x_n, v_n; F]^{-1}([x_n, v_n; F] - [x_n, x^*; F])(x_n - x^*), \\ \|y_n - x^*\| &\leq \frac{\varphi(\|x_n - x^*\|, \|v_n - x^*\|)\|x_n - x^*\|}{1 - \varphi_0(\|x_n - x^*\|, \|v_n - x^*\|)}, \\ v_n - x^* &= (I + [x_n, x^*; F])(x_n - x^*), \\ \bar{v}_n - x^* &= (I + [y_n, x^*; F])(y_n - x^*), \end{aligned}$$

$$\begin{aligned} \|(2F'(x^*))^{-1}(D_n - 2F'(x^*))\| &\leq \|(2F'(x^*))^{-1}([x_n, v_n; F] + [y_n, \bar{v}_n; F] - 2F'(x^*))\| \\ &\leq \frac{1}{2}(\|F'(x^*)^{-1}([x_n, v_n; F] - F'(x^*))\| \\ &\quad + \|F'(x^*)^{-1}([y_n, \bar{v}_n; F] - F'(x^*))\|, \\ &\leq \frac{1}{2}(\varphi_0(\|x_n - x^*\|, \|v_n - x^*\|) + \varphi_0(\|y_n - x^*\|, \|\bar{v}_n - x^*\|) \\ &= \bar{q}_n \leq q_n, \end{aligned}$$

$$\|D_n^{-1}F'(x^*)\| \leq \frac{1}{2(1-\bar{q}_n)} \leq \frac{1}{2(1-q_n)},$$

$$\begin{aligned}
z_n - x^* &= x_n - x^* - 2D_n^{-1}F(x_n) \\
&= D_n^{-1}(D_n - 2[x_n, x^*; F])(x_n - x^*), \\
\|z_n - x^*\| &\leq \|D_n^{-1}F'(x^*)\|\|F'(x^*)^{-1}(([x_n, v_n; F] \\
&\quad -[x_n, x^*; F]) + ([y_n, \bar{v}_n; F] - [x_n, x^*; F])\| \\
&\leq \frac{\bar{p}_n}{1-\bar{q}_n}\|x_n - x^*\|, \\
p_n &= \varphi(\|x_n - x^*\|, \|v_n - x^*\|) + \varphi_1(\|x_n - x^*\|, \|y_n - x^*\|, \|\bar{v}_n - x^*\|),
\end{aligned}$$

$$x_{n+1} - x^* = [y_n, \bar{v}_n; F]^{-1}([y_n, \bar{v}_n; F] - [z_n, x^*; F])(z_n - x^*)$$

and

$$\|x_{n+1} - x^*\| \leq \frac{\varphi_1(\|y_n - x^*\|, \|\bar{v}_n - x^*\|, \|z_n - x^*\|)}{1 - \varphi_0(\|y_n - x^*\|, \|x_n - x^*\|)})\|z_n - x^*\|.$$

Hence, the majorizing functions can be chosen to be

$$h_1(t) = \frac{\varphi(t, \gamma(t)t)}{1 - \varphi_0(t, \gamma(t)t)},$$

$$h_2(t) = \frac{\varphi(t, \gamma(t)t) + \varphi_1(t, h_1(t)t, \gamma(h_1(t)t)h_1(t)t)}{2(1 - \frac{1}{2}q(t))}$$

and

$$h_3(t) = \frac{\varphi_1(h_1(t)t, \gamma(h_1(t)t)h_1(t)t, h_2(t)t))h_2(t)}{1 - \varphi_0(h_1(t)t, \gamma(h_1(t)t)h_1(t)t)}$$

where

$$R = \min\{r, \gamma(r)r\}, r = \min\{r_1, r_2, r_3\}$$

and r_i are the smallest positive solutions of equations $h_i(t) - 1 = 0, i = 1, 2, 3$.

Local (34.3)

In view of the appearance of the operator G the following conditions are imposed

$$\|F'(x^*)^{-1}([x, G(x); F] - [x, x^*; F])\| \leq a(\|x - x^*\|),$$

$$\|F'(x^*)^{-1}([x, G(x); F] - F'(x^*))\| \leq b(\|x - x^*\|),$$

$$\|F'(x^*)^{-1}([y, G(y); F] - [x, G(x); F])\| \leq c(\|x - x^*\|),$$

for each $x \in \Omega$. This way we have the following estimates, in turn, assuming that the interates exist:

$$\begin{aligned}
y_n - x^* &= x_n - x^* - \frac{1}{2}A_n^{-1}F(x_n) \\
&= \frac{1}{2}A_n^{-1}[A_n - [x_n, x^*; F])(x_n - x^*) + \frac{1}{2}I,
\end{aligned}$$

$$\begin{aligned}
\|y_n - x^*\| &\le \frac{1}{2}\left[\frac{a(\|x_n - x^*\|)}{1 - b(\|x_n - x^*\|)} + 1\right]\|x_n - x^*\| \\
&\le g_1(\|x_n - x^*\|\|x_n - x^*\| \le \|x_n - x^*\|, \\
z_n - x^* &= y_n - x^* + \frac{1}{2}(A_n^{-1} - B_n^{-1})F(x_n) + \frac{1}{2}B_n^{-1}F(x_n) \\
&= y_n - x^* + \frac{1}{2}A_n^{-1}(B_n - A_n)B_n^{-1}F(x_n) + \frac{1}{2}B_n^{-1}F(x_n) \\
d(\|x_n - x^*\|) &= \frac{c(\|x_n - x^*\|)}{(1 - b(\|x_n - x^*\|)(1 - b(\|y_n - x^*\|))},
\end{aligned}$$

$$\begin{aligned}
\|z_n - x^*\| &\le \|y_n - x^*\| + \frac{1}{2}(d(\|x_n - x^*\| + \frac{1}{1 - b(\|y_n - x^\|)}) \\
&\quad (1 + \int_0^1 w_0(\theta\|x_n - x^*\|)d\theta)\|x_n - x^*\| \\
&\le g_2(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|,
\end{aligned}$$

$$x_{n+1} - x^* = z_n - x^* - (B_n^{-1} - A_n^{-1})F(z_n) - B_n^{-1}F(z_n)$$

and

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le [1 + d(\|x_n - x^*\|) + \frac{1}{1 - b(\|y_n - x^*\|)}] \\
&\quad (1 + \int_0^1 w_0(\theta\|z_n - x^*\|)d\theta)\|z_n - x^*\| \\
&\le g_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|,
\end{aligned}$$

provided that

$$g_1(t) = \frac{1}{2}(1 + \frac{a(t)}{1 - b(t)}),$$

$$d(t) = \frac{c(t)}{(1 - b(t))(1 - b(g_1(t)t))},$$

$$g_2(t) = (1 + d(t) + \frac{1}{1 - b(g_1(t)t)})(1 + \int_0^1 w_0(\theta g_2(t)t)d\theta).$$

and

$$g_3(t) = g_1(t) + \frac{1}{2}(d(t) + \frac{1}{1 - b(g_1(t)t)})(1 + \int_0^1 w_0(\theta t)d\theta)g_2(t).$$

Local (34.4)

$$\begin{aligned}
g_1(t) &= \frac{\varphi(\gamma_0(t)t, \gamma(t)t, t)}{1 - p(t)}, \\
p(t) &= \varphi_0(\gamma_0(t)t, \gamma(t)t), \\
g_2(t) &= \frac{(\varphi_1(t, g_1(t)t, \gamma_0(t)t, \gamma(t)t) + \varphi(t, g_1(t)t, g_1(t)t))g_1(t)}{1 - q(t)}, \\
q(t) &= 2\varphi_0(t, g_1(t)t) + \varphi_0(\gamma_0(t)t, \gamma(t)t),
\end{aligned}$$

and

$$g_3(t) = \frac{(\varphi_1(t, g_1(t)t, \gamma_0(t)t, \gamma(t)t) + \varphi(t, g_1(t)t, g_2(t)t))g_2(t)}{1 - q(t)}.$$

The estimates are:

$$\begin{aligned}
y_n - x^* &= [v_n, w_n; F]^{-1}([v_n, w_n; F] - [x_n, x^*; F])(x_n - x^*)\\
v_n - x^* &= (I + [x_n, x^*; F])(x_n - x^*),\\
w_n - x^* &= (I - [x_n, x^*; F])(x_n - x^*),\\
\|F'(x^*)^{-1}(M_n - F'(x^*))\| &\leq 2\|F'(x^*)^{-1}([y_n, x_n; F] - F'(x^*))\|\\
&\quad + \|F'(x^*)^{-1}([v_n, w_n; F] - F'(x^*))\|\\
&\leq 2\varphi_0(\|y_n - x^*\|, \|x_n - x^*\|) + \varphi_0(\|v_n - x^*\|, \|w_n - x^*\|)\\
&\leq \bar{q}_n \leq q_n < 1,
\end{aligned}$$

$$\begin{aligned}
z_n - x^* &= M_n^{-1}(M_n - [y_n, x^*; F])(y_n - x^*)\\
&\quad \|F'(x^*)^{-1}(M_n - [y_n, x^*; F])\|\\
&\leq \|F'(x^*)^{-1}(([y_n, x_n; F] - [v_n, w_n; F])\\
&\quad + ([y_n, x_n; F] - [y_n, x^*; F])\|\\
&\leq \varphi_1(\|x_n - x^*\|, \|y_n - x^*\|, \|v_n - x^*\|, \|w_n - x^*\|)\\
&\quad + \varphi(\|x_n - x^*\|, \|y_n - x^*\|, \|y_n - x^*\|),\\
x_{n+1} - x^* &= M_n^{-1}(M_n - [z_n; x^*; F])(z_n - x^*)\\
\text{and}&\\
\|x_{n+1} - x^*\| &\leq \Delta\|z_n - x^*\|,
\end{aligned}$$

where $\Delta = \dfrac{1}{1-\bar{q}_n}(\varphi_1(\|x_n - x^*\|, \|y_n - x^*\|, \|v_n - x^*\|, \|w_n - x^*\|) + \varphi(\|x_n - x^*\|, \|y_n - x^*\|, \|z_n - x^*\|))$.

Local (34.5)

$$g_1(t) = \frac{\varphi(t, t, \gamma_0(t)t)}{1 - \varphi_0(t, \gamma_0(t)t)},$$

$$b(t) = [1 + \varphi_0(\gamma_0(t)t, t) + \varphi(t, g_1(t)t, \gamma_0(t)t)]\delta(g_1(t)t),$$

$$g_2(t) = (1 + \frac{b(t)}{(1 - \varphi_0(t, g_1(t)t))^2})g_1(t),$$

$$g_3(t) = [1 + \frac{\delta(g_2(t)t)}{1 - c(t)}]g_2(t)$$

where

$$c(t) = \varphi(t, g_1(t)t, g_2(t)t) + \varphi_1(g_1(t)t, g_2(t)t).$$

We use the estimates

$$\begin{aligned}
y_n - x^* &= [v_n, x_n; F]^{-1}([v_n, x_n; F] - [x_n, x^*; F])(x_n - x^*), \\
z_n - x^* &= y_n - x^* - [y_n, x_n; F]^{-1} T_n [y_n, x_n; F]^{-1} F(y_n), \\
& \quad \|F'(x^*)^{-1} T_n\| \|F'(x^*)^{-1}[y_n, x^*; F]\| \\
&\leq (1 + \varphi_0(\|v_n - x^*\|, \|x_n - x^*\|) \\
& \quad + \varphi(\|x_n - x^*\|, \|y_n - x^*\|, \|v_n - x^*\|))\delta(\|y_n - x^*\|) \\
&= \bar{b}_n \leq b_n, \\
\|F'(x^*)^{-1}(Q_n - F'(x^*))\| &\leq \varphi_1(\|x_n - x^*\|, \|y_n - x^*\|, \|z_n - x^*\|) \\
& \quad + \varphi_0(\|y_n - x^*\|, \|z_n - x^*\|) \\
&= \bar{c}_n \leq c_n
\end{aligned}$$

and

$$\|x_{n+1} - x^*\| \leq (1 + \frac{\delta(\|z_n - x^*\|}{1 - \bar{c}_n})\|z_n - x^*\|.$$

Semi-local Convergence
Method (34.2)

$$\begin{aligned}
a_n &= \psi_1(t_n, s_n, \gamma_0(t_n)t_n + \eta_0, \gamma(t_n)t_n + \eta_0) + \psi_0(t_n, s_n), \\
u_n &= s_n + \frac{\psi_1(t_n, s_n, \gamma_0(t_n)t_n + \eta_0, \gamma(t_n)t_n + \eta_0)}{1 - a_n}(s_n - t_n), \\
t_{n+1} &= u_n + \frac{1}{1 - a_n}(1 + \psi_0(u_n, s_n))(u_n - s_n) \\
& \quad + \psi_1(t_n, s_n, \gamma_0(t_n)t_n + \eta_0, \gamma(t_n)t_n + \eta_0)
\end{aligned}$$

and

$$\begin{aligned}
s_{n+1} &= t_{n+1} + \frac{1}{1 - \psi_0(\gamma_0(t_{n+1})t_{n+1} + \eta_0, \gamma(t_{n+1})t_{n+1} + \eta_0)} \\
& \quad \times (1 + \psi_0(\gamma_0(t_n)t_n + \eta_0, \gamma(t_n)t_n + \eta_0))(s_n - t_n) \\
& \quad + (1 + \psi_0(t_n, t_{n+1})(t_{n+1} - t_n).
\end{aligned}$$

The estimates are:

$$\begin{aligned}
\|F'(x_0)^{-1}(D_n - F'(x_0))\| &\leq \psi_1(\|x_n - x_0\|, \|y_n - x_0\|, \|v_n - x_0\|, \|w_n - x_0\|) \\
& \quad + \psi_0(\|x_n - x_0\|, \|y_n - x_0\|) \\
&= \bar{a}_n \leq a_n < 1, \\
\|z_n - y_n\| &\leq \|D_n^{-1} F'(x_0)\| \|F'(x_0)^{-1} F(y_n)\| \\
&\leq \frac{\psi_1(\|x_n - x_0\|, \|y_n - x_0\|, \|v_n - x_0\|, \|w_n - x_0\|)}{1 - \bar{a}_n} \\
& \quad \times \|y_n - x_n\|,
\end{aligned}$$

$$F(z_n) = F(z_n) - F(y_n) + F(y_n) = [z_n, y_n; F](z_n - y_n) + F(y_n),$$

$$\begin{aligned}\|x_{n+1}-z_n\| &\leq \|D_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(z_n)\| \\ &\leq \frac{1}{1-\bar{a}_n}(1+\psi_0(\|z_n-x_0\|,\|y_n-x_0\|))\|z_n-y_n\| \\ &\quad +\psi_1(\|x_n-x_0\|,\|y_n-x_0\|,\|v_n-x_0\|,\|w_n-x_0\|)\|y_n-x_n\|,\end{aligned}$$

$$F(x_{n+1}) = [x_{n+1},x_n;F](x_{n+1}-x_n)-[v_n,w_n;F](y_n-x_n)$$

and

$$\|y_{n+1}-x_{n+1}\| \leq \|D_{n+1}^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\|.$$

Method (34.3)

Under the conditions of the local case with $x^* = x_0$, we obtain the estimates

$$z_n - y_n = \frac{1}{2}(A_n^{-1}-B_n^{-1})F(x_n)-\frac{1}{2}B_n^{-1}F(x_n),$$

$$\|z_n-y_n\| \leq \frac{1}{2}(d_n+\frac{1}{1-b(s_n)})e_n = u_n - s_n,$$

since

$$F(x_n) = -2A_n(y_n-x_n),$$

$$\|F'(x_0)^{-1}F(x_n)\| \leq 2(1+b(t_n))(s_n-t_n) = e_n$$

and

$$d_n = \frac{c(t_n)}{(1-b(t_n))(1-b(s_n))},$$

$$\begin{aligned}\|x_{n+1}-z_n\| &= \|(-B_n^{-1}(D_n-B_n)D_n^{-1}-B_n^{-1})F(z_n)\| \\ &\leq (d_n+\frac{1}{1-b(s_n)})f_n = t_{n+1}-u_n,\end{aligned}$$

since $F(z_n) = F(z_n)-F(x_n)-2D_n(y_n-x_n)$,

$$\begin{aligned}\|F'(x_0)^{-1}F(z_n)\| &\leq \int_0^1 w(t_n+\theta(u_n-t_n))d\theta(u_n-t_n) \\ &\quad +2(1+b(t_n))(s_n-t_n) = f_n, \\ \|F'(x_0)^{-1}F(x_{n+1})\| &= \|F'(x_0)^{-1}(F(x_{n+1})-F(x_n)-2D_n(y_n-x_n))\| \\ &\leq (1+\int_0^1 w_0(t_n+\theta(t_{n+1}-t_n))d\theta)(t_{n+1}-t_n) \\ &\quad +2(1+b(t_n))(s_n-t_n) = h_{n+1}\end{aligned}$$

and

$$\begin{aligned}\|y_{n+1}-x_{n+1}\| &\leq \frac{1}{2}\|D_{n+1}^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\| \\ &\leq \frac{1}{2}\frac{h_{n+1}}{1-b(t_{n+1})} = s_{n+1}-t_{n+1}.\end{aligned}$$

Method (34.4)

$$\begin{aligned}
\|F'(x_0)^{-1}(M_n - F'(x_0))\| &\leq \|F'(x_0)^{-1}([y_n,x_n;F]-[v_n,w_n;F])\| \\
&\quad + \|F'(x_0)^{-1}([y_n,x_n;F]-F'(x_0))\| \\
&\leq \psi(s_n,t_n,\gamma_0(t_n)t_n,\gamma(t_n)t_n+\psi_0(s_n,t_n)=a_n \\
F(y_n) &= F(y_n)-F(x_n)+F(x_n) \\
&= \int_0^1 F'(x_n+\theta(y_n-x_n))d\theta(y_n-x_n) \\
&\quad -[v_n,x_n;F](y_n-x_n),
\end{aligned}$$

$$\begin{aligned}
\|F'(x_0)^{-1}F(y_n) &\leq (1+\psi_0(s_n,t_n))(s_n-t_n) \\
&\quad (1+\psi_0(t_n,\gamma_0(t_n)t_n)(s_n-t_n)=b_n \\
\|z_n-y_n\| &\leq \|M_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_n)\| \\
&\leq \frac{b_n}{1-a_n}=u_n-s_n, \\
F(z_n) &= F(z_n)-F(x_n)+F(x_n),
\end{aligned}$$

$$\begin{aligned}
\|F'(x_0)^{-1}F(z_n)\| &\leq (1+\psi_0(t_n,u_n))(u_n-t_n) \\
&\quad +(1+\psi_0(t_n,\gamma_0(t_n)t_n)(s_n-t_n)=c_n, \\
\|x_{n+1}-z_n\| &\leq \|M_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(z_n)\| \\
&\leq \frac{c_n}{1-a_n}=t_{n+1}-u_n,
\end{aligned}$$

$$F(x_{n+1})=F(x_{n+1})-F(x_n)-[v_n,x_n;F](y_n-x_n),$$

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq (1+\psi_0(t_n,t_{n+1}))(t_{n+1}-t_n) \\
&\quad +(1+\psi_0(t_n,\gamma_0(t_n)t_n))(s_n-t_n)=d_{n+1}
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1}-x_{n+1}\| &\leq \frac{d_{n+1}}{1-\psi_0(t_n,\gamma_0(t_n)t_n)} \\
&= s_{n+1}-t_{n+1}.
\end{aligned}$$

Method (34.5)

$$\begin{aligned}
u_n &= s_n+\frac{\mu_n}{(1-\psi_0(t_n,s_n))^2}, \\
\mu_n &= 1+2\psi(t_n,s_n,\gamma_0(t_n)t_n+\eta_0)+\psi_0(t_n,\gamma_0(t_n)t_n+\eta_0)), \\
t_{n+1} &= u_n+\frac{\ell_n}{1-\lambda_n}, \\
\ell_n &= \psi(t_n,s_n,u_n)+\psi_0(u_n,s_n)), \\
\sigma_{n+1} &= (1+\psi_0(t_n,t_{n+1}))(t_{n+1}-t_n) \\
&\quad +(1+\psi_0(t_n,\gamma_0(t_n)t_n+\eta_0))(s_n-t_n)
\end{aligned}$$

and

$$s_{n+1}=t_{n+1}+\frac{\sigma_{n+1}}{1-\psi_0(\gamma_0(t_{n+1})t_{n+1}+\eta_0,\gamma(t_{n+1})t_{n+1}+\eta_0)}.$$

The estimates are:

$$\begin{aligned}
\|F'(x_0)^{-1}T_n\| &\leq (1+\psi(\|x_n-x_0\|,\|y_n-x_0\|,\|v_n-x_0\|)\\
&\quad +\psi_0(\|x_n-x_0\|,\|v_n-x_0\|)\\
&\quad +\psi(\|x_n-x_0\|,\|y_n-x_0\|,\|v_n-x_0\|)\\
&= \bar{\mu}_n \leq \mu_n,\\
\|F'(x_0)^{-1}(Q_n-F'(x_0))\| &\leq \psi(\|x_n-x_0\|,\|y_n-x_0\|,\|z_n-x_0\|)\\
&\quad +\psi_0(\|z_n-x_0\|,\|x_n-x_0\|)\\
&= \bar{\lambda}_n \leq \lambda_n < 1,\\
\|z_n-y_n\| &\leq \frac{\mu_n}{(1-\psi_0(t_n,s_n))^2},
\end{aligned}$$

$$F(z_n)=F(z_n)-F(y_n)+F(y_n)=[z_n,y_n;F](z_n-y_n)+F(y_n),$$

$$\begin{aligned}
\|F'(x_0)^{-1}F(z_n)\| &\leq (1+\psi_0(\|y_n-x_0\|,\|z_n-x_0\|))\|z_n-y_n\|\\
&\quad +\psi(\|x_n-x_0\|,\|y_n-x_0\|,\|v_n-x_0\|)\|y_n-x_0\|\\
&= \bar{\ell}_n \leq \ell_n,
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1}-z_n\| &\leq \|Q_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(z_n)\|\\
&\leq \frac{\bar{\ell}_n}{1-\bar{\lambda}_n} \leq \frac{\ell_n}{1-\lambda_n},
\end{aligned}$$

$$F(x_{n+1})=[x_{n+1},x_n;F](x_{n+1}-x_n)-[v_n,x_n;F](y_n-x_n),$$

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq (1+\psi_0(\|x_n-x_0\|,\|x_{n+1}-x_0\|))\|x_{n+1}-x_n\|\\
&\quad +(1+\psi_0(\|x_n-x_0\|,\|v_n-x_0\|))\|y_n-x_n\|\\
&= \bar{\sigma}_{n+1} \leq \sigma_{n+1}
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1}-x_{n+1}\| &\leq \|F'(x_{n+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\|\\
&\leq \frac{\sigma_{n+1}}{1-\psi_0(\gamma_0(t_n)t_n+\eta_0,t_n)},
\end{aligned}$$

where we also used

$$\begin{aligned}
\|v_n-x_0\| &= \|(I+[x_n,x_0;F])(x_n-x_0)+F(x_0)\|\\
&\leq \gamma_0(\|x_n-x_0\|)\|x_n-x_0\|+\|F(x_0)\|\\
&\leq \gamma_0(\|x_n-x_0\|)\|x_n-x_0\|+\eta_0 \ \ (\|F(x_0)\leq \eta_0).
\end{aligned}$$

Bibliography

[1] Argyros, I.K., *Computational Theory of Iterative Methods Methods*. In: Chui, C.K., Wuytack, L. (eds.) Series: Studies in Computational Mathematics, vol. 15., Elsevier Publ. Co., New York (2007).

[2] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics (MDPI)*, 2021, 9, (16), 1942. https://doi.org/10.3390/math9161942.

[3] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, Engineering Series, CRC Press, Taylor and Francis Group, 2022.

[4] Regmi, S., Argyros, I.K., George, S., Argyros, C.I., Extended Convergence of Three Step Iterative Methods for Solving Equations in Banach Space with Applications, *Symmetry*, 14, (7), 1484.

[5] Regmi, S., Argyros, I.K., George, S., Argyros, C.I., Numerical Processes for Approximating Solutions of Nonlinear Equations, *Axioms*, 11, (7), 307.

[6] Singh, S., Martinez, E., Maroju, P., Behl, R., A study of the local convergence of a fifth order iterative method, *Indian Journal of pure and Applied mathematics*, 51, 2,(2020), 439-455.

[7] Sharma, J.R., Gupandee, P., A study of the local convergence of a derivative free method in Banach spaces, *The J. Analysis*, (2022).

[8] Sharma, J.R., Arora, H., A novel derivative free algorithm with seventh order convergence for solving systems of nonlinear equations, *Numer. Algor.*, 4, (2014), 917933.

Chapter 35

High Convergence Order Methods

1. Introduction

A plethora of iterative methods for solving nonlinear equations in a Banach space is presented together with their majorizing sequences. The convergence conditions relating these sequences with the corresponding methods have been given in the preface. The local convergence analysis of these methods has been reported in the corresponding references. But we present the more interesting semi-local case under conditions involving the first derivative or divided difference of order one. This approach is in contrast to the local convergence case where derivatives or divided differences of order higher than one are required to show convergence. Hence, the applicability of these methods is extended.

2. Semi-Local Convergence

As before the method is presented followed by its majorizing sequence and the estimates motivating these choices.

Method [1, 2, 3, 4]:

$$y_n = x_n - F'(x_n)^{-1}F(x_n)$$

and

$$x_{n+1} = y_n - (2F'(x_n)^{-1} - F'(x_n)^{-1}F'(y_n)F'(x_n)^{-1})F(y_n). \tag{35.1}$$

The majorizing sequence is defined for $t_0 = 0, s_0 = \Omega \geq 0$ by

$$t_{n+1} = s_n + \frac{1}{1-w_0(t_n)}\left(1+\frac{v_n}{1-w_0(t_n)}\right)\int_0^1 w\left[(1-\theta)(s_n-t_n)\right]d\theta(s_n-t_n)$$

and

$$s_{n+1} = t_{n+1} + \frac{a_{n+1}}{1-w_0(t_n)},$$

where

$$v_n = \begin{cases} w(s_n-t_n) \\ or \\ w_0(t_n)+w_0(s_n) \end{cases}$$

and

$$a_{n+1}=\int_0^1 w((1-\theta)(t_{n+1}-t_n))d\theta(t_{n+1}-t_n)+(1+w_0(t_n))(t_{n+1}-s_n).$$

The estimates leading to these choices are, respectively

$$\begin{aligned}
x_{n+1}-y_n &= -F'(x_n)^{-1}F(y_n)-F'(x_n)^{-1}(I-F'(y_n)F'(x_n)^{-1})F(y_n)\\
&= -F'(x_n)^{-1}F(y_n)-F'(x_n)^{-1}\\
&\quad \times(F'(x_n)-F'(y_n))F'(x_n)^{-1}F(y_n),\\
F(y_n) &= F(y_n)-F(x_n)-F'(x_n)(y_n-x_n)\\
&= \int_0^1 F'((1-\theta)\|y_n-x_n\|)d\theta(y_n-x_n),\\
\|x_{n+1}-y_n\| &\le \frac{1}{1-w_0(\|x_n-x_0\|)}\left(1+\frac{\overline{v_n}}{1-w_0(\|x_n-x_0\|)}\right)\\
&\quad \times\int_0^1 w((1-\theta)\|y_n-x_n\|)d\theta\|y_n-x_n\|\\
&\le t_{n+1}-s_n,\\
F(x_{n+1}) &= F(x_{n+1})-F(x_n)-F'(x_n)(y_n-x_n)\\
&\quad -F'(x_n)(x_{n+1}-x_n)+F'(x_n)(x_{n+1}-x_n)\\
&= F(x_{n+1})-F(x_n)-F'(x_n)(x_{n+1}-x_n)\\
&\quad -F'(x_n)(x_{n+1}-y_n)
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1}-x_{n+1}\| &\le \frac{\overline{a_{n+1}}}{1-w_0(\|x_{n+1}-x_0\|)}\\
&\le \frac{a_{n+1}}{1-w_0(t_{n+1})}\\
&= s_{n+1}-t_{n+1}.
\end{aligned}$$

A convergence criterion for the sequence (35.1) is $w_0(t_n)<1$ and $t_n<\tau$ for all $n=0,1,2,\ldots$ and some $\tau>0$. The proof that sequence $\{t_n\}$ majorizes method (35.1) is standard and is given as in previous chapters.
Method [1,2,3,4]:

$$\begin{aligned}
y_n &= x_n-F'(x_n)^{-1}F'(x_n)\\
x_{n+1} &= x_n-A_{n,k}^{-1}F(x_n),
\end{aligned}\tag{35.2}$$

and

$$A_{n,k} = \sum_{j=1}^{k} c_i F'(x_n-d_jF'(x_n)^{-1}F(x_n)), A_{n,k}=A_n, \sum_{j=1}^{k} c_j=1,$$

k is fixed and $c_j,d_j\in\mathbb{R}$.

The majorizing sequence is for $t_0 = 0, s_0 = \Omega$,

$$
\begin{aligned}
p_n &= \sum_{j=1}^{k} |c_j| w(d_j(s_n - t_n)),\\
q_n &= \sum_{j=1}^{k} |c_j| w_0(t_n + |d_j|(s_n - t_n)),\\
t_{n+1} &= s_n + \frac{p_n(s_n - t_n)}{1 - q_n}
\end{aligned}
$$

and

$$
s_{n+1} = t_{n+1} + \frac{a_{n+1}}{1 - w_0(t_{n+1})},
$$

where a_{n+1} is given in method (35.1).

The estimates are:

$$
\begin{aligned}
\|F'(x_0)^{-1}(A_n - F'(x_0))\| &= \|F'(x_0)^{-1} \sum_{j=1}^{k} c_j (F'(x_n - d_j F'(x_n)^{-1} F(x_n) - F'(x_0)))\|\\
&\quad (\text{since } \sum_{j=1}^{k} c_j F'(x_0) = F'(x_0))\\
&\leq \sum_{j=1}^{k} |c_j| w_0(\|x_n - x_0\| + |d_j| \|y_n - x_n\|) = \overline{q_n} \leq q_n = 1,\\
\|A_n^{-1} F'(x_0)\| &\leq \frac{1}{1 - q_n},\\
x_{n+1} &= y_n + (F'(x_n)^{-1} - A_n^{-1}) F(x_n)\\
&= y_n - (A_n^{-1} - F'(x_n)^{-1}) F(x_n),\\
x_{n+1} - y_n &= -A_n^{-1}(F'(x_n) - A_n) F'(x_n)^{-1} F(x_n)\\
A_n - F'(x_n) &= \sum_{j=1}^{k} c_j (F'(x_n - d_j F'(x_n)^{-1} F(x_n)) - F'(x_n)),\\
\|F'(x_0)^{-1}(A_n - F'(x_n))\| &\leq \sum_{j=1}^{k} |c_j| w(|d_j| \|y_n - x_n\|) = \overline{p_n} \leq p_n,\\
\|x_{n+1} - y_n\| &\leq \frac{\overline{p_n} \|y_n - x_n\|}{1 - \overline{q_n}} \leq \frac{p_n(s_n - t_n)}{1 - q_n} = t_{n+1} - s_n
\end{aligned}
$$

and

$$
\|y_{n+1} - x_{n+1}\| \leq s_{n+1} - t_{n+1}.
$$

The last estimate is given in method (35.1).

Method [1,2,3,4] :

$$
\begin{aligned}
y_n &= x_n - \alpha F'(x_n)^{-1} F(x_n),\\
z_n &= x_n - F'(x_n)^{-1}(F(y_n) + \alpha F(x_n)) \qquad (35.3)
\end{aligned}
$$

and

$$
x_{n+1} = x_n - F'(x_n)^{-1}(F(z_n) + F(y_n) + \alpha F(x_n)).
$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = |\alpha|\Omega$,

$$\begin{aligned}
p_n &= \int_0^1 w((1-\theta)(s_n-t_n))d\theta(s_n-t_n)+|1-\frac{1}{\alpha}|(1+w_0(t_n))(s_n-t_n),\\
u_n &= s_n+\frac{p_n}{1-w_0(t_n)},\\
q_n &= \left(1+\int_0^1 w_0(s_n-t_n+\theta(u_n-s_n))d\theta\right)(u_n-s_n)+p_n,\\
t_{n+1} &= u_n+\frac{q_n}{1-w_0(t_n)},\\
b_{n+1} &= \int_0^1 w((1-\theta)(t_{n+1}-t_n))d\theta(t_{n+1}-t_n)+|1-\frac{1}{\alpha}|(1+w_0(t_n))(s_n-t_n)\\
&\quad +(1+w_0(t_n))(t_{n+1}-s_n)
\end{aligned}$$

and

$$s_{n+1} = t_{n+1}+\frac{b_{n+1}}{1-w_0(t_{n+1})}.$$

The estimates are:

$$\begin{aligned}
F(y_n) &= F(y_n)-F(x_n)-\frac{1}{\alpha}F'(x_n)(y_n-x_n)-F'(x_n)(y_n-x_n)\\
&\quad +F'(x_n)(y_n-x_n),\\
\|F'(x_0)^{-1}F(y_n)\| &\leq \int_0^1 w((1-\theta)\|y_n-x_n\|)d\theta\|y_n-x_n\|\\
&\quad +|1-\frac{1}{\alpha}|(1+w_0(\|x_n-x_0\|))\|y_n-x_n\|\\
&= \overline{p_n}\leq p_n,\\
z_n &= x_n-\alpha F(x_n)-F'(x_n)^{-1}F(y_n)\\
&= y_n-F'(x_n)^{-1}F(y_n)\\
z_n-y_n &= -F'(x_n)^{-1}F(y_n)\\
\|z_n-y_n\| &\leq \frac{\overline{p_n}}{1-w_0(\|x_n-x_0\|)}\leq\frac{p_n}{1-w_0(t_n)}=u_n-s_n.\\
x_{n+1} &= x_n-F'(x_n)^{-1}(F(y_n)+\alpha F(x_n))-F'(x_n)^{-1}F(z_n)\\
&= z_n-F'(x_n)^{-1}F(z_n)\\
F(z_n) &= F(z_n)-F(y_n)+F(y_n),\\
\|F'(x_0)^{-1}F(z_n)\| &\leq \left(1+\int_0^1 w_0(\|y_n-x_0\|+\theta\|z_n-y_n\|)d\theta\right)\|z_n-y_n\|+\overline{p_n},\\
x_{n+1}-z_n &= -F'(y_n)+F(y_n),\\
\|x_{n+1}-z_n\| &\leq \frac{\overline{q_n}}{1-w_0(\|x_n-x_0\|)}\leq\frac{q_n}{1-w_0(t_n)}=t_{n+1}-u_n,\\
F(x_{n+1}) &= F(x_{n+1})-F(x_n)-F'(x_n)(x_{n+1}-x_n)\\
&\quad +\left(1-\frac{1}{\alpha}\right)F'(x_n)(y_n-x_n)+F'(x_n)(x_{n+1}-y_n),
\end{aligned}$$

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq \int_0^1 w((1-\theta)\|x_{n+1}-x_n\|)d\theta\|x_{n+1}-x_n\| \\
&+|1-\frac{1}{\alpha}|(1+w_0(\|x_n-x_0\|))\|y_n-x_n\| \\
&+(1+w_0(\|x_n-x_0\|))\|x_{n+1}-y_n\| \\
&= \overline{a_{n+1}} \leq a_{n+1}, \\
\|y_{n+1}-x_{n+1}\| &\leq \frac{\overline{a_{n+1}}}{1-w_0(\|x_{n+1}-x_0\|)} \leq \frac{a_{n+1}}{1-w_0(t_{n+1})} = s_{n+1}-t_{n+1}.
\end{aligned}$$

Method $[1,2,3,4]$:

$$\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n), \\
z_n &= x_n - 2(F'(y_n)+F'(x_n))^{-1}F(x_n) \qquad (35.4)
\end{aligned}$$

and

$$x_{n+1} = z_n - F'(y_n)^{-1}F(z_n).$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = \Omega$ by

$$q_n = \frac{1}{\alpha}(w_0(s_n)+w_0(t_n))$$

and

$$u_n = s_n + \frac{w(s_n-t_n)(s_n-t_n)}{2(1-q_n)}.$$

The iterate t_{n+1} is given in method (35.3), whereas the iterate s_{n+1} in method (35.1).

The estimates are,

$$\begin{aligned}
\|(2F'(x_0))^{-1}(F'(y_n)+F'(x_n)-2F'(x_0))\| &\leq \frac{1}{2}(w_0(\|y_n-x_0\|)+w_0(\|x_n-x_0\|)) \\
&= \overline{q_n} \leq q_n < 1,
\end{aligned}$$

so

$$\begin{aligned}
\|(F'(y_n)+F'(x_n))^{-1}F'(x_0)\| &\leq \frac{1}{2(1-q_n)}, \\
z_n - y_n &= -(2(F'(y_n)+F'(x_n))^{-1}-F'(x_n)^{-1})F(x_n) \\
&= -(F'(y_n)+F'(x_n))^{-1}(2F'(x_n)-F'(y_n)-F'(x_n)) \\
&\quad \times F'(x_n)^{-1}F(x_n) \\
\|z_n-y_n\| &\leq \frac{w(\|y_n-x_n\|)\|y_n-x_n\|}{2(1-q_n)} = u_n - s_n.
\end{aligned}$$

Method $[1,2,3,4]$:

$$\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n), \\
z_n &= x_n - F'(y_n)^{-1}F(x_n) \qquad (35.5)
\end{aligned}$$

and

$$x_{n+1} = z_n - (2F'(y_n)^{-1}-F'(x_n)^{-1})F(z_n).$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = \frac{1}{2}\Omega$ by

$$
\begin{aligned}
u_n &= s_n + \frac{1}{1-w_0(s_n)}\left(1+\frac{w(s_n-t_n)}{1-w_0(t_n)}\right)(1+w_0(t_n))(s_n-t_n),\\
a_n &= \left(1+\int_0^1 w_0(t_n+\theta(u_n-t_n))d\theta\right)(u_n-t_n)+2(1+w_0(t_n))(s_n-t_n),\\
t_{n+1} &= u_n + \frac{1}{1-w_0(s_n)}\left(1+\frac{w(s_n-t_n)}{1-w_0(t_n)}\right)a_n,\\
b_{n+1} &= \left(1+\int_0^1 w_0(t_n+\theta(t_{n+1}-t_n))d\theta\right)(t_{n+1}-t_n)+2(1+w_0(t_n))(s_n-t_n)
\end{aligned}
$$

and

$$
s_{n+1} = t_{n+1} + \frac{b_{n+1}}{2(1-w_0(t_{n+1}))}.
$$

The estimates are:

$$
\begin{aligned}
\|z_n-y_n\| &\leq \|\left[\frac{1}{2}(F'(x_n)^{-1}-F'(y_n)^{-1}-\frac{1}{2}F'(y_n)^{-1})\right]F(x_n)\|\\
&\leq \frac{1}{1-w_0(\|y_n-x_0\|)}\left[1+\frac{w(\|y_n-x_n\|)}{1-w_0(\|x_n-x_0\|)}\right]\\
&\quad \times(1+w_0(\|y_n-x_0\|))\|y_n-x_n\|\\
&\leq u_n-s_n,\\
F(z_n) &= F(z_n)-F(x_n)-2F'(x_n)(y_n-x_n),\\
\|F'(x_0)^{-1}F(z_n)\| &\leq \left(1+\int_0^1 w_0(\|x_n-x_0\|+\theta\|z_n-x_n\|)d\theta\right)\|z_n-x_n\|\\
&\quad +2(1+w_0(\|x_n-x_0\|))\|y_n-x_n\| = \overline{a_n} \leq a_n\\
\|x_{n+1}-z_n\| &= \|(F'(y_n)^{-1}+(F'(y_n)^{-1}-F'(z_n)^{-1}))F(z_n)\|\\
&\leq \frac{1}{1-w_0(\|y_n-x_0\|)}\left[1+\frac{w(\|y_n-x_n\|)}{1-w_0(\|x_n-x_0\|)}\right]\overline{a_n} \leq t_{n+1}-u_n,\\
F(x_{n+1}) &= F(x_{n+1})-F(x_n)-2F'(x_n)(y_n-x_n),\\
\|F'(x_n)^{-1}F(x_{n+1})\| &\leq \left(1+\int_0^1 w_0(\|x_n-x_0\|+\theta\|x_{n+1}-x_n\|)d\theta\right)\|x_{n+1}-x_n\|\\
&\quad +2(1+w_0(\|x_n-x_0\|))\|y_n-x_n\| = \overline{b_{n+1}} \leq b_{n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
\|y_{n+1}-x_{n+1}\| &\leq \frac{1}{2}\|F'(x_{n+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\|\\
&\leq \frac{\overline{b_{n+1}}}{2(1-w_0(\|x_{n+1}-x_0\|)} \leq \frac{b_{n+1}}{2(1-w_0(t_{n+1}))} = s_{n+1}-t_{n+1}.
\end{aligned}
$$

Method [1,2,3,4]:

$$y_n = x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n), \tag{35.6}$$

$$z_n = x_n - \frac{1}{2}(3F'(y_n)-F'(x_n))^{-1}(3F'(y_n)+F'(x_n))F'(x_n)^{-1}F(x_n)$$

and

$$x_{n+1} = z_n - 2(3F'(y_n)-F'(x_n))^{-1}F(z_n).$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = \frac{2}{3}\Omega$ by

$$\begin{aligned}
q_n &= \frac{1}{2}(3w_0(s_n)+w_0(t_n)),\\
u_n &= s_n + \frac{1}{8(1-q_n)}(3w(s_n-t_n)+4(1+w_0(t_n)))(s_n-t_n),\\
a_n &= \left(1+\int_0^1 w_0(t_n+\theta(u_n-t_n))d\theta\right)(u_n-t_n)+\frac{3}{2}(1+w_0(t_n))(s_n-t_n),\\
t_{n+1} &= u_n\frac{a_n}{1-q_n},\\
b_{n+1} &= \left(1+\int_0^1 w_0(t_n+\theta(t_{n+1}-t_n))d\theta\right)(t_{n+1}-t_n)+\frac{3}{2}(1+w_0(t_n))(s_n-t_n),
\end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{2b_{n+1}}{3(1-w_0(t_{n+1}))}.$$

The estimates are for $A_n = 3F'(y_n) - F'(x_n)$,

$$\begin{aligned}
\|(2F'(x_0))^{-1}(A_n - 2F'(x_0))\| &\leq \frac{1}{2}(3w_0(\|y_n - x_0\|) + \|x_n - x_0\|) \\
&= \overline{q_n} \leq q_n < 1, \\
\|A_n^{-1}F'(x_0)\| &\leq \frac{1}{2(1-q_n)}, \\
\|z_n - y_n\| &= \frac{1}{6}\|A_n^{-1}(4(3F'(y_n) - 7F'(x_n)))F'(x_n)^{-1}F(x_n)\| \\
&\leq \frac{1}{12(1-q_n)}[3w(\|y_n - x_n\|) \\
&\quad +4(1 + w_0(\|x_n - x_0\|))]\frac{3}{2}\|y_n - x_n\| \\
&\leq u_n - s_n, \\
F(z_n) &= F(z_n) - F(x_n) - \frac{2}{3}F'(x_n)(y_n - x_n), \\
\|F'(x_0)^{-1}F(z_n)\| &\leq \left(1 + \int_0^1 w_0(\|x_n - x_0\| + \theta\|z_n - x_n\|)d\theta\right)\|z_n - x_n\| \\
&\quad + \frac{3}{2}(1 + w_0(\|x_n - x_0\|))\|y_n - x_n\| = \overline{b_n} \leq b_n. \\
\|x_{n+1} - z_n\| &\leq 2\|A_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(z_n)\| \\
&\leq \frac{2\overline{a_n}}{2(1-\overline{q_n})} \leq \frac{a_n}{1-q_n} = t_{n+1} - u_n, \\
F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - \frac{3}{2}F'(x_n)(y_n - x_n), \\
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq \left(1 + \int_0^1 w_0(\|x_n - x_0\| + \theta\|x_{n+1} - x_n\|)d\theta\right)\|x_{n+1} - x_n\| \\
&\quad + \frac{3}{2}(1 + w_0(\|x_n - x_0\|))\|y_n - x_n\| = \overline{b_{n+1}} \leq b_{n+1}
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1} - x_{n+1}\| &\leq \frac{2}{3}\|F'(x_{n+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\|, \\
&\leq \frac{2}{3}\frac{\overline{b_{n+1}}}{1 - w_0(\|x_{n+1} - x_0\|)} \\
&\leq \frac{2}{3}\frac{b_{n+1}}{(1 - w_0(t_{n+1}))} = s_{n+1} - t_{n+1}.
\end{aligned}$$

Method $[1,2,3,4]$:

$$\begin{aligned}
y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n), A_n = 3F'(y_n) - F'(x_n), \\
z_n &= x_n - \frac{1}{2}A_n^{-1}(3F'(y_n) + F'(x_n))F'(x_n)^{-1}F(x_n) \qquad (35.7)
\end{aligned}$$

and

$$x_{n+1} = z_n - \left[\left(\frac{1}{2}A_n^{-1}(3F'(y_n) + F'(x_n))\right)^2\right]F'(x_n)^{-1}F(z_n).$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = \frac{2}{3}\Omega$ as follows: The iterates u_n, s_{n+1} are as given in method(35.6) and

$$t_{n+1} = u_n + \frac{1}{16}\frac{(3(1+w_0(s_n)) + (1+w_0(t_n)))^2 a_n}{(1-q_n)^2(1-w_0(t_n))}.$$

The estimate is

$$\begin{aligned} \|x_{n+1} - z_n\| &\leq \frac{1}{4}\|A_n^{-1}F'(x_0)\|^2\|F'(x_0)^{-1}(3F'(y_n) + F'(x_n))\|^2 \\ &\quad \times \|F'(x_n)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(z_n)\| \\ &\leq \frac{(3(1+w_0(\|y_n - x_0\|)) + (1+w_0(\|x_n - x_0\|)))^2 a_n}{16(1-q_n)^2(1-w_0(t_n))} \\ &\leq t_{n+1} - u_n. \end{aligned}$$

Method [5]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \qquad (35.8) \\ z_n &= y_n + \frac{1}{3}(F'(x_n)^{-1} + 2A_n^{-1})F(x_n), A_n = F'(x_n) - 3F'(y_n) \end{aligned}$$

and

$$x_{n+1} = z_n + \frac{1}{3}(A_n^{-1} - F'(x_n)^{-1})F(z_n).$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = \Omega$ by

$$\begin{aligned} q_n &= \frac{1}{2}[w(s_n - t_n) + 2w_0(s_n)], \\ u_n &= s_n + \frac{w(s_n - t_n)(s_n - t_n)}{2(1-q_n)}, \\ p_n &= \left(1 + \int_0^1 w_0(s_n + \theta(u_n - s_n))d\theta\right)(u_n - s_n) \\ &\quad + \int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n) \end{aligned}$$

and

$$t_{n+1} = u_n + \frac{(1+w_0(s_n))p_n}{2(1-q_n)(1-w_0(t_n))}.$$

The iterate s_{n+1} is as defined in method (35.1).

The estimates are:

$$
\begin{aligned}
\|(-2F'(x_0))^{-1}(A_n - F'(x_0))\| &\leq \frac{1}{2}\left[F'(x_0)^{-1}(F'(x_n)-F'(y_n))\|\right. \\
&\quad \left. + 2\|F'(x_0)^{-1}(F'(y_n)-F'(x_0))\right] \\
&\leq \overline{q_n} \leq q_n < 1, \\
\|A_n^{-1}F'(x_0)\| &\leq \frac{1}{2(1-q_n)}, \\
\|z_n - y_n\| &= \frac{1}{3}\|A_n^{-1}(2F'(x_n)+A_n)F'(x_n)^{-1}F(x_n)\| \\
&\leq \frac{\|F'(x_0)^{-1}(F'(x_n)-F'(y_n))\|\|y_n - x_n\|}{2(1-\overline{q_n})} \\
&\leq \frac{w(\|y_n - x_n\|)\|y_n - x_n\|}{2(1-\overline{q_n})} \\
&\leq \frac{w(s_n - t_n)(s_n - t_n)}{2(1-q_n)} = u_n - s_n, \\
x_{n+1} - z_n &= \frac{1}{3}A_n^{-1}F'(y_n)F'(x_n)^{-1}F(z_n), \\
F(z_n) &= F(z_n) - F(y_n) + F(y_n) \\
&= \int_0^1 F'(y_n + \theta(z_n - y_n))d\theta(z_n - y_n) + F(y_n), \\
\|F'(x_0)^{-1}F(z_n)\| &\leq (1 + \int_0^1 w_0(\|y_n - x_0\| + \theta\|z_n - y_n\|))d\theta\|z_n - y_n\| \\
&\quad + \int_0^1 w((1-\theta)\|y_n - x_n\|)d\theta\|y_n - x_n\| \\
&= \overline{p_n} \leq p_n
\end{aligned}
$$

and

$$
\|x_{n+1} - z_n\| \leq \frac{(1 + w_0(\|y_n - x_0\|))\overline{p_n}}{2(1-\overline{q_n})(1 - w_0(\|x_n - x_0\|))} \leq t_{n+1} - u_n.
$$

Finally, the iterate s_{n+1} is as given in method (35.1).

Method [12] :

$$
y_n = x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n) \tag{35.9}
$$

and

$$
x_{n+1} = x_n - \frac{1}{2}\left(-I + \frac{9}{4}F'(y_n)^{-1}F'(x_n)\right) + \frac{3}{4}(F'(x_n)^{-1}F(y_n))F'(x_n)^{-1}F(x_n).
$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = \frac{2}{3}\Omega$,

$$\begin{aligned}
t_{n+1} &= s_n + \frac{3}{2}\left(\frac{1}{3} + \frac{9w(s_n - t_n)}{8(1 - w_0(s_n))} + \frac{3w(s_n - t_n)}{8(1 - w_0(t_n))}\right)(s_n - t_n),\\
a_{n+1} &= \left(1 + \int_0^1 w_0(t_n + \theta(t_{n+1} - t_n))d\theta\right)(t_{n+1} - t_n) + \frac{3}{2}(1 + w_0(t_n))(s_n - t_n)
\end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{2}{3}\frac{a_{n+1}}{(1 - w_0(t_{n+1}))}.$$

The estimates are:

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \|\left[\frac{2}{3}I + \frac{1}{2}I - \frac{1}{2}\left(\frac{9}{4}F'(y_n)^{-1}F'(x_n)\right)\right.\\
&\quad \left. + \frac{3}{4}F'(x_n)^{-1}F'(y_n)\right]F'(x_n)^{-1}F(x_n)\|,\\
&= \|\left[-\frac{1}{3}I + \frac{9}{8}(I - F'(y_n)^{-1}F'(x_n))\right.\\
&\quad \left. + \frac{3}{8}(I - F'(x_n)^{-1}F'(y_n))\right]\|\|F'(x_n)^{-1}F(x_n)\|\\
&\leq \frac{3}{2}\left(\frac{1}{3} + \frac{9w(\|y_n - x_n\|)}{8(1 - w_0(\|y_n - x_0\|))}\right.\\
&\quad \left. + \frac{3w(\|y_n - x_n\|)}{8(1 - w_0(\|x_n - x_0\|))}\right)\|y_n - x_n\|\\
&\leq t_{n+1} - s_n,\\
F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - \frac{3}{2}F'(x_n)(y_n - x_n),\\
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq \left(1 + \int_0^1 w_0(\|x_n - x_0\| + \theta\|x_{n+1} - x_n\|)d\theta\right)\|x_{n+1} - x_n\|\\
&\quad + \frac{3}{2}(1 + w_0(\|x_n - x_0\|))\|y_n - x_n\|\\
&= \overline{a_{n+1}} \leq a_{n+1}
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1} - x_{n+1}\| &\leq \frac{2}{3}\|F'(x_{n+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\|\\
&\leq \frac{2}{3}\frac{\overline{a_{n+1}}}{1 - w_0(\|x_{n+1} - x_0\|)} \leq \frac{2}{3}\frac{a_{n+1}}{1 - w_0(t_{n+1})} = s_{n+1} - t_{n+1}.
\end{aligned}$$

Method [12] :

$$\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n), \qquad (35.10)\\
z_n &= x_n - F'(x_n)^{-1}(F(x_n) + F(y_n))
\end{aligned}$$

and

$$x_{n+1} = x_n - A_n^{-1}F(x_n), A_n = \frac{1}{6}F'(x_n) + \frac{2}{3}F'(\frac{x_n + z_n}{2}) + \frac{1}{6}F'(z_n).$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = \Omega_0$ by,

$$\begin{aligned}
u_n &= s_n + \frac{\int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n)}{1 - w_0(t_n)}, \\
a_n &= \frac{2}{3}w(\frac{u_n - t_n}{2}) + \frac{1}{6}w(u_n - t_n), \\
q_n &= \frac{1}{6}w_0(t_n) + \frac{2}{3}w_0(\frac{u_n + t_n}{2}) + \frac{1}{6}w_0(u_n)
\end{aligned}$$

and

$$t_{n+1} = u_n + \left[\frac{a_n}{1-q_n} + \frac{\int_0^1 w((1-\theta)(s_n - t_n))d\theta}{1 - w_0(t_n)}\right](s_n - t_n).$$

The iterate y_{n+1} is as given in method (35.1).

The estimates are:

$$\begin{aligned}
\|z_n - y_n\| &\leq \|F'(x_n)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_n)\| \\
&\leq \frac{\int_0^1 w((1-\theta)\|y_n - x_n\|)d\theta\|y_n - x_n\|}{1 - w_0(\|x_n - x_0\|)} \leq u_n - s_n, \\
\|F'(x_0)^{-1}(A_n - F'(x_n))\| &\leq \frac{2}{3}w\left(\frac{\|z_n - x_n\|}{2}\right) \\
&\quad + \frac{1}{6}w_0(\|z_n - x_n\|) \\
&= \overline{a_n} \leq a_n, \\
\|F'(x_0)^{-1}(A_n - \frac{6}{6}F'(x_0))\| &\leq \frac{1}{6}w_0(\|x_n - x_0\|) \\
&\quad + \frac{2}{3}w_0\left(\frac{\|z_n - x_0\| + \|x_n - x_0\|}{2}\right) + \frac{1}{6}w_0(\|z_n - x_0\|) \\
&= \overline{q_n} \leq q_n < 1, \\
\|A_n^{-1}F'(x_0)\| &\leq \frac{1}{1-q_n}, \\
\|x_{n+1} - z_n\| &= \|(F'(x_n)^{-1} - A_n^{-1})F(x_n) + F'(x_n)^{-1}F(y_n)\| \\
&\leq \|A_n^{-1}(F'(x_n) - A_n)F'(x_n)^{-1}F(x_n)\| \\
&\quad + \|F'(x_n)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_n)\| \\
&\leq \frac{a_n(s_n - t_n)}{1-q_n} + \frac{\int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n)}{1 - w_0(t_n)} \\
&= t_{n+1} - u_n.
\end{aligned}$$

Method [12]:

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1}F(x_n) \qquad (35.11)\\
z_n &= x_n - 2A_n^{-1}F(x_n), A_n = F'(x_n) + F'(y_n)\\
v_n &= z_n - B_nF'(x_n)^{-1}F(x_n),\\
B_n &= \left[\frac{7}{2}I - 4F'(x_n)^{-1}F'(y_n) + \frac{3}{2}(F'(x_n)^{-1}F'(y_n))^2\right]\\
u_n &= v_n - B_nF'(x_n)^{-1}F(v_n)\\
x_{n+1} &= u_n - B_nF'(x_n)^{-1}F(u_n).
\end{aligned}
$$

The majorizing sequence $\{a_n\}$ is defined for $a_0 = 0, b_0 = \Omega$ by

$$
\begin{aligned}
q_n &= \frac{1}{2}(w_0(a_n) + w_0(b_n)),\\
c_n &= b_n + \frac{w(b_n - a_n)(b_n - a_n)}{2(1 - q_n)}\\
p_n &= \frac{w(b_n - a_n)}{1 - w_0(a_n)},\\
h_n &= \left(1 + \int_0^1 w_0(a_n + \theta(c_n - a_n))d\theta\right)(c_n - a_n) + (1 + w_0(a_n))(b_n - a_n),\\
\alpha_n &= 4 + 2p_n + 3p_n^2,\\
d_n &= c_n + \frac{1}{2}\frac{\alpha_n}{1 - w_0(a_n)},\\
e_n &= d_n + \alpha_n\left(1 + \int_0^1 w_0(c_n + \theta(d_n - c_n))d\theta\right)(d_n - c_n) + h_n\\
\sigma_{n+1} &= \int_0^1 w((1-\theta)(a_{n+1} - a_n))d\theta(a_{n+1} - a_n) + (1 + w_0(a_n))(a_{n+1} - b_n),
\end{aligned}
$$

and

$$
b_{n+1} = a_{n+1} + \frac{\sigma_{n+1}}{1 - w_0(a_{n+1})}.
$$

The estimates are:

$$\begin{aligned}
\|2F'(x_0)^{-1}(A_n-2F'(x_0)) &\leq \frac{1}{2}(w_0(\|x_n-x_0\|)+w_0(\|y_n-x_0\|))\\
&\leq \frac{1}{2}(w_0(a_n)+w_0(b_n))=q_n<1,\\
\|A_n^{-1}F'(x_0)\| &\leq \frac{1}{2(1-q_n)},\\
\|z_n-y_n\| &\leq \|A_n^{-1}(2F'(x_n)-A_n)\|\|F'(x_n)^{-1}F(x_n)\|\\
&\leq \|A_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}(F'(x_n)-F'(y_n))\|\\
&\quad\times\|F'(x_n)^{-1}F(x_n)\|\\
&\leq \frac{w(b_n-a_n)(b_n-a_n)}{2(1-q_n)}=d_n-c_n,\\
\|v_n-z_n\| &\leq \frac{1}{2}\|4I+2(I-F'(x_n)^{-1}F'(y_n))\\
&\quad+3(I-F'(x_n)^{-1}F'(y_n))^2\|\|F'(x_n)^{-1}F(z_n)\|\\
&\leq \frac{1}{2}\frac{\alpha_n h_n}{1-w_0(a_n)}=d_n-c_n,
\end{aligned}$$

$$\begin{aligned}
\overline{h_n} &= \left(1+\int_0^1 w_0(\|x_n-x_0\|+\theta\|z_n-x_n\|)d\theta\right)\|z_n-x_n\|\\
&\quad+(1+w_0(\|x_n-x_0\|))\|y_n-x_n\|\leq h_n,\\
\overline{p_n} &= \frac{w(\|y_n-x_n\|)}{1-w_0(\|x_n-x_0\|)}\leq\frac{w(b_n-a_n)}{1-w_0(a_n)}=p_n,
\end{aligned}$$

$$\begin{aligned}
\|u_n-v_n\| &\leq \frac{\alpha_n(1+\int_0^1 w_0(\|z_n-x_0\|+\theta\|v_n-z_n\|)d\theta\|v_n-z_n\|+h_n)}{1-w_0(\|x_n-x_0\|)}\\
&\leq e_n-d_n,\\
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq \int_0^1 w((1-\theta)\|x_{n+1}-x_n\|)d\theta\|x_{n+1}-x_n\|\\
&\quad+(1+w_0(\|x_n-x_0\|))\|x_{n+1}-y_n\|\\
&\leq \overline{\sigma_{n+1}},
\end{aligned}$$

So,

$$\|y_{n+1}-x_{n+1}\| \leq \frac{\overline{\sigma_{n+1}}}{1-w_0(a_{n+1})}\leq\frac{\sigma_{n+1}}{1-w_0(a_{n+1})}=b_{n+1}-a_{n+1}.$$

Method [12]:

$$\begin{aligned}
y_n &= x_n-\frac{1}{2}F'(x_n)^{-1}F(x_n) \qquad (35.12)\\
z_n &= \frac{1}{3}(4y_n-x_n)\\
v_n &= y_n+A_n^{-1}F(x_n), A_n=F'(x_n)-3F'(z_n),\\
x_{n+1} &= v_n+A_n^{-1}F(z_n).
\end{aligned}$$

The majorizing sequence $\{a_n\}$ is defined for $a_0 = 0, b_0 = \frac{1}{2}\Omega$ by

$$
\begin{aligned}
c_n &= b_n + \frac{1}{3}(b_n - a_n), \\
q_n &= \frac{1}{2}(3w_0(c_n) + w_0(a_n)), \\
d_n &= c_n + \frac{1}{4}(c_n - a_n) + \frac{3(1+w_0(a_n))(b_n - a_n)}{2(1-q_n)}, \\
a_{n+1} &= d_n + \frac{1}{2(1-q_n)}\left(1 + \int_0^1 w_0(a_n + \theta(c_n - a_n))d\theta\right)(c_n - a_n) \\
&\quad +2(1+w_0(a_n))(b_n - a_n) \\
\sigma_{n+1} &= \left(1 + \int_0^1 w_0(a_n + \theta(a_{n+1} - a_n))d\theta\right)(a_{n+1} - a_n)
\end{aligned}
$$

and

$$
b_{n+1} = a_{n+1} + \frac{\sigma_{n+1}}{1 - w_0(a_{n+1})}.
$$

The estimates are,

$$
\begin{aligned}
\|z_n - y_n\| &= \|\frac{4y_n}{3} - \frac{1}{3}x_n - y_n\| \\
&= \frac{1}{3}\|y_n - x_n\| \le \frac{1}{3}\|b_n - a_n\| = c_n - b_n, \\
\|(2F'(x_0))^{-1}(A_n - 2F'(x_0))\| &\le \frac{1}{2}(3w_0(\|z_n - x_0\|) + w_0(\|x_n - x_0\|)) = \overline{q_n} \le q_n, \\
\|A_n^{-1}F'(x_0)\| &\le \frac{1}{2(1-q_n)}, \\
\|x_{n+1} - v_n\| &\le \frac{\left(1 + \int_0^1 w_0(\|x_n - x_0\| + \theta\|z_n - x_0\|)d\theta\right)\|z_n - x_n\|}{2(1-q_n)} \\
&\quad + \frac{2(1+w_0(\|x_n - x_0\|))\|y_n - x_n\|}{2(1-q_n)} \\
&\le u_{n+1} - d_n, \\
F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - 2F'(x_n)(y_n - x_n), \\
\|F'(x_0)^{-1}F(x_{n+1})\| &\le \left(1 + \int_0^1 w_0(\|x_n - x_0\| + \theta\|x_{n+1} - x_n\|)d\theta\right) \\
&\quad \times\|x_{n+1} - x_n\| + 2(1+w_0(\|x_n - x_0\|))\|y_n - x_n\| \\
&= \overline{\sigma_{n+1}} \le \sigma_{n+1}, \\
\|y_{n+1} - x_{n+1}\| &\le \|F'(x_{n+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\| \\
&\le \frac{\overline{\sigma_{n+1}}}{1 - w_0(\|x_{n+1} - x_0\|)} \\
&\le \frac{\sigma_{n+1}}{1 - w_0(a_{n+1})} = b_{n+1} - a_{n+1}.
\end{aligned}
$$

Method [13]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - F'(y_n)^{-1}F(y_n) \\ x_{n+1} &= z_n - (I + \frac{1}{2}L_n)F'(y_n)^{-1}F(z_n) \\ L_n &= F'(y_n)^{-1}F''(y_n)F'(y_n)^{-1}F(y_n). \end{aligned} \tag{35.13}$$

The majorizing sequence $\{t_n\}$ is given for $t_0 = 0, s_0 = \Omega$ by

$$\begin{aligned} u_n &= s_n + \frac{\int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n)}{1 - w_0(s_n)} \\ l_n &= \frac{(b + v(s_n))\int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n)}{(1 - w_0(s_n))^2} \\ t_{n+1} &= u_n + \frac{d_n}{1 - w_0(s_n)}\left(1 + \frac{l_n}{2}\right), \\ d_n &= \left(1 + \int_0^1 w_0(s_n + \theta(u_n - s_n))d\theta\right)(u_n - s_n) \\ &\quad + \int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n) \end{aligned}$$

The iterate s_{n+1} is as given in method (35.1).

In the estimate, we also use the additional conditions

$$\|F'(x_0)^{-1}(F''(x) - F''(x_0))\| \le v(\|x - x_0\|)$$

and

$$\|F'(x_0)^{-1}F''(x_0)\| \le b$$

where the function v is as w and $b > 0$.

Thus we obtain

$$\begin{aligned} \|z_n - y_n\| &\le \frac{\int_0^1 w((1-\theta)\|y_n - x_n\|)d\theta\|y_n - x_n\|}{1 - w_0(\|y_n - x_0\|)} \le u_n - s_n, \\ \|L_n\| &\le \frac{\|F'(x_0)^{-1}(F''(y_n) - F''(x_0) + F''(x_0)\|\|F'(x_0)^{-1}F(y_n)\|}{(1 - w_0(\|y_n - x_0\|))^2} \\ &\le \frac{(b + v(\|y_n - x_0\|))\int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n)}{(1 - w_0(s_n))^2} \\ &= \overline{l_n} \le l_n, \\ \|x_{n+1} - z_n\| &\le \|F'(y_n)^{-1}F(z_n)\| + \frac{1}{2}\|L_n\|\|F'(y_n)^{-1}F(z_n)\| \\ &\le \frac{1}{1 - w_0(\|y_n - x_0\|)}\left(1 + \frac{\overline{l_n}}{2}\right)\overline{d_n} \\ &\le \frac{1}{1 - w_0(s_n)}\left(1 + \frac{l_n}{2}\right)d_n = t_{n+1} - u_n. \end{aligned}$$

Method [11]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), A_n = 2[x_n, y_n; F] - F'(x_n) \\ x_{n+1} &= y_n - A_n^{-1}F(y_n). \end{aligned} \tag{35.14}$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = \Omega$ by

$$\begin{aligned} t_{n+1} &= s_n + \frac{\int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n)}{1 - q_n} \\ q_n &= \psi(t_n, s_n) + \psi_0(t_n, s_n) \end{aligned}$$

The iterate s_{n+1} is as given in method (35.1).

The estimates are:

$$\begin{aligned} \|F'(x_0)^{-1}(A_n - F'(x_0))\| &\leq \|F'(x_0)^{-1}([x_n, y_n; F] - F'(x_n))\| \\ &\quad + \|F'(x_0)^{-1}([x_n, y_n; F] - F'(x_0))\| \\ &\leq \psi(\|x_n - x_0\|, \|y_n - x_0\|) + \psi_0(\|x_n - x_0\|, \|y_n - y_0\|) \\ &= \overline{q_n} \leq q_n < 1, \\ \|A_n^{-1}F'(x_0)\| &\leq \frac{1}{1 - q_n} \end{aligned}$$

and

$$\|x_{n+1} - y_n\| \leq \|A_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_n)\| \leq t_{n+1} - s_n.$$

Method [7]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n - (2[x_n, y_n; F]^{-1} - F'(x_n)^{-1})F(y_n). \end{aligned} \tag{35.15}$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0$ and $s_0 = \Omega$ by

$$t_{n+1} = s_n + \left(\frac{2}{1 - \psi_0(t_n, s_n)} + \frac{1}{1 - w_0(t_n)}\right) \int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n).$$

The iterate s_{n+1} is as defined in method (35.1).

The estimates are

$$\begin{aligned} \|F'(x_0)^{-1}([x_n, y_n; F] - F'(x_0))\| &\leq \psi_0(\|x_n - x_0\|, \|y_n - x_0\|) \\ &\leq \psi_0(t_n, s_n) < 1, \\ \|[x_n, y_n; F]^{-1}F'(x_0) &\leq \frac{1}{1 - \psi_0(t_n, s_n)}, \end{aligned}$$

and

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq \left(\frac{2}{1 - \psi_0(\|x_n - x_0\|, \|y_n - x_0\|)} + \frac{1}{1 - w_0(\|x_n - x_0\|)}\right) \\ &\quad \times \int_0^1 w((1-\theta)\|y_n - x_n\|)d\theta\|y_n - x_n\| \leq t_{n+1} - s_n. \end{aligned}$$

Method [13]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= y_n - (2[x_n,y_n;F]^{-1} - F'(x_n)^{-1})F(y_n) \\ x_{n+1} &= z_n - (2[x_n,y_n;F]^{-1} - F'(x_n)^{-1})F(z_n). \end{aligned} \tag{35.16}$$

The majorizing sequence $\{t_n\}$ is given for $t_0 = 0, s_0 = \Omega$ by

$$\begin{aligned} u_n &= s_n + \left(\frac{2}{1-\psi_0(t_n,s_n)} + \frac{1}{1-w_0(t_n)}\right) \\ &\quad \int_0^1 w((1-\theta)(s_n-t_n))d\theta(s_n-t_n) \\ b_n &= \left(1+\int_0^1 w_0(s_n+\theta(u_n-s_n))d\theta\right)(u_n-s_n) \\ &\quad + \int_0^1 w((1-\theta)(s_n-t_n))d\theta(s_n-t_n) \\ t_{n+1} &= u_n + \left(\frac{2}{1-\psi_0(t_n,s_n)} + \frac{1}{1-w_0(t_n)}\right)b_n, \end{aligned}$$

and the iterate s_{n+1} is as defined in method (35.1).

The estimates are given in the previous method with the exception of

$$\begin{aligned} \|x_{n+1}-z_n\| &\leq \|(2[x_n,y_n;F]^{-1} - F'(x_n)^{-1})F(z_n)\| \\ &\leq \left(\frac{2}{1-\psi_0(t_n,s_n)} + \frac{1}{1-w_0(t_n)}\right)b_n, \end{aligned}$$

where we used

$$F(z_n) = F(z_n) - F(y_n) + F(y_n),$$

so

$$\begin{aligned} \|F'(x_0)^{-1} &\leq \left(1+\int_0^1 w_0(\|y_n-x_0\|+\theta\|z_n-y_n\|)\right)d\theta \\ &\quad + \|F'(x_0)^{-1}F(y_n)\| = \overline{b_n} \leq b_n. \end{aligned}$$

Method [13]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= x_n - F'(x_n)^{-1}(F(x_n)+F(y_n)) \end{aligned} \tag{35.17}$$

The majorizing sequence $\{t_n\}$ is given for $t_0 = 0, s_0 = \Omega$ by

$$t_{n+1} = s_n + \frac{\int_0^1 w((1-\theta)(s_n-t_n))d\theta(s_n-t_n)}{1-w_0(t_n)}$$

The iterate s_{n+1} is as defined in method (35.1).

The estimates are

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) - F'(x_n)^{-1}F(y_n) = y_n - F'(x_n)^{-1}F(y_n),$$

so

$$\|x_{n+1}-y_n\| \leq \frac{\int_0^1 w(1-\theta)(s_n-t_n)d\theta(s_n-t_n)}{1-w_0(t_n)} = t_{n+1}-s_n.$$

Method [7]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= x_n - F'(x_n)^{-1}F(x_n) - 2F'(x_n)^{-1}F(y_n) \\ &\quad +F'(x_n)^{-1}F'(y_n)F'(y_n)F'(x_n)^{-1}F(y_n) \end{aligned} \tag{35.18}$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = \Omega$ by

$$\begin{aligned} \delta_n &= 2 + \frac{(w_0(s_n)+1)^2}{1-w_0(t_n)} \\ t_{n+1} &= s_n + \frac{\delta_n \int_0^1 w((1-\theta)(s_n-t_n))d\theta(s_n-t_n)}{1-w_0(t_n)} \end{aligned}$$

The iterate s_{n+1} is given in method (35.1).

Method [7]:

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_{i,n} &= x_n - d_i F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= x_n - F'(x_n)^{-1}\sum_{i=1}^{k} a_i F(z_{i,n}) \end{aligned} \tag{35.19}$$

The majorizinfg sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = \Omega$ by

$$\begin{aligned} a &= \sum_{i=1}^{k} a_i, \\ p_{i,n} &= \sum_{i=1}^{k} |a_i|(1+\int_0^1 w_0(t_n+\theta|d_i|(s_n-t_n))d\theta)|d_i|(s_n-t_n) \\ t_{n+1} &= s_n + |1-a|(s_n-t_n) + \frac{p_{i,n}(s_n-t_n)}{1-w_0(t_n)} \end{aligned}$$

The iterate s_{n+1} is given in method (35.1).

The estimates are:

$$\begin{aligned} x_{n+1} &= y_n + F'(x_n)^{-1}F(x_n) - F'(x_n)^{-1}\left(\sum_{i=1}^{k} a_i F(z_{i,n})\right), \\ x_{n+1}-y_n &= (1-a)F'(x_n)^{-1}F(x_n) \\ &\quad +F'(x_n)^{-1}\sum_{i=1}^{k} a_i(F(x_n)-F(z_{i,n})) \\ \|x_{n+1}-y_n\| &\leq |1-a|\|y_n-x_n\| + \frac{\overline{p_{i,n}}(s_n-t_n)}{1-w_0(t_n)} \leq t_{n+1}-s_n, \end{aligned}$$

where we also used

$$\begin{aligned}
& \|F'(x_0)^{-1}(F(x_n)-F(z_{i,n}))\| \\
\leq & \sum_{i=0}^{k}\left(1+\int_0^1 w_0(\|x_n-x_0\|+\theta\|z_{i,n}-x_n\|)d\theta\right)\|z_{i,n}-x_n\| \\
= & \overline{p_{i,n}}\|y_n-x_n\| \leq p_{i,n}(s_n-t_n).
\end{aligned}$$

Method [8]:

$$\begin{aligned}
y_n &= Q(x_n) \\
x_{n+1} &= Q(x_n)-F'(x_n)^{-1}F(y_n),
\end{aligned} \tag{35.20}$$

where Q is a method of order p.

The majorizing sequence $\{t_n\}$ is given for $t_0=0, s_0=\Omega$ by

$$\begin{aligned}
p_n &= \left(1+\int_0^1 w_0(\theta(s_n))d\theta\right)s_n+\Omega \\
t_{n+1} &= s_n+\frac{p_n}{1-w_0(t_n)} \\
s_{n+1} &= t_{n+1}+h_{n+1}.
\end{aligned}$$

In suppose

$$\begin{aligned}
\|y_{n+1}-x_{n+1}\| &= \|Q(x_{n+1})-x_{n+1}\| \\
&\leq h(\|x_n-x_0\|,\|y_n-x_0\|, \\
&\quad \|x_{n+1}-x_0\|,\|x_{n+1}-y_n\|,\|x_{n+1}-x_n\|), \\
&= \overline{h_n}
\end{aligned}$$

where h is a continuous and nondecreasing function in each variable.

The estimates are

$$\begin{aligned}
F(y_n) &= F(y_n)-F(x_0)+F(x_0) \\
&= \int_0^1 \left[F'(x_0+\theta(y_n-x_0))d\theta-F'(x_0)\right. \\
&\quad \left.+F'(x_0)\right](y_n-x_0)+F(x_0), \\
\|F(x_0)^{-1}F(y_n)\| &\leq \left(1+\int_0^1 w_0(\theta\|y_n-x_0\|)d\theta\right)\|y_n-x_0\|+\Omega \\
&= \overline{p_n} \leq p_n.
\end{aligned}$$

The iterate s_{n+1} is defined by the additional condition.

Method [1,2,3,4]:

$$\begin{aligned}
y_n &= x_n-\lambda F'(x_n)^{-1}F(x_n), \lambda=\frac{\sqrt{5}-1}{2} \\
z_n &= x_n-\mu F'(x_n)^{-1}F(y_n), \lambda=\frac{\sqrt{5}+3}{2} \\
x_{n+1} &= z_n-F'(x_n)^{-1}F(z_n).
\end{aligned} \tag{35.21}$$

The majorizing sequence $\{t_n\}$ is defined for $t_0 = 0, s_0 = |\lambda|\Omega$ by

$$\begin{aligned}
b_n &= \left(1 + \int_0^1 w_0(t_n) + \theta(s_n - t_n)\right) d\theta(s_n - t_n) \\
&\quad + \frac{1}{|\lambda|}(1 + w_0(t_n))(s_n - t_n), \\
u_n &= 2s_n - t_n + \frac{|\mu| b_n}{1 - w_0(t_n)}, \\
c_n &= \left(1 + \int_0^1 w_0(s_n + \theta(u_n - s_n)) d\theta\right)(u_n - s_n) + b_n, \\
t_{n+1} &= u_n + \frac{c_n}{1 - w_0(t_n)}, \\
\sigma_{n+1} &= \left(1 + \int_0^1 w_0(t_n + \theta(t_{n+1} - t_n)) d\theta\right)(t_{n+1} - t_n) \\
&\quad \frac{1}{|\lambda|}(1 + w_0(t_n))(s_n - t_n). \\
s_{n+1} &= t_{n+1} + \frac{\sigma_{n+1}}{1 - w_0(t_{n+1})}
\end{aligned}$$

The estimates are:

$$\begin{aligned}
F(y_n) &= F(y_n) - F(x_n) + F(x_n) \\
\|F'(x_0)^{-1} F(y_n)\| &\leq \left(1 + \int_0^1 w_0(\|x_n - x_0\| + \theta\|y_n - x_n\|) d\theta\right) \|y_n - x_n\| \\
&\quad + \frac{1}{|\lambda|}(1 + w_0(\|x_n - x_0\|))\|y_n - x_n\| = \overline{b_n} \leq b_n, \\
\|z_n - y_n\| &\leq \|\lambda F'(x_n)^{-1} F(x_n)\| + |\mu| \|F'(x_n)^{-1} F(y_n)\| \\
&\leq \|y_n - x_n\| + \frac{|\mu| b_n}{1 - w_0(\|x_n - x_0\|)} \leq u_n - s_n. \\
F(z_n) &= F(z_n) - F(y_n) + F(y_n), \\
\|F'(x_0)^{-1} F(z_n)\| &\leq \left(1 + \int_0^1 w_0(\|y_n - x_0\| + \theta\|z_n - y_n\|) d\theta\right) \\
&\quad \times \|z_n - y_n\| + \overline{b_n} = \overline{c_n} \leq c_n, \\
\|x_{n+1} - z_n\| &\leq \frac{\overline{c_n}}{1 - w_0(\|x_n - x_0\|)} \leq \frac{c_n}{1 - w_0(t_n)} = t_{n+1} - u_n, \\
F(x_{n+1}) &= F(x_{n+1}) - F(x_n) - \frac{1}{\lambda} F'(x_n)(y_n - x_n), \\
\|F'(x_0)^{-1} F(x_{n+1})\| &\leq \left(1 + \int_0^1 w_0(\|x_n - x_0\| + \theta\|x_{n+1} - x_n\|) d\theta\right) \\
&\quad \times \|x_{n+1} - x_n\| \\
&\quad + \frac{1}{|\lambda|}(1 + w_0(\|x_n - x_0\|))\|y_n - x_n\| = \overline{s_{n+1}} \\
&\leq \sigma_{n+1}
\end{aligned}$$

and

$$\|y_{n+1} - x_{n+1}\| \leq \frac{\sigma_{n+1}}{1 - w_0(t_{n+1})} = s_{n+1} - t_{n+1}.$$

Bibliography

[1] Cordero, A., Soleymani, F. Torregrosa, J.R., Dynamical analysis of iterative methods for nonlinear systems or how to deal with the dimension?, *Applied Mathematics and Computation* 244 (2014) 398-412.

[2] Cordero, A., Hueso, J.L., Martinez, E., Torregrosa, J.R., Efficient high-order methods based on golden ratio for nonlinear equations, *Applied Mathematics and Computation* 217 (2011) 4548-4556.

[3] Cordero, A., Torregrosa, J.R., Variants of Newton method using fifth-order quadrature formulae, *Appl. Math. Comput.* 190 (2007) 686-698.

[4] Cordero, A., Hueso, J.L., Martinez, E., Torregrosa, J.R., A modified Newton-Jarratts composition, *Numer. Algor.* 55 (2010) 87-99.

[5] Esmaeili, H., Ahmadi, M., An efficient three-step method to solve system of nonlinear equations, *Applied Mathematics and Computation* 266 (2015) 1093-1101.

[6] Frontini, M., Sormani, E., Third-order methods from quadrature formulae for solving systems of nonlinear equations, *Applied Mathematics and Computation* 149 (2004) 771-782.

[7] Grau-Sanchez, M., Grau, A., Noguera, M., Ostrowski type methods for solving systems of nonlinear equations, *Appl. Math. Comput.* 218(2011) 2377-2385.

[8] Khan, W.A., Noor, K.I., Bhatti, K., Ansari, F.A., A new fourth Newton-type method for solution of system of nonlinear equations, *Applied Mathematics and Computation* 270 (2015) 724-730.

[9] Lotfi, T., Bakhtiari, P., Cordero, A., Mahdiani, K., Torregrosa, J.R., Some new efficient multipoint iterative methods for solving nonlinear system of equations, *International Journal of Computer Mathematics*, 2015, Vol.92, No. 9, 1921-1934, https://dx.doi.org/10.1080/00207160.2014.946412.

[10] Sharma, J.R., Gupta, P., An efficient fifth order method for solving system of nonlinear equations, *Comput. Math. Appl.* 67 (2014) 591-601.

[11] Soleymani, F., Lotfi, T., Bakhtiari, P., A multi-step class of iterative methods for nonlinear systems, *Optim. Lett.* 8 (2014) 1001-1015.

[12] Sharma, J.R., Guha, R.K., Sharma, R., An efficient fourth order weighted-Newton method for systems of nonlinear equations, *Numer. Algor.* (2013) 62:307-323. DOI 10.1007/s11075-012-9585-7.

[13] Sharma, J.R., Kumar, D., A fast and efficient composite Newton- Chebyshev method for systems of nonlinear equations, *Journal of Complexity* 49 (2018) 56-73.

Chapter 36

On a Family of Optimal Eighth Order Methods for Equations in Banach Space

1. Introduction

Consider the nonlinear equation

$$F(x) = 0, \tag{36.1}$$

where $F : D \subset X_1 \longrightarrow X_2$ is a Fréchet differentiable operator between the Banach spaces X_1 and X_2. Here D is an open convex subset of X_1. Since closed-form solution is possible only in special cases, iterative methods are used to approximate the solution x^* of (36.1). In this study, we consider the iterative method defined for all $n = 0, 1, 2, \ldots$, by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - A_nF'(x_n)^{-1}F(y_n) \\ \text{and} & \\ x_{n+1} &= z_n - B_nF'(x_n)^{-1}F(z_n), \\ \text{where} & \\ A_n &= A(x_n, y_n), A : D \times D \longrightarrow L(X_1, X_2), \\ B_n &= B(y_n, x_n, z_n), B : D \times D \times D \longrightarrow L(X_1, X_2). \end{aligned} \tag{36.2}$$

The local convergence order eighth was shown using condition on the ninth derivative when $X_1 = X_2 = \mathbb{R}$ [11]. In this Chapter, the local is presented in the more general setting of a Banach space. Moreover, conditions only on the operators appearing in these methods are used in the analysis. Hence, the applicability of these methods is extended.

2. Local Analysis

We develop functions and parameters to be used in the local convergence analysis of the method (36.2). Set $M = [0, \infty)$. Suppose function:

(i) $\omega_0(t) - 1$ has a minimal zero $\delta_0 \in M - \{0\}$ for some fucntion $\omega_0 : M \longrightarrow M$ which is continuous and nondecreasing. Set $M_0 = [0, \delta_0)$.

(ii) $g_1(t) - 1$ has a minimal zero $d_1 \in B_0 - \{0\}$ for some fucntion $w : M_0 \longrightarrow M$ which is continuous and nondecreasing and $g_1 : B_0 \longrightarrow M$ defined by

$$g_1(t) = \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1-\omega_0(t)}.$$

(iii) $\omega_0(g_1(t)t) - 1$ ha a minimal zero $\delta_1 \in B_0 - \{0\}$. Set

$$\delta_2 = \min\{\delta_0, \delta_1\}$$

and $M_1 = [0, \delta_2)$.

(iv) $g_2(t) - 1$ has a minimal zero $d_2 \in M_1 - \{0\}$ for some functions $\omega_1 : M - 1 \longrightarrow M, p : M_1 \longrightarrow M$ which are continuous and nondecreasing and $g_2 : M_1 \longrightarrow M$ defined by

$$\begin{aligned} g_2(t) &= [g_1(g_1(t)t) + \frac{(\omega_0(t) + \omega_1(g_1(t)t)) \int_0^1 \omega_1(\theta g_1(t)t)d\theta}{(1-\omega_0(t))(1-\omega_0(g_1(t)t))} \\ &+ \frac{p(t)\int_0^1 \omega_1(\theta g_1(t)t)d\theta}{1-\omega_0(t)}]g_1(t). \end{aligned}$$

(v) $\omega_0(g_2(t)t) - 1$ has a minimal zero $\delta_3 \in M_1 - \{0\}$. Set $\delta = \min\{\delta_2, \delta_3\}$ and $M_2 = [0, \delta)$.

(vi) $g_3(t) - 1$ has a minimal zero $d_3 \in M_2 - \{0\}$ for some function $q : M_2 \longrightarrow M$ defined by

$$\begin{aligned} g_3(t) &= [g_1(g_2(t)t) + \frac{(\omega_0(t) + \omega_1(g_2(t)t)) \int_0^1 \omega_1(\theta g_2(t)t)d\theta}{(1-\omega_0(t))(1-\omega_0(g_2(t)t))} \\ &+ \frac{q(t)\int_0^1 \omega_1(\theta g_2(t)t)d\theta}{1-\omega_0(t)}]g_2(t). \end{aligned}$$

Define

$$d = min\{d_k\},\ k = 1,2,3. \tag{36.1}$$

It shall be shown that d is a radius of convergence for method (36.2).

Denote by $U[x,\mu]$ the closure of the open ball $U(x,\mu)$ of center $x \in D$ and radius $\mu > 0$.

The hypotheses (H) are developed provided x_* is a simple zero of F and functions ω, p, q are as previously given.

Suppose:

(H1) For each $x \in D$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq \omega_0(\|x - x^*\|).$$

Set $\Omega_0 = U(x^*, \delta_0) \cap D$.

(H2) For each $x, y \in \Omega_0$, $y = x - F'(x)^{-1}F(x)$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \omega_0(\|x - y\|),$$

$$\|F'(x^*)^{-1}F'(x)\| \leq \omega_1(\|x - x^*\|),$$

$$\|I - A(x,y)\| \leq p(\|x - x^*\|)$$

and

$$\|I - B(x,y)\| \leq q(\|x - x^*\|).$$

(H3) $U[x^*, d] \subset D$,
and

(H5) There exists $d_* \geq d$ satisfying

$$\int_0^1 \omega_0(\theta d_*)d\theta < 1.$$

Set $\Omega_1 = U[x^*, d_*] \cap D$.

Next, the local convergence analysis of method (36.2) is given using hypotheses H.

Theorem 42. Suppose hypotheses (H) hold. Then, the following assertions hold provided that $x_0 \in U(x^*, d) - \{x^*\}$:

$$\{x_n\} \subset U(x^*, d), \tag{36.2}$$

$$\|y_n - x^*\| \leq g_1(e_n)e_n \leq e_n < d, \tag{36.3}$$

$$\|z_n - x^*\| \leq g_2(e_n)e_n \leq e_n < d, \tag{36.4}$$

$$e_{n+1} \leq g_3(e_n)e_n \leq e_n, \tag{36.5}$$

where $e_n = \|x_n - x^*\|$, radius d is defined by (36.1) and the functions g_k are as given previously. Moreover, the only zero of F in the set Ω_1 is x^*.

Proof. Set $M - 3 = [0, d)$. It follows from the definition of d that for each $t \in M_3$ the following hold

$$0 \leq \omega_0(t) < 1 \tag{36.6}$$

$$0 \leq \omega_0(g_1(t)t) < 1 \tag{36.7}$$

$$0 \leq \omega_0(g_2(t)t) < 1 \tag{36.8}$$

and

$$0 \leq g_k(t) < 1. \tag{36.9}$$

We have that (36.2) holds for $n = 0$. Let $u \in U(x^*, d) - \{x^*\}$. Using (36.1), (36.6) and (H1), we have in turn

$$\|F'(x^*)^{-1}(F'(u) - F'(x^*))\| \leq \omega_0(\|u - x^*\|) \leq w_0(d) < 1. \tag{36.10}$$

Then, (36.10) together with the Banach lemma on linear operators with inverses [4] give $F'(u)^{-1} \in L(X_2, X_1)$ with

$$\|F'(u)^{-1}F'(x^*)\| \leq \frac{1}{1-\omega_0(\|u-x^*\|)}. \tag{36.11}$$

Notice that iterates y_0, z_0, x_1 are well defined by method (36.2) for $n=0$ and we can write

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= (F'(x_0)^{-1}F(x^*)) \\ &\quad \times (\int_0^1 F'(x^*)^{-1}(F'(x_0) - F'(x^* + \theta(x_0 - x^*)))d\theta(x_0 - x^*). \end{aligned} \tag{36.12}$$

By (36.1), (36.11) (for $u = x_0$,(36.9)(for $k=1$), (H2) and (36.12) we get in turn that

$$\|y_0 - x^*\| \leq \frac{\omega((1-\theta)\|x_0-x^*\|)d\theta\|x_0-x^*\|}{1-\omega_0(\|x_0-x^*\|)} \tag{36.13}$$

$$\leq g_1(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\| < d, \tag{36.14}$$

showing (36.2) and (36.3) for $n=0$.

We can also write by the second substep of method (36.2) that

$$\begin{aligned} z_0 - x^* &= y_0 - x^* - F'(y_0)^{-1}F(y_0) \\ &\quad + (F'(y_0)^{-1} - F'(x_0)^{-1})F(y_0) \\ &\quad + (I - A_0)F'(x_0)^{-1}F(y_0) \\ &= y_0 - x^* - F'(y_0)^{-1}F(y_0) + F'(y_0)^{-1}(F'(x_0) - F'(y_0))F'(x_0)^{-1}F(y_0) \\ &\quad + (I - A_0)F'(x_0)^{-1}F(y_0). \end{aligned} \tag{36.15}$$

Using (36.1), (36.7), (36.9) (for $k=2$), (H2), (H3), (36.11) (for $u = y_0$), (36.14), (36.15) and the triangle inequality, we obtain in turn

$$\begin{aligned} \|z_0 - x^*\| &\leq [g_1(\|y_0 - x^*\|) \\ &\quad + \frac{(\omega_0(\|x_0-x^*\|) + \omega_0(\|y_0-x^*\|)) \int_0^1 \omega_1(\theta\|y_0-x^*\|)d\theta}{(1-\omega_0(\|x_0-x^*\|))(1-\omega_0(\|y_0-x^*\|))} \\ &\quad + \frac{p(\|x_0-x^*\|) \int_0^1 \omega_1(\theta\|y_0-x^*\|)d\theta}{1-\omega_0(\|x_0-x^*\|}]\|y_0-x^*\| \\ &\leq g_2(\|x_0-x^*\|)\|x_0-x^*\| \leq \|x_0-x^*\|, \end{aligned} \tag{36.16}$$

showing (36.2) and (36.3) for $n=0$.

Moreover, by the third substep of method (36.2) for $n=0$, we can write analogously

$$\begin{aligned} x_1 - x^* &= z_0 - x^* - F'(z_0)^{-1}F(z_0) \\ &\quad + (F'(z_0)^{-1}(F'(x_0) - F'(z_0))F'(x_0)^{-1})F(z_0) \\ &\quad + (I - B_0)F'(x_0)^{-1}F(z_0). \end{aligned} \tag{36.17}$$

Using (36.1), (36.8), (36.9) (for $k = 3$), (H2), (H3), (36.11) (for $u = z_0$), (36.18), (36.17) and the triangle inequality, we obtain in turn

$$\begin{aligned}\|x_1 - x^*\| &\leq [g_1(\|z_0 - x^*\|) \\ &+ \frac{(\omega_0(\|x_0 - x^*\|) + \omega_0(\|z_0 - x^*\|)) \int_0^1 \omega_1(\theta\|z_0 - x^*\|)d\theta}{(1 - \omega_0(\|x_0 - x^*\|))(1 - \omega_0(\|z_0 - x^*\|))} \\ &+ \frac{q(\|x_0 - x^*\|) \int_0^1 \omega_1(\theta\|z_0 - x^*\|)d\theta}{1 - \omega_0(\|x_0 - x^*\|}]\|z_0 - x^*\| \\ &\leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|, \qquad (36.18)\end{aligned}$$

showing (36.2) and (36.5) for $n = 0$. Simply substitute x_j, y_j, z_j, x_{j+1} for x_0, y_0, z_0, x_1 in the previous calculations, we finish the induction for assertions (36.2)- (36.5). It follows from the estimation

$$e_{j+1} \leq \gamma e_j < d, \qquad (36.19)$$

where $\gamma = g_3(\|x_0 - x^*\|)$, that (36.2) and (36.6) hold. Set $T = \int_0^1 F'(x^* + \theta(b - x^*))d\theta$, for some $b \in \Omega_1$ with $F(b) = 0$. Then, in view of (H1) and (H4), we get in turn

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \leq \int_0^1 \omega_0(\theta\|b - x^*\|)d\theta \leq \int_0^1 \omega_0(\theta d_*)d\theta < 1,$$

leading to $x^* = b$, since $T^{-1} \in L(X_2, X_1)$ and $T(b - x^*) = F(b) - F(x^*) = 0$. □

Bibliography

[1] Grau-Sánchez, M., Grau, A., Noguera, M., Frozen divided difference solving systems of nonlinear equations, *J. Comput. Appl. Math.*, 235, (2011), 1739-1743.

[2] Grau-Sánchez, M., Noguera, V., A technique to choose the most efficient method between secant method and some variants, *Appl. Math. Comput.*, 218, (2012), 6415-6426.

[3] Kelley, C.T., Solving nonlinear equations with Newton's Method. *SIAM*, Philadelphia, 2003.

[4] Potra, F.A., Pták, V., *Nondiscrete Induction and Iterative Processes*, Pitman Publishing, Boston, 1984.

[5] Ren, H., Wu, A., Bi, W., A class of two-step Steffensen type methods with fourth-order convergence, *Appl. Math. Comput.*, 209, (2009), 206-210.

[6] Sharma, J.R., Guha, R.K., Gupta, P., Some efficient derivative free methods with memory for solving nonlinear equations, *Appl. Math. Comput.*, 219, (2012), 699-707.

[7] Steffensen, J.F., Remarks on iteration, *Skand. Aktuar Tidskr.*, 16, (1933), 64-72.

[8] Traub, J.F., *Iterative Methods for the Solution of Equations*, Prentice-Hall, New Jersey, 1964.

[9] Wang, X., Zhang, T., A family of Steffensen type methods with seventh-order convergence, *Numer. Algor.*, 62, (2013), 229-444.

[10] Wolfram, S., *The Mathematica Book*, fifth ed. Wolfram Media, 2003.

[11] Sharma, J.R., Arora, H., An efficient derivative free iterative method for solving systems of nonlinear equations, *Algorithm*, 2022, 390-403.

Chapter 37

Extended Ball Convergence of a Xiao-Yin Fifth Order Scheme for Equations

The local convergence for a Xiao-Yin method of order five is studied using assumptions only on the first derivative of the operator involved. The convergence of this method was shown by assuming that the sixth order derivative of the operator not on the method exists and hence it is limiting its applicability. Moreover, no computational error bounds or uniqueness of the solution are given. We address all these problems using only the first derivative that appears on the method. Hence, we extend the applicability of the method. Our techniques can be used to obtain the convergence of other similar higher-order methods using assumptions on the first derivative of the operator involved.

1. Introduction

Let $F : D \subset E_1 \longrightarrow E_2$ be a nonlinear operator acting between Banach spaces E_1 and E_2 and $D \neq \emptyset$ be an open set. Consider the problem of solving the nonlinear equation

$$F(x) = 0. \tag{37.1}$$

Iterative methods are used to approximate a solution x^* of the equation (37.1). The following iterative method was studied in [26],

$$\begin{aligned} y_n &= x_n - \beta F'(x_n)^{-1}F(x_n), \\ z_n &= x_n - \frac{1}{4}(3F'(y_n)^{-1} + F'(x_n)^{-1})F(x_n) \end{aligned}$$

and

$$x_{n+1} = z_n - \frac{1}{2}(3F'(y_n)^{-1} - F'(x_n)^{-1})F(z_n).$$

In this chapter, we study the convergence of method (37.2) using assumptions only on the first derivative of F, unlike earlier studies [26] where the convergence analysis required assumptions on the derivatives of F up to the order six. This method can be used on other methods and relevant topics along the same lines [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15, 16,17,18,19,20,21,22,23,24,25,26,27,28,29].

For example: Let $X = Y = \mathbb{R}$, $D = [-\frac{1}{2}, \frac{3}{2}]$. Define f on D by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we have $f(1) = 0$,

$$f'''(t) = 6\log t^2 + 60t^2 - 24t + 22.$$

Obviously, $f'''(t)$ is not bounded by D. So, the convergence of the method (37.2) is not guaranteed by the analysis in [26].

Throughout the chapter $U(x_0, R) = \{x \in X : \|x - x_0\| < R\}$ and $U[x_0, R] = \{x \in X : \|x - x_0\| \leq R\}$ for some $R > 0$.

The chapter contains local convergence analysis in Section 2, and the numerical examples are given in Section 3.

2. Ball Convergence

The convergence uses real functions and parameters. Set $A = [0, \infty)$. Suppose functions:

(1) $h_0 : A \longrightarrow A$ is continuous, non-decreasing and such that $h_0(t) - 1$ has a smallest zero $\rho_0 \in A - \{0\}$. Set $A_0 = [0, \rho_0)$.

(2) $h_0 : A_0 \longrightarrow A, h_1 : A_0 \longrightarrow A$ are continuous, non-decreasing and such that $f_1(t) - 1$ has a smallest zero $R_1 \in A_0 - \{0\}$, where $f_1 : A_0 \longrightarrow A$ is defined by

$$f_1(t) = \frac{\int_0^1 h((1-\theta)t)d\theta + |1-\beta| \int_0^1 h_1(\theta t)d\theta}{1 - h_0(t)}.$$

(3) $h_0(f_1(t)) - 1$ has a smallest zero $\rho_1 \in A_0 - \{0\}$. Set $\rho_2 = \min\{\rho_0, \rho_1\}$ and $A_1 = [0, \rho_2)$.

(4) $f_2(t) - 1$ has a smallest zero $R_2 \in A_1 - \{0\}$, where f_2 is defined by

$$f_2(t) = \frac{\int_0^1 h((1-\theta)t)\,d\theta}{1 - h_0(t)} + \frac{3}{4}\frac{(h_0(t)) + h_0(f_1(t)t)\int_0^1 h_1(\theta t)d\theta}{(1 - h_0(t))(1 - h_0(f_1(t)t))}.$$

(5) $h_0(f_2(t)t) - 1$ has a smallest zero $\rho \in A_1 - \{0\}$,

(6) $f_3(t) - 1$ has a smallest zero $R_3 \in A_1 - \{0\}$, where

$$\begin{aligned} f_3(t) \quad = \quad & \left[\frac{\int_0^1 h((1-\theta)f_2(t)t)d\theta}{1 - h_0(f_2(t)t)} \right. \\ & + \frac{(h_0(f_2(t)t) + h_0(f_1(t)t)) \int_0^1 h_1(\theta f_2(t)t)d\theta}{(1 - h_0(f_2(t)t))(1 - h_0(f_1(t)t))} \\ & \left. + \frac{(h_0(t) + h_0(f_1(t)t)) \int_0^1 h_1(\theta f_2(t)t)d\theta}{2(1 - h_0(t))(1 - h_0(f_1(t)t))} \right] f_2(t). \end{aligned}$$

The parameter

$$R = \min\{R_j\},\ j = 1,2,3 \tag{37.2}$$

shall be shown to be a convergence radius for the scheme (37.2). The definition of R implies that for all $t \in [0,R)$

$$0 \le h_0(t) < 1, \tag{37.3}$$

$$0 \le h_0(f_1(t)t) < 1, \tag{37.4}$$

$$0 \le h_0(f_2(t)t) < 1 \tag{37.5}$$

and

$$0 \le f_j(t) < 1. \tag{37.6}$$

Moreover, hypotheses (H) shall be used.
Suppose:

(H1) Element $x^* \in D$ is a simple solution of equation (37.1).

(H2) $\|F'(x^*)^{-1}(F'(w) - F'(x^*))\| \le h_0(\|w - x^*\|$ for all $w \in D$. Set $D_0 = U(x^*, \rho_0) \cap D$.

(H3) $\|F'(x^*)^{-1}(F'(w) - F'(v))\| \le h(\|w - v\|$ and

$$\|F'(x^*)^{-1}F'(w)\| \le h_1(\|w - x^*\|)$$

for all $w, v \in D_0$.

(H4) $U[x^\star, R] \subset D$.

Next, the main ball convergence for the scheme (37.2) follows.

Theorem 43. Suppose hypotheses (H) hold, and choose $x_0 \in U(x^\star, R) - \{x^\star\}$. Then, sequence $\{x_n\}$ generated by scheme (37.2) is such that $\lim_{n\to\infty} x_n = x^\star$.

Proof. Let $u \in U(x^\star, R)$. Using (H_2) one gets

$$\|F'(x^\star)^{-1}(F'(u) - F'(x^\star))\| \le h_0(\|u - x^\star\|) \le h_0(R) < 1,$$

so $F'(u)^{-1} \in \mathbb{L}(B_2, B_1)$ and

$$\|F'(u)^{-1}F'(x^\star)\| \le \frac{1}{1 - h_0(\|u - x^\star\|)} \tag{37.7}$$

follow by a standard perturbation lemma due to Banach [15]. In particular, if $u = x_0$, iterate y_0 exists by the first sub-step of scheme (37.2) for $n = 0$. Moreover, one gets

$$y_0 - x^\star = x_0 - x^\star - F'(x_0)^{-1}F(x_0) + (1 - \beta)F'(x_0)^{-1}F(x_0). \tag{37.8}$$

By (2.5) for $j=0$, (37.7) for $u=x_0$, (H_3) and (37.8) one has

$$\begin{aligned}
\|y_0-x^\star\| &\le \|F'(x_0)^{-1}F'(x^\star)\| \\
&\quad \times \left\| \int_0^1 F'(x^\star)^{-1}(F'(x^\star+\theta(x_0-x^\star))d\theta - F'(x_0))(x_0-x^\star)\right\| \\
&\quad +|1-\beta|\|F'(x_0)^{-1}F'(x^\star)\|\|F'(x^\star)^{-1}F(x_0)\| \\
&\le \frac{(\int_0^1 h((1-\theta)\|x_0-x^\star\|)d\theta + |1-\beta|\int_0^1 h_1(\theta\|x_0-x^\star\|)d\theta)\|x_0-x^\star\|}{1-h_0(\|x_0-x^\star\|)} \\
&= f_1(\|x_0-x^\star\|)\|x_0-x^\star\| \le \|x_0-x^\star\| < R.
\end{aligned}$$

So, $y_0 \in U(x^\star,R)$, and z_0, r_1 are well defined by scheme (37.2) for $n=0$.

Furthermore, one can write

$$z_0-x^\star = x_0-x^\star-F'(x_0)^{-1}F(x_0)+\frac{3}{4}F'(x_0)^{-1}(F'(y_0)-F'(x_0))F'(y_0)^{-1}F(x_0). \quad (37.9)$$

In view of (2.5) (for $j=2$), (H_3), (37.7) for $u=x_0,y_0$, (37.9) and (37.9) one obtains

$$\begin{aligned}
\|z_0-x^\star\| &\le \|x_0-x^\star-F'(x_0)^{-1}F'(x_0)\| \\
&\quad +\frac{3}{4}\|F'(x_0)^{-1}F'(x^\star)\|\|F'(x^\star)^{-1}(F'(y_0)-F'(x_0))\|\|F'(y_0)^{-1}F'(x^\star)\| \\
&\quad \times\|F'(x^\star)^{-1}F(x_0)\| \\
&\le \Bigg[\frac{\int_0^1 h((1-\theta)\|x_0-x^\star\|)d\theta}{1-h_0(\|x_0-x^\star\|)} \\
&\quad +\frac{3}{4}\frac{(h_0(\|x_0-x^\star\|)+h_0(\|y_0-x^\star\|))\int_0^1 h_1(\theta\|x_0-x^\star\|)d\theta}{(1-h_0(\|x_0-x^\star))(1-h_0(\|y_0-x^\star\|))}\Bigg]\|x_0-x^\star\| \\
&\le f_2(\|x_0-x^\star\|)\|x_0-x^\star\| \le \|x_0-x^\star\|,
\end{aligned}$$

so $z_0 \in U(x^\star,R)$. By the third sub-step of scheme (37.2) similarly one can write

$$\begin{aligned}
x_1-x^\star &= z_0-x^\star-F'(z_0)^{-1}F(z_0)+\frac{1}{2}\Big[2F'(z_0)^{-1}(F'(y_0)-F'(z_0))F'(y_0)^{-1} \\
&\quad +F'(x_0)^{-1}(F'(y_0)-F'(x_0))F'(y_0)^{-1}\Big]F(z_0).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\|x_1-x_\star\| &\le \Bigg[\frac{\int_0^1 h((1-\theta)\|z_0-x^\star\|)d\theta}{1-h_0(\|z_0-x^\star\|)} \\
&\quad +\frac{(h_0(\|z_0-x^\star\|)+h_0(\|y_0-x^\star\|))\int_0^1 h_1(\theta\|z_0-x^\star\|)d\theta}{(1-h_0(\|z_0-x^\star))(1-h_0(\|y_0-x^\star\|))} \\
&\quad +\frac{1}{2}\frac{(h_0(\|x_0-x^\star\|)+h_0(\|y_0-x^\star\|))\int_0^1 h_1(\theta\|z_0-x^\star\|)d\theta}{(1-h_0(\|x_0-x^\star))(1-h_0(\|y_0-x^\star\|))}\Bigg]\|z_0-x^\star\| \\
&\le f_3(\|x_0-x^\star\|)\|x_0-x^\star\| \le \|x_0-x^\star\|.
\end{aligned}$$

Then, we have $x_1 \in U(x^\star, R)$. By repeating the preceding computations with x_0, y_0, z_0, x_1 replaced by x_m, y_m, z_m, x_{m+1} we get

$$\|x_{m+1} - x^\star\| \leq c\|x_m - x^\star\| < R, \tag{37.10}$$

where $c = f_3(\|x_0 - x^\star\|) \in [0,1)$, one deduces that $x_{m+1} \in U(x^\star, R)$ and $\lim_{m\to\infty} x_m = x^\star$. □

Proposition 25. *Suppose:*

(1) The element $x^* \in U(x^*, s^*)$, subset of D for some $s^* > 0$ is a simple solution of (37.1), and (H2) holds.

(2) There exists $\delta \geq s^*$ so that

$$K_0(s^* + \delta) < 2. \tag{37.11}$$

Set $D_1 = D \cap U[x^*, \delta]$. Then, x^* is the unique solution of equation (37.1) in the domain D_1.

Proof. Let $q \in D_1$ with $F(q) = 0$. Define $S = \int_0^1 F'(q + \theta(x^* - q))d\theta$. Using (H2) and (37.11) one obtains

$$\begin{aligned}
\|F'(x_0)^{-1}(S - F'(x_0))\| &\leq K_0 \int_0^1 ((1-\theta)\|q - x_0\| + \theta\|x^* - x_0\|)d\theta \\
&\leq \frac{K_0}{2}(s^* + \delta) < 1,
\end{aligned}$$

so $q = x^*$, follows from the invertibility of S and the identity $S(q - x^*) = F(q) - F(x^*) = 0 - 0 = 0$. □

3. Numerical Experiments

We compute the radius of convergence in this section.

Example 7. Let $E_1 = E_2 = \mathbb{R}^3, D = B[0,1], x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, we get

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so for $h_0(t) = (e-1)t, h(t) = e^{\frac{1}{e-1}}t$ and $h_1(t) = e^{\frac{1}{e-1}}$, we have for $\beta = \frac{1}{2}$,

$$R_1 = 0.0403 = R, R_2 = 0.0876, R_3 = 0.0612.$$

Example 8. Let $B_1 = B_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ be equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{37.12}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D.$$

Then, for $x^* = 0$, we get $h_0(t) = 7.5t, h(t) = 15t$ and $h_1(t) = 2$. Then, for $\beta = \frac{1}{4}$, the radii are

$$R_1 = 0.0333, R_2 = 0.0167, R_3 = 0.0136 = R.$$

Example 9. Returning back to the motivational example at the introduction of this study, we have $h_0(t) = h(t) = 96.6629073t$ and $h_1(t) = 2$.

Then, for $\beta = \frac{1}{4}$, the radii are

$$R_1 = 0.0034, R_2 = 0.0014, R_3 = 0.0011 = R.$$

Bibliography

[1] Argyros, I.K., On the Newton - Kantorovich hypothesis for solving equations, *J. Comput. Math.*, 169, (2004), 315-332.

[2] Argyros, I.K., *Computational theory of iterative methods. Series: Studies in Computational Mathematics*, 15, Editors: C.K. Chui and L. Wuytack, Elsevier Publ. Co. New York, U.S.A, 2007.

[3] Argyros, I.K. *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[4] Argyros, I.K., Hilout, S., Weaker conditions for the convergence of Newton's method. *J. Complexity*, 28, (2012), 364–387.

[5] Argyros, I.K., Hilout, S., On an improved convergence analysis of Newton's methods, *Applied Mathematics and Computation*, 225, (2013), 372-386.

[6] Argyros, I.K., Magréñan, A.A., *Iterative methods and their dynamics with applications*, CRC Press, New York, USA, 2017.

[7] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative methods*, Elsevier (Academic Press), New York, 2018.

[8] Arroyo, V., Cordero, A., Torregrosa, J.R., Approximation of artificial satellites preliminary orbits: The efficiency challenge. *Mathematical and Computer Modeling*, 54, (7-8), (2011), 1802-1807.

[9] Behl, R., Maroju, P., Martinez, E., Singh, S., A study of the local convergence of a fifth order iterative method, *Indian J. Pure Appl. Math.*, 51, 2, (2020), 439-455.

[10] Cătinaş, Emil The inexact, inexact perturbed, and quasi-Newton methods are equivalent models, *Math. Comp.*, 74, (2005), 291-301.

[11] Ezquerro, J.A., Gutiérrez, J. M., Hernández, M. A., Romero, N., Rubio, M.J., The Newton method: from Newton to Kantorovich(Spanish), *Gac. R. Soc. Mat. Esp.*, 13, (2010), 53-76.

[12] Ezquerro, J.A., Hernandez, M.A., *Newton's method: An updated approach of Kantorovich's theory*, Cham Switzerland, (2018).

[13] Grau-Sánchez, Miquel and Grau, Àngela and Noguera, Miquel, Ostrowski type methods for solving systems of nonlinear equations, *Appl. Math. Comput.*, 281, (2011), 2377-2385.

[14] Homeier, H.H.H., A modified Newton method with cubic convergence: the multivariate case, *Journal of Computational and Applied Mathematics*, 169, (2004), 161-169.

[15] Kantorovich, L.V., Akilov, G.P. *Functional Analysis*, Pergamon Press, Oxford, (1982).

[16] Magréñan, A.A., Argyros, I.K., Rainer, J.J., Sicilia, J.A., Ball convergence of a sixth-order Newton-like method based on means under weak conditions, *J. Math. Chem.*, (2018), 56, 2117-2131, https://doi.org/10.1007/ s10910-018-0856-y.

[17] Magréñan, A.A., Gutiérrez, J.M., Real dynamics for damped Newton's method applied to cubic polynomials, *J. Comput. Appl. Math.*, 275, (2015), 527–538.

[18] Ortega, L.M., Rheinboldt, W.C., *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, (1970).

[19] Ostrowski, A.M., *Solution of equations in Euclidean and Banach spaces*, Elsevier, 1973.

[20] Potra, F.A. and Pták, V., Nondiscrete induction and iterative processes. *Research Notes in Mathematics*, 103. Pitman (Advanced Publishing Program), Boston,MA. (1984).

[21] Proinov, P.D., General local convergence theory for a class of iterative processes and its applications to Newton's method, *J. Complexity*, 25, (2009), 38-62.

[22] Proinov P.D., New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems, *J. Complexity*, 26, (2010), 3-42.

[23] Rheinboldt, W.C. An adaptive continuation process of solving systems of nonlinear equations. Polish Academy of Science, *Banach Ctr. Publ.* 3,(1978), 129-142.

[24] Shakhno, S.M., Gnatyshyn, O. P., On an iterative algorithm of order 1.839... for solving nonlinear least squares problems, *Appl. Math. Applic.*, 161, (2005), 253-264.

[25] Shakhno, S.M., Iakymchuk, R. P., Yarmola, H. P., Convergence analysis of a two step method for the nonlinear squares problem with decomposition of operator, *J. Numer. Appl. Math.*, 128, (2018), 82-95.

[26] Xiao, X., Yin, H., Accelerating the convergence speed of iterative methods for solving equations, *Appl. Math. Comput.*, 333, (2018), 8-19.

[27] Sharma, J.R, Kumar, S., A class of accurate Newton-Jarratt like methods with applications to nonlinear models, *Comput. Appl. Math*, (2022).

[28] Traub, J.F., *Iterative methods for the solution of equations*, Prentice Hall, New Jersey, U.S.A, (1964).

[29] Verma, R., *New Trends in Fractional Programming*, Nova Science Publishers, New York, USA, (2019).

Chapter 38

On the Semi-Local Convergence of King's Fourth Order Method for Solving Equations

We provide a semi-local convergence analysis of King's fourth order method for solving nonlinear equations using majorizing sequences. Earlier studies on the local convergence have used conditions on the fifth derivative which is not on the method. But we only use the first derivative. Hence we extend the usage of this method in the more interesting semi-local convergence case.

1. Introduction

A Plethora of problems in diverse disciplines such as Applied Mathematics, Mathematical Programming, Economics, Physics, Engineering, and Transport theory to mention a few can be formulated as

$$F(x) = 0, \tag{38.1}$$

where $F : \Omega \subset S \to T$ Ω open, $S = \mathbb{R}$ or $T = \mathbb{C}$. Iterative methods are mostly used to generate a sequence converging to a solution x^* of (38.1). Consider

King's Fourth Order Method

$$y_n = x_n - F'(x_n)^{-1}F(x_n)$$

and (38.2)

$$x_{n+1} = y_n - A_n^{-1}B_nF'(x_n)^{-1}F(y_n),$$

where (38.3)

$$A_n = F(x_n) + (\gamma - 2)F(y_n)$$
$$B_n = F(x_n) + \gamma F(y_n), \gamma \in T.$$

In this chapter, we study the more interesting semi-local case. Moreover, we use conditions only on the first derivative appearing on (38.3). Hence, we extend its applicability. The local convergence of this method was shown [6, 8, 9] using conditions reaching the fifth derivative which is not on (38.3).

But these restrictions limit the applicability of the method (38.3) although they may converge.

For example: Let $S = T = \mathbb{R}$, $\Omega = [-0.5, 1.5]$. Define Ψ on Ω by

$$\Psi(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\, t \neq 0 \\ 0 & if\, t = 0. \end{cases}$$

Then, we get $t^* = 1$, and

$$\Psi'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

Obviously $\Psi'''(t)$ is not bounded on Ω. Hence, the results in [6, 8, 9] utilizing the fifth derivative in the local case cannot guarantee convergence. We use only conditions on the first derivative and in the more interesting semi-local case. The technique is very general. Hence, it can be used to extend the applicability of other methods [1, 2, 3, 4, 5, 6, 7, 8, 9]. Majorizing sequences for method (38.3) are given in Section 2, the analysis of the method in Section 3, and the examples in Section 4.

2. Scalar Majorizing Sequences

Let $L_0, L, L_1, L_2, L_3, L_4, L_5 > 0$ and $\eta \geq 0$ be parameters. Set $L_6 = \frac{L}{2}, L_7 = L_1 L_5 + L L_4, L_8 = \frac{|\gamma|}{2} L L_5$ and $L_9 = \frac{|\gamma|}{2} L_3$. Define scalar sequences $\{t_n\}, \{s_n\}$ by $t_0 = 0, t_1 = \eta, s_1 = \eta + L_6 (L_3 + L_9 \eta) \eta^3$,

$$\begin{aligned} s_{n+1} &= t_{n+1} + \frac{\alpha_{n+1}(t_{n+1} - s_n)}{1 - L_0 t_{n+1}}, \\ t_{n+2} &= s_{n+1} + \frac{L_6 (L_3 + L_9 (s_{n+1} - t_{n+1}))(s_{n+1} - t_{n+1})^3}{(1 - p_{n+1})(1 - L_0 t_{n+1})}, \end{aligned} \tag{38.4}$$

where,

$$\begin{aligned} p_n &= L_3 (t_n + |\gamma - 2| (s_n - \eta)), \\ b_n &= \frac{1}{1 - \gamma_n}, \\ \gamma_n &= L_4 (t_n + |\gamma| (s_n - \eta)), \\ \alpha_{n+1} &= L_6 (t_{n+1} - s_n) + b_n (L_7 + L_8 (s_n - t_n))(s_n - t_n)^2 \end{aligned}$$

and

$$\overline{\alpha_{n+1}} = L_6 (t_{n+1} - s_n) + M (s_n - t_n),$$

where, $M = 2(L_7 + L_8 \eta)\eta$. Sequence $\{t_n\}$ shall be shown in Section 3 to be majorizing for method (38.3).

Next, we present a general convergence result for a sequence $\{t_n\}$.

Lemma 42. Suppose,

$$L_0 t_{n+1} < 1, p_{n+1} < 1 \text{ and } \gamma_n < 1. \tag{38.5}$$

Then, the following assertions hold for sequence $\{t_n\}$

$$0 \leq t_n \leq s_n \leq t_{n+1} \tag{38.6}$$

$$\text{and } \lim_{n\to\infty} t_n = t^*, \tag{38.7}$$

where t^* is the least upper bound of sequence $\{t_n\}$ satisfying $t^* \in [\eta, \frac{1}{L_0}], (L_0 \neq 0)$.

Proof. It follows by the definitions of sequences $\{t_n\}$ and (38.5) that it is non-decreasing and bounded from above by $\frac{1}{L_0}$ and as such it converges to t^*. □

Next, we present another convergence result for a sequence $\{t_n\}$ under stronger than (38.5) conditions but which are easier to verify.
But first, it is convenient to define polynomials on the interval $[0,1)$ by

$$\begin{aligned} h_n^{(1)}(t) &= L_6 t^n \eta + M t^{n-1}\eta + L_0(1+t+\ldots+t^{k+1})\eta - 1, \\ g_1(t) &= L_6 t^2 - L_6 t + Mt - M + L_0 t^3, \\ h_n^{(2)}(t) &= N t^{n+1}\eta + L_0(1+t+\ldots+t^{n+1})\eta - 1 \end{aligned}$$

and

$$g_2(t) = (N+L_0)t - N,$$

where $N = 2L_6(L_3 + L_9\eta)\eta$.
Notice that $g_1(0) = -M$ and $g_1(1) = L_0$. Then, it follows by the intermediate value theorem that polynomial g_1 has zeros in $(0,1)$. Denote by δ_1 the smallest such zero and let $\delta_2 = \frac{N}{N+L_0}$. Parameters M and N can be chosen to be independent of η as follows. Pick $\varepsilon_1, \varepsilon_2 \in (0,1)$ such that

$$M \leq \varepsilon_1 \text{ and } N \leq \varepsilon_2.$$

Then, δ_1 and δ_2 become independent of η. Moreover, set

$$a = \frac{\alpha_1(t - s_0)}{1 - L_0 t}, b = L_6(L_3 + L_0\eta)\eta^2,$$

$$c = max\{a,b\}, \delta_3 = min\{\delta_1, \delta_2\}, \delta = max\{\delta_1, \delta_2\},$$

$$\lambda_1 = 1 - L_0\eta, \lambda_2 = \frac{1 - 2L_4\eta}{1 + 2L_4\eta|\gamma|}, \lambda_3 = \frac{1 - 2L_3\eta}{1 + 2L_3\eta|\gamma - 2|}$$

and $\lambda = min\{\lambda_1, \lambda_2, \lambda_3\}$.

Next, use this terminology to show a second convergence for a sequence $\{t_n\}$. But first it is convenient to define polynomials on the interval $[0,1)$ by

$$h_n^{(1)}(t) = L_6 t^n \eta + M t^{n-1}\eta + L_0(1+t+\ldots+t^{n+1})\eta - 1.$$

Lemma 43. Suppose

$$c \le \delta_3 \le \delta < \lambda < 1. \tag{38.8}$$

Then, the conclusions of Lemma 42 hold for sequence $\{t_n\}$, where $t^* \in [\eta, \frac{\eta}{1-\delta}]$. Moreover, the following estimates hold

$$0 \le t_{n+1} - s_n \le \delta(s_n - t_n) \le \delta^{2n+1}(s_0 - t_0) = \delta^{2n+1}\eta \tag{38.9}$$

and

$$0 \le s_n - t_n \le \delta(t_n - s_{n-1}) \le \delta^{2n}\eta. \tag{38.10}$$

Proof. Mathematical Induction is used to show

$$0 \le \frac{\alpha_{k+1}}{1 - L_0 t_{k+1}} \le \delta, \tag{38.11}$$

$$0 \le \frac{L_6(L_3 + L_9(s_{k+1} - t_{k+1}))(s_{k+1} - t_{k+1})^2}{(1 - p_{k+1})(1 - L_0 t_{k+1})} \le \delta, \tag{38.12}$$

$$2L_4(t_k + s_k + |\gamma|(s_k - \eta)) \le 1 \tag{38.13}$$

and

$$2L_3(t_k + |\gamma - 2|(s_k - t_k)) \le 1. \tag{38.14}$$

If $k = 0$ estimates (38.11), (38.12) hold by the choice of $a, b, \lambda_2, \lambda_3$, respectively. Notice that by (38.13) and (38.14) , $\frac{1}{1 - p_{k+1}} \le 2$ and $b_k \le 2$.
So, 38.9 and 38.10 hold for $k = 0$. Suppose these estimates hold for all integers smaller than n. Then, by (38.9) and (38.10), we have

$$\begin{aligned} s_k &\le t_k + \delta^{2k}\eta \le s_{k-1} + \delta^{2k-1}\eta + \delta^{2k}\eta \\ &\le \cdots \le \eta + \delta\eta + \cdots + \delta^{2k}\eta = \frac{1 - \delta^{2k+1}}{1 - \delta}\eta < \frac{\eta}{1 - \delta} = t^{**} \end{aligned} \tag{38.15}$$

and

$$\begin{aligned} t_{k+1} &\le s_k + \delta^{2k+1}\eta \le t_k + \delta^{2k}\eta + \delta^{2k+1}\eta \\ &\le \cdots \le \eta + \delta\eta + \cdots + \delta^{2k+1}\eta = \frac{1 - \delta^{2k+2}}{1 - \delta}\eta < t^{**}. \end{aligned} \tag{38.16}$$

Then, by the induction hypotheses and since $\delta \in [0, 1)$, (38.11) certainly holds if

$$0 \le \frac{L_6(t_{k+1} - s_k) + M(s_k - t_k)}{1 - L_0 t_{k+1}} \le \delta \tag{38.17}$$

or since $\delta \in [0, 1)$,

$$L_6\delta^{2k+1}\eta + M\delta^{2k}\eta + L_0\delta\frac{1 - \delta^{2k+2}}{1 - \delta}\eta - \delta \le 0$$

or

$$h_k^1(t) \le 0 \text{ at } \quad t = \delta_1. \tag{38.18}$$

A relationship between two consecutive polynomials h_k^1 is needed. By the definition of these polynomials

$$\begin{aligned} h_{k+1}^1(t) &= L_6 t^{k+1}\eta + M t^k \eta + L_0(1+t+\ldots+t^{k+2})\eta - 1 \\ &+ h_k^1(t) - L_6 t^k \eta - M t^{k-1}\eta - L_0(1+t+\ldots t^{k+1})\eta + 1 \\ &= h_k^1(t) + g_1(t) t^{k-1}\eta. \end{aligned} \tag{38.19}$$

In particular, we have

$$h_{k+1}^1(\delta_1) = h_k^1(\delta_1) \tag{38.20}$$

by the definition of δ_1.
Define function

$$f_\infty^1(t) = \lim_{k\to\infty} h_k^1(t). \tag{38.21}$$

By the definition of h_k^1 and (38.21) we get,

$$f_\infty^1(t) = \frac{L_0\eta}{1-t} - 1. \tag{38.22}$$

Hence, (38.18) holds if $f_\infty^1(t) \le 0$ at $t = \delta_1$, which holds by (38.8). Similarly, (38.12) holds provided that

$$\frac{2L_6(L_3 + L_9(s_{k+1} - t_{k+1}))(s_{k+1} - t_{k+1})^2}{1 - L_0 t_{k+1}} \le \delta \tag{38.23}$$

or

$$\frac{N(s_{k+1} - t_{k+1})}{1 - L_0 t_{k+1}} \le \delta$$

or

$$N\delta^{2k+2}\eta + \delta L_0\left(\frac{1-\delta^{2k+2}}{1-\delta}\right)\eta - \delta \le 0$$

or

$$h_k^2(t) \le 0 \ \text{ at } \ t = \delta_2 \tag{38.24}$$

This time we have,

$$\begin{aligned} h_{k+1}^2(t) &= h_{k+1}^2(t) - h_k^2(t) + h_k^2(t) \\ &= h_k^2(t) + N t^{k+2}\eta + L_0(1+t+\ldots+t^{k+2})\eta - 1 \\ &- N t^{k+1}\eta - L_0(1+t+\ldots+t^{k+1})\eta + 1 \\ &= h_k^2(t) + g_2(t) t^{k+1}\eta, \end{aligned} \tag{38.25}$$

Hence, we get

$$h_{k+1}^2(t) = h_k^2(t) + g_2(t) t^{k+1}\eta. \tag{38.26}$$

In particular, we have

$$h_{k+1}^2(\delta_2) = h_k^2(\delta_2)$$

by the definition of δ_2. Define function

$$h_\infty^2(t) = \lim_{k\to\infty} h_k^2(t).$$

Then, we get

$$h_\infty^2(t) = \frac{L_0\eta}{1-t} - 1.$$

So, $h_\infty^2(t) \le 0$ at $t = \delta_2$ holds if $h_\infty^2(t) \le 0$ at $t = \delta_2$ which is true by (38.8).
Instead of (38.13), we can show

$$2L_4(\frac{\eta}{1-\delta} + |\gamma|(\frac{\eta}{1-\delta}) \le 1,$$

or $\delta \le \lambda_2$, which is true by (38.8). Similarly, we can show instead of (38.14)

$$2L_3(\frac{\eta}{1-\delta} + |\gamma - 2|(\frac{\eta}{1-\delta})) \le 1,$$

or $\delta \le \lambda_3$, which is also true by (38.8). The induction for items (38.11) - (38.14) is completed. It follows that sequence $\{t_n\}$ is non decreasing and bounded above by $\frac{\eta}{1-\delta}$ and as such it converges to t^*. □

3. Analysis

The semi-local convergence analysis of method (38.3) is based on majorizing sequence $\{t_n\}$ and hypothesis C.
Suppose :

(C1) There exist $x_0 \in \Omega, \eta \ge 0, b > 0$ such that $F'(x_0)^{-1} \in L(Y,X), B_0^{-1} \in L(Y,X), \|B_0^{-1}\| \le b$ and $\|F'(x_0)^{-1}F(x_0)\| \le \eta$.

(C_2) There exists $L_0 > 0$ such that for all $w \in \Omega$,

$$\|F'(x_0)^{-1}(F'(w) - F'(x_0))\| \le L_0\|w - x_0\|.$$

Define $\Omega_0 = U(x_0, \frac{1}{L_0}) \cap \Omega$, $(L_0 \neq 0)$.

(C3) There exists $L, L_1, L_2, L_3, L_4, L_5 > 0$ such that for all $z, w \in \Omega_0$

$$\|(F'(z) - F'(w))\| \le L\|z - w\|,$$

$$\|F'(x_0)^{-1}F'(z)\| \le L_1,$$

$$\|F'(x_0)^{-1}(F'(z) - F'(w))\| \le L_2\|z - w\|,$$

$$\|A_0^{-1}F'(z)\| \le L_3,$$

$$\|B_0^{-1}F'(z)\| \le L_4$$

and

$$\|B_0^{-1}(F'(z) - F'(w))\| \le L_5\|z - w\|.$$

(C4) Hypotheses of lemma 42 or lemma 43 hold.

(C_5) $U[x_0,t^*] \subseteq \Omega$ (or $U[x_0,t^**] \subset \Omega$),where $t^** = \frac{1}{L_0}$ (or $t^{**} = \frac{\eta}{1-\eta}$) in the case of Lemma 42 (or Lemma 43)

4. Convergence Result for Method 38.3

Theorem 44. Suppose conditions C hold.Then,iteration $\{x_n\}$ generated by method(38.3) exists in $U(x_0,t^*)$, stays in $U[x_0,t^*]$ for all $n = 0,1,2,\ldots$ and converges to a solution $x^* \in U[x_0,t^*]$ of equation $F(x) = 0$ so that

$$\|x^* - x_n\| \leq t^* - t_n \tag{38.27}$$

Proof. Estimates

$$\|y_k - x_k \leq s_k - t_k \tag{38.28}$$

and

$$\|x_{k+1} - y_k\| \leq t_{k+1} - s_k \tag{38.29}$$

are shown using induction.It follows from (C_1) and (38.4) for $n = 0$ that

$$\|y_0 - x_0\| = \|F'(y_0)^{-1}F(x_0)\| \leq \eta = s_0 - t_0 = s_0 \leq t^*,$$

so (38.28) holds for $k = 0$ and $y_0 \in U(x_0,t^*)$. Suppose (38.28) holds for integer values up to k. We can write

$$\begin{aligned} F(x_n) &= -F'(x_k)(y_k - x_k) \\ F(y_k) &= F(y_k) - F(x_k) - F'(x_k)(y_k - x_k). \end{aligned}$$

Hence, by (C_3), we get

$$\begin{aligned} \|F'(x_0)^{-1}F(y_k)\| &\leq \|\int_0^1 F'(x_0)^{-1}\left(F'(x_k + \theta(y_k - x_k)) - F'(x_k)\right) d\theta(y_k - x_k)\| \\ &\leq \frac{L}{2}\|y_k - x_k\|^2 \\ &\leq \frac{L}{2}(s_k - t_k)^2. \end{aligned} \tag{38.30}$$

By the definition of A_k and (C_3),we have

$$\begin{aligned}
\|A_0^{-1}(A_k - A_0)\| &\leq \|A_0^{-1}(F(x_k) + (\gamma-2))F(y_k) - F(x_0) - (\gamma-2)F(y_0)\| \\
&= \|A_0^{-1}[(F(x_k) - F(x_0)) + (\gamma-2)(F(y_k - F(y_0)))]\| \\
&\leq \|\int_0^1 A_0^{-1}(F'(x_0 + \theta(x_k - x_0)))\| \\
&\quad + |\gamma-2|\|\int_0^1 A_0^{-1}(F'(y_0 + \theta(y_k - y_0)))d\theta(y_k - y_0)\| \\
&\leq L_3(\|x_k - x_0\| + |\gamma-2|\|y_k - y_0\|) \\
&\leq L_3(t_\alpha - t_0 + |\gamma-2|(s_k - s_0)) \\
&\leq L_3(t^* + |\gamma-2|(t^* - \eta)) \\
&= L_3\left(\frac{\eta}{1-\delta} + |\gamma-2|\left(\frac{\eta}{1-\delta} - \eta\right)\right) \\
&< 1. \qquad (38.31)
\end{aligned}$$

It follows from Banach Lemma on invertible functions [1, 2, 3, 4] and (38.31) that $A_k \neq 0$ and

$$\|A_k^{-1}A_0\| \leq \frac{1}{1-p_k}. \qquad (38.32)$$

Using (C_1) and (C_2) for $z \in U(x_0, t^*)$, we get

$$\|F'(x_0)^{-1}(F'(z) - F'(x_0))\| \leq L_0\|z - x_0\| < L_0 t^* < 1,$$

so

$$\|F'(z)^{-1}F'(x_0)\| \leq \frac{1}{1 - L_0\|z_0 - x_0\|}. \qquad (38.33)$$

It follows from (38.33) (for $z = x_0$),(38.32),(C_3),(38.30) and (1.2) that

$$\begin{aligned}
\|x_{k+1} - y_k\| &\leq \|A_k^{-1}A_0\|[\|A_0^{-1}F(x_k)\| + |\gamma|\|A_0^{-1}F(y_k)]\| \qquad (38.34) \\
&\quad \|F'(x_k)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_k)\| \\
&\leq \frac{\left(L_3\|y_k - x_k\| + \frac{|\gamma|}{2}L_3\|y_k - x_k\|^2\right)\frac{L}{2}\|y_k - x_k\|^2}{(1-p_k)(1-L_0 t_k)} \\
&\leq \frac{L_6(L_3(s_k - t_k) + L_0(s_k - t_k)^2)(s_k - t_k)^2}{(1-p_k)(1-L_0 t_k)} \\
&= t_{k+1} - s_k, \qquad (38.35)
\end{aligned}$$

where we also used method (38.3) to obtain the estimate

$$F(x_k) = F'(x_k)(y_k - x_k).$$

So

$$\begin{aligned}
\|A_0^{-1}F(x_k)\| &= \|A_0^{-1}F'(x_k)(y_k - x_k)\| \\
&\leq \|A_0^{-1}F'(x_k)\|\|y_n - x_n\| \\
&\leq L_3\|y_k - x_k\| \\
&= L_3(s_k - t_k). \qquad (38.36)
\end{aligned}$$

We also have,

$$\|x_{k+1}-x_0\| \leq \|x_{k+1}-y_k\|+\|y_k-x_0\| \leq t_{k+1}-s_k+s_k-t_0=t_{k+1}<t^*, \qquad (38.37)$$

so $x_{k+1} \in U(x_0,t^*)$.By (C_3) we get as in (38.31) estimate

$$\begin{aligned}
\|B_0^{-1}(B_k-B_0)\| &\leq \|B_0^{-1}(F(x_k)-F(x_0))\|+|\gamma|\|B_0^{-1}(F'(y_k)-F'(y_0))\| \\
&\leq L_4(t_k+|\gamma|(s_k-s_0)) \\
&\leq L_4\left(\frac{\eta}{1-\delta}+|\gamma|\left(\frac{\eta}{1-\delta}-\eta\right)\right) \\
&< 1.
\end{aligned}$$

That is

$$\|B_k^{-1}B_0\| \leq \frac{1}{1-\gamma_k}=b_k. \qquad (38.38)$$

By method (1.2), we get

$$\begin{aligned}
F(x_{k+1}) &= F(x_{k+1})-F(y_k)-F'(y_k)(x_{k+1}-y_k) \\
&\quad +F'(y_k)(x_{k+1}-y_k)-A_nB_n^{-1}F'(x_k)(x_{k+1}-y_k) \\
&= \int_0^1 (F'(y_k+\theta(x_{k+1}-y_k))d\theta-F'(y_k))(x_{k+1}-y_k) \\
&\quad +B_k^{-1}(B_kF'(y_k)-A_kF'(x_k))(x_{k+1}-y_k). \qquad (38.39)
\end{aligned}$$

We can write

$$\begin{aligned}
T_k &:= B_kF'(y_k)-A_kF'(x_k) \\
&= (F(x_k)+\gamma F(y_k))F'(y_k)-(F(x_k)+(\gamma-2)F(y_k))F'(x_k) \\
&= F(x_k)(F'(y_k)-F'(x_k))+\gamma F(y_k)(F'(y_k)-F'(x_k)) \\
&\quad +2F(y_k)F'(x_k). \qquad (38.40)
\end{aligned}$$

It then follows from (C_3),(38.28), and (38.40) that

$$\begin{aligned}
\|F'(x_0)^{-1}B_0^{-1}T_k\| &\leq \|F'(x_0)^{-1}F_(x_k)\|\|B_0^{-1}(F'(y_k)-F'(x_k))\| \\
&\quad +|\gamma|\|F'(x_0)^{-1}F(y_k)\|\|B_0^{-1}(F'(y_k)-F'(x_k))\| \\
&\quad +2\|F'(x_0)^{-1}F(y_k)\|\|B_0^{-1}F'(x_k)\| \\
&\leq L_1L_5\|y_k-x_k\|^2+\frac{|\gamma|L}{2}L_5\|y_k-x_k\|^3 \\
&\quad +2\frac{LL_4}{2}\|y_k-x_k\|^2 \\
&\leq (L_7+L_8(s_k-t_k))(s_k-t_k)^2. \qquad (38.41)
\end{aligned}$$

It then follows from (38.30),(38.33),(38.39),(38.4) and (38.41) that

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{k+1})\| &\leq \frac{L}{2}\|y_{k+1}-x_{k+1}\|^2+\|F'(x_0)^{-1}B_0^{-1}T_k\|\|x_{k+1}-y_k\| \\
&\leq (\frac{L}{2}(s_{k+1}-t_{k+1})+(L_7+L_8(s_k-t_k))(s_k-t_k)^2) \\
&\quad \times(s_{k+1}-t_{k+1})^2
\end{aligned}$$

(38.42)

Hence we have by (38.4),(38.33) and (38.42)

$$\|y_{k+1}-x_{k+1}\| \leq \|F'(x_{k+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{k+1})\| \leq s_{k+1}-t_{k+1}. \tag{38.43}$$

Moreover, we can obtain,

$$\|y_{k+1}-x_0\| \leq \|y_{k+1}-x_{k+1}\|+\|x_{k+1}-x_0\| \leq s_{k+1}-t_{k+1}+t_{k+1}-t_0 = s_{k+1} < t^*. \tag{38.44}$$

So, $y_{k+1} \in \cup(x_0,t^*)$. The induction for $x_k, y_k \in \cup(x_0,t^*)$, and (38.29) is completed. Sequence $\{t_k\}$ is Cauchy is convergent. Therefore, sequence $\{x_k\}$ is Cauchy too in S, so it converges to some $x^* \in \cup[x_0,t^*]$. (Since $\cup[x_0,t^*]$ is a closed set.) By letting $k \to \infty$ in (38.42) we get $F(x^*) = 0$. □

Next, the uniqueness of the solution x^* result is provided.

Proposition 26. *Suppose,*

(i) *The point $x^* \in U(x_0,r) \subset \Omega$ is a solution of equation $F(x) = 0$ for some $r > 0$.*

(ii) *Condition (C_2) holds.*

(iii) *There exists $r_1 \geq r$ such that*

$$L_0(r_1+r) < 2. \tag{38.45}$$

Set $\Omega_1 = U(x_0,r_1) \cap \Omega$. Then, the only solution of equation $F(x) = 0$ in Ω_1 is x^*.

Proof. Define $Q = \int_0^1 F'(x^*+\theta(\bar{x}-x^*))d\theta$. Using (C_2) and (38.45), we obtain in turn that

$$\begin{aligned} \|F'(x_0)^{-1}(Q-F'(x_0))\| &\leq \int_0^1 ((1-\theta)\|x^*-x_0\|+\theta\|y^*-x_0\|)d\theta \\ &\leq \frac{L_0}{2}(r+r_1) < 1, \end{aligned}$$

So $\bar{x} = x^*$, since $Q \neq 0$ and $Q(\bar{x}-x^*) = F(\bar{x})-F(x^*) = 0$. □

Remark. We only assumed (C2) from conditions C. But if all conditions are assumed, then, we can set $r = t^*$.

5. Applications

Example 10. Let $S = T = \mathbb{R}, U = [q, 2-q]$ for $q \in M = (0,1)$, $q = 1$ and $t_0 = 1$. Define real function F on Ω as

$$F(t) = t^3 - q. \tag{38.46}$$

Then, the parameters are $\eta = \frac{1}{3}(1-q)$, $L_0 = 3-q$. Moreover, one gets $\Omega_0 = U(1,1-q) \cap U(1,\frac{1}{L_0}) = U(1,\frac{1}{L_0})$, so $L = 6(1+\frac{1}{3-q})$.

Set $\gamma = 0$, $b = (1-q)^{-1}$ then

$L_1 = (2-q)^2, L_2 = L/3, L_3 = 3(2-q)^2[1-q-2((2+q)/3))^3-1)]^{-1}, L_4 = 3b(2-q)^2, L_5 = Lb.$

Then for $q = 0.98$, we have

Hence, the conditions of Lemma 42 and Theorem 44 hold.

Table 38.1. Sequence (38.4) and condition (38.5)

n	1	2
s_{n+1}	0.0067	0.0067
t_{n+2}	0.0067	0.0067
$L_0 t_{n+1}$	0.0069	0.0069
p_{n+1}	0.0	0.0
γ_n	0.0	0.0

Bibliography

[1] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics*, 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[2] Argyros I.K., Hilout S., Weaker conditions for the convergence of Newto's method, *J. Complexity*, 28, 3, (2012), 364-387.

[3] Argyros I.K., Hilout S., On an improved convergence analysis of Newton's method, *Applied Mathematics and Computation*, 225, (2013), 372-386.

[4] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative procedures*, Elsevier (Academic Press), New York, 2018.

[5] Argyros, I.K., George, S., *Mathematical modeling for the solution of equations and systems of equations with applications*, Volume-IV, Nova Science Publishers, NY, 2021.

[6] Chun, C., Neta, B., Comparative study of eight order methods for finding simple roots of nonlinear equations, *Numer. Algor.*, 74, (2017), 1169 -1201.

[7] Kantorovich, L.V., Akilov, G.P., *Functional analysis in normed Spaces*, The Macmillan Co, New York, (1964).

[8] Neta, B., Chun, C., Scott, M., Basins of attractions for optimal eight order methods to find simple roots of nonlinear equations, *Appl. Math. Comp.*, 227, (2014), 567-592.

[9] Petkovic, M.S., Neta, B., Petkovic, L.D., Dzonic, J., *Multipoint methods for solving nonlinear equations*, Elsevier, New York,(2013).

Chapter 39

On the Convergence of Two Eighth Order Methods with Divided Differences and Derivatives

1. Introduction

We solve the nonlinear equation

$$F(x) = 0, \tag{39.1}$$

where $F : D \subset X \longrightarrow Y$ is a Fréchet-differentiable operator, X, Y are Banach spaces and D is a nonempty open and convex set. A solution x^* of the equation (40.1) is found using the two three step iterative methods defined for all $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - (3I - 2F'(x_n)^{-1}[y_n, x_n; F])F'(x_n)^{-1}F(y_n), \\ C_n &= 2[z_n, y_n; F] - [z_n, x_n; F], \\ D_n &= F'(x_n) - [y_n, x_n; F] + [z_n, y_n; F] \end{aligned} \tag{39.2}$$

and

$$x_{n+1} = z_n - C_n^{-1}D_nF'(x_n)^{-1}F(z_n)$$

and

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - A_n^{-1}F(y_n), A_n = 2[y_n, x_n; F] - F'(x_n), \end{aligned} \tag{39.3}$$

and

$$x_{n+1} = z_n - [z_n, x_n; F]^{-1}[z_n, y_n; F]B_n^{-1}F(z_n),$$

where

$$B_n = 2[z_n, y_n; F] - [z_n, x_n; F].$$

The eighth order of local convergence for these methods was established in [11, 12], respectively, when $X = Y = \mathbb{R}$. But condition on the ninth derivative were used not on these methods. We only use conditions on the operators appearing on these methods and in a Banach space setting.

Moreover, computable error bounds and uniqueness of the solution results not given earlier are also provided. Hence, we extend the applicability of these methods in both the local and the semi-local convergence (which is also not given before).

We use the same notation for the majorizing "w" and φ" in both types of convergence. But notice that in the local case they relate to x^* whereas in the semi-local case they relate to x_0.

2. Local Convergence

The auxiliary functions are needed first for the method (40.2):

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)},$$

$$g_2(t) = \left[1 + \frac{(1+2\bar{\varphi}(t))(1+\int_0^1 w_0(\theta g_1(t)t)d\theta)}{1-w_0(t)}\right] g_1(t),$$

$$\begin{aligned} g_3(t) \quad = \quad & \left[\frac{\int_0^1 w((1-\theta)g_2(t)t)d\theta}{1-w_0(g_2(t)t)} + \frac{\bar{w}(t)}{(1-w_0(t))(1-w_0(g_2(t)t))}\right. \\ & \left. + \frac{a(t)(1+\int_0 61 w_0(\theta g_2(t)t)d\theta)}{(1-b(t))(1-w_0(t))}\right] g_2(t), \end{aligned}$$

where

$$\bar{\varphi}(t) = \begin{cases} \varphi_1(t, g_1(t)t) \\ w_0(t) + \varphi_0(t, g_1(t)t), \end{cases}$$

$$\bar{w}(t) = \begin{cases} w(1+g_1(t)t) \\ w_0(t) + w_0(g_1(t)t), \end{cases}$$

$$a(t) = \varphi(t, g_1(t)t, g_2(t)t + \varphi_1(t, g_1(t)t)$$

and

$$b(t) = \varphi(t, g_1(t)t, g_2(t)t) + \varphi_0(g_1(t)t, g_2(t)t).$$

Suppose that there exists a smallest positive number ρ such that

$$0 \le w_0(t) < 1,$$

$$0 \le b(t) < 1$$

and

$$0 \le w_0(g_2(t)t) < 1$$

for all $t \in [0, \rho)$. Then, the functions g_i, $i = 1, 2, 3$ are well defined on the interval $[0, \rho)$. Moreover, suppose that there exist smallest positive solutions of the equations

$$g_i(t) - 1 = 0$$

in the interval $[0,\rho)$. Denote such solutions by r_i, respectively. Then, the parameter r efined by

$$r = \min\{r_i\} \tag{39.1}$$

shall be shown to be a radius of convergence for the method (40.2). It follows by this definition that for all $t \in [0,r)$

$$0 \leq g_i(t) < 1. \tag{39.2}$$

Notice that in practice the smallest version of the bar functions shall be used.

The motivational estimates for the definition of the functions g_i are given, respectively by

$$\begin{aligned}
\|y_n - x^*\| &\leq \|F'(x_n)^{-1}F'(x^*)\| \\
&\quad \times \|\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_n - x^*))d\theta - F'(x_n))(x_n - x^*)\| \\
&\leq \frac{\int_0^1 w((1-\theta)\|x_n - x^*\|)d\theta\|x_n - x^*\|}{1 - w_0(\|x_n - x^*\|)} \\
&= g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \\
\|z_n - x^*\| &= \|y_n - x^* - 2F'(x_n)^{-1}(F'(x_n) - [y_n, x_n; F])F'(x_n)^{-1}F(y_n)\| \\
&\leq [1 + \frac{(1+2\bar{\varphi}_n)(1 + \int_0^1 w_0(\theta\|y_n - x^*\|)d\theta)}{1 - w_0(\|x_n - x^*\|)}]\|y_n - x^*\| \\
&\leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|,
\end{aligned}$$

$$\begin{aligned}
C_n - D_n &= 2[z_n, y_n; F] - [z_n, x_n; F] - F'(x_n) - [y_n, x_n; F] - [z_n, y_n; F] \\
&= [z_n, y_n; F] + [y_n, x_n; F] - [z_n, x_n; F] - F'(x_n), \\
\|F'(x^*)^{-1}(C_n - D_n)\| &\leq \varphi(\|x_n - x^*\|, \|y_n - x^*\|, \|z_n - x^*\|) \\
&\quad + \varphi_1(\|x_n - x^*\|, \|y_n - x^*\|) + \bar{a}_n \leq a_n, \\
&\quad \|F'(x^*)^{-1}(2[z_n, y_n; F] - [z_n, x_n; F] - F'(x^*))\| \\
&\leq \varphi(\|x_n - x^*\|, \|y_n - x^*\|, \|z_n - x^*\|) \\
&\quad + \varphi_0(\|y_n - x^*\|, \|z_n - x^*\|) = \bar{b}_n = b_n < 1
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|z_n - x^* - F'9z_n)^{-1} + (F'(z_n)^{-1} - F'(x_n)^{-1})F(z_n)\| \\
&\quad + C_n^{-1}(C_n - D_n)F'(x_n)^{-1}F(z_n)\| \\
&\leq \left[\frac{\int_0^1 w((1-\theta)\|z_n - x^*\|)d\theta}{1 - w_0(\|z_n - x^*\|)} + \frac{\bar{w}_n}{(1 - w_0(\|x_n - x^*\|))(1 - w_0(\|z_n - x^*\|))}\right. \\
&\quad \left. + \frac{\bar{a}_n(1 + \int_0^1 w_0(\theta\|z_n - x^*\|)d\theta)}{(1 - b_n)(1 - w_0(\|x_n - x^*\|))}\right] \|z_n - x^*\| \\
&\leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|.
\end{aligned}$$

Hence, we arrive at:

Theorem 45. Under the conditions stated previously, the method (40.2) is convergent to x^*.

Similarly, we present the local convergence of the method (39.3). The functions g_2 and g_3 are defined by

$$\begin{aligned} g_2(t) &= \left[\frac{\int_0^1 w((1-\theta)g_1(t)t)d\theta}{1-w_0(g_1(t)t)}\right. \\ &\quad \left. + \frac{\alpha(t)(1+\int_0^1 w_0(\theta g_1(t)t)d\theta)}{(1-b(t))(1-w_0(g_1(t)t))}\right] g_1(t), \\ g_3(t) &= \left[1+\frac{\gamma(t)(1+\int_0^1 w_0(\theta g_2(t)t)d\theta)}{(1-h(t))(1-\varphi_0(t,g_2(t)t))}\right] g_2(t), \end{aligned}$$

where

$$\beta(t) = \varphi_1(t,g_1(t)t)+\varphi_0(t,g_1(t)t),$$
$$\alpha(t) = 2\varphi_1(t,g_1(t)t),$$
$$\gamma(t) = 1+\varphi_0(t,g_2(t)t)$$

and

$$h(t) = \varphi(t,g_1(t)t,g_2(t)t)+\varphi_0(g_1(t)t,g_2(t)t).$$

This time the conditions are: There exist a smallest positive number ρ such that

$$\begin{aligned} 0 &\le w_0(t) < 1, \\ 0 &\le w_0(g_1(t)t) < 1, \\ 0 &\le \beta(t) < 1, \\ 0 &\le \varphi_0(t,g_2(t)t) < 1 \end{aligned}$$

and

$$0 \le h(t) < 1$$

for all $t \in [0,\rho)$.

The motivation is respectively

$$\begin{aligned} \|z_n-x^*\| &= \|y_n-x^*-F'(y_n)^{-1}F(y_n)+F'(y_n)^{-1}(A_n-F'(y_n))F(y_n)\| \\ &\le \left[\frac{\int_0^1 w((1-\theta)\|y_n-x^*\|)d\theta}{1-w_0(\|y_n-x^*\|)}\right. \\ &\quad \left. + \frac{\bar{\alpha}_n(1+\int_0^1 w_0(\theta\|y_n-x^*\|)d\theta}{(1-\bar{\beta}_n)(1-w_0(\|y_n-x^*\|))}\right] \|y_n-x^*\| \\ &\le g_2(\|x_n-x^*\|)\|x_n-x^*\| \le \|x_n-x^*\| < r, \\ \|x_{n+1}-x^*\| &\le \left[1+\frac{\bar{\gamma}_n(1+\int_0^1 w_0(\theta\|z_n-x^*\|)d\theta}{(1-\bar{\beta}_n)(1-\varphi_0(\|x_n-x^*\|,\|z_n-x^*\|))}\right] \|z_n-x^*\| \\ &\le g_3(\|x_n-x^*\|)\|x_n-x^*\| \le \|x_n-x^*\|, \end{aligned}$$

where we also used

$$\begin{aligned}
A_n - F'(y_n) &= 2[y_n, x_n; F] - F'(x_n) - F'(y_n), \\
\|F'(x^*)^{-1}(A_n - F'(x^*))\| &\leq \varphi_1(\|x_n - x^*\|, \|y_n - x^*\|) \\
&\quad + \varphi_0(\|x_n - x^*\|, \|y_n - x^*\|) = \bar{\beta}_n < 1, \\
\|F'(x^*)^{-1}(A_n - F'(y_n))\| &\leq 2\varphi_1(\|x_n - x^*\|, \|y_n - x^*\|)\bar{\alpha}_n \\
\|F'(x^*)^{-1}[z_n, x_n; F]\| &\leq 1 + \varphi_0(\|x_n - x^*\|, \|z_n - x^*\|) = \bar{\gamma}_n
\end{aligned}$$

and

$$\begin{aligned}
\|F'(x^*)^{-1}B_n\| &\leq \varphi(\|x_n - x^*\|, \|y_n - x^*\|, \|z_n - x^*\|) \\
&\quad + \varphi_0(\|y_n - x^*\|, \|z_n - x^*\|) = \bar{h}_n < 1.
\end{aligned}$$

Then, the conclusions of Theorem 47 hold for method (39.3) under these modifications.

3. Semi-Local Convergence

The majorizing sequence for the methods (40.2) and (39.3) are given, respectively for $t_0 = 0, s_0 = \eta \geq 0$ by

$$\begin{aligned}
d_n &= \int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n), \\
u_n &= s_n + \frac{1}{1 - w_0(t_n)}(1 + \frac{2\varphi_1(t_n, s_n)}{1 - w_0(t_0)})d_n, \qquad (39.1) \\
q_n &= \varphi(t_n, s_n, u_n) + \varphi_0(u_n, s_n), \\
h_n &= 1 + w_0(t_n) + \varphi(t_n, s_n, u_n), \\
p_n &= (1 + \int_0^1 w_0(s_n + \theta(u_n - s_n))d\theta)(u_n - s_n) + d_n, \\
t_{n+1} &= u_n + \frac{h_n p_n}{(1 - q_n)(1 - w_0(t_n))}
\end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{\int_0^1 w((1-\theta)(t_{n+1} - t_n))d\theta(t_{n+1} - t_n) + (1 + w_0(t_n))(t_{n+1} - s_n)}{1 - w_0(t_{n+1})}$$

and

$$\begin{aligned}
u_n &= s_n + \frac{\int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n)}{1 - \delta_n}, \qquad (39.2) \\
e_n &= \varphi(t_n, s_n, u_n) + \varphi_0(s_n, u_n) \\
\delta_n &= \varphi_1(t_n, s_n) + \varphi_0(s_n, u_n) \\
\Theta_n &= (1 + \int_0^1 w_0(s_n + \theta(u_n - s_n))d\theta)(u_n - s_n) \\
&\quad + \int_0^1 w((1-\theta)(s_n - t_n))d\theta(s_n - t_n) \\
t_{n+1} &= u_n + \frac{(1 + \varphi_0(u_n, s_n))\Theta_n}{(1 - e_n)(1 - \varphi_0(t_n, s_n))},
\end{aligned}$$

and

$$s_{n+1} = t_{n+1} + \frac{\int_0^1 w((1-\theta)(t_{n+1}-t_n))d\theta(t_{n+1}-t_n) + (1+w_0(t_n))(t_{n+1}-s_n)}{1-w_0(t_{n+1})}.$$

Next, we present a local convergence result first for method (40.2).

Lemma 44. Suppose that for all $n = 0,1,2,\ldots$

$$w_0(t_n) < 1, b(t_n) < 1, \; w_0(u_n) < 1 \tag{39.3}$$

and

$$t_n < \tau \text{ for some } \tau > 0. \tag{39.4}$$

Then, the sequence $\{t_n\}$ given by the formula (40.1) is nondecreasing and converging to its unique least upper bound $t^* \in [0,\tau]$.

Proof. It is immediate by (40.1), (40.3) and (40.4). □

Lemma 45. Suppose that for all $n = 0,1,2,\ldots$

$$\delta(t_n) < 1, w_0(t_n) < 1, e(t_n) < 1, \; \varphi_0(t_n,s_n) < 1 \tag{39.5}$$

and

$$\tau_n < \tau. \tag{39.6}$$

Then, the sequence $\{t_n\}$ given by the formula (40.2) is nondecreasing and converging to its unique least upper bound $t^* \in [0,\tau]$.

Proof. It is immediate by (40.2), (40.5) and (40.6). □

The motivation for the construction of the majorizing sequences are respectively

$$\begin{aligned}
\|z_n - y_n\| &= \| -F'(x_n)^{-1}F(z_n) \\
&\quad +2F'(x_n)^{-1}(F'(x_n) - [y_n,x_n;F])F'(x_n)^{-1}F(y_n)\| \\
&\leq \frac{1}{1-w_0(\|x_n-x_0\|)}\left[1+\frac{2\varphi_1(\|x_n-x_0\|,\|y_n-x_0\|)}{1-w_0(\|x_n-x_0\|)}\right]\bar{d}_n \\
&\leq u_n - s_n, \\
F(y_n) &= F(y_n) - F(x_n) - F'(y_n)(y_n - x_n), \\
F(z_n) &= F(z_n) - F(y_n) + F(y_n) \\
&= \int_0^1 F'(y_n + \theta(z_n - y_n))d\theta(z_n - y_n),
\end{aligned}$$

$$\begin{aligned}
\|F'(x_0)^{-1}F(z_n)\| &\leq (1+\int_0^1 w_0(\|y_n-x_0\|+\theta\|z_n-y_n\|)d\theta)\|z_n-y_n\| \\
&\quad +\int_0^1 w((1-\theta)\|y_n-x_n\|)d\theta\|y_n-x_n\| \\
&= \bar{p}_n \leq p_n, \\
\|x_{n+1}-z_n\| &\leq \|C_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}D_n\| \\
&\quad \times\|F'(x_n)^{-1}F')x_0)\|\|F'(x_0)^{-1}F9z_n)\| \\
&\leq t_{n+1} - u_n,
\end{aligned}$$

since

$$\begin{aligned}
\|F'(x_0)^{-1}D_n\| &\leq \|F'(x_0)^{-1}(F'(x_n)-F'(x_0)+[z_n,y_n;F] \\
&\quad -[y_n,x_n;F]+F'(x_0))\| \\
&\leq 1+w_0(\|x_n-x_0\|)+\varphi(\|x_n-x_0\|,\|y_n-x_0\|,\|z_n-x_0\|) \\
&= \bar{h}_n \leq h_n, \\
\bar{d_n} &= \int_0^1 w((1-\theta)\|y_n-x_n\|)d\theta\|y_n-x_n\| \leq d_n, \\
\|F'(x_0)^{-1}C_n\| &\leq \varphi(\|x_n-x_0\|,\|y_n-x_0\|,\|z_n-x_0\|) \\
&\quad +\varphi_0(\|z_n-x_0\|,\|y_n-x_0\|) \\
&= \bar{q}_n \leq q_n < 1
\end{aligned}$$

and

$$\|y_{n+1}-x_{n+1}\| \leq \frac{\|F'(x_0)^{-1}F(x_{n+1})\|}{1-w_0(\|x_{n+1}-x_0\|)} \leq s_{n+1}-t_{n+1},$$

since

$$F(x_{n+1}) = F(x_{n+1})-F(x_n)-F'(x_n)(x_{n+1})+F'(x_n)(x_{n+1}-y_n),$$

so

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq \int_0^1 w((1-\theta)\|x_{n+1}-x_n\|)d\theta\|x_{n+1}-x_n\| \\
&\quad +\|F'(x_0)^{-1}(F'(x_0)+F'(x_n)-F'(x_0))\|\|x_{n+1}-y_n\| \\
&\leq \int_0^1 w((1-\theta)(t_{n+1}-t_n))d\theta(t_{n+1}-t_n) \\
&\quad +(1+w_0(t_n))(t_{n+1}-s_n).
\end{aligned}$$

Hence, we arrive at:

Theorem 46. Under the conditions of Lemma 46 and the conditions connected the "φ" and "w" functions to the operators on the method (40.2) this sequence converges to a solution $x^* \in U[x_0,t^*]$ of the equation $F(x)=0$. Moreover, the following error bounds hold for all $n=0,1,2,\ldots$

$$\|y_n-x_n\| \leq s_n-t_n,$$

$$\|z_n-y_n\| \leq u_n-s_n$$

and

$$\|x_{n+1}-z_n\| \leq t_{n+1}-u_n.$$

Similarly, the motivation for the method (40.2) is:
Similarly, for the method (39.3) we have

$$\begin{aligned}
\|F'(x_0)^{-1}(A_n-F'(x_0))\| &\leq \varphi_1(\|x_n-x_0\|,\|y_n-x_0\|) \\
&\quad +\varphi_0(\|x_n-x_0\|,\|y_n-x_0\|) = \bar{\delta}_n \leq \delta_n < 1, \\
\|z_n-y_n\| &\leq \|A_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(y_n)\| \\
&\leq \frac{\int_0^1 w((1-\theta)\|y_n-x_n\|)d\theta\|y_n-x_n\|}{1-\bar{\delta}_n} \leq u_n-s_n,
\end{aligned}$$

$$
\begin{aligned}
\bar{e}_n &= \varphi(\|x_n - x_0\|, \|y_n - x_0\|, \|z_n - x_0\|) \\
&\quad + \varphi_0(\|y_n - x_0\|, \|z_n - x_0\|) \leq e_n < 1, \\
F(z_n) &= F(z_n) - F(y_n) + F(y_n), \\
\|F'(x_0)^{-1}F(z_n)\| &\leq (1 + \int_0^1 w_0(\|y_n - x_0\| + \theta\|z_n - y_n\|)d\theta\|z_n - y_n\| \\
&\quad + \int_0^1 w((1-\theta)\|y_n - x_n\|)d\theta\|y_n - x_n\|
\end{aligned}
$$

$$
\begin{aligned}
\|x_{n+1} - z_n\| &\leq \|[z_n, y_n; F]^{-1}F'(x_0)\|\|F'(x_0)^{-1}[z_n, y_n; F]\| \\
&\quad \times \|B_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(z_n)\| \\
&\leq \frac{(1 + \varphi_0(\|z_n - x_0\|, \|y_n - x_0\|))\bar{\Theta}_n}{(1 - \varphi_0(\|x_n - x_0\|, \|y_n - x_0\|))(1 - \bar{e}_n)} \\
&\leq t_{n+1} - u_n.
\end{aligned}
$$

The estimate for the iterate s_{n+1} is given in the previous method. Then, the conclusions of Theorem 46 hold for the method (39.3) under these modifications.

Bibliography

[1] Argyros, I.K., Hilout, S., Inexact Newton-type procedures, *J. Complex*, 26, (6), (2010), 577-590.

[2] Argyros, I.K., *Convergence and applications of Newton-type iterations*. Springer, New York, (2008).

[3] Argyros, I.K., Magréñan, A.A., *A contemporary study of iterative procedures*, Elsevier (Academic Press), New York, 2018.

[4] Argyros, I.K., George, S., *Mathematical modeling for the solution of equations and systems of equations with applications*, Volume-IV, Nova Publisher, NY, 2021.

[5] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics*, 2021, 9(16), 1942; https://doi.org/10.3390/math9161942.

[6] Argyros, I.K., Chen, D., Results on the Chebyshev method in Banach spaces, *Proyecciones*, 12, (2), (1993), 119128.

[7] Argyros, I.K., Magrenan, A.A., Orcos, L., Local convergence and a chemical application of derivative free root finding methods with one parameter based on interpolation, *J. Math. Chem.*, 2016, 54, 14041416.

[8] Babajee, D.K.R., Madhu, K., Jayaraman, J., A family of higher order multi-point iterative methods based on power mean for solving nonlinear equations, *Afrika Matematika*, 2016, 27, 865876.

[9] Liu, L., Wang, X., Eighth-order methods with high efficiency index for solving nonlinear equations, *Appl. Math. Comput.*, 2010, 215, 34493454.

[10] Madhu, K., Jayaraman, J., Higher order methods for nonlinear equations and their basins of attraction, *Mathematics*, 2016, 4, 22.

[11] Sharma, J.R., Arora, H., An efficient family of weighted-newton methods with optimal eighth order convergence, *Appl. Math. Lett.*, 2014, 29, 16.

[12] Sharma, J.R., Arora, H., A new family of optimal eighth order methods with dynamics for nonlinear equations, *Appl. Math. Comput.*, 2016, 273, 924933.

Chapter 40

Bilinear Operator Free Method for Solving Nonlinear Equations

1. Introduction

The aim of this chapter is to extend the applicability of a third order Newton-like method free of bilinear operators for solving Banach space valued equations. Using majorant functions that are actually needed, we present a finer convergence analysis than in earlier studies.

Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces, $\Omega \subset \mathcal{B}_1$ be a nonempty open set and $F : \Omega \longrightarrow \mathcal{B}_2$ be a $C^1(\Omega)$ operator. A plethora of applications from computational sciences can be formulated like nonlinear equation

$$\mathcal{G}(x) = 0. \tag{40.1}$$

The task of finding a solution $x_* \in \Omega$ in closed form is achieved only in special cases. That is why iterative procedures are mostly utilize to generate iterates approximating x_*. A survey of iterative procedures can be found in [1] and the references there in (see also [2,3,4]).

In particular, consider the following third-order Newton-like procedure defined for $x_0 \in \Omega$ and each $n = 0,1,2,\ldots$ by

$$\begin{aligned} y_n &= x_n + \mathcal{G}'(x_k)^{-1}\mathcal{G}(x_k) \\ \text{and} \\ x_{n+1} &= x_n - A_n^{-1}\mathcal{G}(x_n), \end{aligned} \tag{40.2}$$

has been used [1], where $A_n = \mathcal{G}'(x_n)[x_n, y; \mathcal{G}]^{-1}\mathcal{G}(x_n)$, $[.,.;\mathcal{G}] : \Omega \times \Omega \longrightarrow L(\mathcal{B}_1, \mathcal{B}_2)$ is a divided difference of order one [2,3,4]. The motivation, derivation and benefits out of using procedure (40.2) over third-order ones such as e.g. Chebyshev's (containing a bilinear operator) have been also well explained in [2,3,4].

The semi-local convergence analysis was based on generalized continuity conditions, and the results are given in non-affine invariant form. Motivated by optimization considerations we present a finer semi-local convergence analysis with benefits: results are presented in affine invariant form and under weaker conditions. Relevant work can be found in [6,7,8,9,10,11,12,13,14].

The results in [1] are presented in the next Section to make the chapter as self contained as possible and for comparison. More details can be found in [1]. The new semi-local convergence analysis can be found in Section 3. The examples appear in Section 4 followed by the conclusion in Section 5.

2. History of the Procedure

Throughout the chapter $U(x_0,\rho) = \{x \in \mathcal{B}_1 : \|x - x_0\| < \rho\}$ and $U[x_0,\rho] = \{x \in \mathcal{B}_1 : \|x - x_0\| \leq \rho\}$ for some $\rho > 0$.

The following semi-local convergence result for procedure (40.2) was shown in [1,2, Theorem 3].

Theorem 47. Suppose:

(1)
$$\|[x,y;\mathcal{G}] - [\bar{x},\bar{y};\mathcal{G}]\| \leq f(\|x-\bar{x}\|,\|y-\bar{y}\|) \tag{40.1}$$

for all $x,\bar{x},y,\bar{y} \in \Omega$, where $f : S \times S \longrightarrow S$ is a non-decreasing and continuous function satisfying

$$f(0,t) = f(t,0) = \frac{1}{2}f(t,t). \tag{40.2}$$

(2) $\|\mathcal{G}'(x_0)^{-1}\| \leq \beta$ for some $x_0 \in \Omega$ with $\mathcal{G}'(x_0)^{-1} \in L(\mathcal{B}_2,\mathcal{B}_1)$.

(3) $Max\{\|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)\|, \|A_0^{-1}\mathcal{G}(x_0)\|\} \leq a$.

(4) Equation $\left(1 - \dfrac{M_0}{1-2\beta f(t,t)}\right)t - 1 = 0$ has a smallest solution T, where $M_0 = \beta f(a,a)$.

(5) If $\beta f(T,T) < \dfrac{1}{3}$ and $U[x_0,T] \subset \Omega$, then $M = \dfrac{M_0}{1-2\beta f(T,T)} \in (0,1)$. Then, iteration $\{x_n\}$ given by method (40.2) exists in $U(x_0,T)$, stays in $U(x_0,T)$ for each $n = 0,1,2,\ldots$, $\lim_{n\longrightarrow\infty} x_n = x_* \in U[x_0,T]$ and $F(x_*) = 0$. Moreover, the limit point x_* is the only solution of equation $F(x) = 0$ in $U[x_0,T]$.

Remark. Conditions (40.1) and (40.2) are in general restrictive and may not hold although method can converge. Results are given in non-affine invariant form. The disadvantages over affine invariant results are well explained in the literature [2,3,4,8,9]. The error bounds on $\|x_{n+1} - x_n\|, \|x_* - x_n\|$ can be tighter, if function f is replaced by a tighter one. The uniqueness of solution may be extended. We positively answer to all these concerns in Section 3.

3. Convergence Analysis

The semi-local convergence analysis is based on some constants and real functions. Let λ and μ be non-negative constants. Set $S = [0,\infty)$.

Suppose there exists:

(i) Function $g_0 : S \longrightarrow S$ non-decreasing and continuous such that equation

$$g_0(t) - 1 = 0$$

has a smallest solution $\rho_0 \in S - \{0\}$ with $\rho_0 \geq \lambda$. Set $S_0 = [0, \rho_0)$.

(ii) Function $h_0 : S_0 \times S_0 \longrightarrow S$ non-decreasing and continuous in both variables such that equation

$$h_0(t,t) - 1 = 0$$

has a smallest solution $\rho_1 \in S_0 - \{0\}$ with $\rho_1 \geq \mu$. Set $\rho_2 = \min\{\rho_0, \rho_1\}, \rho_3 = \max\{\rho_0, \rho_1\}$ and $S_1 = [0, \rho_2)$.

(iii) Equation

$$p(t) - 1 = 0$$

has a smallest solution $\rho \in S_1 - \{0\}$ with $\rho \geq \rho_3$. Set $S_2 = [0, \rho)$. Define function $q : S_2 \longrightarrow S$ by

$$q(t) = \frac{1}{1 - p(t)},$$

where

$$p(t) = \frac{g_0(t)^2 + 2g_0(t) + h_0(t,t)}{1 - h_0(t,t)}.$$

Consider function $h : S_2 \longrightarrow S, h_1 : S_2 \longrightarrow S$ to be non-decreasing and continuous. Define constants

$$\alpha_1 = h(\lambda) + \frac{(1 + g_0(0))h_1(\mu)}{1 - h_0(0, \mu)},$$

$$\gamma_1 = \frac{\alpha_1}{1 - g_0(\lambda)}, \; p_1 = \frac{g_0(\lambda)^2 + 2g_0(\lambda) + h_0(0, \mu)}{1 - h_0(0, \mu)},$$

$$\delta_1 = q_1\alpha_1, \; q_1 = \frac{1}{1 - p_1},$$

functions on the interval S_2 with values in S by

$$\alpha(t) = h(\lambda) + \frac{(1 + g_0(t))h_1(\gamma_1\mu)}{1 - h_0(t,t)},$$

$$\gamma(t) = \frac{\alpha(t)}{1 - g_0(t)}, \; \delta(t) = q(t)\alpha(t),$$

$$\varphi(t) = (\gamma_1 + \frac{\delta^2(t)}{1 - \delta(t)} + \delta_1 + 1)\lambda - 1.$$

(iv) Equations

$$\gamma(t) - 1 = 0$$

and

$$\delta(t) - 1 = 0$$

have smallest solutions $\rho_4, \rho_5 \in S_1 - \{0\}$, respectively with $\rho_4 \geq \rho_3$ and $\rho_5 \geq \rho_3$ and

(v) Equation

$$\varphi(t) = 0$$

has a smallest solution $r \in S_2 - \{0\}$. Set $S_3 = [0, r)$. Then, it follows from these definitions that

$$0 \le g_0(\lambda) < 1,\ 0 \le g_0(t) < 1, \tag{40.1}$$

$$0 \le h_0(0, \mu) < 1, \tag{40.2}$$

$$0 \le p_1 < 1, \tag{40.3}$$

$$0 \le p(t) < 1, \tag{40.4}$$

$$0 \le \gamma(t) < 1, \tag{40.5}$$

$$0 \le \delta(t) < 1, \tag{40.6}$$

and

$$0 \le \varphi(t) < 1, \tag{40.7}$$

hold for all $t \in S_3$. Additionally the following conditions are used:

(vi) There exists $x_0 \in \Omega$, $\lambda \ge 0, \mu \ge 0$ such that $\mathcal{G}'(x_0)^{-1}, A_0^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$,

$$\|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)\| \le \mu,$$

$$A_0^{-1}\mathcal{G}(x_0)\| \le \lambda.$$

and

$$\|\mathcal{G}'(x_0)^{-1}([x, y; \mathcal{G}] - \mathcal{G}'(x_0))\| \le h_0(\|x - x_0\|, \|y - x_0\|).$$

(vii) $\|\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(x) - \mathcal{G}'(x_0))\| \le g_0(\|x - x_0\|)$ for each $x \in \Omega$. Set $\Omega_0 = U(x_0, \rho_0) \cap \Omega$.

(viii) $\|\mathcal{G}'(x_0)^{-1}([y, x; \mathcal{G}] - \mathcal{G}'(x))\| \le h(\|y - x\|)$ and

$$\|\mathcal{G}'(x_0)^{-1}([x, y; \mathcal{G}] - \mathcal{G}'(x))\| \le h_1(\|y - x\|)$$

for all $x, y \in \Omega_0$ and

(ix) $U[x_0, r] \subset \Omega$.

Next, we need some relations connecting the iterates $\{x_n\}, \{y_n\}$.

Lemma 46. Suppose iterates $\{x_n\}, \{y_n\}$ exist for each $n = 0, 1, 2, \ldots$. Then, the following assertions hold:

$$\begin{aligned} B_{n+1} := [x_{n+1}, x_n; \mathcal{G}] - A_n & \\ = \ & [x_{n+1}, x_n; \mathcal{G}] - \mathcal{G}'(x_n) \\ & + \mathcal{G}'(x_n)[x_n, y_n; \mathcal{G}]^{-1}([x_n, y_n; \mathcal{G}] - \mathcal{G}'(x_n)), \end{aligned} \tag{40.8}$$

$$\begin{aligned}
\mathcal{G}'(x_0)^{-1}(A_{n+1}-\mathcal{G}'(x_0)) &= \mathcal{G}'(x_0)^{-1}(\mathcal{G}(x_{n+1})-\mathcal{G}'(x_0)) \\
&\quad \times[x_n,y_n;\mathcal{G}]^{-1}\mathcal{G}'(x_0)\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0)) \\
&\quad +\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0))[x_n,y_n;\mathcal{G}]^{-1}\mathcal{G}'(x_0) \\
&\quad +[x_n,y_n;\mathcal{G}]^{-1}\mathcal{G}'(x_0)\mathcal{G}'(x_0)^{-1} \\
&\quad \times(\mathcal{G}'(x_{n+1})-[x_n,y_n;\mathcal{G}]) \qquad (40.9)\\
\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_{n+1}) &= \mathcal{G}'(x_0)^{-1}B_{n+1}(x_{n+1}-x_n) \qquad (40.10)
\end{aligned}$$

and

$$\begin{aligned}
x_{n+2}-x_{n+1} &= A_{n+1}^{-1}\mathcal{G}(x_{n+1}) \\
&= A_{n+1}^{-1}B_{n+1}(x_{n+1}-x_n) \\
&= A_{n+1}^{-1}\mathcal{G}'(x_0)\mathcal{G}'(x_0)^{-1}B_{n+1}(x_{n+1}-x_n). \qquad (40.11)
\end{aligned}$$

Proof. By the definition of A_n and B_{n+1}, we have in turn that

$$\begin{aligned}
B_{n+1} &= [x_{n+1},x_n;\mathcal{G}]-\mathcal{G}'(x_n) \\
&\quad +\mathcal{G}'(x_n)-\mathcal{G}'(x_n)[x_n,y_n;\mathcal{G}]^{-1}\mathcal{G}'(x_n) \\
&= [x_{n+1},x_n;\mathcal{G}]-\mathcal{G}'(x_n)+\mathcal{G}'(x_n)(I-[x_n,y_n;\mathcal{G}]^{-1}\mathcal{G}'(x_n)) \\
&= [x_{n+1},x_n;\mathcal{G}]-\mathcal{G}'(x_n) \\
&\quad +\mathcal{G}'(x_n)[x_n,y_n;\mathcal{G}]^{-1}([x_n,y_n;\mathcal{G}]-\mathcal{G}'(x_n)),
\end{aligned}$$

showing (40.8). Using the definition of A_{n+1}, we get in turn that

$$\begin{aligned}
A_{n+1}-\mathcal{G}'(x_0) &= (\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0))[x_n,y_n;\mathcal{G}]^{-1}\mathcal{G}'(x_{n+1}) \\
&\quad +\mathcal{G}'(x_0)[x_n,y_n;\mathcal{G}]^{1}\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0) \\
&= (\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0))[x_n,y_n;\mathcal{G}]^{-1}(\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0)) \\
&\quad +(\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0))[x_n,y_n;\mathcal{G}]^{1}(\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0)) \\
&\quad +(\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0))[x_n,y_n;\mathcal{G}]^{-1}\mathcal{G}'(x_0) \\
&\quad +\mathcal{G}'(x_0)[x_n,y_n;\mathcal{G}]^{1}\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0),
\end{aligned}$$

so

$$\begin{aligned}
\mathcal{G}'(x_0)^{-1}(A_{n+1}-\mathcal{G}'(x_0)) &= \mathcal{G}'(x_0)^{-1}(\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0)) \\
&\quad \times[x_n,y_n;\mathcal{G}]^{1}\mathcal{G}'(x_0)\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0)) \\
&\quad +\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(x_{n+1})-\mathcal{G}'(x_0))[x_n,y_n;\mathcal{G}]^{1}\mathcal{G}'(x_0) \\
&\quad +[x_n,y_n;\mathcal{G}]\mathcal{G}'(x_0)\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(x_{n+1})-[x_n,y_n;\mathcal{G}],
\end{aligned}$$

showing (40.9). Moreover, by the definition of method (40.2) we can write

$$\begin{aligned}
\mathcal{G}(x_{n+1}) &= \mathcal{G}(x_{n+1})-\mathcal{G}(x_n)+\mathcal{G}(x_n) \\
&= ([x_{n+1},x_n;\mathcal{G}]-A_{n+1})(x_{n+1}-x_n) \\
&= B_{n+1}(x_{n+1}-x_n),
\end{aligned}$$

so

$$\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_{n+1}) = \mathcal{G}'(x_0)^{-1}B_{n+1}(x_{n+1}-x_n),$$

showing (40.10). Furthermore, by the second substep of method (40.2) and (40.10) we obtain

$$\begin{aligned} x_{n+2}-x_{n+1} &= A_{n+1}^{-1}\mathcal{G}(x_{n+1}) \\ &= A_{n+1}^{-1}B_{n+1}(x_{n+1}-x_n) \\ &= A_{n+1}^{-1}\mathcal{G}'(x_0)\mathcal{G}'(x_0)^{-1}B_{n+1}(x_{n+1}-x_n). \end{aligned}$$

□

Next, we show the semi-local convergence result for method (40.2).

Theorem 48. Suppose conditions (i)-(ix) hold. Then, iterations $\{x_n\},\{y_n\}$ initiated at $x_0 \in \Omega$ and produced by method (40.2) exist in $U(x_0,r)$, stay in $U(x_0,r)$ for each $n = 0,1,2,\ldots$, $\lim_{n\longrightarrow\infty} x_n = x_*$ and $\mathcal{G}(x_*) = 0$.

Proof. It follows from (vi), the definition of r that y_0 is well defined, and

$$\|y_0 - x_0\| = \|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)\| \leq \mu < r.$$

Thus, $y_0 \in U(x_0,r)$. Iterate x_1 is well defined by method (40.2) for $n = 0$ and $\|x_1 - x_0\| \leq \mu < r$, so $x_1 \in U(x_0,r)$. Let $w \in U(x_0,r)$. Then, by (vii)

$$\|\mathcal{G}'9x_0)^{-1}(\mathcal{G}'(w) - \mathcal{G}'(x_0))\| \leq g_0(\|w-x_0\|) \leq g_0(r) < 1. \tag{40.12}$$

Hence, $\mathcal{G}'(w)^{-1} \in L(\mathcal{B}_2,\mathcal{B}_1)$, and

$$\|\mathcal{G}'(w)^{-1}\mathcal{G}'(x_0)\| \leq \frac{1}{1-g_0(\|w-x_0\|)} \tag{40.13}$$

follow by a Lemma due to Banach for invertible linear operators [3,9,11]. By (40.9) and (viii), we get

$$\begin{aligned} \|\mathcal{G}'(x_0)(A_{k+1} - \mathcal{G}'(x_0))\| &\leq \frac{g_0(\|x_{k+1}-x_0\|)^2}{1-h_0(\|x_k-x_0\|,\|y_k-x_0\|)} \\ &\quad + \frac{g_0(\|x_{k+1}-x_0\|}{1-h_0(\|x_k-x_0\|,\|y_k-x_0\|} \\ &\quad + \frac{g_0(\|x_{k+1}-x_0\|)+h_0(\|x_k-x_0\|,\|y_k-x_0\|)}{1-h_0(\|x_k-x_0\|,\|y_k-x_0\|)} \\ &= p_{k+1} < 1, \end{aligned}$$

so

$$\|A_{k+1}^{-1}\mathcal{G}'(x_0)\| \leq q_{k+1}, \tag{40.14}$$

and iterate x_{k+2} is well defined by the second substep of method (40.2). By the definition of B_{k+1} in (40.8) and α_{k+1} we get in turn that

$$\begin{aligned} \|\mathcal{G}'(x_0)^{-1}B_{k+1}\| &\leq h(\|x_{k+1}-x_k\| \\ &\quad + \frac{(1+g_0(\|x_k-x_0\|))h_1(\|y_k-x_k\|)}{1-g_0(\|x_k-x_0\|,\|y_k-x_0\|)} = \alpha_{k+1}. \end{aligned} \tag{40.15}$$

It then follows from (40.11), (40.14) and (40.15)

$$\begin{aligned} \|x_{k+2}-x_{k+1}\| &\leq \|A_{k+1}^{-1}\mathcal{G}'(x_0)\|\|\mathcal{G}'(x_0)^{-1}B_{k+1}\|\|x_{k+1}-x_k\| \\ &\leq q_{k+1}\alpha_{k+1}\|x_{k+1}-x_k\| \\ &\leq \bar{\delta}(r)\|x_{k+1}-x_k\|, \end{aligned} \tag{40.16}$$

where

$$\bar{\delta}(r) = \begin{cases} \delta_1 & \text{if } k=0 \\ \delta(r) & \text{if } k=1,2,\ldots. \end{cases}$$

Similarly, by the first substep of method (40.2), (40.13) (for $w = x_{k+1}$) and (40.8), we have the estimate

$$\begin{aligned} \|y_k - x_k\| &\leq \|\mathcal{G}'(x_0)^{-1}\mathcal{G}'(x_0)\|\|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_k)\| \\ &\leq \frac{\alpha_k}{1-g_0(\|x_k-x_0\|)}\|x_k-x_{k-1}\| \\ &\leq \bar{\gamma}(r)\|x_k-x_{k-1}\|, \end{aligned} \tag{40.17}$$

where

$$\bar{\gamma}(r) = \begin{cases} \gamma_1 & \text{if } k=1 \\ \gamma(r) & \text{if } k=2,\ldots. \end{cases}$$

Notice that we also used

$$\begin{aligned} \|x_{k+1}-x_0\| &\leq \|x_{k+1}-x_k\|+\|x_k-x_{k-1}\|+\ldots+\|x_2-x_1\|+\|x_1-x_0\| \\ &\leq \delta^k\lambda+\delta^{k-1}\lambda+\ldots+\delta^2\lambda+\delta_1\lambda+\lambda \\ &= (\delta^2\frac{1-\delta^{k-1}}{1-\delta}+\delta_1+1)\lambda \\ &\leq (\frac{\delta^2}{1-\delta}+\delta_1+1)\lambda < r \end{aligned} \tag{40.18}$$

(by (v)), so $x_{k+1} \in U(x_0, r)$. Similarly,

$$\begin{aligned} \|y_k-x_0\| &\leq \|y_k-x_k\|+\|x_k-x_0\| \\ &\leq \gamma^{k-1}\gamma_1\lambda+\delta^{k-1}\lambda+\ldots+\delta^2\lambda+(\frac{\delta^2}{1-\delta}+\delta_1+1)\lambda \\ &\leq (\gamma_1+\frac{\delta^2}{1-\delta}+\delta_1+1)\lambda < r. \end{aligned} \tag{40.19}$$

Hence, iterate $y_k \in U(x_0, r)$. Similarly, we have $x_{k+2}, y_{k+1} \in U(x_0, r)$. It follows from (40.16) that

$$\|x_{k+2}-x_{k+1}\| \leq \delta^{k+1}\lambda. \tag{40.20}$$

Let $m \geq 1$ be a positive integer. Then, it follows by (40.20) that

$$\begin{aligned} \|x_{k+m}-x_{k+1}\| &\leq \|x_{k+m}-x_{k+m-1}\| \\ &\quad +\|x_{k+m-1}-x_{k+m-2}\|+\ldots+\|x_{k+2}-x_{k+1}\| \\ &\leq (\delta^{k+m-1}+\delta^{k+m-2}+\ldots+\delta^{k+1})\lambda \\ &= (\delta^{m-2}+\delta^{m-3}+\ldots+1)\delta^{k+1}\lambda \\ &= \frac{1-\delta^{m-1}}{1-\delta}\delta^{k+1}\lambda \leq \delta^{k+1}\lambda. \end{aligned} \tag{40.21}$$

It follows that sequence $\{x_k\}$ is complete in a Banach space $\mathcal{B}_1$, and such it converges to some $x_* \in U[x_0, r]$, (since $U[x_0, r]$ is a closed set.). Moreover, using (40.10) and (40.15), we obtain

$$\|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_{k+1})\| \leq \alpha(r)\|x_{k+1} - x_k\|. \tag{40.22}$$

By letting $k \longrightarrow \infty$ in (40.22) and using the continuity of $\mathcal{G}$ we deduce $\mathcal{G}(x_*) = 0$. Furthermore, by letting $m \longrightarrow \infty$ in (40.21), we conclude

$$\|x_* - x_k\| \leq \delta^k \lambda. \tag{40.23}$$

□

Next, we present a uniqueness of the solution result by not necessarily using all the conditions of Theorem 48.

Proposition 27. *Suppose*

$$\|\mathcal{G}^{-1}(x_0)(\mathcal{G}'(x) - \mathcal{G}'(x_0))\| \leq g_0(\|x - x_0\|)$$

for all $x \in \Omega$, the point x_* is a solution of equation $\mathcal{G}(x) = 0$ with $x_* \in U[x_0, b] \subset \Omega$ for some $b > 0$ and these exists $c \geq b$ such that

$$\int_0^1 g_0((1-\theta)b + \theta c)d\theta < 1.$$

Define $\Omega_1 = U[x_0, c] \cap \Omega$. Then, the point x_* is the only solution of equation $\mathcal{G}(x) = 0$ in the set Ω_1.

Proof. Let $y_* \in \Omega_1$ with $\mathcal{G}(y_*) = 0$. Define operator $Q = \int_0^1 \mathcal{G}'(x_* + \theta(y_* - x_*))d\theta$. Then, we obtain in turn that

$$\begin{aligned} \|\mathcal{G}'(x_0)^{-1}(Q - \mathcal{G}'(x_0))\| &\leq \int_0^1 g_0((1-\theta)\|x_* - x_0\| + \theta\|y_* - x_)\|]d\theta \\ &\leq \int_0^1 g_0((1-\theta)b + \theta c)d\theta < 1. \end{aligned}$$

So, we conclude $y_* = x_*$ by the using the identity $0 = \mathcal{G}(y_*) - \mathcal{G}(x_*) = Q(y_* - x_*)$, and the invertibility of linear operator Q. □

Remark. The concerns in Remark 2. were addressed by Theorem 48. In particular, notice that majorant functions are tighter than f, g and the latter implies the new functions but not necessarily vice versa.

4. Numerical Experiments

We compute the radius of convergence in this section.

Example 11. Let us consider a scalar function $\mathcal{G}$ defined on the set $D = U[u_0, 1-s]$ for $s \in (0,1)$ by

$$\mathcal{G}(x) = x^3 - s.$$

Choose $x_0 = 1$. The conditions of Theorem 48 are satisfied provided that $\mu = \frac{1-s}{3}$, $g_0(t) = (3-s)t$, $h_0(u,t) = \frac{1}{2}(3-s)(u+t)$, $h(t) = h_1(t) = (1 + \frac{1}{3-s})t$, $\lambda = \frac{1-s}{y_0^2 + x_0 y_0 + x_0^2}$, $y_0 = x_0 - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)$. Then, we have

$$\rho_0 = \rho_1 = \rho_2 = \rho_3 = 0.4878, \rho = 0.1152, \rho_4 = 0.3565, \rho_5 = 0.1079 \text{ and } r = 0.0962.$$

Bibliography

[1] Amat, S., Bermudez, C., Busquier, S., Plaza, S., On a third order Newton-type method free of bilinear operators, *Numer. linear Algebra Appl.*, 17, (2010), 639-653.

[2] Argyros, I.K,. *Convergence and Applications of Newton-type Iterations*, Springer Verlag, Berlin, Germany, (2008).

[3] Argyros, I.K., Unified convergence criteria for Banach space valued methods with applications, *Mathematics*, 9, 16, (2021),1942.

[4] Argyros, I.K., *The theory and applications of iteration methods*, 2nd Edition, CRC Press, Engineering Series, Taylor and Francis Group, 2022.

[5] Argyros, I.K., Unified Convergence Criteria for Iterative Banach Space Valued Methods with Applications, *Mathematics 2021*, 9(16), 1942; https://doi.org/10.3390/math9161942.

[6] Behl, R., Maroju, P., Martinez, E., Singh, S., A study of the local convergence of a fifth order iterative method, *Indian J. Pure Appl. Math.*, 51, 2, (2020), 439-455.

[7] Ezquerro J.A, Hernández M.A., Halleys method for operators with unbounded second derivative, *Applied Numerical Mathematics*, 57, (3), 2007, 354-360.

[8] Gutiérrez J.M., A new semi-local convergence theorem for Newtons method, *Journal of Computational and Applied Mathematics*, 79, 1997, 131-145.

[9] Hernández M.A., Rubio M.J., Semilocal convergence of the secant method under mild convergence conditions of differentiability, *Computers and Mathematics with Applications*, 44, (34), 2002, 277285.

[10] Kantorovich L.V., Akilov, G.P., *Functional Analysis*, Pergamon Press: Oxford, 1982.

[11] Kou J., Li Y., Wang X., A modification of Newtons method with third-order convergence, *Applied Mathematics and Computation*, 181, 2007, 1106-1111.

[12] Hernández M.A., Romero N., General study of iterative processes of R-order at least three under weak convergence conditions, *Journal of Optimization Theory and Applications*, 133, (2), 2007, 163-177.

[13] Traub, J.F., *Iterative methods for the solution of equations*, Prentice Hall, New Jersey, U.S.A., (1964).

[14] Verma, R.U., *Next Generation Newton-Type Methods*, Nova Publisher, NY, 2019.

Index

A

algorithm, 26, 36, 52, 62, 88, 101, 255, 315, 345, 377, 386
applied mathematics, 25, 26, 35, 36, 51, 52, 61, 62, 87, 99, 107, 163, 199, 281, 305, 369, 385, 386, 387, 399, 421
approximation, 25, 35, 51, 61, 76, 87, 99, 191, 213, 385

B

ball convergence, 89, 90, 91, 93, 95, 97, 379, 380, 381, 383
Banach Lemma, 3, 121, 185, 204, 225, 260, 394
Banach space valued equations, 411
Banach space(s), 1, 3, 5, 7, 13, 14, 17, 19, 21, 24, 26, 29, 32, 34, 36, 39, 48, 52, 55, 57, 59, 62, 63, 72, 75, 77, 83, 84, 87, 88, 89, 99, 100, 103, 104, 107, 109, 114, 115, 117, 119, 121, 123, 125, 127, 137, 142, 153, 163, 164, 165, 168, 171, 173, 176, 181, 183, 185, 189, 190, 191, 199, 201, 204, 207, 208, 209, 211, 215, 217, 221, 223, 225, 233, 235, 240, 243, 245, 249, 257, 260, 263, 265, 271, 281, 283, 285, 290, 305, 315, 317, 323, 325, 328, 333, 335, 345, 347, 371, 374, 379, 381, 386, 394, 399, 401, 402, 409, 411, 416, 418, 421
bilinear operators, 411, 421
bounds, 19, 29, 39, 55, 64, 77, 97, 109, 119, 167, 169, 215, 233, 326, 379, 402, 407, 412

C

Chebyshev-Halley method, 6, 18, 76, 100, 181, 189, 207
closed ball, 130, 203, 248, 259, 267, 284, 327
convergence analysis, 12, 20, 25, 30, 35, 40, 51, 55, 56, 61, 63, 78, 99, 117, 126, 127, 171, 173, 181, 183, 189, 199, 207, 222, 244, 325, 379, 385, 399, 411, 412
convergence criteria, 5, 8, 17, 20, 40, 41, 69, 75, 85, 87, 117, 125, 151, 161, 166, 171, 181, 189, 199, 207, 221, 243, 263, 271, 281, 305, 315, 320, 323, 345, 399, 409, 421
convergence radius, 2, 101, 285, 325, 327, 329, 331, 381
convergence region, 181, 189, 207

D

derivative, 5, 7, 9, 11, 13, 15, 19, 20, 29, 30, 39, 40, 55, 63, 65, 67, 69, 71, 73, 77, 78, 100, 101, 107, 109, 111, 113, 115, 127, 137, 139, 165, 167, 173, 174, 183, 185, 187, 191, 201, 203, 205, 208, 209, 211, 214, 245, 247, 249, 251, 253, 255, 265, 271, 283, 285, 287, 289, 291, 293, 295, 297, 299, 301, 303, 307, 309, 311, 313, 315, 316, 317, 319, 321, 323, 325, 326, 335, 337, 339, 341, 343, 345, 371, 377, 379, 387, 388, 401, 409, 421
derivative free method, 7, 9, 11, 13, 15, 265, 285, 287, 289, 291, 293, 295, 299, 301, 303, 309, 311, 313, 317, 319, 321, 335, 337, 339, 341, 343, 377

E

eighth order methods, 231, 409

F

fifth derivative, 209, 283, 387
fifth-order method, 39
fixed point, 192
fourth convergence order methods, 183

G

generalized condition, 76, 165, 171, 173
generalized equation, 199, 200
generalized nonlinear equation (GNE), 191

H

Halley's method, 6, 18, 75, 76, 99, 100

I

inverses, 3, 13, 83, 225, 260, 374
iterative methods, 5, 17, 25, 35, 51, 61, 62, 87, 89, 99, 107, 117, 125, 153, 155, 157, 159, 161, 163, 164, 174, 200, 213, 221, 231, 243, 244, 245, 255, 257, 271, 281, 295, 297, 305, 306, 315, 323, 325, 326, 333, 335, 345, 347, 369, 371, 377, 385, 386, 401

J

Jarratt, 26, 36, 52, 62, 88, 153, 173, 175, 177, 179, 181, 190, 231, 305, 333, 386
Jarratt-like, 173, 175, 177, 179, 181, 190, 305

K

Kantorovich Theory, 193, 195, 197

L

Lipschitz conditions, 165, 166
local analysis, 56, 130, 139, 216, 371
local convergence, 1, 2, 3, 20, 25, 26, 29, 35, 36, 51, 52, 61, 62, 75, 77, 87, 88, 101, 125, 127, 129, 131, 133, 137, 139, 174, 183, 201, 209, 211, 213, 215, 217, 219, 221, 223, 225, 234, 235, 243, 257, 259, 265, 267, 274, 283, 284, 285, 287, 292, 297, 300, 308, 317, 318,326, 327, 328, 336, 345, 347, 371, 373, 379, 380, 385, 386, 387, 388, 401, 402, 404, 406, 421
local convergence analysis, 2, 20, 137, 201, 209, 223, 234, 259, 267, 284, 285, 292, 317, 326, 327, 371, 373, 380

M

majorizing sequences, 8, 102, 110, 388
mathematical induction, 121, 176, 260, 285, 390
mathematical modeling, 165, 255, 263, 271

N

Newton, 5, 6, 17, 18, 25, 26, 35, 36, 51, 52, 61, 62, 75, 87, 88, 89, 99, 125, 135, 153, 163, 165, 167, 169, 171, 181, 189, 196, 199, 200, 207, 213, 221, 243, 244, 245, 255, 281, 295, 305, 333, 369, 370, 377, 385, 386, 399, 409, 411, 421, 422
Newton Method, 167, 169, 245
Newton-like method, 5, 17, 26, 36, 52, 62, 88, 181, 189, 199, 207, 244, 333, 386
nonlinear equations, 26, 36, 37, 52, 62, 63, 65, 75, 87, 99, 101, 107, 153, 163, 164, 189, 190, 200, 207, 208, 213, 231, 244, 255, 263, 281, 295, 305, 315, 323, 345, 347, 369, 370, 377, 386, 387, 399, 409
numerical example, 151, 330

P

parameters, 1, 2, 8, 20, 30, 41, 56, 63, 78, 79, 90, 102, 110, 111, 115, 119, 120, 175, 184, 192, 201, 209, 215, 216, 217, 223, 224, 234, 238, 244, 246, 257, 265, 300, 330, 371, 380, 388, 389, 396

Q

quadratic, 333

R

radius of convergence, 85, 109, 127, 129, 130, 202, 210, 216, 219, 224, 235, 247, 258, 267, 284, 298, 300, 325, 327, 372, 383, 403, 418

S

scalar function, 64, 85, 130, 151, 201, 203, 223, 238, 284, 419
scalar sequences, 68, 102, 175, 184, 227, 238, 301, 388
secant type method (STM), 215, 216, 217
second derivative, 17, 64, 75, 76, 87, 99, 100
second substep of method, 3, 32, 48, 83, 103, 122, 186, 205, 211, 218, 249, 251, 252, 260, 290, 328, 374, 416
semi-local convergence, 7, 9, 11, 13, 15, 19, 21, 23, 27, 29, 30, 31, 33, 37, 39, 41, 43, 45, 47, 49, 53, 55, 56, 57, 59, 64, 65, 66, 68, 71, 77, 78, 109, 111, 113, 115, 117, 119, 120, 121, 123, 126, 127, 137, 139, 141, 143, 145, 147, 149, 151, 155, 165, 166, 167, 169, 173, 174, 175, 177, 179, 183, 184, 185, 187, 192, 194, 222, 233, 238, 239, 273, 277, 283, 288, 289, 297, 300, 307, 311, 319, 336, 347, 387, 389, 391, 392, 393, 395, 397, 402, 405, 411, 412, 416, 421
seventh order methods, 247, 249, 251, 253

T

Taylor expansion, 174, 325
Taylor series, 101, 139
third order Newton-like method, 411
third-order method, 75, 87, 99
Traub Method, 245

About the Authors

Christopher I. Argyros

Researcher
Department of Computing and Mathematical Sciences
Cameron University
Lawton, OK, USA
Email: christopher.argyros@cameron.edu

Samundra Regmi

Researcher
University of Houston
Houston, TX, USA
Email: sregmi5@uh.edu

Ioannis K. Argyros

Professor
Department of Computing and Mathematical Sciences
Cameron University
Lawton, OK, USA
Email: iargyros@cameron.edu

Dr. Santhosh George

Department of Mathematical and Computational Sciences
National Institute of Technology
Karnataka, India
Email: sgeorge@nitk.edu.in

CHRISTOPHER I. ARGYROS
SAMUNDRA REGMI
IOANNIS K. ARGYROS, PHD
SANTHOSH GEORGE, PHD

CONTEMPORARY ALGORITHMS

THEORY AND APPLICATIONS

VOLUME III

nova science publishers
www.novapublishers.com
ISBN-13: 979-8-88697-985-5
9 798886 979855