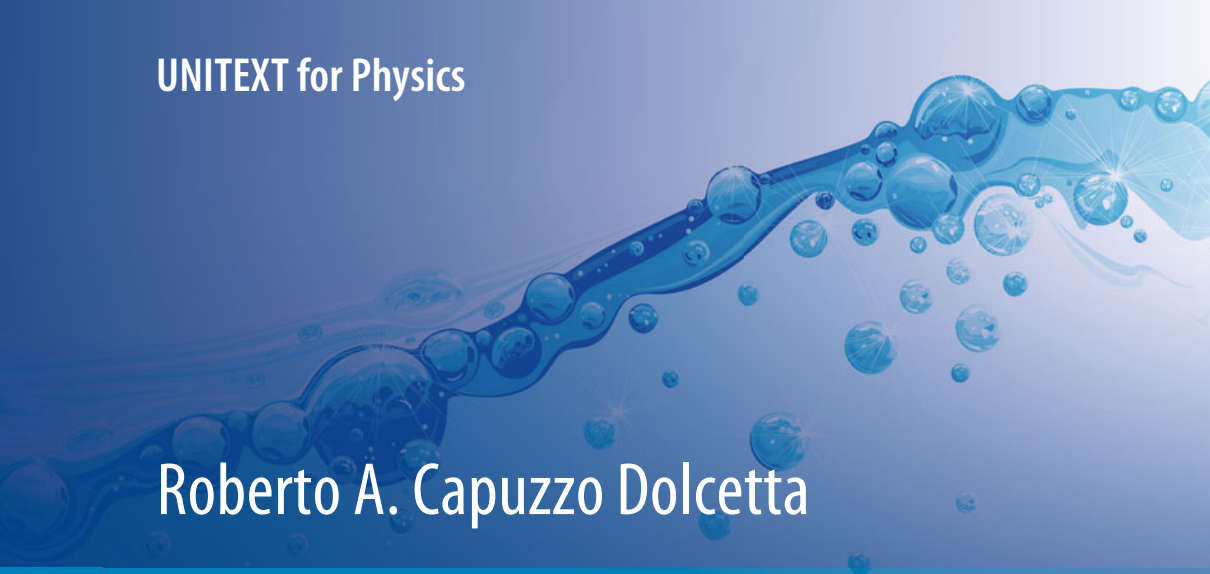
UNITEXT for Physics



Roberto A. Capuzzo Dolcetta

Physics of Fluids

Springer

UNITEXT for Physics

UNITEXT for Physics series publishes textbooks in physics and astronomy, characterized by a didactic style and comprehensiveness. The books are addressed to upper-undergraduate and graduate students, but also to scientists and researchers as important resources for their education, knowledge, and teaching.

Roberto A. Capuzzo Dolcetta

Physics of Fluids

Springer

Roberto A. Capuzzo Dolcetta
Dipartimento di Fisica
Sapienza Università di Roma
Rome, Italy

ISSN 2198-7882 ISSN 2198-7890 (electronic)
UNITEXT for Physics
ISBN 978-3-031-30749-2 ISBN 978-3-031-30750-8 (eBook)
https://doi.org/10.1007/978-3-031-30750-8

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

This book is dedicated to Alvio, Daniela and Flavio.

Preface

When I meet God, I am going to ask him two questions: why relativity? And why turbulence? I really believe he will have an answer for the first.

—Werner Heisenberg

The scope of this book is to provide readers having a knowledge of calculus and Physics at the college level with an introduction to physics of fluids.

That of fluid is one of the various states of matter, the most common in nature both on the Earth and in the astronomical context.

Fluids, which include liquids, gases and plasmas, are essential for life: the air we breathe is a gas, the atmosphere which shields us from harmful radiations is a gas mixture, the water we drink and feeds plants is a liquid, and the salted water of seas and oceans is a liquid. So, a basic knowledge of the behavior of gases and liquids should actually constitute a natural value for everyone's culture. Such a knowledge might be helpful, for instance, in distinguishing better what is true and what is false among the many assessments that are commonly stated about things like global warming, greenhouse effect, ice retreat and so on.

I wrote this book with the aim of coupling rigor in mathematical developments to a clear and intuitive representation of the various phenomena characterizing the often very complicated behavior of fluids in nature, in terrestrial as well as in astronomical environments. I tried to do this in a short volume which can be easily understood and handled. I proposed also a few developed exercises which should be helpful in a better understanding of the topics.

The book is mainly thought for an audience of undergraduate and graduate students in Physics, Mathematics, Geology, Engineering as well as in Astronomy and Astrophysics. Anyway, it should be appreciated by any reader with a knowledge of mathematics and physics at the college level.

Rome, Italy Roberto A. Capuzzo Dolcetta
March 2023

Acknowledgements I wrote some parts of this book while I was hosted at the Institut für Theoretische Physik in Heidelberg (DE), which I thank for the hospitality.

Contents

Symbols

M, L, T	They indicate the fundamental mass, length and time physical dimensions.
amu	Atomic mass unit, defined as 1/12 the mass of a Carbon 12 atom, 1 amu $= 1.66054 \times 10^{-27}$ kg.
m_H	Hydrogen mass, $m_H = 1.00784$ amu.
K	It indicates kelvin, primary unit of temperature in the international system of units.
$M_\odot$	Value of the Sun mass, $M_\odot = 1.989 \times 10^{30}$ kg.
k	Boltzmann constant, $k = 1.380649 \times 10^{-16}$ erg/K.
G	Newtonian constant of gravitation, $G = 6.67430 \times 10^{-11}$ N kg^{-2} m^2.
c	It indicates the speed of light in vacuum, $c = 2.99792 \times 10^5$ km s^{-1}
AU	Astronomical Unit, i.e. the average Sun–Earth distance 1 AU $=$ 1.49598×10^8 km.
$\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}^{\pm}$	With these symbols we refer, respectively, to the set of real, integer and positive (+) or negative (−) integer numbers.
v, $\lvert\mathbf{v}\rvert$, $\lVert\mathbf{v}\rVert$	With the symbols v, $\lvert\mathbf{v}\rvert$ or $\lVert\mathbf{v}\rVert$ we intend the Euclidean norm of a vector $\mathbf{v}$, i.e., in $\mathbb{R}^n$, $v \equiv \lvert\mathbf{v}\rvert \equiv \lVert\mathbf{v}\rVert \equiv \left(\sum_{i=1}^{n} v_i^2\right)^{1/2}$.
$\mathbf{e}_i$	This symbol refers to the unit vector in the ith coordinate direction in $\mathbb{R}^n$.
$\mathbf{r}$	It represents the position vector in $\mathbb{R}^n$, i.e. $\mathbf{r} = \sum_{i=1}^{n} x_i \mathbf{e}_i$.
$\mathbf{e}_v$	Unit vector in the direction of $\mathbf{v}$, $\mathbf{e}_v \equiv \mathbf{v}/v$.
$\mathbf{n}$	This symbol indicates the outward unit vector in a point of a surface.
∇	With this symbol we intend, in $\mathbb{R}^n$, the formal vector operator $\nabla \equiv \sum_{i=1}^{n} \mathbf{e}_i \frac{\partial}{\partial x_i}$.
$\dot{\cdot}$	The dot over a function represents its total time derivative, i.e. it stands for $\frac{\mathrm{d}}{\mathrm{d}t}$.

$\cdot, \wedge$	These symbols represent, respectively, scalar ($\cdot$) and vector ($\wedge$) product between vectors.
$\det(\mathbf{A})$	Determinant of matrix $\mathbf{A}$.
r, θ, φ	These symbols represent spherical polar coordinates in $\mathbb{R}^3$.
R, θ, z	These symbols represent cylindrical coordinates in $\mathbb{R}^3$.
$\mathrm{S}_R(\mathbf{r})$	With this symbol we indicate, in $\mathbb{R}^n$, the spherical hypersurface of radius R and center in $\mathbf{r}$.
$x_i v_i$	We adopt Einstein's summation convention that implies summation over repeated indexes, i.e., in $\mathbb{R}^n$, $x_i v_i \equiv \sum_{i=1}^{n} x_i v_i$.
$\binom{n}{k}$	For integers n and k, $0 \leq k \leq n$, it gives the usual binomial coefficient $\frac{n!}{k!(n-k)!}$.
e^x	It indicates the exponential of x (e is Euler's number, $e \simeq 2.71828$).
ln	It represents the natural logarithm (in base e).
log, Log	Both represent decimal logarithm (in base 10).
$\sinh x$, $\cosh x$	They represent the hyperbolic sine and cosine functions.
∞^k	It indicates that there are k independent arbitrary choices for a parameter.
$F\|_a^b$	It indicates the difference $F(b) - F(a)$, where $F(x)$ is the primitive function of another function $f(x)$.
lhs, rhs	They denote the left- and right-hand sides of an equation.
$\Longrightarrow$	Logical implication: $a \Longrightarrow b$ means *a implies b*, i.e. that if proposition a is true then proposition b is also true.
$\Longleftarrow$	Inverse logical implication: $a \Longleftarrow b$ means *b implies a*, i.e. that if proposition b is true then proposition a is also true.
$\Longleftrightarrow$	Double implication, which means identity.

Chapter 1
Fluids and Their Fundamental Aspects

This book gives an introduction to the physics of fluids, with special attention to fluid dynamics.

Everyone knows that a large part of the physical matter is not in the *solid* state but, rather, in a *fluid* state. Actually, about two-thirds of the Earth's surface is covered by oceans, and so by water, which is a liquid, one of the ways a fluid can appear in nature. Moreover, the Earth is embedded in its *gaseous* atmosphere.

A fluid is a state of matter which corresponds to a weak resistance to a deformation force (a *shear* stress) so that the constituent molecules are almost free to move relative to each other. On the other hand, a solid is robust to external solicitations so that the mutual distances among its elementary components keep unchanged until, eventually, there is a sudden breaking (when the solid is at all effects broken). Given the above, rough, definition of fluid, it is clear that the fluid category includes an internal subdivision into *liquids* and *gases* and, to be even more specific, *plasmas*.

In more sophisticated terms, physicists speak of four different classical states of matter: (1) solid, (2) liquid, (3) gaseous and (4) plasma (see Fig. 1.1). In a more modern vision, several other (non-classical) states of matter have been taken into consideration, like that of the man-made so-called Bose–Einstein condensate. We just briefly refer to this in the last chapter of this book.

In this book, we deal with the states (2), (3) and (4) as different ways of appearance of a fluid, so the title of this book as Physics of Fluids refers to the classical physics of these 3 states of matter, reserving the sixth chapter to a succinct outline of the special relativistic approach and the exotic state of *superfluid* and of Bose–Einstein condensate.

Although rough and, on many aspects, unsatisfactory, the distinction among solids, liquids and gases usually provided in the preparatory schools as, for solids—'a substance which has both a definite volume and shape' while for liquids— 'a substance which has a definite volume but not a definite shape because it assumes that of its container' and, finally, for gases—'a substance which has neither a definite volume nor a definite shape', gives an acceptable and practical idea. Of course, these definitions of the two types of fluids can be criticized in that it seems to require the actual

R. A. Capuzzo Dolcetta, *Physics of Fluids*, UNITEXT for Physics,
https://doi.org/10.1007/978-3-031-30750-8_1

existence of a physical container for a liquid (like a glass for water, the ocean basin for ocean water, etc.), and so it is not self-consistent and does not apply to liquids in a more general situation than in the physics on the Earth's ground. Moreover, the above rough definition does not account for the fourth state of matter, i.e. the substate of fluids called plasma.

A plasma is a gas whose atoms have been partially or even fully ionized, i.e. some or all electrons have been stripped away from their parent atomic nuclei. At first sight, a plasma looks like a gas, but its behavior is very different from that of a normal gas. The main difference is because a plasma contains a significant fraction of free electrons that make the gas electrically conductive so that long-range electromagnetic force may become dominant in its behavior. Although plasmas are not particularly common in earth's physics, because their existence requires ionization of a preexistent gas (a thing that happens, for example, in nature due to lightnings, or via man-made ionization as that produced on neon and xenon in the TV plasma screen), they are extremely common in the astrophysical context. As a matter of fact, stars (including Sun) are self-contained (by gravity) enormous masses of plasmas. Moreover, in a cosmic environment, many other objects are constituted by partially or fully ionized gas. It is estimated that more than 99% of the matter in the Universe is comprised of plasma.

It is relevant noting that the distinction between the various states of matter is not invariable. A certain element of matter can show a solid behavior under some external physical conditions to transform into a fluid behavior when conditions vary, and, when behaving as a fluid, show, indeed, specific liquid, gas or plasma characteristics at different stages. Everyone knows that the heating of a piece of lead conduces it to melt and pass from a solid to a liquid state. Giving much more heat (energy), it is also possible to set the lead into vapor (gaseous) state. Additional injection of energy would lead to ionization and consequent plasma behavior. As indicated in Fig. 1.1, these subsequent phases are reached by the processes of melting, vaporization, ionization and their inverse, i.e. deionization (recombination), condensation and freezing (solidification). The progression from state 1 (solid) to state 4 (plasma)

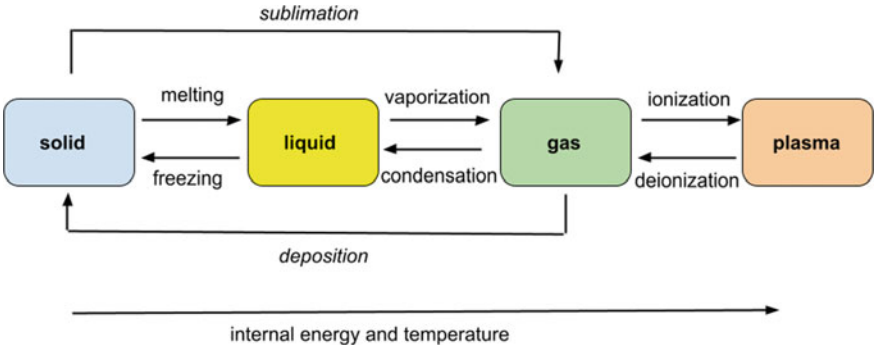


Fig. 1.1 States of matter

is a sequence of what in physics are called *phase transitions*, and are characterized by the increasing quantity of energy given to the system to make the transition. Practically speaking, this corresponds to the fact that the energy injected in the system goes into kinetic energy of its constituent elements, tending to overcome the mutual bounding links. At a certain threshold, these links are broken and the phase transition occurs in a quite discontinuous fashion.

Another usual characteristic attributed to a fluid is its *volatility* which relates to the spontaneous tendency of a liquid to turn into a gas or vapor phase at normal temperature. This occurs usually at the free surface of a liquid, and liquid parfums are typical examples of high volatility.

1.1 Newtonian and Non-Newtonian Fluids

Another fundamental difference among types of fluids is their different reaction to deformation stress. The main role in this is played by the fluid *viscosity*.

The viscosity of a fluid is a measure of its internal resistance to flow. Viscosity is a global characteristic of a fluid which expresses the internal friction among the elementary constituents of a fluid: strong intermolecular forces imply high viscosity. As examples, honey is highly viscous, olive oil is less viscous than honey but more viscous than water that, in its turn, is more viscous than ethyl alcohol. Here, we do not deepen this topic (which will be more extensively studied in Chap. 2) but just say that fluid viscosity is a function of state variables, like pressure, p, matter density, ρ, and temperature, T. This is because viscosity depends on the strength of the intermolecular forces within the fluid, and these clearly depend on the number of molecules per unit volume (density) and their kinetic energy (proportional to temperature). So, it is intuitive that a dilute and hot liquid is less viscous than a denser and colder one. Everyone knows that honey tends to solidify when put into a fridge and gets liquid again when heated.

Every real fluid shows some viscosity, and the limit of zero viscosity is that of the so-called *perfect* or *ideal* fluid (and of the superfluids; see Chap. 6).

The behavior of internal friction, and so of viscosity, defines a fluid as *Newtonian* or *non-Newtonian*.

The study of viscosity characteristics of fluids was pioneered by Newton. A Newtonian fluid is one for which the Newton–Stokes law (independently obtained by the two scientists in different ways) of viscosity holds. The Newton–Stokes viscosity law states that the shear stress between adjacent fluid layers is proportional to the velocity gradients between the two layers. In its original form, valid for *laminar* flows at low Reynolds numbers (non-turbulent flows),[1] the Newton–Stokes law states a linear relation between the stress acting on a fluid layer and its resulting deformation. The *shear stress* is the value of the force (per unit area) acting parallel to a given

[1] Precise definitions of laminar and turbulent flows as well as the Reynolds number will be given in Chap. 2. In brief, a fluid flow is laminar when its particle trajectories are rectilinear and parallel.

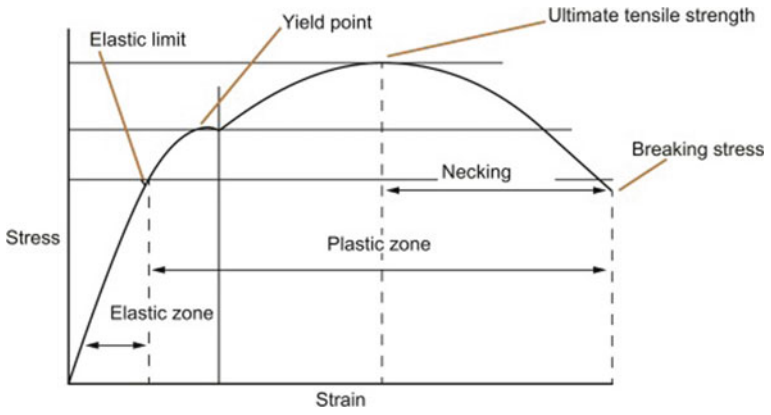


Fig. 1.2 Stress behavior in various regimes of strain. From https://www.sciencedirect.com/topics/engineering/stress-strain-curve

surface cross section in a fluid. It is represented as a tensor[2] of order 2, τ_{ij}. The velocity gradients between the two flowing layers give the *shear rate* (or *strain* rate) tensor that has the dimensions of the inverse of a time. A more accurate definition for the rate of shear (strain) tensor is

$$\dot{\gamma}_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \tag{1.1}$$

where x_k and v_k indicate the kth component of the position vector, $\mathbf{r}$, and velocity, $\mathbf{v}$, respectively, and the dot indicates time derivative. The Newtonian second law of dynamics implies that the shear rate is the motion reaction (speed of deformation) of a fluid under a certain shear stress, which for isotropic fluids is written simply as

$$\eta\dot{\gamma} = \tau, \tag{1.2}$$

where η is the dynamic shear viscosity. The equation above can also be considered as a definition of viscosity in the form of the ratio between the shear stress and the shear (strain) rate. A fluid is called Newtonian if its viscosity η is constant (in the sense it depends on the type of fluid and on its density and temperature but not on the quantity of stress), and so the rate of shear is linearly proportional to the shear stress: the larger the shear stress acting on a portion of fluid the faster its shear deformation, in a linear way (see left part of Fig. 1.2).

Conversely, non-Newtonian fluids are those for which the viscosity is not constant (and so the response of the fluid to the stress solicitation is non-linear) but rather depends upon the quantity of shear stress the fluid is submitted to. On earth,

[2] For the definition of tensors, see Appendix A.5.

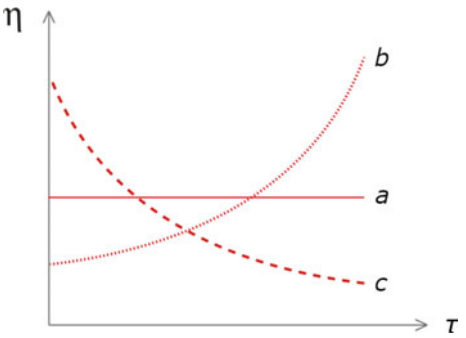


Fig. 1.3 Viscosity versus shear stress for a Newtonian fluid (**a**), a shear-thickening one (dotted curve **b**) and a shear-thinning fluid (dashed curve **c**)

Newtonian fluids are by far more common: water is a Newtonian fluid (liquid) so is mineral oil, for example.

The viscosity variation under increasing shear stress (see Fig. 1.3) may consist in both its increase (shear- *thickening* or dilatant or rheopectic fluid) or decrease (shear *thinning* or pseudoplastic or thixotropic fluid). As a matter of fact, the variation of viscosity is easily tested: shaking rapidly a viscosity thickening fluid its viscosity increases abruptly, to reduce as soon as the external solicitation (shear stress) decays.

Simple examples of shear-thickening fluids are given by various solutions like water and amide (for instance maizena) water and potato starch, etc. Examples of shear thinning fluids are ketchup and mayonnaise sauce, paint and blood.

While the shear thinning phenomenon is still out of complete physical comprehension, for shear thickening a convincing explanation bases on that such a fluid is composed by a sort of suspension of closely packed solid particles in a smoother liquid phase. In situations of moderate shear stress and shear rate, the liquid fills adequately the interparticle distance so that the global fluid movement is dominated by the liquid phase, but when the shear stress is high the liquid phase is moved quickly away and so it is no more able to fill the space between solid particles which collide against each other giving to the surface of the fluid subjected to this high stress a temporary behavior similar to a solid surface.

1.2 Plasma as Peculiar Fluid

On earth, liquids and gases are usually *neutral*, i.e. non-ionized. The elementary constituents are neutral particles (atoms and/or molecules) which express their interactions in different ways, as better specified by the equation of state,[3] but this interaction is limited to closest neighbors (*short*- range interaction). Actually, the interaction between neutral particles is not long range if systems are not self-gravitating, as it happens in fluids on earth, but it just occurs when two or more particles approach

[3] The equation of state is an algebraic relation among state variables characterizing the fluid.

each other at molecular size distance. This is however unlikely in a gas which, by its definition, is a state of fluid where the constituents' mean free path is much larger than the average distance between neighbors.

Another is the case of an ionized gas, which is commonly called *plasma* and, as we anticipated in Sect. 1.1, is indeed a very particular case of fluid. Ionization of a gas means that some (partial ionization) or all (full ionization) the electrons have been stripped from the gas atoms. Consequently, the gas is composed of a mixture of positively charged ions and electrons. Globally, the gas is neutral, but locally the interaction among ions and electrons as well as among ions and electrons with themselves is relevant and, due to the Colombian interaction between charged particles, it is long range. This is in common with gravity, whose intensity scales with the same inverse square of the distance between particles as the Colombian interaction, with the crucial difference that gravity is always attractive while Coulomb force is either attractive or repulsive. The presence of free electrons in plasma makes it electrically conductive.

The identification of this peculiar state of matter was done, first, by Sir W. Crookes in 1879, while its denomination as 'plasma' was given by I. Langmuir in 1928. Crookes was able to transform a normal gas into a partially ionized one by a device (called after him Crookes tube) consisting of a partially evacuated glass bulb where a cathode and an anode are placed on the two opposite sides of the bulb. When a high potential difference (voltage) is applied between them, an electronic current starts flowing from the cathode to the anode. Along their way from one electrode to the other, electrons happen to hit atoms of the gas contained in the tube and ionize them (strip one or more electrons), producing, so, a plasma.

Being an electrically charged gas, a plasma is submitted to collective effects because it naturally produces a global electromagnetic field, and it is highly sensitive to external fields. This response of plasma to applied electrical fields is at the base, for instance, of the TV plasma displays or of the plasma etching technique used to fabricate integrated circuits.

The presence of plasmas on earth is rare, being produced in nature just during strong electric exchange due to lightning, and in northern lights (aurora borealis, as named by Galileo Galilei in 1619), these latter being regions of the atmosphere partially ionized by incoming charged particles from the Sun. Plasma can be produced by human intervention, via electron beam production by cathodic tubes of the kind once used for television sets, or in neon signs, where tenuous light is produced by long gas-discharge tubes containing a rarefied gas, like neon. On the contrary, in the astronomical context, plasma constitutes about 99% of ordinary matter. Stars are made of plasma because their density and temperature are high, although decreasing from the center (where the gas is fully ionized) to the surface (where the gas is only partially ionized). Also, many astronomical nebulae are fully or partially ionized tenuous agglomerates (gas clouds) of basic elements like hydrogen or carbon, usually mixed with dust.

1.3 The Continuum Hypothesis

The quantitative study of fluids implies various simplifying hypotheses and frameworks. The first fundamental hypothesis is the *continuum* hypothesis.

What is it and what is it based on?
In principle, a full description of the dynamics of a piece of fluid would consist in the solution of the equations of motion of all its *elementary* constituents.

This problem requires, first, a clear definition of what the elementary constituents of a fluid are. Because we deal with macroscopic portions of a fluid subjected to internal and external stresses to which the fluid reacts with a global motion and an internal motion, we can reasonably consider fluid molecules as the elementary components, because it is their mutual force interaction on the intermolecular distance scale which determines the fluid response to an external solicitation, and not the smaller scale of interactions among electrons and nuclei in the atoms. Of course, for ionized fluids (plasmas) the situation is a bit different and for them collective phenomena (interaction at a large scale) can become very important.

A piece of fluid is composed of an enormous number of 'elements'. In a liter of water, the number, N, of H_2O molecules is $N = M/m_{H_2O} \sim 3.34 \times 10^{25}$, where M is the mass of a liter of water, and $m_{H_2O} = 18.02$ amu $\simeq 2.99 \times 10^{-23}$ g is the H_2O molecule mass. This is an exceedingly large number if we aim at studying the collective motion of a liter of water as a combination of the motion of all its individual molecules. A fully mechanical, particle-by-particle view of a fluid motion will require, indeed, writing and solving an extremely large and complicated system of dynamical equations composed of $3N$, second order, differential equations strictly coupled by the interaction forces among the molecules. This task is out of any practical possibility, so a different scheme must be adopted.

The basic scheme to adopt is that of the *continuum* assumption. Qualitatively speaking, this corresponds to assuming that the number of elements constituting the fluid is so large ($N \to \infty$) that there is practically no empty space between them, so that any individual characteristics of the fluid components is actually lost. The adoption of such an assumption implies the adoption of a *statistical* treatment of fluids. This means that every part of fluid under study, which in principle we would like to be small in size in order to have information about its local state, cannot be too small because the average values we want to measure should not be too uncertain to be unreliable, as it happens when the sampling is so poor that its number of elements, that we keep calling as N, is so small to make the (statistical) local properties of the fluid too fluctuating to give them physical reliability. Quantitatively speaking, the condition for a piece of fluid to be treated in the continuum assumption is to be comprised of a sufficiently large number of 'particles' such that the relative *uncertainty*, measured by $\delta_N \equiv \sqrt{N}/N = 1/\sqrt{N}$ is much less than 1. This condition guarantees that fluctuations are small.

Actually, if we consider a piece of fluid with N such that $\delta_N \ll 1$, we can say that, given a physical boundary enclosing this piece of fluid, the distribution of the

impulses released by the fluid particles to the boundary is almost constant in time, because the Poissonian fluctuations over the mean are extremely small.

In the above-discussed example of a liter of water $\delta_N \simeq 1.73 \times 10^{-13}$, a so small number to guarantee that fluctuations are totally negligible and that the continuum hypothesis is at all valid. Another way to evaluate the validity of the continuum hypothesis is via considering the *collisional mean free path*, λ, as compared to the macroscopic size of the piece of fluid, L. Obviously, if $\lambda \ll L$ the piece of fluid can be considered as fully relaxed and mixed, to be well represented under the continuum hypothesis provided that sufficient time has passed to have actually allowed collisions among runaway molecules in the fluid to happen. Given n and σ as the fluid particle number density and collisional cross section (whose physical dimensions are L^{-3} and L^2, respectively), we have

$$\lambda = \frac{1}{n\sigma}. \tag{1.3}$$

In the case of water, $n = \rho/m_{\mathrm{H_2O}} \simeq 3.34 \times 10^{22}$ cm^{-3}, ρ being the water mass density ($\rho = 1$ g cm^{-3}) and $m_{\mathrm{H_2O}}$ the H_2O molecule mass. The collisional cross section is estimated as $\sigma = 4\pi R_{\mathrm{H_2O}}^2 \simeq 1.26 \times 10^{-15}$ cm^2, having assumed $R_{\mathrm{H_2O}} \simeq 1.375 \times 10^{-8}$cm as the value of the H_2O molecule radius.

These values lead to $\lambda \simeq 1.26 \times 10^{-8}$ cm, which is much smaller than any interesting macroscopic piece of fluid size L (L of the order of 10 cm for a liter of water, so that $\lambda/L \simeq 1.26 \times 10^{-9}$). To be rigorous, this effectiveness of collisions is guaranteed only if the fluid is 'old' enough with respect to the average inter-collision time, τ_c, which can be measured as $\tau_c = \lambda/v_{rms}$, where v_{rms} is the root mean square velocity of the fluid particles. Referring, again, to water at room temperature ($T = 295$ K), we have

$$\tau_c = \frac{1}{n\sigma v_{rms}} = \frac{\sqrt{\frac{\mu m_H}{3kT}}}{n\sigma} \simeq 1.97 \times 10^{-13}\mathrm{s}, \tag{1.4}$$

with μ the fluid average molecular weight ($\mu = \langle m \rangle / m_H$, where $\langle m \rangle$ is the mean mass of the fluid 'particles'). The above value of τ_c is very small, so we have the confirmation that water at room temperature is actually a *collision-dominated* system. The characteristic to be collision-dominated can be actually considered in itself as a definition of a fluid.

1.4 The Lagrangian and Eulerian Descriptions of Fluids

There are two ways to approach the study of the motion of a fluid: *Eulerian* and *Lagrangian*.

The Eulerian *description* (or specification) corresponds to a *field* view, and the relevant fields are vector or scalar functions of position and time characterizing the fluid velocity, density, pressure and internal energy. It is like choosing arbitrary positions in space and, as time runs, pointing the attention to the evolution of the

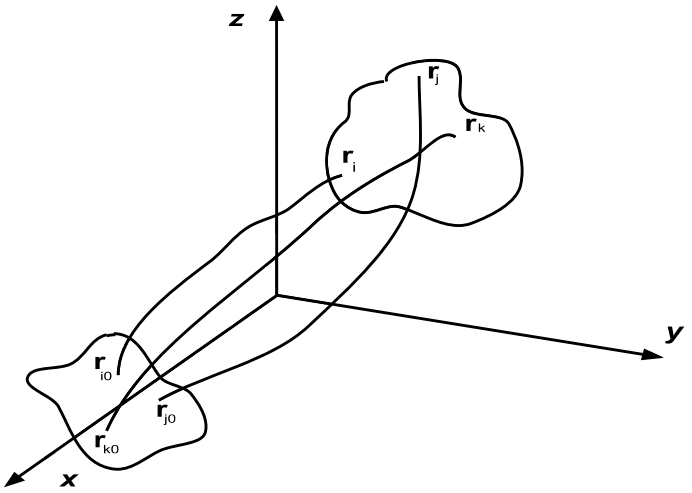


Fig. 1.4 A set of three fluid particles (indexed by i, j, k) and their path from position at time t_0 to that at time t

above-mentioned physical properties of the fluid thought as functions of the position and time, let's say, for example, $\rho(\mathbf{r}; t)$ for the mass density.

On the other hand, the Lagrangian description consists in looking at fluid motion where the hypothetical 'observer' follows an individual fluid *parcel* (a representative, and small, piece of fluid) as it moves through space and time. The Lagrangian specification in its 'rigorous' version, which is the one in which every individual fluid component over a total of N is followed along its motion (see Fig. 1.4), is, a priori, the best to adopt to describe fluid motion. Indeed, a full Lagrangian description of a fluid motion would correspond to the knowledge of the N position vectors of the fluid particles, $\mathbf{r}_i(\mathbf{r}_{i0}; t)$, which exploit the path in time of the ith particle ($i = 1, ..., N$) that at time $t_0 = 0$ was at position $\mathbf{r}_{i0}$. This corresponds to labeling all the fluid particles and follow all their paths. Such a tracking yields also the velocity of all particles, given by $\mathbf{v}_i(\mathbf{r}_{i0}; t)) = \dot{\mathbf{r}}_i(\mathbf{r}_{i0}; t)$. This complete specification is practically impossible to adopt, because of the enormous number of fluid particles, as seen in the previous section.

From a dynamical point of view, a Lagrangian approach would mathematically consist in writing the classical Newtonian equations of motion of an ensemble of N particles

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i, \qquad (i = 1, 2, \ldots, N), \tag{1.5}$$

where $\mathbf{F}_i$, is the resultant force acting on the ith particle of mass m_i, subjected to proper initial conditions in position and velocity, $\mathbf{r}_{i0}$ and $\dot{\mathbf{r}}_{i0}$. In principle, the force acting on every fluid particle depends upon the coordinates of all the other fluid

particles and, possibly, also explicitly on time, so that $\mathbf{F}_i = \mathbf{F}_i(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N; t)$. This would involve an overwhelming evaluation of $O(N^2)$ pair interactions.[4]

Let us think about a one-liter bottle of water as constituted by H_2O molecules considered as fluid elementary particles. As we said above, the number of water molecules in a liter of water is $N \simeq 3.34 \times 10^{25}$, so that the number of force interactions to compute is $N(N-1)/2 \sim 5.56 \times 10^{50}$. This means that a perfectly deterministic particle-by-particle motion calculation is absolutely unviable. Of course, if the particle–particle interaction is sufficiently short-range, the dependence of $\mathbf{F}_i$ is just only upon the coordinates of particles which are in a proper spherical neighbor around the ith particle, whose radius (radius of *influence*) is defined as that within which the interaction strength is quantitatively relevant with respect to the rest of the fluid and to the contribution of the external force. The contribution of all the other particles, as that of any other external contribution, would come by a proper averaging procedure. This would reduce, but surely not eliminate, the overwhelming computational complexity[5] of the problem due to the exceedingly large values of N in all practical situations. For this reason, practical use of a Lagrangian approach to fluid dynamics requires the adoption of a hypothetical subdivision of the fluid in a set of $\tilde{N} \ll N$ parcels considered as representative (through a proper averaging) of the characteristics of the fluid, thus reducing a lot the *dimensionality* of the problem. This is the approach, for example, of the so-called *smoothed particle hydrodynamics* (SPH) method, widely used when the fluid is characterized by the presence of a significant, non-constant, body force (for a precise definition of body force, see Sect. 2.2).

1.4.1 From the Lagrangian to the Eulerian Description and Vice Versa

As we said, the Lagrangian description corresponds to the knowledge, at any time t, of the functions $\mathbf{r}_i(\mathbf{r}_{i0}; t)$ where $\mathbf{r}_{i0}$ indicates the position of the generic, ith, fluid particle at $t = t_0$ (hereafter, we omit for brevity the generic particle index i). This corresponds to the knowledge, for every particle, of the 3 scalar functions

$$\begin{cases} x = x(x_0, y_0, z_0; t), \\ y = y(x_0, y_0, z_0; t), \\ z = z(x_0, y_0, z_0; t), \end{cases} \tag{1.6}$$

which are supposed continuous for every t.

[4] The number of distinct pairs of particles is $\binom{N}{2} = \frac{N(N-1)}{2}$.

[5] For computational complexity of a problem, it is intended the amount of computational resources needed to solve the problem.

The time derivatives of the functions in Eq. 1.6

$$\begin{cases} \dot{x} = \dot{x}(x_0, y_0, z_0; t), \\ \dot{y} = \dot{y}(x_0, y_0, z_0; t), \\ \dot{z} = \dot{z}(x_0, y_0, z_0; t) \end{cases} \tag{1.7}$$

give the velocity of every particle at any time as a function of its initial position. The mathematical procedure of elimination of x_0, y_0, z_0 in the rhs of Eqs. 1.7 – done (if possible) by inversion in Eqs. 1.6 – so to express $\dot{x}$, $\dot{y}$, $\dot{z}$ as functions of x, y, z and t, allows passing from Lagrangian to Eulerian specification, and corresponds to obtaining the velocity field (fluid velocity *flow*) $\mathbf{v}(\mathbf{r}; t) = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ at any time t and in any point x, y, z of the 3D space, where

$$v_x = \dot{x} = \dot{x}(x, y, z; t), \qquad v_y = \dot{y} = \dot{y}(x, y, z; t), \qquad v_z = \dot{z} = \dot{z}(x, y, z; t). \tag{1.8}$$

The inverse procedure, i.e. from Eulerian to Lagrangian specification, consists in obtaining the Lagrangian particle paths $\mathbf{r}(\mathbf{r}_0; t)$ by integration of the velocity field. See Exercises 1.1 and 1.2 for practical application examples.

An important formal step as a connection between the Lagrangian and Eulerian view is the *Lagrangian derivative*.

Consider the usual total time derivative of a scalar function $\phi(\mathbf{r}; t)$, or a vector function $\mathbf{A}(\mathbf{r}; t)$. Applying the chain rule

$$\dot{\phi} \equiv \frac{\mathrm{d}\phi(\mathbf{r}; t)}{\mathrm{dt}} = \frac{\partial\phi}{\partial t} + \dot{\mathbf{r}} \cdot \nabla\phi = \frac{\partial\phi}{\partial t} + (\dot{\mathbf{r}} \cdot \nabla)\phi, \tag{1.9}$$

and

$$\dot{\mathbf{A}} \equiv \frac{\mathrm{d}\mathbf{A}(\mathbf{r}; t)}{\mathrm{dt}} = \frac{\partial\mathbf{A}}{\partial t} + \dot{\mathbf{r}} \cdot \nabla\mathbf{A} = \frac{\partial\mathbf{A}}{\partial t} + (\dot{\mathbf{r}} \cdot \nabla)\mathbf{A}, \tag{1.10}$$

where ∇ (nabla) is the usual formal operator $\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ in Cartesian coordinates, so as $\nabla\phi$ is simply the gradient of the scalar function ϕ, while $\nabla\mathbf{A}$ is a matrix whose 3 rows are the 3 partial space derivatives of the x, y, z components of $\mathbf{A}$, respectively (see Appendix A.1). In other words, this matrix has rows which are the 3 vectors ∇A_x, ∇A_y and ∇A_z. As clarified by the identity between $\dot{\mathbf{r}} \cdot \nabla\mathbf{A}$ and $(\dot{\mathbf{r}} \cdot \nabla)\mathbf{A}$, the dot (scalar) product of the vector $\dot{\mathbf{r}}$ and the matrix $\mathbf{A}$ is a vector whose components are the dot product of $\dot{\mathbf{r}}$ with the 3 rows of the matrix $\mathbf{A}$. This is the usual 'line by column' product of a matrix with a vector.

Of course, $\partial\mathbf{A}/\partial t$ is a vector whose components are the time derivatives of the components of $\mathbf{A}$.

The Lagrangian (also called material, substantial, convective, etc.) derivative is obtained when letting $\dot{\mathbf{r}} = \mathbf{v}$ in Eqs. 1.9 and 1.10, where $\mathbf{v}$ is the actual flow velocity. This means derivatives of the functions ϕ and $\mathbf{A}$ done *following the fluid motion*. Formally, Eqs. 1.9 and 1.10 transform into the Lagrangian derivatives

$$\frac{\mathrm{D}\phi(\mathbf{r};t)}{\mathrm{Dt}} = \frac{\partial\phi}{\partial t} + \mathbf{v}\cdot\nabla\phi = \frac{\partial\phi}{\partial t} + (\mathbf{v}\cdot\nabla)\phi, \tag{1.11}$$

$$\frac{\mathrm{D}\mathbf{A}(\mathbf{r};t)}{\mathrm{Dt}} = \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{A} = \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A}. \tag{1.12}$$

In Eqs. 1.11 and 1.12, the spatial terms $\mathbf{v}\cdot\nabla$ are called *convection* terms (more precisely: when applied to a scalar function they are called *advection* terms).

The Lagrangian derivative constitutes a bridge between Lagrangian and Eulerian descriptions of fluid motion. Given a scalar ϕ quantity characterizing a fluid (for example temperature) or a vector quantity $\mathbf{A}$:

- the $\mathrm{D}\phi/\mathrm{D}t$ and $\mathrm{D}\mathbf{A}/\mathrm{D}t$ derivatives are the Lagrangian derivatives;
- the $\partial\phi/\partial t$, $\partial\mathbf{A}/\partial t$ and $\partial\phi/\partial x_i$, $\partial\mathbf{A}/\partial x_i$ partial derivatives are Eulerian derivatives;
- $\mathbf{v}$ in Eqs. 1.11 and 1.12 is the velocity field as measured by the Eulerian observer.

1.5 Stream Lines and Flux Tubes

Given a velocity field $\mathbf{v}(\mathbf{r};t)$, the *stream lines* are defined as those curves having as a tangent in every point the direction of $\mathbf{v}$ in that point. This geometrical condition the flow lines must satisfy translates into the differential condition

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = c\mathbf{v}, \tag{1.13}$$

where s is a curvilinear abscissa in the parametric expression $\mathbf{r}(s)$ of the unknown stream line, and $c \neq 0$ an arbitrary constant. The above ∞^1 vector conditions correspond to these scalar conditions

$$\frac{\mathrm{d}x}{v_x(x,y,z;t)} = \frac{\mathrm{d}y}{v_y(x,y,z;t)} = \frac{\mathrm{d}z}{v_z(x,y,z;t)}. \tag{1.14}$$

Integration of the above system of equations at every time t leads to the ∞^1 stream lines. Clearly, if the velocity field is time-dependent (*unsteady* flow), the flow lines *do* not correspond to fluid particle trajectories, while they do in case of time-independent velocity field (*steady* flow). In the latter case, indeed, flow lines don't change shape in time and correspond to fluid motion trajectories.

Another useful concept is that of *flux tube* (Fig.1.5).

The original description of a flux tube is due to J. C. Maxwell ("On Physical Lines of Force", Philosophical Magazine and Journal of Science, 4, 1861) who described flux tubes this way: "If upon any surface which cuts the lines of fluid motion we draw a closed curve, and if from every point of this curve we draw lines of motion, these lines of motion will generate a tubular surface which we may call a tube of fluid motion." Although lacking mathematical rigor, this can be indeed considered

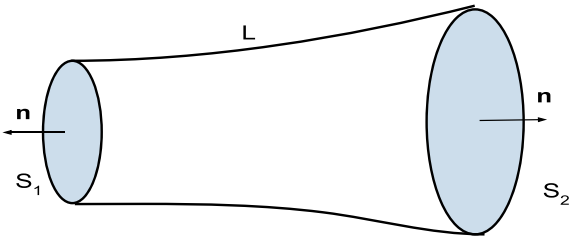


Fig. 1.5 Schematization of a flux tube with cross sections S_1 and S_2 with outward normal unit vectors **n**. L indicates the tube *lateral* side, surface that, by definition, is never crossed by the vector field

a good operational definition of a flow tube. We do not go here into deeper details on the concepts of stream lines and flux tubes, which will appear in their importance in the next chapter of this book where the fundamental equations of fluid mechanics are derived and discussed.

1.6 Solved Exercises

Exercise 1.1 A 2D flow is described in the Eulerian view as

$$\mathbf{v}(x, y; t) = (x + y + 2t)\mathbf{i} + (2y + t)\mathbf{j}. \tag{1.15}$$

Determine the Lagrangian coordinates as functions of the initial positions $\mathbf{r}_0 = (x_0, y_0)$ and time t.

Solution
Velocity components can be expressed in the form of linear differential equations

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = x + y + 2t, \qquad v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = 2y + t. \tag{1.16}$$

The two differential equations above are subjected to the initial conditions $x(0) = x_0$ and $y(0) = y_0$, and may be solved with usual techniques valid for first order ordinary, linear and non-homogeneous differential equations. The solution for $y(t)$ of the second equation in 1.16 above is obtained via the formula

$$y(t) = e^{\int_0^t 2\mathrm{d}t} \left(y_0 + \int_0^t t e^{-2t} \mathrm{d}t \right)$$

that leads to

$$y(t) = \left(y_0 + \frac{1}{4} \right) e^{2t} - \frac{1}{4}(2t + 1)$$

that, inserted in the first and adopting the same resolution procedure, leads to the solution for x

$$x(t) = \left[(x_0 - y_0 + 1)e^t + (y_0 + \frac{1}{4})e^{2t} - \frac{1}{4}(6t + 5)\right].$$

Finally, the searched Lagrangian expression for the flow is

$$\begin{aligned}\mathbf{r}(\mathbf{r}_0; t) = \left[(x_0 - y_0 + 1)e^t + (y_0 + \frac{1}{4})e^{2t} - \frac{1}{4}(6t + 5)\right]\mathbf{i} + \\ + \left[(y_0 + \frac{1}{4})e^{2t} - \frac{1}{4}(2t + 1)\right]\mathbf{j}.\end{aligned} \tag{1.17}$$

Exercise 1.2 A fluid flow has the Lagrangian representation

$$\mathbf{r}(\mathbf{r}_0; t) = x_0 e^t \mathbf{i} + y_0 e^{-t}\mathbf{j} + z_0\mathbf{k}. \tag{1.18}$$

Convert it into Eulerian representation.

Solution
By elimination of time in x and y, the trajectory of motion is found in Cartesian form

$$y = \frac{x_0 y_0}{x}, \qquad z = z_0,$$

which represent hyperbolas on the $z = z_0$ plane, in the first and third quadrant if $x_0 y_0 > 0$ or in the second and fourth if $x_0 y_0 < 0$. The velocity field (Eulerian representation) is simply

$$\mathbf{v}(\mathbf{r}; t) = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} = x\mathbf{i} - y\mathbf{j}, \tag{1.19}$$

and is 2D and stationary. The stream lines are obtained (see Sect. 1.5) by integration of

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} \Longrightarrow \frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}y}{y},$$

whose solution is $\ln|y| = -\ln|x| + c$ or, equivalently, $|y||x| = e^c > 0$ which are indeed hyperbolas. This is not a surprise because the velocity field Eq. 1.19 is stationary and so stream lines and trajectories coincide.

Exercise 1.3 Determine the expression of the stream lines for the 2D flow

$$\mathbf{v} = e^x \cosh y\mathbf{i} - e^x \sinh y\mathbf{j}. \tag{1.20}$$

Solution
The equations of the stream lines are

$$\frac{\mathrm{d}x}{e^x \cosh y} = -\frac{\mathrm{d}y}{e^x \sinh y},$$

or

$$\mathrm{dx} + \coth \mathrm{dy} = 0,$$

whose integration gives

$$x + \ln \sinh y = \ln c,$$

where $c > 0$ is an integration constant. The above expression can be also written as

$$\sinh y = ce^{-x}. \tag{1.21}$$

Historical Note

Euler and Lagrange, both eminent mathematicians, are undoubtedly two of the founders of modern fluid mechanics.

Leonhard Euler (b. 1707 in Basel, Switzerland, d. 1783 in St. Petersburg, Russia) has given enormous contributions in many fields of mathematics and mathematics applied to physics. His father Paul was a protestant minister who had been living at Jacob Bernoulli's home and attended his lectures while studying in Basel. Jacob Bernoulli (b. 1655, d. 1705) was a great mathematician, the first to introduce the term 'integral' in calculus, and a member of the Bernoulli family, which also mathematicians and physicists Johann Bernoulli (b. 1667, d. 1748) and Daniel Bernoulli (b. 1700, d. 1802) belonged to. As we will see in Chap. 2 of this book, the very important inverse relation between pressure and speed of a moving fluid has been named after Daniel Bernoulli who proposed it first in 1738 in his book Hydrodynamica. It was Euler, indeed, who derived Bernoulli's equation in its commonly used form.

The formation of L. Euler was profoundly influenced by Johann Bernoulli with whom he had several discussions and talks when he was very young, so to lead him toward mathematics and physics rather than to theology as his father would prefer. Leonhard Euler completed his studies in Basel at the age of 19 and published a paper on isochronous curves in a resisting medium. Part of his life and career was spent in St. Petersburg where for years he cohabited with Daniel Bernoulli whose senior chair in mathematics he herited when D. Bernoulli returned to Basel in 1733. Euler, on his side, left St. Petersburg in 1741 to Berlin, keeping strict contacts with Russia where he made a return in 1766. In the last part of his life, he was almost completely blind but in spite of this his scientific production was huge. He died in St. Petersburg in 1783. Apart from the overwhelming contributions to mathematics and theoretical mechanics, Euler's contribution to fluid dynamics dated mainly around the 1750s when he settled

up the main mathematical features of fluid mechanics, from the continuity equation to the equation of motion for an inviscid, incompressible fluid, which was named after him (see Chap. 2 of this book). In this regard, in 1752 he wrote: *However sublime are the researches on fluids which we owe to Messrs Bernoulli, Clairaut and Le Ronde d'Alembert, they flow so naturally from my two general formulae that one cannot sufficiently admire this accord of their profound meditations with the simplicity of the principles from which I have drawn my two equations ....*

Joseph-Louis Lagrange (b. 1736 in Turin, Italy, d. in 1813 in Paris, France) was born in Italy and baptized with the name Giuseppe Lodovico Lagrangia. At that time, Turin was the capital of the kingdom of Sardinia. He started his studies at the College of Turin, and, initially, he was not oriented toward scientific subjects. The financial difficulties of his family pushed him from law studies, as his father preferred for him, to an engagement in mathematics. He had not the chance to have particularly relevant teachers, and initially his scientific work was not particularly brilliant. However, he got in touch with Euler (who was in Berlin at that time) to discuss results which constituted basis of what Euler called "calculus of variations" in 1766. Thanks to Euler and Pierre Louis Moreau de Maupertuis (b. in 1698 in Saint Malo, France, d. in 1759 in Basel, Switzerland) who was the president of the Berlin academy, he was offered a position which he politely refused to remain in Turin until 1766 when he finally succeeded Euler as Director of Mathematics at the Berlin Academy. Before leaving Turin, he was one of the founders of the Royal Academy of Sciences of Turin.

A significant part of Lagrange's contribution to mathematics and physics appeared in the new journal Mélanges de Turin published by the Turin Royal Academy of Sciences. In the 3rd volume of the Mélanges de Turin, he applied some of his own results on the theory of differential equations to topics of fluid dynamics. In 1787, he left Berlin to enter the Académie des sciences in Paris city where he lived until his death. It was in Paris that his masterpiece, the book Mécanique analytique, was published, although it was written while Lagrange worked in Berlin. This book formalized mechanics in a full way, by adopting a description based on differential equations, whose theory has been widely developed by himself. He was the first professor of analysis in the newborn (1794) École Polytechnique. In Paris, he passed through the experience of the resolution and the following Napoleon empire. Napoleon named him to the Legion of Honour and Count of the Empire in 1808, and in 1813, a few weeks before his death, he was awarded the Grand Croix of the Ordre Impérial de la Réunion.

1.7 Further Readings

Good reference books for fluid dynamics generalities are [1–3].

Chapter 2
The Basic Equations for Fluid Motion

Fluid dynamics bases on a set of equations that, upon different hypotheses and approximations, 'govern' the evolution of the fluid system under study. They are also called *constitutive* equations.

Their explicit and specific mathematical form depends on the framework adopted (Eulerian or Lagrangian), and they derive essentially by simple requirements of *conservation* of some characteristic quantities that depend on the unknown quantities (density, pressure, velocity, internal energy, etc.) we look for.

The conservations laws which the basic fluid dynamics equations are based upon are

- *mass* conservation;
- *momentum* conservation (Newton's second law);
- *energy* conservation.

Being conservation laws, they express that mass, momentum and energy remain constant along the flow: their rates of variation along the fluid motion is zero. The rate of variation means time derivative, and so the conservation equations are ordinary differential equations (ODEs) in the Lagrangian view and partial differential equations (PDEs) in the Eulerian one. As we will see, mass conservation gives a link between the (scalar) mass density ρ and velocity $\mathbf{v}$ (vector) along time t, while momentum conservation gives a link among ρ, $\mathbf{v}$ and pressure p along time. Finally, energy conservation gives a link among ρ, $\mathbf{v}$, p and internal energy density e. So, the number of unknown functions is six, because $\mathbf{v}$ is a vector quantity. This means that the resulting (after some needed considerations) system of five ordinary or partial differential equations expressing conservation laws does not suffice to determine, once explicitly or numerically solved under proper boundary and initial conditions, the whole fluid characteristics because it is undetermined: the number of equations, 5, is less than the number, 6, of unknowns. This indeterminacy is solved, and the system is *closed*, by the further assumption of a mathematical link among some of the unknown quantities, usually called *equation of state* (EOS). We shall discuss this later in this chapter.

R. A. Capuzzo Dolcetta, *Physics of Fluids*, UNITEXT for Physics,
https://doi.org/10.1007/978-3-031-30750-8_2

2.1 The Continuity Equation

The first constitutive equation is the one which mathematically represents the conservation of mass of a fluid along its flow. As we said, this means a relation between the mass density, $\rho(x, y, z; t)$, and velocity, $\mathbf{v}(x, y, z; t)$.

By definition, the fluid mass density at a given time is a point function defined as

$$\rho(x, y, z; t) = \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V}, \tag{2.1}$$

where ΔV is a sampling volume around (x, y, z) and ΔM its mass. At time t, under regularity hypotheses, the limit in the above expression is the derivative of mass with respect to volume, $\rho(x, y, z; t) = dM/dV$, evaluated at time t in the given point in space. According to the above definition of mass density, the mass that at time t is contained in an arbitrary volume ΔV around (x, y, z) is given by the volume integral

$$\Delta M = \int_{\Delta V} \rho \, \mathrm{d}V = \iiint_{\Delta V} \rho \, \mathrm{d}^3\mathbf{r}. \tag{2.2}$$

The principle of *mass conservation* states that if mass is neither locally created nor destroyed (neither *sources* nor *sinks* are present), the mass contained in the arbitrary volume may vary just due to the fluid flow across the boundary of ΔV, the latter represented by the symbol $\partial \Delta V$. In the unit of time, the mass flow across this boundary is given by the surface integral

$$\int_{\partial \Delta V} \rho \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}\sigma, \tag{2.3}$$

where $\mathbf{n}$ is the unit vector orthogonal to $\partial \Delta V$ in the outward direction. Of course, if $\mathbf{v} \cdot \mathbf{n} > 0$ the mass is locally flowing out of the sampling volume, while if $\mathbf{v} \cdot \mathbf{n} < 0$ the mass is flowing in the sampling volume. Consequently, with the expression

$$\frac{\mathrm{d}M}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Delta V} \rho \, \mathrm{d}V = - \int_{\partial \Delta V} \rho \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}\sigma, \tag{2.4}$$

we represent the global rate of change of the mass in the volume element. Equation 2.4 is an *integral* form of the continuity equation.

We can get a differential form with a few passages.

Actually, because the volume ΔV is fixed in time, we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Delta V} \rho \, \mathrm{d}V = \int_{\Delta V} \frac{\partial \rho}{\partial t} \mathrm{d}V, \tag{2.5}$$

which, in the hypothesis of validity of the divergence theorem (see Appendix A.4), turns Eq. 2.4 into

$$\int_{\Delta V} \frac{\partial \rho}{\partial t} \mathrm{d}V = -\int_{\Delta V} \nabla \cdot \rho \mathbf{v} \, \mathrm{d}V. \tag{2.6}$$

The equality above constitutes another integral form of the continuity equation that is also written as

$$\int_{\Delta V} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} \right) \mathrm{d}V = 0. \tag{2.7}$$

Given the arbitrarity of the choice for ΔV, satisfaction of Eq. 2.7 requires the integrand to vanish identically, i.e. ρ and $\mathbf{v}$ must satisfy the partial differential equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0. \tag{2.8}$$

Equation 2.8 says that for a steady flow, $\frac{\partial \rho}{\partial t} = 0$, it is $\nabla \cdot \rho \mathbf{v} = 0$. Mathematically, this corresponds to *solenoidality* of the vector field $\rho \mathbf{v}$, whose physical dimensions are $\mathrm{ML}^{-2}\mathrm{T}^{-1}$ and which physically represents the density of momentum transported by the fluid at time t at position $\mathbf{r}$.

If there are sources or sinks of fluid matter, Eq. 2.8 would carry an extra term in the rhs, $\Sigma \equiv \mathrm{S} - \mathrm{s}$, where $\mathrm{S} \geq 0$ and $\mathrm{s} \geq 0$ account for source and sink, respectively, so that it generalizes to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = \Sigma, \tag{2.9}$$

where Σ represents the net rate of mass injected and ejected, per unit volume, into the fluid.

By using the property of the divergence of the product between a scalar and a vector function (see Exercise 2.1 and Appendix A.3.2), we have $\nabla \cdot \rho \mathbf{v} = \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho$ so that Eq. 2.8 turns into

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho = 0. \tag{2.10}$$

Recalling the expression for the Lagrangian derivative, the above equation is found equivalent to

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{v} = 0. \tag{2.11}$$

By simple manipulation of Eq. 2.11, the further expression,

$$\frac{1}{\rho} \frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\mathrm{D} \ln \rho}{\mathrm{D}t} = -\nabla \cdot \mathbf{v}, \tag{2.12}$$

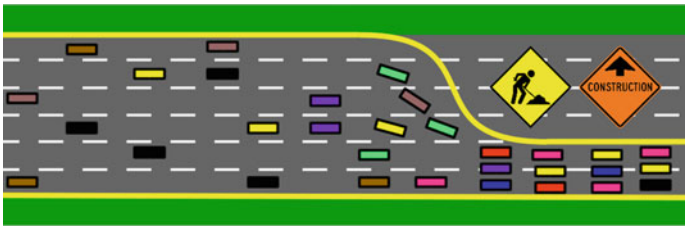


Fig. 2.1 The example of an increasing car density due to a bottleneck. In the vicinity of the bottleneck, cars slow down their motion to avoid accidents and this means $\nabla \cdot \mathbf{v} < 0$ and so the car density grows locally

is obtained. The above expression states that the velocity field of an incompressible fluid ($\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$) is necessarily *solenoidal*, i.e. $\nabla \cdot \mathbf{v} = 0$. On the other hand, it says also that an expansion of the fluid ($\nabla \cdot \mathbf{v} > 0$) implies a density reduction ($\frac{\mathrm{D}\rho}{\mathrm{D}t} < 0$) while a compression ($\nabla \cdot \mathbf{v} < 0$) leads to an increasing density ($\frac{\mathrm{D}\rho}{\mathrm{D}t} > 0$). A good representative example of a continuity equation at work is the development of a traffic jam on a freeway (Fig. 2.1).

A relevant result obtainable by a straight application of the continuity equation and the divergence theorem to an incompressible flow is the following.

In the fluid flow, let us consider a flux tube T whose orthogonal cross sections are indicated by S_1 and S_2 (see Fig. 1.5). If in that region the flow is incompressible, by applying the divergence theorem (see Appendix A.4) it is

$$\int_T \nabla \cdot \mathbf{v}\, \mathrm{d}V = 0 = \int_{\partial T} \mathbf{v} \cdot \mathbf{n}\, \mathrm{d}\sigma = \int_{S_1 \cup S_2} \mathbf{v} \cdot \mathbf{n}\, \mathrm{d}\sigma + \int_L \mathbf{v} \cdot \mathbf{n}\, \mathrm{d}\sigma, \tag{2.13}$$

where ∂T is the surface of the whole flux tube, given by the union of S_1, S_2 and the lateral, L, side. By definition of flux tube, the rightmost integral vanishes because $\mathbf{v} \cdot \mathbf{n} = 0$ identically over L, so that the above equation reduces to

$$0 = \int_{S_1 \cup S_2} \mathbf{v} \cdot \mathbf{n}\, \mathrm{d}\sigma = \int_{S_1} \mathbf{v} \cdot \mathbf{n}\, \mathrm{d}\sigma + \int_{S_2} \mathbf{v} \cdot \mathbf{n}\, \mathrm{d}\sigma, \tag{2.14}$$

or equivalently

$$\int_{S_1} \mathbf{v} \cdot \mathbf{n}\, \mathrm{d}\sigma = -\int_{S_2} \mathbf{v} \cdot \mathbf{n}\, \mathrm{d}\sigma. \tag{2.15}$$

Taking into account that $\mathbf{v} \cdot \mathbf{n}$ is negative in the *entrance* cross-sectional side (assumed as S_1) and positive on the *exit* side (S_2), a straightforward application of the integral mean value theorem leads to

$$\langle v_1 \rangle area(S_1) = \langle v_2 \rangle area(S_2), \tag{2.16}$$

where $\langle v_1 \rangle$ and $\langle v_2 \rangle$ are average values of the modulus of the normal components of $\mathbf{v}$ over the two cross-sectional surfaces. The relation Eq. 2.16 indicates to a higher fluid speed across the smaller cross section with respect to the larger one in a measure which scales with the ratio of the two cross-sectional areas.

2.1.1 *Lagrangian Form of Continuity Equation*

Consider Fig. 1.4 and denote by C_0 the volume domain occupied by a set of fluid particles at time t_0 and by C_t the volume domain occupied by the same particles at time t. The conservation of mass states that the mass contained in C_t is equal to that in C_0

$$M_0 \equiv \iiint_{C_0} \rho(\mathbf{r}_0; t_0)\, \mathrm{d}^3\mathbf{r}_0 = M_t = \iiint_{C_t} \rho(\mathbf{r}; t)\, \mathrm{d}^3\mathbf{r}. \tag{2.17}$$

The differential relation satisfied by the coordinate transformation $\mathbf{r}_0 \to \mathbf{r}$ is

$$\mathrm{d}^3\mathbf{r} = \mathrm{J}(\mathbf{r}_0, \mathbf{r})\mathrm{d}^3\mathbf{r}_0, \tag{2.18}$$

where

$$\mathrm{J}(\mathbf{r}_0, \mathbf{r}) = \det\left(\frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)}\right) = \begin{vmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} & \dfrac{\partial x}{\partial z_0} \\ \dfrac{\partial y}{\partial x_0} & \dfrac{\partial y}{\partial y_0} & \dfrac{\partial y}{\partial z_0} \\ \dfrac{\partial z}{\partial x_0} & \dfrac{\partial z}{\partial y_0} & \dfrac{\partial z}{\partial z_0} \end{vmatrix}, \tag{2.19}$$

is the determinant of the Jacobian matrix of the $\mathbf{r}_0$ to $\mathbf{r}$ transformation. Substituting by Eq. 2.18 in the second integral in Eq. 2.17, we have

$$\iiint_{C_0} \rho(\mathbf{r}_0; t_0)\mathrm{d}^3\mathbf{r}_0 = \iiint_{C_0} \rho(\mathbf{r}; t)\mathrm{J}(\mathbf{r}_0, \mathbf{r})\mathrm{d}^3\mathbf{r}_0, \tag{2.20}$$

which, due to the arbitrary choice of C_0, to be satisfied requires

$$\rho(\mathbf{r}_0; t_0) = \rho(\mathbf{r}; t)\mathrm{J}(\mathbf{r}_0, \mathbf{r}). \tag{2.21}$$

Equation 2.21 is known as the equation of continuity in Lagrangian form. In the following Exercise 2.2, it will be shown how the two forms (Eulerian, Eq. 2.10 and Lagrangian, Eq. 2.21) are actually equivalent.

2.2 The Motion Equation for Fluids

We now work out the formal expressions for the equations describing the motion of a fluid in a classic, Newtonian, non-relativistic, scheme.

Basically, one should adapt to fluids the Newton's second law of dynamics originally apt to describe the motion of solid bodies.

As is well known, Newton's second law of dynamics is written as

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{F}, \tag{2.22}$$

where $\mathbf{q} = m\mathbf{v}$ is the momentum of the particle of mass m (either point like or extended rigid body) and $\mathbf{F}$ is the resultant net force acting on it. In the case of an extended rigid body, the above differential equation describes the motion of the body's center of mass (c.o.m.) and cannot account for possible rotations of the body around it. In the case of constant mass, Eq. 2.22 is equivalent to three scalar, second order, ordinary differential equations for the coordinates x, y, z of the point-like particle or the c.o.m. of the extended rigid object.

The adaptation of the above Eq. 2.22 to a fluid requires, first, a full understanding of the concept of the total force acting on a fluid.

For a fluid, the acting forces are usually distinguished into two categories:

- *body* forces;
- *surface* forces.

The body (or volume) forces are those that act throughout the whole volume of the fluid, at least over a certain range of lengths, so they are usually *long-range* forces. Typical examples are gravity and electromagnetic forces. Due to its *pervasive* nature and, usually, the slow decrease of intensity with distance, the total body force acting on a sufficiently small fluid parcel is, quite intuitively, proportional to its volume, because it is about the same on every element of the parcel. On the other hand, a surface force is thought of as one acting on the (ideal) surface of the separation of two portions of fluid. They are, indeed, a type of *contact* forces that are those, in the realm of solids, acting on the surface of contact between two bodies. A typical example is the friction force. Due to the need for contact to occur, it is clear that the origin of these forces is microscopic, so that they are *short-range* forces. In other words, their decay with distance is extremely rapid so to practically vanish over a distance of a few average intermolecule separations.

It is usual to decompose the vector representing the surface force acting on a point of a surface into the normal and tangential (to the surface) components. They are called, respectively, *normal* and *shear* forces. The intensity of the normal force per unit area is, by definition, the pressure, p. On the other hand, the tangential (shear) force per unit area is called shear stress, usually referred to as τ. Clearly, the shear stress is what causes a deformation (strain) of a material in a direction parallel to that of the acting stress (see Sect. 1.1 of Chap. 1).

Given these definitions, we can progress in stating the form of the fluid motion equation as a generalization of Eq. 2.22. Assuming now as $\mathbf{Q}$ the local momentum density per unit volume of the fluid, we have $\mathbf{Q} = \rho\mathbf{v}$. Adopting a Lagrangian view and integrating $\mathbf{Q}$ over an arbitrary control parcel of fluid of volume V (which conserves its mass along the motion) to have the total parcel momentum, and making its Lagrangian time derivative

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \mathbf{Q}\,\mathrm{d}V = \int_V (\mathbf{F}_S + \mathbf{F}_B)\,\mathrm{d}V, \tag{2.23}$$

where $\mathbf{F}_S$ and $\mathbf{F}_B$ are the surface and body (volume) forces per unit volume acting on the moving 'fluid particle', we have that, due to the conservation of mass of the fluid parcel along its motion and due to the arbitrariness of the control parcel of fluid, the rate of change of the density of momentum is

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V \mathbf{Q}\,\mathrm{d}V = \rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \mathbf{F}_S + \mathbf{F}_B. \tag{2.24}$$

The above equation can be turned into its Eulerian expression by means of Eq. 1.12, giving

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = \frac{1}{\rho}\mathbf{F}_S + \frac{1}{\rho}\mathbf{F}_B. \tag{2.25}$$

2.2.1 For Ideal Fluids

In the hypothesis that the fluid is *ideal*,[1] due to zero viscosity there is no resistance to shear force in the fluid and so the acceleration is due just to the body force and to the normal part of the surface force (pressure contribution).

To obtain a proper expression of the equation of motion for such a fluid, we have to specify the rhs of Eq. 2.25 by considering a 'representative', arbitrary, element of fluid C and make some kind of integration, as we now explain.

In the absence of active shear, the total surface force acting on the fluid element C as due to the rest of the fluid is given by

$$\mathbf{F}_{S,C} = -\int_{\partial C} p\,\mathbf{n}\,\mathrm{d}\sigma, \tag{2.26}$$

where the pressure p is, in general, a function of the position vector $\mathbf{r}$ and of time, and $\mathbf{n}$ is the normal outward unit vector to the surface ∂C, contour of the fluid parcel C.

[1] An ideal fluid is an incompressible and *fully* adiabatic fluid, i.e. a fluid where there is no heat exchange between different parts of it. This means non-conductive and inviscid.

The minus sign is because we must consider the force acting on the C boundary due to the external fluid. The above surface integral may be transformed into a volume integral by the application of the divergence theorem (see Theorem A.1 and Corollary A.1 in Appendix A.4) leading to

$$\mathbf{F}_{S,C} = -\int_C \nabla p \, \mathrm{d}V. \tag{2.27}$$

The choice of C being arbitrary, the above implies that in any point of the fluid the local surface force per unit volume $\mathbf{F}_S$ is equal to $-\nabla p$, so that the fluid motion equation is

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{F}_B. \tag{2.28}$$

The above equation, in the case of null body force, is the classic Euler's equation for an ideal (inviscid) fluid, first obtained by L. Euler in 1755. Of course, the Lagrangian form of the equation of motion is

$$\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{F}_B. \tag{2.29}$$

Note that the hypothesis of incompressibility has not been used in the deduction of Eqs. 2.28 and 2.29, so they can be considered valid also for compressible, fully adiabatic (inviscid) fluids. Actually, as we will see in Sect. 2.2.2 of this chapter, viscosity acts also in pure compression and dilation of a fluid and not only along local shear, so that incompressibility and zero shear viscosity assume the same role of zero bulk viscosity and zero shear viscosity to lead to the same Eq. 2.28 equation of motion.

The role of the volume force is, in terrestrial fluids, often easy to handle. For a flowing liquid, like a water river, the volume force per unit mass, $\mathbf{F}_B/\rho$, is simply the constant gravitational acceleration on the Earth's surface, $\mathbf{g}$. In an astrophysical context, the situation is usually much more complicated, because the gravitational force field cannot be considered as a constant vector along the fluid size, because both the intensity and direction are position-dependent and so it must be self-consistently deduced by a proper field equation (Poisson's equation, in the case of gravity (see Sect. 3.2)).

Equations 2.28 and 2.29 are submitted to the proper boundary and initial conditions. If the fluid is physically *contained* (there is a boundary which separates the fluid from an external environment), the condition $\mathbf{v} \cdot \mathbf{n} = 0$ holds on the whole surface S of the boundary (as usual $\mathbf{n}$ is the outward normal unit vector). This means that the flow does not cross the boundary. Note that, in the case of a viscous fluid, the boundary condition $\mathbf{v} \cdot \mathbf{e}_t = 0$ with $\mathbf{e}_t$ local unit vector tangent to S must be assumed, too, and it is referred to as *no-slip* condition.

2.2.1.1 Euler's Equation in Terms of Velocity only

The first principle of thermodynamics, with specific (per unit mass) variables (intensive quantities), is written in infinitesimal form as

$$\mathrm{d}q^* = \mathrm{d}e + p\mathrm{d}V^*, \tag{2.30}$$

where q^* and e are,[2] respectively, the quantity of heat and internal energy per unit mass, and V^* (specific volume) is the volume occupied by the unit mass, i.e. $V^* = 1/\rho$.

Upon definition of a further thermodynamic intensive quantity,[3] the *enthalpy* $w = e + pV^*$, by its differentiation and taking into account that in the adiabatic case $dq^* = 0$, we have

$$\mathrm{d}w = \mathrm{d}e + p\mathrm{d}V^* + V^*\mathrm{d}p = V^*\mathrm{d}p = \frac{\mathrm{d}p}{\rho}. \tag{2.31}$$

The above relation, due to that $\mathrm{d}w = \nabla w \cdot \mathbf{r}$ and $\mathrm{d}p = \nabla p \cdot \mathbf{r}$, is equivalent to $\nabla w = \nabla p/\rho$, so that Euler's Eq. 2.28 in the assumption of conservative body force, $\mathbf{F}_B/\rho = \nabla U$ where U is the potential function, turns into

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\nabla w + \nabla U. \tag{2.32}$$

In the above equation, the term $(\mathbf{v} \cdot \nabla)\,\mathbf{v}$ can be eliminated by the vector identity (see Appendix A.3.2)

$$\frac{1}{2}\nabla v^2 = \mathbf{v} \wedge (\nabla \wedge \mathbf{v}) + (\mathbf{v} \cdot \nabla)\,\mathbf{v}, \tag{2.33}$$

to obtain

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla v^2 - \mathbf{v} \wedge (\nabla \wedge \mathbf{v}) = -\nabla w + \nabla U, \tag{2.34}$$

which, in its turn, can be written as

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \wedge (\nabla \wedge \mathbf{v}) = -\nabla\left(\frac{1}{2}v^2 + w - U\right). \tag{2.35}$$

Taking the curl of both sides of the above equation and then swapping the curl with partial derivative with respect to t in the lhs,

---

[2] Symbols e for specific internal energy and for Euler's number should not be confused.

[3] An intensive quantity has a value that *does not* depend on the mass of the system it refers to. Mass density, pressure, specific internal energy and viscosity are examples of intensive quantities.

$$\frac{\partial}{\partial t}(\nabla \wedge \mathbf{v}) - \nabla \wedge [\mathbf{v} \wedge (\nabla \wedge \mathbf{v})] = -\nabla \wedge \nabla \left(\frac{1}{2} v^2 + w - U \right) = 0, \tag{2.36}$$

the rightmost equality due to $\nabla \wedge \nabla f = 0$ for any scalar function $f(\mathbf{r}; t)$ (see also Appendix A.3.2). Consequently Euler's equation can be written in the simple form

$$\frac{\partial}{\partial t}(\nabla \wedge \mathbf{v}) = \nabla \wedge [\mathbf{v} \wedge (\nabla \wedge \mathbf{v})], \tag{2.37}$$

which involves the velocity flow, only.

Additionally, if the flow is irrotational, i.e. its vorticity $\boldsymbol{\omega} \equiv \nabla \wedge \mathbf{v} = 0$, it admits a velocity potential, so that $\mathbf{v} = \nabla \Phi$ where $\Phi(\mathbf{r}; t)$ is a scalar function (called velocity potential); consequently Eq. 2.34 can be written as

$$\frac{\partial}{\partial t} \nabla \Phi + \frac{1}{2} \nabla v^2 = -\nabla w + \nabla U, \tag{2.38}$$

which, swapping $\partial / \partial t$ with ∇, leads to

$$\nabla \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} v^2 + w - U \right) = 0, \tag{2.39}$$

implying

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} v^2 + w - U = f(t), \tag{2.40}$$

where $f(t)$ is an arbitrary function of time, which can be assumed zero, as it comes out by the transformation $\Psi = \Phi - \int_{t_0}^{t} f(t) \mathrm{d}t$, which keeps unaltered the flow velocity because $\mathbf{v} = \nabla \Phi = \nabla \Psi$.

If, additionally, the flow is stationary, $\frac{\partial \Phi}{\partial t} = 0$,[4] and $f(t) = const.$, and Eq. 2.40 leads to the standard form of *Bernoulli's* equation

$$\frac{1}{2} v^2 + w - U = const., \tag{2.41}$$

which finds several practical applications, among which the most known is to the hydrodynamic and aerodynamic lift (see Fig. 2.2).

[4] If the flow is stationary, also v and w don't depend on time.

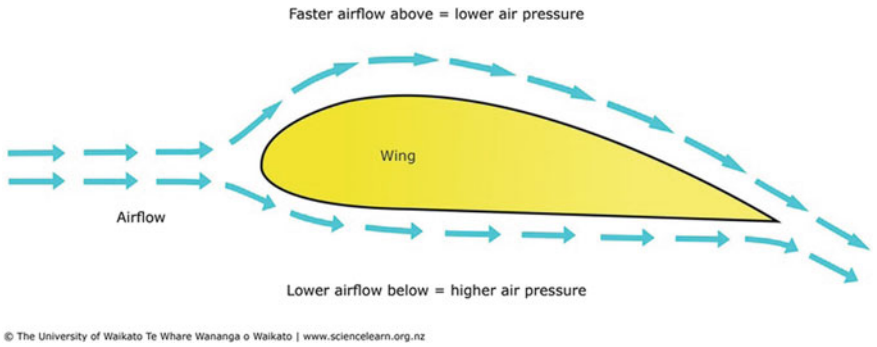


Fig. 2.2 Cross-sectional view of an air wing

2.2.2 For Non-ideal Fluids

The continuity Eq. 2.8 can be written in component form as

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_k}{\partial x_k} = 0, \tag{2.42}$$

where the Einstein convention of summation over the repeated index k is adopted. Also Euler's (motion) Eq. 2.28 can be written in component form

$$\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}, \tag{2.43}$$

where we dropped, for simplicity, the body force in Eq. 2.28 rhs.

Now, in the expression by components of the partial time derivative of the density of momentum

$$\frac{\partial \rho v_i}{\partial t} = \rho \frac{\partial v_i}{\partial t} + v_i \frac{\partial \rho}{\partial t}, \tag{2.44}$$

we can express the first addend in the rhs by Euler's Eq. 2.43 and the second by the continuity Eq. 2.42, obtaining

$$\begin{aligned} \frac{\partial \rho v_i}{\partial t} &= \rho \left(-v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \right) + v_i \left(-\frac{\partial \rho v_k}{\partial x_k} \right) = -\rho v_k \frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - v_i \frac{\partial \rho v_k}{\partial x_k} = \\ &= -\frac{\partial p}{\partial x_i} - \frac{\partial \rho v_i v_k}{\partial x_k}. \end{aligned} \tag{2.45}$$

By the use of the unitary (diagonal) tensor, called also Kronecker's delta,

$$\delta_{ik} = \begin{cases} 0, & \text{if } i \neq k, \\ 1, & \text{if } i = k, \end{cases} \tag{2.46}$$

we can write $\frac{\partial p}{\partial x_i} = \delta_{ik}\frac{\partial p}{\partial x_k}$ and substitute it into Eq. 2.45, to have

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial}{\partial x_k}\left(\delta_{ik}p + \rho v_i v_k\right) \equiv -\frac{\partial \Pi_{ik}}{\partial x_k}, \tag{2.47}$$

having defined the (symmetric) tensor

$$\Pi_{ik} \equiv \delta_{ik}p + \rho v_i v_k, \tag{2.48}$$

which is called *Reynolds* tensor or stress tensor or momentum flux tensor. In vector notation, the above equation is written as

$$\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \cdot \underline{\Pi}, \tag{2.49}$$

where $\underline{\Pi}$ represents the Π_{ik} tensor in vector notation (see Appendix A.5).

The denomination of momentum flux tensor is explained by integrating over a given, fixed, volume of fluid the leftmost and rightmost sides of Eq. 2.47,

$$\int_V \frac{\partial \rho v_i}{\partial t} \mathrm{d}V = \frac{\partial}{\partial t}\int_V \rho v_i \mathrm{d}V = -\int_V \frac{\partial \Pi_{ik}}{\partial x_k}\mathrm{d}V = -\int_{\partial V} \Pi_{ik} n_k \mathrm{d}\sigma, \tag{2.50}$$

where the last equality derives from the application of the divergence theorem. Clearly, the second integral in the above equations is the rate of variation of the ith component of the density of momentum within V, so that the surface integral at the rightmost side is the flux of momentum that, per unit time, flows across the surface ∂V bounding the volume V. Note the analogy between the rate of change of mass (lhs of Eq. 2.4) as due to the surface integral (rhs of Eq. 2.4) of the momentum density vector (first order tensor) and the rate of change of the momentum density vector (lhs of Eq. 2.50) as due to the surface integral (rhs of Eq. 2.50) of the second order Reynolds stress tensor.

So far, we neglected viscosity and every other dissipative phenomenon. Consequently, the above-given stress tensor represents a reversible situation, describing just the mechanical transport of momentum from a point to another and the contribution of pressure. Aiming at the description of a viscous fluid, it is needed to modify accordingly the stress tensor to account for the addition of an irreversible process.

To get a proper expression of the stress tensor accounting for viscosity in Newtonian fluids, we will follow a heuristic procedure. Recalling the dissipative action of friction on a solid body motion as representable with the addition to the conservative acceleration on the moving body of a deceleration term opposite to the velocity direction and linear in v ($-\lambda^2\mathbf{v}$), schematization valid also for a solid body moving in a fluid (Stokes law; see Sect. 2.5), we assume that irreversible processes can be accounted by adding them to the *reversible* tensor Π_{ik} and another tensor σ_{ik}

$$\widetilde{\Pi}_{ik} = \Pi_{ik} - \sigma_{ik}, \tag{2.51}$$

where σ_{ik} would represent the *viscosity* stress tensor.

The way to determine the actual expression of σ_{ik} is not unique, but passes through some hypotheses and approximations. First of all, viscosity is simply a conceptual generalization to a fluid of the friction acting on the surface of contact of bodies (see Sect. 1.1). Viscous dissipation, thus, exists only when there is a relative motion within the fluid (a layer of fluid moves with respect to the adjacent). So, viscosity must depend upon spatial derivatives of the internal fluid velocity field. In the hypothesis of small velocity variations, it is natural that the momentum transfer by viscosity depends only linearly upon the first partial spatial derivatives of the velocity and that the dependence on velocity space derivatives of order higher than one is negligible, $\sigma_{ik} \propto \partial v_i/\partial x_k$. Of course, the condition $\sigma_{ik} = 0$ when $\mathbf{v} = const.$ is imposed (no velocity means obviously no internal shear): this means that σ_{ik} cannot contain terms independent of $\partial v_i/\partial x_k$.

The viscosity tensor is constrained also by the condition it should vanish for a fluid in uniform (rigid) rotation, because even if the velocity field is not constant in a rotation being $\mathbf{v} = \boldsymbol{\omega} \wedge \mathbf{r}$, when the rotation is uniform ($\boldsymbol{\omega} = const.$) there is no relative motion within the fluid and so viscosity cannot act. This condition translates into a series of conditions on the space partial derivatives of the velocity field, which can be resumed by $\dfrac{\partial v_i}{\partial x_k} + \dfrac{\partial v_k}{\partial x_i} = 0$, for $i, k = 1, 2, 3$.

The demonstration is easy. If the fluid is uniformly rotating (rigid rotation), the velocity field is

$$\mathbf{v} = \boldsymbol{\omega} \wedge \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = (\omega_y z - \omega_z y)\mathbf{i} - (\omega_x z - \omega_z x)\mathbf{j} + (\omega_x y - \omega_y x)\mathbf{k}, \tag{2.52}$$

which implies

$$\begin{aligned} &\frac{\partial v_x}{\partial x} = 0, \quad \frac{\partial v_x}{\partial y} = -\omega_z, \quad \frac{\partial v_x}{\partial z} = \omega_y, \\ &\frac{\partial v_y}{\partial x} = \omega_z, \quad \frac{\partial v_y}{\partial y} = 0, \quad \frac{\partial v_y}{\partial z} = -\omega_x, \\ &\frac{\partial v_z}{\partial x} = -\omega_y, \quad \frac{\partial v_z}{\partial y} = \omega_x, \quad \frac{\partial v_z}{\partial z} = 0, \end{aligned} \tag{2.53}$$

which summarizes in the general relations

$$\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} = 0, \tag{2.54}$$

with the usual convention $x_1 = x, x_2 = y, x_3 = z$; $v_1 = v_x$, $v_2 = v_y$, $v_3 = v_z$. Eq. 2.53 show that in rigid rotation $\nabla \cdot \mathbf{v}$ is identically null.

Given that by the above considerations the heuristic expression of σ_{ik} would contain only linear terms in $\dfrac{\partial v_i}{\partial x_k}$, it is worth writing

$$\begin{aligned}\frac{\partial v_i}{\partial x_k} &= \frac{1}{2}\frac{\partial v_i}{\partial x_k} + \frac{1}{2}\frac{\partial v_i}{\partial x_k} + \frac{1}{2}\frac{\partial v_k}{\partial x_i} - \frac{1}{2}\frac{\partial v_k}{\partial x_i} + \frac{1}{3}\delta_{ik}\frac{\partial v_j}{\partial x_j} - \frac{1}{3}\delta_{ik}\frac{\partial v_j}{\partial x_j} = \\ &= \frac{1}{3}\delta_{ik}\frac{\partial v_j}{\partial x_j} + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_j}{\partial x_j}\right) = \\ &\equiv A_{ik} + B_{ik} + C_{ik}.\end{aligned} \tag{2.55}$$

The first addend, $A_{ik} = \frac{1}{3}\delta_{ik}\frac{\partial v_j}{\partial x_j}$, in the last equality is a symmetric tensor ($A_{ik} = A_{ki}$) identically null in rigid rotation and having trace[5] equal to $\nabla \cdot \mathbf{v}$. The second one, $B_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\right)$, is an antisymmetric tensor ($B_{ik} = -B_{ik}$) that does not vanish in rigid rotation and with trace equal to zero. The last one, $C_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_j}{\partial x_j}\right)$, is, a symmetric tensor with trace equal to zero and identically null in rigid rotation. Because the tensor B_{ik} does not vanish in rigid rotation, as required by the concept of viscosity, the linear combinations of the first order space partial derivatives the viscosity tensor may depend upon are of the type $\alpha A_{ik} + \beta B_{ik} + \gamma C_{ik}$, with $\beta = 0$.

In the following, we indicate by $\zeta \equiv \alpha/3$ and $\eta \equiv \gamma/2$ the so-called *bulk* and *shear* viscosity, respectively. The name bulk viscosity is because ζ is the coefficient of the tensor A_{ik} whose trace is $\nabla \cdot \mathbf{v}$ which, by the equation of continuity, is a measure of the fluid compressibility. Analogously, the name shear viscosity is associated with trace$(C_{ik}) = 0$. Note that the physical dimensions of ζ and η are $[\zeta] = [\eta]$=ML^{-1}T^{-1}, which include M, justifying the adjectivation of *dynamic* viscosity coefficients. To the dynamic viscosities correspond *kinematic* viscosities $\tilde{\zeta} = \zeta/\rho$ and $\tilde{\eta} = \eta/\rho$ whose dimensions do not include M. It can be said that dynamic viscosity gives information on the force needed to make the fluid flow at a certain rate, while kinematic viscosity tells how fast the fluid is moving when a certain force is applied.

As a consequence of what was said above, the heuristic form of the viscosity stress tensor is

$$\sigma_{ik} = \zeta\delta_{ik}\frac{\partial v_j}{\partial x_j} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_j}{\partial x_j}\right). \tag{2.56}$$

In Table 2.1, some values of shear kinematic viscosities of some liquids and gases are reported at various temperatures in the $0° \div 100°$ C range. Note how for liquids and the saturated water vapor, viscosity decreases at higher temperatures while the opposite occurs for methane and air, which are two gases. This different behavior of viscosity for liquids and gases is interpreted in terms of the efficiency of internal collisions: in a gas a higher temperature implies larger thermal motion of particles and so a stronger interaction, while in a liquid the intermolecule interaction force

[5] The trace of a square matrix is the sum of the elements on its diagonal.

Table 2.1 Values of shear kinematic viscosities, η/ρ, for various fluids at selected temperatures (in degrees Celsius). Viscosity values in 10^{-7} m^2/s (from https://www.engineerplant.it/dtec/)

Fluid	0° C	10° C	20° C	50° C	100° C
Mercury	1.3	1.2	1.2	1.1	–
Ammonia	3.1	2.9	2.7	2.0	–
Gasoline	8.0	7.0	6.0	4.5	–
Water	19.0	14.0	11.0	5.5	3.0
Salt water	25.0	18.0	12.0	8.0	4.5
Kerosene	42.0	28.0	24.0	13.0	9.0
Gear oil (SAE 30)	2×10^4	6×10^3	2.8×10^3	550	120
Methane	120	140	150	180	220
Air	120	21	23	26	35
Saturated H_2O vapor	1.8×10^4	10^4	5.1×10^3	1.1×10^3	210

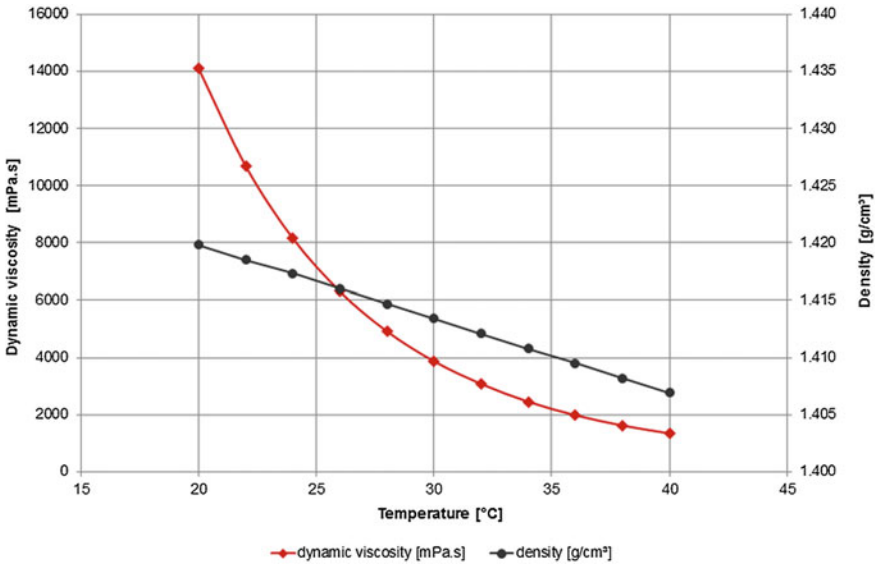


Fig. 2.3 Dependence on temperature of honey density (in black) and dynamic viscosity (in red) (from https://wiki.anton-paar.com/en/flower-honey-blended/)

weakens with increasing temperature. Note that a priori η and ζ could be $\lessgtr 0$, but the dissipative nature of viscosity implies they are both ≥ 0 (see Sect. 2.3.2). Figure 2.3 shows the dependence of both density and dynamic shear viscosity η of a mixed flower honey.

Given the expressions in Eq. 2.51 and 2.56, the equation of motion is written as

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial \widetilde{\mathbf{\Pi}}_{ik}}{\partial x_k} = -\frac{\partial}{\partial x_k}\left(\Pi_{ik} - \sigma_{ik}\right) = \\ = -\frac{\partial}{\partial x_k}\left[\delta_{ik} p + \rho v_i v_k - \zeta \delta_{ik} \frac{\partial v_j}{\partial x_j} - \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_j}{\partial x_j}\right)\right], \tag{2.57}$$

which is the general expression for the equation of motion of a viscous fluid under the above-described approximations.

In principle, both η and ζ are functions of the state variables ρ and T, which vary across the fluid. Anyway, in a limited range of ρ and T, η and ζ may be assumed constant (Newtonian fluid), and Eq. 2.57 simplifies to

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial}{\partial x_k}\left(\delta_{ik} p + \rho v_i v_k\right) + \zeta \frac{\partial}{\partial x_k}\left(\delta_{ik}\frac{\partial v_j}{\partial x_j}\right) + \\ + \eta \frac{\partial}{\partial x_k}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_j}{\partial x_j}\right). \tag{2.58}$$

In vector form, the above equation is

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \mathbf{v}\nabla \cdot \rho \mathbf{v} + (\rho \mathbf{v} \cdot \nabla)\,\mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} + \left(\frac{\eta}{3} + \zeta\right)\nabla \nabla \cdot \mathbf{v}, \tag{2.59}$$

which is the extended expression for

$$\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \cdot \underline{\widetilde{\mathbf{\Pi}}}, \tag{2.60}$$

where $\underline{\widetilde{\mathbf{\Pi}}} = \underline{\mathbf{\Pi}} - \underline{\boldsymbol{\sigma}}$ is the vector representation of the tensor given in Eq. 2.51.

Using the equation of continuity to simplify the lhs of Eq. 2.59, it reduces to the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v} + \left(\frac{1}{3}\frac{\eta}{\rho} + \frac{\zeta}{\rho}\right)\nabla \nabla \cdot \mathbf{v}, \tag{2.61}$$

called Navier–Stokes (N-S) equation. It is formally identical to Euler's equation (to which it reduces when $\eta = \zeta = 0$) but for the addition of the viscosity terms in the rhs. These terms describe, actually, the role played by viscosity on the *diffusion*[6] of momentum along the fluid motion, in some opposition to the ordered role of the convection term at the lhs.

For an incompressible fluid, $\nabla \cdot \mathbf{v} = 0$, and so Eq. 2.61 assumes the most common form of the *Navier–Stokes* equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{v}. \tag{2.62}$$

[6] Diffusion is the physical phenomenon of spreading out a quantity around a point.

The possible presence of an external force, i.e. a body force (per unit mass), can be accounted for by adding it at the rhs side of the above equation.

For the sake of completeness, we give also the equivalent of Eq. 2.61 written in component form

$$\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\eta}{\rho}\frac{\partial}{\partial x_k}\frac{\partial v_i}{\partial x_k} + \left(\frac{1}{3}\frac{\eta}{\rho} + \frac{\zeta}{\rho}\right)\frac{\partial}{\partial x_i}\frac{\partial v_k}{\partial x_k}. \tag{2.63}$$

2.2.2.1 The Navier–Stokes Equation for Incompressible Fluids

As was said, the Navier–Stokes equation for incompressible fluids assumes the expression in Eq. 2.62. Recalling the vector identity 2.33 to substitute for $(\mathbf{v}\cdot\nabla)\mathbf{v}$ in Eq. 2.62, taking the curl of both sides of the obtained equation, and assuming constant the kinematic shear viscosity $\widetilde{\eta} \equiv \eta/\rho$, Eq. 2.62 transforms into

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \wedge \left(\frac{1}{2}\nabla v^2 - \mathbf{v} \wedge \boldsymbol{\omega}\right) = -\nabla \wedge \frac{\nabla p}{\rho} + \widetilde{\eta}\,\nabla^2\boldsymbol{\omega}. \tag{2.64}$$

In the equation above, $\nabla \wedge \frac{1}{2}\nabla v^2$ is identically zero. Moreover, we may use the vector identity (see Appendix A.3.2)

$$\nabla \wedge (\mathbf{v} \wedge \boldsymbol{\omega}) = (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} + \mathbf{v}\nabla\cdot\boldsymbol{\omega} - \boldsymbol{\omega}\nabla\cdot\mathbf{v}, \tag{2.65}$$

where the two rightmost terms are zero because $\nabla\cdot\boldsymbol{\omega} = 0$ being the divergence of a curl and $\nabla\cdot\mathbf{v} = 0$ for incompressibility; given all this, and by the development of $\nabla \wedge (\nabla p/\rho)$, Eq. 2.64 is written as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} = \frac{1}{\rho^2}\nabla\rho \wedge \nabla p + \widetilde{\eta}\,\nabla^2\boldsymbol{\omega}. \tag{2.66}$$

In terms of Lagrangian derivative, the above equation (*vorticity* equation) is

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} + \frac{1}{\rho^2}\nabla\rho \wedge \nabla p + \widetilde{\eta}\,\nabla^2\boldsymbol{\omega}. \tag{2.67}$$

In the latter equation, the first term in the rhs is called *vortex stretching* because it increases vorticity according to the corresponding increase of the component of vorticity in the stretching direction. In a 2D flow, vortex stretching is zero, because $\mathbf{v}$ is confined on a plane and $\boldsymbol{\omega}$ is orthogonal to that plane. The second term in the rhs of Eq. 2.67 is the *baroclinic* term, which goes to modify vorticity as much as directions of density and pressure variations are inclined each other (the term is null when $\nabla\rho \parallel \nabla p$). The baroclinic term is identically null when the EOS is barotropic, $p = p(\rho)$, because in this case

$$\nabla p = \frac{\mathrm{d}p}{\mathrm{d}\rho}\nabla\rho, \tag{2.68}$$

so to give $\nabla\rho \wedge \nabla p = 0$. While the vortex stretching term is connected with the development of turbulence in fluids, the baroclinic term is concerned mainly with atmospheric physics. Baroclinic instabilities are indeed primary causes of the formation and development of cyclones, typhoons and tornadoes.

The last contribution to vorticity rate of change, $\widetilde{\eta}\,\nabla^2\boldsymbol{\omega}$, corresponds to molecular diffusion of vorticity, governed by kinematic shear viscosity, being zero only for inviscid fluids.

2.2.2.2 An Exact Solution of Navier–Stokes Equation for Incompressible Fluids

Due to their strong non-linearity, no general solutions of neither Euler's nor N-S equation are known.

Actually, some solutions to the full non-linear N-S equation do exist (Jeffery–Hamel flow, von Kármán swirling flow, stagnation point flow, Landau–Squire jet, and Taylor–Green vortex) whose stability is anyway hard to be stated, being likely subjected to turbulence development at high Reynolds numbers.[7]

Anyway, under strong simplifying assumptions, which mainly correspond to erasing non-linear terms, other, simpler, exact solutions of the N-S equation are found (Hagen–Poiseuille flow, , Couette flow and oscillatory Stokes boundary layer).

For sake of an example, we deal here with the so-called *Hagen–Poiseuille flow*.

The physical scheme is that of a viscous and incompressible fluid flowing steadily in a cylindrical pipe of circular section of radius R_p significantly smaller than the pipe length L_p. As it will be shown later in this chapter, the condition for the flow to be laminar and not turbulent is that the fluid speed and the pipe cross section are not too large for a given fluid kinematic viscosity.

The clear symmetry of the problem suggests the use of cylindrical coordinates, (R, θ, z), where z is taken along the pipe axis and pointed in the flow direction and R and θ are the usual radial and azimuthal coordinates on the equatorial plane.

Mathematically, the assumptions are as follows:

- the fluid is incompressible: $\nabla \cdot \mathbf{v} = 0$;
- steady fluid: all partial derivatives with respect to time are null;
- laminar flow: $v_\theta = v_R = 0$, $v_z > 0$;
- axisymmetric flow: all partial derivatives with respect to θ are null;
- fully developed flow: $\frac{\partial v_z}{\partial z} = 0$;
- the pipe is horizontal and the external force is terrestrial gravity alone: $(\mathbf{F}_B/\rho)_z = 0$).

Upon above conditions, the equation of continuity in cylindrical coordinates (see Appendix A.2.2) reduces to

[7] The concept of turbulence and its relation with the Reynolds number is developed in Sect. 2.5.

$$\frac{\partial \rho v_z}{\partial z} = v_z \frac{\partial \rho}{\partial z} = 0, \tag{2.69}$$

implying $\frac{\partial \rho}{\partial z} = 0$.

The radial N-S equation in cylindrical coordinates (see again Appendix A.2.2) reduces to $\frac{\partial p}{\partial R} = 0$, meaning that p depends on z, only. The azimuthal N-S equation is identically satisfied, so the only non-trivial equation left is the axial equation which here reduces to

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\eta}{\rho}\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial v_z}{\partial R}\right) = 0. \tag{2.70}$$

Because $p = p(z)$ and $v_z = v_z(R)$, in order for a function of z alone and one of R alone to be equal they must be equal to the same constant, which we call A.

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{\eta}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}v_z}{\mathrm{d}R}\right) = A. \tag{2.71}$$

The equation $\mathrm{d}p/\mathrm{d}z = A$ is integrated giving $p(z) = Az + c_0$, with c_0 integration constant. The pressure difference on two generic orthogonal sides of the pipe is $\Delta p \equiv p(z_1) - p(z_2) = A(z_1 - z_2)$, so that $\Delta p > 0$ if the fluid flows from face 1 (identified by a z_1 where the flow is already fully developed) to face 2. Consequently

$$A = -\frac{\Delta p}{L_p} < 0. \tag{2.72}$$

By two successive integrations with respect to R, the solution for v_z of Eq. 2.71 is

$$v_z(R) = \frac{A}{4\eta}R^2 + c_1 \log R + c_2. \tag{2.73}$$

Non-singularity of the solution for $R = 0$ requires $c_1 = 0$, while c_2 is determined by the no-slip boundary condition, $v_z(R_p) = 0$, which gives $c_2 = -(AR_p^2)/(4\eta)$. The final parabolic velocity profile (see Fig. 2.4) is so

$$v_z(R) = \frac{A}{4\eta}(R^2 - R_p^2). \tag{2.74}$$

As intuitively expected, the maximum speed is along the z-axis, $R = 0$, and is $v_{z,max} = -AR_p^2/(4\eta)$.

A practical way to measure the pressure drop due to viscosity in the Hagen–Poiseuille pipe is by a measure of the average volumetric flow rate $Q = \pi R_p^2 \langle v_z \rangle$ (whose dimensions are $\mathrm{L}^3\mathrm{T}^{-1}$ and represents an average measure of the volume of

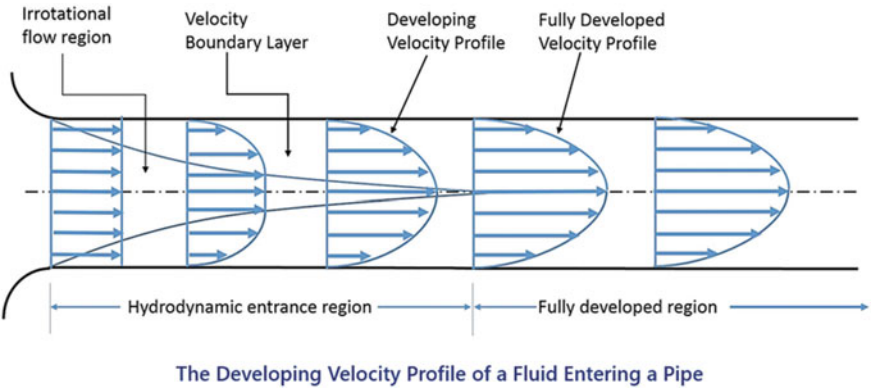


Fig. 2.4 Flow development within a cylindrical pipe. The dot-dashed line is along the z axis (Figure taken from https://skill-lync.com/student-projects/week-11-simulation-of-flow-through-a-pipe-in-openfoam-232)

fluid flowing through the pipe in the unit of time) where the average speed $\langle v_z \rangle$ is obtained as

$$\langle v_z \rangle = \frac{1}{\pi R_p^2} \int_0^{R_p} v_z(R) 2\pi R \mathrm{d}R = -\frac{A R_p^2}{8\eta} = \frac{1}{2} v_{z,max}, \tag{2.75}$$

which leads to $Q = -(\pi A R_p^4)/(8\eta)$, and so $A = -(8\eta Q)/(\pi R_p^4)$ that inserted into Eq. 2.72 leads to the pressure drop

$$\Delta p = -\frac{8\eta Q L_p}{\pi R_p^4}, \tag{2.76}$$

usually referred to as the Hagen–Poiseuille equation or law, determined experimentally by both Gotthilf Heinrich Ludwig Hagen and Jean Leonard Marie Poiseuille in 1838 and theoretically justified later by George Stokes in 1845. It is interesting noting that this law successfully applies to the flow of blood in the vascular system, as well as to the airflow in lung alveoli and liquid flow along a drinking straw or liquid flow along a hypodermic needle.

2.2.3 *The Stream Function*

If the velocity field can be written as $\mathbf{v} = \nabla \wedge \boldsymbol{\psi}$, where $\boldsymbol{\psi}$ is a *vector potential*, [8] it results in the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ being automatically satisfied

[8] A vector potential is a vector field whose curl is another, given, vector field, while a scalar potential is a scalar field whose gradient is another, given, vector field.

as a divergence of a curl. In addition, if the flow is in 2D (on the (x, y) plane), it is $v_z = 0$ and

$$\mathbf{v} = \left(\frac{\partial \psi_z}{\partial y} - \frac{\partial \psi_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial \psi_z}{\partial x} - \frac{\partial \psi_x}{\partial z}\right)\mathbf{j}. \tag{2.77}$$

If $\boldsymbol{\psi}$ is a 1D field, $\boldsymbol{\psi} = \psi_z \mathbf{k} \equiv \psi \mathbf{k}$, the above expression of $\mathbf{v}$ simplifies to

$$v_x = \frac{\partial \psi}{\partial y}, \qquad v_y = -\frac{\partial \psi}{\partial x}. \tag{2.78}$$

The function ψ is, a priori, also dependent on t and is called *stream* function. It can be defined in 2D flows, and, in 3D, just for axisymmetric flows.

An important characteristic of ψ is being constant along stream lines. Recalling that a stream line is a curve such that $d\mathbf{r}/ds = c\mathbf{v}$ (Sect. 1.5), we have that along a stream line it results in

$$\frac{d\psi}{ds} = \frac{\partial \psi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial \psi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} = c\left(\frac{\partial \psi}{\partial x}v_x + \frac{\partial \psi}{\partial y}v_y\right) = c\left(\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x}\right) = 0. \tag{2.79}$$

Now, note that the vorticity of a 2D flow $\mathbf{v}$ admitting a vector potential $\boldsymbol{\psi}$ is

$$\boldsymbol{\omega} = \nabla \wedge \mathbf{v} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{k} = \left(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right)\mathbf{k} = -\nabla^2 \psi\, \mathbf{k}. \tag{2.80}$$

This expression says that in an irrotational flow, the stream function is harmonic,[9] $\nabla^2 \psi = 0$. If the flow is irrotational, $\mathbf{v}$ admits a scalar potential ϕ, $\mathbf{v} = \nabla\phi$, and, if incompressible, the condition $\nabla \cdot \mathbf{v} = 0$ implies that also the velocity potential is harmonic, $\nabla^2\phi = 0$. In the general $\boldsymbol{\omega} \neq 0$ case, using Eq. 2.80 we can write the vorticity evolution Eq. 2.67 for a barotropic fluid in the form

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} + \mathbf{k}\,\widetilde{\eta}\,\nabla^2\nabla^2\psi = 0, \tag{2.81}$$

which, by developing the Lagrangian derivative, transforms into

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y} - \widetilde{\eta}\,\nabla^2\nabla^2\psi = 0, \tag{2.82}$$

where $\nabla^2\nabla^2 \equiv \nabla^4$ is the square of the Laplacian operator, i.e. the fourth power of the nabla operator, that in Cartesian coordinates in n dimensions, is (assuming Einstein convention)

$$\nabla^4 = \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}. \tag{2.83}$$

[9] A *harmonic* function is a twice continuously differentiable function $f(x_1, \ldots, x_n)$ which satisfies Laplace's equation, $\nabla^2 f = 0$.

Equation 2.82 has the advantage of being a single scalar equation in the unknown function ψ, with difficult to determine boundary conditions.

If viscous forces dominate over inertia forces (Stokes flow, low Reynolds numbers; see Sect. 2.5), $\mathrm{D}\boldsymbol{\omega}/\mathrm{D}t \simeq 0$ so satisfaction of Eq. 2.81 requires ψ to be a *biharmonic* function, $\nabla^2\nabla^2\psi \equiv \nabla^4\psi = 0$.

Finally: (i) 2D irrotational flows are characterized by a velocity field which derives from a harmonic stream function; (ii) 2D rotational, high viscosity, flows, instead, are characterized by a biharmonic stream function.

2.3 The Energy Equation

In the continuum view, a fluid is characterized by both an 'ordered' (kinetic) energy and a 'disordered' (internal) energy. This essentially means two equations to be satisfied by two scalar quantities (the kinetic and the internal energies). In an ideal fluid, which by definition is fully adiabatic, there are no dissipative processes of any kind (viscosity, heat conduction, convection, acoustic dissipation, irradiation, etc.), so its entropy, S, is constant everywhere: $dS/T = 0$. If we denote by s the entropy per unit mass (intensive quantity), its conservation is expressed by

$$\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s = 0, \tag{2.84}$$

which, together with the continuity equation in the form of Eq. 2.8, transforms (see Exercise 2.4) into

$$\frac{\partial \rho s}{\partial t} + \nabla\cdot\rho s\mathbf{v} = 0, \tag{2.85}$$

which has indeed the same form of the conservation of mass (continuity) Eq. 2.8 with the substitution of ρ with ρs, and corresponds to conservation of the whole fluid entropy $\int_V \rho s dV$ instead of the fluid mass $\int_V \rho dV$. The vector quantity $\rho s\mathbf{v}$ is called 'entropy flux'. Equation 2.85 can be considered as an Eulerian form of the energy equation for an ideal fluid.

Let us obtain another, Lagrangian, expression for the energy equation of an ideal fluid. To do this, we use the intensive quantities e, V^* and q^* introduced in Sect. 2.2.1.1 that are specific internal energy,[10] specific volume and heat, all per unit mass. By its definition, $V^* = 1/\rho$, and so the first principle of thermodynamics implies that $dq^* = de + pdV^* = de + pd(1/\rho) = de - (p/\rho^2)d\rho$, which translates into the differential Lagrangian form

$$\frac{\mathrm{D}q^*}{\mathrm{D}t} = \frac{\mathrm{D}e}{\mathrm{D}t} - \frac{p}{\rho^2}\frac{\mathrm{D}\rho}{\mathrm{D}t}. \tag{2.86}$$

[10] Given the context where it is used, e cannot be confused with Euler's number represented with the same letter.

Of course, if heat is injected in the fluid $\mathrm{D}q^*/\mathrm{D}t > 0$, while the opposite, $\mathrm{D}q^*/\mathrm{D}t < 0$, holds if heat flows out of the fluid. These situations are caused by phenomena like

- chemical endo/exo-thermic chemical reactions;
- nuclear reactions;
- energy transport and deposition by conduction, convection and/or by radiation;
- viscosity dissipation;
- sound absorption.

Coupling Eq. 2.86 to mass continuity equation yields

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p\nabla \cdot \mathbf{v} + \rho \frac{\mathrm{D}q^*}{\mathrm{D}t}, \tag{2.87}$$

which quantitatively expresses the rate of change of the internal energy due to the contemporary action of the pressure work in unit time ($p\nabla \cdot \mathbf{v}$) and of the chemo-physical processes identified above ($\rho \mathrm{D}q^*/\mathrm{D}t$). In astrophysics, for instance, radiative processes are very relevant. Their role in the energy balance is mediated by the characteristics of the matter–radiation interaction. When radiative energy is transferred to matter $\mathrm{D}q^*/\mathrm{D}t > 0$, the opposite happens when part of the matter energy is transformed into radiative energy. In the *particle* view of light, these two cases correspond, respectively, to photons *absorbed* by matter (going to increase its temperature, i.e. a *heating* process) or to photons *emitted* by matter (leading to lowering its temperature, i.e. a *cooling* process). Usually, heating and cooling are accounted for in Eq. 2.87 by two non-negative *heating* and *cooling* functions, Γ and Λ respectively, such that

$$\rho \left(\frac{\mathrm{D}q^*}{\mathrm{D}t} \right)_{rad} \equiv \rho \left(\Gamma - \Lambda \right) \tag{2.88}$$

gives the non-adiabatic radiation contribution to the internal energy evolution of the fluid. The exact dependence of Γ and Λ upon state variables is extremely complex because they relate to both the chemical composition of matter and its excitation and ionization state. Under various simplifications, including that of matter–radiation equilibrium such that kinetic and radiation temperatures are the same, these dependences might be resumed into a dependence on ρ and T, only, so that $\Gamma = \Gamma(\rho, T)$ and $\Lambda = \Lambda(\rho, T)$, often expressed in tabular form.

Finally, an often used form of the Lagrangian energy equation is

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p\nabla \cdot \mathbf{v} + \rho(\Gamma - \Lambda) + \rho \left(\frac{\mathrm{D}q^*}{\mathrm{D}t} \right)_{non-rad}, \tag{2.89}$$

where both the radiative and non-radiative contributions are considered as known (although approximated) functions of ρ and T. Of course, if the fluid is ideal (fully adiabatic) the above equation reduces to

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p\nabla \cdot \mathbf{v}, \tag{2.90}$$

where only the work (per unit time) done by pressure goes into either increasing internal energy (compression, i.e. $\nabla \cdot \mathbf{v} < 0$) or decreasing it (rarefaction, i.e. $\nabla \cdot \mathbf{v} > 0$) in perfect agreement with the standard adiabatic form of the first principle of thermodynamics.

2.3.1 *Kinetic Energy of the Fluid*

The fluid total energy accounts for both the internal energy discussed above (that on microscopic space scales) and the kinetic energy, the one pertaining to the global movement of the fluid, i.e. that on the large, translational, space scale. The time evolution of the fluid kinetic energy can be thus obtained, in the simplest case of an ideal fluid, by the Euler's equation. A scalar multiplication of both sides of Eq. 2.29 by the fluid velocity $\mathbf{v}$ gives

$$\rho \mathbf{v} \cdot \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\mathbf{v} \cdot \nabla p + \mathbf{v} \cdot \mathbf{F}_B, \tag{2.91}$$

which is equivalent to

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \frac{1}{2} v^2 = -\mathbf{v} \cdot \nabla p + \mathbf{v} \cdot \mathbf{F}_B. \tag{2.92}$$

The above equation represents kinetic energy evolution, stating that kinetic energy per unit mass varies due to work done in the unit time by the pressure (surface) forces (first term in the rhs) and by the external, body, forces (second term in the rhs).

A straight side-by-side summation of Eqs. 2.90 and 2.92 leads to the equation

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{1}{2} v^2 + e \right) = -\mathbf{v} \cdot \nabla p - p\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{F}_B = -\nabla \cdot p\mathbf{v} + \mathbf{v} \cdot \mathbf{F}_B, \tag{2.93}$$

which defines the rate of change of the total (internal+kinetic) energy per unit mass along the fluid motion, in Lagrangian form.

The Eulerian form of the energy equation is obtained by recalling (see again Exercise 2.4) that for a sufficiently regular scalar function $\phi(\mathbf{r}, \mathbf{v}; t)$, under validity of continuity equation, it results in

$$\rho \frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\partial \rho \phi}{\partial t} + \nabla \cdot \rho \phi \mathbf{v}, \tag{2.94}$$

so that, letting $\phi \equiv v^2/2 + e$, Eq. 2.93 converts into Eulerian form

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2+e\right)\right]+\nabla\cdot\rho\left(\frac{1}{2}v^2+e\right)\mathbf{v}=-\nabla\cdot p\mathbf{v}+\mathbf{v}\cdot\mathbf{F}_B, \tag{2.95}$$

which may be rewritten as

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{1}{2}v^2+e\right)\right]+\nabla\cdot\rho\left(\frac{1}{2}v^2+e+\frac{p}{\rho}\right)\mathbf{v}=\mathbf{v}\cdot\mathbf{F}_B, \tag{2.96}$$

which we definitely consider as the Eulerian energy equation.

Note that the second addend in the lhs can be written as $\nabla\cdot\rho\left(v^2/2+w\right)\mathbf{v}$, where $w\equiv e+p/\rho$ is the specific (per unit mass) enthalpy of the fluid.[11]

Integrating both sides of Eq. 2.96 over a generic portion of volume, V, of the fluid, swapping the derivative with respect to t with the integral on the lhs and moving the integral of the divergence to the rhs

$$\frac{\partial}{\partial t}\int_V\rho\left(\frac{1}{2}v^2+e\right)\mathrm{d}V=-\int_V\nabla\cdot\rho\left(\frac{1}{2}v^2+w\right)\mathbf{v}\,\mathrm{d}V+\int_V\mathbf{v}\cdot\mathbf{F}_B\mathrm{d}V, \tag{2.97}$$

which, by applying the divergence theorem to the first integral in the rhs, transforms into

$$\frac{\partial}{\partial t}\int_V\rho\left(\frac{1}{2}v^2+e\right)\mathrm{d}V=-\int_{\partial V}\rho\left(\frac{1}{2}v^2+w\right)\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}\sigma+\int_V\mathbf{v}\cdot\mathbf{F}_B\mathrm{d}V. \tag{2.98}$$

In Eq. 2.98 the lhs is the rate of change of the fluid energy of the given volume V while at rhs the integral over the surface surrounding V represents, with its negative sign, the inward flux of energy, hence the name of energy flux density vector for the quantity $\rho\left(v^2/2+w\right)\mathbf{v}$. Note that the net change of energy of the fluid is contributed by the body force work (rightmost term in the rhs above) and by both the flux of kinetic + internal energy and by the work done by pressure force on the fluid within the surface surrounding V. This explains why in the energy flux density vector appears the enthalpy $w=e+p/\rho$.

2.3.2 The Energy Equation for a Dissipative Fluid

In the case of a dissipative fluid, we have obtained the equation of motion in the Navier–Stokes form (Eq. 2.61). Operating in the same way followed above for the Euler's equation, the Eulerian energy equation in the N-S case is

[11] Note that for an ideal gas, enthalpy depends only upon temperature and not on pressure and density.

$$\frac{\partial}{\partial t}\rho\left(\frac{1}{2}v^2+e\right)+\nabla\cdot\rho\left(\frac{1}{2}v^2+e+\frac{p}{\rho}\right)\mathbf{v}=\left[\eta\nabla^2\mathbf{v}+\left(\frac{1}{3}\eta+\zeta\right)\nabla\nabla\cdot\mathbf{v}\right]\cdot\mathbf{v}+$$
$$+\rho\frac{\mathrm{D}q_*}{\mathrm{D}t}+\mathbf{v}\cdot\mathbf{F}_B. \tag{2.99}$$

The above equation shows that the work per unit time of viscosity acts, if both η and ζ are non-negative, in such a way as to reduce (dissipate to smaller scales) the kinetic energy of the fluid. We can see it easily in the case of an incompressible fluid ($\nabla\cdot\mathbf{v}=0$).

Actually, it is simple to follow the same procedure adopted to obtain the Eulerian form of the kinetic energy per unit mass evolution of an ideal fluid, to obtain, in the incompressible N-S case and neglecting $\rho\mathrm{D}q^*/\mathrm{D}t$ and $\mathbf{v}\cdot\mathbf{F}_B$,

$$\frac{\partial}{\partial t}\rho\frac{v^2}{2}+\nabla\cdot\rho\left(\frac{1}{2}v^2+\frac{p}{\rho}\right)\mathbf{v}=\eta\nabla^2\mathbf{v}\cdot\mathbf{v}, \tag{2.100}$$

which, as expected, is identical to Eq. 2.99 where we let $e=0$. Now, an integration of both sides of the above equation over the whole volume V occupied by the fluid gives the rate of change of the fluid kinetic energy

$$\dot{E}_{kin}=\frac{\partial}{\partial t}\int\limits_V\frac{1}{2}\rho v^2\mathrm{d}V=-\int\limits_V\nabla\cdot\rho\left(\frac{1}{2}v^2+\frac{p}{\rho}\right)\mathbf{v}\mathrm{d}V+\eta\int\limits_V\nabla^2\mathbf{v}\cdot\mathbf{v}\mathrm{d}V. \tag{2.101}$$

Substituting for $\nabla^2\mathbf{v}\cdot\mathbf{v}$ in the rhs of the above relation by means of the vector identity $\nabla\cdot(\mathbf{v}\nabla\mathbf{v})=\mathbf{v}\cdot\nabla^2\mathbf{v}+\nabla\mathbf{v}\cdot\nabla\mathbf{v}$, [12] we get

$$\dot{E}_{kin}=\frac{\partial}{\partial t}\int\limits_V\frac{1}{2}\rho v^2\mathrm{d}V=-\int\limits_V\nabla\cdot\rho\left(\frac{1}{2}v^2+\frac{p}{\rho}\right)\mathbf{v}\mathrm{d}V+$$
$$+\eta\int\limits_V\nabla\cdot(\mathbf{v}\nabla\mathbf{v})\,\mathrm{d}V-\eta\int\limits_V\nabla\mathbf{v}\cdot\nabla\mathbf{v}\mathrm{d}V. \tag{2.102}$$

The divergence theorem can now be used to transform the first two volume integrals in the rhs of the above relation into surface integrals which vanish (due to boundary conditions), so that Eq. 2.102 reduces to

$$\dot{E}_{kin}=-\eta\int\limits_V\nabla\mathbf{v}\cdot\nabla\mathbf{v}\mathrm{d}V\le 0, \tag{2.103}$$

[12] This identity can be also obtained by the formal substitution $f\to\mathbf{v}$ and $\mathbf{v}\to\nabla\mathbf{v}$ in the 10th of the identities A.45 of Appendix A.3.

whenever $\eta \geq 0$, because $\nabla \mathbf{v} \cdot \nabla \mathbf{v}$ is, obviously, always non-negative. The above result can be also considered as a demonstration that the shear viscosity coefficient must be non-negative due to the intrinsic dissipational nature of fluid viscosity.

2.4 Thermodynamics of Fluids and Equations of State

As we said, in the Eulerian specification of a fluid the relevant quantities are the velocity vector field, $\mathbf{v}$, and the (scalar) mass density, pressure, and internal energy fields, ρ, p and e. This means there are six unknown quantities, leading to an undetermined situation because the governing equations (continuity, motion, energy) are just five (two scalar and one vectorial).

As a matter of fact, the (intensive) state variables (density, pressure, temperature and internal energy) are, indeed, scalar because their spatial resolution, in the sense of assigning to each of them a vector, is unrealistic. They refer to the small scale structure of the fluid and can be related to each other by thermodynamical concepts. The most relevant is the link among small scale variables provided by the equation of state, adopting, indeed, the terminology of *state* variables for ρ, p, T, e. The equation of state is thought of as a functional connection among the state variables which, in the most general form, can be expressed as $f(\rho, p, e) = 0$. Apart from mathematical complications related to the actual explicit expression, one can in principle obtain one variable once the others are known. Note that the EOS is not a differential equation but rather an algebraic connection among the state variables, needed to 'close' the system of differential equations governing the time evolution of the fluid.

There is a variety of EOSs, each describing a particular situation of a fluid, valid in different ranges of density, pressure and internal energy.

The simplest is the classical EOS of *ideal* gases,[13] $p = nkT$, where n is the fluid number density, $n = \rho/\langle m \rangle$, and k is Boltzmann's constant. Another often used form of the EOS is the *polytropic* form, which is a particular, power law, form of *barotropic* EOS, which is one of the type $p = p(\rho)$.
A polytropic EOS is expressed as

$$p = c_\gamma \rho^{1+1/n}, \tag{2.104}$$

where $c_\gamma > 0$ is a constant dependent on the parameter $\gamma = 1 + 1/n$, where n is the polytropic index which in principle can be either positive or negative.[14] Between the polytropic index and the specific heat c, the relation $\gamma = (c_p - c)/(c_V - c)$ holds, where c_p and c_V are the specific heats at constant pressure and volume, respectively.

[13] In this book, we refer to an *ideal* gas as that obeying the $p = nkT$ EOS. We also refer to a *perfect* gas as an ideal gas whose heat capacity is independent of T.

[14] In an ideal gas γ is related to the number of degrees of freedom per molecule, n_f, according to $\gamma = (n_f + 2)/n_f$. A monoatomic gas has $n_f = 3$, a diatomic $n_f = 5$ and a polyatomic $n_f = 6$.

A polytropic EOS is apt to describe the link between ρ and p in a polytropic process of expansion and/or compression of a fluid. By its definition, an isothermal gas is represented by $n \to \infty$ ($\gamma \to 1$). Indeed, the ideal gas EOS, $p = \rho kT/\langle m \rangle$, with $T = const.$ (isothermality condition), gives $p \propto \rho$ and this is indeed obtained in the limit $n \to \infty$ in the polytropic EOS.

As we know, if the fluid is ideal, and so characterized by its full adiabaticity (no internal and external exchange of heat), we can write the first principle of thermodynamics in the specific form $dq^* = de + pdV^* = 0$, where $V^* = 1/\rho$. Consequently,

$$\mathrm{d}e = \frac{p}{\rho^2}\,\mathrm{d}\rho, \tag{2.105}$$

which, in the case of a polytropic EOS, by integration leads to

$$e = c_\gamma \int \rho^{\gamma-2}\mathrm{d}\rho = \frac{c_\gamma}{\gamma - 1}\rho^{\gamma-1} + a, \tag{2.106}$$

where a is an integration constant. Letting the constant a to zero, the EOS assumes the equivalent form (link between pressure, internal energy and density)

$$p = (\gamma - 1)\rho e. \tag{2.107}$$

2.5 Turbulence

When a fluid flows in a way such that its layers move parallel to each other without disturbances of the velocity field, the flow is called *laminar*. On the other hand, there are situations where the fluid motion appears irregular, with local abrupt changes in fluid characteristic variables (pressure, density, velocity, etc.). Everyday experience suggests that laminar flows correspond to fluids calmly flowing, at low enough velocity, while turbulent flows are associated with high fluid speed.

Actually, examples of turbulence in fluids are the wave breaking of water near a beach in conditions of strong wind, the rapid flow of water in a mountain river, storming clouds, etc. This connection of the rise of turbulence with the flow rapidity leads to the intuition that turbulence should derive from insufficient efficiency of viscosity diffusion to dissipate (i.e. diffuse) a large amount of kinetic energy of the fluid.

Historically, already in the sixteenth century Leonardo da Vinci worried about some strange configurations assumed by fluids passing through obstacles. The first significant experiments about the different ways for fluids (in particular, liquids) to move are due to Osborn Reynolds (Belfast, 1842–Watchet, 1912) who was the first to note that the motion of a liquid in a pipe transits from a laminar (regular, almost rectilinear stream lines) to a chaotic behavior by varying some of the characteristic parameters of the flow. Essentially, the relevant parameters he studied regarded both

the liquid, characterized by its density ρ, speed U and dynamic viscosity η, and the diameter D of the cylindrical pipe where the liquid was made to move in. The length, L, of the cylinder was checked by Reynolds to be irrelevant regarding the study of laminarity breaking as long as $D \ll L$. Assuming U, ρ and η as constant in the moving liquid, he noted, by looking at the shape of a dyed substance injected in the pipe acting as tracer of the liquid movement, that the fluid motion remained laminar until the dimensionless quantity,

$$\mathrm{Re} = \frac{\rho\, U\, D}{\eta}, \tag{2.108}$$

(called after him *Reynolds number*) stayed at low values, while for large values of Re, obtained simply by increasing the liquid speed U at given η and D, a clear *chaotic* behavior of the liquid stream lines was observed. He noted that the transition from the laminar to the chaotic (turbulent) regime occurred for values of Re around 2300, the liquid becoming fully turbulent at $\mathrm{Re} \simeq 2900$.

Note that the Reynolds number quantifies the relative role of inertia forces and viscous forces in a fluid, in what ρU is the density of momentum transported by the fluid (inertia) and η/D is the density of momentum 'damped' by viscosity. If inertia is too large, there is no chance for viscosity to damp local irregularities in the fluid motion and so turbulence appears.

It is easily seen that, once given the set of dimensional quantities ρ, U, η and D, the only way to combine them to get a dimensionless parameter is the one in Eq. 2.108, in the sense that any other dimensionless parameter built with ρ, U, η and D can be written as a function of Re.

For the sake of exercise, we give here the constructive method to produce the Reynolds number. Given that the physical dimensions of ρ, U, η and D are

$$[\rho] = \mathrm{ML}^{-3}, \quad [U] = \mathrm{LT}^{-1}, \quad [\eta] = \mathrm{ML}^{-1}\mathrm{T}^{-1}, \quad [D] = \mathrm{L}, \tag{2.109}$$

a dimensionless quantity, Re, can be obtained as a combination of the type

$$\rho^{\alpha} U^{\beta} \eta^{\gamma} D^{\delta}, \tag{2.110}$$

where α, β, γ, δ must satisfy the *dimensional* equation

$$\mathrm{M}^{\alpha+\gamma}\mathrm{L}^{-3\alpha+\beta-\gamma+\delta}\,\mathrm{T}^{-\beta-\gamma} = \mathrm{M}^{0}\mathrm{L}^{0}\mathrm{T}^{0}, \tag{2.111}$$

which gives an underdetermined linear system of 3 equations for 4 unknowns

$$\begin{cases} \alpha + \gamma = 0 \\ -3\alpha + \beta - \gamma + \delta = 0 \\ -\beta - \gamma = 0, \end{cases} \tag{2.112}$$

which admits ∞^1 solutions. Assuming δ as a parameter, the solution is $\alpha = \beta = -\gamma = \delta$, so that the Reynolds number is obtained by assuming $\delta = 1$.

The role of the Reynolds number for turbulence becomes clear when dealing with the Navier–Stokes equation, which is the equation of motion in the presence of viscosity. First, we note that Euler's equation for ideal fluids in the absence of body forces does not contain any physical constant, a thing that, together with the assumption of barotropic EOS, implies its scale invariance. We don't give a straight demonstration of this but just rely on the intuition that in the absence of physical constants involving pressure, density and velocity flow there is no reason to believe that there is any preferred length or time scale, and so there is no reason to believe that scale invariance is broken along the fluid motion.

On the contrary, in the N-S equation, even in absence of external body forces, viscosity implies the existence of one or more specific physical parameters (the shear and bulk viscosity coefficients) in the governing of fluid motion. And this breaks the scale invariance. This corresponds to an intrinsic big difference between the motion of ideal fluids and viscous fluids, in which the Reynolds number plays a relevant role.

We can see it by writing the N-S Eq. 2.61 in non-dimensional form. To do this, we assume new non-dimensional variables

$$\mathbf{r}' = \frac{\mathbf{r}}{D},\ \mathbf{v}' = \frac{\mathbf{v}}{U},\ t' = \frac{t}{T},\ \rho' = \frac{\rho}{\overline{\rho}},\ p' = \frac{p}{P}, \tag{2.113}$$

where $D, U, T, \overline{\rho}$ and P are arbitrary dimensional constants. Choosing for D and U a typical length and speed, respectively, the time constant T is naturally adopted as $T = D/U$. Similarly, the choice of the typical fluid density $\overline{\rho}$ leads to the natural choice of $P = \overline{\rho}U^2$. Considering that, upon the above time and space transformations, derivatives transform as

$$\frac{\partial}{\partial t} = \frac{1}{D/U}\frac{\partial}{\partial t'},\ \nabla_i \equiv \frac{\partial}{\partial x_i} = \frac{1}{D}\frac{\partial}{\partial x_i'} \equiv \frac{1}{D}\nabla_i' \tag{2.114}$$

so that $\nabla^2 = \nabla'^2/D^2$, the N-S equation, where we account at its rhs for a possible body force per unit volume $\mathbf{F}_B$, is written in dimensionless form as

$$\frac{\partial \mathbf{v}'}{\partial t'} + \left(\mathbf{v}' \cdot \nabla'\right)\mathbf{v}' = -\frac{1}{\rho'}\nabla' p' + \frac{\eta}{\overline{\rho}DU}\frac{1}{\rho'}\nabla'^2\mathbf{v}' + \frac{1}{\overline{\rho}DU}\frac{1}{\rho'}\left(\frac{\eta}{3} + \zeta\right)\nabla'\nabla' \cdot \mathbf{v}' + \frac{D}{\overline{\rho}U^2}\frac{1}{\rho'}\mathbf{F}_B. \tag{2.115}$$

The coefficients of the diffusion terms in the above equation can be written in terms of Re, and, finally, the non-dimensional N-S equation is

$$\frac{\partial \mathbf{v}'}{\partial t'} + \left(\mathbf{v}' \cdot \nabla'\right)\mathbf{v}' = -\frac{1}{\rho'}\nabla' p' + \frac{1}{\mathrm{Re}}\frac{1}{\rho'}\nabla'^2\mathbf{v}' + \frac{1}{\mathrm{Re}}\frac{1}{\rho'}\left(\frac{1}{3} + \frac{\zeta}{\eta}\right)\nabla'\nabla' \cdot \mathbf{v}' + \frac{1}{\rho'}\mathbf{F}_B', \tag{2.116}$$

where $\mathbf{F}_B' = \left[D/(\overline{\rho}U^2)\right]\mathbf{F}_B$ is the dimensionless body force per unit volume.

If we consider a stationary flow, the ratio (in norm) between the convection term and the diffusion terms in the equation above is

$$\frac{\|(\mathbf{v}'\cdot\nabla')\,\mathbf{v}'\|}{\left\|\frac{1}{\mathrm{Re}}\frac{1}{\rho'}\nabla'^2\mathbf{v}'+\frac{1}{\mathrm{Re}}\frac{1}{\rho'}\left(\frac{1}{3}+\frac{\zeta}{\eta}\right)\nabla'\nabla'\cdot\mathbf{v}'\right\|}=\mathrm{Re}\frac{\|(\mathbf{v}'\cdot\nabla')\,\mathbf{v}'\|}{\left\|\frac{1}{\rho'}\nabla'^2\mathbf{v}'+\frac{1}{\rho'}\left(\frac{1}{3}+\frac{\zeta}{\eta}\right)\nabla'\nabla'\cdot\mathbf{v}'\right\|}, \quad (2.117)$$

which is of the order of Re whenever ζ/η is not far from 1 because, for the proper choice of non-dimensionalization constants, the ratio of the norms in the rhs of the equation above is of the order of 1. This corresponds to that when Re is large, diffusion is unable to spread out the fluid momentum 'carried' by the convection term and so it is impossible to *damp* local perturbation to a laminar flow which, consequently, turns out into a turbulent flow. Of course, if the fluid is incompressible, the convection-to-diffusion term ratio is of order Re independently of ζ/η.

Note that the way to write the N-S equation in dimensionless form is not unique. If, for instance, the pressure scale P is taken as $P=(\eta U)/D$, the dimensionless N-S equation would be written as

$$\frac{\partial\mathbf{v}'}{\partial t'}+(\mathbf{v}'\cdot\nabla')\,\mathbf{v}'=-\frac{1}{\mathrm{Re}}\frac{1}{\rho'}\nabla' p'+\frac{1}{\mathrm{Re}}\frac{1}{\rho'}\nabla'^2\mathbf{v}'+\frac{1}{\mathrm{Re}}\frac{1}{\rho'}\left(\frac{1}{3}+\frac{\zeta}{\eta}\right)\nabla'\nabla'\cdot\mathbf{v}'+\frac{1}{\rho'}\mathbf{F}'_B, \quad (2.118)$$

where Re is the same as Eq. 2.108 and, again, $\mathbf{F}'_B=\left[D/(\overline{\rho}U^2)\right]\mathbf{F}_B$. The above equation can be also written as

$$\mathrm{Re}\left(\frac{\partial\mathbf{v}'}{\partial t'}+(\mathbf{v}'\cdot\nabla')\,\mathbf{v}'\right)=-\frac{1}{\rho'}\nabla' p'+\frac{1}{\rho'}\nabla'^2\mathbf{v}'+\frac{1}{\rho'}\left(\frac{1}{3}+\frac{\zeta}{\eta}\right)\nabla'\nabla'\cdot\mathbf{v}'+\frac{1}{\rho'}\mathbf{F}''_B, \quad (2.119)$$

where $\mathbf{F}''_B=\mathrm{Re}\mathbf{F}'_B=\left[D^2/(\eta U)\right]\mathbf{F}_B$.

It is evident from the above equation that, in the limit $\mathrm{Re}\ll 1$, its lhs, representing the fluid inertia acceleration, is negligible and so this equation together with the incompressibility condition approximates to

$$\begin{cases}\nabla'^2\mathbf{v}'-\nabla' p'+\mathbf{F}''_B=0\\ \nabla'\cdot\mathbf{v}'=0,\end{cases} \quad (2.120)$$

or, in the original dimensional variables

$$\begin{cases}\eta\nabla^2\mathbf{v}-\nabla p+\mathbf{F}_B=0\\ \nabla\cdot\mathbf{v}=0,\end{cases} \quad (2.121)$$

which represent the *Stokes* flows, also named *creeping* flows. This situation is not uncommon in applications when dealing with very small velocities in the fluid or very large viscosity or very small fluid length scales. Equations 2.120 and 2.121 constitute a relevant simplification of the N-S equation, because they are both time-independent (the dependence on time being limited to possible time-dependent boundary conditions) and linear (having neglected the non-linear inertial acceleration). The existence and uniqueness of solutions for such equations may be proven, and many solution techniques suited for linear partial differential equations can be used. An important consequence of the steadiness and linearity of Stokes flows is their time-reversibility.

2.6 Stokes Law and Falling Sphere Viscometer

There are several ways to measure viscosity of a fluid, the oldest of which is based on the application of Stokes law to a creeping flow. Stokes law was deduced in 1851 by G. G. Stokes as an expression of the resistive (drag) force exerted on a spherical object by the highly viscous fluid it moves in. This law constitutes a generalization to fluids of the common frictional force acting on the contact surface of the separation of two solid bodies in relative motion. The expression, obtained by use of the linear N-S equation valid for creeping (Stokes) flows, gives the drag force, $\mathbf{F}_d$, on the sphere as

$$\mathbf{F}_d = -6\pi\eta R\mathbf{v}, \tag{2.122}$$

where η is the fluid dynamic shear viscosity, R the sphere radius and $\mathbf{v}$ its velocity relative to the fluid. The total force acting on a spherical object falling in the fluid by its own gravity is the vectorial sum of the gravity downward force, $\mathbf{F}_g$, the buoyant, $\mathbf{F}_b$ depending on the fluid density ρ_f and the drag, $\mathbf{F}_d$, forces, these latter both upward (see Fig. 2.5). Because the intensity of the drag force increases with falling sphere speed, if the path of the sphere in the fluid is long enough it can reach a velocity (called *terminal* velocity, v_t) such that the resultant force vanishes, $\mathbf{F} = \mathbf{F}_g + \mathbf{F}_b + \mathbf{F}_d = 0$. The following motion is rectilinear and uniform. Taking into account that $\mathbf{F}_g = \rho_s(4\pi/3)R^3\mathbf{g}$, where ρ_s is the density of the falling sphere, and that $\mathbf{F}_b = -\rho_f(4\pi/3)R^3\mathbf{g}$ it easily results in

$$v_t = \frac{2}{9}\frac{gR^2}{\eta}\left(\rho_s - \rho_f\right). \tag{2.123}$$

Provided that the spherical surface of the falling ball is very smooth, a device such as to permit to estimate with good precision the terminal velocity of the ball would allow the determination of η by Eq. 2.123 above.

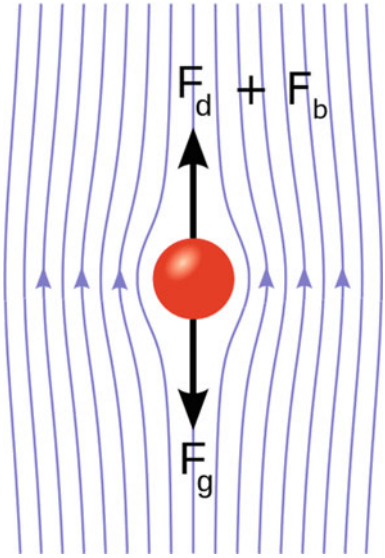


Fig. 2.5 Ball falling down in a fluid, subjected to gravity ($\mathbf{F}_g$), buoyancy ($\mathbf{F}_b$) and drag ($\mathbf{F}_d$)

Note

The Navier–Stokes equation is a generalization to viscous fluids of the equation introduced by Euler who obtained the vector, partial differential equation describing the motion of frictionless fluids. The N-S equation is a non-linear parabolic equation for which it has not been proven so far whether smooth solutions always exist in three dimensions. It is a problem of existence and smoothness.

This mathematical complexity makes the N-S equation (even in the simpler version for incompressible fluids) one of the most relevant issues of mathematical physics, such to merit to be included by the Clay Mathematics Institute in the list of the seven most important open problems in mathematics. The Clay Institute offered a 1 million US dollar prize for the first correct solution to each problem. At present, only one prize has been awarded for the solution of the Poincaré conjecture.

The dimensionless Navier-Stokes equation contains a numerical parameter (the Reynolds number) whose magnitude is an indication of the transition from a laminar flow to a turbulent one. Unfortunately, due to the intrinsic complication of the Navier-Stokes equation, a detailed description of turbulence is far to be known. Actually, although particular solutions to the N-S equation can be found, they are unstable to finite perturbations when the Reynolds number is large. Due to this instability, the fluid flow becomes irregular both in space and time forcing to some statistical (average) treatment.

2.7 Solved Exercises

Exercise 2.1 Given a scalar field f and a vector field $\mathbf{v}$ defined and continuous in $\mathbb{R}^n$, demonstrate the validity of the vector identity

$$\nabla \cdot f\mathbf{v} = f\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla f, \tag{2.124}$$

used to transform the equation of continuity from its expression Eq. 2.8 into Eq. 2.11.

Solution Assuming coordinates $x_1, x_2, \ldots, x_n$, for which $\mathbf{v} = v_i \mathbf{e}_i$, it is

$$\nabla \cdot f\mathbf{v} = \frac{\partial f v_i}{\partial x_i} = f\frac{\partial v_i}{\partial x_i} + v_i \frac{\partial f}{\partial x_i},$$

and the rightmost side above is clearly equal to $f\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla f$, as required to show.

Exercise 2.2 Show that the Eulerian form of continuity equation in Eq. 2.8 and the Lagrangian form in Eq. 2.21 are equivalent.

Solution

The proof of equivalence requires that Eq. 2.8 implies Eq. 2.21 and vice versa. Here we limit to show that Eq. 2.21 implies Eq. 2.8.

By differentiation with respect to time of Eq. 2.21 (for brevity, we use here the dot above a quantity to represent its Lagrangian derivative $\frac{\mathrm{D}}{\mathrm{D}t}$),

$$\dot{\rho}\mathrm{J} + \rho\dot{\mathrm{J}} = 0, \tag{2.125}$$

where

$$\dot{\mathrm{J}} = \begin{vmatrix} \frac{\partial^2 x}{\partial t \partial x_0} & \frac{\partial^2 x}{\partial t \partial y_0} & \frac{\partial^2 x}{\partial t \partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial^2 y}{\partial t \partial x_0} & \frac{\partial^2 y}{\partial t \partial y_0} & \frac{\partial^2 y}{\partial t \partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial^2 z}{\partial t \partial x_0} & \frac{\partial^2 z}{\partial t \partial y_0} & \frac{\partial^2 z}{\partial t \partial z_0} \end{vmatrix}$$

is the time derivative of the Jacobian determinant in Eq. 2.19. In the hypothesis of continuity of both first and second partial derivatives in the above expression, the order of derivation with respect to t and x_0, y_0, z_0 can be interchanged and, because $\dot{x} = v_x$, $\dot{y} = v_y$ and $\dot{z} = v_z$, the above time derivative of the Jacobian can be written as

$$\dot{\mathrm{J}} = \begin{vmatrix} \frac{\partial v_x}{\partial x_0} & \frac{\partial v_x}{\partial y_0} & \frac{\partial v_x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial v_y}{\partial x_0} & \frac{\partial v_y}{\partial y_0} & \frac{\partial v_y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{vmatrix} + \begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial x_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial v_z}{\partial x_0} & \frac{\partial v_z}{\partial y_0} & \frac{\partial v_z}{\partial z_0} \end{vmatrix},$$

or

$$\dot{\mathrm{J}} = \left| \left(\frac{\partial(v_x, y, z)}{\partial(x_0, y_0, z_0)} \right) \right| + \left| \left(\frac{\partial(x, v_y, z)}{\partial(x_0, y_0, z_0)} \right) \right| + \left| \left(\frac{\partial(x, y, v_z)}{\partial(x_0, y_0, z_0)} \right) \right|. \tag{2.126}$$

Now, it is

$$\begin{aligned} \frac{\partial v_x}{\partial x_0} &= \frac{\partial v_x}{\partial x}\frac{\partial x}{\partial x_0} + \frac{\partial v_x}{\partial y}\frac{\partial y}{\partial x_0} + \frac{\partial v_x}{\partial z}\frac{\partial z}{\partial x_0}, \\ \frac{\partial v_x}{\partial y_0} &= \frac{\partial v_x}{\partial x}\frac{\partial x}{\partial y_0} + \frac{\partial v_x}{\partial y}\frac{\partial y}{\partial y_0} + \frac{\partial v_x}{\partial z}\frac{\partial z}{\partial y_0}, \\ \frac{\partial v_x}{\partial z_0} &= \frac{\partial v_x}{\partial x}\frac{\partial x}{\partial z_0} + \frac{\partial v_x}{\partial y}\frac{\partial y}{\partial z_0} + \frac{\partial v_x}{\partial z}\frac{\partial z}{\partial z_0}, \end{aligned}$$

whose solutions for $\partial v_x/\partial x$, $\partial v_y/\partial y$ and $\partial v_z/\partial z$ are

$$\begin{aligned} \frac{\partial v_x}{\partial x}\mathrm{J} &= \left| \left(\frac{\partial(v_x, y, z)}{\partial(x_0, y_0, z_0)} \right) \right|, \\ \frac{\partial v_y}{\partial y}\mathrm{J} &= \left| \left(\frac{\partial(x, v_y, z)}{\partial(x_0, y_0, z_0)} \right) \right|, \\ \frac{\partial v_z}{\partial z}\mathrm{J} &= \left| \left(\frac{\partial(x, y, v_z)}{\partial(x_0, y_0, z_0)} \right) \right|. \end{aligned}$$

These latter relations inserted into Eq. 2.126 lead to

$$\dot{\mathrm{J}} = \mathrm{J}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = \mathrm{J}\nabla \cdot \mathbf{v},$$

which, once inserted into Eq. 2.125, gives

$$\dot{\rho}\mathrm{J} + \rho\mathrm{J}\nabla \cdot \mathrm{v} = 0,$$

which implies

$$\dot{\rho} + \rho\nabla \cdot \mathrm{v} = 0,$$

as we wanted to show.

Exercise 2.3 A *protostar* is considered as a point-like massive object with mass $M = 1\ \mathrm{M}_\odot$ ($\mathrm{M}_\odot$ is the mass of the Sun, $\mathrm{M}_\odot \simeq 1.989 \times 10^{30}$ kg). It is embedded in a diffuse spherical cloud of homogeneous gas of density $\rho_0 = 10^{-18}$ g cm^{-3} which accretes radially onto the protostar at constant speed. Assume the cloud radius $r_0 = 1$AU (AU = astronomical unit represents the average Earth–Sun distance and is an astronomical unit of distance; 1 AU $= 1.49598 \times 10^8$ km).

What is the radial profile of the accreting gas mass density?

What is the distance to the protostar where the accreting gas density reaches the water density value?

Solution The results come from the direct application of the continuity equation in spherical polar coordinates (see Appendix A.2.2) in the case of steady state and radial symmetry

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v_r) = 0,$$

which implies $r^2 \rho v_r = const. = r_0^2 \rho_0 v_{r0}$, where $v_r = v_{r0}$ (constant accretion speed). The density profile is so independent of v_r and is $\rho(r) = \rho_0 (r_0/r)^2$, i.e. an inverse parabolic law. For the given values of r_0 and ρ_0, the distance $\bar{r}$ where the density reaches the value $\rho(\bar{r}) = 1$ g cm^{-3} is $\bar{r} = r_0\sqrt{\rho_0/\rho(\bar{r})} = 1.5 \times 10^4$ cm $= 0.15$ km.

Exercise 2.4 Given a continuous scalar field φ and a fluid of density ρ flowing with velocity field $\mathbf{v}$ respecting the equation of continuity, show that

$$\rho \frac{\mathrm{D}\varphi}{\mathrm{D}t} = \frac{\partial \rho\varphi}{\partial t} + \nabla \cdot \rho\varphi\mathbf{v}. \tag{2.127}$$

Solution

A quick way to obtain the above relation is by writing

$$\rho \frac{\mathrm{D}\varphi}{\mathrm{D}t} = \frac{\mathrm{D}\rho\varphi}{\mathrm{D}t} - \varphi \frac{\mathrm{D}\rho}{\mathrm{D}t}. \tag{2.128}$$

The rightmost term, in virtue of continuity equation Eq. 2.11, is

$$-\varphi \frac{\mathrm{D}\rho}{\mathrm{D}t} = \varphi\rho\nabla \cdot \mathbf{v},$$

which inserted in Eq. 2.128 yields

$$\rho \frac{\mathrm{D}\varphi}{\mathrm{D}t} = \frac{\mathrm{D}\rho\varphi}{\mathrm{D}t} + \rho\varphi\nabla \cdot \mathbf{v} = \frac{\partial \rho\varphi}{\partial t} + \mathbf{v} \cdot \nabla(\rho\varphi) + \rho\varphi\nabla \cdot \mathbf{v} = \frac{\partial \rho\varphi}{\partial t} + \nabla \cdot \rho\varphi\mathbf{v},$$

as we wanted to show.

Another way to obtain the expression Eq. 2.127 is by developing its rhs by grouping $\varphi\mathbf{v}$ in the divergence and making use of both the 10th identity in Eq. A.45 of

Appendix A.3.2 and the Eulerian equation of continuity Eq. 2.8 with no sources or sinks

$$\begin{aligned}\frac{\partial \rho\varphi}{\partial t} + \nabla \cdot \rho\varphi\mathbf{v} &= \rho\frac{\partial \varphi}{\partial t} + \varphi\frac{\partial \rho}{\partial t} + \rho\nabla \cdot \varphi\mathbf{v}. + \varphi\mathbf{v} \cdot \nabla\rho = \\ &= \rho\frac{\partial \varphi}{\partial t} + \varphi\frac{\partial \rho}{\partial t} + \rho(\varphi\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla\varphi) + \varphi\mathbf{v} \cdot \nabla\rho = \\ &= \varphi\left(\frac{\partial \rho}{\partial t} + \rho\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla\rho\right) + \rho\left(\frac{\partial \varphi}{\partial t} + \mathbf{v} \cdot \nabla\varphi\right) = \\ &= \varphi\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\mathbf{v}\right) + \rho\frac{\mathrm{D}\varphi}{\mathrm{D}t} = \rho\frac{\mathrm{D}\varphi}{\mathrm{D}t}.\end{aligned}$$

Exercise 2.5 Find the pressure field $p(x, y, z)$ such that the stationary velocity field of a homogeneous fluid

$$\mathbf{v} = A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\mathbf{i} + A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\mathbf{k}, \tag{2.129}$$

where $A \neq 0$ and $a \neq 0$ are constants, satisfies Euler's equation in the absence of body force in an infinite rigid tube $-a \leq x \leq a, 0 \leq z \leq 2a$.

Solution Euler's equation of motion Eq. 2.28 with $\mathbf{F}_B = 0$ and accounting for $v_y = 0$ when projected on the axes gives

$$\begin{cases} v_x\dfrac{\partial v_x}{\partial x} + v_z\dfrac{\partial v_x}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}, \\ v_x\dfrac{\partial v_z}{\partial x} + v_z\dfrac{\partial v_z}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}. \end{cases}$$

The second equation above implies $p = p(x, z)$. While the other two equations, given the data of the problem, are specified as

$$\begin{cases} \left(A\cos\dfrac{\pi x}{2a}\cos\dfrac{\pi z}{2a}\right)\left(-\dfrac{\pi A}{2a}\sin\dfrac{\pi x}{2a}\cos\dfrac{\pi z}{2a}\right) + \\ \qquad + \left(A\sin\dfrac{\pi x}{2a}\sin\dfrac{\pi z}{2a}\right)\left(-\dfrac{\pi A}{2a}\cos\dfrac{\pi x}{2a}\sin\dfrac{\pi z}{2a}\right) = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ \left(A\cos\dfrac{\pi x}{2a}\cos\dfrac{\pi z}{2a}\right)\left(\dfrac{\pi A}{2a}\cos\dfrac{\pi x}{2a}\sin\dfrac{\pi z}{2a}\right) + \\ \qquad + \left(A\sin\dfrac{\pi x}{2a}\sin\dfrac{\pi z}{2a}\right)\left(\dfrac{\pi A}{2a}\sin\dfrac{\pi x}{2a}\cos\dfrac{\pi z}{2a}\right) = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}. \end{cases}$$

Thanks to the trigonometric identities $\cos^2\frac{\pi x}{2a} + \sin^2\frac{\pi x}{2a} = \cos^2\frac{\pi z}{2a} + \sin^2\frac{\pi z}{2a} = 1$, the above equations simplify into

$$\begin{cases} -\dfrac{\pi A^2}{2a}\cos\dfrac{\pi x}{2a}\sin\dfrac{\pi x}{2a} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ \dfrac{\pi A^2}{2a}\cos\dfrac{\pi z}{2a}\sin\dfrac{\pi z}{2a} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}. \end{cases}$$

By successive integrations with respect to x and z, the pressure field is found as

$$p(x,z) = \frac{1}{2}\rho A^2\left(\cos^2\frac{\pi z}{2a} - \cos^2\frac{\pi x}{2a}\right) + c, \tag{2.130}$$

where c is an integration constant.

Exercise 2.6 Water flows from left to right in the horizontal pipe as schematized in Fig. 2.6, where the area of the entrance cross section S_1 is A_1 and that of the exit cross section S_2 is A_2. Pressure and speed of the liquid through S_1 are p_1 and v_1 and through S_2 they are p_2 and v_2. Assuming $p_1 = 10^5$ Pa, $v_1 = 4$ ms^{-1} and measuring $p_2 = 0.2 \times 10^5$ Pa, it is asked to determine v_2 and A_2/A_1.

Solution Bernoulli's equation 2.41 gives

$$\frac{1}{2}v_1^2 + e_1 + \frac{p_1}{\rho_1} + gh_1 = \frac{1}{2}v_2^2 + e_2 + \frac{p_2}{\rho_2} + gh_2,$$

in which $\rho_1 = \rho_2$ (water is incompressible), $h_1 = h_2$ (the pipe is horizontal) and it is reasonable to assume $e_1 = e_2$ (water temperature varies very little along the flow). Consequently

$$v_2 = \sqrt{v_1^2 + \frac{2}{\rho}(p_1 - p_2)},$$

which, with the data values of the problem and given the density of water $\rho = 1$ gcm^{-3}, leads to $v_2 = 4\sqrt{11}$ ms^{-1} $\simeq 13.28$ ms^{-1}.

On the other hand, being water incompressible, it is $A_1 v_1 = A_2 v_2$ and so $A_2/A_1 = v_1/v_2 = 1/\sqrt{11} \simeq 0.3015$.

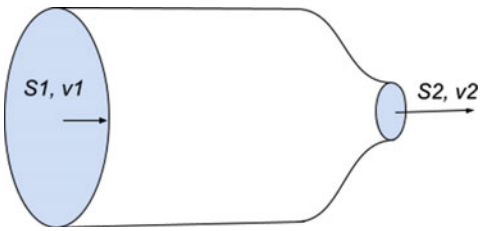


Fig. 2.6 Water crosses the plane surface S_1 at speed v_1 and exits the pipe at S_2 at speed v_2

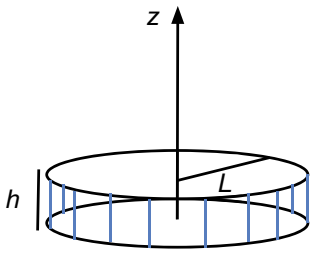


Fig. 2.7 Fluid (blue lines) is pressed by two approaching parallel disks of radius L at slowly varying distance h

Exercise 2.7 Two material plane and circular disks of radius L are superposed at a small initial distance h. The disks are parallel among themselves and the ground. The space between the disks is filled by a viscous, incompressible fluid. The disks approach each other at a constant velocity u, displacing the fluid (see Fig. 2.7).

Questions
Determine the resistant force to the disk motion.

Solution Due to the clear axisymmetry of the problem around the line perpendicular to the disks and passing through their centers (the disks are non-rotating), the choice of cylindrical (R, θ, z) coordinates with origin in the center of the bottom disk in Fig. 2.7 is natural. The fluid layer being thin ($h \ll L$), its flow is mostly radial $v_z \ll v_R$, while $v_\theta = 0$ for the axisymmetry. Again, due to axisymmetry, $\frac{\partial v_R}{\partial \theta} = \frac{\partial v_\theta}{\partial \theta} = 0$.

Taking into account the symmetry conditions and the characteristics of the problem, the N-S equation for the incompressible fluids, in stationary conditions (the disk approaching velocity is assumed constant), keeps, in cylindrical coordinates, just the radial, R, and vertical, z part[15]

$$\begin{cases} 0 = -\dfrac{\partial p}{\partial R} + \eta \dfrac{\partial^2 v_R}{\partial z^2} \\ 0 = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g, \end{cases} \tag{2.131}$$

where η is the shear dynamic viscosity and g is the earth gravity acceleration, while the continuity equation is written as

$$\frac{1}{R}\frac{\partial (R v_R)}{\partial R} + \frac{\partial v_z}{\partial z} = 0. \tag{2.132}$$

[15] In the R part of N-S equation, we eliminated v_R/R^2 because

$$\frac{v_R}{R^2} = \frac{1}{R}\left[\frac{1}{R}\frac{\partial (R v_R)}{\partial R} - \frac{\partial v_R}{\partial R}\right] \approx \frac{1}{R^2}\frac{\partial (R v_R)}{\partial R} = -\frac{1}{R}\frac{\partial v_z}{\partial z} \simeq 0,$$

having made use of the continuity equation, and of that the flow is fully developed and the thickness, h, of the fluid is very small.

The equations above are subjected to the no-slip and impermeability boundary conditions for velocity and to the pressure equilibrium condition at the boundary

$$\begin{cases} v_R(z=0) = v_z(z=0) = 0, \\ v_R(z=h) = 0,\ v_z(z=h) = -u, \\ p(R=L) = p_{ext}, \end{cases} \tag{2.133}$$

where p_{ext} is the value of the external atmospheric pressure. A double integration in z of the first equation in Eq. 2.131 accounting for the above boundary conditions gives (in the hypothesis that neither η nor $\partial p/\partial R$ depend on z)

$$v_R = \frac{1}{2\eta}\frac{\partial p}{\partial R} z(z-h).$$

Inserting the above expression for v_r in the continuity Eq. 2.132 and integrating over z, we get

$$v_z = -\frac{1}{2\eta R}\left[\frac{\partial}{\partial R}\left(R\frac{\partial p}{\partial R}\right)\right]\left(\frac{z^3}{3} - h\frac{z^2}{2}\right) + a,$$

where the integration constant a must be zero to satisfy $v_z(z=0) = 0$. Because $v_z(z=h) = -u$, the above relation gives

$$u = -\frac{h^3}{12\eta R}\frac{\partial}{\partial R}\left(R\frac{\partial p}{\partial R}\right),$$

which, by a double integration over R and noting that the first integration constant must be zero to avoid singularity in pressure derivative for $R \to 0$ and the second must respect the last boundary condition in Eq. 2.133, leads to

$$p = \frac{3\eta u}{h^3}\left(L^2 - R^2\right) + p_{ext}.$$

The global force acting over the lower disk surface S defined by $0 \le R \le L, \theta \le \theta \le 2\pi, z = 0$, is found by integration of the above expression of p to get

$$F_g = \int_S p \mathrm{d}\sigma = \frac{3\eta u \pi L^4}{2h^3} + \pi L^2 p_{ext},$$

where the last contribution is due to the atmospheric pressure, so that the resistance force is simply

$$F_r = F_g - \pi L^2 p_{ext} = \frac{3\eta u \pi L^4}{2h^3}. \tag{2.134}$$

2.8 Further Readings

There are plenty of books on basic fluid dynamics that introduce fundamental governing equations. We suggest the classic textbooks [1, 2]. Whoever wants to deepen the topic of Navier–Stokes equations can read [4]. Energy and heat transfer is well treated in [5]. Turbulence is well described in [6, 7].

Chapter 3
Fluid Flows in Different Environments

As we said, fluids present themselves in various forms in dependence on their microscopic structure which, in its turn, depends also on the environment. A liquid is such whenever its temperature is below the vaporization temperature above which it moves from the liquid to the gaseous state. For example, a certain quantity of water, which is a collection of H_2O molecules, when put in a hot environment undergoes transition into water vapor, still consisting of H_2O molecules interacting much more weakly. A further increment of temperature of the environment would lead to a dissociation of H_2O into H and O atoms with a reduction in the number of degrees of freedom of the elementary constituent to just the 3 translational ones, at least if we neglect the role of the bound electrons. A further temperature increase leads to ionization of one or both the atomic species to the plasma state which has the same, 3, number of degrees of freedom.

In these transformations also the environment plays a role, even if significantly less important.

3.1 Fluids in Terrestrial and Astrophysical Contexts

The main difference between fluids in terrestrial and astrophysical contexts is given by (i) the range of variation of density, pressure and temperature involved and (ii) the role played by the body forces.

Regarding (i), on Earth the densities, pressures and temperatures are limited to a relatively narrow range around the characteristic environment values although, for example, air density reduces a lot at high altitudes, by a factor ten when going from ground up to 17 km of altitude, as well as the temperature of lava in volcanoes exceeds 1000 °C, against the ∼20 °C of environmental air. As another example, the pressure in water depends of course on the depth of water where we measure it. In

R. A. Capuzzo Dolcetta, *Physics of Fluids*, UNITEXT for Physics,
https://doi.org/10.1007/978-3-031-30750-8_3

the deepest regions of oceans (depth around 11, 000 meters under the sea surface), pressure reaches 1100 kg cm^{-2} and temperature drops to just little above 0 °C.

On the other hand, in astrophysical fluids the range of variation of the state variables is very huge, with densities going from almost nuclear values, $\sim 10^{14}$ g cm^{-3} in the center of collapsed, dead, stars like neutron stars to values below 10^{-21} g cm^{-3} (i.e. $\sim 10^3$ particles per cubic centimeter) in the giant molecular clouds (GMCs), sites of star formation. Temperatures vary from a few Kelvins in the GMC to the $\sim 10^8$ K in the nuclei of helium burning stars. Analogously, pressures span more than eight orders of magnitude.

There are also other characteristics of fluids which are very different in the terrestrial and astrophysical cases. The liquid state is the prominent one in the terrestrial case, while the gaseous (and plasma) state dominates astrophysical fluids. Compressibility is usually much more important in astrophysical fluids and plasmas than in terrestrial context. The role of viscosity is more difficult to determine in what viscosity itself is a concept that in astrophysical frame is hard to define.

3.2 Self-gravitating Fluids

A natural question arises:

—Why such huge differences between terrestrial and extraterrestrial fluids?

An immediate answer is impossible, but it is clear that on earth flows are essentially determined by the combined roles of local, small scale, forces and the large scale due to gravity and Coulomb forces. On earth, gravity is at every practical effect a body force of constant radial direction and intensity. We can say that liquids like lakes and rivers are gravitating in an external *fixed* field. The flow of a river is caused by its (approximate) free fall (out of equilibrium) from the source to the mouth, while lakes have just local movements around an equilibrium between the terrestrial gravity and the binding reaction of the ground. Astrophysical fluids, instead, are such that the body forces may vary significantly within their body. Even considering gravity, only, it is different from one point to another. Astrophysical fluids are essentially self-gravitating fluids while terrestrial are not. We can clarify this better.

As we know, in a fluid all the constituent particles exert forces on each other, and the distinction between short-range forces and long-range forces leads to the distinction between surface and body forces. Newtonian gravity is long-range and so every ith fluid particle interacts with any other jth via a force scaling as r_{ij}^{-2} where r_{ij} is the distance between the objects in the pair. Thus, the generic ith fluid particle is subjected to an 'internal' gravity acceleration due to the presence/interaction of/with the other $N-1$ particles summed to an 'external' gravity acceleration due to the presence of external massive bodies. In formulas

$$\ddot{\mathbf{r}}_i = G \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j}{r_{ij}^3} (\mathbf{r}_j - \mathbf{r}_i) + \ddot{\mathbf{r}}_{ext,i}. \tag{3.1}$$

Clearly, if for every particle i the acceleration induced by the other particles of the system is large with respect to the external acceleration, the fluid is said to be *self-gravitating*, and vice versa. The self-gravity condition can be tested by checking an energetic condition. The energetic condition consists in an approximated evaluation of the dimensionless parameter $\alpha = |\Omega_{int}|/|\Omega_{ext}|$ where Ω_{int} and Ω_{ext} stand, respectively, for the values of gravitational internal and external energy. The limit $\alpha \ll 1$ would be one of non-self-gravitating fluids.

The precise evaluation of Ω_{int} is practically impossible because it involves a double summation over all the particles of the fluids

$$\Omega_{int} = -\frac{1}{2} \sum_{\substack{(i,j)=1 \\ i \neq j}}^{N} G \frac{m_i m_j}{r_{ij}}, \tag{3.2}$$

so for practical use it is necessary to resort to an approximate evaluation, which we give here as an example.

Consider a lake like the big Lake Victoria crossing Kenia, Uganda and Tanzania in Africa (see Fig. 3.1). Its basin has a volume of 2760 km^3 $= 2.760 \times 10^{18}$ cm^3. This means a mass of water of $M = 2.760 \times 10^{18}$ g. A rough, but valid, evaluation of the energy of self-gravitation of the lake can be obtained by calculating the gravitational energy of the same mass of water enclosed in a sphere whose radius is, obviously, $R = [3M/(4\pi\rho)]^{1/3} \simeq 8.7018$ km, where $\rho = 1$ g cm^{-3} is the density of water. A straightforward calculation of the integral giving the gravitational energy of a homogeneous sphere of mass M and radius R gives

$$\Omega_{int} = -\frac{3}{5} \frac{GM^2}{R} \simeq -3.50 \times 10^{23} \text{erg} = -3.50 \times 10^{16} \text{J}, \tag{3.3}$$

while the external (due to Earth) gravitational energy is

$$\Omega_{ext} = -\frac{GMM_E}{R_E} \simeq -1.73 \times 10^{30} \text{erg} = -1.73 \times 10^{23} \text{J}, \tag{3.4}$$

where M_E and R_E are the mass and equatorial radius of the Earth. Consequently,

$$\alpha = \frac{3}{5} \frac{M}{M_E} \frac{R_E}{R}, \tag{3.5}$$

which, assuming $M_E = 5.972 \times 10^{24}$ kg and $R_E = 6371$ km, gives $\alpha \simeq 2.0302 \times 10^{-7}$ which is $\ll 1$. So, fluids on Earth are actually *non-self-gravitating*. On the contrary, astrophysical objects like stars are self-gravitating because they are all

composed of fluid (essentially plasma, in a star like the Sun) which is 'contained' by its own internal gravity while the external gravity is negligible. The real difference is, indeed, that in stars there is not a solid/rigid container of a fluid that, for terrestrial gases, is necessary to confine the gas against its pressure gradient, but, instead, this confinement role is played by the strong self-gravitational force.

Self-gravity represents a relevant complication in the study of fluids because in the governing equations, it gives a body force field which is not a constant vector but depends on the position and, in non-stationary cases, on time also. This vector field, to be determined, needs the solution of the proper field equation which links the source of field (the matter density ρ) with the potential it generates, $U(\mathbf{r})$. In the case of gravity, this is the Poisson's equation

$$\nabla^2 U = -4\pi G\rho, \tag{3.6}$$

which must be added to the whole system of constitutive equations. Consequently, the full system of constitutive equations for an astrophysical fluid—when neglecting dynamical electromagnetic effects—is written, in Eulerian form, as

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \\ \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\dfrac{1}{\rho}\nabla p + \dfrac{\eta}{\rho}\nabla^2 \mathbf{v} + \left(\dfrac{1}{3}\dfrac{\eta}{\rho} + \dfrac{\zeta}{\rho}\right)\nabla\nabla \cdot \mathbf{v} + \nabla U \\ \dfrac{\partial}{\partial t}\rho\left(\dfrac{1}{2}v^2 + e\right) + \nabla \cdot \rho\left(\dfrac{1}{2}v^2 + e + \dfrac{p}{\rho}\right)\mathbf{v} = \\ \qquad = \left[\eta\nabla^2 \mathbf{v} + \left(\dfrac{1}{3}\eta + \zeta\right)\nabla\nabla \cdot \mathbf{v}\right] \cdot \mathbf{v} + \rho\dfrac{\mathrm{D}q^*}{\mathrm{D}t} \\ \nabla^2 U = -4\pi G\rho \\ f(\rho, p, e) = 0, \end{cases} \tag{3.7}$$

which are six PDEs and a functional relation for seven unknowns (the four scalar fields ρ, p, e, U and the vector $\mathbf{v}$ field) subjected to proper initial and boundary conditions (Fig. 3.1).

The presence of self-gravity is not the unique difference in the treatment of astrophysical fluids and flows with respect to earth (two examples of self-gravitating systems are shown in Fig. 3.2). Another relevant difference is the energy generation and transport in the fluid.

In stars, energy is generated by nuclear reactions in their innermost region which is characterized by high pressure and temperature. This energy is released slowly in time but in huge and long-lasting quantity. The energy output of nuclear reactions is in the form of radiation and kinetic energy of resulting particles and neutrinos. The energy transport, crucially depending on the interaction with the surrounding

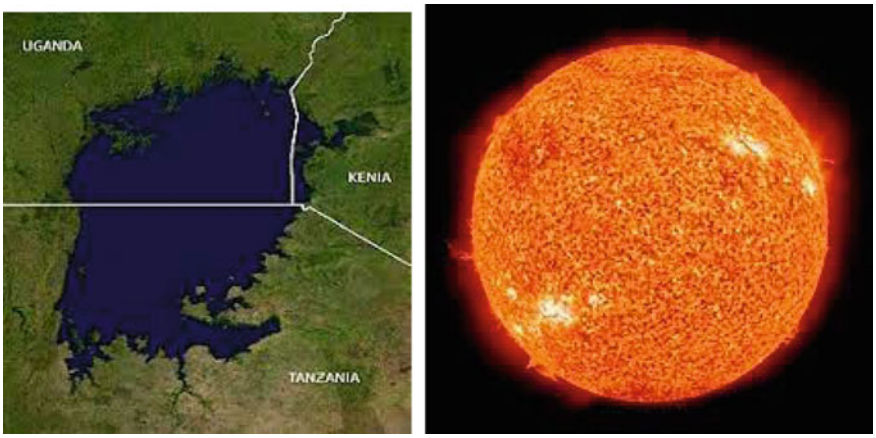


Fig. 3.1 A typical non-self-gravitating fluid (Lake Victoria in Uganda on the left) and a typical self-gravitating one (the Sun, on the right)

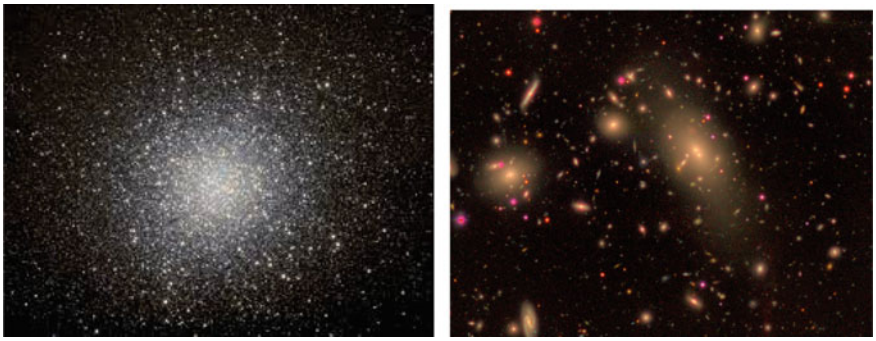


Fig. 3.2 The M13 globular cluster (left) and a bright galaxy cluster (right)

matter, determines the internal structure of the star. So, both in stars and in other astrophysical environments, the energy transport should be properly accounted for in the energy equation of the fluid, as outlined in Sect. 2.3.

As a matter of fact, while the fundamental equations of fluid dynamics are common on the Earth and in astrophysical (non-relativistic) environments, in the astrophysical context fluid dynamics involves a series of specificities that characterize what is commonly known as Astrophysical fluid dynamics. We give below important examples of such peculiarities.

3.2.1 Equilibrium of Self-gravitating Fluids

The inclusion of Poisson's equation as coupled to Euler's equation of motion, together with the assumption of spherical symmetry, leads to

$$\begin{cases} \dfrac{\mathrm{D}v_r}{\mathrm{D}t} = -\dfrac{1}{\rho}\dfrac{\mathrm{d}p}{\mathrm{d}r} + \dfrac{\mathrm{d}U}{\mathrm{d}r} \\ \dfrac{\mathrm{d}U}{\mathrm{d}r} = -G\dfrac{M(r)}{r^2} \end{cases} \tag{3.8}$$

where v_r is the radial component of the velocity field (here $v_\theta = v_\phi = 0$) and $M(r)$ is the mass enclosed in the sphere of radius r, namely

$$M(r) = 4\pi \int_0^r \rho r^2 \, \mathrm{d}r. \tag{3.9}$$

.

At equilibrium, $\mathrm{D}v_r/\mathrm{D}t = 0$ and, Eq. 3.8 reduce to the single integro-differential equation

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} = -G\frac{M(r)}{r^2}. \tag{3.10}$$

This equation can be transformed into a second order differential equation by multiplying both sides by r^2 followed by a differentiation with respect to r, to obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = -4\pi G\rho r^2, \tag{3.11}$$

which is still in two unknowns (ρ and p). As usual, the reduction to just one unknown is done by using an EOS, provided it is *barotropic*, i.e. in the form $p = p(\rho)$.

Under such assumption, Eq. 3.11 becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -4\pi G\rho r^2. \tag{3.12}$$

Let us examine two particular cases of EOS: the isothermal case ($T = const. = T_0$) and the polytropic case $p \propto \rho^\gamma$, which indeed contains, for $\gamma = 1$, the isothermal case (see also Sect. 2.4). The EOS of an astrophysical gas, whenever pressure is not too high and for sufficiently high temperature, can be acceptably represented with its ideal gas expression $p = nkT = \rho kT/(\mu m_H)$.

In the (spatial) isothermality assumption, the ideal gas EOS becomes linear in ρ because μ shows a much weaker dependence on ρ and the specific abundances of various chemical species than on T, so that μ can be considered constant once T_0 is fixed, and the equilibrium equation 3.12 is written as

$$\frac{kT_0}{\mu_0 m_H}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -4\pi G\rho r^2, \tag{3.13}$$

which can be also expressed in logarithmic form as

$$\frac{kT_0}{\mu_0 m_H}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r}\right) = -4\pi G\rho r^2. \tag{3.14}$$

An explicit solution of the above equation can be found in power law form, $\rho = a^2 r^\alpha$, where a and α have to be determined (to have equilibrium, α must be negative[1]). Substituting for $\rho = a^2 r^\alpha$ in Eq. 3.13 (or equivalently $\ln\rho = \ln a^2 + \alpha \ln r$ in Eq. 3.14), it becomes

$$\frac{kT_0}{\mu_0 m_H}\alpha = -4\pi G a^2 r^{\alpha+2}, \tag{3.15}$$

which requires $\alpha = -2$. The constant a^2 is consequently determined by the same Eq. 3.15 as

$$a^2 = \frac{kT_0}{2\pi G\mu_0 m_H}. \tag{3.16}$$

The obtained solution, $\rho(r) = a^2 r^{-2}$, is called *singular* isothermal sphere, because it is not defined at the center, being $\lim_{r\to 0}\rho(r) = \infty$. In spite of the central density singularity, the mass contained in any spherical neighborhood of the center is finite

$$M(\epsilon) = 4\pi a^2 \int_0^\epsilon r^{-2} r^2\,\mathrm{d}r = 4\pi a^2\epsilon, \quad \forall\epsilon > 0, \tag{3.17}$$

so that the real physical problem with this solution is not in the origin but on the large scale, because if the sphere has no finite boundary its mass diverges linearly with r.

The solution for an isothermal sphere with a finite central density ($\rho(0) = \rho_0 > 0$) can, as for any other polytropic spheres of arbitrary value of the exponent γ in the EOS, be obtained numerically, according to the following procedure.

Assuming $p = c_\gamma \rho^\gamma$ with c_γ a positive constant, the equilibrium equation 3.10 becomes

[1] If $\alpha > 0$,

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{kT_0}{\mu_0 m_H} a^2 \alpha \rho^{\alpha-1} > 0,$$

contrary to the necessary condition for equilibrium, $\mathrm{d}p/\mathrm{d}r < 0$.

$$c_\gamma \gamma \rho^{\gamma-2} \frac{\mathrm{d}\rho}{\mathrm{d}r} = -G\frac{M(r)}{r^2}, \tag{3.18}$$

which, to allow an easy numerical treatment, should not be transformed into a second order differential equation, as done to obtain Eq.3.11 with a further development that would lead to the so-called Lane–Emden equation,[2] but rather transformed into a set of two first order ODEs by the introduction of the auxiliary variable $u = GM(r)$

$$\begin{cases} c_\gamma \gamma \rho^{\gamma-2} \dfrac{\mathrm{d}\rho}{\mathrm{d}r} = -\dfrac{u}{r^2} \\ \\ \dfrac{\mathrm{d}u}{\mathrm{d}r} = 4\pi G \rho r^2 \end{cases} \tag{3.19}$$

whose initial conditions are $\rho(0) = \rho_0 > 0$ and $u(0) = 0$. The condition of centrally vanishing $u(r)$ is naturally implied by the finiteness of the supposedly spatially continuous density, because

$$M(0) = \lim_{r\to 0} 4\pi \int_0^r \rho(t) t^2 \,\mathrm{d}t = 4\pi \lim_{r\to 0} \rho(\xi_r)\frac{r^3}{3} = \frac{4\pi}{3}\rho_0 \lim_{r\to 0} r^3 = 0, \tag{3.20}$$

where we applied the Lagrange mean value theorem to the integral, with $0 \le \xi_r \le r$.

Once the equilibrium condition is set in the form of the first order ODE system Eq. 3.19, every standard method of numerical integration of first order ODEs and systems of ODEs can be applied.

Finally, we note that the previously studied case of isothermal, finite density, sphere previously studied is obtained letting $\gamma = 1$ and $c_1 = kT_0/(\mu_0 m_H)$ in the system Eq. 3.19.

3.2.2 *Gravitating Systems Out of Equilibrium*

Consider a gas in a spherically symmetric configuration of radius R and whose pressure is spatially uniform ($p = const.$). Suppose also that at the center there is a point-like object of mass $M \gg M_g$ where M_g is the total gas mass.

Under such conditions, the Euler's equation is written as

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} = \nabla U, \tag{3.21}$$

[2] J. H. Lane and R. Emden in the second half of nineteenth century developed the theory of polytropic spheres at equilibrium, which join the equilibrium condition and Poisson's equation for the gravitational potential of a polytropic gas into a single second order dimensionless equation called, after them, Lane–Emden equation.

where, in our case, $\nabla U = -(GM/r^2)\mathbf{e}_r$ because the gas is not self-gravitating ($M_g \ll M$). If the flow is stationary, the above equation can be integrated for the velocity of the generic gas particle that initially was at distance r_0 from the attraction center with initial velocity v_{r0}, giving

$$v_r(r) = \pm\sqrt{v_{r0}^2 + 2\frac{GM}{r_0}\left(\frac{r_0}{r} - 1\right)}. \tag{3.22}$$

If $v_{r0} = 0$, the minus sign in the above equation is taken, because of the attractive nature of the point mass gravity.[3]

By means of Eq. 3.22, the time needed to reach the origin (*free-fall* time) is obtained as

$$t_{ff}(r_0) = \int_{r_0}^{0} \frac{\mathrm{d}r}{v_r} = \frac{r_0^{3/2}}{\sqrt{2GM}} \int_0^1 \frac{\mathrm{d}x}{\sqrt{\frac{1}{x} - 1}}, \tag{3.23}$$

having set $x = r/r_0$. The rightmost integral in Eq. 3.23 which we denote by I, by letting $x = \sin^2\theta$, transforms into

$$I = 2\int_0^{\pi/2} \sin^2\theta \mathrm{d}\theta, \tag{3.24}$$

which is solved by parts, giving $I = \pi/2$ so that eventually

$$t_{ff}(r_0) = \frac{\sqrt{2}}{4}\pi\frac{r_0^{3/2}}{\sqrt{GM}}. \tag{3.25}$$

Note that t_{ff} is an increasing function of r_0 so to justify as a definition of free-fall time of the whole gas cloud its maximum value $t_{ff}(R)$.

In a similar way, it is possible to evaluate the free-fall time in the somehow opposite case when the spherical gas cloud, supposed uniform in density, $\rho = const.$, does not carry a compact point-like mass M at its center or, similarly, when the gas mass M_g is overabundant with respect to a possible central point-like mass. In such a case, the gas is self-gravitating and undergoes collapse due to its own gravitation.

The way to obtain the free-fall time in this case is the same as before but with

$$\nabla U = -\frac{GM_g}{R^3} r\ \mathbf{e}_r, \tag{3.26}$$

so to obtain ($v_{r0} = 0$, and letting again $x = r/r_0$)

[3] Note that v_r is diverging for $r \to 0$.

$$t_{ff}(r_0) = \frac{R^{3/2}}{\sqrt{GM_g}} \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{R^{3/2}}{\sqrt{GM_g}} \frac{\pi}{2}. \tag{3.27}$$

The time to reach the origin in this case of uniform density cloud is independent of r_0 and is so assumed as global free-fall time. In the assumption that the gas mass is equal to what was assumed to be the central point-like mass, $M_g = M$, we see that the free-fall time of Eq. 3.27 is $\sqrt{2}$ longer than that of Eq. 3.25 where $r_0 = R$ is set.

3.3 Jeans Theory of Gravitational Instability

How are the stars born is, in astrophysics, a question still partially unanswered.

That stars originated from a diffuse gaseous medium is certain and this constitutes a sort of phase transition, but the actual modes that lead from a dilute medium to a compact state are not deeply known. As every phase transition, it depends on the onset of some kind of instability which, over a certain time scale, leads the system from one equilibrium to another. A local portion of a dilute gas undergoes a collapse instability until a sort of equilibrium is reached in an almost spherical, compact, configuration. The specific mechanism which triggers the instability is not uniquely determined and is likely different from case to case.

Surely the formation of a new star would involve processes different than those characterizing the formation of a planet. Observations indeed show that stars are formed in the so-called *cores* of giant tenuous and cold nebulae mainly composed of molecular hydrogen (the Large Molecular Clouds, LMCs), which are irregular in shape, while planets are found around stars on almost coplanar orbits, suggesting their origin from some accretion mechanism of gas left from the parent star formation and essentially confined in the star equatorial plane. The role of gravity is undoubtedly important in both such example cases, but surely in a different way.

Let us concentrate on the case of a diffuse, large and cold extension of H_2 gas (LMC). The estimated kinetic temperature of these clouds is around 10 K, and the matter density is very low ($\sim$100 molecules per cubic centimeter). In spite of the low density, these clouds are so spatially extended that their mass overcomes 10^5 $M_\odot$ and so, being very cold, their gravitational energy is large enough to make them intrinsically unstable against gravity. For such kinds of systems, it is so worth studying the so-called *gravitational instability* mechanism proposed by J. H. Jeans in 1902.

Let's consider the, idealized, case of a spatially infinite extension of an ideal fluid, homogeneous in density and at rest. The fluid is, so, characterized by $\rho = \rho_0 > 0$, $p = p_0 > 0$, $\mathbf{v} = \mathbf{v}_0 = 0$. It is straightforward to verify that this set of constant values satisfies the governing equations (continuity, Euler's equation and isentropic equation) if body forces are absent.

Now, let us consider a perturbation of the global equilibrium state of the fluid by additive quantities $\delta\rho(\mathbf{r};t)$, $\delta p(\mathbf{r};t)$, $\delta\mathbf{v}(\mathbf{r};t)$ such that $\rho = \rho_0 + \delta\rho$, $p = p_0 + \delta p$, $\mathbf{v} = \delta\mathbf{v}$. In order to linearize the governing equations, it is required that perturbations are 'small', that is, $|\delta\rho/\rho_0| \ll 1$ and $|\delta p/p_0| \ll 1$. Also $\delta\mathbf{v}$ must be small with respect to some typical velocity characterizing the fluid (*not* with respect to $\mathbf{v}_0$ which is zero). To quantify this in the given context, the request is $\|\delta\mathbf{v}\| \ll c_s$, where c_s is the fluid sound speed

$$c_s = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_{ad}}, \tag{3.28}$$

which, evaluated at time zero, gives

$$c_{s0} = \sqrt{\gamma \frac{p_0}{\rho_0}}, \tag{3.29}$$

upon the adoption of the $p \propto \rho^\gamma$ EOS with constant γ.

Inserting $\rho = \rho_0 + \delta\rho$, $p = p_0 + \delta p$, $\mathbf{v} = \delta\mathbf{v}$ in the continuity and Euler's equations, in the absence of body forces, and keeping only linear terms we have

$$\begin{cases} \dfrac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta\mathbf{v} = 0, \\ \dfrac{\partial \delta\mathbf{v}}{\partial t} = -\dfrac{1}{\rho_0}\nabla \delta p. \end{cases} \tag{3.30}$$

Moreover, at first order,

$$\delta p = p(\rho_0 + \delta\rho) - p(\rho_0) = \left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)_0 \delta\rho, \tag{3.31}$$

and so $c_s^2 = \delta p/\delta\rho$. Taking the partial derivative with respect to t of both sides of the first equation in Eq. 3.30 and taking the divergence of the second equation after the substitution $\delta p = c_{s0}^2 \delta\rho$ in its right-hand side, and, finally, making a side-by-side subtraction of the obtained second equation from the first equation, we get eventually the single scalar PDE (for simplicity, we indicate c_{s0} as c_s)

$$\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\frac{\delta\rho}{\rho_0} - \nabla^2 \frac{\delta\rho}{\rho_0} = 0, \tag{3.32}$$

which is the classical linear hyperbolic equation of wave propagation in the unknown quantity $\delta\rho/\rho_0$.

A solution of Eq. 3.32 is

$$\left(\frac{\delta\rho}{\rho_0}\right)_k = A_k e^{i(\mathbf{k}\cdot\mathbf{r}\pm\omega t)}, \tag{3.33}$$

where A_k is the amplitude of the k-wave component ($|A_k| < 1$ in the linear regime), i is the imaginary unit, $\mathbf{k}$ is the wave vector and $\omega = 2\pi\nu$ is the angular frequency, $\nu = c_s/\lambda$ being the ordinary frequency. The vector $\mathbf{k}$, whose magnitude is $k = 2\pi/\lambda$, points in the direction of phase velocity, which is usually normal to the propagating wavefront.

It is easily seen by substitution of Eq. 3.33 in Eq. 3.32 that satisfaction of Eq. 3.32 is subjected to satisfaction of the *dispersion* relation

$$\frac{\omega^2}{c_s^2} - \mathbf{k}\cdot\mathbf{k} = 0, \tag{3.34}$$

that is, $\omega^2 = c_s^2 k^2$.

Due to the linearity of the wave equation, any superposition of k-wave components is still a solution. Note that in the solution Eq. 3.33 if ω is imaginary the choice of $+$ in the argument of the exponential gives a *damping* solution (it tends to zero for t going to infinite) while the choice of $-$ gives a *growing* in time solution which, in our linearization, would lose sense when $\delta\rho/\rho_0$ overcomes 1. Anyway, in the present case of the absence of body forces the dispersion relation is $\omega^2 = c_s^2 k^2 > 0$, so that the angular frequency ω is real. As a consequence, all the possible wave components of the general solution are bound in time and, indeed, the general solution is a superposition of sinusoidal oscillations. This situation corresponds to the actual *stability* of the fluid upon small perturbations.

The situation changes when considering a body force in Euler's equation. In this case, the evolutive equation 3.32 of density perturbations becomes

$$\frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\frac{\delta\rho}{\rho_0} - \nabla^2\frac{\delta\rho}{\rho_0} + \frac{1}{c_s^2}\nabla\cdot\frac{\delta\mathbf{F}_B}{\rho_0} = 0, \tag{3.35}$$

where $\delta\mathbf{F}_B$ is a perturbative body force (per unit volume). Let us assume a linear dependence on $\delta\rho/\rho_0$ of the body force contribution in the above equation, i.e.

$$\nabla\cdot\frac{\delta\mathbf{F}_B}{\rho_0} = a\frac{\delta\rho}{\rho_0}, \tag{3.36}$$

where $a \neq 0$ is a constant.

Upon this assumption, the plane wave function of Eq. 3.33 still satisfies the wave equation Eq. 3.35 provided that the new dispersion relation

$$\frac{\omega^2}{c_s^2} - \mathbf{k}\cdot\mathbf{k} - \frac{a}{c_s^2} = 0 \tag{3.37}$$

is satisfied. Clearly, if $a \geq 0$, $\omega^2 = c_s^2 k^2 + a \geq 0$ for every k. When $a < 0$, it results in ω^2 being less, equal or greater than zero when $k^2 < -a/c_s^2$, $k^2 = -a/c_s^2$ and $k^2 > -a/c_s^2$, respectively. Denoting as $k_c^2 \equiv -a/c_s^2 > 0$ the critical (separation) value, we have that ω is imaginary for $k^2 < k_c^2$ and real for $k^2 \geq k_c^2$. As a consequence, in the general solution of Eq. 3.35 all the wave components with $k < k_c$ are 'unstable' in that they correspond to one exponentially diverging in time and one exponentially vanishing in time solution, in pair. On the other hand, all the large wave numbers ($k \geq k_c$) correspond to limited oscillations, as it happens in the absence of body force.

Now the question is: how is a determined in the case of the hypothetical infinite, homogeneous medium initially at rest studied by Jeans?

To answer this, we have to assume that the local body perturbation is of gravitational origin. In a uniform density, infinitely extended medium, gravity is balanced everywhere, so every point in the medium is an equilibrium point. But a local perturbation of density causes an unbalance: intuitively, a local $\delta\rho > 0$ (overdensity) in a point makes it an 'attraction' point while the opposite happens when $\delta\rho < 0$ (underdensity).

Due to that, letting as reasonable a spherical symmetry in a neighborhood of the considered point, the resulting perturbing force $\delta\mathbf{F}$ is radially oriented, naturally inward if $\delta\rho > 0$ and outward if $\delta\rho < 0$. This can be expressed by $\delta\mathbf{F} = \pm A^2 \mathbf{e}_r$ ($\mathbf{e}_r$ is the unit vector pointing radially outward from the considered point), taking $-$ (attraction) in overdensity and $+$ (repulsion) in underdensity.

Consequently

$$\nabla \cdot \frac{\delta\mathbf{F}}{\rho_0} = a\frac{\delta\rho}{\rho_0} = \pm A^2 \nabla \cdot \mathbf{e}_r = \pm A^2 \frac{2}{r}, \tag{3.38}$$

the first equality above standing for the given assumption on body force divergence-density perturbation linearity (Eq. 3.36). Equation 3.38 can be satisfied only if $a < 0$: in such a case, indeed, a positive $\delta\rho$ gives the right '$-$' sign in the body force perturbation divergence and a negative $\delta\rho$ gives the right '$+$' sign.

As a matter of fact, not only the sign of a is determined but also its absolute value, which simply derives from the assumed validity of Poisson's equation that links matter density and the resulting gravitational potential. As we said above, in the case of an infinite and homogeneous fluid, $\rho = \rho_0 = \text{const.}$, the gravitational force $\mathbf{F}_0$ is zero in every point, so that if $\mathbf{F}_0$ admits a potential function U it would be $\mathbf{F}_0 = \nabla U_0 = 0$, implying $(\nabla \cdot \nabla U)_0 = (\nabla^2 U)_0 = 0$ (satisfying Laplace's equation) while Poisson's equation would require instead $(\nabla^2 U)_0 = -4\pi G\rho_0 \neq 0$. This consideration rules out the validity of Poisson's equation in the original unperturbed situation. If, anyway, we force the assumption of its validity (Jeans' *swindle*), we can fruit of linearity of Poisson's equation to obtain in the perturbed case

$$\nabla \cdot (\mathbf{F}_0 + \delta\mathbf{F}) = -4\pi G(\rho_0 + \delta\rho) \Rightarrow \nabla \cdot \delta\mathbf{F} = -4\pi G\delta\rho, \tag{3.39}$$

leading to $a = -4\pi G\rho_0$ as the value of the quantity a in Eq. 3.36 that we have previously obtained as necessarily negative on a heuristic base.

Now, basing on the dispersion relation in Eq. 3.37, we have two exponential solutions in time provided $k^2 < k_c^2$ (and so ω imaginary), because in this case $\omega = i|\omega|$, and so the solutions Eq. 3.33 are written as

$$\left(\frac{\delta\rho}{\rho_0}\right)_{\pm} = \begin{cases} A_k e^{i\mathbf{k}\cdot\mathbf{r}} e^{-|\omega|t}, \\ A_k e^{i\mathbf{k}\cdot\mathbf{r}} e^{|\omega|t}, \end{cases} \tag{3.40}$$

taking in the exponential of Eq. 3.33 the plus and minus signs.

Of course, also the sum $(\delta\rho/\rho_0)_+ + (\delta\rho/\rho_0)_-$ still solves the wave equation.

The condition $k^2 < k_c^2$ reflects into $\lambda^2 > 4\pi^2/k_c^2$, that is, $\lambda^2 > \pi\gamma k T_0/(G\rho_0)$ because $c_s^2 = \gamma p/\rho_0$ and because of the assumption of the ideal gas EOS, $p = k\rho T/(\mu m_H)$. In conclusion, the instability condition translates into a critical wavelength

$$\lambda_c = \sqrt{\frac{\pi\gamma k T_0}{\mu_0 m_H G\rho_0}}, \tag{3.41}$$

which naturally corresponds to a critical mass scale

$$M_c = \frac{4\pi}{3}\rho_0 \left(\frac{\lambda_c}{4}\right)^3, \tag{3.42}$$

as the one contained in the compression (rarefaction) region and so undergoing gravitational instability (other authors use $\lambda_c/2$ instead of $\lambda_c/4$ in Eq. 3.42). The above expression puts in evidence the dependence of M_c on $\rho_0^{-1/2} T_0^{3/2}$, marking the opposite role of gravity *against* thermal pressure in the instability.

In its limitations, Jeans' theory robustly supports a scenario where large, almost uniform clouds of gas are indeed prone to gravitational instability even if their mass density is low whenever they are cold enough. The process of local collapse likely involves different zones of the extended low-temperature region leading to a sort of cascade of further 'fragmentations' of the dilute medium. It is worth reminding, however, that Jeans' is a linear theory and so it loses predictive validity when the amplitude of the absolute value of the density contrast $\delta\rho/\rho_0$ approaches unity from lower values. A full study of the collapse of gaseous clouds requires thus a complete hydrodynamical and energetic treatment.

To give an astrophysical example, in the disk of the Milky Way there are several Large Molecular Clouds composed mainly of molecular hydrogen (H_2) with the mass of thousands of solar masses over a spatial size in the range 5–200 pc with a mass density up to 1000 particles per cubic centimeter (the average density in the solar neighborhood is about 1 particle per cubic centimeter). The typical temperature is 10 K. If we take, as an example, $T = 10$ K, $\rho_0 = 10^{-22}$ g cm^{-3} and $\gamma = 7/5$, the critical mass is $M_c \approx 8.80\ \mathrm{M}_\odot$, which means that almost all LMCs with this temperature and density are unstable against density perturbations.

3.4 The Role of Viscosity in Astrophysical Environments

Whenever there is a shear (gradient in velocity) in a fluid, there are collisions among neighboring particles giving rise to a net transport of momentum which drives the system toward equilibrium. This means a reduction of velocity gradients, toward a no-shear situation. 'Viscosity' in fluids operates this resistance to shear by means of the effectiveness of collisions. Viscosity is a sort of global view of this particle-by-particle interaction, which is difficult to describe precisely on a small scale. Together with the efficiency of collisions in reducing the shear by net transport of momentum, there is also another approach to equilibrium given by the transport of energy induced by collisions. In this case, the global quantity which measures the efficiency of the process is the thermal conductivity. So, both viscosity and thermal conduction are parameters tuning the transport efficiency, and so they are also called *transport* coefficients.

The consistent deduction of these coefficients is hard to exploit.

A good formal approach which leads to their expression in terms of collisional cross section, σ, fluid temperature, T, and particle mass, m, derives from the Chapman–Enskog expansion. It is complicated and out of the purposes of this introductory book, but the interested reader can find it in textbooks like *Statistical Mechanics* by H. Kuang or *The Mathematical Theory of Non-Uniform Gases* by S. Chapman and T. Cowling. The Chapman–Enskog expansion for a pure monoatomic gas leads to

$$\eta = \frac{a}{\sigma^2}\frac{\sqrt{\mu T}}{\Omega(T)}, \quad \kappa = \frac{5}{2}\eta c_V, \tag{3.43}$$

where a is a numerical constant characteristic of the fluid, μ the gas molecular weight, σ the collisional cross section, $\Omega(T)$ a weak function of T approximable to 1, κ the thermal conductivity and c_V the specific (per unit mass) heat. For a given gas (a and σ are fixed), η and κ are at all practical aspects functions of T as square roots. This is partially true for multicomponent gas mixture, but surely not true for liquids for which, at least in a range of temperatures, viscosity decreases with increasing T. But, as we said, for gases it is a good approximation, and so it can be used in an astrophysical context. In environments like molecular clouds, where the temperature is of the order of 10 K, viscosity is expected to play a minor role, indeed. This means that MCs are strongly unstable systems because of almost zero shear viscosity in the gas. The almost zero viscosity permits to neglect it in both the motion and the energy equations, leading to some simplification which, in numerical hydrodynamics, is overcome by the onset of local instabilities caused by the finite resolution of any numerical scheme. This problem is often cured by introducing an *artificial* viscosity which helps in smoothing out the instabilities rising on the scale of the spatial grid.

3.5 Peculiarities of Astrophysical Equations of State

The equation of state is crucial for the closure of the system of equations describing fluids. Astrophysical gases are often well described by the ideal gas law. Even for a fully ionized gas, the interparticle forces (Coulomb force) can typically be neglected (i.e. the potential energies involved are typically $< 10\%$ of the kinetic energies). Ideal gas law breaks down for dense and cool gases, such as those present in gaseous planets. Also in the very dense environments of dead stars (those where nuclear burning is over) like white dwarves and neutron stars, the EOS is very different from that of perfect gases. This is indeed one of the crucial aspects of astrophysical fluid dynamics: understanding what the proper EOS to use is, because of the vast range of variations of the state variables involved. A further complication is that in time-dependent situations, like those of an approach to a possible equilibrium, the gas can undergo internal changes of dissociation and ionization, out of equilibrium. This means a dynamical evolution of the EOS with the need to evaluate the molecular and atomic abundances and their ionization state 'along the way' by means of the time integration of the *rate* equations which are needed to determine the proper values of the abundances of the various elements and the free electrons. This procedure implies a significant complication in numerical modelization, because the changes in the element abundances occur on time scales which are very short with respect to the global dynamical time scales. So, to follow the abundance evolution, implicit schemes are required.

3.6 Solved Exercises

Exercise 1 A stone is left to fall freely (this means that the air resistance is negligible) in a well 8 meter deep.

How long does it take to hear the sound of the stone hitting the bottom of the well? (note: assume air as an ideal $\gamma = 7/5$ gas at temperature $T = 20\,^{\circ}\mathrm{C}$, and assume its mean molecular weight as $\mu = 29$).

Solution

The requested time, τ, is the sum of the time, t_{ff}, needed by the stone to fall freely to hit the well bottom plus the time for the sound of the hit to reach the top of the well, t_s. The stone fall is governed by the equation of the uniformly accelerated motion $z(t) = z(0) + \dot{z}(0)t + \frac{1}{2}gt^2$ where a vertical, downward pointing, z-axis with origin at the open mouth of the well is adopted. For the stone, $z(0) = \dot{z}(0) = 0$ and, consequently, $t_{ff} = \sqrt{2h/g}$, where $h = 8$ m is the depth of the well and $g = 9.81$ ms^{-2} is the Earth's gravitational acceleration. It results in $t_{ff} = 1.277$ s.

After hitting the soil, the sound propagates at speed $c_s = \sqrt{\gamma p/\rho} = \sqrt{(7/5)k(T + 273)/(\mu m_H)} \simeq 343$ ms^{-1}, so that it reaches the well mouth after $t_s = h/c_s = 0.023$ s. Finally, the searched time is $\tau = 1.277 + 0.023 = 1.300$ s.

Exercise 2 Show that the equilibrium equation 3.10 for a monoatomic gas embedded in a medium exerting on its boundary a pressure p_{ext} admits as first integral[4] the quantity $2E_i - p_{ext}4\pi R^3 + \Omega = 0$, where E_i is the total internal energy of the self-gravitating fluid and Ω its gravitational energy (R is the radius of the equilibrium spherical configuration, assumed as finite, as finite is its mass M).

Solution
Let us write Eq. 3.10 in the form

$$\mathrm{d}p = G\rho\frac{\mathrm{d}U}{\mathrm{d}r}\mathrm{d}r,$$

and multiply both its sides by $4\pi r^3$ and integrate over the whole spherical configuration, to obtain

$$\int_0^R 4\pi r^3 \mathrm{d}p = \int_0^R \frac{\mathrm{d}U}{\mathrm{d}r} r 4\pi\rho r^2 \mathrm{d}r. \tag{3.44}$$

The lhs integral above may be integrated by parts[5] letting $u(r) = r^3$ and $\mathrm{d}v = \mathrm{d}p$

$$\int_0^R 4\pi r^3 \mathrm{d}p = 4\pi\left[r^3 p\Big|_0^R - 3\int_0^R pr^2\mathrm{d}r\right],$$

that is,

$$\int_0^R 4\pi r^3 \mathrm{d}p = 4\pi\left[R^3 p(R) - \lim_{r\to 0^+} r^3 p(r) - 3\int_0^R pr^2\mathrm{d}r\right]. \tag{3.45}$$

The equilibrium condition requires that at the border $p(R) = p_{ext}$ where p_{ext} is the pressure exerted by the external medium, so that Eq. 3.45, assuming $\lim_{r\to 0^+} r^3 p(r) = 0$ and accounting for the EOS $p = (\gamma - 1)\rho e$, becomes

$$\int_0^R 4\pi r^3 \mathrm{d}p = p_{ext}4\pi R^3 - 3(\gamma - 1)E_i, \tag{3.46}$$

where

$$E_i = \int_0^R \rho e 4\pi r^2 \mathrm{d}r = \int_M e\,\mathrm{d}m$$

[4] A first integral is a function independent of t which remains constant along the fluid evolution.

[5] Integration by parts bases on $\int_a^b u\mathrm{d}v = uv\Big|_a^b - \int_a^b v\mathrm{d}u$.

is the total internal energy.

Inserting the above expression Eq. 3.46 into Eq. 3.44 results in

$$p_{ext} 4\pi R^3 - 3(\gamma - 1)E_i = G \int_M \frac{\mathrm{d}U}{\mathrm{d}r} r \mathrm{d}m,$$

and the integral in the rhs is clearly the gravitational energy, Ω,[6] finally leading to

$$3(\gamma - 1)E_i - p_{ext} 4\pi R^3 + \Omega = 0. \tag{3.47}$$

For an equilibrium monoatomic gas ($\gamma = 5/3$), the above relation reduces to $2E_i - p_{ext} 4\pi R^3 + \Omega = 0$. If the sphere is in vacuum ($p_{ext} = 0$) or extends to infinity, the relation simplifies to $2E_i + \Omega = 0$.

Historical Note

The developer of the theory of gravitational instability was James Jeans (b. 1877 in Ormskirk, UK, d. 1946 in Dorking, UK). Initially, Jeans took interest in classic studies but later, also thanks to a very good mathematics teacher, he turned his attention to mathematics. He was particularly fascinated by numbers and their theory. At the age of 19, he went to the Trinity College in Cambridge (the same where Isaac Newton worked some 250 years before) with a mathematics scholarship. Among his teachers, there were A. N. Whitehead and E. T. Whittaker. Jeans was awarded an Isaac Newton Studentship in astronomy and optics, then in 1901 he was elected a Fellow of Trinity. During those times, he turned his attention from number theory to physics, in both its theoretical and experimental aspects. In 1904, he published The Dynamical Theory of Gases, his first important contribution to gas and fluid dynamics. That same year, Jeans was appointed a Lecturer in Mathematics at Cambridge. In 1905, he started lectures as a professor of Applied Mathematics at Princeton (USA) until 1909. After writing another book, Theoretical Mechanics, and after developing (together with J. W. S. Rayleigh, b. 1842 at Langford Grove, UK, d. in 1919 at Witham, UK) a theory of black-body radiation, he pointed his scientific interest toward cosmology and astrophysics. In these fields, he gave fundamental theoretical insights, although he was wrong when he conjectured a steady state cosmology, i.e. a universe where matter is continuously created. Another failure occurred when in a controversy with the other British scientist A. Eddington (b. 1882 in Kendal, UK, d. 1944 in Cambridge, UK) he defended his own idea that the origin of the energy released by the Sun was gravitational contraction, while Eddington was right in saying that the energy source is slow nuclear fusion in the inner solar region. In spite of these two failures, Jeans achieved many important and original results, the most relevant being probably

[6] The integrand is indeed the work per unit mass done against gravitational (radial) force.

the theory of gravitational instability as explaining the primary reason for the contraction of interstellar dilute nebulae toward the process of star formation. Another, less known and acknowledged, great result was the deduction of the set of partial differential equations describing in a compact way the motion of stars in a gravitational field using a method deriving by gas dynamics. This set of equations has been called 'Jeans equations' after him.

3.7 Further Readings

Fundamentals of astrophysical fluid dynamics are found in [8–10]. A useful introduction to self-gravitating fluids is in [11]. A detailed analysis of Jeans' instability is found in [12]. The physics of dilute astrophysical medium is studied in [13].

Chapter 4
Discontinuities in Fluid Flows

As we saw in the previous chapters, the motion of fluids is governed by a set of differential equations, both in the Lagrangian and Eulerian views. This implicitly requires that the unknown variables to determine are regular enough. Anyway, discontinuities in some (or all) of the variables which describe the flow should not be excluded a priori, as the experience suggests with special reference to gases. Given this, to account for possible discontinuities arising in fluid flows a special treatment is indeed needed.

4.1 Jump Conditions

The concept of *jump* across a surface S in a point $\mathbf{r}_0$ of this surface is crucial to deal properly with flow discontinuities.

We define as jump of a characteristic variable X (for example, the density ρ or the pressure p) the quantity

$$[X(\mathbf{r}_0)] \equiv X_1 - X_2, \tag{4.1}$$

where X_1 and X_2 are the values of the limit of X approaching the point from one side and the other of the given surface. If the quantity X is spatially continuous, obviously $[X] = 0$ in the given point. On the other hand, if $[X(\mathbf{r}_0)] \neq 0$ there is a local discontinuity. If $[X] \neq 0$ in every point of the surface S, this is called a *surface of discontinuity* for the variable X.

Now let us apply this concept to fluid flows.

We will deal with stationary flows; when the flow is not stationary, the possible surface of discontinuity moves along the flow. If we point our attention to a local discontinuity involving a neighborhood of $\mathbf{r}_0$, we can define the normal unit vector

R. A. Capuzzo Dolcetta, *Physics of Fluids*, UNITEXT for Physics,
https://doi.org/10.1007/978-3-031-30750-8_4

n in that point by a choice of its direction and approximating locally the surface with its osculating plane (obviously perpendicular to **n** in $\mathbf{r}_0$). At this point, we can write the fluid fundamental equations (for an inviscid fluid) at stationarity

$$\begin{cases} \nabla \cdot \rho \mathbf{v} = 0, \\ \nabla \cdot \underline{\mathbf{\Pi}} = 0, \\ \nabla \cdot \rho \left(\frac{1}{2} v^2 + w \right) \mathbf{v} = 0, \end{cases} \tag{4.2}$$

where we remind that $\underline{\mathbf{\Pi}}$ is Reynold stress tensor whose components are

$$\Pi_{ik} = \delta_{ik} p + \rho v_i v_k, \tag{4.3}$$

and $w = e + p/\rho$ is the enthalpy.

The set of equations 4.2 represents conservation laws: in our 1D vision, we assume an x-axis along the normal to the surface (and to the osculating plane) with origin in the given point (and so comoving with the surface) and $x > 0$ in the **n** direction, so that we would contain in the divergence only partial derivatives with respect to $x \equiv x_1$. Naming $x \equiv x_1$, $y \equiv x_2$ and $z \equiv x_3$ and $\mathbf{v} \equiv (v_1, v_2, v_3)$, the system Eq. 4.2 reduces to

$$\begin{cases} \frac{\partial}{\partial x_1} \rho v_1 = 0, \\ \frac{\partial}{\partial x_1} \Pi_{i1} = 0, \qquad i = 1, 2, 3, \\ \frac{\partial}{\partial x_1} \rho \left(\frac{1}{2} v^2 + w \right) v_1 = 0. \end{cases} \tag{4.4}$$

Indicating with 1 the region of space where the unit vector **n** points to and with 2 the other, indexing accordingly ρ, v_1, v_2, v_3, w (in the velocity components, the second index will refer to semispace 1 or 2), and using Eq. 4.3, in terms of 'jumps', the above Eq. 4.4 corresponds to four continuity conditions

$$\begin{cases} [\rho v_1] = \rho_1 v_{11} - \rho_2 v_{12} = 0, \\ \left[p + \rho v_1^2 \right] = p_1 + \rho v_{11}^2 - p_2 - \rho v_{12}^2 = 0, \\ [\rho v_1 v_2] = \rho_1 v_{11} v_{21} - \rho_2 v_{12} v_{22} = 0, \\ [\rho v_1 v_3] = \rho_1 v_{11} v_{31} - \rho_2 v_{12} v_{32} = 0, \\ \left[\rho \left(\frac{1}{2} v^2 + w \right) v_1 \right] = \rho_1 \left(\frac{1}{2} (v_{11}^2 + v_{21}^2 + v_{31}^2) + w_1 \right) v_{11} + \\ \qquad\qquad - \rho_2 \left(\frac{1}{2} (v_{12}^2 + v_{22}^2 + v_{32}^2) + w_2 \right) v_{12} = 0. \end{cases} \tag{4.5}$$

The above conditions involve combinations of the fluid variables ρ, p, w, **v** such that, individually, some of them can be discontinuous (non-zero jump) and other con-

tinuous (zero jump). An additional specification that allows to better define possible discontinuities involves the distinction between flows such that

(i) matter *does not flow across* the surface,
or
(ii) matter *flows across* the surface.

In the first case, discontinuities are called *tangential*, in the second they are called *shocks*.

4.2 Tangential Discontinuities

By the definition above, tangential discontinuities are those for which matter does not flow across the surface. This means that the first jump condition $[\rho v_1] = 0$ implies (being ρ_1 and ρ_2 both > 0) $v_{11} = v_{12} = 0$. This leads automatically to satisfy the third, fourth and fifth jump conditions in Eq. 4.5 $[\rho v_1 v_2] = 0$, $[\rho v_1 v_3] = 0$ and $\left[\rho\left(\frac{1}{2}v^2 + w\right)v_1\right] = 0$. This means that v_2, v_3, ρ and w may be discontinuous.

On the other hand, the second jump condition of Eq. 4.5, $\left[p + \rho v_1^2\right] = 0$, reduces to $[p] = 0$, which means that pressure is continuous.

When the tangential velocity components (v_2, v_3) are continuous (and ρ is discontinuous), we refer to it as *contact* discontinuity. For an ideal fluid, it can be shown that when, instead, v_2 and/or v_3 are discontinuous, the flow is unstable around the surface of discontinuity (*Kelvin–Helmholtz* (K-H) instability, Figs. 4.1 and 4.2). This instability results in a turbulent flow.

When, instead, the tangential discontinuity concerns a density discontinuity, as it happens when in the earth gravitational field a liquid like water is suspended over a less dense liquid, like oil, the *Rayleigh–Taylor* (R–T) instability appears (Fig. 4.3). Such instability is characteristic, also, of mushroom clouds like those on top of volcanoes or coming from a nuclear explosion.

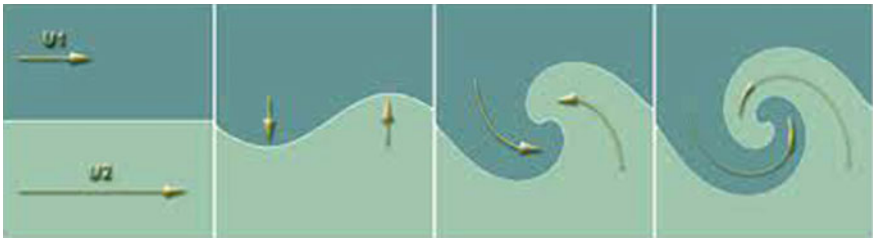


Fig. 4.1 Kelvin–Helmholtz instability development (from https://www.brockmann-consult.de/CloudStructures/images/kelvin-helmholtz-instab/k-w-system.gif)

Fig. 4.2 Kelvin–Helmholtz clouds

4.3 Shock Waves

The *Mach* number is defined as the ratio of the average speed of a fluid to the local sound velocity, so in the case of a perturbed and a still unperturbed region, two values of the Mach number are computed, $M_i = v_i/c_{si}$, $i = 1, 2$.

The introduction of the Mach number is of enormous importance in technical applications (especially in aeronautics and aerodynamics) because the flow of a gas is completely different in nature when it is *subsonic* ($M < 1$) or *supersonic* ($M > 1$). Flows characterized by $M > 5$ are considered *hypersonic*.

The sonic threshold is important because when the velocity of a moving fluid becomes comparable or greater than that of sound, the compressibility of the fluid starts to play a significant role. Practically speaking, these situations can occur in gases and not in liquids, so the proper context is that of *gas dynamics*.

In gas dynamics, the Reynolds number is always very large. This because the kinematic shear viscosity of a gas, $\tilde{\eta}$, is (from the kinetic theory of gases) of the order of the mean free path, λ, of the gas molecules times the average velocity of their thermal motion. This velocity is of the same order as the sound velocity, c_s, and consequently $\tilde{\eta} \sim \lambda c_s$. If the characteristic velocity of the gas dynamical problem in consideration, U, is also of the order of c_s, the Reynolds number results in $\mathrm{Re} = UD/\tilde{\eta} \sim D/\lambda \gg 1$. So, Re being very large, the actual role of viscosity is negligible so that, at all effects, the gas can be considered, as we will do in the following unless explicitly contrary stated, as an ideal fluid.

After these preliminary considerations, let us now analyze some aspects of shock waves in a gas.

When the matter flux through the surface is not zero, also v_{11} and v_{12} are not zero. Consequently the jump condition $[\rho v_1 v_2] = 0$ corresponds to $[v_2] = 0$, meaning that v_2 is continuous through the surface. By the same token, also v_3 is continuous.

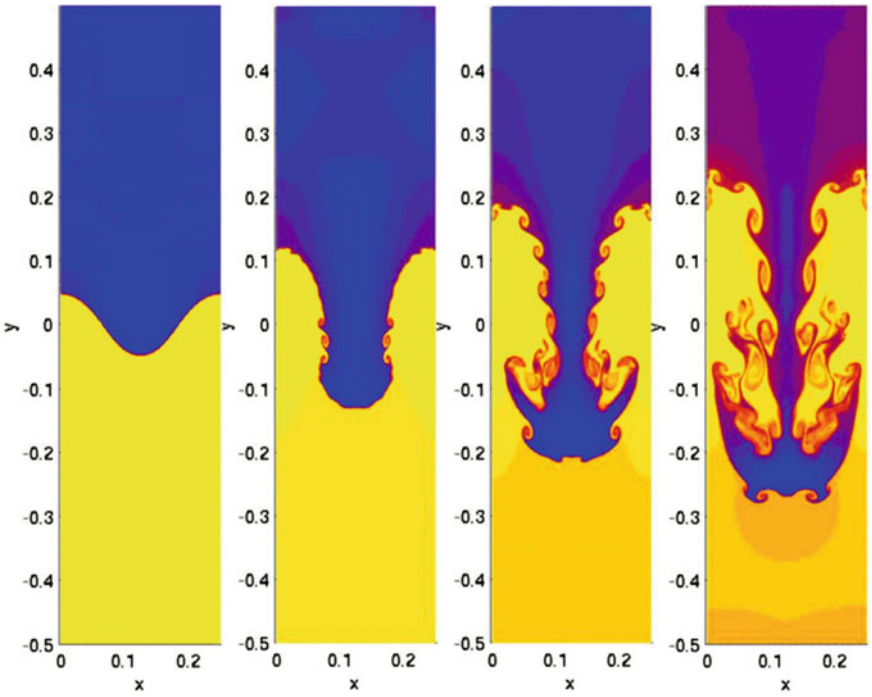


Fig. 4.3 Hydrodynamics simulation of a single 'finger' of the Rayleigh–Taylor instability. Note the formation of Kelvin–Helmholtz instabilities, in the second (from left) and later snapshots shown (starting initially around the level $y = 0$, as well as the formation of a 'mushroom cap' at a later stage in the third and fourth frame in the sequence (this information has been authored by an employee or employees of the Triad National Security, LLC, operator of the Los Alamos National Laboratory with the U.S. Department of Energy. The U.S. Government has the rights to use, reproduce, and distribute this information. The public may copy and use this information without charge, provided that this Notice and any statement of authorship are reproduced on all copies. Neither the Government nor Triad makes any warranty, express or implied, or assumes any liability or responsibility for the use of this information)

Because of the continuity of v_2 and v_3, the jump condition for the energy flux in the present case reduces to $\left[v_1^2/2 + e + p/\rho\right] = 0$. At the same time, v_1, e, p, ρ can be discontinuous. Such a discontinuity is usually called a *shock wave*, and we now go to deepen this topic here and in the following chapter.

When talking of shock waves, the jump conditions, named *Rankine–Hugoniot* conditions after the two scientists of the nineteenth century Pierre-Henry Hugoniot and William Rankine, are

$$\begin{cases} [\rho v_1] = 0 \Longleftrightarrow \rho_1 v_{11} = \rho_2 v_{12} \equiv j \neq 0, \\ \left[p + \rho v_1^2\right] = p_1 + \rho_1 v_{11}^2 - p_2 - \rho_2 v_{12}^2 = 0, \\ [v_2] = v_{21} - v_{22} = 0, \\ [v_3] = v_{31} - v_{32} = 0, \\ \left[\frac{1}{2} v_1^2 + e + \frac{p}{\rho}\right] = \frac{1}{2} v_{11}^2 + e_1 + \frac{p_1}{\rho_1} - \frac{1}{2} v_{12}^2 - e_2 - \frac{p_2}{\rho_2} = 0. \end{cases} \tag{4.6}$$

In this book, we call *shock front* the advancing edge of the *shock wave* behind it.

When the front is perpendicular to the shock's medium flow, the shock wave is named *normal*, otherwise is named *oblique*.

For simplicity, let us assume that matter travels through the front from region 2 (behind the shock, named also post-shock region) to region 1 (unperturbed region in front of the shock, named also pre-shock region) so as $j > 0$ in the first of Eq. 4.6, and choose coordinates such that the transversal velocity components (v_2 and v_3) are zero on both sides of the front (this is not a limitation because v_2 and v_3 are continuous across the front). Consequently, we can omit for simplicity the coordinate index in the velocity (i.e. put $v_1 = v_{11}$ and $v_2 = v_{12}$) to have, from the first of Eq. 4.6

$$v_1 = j V_1^* \quad \text{and } v_2 = j V_2^*, \tag{4.7}$$

where $V_1^* = 1/\rho_1$ and $V_2^* = 1/\rho_2$ are the specific volumes.

The above relations inserted in the second jump condition of Eq. 4.6 give

$$p_1 + j^2 V_1^* = p_2 + j^2 V_2^*, \tag{4.8}$$

equivalent to

$$j^2 = \frac{p_2 - p_1}{V_1^* - V_2^*}. \tag{4.9}$$

Due to $j^2 > 0$, the above relation is satisfied when $p_2 > p_1$ and $V_1^* > V_2^*$ (that is, $\rho_1 < \rho_2$) or when $p_2 < p_1$ and $V_1^* < V_2^*$ (that is, $\rho_1 > \rho_2$). Calling $\Delta p = p_2 - p_1$ and $\Delta\rho = \rho_2 - \rho_1$ means that Δp and $\Delta\rho$ must have the same sign.

The question is: can both cases (Δp and $\Delta\rho$ both positive or both negative) be realized in a shock wave or just one (positive *or* negative) and which of the two?

We will answer this question later (Sect. 4.3.3) as a corollary of what we will see in the following. Now, using Eq. 4.7 in the fifth jump condition of Eq. 4.6 results in

$$\frac{1}{2} j^2 V_1^{*2} + w_1 = \frac{1}{2} j^2 V_2^{*2} + w_2, \tag{4.10}$$

which, by means of Eq. 4.9, translates into

$$\frac{1}{2}\frac{p_2 - p_1}{V_1^* - V2^*}\left(V_1^{*2} - V_2^{*2}\right) + w_1 - w_2 = \\ = \frac{1}{2}\left(p_2 - p_1\right)\left(V_1^* + V_2^*\right) + w_1 - w_2 = 0. \tag{4.11}$$

The above equation can be rewritten accounting for the polytropic EOS, $p = (\gamma - 1)\rho e$, leading to

$$w = e + \frac{p}{\rho} = \frac{1}{\gamma - 1}\frac{p}{\rho} + \frac{p}{\rho} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}, \tag{4.12}$$

which (with proper indexing) transforms Eq. 4.11 into

$$\frac{V_2^*}{V_1^*} = \frac{\rho_1}{\rho_2} = \frac{\gamma + 1 + (\gamma - 1)\dfrac{p_2}{p_1}}{\gamma - 1 + (\gamma + 1)\dfrac{p_2}{p_1}}. \tag{4.13}$$

It is easy to see that

$$\lim_{(p_2/p_1)\to\infty} \frac{V_2^*}{V_1^*} = \lim_{(p_2/p_1)\to\infty} \frac{\rho_1}{\rho_2} = \frac{\gamma - 1}{\gamma + 1}, \tag{4.14}$$

so that $(\gamma - 1)/(\gamma + 1) < V_2^*/V_1^* = \rho_1/\rho_2 \leq (\gamma + 1)/(\gamma - 1)$, because V_2^*/V_1^* is a decreasing function of $p_2/p1$, as seen by its negative derivative

$$\frac{\mathrm{d}}{\mathrm{d}(p_2/p_1)}\frac{V_2^*}{V_1^*} = -\frac{4\gamma}{\left[\gamma - 1 + (\gamma + 1)\dfrac{p_2}{p_1}\right]^2} < 0, \tag{4.15}$$

so that its maximum is taken for $p_2/p_1 = 0$ and is equal to $(\gamma + 1)/(\gamma - 1)$.

Figure 4.4 shows the behavior of V_2^*/V_1^* vs p_2/p_1 for various values of $\gamma \geq 1$.

Assuming the equation of the state of ideal gases, the temperature ratio across the front is

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{V_2^*}{V_1^*}\frac{\mu_2}{\mu_1} = \frac{\mu_2}{\mu_1}\frac{p_2}{p_1}\frac{\gamma + 1 + (\gamma - 1)\dfrac{p_2}{p_1}}{\gamma - 1 + (\gamma + 1)\dfrac{p_2}{p_1}}, \tag{4.16}$$

immediately showing that its asymptotic (for $p_2/p_2 \gg 1$) behavior is

$$\frac{T_2}{T_1} \sim \frac{\mu_2}{\mu_1}\frac{\gamma - 1}{\gamma + 1}\frac{p_2}{p_1}, \tag{4.17}$$

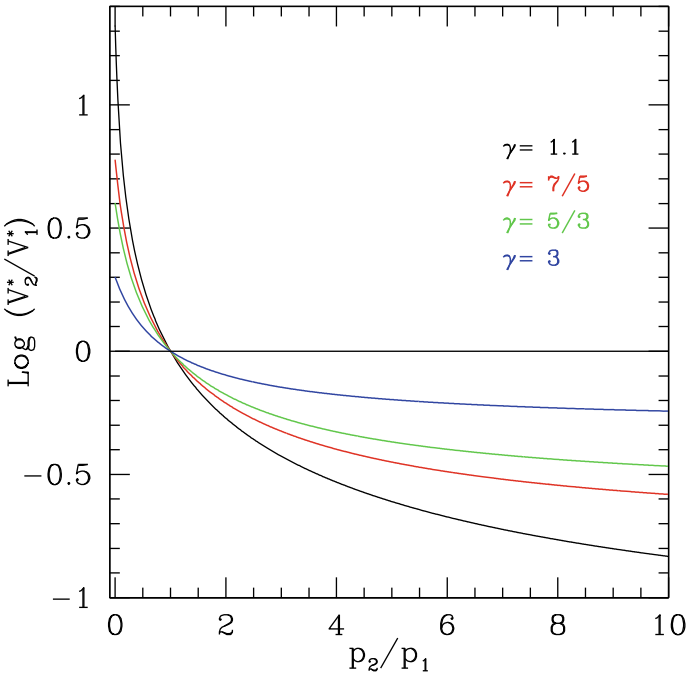


Fig. 4.4 Behavior of Log (V_2^*/V_1^*) versus p_2/p_1 for various values of γ as labeled. The horizontal line corresponds to $V_2^* = V_1^*$ $(\rho_1 = \rho_2)$

so that $\lim_{(p_2/p_1)\to\infty} (T_2/T_1) = +\infty$. This means that contrary to the density ratio, which we showed above to be limited, the temperature ratio is unbound (from this the term 'fire ball' for the region contained within the shock front).

4.3.1 Jumps in Terms of Mach Numbers

The expression of the various quantities as a function of the *Mach* number has high relevance for aerodynamic applications, because in aerodynamics the Mach number is used as a unit of measure for the object (an aircraft) in motion in the surrounding medium,[1] whose sound velocity can vary significantly due to temperature variation with the altitude.

Eliminating V_2^* into the denominator of Eq. 4.9 by means of Eq. 4.13, we get

[1] Of course, in this case the fluid velocity to compare to the sound velocity is that of the aircraft with respect to the fluid.

$$j^2 = \frac{p_1 - p_2}{V_1^* \left[\dfrac{(\gamma+1)p_1 + (\gamma-1)p_2}{(\gamma+1)p_2 + (\gamma-1)p_1} - 1 \right]} = \frac{(\gamma+1)p_2 + (\gamma-1)p_1}{2V_1^*}. \tag{4.18}$$

Remembering that $c_{s1}^2 = \gamma p_1/\rho_1$, the above expression allows the expression of v_1^2 in terms of c_{s1}^2 and p_2/p_1 as

$$v_1^2 = j^2 V_1^{*2} = \frac{V_1^*}{2}\left[(\gamma+1)p_2 + (\gamma-1)p_1\right] = \frac{1}{2}\frac{c_{s1}^2}{\gamma}\left[(\gamma+1)\frac{p_2}{p_1} + \gamma - 1\right]. \tag{4.19}$$

Analogous procedure, and assuming $\gamma = const.$ through the shock front, leads to

$$v_2^2 = \frac{1}{2}\frac{c_{s2}^2}{\gamma}\left[(\gamma+1)\frac{p_1}{p_2} + \gamma - 1\right]. \tag{4.20}$$

Recalling the definition of the Mach numbers in the shocked, 2, and unshocked, 1, regions, and using the two above relations results in

$$\begin{aligned} M_1^2 &= \frac{1}{2\gamma}\left[(\gamma+1)\frac{p_2}{p_1} + \gamma - 1\right], \\ M_2^2 &= \frac{1}{2\gamma}\left[(\gamma+1)\frac{p_1}{p_2} + \gamma - 1\right]. \end{aligned} \tag{4.21}$$

It is so possible writing M_2 in terms of M_1 and, by a simple exchange of indexes, the vice versa

$$\begin{aligned} M_2^2 &= \frac{2 + (\gamma-1)M_1^2}{2\gamma M_1^2 - (\gamma-1)}, \\ M_1^2 &= \frac{2 + (\gamma-1)M_2^2}{2\gamma M_2^2 - (\gamma-1)}. \end{aligned} \tag{4.22}$$

The above expressions and the assumption $\mu_2/\mu_1 = 1$ allow writing ρ_2/ρ_1, p_2/p_1 and T_2/T_1 as functions of M_1^2

$$\begin{aligned} \frac{\rho_2}{\rho_1} &= \frac{v_1}{v_2} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2}, \\ \frac{p_2}{p_1} &= \frac{2\gamma M_1^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}, \\ \frac{T_2}{T_1} &= \frac{\left[2\gamma M_1^2 - (\gamma-1)\right]\left[(\gamma-1)M_1^2 + 2\right]}{(\gamma+1)^2 M_1^2}, \end{aligned} \tag{4.23}$$

which in the limit of strong shock, $M_1^2 \gg 1$, become

$$\begin{aligned}
\frac{\rho_2}{\rho_1} &= \frac{v_1}{v_2} \sim \frac{\gamma+1}{\gamma-1}, \\
\frac{p_2}{p_1} &\sim \frac{2\gamma M_1^2}{\gamma+1} \to \infty, \\
\frac{T_2}{T_1} &\sim \frac{2\gamma(\gamma-1)M_1^2}{(\gamma+1)^2} \to \infty.
\end{aligned} \tag{4.24}$$

So, as we knew, in the shocked zone both p and T are unbound for every γ while the density contrast is limited.

4.3.2 Weak Shocks

In the limit of *weak* shock, the pressure ratio across the front is slightly greater than 1, so that defining $z \equiv (p_2 - p_1)/p_1 = p_2/p_1 - 1$ it is $z \ll 1$, and so, expressing M_1^2 in terms of p_2/p_1 by means of the first relation in Eq. 4.21 and eliminating p_2/p_1 by means of z

$$M_1^2 = \frac{\gamma+1}{2\gamma}(z+1) + \frac{\gamma-1}{2\gamma} = 1 + \frac{\gamma+1}{2\gamma}z, \tag{4.25}$$

which, at first order in z, gives

$$M_1 \simeq 1 + \frac{\gamma+1}{4\gamma}z. \tag{4.26}$$

A substitution of the above expression of M_1^2 in the density ratio in Eq. 4.23 yields, at second order in z,

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma+1)\left(1+\dfrac{\gamma+1}{2\gamma}z\right)}{(\gamma-1)\left(1+\dfrac{\gamma+1}{2\gamma}z\right)+2} \simeq 1 + \frac{z}{\gamma} - \frac{\gamma-1}{2\gamma}z^2. \tag{4.27}$$

4.3.3 Physical Meaning of Shock Waves

Equation 4.23 says that when $M_1 = 1$ it results in $\rho_2 = \rho_1$, $v_1 = v_2$, $p_2 = p_1$ and $T_2 = T_1$, so there are no discontinuities at all. Now, because

$$\frac{\mathrm{d}}{\mathrm{d}M_1}\left(\frac{v_2}{v_1}\right) = 2M_1\frac{\mathrm{d}}{\mathrm{d}M_1^2}\left(\frac{v_2}{v_1}\right) = -\frac{4(\gamma+1)M_1}{\left[(\gamma+1)M_1^2\right]^2} < 0, \tag{4.28}$$

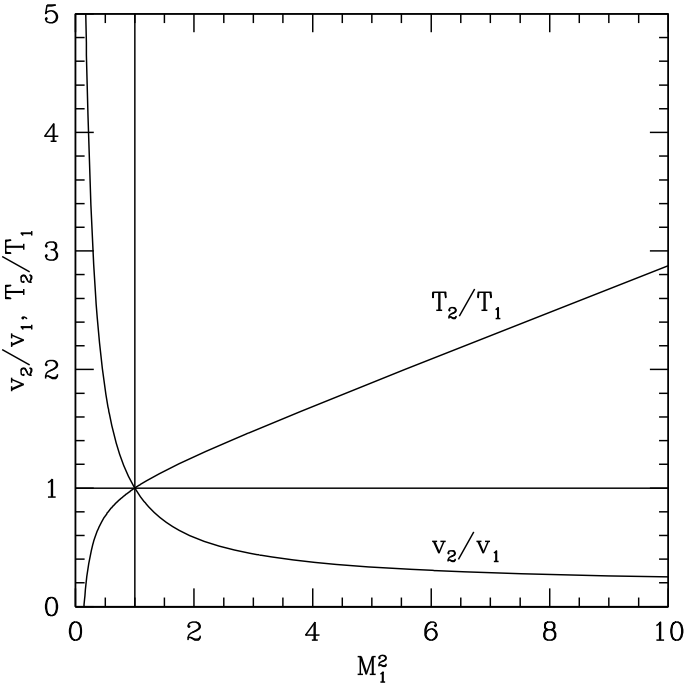


Fig. 4.5 The ratios v_2/v_1 and T_2/T_1 (as labeled) versus M_1^2 for $\gamma = 7/5$. The vertical line separates the subsonic ($M_1 < 1$) from the supersonic ($M_1 > 1$) region. The horizontal line corresponds to $v_2 = v_1$ and $T_2 = T_1$

the velocity ratio v_2/v_1 is a decreasing function of the Mach number M_1 (Fig. 4.5); consequently, being $v_2/v_1 = 1$ for $M_1 = 1$, it results in $v_2/v_1 > 1$ for $M_1 < 1$ and $v_2/v_1 < 1$ for $M_1 > 1$.

On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}M_1}\left(\frac{T_2}{T_1}\right) = \frac{4(\gamma-1)\left(\gamma M_1^4 + 1\right)}{(\gamma+1)^2 M_1^3}, \tag{4.29}$$

which is always > 0 because $\gamma > 1$. Being, again, $T_2 = T_1$ for $M_1 = 1$, the monotonically increasing dependence of T_2/T_1 on M_1 implies that $T_2/T_1 < 1$ for $M_1 < 1$ and $T_2/T_1 > 1$ for $M_1 > 1$ (Fig. 4.5).

Given all this, the case $M_1 < 1$ cannot give rise to a shock wave because it would imply that through the hypothetical shock surface unordered 'thermal' energy (measured by temperature) is transformed into 'ordered' kinetic energy (measured by velocity squared) violating the second principle of thermodynamics. This means that a physical shock can occur only in the supersonic, $M_1 > 1$, case.

The above considerations allows the answer to the question raised before in Sect. 4.3 about $\Delta p/\Delta\rho$ across the front: due to $\Delta\rho > 0$, also Δp is necessarily > 0.

4.4 Solved Exercises

Exercise 4.1 The Rankine–Hugoniot conditions (Eq. 4.6) can be synthesized as conservations across the shock boundary of mass, momentum and energy

$$\begin{cases} \rho v = j \neq 0, \\ p + \rho v^2 = \alpha \neq 0, \\ \dfrac{1}{2}v^2 + e + \dfrac{p}{\rho} = \beta \neq 0. \end{cases} \tag{4.30}$$

Questions
Assuming a polytropic EOS, upon definition of a proper reference velocity $\bar{v}$, and by using a dimensionless variable $\lambda = v/\bar{v}$:

1. derive as functions of λ the expressions for β, for the internal energy (per unit mass), e, and for the Mach number;
2. show that the quantity

$$\sigma = \frac{j^{\gamma-1}}{\bar{v}^{\gamma+1}} \frac{p}{\rho^{\gamma}} \tag{4.31}$$

is a dimensionless, monotonic (invertible) function of the specific entropy s, which can be put in the form $\sigma = \lambda^{\gamma}(1-\lambda)$.

Solution
1. The sound speed squared is $c_s^2 = \gamma p/\rho$, so that the momentum conservation yields the quadratic equation for v

$$\frac{c_s^2}{\gamma} + v^2 - \frac{\alpha}{\rho} = 0. \tag{4.32}$$

Now, the mass conservation implies $\rho = j/v$, so that $\alpha/\rho = v(\alpha/j)$; consequently, the quantity $\bar{v} = \alpha/j$ appears to be the proper reference velocity, implying that Eq. 4.32 can be written as

$$v^2 + \frac{c_s^2}{\gamma} - v\bar{v} = 0,$$

whose two roots represent the velocity on either side of the shock front. The above equation gives

$$c_s^2 = \gamma v(\bar{v} - v)$$

by which we eliminate c_s^2 in the third jump condition of Eq. 4.30 and write

$$\beta = v\left[v\left(\frac{1}{2} - \frac{\gamma}{\gamma-1}\right) + \frac{\gamma}{\gamma-1}\bar{v}\right],$$

having expressed the specific internal energy as

$$e = \frac{c_s^2}{\gamma(\gamma-1)} = \frac{1}{\gamma-1} v(\bar{v}-v).$$

The two expressions above are written in terms of the dimensionless velocity $\lambda = v/\bar{v}$

$$\beta = \bar{v}^2 \lambda \left[\lambda \left(\frac{1}{2} - \frac{\gamma}{\gamma-1} \right) + \frac{\gamma}{\gamma-1} \right],$$

and

$$e = \bar{v}^2 \frac{1}{\gamma-1} \lambda(1-\lambda).$$

Finally, the Mach number squared, $M^2 = v^2/c_s^2$, is

$$M^2 = \frac{\lambda}{\gamma(\lambda-1)}. \tag{4.33}$$

2. The specific entropy of an ideal gas is given by

$$s = a \ln \frac{p}{\rho^\gamma} + b,$$

where a and b are constants. The above relation implies

$$\frac{p}{\rho^\gamma} = e^{(s-b)/a},$$

which is a monotonic (increasing) function of s that we call s' and whose variations can be considered instead of those of s.

A dimensionless quantity keeping characteristics of specific entropy can be obtained. Actually, using the polytropic E.O.S.

$$s' = \frac{1}{\rho^{\gamma-1}} \frac{p}{\rho} = \frac{1}{\rho^{\gamma-1}} (\gamma-1) e,$$

which may be written in terms of λ as

$$s' = \frac{1}{\rho^{\gamma-1}} \bar{v}^2 \lambda(1-\lambda).$$

As a consequence, the quantity

$$\rho^{\gamma-1} \frac{s'}{\bar{v}^2} = \left(\frac{j}{v} \right)^{\gamma-1} \frac{s'}{\bar{v}^2} = \lambda(1-\lambda),$$

where we used mass conservation from Eq. 4.30 to eliminate ρ, is clearly non-dimensional and by simple multiplication of the rightmost two sides of the above relation by $(v/\bar{v})^{\gamma-1} = \lambda^{\gamma-1}$ we get σ as a non-dimensional and monotonic function of the original specific entropy s through s'

$$\sigma = \left(\frac{j}{\bar{v}}\right)^{\gamma-1} \frac{s'}{\bar{v}^2} = \lambda^{\gamma}(1-\lambda). \tag{4.34}$$

4.5 Further Readings

A good general reference book for discontinuities in fluid flows is [2]. More specifically devoted to gas dynamics discontinuities is [14].

Chapter 5
Blast Waves

In this chapter we study the properties of strong supersonic explosions leading to a blast wave propagation. The so-called Sedov–Taylor–von Neumann (S–T–vN) approach is followed, which consists in finding a formal or numerical approximation in a self-similar schematization of the problem. Once the spherical propagation front time dependence has been found via a dimensional reasoning, the values of the internal quantities are determined by the solution of a system of ODEs. Application to the determination of the unknown amount of energy released by a bomb or a Supernova (SN) follows.

5.1 Propagation of a Blast Wave

A sudden release of energy, ΔE, in a small volume over a small time, Δt, corresponds to an explosion. We can call it explosion whenever $\Delta E/\Delta t$, which is a power, is very large. The explosion often produces a quick and strong compression front which propagates in the environment at supersonic speed. This compression front is called *blast wave* and its moving surface, which is expected to be spherical if the environment is homogenous, actually constitutes a discontinuity surface across which some of the fluid variables experience jumps across it. The Rankine–Hugoniot jump conditions discussed in the previous chapter define the characteristics of the discontinuities of the fluid variables across the shock front.

The physical sources of explosions are of different types. Talking of explosions on earth, it is useful to distinguish between *deflagration* and *detonation*. A deflagration is an explosion which causes a fireball to move at subsonic speed, while a detonation induces a supersonic fireball and, so, an actual shock wave. Deflagrations are sort of non-destructive, controlled, explosions where the fire release occurs on a time scale

R. A. Capuzzo Dolcetta, *Physics of Fluids*, UNITEXT for Physics,
https://doi.org/10.1007/978-3-031-30750-8_5

which is not too short, causing a modest overpressure ($\lesssim 0.5$ bar). Typical examples of deflagrations are those obtained by fossil fuel combustion in an engine like those used to power vehicles. This control of deflagrations makes them useful tools to convert multiple explosions into ordered work.

On the other hand, detonations (from Latin *detonare* = 'to thunder down') are stronger explosions, able to produce a flame front at supersonic speed. A detonation, so, corresponds to a $(\Delta E/\Delta t)_{det} \gg (\Delta E/\Delta t)_{def}$, where the subscript refers to detonation and deflagration. A detonation can be obtained by a chemical reaction in which usually a decomposition reaction is followed by recombination: unstable molecules dissociate, and the resulting constituents form rapidly new molecules releasing a high quantity of energy. Dynamite is a typical example of detonating chemical product. The practical use of detonations is, on a civic side, that of destroying rapidly things that would, otherwise, require a lot of human work and resources. For example, the construction of a road in a rocky environment is highly facilitated by controlled explosions to destroy big rocks. Detonations are also used to cause the collapse of old buildings, often via a set of combined explosions. Due to their destructive power, unfortunately, detonations have found heavy military applications, from classic grenades up to nuclear bombs. In the astronomical context, the most common example is the explosion of a Supernova. A Supernova is a massive star that at the end of its life as a nuclear burner has an inner core of ^{56}Fe, which is the most stable nuclide in nature, defining the minimum of the binding energy per nucleon curve. At that time nuclear reactions stop, so that gravity overwhelms the reduced pressure gradient, causing a sudden instability with the catastrophic consequence of a gigantic explosion involving energies up to 10^{53} erg emitted in various forms (kinetic, neutrinos, radiation) and different fractions.

Let us now approach quantitatively the problem of the blast wave caused by a detonation. Suppose that an amount of energy E is suddenly released at time $t = 0$ in a generic point of an adiabatic environment with adiabatic index γ and characterized by uniform values of density and pressure, referred to as ρ_1 and p_1, respectively. In the further hypothesis that the environment temperature is very low, $T_1 \simeq 0$, we have from the ideal gas EOS that the sound speed

$$c_{s1} = \sqrt{\gamma \frac{p_1}{\rho_1}} = \sqrt{\gamma \frac{kT_1}{\mu_1 m_H}} \tag{5.1}$$

is very small, too, and this guarantees that the flow perturbation caused by the explosion is supersonic ($M_1 > 1$) or even hypersonic ($M_1 \gg 1$). In the latter case, the shock is in the *strong* regime; the hypothesis of strong shock will be used heavily in the following.

Now, following the way proposed in the middle of the twentieth century by L. I. Sedov, G. I. Taylor and J. von Neumann (see the historical note), we make use of dimensional considerations to deduce an expression for the time dependence of the spherical blast wave front radius, $R_S(t)$; the assumption of a spherical front is a straight consequence of the localized point-like explosion in a uniform environ-

ment. The front expansion either than on time should depend also upon E, p_1, ρ_1 and γ as parameters, where p_1 and ρ_1 are linked by the adiabatic EOS so as to be interdependent. A quantity with the requested dimensions of a length can be built in the form of powers of t, E and ρ_1 as characterizing dimensional parameters, letting

$$R_S = k_\gamma E^\alpha \rho_1^\beta t^\delta, \tag{5.2}$$

where k_γ is a dimensionless quantity dependent on γ whose determination will be discussed later, and α, β and δ are exponents to determine by the condition that the resulting power law product has the dimension of length.

Given that $[E] = \mathrm{ML}^2\mathrm{T}^{-2}$, $[\rho_1] = \mathrm{ML}^{-3}$, $[t] = \mathrm{T}$, the dimensional equation deriving from Eq. 5.2 is

$$\mathrm{L} = \mathrm{M}^{\alpha+\beta}\mathrm{L}^{2\alpha-3\beta}\mathrm{T}^{-2\alpha+\delta}, \tag{5.3}$$

equivalent to the system

$$\begin{cases} \alpha + \beta = 0, \\ 2\alpha - 3\beta = 1, \\ -2\alpha + \delta = 0, \end{cases} \tag{5.4}$$

whose unique solution is $\alpha = -\beta = 1/5, \delta = 2\alpha = 2/5$. Consequently, we can write

$$R_S(t) = k_\gamma E^{1/5} \rho_1^{-1/5} t^{2/5} = k_\gamma \left(\frac{E t^2}{\rho_1}\right)^{1/5}, \tag{5.5}$$

whose time derivative

$$v_S \equiv \dot{R}_S = \frac{2}{5} k_\gamma \left(\frac{E}{\rho_1}\right)^{1/5} t^{-3/5} = \frac{2}{5}\frac{R_S}{t}, \tag{5.6}$$

can be taken as the shock front speed, which diverges as $t^{-3/5}$ for $t \to 0$ and goes to zero for $t \to \infty$. Figure 5.1 gives an idea of the scaling with the time of the shock front size expansion, while Fig. 5.2 shows snapshots of the Trinity bomb exploded on July 16, 1945 as a test in the frame of the Manhattan project. Figure 5.3 indicates how good is Eq. 5.5 in representing the Trinity bomb blast front expansion.

Anyway, there are two notes about the validity of the obtained front expansion in Eqs. 5.5 and 5.6. First, Eq. 5.6 leads to an intrinsic lower limiting time, $t_{min} = [2k_\gamma/(5c)]^{5/3}(E/\rho_1)^{1/3}$ where c is the speed of light in vacuum, such that for $t < t_{min}$ the shock front expansion would be superluminal ($\dot{R}_S > c$). The second point to note is that the front expansion Eq. 5.5 is decelerated, being

$$\ddot{R}_S(t) = -\frac{6}{25} k_\gamma \left(\frac{E}{\rho_1}\right)^{1/5} t^{-8/5} < 0, \tag{5.7}$$

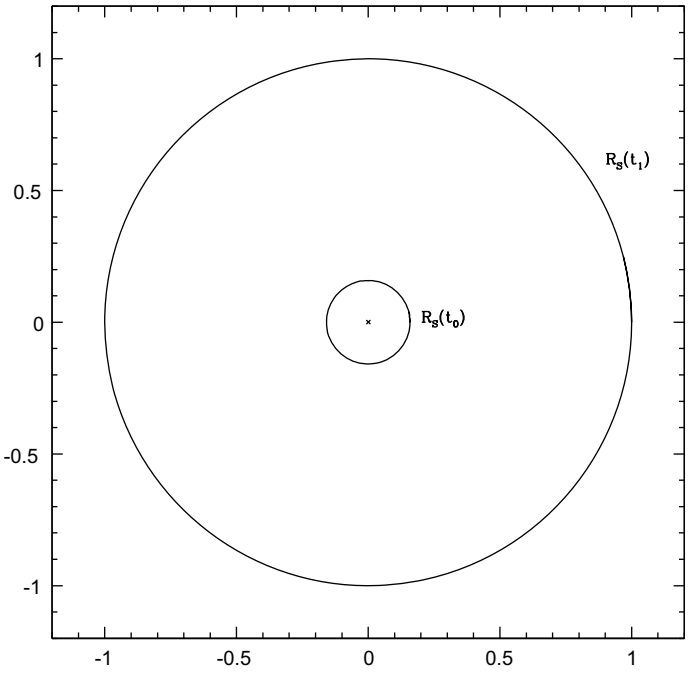


Fig. 5.1 Radii (on scale) of the shock fronts at times t_0 and $t_1 = 100t_0$

that does not fitting the expected free expansion shortly after the explosion.

As we know from Chap. 4, the, likely, assumption of strong shock ($M_1 \gg 1$) leads to the density jump across the discontinuity surface (with the usual convention of labeling with 1 the pre-shock and with 2 the post-shock regions, respectively)

$$\rho_2 = \frac{\gamma + 1}{\gamma - 1}\rho_1. \tag{5.8}$$

The jump condition $[\rho v] = 0$ in the case of mass transfer across the discontinuity surface corresponds to $v_2/v_1 = \rho_1/\rho_2$ in a reference frame fixed on the shock front, and so, calling $\bar{v}_1$ and $\bar{v}_2$ the velocities with respect to a fixed frame (absolute velocities) where the front moves at speed v_S

$$\bar{v}_2 - v_S = \frac{\gamma - 1}{\gamma + 1}(\bar{v}_1 - v_S). \tag{5.9}$$

For simplicity of notation, hereafter we will omit the bar over the v symbol to refer to the absolute velocity. Over a wide time interval, it is $v_1 \ll v_S$ so that Eq. 5.9 approximates to

$$v_2 \simeq \frac{2}{\gamma + 1}v_S, \tag{5.10}$$

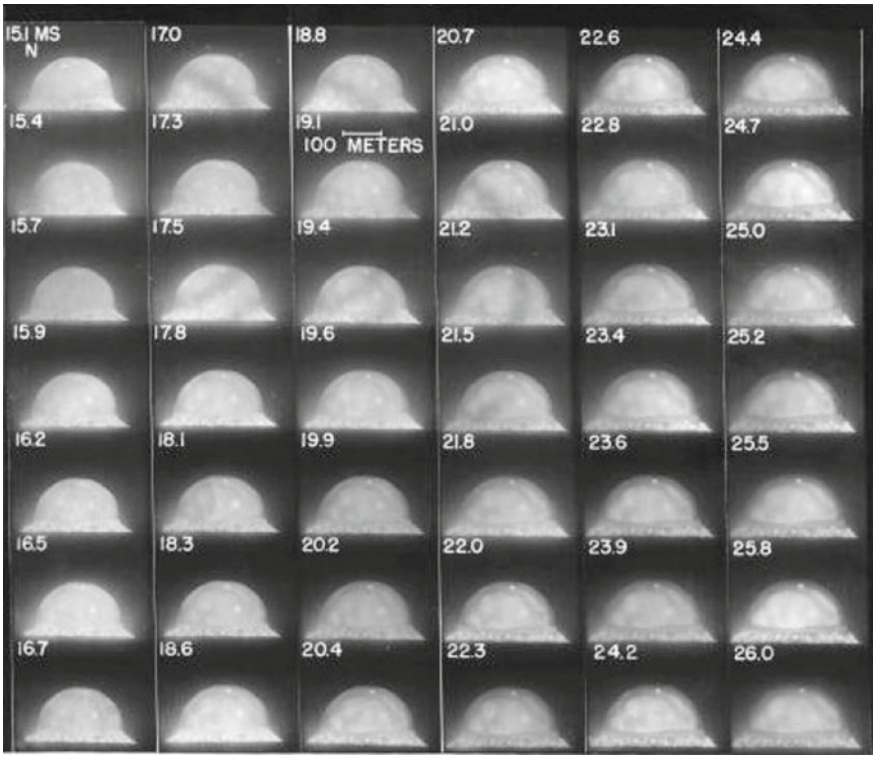


Fig. 5.2 Snapshots of the Trinity bomb explosion taken at different times (as labeled)

showing that in the case of strong shock the flow speed behind the front (v_2) depends only on the front speed (and on γ) but not on v_1. Equation 5.10 implies the limitation $v_2/v_S \leq 1$, because $\gamma \geq 1$. For pressure, the second relation in Eq. 4.23 gives in the hypersonic regime

$$p_2 \simeq \frac{2\gamma}{\gamma+1} M_1^2 p_1, \tag{5.11}$$

which, because $M_1^2 = (v_1/c_{s1})^2$ and eliminating c_{s1}^2 by Eq. 5.1 in the regime $v_1 \ll v_S$, approximates to

$$p_2 \simeq \frac{2\rho_1 v_S^2}{\gamma+1}, \tag{5.12}$$

which states a dependence, in the shocked region, of pressure on the scale pressure $\rho_1 v_S^2$, without dependence on the actual pressure, p_1, of the unperturbed fluid. Being $\gamma \geq 1$, the obtained limitation is $0 < p_2 \leq \rho_1 v_S^2$.

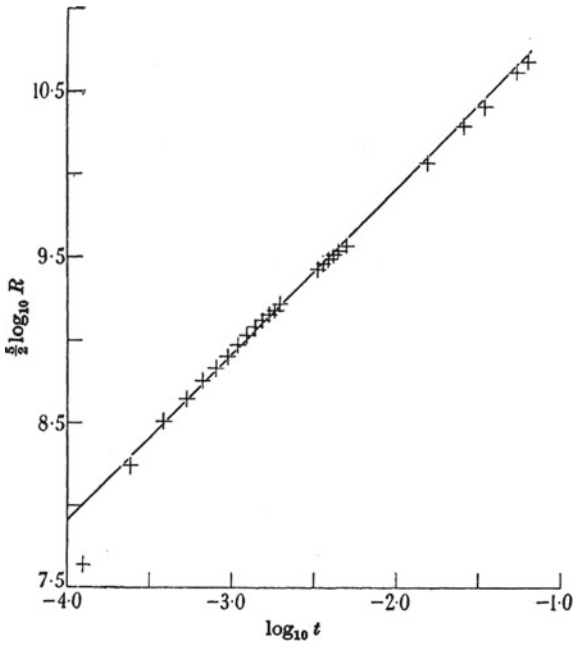


Fig. 5.3 This plot (from Taylor, Proc. of the Roy. Soc. of London, Ser. A, Math. and Phys. Sciences, vol. 201, issue 1065, 1950) shows the excellent agreement of data of the Trinity blast front expansion (crosses) with the theoretical expression of $R_S(t)$ of Eq. 5.5

5.2 A Similarity Solution

In some cases it is possible, and worth, looking for families of solutions of a physical problem which have the property to be *similar*.

Of course, the concept of *similarity* is not as uniquely defined as that of *equality*, and it must be adapted to the category of cases under study.

For example, in Euclidean geometry two figures are similar if one can be obtained by the other by simple size scaling together with, if needed, additional translation, rotation and/or reflection. Given this definition, all circles are similar to each other, so they are *self-similar* figures. On the other hand, neither ellipses nor rectangles nor isosceles triangles are self-similar. In the case of differential equations, like those of fluid dynamics, similar or self-similar solutions could be considered those solutions which can be obtained one from the other by a simple scaling given by a factor.

The procedure to get these solutions usually passes through the introduction of a proper dimensionless variable which resumes both the space and time dependence. The introduction of such a variable allows, usually, the transformation of the original set of partial differential equations into a set of ordinary differential equations, much easier to solve numerically.

In our specific case, the idea behind the whole procedure is to seek the dependence of the fluid variables ρ, v, p behind the shock front, called $\rho_2(r;t)$, $v_2(r;t)$, $p_2(r;t)$ for $r \leq R_s(t)$, via scaling of the discontinuities with dimensionless functions $R(r;t)$, $P(r;t)$ and $V(r;t)$:

$$\begin{aligned}\rho_2(r;t) &= \frac{\gamma+1}{\gamma-1}\rho_1 R(r;t),\\ v_2(r;t) &= \frac{2v_S}{\gamma+1}V(r;t),\\ p_2(r;t) &= \frac{2\rho_1 v_S^2}{\gamma+1}P(r;t),\end{aligned} \tag{5.13}$$

having taken into account Eqs. 5.8, 5.10 and 5.12.

The sought dimensionless abscissa variable to use for seeking self-similar solutions can be constructed in the usual dimensional way using the four parameters available, ρ_1, E, t and r, looking for exponents α, β, δ and ϵ such that the combination

$$\xi = E^{\alpha}\rho_1^{\beta}t^{\delta}r^{\epsilon} \tag{5.14}$$

is dimensionless. The resulting system of three equations in the four unknowns has, of course, ∞^1 solutions, which (taking ϵ as parameter) are

$$\begin{cases}\alpha = -\dfrac{\epsilon}{5},\\ \beta = \dfrac{\epsilon}{5},\\ \delta = -\dfrac{2}{5}\epsilon,\end{cases} \tag{5.15}$$

which, with the choice $\epsilon = 1$, leads to $\xi = \Gamma_\gamma E^{-1/5}\rho_1^{1/5}t^{-2/5}r$ as non-dimensional variable, with Γ_γ a numerical constant depending on γ. This expression is indeed equivalent to $\xi = r/R_S(t)$ when letting, without loss of generality, $\Gamma_\gamma = 1/k_\gamma$. This dimensionless variable $\xi(r;t)$ allows rewriting the Eq. 5.13 in the form

$$\begin{aligned}\rho_2(r;t) &= \frac{\gamma+1}{\gamma-1}\rho_1 R(\xi),\\ v_2(r;t) &= \frac{2v_S}{\gamma+1}V(\xi),\\ p_2(r;t) &= \frac{2\rho_1 v_S^2}{\gamma+1}P(\xi),\end{aligned} \tag{5.16}$$

where the unknown dimensionless functions R, V and P must satisfy the boundary conditions $R(1) = V(1) = P(1) = 1$.

The system of fluid dynamical equations which R, V and P must satisfy is, in the ideal case and exploiting spherical symmetry ($v = v_r$),

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 \rho v_r) & = 0 \\ \dfrac{\partial v_r}{\partial t} + v_r \dfrac{\partial v_r}{\partial r} & = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} \\ \dfrac{\partial s}{\partial t} + v_r \dfrac{\partial s}{\partial r} & = 0. \end{cases} \tag{5.17}$$

Taking into account that for a reversible polytropic transformation (i.e. isoentropic), the specific entropy can be written as

$$s = a \ln \frac{p}{\rho_\gamma}, \tag{5.18}$$

where a is a constant, the entropy conservation (last of Eq. 5.17) is written as

$$a \left(\frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} \right) \ln \frac{p}{\rho^\gamma} = 0. \tag{5.19}$$

After substitution of the above equation and of relations Eq. 5.16 into the set Eq. 5.17, it turns into a system of first order ODEs

$$\begin{cases} \left(\dfrac{2}{\gamma+1} V - \xi \right) R' + \dfrac{2}{\gamma+1} R V' & = -\dfrac{4}{\gamma+1}\dfrac{RV}{\xi} \\ \left(\dfrac{4}{\gamma+1} V - 2\xi \right) V' + 2\dfrac{\gamma-1}{\gamma+1}\dfrac{P'}{R} & = 3V \\ \gamma \left(\xi - \dfrac{2}{\gamma+1} V \right) \dfrac{R'}{R} + \left(\dfrac{2}{\gamma+1} V - \xi \right) \dfrac{P'}{P} & = 3, \end{cases} \tag{5.20}$$

where the apex, $'$, denotes differentiation with respect to ξ.

Although the system Eq. 5.20 subjected to the conditions $R(1) = V(1) = P(1) = 1$ admits solutions in closed form, a numerical approach provides a more general treatment.

The first step in the direction of finding a numerical solution is the transformation of the system Eq. 5.20 in *normal* form, which means a form where all the first order derivatives of the unknown, $R'(\xi)$, $V'(\xi)$, and $P'(\xi)$, are explicated in terms of the independent variable ξ and of the (non-differentiated) unknowns R, V and P. This can be done by solving the linear system

$$\mathbf{A}\mathbf{x}' = \mathbf{b}, \tag{5.21}$$

where $\mathbf{x}'$ is the column vector of the unknown derivatives having components

$$\begin{aligned} x_1' &= R', \\ x_2' &= V', \\ x_3' &= P', \end{aligned} \tag{5.22}$$

while $\mathbf{b}$ is the column vector whose components are

$$\begin{aligned} b_1 &= -\frac{4}{\gamma+1}\frac{RV}{\xi}, \\ b_2 &= 3V, \\ b_3 &= 3, \end{aligned} \tag{5.23}$$

and $\mathbf{A}$ is the 3×3 matrix

$$\begin{pmatrix} \dfrac{2V}{\gamma+1} - \xi & \dfrac{2R}{\gamma+1} & 0 \\ 0 & \dfrac{4V}{\gamma+1} - 2\xi & \dfrac{\gamma-1}{\gamma+1}\dfrac{2}{R} \\ \left(\xi - \dfrac{2V}{\gamma+1}\right)\dfrac{\gamma}{R} & 0 & \left(\dfrac{2}{\gamma+1}V - \xi\right)\dfrac{1}{P} \end{pmatrix}. \tag{5.24}$$

Applying the rule of Sarrus, the determinant of $\mathbf{A}$ is

$$\det(\mathbf{A}) = 2\left(\frac{2V}{\gamma+1} - \xi\right)\left[\left(\frac{2V}{\gamma+1} - \xi\right)\left(\frac{2}{\gamma+1}V - \xi\right)\frac{1}{P} - \gamma\frac{\gamma-1}{(\gamma+1)^2}\frac{2}{R}\right]. \tag{5.25}$$

The Rouché–Capelli theorem guarantees a solution which can be found, for instance, applying Cramer's rule, that is

$$x_i' = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}, \quad i = 1, 2, 3 \tag{5.26}$$

where $\mathbf{A}_i$ is the matrix formed by replacing the ith column of A by the column vector $\mathbf{b}$.

As a result, we find the normal form for our system of first order ODEs

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}; \xi), \tag{5.27}$$

where $\mathbf{F}$ is a vector function depending upon $\mathbf{x}$ and ξ.

Another way to put the system Eq. 5.20 of ODEs in normal form is via successive substitutions. We follow this way in Exercise 5.1.

The system Eq. 5.27 subjected to initial (at $\xi = 1$) conditions $x_1(1) \equiv R(1) = 1, x_2(1) \equiv V(1) = 1, x_3(1) \equiv P(1) = 1$ may be now numerically *backward* integrated with a standard numerical algorithm for systems of first order ODEs.

Figure 5.4 shows solutions for $\gamma = 7/5$.

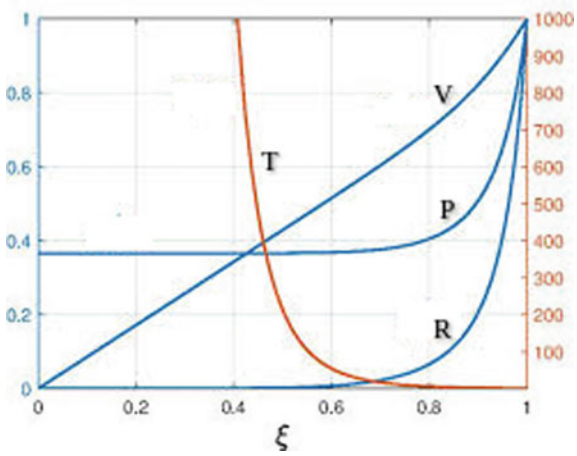


Fig. 5.4 Behaviors of R, P, V (see left ordinate scale). In this plot T is the ratio of the internal temperature and the one at boundary (right ordinate scale). Plot adapted from commons:wikimedia.org/wiki/File:TNS_blast_wave.pdf

5.2.1 Determination of the Dimensionless Constant k_γ

Once the self-similar solutions of the fluid-dynamical equations in the shocked gas have been found, it remains to determine the dimensionless constant k_γ in Eq. 5.5 which eventually solves completely the problem, allowing also to calculate the released energy E once γ is known.

The way to get k_γ is via imposing energy conservation. This is done by letting E equal to the integral which gives, at every time t, the total energy from the kinetic + internal energy density

$$E = \int_0^{R_S} \rho \left(\frac{1}{2} v^2 + e \right) 4\pi r^2 \, \mathrm{d}r. \tag{5.28}$$

Given the adiabatic EOS, $p = (\gamma - 1)\rho e$, and the relations Eqs. 5.13, 5.28 transforms into

$$E = \frac{8\pi v_S^2 \rho_1 R_S^3}{(\gamma + 1)(\gamma - 1)} \int_0^1 \left[R(\xi) V^2(\xi) + P(\xi) \right] \xi^2 \, \mathrm{d}\xi, \tag{5.29}$$

which, by elimination of v_S with Eq. 5.6, leads to

$$E = \frac{32\pi}{25(\gamma^2 - 1)} \frac{\rho_1 R_S^5}{t^2} \int_0^1 \left[R(\xi) V^2(\xi) + P(\xi) \right] \xi^2 \, \mathrm{d}\xi. \tag{5.30}$$

Letting I_γ the integral in the above relation and imposing $E = (R_S/k_\gamma)^5(\rho_1/t^2)$, as given by Eq. 5.5, the constant k_γ is finally given by

Table 5.1 Values of k_γ for a selected set of γ

γ	k_γ
$5/3 \simeq 1.6667$	1.15167
$3/2 = 1.5000$	1.08231
$7/5 = 1.4000$	1.03278
$4/3 \simeq 1.3333$	0.99467
$9/7 \simeq 1.2857$	0.96392
$5/4 = 1.2500$	0.93827
$11/9 \simeq 1.2222$	0.91634
$6/5 = 1.2$	0.89723
$13/11 \simeq 1.1818$	0.88035
$7/6 \simeq 1.1667$	0.86524
$15/13 \simeq 1.1538$	0.85160
$152/150 \simeq 1.0133$	0.52434
$202/200 \simeq 1.01$	0.49512

$$k_\gamma = \left[\frac{25(\gamma^2 - 1)}{32\pi I_\gamma}\right]^{1/5}. \tag{5.31}$$

The coefficient k_γ depends upon γ both explicitly (numerator) and implicitly (denominator) via the integral I_γ and can be obtained once the solutions $R(\xi)$, $V(\xi)$ and $P(\xi)$, which all depend on γ, have been obtained. As expected on the elementary basis, the values of k_γ are around 1 for reasonable values of γ, as shown in Table 5.1, which shows an increasing trend of k_γ with γ. The expansion of the shock front is favored by larger values of the exponent of the polytropic EOS. Moreover, $k_\gamma \to 0$ for $\gamma \to 1$ as suggested by Eq. 5.30 since I_γ is strictly positive and limited due to the flat behavior of $P(\xi)$ in the neighborhood of $\xi = 0$ (see Fig. 5.4). This means that the explosion front remains *frozen* in the place of the explosion in an isothermal ($\gamma = 1$) environment. Numerically, it is found that the decrease to zero of k_γ is quite steep being (see Table 5.1) $k_\gamma < 0.5$ only for $\gamma \lessapprox 1.01$.

On the other hand, for $\gamma \gg 1$ it is $k_\gamma \sim \gamma^{2/5}$, because I_γ is limited for every γ.

5.3 Supernova Explosions

The Sedov–Taylor–von Neumann solution finds an interesting application in a particular phase following the Supernova blast wave caused by the explosion occurring in the central region of some types of stars.

Stars are self-gravitating gaseous structures deriving by gravitational instability of a diffuse cloud which starts a collapse until an almost stationary *core* forms that

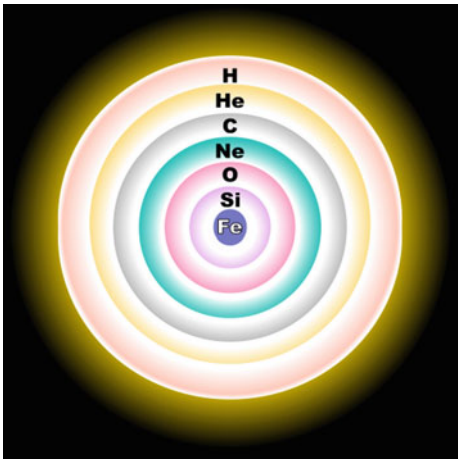


Fig. 5.5 The internal structure of a massive ($M > 8\ M_\odot$) pre-SN star (from Ursus & Hall, CC BY 2.5, https://commons.wikimedia.org/w/index.php?curid=2565862)

continues to accrete its mass by infalling matter so that its inner part becomes so dense and hot that nuclear fusions begin and produce sufficient energy to support the structure against its self-gravitation, reaching hydrostatic equilibrium. This equilibrium slowly evolves through successive types of nuclear burnings, from lighter to heavier elements formed in the inner star region. The speed of these nuclear burning phases depends mostly on the star mass: the larger the mass of the star, the quicker the various burning phases and the higher the rate of energy (luminosity) emitted by the star.

The so-called *core collapse* SN are those stars whose mass is so large that the sequence of nuclear burnings continues until an inner region composed by Iron 56 (Fe^{56}) is produced (see Fig. 5.5).

Considering the (negative) energy of a given nucleus composed of a certain number of components (nucleons),[1] it is seen that its relation with the number of nucleons in the nucleus has indeed its minimum for Fe^{56}. This means that lighter elements than Fe^{56}, when they merge, give a fusion product with lower energy than the sum of reactants, corresponding to an emission of the energy excess: these are *exothermic* fusion reactions. On the contrary, on the right side of the energy-number of nucleons curve, the energy increases, meaning that only fission nuclear reactions are exothermic while fusions are endothermic. As a consequence, the Fe^{56} stellar core steadily grows in mass until it reaches a value of mass (called Chandrasekhar mass, $\simeq 1.4\ M_\odot$) when equilibrium is broken and a sudden collapse occurs which can give rise to the SN explosion and subsequent blast wave.

The blast wave corresponds to a shock front which expands supersonically in the surrounding, tenuous, interstellar medium; its expansion and internal characteristics can be represented over a limited amount of time by the previously discussed S–T–vN solution. This phase (S–T–vN or adiabatic, energy-conserving, phase) starts about

[1] Note that the *binding energy* usually referred to in the literature as the amount of energy to give to a specific nucleus to disperse all its elements (nucleons) is the absolute value of the nucleus energy at rest.

200 yr after the explosion and is characterized by the expansion of the radius of the shock according to the S–T–vN law $R_S \propto t^{2/5}$, which follows the first phase of free (no deceleration) $R_S \propto t$ expansion. The S–T–vN phase ends about 3×10^4 yr after the explosion, when energy dissipation of the expanding remnant, essentially due to radiative processes, begins. In this case, there is a very dense and thin shell which radiates efficiently the energy of the shock in the surrounding medium. This third phase, lasting about 2×10^5 yr, is called *snow plough* phase and $R_S \propto t^{1/4}$. Physically speaking, while the S–T–vN phase corresponds to a constant energy phase, this third phase corresponds to an almost constant momentum stage. The fourth, and final, phase is that of remnant dilution into the interstellar medium, i.e. $R_S \propto t^0 = const.$.

5.4 Solved Exercises

Exercise 5.1 Reduce the system of ODEs (5.20) in the normal form via successive substitutions.

Solution
The system (5.20) can be easily converted into the semi-explicit form

$$\begin{cases} \dfrac{R'}{R} = -\dfrac{2}{\gamma+1}\dfrac{\dfrac{V'}{V}+\dfrac{2}{\xi}}{\dfrac{2}{\gamma+1}-\dfrac{\xi}{V}} \\ \dfrac{V'}{V} = \dfrac{3-2\dfrac{\gamma-1}{\gamma+1}\dfrac{P'}{RV}}{\dfrac{4}{\gamma+1}V-2\xi} \\ \dfrac{P'}{P} = \dfrac{3}{\dfrac{2}{\gamma+1}V-\xi}+\gamma\dfrac{R'}{R}. \end{cases} \tag{5.32}$$

It is now possible, for instance, to express V'/V (second equation in the system Eq. 5.32) eliminating P' with the third relation where, in its turn, the first has been used to express R'/R in terms of V, V' and ξ. Finally, we have

$$\frac{V'}{V} = \frac{3\left(\dfrac{2}{\gamma+1}V-\xi\right)+2\dfrac{\gamma-1}{\gamma+1}\left(-3+4\dfrac{\gamma}{\gamma+1}\dfrac{V}{\xi}\right)\dfrac{P}{RV}}{2\left(\dfrac{2}{\gamma+1}V-\xi\right)^2-4\gamma\dfrac{\gamma-1}{(\gamma+1)^2}\dfrac{P}{R}} \equiv f_\gamma(R,P,V;\xi). \tag{5.33}$$

The above expression, inserted into the first in Eq. 5.32, leads to

$$\frac{R'}{R} = -\frac{2}{\gamma+1}\frac{f_\gamma + \dfrac{2}{\xi}}{\dfrac{2}{\gamma+1} - \dfrac{\xi}{V}} \equiv g_\gamma(R, P, V; \xi), \tag{5.34}$$

and its final insertion in the third of Eq. 5.32 leads to the last searched equation

$$\frac{P'}{P} = \frac{3}{\dfrac{2}{\gamma+1}V - \xi} + \gamma g_\gamma, \equiv h_\gamma(R, P, V; \xi). \tag{5.35}$$

Exercise 5.2 The Crab nebula is the remnant of a Supernova whose explosion is dated 1054 as from ancient Chinese observations.

Assuming as observational data its present radius ~ 5.5 ly (~ 1.7 pc) and density of the interstellar medium, $\rho_1 \sim 10^{-24}$ g cm^{-3}, give an estimate of the energy, E, released by the SN explosion.

Solution

From what is said in Sect. 5.3 of this chapter, the Crab nebula is likely, at present, in the S–T–vN phase after 968 yr from the SN explosion. Adopting the R_S versus time relation in Eq. 5.5, where for simplicity we put $k_\gamma = 1$, and using for the present radius and environmental density the input data above, a rough estimate of E is $E \simeq 4.09 \times 10^{46}$ erg, which is an energy about 9.45×10^{25} times that released by the Trinity atomic bomb! (see the following historical note).

Taking into account that the maximum energy that can be extracted by a mass $m = \alpha\ \mathrm{M}_\odot$ is the rest mass energy $E_m = mc^2 = \alpha 1.788 \times 10^{54}$ erg where c is the speed of light in vacuum, and $\alpha \geq 1.4$ (the Chandrasekhar mass is $\sim 1.4\ \mathrm{M}_\odot$), the previous evaluation of E corresponds to a rest mass energy conversion efficiency $E/E_m = (2.29 \times 10^{-8})/\alpha \leq 1.64 \times 10^{-8}$.

Historical Note

The story behind the work independently done by J. von Neumann in the U.S.A., G. I. Taylor in the UK and L. I. Sedov in the U.S.S.R. about the so-called 'point source model' (as it was termed by J. von Neumann) for a strong explosion is intriguing. It seems that all three of them were almost contemporarily (1941) asked by their governments to investigate the likely effects of a powerful bomb whose energy release is caused by a sudden process of nuclear fission. Note that in 1941 all these countries were deeply fighting in World War 2, and that J. von Neumann was involved in the Manhattan Project, directed by R. Oppenheimer and that lead to the construction of the first atomic bomb in 1945. Neither Taylor nor Sedov, for what is known, were directly involved in nuclear projects.

The three scientists approached the problem in almost the same way, i.e. searching for an interpretation of the development of a spherical shock front (as expected from a point-like explosion in a homogeneous environment) in terms of a function of time deduced by a mere dimensional analysis which necessarily left as undetermined a numerical constant (k_γ in this book) naturally dependent on the environment characteristics, essentially the exponent γ of the EOS, supposedly polytropic. It appeared clear to all of them that the real feedback between the energy released by the strong explosion and the shock front advancement could be determined only upon the determination of such a constant. While von Neumann and Sedov, on their sides, independently arrived at complicated, somewhat implicit, exact solutions to the problem of the shape of radial velocity, density and pressure within the spherical shock front, Taylor proposed just approximations.

Taylor and von Neumann both submitted their reports to their military structures at the end of June 1941. Their work remained likely secreted until 1950 when two papers by Taylor were published in the Proceedings of the Royal Society, titled "The formation of a blast wave by a very intense explosion: I Theoretical discussion and II The atomic explosion of 1945". It was only in 1963 that, seven years after his death, the paper "The point source solution" by von Neumann was published in a volume of his collected works. On the other hand, Sedov was able to publish earlier his results in the Russian language in 1946.

In 1947, two years after the end of World War 2, images of the Trinity test of the atomic bomb explosion in New Mexico were declassified and released by the U.S. Atomic Energy Commission and freely circulated (even published in Life magazine). Basing on a sufficiently detailed set of frames accompanied by the time of the shots and the length scale, Taylor, in his second paper of 1950, was able to check the validity of the dimensionally derived $R_S \propto t^{2/5}$ and, by means of a determination of k_γ based upon energy conservation constraint under the assumption $\gamma = 7/5$ and with a reasonable choice of the air density at the test location, to obtain $E = 16.8$ kiloton as the energy released by the Trinity test explosion. Actually, due to that the real physical situation after the phase of energy conserving expansion is that of a phase of energy dissipation by radiation, the estimate of the bomb released energy in the S–T–vN framework is intrinsically an underestimate of the real bomb energy.

The exact value of the energy actually released in that experiment is not known but an official report dated two days after the experiment by General L. Groves, military commander of the Manhattan Project, speaks of a conservative estimate of a release of energy between 15 and 20 kiloton, in good agreement with Taylor's estimate.

5.5 Further Readings

This chapter presented the dimensional-similarity method originally developed independently by von Neumann, Sedov and Taylor to investigate blast wave propagation. The book [15] by Sedov is surely an important reference. Among many other textbooks on the subject, I cite [16, 17].

Chapter 6
Peculiar Fluid Dynamics

6.1 Relativistic Fluid Dynamics

As is well known, classical physics goes in trouble when dealing with submicroscopic scales (atomic and subatomic scales) where quantum mechanics provides the necessary tools of investigation, and, also, when dealing with velocities comparable with that of light in vacuum, i.e. out of the classical regime $v/c \ll 1$, or when in the presence of strong gravitational fields. In the latter two cases, the *special* and *general* relativistic regime is entered, respectively. Of course, these situations of very large velocities and very strong gravitational fields are not likely to occur on the Earth, but may happen in some astrophysical environments.

For the purposes of this book, we will say something just about the special relativistic regime for fluid dynamics. It is preliminarily worth noting that the presence of relativistic effects is expected not only when the macroscopic velocity of the flow is large but also when the microscopic velocities of the fluid particles are large.

6.1.1 The Energy–Momentum Tensor

To obtain the equations governing a fluid in the special relativistic regime, it is, first of all, necessary to introduce the energy–momentum tensor (also called stress–energy or stress–energy–momentum tensor), T^{ik} (in contravariant form) for the fluid.[1] The relativistic T^{ik} tensor generalizes the Newtonian stress tensor and describes the density and flux of energy and momentum in the 4D space–time.

[1] Here, we adopt the notation of *The classical theory of fields* by Landau and Lifshits [2], that is, Latin suffices like i and k assume the values 0, 1, 2, 3 referring to the whole space–time, while Greek suffices (e.g. α and β) refer to the space indexes 1, 2, 3. Special relativity corresponds to a diagonal metric tensor, g_{ik}, such that $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$ (Galilean metric).

R. A. Capuzzo Dolcetta, *Physics of Fluids*, UNITEXT for Physics,
https://doi.org/10.1007/978-3-031-30750-8_6

The energy–momentum tensor is of order 2, so it can be represented as a 4×4 matrix

$$T^{ik} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}. \tag{6.1}$$

The energy–momentum tensor of a perfect fluid in the proper (rest) frame is written as

$$T^{ik} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{6.2}$$

where $\epsilon = \rho e$ is the internal energy per unit volume (including the rest mass energy) and p is the pressure.

Starting from the above expression Eq. 6.2 of the T^{ik} tensor in the rest frame, its expression in a generic frame of reference is

$$T^{ik} = \overline{w} u^i u^k - p g^{ik}, \tag{6.3}$$

where u^i is the fluid 4-velocity,[2] $\overline{w} = \rho w$ is the enthalpy per unit volume and g^{ik} is the contravariant metric tensor. Given the expression of the 4-velocity in terms of the usual 3-velocity, the T^{ik} tensor has *time* components

$$T^{00} = \frac{\overline{w}}{1 - v^2/c^2} - p = \frac{\epsilon + p v^2/c^2}{1 - v^2/c^2}, \qquad T^{0\alpha} = \frac{\overline{w} v_\alpha}{c(1 - v^2/c^2)}, \tag{6.4}$$

and *space* components

$$T^{\alpha\beta} = \frac{\overline{w} v_\alpha v_\beta}{c^2(1 - v^2/c^2)} + \delta_{\alpha\beta} p. \tag{6.5}$$

After passing T^{ik} to covariant form (i.e. 'lowering' indexes), it is easy to see that in the limit of both small fluid velocity ($v \ll c$) and small microscopic motion of the fluid particles, the usual (non-relativistic) expression of the Reynolds tensor $T_{\alpha\beta} = \rho v_\alpha v_\beta + \delta_{\alpha\beta} p$ is recovered.

[2] Given the 4 space time coordinates x^i as functions of the proper time τ, the 4-velocity is the 4D vector $u^i = dx^i/d\tau$.

On the other hand, when passing to the non-relativistic regime, one must bear in mind that the relativistic internal energy includes the rest energy per unit volume ρc^2, so that the limiting value of T^{00} is $\rho c^2 + \frac{1}{2}\rho v^2 + \epsilon$.

6.1.2 Relativistic Equations of Fluid Motion

The equations of fluid motion are compactly expressed by the mass, energy and momentum conservation laws

$$\begin{cases} \nabla_k J^k = 0, \\ \nabla_k T^{ik} = 0, \end{cases} \tag{6.6}$$

where ∇_k stands for the formal vector representing covariant derivative, i.e. a derivative along tangent vectors of a manifold, and $J^k = \rho u^k$ is the rest mass baryon density current.

In the perfect fluid case, the above equations are written as

$$\begin{cases} \nabla_k J^k = \rho \nabla_k u^k + u^k \nabla_k \rho = 0, \\ \nabla_k T^{ik} = \nabla_k \left[(\epsilon + p) u^k u^i + g^{ki} p \right] = 0. \end{cases} \tag{6.7}$$

Equation 6.7 are 5 PDEs of hyperbolic type in the 6 unknown quantities that are the 3 components of the 4-velocity (the time component one is determined by the normalization condition $\|\mathbf{u}\|^2 = c^2$), the rest mass density, the pressure and the internal energy density. Of course to close the system another constraint is needed, provided by an equation of state in the form $p = p(\rho, \epsilon)$.

Assuming small values of macroscopic and microscopic velocities, as well as a pressure small with respect to volumetric energy density (the latter assumed dominated by the rest mass energy density), Eq. 6.7 lead to the classical continuity and Euler's equations. In a way partly similar to that followed in Chap. 2, it is possible also in the relativistic case to modify the momentum–energy tensor to account for dissipative processes (viscosity, thermal conduction, etc.), writing (in contravariant form)

$$\overline{T}^{ik} = T^{ik} - \tau^{ik}, \tag{6.8}$$

where τ^{ik} accounts for the non-ideal (dissipative) fluid contribution and

$$\bar{J}^k = J^k + J^k_{ni} \tag{6.9}$$

is the modification of the rest mass density current to account for non-ideal (ni) fluid structure. There are several ways to obtain explicit forms for both τ^{ik} and J^k_{ni}, procedures that are out of the scope of this introductory book, and for which we point

the attention of the reader to the suggested further readings. Formally, the motion equations keep the conservation form

$$\begin{cases} \nabla_k \bar{J}^k = 0, \\ \nabla_k \overline{T}^{ik} = 0. \end{cases} \tag{6.10}$$

6.2 Superfluids

When temperature is reduced drastically, most of the liquids solidify. There is the notable exception of helium-4 (^{4}He) which can keep its liquid characteristics even at so low temperatures that quantum effects appear. In such situations, the helium fluid behaves in a peculiar way, which is called *superfluidity*. Superfluidity is an extreme state of matter in which viscosity is totally absent as well as entropy and thermal capacity are infinite, which means that a superfluid moves without any dissipation of kinetic energy and it is impossible to increase its temperature.

Superfluidity was discovered as a characteristic of ^{4}He at cryogenic temperatures in 1937 by P. Kapitsa (1978 Nobel prize for his studies on the behavior of matter at very low temperatures) and, independently, by J. F. Allen and D. Misener. They noted that below a certain threshold of temperature ($T \simeq 2.172$ K in standard conditions), liquid ^{4}He undergoes a particular phase transition which leaves ^{4}He in its liquid state but at vanishing viscosity.

Another Nobel prize was awarded in 1996 to D. M. Lee, D. D. Osheroff and R. C. Richardson for the discovery of superfluidity in the helium-3 isotope (^{3}He) at much lower temperature ($T \simeq 0.002$ K) than for ^{4}He

A theoretical interpretation of superfluidity was attempted, first, by F. London in 1938 basing on considerations about Bose–Einstein condensates (BEC). But neither all superfluids are Bose–Einstein condensates nor all Bose–Einstein condensates are superfluids, so a better theory of superfluidity should not base on BEC. Such a theory was developed by L. D. Landau which earned him a Nobel prize in Physics in 1962.

Another fundamental aspect of superfluids is that quantum effects are present. When quantum effects are considered, the superfluid continuity and momentum conservation equations are written as

$$\begin{aligned} &\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{v} = 0 \\ &\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = \nabla U + \nabla \mu_c \end{aligned} \tag{6.11}$$

where n is the number density and μ_c is the *chemical potential* which is in this case

$$\mu_c(n) = gn - \frac{\hbar^2}{2m^2}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}, \tag{6.12}$$

where $\hbar = h/(2\pi)$ is the reduced Planck constant and

$$g = \frac{4\pi\hbar^2 a_s}{m} \tag{6.13}$$

is a coupling constant (m is the mass of the boson and a_s the s-wave scattering length of two interacting bosons).

6.3 Further Readings

I suggest [2, 18] for deeper insight into relativistic fluid dynamics. Reference [19] provides an insight into the topic of superfluidity.

Appendix A
Basic Vector Analysis and Calculus

On the physical side, a prerequirement to understand properly this book is the knowledge of classical mechanics and thermodynamics, at the level of undergraduate classes. On the mathematical side, in this book we use some concepts of differential and integral calculus as well as vector analysis and fundamentals of analytic geometry. For the sake of a better comprehension of the text, in this Appendix we make a short summary of vector analysis and calculus in the formalism adopted throughout the book.

We consider real vector spaces in n dimensions., i.e. subsets of $\mathbb{R}^n$ whose elements are n real numbers which are usually considered as the *n coordinates* of a vector in an assumed coordinate system. In this book, a generic vector in $\mathbb{R}^n$ is represented as a boldfaced letter

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i \equiv v_i \mathbf{e}_i \,, \tag{A.1}$$

where $\mathbf{e}_i$, $i = 1, \ldots, n$, are the unitary vectors (versors) of the basis over which v_i, $i = 1, \ldots, n$, are the *components* of the vector $\mathbf{v}$. The rightmost expression in the above relation Eq. A.1 represents the Einstein *summation convention*[1] which assumes implicit summation over the values referring to a repeated index in a formula (i in this case).

In this book, we will generally refer to fluids moving in time in a three-dimensional (3D) space, so that the position of the generic fluid particle is identified by its 3D *position* vector $\mathbf{r}$ (called also radius vector). In the commonly adopted Cartesian reference frame, whose basis vectors are identified by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, it is

[1] This convention, also known as Einstein notation, was introduced by A. Einstein in his 1916 paper "The Foundation of the General Theory of Relativity".

R. A. Capuzzo Dolcetta, *Physics of Fluids*, UNITEXT for Physics,
https://doi.org/10.1007/978-3-031-30750-8

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \tag{A.2}$$

where x, y, z are the Cartesian coordinates of the point.

Assuming the fluid particle is moving in a collective flow, its radius vector varies with time t, so that $\mathbf{r}(t)$ represents the first example of 3D *vector field* function of the variable t.

Being $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ an orthonormal basis, it results in

$$\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1, \tag{A.3}$$

where the symbol $\|\mathbf{v}\|$ represents the norm of a vector $\mathbf{v}$, which we shortly indicate by v and that is the 'length' of the vector. Throughout this book, we adopt as a norm of an n-dimensional vector its *Euclidean* norm

$$\|\mathbf{v}\| \equiv v = \sqrt{\sum_{i=1}^{n} v_i^2} = \sqrt{v_i v_i}. \tag{A.4}$$

The *scalar* (or *dot*) product between two vectors is defined as

$$\mathbf{v} \cdot \mathbf{w} = v_i w_i = \mathbf{w} \cdot \mathbf{v}, \tag{A.5}$$

where we employed Einstein's convention, so that it clearly results in

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}. \tag{A.6}$$

When using Cartesian coordinates, the definition Eq. A.5 above is equivalent to $\mathbf{v} \cdot \mathbf{w} = vw \cos \widehat{\mathbf{vw}}$, where $0 \leq \widehat{\mathbf{vw}} \leq \pi$ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

An important application of the scalar product is to find the component of a vector along a given direction, represented by a unit vector in that direction. For example, the component of $\mathbf{v}$ along the direction of $\mathbf{w}$ is

$$\mathbf{v} \cdot \mathbf{e}_w = v \cos \widehat{\mathbf{vw}}, \tag{A.7}$$

where $\mathbf{e}_w = \mathbf{w}/w$ is the unit vector in the direction of $\mathbf{w}$.

From what we said above, clearly two orthogonal vectors give null scalar product ($\cos \widehat{\mathbf{vw}} = 0$ in this case), so the orthogonality condition for the Cartesian basis corresponds to

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0. \tag{A.8}$$

Another product between vectors is the *vector* product (sometimes called *cross* product). While the result of a scalar product is a scalar, the result of a vector product is a vector of the same dimension as the two vector factors. In this book, the vector product is represented with the symbol $\wedge$ instead of the often-used $\times$ symbol to avoid ambiguity with some notations where $\times$ is used instead for the scalar product.

Contrary to the scalar product, which is defined for arbitrary n dimension vectors, the vector product is defined in 3D space only. Its definition is

$$\mathbf{v} \wedge \mathbf{w} = vw \sin \widehat{\mathbf{vw}} \mathbf{e}_{vw} = -\mathbf{w} \wedge \mathbf{v}, \tag{A.9}$$

where $\mathbf{e}_{vw}$ is a unit vector perpendicular to the plane identified by $\mathbf{v}$ and $\mathbf{w}$ and oriented such that the set $(\mathbf{v}, \mathbf{w}, \mathbf{e}_{vw})$ identifies a frame of reference where the superposition of $\mathbf{v}$ over $\mathbf{w}$ upon rotation according to the smaller of the 2 angles formed by $\mathbf{v}$ and $\mathbf{w}$ corresponds to a counterclockwise rotation around $\mathbf{e}_{vw}$. This definition corresponds to the often referred to as *right-hand* rule. Note that the norm of the vector product geometrically represents the area of the parallelogram with sides v and w.

In a Cartesian 3D reference system with basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, the vector product between $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$ and $\mathbf{w} = w_x\mathbf{i} + w_y\mathbf{j} + w_z\mathbf{k}$ can be written as the following determinant:

$$\mathbf{v} \wedge \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = (v_y w_z - v_z w_y)\mathbf{i} - (v_x w_z - v_z w_x)\mathbf{j} + (v_y w_y - v_y w_x)\mathbf{k}. \tag{A.10}$$

Note that

$$\mathbf{i} \wedge \mathbf{j} = \mathbf{k},\ \mathbf{i} \wedge \mathbf{k} = -\mathbf{j},\ \mathbf{j} \wedge \mathbf{k} = \mathbf{i}, \tag{A.11}$$

and

$$\mathbf{i} \wedge \mathbf{i} = \mathbf{j} \wedge \mathbf{j} = \mathbf{k} \wedge \mathbf{k} = 0. \tag{A.12}$$

While the scalar product between two vectors gives the components of one vector along the other (parallel projection), the vector product gives the orthogonal components (orthogonal projection). This reflects the condition of parallelism as $\mathbf{v} \wedge \mathbf{w} = 0$, counterpart of the condition of orthogonality $\mathbf{v} \cdot \mathbf{w} = 0$.

A.1 The Formal Vector Operator ∇

In this book, we have made extensive use of the operator nabla, ∇, which is defined as a (formal) differential vector. In n dimensions and Cartesian coordinates, it is written as

$$\nabla \equiv \mathbf{e}_i \frac{\partial}{\partial x_i}, \tag{A.13}$$

(Einstein convention is used) that in 3D specifies to

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{A.14}$$

This *formal* vector ∇ can be operated as a real vector so that by adopting the available operations of products with a scalar function $f(x_1, \ldots, x_n)$, we have

$$\nabla f = \mathbf{e}_i \frac{\partial f}{\partial x_i} = \text{grad}\, f, \tag{A.15}$$

which is the usual *gradient* of the function f, while making the scalar product with a vector $\mathbf{v}$

$$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i} = \text{div}\, \mathbf{v}, \tag{A.16}$$

which is the usual *divergence* of the nD vector field $\mathbf{v}$. Finally, the vector product of ∇ with $\mathbf{v}$ leads to (here we are in 3D)

$$\nabla \wedge \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \text{curl}\, \mathbf{v}, \tag{A.17}$$

which is the usual *curl* of the vector field $\mathbf{v}$.

The operator ∇ acts also on vector fields. The gradient of a vector, $\nabla \mathbf{v}$, is defined as the matrix whose rows are the spatial derivatives of the vector components. In Cartesian 3D coordinates,

$$\nabla \mathbf{v} = \begin{pmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_x}{\partial y} & \dfrac{\partial v_x}{\partial z} \\ \dfrac{\partial v_y}{\partial x} & \dfrac{\partial v_y}{\partial y} & \dfrac{\partial v_y}{\partial z} \\ \dfrac{\partial v_z}{\partial x} & \dfrac{\partial v_z}{\partial y} & \dfrac{\partial v_z}{\partial z} \end{pmatrix}. \tag{A.18}$$

In general, given a vector field $\mathbf{v} : \mathbb{R}^m \to \mathbb{R}^n$, its gradient is an $n \times n$ matrix with elements

$$(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j}, \quad (i = 1, \ldots, n; \;\; j = 1, \ldots, m). \tag{A.19}$$

Note that the matrix representing the gradient of a vector field is the Jacobian matrix of the $(x, y, z) \to (v_x, v_y, v_z)$ transformation, which we have introduced in Sect. 2.1.1.

As a consequence of the above definition, the Laplacian of a vector (which appears, for instance, in the rhs of the Navier–Stokes equation), $\nabla^2 \mathbf{v} = \nabla \cdot \nabla \mathbf{v} = (\nabla \cdot \nabla)\mathbf{v}$, is the vector whose components are the Laplacians of the components of $\mathbf{v}$. In Cartesian 3D,

$$\nabla^2 \mathbf{v} = \nabla^2 v_x \mathbf{i} + \nabla^2 v_y \mathbf{j} + \nabla^2 v_z \mathbf{k}. \tag{A.20}$$

A.2 Coordinate Systems

In 3D, it is usual to adopt *Cartesian, spherical polar* and *cylindrical* coordinates. In fluid dynamics, like in other fields of Physics, the choice of one or another system of coordinates is a matter of convenience, which commonly depends upon possible symmetries, for instance radial or axial symmetry, in the system under study.

We give here the laws of transformation between these three coordinate systems.

The transformation from spherical polar (r, θ, φ) to Cartesian coordinates (x, y, z) is[2]

$$\begin{cases} x = r \sin\varphi \cos\theta, \\ y = r \sin\varphi \sin\theta, \\ z = r \cos\varphi, \end{cases} \quad \text{with} \quad r \geq 0, \quad 0 \leq \varphi \leq \pi \quad \text{and} \quad 0 \leq \theta < 2\pi, \tag{A.21}$$

which has the inverse transformation

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = 2 \arctan \frac{y}{\sqrt{x^2 + y^2} + x}, \quad \varphi = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}. \tag{A.22}$$

A vector in a spherical polar frame is written as

$$\mathbf{v} = v_r \mathbf{e}_r + v_\theta \, \mathbf{e}_\theta + v_\varphi \, \mathbf{e}_\varphi, \tag{A.23}$$

where $v_r = \mathbf{v} \cdot \mathbf{e}_r$, $v_\theta = \mathbf{v} \cdot \mathbf{e}_\theta$, $v_\varphi = \mathbf{v} \cdot \mathbf{e}_\varphi$ are the components of $\mathbf{v}$ on the spherical polar basis, whose unit vectors are

$$\begin{cases} \mathbf{e}_r = \dfrac{\mathbf{r}}{r} = \sin\varphi \cos\theta \, \mathbf{i} + \sin\varphi \sin\theta \, \mathbf{j} + \cos\varphi \, \mathbf{k}, \\ \mathbf{e}_\theta = -\sin\theta \, \mathbf{i} + \cos\theta \, \mathbf{j}, \\ \mathbf{e}_\varphi = \cos\theta \cos\varphi \, \mathbf{i} + \cos\varphi \sin\theta \, \mathbf{j} - \sin\varphi \, \mathbf{k}. \end{cases} \tag{A.24}$$

The transformation between Cartesian and cylindrical coordinates (R, θ, z) is expressed by

$$\begin{cases} x = R \cos\theta, \\ y = R \sin\theta, \\ z = z. \end{cases} \quad \text{with} \quad 0 \leq \theta < 2\pi, \tag{A.25}$$

which has the inverse transformation

$$R = \sqrt{x^2 + y^2}, \quad \theta = \arctan \frac{y}{x}, \quad z = z. \tag{A.26}$$

The basis unit vectors are

[2] Note that in this book we use symbol θ for the angle in the equatorial, $x - y$, plane and φ for the *colatitude*, i.e. the angle between $\mathbf{r}$ and the positive z-axis, differently from other textbooks (see Fig. A.1).

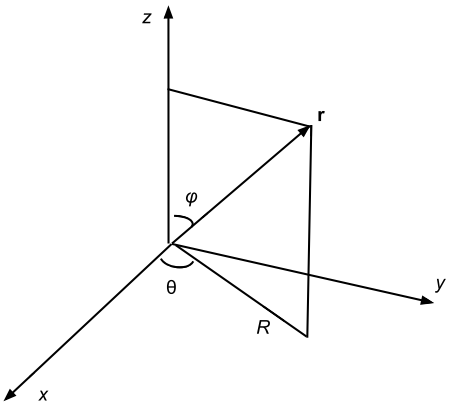


Fig. A.1 Sketch of Cartesian (x, y, z), spherical polar (r, θ, φ) and cylindrical (R, θ, z) coordinates of a point with position vector $\mathbf{r}$

$$\begin{cases} \mathbf{e}_R = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \\ \mathbf{e}_\theta = \dfrac{\partial \mathbf{r}}{\partial \theta}\dfrac{1}{\dfrac{ds}{d\theta}} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}, \\ \mathbf{e}_z = \mathbf{k}. \end{cases} \tag{A.27}$$

In such a basis, a generic vector $\mathbf{v}$ is written as

$$\mathbf{v} = v_R\,\mathbf{e}_R + v_\theta\,\mathbf{e}_\theta + v_z\,\mathbf{e}_z, \tag{A.28}$$

where $v_R = \mathbf{v}\cdot\mathbf{e}_R$, $v_\theta = \mathbf{v}\cdot\mathbf{e}_\theta$, $v_z = \mathbf{v}\cdot\mathbf{e}_z$ are the components of $\mathbf{v}$ on the cylindrical coordinate basis.

A.2.1 Gradient, Divergence, Laplacian and Curl in Different Coordinate Systems

Taking into account the above-described coordinate transformations, the formal vector ∇ is written as

$$\nabla = \begin{cases} \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}, & \text{Cartesian coordinates,} \\ \mathbf{e}_r\dfrac{\partial}{\partial r} + \mathbf{e}_\theta\dfrac{1}{r\sin\varphi}\dfrac{\partial}{\partial\theta} + \mathbf{e}_\varphi\dfrac{1}{r}\dfrac{\partial}{\partial\varphi}, & \text{spherical polar coordinates,} \\ \mathbf{e}_R\dfrac{\partial}{\partial R} + \mathbf{e}_\theta\dfrac{1}{R}\dfrac{\partial}{\partial\theta} + \mathbf{e}_z\dfrac{\partial}{\partial z}, & \text{cylindrical coordinates.} \end{cases} \tag{A.29}$$

From the above relations, it results after some simple calculations that the divergence of a vector field $\mathbf{F}$ is written as

$$\nabla \cdot \mathbf{F} = \begin{cases} \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}, & \text{Cartesian,} \\ \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 F_r\right) + \dfrac{1}{r\sin\varphi}\dfrac{\partial F_\theta}{\partial \theta} + \dfrac{1}{r\sin\varphi}\dfrac{\partial}{\partial \varphi}\left(\sin\varphi\, F_\varphi\right) & \text{spherical polar,} \\ \dfrac{1}{R}\dfrac{\partial}{\partial R}\left(R F_R\right) + \dfrac{1}{R}\dfrac{\partial F_\theta}{\partial \theta} + + \dfrac{\partial F_z}{\partial z}, & \text{cylindrical.} \end{cases} \tag{A.30}$$

Assuming $\mathbf{F} = \nabla f$, a substitution in relations Eqs. A.29 and A.30 gives the Laplacian, $\nabla^2 f = \nabla \cdot \mathbf{F}$, of the scalar function f

$$\nabla^2 f = \begin{cases} \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}, & \text{Cartesian,} \\ \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2\sin^2\varphi}\dfrac{\partial^2 f}{\partial \theta^2} + \\ \qquad + \dfrac{1}{r^2\sin\varphi}\dfrac{\partial}{\partial \varphi}\left(\sin\varphi\dfrac{\partial f}{\partial \varphi}\right), & \text{spherical polar,} \\ \dfrac{1}{R}\dfrac{\partial}{\partial R}\left(R\dfrac{\partial f}{\partial R}\right) + \dfrac{1}{R^2}\dfrac{\partial^2 f}{\partial \theta^2} + \dfrac{\partial^2 f}{\partial z^2}, & \text{cylindrical.} \end{cases} \tag{A.31}$$

The curl of $\mathbf{v}$ is calculated as $\mathrm{curl}\mathbf{v} = \nabla \wedge \mathbf{v}$, to obtain

$$\mathrm{curl}\mathbf{v} = \begin{cases} \left(\dfrac{\partial v_z}{\partial y} - \dfrac{\partial v_y}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y}\right)\mathbf{k}, & \text{Cartesian,} \\ \dfrac{1}{r\sin\varphi}\left(\dfrac{\partial}{\partial \varphi}(v_\theta \sin\varphi) - \dfrac{\partial v_\varphi}{\partial \theta}\right)\mathbf{e}_r + \\ + \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}(r v_\varphi) - \dfrac{\partial v_r}{\partial \varphi}\right)\mathbf{e}_\theta + \dfrac{1}{r}\left(\dfrac{1}{\sin\varphi}\dfrac{\partial v_r}{\partial \theta} - \dfrac{\partial}{\partial r}(r v_\theta)\right)\mathbf{e}_\varphi, & \text{spherical polar,} \\ \left(\dfrac{1}{R}\dfrac{\partial v_z}{\partial \theta} - \dfrac{\partial v_\theta}{\partial z}\right)\mathbf{e}_R + \left(\dfrac{\partial v_R}{\partial z} - \dfrac{\partial v_z}{\partial R}\right)\mathbf{e}_\theta + \dfrac{1}{R}\left(\dfrac{\partial R v_\theta}{\partial R} - \dfrac{\partial v_R}{\partial \theta}\right)\mathbf{e}_z, & \text{cylindrical.} \end{cases} \tag{A.32}$$

Accounting for Eq. A.30, the Lagrangian derivative of the components of a generic vector $\mathbf{F} = F_k \mathbf{e}_k$ in the three coordinate systems is written as

$$\frac{\mathrm{D}F_i}{\mathrm{D}t} = \begin{cases} \dfrac{\partial F_i}{\partial t} + v_x\dfrac{\partial F_i}{\partial x} + v_y\dfrac{\partial F_i}{\partial y} + v_z\dfrac{\partial F_i}{\partial z}, & \text{Cartesian,} \\ \dfrac{\partial F_i}{\partial t} + v_r\dfrac{\partial F_i}{\partial r} + \dfrac{v_\theta}{r\sin\varphi}\dfrac{\partial F_i}{\partial \theta} + \dfrac{v_\varphi}{r}\dfrac{\partial F_i}{\partial \varphi}, & \text{spherical polar,} \\ \dfrac{\partial F_i}{\partial t} + v_R\dfrac{\partial F_i}{\partial R} + \dfrac{v_\theta}{R}\dfrac{\partial F_i}{\partial \theta} + v_z\dfrac{\partial F_i}{\partial z}, & \text{cylindrical.} \end{cases} \tag{A.33}$$

In the Lagrangian derivative of a vector, it appears the convection operator $\mathbf{v}\cdot\nabla$ (see Eq. 1.13) that is written as

$$\mathbf{v}\cdot\nabla=\begin{cases} v_x\dfrac{\partial}{\partial x}+v_y\dfrac{\partial}{\partial y}+v_z\dfrac{\partial}{\partial z}, & \text{Cartesian,}\\ v_r\dfrac{\partial}{\partial r}+\dfrac{v_\theta}{r\sin\varphi}\dfrac{\partial}{\partial\theta}+\dfrac{v_r}{r}\dfrac{\partial}{\partial\varphi}, & \text{spherical polar,}\\ v_R\dfrac{\partial}{\partial R}+\dfrac{v_\theta}{R}\dfrac{\partial}{\partial\theta}+v_z\dfrac{\partial}{\partial z}, & \text{cylindrical.}\end{cases} \tag{A.34}$$

Applying the above operator to $\mathbf{v}$, in Cartesian coordinates the result is simply[3]

$$\begin{aligned}(\mathbf{v}\cdot\nabla)\mathbf{v}=&v_x\frac{\partial\mathbf{v}}{\partial x}+v_y\frac{\partial\mathbf{v}}{\partial y}+v_z\frac{\partial\mathbf{v}}{\partial z}=\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)\mathbf{i}+\\&+\left(v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)\mathbf{j}+\left(v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)\mathbf{k}.\end{aligned} \tag{A.35}$$

In spherical polar and cylindrical coordinates, considering the expressions Eqs. A.23 and A.28 for the velocity and Eqs. A.24 and A.27 for the basis vectors (whose derivatives have to be made), the result (after some calculation) is in **spherical polar coordinates**:

$$\begin{aligned}(\mathbf{v}\cdot\nabla)\mathbf{v}=&\left(v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r\sin\varphi}\frac{\partial v_r}{\partial\theta}+\frac{v_\varphi}{r}\frac{\partial v_r}{\partial\varphi}-\frac{v_\theta^2+v_\varphi^2}{r}\right)\mathbf{e}_r+\\&+\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r\sin\varphi}\frac{\partial v_\theta}{\partial\theta}+\frac{v_\varphi}{r}\frac{\partial v_\theta}{\partial\varphi}+\frac{v_r v_\theta}{r}+\frac{v_\theta v_\varphi}{r}\cot\varphi\right)\mathbf{e}_\theta+\\&+\left(v_r\frac{\partial v_\varphi}{\partial r}+\frac{v_\theta}{r\sin\varphi}\frac{\partial v_\varphi}{\partial\theta}+\frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial\varphi}+\frac{v_r v_\varphi}{r}-\frac{v_\theta^2}{r}\cot\varphi\right)\mathbf{e}_\varphi,\end{aligned} \tag{A.36}$$

and in **cylindrical coordinates**:

$$\begin{aligned}(\mathbf{v}\cdot\nabla)\mathbf{v}=&\left(v_R\frac{\partial v_R}{\partial R}+\frac{v_\theta}{R}\frac{\partial v_R}{\partial\theta}+v_z\frac{\partial v_R}{\partial z}-\frac{v_\theta^2}{R}\right)\mathbf{e}_R+\\&+\left(v_R\frac{\partial v_\theta}{\partial R}+\frac{v_\theta}{R}\frac{\partial v_\theta}{\partial\theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_R v_\theta}{R}\right)\mathbf{e}_\theta+\\&+\left(v_R\frac{\partial v_z}{\partial R}+\frac{v_\theta}{R}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}\right)\mathbf{e}_z.\end{aligned} \tag{A.37}$$

[3] We use here the convention

$$\frac{\partial\mathbf{v}}{\partial x}=\mathbf{i}\frac{\partial v_x}{\partial x}+\mathbf{j}\frac{\partial v_y}{\partial x}+\mathbf{k}\frac{\partial v_z}{\partial x},$$

and similarly for derivatives with respect to y and z.

A.2.2 Continuity and Motion Equation in Spherical Polar and Cylindrical Coordinates

Thanks to the relations above and to the proper expression of velocity in spherical polar and cylindrical coordinates, it is easy to obtain the expression of both continuity and Euler's equations in these coordinate systems.

Spherical Polar Coordinates
In spherical polars (r, θ, φ), the equation of continuity Eq. 2.8 is written as

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 \rho v_r)}{\partial r} + \frac{1}{r \sin \varphi}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{1}{r \sin \varphi}\frac{\partial (\sin \varphi \rho v_\varphi)}{\partial \varphi} = 0, \tag{A.38}$$

and Euler's equation 2.29 is written as the system

$$\begin{cases} \dfrac{\mathrm{D}v_r}{\mathrm{D}t} - \dfrac{v_\theta^2 + v_\varphi^2}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} + \dfrac{1}{\rho}F_{Br}, \\ \dfrac{\mathrm{D}v_\theta}{\mathrm{D}t} + \dfrac{v_\theta v_r}{r} + \dfrac{v_\varphi v_\theta \cot \varphi}{r} = -\dfrac{1}{\rho}\dfrac{1}{r \sin \varphi}\dfrac{\partial p}{\partial \theta} + \dfrac{1}{\rho}F_{B\theta}, \\ \dfrac{\mathrm{D}v_\varphi}{\mathrm{D}t} + \dfrac{v_\varphi v_r}{r} - \dfrac{v_\theta^2 \cot \varphi}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial \varphi} + \dfrac{1}{\rho}F_{B\varphi}. \end{cases} \tag{A.39}$$

Cylindrical coordinates
In cylindrical coordinates (R, θ, z), the equation of continuity Eq. 2.8 is written as

$$\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial (R \rho v_R)}{\partial R} + \frac{1}{R}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0, \tag{A.40}$$

and Euler's equation 2.29 is written as the system

$$\begin{cases} \dfrac{\mathrm{D}v_R}{\mathrm{D}t} - \dfrac{v_\theta^2}{R} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial R} + \dfrac{1}{\rho}F_{BR}, \\ \dfrac{\mathrm{D}v_\theta}{\mathrm{D}t} + \dfrac{v_\theta v_R}{R} = -\dfrac{1}{\rho}\dfrac{1}{R}\dfrac{\partial p}{\partial \theta} + \dfrac{1}{\rho}F_{B\theta}, \\ \dfrac{\mathrm{D}v_z}{\mathrm{D}t} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + \dfrac{1}{\rho}F_{Bz}. \end{cases} \tag{A.41}$$

Note that by eliminating in Eqs. A.39 and A.41 the Lagrangian derivatives with respect to t by means of Eq. A.33 leads to Eulerian expressions of Euler's equation.

As shown in Chap. 2, the N-S equation in the incompressible case is simply obtained by adding the term $(\eta/\rho)\nabla^2 \mathbf{v}$ in the rhs of the Euler's equation. Taking into account that $\nabla^2 = \nabla \cdot \nabla$, the N-S equation for incompressible fluids is obtained in spherical polar and cylindrical coordinates by simply adding in the rhs of Eqs. A.39 and A.41 the following N-S terms:

Spherical Polar Coordinates

$$\begin{cases} \dfrac{1}{\rho}F_{NSr} = \dfrac{\eta}{\rho}\left(\nabla^2 v_r - 2\dfrac{v_r}{r^2} - \dfrac{2}{r^2}\dfrac{\partial v_\varphi}{\partial \varphi} - 2\dfrac{v_\varphi \cot\varphi}{r^2} - \dfrac{2}{r^2\sin\varphi}\dfrac{\partial v_\theta}{\partial \theta}\right), \\ \dfrac{1}{\rho}F_{NS\theta} = \dfrac{\eta}{\rho}\left(\nabla^2 v_\theta + \dfrac{2}{r^2\sin\varphi}\dfrac{\partial v_r}{\partial \theta} - \dfrac{v_\theta}{r^2\sin^2\varphi} + 2\dfrac{\cot\varphi}{r^2\sin\varphi}\dfrac{\partial v_\varphi}{\partial \theta}\right), \\ \dfrac{1}{\rho}F_{NS\varphi} = \dfrac{\eta}{\rho}\left(\nabla^2 v_\varphi + \dfrac{2}{r^2}\dfrac{\partial v_r}{\partial \varphi} - \dfrac{v_\varphi}{r^2\sin^2\varphi} - 2\dfrac{\cot\varphi}{r^2\sin\varphi}\dfrac{\partial v_\theta}{\partial \theta}\right). \end{cases} \tag{A.42}$$

Cylindrical Coordinates

$$\begin{cases} \dfrac{1}{\rho}F_{NSR} = \dfrac{\eta}{\rho}\left(\nabla^2 v_R - \dfrac{v_R}{R^2} - \dfrac{2}{R^2}\dfrac{\partial v_\theta}{\partial \theta}\right), \\ \dfrac{1}{\rho}F_{NS\theta} = \dfrac{\eta}{\rho}\left(\nabla^2 v_\theta - \dfrac{v_\theta}{R^2} + \dfrac{2}{R^2}\dfrac{\partial v_R}{\partial \theta}\right), \\ \dfrac{1}{\rho}F_{NSz} = \dfrac{\eta}{\rho}\nabla^2 v_z. \end{cases} \tag{A.43}$$

A.3 Some Useful Vector Formulas

A.3.1 Identities of Vector Algebra

For given sufficiently regular vector fields $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ in $\mathbb{R}^3$, these relations hold [4]

$$\begin{aligned} &\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}, \\ &\mathbf{A}\wedge\mathbf{B} = -\mathbf{B}\wedge\mathbf{A}, \\ &\mathbf{A}\cdot(\mathbf{B}\wedge\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\wedge\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\wedge\mathbf{B}) = \det(\mathbf{ABC}), \\ &\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}, \\ &(\mathbf{A}\wedge\mathbf{B})\wedge(\mathbf{C}\wedge\mathbf{D}) = \det(\mathbf{ABD})\mathbf{C} - \det(\mathbf{ABC})\mathbf{D} = \det(\mathbf{ACD})\mathbf{B} - \det(\mathbf{BCD})\mathbf{A}, \\ &\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C}) + \mathbf{C}\wedge(\mathbf{A}\wedge\mathbf{B}) + \mathbf{B}\wedge(\mathbf{C}\wedge\mathbf{A}) = 0, \\ &(\mathbf{A}\wedge\mathbf{B})\cdot(\mathbf{C}\wedge\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D}), \\ &\|\mathbf{A}\wedge\mathbf{B}\|^2 = (\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{B})^2. \end{aligned} \tag{A.44}$$

[4] $\det(\mathbf{ABC})$ is the determinant of the 3×3 matrix whose 1st, 2nd and 3rd columns are (A_x, A_y, A_z), (B_x, B_y, B_z) and (C_x, C_y, C_z), respectively.

A.3.2 Identities Involving ∇

Given the scalar fields f and g and the vector fields $\mathbf{v}$ and $\mathbf{w}$, the following identities hold:

$$
\begin{aligned}
&\nabla(f+g) = \nabla f + \nabla g,\\
&\nabla(fg) = f\nabla g + g\nabla f,\\
&\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2},\\
&\nabla^2(fg) = f\nabla^2 g + 2\nabla f\cdot\nabla g + g\nabla^2 f,\\
&\nabla^2\left(\frac{f}{g}\right) = \frac{g\nabla^2 f - 2g\nabla\left(\frac{f}{g}\right) - f\nabla^2 g}{g^2},\\
&\nabla\wedge\nabla f = 0,\\
&\nabla(\mathbf{v}+\mathbf{w}) = \nabla\mathbf{v} + \nabla\mathbf{w},\\
&\nabla(\mathbf{v}\cdot\mathbf{w}) = (\mathbf{v}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v} + \mathbf{v}\wedge(\nabla\wedge\mathbf{w}) + \mathbf{w}\wedge(\nabla\wedge\mathbf{v}),\\
&\nabla\cdot(\mathbf{v}+\mathbf{w}) = \nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{w},\\
&\nabla\cdot f\mathbf{v} = f\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla f,\\
&\nabla\cdot(\mathbf{v}\nabla\mathbf{v}) = \mathbf{v}\cdot\nabla^2\mathbf{v} + \nabla\mathbf{v}\cdot\nabla\mathbf{v},\\
&\nabla\wedge(\mathbf{v}+\mathbf{w}) = \nabla\wedge\mathbf{v} + \nabla\wedge\mathbf{w},\\
&\nabla\wedge f\mathbf{v} = f\nabla\wedge\mathbf{v} - \mathbf{v}\wedge\nabla f,\\
&\nabla\wedge(\mathbf{v}\wedge\mathbf{w}) = (\mathbf{w}\cdot\nabla)\mathbf{v} - (\mathbf{v}\cdot\nabla)\mathbf{w} + \mathbf{v}\nabla\cdot\mathbf{w} - \mathbf{w}\nabla\cdot\mathbf{v}\\
&\nabla\wedge(\nabla\wedge\mathbf{v}) = \nabla\nabla\cdot\mathbf{v} - \nabla^2\mathbf{v},\\
&\nabla\frac{1}{2}v^2 = \mathbf{v}\wedge(\nabla\wedge\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{v}.
\end{aligned}
\tag{A.45}
$$

Let us demonstrate the last one in the above sequence due to its importance in giving an alternative way of writing the Euler's equation (see Sect. 2.2.1.1). It comes from the identity

$$
\nabla(\mathbf{v}\cdot\mathbf{w}) = (\mathbf{v}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v} + \mathbf{v}\wedge(\nabla\wedge\mathbf{w}) + \mathbf{w}\wedge(\nabla\wedge\mathbf{v}), \tag{A.46}
$$

where $\mathbf{v}$ and $\mathbf{w}$ are continuously differentiable vector fields, in which we will specialize $\mathbf{w} = \mathbf{v}$. The above identity is verified as follows.

First, we write the lhs as

$$\nabla(\mathbf{v}\cdot\mathbf{w}) = \mathbf{i}\frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{w}) + \mathbf{j}\frac{\partial}{\partial y}(\mathbf{v}\cdot\mathbf{w}) + \mathbf{k}\frac{\partial}{\partial z}(\mathbf{v}\cdot\mathbf{w}) =$$
$$= \mathbf{i}\left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial x} + \mathbf{w}\cdot\frac{\partial\mathbf{v}}{\partial x}\right) + \mathbf{j}\left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial y} + \mathbf{w}\cdot\frac{\partial\mathbf{v}}{\partial y}\right) + \mathbf{k}\left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial z} + \mathbf{w}\cdot\frac{\partial\mathbf{v}}{\partial z}\right). \tag{A.47}$$

Given the second identity in the set Eq. A.36 in which we put $\mathbf{A} = \mathbf{v}$, $\mathbf{B} = \mathbf{i}$ and $\mathbf{C} = \dfrac{\partial\mathbf{w}}{\partial x}$, we obtain

$$\left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial x}\right)\mathbf{i} = \mathbf{v}\wedge\left(\mathbf{i}\wedge\frac{\partial\mathbf{w}}{\partial x}\right) + (\mathbf{v}\cdot\mathbf{i})\,\frac{\partial\mathbf{w}}{\partial x}. \tag{A.48}$$

Analogously, letting once $\mathbf{B} = \mathbf{j}$ and $\mathbf{C} = \dfrac{\partial\mathbf{w}}{\partial y}$, and once $\mathbf{B} = \mathbf{k}$ and $\mathbf{C} = \dfrac{\partial\mathbf{w}}{\partial z}$, we obtain, respectively,

$$\left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial y}\right)\mathbf{j} = \mathbf{v}\wedge\left(\mathbf{j}\wedge\frac{\partial\mathbf{w}}{\partial y}\right) + (\mathbf{v}\cdot\mathbf{j})\,\frac{\partial\mathbf{w}}{\partial y}, \tag{A.49}$$

and

$$\left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial z}\right)\mathbf{k} = \mathbf{v}\wedge\left(\mathbf{k}\wedge\frac{\partial\mathbf{w}}{\partial z}\right) + (\mathbf{v}\cdot\mathbf{k})\,\frac{\partial\mathbf{w}}{\partial z}. \tag{A.50}$$

Adding side by side Eqs. A.48, A.49 and A.50 and considering that $\mathbf{v}\cdot\mathbf{i} = v_x$, $\mathbf{v}\cdot\mathbf{j} = v_y$ and $\mathbf{v}\cdot\mathbf{k} = v_z$, we get

$$\left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial x}\right)\mathbf{i} + \left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial y}\right)\mathbf{j} + \left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial z}\right)\mathbf{k} =$$
$$= \mathbf{v}\wedge\left(\mathbf{i}\wedge\frac{\partial\mathbf{w}}{\partial x} + \mathbf{j}\wedge\frac{\partial\mathbf{w}}{\partial y} + \mathbf{k}\wedge\frac{\partial\mathbf{w}}{\partial z}\right) + v_x\frac{\partial\mathbf{w}}{\partial x} + v_y\frac{\partial\mathbf{w}}{\partial y} + v_z\frac{\partial\mathbf{w}}{\partial z}, \tag{A.51}$$

which can be written in the more compact form as

$$\left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial x}\right)\mathbf{i} + \left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial y}\right)\mathbf{j} + \left(\mathbf{v}\cdot\frac{\partial\mathbf{w}}{\partial z}\right)\mathbf{k} = \mathbf{v}\wedge(\nabla\wedge\mathbf{w}) + (\mathbf{v}\cdot\nabla)\,\mathbf{w}. \tag{A.52}$$

Similarly, interchanging $\mathbf{v}$ and $\mathbf{w}$,

$$\left(\mathbf{w}\cdot\frac{\partial\mathbf{v}}{\partial x}\right)\mathbf{i} + \left(\mathbf{w}\cdot\frac{\partial\mathbf{v}}{\partial y}\right)\mathbf{j} + \left(\mathbf{w}\cdot\frac{\partial\mathbf{v}}{\partial z}\right)\mathbf{k} = \mathbf{w}\wedge(\nabla\wedge\mathbf{v}) + (\mathbf{w}\cdot\nabla)\,\mathbf{v}. \tag{A.53}$$

Finally, substituting via Eqs. A.52 and A.53 in the rightmost side of Eq. A.47, we obtain

$$\nabla\,(\mathbf{v}\cdot\mathbf{w}) = (\mathbf{v}\cdot\nabla)\,\mathbf{w} + (\mathbf{w}\cdot\nabla)\,\mathbf{v} + \mathbf{v}\wedge(\nabla\wedge\mathbf{w}) + \mathbf{w}\wedge(\nabla\wedge\mathbf{v})\,, \tag{A.54}$$

which is exactly Eq. A.46 that, letting $\mathbf{w} = \mathbf{v}$, gives the sought last relation in the set Eq. A.45.

Some other useful vector analysis results:

- the divergence of the gradient of a scalar function f is the Laplacian of f: $\nabla \cdot \nabla f = \nabla^2 f = \Delta f$;
- the divergence of the curl is zero for every continuously differentiable vector field: $\nabla \cdot \nabla \wedge \mathbf{v} = 0$;
- the curl of gradient of every scalar function continuously twice-differentiable is zero: $\nabla \wedge \nabla f = 0$;
- the curl of every vector function continuously twice-differentiable is (see Eq. A.45) $\nabla \wedge (\nabla \wedge \mathbf{v}) = \nabla \nabla \cdot \mathbf{v} - \nabla^2 \mathbf{v}$.

A.4 Basic Concepts and Theorems of Vector Calculus

Theorem A.1, *divergence* theorem (also known as Gauss's or Ostrogradsky's theorem), Theorem A.2, *gradient* theorem, and Theorem A.3, *Stokes* theorem, are all relevant to fluid dynamics and in this book we have used them. Another relevant theorem is *Green's* theorem, which is actually a corollary of the divergence theorem in 2D.

They are stated as follows:

Theorem A.1 (Divergence theorem) *Let's give C as a subset of $\mathbb{R}^3$ which is compact and delimited by a piecewise smooth boundary $S \equiv \partial C$. If $\mathbf{A}$ is a continuously differentiable vector field in a set $N \supset C$, then*

$$\int_C \nabla \cdot \mathbf{A} \mathrm{d}V = \int_S \mathbf{A} \cdot \mathbf{n} \, \mathrm{d}\sigma, \tag{A.55}$$

where $\mathbf{n}$ is the outward unit vector in each point of S.

Because every scalar field can be written as a divergence of a vector field, a consequence of Theorem A.1 is the statement in the next corollary.

Corollary A.1 *Given a continuously differentiable scalar field ϕ defined in a domain containing C, it results in*

$$\int_S \phi \mathbf{n} \, \mathrm{d}\sigma = \int_C \nabla \phi \, \mathrm{d}V. \tag{A.56}$$

Proof Put $\mathbf{A} = \phi \mathbf{a}$, where $\mathbf{a}$ is an arbitrary constant non-zero vector. By the thesis of the divergence theorem and in virtue of the tenth relation in the set Eq. A.45,

$$\int_S \phi \mathbf{a} \cdot \mathbf{n} \, \mathrm{d}\sigma = \int_C \nabla \cdot \phi \mathbf{a} \, \mathrm{d}V = \int_C (\phi \nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla \phi) \, \mathrm{d}V. \tag{A.57}$$

In the rightmost side of the above equation, $\nabla \cdot \mathbf{a} = 0$ because $\mathbf{a}$ is a constant vector; consequently the equality in Eq. A.57 reduces to

$$\mathbf{a} \cdot \left(\int_S \phi \mathbf{n} \mathrm{d}\sigma - \int_S \nabla \phi \mathrm{d}V \right) = 0. \tag{A.58}$$

Given the arbitrarity of $\mathbf{a}$, the above equation implies what we wanted to prove, i.e. Eq. A.56. □

Theorem A.2 (Gradient theorem) *If* $\mathbf{F}$ *is a differentiable vector field defined in a set* C *such that* $\mathbf{F} = \nabla \Phi$*, where* Φ *is a continuously differentiable scalar function defined in* C *(i.e.* $\mathbf{F}$ *is conservative), then, given any regular curve* γ *connecting two generic points* $\mathbf{r}_1 = (x_1, y_1, z_1)$ *and* $\mathbf{r}_1 = (x_2, , y_2, z_2)$ *belonging to* C *and oriented from* $\mathbf{r}_1$ *to* $\mathbf{r}_2$*, it results in*

$$\int_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_\gamma \nabla \Phi \cdot \mathrm{d}\mathbf{r} = \Phi(x_2, , y_2, z_2) - \Phi(x_1, y_1, z_1). \tag{A.59}$$

Theorem A.3 (Stokes' theorem) *Let us consider a simple, closed curve* γ *in* $\mathbb{R}^3$ *and a generic smooth oriented surface,* S*, having* γ *as boundary. If* $\mathbf{A}$ *is a continuously differentiable vector field in a region containing* S*, then*

$$\int_S \nabla \wedge \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}\sigma = \int_\gamma \mathbf{F} \cdot \mathrm{d}\mathbf{r}. \tag{A.60}$$

The link between Stokes' and the gradient theorem is evident.

A.5 Tensors

As we said, a vector in an n-dimensional space is represented as a 1D array (column or line array) composed by n elements with respect to a given basis.

Tensors are a generalization of vectors from 1D to multi-dimensional arrays. The total number of indices of a tensor is the *order* of the tensor (called also rank or degree). For example, a second *order* tensor can be denoted, in component form, by a pair of indices, like we did for the Reynolds tensor Π_{ij}, with i and j both running from 1 to n. We can think of a second order tensor as a $n \times n$ matrix A_{ij} where i identifies the line and j the column. Examples of order two tensors met in this book

are, other than the Reynolds tensor, the viscosity tensor σ_{ij}, the stress tensor τ_{ij} and the rate of strain tensor $\dot{\gamma}_{ij}$.

Tensors can be represented with indices in different positions, like T^{ij} or T_i^j, according to different laws of transformation upon change of basis. When the index is low we talk of *covariant* tensor and of *contravariant* when index is up. In this book, we have met covariant and contravariant tensors only in the short chapter 6, when dealing with relativistic fluid dynamics. It is so not worth deepening here these concepts, while it is instead important mentioning that in many books a *vectorial* notation is used for tensors instead of the component one, in similarity with what is done for vectors (1D tensors). This means that the Reynolds tensor, for instance, can be represented both as Π_{ij} and $\underline{\mathbf{\Pi}}$.

Note that if the divergence of a vector (1D tensor) is a scalar, the divergence of a 2D tensor is a vector (in general, the results of the divergence operation reduces the tensor order by one):

$$\nabla \cdot \underline{\mathbf{\Pi}} = \frac{\partial \Pi_{ik}}{\partial x_k}, \tag{A.61}$$

where, as usual, Einstein's summation convention is adopted.

A.6 Further Readings

There are plenty of textbooks on vector analysis, and calculus. I limit to cite a book of myself, [11], which is quite specific to the content of this introductory book on fluid dynamics. Deeper insight on the topics is given in the classic [20] book.

References

1. Batchelor, G. K.: An Introduction to Fluid Dynamics, 2nd revised edition. Cambridge University Press, UK (2000)
2. Landau, L.D., Lifshits, E.M.: Fluid Dynamics, Course of Theoretical Physics, vol. 6, 2nd rev. edition, Elsevier Pergamon Press, The Netherland (2013)
3. Kundu, P.K., Cohen, I.M.: Fluid Mechanics. Academic Press, Elsevier, The Netherland (2008)
4. Łukaszewicz, G., Kalita, P., Navier-Stokes Equations: An Introduction with Applications, Advances in Mechanics and Mathematics, vol. 34. Springer International Publishing, Cham (2016)
5. Kay, J.M., Nedderman, R.M.: An Introduction to Fluid Mechanics and Heat Transfer, 3rd edn. Cambridge University Press, UK (1974)
6. Tennekes, H., Lumley, J.L.: A First Course in Turbulence. MIT Press, USA (1972)
7. Davidson, P., Turbulence: An Introduction for Scientists and Engineers, 2nd edn. Oxford University Press, UK (2015)
8. Shu, F.H.: The Physics of Astrophysics, vols. 1 and 2, A Series of Books in Astronomy. University Science Books, USA (1991)
9. Clarke, J.K, Carlswell, F.: Principles of Astrophysical Fluid Dynamics. Cambridge University Press, UK (2007)
10. Shore, S.N.: Astrophysical Hydrodynamics, An Introduction, 2nd, revised edn. Wiley-VCH Verlag GmbH & Co, KGaA (DE) (2007)
11. Capuzzo Dolcetta, R.: Classical Newtonian Gravity. Unitext for Physics. Springer Nature, Cham (2019)
12. Fridman, A.M., Polyachenko, V.L.: Physics of Gravitating Systems I: Equilibrium and Stability. Springer, USA (1984)
13. Carraro, G.: Astrophysics of the Interstellar Medium. Unitext for Physics. Springer Nature, Cham (2021)
14. Wallace, D.H.: Gasdynamic Discontinuities, vol. 1875 in the Series Princeton Legacy Library. Princeton University Press, USA (1960)
15. Sedov, L.I.: Similarity and Dimensional Methods in Mechanics, 10th edn. CRC Press, Taylor and Francis Group, USA (1993)
16. Zel'Dovich, Y.B., Raizer, Y.P.: Physics of Shock Waves and High-temperature Hydrodynamic Phenomena. Courier Corporation, USA (2012)
17. Courant, R., Friedrichs, K.O.: Supersonic Flow and Shock Waves. Ulan Press (DE) (2012)

R. A. Capuzzo Dolcetta, *Physics of Fluids*, UNITEXT for Physics,
https://doi.org/10.1007/978-3-031-30750-8

18. Rezzolla, L., Zanotti, O.: Relativistic Hydrodynamics. Oxford University Press, UK (2013)
19. Schmitt, A.: Introduction to Superfluidity: Field-theoretical Approach and Applications, Lecture Notes in Physics, Band 888. Springer, Cham (2014)
20. Kemmer, N.: Vector Analysis A Physicist's Guide to the Mathematics of Fields in Three Dimensions. Cambridge University Press, UK (1977)

Index

R. A. Capuzzo Dolcetta, *Physics of Fluids*, UNITEXT for Physics,
https://doi.org/10.1007/978-3-031-30750-8