IV.

DYNAMICS AND SCATTERING-POWER OF BORN'S ELECTRON.

(From the Dublin Institute for Advanced Studies.)

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This paper is the continuation of another one,[1] to be quoted here as N.O. Yet I try to resume the subject in a way so as not to make the knowledge, at any rate not the thorough study, of the previous paper indispensable. The complex presentation of Born's theory is *not* required here.—Towards the end of N.O. (sect. 11) it became clear that even the scattering by a Born-singularity *fixed in space* is a very difficult mathematical problem, because it includes the solution of the differential equations (1, 5) and (1, 7) below. Additional complications are introduced when you drop the fiction of fixation of the singularity and allow it to yield to the field the way it has to for safeguarding the conservation laws.

Pending the solution of those two ordinary linear homogeneous differential equations of the second order, I have tried to get ahead by approximation methods, less in the idea of producing already by them very valuable results—for they cannot possibly carry you into the truly interesting region of very short light waves—than with the scope of getting better acquainted with the problem and facilitating its true solution by knowing precisely *what* about the mathematical solution is physically required. A fascinating aspect of the equation of motion has been encountered on the way (sect. 5).

As long as the wave-length of the light is large compared with $r_0 (= 2{,}28\ 10^{-13}$ cm.), Born's theory appears to differ from Lorentz's only

[1] Nonlinear Optics, Proc. R.I.A., 47, 77, 1942.

by terms containing some positive power of r_0/λ as a factor. To many a reader it may seem lunatic to bother about corrections of that small order by classical methods, whilst the Compton-effect shows us that everything goes wrong already with a wave-length of the order $h/2\pi mc \propto 110\, r_0$, unless wave-mechanics, including quantisation of the electro-magnetic field, is introduced.—But, first of all, as has often been observed, the cross-ratio of the two ratios in question is not so huge, that one could declare a theory which is apparently connected with one of them to be unfit ever to account for such features of observation as are apparently controlled by the other one (see e.g. the remark on the reciprocal fine-structure-constant at the end of N.O.). Secondly, though quite convinced that something like field quantisation is unavoidable, I have as yet come across no case where it allowed us to skip the classical treatment altogether—if the case *had* a classical analogue, which it mostly has. The Compton-effect is no exception. For Dirac's equation, employed in deducing the Klein-Nishina formula, is but an ingenious translation into wave-mechanics of Lorentz's classical Hamiltonian. Many cases could be quoted where a detailed knowledge of the classical aspect led to the discovery of a phenomenon, and served as a reliable guide in its quantum-mechanical description, after which that phenomenon was then declared—and perhaps with due right—to be entirely ununderstandable along classical lines of thought. Therefore I believe, if Born's electrodynamics has any bearing on facts at all, its classical understanding will have to precede its quantum-mechanical understanding.

1. The proper vibrations of the perturbation field.

In Part II, sect. 8 of N.O., it was proved, that in Born's nonlinear electrodynamics the following holds. Whenever an electromagnetic field can be regarded as the *sum* of a *weak* field E, B, D, H and a purely electric field E°, D°, which is allowed to be strong, is supposed to be known, and *has to be* an exact solution of the nonlinear field equations, then the *weak* field is, in first approximation as regards its weakness,[2] controlled by Maxwell's equations for an inhomogeneous, anisotropic magnetizable dielectric

$$\begin{aligned} &\operatorname{curl} E + \dot{B} = 0 \qquad && \operatorname{div} B = 0 \\ &\operatorname{curl} H - \dot{D} = 0 && \operatorname{div} D = 0 \end{aligned}$$

with

$$\begin{aligned} &D_\parallel = A^{-3} E_\parallel \qquad && B_\parallel = A^{-1} H_\parallel \\ &D_\perp = A^{-1} E_\perp && B_\perp = A H_\perp \end{aligned} \tag{1, 1}$$

$$A = \sqrt{1 - E^{\circ 2}}.$$

[2] *This* point of view of approximation is *not* extended in the present paper.

The notation $\parallel$ and $\perp$ refers to the direction of E°, which, by the way, equals AD°. The weak field is, properly speaking, a *perturbation* field, the equations (1, 1) are perturbational field equations. But we shall occasionally drop the term perturbation(al). To E°, D° we refer as *main field*, or occasionally as the *field of reference*, whereas the sum of the two is called *total* or *true* field. In this paper the field of reference is always going to be that of a Born-electron at rest, even though the contemplated Born-electron may not be permanently at rest. Thus

$$A = \frac{r^2}{\sqrt{1 + r^4}}, \tag{1, 1)'}$$

where r is the distance from the singularity of D°.

The idea of *perturbational field equations* is entirely different from the perturbation theories (p.th.) I began to employ in 1926, though they also were p.th. of a field equation, viz. either of the eigenvalue problem engendered by a certain linear homogeneous partial differential equation (1st method) or of the initial-value-problem of that equation (2nd method). *There* the perturbation consisted in an additional linear and homogeneous term of the equation, causing a *definite* modification of *any* eigensolution (1st method) or of the unrollment of *any* initial state (2nd method). *Here* the perturbation consists in the quadratic terms of the original equations and creates linear and homogeneous field laws for *any* modification or modulation superposed on a *definite* exact solution of the original equations. The perturbational field laws, though subjected to the suzerainty of the original equations and even of a definite solution thereof, engender an eigenvalue problem of their own. Or rather, it is only they that create one, the only kind to which the original field laws give rise. In their non-linear form they have none, at any rate not in the familiar sense of the word.

For the fictitious case, when the electron is supposed to be permanently at rest in the origin (immovable singularity), the complete formal solution of (1, 1) was communicated *in nuce* in sect. 11 of N.O. The proper vibrations resemble the corresponding Maxwellian ones for empty space: standing spherical waves of all possible electric (I) and magnetic (II) multipole types, which are conveniently described by *vector-potentials*, of which the time-component is always zero. We call the components ϕ_k in case I and ψ_k in case II, whereby the subscripts 1, 2, 3 shall refer to the polar coordinates r, θ, ϕ, in this order. The choice of different letters, ϕ and ψ, just avoids cumbersome subscripts "I" and "II," otherwise it is the same physical quantity, from which the field, notably the field E, B, *not* D, H, is derived in the familiar way. Writing for the moment x_1, x_2, x_3, x_4 for r, θ, ϕ, t and putting

$$\phi_{kl} = \frac{\partial \phi_l}{\partial x_k} - \frac{\partial \phi_k}{\partial x_l}, \qquad \psi_{kl} = \frac{\partial \psi_l}{\partial x_k} - \frac{\partial \psi_k}{\partial x_l}, \tag{1, 2}$$

the following six equations give the field in case I,

$$\left.\begin{matrix} E_r & E_\theta & E_\phi \\ \\ B_r & B_\theta & B_\phi \end{matrix}\right\} = \left\{\begin{matrix} \phi_{14} & \dfrac{\phi_{24}}{r} & \dfrac{\phi_{34}}{r\sin\theta} \\ \\ \dfrac{\phi_{23}}{r^2\sin\theta} & \dfrac{\phi_{31}}{r\sin\theta} & \dfrac{\phi_{12}}{r} \end{matrix}\right. \tag{1, 3}$$

whereas in case II the letter ϕ has to be replaced by ψ. The E_r, etc., are field components in the ordinary experimentalist's sense, all of the same physical dimensions, and so are the components of D, H, to be obtained from the "material equations" out of (1, 1).—The components ϕ_k and ψ_k themselves are, by the way, defined in the sense of generalized metrics, which greatly simplified my calculations and slightly simplifies the formulae (1, 2), (1, 4) and (1, 6).

The electric multipole vibration associated with the "associated Legendre function" $P_n^m(\theta)$, $n = 1, 2, 3, \cdots\cdots$, is obtained by putting

$$\begin{aligned}
\phi_1 &= \frac{n(n+1)A^3}{r^2} F_n(r)\, P_n^m(\theta) \frac{\sin}{\cos} m\phi\, e^{i\omega t} \\
\phi_2 &= A \frac{dF_n(r)}{dr} \frac{dP_n^m(\theta)}{d\theta} \frac{\sin}{\cos} m\phi\, e^{i\omega t} \\
\phi_3 &= A \frac{dF_n(r)}{dr} P_n^m(\theta) \frac{d}{d\phi}\left(\frac{\sin}{\cos} m\phi\right) e^{i\omega t} \\
\phi_4 &= 0
\end{aligned} \tag{1, 4}$$

where F_n has to satisfy

$$\frac{d^2F_n}{dr^2} + \frac{2}{r(1+r^4)} \frac{dF_n}{dr} + \left(\omega^2 - \frac{n(n+1)r^2}{1+r^4}\right) F_n = 0. \tag{1, 5}$$

The magnetic multipole vibration, associated with the same P_n^m (but, of course, wholly independent of the aforestanding electric one), is obtained by putting

$$\begin{aligned}
\psi_1 &= 0 \\
\psi_2 &= G_n(r)(\sin\theta)^{-1} P_n^m(\theta) \frac{d}{d\phi}\left(\frac{\sin}{\cos} m\phi\right) e^{i\omega t} \\
\psi_3 &= -\, G_n(r) \sin\theta \frac{d}{d\theta} P_n^m(\theta) \frac{\sin}{\cos} m\phi\, e^{i\omega t} \\
\psi_4 &= 0,
\end{aligned} \tag{1, 6}$$

where G_n has to satisfy

$$\frac{d^2G_n}{dr^2} - \frac{2}{r(1+r^4)} \frac{dG_n}{dr} + \left(\omega^2 - \frac{n(n+1)r^2}{1+r^4}\right) G_n = 0. \tag{1, 7}$$

It is a matter of straightforward calculation to show that in both cases equ. (1, 1) *are* fulfilled. The substitution, paying attention to the well-known differential equations for $\frac{\sin}{\cos} m\phi$ and P_n^m, leads you to demand (1, 5) and (1, 7) for F_n and G_n respectively, and nothing more.

But it turns out that with F_n and G_n *arbitrary* solutions of their respective equations the dielectric displacement D would have a singularity r^{-3} in the origin, and that in consequence of this fact the perturbation equations (1, 1), though they are satisfied all right, are no longer *competent*, the perturbation being, *near* the origin, stronger than the main field (actually the *total* field would violate Born's equations there).

Equ. (1. 5) has at $r = 0$ the exponents -1 and 0. Only the fundamental solution with exponent 0 reduces the singularity of D to r^{-2}, and thereby avoids the deficiency just mentioned.

Equ. (1, 7) has at $r = 0$ the exponents 0 and $+3$. Only the fundamental solution with exponent 3 (by removing the singularity of D altogether) avoids the deficiency.—*These two remarks complete the definition of the proper vibrations.*

For future use we put down the components of E and B, obtained by inserting (1, 4) or (1, 6) in (1, 2) and the result in (1, 3), whereby considerable simplifications take place in virtue of the differential equations for P_n^m, F_n, G_n. Indicating, for shortness, the derivatives with respect to r, θ (*not* $\cos\theta$) and ϕ, occurring on the right hand sides, *by subscripts*, we get in the *electric* case (i.e. with ϕ_k):

$$\left.\begin{aligned}
E_r &= -\frac{i\omega\, n(n+1)A^3}{r^2} F_n\, P_n^m \begin{matrix}\sin\\ \cos\end{matrix} m\phi\; e^{i\omega t}\\
E_\theta &= -\frac{i\omega A}{r} (F_n)_r\, (P_n^m)_\theta \begin{matrix}\sin\\ \cos\end{matrix} m\phi\; e^{i\omega t}\\
E_\phi &= -\frac{i\omega A}{r\sin\theta} (F_n)_r\, P_n^m \left(\begin{matrix}\sin\\ \cos\end{matrix} m\phi\right)_\phi e^{i\omega t}\\
B_r &= 0\\
B_\theta &= \frac{\omega^2 A}{r\sin\theta} F_n\, P_n^m \left(\begin{matrix}\sin\\ \cos\end{matrix} m\phi\right)_\phi e^{i\omega t}\\
B_\phi &= -\frac{\omega^2 A}{r} F_n\, (P_n^m)_\theta \begin{matrix}\sin\\ \cos\end{matrix} m\phi\; e^{i\omega t}.
\end{aligned}\right\} \qquad (1, 8)$$

In the *magnetic* case (i.e. with ψ_k):

$$\left.\begin{aligned}
E_r &= 0\\
E_\theta &= -\frac{i\omega}{r\sin\theta} G_n\, P_n^m \left(\begin{matrix}\sin\\ \cos\end{matrix} m\phi\right)_\phi e^{i\omega t}\\
E_\phi &= \frac{i\omega}{r} G_n\, (P_n^m)_\theta \begin{matrix}\sin\\ \cos\end{matrix} m\phi\; e^{i\omega t}\\
B_r &= \frac{n(n+1)}{r^2} G_n\, P_n^m \begin{matrix}\sin\\ \cos\end{matrix} m\phi\; e^{i\omega t}\\
B_\theta &= \frac{1}{r} (G_n)_r\, (P_n^m)_\theta \begin{matrix}\sin\\ \cos\end{matrix} m\phi\; e^{i\omega t}\\
B_\phi &= \frac{1}{r\sin\theta} (G_n)_r\, P_n^m \left(\begin{matrix}\sin\\ \cos\end{matrix} m\phi\right)_\phi e^{i\omega t}.
\end{aligned}\right\} \qquad (1, 9)$$

A stands for the function (1, 1). An individual factor, a power of A, evident from (1, 1), is to be added to every component of E, B, if the corresponding component of D, H is desired. Behold the relatively complete symmetry between the two cases, but also that it is broken not only by F_n and G_n being entirely different functions, but also by entirely different powers of A entering. The statements made above about the nature of the singularities in the origin are now easy to check.

With $A = 1$ and with F_n or G_n replaced by $\sqrt{r}\, J_{n+\frac{1}{2}}(\omega r)$ the potentials (1, 4) with the field-components (1, 8) or the potentials (1, 6) with the field-components (1, 9) describe an ordinary Maxwellian vibration of empty space, *without* an electron at the origin. This vibration is the superposition of an ingoing spherical wave and an outgoing spherical wave, corresponding to the splitting of $\sqrt{r}\, J_{n+\frac{1}{2}}(\omega r)$ into $\sqrt{r}\, H^{(1)}{}_{n+\frac{1}{2}}(\omega r)$ and $\sqrt{r}\, H^{(2)}{}_{n+\frac{1}{2}}(\omega r)$. The latter products, in which H means the Hankel-function, approach, apart from numerical factors, to $e^{i\omega r}$ and $e^{-i\omega r}$, when r becomes large. Hence, if at large distance from the origin, where $A \rightarrow 1$ anyhow, one of these exponentials appears in lieu of F_n or G_n (or forms part of that F_n or G_n that does appear there), then that means an ingoing or outgoing spherical wave, according to the *sign* of the exponent. We shall use that in special cases.

On the other hand, slight and pretty obvious *generalisations*, needed in Section 5, are obtained by replacing *everywhere* in (1, 4)–(1, 9) the product $F_n(r)\, e^{i\omega t}$ by $F_n(r, t)$ and the product $G_n(r)\, e^{i\omega t}$ by $G_n(r, t)$ *and* $i\omega$ by $\frac{\partial}{\partial t}$, so that (1, 5) and (1, 7) turn into *partial* differential equations, controlling the functions of two arguments $F_n(r, t)$ and $G_n(r, t)$ respectively. It is easy to check directly, that this procedure yields solutions of (1, 1), more general as regards their time-dependence. But a moment's reflection on Fourier-analysis, or rather Fourier-synthesis with respect to t, is sufficient to render an actual check superfluous.

2. Scattering by an immovable singularity.

Although the point of genuine interest is, naturally, the total response of the electron to an external field, including its being set in motion, we shall first deal in full with the fictitious case, which consists in replacing the electron by an *immovable* singularity at the origin. That is useful for various reasons. First of all, we thus get the effect of non-linearity, so to speak, in pure breed: for a Lorentz electron would under these circumstances not scatter at all. Secondly, the greatest part of the calculus can be taken over without any change to the actual dynamical case. Yet the latter offers, as we shall see later on, one peculiar difficulty, which it is very agreeable to have well separated from the mere technicalities, always involved in a scattering-calculation.—In sects. 9 and 10 of N.O. we dealt with the fictitious case only, and only in first approximation. The results we, rather frivolously, compounded with the known results about the electronic motion, a procedure which, to my amazement, seems to introduce a wrong factor 2 in the Born-correction (see sect. 7, below). So we now remove, one by one, the two deficiencies of the N.O. treatment, viz., (i) the restriction to long waves, (ii) the taking the dynamics for granted.

Supposing the relevant solutions of (1, 5) and (1, 7), viz., the F_n with exponent zero at $r = 0$ and the G_n with exponent three at $r = 0$, to be known—which they are not—the tackling of the fictitious case is straightforward. Like any solution of either equation, ours must approach for $r >> 1$ to Bessel functions, index $n + \frac{1}{2}$, argument ωr, multiplied by $\sqrt{r}$, which we express thus:

$$\begin{aligned} F_n &\rightarrow A_n \sqrt{\omega r}\, J_{n+\frac{1}{2}}(\omega r) + A_{-n} \sqrt{\omega r}\, J_{-n-\frac{1}{2}}(\omega r) \\ G_n &\rightarrow B_n \sqrt{\omega r}\, J_{n+\frac{1}{2}}(\omega r) + B_{-n} \sqrt{\omega r}\, J_{-n-\frac{1}{2}}(\omega r). \end{aligned} \qquad (2, 1)$$

Naturally, the equations only determine the *ratios* A_{-n}/A_n and B_{-n}/B_n. *We assume them to be known for every n and* ω.

Let us now assume a plane, linearly polarized incident wave, consisting —if the electron were not there—solely of the one component of vector-potential

$$A_y = \frac{ia}{\omega} e^{i\omega(t+z)} \qquad (2, 2)$$

and having thus in the origin—if the electron were not there—the electric vector

$$E_y^{incident} = a\, e^{i\omega t}. \qquad (2, 2)'$$

It would, in this case, be exactly represented by the following ϕ_k's and ψ_k's, where *now*, of course, only the *sums* $\phi_k + \psi_k$ have a physical meaning:

$$\begin{aligned} \phi_1 &= a\omega^{-3} e^{i\omega t} \sqrt{\frac{\pi}{2}} \sum_1^\infty i^n (2n+1) P_n^1 \sin\phi\, r^{-2} \sqrt{\omega r}\, J_{n+\frac{1}{2}}(\omega r) \\ \phi_2 &= a\omega^{-3} e^{i\omega t} \sqrt{\frac{\pi}{2}} \sum_1^\infty i^n \frac{2n+1}{n(n+1)} (P_n^1)_\theta \sin\phi \left(\sqrt{\omega r}\, J_{n+\frac{1}{2}}(\omega r)\right)_r \\ \phi_3 &= a\omega^{-3} e^{i\omega t} \sqrt{\frac{\pi}{2}} \sum_1^\infty i^n \frac{2n+1}{n(n+1)} P_n^1 \cos\phi \left(\sqrt{\omega r}\, J_{n+\frac{1}{2}}(\omega r)\right)_r \\ \psi_1 &= 0 \\ \psi_2 &= a\omega^{-2} e^{i\omega t} \sqrt{\frac{\pi}{2}} \sum_1^\infty i^{n+1} \frac{2n+1}{n(n+1)} (\sin\theta)^{-1} P_n^1 \sin\phi \sqrt{\omega r}\, J_{n+\frac{1}{2}}(\omega r) \\ \psi_3 &= a\omega^{-2} e^{i\omega t} \sqrt{\frac{\pi}{2}} \sum_1^\infty i^{n+1} \frac{2n+1}{n(n+1)} \sin\theta\, (P_n^1)_\theta \cos\phi \sqrt{\omega r}\, J_{n+\frac{1}{2}}(\omega r). \end{aligned} \qquad (2, 3)$$

Let me skip re-stating this well-known expansion, though it may appear here in a slightly simpler form than usual.

This expansion *could be imitated exactly*, though only asymptotically for $r \rightarrow \infty$, by inserting (2, 1) in (1, 4) and (1, 6), then combining the single multipole waves with the coefficients A_n and B_n demanded by (2, 3)—*if* the coefficients A_{-n} and B_{-n} were all zero. Thus, *except for them*, the presence of the electron would remain unnoticed at large distance. *The non-vanishing ratios* A_{-n}/A_n and B_{-n}/B_n *alone are responsible for the scattering.*

To render mathematically the assumption, that the incident wave consist precisely of (2, 2), it would be nearly correct, but not quite correct to choose, e.g. for A_n the value

$$a\,\omega^{-3}\sqrt{\frac{\pi}{2}}\;i^n\,\frac{2n+1}{n(n+1)},$$

indicated by comparing (2, 3) with (1, 4). It is perhaps quite interesting to mention that this rough procedure, when applied to the ordinary Rayleigh scattering, just obliterates the effect of radiation damping. The correct procedure is well known to be as follows. With r large

$$\begin{aligned}\sqrt{\omega r}\,J_{n+\frac{1}{2}}(\omega r) &\to \sqrt{\frac{2}{\pi}}\cos\left(\omega r-\frac{(n+1)\pi}{2}\right)=\frac{i^{-n-1}}{\sqrt{2\pi}}e^{i\omega r}+\frac{i^{n+1}}{\sqrt{2\pi}}e^{-i\omega r}\\ \sqrt{\omega r}\,J_{-n-\frac{1}{2}}(\omega r) &\to \sqrt{\frac{2}{\pi}}\cos\left(\omega r+\frac{n\pi}{2}\right)=\frac{i^{n}}{\sqrt{2\pi}}e^{i\omega r}+\frac{i^{-n}}{\sqrt{2\pi}}e^{-i\omega r}.\end{aligned}\tag{2, 4}$$

The parts with the "positive" exponentials, when combined with $e^{i\omega t}$ are obviously *ingoing* waves, the others *outgoing* waves. *The former have to be matched. The supernumerary parts of the latter are the scattered radiation.* Hence our constants have to satisfy the relations

$$\begin{aligned}\frac{i^{-n-1}}{\sqrt{2\pi}}A_n+\frac{i^n}{\sqrt{2\pi}}A_{-n} &= a\omega^{-3}\sqrt{\frac{\pi}{2}}\;i^n\frac{2n+1}{n(n+1)}\frac{i^{-n-1}}{\sqrt{2\pi}}\\ \frac{i^{-n-1}}{\sqrt{2\pi}}B_n+\frac{i^n}{\sqrt{2\pi}}B_{-n} &= a\omega^{-2}\sqrt{\frac{\pi}{2}}\;i^{n+1}\frac{2n+1}{n(n+1)}\frac{i^{-n-1}}{\sqrt{2\pi}},\end{aligned}\tag{2, 5}$$

whereas the potentials of the scattered waves for large r are obtained by putting $A = 1$ and $m = 1$ in (1, 4) and (1, 6) and inserting there for F_n and G_n the exponential $e^{-i\omega r}$, multiplied by two other constants, which we call a_n and b_n respectively, and which are determined by

$$\begin{aligned}\frac{i^{n+1}}{\sqrt{2\pi}}A_n+\frac{i^{-n}}{\sqrt{2\pi}}A_{-n}-a\omega^{-3}\sqrt{\frac{\pi}{2}}\frac{2n+1}{n(n+1)}\frac{i^{n+1}}{\sqrt{2\pi}} &= a_n\\ \frac{i^{n+1}}{\sqrt{2\pi}}B_n+\frac{i^{-n}}{\sqrt{2\pi}}B_{-n}-a\omega^{-2}\sqrt{\frac{\pi}{2}}\frac{2n+1}{n(n+1)}\frac{i^{n+1}}{\sqrt{2\pi}} &= b_n.\end{aligned}\tag{2, 6}$$

It is convenient to introduce by

$$\frac{A_{-n}}{A_n}=\tan\delta_n\,; \qquad \frac{B_{-n}}{B_n}=\tan\delta_n' \tag{2, 7}$$

the *real* constants δ_n and δ_n', which we call the *phase-shifts*. They are of course, functions of ω, determined unequivocally, but in a *very* complicated way by the equations (1, 5) and (1, 7) alone. By working out (2, 5) and (2, 6) we get

$$\begin{aligned} a_n &= \frac{a\omega^{-3} \tan \delta_n}{1 + (-1)^n i \tan \delta_n} \frac{2n+1}{n(n+1)} \\ b_n &= \frac{ia\omega^{-2} \tan \delta_n'}{1 + (-1)^n i \tan \delta_n'} \frac{2n+1}{n(n+1)} \end{aligned} \tag{2, 8}$$

The computation of the total energy flow includes nothing worth speaking of, since the field components (1, 8) and (1, 9) asymptotically coincide with the familiar ones. The scattered energy per unit time, integrated over the sphere, is given by the expressions

$$e_n = \frac{2n+1}{4} a^2\omega^{-2} \sin^2 \delta_n\,; \qquad e_n' = \frac{2n+1}{4} a^2\omega^{-2} \sin^2 \delta_n', \tag{2, 9}$$

where n is the order of the multipole and the prime indicates the magnetic case. The *total* scattering is

$$e_{total} = \sum_1^\infty (e_n + e_n'). \tag{2, 10}$$

There is, of course, an interference effect, because all the multipole radiations are coherent. But it cancels out on integrating over all angles. The peculiar angular distribution it produces was discussed in N.O., sect. 10b, for long waves, when the two kinds of dipole-scattering ($n = 1$) prevail and are, in the case of the fixed electron contemplated here, of the same order of magnitude.

The factor $8\pi/a^2$, being the reciprocal of the energy flow in the incident wave, has to be added in (2, 9), in order to obtain the individual *scattering cross-sections.*

Though we have thus completed the formal theory of scattering of light of arbitrary wave length by a fixed Born-singularity, yet already the complete *identity* of the two expressions (2, 9) can tell us that they do not yet contain anything characteristic of our particular problem, and would, indeed, be the same, whatever "mechanism" near the origin produced the phase-shifts. The only non-trivial statement is, that and precisely in what way the two sets of equations (1, 5) and (1, 7) represent this mechanism in our case. To find from them the two infinite sets of functions of ω which they thus determine, *that* is truly *our* scattering problem. The next section explains a method of obtaining at least their *exponents* at $\omega = 0$ and, at the expense of much labour, the early parts of their power series.

3. Determination of the phase-shifts.

(a) *General method.*

To fix the ideas, we concentrate in the first three subsections on the electric case, equ. (1, 5). Yet everything applies, with slight but relevant changes, also to the other equation, which has only *one sign* different. The changes are occasionally mentioned forthwith, but dealt with in full in subsect. (d). If you wish to save turning leaves whilst reading the following paragraph, keep your eye on (3, 2) below, which is the form of (1, 5) obtained by the transformation (3, 1). But, for the moment, the exact wording still refers to (1, 5).

Out of many attempts I made, the only one to meet with any success at all was the obvious and well-known one of expanding *the solution itself* into a series of ascending powers of ω^2. It is a method of iterated integrations. The absolute term in the series is a solution of the equation with $\omega^2 = 0$ (let us always call that *the homogeneous one,* for shortness), notably *that* solution which itself satisfies the requirement at $r = 0$; the coefficient of ω^{2k} is always a solution of the corresponding *inhomogeneous* equation with second member: minus the coefficient of ω^{2k-2}; notably *that* solution thereof which *does not impair* the requirement; the requirement, it may be remembered, is: exponent zero at $r = 0$ (exponent three in the magnetic case).

But that is not all. The second, and indeed the most delicate, part of our investigation will consist in examining the solution thus obtained, with regard to the *proportion* in which it "contains" the two Bessel functions—see the statement (2, 1). We come back to that point very soon, but must first give details about the iteration process.

The simple transformation

$$z = r^4, \qquad y(z) = F_n(r) \tag{3, 1}$$

by turning (1, 5) into

$$y'' + \frac{5+3z}{4z(1+z)}\, y' - \frac{n(n+1)}{16z(1+z)}\, y + \frac{\omega^2}{16z^{\frac{3}{2}}}\, y = 0, \tag{3, 2}$$

greatly facilitates both parts of our enterprise, because the "homogeneous" equation is now of hypergeometric type. Its two fundamental solutions at $r = 0$ are

$$\left.\begin{aligned} y_1 &= F\left(\frac{n}{4},\ -\frac{n+1}{4};\ \frac{5}{4};\ -z\right) \\ y_2 &= z^{-\frac{1}{4}}\, F\left(\frac{n-1}{4},\ -\frac{n+2}{4};\ \frac{3}{4};\ -z\right), \end{aligned}\right\} \tag{3, 3}$$

where F means the Gaussian series, this notation not to be confounded with F_n. On the other hand, I beg the reader to *remember* that y, y_1, y_2 carry a "silent" subscript n which, particularly when you have to specify it, is actually *easier* to remember, than to disentangle from the other one.

The function from which to start our development in powers of ω^2 is obviously y_1. Let us put

$$y = y_1 + \omega^2 w_1 + \omega^4 w_2 + \ldots + \omega^{2k} w_k + \ldots \qquad (3, 4)$$

The reciprocal of the Jacobian of y_1 and y_2 is easily found to be

$$\Delta^{-1} = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}^{-1} = -4 z^{\frac{5}{4}} (1+z)^{-\frac{1}{2}}.$$

Hence from the well-known expression for a solution of the inhomogeneous equation we get

$$\begin{aligned} w_1 &= \tfrac{1}{4} \left\{ - y_1 \int_0^z z^{-\frac{1}{4}} (1+z)^{-\frac{1}{2}} y_1 y_2 \, dz + y_2 \int_0^z z^{-\frac{1}{4}} (1+z)^{-\frac{1}{2}} y_1^2 \, dz \right\} \\ w_2 &= \tfrac{1}{4} \left\{ - y_1 \int_0^z z^{-\frac{1}{4}} (1+z)^{-\frac{1}{2}} w_1 y_2 \, dz + y_2 \int_0^z z^{-\frac{1}{4}} (1+z)^{-\frac{1}{2}} w_1 y_1 \, dz \right\} \\ & \ldots \ldots \ldots \ldots \ldots \ldots \end{aligned} \qquad (3, 6)$$

The lower limits of the integrals are, to begin with, *all* arbitrary. For the *second* integral in every line the limit zero is uniquely imposed by the singularity $z^{-\frac{1}{4}}$ of y_2, which would otherwise infect the w in question. In the case of the *first* integral in every line the choice is actually free again and again, but adds nothing to generality, because to change one of these limits amounts to nothing more than to adding a *constant* multiplier to the total solution (3, 4). The value zero, always permissible with regard to convergence, is recommended by simplicity.

Before embarking on actual computations it is necessary to scrutinize further what I called the more delicate part of our task, which consists in finding out from an expansion like (3, 4) the ratio A_{-n}/A_n which the function, represented by (3, 4), will exhibit, when decomposed according to (2, 1), this decomposition holding only asymptotically for $r >> 1$. The trouble is that, naturally, only a few initial terms of (3, 4) can be actually computed, by far too few for comparing them directly with the asymptotic behaviour (2, 4) the Besselfunctions exhibit for $\omega r >> 1$. Hence, as far as I can see, only the permanently convergent power series are available for the latter. Since a power series, though permanently convergent, is unfit for practical use with large values of its argument, the salvation lies only in the simple fact, that with ω small you will find a region where ωr is still small or at least moderate, so that the power series are of avail, yet r itself is already large enough, to justify the asymptotic decomposition into Besselfunctions, and also, as we shall see, to obtain manageable expressions for the coefficient-functions w_k in (3, 4).

Assume first ω to be *arbitrarily* small, so that, from (3, 4), (3, 3) and (3, 1), the function

$$y = y_1 = F\left(\frac{n}{4}, \ -\frac{n+1}{4}; \ \frac{5}{4}; \ -z\right) \tag{3, 7}$$

is a sufficient approximation for $F_n(r)$. In the point $z^{-1} = 0$, that is to say at infinity, y_1 is a certain well-known mixture (see equ. (3, 9) below) of the two fundamental solutions *there,* which are represented by two power series in z^{-1}, valid for $z > 1$ and having the *leading powers* $z^{-\frac{n}{4}}$ and $z^{\frac{n+1}{4}}$ (or r^{-n} and r^{n+1}) respectively. Now these are precisely the leading powers of the two permanently converging power series that represent the two r-functions in the decomposition (2, 1), one of which—let that be noted—is always an even function, the other an odd one. Hence, in virtue of the existence of the *intermediate* range of r described above, there is no doubt that in this case the desired ratio A_{-n}/A_n can be read off by equating to unity the cross-ratio of the two couples of leading coefficients.

(b) *First approximation.*

Let us carry out this first approximation at once, to see how it works. The relevant transition relation for the hypergeometric function, written with $-z$, to serve our purpose, reads[3]

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\gamma)} F(\alpha, \beta; \gamma; -z) = \frac{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)} z^{-\alpha} F(\alpha, 1-\gamma+\alpha; 1-\beta+\alpha; -z^{-1})$$
$$+ \frac{\Gamma(\beta)\,\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)} z^{-\beta} F(\beta, 1-\gamma+\beta; 1-\alpha+\beta; -z^{-1}). \tag{3, 8}$$

Using this for (3, 7) we get

$$y_1 = \frac{\Gamma\left(-\frac{2n+1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(-\frac{n-5}{4}\right)\Gamma\left(-\frac{n+1}{4}\right)} z^{-\frac{n}{4}} F\left(\frac{n}{4}, \ \frac{n-1}{4}; \ \frac{2n+5}{4}; \ -z^{-1}\right) +$$
$$+ \frac{\Gamma\left(\frac{2n+1}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{n+6}{4}\right)\Gamma\left(\frac{n}{4}\right)} z^{\frac{n+1}{4}} F\left(-\frac{n+1}{4}, \ -\frac{n+2}{4}; \ -\frac{2n-3}{4}; \ -z^{-1}\right) \tag{3, 9}$$

On the other hand, from the well-known power series for the Bessel-

[3] See Whittaker and Watson, Modern Analysis, 4th edition, p. 289, where, by the way, a slight flaw in some of the Γ-functions has to be emended.—That our argument is not z, but $-z$ is vital, otherwise the third singular point would block the transition on the real positive axis, where we need it.

functions (x is a neutral variable, used only for this moment)

$$\sqrt{x}\, J_{n+\frac{1}{2}}(x) = \frac{x^{n+1}}{2^{n+\frac{1}{2}}\, \Gamma\left(\frac{2n+3}{2}\right)} \left(1 + O(x^2)\right)$$

$$\sqrt{x}\, J_{-n-\frac{1}{2}}(x) = \frac{x^{-n}}{2^{-n-\frac{1}{2}}\, \Gamma\left(-\frac{2n-1}{2}\right)} \left(1 + O(x^2)\right). \qquad (3, 10)$$

Hence from (2, 1)

$$F_n \to \frac{A_n\,(\omega r)^{n+1}}{2^{n+\frac{1}{2}}\, \Gamma\left(\frac{2n+3}{2}\right)} \left(1 + O(\omega^2 r^2)\right) +$$

$$+ \frac{A_{-n}\,(\omega r)^{-n}}{2^{-n-\frac{1}{2}}\, \Gamma\left(-\frac{2n-1}{2}\right)} \left(1 + O(\omega^2 r^2)\right). \qquad (3, 11)$$

Remembering $z^{\frac{1}{2}} = r$ and equating, according to plan, the ratios of the leading coefficients in (3, 9) and (3, 11) we get

$$\tan \delta_n = \frac{A_{-n}}{A_n} = \left(\frac{\omega}{2}\right)^{2n+1} \frac{\Gamma\left(-\frac{2n-1}{2}\right)\Gamma\left(-\frac{2n+1}{4}\right)\Gamma\left(\frac{n+6}{4}\right)\Gamma\left(\frac{n}{4}\right)}{\Gamma\left(\frac{2n+3}{2}\right)\Gamma\left(\frac{2n+1}{4}\right)\Gamma\left(-\frac{n-5}{4}\right)\Gamma\left(-\frac{n+1}{4}\right)}. \qquad (3, 12)$$

You observe that the ratio is *never infinite.* If n is *odd* and > 1, either $n - 5$ or $n + 1$ is a non-negative integral multiple of 4, hence in this case $\tan \delta_n$ is *zero*—in our present approximation, not necessarily in the higher ones, which proceed, of course, *always* in odd powers of ω. So we can only say

$$\tan \delta_{2k+1} = O(\omega^{4k+5}); \qquad k = 1, 2, 3, 4 \ldots, \qquad (3, 13)$$

where $O(\omega^s)$ means: of order ω^s *or smaller.*

The case $n = 1$ is exceptional, and we take it in advance, in order to have done with *all* odd n. (The *magnetic* dipole, by the way, will prove to be *not* exceptional, because it will turn out that in the magnetic case it is the *even* n's for which the first approximation gives nothing.) With $n = 1$

$$\tan \delta_1 = \left(\frac{\omega}{2}\right)^3 \frac{\Gamma(-\frac{1}{2})\,\Gamma(-\frac{3}{4})\,\Gamma(\frac{7}{4})\,\Gamma(\frac{1}{4})}{\Gamma(\frac{5}{2})\,\Gamma(\frac{3}{4})\,\Gamma(1)\,\Gamma(-\frac{1}{2})}. \qquad (3, 14)$$

An easy reduction gives you

$$\tan \delta_1 = -\,\omega^3 \frac{\Gamma(\frac{1}{4})^2}{6\sqrt{\pi}}$$

$$= -\frac{2\gamma}{3}\omega^3 = -\tfrac{2}{3}\cdot 1{\cdot}8541\,\omega^3. \qquad (3, 15)$$

Here we have re-introduced Born's constant γ,—that it is a namesake of the customary third argument in Gauss' series is an accident. Originally defined by the complete elliptic integral

$$\gamma = \int_0^\infty \frac{dr}{\sqrt{1+r^4}},$$

it can be transformed, by $r^4 = z$, thus

$$\gamma = \tfrac{1}{4}\int_0^\infty z^{-\frac{3}{4}}(1+z)^{-\frac{1}{2}}\,dz = \frac{\Gamma(\frac{1}{4})^2}{4\sqrt{\pi}}, \qquad (3,\ 16)$$

where we have used

$$\int_0^\infty z^{p-1}(1+z)^{-p-q}\,dz = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}.$$

By the way: $\Gamma(\frac{1}{4})$, more especially just γ, is the only transcendental number, besides π, that I have met with hitherto in developing Born's theory.

To check (3, 15) with previous findings we calculate e_1 from (2, 9) and multiply it by $8\pi/a^2$, to get the corresponding cross-section. We obtain for it $8\pi\gamma^2\omega^4/3$, which corresponds exactly to the polarizability γ found in N.O. by a different method (for checking, use (10, 3) l.c. and consider that $\omega = 2\pi/\lambda$, in our units).—

If n is *even*, the Γ-quotient in (3, 12) is reduced by first eliminating *those* Γ's in which n occurs with a minus-sign, with the help of $\Gamma(p)\Gamma(1-p) = \pi/\sin p\pi$, then using the formula[4]

$$\Gamma(p)\ \Gamma\left(p+\frac{1}{s}\right)\Gamma\left(p+\frac{2}{s}\right)\ldots\ \Gamma\left(p+\frac{s-1}{s}\right) = (2\pi)^{\frac{s-1}{2}}\ s^{\frac{1}{2}-sp}\ \Gamma(sp)$$

for $s = 4$ and $p = \dfrac{n}{4}$. After some obvious reductions you get

$$\tan\delta_n = -\left(\frac{\omega}{4}\right)^{2n+1}\frac{4\pi^{\frac{3}{2}}\ \Gamma(n+3)}{n(n-1)\ \Gamma\left(\frac{2n+3}{2}\right)^2\Gamma\left(\frac{2n+1}{4}\right)};\quad (n = \text{even}). \qquad (3,\ 17)$$

The coefficient can be expressed by γ alone, which this time is in the denominator. E.g.

$$\tan\delta_2 = -\frac{4}{75\gamma}\,\omega^5. \qquad (3,\ 18)$$

The *negative sign* of all these δ's must mean something similar to what for $n = 1$ we express by saying the polarizability is positive. To interpret the meaning for $n > 1$ precisely, one would have to return to the considerations which led up to (2, 5) and (2, 6); but it is not interesting.

[4] Whittaker and Watson, Modern Analysis, 4th ed., p. 240.

The order of magnitude of the *scattering* decreases rapidly for the higher multipoles, *even more rapidly* than the power ω^{2n+1} seems to indicate. For even the *next term* in $\tan \delta_1$, which is expected to be, and actually is, of order ω^5, the same as $\tan \delta_2$, still makes a larger contribution to the total scattering than the whole quadrupole effect. For, in *co-operation* with ω^3, it produces in (2, 9) the power ω^6, whereas the unaided ω^5 of $\tan \delta_2$ only produces ω^8.—This situation is still more marked in the case of the *yielding electron*; for the yielding, as we shall see, produces in $\tan \delta_1$ a term with the *first* power of ω, preceding the one with ω^3, while all the other phase-shifts remain the same, up to all orders in ω.

(c) *Higher approximations.*

The general procedure will be illustrated by the second approximation for the dipole, $n = 1$. The solutions of what we called the "homogeneous equation," which were given in (3, 3) for arbitrary n, are for $n = 1$ the following:

$$y_1 = \tfrac{1}{3}\left(\sqrt{1+z} + \tfrac{1}{2} z^{-\frac{1}{4}} I\right); \quad \text{with} \quad I(z) = \int_0^z z^{-\frac{3}{4}} (1+z)^{-\frac{1}{2}} dz \qquad (3, 19)$$
$$y_2 = z^{-\frac{1}{4}}.$$

The second is obvious, because the second Gauss-series in (3, 3) equals 1 for $n = 1$. From y_2 y_1 is easily made out directly. From (3, 16) it will be seen that

$$I(\infty) = 4\gamma. \qquad (3, 20)$$

To obtain w_1 we insert (3, 19) in the first of the recurrent formulae (3, 6), giving

$$w_1 = \frac{1}{36}\left\{ -\left(\sqrt{1+z} + \tfrac{1}{2} z^{-\frac{1}{4}} I\right) \int_0^z z^{-\frac{1}{2}} (1+z)^{-\frac{1}{2}} \left(\sqrt{1+z} + \tfrac{1}{2} z^{-\frac{1}{4}} I\right) dz + \right.$$
$$\left. + z^{-\frac{1}{4}} \int_0^z z^{-\frac{1}{4}} (1+z)^{-\frac{1}{2}} \left(\sqrt{1+z} + \tfrac{1}{2} z^{-\frac{1}{4}} I\right)^2 dz \right\}. \qquad (3, 21)$$

The *five* integrals with which we are faced here have to be treated, more or less each separately, by partial integrations—the *first* scope being to express them as *simple* integrals, which brings them in direct reach of tables; the *second* scope, to transform those of the simple integrals which *diverge* at $z^{-1} = 0$, where the behaviour of w_1 interests us, into convergent ones. The result of this reduction is

$$w_1 = \frac{1}{9}\left\{ -\left(\sqrt{1+z} + \tfrac{1}{2} z^{-\frac{1}{4}} I\right)\left(\tfrac{1}{2}\sqrt{z} + \frac{1}{16} I^2\right) + \right.$$
$$+ z^{-\frac{1}{4}} \left(\frac{4}{5} \int_0^z z^{-\frac{1}{4}} (1+z)^{-\frac{3}{2}} dz - \frac{8}{5} z^{\frac{3}{4}} (1+z)^{-\frac{1}{2}} + \frac{1}{5} z^{\frac{3}{4}} (1+z)^{\frac{1}{2}} + \right. \qquad (3, 22)$$
$$\left. + \frac{1}{2} z^{\frac{1}{2}} I + \frac{1}{48} I^3 \right\}.$$

This is now easy to expand at infinity; for instance

$$
\begin{aligned}
I(z) &= 4\gamma - 4z^{-\frac{1}{4}} + \frac{2}{5}z^{-\frac{5}{4}} - \frac{1}{6}z^{-\frac{9}{4}} + - \dots \\
\int_0^z z^{-\frac{1}{4}}(1 + z)^{-\frac{3}{2}}\,dz &= \frac{\pi}{\gamma} - \frac{4}{3}z^{-\frac{3}{4}} + \frac{6}{7}z^{-\frac{7}{4}} - \frac{15}{22}z^{-\frac{11}{4}} + - \dots,
\end{aligned}
\tag{3, 23}
$$

whilst everything else is still more obvious. Let us *imagine* this expansion to be performed, and indicate it, for shortness, by $\{w_1\}_{expand}$. We must insert it in (3, 4), where we also have to insert for y_1 the expansion (3, 9), already used before, to be specialized now for $n = 1$, when it reads

$$
y_1 = \frac{2\gamma}{3} z^{-\frac{1}{4}} + \tfrac{1}{3} z^{\frac{1}{2}} F(-\tfrac{1}{2}, -\tfrac{3}{4}; \tfrac{1}{2}; -z^{-1}). \tag{3, 24}
$$

Hence the second approximation to the required solution of (1, 5) or (3, 2), for $n = 1$, reads

$$
y(z) = F_1(r) = \frac{2\gamma}{3} z^{-\frac{1}{4}} + \tfrac{1}{3} z^{\frac{1}{2}}[1 + O(z^{-1})] + \omega^2 \{w_1\}_{expand}. \tag{3, 25}
$$

On the other hand, (3, 11) gives in this case

$$
\begin{aligned}
F_1(r) \rightarrow &- A_{-1}\,\omega^{-1} r^{-1} \sqrt{\frac{2}{\pi}} \left[1 + O(\omega^2 r^2)\right] + \\
&+ A_1\,\omega^2 r^2\, \tfrac{1}{3} \sqrt{\frac{2}{\pi}} \left[1 + O(\omega^2 r^2)\right].
\end{aligned}
\tag{3, 26}
$$

From the confrontation of the last two expansions we must extract second order information about the ratio A_{-n}/A_n, the ultimate scope of all our striving.—One feels tempted again just to compare the leading terms as before, which have remained the same in (3, 26), but have received a small correction, coming from $\{w_1\}_{expand}$ in (3, 25). It so happens that, proceeding thus thoughtlessly, we should commit no fault in the present example, but we would in general and in principle, if the case is to stand as a paradigm of the general method of expanding the phase-shifts in powers of ω^2. The mere fact that (3, 25) needed emendation by the w_1-term, must make us suspect that (3, 26) wants overhauling as well. Indeed, as it stands, it is asymptotic only. This circumstance might, and indeed does, falsify its leading coefficients in the proportion of $[1 + O(\omega^s)] : 1$. We shall find $s = 4$, innocuous in the present case.

Let me expose the method very succinctly. It consists in solving our equation (1, 5) by another iteration process, very similar to the one carried out above in the variable $z = r^4$. This time we have to use the variables

$$
x = \omega r \quad ; \qquad f(x) = F_n(r) \tag{3, 27}
$$

and have to regard as the main part of the operator of the equation *that* part which, standing by itself, would produce the Besselsolutions

$\sqrt{x}\, J_{n+\frac{1}{2}}$, $\sqrt{x}\, J_{-n-\frac{1}{2}}$, whereas everything that *from this point of view* is superabundant is treated as "perturbation" or "second member." Throwing it actually to the right, the equ. (1, 5) reads

$$f'' + \left(1 - \frac{n(n+1)}{x^2}\right) f = -\omega^4 \frac{2xf' + n(n+1)f}{x^2(\omega^4 + x^4)} \qquad (3, 28)$$

$$= -\omega^4 x^{-6}(2xf' + n(n+1)f)[1 + O(\omega^4)].$$

The underlying idea in the last line is, that x be $\gg \omega$ (that is $r \gg 1$), though x itself may be fairly small. If you put, considering (2, 1),

$$\begin{aligned} f_1 &= \sqrt{x}\, J_{n+\frac{1}{2}}(x), \qquad f_2 = \sqrt{x}\, J_{-n-\frac{1}{2}}(x) \\ f_0 &= A_{-n} f_2 + A_n f_1 \\ f &= f_0 + \omega^4 u_1 + \omega^8 u_2 + \ldots, \end{aligned} \qquad (3, 29)$$

you easily get

$$u_1 = (-1)^{n+1} \frac{\pi}{2} \left\{ - f_1 \int_x^\infty x^{-6} f_2 (2xf' + n(n+1)f)\, dx + \right. \qquad (3, 30)$$

$$\left. + f_2 \int_x^\infty x^{-6} f_1 (2xf' + n(n+1)f)\, dx \right\},$$

and so on—but if one actually intended to continue the iterations, a sufficient amount of the $O(\omega^4)$-terms out of (3, 28) would have to be included *even already* in the formula we have just written down.—The constant limits of the integrals are this time dictated by the necessity of annihilating the u_k's for $x \to \infty$, since f_0 is to be the asymptotic solution.—We need not continue. It is clear that (3, 30) would have to be worked out and taken into account *only at the next step*, the step that would include w_2 in (3, 4).—

Returning after this sort of digression to the actual comparison of the characteristic coefficients in (3, 25) and (3, 26), we need the terms with $z^{-\frac{1}{4}}$ and $z^{\frac{3}{4}}$ out of the expansion abbreviated by $\{w_1\}_{expand.}$ From (3, 22) they work out thus

$$\{w_1\}_{expand.} = \ldots - z^{\frac{3}{4}} \frac{\gamma^2}{9} + \ldots + z^{-\frac{1}{4}} \left(-\frac{5}{27}\gamma^3 + \frac{4\pi}{45\gamma}\right) + \ldots \qquad (3, 31)$$

Hence the characteristic terms in (3, 25) read

$$y(z) = F_1(r) = \ldots z^{\frac{3}{4}} \left(\frac{1}{3} - \omega^2 \frac{\gamma^2}{9}\right) + \ldots +$$

$$+ z^{-\frac{1}{4}} \left[\frac{2\gamma}{3} + \omega^2 \left(-\frac{5}{27}\gamma^3 + \frac{4\pi}{45\gamma}\right)\right] + \ldots \qquad (3, 32)$$

Equating their ratio with the one in (3, 26) you get

$$\tan \delta_1 = \frac{A_{-1}}{A_1} = -\frac{2\gamma}{3}\omega^3 - \left(\frac{4\pi}{45\gamma} + \frac{1}{27}\gamma^3\right)\omega^5 + \ldots$$

$$= -1{\cdot}2361\, \omega^3 - 0{\cdot}3867\, \omega^5 + \ldots \qquad (3, 33)$$

(d) *Magnetic multipole waves.*

The *magnetic* case can now be explained in comparative shortness, because from its electric counterpart, dealt with sub (*a*), (*b*), (*c*), it differs only by the fact that the *hypergeometric equation* which serves as an approximation, for $\omega \rightarrow 0$, to the *now* competent equation

$$y'' + \frac{1 + 3z}{4z(1 + z)} y' - \frac{n(n + 1)}{16z(1 + z)} y + \frac{\omega^2}{16z^{\frac{3}{2}}} y = 0 \qquad (3, 34)$$

—obtained from the now competent (1, 7) by the substitution

$$z = r^4, \qquad y(z) = G_n(r) \quad — \qquad (3, 35)$$

—I say, the difference lies only in the fact, that the approximating hypergeometric equation has the third argument of its standard solution $\frac{1}{4}$ instead of $\frac{5}{4}$; and in the further fact, that of its two fundamental solutions at $z = 0$, viz.,

$$\left.\begin{aligned} y_1 &= z^{\frac{3}{4}} F\left(\frac{n + 3}{4}, -\frac{n - 2}{4}; \frac{7}{4}; -z\right) \\ y_2 &= F\left(\frac{n}{4}, -\frac{n + 1}{4}; \frac{1}{4}; -z\right) \end{aligned}\right\} \qquad (3, 36)$$

the physical problem selects y_1 to be the approximation function, because the physical problem demands of G_n the exponent 3, not 0, as it did of F_n. Apart from these two changes our present investigation runs completely parallel to the previous one. To emphasize the parallelism, I have used the same letters y, y_1, y_2 as before, and so will I do with $w_1, w_2, w_3, \cdots\cdots$. I hope that will not be confusing. Each of these symbols stood for an *infinite family* of functions anyhow, and now it stands for a second one—viz., for the magnetic counterpart of the first.

This notation agreed upon, the expansion (3, 4) reads alike and we waive copying it. But the reciprocal Jacobian of (3, 36) is

$$\Delta^{-1} = \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}^{-1} = -\tfrac{4}{3} z^{\frac{1}{4}} (1 + z)^{\frac{1}{2}}, \qquad (3, 37)$$

different from before, but again independent of n. That alone causes a difference in the iteration formulae, corresponding to (3, 6), which read

$$\left.\begin{aligned} w_1 &= \frac{1}{12}\left\{- y_1 \int_0^z z^{-\frac{5}{4}} (1 + z)^{-\frac{1}{2}} y_1 y_2 \, dz + y_2 \int_0^z z^{-\frac{5}{4}} (1 + z)^{-\frac{1}{2}} y_1^2 \, dz\right\} \\ w_2 &= \frac{1}{12}\left\{- y_1 \int_0^z z^{-\frac{5}{4}} (1 + z)^{-\frac{1}{2}} w_1 y_2 \, dz + y_2 \int_0^z z^{-\frac{5}{4}} (1 + z)^{-\frac{1}{2}} w_1 y_1 \, dz\right\} \end{aligned}\right\} \qquad (3, 38)$$

. . . .

Applying the general transition-relation (3, 8) to our present y_1 of (3, 36) we get

$$y_1 = \frac{\Gamma\left(-\frac{2n+1}{4}\right)\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(-\frac{n-4}{4}\right)\Gamma\left(-\frac{n-2}{4}\right)} z^{-\frac{n}{4}} F\left(\frac{n+3}{4},\ \frac{n}{4};\ \frac{2n+5}{4};\ -\frac{1}{z}\right) +$$

$$+ \frac{\Gamma\left(\frac{2n+1}{4}\right)\Gamma\left(\frac{7}{4}\right)}{\Gamma\left(\frac{n+5}{4}\right)\Gamma\left(\frac{n+3}{4}\right)} z^{\frac{n+1}{4}} F\left(-\frac{n-2}{4},\ -\frac{n+1}{4};\ -\frac{2n-3}{4};\ -\frac{1}{z}\right), \tag{3, 39}$$

corresponding to (3, 9). The exponents at $z^{-1} = 0$ are seen to be $\frac{n}{4}$ and $-\frac{n+1}{4}$, the same as before. Equating the ratio of the coefficients of these powers again to the one in (3, 11)—which need not be re-written, just think G_n, B_n, B_{-n} instead of F_n, A_n, A_{-n}—you obtain as the *first approximation,* corresponding to (3, 12),

$$\tan \delta_n' = \frac{B_{-n}}{B_n} = \left(\frac{\omega}{2}\right)^{2n+1} \frac{\Gamma\left(-\frac{2n-1}{2}\right)\Gamma\left(-\frac{2n+1}{4}\right)\Gamma\left(\frac{n+5}{4}\right)\Gamma\left(\frac{n+3}{4}\right)}{\Gamma\left(\frac{2n+3}{2}\right)\Gamma\left(\frac{2n+1}{4}\right)\Gamma\left(-\frac{n-4}{4}\right)\Gamma\left(-\frac{n-2}{4}\right)} \tag{3, 40}$$

Again, the ratio is *never infinite.* With *n even* ($= 2, 4, 6, \ldots$) either $n - 4$ or $n - 2$ is a non-negative multiple of 4, hence the ratio vanishes this time for even n. That means

$$\tan \delta_{2k}' = O\left(\omega^{4k+3}\right); \qquad k = 1, 2, 3, \ldots \tag{3, 41}$$

The case $n = 1$ is this time not exceptional, it can be treated along with *n odd.* The reduction of the Γ-quotient gives in this case

$$\tan \delta_n' = \left(\frac{\omega}{4}\right)^{2n+1} \frac{\pi^{\frac{3}{2}}\,\Gamma(n+2)}{n\,\Gamma\left(\frac{2n+1}{2}\right)^2 \Gamma\left(\frac{2n+5}{4}\right)^2}; \quad (n = \text{odd}). \tag{3, 42}$$

These are the magnetic phase-shifts for odd n in first approximation. For $n = 1$ you get, paying attention to (3, 16):

$$\tan \delta_1' = \frac{4\gamma}{9\pi}\,\omega^3. \tag{3, 43}$$

To show the agreement with the magnetic polarizability p_m, obtained in N.O. by a different method, we observe that the complete elliptic integral occurring in (9, 22) l.c., when transformed by the same substitution as used above, in (3, 16), for Born's integral, gives

$$p_m = -\frac{2}{3\pi}\,\gamma. \tag{3, 44}$$

[11*]

The simple ratio $-2/3\pi$ between the two polarizabilities had escaped my notice, until N.O. was under press. It is the same as that of $\tan \delta_1'$ and $\tan \delta_1$ by (3, 43) and (3, 15) above. That proves the agreement with N.O. for the former, because for the latter it has already been checked. The *positive sign* of $\tan \delta_1'$ indicates *dia*magnetism.

Since the magnetic dipole scattering is of the same order of magnitude as the electric one, I have, here too, computed the next approximation, which proves even more laborious than in the other case. Skipping the details, let me only mention that for $n = 1$ the fundamental solutions (3, 36) read

$$
\begin{aligned}
y_1 &= \tfrac{3}{4}\sqrt{1+z}\int_0^z z^{-\frac{1}{4}}(1+z)^{-\frac{3}{2}}\,dz \\
y_2 &= \sqrt{1+z}.
\end{aligned}
\qquad (3, 46)
$$

The integral, by the way, is the same that occurred in (3, 22) and (3, 23) and also the same as the elliptic integral mentioned in the last paragraph. The result of a computation, cramming five pages in a full-size quarto note-book, is that the next term in $\tan \delta_1'$ has the coefficient $\pi/18\gamma$. Thus

$$
\begin{aligned}
\tan \delta_1' &= \frac{4\gamma}{9\pi}\,\omega^3 + \frac{\pi}{18\gamma}\,\omega^5 + \ldots \\
&= 0{,}26230\,\omega^3 + 0{,}09413\,\omega^5 + \ldots
\end{aligned}
\qquad (3, 47)
$$

I wonder whether this result could be obtained in a few lines, once the theory of the equation (1, 7) were worked out.

Both results, (3, 33) and (3, 47), are in the direction, that with *de*creasing wave-length the scattering increases at first *more* rapidly than the fourth power of $\omega = 2\pi r_0/\lambda$.—Actually, of course, all this was only a preparation for the "real" case, when the singularity is not fixed, but is allowed to follow the impulse of the incident wave. Let us now turn to *this* problem.

4. The kinetic conditions.

As Born and his collaborators, mainly Pryce,[5] recognized, the field equations alone do not yet determine any *dynamics* of the electron, any definite connection, that is, between the motion of a point-singularity and the field around it. This fact is corroborated by the existence of solutions, already indicated in sect. 9 of N.O., which had a point singularity permanently at rest at the origin, yet went over at a moderate distance into a homogeneous electrostatic field of arbitrary, only not too great,

[5] Pryce, Proc. Roy. Soc., A, 155, p. 597, 1936.

strength, thus a field which we would expect to be compatible only with an *accelerated* singularity. We feel naïvely it "ought to set the electron in motion."

Hence, some condition has to be added to the field-equations, something *they do not yet contain.* It must be a condition the solution just alluded to and the more general one investigated on the preceding pages of the present paper *do not comply with.* Moreover, it must obviously be a condition that refers to a mathematically infinitesimal neighbourhood of the singularity. For surely it would not do in a field theory like this to have the shape of the world line of the singularity determined by *actio in distans.*

There can be little doubt as to the nature of the additional demand. The divergence of Born's energy-momentum-tensor vanishes everywhere where it is at all defined. From this, by Gauss' theorem, follow the conservation laws. But if the volume to which the partial integration is to be applied contains a singularity, the latter has to be excluded by a small closed surface, which afterwards has to be contracted to infinitesimal smallness towards the singularity. If and only if the residual surface integrals vanish in this limit, do the conservation laws hold for *any* closed surface. That we wish them to hold, furnishes the condition:

The surface integrals of stress and of energy flow have to vanish in the limit for any closed surface contracted to infinitesimal smallness towards the singularity.—For shortness let us call these four limiting values or residues, since the condition of their vanishing determines the motion, the *kinetic integrals.*

This postulate is not only materially identical with that of Born and Pryce, also the present formulation was known to them. But they stressed other formulations, as the vanishing of the variation of a certain fourdimensional integral; or, alternatively, the volume integral of a certain density of force, essentially the Lorentz one, to vanish. That comes dangerously near to the condition used with Lorentz' electron, where it was admittedly but an *asylum ignorantiae* and clearly involved *actio in distans* inasmuch as the body of the electron was supposed to be rigid or quasi-rigid and its motion as a whole was supposed to be determined by the simultaneous values of the field in the whole region which the body of the electron occupied. It was this quasi-rigidity which gave rise to all the trouble, violation of the energy principle, factor 4/3 in the mass energy relation, etc., difficulties which are probably the reason, why the Lorentz electron, to say the truth, has disappeared from modern theory and is replaced by a somnambulistic tampering with the mathematical-point-electron.—I have no objection to the Born-Pryce formulations, though I have no use for them myself. But it must be clearly understood, that they do not detract from the fact, that the motion of the singularity is exclusively determined by a condition imposed on the field in its immediate neighbourhood; just as the Hamiltonian principle in classical mechanics does not detract from the fact that the dependent variables there are determined by *differential* equations.

After adopting this new principle—new inasmuch as it is not yet contained in the field-equations—I examined the four kinetic integrals for the solution developed in sects. 2 and 3. Since the integrals obviously vanish for the *main field* and since the square of the *perturbation field* is neglected in principle, only the *bilinear* terms deserve attention. Hence every multipole produces, in bilinear co-operation with the main field, its own set of four kinetic integrals, which can be calculated each separately, without referring to the manner in which the multipoles combine to build up this or that particular solution, be it the one we have investigated or a different one. It need hardly be said that the two quasistatic cases dealt with in N.O. are included in the multipole-solutions—viz., for $n = 1$ and $\omega \rightarrow 0$.

A detailed calculus shows that in the *magnetic* case (ψ) *all* the kinetic integrals vanish already in virtue of the comparatively weak singularity, viz. of order r^{-1}, which the bilinear parts of the components of stress and energy-flow display as r approaches zero. As regards the *energy-flow,* the same is true in the *electric* case. But the bilinear part of the *stress* reaches in this case the order r^{-2}, just enough to produce a finite residue. Yet *all but one* of these 3 times $\aleph_0$ residues vanish by symmetry on integrating over the angles θ, ϕ, a sphere being taken as the surface of integration, *for convenience.* The non-vanishing residue is that of the y-component of stress for $n = 1$. y is the direction of the electric field at large distance.—Naturally and trivially, the other two electric-dipole-solutions, which do not turn up in the plane-wave-solution, also have non-vanishing stress-residues, but they need not interest us here.

This result is of great importance to us. An arbitrary field, prescribed at moderate distance from the singularity, can be resolved into plane waves. In trying to resolve these further into the eigensolutions of the singularity we find among the latter, as determined in the preceding sections, only just *one* type that needs emendation, viz., the electric-dipole-type. All the other eigensolutions are available, and the resolution with respect to *them* is to be carried out exactly in the same way as in the "fictitious case."

The actual value of the non-vanishing residue is not of interest. It has nothing to do with the "pull exerted on the electron by this solution." Indeed, the phrase in inverted commas is meaningless, because the solution of which it speaks is, from our revised point of view, physically inadmissible and wrong, we have to cast it away and replace it by a good one. The condition for the latter is obvious. Since the angle-integration does not annihilate the residue, the stress components must have a weaker singularity than r^{-2}, very probably of the order r^{-1} only. One easily finds that this means the perturbational field components must not reach the order r^{-2}, but very probably the order r^{-1} only.

At first sight that seems embarrassing. For, as we know, *all but one* of the solutions of equ. (1, 5) engender field-components of order r^{-3}. We could not do better than always choose that *one* which produces but r^{-2}. But we have just found that for $n = 1$ not even that suffices, and that we have to cast away our precious solution in this case. A further reduction of the degree of singularity is required, which seems a rather exacting demand! Yet it can be fulfilled.

5. ACCELERATING ELECTRON AT REST.

For the linear perturbation equations (1, 1) to hold, the perturbation field has to mean the deviation from a purely electric field of reference, which has to satisfy Born's non-linear field equations. Not only would the perturbation theory of a *more general kind of main field* be much more complicated and, as a rule, not amenable to a general solution, but *we have none* to which the generalized theory could be applied, the motion of the electron being unknown, being the very object of our investigation. So there is no point in contemplating such a generalization One might think of using as the field of reference at least a Lorentz-transformed of Born's static, spherically symmetric solution, instead of that solution itself. But it will soon become clear that this would have no advantage, only entail quite unnecessary complication.

On the other hand, of course, the actual field-variation that occurs, when the singularity moves on to a different point in space, cannot possibly be regarded as a *small perturbation*. Hence the perturbation theory is now certainly not applicable to a finite interval of time, only just *to one moment*, giving but a snapshot of the situation. Or perhaps one ought not to say just to *one* moment, but to two infinitesimally adjacent moments. The infinitesimal interval begins with the singularity at rest in the origin and extends to a moment when the singularity is still in the origin, but no longer at rest—it has acquired an infinitesimal velocity. We shall see that this snapshot is sufficient to deduce at least the law of quasistationary motion, including the effect usually referred to as *radiation damping*.

What we are contemplating may be called an *accelerating electron at rest* (in German I would say "beschleunigt ruhendes Elektron"). We are using for the moment in question—and that can be any moment during the motion—*that* Lorentz frame in which the electron is in the situation of a tennis ball. thrown vertically up into the air and *just beginning to fall again*. The task is to find a solution of equations (1, 1) to depict this situation.

Using for a moment the *customary* notion of a moving point-charge, let me recall that its field, when transformed to the rest-frame, is *not even in its immediate neighbourhood* the same as that of a point-charge permanently at rest, though both fields are there purely electric. The deviation is not in itself infinitesimal, not vanishingly small, but small only inasmuch as the *acceleration* is regarded as a small quantity. That can be seen from the following consideration, in which we concentrate attention to the closest vicinity of the charge.

A moment *earlier* the charge was still in motion in some direction, though this motion was already infinitely slow. It was surrounded by infinitesimally weak circular magnetic field lines. A moment *later* the charge *will* be moving in the opposite direction, the magnetic circles re-appearing, but arrowed the other way round. In the moment of actual rest which we are contemplating, the magnetic field is zero, but *its time derivative* is not zero and is also not infinitesimal. It is obviously proportional to the *acceleration.* That entails a finite value of the electric *curl,* and thus a well defined finite addition to the electric field, over and above the static field of the same point charge, when permanently at rest.—We must determine this additional field for Born's electron. For it will give us the clue to the solution.

By transforming the Born solution to a moving frame you find that for a velocity *of the electron*

$$v = \tanh a \tag{5, 1}$$

in the direction[6] of z, the field components in any point P of the plane $x = 0$ are the following, if θ be the angle between OP and the z-axis:

$$\begin{aligned}
E_x^0 &= 0 & D_x^0 &= 0 \\
E_y^0 &= \frac{\sin\theta\cosh a}{\sqrt{1+r^4}} & D_y^0 &= \frac{\sin\theta\cosh a}{r^2} \\
E_z^0 &= \frac{\cos\theta}{\sqrt{1+r^4}} & D_z^0 &= \frac{\cos\theta}{r^2} \\
B_x^0 &= -\frac{\sin\theta\sinh a}{\sqrt{1+r^4}} & H_x^0 &= -\frac{\sin\theta\sinh a}{r^2} \\
B_y^0 &= 0 & H_y^0 &= 0 \\
B_z^0 &= 0 & H_z^0 &= 0.
\end{aligned} \tag{5, 2}$$

From the value of B^0 we infer that an electron *at rest* with quasistationary *acceleration* $\dot{v}$ in the direction of the z-axis would have the following value of $-\dot{B}_x^0$ for which we prefer to write $\dot{B}_\phi^0$:

$$\dot{B}_\phi^0 = \frac{\dot{v}\sin\theta}{\sqrt{1+r^4}}, \tag{5, 3}$$

and the corresponding *curl* of E^0, whence the $\dot{v}$-term of E^0 itself could easily be deduced, but we shall not need it.

Regarding these additions, due to and proportional to the acceleration, as a *perturbation field*—in which we are largely justified, because only tremendously large accelerations would *not* be very small in our units[7]—we can now indicate a solution of (1, 5), for $n = 1$, a solution which engenders a field that in a given moment of time has precisely these features. But since the present investigation has nothing to do with a periodic phenomenon, we prefer to use (1, 5) in the slightly generalized

[6] It is a trifle more convenient to use the z-direction now.—This z and x will not be confounded with the variables $z = r^4$ and $x = \omega r$, used elsewhere.

[7] $c = 1$, $r_0 = 1$.

form, pointed out on p. 96, small print, second paragraph, viz. in the form

$$\frac{\partial^2 F_{1,a}}{\partial r^2} + \frac{2}{r(1+r^4)}\frac{\partial F_{1,a}}{\partial r} - \frac{2r^2}{1+r^4}F_{1,a} - \frac{\partial^2 F_{1,a}}{\partial t^2} = 0. \tag{5, 4}$$

The additional subscript a in F shall remind us of the notion "accelerating."—The field components are to be read off (1, 8), with $n = 1$ and with the changes indicated in the paragraph just referred to. With the further specification $m = 0$

$$B_\phi = -\frac{r\sin\theta}{\sqrt{1+r^4}}\frac{\partial^2 F_{1,a}}{\partial t^2}$$

thus

$$\dot{B}_\phi = -\frac{r\sin\theta}{\sqrt{1+r^4}}\frac{\partial^3 F_{1,a}}{\partial t^3}. \tag{5, 5}$$

To match with (5, 3) we must put

$$\frac{\partial^3 F_{1,a}}{\partial t^3} = -\frac{\dot{v}}{r} \tag{5, 6}$$

for the moment in question, call it $t = 0$. An *exact* solution of (5, 4), complying with this demand and introducing, as we shall see, no spurious field-perturbations, is obtained by putting

$$F_{1,a} = w(r^4)\,t - \frac{1}{6}\frac{\dot{v}}{r}t^3, \tag{5, 7}$$

with $w(r^4)$ a function still to be determined.[8] You readily test, if you do not remember it from (3, 19), that r^{-1} is itself a solution of (5, 4). Hence $\dot{v}t^3/6r$ only gives the contribution $\dot{v}t/r$. The demand on $w(r^4)$ is therefore

$$\frac{d^2w}{dr^2} + \frac{2}{r(1+r^4)}\frac{dw}{dr} - \frac{2r^2}{1+r^4}w = \frac{\dot{v}}{r}. \tag{5, 8}$$

Introducing here again $z = r^4$, we obtain, for pretty perspicuous reasons, exactly *that* inhomogeneous equation that controlled the iteration process (3, 6) for $n = 1$, for which case the solutions of the corresponding *homogeneous* equation are indicated in (3, 19). Formally the solution runs completely parallel to the first line in (3, 6), but intrinsically there is the big difference, first that this time it is not y_1, but precisely the *singular* $y_2 = z^{-\frac{1}{4}} = r^{-1}$ which constitutes the "second member"; secondly, that the lower limits of the integrals are, this time, prescribed by the demand, that the field components, engendered by (5, 7) must, for $t = 0$ at $r \to 0$, not exceed the order r^{-1}. The result is rapidly worked out. I give it

[8] Calling it $w(r^4)$, not just $w(r)$, is a little pedantry, intended to emphasize the intimate, but not quite simple family-relation of this *one* function to the $(n=1)$-branch of the $w_k(z)$-family in (3, 4).

both in the variable r and z, the first of which is more suitable for physical reflexions, the second for mathematical use:[9]

$$F_{1,a} = \frac{\dot{v}}{3}\left\{\left[-\sqrt{1+r^4}\int_0^r \frac{dr}{\sqrt{1+r^4}} - \frac{1}{r}\left(\int_0^r \frac{dr}{\sqrt{1+r^4}}\right)^2 + \frac{r}{2}\right] t - \frac{1}{2}\frac{t^3}{r}\right\}$$
$$= \frac{\dot{v}}{3}\left\{\left(-\tfrac{1}{4} I\sqrt{1+z} - \tfrac{1}{16} I^2 z^{-\frac{1}{4}} + \tfrac{1}{2} z^{\frac{1}{4}}\right) t - \tfrac{1}{2} z^{-\frac{1}{4}} t^3\right\} \qquad (5, 9)$$

I is the integral explained in (3, 19).

We said that no *spurious* fields were introduced. To test that point, using (1, 8) in the manner described in the second small-print-paragraph on p. 96, we state that for $t = 0$ there is, indeed, no magnetic field. In this moment the frightening t^3-term in (5, 9) is altogether innocuous. The *electric* field comes from the t-term alone and is bound to include the "curly" part connected with $\dot{B}$. But it contains something much more important. The term

$$-\frac{\dot{v}}{3}\sqrt{1+r^4}\int_0^r \frac{dr}{\sqrt{1+r^4}} = -\frac{\dot{v}}{12} I\sqrt{1+z}, \qquad (5, 10)$$

which for $r >> 1$ (meaning $r >> r_0$) reads, by (3, 20),

$$-\frac{\gamma}{3}\dot{v} z^{\frac{1}{2}} = -\frac{\gamma}{3}\dot{v} r^2, \qquad (5, 11)$$

gives, by (1, 8), a *homogeneous* field $2\gamma\dot{v}/3$ in the direction of the acceleration $\dot{v}$ which we had imparted to the singularity.

This way of obtaining the equation of motion is rather interesting. The usual way in such cases is, to assume a homogeneous field at $r >> r_0$ and to show that it entails acceleration. Here we have, inversely, assumed acceleration and have revealed the necessity of its being supported by a homogeneous field. The mass $2\gamma/3$ agrees with Born's value. Since we have used the rest-frame, which we could now change, the relativistic variation of mass and the magnetic part of the force are, of course, included. In addition, we shall later on get indirect evidence that our solution also takes proper care of what is usually called *radiation damping.*

That radiation damping be included is, at first sight, amazing, because the second derivative, $\ddot{v}$, which is usually regarded to be responsible for it, has not entered the considerations by which our superpotential function (5, 9) has been derived. What happens is, that the equation of motion which we have read off this solution does not correspond with the elementary Lorentz-equation-of-motion, but with equ. (22)

$$m\,\dot{v}_\mu = e\,v_\nu f_\mu^\nu \qquad (22)\ \text{(Dirac, l.c.)}$$

[9] It is well known that in dealing with the functions of the *lemniscate*, which are the Jacobian elliptic function for modul $1/\sqrt{2}$, you can draw still on a second large mathematical theory, viz., the theory of the Γ- and the hypergeometric function. See Whittaker and Watson, Modern Analysis, p. 524, where Born's constant γ is quoted to eight decimal places.

in a much discussed paper of Dirac's on the classical theory of radiating electrons,[10] to which the same author has quite recently given an extremely abstract and extremely fascinating continuation.[11] Our homogeneous field $2\gamma\dot{v}/3$, supporting the acceleration, is not, as in the elementary Lorentz-equation-of-motion, the difference between the actual field and the electron's own retarded field; our homogeneous field $2\gamma\dot{v}/3$ is to be regarded, just like Dirac's f_{μ}^{ν}, as the difference between the actual field and the arithmetical mean of the electron's retarded and *advanced* fields. Indeed, the t^3-term in (5, 9), could be shown to indicate this mean value rather than the retarded field. (It is true that this t^3 term plays no roll *here*, but it will in the next section, to wit, in (6, 1)).

It would be lunatic to try and pretend that our solution were the only one compatible with the assumed acceleration $\dot{v}$. It is only right that it is not. Indeed, powerful radiations might have been let loose on our electron and might have penetrated towards the singularity even to a distance that were only a fraction of r_0. Yet before they have actually reached it, they must not influence the motion *of the singularity*—not according to the point of view taken here.

What is the meaning of the term with t^3 in (5, 9)? Though we are to use this (super-)potential only for $t = 0$, when the t^3-term contributes nothing to the field, it is bewildering to see a thing turning up which for any $t \gtrless 0$ would engender field-singularities of the entirely forbidden order r^{-3}, to see it turning up as a necessary consequence of our obligation to reduce the actual singularities of the perturbation field from r^{-2} to r^{-1}.

But the term in question is the most natural thing in the world! We must not forget that our field of reference is, and always remains, the spherically symmetric Born field at rest. The fact that the singularity has only just come to rest and is just beginning to shift again, must therefore be expressed in the *perturbation*. It is not difficult to show that the term in question describes (for $t > 0$) precisely the first re-commencement of this shift. In other words the field this term engenders is exactly

$$-\frac{\dot{v}t^2}{2}\frac{\partial}{\partial z} \text{ (centrally symmetric Born field),}$$

where z is not r^4, but the Cartesian coordinate in the direction of the shift.

6. Oscillating-dipole-potential for the yielding electron.

From the results of the preceding section we must get a *substitute* for the proper vibrations belonging to $n = 1$, that is to say, for the electric-dipole-superpotential F_1, ruled out for violating one of the kinetic conditions. Naturally, the substitute must not hold for one moment only, but have the time factor $e^{i\omega t}$ like the rest.

We must once and for all abandon the idea that the thing we are out for be a solution of the perturbational field equations (1, 1). For,

[10] P. A. M. Dirac, Proc. Roy. Soc., **A**, vol. 167, p. 148, 1938.

[11] P. A. M. Dirac, Proc. Roy. Soc., **A**, vol. 180, p. 1, 1942.

except in the snapshot-meaning of the last section, *there are no perturbation equations in the region where the strong field shifts appreciably.* The solution of a non-existing equation is no object of investigation.

From the periodic nature of the whole phenomenon and from the equation of motion deduced above, it is a safe trial to impart to the singularity a harmonic motion in the direction of the electric field of the incident wave, without prejudice as to the, possibly complex, amplitude of the former, except that it must obviously vanish together with the amplitude of the latter. The next suggestion is to take the snapshot solution (5, 9) in every moment of time and to transform the lot of them to a common origin and to a common Lorentz-frame. But we soon discover that the familiar part of these transformations can be waived, because we are neglecting by principle the *square* of the amplitude of the incident wave and both the snapshot-perturbation and the amplitude of the vibration of the singularity are proportional to the first power thereof. The only relevant part in the reduction of all the snapshots to a common aspect is not that the *frames* of reference but that the *fields* of reference are different. For the field of reference is in every moment the spherically symmetric Born-field in the momentary rest system of the singularity and centred on its momentary position. Now for *this* field the latter two circumstances are not negligible, because its own strength is independent of the amplitude of the incident wave. Hence this field is appreciably different for the different snapshots and has therefore to be added to the snapshot fields. It is convenient to subtract from them at the same time the Born field centered on, and at rest with respect to, the *mean* position of the singularity, which we adopt as the common origin.

Proceeding carefully along these lines, starting from the snapshot-solution (5, 9), using the formulae (5, 2) for the Lorentz-transformed of the Born-field, we find that the field circumstantially described in the preceding paragraph is derived from the following superpotential:

$$F_{1,s}(r)e^{\imath\omega t} = \frac{\imath\omega b}{3}\, e^{\imath\omega t}\left\{-\frac{1}{4} I\sqrt{1+z} - \frac{1}{16} I^2 z^{-\frac{1}{4}} + \frac{1}{2} z^{\frac{1}{4}} + \frac{3}{\omega^2} z^{-\frac{1}{4}}\right\}. \quad (6,\ 1)$$

The understanding is, that $be^{\imath\omega t}$ be the *elongation* of the singularity, thus

$$b = \text{amplitude of electrons vibration.} \quad (6,\ 2)$$

The variable $z = r^4$. The subscript s in $F_{1,s}$ is to remind us of "substitute." *From the considerations which led to (6, 1)* it follows that this function has a physical meaning only outside a certain region around the origin and must certainly *never* be used at $z = 0$. Hence its singularity there, otherwise appalling, does not embarrass us. Again from those considerations (not from the nature of the singularity, which is physically meaningless) the excluded region tends to zero, as b tends to zero. Since on the other hand b is only a constant multiplier in $F_{1,s}$, it is not very astonishing to find that, as a matter of fact, our function fulfils, just like F_1, which it supersedes, the equation (1, 5) or (3, 2) for

$n = 1$; not exactly, but about in the way some initial trunk of the series (3, 4) does. In fact, where it is at all defined, that is, outside the excluded region, our field again deserves the name of perturbation field full well.

Yet without the preceding considerations we would have had no guide for adopting, nor could we have had the boldness of admitting that particular function. For it belongs, of course, to the lot which engender a field of the order r^{-3} at the origin, inadmissible not only according to the more exacting demands (r^{-1}) of sect. 4, but already on those (r^{-2}) of sect. 1. The actual fulfilment of the exacting demands cannot be read off the function, because the latter ceases to be competent, before the point to which the demands refer is reached; the fulfilment is warranted by the considerations that gave us the function for using it at a little greater distance.

7. Scattering by the yielding electron.

From sect. 4 it follows that the phase-shifts and therefore the intensities of all the scattered multipole waves are the same as in the case of the immovable singularity, discussed in sect. 3, *with the only exception of the electric dipole,* whose superpotential F_1 is superseded by $F_{1,s}$; otherwise the analytic procedure remains the same also in this case.

Obviously the expression (6, 1) which we found for this substituting superpotential is only an approximation, in which higher powers of ω are to follow, just as they do in (3, 4). The question of improving it will be discussed later. The function to which it approximates will *outside* the excluded region near the origin—of which the *size* is insignificant, because it vanishes with vanishing amplitude of the incident wave—necessarily be an *exact* solution of (1, 5) or (3, 2), because the conditions for our perturbation method (1, 1) are there fulfilled. Hence we are justified in copying the asymptotic decomposition (3, 26) for it, just adding the subscript s (for "substitute") in the proper places:

$$F_{1,s}(r) \rightarrow -A_{-1,s}\,\omega^{-1}r^{-1}\sqrt{\frac{2}{\pi}}\left[1 + O(\omega^2 r^2)\right] + \\ + A_{1,s}\,\omega^2 r^2 \tfrac{1}{3}\sqrt{\frac{2}{\pi}}\left[1 + O(\omega^2 r^2)\right]. \qquad (7, 1)$$

In drawing the expansion in descending powers of z, which we need, from (6, 1), the first line of (3, 23) is useful. Again, as in (3, 32), we write out only the characteristic terms:

$$F_{1,s}(r) = \frac{i\omega b}{3}\left\{\ldots + \frac{3}{\omega^2}z^{-\frac{1}{2}} - \gamma^2 z^{-\frac{1}{2}} + \ldots - \gamma z^{\frac{1}{2}} + \ldots\right\}. \quad (7, 2)$$

Equating the ratios of the coefficients of these terms in (7, 1) and in (7, 2) we obtain

$$\tan \delta_{1,s} = \frac{\omega}{\gamma} - \frac{\gamma}{3}\omega^3. \qquad (7, 3)$$

(Compare with (3, 15) and behold the missing factor 2. See below.)

We defer the discussion of this result. For, this time, since b has the palpable meaning of the electrons amplitude, we are interested in the coefficients themselves, not only in their ratio. Equating those of $z^{\frac{1}{2}}$ in (7, 2) and r^2 in (7, 1) we get

$$b = \frac{A_{1,s}\, i\, \omega}{\gamma} \sqrt{\frac{2}{\pi}}\,. \tag{7, 4}$$

Remember on the other hand the equations (2, 5), which resulted from matching by proper-vibrations the ingoing wavelets contained in the incident wave. The first of these equations, taken for $n = 1$, must now be satisfied by our substitute amplitudes $A_{1,s}$ and $A_{-1,s}$. It can be written

$$A_{1,s}(1 - i \tan \delta_{1,s}) = \frac{3\, i\, a}{2\, \omega^3} \sqrt{\frac{\pi}{2}}\,. \tag{7, 5}$$

Thus

$$b = -\frac{3\, a}{2\, \omega^2 \gamma} (1 - i \tan \delta_{1,s})^{-1}\,. \tag{7, 6}$$

If we insert here at first only the rough approximation ω/γ for $\tan \delta_{1,s}$, we get

$$b = -\frac{a}{\frac{2\gamma}{3}\omega^2 - \frac{2}{3} i\omega^3} \tag{7, 7}$$

That is precisely the phase- and amplitude-relation between the field amplitude a and the complex electronic amplitude b of a Lorentz electron with mass $2\gamma/3$ and charge unity. In ordinary units the well-known relation reads

$$b = -\frac{a}{m\,\omega^2 - \frac{2\,e^2}{3\,c^3} i\omega^3}\,.$$

By giving the imaginary part of the denominator correct, *already our roughest approximation includes the ordinary radiation damping.*

We now turn to the full discussion of (7, 3). The term with ω^3 is already peculiar to Born's electron. Being negative, it slightly reduces the small phase-lag of a behind $-b$, ascribed to radiation damping. So one would say the Born-term *counteracts* radiation damping. This correction is only of the relative order ω^2, meaning $(2\pi r_0/\lambda)^2$.

But looking upon things in another way, the Born-term *re-inforces* the effect of radiation damping. Notably it reduces the amount of scattered radiation still further, and this reduction is of the *same order of magnitude* as the well-known reduction ascribed to radiation damping. I shall prove and discuss this statement immediately. But let me say before, that in wording these phrases I have adopted the customary abbreviated expression which compares the *actual* result about the scattered radiation with the result of a truncated theory, which gives *no phase-lag at all. There is not much point in this comparison,* because the second result must be faked. For in a consistent *general* theory—regardless of what causes the scattering—*there is no scattering without phase-shift,* as can be seen from equations (2, 9), which embody that *general* theory.

To prove the statement made above, we observe that what is called the effect of radiation damping on the amount of scattered radiation is represented by the simple fact, that in the formulae (2, 9) stands $\sin^2\delta$, and not just $\tan^2\delta$. Now in sufficient approximation for our present purpose $\sin^2\delta = \tan^2\delta - \tan^4\delta$. Hence from (7, 3), *omitting* at first the Born-term,

$$\sin^2 \delta_{1,s} = \frac{\omega^2}{\gamma^2} - \frac{\omega^4}{\gamma^4}, \tag{7, 8}$$

but *including* it

$$\sin^2\delta_{1,s} = \frac{\omega^2}{\gamma^2} - \frac{\omega^4}{\gamma^4} - \frac{2}{3}\,\omega^4. \tag{7, 9}$$

That proves the statement. By the way, be not astonished to find Born's constant γ appearing in the customary terms, but absent in the Born-term. It stands for both the electrical *polarizability* (which is one half of it) and the *mass* (which is two-thirds of it).

I just said, anticipating what is to follow now, that the polarizability is only one half of γ. In N.O. we found γ for it, and this result has been confirmed in sect. 3b here. Let us calculate the cross-section for electric-dipole-scattering from (7, 9) and (2, 9), where $n = 1$, $\delta_{1,s}$ replaces δ_1 and the multiplier $8\pi/a^2$ is to be added. We get

$$s_{el.\ dipole} = \frac{8\pi}{3}\left(\frac{2\gamma}{3}\right)^{-2}\left[1 - \left(\gamma^{-2} + \frac{2}{3}\gamma^2\right)\omega^2 \ldots\right. \tag{7, 10}$$

In N.O. we obtained for the same physical quantity, under the preliminary assumption that Rayleigh-scattering and structural scattering can be just superposed (interfering, of course, but without mutually disturbing their mechanisms), the equ. (10, 10) l.c., which in our simplifying units and with omission of the ω^4-terms reads

$$s_e = \frac{8\pi}{3}\,m^{-2}\left[1 - \left(\gamma^{-2} + \frac{4}{3}\,\gamma^2\right)\omega^2 \ldots\right]. \qquad (10, 10)\ \text{N.O.}$$

The only discrepancy is that *the Born-correction is doubled,* which we have to declare as an *error,* caused by the illegitimacy of that preliminary assumption. Since the correction-term in question is a *bilinear* Rayleigh-Born one, the correct Born-polarizability is to be halved, thus $\gamma/2$, not γ. The "destruction" of half of the Born-correction occurs in the most essential step, viz., on solving the inhomogeneous equation (5, 8); it can so to speak be watched. From there on I can see no possibility for one of those stupid little mistakes which so often let you drop a factor, usually 2 or 1/2.

I think one does best to describe the thing the way I did, viz., that the *yielding* electron displays only half the electric polarizability of a fixed singularity. But probably it is logically impossible to distinguish between the influence of the motion on the polarization and that of the polarization on the motion. At any rate there is nothing contradictory in the different behaviour of the yielding and the fixed electron. The two cases cannot be reduced to each other by a change of frame, because it is a question of *acceleration*. And, after all, the second case is only fictitious.

Owing to the presence of the comparatively large Rayleigh-term ω/γ in the electrical-dipole-scattering-constant $\tan \delta_{1,s}$ the expansion of this constant in powers of ω^2 would have to be driven *two* steps further, to include the power ω^7, in order to get for the scattering cross-section the same precision, viz., inclusive of ω^6, as was reached for the fictitious case in sect. 3c with comparative ease.

It is obvious that the *progressive refinement* of the formula (6, 1) for our potential $F_{1,s}$ would have to proceed by the "hypergeometric" iteration process (3, 6), the terms without ω in the curved bracket constituting the "second member" for the *next* step. Since that step already deals with the *relative* order ω^4, the *other* iteration process, (3, 29) and (3, 30), which might be called the "Bessel" iteration, would now have to be consulted as well. The only question of principle that arises is the fixation of the lower limits of the couple of integrals in the "hypergeometric" iteration. For it seems a bit daring to impose a *physical* condition (viz., field not to exceed order r^{-1} at $r = 0$) in a point where admittedly the mathematical function in question never represents the physical quantity in question; moreover, to impose it on the correction terms, whilst the principal term (viz., $3z^{-\frac{1}{2}}/\omega^2$) is excused from it for that very reason.—Yet I think the demand *is* correct.

It is well to remember that these iterations, however far you carry them out, have nothing to do with the restriction to linear terms only of the perturbation field, a restriction that would not be removed, even if the approximation were superseded by an exact solution of (1, 5) and (1, 7). If the classical treatment were intended to give the ultimate description, the omission of all quadratic terms would be a grave deficiency; for among them there must be such as ought really never to be neglected, because they demand an ever increasing velocity of the electron in the direction of propagation of the incident wave, the well-known classical analogue of the Compton effect.

To work it out from Born's theory, beyond the re-statement of trivial classical results, would not only be extremely difficult, but would almost certainly fail to provide us with any useful information.

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