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Geometric Harmonic Analysis IV

Boundary Layer Potentials in Uniformly Rectifiable Domains, and Applications to Complex Analysis

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Aims and Scope

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Dedicated with love to our parents

Prefacing the Full Series

The current work is part of a series, comprised of five volumes, [68], [69], [70], [71], [72]. In broad terms, the principal aim is to develop tools in Real and Harmonic Analysis, of geometric measure theoretic flavor, capable of treating a broad spectrum of boundary value problems formulated in rather general geometric and analytic settings.

In Volume I ([68]) we establish a sharp version of Divergence Theorem (aka Fundamental Theorem of Calculus) which allows for an inclusive class of vector fields whose boundary trace is only assumed to exist in a nontangential pointwise sense.

Volume II ([69]) is concerned with function spaces measuring size and/or smoothness, such as Hardy spaces, Besov spaces, Triebel-Lizorkin spaces, Sobolev spaces, Morrey spaces, Morrey-Campanato spaces, spaces of functions of Bounded Mean Oscillations, etc., in general geometric settings. Work here also highlights the close interplay between differentiability properties of functions and singular integral operators.

The topic of singular integral operators is properly considered in Volume III ([70]), where we develop a versatile Calderón-Zygmund theory for singular integral operators of convolution type (and with variable coefficient kernels) on uniformly rectifiable sets in the Euclidean ambient, and the setting of Riemannian manifolds. Applications to scattering by rough obstacles are also discussed in this volume.

In Volume IV ([71]) we focus on singular integral operators of boundary layer type which enjoy more specialized properties (compared with generic, garden variety singular integral operators treated earlier in Volume III). Applications to Complex Analysis in several variables are subsequently presented, starting from the realizations that many natural integral operators in this setting, such as the Bochner-Martinelli operator, are actual particular cases of double layer potential operators associated with the complex Laplacian.

In Volume V ([72]), where everything comes together, finer estimates for a certain class of singular integral operators (of chord-dot-normal type) are produced in a manner which indicates how their size is affected by the (infinitesimal and global) flatness of the "surfaces" on which they are defined. Among the library of double layer potential operators associated with a given second-order system, we then identify those double layers which fall under this category of singular integral operators. It is precisely for this subclass of double layer potentials that Fredholm theory may then be implemented assuming the underlying domain has a compact boundary, which is sufficiently flat at infinitesimal scales. For domains with unbounded boundaries, this very category of double layer potentials may be outright inverted, using a Neumann series argument, assuming the "surface" in question is sufficiently flat globally. In turn, this opens the door for solving a large variety of boundary value problems for second-order systems (involving boundary data from Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, Hardy spaces, Sobolev spaces, BMO, VMO, Morrey spaces, Hölder spaces, etc.) in a large class of domains which, for example, are allowed to have spiral singularities (hence more general than domains locally described as upper-graphs of functions). In the opposite direction, we show that the boundary value problems formulated for systems lacking such special layer potentials may fail to be Fredholm solvable even for really tame domains, like the upper half-space, or the unit disk. Save for the announcement [67], all principal results appear here in print for the first time.

We close with a short epilogue, attempting to place the work undertaken in this series into a broader picture. The main goal is to develop machinery of geometric harmonic analysis flavor capable of ultimately dealing with boundary value problems of a very general nature. One of the principal tools (indeed, the piecè de résistance) in this regard is a new and powerful version of the Divergence Theorem, devised in Volume I, whose very formulation has been motivated and shaped from the outset by its eventual applications to Harmonic Analysis, Partial Differential Equations, Potential Theory, and Complex Analysis. The fact that its footprints may be clearly recognized in the makeup of such a diverse body of results, as presented in Volumes II-V, serves as testament to the versatility and potency of our brand of Divergence Theorem. Alas, our enterprise is multifaceted, so its success is crucially dependent on many other factors. For one thing, it is necessary to develop a robust Calderón-Zygmund theory for singular integrals of boundary layer type (as we do in Volumes III-IV), associated with generic weakly elliptic systems, capable of accommodating a large variety of function spaces of interest considered in rather inclusive geometric settings (of the sort discussed in Volume II). This renders these (boundary-to-domain) layer potentials useful mechanisms for generating lots of null-solutions for the given system of partial differential operators, whose format is compatible with the demands in the very formulation of the boundary value problem we seek to solve. Next, in order to be able to solve the boundary integral equation to which matters are reduced in this fashion, the success of employing Fredholm theory hinges on the ability to suitably estimate the essential norms of the (boundary-to-boundary) layer potentials. In this vein, we succeed in relating the distance from such layer potentials to the space of compact operators to the flatness of the boundary of the domain in question (measured in terms of infinitesimal mean oscillations of the unit normal) in a desirable manner which shows that, in a precise quantitative fashion, *the flatter the domain the smaller the proximity to compact operators*. This subtle and powerful result, bridging between analysis and geometry, may be regarded as a far-reaching extension of the pioneering work of Radon and Carleman in the early 1900's.

Ultimately, ourwork aligns itself with the program stemming from A.P. Calderón's 1978 ICM plenary address in which he advocates the use of layer potentials "*for much more general elliptic systems* [*than the Laplacian*]" – see [9, p. 90], and may be regarded as an optimal extension of the pioneering work of E.B. Fabes, M. Jodeit, and N.M. Rivière in [24] (where layer potential methods have been first used to solve boundary value problems for the Laplacian in bounded $\mathscr{C}^1$ domains). In this endeavor, we have been also motivated by the problem[1] posed by A.P. Calderón on [9, p. 95], asking to identify the function spaces on which singular integral operators (of boundary layer type) are well defined and continuous. This is relevant since, as Calderón mentions, "*A clarification of this question would be very important in the study of boundary value problems for elliptic equations* [*in rough domains*]. *The methods employed so far seem to be insufficient for the treatment of these problems.*" We also wish to mention that our work is also in line with the issue raised as an open problem by C. Kenig in [44, Problem 3.2.2, pp. 116–117], where he asked whether operators of layer potential type may be inverted on appropriate Lebesgue and Sobolev spaces in suitable subclasses on NTA domains with compact Ahlfors regular boundaries.

The task of making geometry and analysis work in unison is fraught with difficulties, and only seldom can a two-way street be built on which to move between these two worlds without loss of information. Given this, it is actually surprising that in many instances we come very close to having optimal hypotheses, almost an accurate embodiment of the slogan *if it makes sense to write it, then it's true.*

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[1] In the last section of [9], simply titled "*Problems*," Calderón singles two directions for further study. The first one is the famous question whether the smallness condition on $\|a'\|_{L^\infty}$ (the Lipschitz constant of the curve $\{(x, a(x)) : x \in \mathbb{R}\}$ on which he proved the L^2-boundedness of the Cauchy operator) may be removed (as is well known, this has been solved in the affirmative by Coifman, McIntosh, and Meyer in [15]). We are referring here to the second (and final) problem formulated by Calderón on [9, p. 95].

Description of Volume IV

The bulk of the results in this volume amounts to a versatile Calderón-Zygmund theory for singular integral operators of layer potential type in open sets with uniformly rectifiable boundaries. As such, we find it appropriate to attempt to place the portion of our work pertaining to this brand of Calderón-Zygmund theory into a broader picture.

Traditionally, the label "Calderón-Zygmund theory" has been applied to a distinguished body of works primarily pertaining to the mapping properties of singular integral operators on Lebesgue spaces, considered in various ambients (ranging from the standard Euclidean setting to arbitrary spaces of homogeneous type). For example, thanks to the fundamental work of G. David and S. Semmes (cf. [19], [20]), it is now understood that, within the Euclidean setting, the most general environment where L^p boundedness results, with $1 < p < \infty$, are available for large classes of singular integral operators is that of uniformly rectifiable subsets[2] of $\mathbb{R}^n$. This being said, demands arising in related areas (e.g., boundary value problems in rough domains) have stimulated interest in expanding the scope of the classical Calderón- Zygmund theory as to include other category of function spaces, such as: Hardy-like spaces, Sobolev-like spaces, Morrey spaces, Morrey-Campanato spaces, Lorentz spaces, BMO spaces, VMO spaces, Hölder spaces, Besov spaces, Triebel-Lizorkin spaces, among many others, while retaining the general nature of the geometric environment. Through the efforts of many people, steady progress has been registered (for example, basic singular integral operators on Lipschitz surfaces are now fairly well-understood) but significant obstacles have limited the success of such an endeavor. Compared with Lebesgue spaces, some of the categorically distinct issues which arise in dealing with singular integral operators acting on many of the scales of spaces listed above are those of *cancelation* and/or *smoothness*. Such aspects are front and center within the realm of Hardy spaces, Sobolev spaces, Morrey- Campanato spaces, BMO spaces, Besov spaces, Triebel-Lizorkin spaces, etc. This calls for technology capable of dealing with the issues of cancelation and smoothness in very general

[2] however, the quality of being uniformly rectifiable loses its central significance if in place of Lebesgue spaces other scales of spaces (e.g., Hölder) are considered.

rough settings, such as sets which are uniformly rectifiable, or even merely Ahlfors regular. The brand of Divergence Theorem developed in Volume I ([68]) goes a long way in addressing this need, by yielding powerful and versatile integration by parts formulas and trace results in very general settings. Succinctly put, various basic aspects of the theory of singular integral operators require subtle cancelations properties which our version of the Divergence Theorem can, for the first time, accommodate.

To offer a concrete example, start in the context of the entire Euclidean space $\mathbb{R}^n$ and consider a linear and continuous mapping $T : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ which extends to a bounded operator on $L^2(\mathbb{R}^n, \mathcal{L}^n)$ and has the property that its Schwartz kernel $K(\cdot,\cdot)$ satisfies

$$K \in L^1_{\rm loc}\big(\mathbb{R}^n \times \mathbb{R}^n \backslash {\rm diag},\ \mathcal{L}^n \otimes \mathcal{L}^n\big) \tag{0.0.1}$$

and there exist some constant $C \in (0,\infty)$ together with some exponent $\gamma \in (0,1]$ such that

$$\begin{aligned}&\text{for every } x, y \in \mathbb{R}^n \text{ with } x \neq y, \text{ and each } z \in \mathbb{R}^n \text{ with } |y-z| < \tfrac{1}{2}|x-y|,\\ &\text{one has } |K(x,y)| \leq \frac{C}{|x-y|^n} \text{ and } |K(x,y) - K(x,z)| \leq \frac{C|y-z|^\gamma}{|x-y|^{n+\gamma}}.\end{aligned} \tag{0.0.2}$$

A classical result in harmonic analysis (see, e.g., the proof of [65, Théorème 3, pp. 237–238]) is that[3]

$$\begin{aligned}&T \text{ extends to a linear and continuous mapping from the Hardy}\\ &\text{space } H^1(\mathbb{R}^n) \text{ into itself if and only if } T^\top(1) = 0, \text{ in the sense}\\ &\text{that } Ta \text{ has mean zero, i.e., } \int_{\mathbb{R}^n} Ta\, d\mathcal{L}^n = 0 \text{ for all } H^1\text{-atoms } a.\end{aligned} \tag{0.0.3}$$

In the entire Euclidean space, there are certain natural ways of checking the cancelation condition required in (0.0.3) is satisfied, such as using the Fourier transform in the case when T is a Fourier multiplier.

It has long know that this type of characterization of boundedness on Hardy spaces is valid in the more general setting of spaces of homogeneous type, in the sense of R. Coifman and G. Weiss. See, e.g., [16, p. 599], [22, Proposition 4.17, p. 104], [22, Theorem 4.27, p. 112]. This being said, when the ambient $\mathbb{R}^n$ is replaced by a uniformly rectifiable set $\Sigma \subseteq \mathbb{R}^n$, an environment in which large classes of singular integral operators are bounded on L^2 with respect to $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$, there are

[3] the same is true with $H^1(\mathbb{R}^n)$ replaced by $H^p(\mathbb{R}^n)$ for each $p \in \left(\frac{n-1}{n-1+\gamma}, 1\right]$.

basically no known examples of singular integral operators bounded on $H^1(\Sigma, \sigma)$ since the actual verification of the crucial cancelation condition

$$\int_{\Sigma} T\, a\, d\sigma = 0 \text{ whenever } a \text{ is an atom on } \Sigma \tag{0.0.4}$$

becomes rather delicate even in the basic case when T is the (transpose) harmonic double layer operator on Σ. As we shall presently see, the brand of the Divergence Theorem developed in Volume I ([68]) can efficiently deal with such an issue.

Ultimately, the picture that emerges is that Calderón-Zygmund theory is a multi-faceted body of results aimed at describing how singular integral operators behave in a multitude of geometric and analytic settings. The final goal becomes understanding the intimate correlation between geometry and analysis from this perspective and, eventually, building a "two-way street" allowing to pass information back and forth between them in an optimal fashion. The study of singular integral operators on Lebesgue spaces becomes a chapter, albeit a fundamental one, in this theory and our own work contributes to this on-going program by vigorously promoting this more general and inclusive point of view. The good news is that, as formidable as this already is, such a version of Calderón-Zygmund theory is yet to reach the height of its splendor. This is a vision worth sharing!

This portion of our work has also been motivated by the problem posed by A.P. Calderón on [9, p. 95], asking to identify the function spaces on which singular integral operators (of boundary layer type) are well defined and continuous. Calderón goes on to mention that: "*A clarification of this question would be very important in the study of boundary value problems for elliptic equations* [*in rough domains*]. *The methods employed so far seem to be insufficient for the treatment of these problems*." We shall employ the body of results established in this volume in the study of boundary value problems in rather inclusive geometric settings and with boundary data in a multitude of function spaces in Volume V ([72]).

Let us now describe the contents of the present volume in greater detail. Chapter 1 deals with singular integral operators of boundary layer type on Lebesgue and Sobolev spaces. One of the main points is that generic Calderón-Zygmund convolution-type SIO's (of the sort considered in [70, Chapter 2]) are not expected to induce well-defined mappings on Sobolev spaces on UR sets. Indeed, for this to happen, the integral kernel must possess a special algebraic structure. For example, this is present in the conormal derivative, or a tangential derivative, of the fundamental solution of a weakly elliptic second-order system, and these are the types of singular integral operators we focus on in this chapter. In addition to the discussion on the history and physical interpretations of the classical harmonic layer potentials from §1.1, topics treated in this chapter include "tangential" singular integral operators, whose kernels exhibit a special algebraic structure, strongly reminiscent of tangential derivatives (in §1.2), volume and integral operators of boundary layer type associated with a given open set of locally finite perimeter and a given weakly elliptic system (in §1.3), a multitude of relevant examples and alternative points of view (in §1.4), a rich function theory of Calderón-Zygmund type for boundary layer potentials associated with a given weakly elliptic system and an open set with a UR boundary (in §1.5), the interpretation of the Cauchy and Cauchy-Clifford operators as double layer potential operators, and the host of consequences naturally derived from such a perspective (in §1.6), the description of kernels and images of singular integral operators of boundary layer type (in §1.7), and how to modify boundary layer potential operators as to increase the class of functions to which they may be applied (in §1.8).

Chapter 2 is largely concerned with layer potential operators on Hardy, BMO, VMO, and Hölder spaces defined on boundaries of UR domains. Once again, it takes a special algebraic structure of the integral kernel for a singular integral operator to map either of these spaces into itself, and the Divergence Theorem devised in Volume I ([68]) plays a crucial role in ensuring this is indeed the case; see the discussion surrounding (0.0.4). In fact, the same type of philosophy prevails in relation to the action of double layer potential operators on Calderón, Morrey-Campanato, and Morrey spaces, studied in Chapter 3, and also for the action of double layer potential operators on Besov and Triebel-Lizorkin spaces, a subject discussed at length in Chapter 4.

The above considerations bring into focus the following fundamental question: *describe the most general classes of singular integral operators on the boundary of an arbitrary given* UR *domain* $\Omega \subseteq \mathbb{R}^n$ *which map Hardy*, BMO, VMO, *Hölder, Besov, and Triebel-Lizorkin spaces defined on* $\partial\Omega$ *boundedly into themselves.* We provide an answer to this basic question in Chapter 5 through the consideration of what we call "generalized double layers." The main attribute of these singular integral operators is the fact that their integral kernels involve the inner product of the outward unit normal (to the "surface" on which this integral operator is defined) with a divergence-free vector-valued function. Such an algebraic structure confers excellent cancelation properties (brought to bear by the Divergence Theorem) which, in turn, allow us to establish boundedness results for these generalized double layers on a multitude of basic scales of function spaces which, in addition to standard Lebesgue spaces (and its Muckenhoupt weighted version), now also includes boundary Sobolev spaces, Hardy spaces, Hölder spaces, the John-Nirenberg space BMO, the Sarason space VMO, Besov spaces, and Triebel-Lizorkin spaces, among others. In the last section of Chapter 5 we take another look at Riesz transforms from the perspecive of generalized double layers.

In Chapter 6 we develop a theory of boundary layer potentials associated with the Stokes system of linear hydrostatics, and related topics. Among other things, we establish Green-type formulas, derive mapping properties for the aforementioned boundary layer potential operators, and prove Fatou-type results, in settings which are sharp from a geometric/analytic point of view. Once again, the brand of Divergence Theorem discussed in Volume I ([68]) plays a prominent role in carrying out this program.

Chapter 7 contains applications of the tools and results developed so far to analysis in several complex variables. It has long been known that Complex Analysis, Geometric Measure Theory, and Harmonic Analysis tightly interface in the complex plane (see, e.g., J. Garnett's book [28] and the earlier references cited there). This is an area of mathematics which continues to undergo major transformations thanks to spectacular contemporary advances, such as G. David's characterization of the L^2 boundedness of the Cauchy operator in terms of Ahlfors regularity, and X. Tolsa's results on analytic capacity, to name just some. However, this rich interplay between these branches of mathematics seems to have been considerably less explored in the higher-dimensional setting, involving several complex variables. The main goal in §7.1-§7.7 is to further our present understanding of this latter aspect. Themes covered in this chapter include: CR-functions and differential forms on boundaries of sets of locally finite perimeter (in §7.1), integration by parts formulas involving the $\bar{\partial}$-operator on sets of locally finite perimeter (in §7.2), the Bochner-Martinelli integral operator (in §7.3), a sharp version of the Bochner-Martinelli-Koppelman formula (in §7.4), the Extension Problem for Hölder CR-functions on boundaries of Ahlfors regular domains (in §7.5), the Extension Problem for Lebesgue and bounded (or vanishing) mean oscillation functions on boundaries of uniformly rectifiable domains (in §7.6), as well as the $\bar{\partial}$-operator and the Dolbeault complex on uniformly rectifiable sets (in §7.7).

Lastly, in Chapter 8 we study Hardy spaces (in which the size is measured via the nontangential maximal operator) for certain second-order weakly elliptic operators in the complex plane. We start with Bitsadze's operator $\partial_{\bar{z}}^2$ in the unit disk of the complex plane in §8.1, and expand the scope of this analysis in §8.2 by considering the more inclusive family of second-order differential operators $L_\lambda := \partial_{\bar{z}}^2 - \lambda^2 \partial_z^2$, indexed by the parameter $\lambda \in \mathbb{C}$. In all cases, the goal is to characterize the space of null-solutions (satisfying approriate size conditions, decsribed in terms of the nontangential maximal operator) and to identify precisely the corresponding spaces of boundary traces.

Contents

Chapter 1
Layer Potential Operators on Lebesgue and Sobolev Spaces

While certain features of singular integral operators are visible from geometric properties in general spaces of homogeneous type (such as the Fractional Integration Theorem, or the abstract boundedness criteria from [16]) other, more delicate properties (typically cancelation sensitive) require fully employing the resourcefulness of the algebraic/geometric ambient and, crucially, involve differential calculus. The boundedness of singular integral operators of boundary layer type on Sobolev spaces falls under the latter category. Indeed, the boundary Sobolev spaces developed in [69, Chapter 11] offer a functional context in which a variety of singular integral operators (SIO's for short) of boundary layer type act in a natural fashion. The main goal of this chapter is to elaborate on this idea. In the process, we shall see that the topics of boundary Sobolev spaces and SIO's are closely intertwined. Specifically, we are going to employ SIO's as a tool to further our understanding of the brand of boundary Sobolev spaces introduced in [69, Chapter 11], ultimately establishing a two-way bridge between these two areas.

1.1 Comments on History and Physical Interpretations of Harmonic Layer Potentials

To put matters in a broader perspective, we begin by making some remarks pertaining to the history and origins of the classical boundary layer potentials associated with the Laplace operator, i.e., when $L = \Delta$. Here we shall also elaborate on ties with mathematical physics, by highlighting connections between these boundary layer potentials and charge distributions on surfaces. Throughout, we shall let $\Omega \subseteq \mathbb{R}^n$ be an Ahlfors regular domain, with geometric measure theoretic outward unit normal ν, and use σ to abbreviate $\mathcal{H}^{n-1} \lfloor \partial\Omega$.

In many ways, solving a PDE amounts to "undoing" (i.e., inverting) a differential operator, and this continues to be the case when dealing with boundary value problems. For example, when working with the Laplacian $\Delta = \partial_1^2 + \cdots + \partial_n^2$ in $\mathbb{R}^n$, the fundamental solution

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$$E_\Delta(x) := \begin{cases} \dfrac{1}{\omega_{n-1}(2-n)}\dfrac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\ \\ \dfrac{1}{2\pi}\ln|x| & \text{if } n = 2, \end{cases} \tag{1.1.1}$$

(where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$) plays a basic role, since this may be used to generate lots of harmonic functions in a given domain $\Omega \subseteq \mathbb{R}^n$, such as

$$\Omega \ni x \longmapsto E_\Delta(x-y) \text{ where } y \in \partial\Omega \text{ is an arbitrary fixed point.} \tag{1.1.2}$$

In fact, one may consider linear combinations of such mappings, i.e.,

$$\Omega \ni x \longmapsto \sum_{j=1}^{N} E_\Delta(x-y_j)\cdot\lambda_j \text{ with } \{y_j\}_{1\leq j\leq N} \subseteq \partial\Omega \text{ and } \{\lambda_j\}_{1\leq j\leq N} \subseteq \mathbb{C}. \tag{1.1.3}$$

More generally, one may consider a "density" $f(y) \in \mathbb{C}$ at each point $y \in \partial\Omega$ and use it to create a weighted infinite "sum" of functions as in (1.1.2). This gives rise to the boundary-to-boundary integral operator

$$\Omega \ni x \longmapsto \int_{\partial\Omega} E_\Delta(x-y)f(y)\,\mathrm{d}\sigma(y). \tag{1.1.4}$$

We are therefore led to consider the classical harmonic single layer operator as the mapping assigning to each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\Big)$ if $n \geq 3$ and each function $f \in L^1\Big(\partial\Omega, \ln(2+|x|)\sigma(x)\Big)$ if $n = 2$ the harmonic function defined at each point $x \in \mathbb{R}^n \setminus \partial\Omega$ by

$$\begin{aligned} \mathscr{S}_\Delta f(x) &:= \int_{\partial\Omega} E_\Delta(x-y)f(y)\,\mathrm{d}\sigma(y) \\ &= \begin{cases} \dfrac{1}{(2-n)\omega_{n-1}}\displaystyle\int_{\partial\Omega}\dfrac{f(y)}{|x-y|^{n-2}}\,\mathrm{d}\sigma(y) & \text{if } n \geq 3, \\ \dfrac{1}{2\pi}\displaystyle\int_{\partial\Omega}(\ln|x-y|)f(y)\,\mathrm{d}\sigma(y) & \text{if } n = 2. \end{cases} \end{aligned} \tag{1.1.5}$$

Of course, one may re-fashion the process which, starting with (1.1.2), has led to (1.1.4), but now taking directional derivatives of E_Δ. For the purpose of dealing with boundary value problems it is most natural to consider the normal derivative of E_Δ, a choice which gives rise to the boundary-to-boundary integral operator

$$\Omega \ni x \longmapsto \int_{\partial\Omega} \partial_{\nu(y)}\big[E_\Delta(x-y)\big]f(y)\,\mathrm{d}\sigma(y), \tag{1.1.6}$$

where ν is the outward unit normal to Ω. This is how the boundary-to-boundary harmonic double layer potential operator comes into play, as the mapping assigning to each $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ the harmonic function

$$\begin{aligned}\mathcal{D}_\Delta f(x) &:= \int_{\partial\Omega} \partial_{\nu(y)}\big[E_\Delta(y-x)\big] f(y)\,\mathrm{d}\sigma(y)\\ &= \frac{1}{\omega_{n-1}}\int_{\partial\Omega} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y) \ \text{ for each } x \in \mathbb{R}^n \setminus \partial\Omega. \end{aligned} \tag{1.1.7}$$

The single layer potential operator has already appeared in the work of Carl Friedrich Gauss as early as 1838. More specifically, in [29] and [30], Gauss introduces boundary integral equations of the first kind and used them to perform numerical computations[1]. Among other things, Gauss has established jump-formulas for the normal derivatives of the harmonic single layer, and re-discovered what is now commonly referred to as Green-Gauss formula.

As with the single layer, the double layer is a mechanism for creating a multitude on null-solutions for the Laplacian in Ω, since for each $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ the function $u := \mathcal{D}_\Delta f$ satisfies

$$u \in \mathscr{C}^\infty(\Omega) \ \text{ and } \ \Delta u = 0 \ \text{ in } \ \Omega. \tag{1.1.8}$$

Moreover, if Ω is a UR domain then the nontangential boundary trace of u may be explicitly computed as

$$u\Big|_{\partial\Omega}^{\text{n.t.}} = (\tfrac{1}{2}I + K_\Delta)f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{1.1.9}$$

where I is the identity and K_Δ is the principal-value singular integral operator

$$(K_\Delta f)(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \partial_{\nu(y)}\big[E_\Delta(x-y)\big] f(y)\,\mathrm{d}\sigma(y) \ \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{1.1.10}$$

As yet another motivation for the format of $\mathscr{S}_\Delta$, recall Isaac Newton's law of universal gravitation according to which two particles, located at $x_0, x \in \mathbb{R}^3$ and having masses $m_0, m \in (0,\infty)$, attract one another with a force of magnitude $cm_0m/|x_0-x|^2$, where c is a universal constant. Hence, in vector notation which also incorporates the direction, the force with which the body located at x is attracted to the body located at x_0 is given by

$$\vec{F}_{x_0,x} := \frac{cm_0m}{|x_0-x|^2}\frac{x_0-x}{|x_0-x|} = cm_0m\frac{x_0-x}{|x_0-x|^3}. \tag{1.1.11}$$

Keeping x_0 fixed and regarding this as a function of x yields a conservative vector field in $\mathbb{R}^3 \setminus \{x_0\}$, since

[1] Gauss actually published extensive tables and graphs of numerical results obtained, at least in part, using such boundary integral equations; see [17] for an informative account on this topic

$$cm_0 m \frac{x_0 - x}{|x_0 - x|^3} = \nabla_x \Big(\frac{cm_0 m}{|x_0 - x|} \Big) \text{ for each } x \in \mathbb{R}^3 \setminus \{x_0\}. \tag{1.1.12}$$

This brings into focus the potential function possessed by the gravitational field, i.e.,

$$U_{x_0}(x) := \frac{cm_0 m}{|x_0 - x|} \text{ for each } x \in \mathbb{R}^3 \setminus \{x_0\}, \tag{1.1.13}$$

which is a scalar-valued function satisfying Laplace's equation

$$\Delta U_{x_0} = 0 \text{ in } \mathbb{R}^3 \setminus \{x_0\}. \tag{1.1.14}$$

Consider now the case when $\Omega \subseteq \mathbb{R}^3$ is an Ahlfors regular domain, x_0 is a point in $\mathbb{R}^3 \setminus \partial\Omega$, and an attractive mass is distributed over the "surface" (or "layer") $\partial\Omega$. Denote by σ the "surface" measure $\mathcal{H}^2 \lfloor \partial\Omega$, and denote by $m(y)$ the mass density at each point $y \in \partial\Omega$. The superposition principle, according to which for all linear systems the net response caused by a family of stimuli is the sum of the responses that would have been caused by each stimulus individually, then shows that the overall potential associated with this system is

$$cm_0 \int_{\partial\Omega} \frac{m(y)}{|x_0 - y|} \, d\sigma(y). \tag{1.1.15}$$

Up to normalization and readjusting notation, this agrees with the harmonic single layer potential operator (1.1.5) (acting on the density m, and evaluated at x_0) in the three-dimensional setting.

The classical harmonic double layer potential operator, recalled in (1.1.7), is typically associated with the names of Carl Neumann and Henri Poincaré. For this reason, (1.1.7) or, rather, its principal-value version (cf. (A.0.101)), is sometimes referred to as the Neumann-Poincaré operator[2]. Following Gauss' pioneering study of the first kind integral equation associated with the single layer potential operator, the next major progress was registered with Carl Neumann's work on the double layer potential operator, a topic on which he has published extensively (including his 1877 monograph [82]). Neumann's crowning achievement was his solution of the second kind integral equation associated with the double layer potential operator in convex domains, via what we presently call a Neumann series. Neumann's proof of the convergence of the method of iterations in this setting uses rather sophisticated geometric and measure theoretic arguments.

In essence, this constitutes an early example of "hard analysis" in potential theory, which set the tone for a great deal of work that followed. Indeed, many subsequent generalizations of Neumann's techniques have also been confined to hard harmonic analysis. For example, in a genuine tour de force, Poincaré has subsequently succeeded (in his long and technical paper [84] published in 1897) to replace the convexity hypothesis by suitable smoothness assumptions. More specifically, Poincaré accomplished the task of finding an alternative proof the convergence of the Neumann series for nonconvex domains, via an approach which required the underlying set

[2] a name first used by T. Carleman in his Ph. D. thesis [12]

to have a $\mathscr{C}^2$ boundary and the functions involved to be fairly regular. Along the way, Poincaré develops in [84] an astounding array of novel techniques. While he himself scarcely employed them further, other mathematicians picked up the mantle and continued to develop C. Neumann and H. Poincaré's groundbreaking work in a multitude of different directions, including Arthur Korn ([46], [47], [48], [49], [50]) Vladimir Steklov ([97], [98], [99], [100], [101]), Stanislaw Zaremba ([106], [107], [108]), Johann Karl August Radon ([86], [87]), and Torsten Carleman ([12]), just to name a few. The classical monographs [31] of N.M. Günter, [43] of O.D. Kellogg's, [91] of F. Riesz and B. Sz.-Nagy, and [95] of S.L. Sobolev are still quite readable, informative accounts on this and related topics.

To elaborate on the physical significance of the harmonic double layer potential operator (1.1.7) work in the three-dimensional setting. Specifically, consider an Ahlfors regular domain $\Omega \subseteq \mathbb{R}^3$ and suppose attractive mass is distributed over the "surface" (or "layer") $\partial\Omega$. As before, we shall let σ denote the "surface" measure $\mathcal{H}^2 \lfloor \partial\Omega$, and denote by $m(y)$ the mass density at each point $y \in \partial\Omega$. Having fixed some (unit) point mass $x \in \Omega$, the gravitational force attracting each point mass y to x is (cf. (1.1.11))

$$\vec{F}_x(y) := c\, m(y) \frac{x-y}{|x-y|^3}. \tag{1.1.16}$$

This gives rise to a vector field $\vec{F}_x : \partial\Omega \to \mathbb{R}^3$, whose flux on the "surface" $\partial\Omega$ is

$$\int_{\partial\Omega} \big\langle \nu(y), \vec{F}_x(y) \big\rangle \, d\sigma(y) = c \int_{\partial\Omega} \frac{\langle \nu(y), x-y \rangle}{|x-y|^3} m(y)\, d\sigma(y) \tag{1.1.17}$$

which, up to normalization and readjusting notation, agrees with the harmonic double layer potential operator (1.1.7) (acting on the density m, and evaluated at x) in the three-dimensional setting.

Here another point of view which provides an alternative physical interpretation of the harmonic double layer (1.1.7) in the three-dimensional setting. The starting point is Coulomb's law describing the field of electrostatic forces in a completely similar manner to Newton's law (1.1.11) (with m_0 and m electric charges in this context). Next, and suppose $\Omega \subseteq \mathbb{R}^3$ is a open set with a sufficiently regular boundary, assumed to be an insulator. Fix $\varepsilon > 0$ and define two[3] "parallel layers," namely $S_\varepsilon^\pm := \{y \pm \varepsilon\nu(y) : y \in \partial\Omega\}$, where ν is the outward unit normal vector to Ω. Assume $S_\varepsilon^\pm$ are conductors charged with distribution[4] $\pm\varepsilon^{-1}\rho(y)$ for each $y \in \partial\Omega$. According to the superposition principle recalled earlier, the electric field generated by these charges is given by ∇U_ε where U_ε is the associated potential function, i.e.,

$$U_\varepsilon(x) := \int_{\partial\Omega} \left\{ \frac{\varepsilon^{-1}\rho(y)}{|x-(y+\varepsilon\nu(y))|} + \frac{-\varepsilon^{-1}\rho(y)}{|x-(y-\varepsilon\nu(y))|} \right\} d\sigma(y) \ \text{ for each } \ x \in \mathbb{R}^3 \setminus \partial\Omega. \tag{1.1.18}$$

Upon letting $\varepsilon \longrightarrow 0^+$, we see that for each $x \in \mathbb{R}^3 \setminus \partial\Omega$ and each $y \in \partial\Omega$ we have

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[3] hence the name, double layer

[4] while attempting to use $\pm\varepsilon^\alpha\rho(y)$ one soon discovers that $\alpha = -1$ is the only exponent for which the limit in (1.1.19) is nontrivial

$$\left\{\frac{\varepsilon^{-1}}{|x-(y+\varepsilon\nu(y))|}+\frac{-\varepsilon^{-1}}{|x-(y-\varepsilon\nu(y))|}\right\}\longrightarrow(-2)\frac{\langle\nu(y),y-x\rangle}{|x-y|^3} \tag{1.1.19}$$

which shows that the harmonic double layer potential can, up to normalization, be thought of as the potential induced by a double layer of charges of opposite signs on $\partial\Omega$.

In parallel to the integral methods used in potential theory, centered around the Laplacian, a similar body of results has been constructed in elasticity theory, corresponding to the Lamé system. Originally, these theories have been largely independent and developed from within, though it has been noted early on that they exhibit many formal analogies. One of the benefits of dealing with boundary layer potentials associated with general weakly elliptic systems, as we do in this volume, is that this builds a robust, inclusive theory which, in particular, subsumes the classical cases of the Laplacian and the Lamé system at once.

1.2 "Tangential" Singular Integral Operators

We begin by considering integral operators whose kernels have the special algebraic structure described in the proposition below, strongly resembling a tangential derivative.

Proposition 1.2.1 *Assume $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\geq 2$) is an open set with the property that $\partial\Omega$ is a* UR *set. Abbreviate $\sigma_*:=\mathcal{H}^{n-1}\lfloor\partial_*\Omega$ and $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, denote by $\nu=(\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω. Next, consider a complex-valued function $b\in L^1_{\rm loc}(\mathbb{R}^n,\mathcal{L}^n)$ with the property that $b\big|_{\mathbb{R}^n\setminus\{0\}}\in\mathscr{C}^N(\mathbb{R}^n\setminus\{0\})$ where $N=N(n)\in\mathbb{N}$ is a sufficiently large number, and such that ∇b is odd and positive homogeneous of degree $1-n$ in $\mathbb{R}^n\setminus\{0\}$. Finally, for each pair of indices $j,k\in\{1,\dots,n\}$ introduce the integral operators acting on each function $f\in L^1\Big(\partial_*\Omega,\frac{\sigma_*(y)}{1+|y|^{n-1}}\Big)$ according to*

$$\mathcal{T}_{jk}f(x):=\int_{\partial_*\Omega}\big\{\nu_j(y)(\partial_k b)(x-y)-\nu_k(y)(\partial_j b)(x-y)\big\}f(y)\,d\sigma_*(y),\quad\forall x\in\Omega, \tag{1.2.1}$$

and, for σ-a.e. $x\in\partial\Omega$,

$$T_{jk}f(x):=\lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial_*\Omega\\|x-y|>\varepsilon}}\big\{\nu_j(y)(\partial_k b)(x-y)-\nu_k(y)(\partial_j b)(x-y)\big\}f(y)\,d\sigma_*(y), \tag{1.2.2}$$

(*which is meaningful thanks to* [70, (2.3.15)]). *Also, for each* $f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ *define at* σ_**-a.e.* $x \in \partial_*\Omega$ (*cf.* [70, (2.3.15)])

$$T^{\#}_{jk} f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big\{\nu_j(x)(\partial_k b)(x-y) - \nu_k(x)(\partial_j b)(x-y)\big\} f(y)\, \mathrm{d}\sigma(y). \tag{1.2.3}$$

Then the following statements are true.

(i) For each aperture parameter $\kappa \in (0,\infty)$ *one has*

$$\begin{aligned} &\mathcal{T}_{jk} f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = T_{jk} f \ \text{ at } \sigma_*\text{-a.e. point on } \partial_*\Omega,\\ &\text{for every function } f \in L^1\Big(\partial_*\Omega, \tfrac{\sigma_*(x)}{1+|x|^{n-1}}\Big). \end{aligned} \tag{1.2.4}$$

Also, for each f *in the weighted boundary Sobolev space* $L^1_1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big)$ (*cf.* (A.0.131)), *each index* $\ell \in \{1,\dots,n\}$, *and each aperture parameter* $\kappa \in (0,\infty)$, *the pointwise nontangential boundary trace*

$$\big(\partial_\ell \mathcal{T}_{jk} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \text{ exists at } \sigma_*\text{-a.e. point on } \partial_*\Omega. \tag{1.2.5}$$

As a consequence of [69, (11.7.22)], *this is true whenever* $f \in L^{p,q}_1(\partial_*\Omega, \sigma_*)$ *with* $p, q \in [1,\infty)$.
In addition, for each $p, q \in [1,\infty)$ *and* $\kappa > 0$ *there exists some finite constant* $C > 0$, *depending only on* $\partial\Omega$, b, n, p, q, *and* κ, *such that for each function* $f \in L^{p,q}_1(\partial_*\Omega, \sigma_*)$ *one has*

$$\big\|\mathcal{N}_\kappa(\mathcal{T}_{jk} f)\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\mathcal{T}_{jk} f)\big\|_{L^q(\partial\Omega,\sigma)} \leq C\|f\|_{L^{p,q}_1(\partial_*\Omega,\sigma_*)} \tag{1.2.6}$$

if $p, q > 1$, *plus similar estimates in the case when* $p = 1$ *or* $q = 1$, *in which scenario the corresponding* L^1*-norms in the left-hand side are now replaced by the quasi-norm* $L^{1,\infty}(\partial\Omega, \sigma)$.

(ii) Fix $p, p' \in (1,\infty)$ *with* $1/p + 1/p' = 1$. *Then the operator*

$$T^{\#}_{jk} : L^p(\partial\Omega, \sigma) \longrightarrow L^p(\partial_*\Omega, \sigma_*) \tag{1.2.7}$$

is well defined, linear, and bounded. Moreover, the transpose of $T^{\#}_{jk}$ *in* (1.2.7) *is*

$$-T_{jk} : L^{p'}(\partial_*\Omega, \sigma_*) \longrightarrow L^{p'}(\partial\Omega, \sigma). \tag{1.2.8}$$

In fact, similar results are valid for the more general scale of Muckenhoupt weighted Lebesgue spaces on $\partial\Omega$, *as well as for Lorentz spaces on* $\partial\Omega$.

(iii) Make the additional assumption that

$$\mathcal{H}^{n-1}\big(\partial_{\text{nta}}\Omega \setminus \partial_*\Omega\big) = 0 \tag{1.2.9}$$

(*a condition automatically satisfied if, e.g.,* Ω *is a* UR *domain to begin with*). *Then the operator*

$$T_{jk} : L_1^{p,q}(\partial_*\Omega, \sigma_*) \longrightarrow L_1^{p,q}(\partial_*\Omega, \sigma_*) \tag{1.2.10}$$

is well defined, linear, and bounded for each $p, q \in (1, \infty)$. *In particular, the operator*

$$T_{jk} : L_1^{p}(\partial_*\Omega, \sigma_*) \longrightarrow L_1^{p}(\partial_*\Omega, \sigma_*) \tag{1.2.11}$$

is well defined, linear, and bounded for each $p \in (1, \infty)$.
Moreover, given $p, q \in (1, \infty)$, *for each function*

$$\begin{aligned} f \in L^p_{\text{loc}}(\partial_*\Omega, \sigma_*) \cap L^1\Big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\Big) \ \textit{satisfying} \\ \partial_{\tau_{i\ell}} f \in L^q_{\text{loc}}(\partial_*\Omega, \sigma_*) \cap L^1\Big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\Big) \ \textit{for all}\ 1 \le i, \ell \le n, \end{aligned} \tag{1.2.12}$$

and each pair of indices $r, s \in \{1, \dots, n\}$ *one has*

$$\partial_{\tau_{rs}}(T_{jk} f) = T^{\#}_{rs}(\partial_{\tau_{jk}} f) \ \textit{at}\ \sigma_*\textit{-a.e. point on}\ \partial_*\Omega, \tag{1.2.13}$$

where $\partial_{\tau_{jk}} f$ *is regarded as a function in* $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ (*where* $T^{\#}_{rs}$ *is defined*) *by extending it by zero outside* $\partial_*\Omega$ *to the entire topological boundary* $\partial\Omega$. *In particular, formula* (1.2.13) *holds for every* $f \in L_1^{p,q}(\partial_*\Omega, \sigma_*)$ *with integrability exponents* $p, q \in (1, \infty)$.

(iv) Retain the additional assumption made in (1.2.9). *Then for each integrability exponent* $p \in (1, \infty)$ *it follows that* $T^{\#}_{jk}$, *originally acting on functions from* $L^p(\partial_*\Omega, \sigma_*)$ (*regarding them as being extended by zero to the entire topological boundary* $\partial\Omega$, *and then applying* $T^{\#}_{jk}$ *in the sense of* (1.2.7)), *further extends uniquely to a linear, bounded operator, from the negative boundary Sobolev space* $L^p_{-1}(\partial_*\Omega, \sigma_*)$ *into itself. Furthermore, if one adopts the same notation* $T^{\#}_{jk}$ *for this extension, then the transpose of* (1.2.11) *is*

$$-T^{\#}_{jk} : L^{p'}_{-1}(\partial_*\Omega, \sigma_*) \longrightarrow L^{p'}_{-1}(\partial_*\Omega, \sigma_*) \tag{1.2.14}$$

where $p' \in (1, \infty)$ *is such that* $1/p + 1/p' = 1$.
More generally, under the additional assumption made in (1.2.9), *for each* $p, q \in (1, \infty)$ *it follows that* $T^{\#}_{jk}$ *acting on functions from* $L^p(\partial_*\Omega, \sigma_*)$ (*in the manner described above*) *further extends uniquely to a linear and bounded operator from the negative off-diagonal boundary Sobolev space* $L^{p,q}_{-1}(\partial_*\Omega, \sigma_*)$ *into itself and, retaining the same notation* $T^{\#}_{jk}$ *for this extension, the transpose of* (1.2.10) *is*

$$-T^{\#}_{jk} : L^{p',q'}_{-1}(\partial_*\Omega, \sigma_*) \longrightarrow L^{p',q'}_{-1}(\partial_*\Omega, \sigma_*) \tag{1.2.15}$$

where $p', q' \in (1, \infty)$ *are such that* $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$.

Proof To justify (1.2.4), first observe that for each $x \in \mathbb{R}^n \setminus \overline{B(0,2)}$ we may use the Fundamental Theorem of Calculus to write

$$b(x) - b\big(\tfrac{x}{|x|}\big) = \int_0^1 (\nabla b)\big(tx + (1-t)\tfrac{x}{|x|}\big) \cdot \big(x - \tfrac{x}{|x|}\big)\, \mathrm{d}t. \tag{1.2.16}$$

Bearing in mind that b is bounded on S^{n-1}, on account of the homogeneity property satisfied by ∇b this readily implies

$$b(x) = O(|x|) \;\text{ as }\; |x| \to \infty. \tag{1.2.17}$$

Since we also have $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{L}^n)$, it follows that b induces a tempered distribution in $\mathbb{R}^n$ via integration against Schwartz functions. Invoking [70, (2.5.19)] for this tempered distribution then yields

$$\xi_k \widehat{(\partial_j b)}(\xi) = \xi_j \widehat{(\partial_k b)}(\xi) \text{ for each } \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}. \tag{1.2.18}$$

Having established (1.2.18), for each given $f \in L^1\Big(\partial_*\Omega, \frac{\sigma_*(y)}{1+|y|^{n-1}}\Big)$ we may now rely on [70, Theorem 2.5.1] to compute

$$\begin{aligned}
&\Big(\big(\mathcal{T}_{jk} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) \\
&\quad = \frac{1}{2\mathrm{i}} \widehat{\partial_k b}\big(\nu(x)\big)\nu_j(x) f(x) - \frac{1}{2\mathrm{i}} \widehat{\partial_j b}\big(\nu(x)\big)\nu_k(x) f(x) \\
&\qquad + \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial_*\Omega \\ |x-y|>\varepsilon}} \Big[\nu_j(y)(\partial_k b)(x-y) - \nu_k(y)(\partial_j b)(x-y)\Big] f(y)\, \mathrm{d}\sigma_*(y) \\
&\quad = (T_{jk} f)(x) \;\text{ at } \sigma_*\text{-a.e. } x \in \partial_*\Omega,
\end{aligned} \tag{1.2.19}$$

finishing the proof of (1.2.4).

Next, pick $f \in L^1_1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big)$. Then, for each point $x \in \Omega$ and $\ell \in \{1, \dots, n\}$, we may write

$$\begin{aligned}
\partial_\ell\big(\mathcal{T}_{jk} f\big)(x) &= -\int_{\partial_*\Omega} \big(\nu_j(y)\partial_{y_k} - \nu_k(y)\partial_{y_j}\big)\big[(\partial_\ell b)(x-y)\big] f(y)\, \mathrm{d}\sigma_*(y) \\
&= -\int_{\partial_*\Omega} \partial_{\tau_{jk}(y)}[(\partial_\ell b)(x-y)] f(y)\, \mathrm{d}\sigma_*(y) \\
&= \int_{\partial_*\Omega} (\partial_\ell b)(x-y)(\partial_{\tau_{jk}} f)(y)\, \mathrm{d}\sigma_*(y).
\end{aligned} \tag{1.2.20}$$

Above, the first equality is differentiation under the integral sign, while the second equality uses the definition of $\partial_{\tau_{jk}(y)}$. In the third equality in (1.2.20) we have integrated by parts on the boundary, as permitted by [69, Lemma 11.1.7], bearing in mind that the function $\varphi := (\partial_\ell b)(x - \cdot)$ satisfies, thanks to our assumptions on b, all properties demanded in [69, Lemma 11.1.7, (11.1.48)]. This finishes the proof of (1.2.20). Once (1.2.20) has been established, we may invoke [70, Theorem 1.4.2] and [70, Theorem 2.5.1] to conclude that $\big(\partial_\ell \mathscr{T}_{jk} f\big)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ_*-a.e. point on $\partial_*\Omega$.

Moreover, in the case when actually $f \in L_1^{p,q}(\partial_*\Omega, \sigma_*)$ with $p, q \in [1, \infty)$ we may rely on (1.2.20), [70, Theorem 1.4.2], and [70, Theorem 2.4.1] to estimate

$$\big\|\mathcal{N}_\kappa(\partial_\ell \mathscr{T}_{jk} f)\big\|_{L^q(\partial\Omega,\sigma)} \leq C\big\|\partial_{\tau_{jk}} f\big\|_{L^q(\partial_*\Omega,\sigma_*)} \text{ if } 1 < q < \infty, \tag{1.2.21}$$

$$\big\|\mathcal{N}_\kappa(\partial_\ell \mathscr{T}_{jk} f)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \leq C\big\|\partial_{\tau_{jk}} f\big\|_{L^1(\partial_*\Omega,\sigma_*)} \text{ if } q = 1. \tag{1.2.22}$$

From (1.2.1) and [70, Theorem 2.4.1] we also know that for each $p \in [1, \infty)$ and $\kappa > 0$ there exists a finite constant $C > 0$ with the property that for every $f \in L^p(\partial_*\Omega, \sigma_*)$ we have

$$\big\|\mathcal{N}_\kappa(\mathscr{T}_{jk} f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p(\partial_*\Omega,\sigma_*)} \text{ if } 1 < p < \infty, \tag{1.2.23}$$

$$\big\|\mathcal{N}_\kappa(\mathscr{T}_{jk} f)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \leq C\|f\|_{L^1(\partial_*\Omega,\sigma_*)} \text{ if } p = 1. \tag{1.2.24}$$

Collectively, (1.2.21)-(1.2.24) prove that the claims in (1.2.6) and the subsequent comment hold. This takes care of item *(i)*.

The claims made in item *(ii)* are consequences of (1.2.2), (1.2.3), [70, Theorem 2.3.2], and [70, (2.3.25)]. On to item *(iii)*. Granted (1.2.4), (1.2.5), (1.2.23), (1.2.6), it follows that for each given function $f \in L_1^{p,q}(\partial_*\Omega, \sigma_*)$ with $p, q \in (1, \infty)$ we may, thanks to (1.2.9), employ [69, Proposition 11.3.2] for the choice $u := \mathscr{T}_{jk} f$ in Ω, to conclude that $T_{jk} f \in L_1^{p,q}(\partial_*\Omega, \sigma_*)$ and, taking into account (1.2.21), write

$$\begin{aligned} \|T_{jk} f\|_{L_1^{p,q}(\partial_*\Omega,\sigma_*)} &\leq C\Big(\big\|\mathcal{N}_\kappa u\big\|_{L^p(\partial_*\Omega,\sigma_*)} + \big\|\mathcal{N}_\kappa(\nabla u)\big\|_{L^q(\partial_*\Omega,\sigma_*)}\Big) \\ &\leq C\|f\|_{L^p(\partial_*\Omega,\sigma_*)} + C\big\|\partial_{\tau_{jk}} f\big\|_{L^q(\partial_*\Omega,\sigma_*)} \\ &\leq C\|f\|_{L_1^{p,q}(\partial_*\Omega,\sigma_*)}, \end{aligned} \tag{1.2.25}$$

for some constant $C \in (0, \infty)$ independent of f. From this, the claims about the operator in (1.2.10)-(1.2.11) are clear.

To justify (1.2.13), fix $r, s \in \{1, \dots, n\}$ along with f as in (1.2.12), for some $p, q \in (1, \infty)$. In particular, $f \in L_1^1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big)$, hence (1.2.20) holds at points $x \in \Omega$ for each $\ell \in \{1, \dots, n\}$. If we now define $u := \mathscr{T}_{jk} f \in \mathscr{C}^{N-1}(\Omega)$, then from (1.2.1), (1.2.20), (1.2.12), and [70, (2.4.8)] we see that the memberships in [69, Proposition 11.3.2, (11.3.24)] are presently satisfied. Also, (1.2.4) and (1.2.5) guarantee that the condition imposed on the nontangential traces in [69, Proposition 11.3.2, (11.3.25)] holds. As such, [69, Proposition 11.3.2] is applicable to the current function u. Together with (1.2.4), (1.2.20), [70, Theorem 2.5.1], [70, (2.5.19)], and (1.2.3), formula [69, (11.3.26)] then implies that at σ_*-a.e. point $x \in \partial_*\Omega$ we have

$$\begin{aligned}
&\partial_{\tau_{rs}}(T_{jk}f)(x) \\
&\quad = \partial_{\tau_{rs}}\Big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) = \nu_r(x)\Big((\partial_s u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) - \nu_s(x)\Big((\partial_r u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) \\
&\quad = \frac{1}{2\mathrm{i}}\nu_r(x)\widehat{\partial_s b}\big(\nu(x)\big)(\partial_{\tau_{jk}}f)(x) \\
&\qquad + \lim_{\varepsilon\to0^+}\int\limits_{\substack{y\in\partial_*\Omega\\|x-y|>\varepsilon}} \nu_r(x)(\partial_s b)(x-y)(\partial_{\tau_{jk}}f)(y)\,\mathrm{d}\sigma_*(y) \\
&\qquad - \frac{1}{2\mathrm{i}}\nu_s(x)\widehat{\partial_r b}\big(\nu(x)\big)(\partial_{\tau_{jk}}f)(x) \\
&\qquad - \lim_{\varepsilon\to0^+}\int\limits_{\substack{y\in\partial_*\Omega\\|x-y|>\varepsilon}} \nu_s(x)(\partial_r b)(x-y)(\partial_{\tau_{jk}}f)(y)\,\mathrm{d}\sigma_*(y) \\
&\quad = \lim_{\varepsilon\to0^+}\int\limits_{\substack{y\in\partial_*\Omega\\|x-y|>\varepsilon}} \big\{\nu_r(x)(\partial_s b)(x-y) - \nu_s(x)(\partial_r b)(x-y)\big\}(\partial_{\tau_{jk}}f)(y)\,\mathrm{d}\sigma_*(y) \\
&\quad = T^{\#}_{rs}(\partial_{\tau_{jk}}f)(x),
\end{aligned} \tag{1.2.26}$$

proving (1.2.13). This takes care of item *(iii)*.

Consider now the claims made in the first half of item *(iv)*. To set the stage, fix two exponents $p, p' \in (1,\infty)$ with $1/p + 1/p' = 1$, and consider the operator

$$\begin{gathered}
\widetilde{T^{\#}_{jk}} : L^{p'}_{-1}(\partial_*\Omega,\sigma_*) \longrightarrow L^{p'}_{-1}(\partial_*\Omega,\sigma_*) \text{ defined by } \big\langle \widetilde{T^{\#}_{jk}}f, g\big\rangle := -\big\langle f, T_{jk}g\big\rangle \\
\text{for all } f \in L^{p'}_{-1}(\partial_*\Omega,\sigma_*) = \big(L^p_1(\partial_*\Omega,\sigma_*)\big)^* \text{ and } g \in L^p_1(\partial_*\Omega,\sigma_*)
\end{gathered} \tag{1.2.27}$$

where the angled brackets $\langle\cdot,\cdot\rangle$ stand for the duality pairing between the space $L^{p'}_{-1}(\partial_*\Omega,\sigma_*) = \big(L^p_1(\partial_*\Omega,\sigma_*)\big)^*$ and $L^p_1(\partial_*\Omega,\sigma_*)$. Thanks to (A.0.136) and (1.2.11) it follows that $\widetilde{T^{\#}_{jk}}$ is a well-defined, linear and bounded operator. To proceed, let us denote by E the mapping extending functions originally defined on $\partial_*\Omega$ by zero to the entire topological boundary $\partial\Omega$. Then, since the transpose of $T^{\#}_{jk}$ in (1.2.7) is the operator $-T_{jk}$ in (1.2.8), we conclude from (1.2.27), and [69, (11.8.4), (11.8.5)] that $\widetilde{T^{\#}_{jk}}$ is the unique extension of the composition $T^{\#}_{jk} \circ E$ (where $T^{\#}_{jk}$ is as in (1.2.7)) to a continuous operator on $L^{p'}_{-1}(\partial_*\Omega, \sigma_*)$. If we now re-denote $\widetilde{T^{\#}_{jk}}$ simply as $T^{\#}_{jk}$, the claims in the first half of item *(iv)* follow. Finally, the claims in the second half of item *(iv)* may be justified in a similar fashion. □

We continue by considering the issue of integration by parts on the boundary, inside a principal value singular integral on a UR set.

Proposition 1.2.2 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) is an open set whose boundary is a* UR *set; in particular, Ω is a set of locally finite perimeter. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma_* := \mathcal{H}^{n-1} \lfloor \partial_*\Omega$.*

Consider a complex-valued function $b \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$, where $N = N(n) \in \mathbb{N}$ is a sufficiently large number, which is even and positive homogeneous of degree $2 - n$. Also, fix some arbitrary indices $j, k \in \{1, \dots, n\}$ and introduce the singular integral operator

$$Bg(x) := \int_{\partial_*\Omega} b(x-y)g(y)\,\mathrm{d}\sigma_*(y) \ \text{ at } \sigma_*\text{-a.e. } x \in \partial_*\Omega, \quad \text{for every function } g \in L^1\Big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-2}}\Big). \tag{1.2.28}$$

Then

$$\text{for each } f \in L^1\Big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\Big) \text{ with } \partial_{\tau_{jk}} f \in L^1\Big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-2}}\Big) \quad \text{one has } B(\partial_{\tau_{jk}} f) = T_{jk} f \ \text{ at } \sigma_*\text{-a.e. point on } \partial_*\Omega, \tag{1.2.29}$$

and

$$\text{for each } g \in L^r_{\mathrm{loc}}(\partial_*\Omega, \sigma_*) \cap L^1\Big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-2}}\Big) \text{ with } r \in (1, \infty) \quad \text{one has } \partial_{\tau_{jk}}(Bg) = T^{\#}_{jk} g \ \text{ at } \sigma_*\text{-a.e. point on } \partial_*\Omega. \tag{1.2.30}$$

Proof For each $g \in L^1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-2}}\big)$ define

$$\mathscr{B}g(x) := \int_{\partial_*\Omega} b(x-y)g(y)\,\mathrm{d}\sigma_*(y), \quad \forall x \in \Omega. \tag{1.2.31}$$

Note that the integrability condition on g ensures that the integral in the right-hand side of (1.2.31) is absolutely convergent for each $x \in \Omega$. Thus, $\mathscr{B}g$ is a well-defined function in Ω. To proceed, recall the operator $\mathcal{T}_{jk}$ from (1.2.1) and fix a function f as in (1.2.29) along with an arbitrary point $x \in \Omega$. Since $\varphi := b(x - \cdot)$ satisfies all demands in [69, Lemma 11.1.7, (11.1.48)] (relative to the present f), we may use [69, (11.1.49) in Lemma 11.1.7] to compute

$$\begin{aligned}\mathcal{T}_{jk}f(x) &= \int_{\partial_*\Omega} \partial_{\tau_{kj}(y)}\big[b(x-y)\big]f(y)\,d\sigma_*(y)\\ &= \int_{\partial_*\Omega} b(x-y)(\partial_{\tau_{jk}}f)(y)\,d\sigma_*(y)\\ &= \mathscr{B}(\partial_{\tau_{jk}}f)(x). \end{aligned}\tag{1.2.32}$$

Having fixed some $\kappa > 0$, we may now rely on (1.2.28), (1.2.32), the last part of [70, Proposition 2.5.39] (also bearing in mind [68, Proposition 8.8.4]), and (1.2.4), to write

$$\begin{aligned}B(\partial_{\tau_{jk}}f)(x) &= \int_{\partial_*\Omega} b(x-y)(\partial_{\tau_{jk}}f)(y)\,d\sigma_*(y) = \big(\mathscr{B}(\partial_{\tau_{jk}}f)\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}(x)\\ &= \big(\mathcal{T}_{jk}f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}(x) = T_{jk}f(x) \text{ for } \sigma_*\text{-a.e. } x \in \partial_*\Omega. \end{aligned}\tag{1.2.33}$$

This establishes (1.2.29).

Moving on, fix a function g as in (1.2.30) and observe that

$$Bg \in L^1_{\text{loc}}(\partial_*\Omega, \sigma_*) \text{ and } T^{\#}_{jk}g \in L^r_{\text{loc}}(\partial_*\Omega, \sigma_*),\tag{1.2.34}$$

thanks to [68, (7.8.5)] and [70, (2.3.17)]. Also, for each $\varphi \in \mathscr{C}^1_c(\mathbb{R}^n)$ we have

$$\begin{aligned}\int_{\partial_*\Omega} Bg(x)(\partial_{\tau_{kj}}\varphi)(x)\,d\sigma_*(x) &= \int_{\partial_*\Omega}\Big(\int_{\partial_*\Omega} b(x-y)g(y)\,d\sigma_*(y)\Big)(\partial_{\tau_{kj}}\varphi)(x)\,d\sigma_*(x)\\ &= \int_{\partial_*\Omega}\Big(\int_{\partial_*\Omega} b(y-x)(\partial_{\tau_{kj}}\varphi)(x)\,d\sigma_*(x)\Big)g(y)\,d\sigma_*(y)\\ &= \int_{\partial_*\Omega}\big(B(\partial_{\tau_{kj}}\varphi)\big)(y)g(y)\,d\sigma_*(y)\\ &= -\int_{\partial_*\Omega} T_{jk}\big(\varphi\big|_{\partial_*\Omega}\big)(y)g(y)\,d\sigma_*(y)\\ &= \int_{\partial_*\Omega}\varphi(x)(T^{\#}_{jk}g)(x)\,d\sigma_*(x). \end{aligned}\tag{1.2.35}$$

Above, the first and third equalities come from (1.2.28), the second equality is a consequence of Fubini's Theorem and the fact that the function b is even, the fourth equality is seen from (1.2.29) applied with $f := \varphi\big|_{\partial_*\Omega}$ and the roles of j, k reversed (cf. [69, (11.1.5)] which accounts for the minus sign), and the fifth equality is implied by [70, (2.3.26)], (1.2.2), (1.2.3).

At this stage, from (1.2.34), (1.2.35), and (A.0.121) we conclude that the claim in (1.2.30) is true. The proof of Proposition 1.2.2 is now complete. □

For future reference, it is useful to specialize Proposition 1.2.2 to the scale of ordinary Lebesgue spaces in the manner made precise in the corollary below.

Corollary 1.2.3 *Retain the assumptions on the set Ω and the function b made in Proposition 1.2.2. Fix some arbitrary indices $j,k \in \{1,\dots,n\}$ along with two integrability exponents, $p \in (1,\infty)$ and $q \in (1,n-1)$, then define*

$$q^* := \big(\tfrac{1}{q} - \tfrac{1}{n-1}\big)^{-1}. \tag{1.2.36}$$

Finally, introduce the singular integral operator

$$\begin{aligned} Bg(x) := \int_{\partial_*\Omega} b(x-y)g(y)\,\mathrm{d}\sigma_*(y) \ \text{ at } \sigma_*\text{-a.e. } x \in \partial_*\Omega, \\ \text{for every function } g \in L^q(\partial_*\Omega,\sigma_*). \end{aligned} \tag{1.2.37}$$

Then

$$B : L^q(\partial_*\Omega,\sigma_*) \longrightarrow L^{q^*,q}_1(\partial_*\Omega,\sigma_*) \tag{1.2.38}$$

is a well-defined, linear and bounded operator, with the property that

$$\begin{aligned} &\text{for each function } f \in L^{p,q}_1(\partial_*\Omega,\sigma_*) \text{ one has} \\ &B(\partial_{\tau_{jk}} f) = T_{jk} f \ \text{ at } \sigma_*\text{-a.e. point on } \partial_*\Omega. \end{aligned} \tag{1.2.39}$$

and such that

$$\begin{aligned} &\text{for every function } g \in L^q(\partial_*\Omega,\sigma_*) \text{ one has} \\ &\partial_{\tau_{jk}}(Bg) = T^{\#}_{jk} g \ \text{ at } \sigma_*\text{-a.e. point on } \partial_*\Omega. \end{aligned} \tag{1.2.40}$$

Moreover, with prime indicating the Hölder conjugate exponent and with the superscript star defined as in (1.2.36), it follows that[5]

$$\begin{aligned} &\text{the transpose of } B : L^q(\partial_*\Omega,\sigma_*) \to L^{q^*}(\partial_*\Omega,\sigma_*) \text{ is} \\ &\text{the operator } B : L^{(q^*)'}(\partial_*\Omega,\sigma_*) \to L^{q'}(\partial_*\Omega,\sigma_*). \end{aligned} \tag{1.2.41}$$

Finally, in the case when $\partial\Omega$ is bounded, the same results are also true for every $q \in [n-1,\infty)$, this time taking $q^* \in (1,\infty)$ arbitrary (and unrelated to q).

Proof All claims, except for (1.2.41), follow from Proposition 1.2.2, bearing in mind [70, (2.5.549)] and the Fractional Integration Theorem (cf. [68, (7.8.7)]). Finally, the claim in (1.2.41) is a direct consequences of the Fractional Integration Theorem and Fubini's Theorem. □

Retain the assumptions on the set Ω and the function b made in Proposition 1.2.2, and recall the operator $\mathscr{B}$ from (1.2.31). We aim to extend the action of this integral operator to negative off-diagonal Sobolev spaces of the following sort. Fix some

[5] note that $(q^*)'$ belongs to $(1,n-1)$ and satisfies $\big((q^*)'\big)^* = q'$

integrability exponent $p \in (1, n-1)$ along with some $q \in (1, \infty)$, and denote by $p' \in \big(\frac{n-1}{n-2}, \infty\big)$ and $q' \in (1, \infty)$ their Hölder conjugate exponents. For each functional $f \in L^{p,q}_{-1}(\partial_*\Omega, \sigma_*) = \big(L^{p',q'}_1(\partial_*\Omega, \sigma_*)\big)^*$ we define

$$\begin{aligned}\mathscr{B}f(x) &:= {}_{L^{p',q'}_1(\partial_*\Omega,\sigma_*)}\Big\langle (b(x-\cdot)\big|_{\partial_*\Omega}, f\Big\rangle_{L^{p,q}_{-1}(\partial_*\Omega,\sigma_*)} \\ &= \int_{\partial_*\Omega} b(x-y) f_0(y)\, \mathrm{d}\sigma_*(y) \\ &\quad + \int_{\partial_*\Omega} \partial_{\tau_{jk}(y)}[b(x-y)] f_{jk}(y)\, \mathrm{d}\sigma_*(y), \qquad \forall x \in \Omega, \end{aligned} \tag{1.2.42}$$

where the functions $f_0 \in L^p(\partial_*\Omega, \sigma_*)$, and $f_{jk} \in L^q(\partial_*\Omega, \sigma_*)$ with $1 \leq j < k \leq n$, are associated with the given functional f as in [69, Proposition 11.8.6].

In this regard, there are three basic remarks we wish to make. First, the version of [69, Lemma 11.1.5] for off-diagonal Sobolev spaces implies (also taking into account [68, Lemma 7.2.1]) that

$$b(x-\cdot)\big|_{\partial_*\Omega} \in L^{p',q'}_1(\partial_*\Omega, \sigma_*), \qquad \forall x \in \Omega. \tag{1.2.43}$$

Second, the final expression in (1.2.42) is unequivocally defined even though the functional f does not determine the functions $\big(f_0, (f_{jk})_{1\leq j,k\leq n}\big)$ uniquely. Third, as is apparent from (1.2.42),

$$\mathscr{B}f \in \mathscr{C}^0(\Omega) \text{ for each } f \in L^{p,q}_{-1}(\partial_*\Omega, \sigma_*). \tag{1.2.44}$$

Proposition 1.2.4 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) is an open set whose boundary is a* UR *set, and abbreviate $\sigma_* := \mathcal{H}^{n-1}\lfloor\partial_*\Omega$, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider a complex-valued function $b \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$, where $N = N(n) \in \mathbb{N}$ is a sufficiently large number, which is even and positive homogeneous of degree $2-n$. Finally, fix $q \in (1, n-1)$ and define $q^* := \big(1/q - 1/(n-1)\big)^{-1}$ so, in particular, if $(q^*)'$ denotes the Hölder conjugate exponent of q^* then $(q^*)' \in (1, n-1)$.*

In this setting, introduce two integral operators, acting on each $f \in L^q(\partial_*\Omega, \sigma_*)$ according to

$$\mathscr{B}f(x) := \int_{\partial_*\Omega} b(x-y) f(y)\, \mathrm{d}\sigma_*(y) \textit{ for each } x \in \Omega, \textit{ and} \tag{1.2.45}$$

$$Bf(x) := \int_{\partial_*\Omega} b(x-y) f(y)\, \mathrm{d}\sigma_*(y) \textit{ for } \sigma_*\textit{-a.e. } x \in \partial_*\Omega. \tag{1.2.46}$$

Then, in relation to these, the following statements are true.

(i) The operator B induces well-defined, linear, and bounded mappings

$$B : L^q(\partial_*\Omega, \sigma_*) \longrightarrow L_1^{q^*,q}(\partial_*\Omega, \sigma_*), \tag{1.2.47}$$

$$B : L_{-1}^{q,q^*}(\partial_*\Omega, \sigma_*) \longrightarrow L^{q^*}(\partial_*\Omega, \sigma_*), \tag{1.2.48}$$

which act in a compatible fashion with one another.

(ii) The (*real*) *transpose of the operator* B *in* (1.2.47) *is the operator* B *in* (1.2.48) *with* q *replaced by* $(q^*)'$.

(iii) The operator B *from* (1.2.48) *further extends to a linear and bounded mapping*

$$B : L_{-1}^{q,p}(\partial_*\Omega, \sigma_*) \longrightarrow L^{q^*}(\partial_*\Omega, \sigma_*) + L^p(\partial_*\Omega, \sigma_*) \tag{1.2.49}$$

for each $p \in (1, \infty)$.

(iv) Fix $\kappa > 0$. *Then for each* $p \in (1, \infty)$ *one has, in a quantitative fashion,*

$$\mathcal{N}_\kappa(\mathscr{B} f) \in L^{q^*}(\partial\Omega, \sigma) + L^p(\partial\Omega, \sigma), \qquad \forall f \in L_{-1}^{q,p}(\partial_*\Omega, \sigma_*). \tag{1.2.50}$$

In particular, there exists a finite constant $C = C(\Omega, b, q, \kappa) > 0$ *such that*

$$\begin{gathered} \big\|\mathcal{N}_\kappa(\mathscr{B} f)\big\|_{L^{q^*}(\partial\Omega,\sigma)} \leq C\|f\|_{L_{-1}^{q,q^*}(\partial_*\Omega,\sigma_*)} \\ \text{for all } \ f \in L_{-1}^{q,q^*}(\partial_*\Omega, \sigma_*). \end{gathered} \tag{1.2.51}$$

(v) For each $\kappa > 0$, *each* $p \in (1, \infty)$, *and each* $f \in L_{-1}^{q,p}(\partial_*\Omega, \sigma_*)$, *the nontangential pointwise trace* $\mathscr{B} f\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ *exists at* σ_**-a.e. point on* $\partial_*\Omega$ *and, in fact, with* Bf *considered in the sense of* (1.2.49) *one has*

$$\mathscr{B} f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = Bf \ \text{ at } \sigma_*\text{-a.e. point on } \partial_*\Omega. \tag{1.2.52}$$

(vi) For each $\kappa > 0$ *there exists a finite constant* $C = C(\Omega, b, q, \kappa) > 0$ *such that for each function* $f \in L^q(\partial_*\Omega, \sigma_*)$ *one has*

$$\big\|\mathcal{N}_\kappa(\nabla\mathscr{B} f)\big\|_{L^q(\partial\Omega,\sigma)} \leq C\|f\|_{L^q(\partial_*\Omega,\sigma_*)} \tag{1.2.53}$$

and

$$(\nabla\mathscr{B} f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists } \sigma_*\text{-a.e. on } \partial_*\Omega. \tag{1.2.54}$$

(vii) If $\partial\Omega$ *is bounded then the same results in items (i)-(vi) are also true for every* $q \in [n-1, \infty)$, *this time taking* $q^* \in (1, \infty)$ *arbitrary* (*and unrelated to* q).

Proof The claim about (1.2.47) has been already noted in (1.2.38). Also, from the Fractional Integration Theorem (cf. [68, (7.8.7)]) we know that

$$B : L^q(\partial_*\Omega, \sigma_*) \longrightarrow L^{q^*}(\partial_*\Omega, \sigma_*) \tag{1.2.55}$$

is well defined, linear and bounded. If q' denotes the Hölder conjugate exponent of q, then Fubini's Theorem shows (bearing in mind that b is even) that the (real) transpose of (1.2.55) is the operator

$$B : L^{(q^*)'}(\partial_*\Omega, \sigma_*) \longrightarrow L^{q'}(\partial_*\Omega, \sigma_*) \tag{1.2.56}$$

whereas the (real) transpose of (1.2.47) is the operator

$$B^\top : L^{(q^*)',q'}_{-1}(\partial_*\Omega, \sigma_*) \longrightarrow L^{q'}(\partial_*\Omega, \sigma_*). \tag{1.2.57}$$

Recall from [69, (11.8.30)] that $L^{(q^*)',q'}_{-1}(\partial_*\Omega, \sigma_*)$ contains $L^{(q^*)'}(\partial_*\Omega, \sigma_*)$, and the latter space is dense in the former. Since the operators (1.2.47), (1.2.55) are compatible with each other, it follows that the operators (1.2.56), (1.2.57) also act in a coherent fashion. The bottom line is that the operator B in (1.2.56) extends uniquely, in a liner and bounded manner, to the larger space $L^{(q^*)',q'}_{-1}(\partial_*\Omega, \sigma_*)$, where it agrees with $B^\top$. Let us also remark that if q is replaced by $(q^*)'$ then q' becomes the Hölder conjugate exponent of $(q^*)'$, i.e., q^*. These considerations simultaneously prove that the operator (1.2.48) is well defined, linear, bounded, acting in a compatible fashion with (1.2.47), and also that the (real) transpose of B in (1.2.47) is B in (1.2.48) with q replaced by $(q^*)'$. The argument so far takes care of the claims made in items *(i)-(ii)*.

Consider next the claim in item *(iii)*. First, from (1.2.55), (A.0.123), and (1.2.40) observe that

$$B\,\phi \in \bigcap_{\substack{\frac{n-1}{n-2}<q_o<\infty \\ 1<p_o<\infty}} L^{q_o,p_o}_1(\partial_*\Omega, \sigma_*), \qquad \forall \phi \in \mathrm{Lip}_c(\partial\Omega). \tag{1.2.58}$$

In particular, if $p \in (1, \infty)$ and $f \in L^{q,p}_{-1}(\partial_*\Omega, \sigma_*) = \big(L^{q',p'}_1(\partial_*\Omega, \sigma_*)\big)^*$, then for each given function $\phi \in \mathrm{Lip}_c(\partial_*\Omega)$ it follows that

$$\text{the pairing } {}_{L^{q,p}_{-1}(\partial_*\Omega,\sigma_*)}\big\langle f, B\,\phi\big\rangle_{L^{q',p'}_1(\partial_*\Omega,\sigma_*)} \text{ is well defined.} \tag{1.2.59}$$

Assuming $f_0 \in L^q(\partial_*\Omega, \sigma_*)$, and $f_{jk} \in L^p(\partial_*\Omega, \sigma_*)$ with $1 \le j < k \le n$, are associated with the given functional f as in [69, Proposition 11.8.6], further work on said pairing yields the following sequence of equalities:

$$
\begin{aligned}
{}_{L^{q,p}_{-1}(\partial_*\Omega,\sigma_*)}\big\langle f,B\,\phi\big\rangle_{L^{q',p'}_{1}(\partial_*\Omega,\sigma_*)}
&= \int_{\partial_*\Omega} f_0 B\phi\,\mathrm{d}\sigma_* + \sum_{1\le j<k\le n}\int_{\partial_*\Omega} f_{jk}\partial_{\tau_{jk}}(B\phi)\,\mathrm{d}\sigma_* \\
&= \int_{\partial_*\Omega} (Bf_0)\phi\,\mathrm{d}\sigma_* + \sum_{1\le j<k\le n}\int_{\partial_*\Omega} f_{jk}T^{\#}_{jk}\phi\,\mathrm{d}\sigma_* \\
&= \int_{\partial_*\Omega} (Bf_0)\phi\,\mathrm{d}\sigma_* - \sum_{1\le j<k\le n}\int_{\partial_*\Omega} \big(T_{jk}f_{jk}\big)\phi\,\mathrm{d}\sigma_* \\
&= \int_{\partial_*\Omega} \Big\{Bf_0 - \sum_{1\le j<k\le n} T_{jk}f_{jk}\Big\}\phi\,\mathrm{d}\sigma_*. \qquad (1.2.60)
\end{aligned}
$$

Above, the first equality uses (1.2.59) and [69, Proposition 11.8.6], the second equality is based on (1.2.41) and (1.2.40), the third equality is a consequence of item *(ii)* in Proposition 1.2.1, while the final equality is plain algebra. This establishes (1.2.60) which, in concert with (1.2.56) and the first claim in item *(ii)* of Proposition 1.2.1, goes to show that

> the operator whose action on each $f \in L^{q,p}_{-1}(\partial_*\Omega,\sigma_*)$ is defined via
> $$\widetilde{B}f := Bf_0 - \sum_{1\le j<k\le n} T_{jk}f_{jk},$$
> assuming the functions $f_0 \in L^q(\partial_*\Omega,\sigma_*)$ and $f_{jk} \in L^p(\partial_*\Omega,\sigma_*)$ with $1 \le j < k \le n$ are associated with the given functional f as in [69, Proposition 11.8.6], is actually a well-defined, linear, and bounded mapping from $L^{q,p}_{-1}(\partial_*\Omega,\sigma_*)$ into $L^{q^*}(\partial_*\Omega,\sigma_*) + L^p(\partial_*\Omega,\sigma_*)$. (1.2.61)

As far as the claim in item *(iii)* is concerned, there remains to observe that the operator $\widetilde{B}$ just defined agrees with B from (1.2.48). However, if $f \in L^{q,p}_{-1}(\partial_*\Omega,\sigma_*)$ and $\phi \in \mathrm{Lip}_c(\partial_*\Omega)$ are arbitrary, based on (1.2.60)-(1.2.61) and the manner in which B has been defined in (1.2.48) we may write

$$
\begin{aligned}
\int_{\partial_*\Omega} (\widetilde{B}f)\phi\,\mathrm{d}\sigma_* &= {}_{L^{q,q^*}_{-1}(\partial_*\Omega,\sigma_*)}\big\langle f, B\,\phi\big\rangle_{L^{q',(q^*)'}_{1}(\partial_*\Omega,\sigma_*)} \\
&= {}_{L^{(q^*)'}(\partial_*\Omega,\sigma_*)}\big\langle Bf, \phi\big\rangle_{L^{q^*}(\partial_*\Omega,\sigma_*)} = \int_{\partial_*\Omega}(Bf)\phi\,\mathrm{d}\sigma_*, \qquad (1.2.62)
\end{aligned}
$$

so the desired conclusion now follows with the help of [68, Corollary 3.7.3].

Next, the claim in item *(iv)* is seen from (1.2.42), [69, Proposition 11.8.6], [70, Theorem 2.4.1], and [70, Proposition 2.5.39]. Also, the boundary trace formula claimed in item *(v)* is a consequence of (1.2.42), [70, Theorem 2.4.1], [70, Proposition 2.5.39], [68, Proposition 8.8.4], and (1.2.4), bearing in mind the coincidence of

B with the operator $\widetilde{B}$ from (1.2.61). The claims in item *(vi)* are direct consequences of [70, Theorem 2.4.1] and [70, Theorem 2.5.1]. Finally, when $\partial\Omega$ is bounded, the addenda [68, (7.8.14)-(7.8.15)] to the Fractional Integration Theorem are in effect, and this ultimately accounts for the claim in made in item *(vii)* of the statement. □

The following technical result is the basic ingredient in the proof of Proposition 1.2.6, stated a little later.

Lemma 1.2.5 *Let* $\Sigma \subseteq \mathbb{R}^n$ *(where* $n \in \mathbb{N}$ *with* $n \geq 2$*) be a closed Ahlfors regular set and denote* $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. *Fix* $\varepsilon \in (0,1)$ *and suppose* $b \in \mathscr{C}^1(\mathbb{R}^n \setminus \{0\})$ *is a function with the property that there exists* $C_0 \in (0,\infty)$ *such that*

$$|b(x)| \leq C_0 \begin{cases} |x|^{2-n+\varepsilon} & \text{if } |x| \geq 1, \\ |x|^{2-n-\varepsilon} & \text{if } |x| \leq 1, \end{cases} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{1.2.63}$$

and

$$|(\nabla b)(x)| \leq C_0 |x|^{1-n} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.2.64}$$

Then for each there exists a constant $C \in (0,\infty)$*, which depends only on* Σ*,* n*,* C_0*, and* ε*, such that*

$$\int_\Sigma \frac{|b(x-y) - b_*(-y)|}{1+|x|^n}\, d\sigma(x) \leq \frac{C}{1+|y|^{n-1-\varepsilon}} \quad \text{for all } y \in \mathbb{R}^n, \tag{1.2.65}$$

where $b_* := b \cdot \mathbf{1}_{\mathbb{R}^n \setminus B(0,1)}$ *in* $\mathbb{R}^n \setminus \{0\}$.

Proof We split our discussion in a number of cases.

Case 1: $|y| \leq 1$. Pick $y \in \mathbb{R}^n$ with $|y| \leq 1$. Then $b_*(-y) = 0$ and the term in the left-hand side of (1.2.65) may be estimated as follows:

$$\begin{aligned} \int_\Sigma \frac{|b(x-y)|}{1+|x|^n}\, d\sigma(x) &= \int\limits_{\substack{x\in\Sigma \\ |x-y|\leq 1}} \frac{|b(x-y)|}{1+|x|^n}\, d\sigma(x) + \int\limits_{\substack{x\in\Sigma \\ |x-y|>1}} \frac{|b(x-y)|}{1+|x|^n}\, d\sigma(x) \\ &\leq C_0 \int\limits_{\substack{x\in\Sigma \\ |x-y|\leq 1}} \frac{d\sigma(x)}{|x-y|^{n-2+\varepsilon}} + C_0 \int\limits_{\substack{x\in\Sigma \\ |x-y|>1}} \frac{d\sigma(x)}{1+|x|^{n-\varepsilon}} \leq C \\ &\leq \frac{C}{1+|y|^{n-1-\varepsilon}}. \end{aligned} \tag{1.2.66}$$

Above, we have used (1.2.63), the two formulas in [68, (7.2.5)] (with $\mu := \sigma$ and ρ the Euclidean distance in $\mathbb{R}^n$), and the fact that $|y| \leq 1$. In particular, the last constant C in (1.2.66) is positive, finite, and depends only on C_0, n, and the Ahlfors regularity constants of Σ. This proves (1.2.65) in the current case.

Case 2: $|y| > 1$. Pick $y \in \mathbb{R}^n$ with $|y| > 1$. First suppose $x \in \Sigma$ is such that $2|x| \leq |y|$. Applying the Mean Value Theorem and (1.2.64) we obtain

$$|b(x-y) - b_*(-y)| \le C_0|x| \sup_{\xi \in [0,x]} |\xi - y|^{1-n} \le C_0|x| \frac{2^{n-1}}{|y|^{n-1}} \tag{1.2.67}$$

since, for every $\xi \in [0, x]$ we have

$$|y| \le |\xi - y| + |\xi| \le |\xi - y| + |x| \le |\xi - y| + |y|/2, \tag{1.2.68}$$

hence $|y|/2 \le |\xi - y|$. Consequently,

$$\begin{aligned}
&\int_{\substack{x\in\Sigma \\ 2|x|\le|y|}} \frac{|b(x-y) - b_*(-y)|}{1+|x|^n} \, d\sigma(x) \\
&\qquad \le C \int_{\substack{x\in\Sigma \\ |y|\ge 2|x|\ge 1}} \frac{|x|}{|y|^{n-1}(1+|x|^n)} \, d\sigma(x) + C \int_{\substack{x\in\Sigma \\ |x|\le 1/2}} \frac{|x|}{|y|^{n-1}(1+|x|^n)} \, d\sigma(x) \\
&\qquad =: \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{1.2.69}$$

Then

$$\mathrm{II} \le C \int_{\substack{x\in\Sigma \\ |x|\le 1/2}} \frac{d\sigma(x)}{|y|^{n-1}} \le \frac{C}{|y|^{n-1}} \le \frac{C}{1+|y|^{n-1-\varepsilon}} \tag{1.2.70}$$

with $C \in (0, \infty)$ dependent only on C_1, n, and the Ahlfors regularity constants of Σ. To estimate I we first observe that there exists $C = C(n, \varepsilon) \in (0, \infty)$ such that

$$\frac{1}{1+|x|^{n-1}} \le \frac{C|y|^{\varepsilon}}{1+|x|^{n-1+\varepsilon}} \quad \text{if } x \in \Sigma \text{ and } |y| \ge 2|x| \ge 1. \tag{1.2.71}$$

Then (1.2.71) and the second formula in [68, (7.2.5)] imply

$$\begin{aligned}
\mathrm{I} &\le \frac{C}{|y|^{n-1}} \int_{\substack{x\in\Sigma \\ |y|\ge 2|x|\ge 1}} \frac{d\sigma(x)}{1+|x|^{n-1}} \\
&\le \frac{C}{|y|^{n-1-\varepsilon}} \int_{\substack{x\in\Sigma \\ |x|\ge 1/2}} \frac{d\sigma(x)}{|x|^{n-1+\varepsilon}} \le \frac{C}{1+|y|^{n-1-\varepsilon}}
\end{aligned} \tag{1.2.72}$$

for some $C \in (0, \infty)$ depending only on C_0, n, the Ahlfors regularity constants of Σ, and ε.

Second, suppose $x \in \Sigma$ is such that $2|x| > |y|$ while maintaining the assumption $|y| > 1$. Then (1.2.63), the second formula in [68, (7.2.5)], and the fact that $|y| > 1$ imply

$$\int_{\substack{x\in\Sigma\\ 2|x|>|y|}} \frac{|b_*(-y)|}{1+|x|^n}\,\mathrm{d}\sigma(x) \leq \frac{C_0}{|y|^{n-2-\varepsilon}} \int_{\substack{x\in\Sigma\\ 2|x|>|y|}} \frac{\mathrm{d}\sigma(x)}{|x|^n}$$

$$\leq C\frac{1}{|y|^{n-2-\varepsilon}}\frac{1}{|y|} = \frac{C}{|y|^{n-1-\varepsilon}}, \tag{1.2.73}$$

where $C \in (0,\infty)$ depends only on C_0, n, and the Ahlfors regularity constants of Σ. Furthermore, invoking again (1.2.63), we may write

$$\int_{\substack{x\in\Sigma\\ 2|x|>|y|}} \frac{|b(x-y)|}{1+|x|^n}\,\mathrm{d}\sigma(x) = \int_{\substack{x\in\Sigma\\ 2|x|>|y|>|x|/2}} \frac{|b(x-y)|}{1+|x|^n}\,\mathrm{d}\sigma(x)$$

$$+ \int_{\substack{x\in\Sigma\\ |y|\leq|x|/2}} \frac{|b(x-y)|}{1+|x|^n}\,\mathrm{d}\sigma(x)$$

$$:= \mathrm{III} + \mathrm{IV}. \tag{1.2.74}$$

Decompose

$$\mathrm{III} = \int_{\substack{x\in\Sigma\\ 2|x|>|y|>|x|/2\\ |x-y|<1}} \frac{|b(x-y)|}{1+|x|^n}\,\mathrm{d}\sigma(x) + \int_{\substack{x\in\Sigma\\ 2|x|>|y|>|x|/2\\ |x-y|\geq 1}} \frac{|b(x-y)|}{1+|x|^n}\,\mathrm{d}\sigma(x)$$

$$\leq C_0 \int_{\substack{x\in\Sigma\\ 2|x|>|y|>|x|/2\\ |x-y|<1}} \frac{\mathrm{d}\sigma(x)}{|x-y|^{n-2+\varepsilon}|(1+|x|^n)}$$

$$+ C_0 \int_{\substack{x\in\Sigma\\ 2|x|>|y|>|x|/2\\ |x-y|\geq 1}} \frac{\mathrm{d}\sigma(x)}{|x-y|^{n-2-\varepsilon}|(1+|x|^n)}$$

$$:= \mathrm{III}_a + \mathrm{III}_b. \tag{1.2.75}$$

Observe that whenever $x \in \Sigma$ and $2|x| > |y| > |x|/2$ we have $|x-y| \leq |x|+|y| \leq 3|y|$ which further gives

$$\mathrm{III}_a \leq C_0 \int_{\substack{x\in\Sigma\\ 2|x|>|y|>|x|/2}} \frac{\mathrm{d}\sigma(x)}{|x-y|^{n-2+\varepsilon}|x|^n} \leq \frac{C}{|y|^n} \int_{\substack{x\in\Sigma\\ |x-y|<3|y|}} \frac{\mathrm{d}\sigma(x)}{|x-y|^{n-2+\varepsilon}}$$

$$\leq \frac{C}{|y|^n}\cdot|y|^{1-\varepsilon} = \frac{C}{|y|^{n-1+\varepsilon}} \leq \frac{C}{|y|^{n-1-\varepsilon}}, \tag{1.2.76}$$

for some $C \in (0, \infty)$ depending only on C_0, n, ε, and the Ahlfors regularity constants of Σ. For the third inequality in (1.2.76) we have used the first formula in [68, (7.2.5)], while the last inequality in (1.2.76) is due to the fact that we are currently assuming $|y| > 1$. Likewise,

$$\mathrm{III}_b \leq C_0 \int\limits_{\substack{x\in\Sigma \\ 2|x|>|y|>|x|/2}} \frac{\mathrm{d}\sigma(x)}{|x-y|^{n-2-\varepsilon}|x|^n} \leq \frac{C}{|y|^n} \int\limits_{\substack{x\in\Sigma \\ |x-y|<3|y|}} \frac{\mathrm{d}\sigma(x)}{|x-y|^{n-2-\varepsilon}}$$
$$\leq \frac{C}{|y|^n} \cdot |y|^{1+\varepsilon} = \frac{C}{|y|^{n-1-\varepsilon}}. \tag{1.2.77}$$

There remain to estimate IV. To do so, we further decompose

$$\mathrm{IV} = \int\limits_{\substack{x\in\Sigma \\ |y|\leq|x|/2 \\ |x-y|<1}} \frac{|b(x-y)|}{1+|x|^n}\,\mathrm{d}\sigma(x) + \int\limits_{\substack{x\in\Sigma \\ |y|\leq|x|/2 \\ |x-y|\geq 1}} \frac{|b(x-y)|}{1+|x|^n}\,\mathrm{d}\sigma(x)$$
$$:= \mathrm{IV}_a + \mathrm{IV}_b. \tag{1.2.78}$$

Note that if $x \in \Sigma$ and $|y| \leq |x|/2$ we have $|x| \leq |x-y| + |y| \leq |x-y| + |x|/2$, hence $|x|/2 \leq |x-y|$. Together with the second formula in [68, (7.2.5)], and the fact that $|y| > 1$ this allows us to write

$$\mathrm{IV}_a \leq C_0 \int\limits_{\substack{x\in\Sigma \\ |y|\leq|x|/2}} \frac{\mathrm{d}\sigma(x)}{|x-y|^{n-2+\varepsilon}|x|^n} \leq C \int\limits_{\substack{x\in\Sigma \\ |y|\leq|x|/2}} \frac{\mathrm{d}\sigma(x)}{|x|^{2n-2+\varepsilon}}$$
$$\leq \frac{C}{|y|^{n-1+\varepsilon}} \leq \frac{C}{|y|^{n-1-\varepsilon}}, \tag{1.2.79}$$

for some $C \in (0, \infty)$ depending only on C_0, n, ε, and the Ahlfors regularity constants of Σ. Similarly,

$$\mathrm{IV}_b \leq C_0 \int\limits_{\substack{x\in\Sigma \\ |y|\leq|x|/2}} \frac{\mathrm{d}\sigma(x)}{|x-y|^{n-2-\varepsilon}|x|^n} \leq C \int\limits_{\substack{x\in\Sigma \\ |y|\leq|x|/2}} \frac{\mathrm{d}\sigma(x)}{|x|^{2n-2-\varepsilon}}$$
$$\leq \frac{C}{|y|^{n-1-\varepsilon}}. \tag{1.2.80}$$

Now estimate (1.2.65) in the case when $|y| > 1$ follows from (1.2.69)-(1.2.70) combined with (1.2.72)-(1.2.80). □

The proposition below abstractly models the behavior of the modified single layer potential operator, something we will consider later down the road.

Proposition 1.2.6 *Let $\Sigma \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) be a closed Ahlfors regular set and denote $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Suppose $b \in \mathscr{C}^1(\mathbb{R}^n \setminus \{0\})$ is a function for which there exist $\varepsilon \in (0,1)$ along with $C_0 \in (0,\infty)$ such that*

$$|b(x)| \leq C_0 \begin{cases} |x|^{2-n+\varepsilon} & \text{if } |x| \geq 1, \\ |x|^{2-n-\varepsilon} & \text{if } |x| \leq 1, \end{cases} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{1.2.81}$$

and

$$|(\nabla b)(x)| \leq C_0 |x|^{1-n} \text{ for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.2.82}$$

Then the operator

$$B_{\text{mod}} : L^1\Big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1-\varepsilon}}\Big) \longrightarrow L^1\Big(\Sigma, \frac{\sigma(x)}{1+|x|^n}\Big) \tag{1.2.83}$$

defined for each function $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1-\varepsilon}}\big)$ as

$$B_{\text{mod}} f(x) := \int_\Sigma \big(b(x-y) - b_*(-y)\big) f(y)\,\mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \Sigma, \tag{1.2.84}$$

where $b_* := b \cdot \mathbf{1}_{\mathbb{R}^n \setminus B(0,1)}$ in $\mathbb{R}^n \setminus \{0\}$, is well defined, linear, and bounded.

Proof Let $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1-\varepsilon}}\big)$ be arbitrary. Making use of Fubini's Theorem and Lemma 1.2.5 we may write

$$\begin{aligned} &\int_\Sigma \Big(\int_\Sigma \frac{|b(x-y) - b_*(-y)||f(y)|}{1+|x|^n}\,\mathrm{d}\sigma(y)\Big)\,\mathrm{d}\sigma(x) \\ &\quad = \int_\Sigma \Big(\int_\Sigma \frac{|b(x-y) - b_*(-y)|}{1+|x|^n}\,\mathrm{d}\sigma(x)\Big)|f(y)|\,\mathrm{d}\sigma(y) \\ &\quad \leq C \int_\Sigma \frac{|f(y)|}{1+|y|^{n-1-\varepsilon}}\,\mathrm{d}\sigma(y) < +\infty, \end{aligned} \tag{1.2.85}$$

for some $C \in (0,\infty)$ independent of f. This goes to show that the integral defining $B_{\text{mod}} f(x)$ in (1.2.84) is absolutely convergent for σ-a.e. point $x \in \Sigma$. Hence, B_{mod} is well defined in the context of (1.2.83), and the estimate derived in (1.2.85) gives

$$\int_\Sigma \frac{|(B_{\text{mod}} f)(x)|}{1+|x|^n}\,\mathrm{d}\sigma(x) \leq C \int_\Sigma \frac{|f(y)|}{1+|y|^{n-1-\varepsilon}}\,\mathrm{d}\sigma(y). \tag{1.2.86}$$

This proves that B_{mod} is indeed a bounded operator. Finally, its linearity is clear from what we have proved so far and (1.2.84). □

1.3 A First Look at Layer Potential Operators

Here we shall introduce volume and boundary integral operators of layer type associated with a given open set and a weakly elliptic system. Specifically, suppose $n \in \mathbb{N}$ with $n \geq 2$ has been fixed. For some $M \in \mathbb{N}$, consider a coefficient tensor

$$A = \left(a_{rs}^{\alpha\beta}\right)_{\substack{1 \leq r,s \leq n \\ 1 \leq \alpha,\beta \leq M}} \tag{1.3.1}$$

with complex entries, with the property that the $M \times M$ second-order system (as always, the summation convention over repeated indices is in effect)

$$L = L_A := \left(a_{rs}^{\alpha\beta} \partial_r \partial_s\right)_{\substack{1 \leq \alpha \leq M \\ 1 \leq \beta \leq M}} \tag{1.3.2}$$

is weakly elliptic (in the sense of [70, (1.3.3) in Definition 1.3.1]). Also, denote by $A^\top$ the transpose of A, i.e., the coefficient tensor

$$A^\top = \left(a_{sr}^{\beta\alpha}\right)_{\substack{1 \leq r,s \leq n \\ 1 \leq \alpha,\beta \leq M}}. \tag{1.3.3}$$

Let $E = (E_{\gamma\beta})_{1 \leq \gamma,\beta \leq M}$ be the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2].

Given an arbitrary open set $\Omega \subseteq \mathbb{R}^n$, define the `Newtonian` (or `volume`) `potential operator` associated with the system L in Ω as the mapping assigning to each Lebesgue measurable function $w : \Omega \to \mathbb{C}^M$ the expression

$$(\Pi_\Omega w)(x) := \int_\Omega E(x-y) w(y)\, \mathrm{d}y \tag{1.3.4}$$

at points $x \in \Omega$ for which the above integral is absolutely convergent. Also, having set $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$, define the action of the boundary-to-domain `single layer potential operator` $\mathscr{S}$ on function

$$f = (f_\alpha)_{1 \leq \alpha \leq M} \in \left[L^1\left(\partial\Omega, \frac{\sigma(x)}{1 + |x|^{n-2}}\right)\right]^M \tag{1.3.5}$$

according to

$$\begin{aligned}\mathscr{S}f(x) &:= \int_{\partial\Omega} E(x-y)f(y)\,\mathrm{d}\sigma(y) = \Big(\int_{\partial\Omega} E_{\gamma\alpha}(x-y)f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M}\\ &= \Big(\int_{\partial\Omega}\Big\langle E_{\gamma\bullet}(x-y),\ f(y)\Big\rangle\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M}\\ &= \int_{\partial\Omega}\big[E_{L^\top}(x-y)\big]^\top f(y)\,\mathrm{d}\sigma(y)\\ &= \Big(\int_{\partial\Omega}\Big\langle\big(E_{L^\top}(x-y)\big)_{\bullet\gamma},\ f(y)\Big\rangle\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M}\quad\text{for each } x\in\Omega. \end{aligned} \tag{1.3.6}$$

Above, $E_{\gamma\bullet}$ denotes the γ-th row of the matrix E. Also, $E_{L^\top}$ is the fundamental solution associated as in [70, Theorem 1.4.2] with the transpose system $L^\top$, and $(E_{L^\top})_{\bullet\gamma}$ denotes the γ-th column of the matrix $E_{L^\top}$. In the two-dimensional case, the weight $(1+|x|^{n-2})^{-1}$ appearing in (1.3.5) should be replaced by $\ln(2+|x|)$, i.e., when $n=2$ in place of (1.3.5) we shall (sometimes tacitly, but always) consider

$$f\in\Big[L^1\Big(\partial\Omega,\ln(2+|x|)\sigma(x)\Big)\Big]^M. \tag{1.3.7}$$

Since it is clear from (1.3.6) and [70, (1.4.20)] that

$$\mathscr{S}f\in\big[\mathscr{C}^\infty(\Omega)\big]^M \text{ and } L(\mathscr{S}f)=0 \text{ in } \Omega, \text{ for each } f \text{ as in (1.3.5)}, \tag{1.3.8}$$

the operator $\mathscr{S}$ may be regarded as a tool for generating a multitude of null-solutions for the given system L in Ω. We will study this operator in detail, starting with Theorem 1.5.1. For now, one thing that we can clarify is the decay at infinity in the case of an exterior domain.

Lemma 1.3.1 *Assume that $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$ satisfies $n\ge 2$) is an exterior domain and that $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ is a finite measure. Consider a weakly elliptic, second-order, homogeneous, constant (complex) coefficient $M\times M$ system in $\mathbb{R}^n$. Associate the single layer potential operator $\mathscr{S}$ with Ω and L as above. Finally, define*

$$u:=\mathscr{S}f \text{ in } \Omega \text{ for some } f\in\big[L^1(\partial\Omega,\sigma)\big]^M. \tag{1.3.9}$$

Then the following statements are true.

(i) Given any multi-index $\alpha\in\mathbb{N}_0^n$, one has $(\partial^\alpha u)(x)=O(|x|^{2-n-|\alpha|})$ as $|x|\to\infty$ provided either $|\alpha|>0$, or $n\ge 3$, or $n=2$ and $\int_{S^1}\big[L(\xi)\big]^{-1}\,\mathrm{d}\mathcal{H}^1(\xi)=0$.

(ii) If $\int_{\partial\Omega} f\,\mathrm{d}\sigma=0$ then $(\partial^\alpha u)(x)=O(|x|^{1-n-|\alpha|})$ as $|x|\to\infty$ for any multi-index $\alpha\in\mathbb{N}_0^n$.

(iii) If $n=2$ then

$$\begin{aligned}&\lim_{x\to\infty} u(x) \text{ exists (in } \mathbb{C}^M) \text{ if and only if } \int_{\partial\Omega} f\,\mathrm{d}\sigma=0 \text{ if and only if}\\ &\lim_{x\to\infty} u(x)=0 \text{ if and only if } u(x)=O(|x|^{-1}) \text{ as } |x|\to\infty.\end{aligned} \tag{1.3.10}$$

Also, in the two-dimensional setting,

$$\begin{aligned} u(x) &= E(x)\Big(\int_{\partial\Omega} f \,\mathrm{d}\sigma\Big) + O(|x|^{-1}) \\ &= O(1) - \frac{\ln|x|}{4\pi^2}\Big(\int_{S^1}\big[L(\xi)\big]^{-1}\,\mathrm{d}\mathcal{H}^1(\xi)\Big)\Big(\int_{\partial\Omega} f\,\mathrm{d}\sigma\Big) \\ &= O(\ln|x|) \ \textit{as}\ |x|\to\infty, \end{aligned} \tag{1.3.11}$$

and, as a consequence,

$$\begin{aligned} &\textit{if } n = 2 \textit{ then } u(x) = o(\ln|x|) \textit{ as } |x|\to\infty \textit{ is equivalent to} \\ &\Big(\textstyle\int_{S^1}\big[L(\xi)\big]^{-1}\,\mathrm{d}\mathcal{H}^1(\xi)\Big)\Big(\textstyle\int_{\partial\Omega} f\,\mathrm{d}\sigma\Big) = 0. \end{aligned} \tag{1.3.12}$$

In particular, if $n = 2$ *and*

$$\textit{the } M\times M \textit{ matrix } \int_{S^1}\big[L(\xi)\big]^{-1}\,\mathrm{d}\mathcal{H}^1(\xi) \textit{ is invertible} \tag{1.3.13}$$

(*which is always the case if the system* L *is assumed to actually satisfy the Legendre-Hadamard strong ellipticity condition, and if* $M = 1$ *this is true if and only if* $L = \nabla\cdot A\nabla$ *for some* $A \in M_2^0$*; cf.* [70, Lemma 1.4.19] *and* [70, (1.4.186)]) *then having* $u(x) = o(\ln|x|)$ *as* $|x|\to\infty$ *is equivalent to having* $\int_{\partial\Omega} f\,\mathrm{d}\sigma = 0$.

Proof For each multi-index $\alpha\in\mathbb{N}_0^n$ we have

$$\partial^\alpha(\mathscr{S}f)(x) = \int_{\partial\Omega}(\partial^\alpha E)(x-y)f(y)\,\mathrm{d}\sigma(y) \ \text{ for all } \ x\in\Omega, \tag{1.3.14}$$

and under the additional condition that $\int_{\partial\Omega} f\,\mathrm{d}\sigma = 0$ we may further express

$$\partial^\alpha(\mathscr{S}f)(x) = \int_{\partial\Omega}\Big\{(\partial^\alpha E)(x-y) - (\partial^\alpha E)(x)\Big\}f(y)\,\mathrm{d}\sigma(y) \tag{1.3.15}$$

for all $x\in\Omega$. The claims in items *(i)-(ii)* are then direct consequences of these observations, [70, (1.4.24)], and the Mean Value Theorem. To treat item *(iii)*, work in the two-dimensional setting. For starters, write

$$\begin{aligned}
u(x) &= E(x)\Big(\int_{\partial\Omega} f(y)\,\mathrm{d}\sigma(y)\Big) + \int_{\partial\Omega} \big\{E(x-y)-E(x)\big\} f(y)\,\mathrm{d}\sigma(y)\\
&= \Big(\Phi(x) - \frac{\ln|x|}{4\pi^2}\int_{S^1}\big[L(\xi)\big]^{-1}\,\mathrm{d}\mathcal{H}^1(\xi)\Big)\Big(\int_{\partial\Omega} f\,\mathrm{d}\sigma\Big) + O(|x|^{-1})\\
&= O(1) - \frac{\ln|x|}{4\pi^2}\Big(\int_{S^1}\big[L(\xi)\big]^{-1}\,\mathrm{d}\mathcal{H}^1(\xi)\Big)\Big(\int_{\partial\Omega} f\,\mathrm{d}\sigma\Big) + O(|x|^{-1})\\
&= O(\ln|x|) \ \text{ as } \ |x|\to\infty, \qquad (1.3.16)
\end{aligned}$$

where we have used [70, (1.4.22), (1.4.24)], the Mean Value Theorem, and the fact that the function $\Phi : \mathbb{R}^2\setminus\{0\}\to\mathbb{C}^{M\times M}$ is bounded (since it is continuous and positive homogeneous of degree 0 in $\mathbb{R}^2\setminus\{0\}$; cf. [70, (1.4.23)]). This proves (1.3.11) which, in turn, implies (1.3.12).

The equivalences in (1.3.10) are seen from the first line in (1.3.11) and [70, (1.4.47)]. The very last equivalence in the statement then follows from (1.3.12) keeping in mind (1.3.13). □

Going further, strengthen the background hypotheses by assuming that the open set Ω is of locally finite perimeter, and denote by $\nu = (\nu_1,\dots,\nu_n)$ its geometric measure theoretic outward unit normal. In this context, we introduce the boundary-to-domain `double layer potential operator` as the integral operator acting on each function

$$f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \qquad (1.3.17)$$

according to (recall that the summation convention over repeated indices is presently in effect)

$$\mathcal{D}f(x) := \Big(-\int_{\partial_*\Omega} \nu_s(y) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y) f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M} \qquad (1.3.18)$$

for every $x\in\Omega$. In relation to this, we make two conventions. First,

> whenever we wish to emphasize the dependence of the double layer operator (1.3.18) on the coefficient tensor A from (1.3.1) we agree to write $\mathcal{D}_A$ in place of $\mathcal{D}$. (1.3.19)

In contrast to the single layer potential operator (1.3.6) which is intrinsically associated with the given system L, to each coefficient tensor A belonging to the "library" of (admissible) coefficient tensors of the system L, defined as

$$\mathfrak{A}_L := \Big\{A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}} : L = L_A := \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}\Big\}, \qquad (1.3.20)$$

there corresponds a double layer $\mathcal{D}_A$. In particular, the terminology *a double layer*, as opposed to *the double layer*, is appropriate whenever referring to $\mathcal{D}$ associated with a generic coefficient tensor $A \in \mathfrak{A}_L$ when no mention (explicit or implicit) of the coefficient tensor A is being made.

Second, typically without any special mention,

$$\text{we agree to retain the same formula for the double layer potential operator, given in (1.3.18), at points } x \text{ belonging to } \mathbb{R}^n \setminus \overline{\Omega} \text{ as well.} \tag{1.3.21}$$

In terms of the conormal derivative operator (A.0.184), the double layer operator (1.3.18) may be expressed more concisely as

$$\begin{aligned}\mathcal{D}f(x) &= \left(\int_{\partial_*\Omega} \left\langle \partial^{A^\top}_{\nu(y)}\Big[\big(E_{L^\top}(x-y)\big)_{\bullet\gamma}\Big],\, f(y)\right\rangle \mathrm{d}\sigma(y)\right)_{1\le\gamma\le M} \\ &= \left(\int_{\partial_*\Omega} \left\langle \partial^{A^\top}_{\nu(y)}\Big[\big(E_{L}(x-y)\big)_{\gamma\bullet}\Big],\, f(y)\right\rangle \mathrm{d}\sigma(y)\right)_{1\le\gamma\le M} \\ &= \int_{\partial_*\Omega} \Big(\partial^{A^\top}_{\nu(y)}\big[E_{L^\top}(x-y)\big]\Big)^{\top} f(y)\,\mathrm{d}\sigma(y), \quad \forall\, x\in\Omega. \end{aligned} \tag{1.3.22}$$

Above, $E_{L^\top}$ is the fundamental solution associated as in [70, Theorem 1.4.2] with the transpose system $L^\top$, and $\big(E_{L^\top}\big)_{\bullet\gamma}$ denotes the γ-th column of the matrix $E_{L^\top}$. Also, $\big(E_L\big)_{\gamma\bullet}$ denotes the γ-th row of the matrix E_L, so the second equality in (1.3.22) becomes a consequence of the first property in [70, (1.4.32)]. Finally, the last equality in (1.3.22) follows from the second, keeping in mind that the conormal derivative acts (in the variable y) as the $M \times M$ first-order system

$$\partial^{A^\top}_{\nu(y)} := \Big(\nu_r(y)a^{\beta\alpha}_{sr}\,\partial_{y_s}\Big)_{1\le\alpha,\beta\le M} \tag{1.3.23}$$

on the $M \times M$ matrix $E_{L^\top}(x-y)$ (in a natural manner which, on the algebraic side, takes into account the ordinary multiplication of $M\times M$ matrices), and the transposed of the resulting $M \times M$ matrix is then applied to the vector $f(y) \in \mathbb{C}^M$.

The double layer operator $\mathcal{D}$ may be regarded as a mechanism for generating lots of null-solutions for the given system L in Ω since, as is apparent from (1.3.18) and item *(2)* in [70, Theorem 1.4.2],

$$\mathcal{D}f \in \big[\mathscr{C}^\infty(\Omega)\big]^M \text{ and } L(\mathcal{D}f) = 0 \text{ in } \Omega, \text{ for each } f \text{ as in (1.3.17).} \tag{1.3.24}$$

Indeed, given any function f as in (1.3.17), for each $x \in \Omega$ we may write

$$\begin{aligned} L\big(\mathcal{D}f\big)(x) &= \Big\{a^{\mu\gamma}_{jk}\,\partial_j\partial_k\big(\mathcal{D}f\big)_\gamma(x)\Big\}_{1\le\mu\le M} \\ &= \left(\int_{\partial_*\Omega} \nu_s(y)a^{\beta\alpha}_{rs}\,\partial_{x_r}\Big[a^{\mu\gamma}_{jk}(\partial_j\partial_k E_{\gamma\beta})(x-y)\Big] f_\alpha(y)\,\mathrm{d}\sigma(y)\right)_{1\le\mu\le M} = 0, \end{aligned} \tag{1.3.25}$$

since the expression inside the brackets vanishes for each $y \in \partial\Omega$, thanks to [70, (1.4.21)].

It is also clear from (1.3.18) and [70, Theorem 1.4.2] that, in the case when Ω is an exterior domain, for each function f as in (1.3.17) and each integer $k \in \mathbb{N}_0$ the following decay condition holds:

$$\nabla^k \mathcal{D}f(x) = \begin{cases} O(|x|^{1-n-k}), \\ O(|x|^{-n-k}) \text{ if } \int_{\partial_*\Omega} f \, \mathrm{d}\sigma = 0, \end{cases} \qquad \text{as } |x| \to \infty. \tag{1.3.26}$$

As we shall see in what follows, in addition to qualities displayed in (1.3.24), (1.3.26) there are other, more specialized, properties that are characteristic to the double layer potential operator defined in (1.3.18).

A fundamental property of the double layer potential operator $\mathcal{D}$ just introduced is the ability of absorbing an arbitrary spatial derivative and eventually relocate it, via integration by parts on the boundary, all the way to the function on which this was applied to begin with. This is made precise in the following basic lemma.

Lemma 1.3.2 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set of locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, for some integer $M \in \mathbb{N}$, consider a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in (1.3.2) is weakly elliptic* (*in the sense of* [70, (1.3.3) in Definition 1.3.1]). *Finally, associate with A and Ω the double layer potential operator $\mathcal{D}$ as in* (1.3.18)*, and consider a function $f = (f_\alpha)_{1\leq\alpha\leq M}$ belonging to the weighted Sobolev space $\big[L^1_1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ (cf.* (A.0.131))*, i.e.,*

$$\begin{gathered} f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \\ \textit{with the property that } \partial_{\tau_{jk}} f_\alpha \in L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \\ \textit{for all } j, k \in \{1, \dots, n\} \textit{ and } \alpha \in \{1, \dots, M\}. \end{gathered} \tag{1.3.27}$$

Then, for each index $\ell \in \{1, \dots, n\}$ and at every $x \in \Omega$, one has

$$\partial_\ell\big(\mathcal{D}f\big)(x) = \Bigg(\int_{\partial_*\Omega} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} f_\alpha)(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\leq\gamma\leq M}. \tag{1.3.28}$$

For example, if $M := 1$ and $L := \Delta$, the Laplace operator in $\mathbb{R}^n$, then (1.3.27)-(1.3.28) amount to saying that for any scalar-valued function

$$
\begin{aligned}
& f \in L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \text{ with the property that} \\
& \partial_{\tau_{jk}} f \in L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \text{ for all } j,k \in \{1,\dots,n\}
\end{aligned}
\tag{1.3.29}
$$

and any index $\ell \in \{1,\dots,n\}$ one has

$$
\partial_\ell(\mathcal{D}f)(x) = \sum_{j=1}^{n} \int_{\partial_*\Omega} (\partial_j E_\Delta)(x-y)(\partial_{\tau_{\ell j}} f)(y)\,\mathrm{d}\sigma(y) \text{ for each } x \in \Omega, \tag{1.3.30}
$$

where $\mathcal{D}$ is the standard boundary-to-domain harmonic double layer operator associated with the set Ω, and E_Δ is the standard fundamental solution for the Laplacian (cf. (A.0.65)).

Proof of Lemma 1.3.2 Since the given function f is as in (1.3.17), we know from (1.3.24) that $\mathcal{D}f$ is well defined and belongs to $\big[\mathscr{C}^\infty(\Omega)\big]^M$. Having fixed an index $\gamma \in \{1,\dots,M\}$ and $x \in \Omega$ we may then compute

$$
\begin{aligned}
\partial_\ell\big(\mathcal{D}f\big)_\gamma(x) &= -\int_{\partial_*\Omega} \nu_s(y) a^{\beta\alpha}_{rs} (\partial_\ell \partial_r E_{\gamma\beta})(x-y) f_\alpha(y)\,\mathrm{d}\sigma(y) \\
&= -\int_{\partial_*\Omega} a^{\beta\alpha}_{rs} \partial_{\tau_{\ell s}(y)}[(\partial_r E_{\gamma\beta})(x-y)] f_\alpha(y)\,\mathrm{d}\sigma(y) \\
&= \int_{\partial_*\Omega} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} f_\alpha)(y)\,\mathrm{d}\sigma(y).
\end{aligned}
\tag{1.3.31}
$$

Above, the first equality comes from (1.3.18) and differentiation under the integral sign, while the second equality uses the definition of $\partial_{\tau_{\ell s}(y)}$ and [70, (1.4.33)]. In the third equality in (1.3.31) we have integrated by parts on the boundary, as permitted by [69, Lemma 11.1.7], given that the function $\varphi := (\partial_r E_{\gamma\beta})(x-\cdot)$ is of class $\mathscr{C}^1$ in a collar neighborhood of $\partial\Omega$ and, thanks to [70, (1.4.24)] and (1.3.27), satisfies

$$
\begin{aligned}
& \int_{\partial_*\Omega} |f||\nabla\varphi|\,\mathrm{d}\sigma < +\infty, \quad \int_{\partial_*\Omega} \big|\partial_{\tau_{\ell s}} f\big||\varphi|\,\mathrm{d}\sigma < +\infty, \\
& \text{as well as } \int_{\partial_*\Omega} |f(x)||\varphi(x)|(1+|x|)^{-1}\,\mathrm{d}\sigma(x) < +\infty.
\end{aligned}
\tag{1.3.32}
$$

This establishes (1.3.31), from which formula (1.3.28) now follows in view of the arbitrariness of $\gamma \in \{1,\dots,M\}$ and $x \in \Omega$. □

In terms of the integral operators introduced above, we may recast the integral representation formula from [70, Theorem 1.5.1] succinctly, as follows.

Theorem 1.3.3 *Let $\Omega \subseteq \mathbb{R}^n$, where*[6] *$n \geq 3$, be an open set with a lower Ahlfors regular boundary, and with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure*

[6] see the last part in the statement for the case $n = 2$

on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Let L be a homogeneous, weakly elliptic, second-order $M\times M$ system in $\mathbb{R}^n$, with complex constant coefficients. Suppose $u \in \big[L^1_{\rm loc}(\Omega,\mathcal{L}^n)\big]^M$ is a vector-valued function satisfying, for some aperture parameter $\kappa > 0$,

$$\begin{gathered} Lu \in \Big[L^1\Big(\Omega, \frac{dy}{1+|y|^{n-2}}\Big)\Big]^M \text{ and } A\nabla u \in \big[L^1_{\rm loc}(\Omega,\mathcal{L}^n)\big]^{M\times n}, \\ u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ and } (A\nabla u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ exist at } \sigma\text{-a.e. point on } \partial_{\rm nta}\Omega, \end{gathered} \tag{1.3.33}$$

(*with all derivatives taken in the sense of distributions, and using the piece of notation introduced in* (A.0.12)). *In addition, assume the following integrability conditions hold:*

$$\int_{\partial\Omega} \frac{(\mathcal{N}_\kappa u)(y)}{1+|y|^{n-1}}\,{\rm d}\sigma(y) < \infty \text{ and } \int_{\partial\Omega} \frac{\big(\mathcal{N}_\kappa(A\nabla u)\big)(y)}{1+|y|^{n-2}}\,{\rm d}\sigma(y) < \infty. \tag{1.3.34}$$

Next, recall the Newtonian potential operator Π_Ω from (1.3.4) *and the single layer potential operator $\mathscr{S}$ from* (1.3.6). *Finally, pick a coefficient tensor A such that $L = L_A$ and consider the potential operator $\mathcal{D}$ and the conormal derivative operator ∂^A_ν associate with A and Ω as in* (1.3.18) *and* (A.0.184), *respectively.*

Then at $\mathcal{L}^n$-a.e. point $x \in \mathbb{R}^n \setminus \partial\Omega$ one has[7]

$$\mathcal{D}\big(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(x) - \mathscr{S}(\partial^A_\nu u)(x) + \Pi_\Omega(Lu)(x) = \begin{cases} u(x) & \text{if } x\in\Omega, \\ 0 & \text{if } x \in \mathbb{R}^n\setminus\overline{\Omega}, \end{cases} \tag{1.3.35}$$

if either Ω is bounded, or $\partial\Omega$ is unbounded. In the case when Ω is an exterior domain, the same conclusion holds under the additional assumption that there exists $\lambda \in (1,\infty)$ such that

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |u|\,{\rm d}\mathcal{L}^n = o(1) \text{ as } R\to\infty. \tag{1.3.36}$$

Finally, similar results are valid in the case when $n = 2$ provided either

$$\int_{S^1} \big[L(\xi)\big]^{-1}\,{\rm d}\mathcal{H}^1(\xi) = 0 \in \mathbb{C}^{M\times M}, \tag{1.3.37}$$

or otherwise one assumes that $\partial\Omega$ is compact, replaces the first membership in (1.3.33) *by*

[7] We agree to retain the formulas for the double layer, single layer, and Newtonian potential operators given, respectively, in (1.3.18), (1.3.6), and (1.3.4), at points x belonging to $\mathbb{R}^n\setminus\overline{\Omega}$ as well. Here we also allow the operator $\mathscr{S}$ to act on functions originally defined only on $\partial_*\Omega$ by extending them by zero to the entire topological boundary $\partial\Omega$; we shall occasionally make use of the latter option in a tacit fashion in the future.

$$\int_{\Omega} |(Lu)(y)| \ln(2 + |y|)\, \mathrm{d}y < +\infty, \tag{1.3.38}$$

and, in the case when Ω *is an exterior domain, replaces* (1.3.36) *by*

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |u| \, \mathrm{d}\mathcal{L}^2 = o\big(\tfrac{1}{\ln R}\big) \ \text{ as } \ R \to \infty. \tag{1.3.39}$$

Proof This is obtained by merely recasting formulas [70, (1.5.4), (1.5.6)], in light of (1.3.18), (1.3.6), (A.0.184), and (1.3.4). □

We wish to augment the above integral representation formula by considering exterior domains in the two-dimensional setting. Corresponding to null-solutions of the given system, we have the following result:

Theorem 1.3.4 *Let* $\Omega \subseteq \mathbb{R}^2$ *be an unbounded open set with a compact lower Ahlfors regular boundary, and with the property that* $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ *is a doubling measure on* $\partial\Omega$*. Denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Let* L *be a homogeneous, weakly elliptic, second-order* $M \times M$ *system in* $\mathbb{R}^2$*, with complex constant coefficients, and pick a coefficient tensor* A *such that* $L = L_A$*. Suppose* u *is a vector-valued function satisfying*

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M \ \text{ and } \ Lu = 0 \ \text{ in } \ \Omega, \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \text{ and } \ (A\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \text{ exist at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega. \end{gathered} \tag{1.3.40}$$

In addition, assume that for some aperture parameter $\kappa > 0$ *and some truncation parameter* $\varepsilon > 0$ *the following integrability conditions hold:*

$$\int_{\partial\Omega} \mathcal{N}_\kappa^\varepsilon u \, \mathrm{d}\sigma < \infty \ \text{ and } \ \int_{\partial\Omega} \mathcal{N}_\kappa^\varepsilon (A\nabla u) \, \mathrm{d}\sigma < \infty, \tag{1.3.41}$$

and assume that there exists $\lambda \in (1, \infty)$ *such that*

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |u| \, \mathrm{d}\mathcal{L}^2 = o(1) \ \text{ as } \ R \to \infty. \tag{1.3.42}$$

Finally, consider the double layer potential operator $\mathcal{D}$ *and the conormal derivative operator* ∂_ν^A *associated with* A *and* Ω *as in* (1.3.18) *and* (A.0.184)*, respectively.*

Then for each point $x \in \mathbb{R}^2 \setminus \partial\Omega$ *one has*

$$\mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) - \mathcal{S}(\partial_\nu^A u)(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^2 \setminus \overline{\Omega}. \end{cases} \tag{1.3.43}$$

Proof This is a direct consequence of [70, Theorem 1.5.7] used with $N = 0$ (in which scenario the polynomial P vanishes identically, since its degree is $\leq \lfloor 0 \rfloor = -1$) and definitions. □

For further use, let us also record here the following remarkable identity involving the double and single layers.

Corollary 1.3.5 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, be an Ahlfors regular domain, which is assumed to have compact boundary if $n = 2$. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, consider a homogeneous, weakly elliptic, second-order $M \times M$ system in $\mathbb{R}^n$, with complex constant coefficients, and denote by $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ the matrix-valued fundamental solution associated with L as in* [70, Theorem 1.4.2].

Next, pick a coefficient tensor A such that $L = L_A$ and recall the double layer potential operator $\mathcal{D}$ and the conormal derivative operator ∂_ν^A associated with A and Ω as in (1.3.18) *and* (A.0.184)*, respectively. Finally, recall the single layer potential operator $\mathcal{S}$ on associated with Ω and L as in* (1.3.6).

Then for each point $x_o \in \Omega$ and each index $\beta \in \{1, \dots, M\}$ one has

$$\mathcal{D}\big(E_{\bullet\beta}(\cdot - x_o)\big|_{\partial\Omega}\big) = \mathcal{S}\big(\partial_\nu^A[E_{\bullet\beta}(\cdot - x_o)]\big) \text{ in } \Omega. \tag{1.3.44}$$

Proof This is a direct consequence of [70, Proposition 1.5.5], [70, Theorem 1.4.2], and definitions. □

It turns out that the boundary-to-domain double layer maps constant functions in a very specific fashion, made precise below.

Proposition 1.3.6 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, be an open set of locally finite perimeter with a compact boundary. Also, consider a homogeneous, weakly elliptic, second-order $M \times M$ system L in $\mathbb{R}^n$, with complex constant coefficients, and denote by $\mathcal{D}$ a double layer potential operator associated with L and Ω as in* (1.3.18)*. Then for each constant $\lambda \in \mathbb{C}^M$ one has*

$$\mathcal{D}\lambda = \begin{cases} \lambda \text{ in } \Omega, & \text{if } \Omega \text{ is bounded,} \\ 0 \text{ in } \Omega, & \text{if } \Omega \text{ is unbounded.} \end{cases} \tag{1.3.45}$$

Furthermore, under the additional assumption that $\partial\Omega$ is a lower Ahlfors regular set and that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$, it follows that for each aperture parameter $\kappa > 0$ and each locally constant $\mathbb{C}^M$-valued function η in Ω which is bounded and whose nontangential boundary trace $\eta\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. on $\partial\Omega$[8] *one has* (*with the convention made in* (1.3.21))

$$\mathcal{D}\big(\eta\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) = \begin{cases} \eta \text{ in } \Omega, & \text{if } \Omega \text{ is bounded,} \\ 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega}, & \text{if } \Omega \text{ is bounded,} \\ \eta - \eta_\infty \text{ in } \Omega, & \text{if } \Omega \text{ is unbounded,} \\ -\eta_\infty \text{ in } \mathbb{R}^n \setminus \overline{\Omega}, & \text{if } \Omega \text{ is unbounded,} \end{cases} \tag{1.3.46}$$

[8] this happens automatically if, e.g., Ω has finitely many connected components which are separated

where $\eta_\infty \in \mathbb{C}^M$ *is the value of* η *in the* (*unique*) *unbounded connected component of* Ω.

Proof Fix $\lambda = (\lambda_\alpha)_{1\leq\alpha\leq M} \in \mathbb{C}^M$ along with $\gamma \in \{1, \dots, M\}$. In a first stage, assume that Ω is bounded and fix an arbitrary point $x \in \Omega$. The idea is to apply [68, Corollary 2.8.8] to a vector field $\vec{F} = (F_s)_{1\leq s\leq n} \in \big[\mathscr{E}'(\mathbb{R}^n)\big]^n$ whose $\mathscr{C}^\infty$-singular support is a compact subset of Ω and with the property that for each $s \in \{1, \dots, n\}$ its scalar component F_s coincides near the compact set $\overline{\Omega}$ with

$$-a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\lambda_\alpha \ \text{ for } \mathcal{L}^n\text{-a.e. } y \in \Omega. \tag{1.3.47}$$

Then [70, (1.4.33)] gives that $\big(\mathrm{div}\vec{F}\big)\big|_\Omega = \lambda_\gamma\delta_x$, where δ_x is the Dirac distribution with mass at x in Ω. In view of this and (1.3.18), the Divergence Formula [68, (2.8.57)] reduces precisely to $(\mathcal{D}\lambda)_\gamma(x) = \lambda_\gamma$, so the version of (1.3.45) when Ω is bounded follows on account of the arbitrariness of $x \in \Omega$ and $\gamma \in \{1, \dots, M\}$. Parenthetically we wish to note that we could have also arrived to this conclusion as a consequence of (1.3.35) used with $u \equiv \lambda$, but under stronger assumptions on the underlying domain.

In the case when Ω is unbounded and $\partial\Omega$ is bounded, [68, Lemma 5.10.10] and [68, (5.6.16)] imply that $\Omega^c := \mathbb{R}^n \setminus \Omega$ is a compact set of locally finite perimeter, satisfying $\partial^*(\mathbb{R}^n \setminus \Omega) = \partial^*\Omega$, and whose geometric measure theoretic outward unit normal is $-\nu$. Having fixed an arbitrary point $x \in \Omega$, apply the De Giorgi-Federer version of the Divergence Theorem (cf. [68, Theorem 1.1.1]) to the set Ω^c and a vector field $\vec{F} \in \big[\mathscr{C}^\infty_c(\mathbb{R}^n)\big]^n$ which coincides in an open neighborhood $\mathcal{O} \subseteq \mathbb{R}^n \setminus \{x\}$ of $\mathbb{R}^n \setminus \Omega$ with

$$\mathcal{O} \ni y \longmapsto \Big\{-a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\lambda_\alpha\Big\}_{1\leq s\leq n} \in \mathbb{C}^n. \tag{1.3.48}$$

Since (1.3.48) and [70, (1.4.33)] imply $\big(\mathrm{div}\vec{F}\big)\big|_{\Omega^c} = 0$, the Divergence Formula [68, (1.1.8)] now simply becomes $(\mathcal{D}\lambda)(x) = 0$, thanks to (1.3.18). The proof of (1.3.45) is therefore complete.

To justify (1.3.46), make the additional assumption that $\partial\Omega$ is a lower Ahlfors regular set and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Also, fix an arbitrary aperture parameter $\kappa > 0$ and pick a locally constant $\mathbb{C}^M$-valued function η in Ω which is bounded, and has a nontangential boundary trace. If Ω is bounded, then the first two cases in (1.3.46) are seen directly from (1.3.35) (used with $u := \eta$). In the case when Ω is unbounded, apply (1.3.35) to the function $u := \eta - \eta_\infty$ (which vanishes at infinity) to obtain

$$\mathcal{D}\Big(\eta\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - \eta_\infty\Big) = \begin{cases} \eta - \eta_\infty & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \tag{1.3.49}$$

From this, the version of (1.3.46) for unbounded sets follows upon invoking formula (1.3.45), written with $\lambda := \eta_\infty$ and both for Ω as well as for $\mathbb{R}^n \setminus \overline{\Omega}$ (bearing in mind that the geometric measure theoretic outward unit normal to the latter set is $-\nu$). $\square$

There is also a boundary version of (1.3.45) involving a brand of principal-value integral in which the radius of the ball excised goes to zero from within a suitably chosen subset of $(0, \infty)$, as described in our next proposition.

Proposition 1.3.7 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ satisfies $n \geq 2$) be a set of locally finite perimeter with compact boundary. Denote by ν its geometric measure theoretic outward unit normal and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, for some $M \in \mathbb{N}$, consider a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in (1.3.2) is weakly elliptic (in the sense of* [70, (1.3.3) in Definition 1.3.1]). *Finally, let $E_{L^\top}$ be the matrix-valued fundamental solution associated with $L^\top$ as in* [70, Theorem 1.4.2].

Then for σ-a.e. point $x \in \partial_*\Omega$ one has

$$\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \partial_{\nu(y)}^{A^\top}\big[E_{L^\top}(x-y)\big]\,\mathrm{d}\sigma(y) = \begin{cases} +\frac{1}{2}I_{M\times M} & \text{if } \Omega \text{ is bounded,}\\ -\frac{1}{2}I_{M\times M} & \text{if } \Omega \text{ is unbounded,}\end{cases} \tag{1.3.50}$$

where the conormal derivative is applied to the columns of $E_{L^\top}$, and $I_{M\times M}$ is the $M \times M$ identity matrix.

Moreover, under the stronger hypothesis that Ω is a Lebesgue measurable set such that $\partial\Omega$ is bounded and $\partial_*\Omega$ is upper Ahlfors regular, there exists some number $N \in (0, \infty)$, depending only on n, L, and the upper Ahlfors regularity constant of $\partial_*\Omega$, with the property that

$$\sup_{\varepsilon>0}\Bigg|\int_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \partial_{\nu(y)}^{A^\top}\big[E_{L^\top}(x-y)\big]\,\mathrm{d}\sigma(y)\Bigg| \leq N \text{ for each } x \in \mathbb{R}^n. \tag{1.3.51}$$

Proof Denote by $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2]. For each pair of indices $\alpha, \gamma \in \{1, \dots, M\}$ define the vector field (recall that the summation convention over repeated indices is presently in effect)

$$\vec{k}_{\alpha\gamma} := -\big(a_{rs}^{\beta\alpha}\partial_r E_{\gamma\beta}\big)_{1\leq s\leq n} \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^n \tag{1.3.52}$$

which is odd and positive homogeneous of degree $1 - n$ (cf. [70, Theorem 1.4.2]), and which satisfies

$$\mathrm{div}\vec{k}_{\alpha\gamma} = -a_{rs}^{\beta\alpha}\partial_r\partial_s E_{\gamma\beta} = 0 \text{ in } \mathbb{R}^n \setminus \{0\}, \tag{1.3.53}$$

thanks to [70, (1.4.33)]. Also, by virtue of the second equality in [70, (1.4.25)],

$$\vartheta_{\alpha\gamma} := \int_{S^{n-1}} \big\langle \omega, \vec{k}_{\alpha\gamma}(\omega) \big\rangle \, d\mathcal{H}^{n-1}(\omega)$$
$$= - \int_{S^{n-1}} \omega_s a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(\omega) \, d\mathcal{H}^{n-1}(\omega) = -\delta_{\alpha\gamma}. \tag{1.3.54}$$

To proceed, observe that for each $y \in \partial^*\Omega$ and $x \in \mathbb{R}^n \setminus \{y\}$ we have

$$\partial^{A^\top}_{\nu(y)} \big[E_{L^\top}(x-y) \big] = \Big(-\nu_r(y) a^{\beta\alpha}_{sr} (\partial_s E_{\gamma\beta})(x-y) \Big)_{1\leq\alpha,\gamma\leq M}$$
$$= \Big(\big\langle \nu(y), \vec{k}_{\alpha\gamma}(x-y) \big\rangle \Big)_{1\leq\alpha,\gamma\leq M}. \tag{1.3.55}$$

Granted this, all desired conclusions follow from [70, Lemma 2.5.9] and [70, (2.5.151)]. □

Moving forward, we retain the assumptions on the $M \times M$ system L from (1.3.2) and consider the case when the open set $\Omega \subseteq \mathbb{R}^n$ has an Ahlfors regular boundary. In such a scenario, [70, (2.5.549)] gives

$$\big[L^p(\partial\Omega,\sigma) \big]^M \hookrightarrow \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}} \Big) \Big]^M \quad \text{whenever } p \in [1, n-1). \tag{1.3.56}$$

In particular, the single layer potential operator (1.3.6) is meaningfully defined on the space $\big[L^p(\partial\Omega,\sigma) \big]^M$ with $p \in [1, n-1)$. Considered as such, it follows from [70, Proposition 2.5.39] (presently used with $\alpha = 1$) that for each $f \in \big[L^p(\partial\Omega,\sigma) \big]^M$ with $p \in [1, n-1)$ and each aperture parameter $\kappa > 0$ there exists a constant $C \in (0,\infty)$, which depends on p and κ but is independent of f, such that

$$\begin{gathered} \big\| \mathcal{N}_\kappa(\mathscr{S} f) \big\|_{L^{p^*}(\partial\Omega,\sigma)} \leq C \|f\|_{[L^p(\partial\Omega,\sigma)]^M} \\ \text{if } p \in (1, n-1) \text{ and } \tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{1}{n-1}, \end{gathered} \tag{1.3.57}$$

and

$$\big\| \mathcal{N}_\kappa(\mathscr{S} f) \big\|_{L^{(n-1)/(n-2),\infty}(\partial\Omega,\sigma)} \leq C \|f\|_{[L^1(\partial\Omega,\sigma)]^M} \text{ if } p = 1. \tag{1.3.58}$$

Moreover, if $\partial\Omega$ is bounded then estimates in the same spirit hold in the range $p \in [n-1,\infty)$. Specifically, [70, (2.5.556)-(2.5.557)] imply that, having fixed $\kappa > 0$, for each $p \in [n-1,\infty)$ there exist finite constants, which now also depend on $\operatorname{diam}(\partial\Omega)$, such that

$$\begin{gathered} \text{if either } n \geq 3, \text{ or } n = 2 \text{ and } \Omega \text{ is bounded, then for all } f \in \big[L^p(\partial\Omega,\sigma) \big]^M \\ \text{we have } \big\| \mathcal{N}_\kappa(\mathscr{S} f) \big\|_{L^\infty(\partial\Omega,\sigma)} \leq C_p \|f\|_{[L^p(\partial\Omega,\sigma)]^M} \text{ if } p \in (n-1,\infty) \text{ and} \\ \big\| \mathcal{N}_\kappa(\mathscr{S} f) \big\|_{L^q(\partial\Omega,\sigma)} \leq C_q \|f\|_{[L^{n-1}(\partial\Omega,\sigma)]^M} \text{ if } p = n-1 \text{ and } q \in (1,\infty). \end{gathered} \tag{1.3.59}$$

Also, corresponding to two-dimensional exterior domains,

$$
\text{if } n = 2 \text{ and } \Omega \text{ is unbounded then for all } f \in \big[L^p(\partial\Omega,\sigma)\big]^M \text{ and } R > 0
$$
$$
\text{we have } \big\|\mathcal{N}_\kappa^{\Omega\cap B(0,R)}(\mathscr{S} f)\big\|_{L^\infty(\partial\Omega,\sigma)} \le C_{R,p}\|f\|_{[L^p(\partial\Omega,\sigma)]^M} \text{ if } p \in (1,\infty) \text{ and}
$$
$$
\big\|\mathcal{N}_\kappa^{\Omega\cap B(0,R)}(\mathscr{S} f)\big\|_{L^q(\partial\Omega,\sigma)} \le C_{R,q}\|f\|_{[L^1(\partial\Omega,\sigma)]^M} \text{ if } p = 1 \text{ and } q \in (1,\infty). \tag{1.3.60}
$$

Let us also remark here that [70, (2.5.552)] implies that for each $p \in [1, n-1)$ there exists $C \in (0,\infty)$ with the property that

$$
\Big\|\delta_{\partial\Omega}^{\frac{n-1}{p}-1}\big|\mathscr{S} f\big|\Big\|_{L^\infty(\Omega,\mathcal{L}^n)} \le C\|f\|_{[L^p(\partial\Omega,\sigma)]^M} \tag{1.3.61}
$$

for all $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$.

Consider next the `boundary-to-boundary single layer operator`

$$
\begin{aligned}
Sf(x) &:= \int_{\partial\Omega} E(x-y)f(y)\,\mathrm{d}\sigma(y) = \Big(\int_{\partial\Omega} E_{\gamma\alpha}(x-y)f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M} \\
&= \Big(\int_{\partial\Omega} \big\langle E_{\gamma\bullet}(x-y),\, f(y)\big\rangle\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M} \\
&= \int_{\partial\Omega} \big[E_{L^\top}(x-y)\big]^\top f(y)\,\mathrm{d}\sigma(y) \\
&= \Big(\int_{\partial\Omega} \big\langle \big(E_{L^\top}(x-y)\big)_{\bullet\gamma},\, f(y)\big\rangle\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M} \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \\
&\quad \text{for each function } f = (f_\alpha)_{1\le\alpha\le M} \in \Big[L^1\Big(\partial\Omega,\, \frac{\sigma(x)}{1+|x|^{n-2}}\Big)\Big]^M,
\end{aligned} \tag{1.3.62}
$$

with the same conventions[9] as in (1.3.6). Then [70, (2.5.554)] ensures that this operator is indeed well defined. By design, S is linear and, thanks to [70, (2.5.549)], acts in a meaningful way on functions belonging to the space $\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p \in [1, n-1)$. From the Fractional Integration Theorem (cf. [68, (7.8.7), (7.8.14)-(7.8.15)]), and the estimates for E from [70, Theorem 1.4.2] it follows that this gives rise to a well-defined linear and continuous mapping

$$
\begin{gathered}
S : \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{p^*}(\partial\Omega,\sigma)\big]^M \\
\text{provided } p \in (1, n-1) \text{ and } \tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{1}{n-1},
\end{gathered} \tag{1.3.63}
$$

and, corresponding to $p = 1$,

$$
S : \big[L^1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{(n-1)/(n-2),\infty}(\partial\Omega,\sigma)\big]^M. \tag{1.3.64}
$$

---

[9] In particular, when $n = 2$ the weight $(1+|x|^{n-2})^{-1}$ should be replaced by $\ln(2+|x|)$. Also, we shall occasionally allow the operator S to act on functions originally defined only on $\partial_*\Omega$ by extending them by zero to the entire topological boundary $\partial\Omega$.

Note that if $S_{A^\top}$ denotes the boundary-to-boundary single layer associated with transpose coefficient tensor $A^\top$ from (1.3.3) in the same manner S has been associated with A in (1.3.62), then

$$\text{for each } p \in (1, n-1), \text{ the real transpose of (1.3.63) is the operator } S_{A^\top} : \big[L^{(p^*)'}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{p'}(\partial\Omega, \sigma)\big]^M, \tag{1.3.65}$$

where $(p^*)' \in (1, n-1)$ and $p' = \big(\frac{1}{(p^*)'} - \frac{1}{n-1}\big)^{-1}$ are the Hölder conjugate exponents of the given p and of $p^* := \big(\frac{1}{p} - \frac{1}{n-1}\big)^{-1}$, respectively. Indeed, the claim in (1.3.65) readily follows by virtue of (1.3.63), (1.3.6), [70, (1.4.32)], and Fubini's Theorem (whose applicability is ensured by the Fractional Integration Theorem; cf. [68, (7.8.12)]).

We also wish to point out that, in the particular case when $\partial\Omega$ is bounded and $p \in [n-1, \infty)$ (a scenario in which $L^p(\partial\Omega, \sigma)$ embeds into any $L^q(\partial\Omega, \sigma)$ with $q \in (1, n-1)$), the operator S from (1.3.62) maps $\big[L^p(\partial\Omega, \sigma)\big]^M$ boundedly into $\big[L^\infty(\partial\Omega, \sigma)\big]^M$ if $p > n-1$ and, corresponding to $p = n-1$, the operator S from (1.3.62) maps $\big[L^{n-1}(\partial\Omega, \sigma)\big]^M$ boundedly into $\big[L^q(\partial\Omega, \sigma)\big]^M$ for any $q \in (0, \infty)$. For future reference, it is useful to note that (1.3.63) implies

$$S : \big[L^\infty(\partial\Omega, \sigma)\big]^M \to \big[L^q(\partial\Omega, \sigma)\big]^M \text{ boundedly provided } \partial\Omega \text{ is a compact set and } q \in (1, \infty). \tag{1.3.66}$$

We also wish to remark that, given any aperture parameter $\kappa > 0$, [70, Proposition 2.5.39] implies (assuming $n \geq 3$ when $\partial\Omega$ is unbounded) that

$$\begin{gathered} \text{for each } f \in \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-2}}\Big)\Big]^M \text{ the boundary trace} \\ \mathscr{S} f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } A_\kappa(\partial\Omega) \text{ and, in fact,} \\ \Big(\mathscr{S} f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = (Sf)(x) \text{ at } \sigma\text{-a.e. point } x \in A_\kappa(\partial\Omega). \end{gathered} \tag{1.3.67}$$

In particular, from [70, (2.5.549)] it follows that the above nontangential boundary trace formula holds for each function $f \in \big[L^p(\partial\Omega, \sigma)\big]^M$ with $p \in [1, n-1)$.

Moving on, assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) is a Lebesgue measurable set, of locally finite perimeter. As in the past, abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$, and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. In such a setting, we define the `boundary-to-boundary double layer potential operator` associated with L and Ω as the principal-value singular integral operators acting on each function f as in (1.3.17) according to

$$Kf(x) := \Bigg(-\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \nu_s(y) a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y) f_\alpha(y)\, d\sigma(y)\Bigg)_{1\leq\gamma\leq M} \tag{1.3.68}$$

for σ-a.e. point $x \in \partial_*\Omega$. From [68, Proposition 5.6.7] we know that this limit exists and Kf is a σ-measurable function on $\partial_*\Omega$. Moreover, the last claim in [68, Proposition 5.6.7] implies that if Ω is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (hence, in particular, if $\partial\Omega$ is a UR set), then for each function f as in (1.3.17) the limit in (1.3.68) actually exists for σ-a.e. $x \in \partial\Omega$ and gives rise to a σ-measurable $\mathbb{C}^M$-valued function on $\partial\Omega$.

Throughout, make the convention that

whenever desirable to emphasize the dependence of the double layer operator (1.3.68) *on the coefficient tensor* (1.3.1) *we shall write K_A in place of K.* (1.3.69)

With the same agreements as in the case of $\mathcal{D}$ (cf. (1.3.22)), for each function f as in (1.3.17) we may refashion (1.3.68) as

$$\begin{aligned} Kf(x) &= \Bigg(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \Big\langle \partial^{A^\top}_{\nu(y)}\Big[\big(E_{L^\top}(x-y)\big)_{\bullet\gamma}\Big], f(y)\Big\rangle\,\mathrm{d}\sigma(y)\Bigg)_{1\leq\gamma\leq M} \\ &= \Bigg(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \Big\langle \partial^{A^\top}_{\nu(y)}\Big[\big(E_{L}(x-y)\big)_{\gamma\bullet}\Big], f(y)\Big\rangle\,\mathrm{d}\sigma(y)\Bigg)_{1\leq\gamma\leq M} \\ &= \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \Big(\partial^{A^\top}_{\nu(y)}\big[E_{L^\top}(x-y)\big]\Big)^\top f(y)\,\mathrm{d}\sigma(y) \end{aligned} \tag{1.3.70}$$

at σ-a.e. $x \in \partial_*\Omega$ if $\Omega \subseteq \mathbb{R}^n$ is a set of locally finite perimeter, or at σ-a.e. $x \in \partial\Omega$ if $\Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure.

Another singular integral operator of major interest, closely related to the one introduced in (1.3.68), is the so-called "`transpose`" `double layer operator` $K^\#$. To define this, assume Ω is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (this is the case if, in particular, $\partial\Omega$ is a UR set). Then we define the action of $K^\#$ on each function

$$f = (f_\gamma)_{1\leq\gamma\leq M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \tag{1.3.71}$$

by setting

$$K^{\#}f(x) := \left(\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_s(x) a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y) f_\gamma(y)\, d\sigma(y) \right)_{1\leq\alpha\leq M} \tag{1.3.72}$$

at σ-a.e. $x \in \partial_*\Omega$. Based on [68, Corollary 5.3.6] we see that this definition is indeed meaningful (here, [68, (5.6.23)] is also helpful). Once again, a similar convention to (1.3.69) is in place for this operator, i.e., whenever desirable to emphasize the dependence of the transpose double layer operator (1.3.72) on the coefficient tensor (1.3.1) we agree to write $K^{\#}_A$ in place of $K^{\#}$. Also, for each function f as in (1.3.71) and σ-a.e. point $x \in \partial_*\Omega$ we may express

$$\begin{aligned} K^{\#}f(x) &= \left(\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \left(\partial^{A^\top}_{\nu(x)}\left[\left(E_{L^\top}(x-y)\right)_{\bullet\gamma}\right]\right)_\alpha f_\gamma(y)\, d\sigma(y) \right)_{1\leq\alpha\leq M} \\ &= \left(\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \left(\partial^{A^\top}_{\nu(x)}\left[\left(E_L(x-y)\right)_{\gamma\bullet}\right]\right)_\alpha f_\gamma(y)\, d\sigma(y) \right)_{1\leq\alpha\leq M}, \end{aligned} \tag{1.3.73}$$

where the second equality in (1.3.73) is a consequence of the first property in [70, (1.4.32)]. Alternatively, we may recast (1.3.72) simply as

$$K^{\#}f(x) = \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \partial^{A^\top}_{\nu(x)}\left[E_{L^\top}(x-y)\right] f(y)\, d\sigma(y) \tag{1.3.74}$$

at σ-a.e. $x \in \partial_*\Omega$.

As hinted in (1.3.19) and (1.3.69), the format of the double layer potential operators $\mathcal{D}$, K, $K^{\#}$ is strongly affected by the choice of a coefficient tensor A used to represent the given system L as L_A (cf. (A.0.139)). More on this later. For now we wish to note that the difference between any two double layer potential operators associated with any two coefficient tensors A_0, A_1 that can be used to write the system L (cf. (1.3.20)) is a linear combination of "tangential" singular integral operators of the sort introduced in (1.2.2). This is made precise in the proposition below.

Proposition 1.3.8 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a set of locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Let L be a homogeneous, second-order, weakly elliptic, $M \times M$ system with constant (complex) coefficients in $\mathbb{R}^n$, and recall the matrix-valued fundamental solution $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ associated with L as in* [70, Theorem 1.4.2]. *For each $\beta, \gamma \in \{1, \dots, M\}$ and $r, s \in \{1, \dots, n\}$ consider the "tangential" singular integral operator acting on each given function $\phi \in L^1\left(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\right)$ according to (compare with (1.2.2))*

$$\big(T^{\gamma\beta}_{rs}\phi\big)(x) \tag{1.3.75}$$

$$:= \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \big\{\nu_r(y)(\partial_s E_{\gamma\beta})(x-y) - \nu_s(y)(\partial_r E_{\gamma\beta})(x-y)\big\}\phi(y)\,\mathrm{d}\sigma(y),$$

which is meaningfully defined at σ-a.e. $x \in \partial_*\Omega$ thanks to the first conclusion in [68, Proposition 5.6.7].

Then for each function

$$f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \tag{1.3.76}$$

and any two coefficient tensors $A_0, A_1 \in \mathfrak{A}_L$ (cf. (1.3.20)), the difference between $K_{A_0}f$ and $K_{A_1}f$ may be expressed as

$$\big(K_{A_0} - K_{A_1}\big)f = \tfrac{1}{2}\big(T^{\gamma\beta}_{rs} b^{\beta\alpha}_{rs} f_\alpha\big)_{1\leq\gamma\leq M} \text{ at } \sigma\text{-a.e. point in } \partial_*\Omega, \tag{1.3.77}$$

where

$$\Big(b^{\alpha\beta}_{rs}\Big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}} := A_0 - A_1. \tag{1.3.78}$$

Furthermore, if $\partial\Omega$ is countably rectifiable (of dimension $n-1$) then the equality in (1.3.77) is valid at σ-a.e. point in $\partial\Omega$.

Proof For each function f as in (1.3.76), we see from (1.3.68) and (1.3.78) that

$$\big(K_{A_0} - K_{A_1}\big)f(x) = \Bigg(-\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \nu_s(y) b^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y) f_\alpha(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\leq\gamma\leq M} \tag{1.3.79}$$

at σ-a.e. point $x \in \partial_*\Omega$. In view of the fact that $A_0, A_1 \in \mathfrak{A}_L$, we conclude from (1.3.78) that

$$b^{\alpha\beta}_{rs} = -b^{\alpha\beta}_{sr} \text{ for all } \alpha,\beta \in \{1,\dots,M\} \text{ and all } r,s \in \{1,\dots,n\}. \tag{1.3.80}$$

Consequently, for each $\alpha, \gamma \in \{1,\dots,M\}$ we may write

$$\begin{aligned}
&\nu_s(y) b^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\\
&\quad = \tfrac{1}{2}\nu_s(y) b^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y) - \tfrac{1}{2}\nu_s(y) b^{\beta\alpha}_{sr}(\partial_r E_{\gamma\beta})(x-y)\\
&\quad = \tfrac{1}{2}\nu_s(y) b^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y) - \tfrac{1}{2}\nu_r(y) b^{\beta\alpha}_{rs}(\partial_s E_{\gamma\beta})(x-y)\\
&\quad = \tfrac{1}{2}\Big(\nu_s(y)(\partial_r E_{\gamma\beta})(x-y) - \nu_r(y)(\partial_s E_{\gamma\beta})(x-y)\Big) b^{\beta\alpha}_{rs}
\end{aligned} \tag{1.3.81}$$

for each $x \in \partial\Omega$ and σ-a.e. point $x \in \partial_*\Omega$. Then (1.3.77) now follows by combining (1.3.79) with (1.3.81) and (1.3.75). The final claim in the statement is a consequence of what we have proved so far and the last part in [68, Proposition 5.6.7]. □

We conclude by recording the following variant of Proposition 1.3.8, for the difference of the transpose double layer corresponding to two choices of the coefficient tensor.

Proposition 1.3.9 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a set of locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Let L be a homogeneous, second-order, weakly elliptic, $M \times M$ system with constant (complex) coefficients in $\mathbb{R}^n$, and recall the matrix-valued fundamental solution $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ associated with L as in* [70, Theorem 1.4.2]. *For each $\beta, \gamma \in \{1, \dots, M\}$ and $r, s \in \{1, \dots, n\}$ consider the "tangential" singular integral operator acting on each given function $\psi \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ according to (compare with (1.2.3))*

$$\big(\widetilde{T}^{\gamma\beta}_{rs}\psi\big)(x) \tag{1.3.82}$$

$$:= \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \big\{\nu_r(x)(\partial_s E_{\gamma\beta})(x-y) - \nu_s(x)(\partial_r E_{\gamma\beta})(x-y)\big\}\psi(y)\,d\sigma(y),$$

which is meaningfully defined at σ-a.e. $x \in \partial_*\Omega$ thanks to the first conclusion in [68, Proposition 5.6.7].

Then for each function

$$f = (f_\gamma)_{1\leq\gamma\leq M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \tag{1.3.83}$$

and any two coefficient tensors $A_0, A_1 \in \mathfrak{A}_L$ (cf. (1.3.20)), the difference between $K^{\#}_{A_0} f$ and $K^{\#}_{A_1} f$ may be expressed as

$$\big(K^{\#}_{A_0} - K^{\#}_{A_1}\big) f = -\tfrac{1}{2}\big(\widetilde{T}^{\gamma\beta}_{rs} b^{\beta\alpha}_{rs} f_\gamma\big)_{1\leq\alpha\leq M} \text{ at } \sigma\text{-a.e. point in } \partial_*\Omega, \tag{1.3.84}$$

where

$$\Big(b^{\alpha\beta}_{rs}\Big)_{\substack{1\leq\alpha,\beta\leq M \\ 1\leq r,s\leq n}} := A_0 - A_1. \tag{1.3.85}$$

Moreover, if $\partial\Omega$ is countably rectifiable (of dimension $n-1$) then the equality in (1.3.84) is valid at σ-a.e. point in $\partial\Omega$.

Proof The same type of argument as in the proof of Proposition 1.3.8, now making use of (1.3.72), yields all desired conclusions. □

Remark 1.3.10 *If the system L given in* (1.3.2) *is actually expressed as*

$$L = A_{rs}\partial_r\partial_s \ \text{ with } \ A_{rs} := \big(a_{rs}^{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M} \in \mathbb{C}^{M\times M} \ \text{ for } 1 \leq r, s \leq n, \tag{1.3.86}$$

then we may recast the action of the boundary-to-domain double layer potential operator on each vector-valued function f as in (1.3.17) *simply as*

$$\mathcal{D}f(x) = -\int_{\partial_*\Omega} \nu_s(y)(\partial_r E)(x-y)A_{rs}f(y)\,\mathrm{d}\sigma(y), \quad \forall\, x \in \Omega. \tag{1.3.87}$$

In particular, this novel algebraic formalism makes it straightforward to see (*bearing in mind that $LE = \delta \cdot I_{M\times M}$ in the sense of distributions in $\mathbb{R}^n$; cf. part* (2) *in* [70, Theorem 1.4.2]) *that $L(\mathcal{D}f) = 0$ in Ω for each f as in* (1.3.17) (*compare with* (1.3.24)-(1.3.25)). *Likewise, in terms of the writing* (1.3.86) *we may express the action of the boundary-to-boundary double layer potential operator K* (*originally associated with L and Ω as in* (1.3.68)) *on vector-valued functions f as in* (1.3.17) *simply as*

$$Kf(x) = -\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \nu_s(y)(\partial_r E)(x-y)A_{rs}f(y)\,\mathrm{d}\sigma(y) \tag{1.3.88}$$

for σ-a.e. point $x \in \partial_*\Omega$. Finally, the action of the "transpose" double layer operator from (1.3.72) *on each vector-valued function f as in* (1.3.71) *may be recast simply as*

$$K^{\#}f(x) = \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_s(x)(A_{rs})^{\top}(\partial_r E)^{\top}(x-y)f(y)\,\mathrm{d}\sigma(y) \tag{1.3.89}$$

at σ-a.e. $x \in \partial_*\Omega$.

1.4 Examples and Alternative Points of View

We begin by looking at a number of basic concrete manifestations of the recipes discussed in §1.3 for associating boundary layer potentials with given sets of locally finite perimeter and weakly elliptic operators.

Example 1.4.1 Let us first consider the case of the two-dimensional Laplacian $L = \Delta$ in a nonempty, Lebesgue measurable, proper subset Ω of $\mathbb{R}^2$, having locally finite perimeter. Denote by ν the geometric measure theoretic outward unit normal to Ω, and abbreviate $\sigma := \mathcal{H}^1\lfloor\partial\Omega$. In this setting, if

$$A := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \widetilde{A} := \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}, \tag{1.4.1}$$

then

$$L_A = L_{\widetilde{A}} = \partial_1^2 + \partial_2^2 = \Delta. \tag{1.4.2}$$

This being said, corresponding to the first choice of a matrix in (1.4.1), the recipes (1.3.18), (1.3.68) presently yield the integral operators acting on each function

$$f \in L^1\Big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|}\Big) \tag{1.4.3}$$

according to

$$\mathcal{D}_A f(x) = \frac{1}{2\pi} \int_{\partial_*\Omega} \frac{\langle \nu(y), y - x\rangle}{|x-y|^2} f(y)\, \mathrm{d}\sigma(y) \text{ for all } x \in \mathring{\Omega}, \tag{1.4.4}$$

$$K_A f(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\partial_*\Omega \setminus \overline{B(x,\varepsilon)}} \frac{\langle \nu(y), y - x\rangle}{|x-y|^2} f(y)\, \mathrm{d}\sigma(y) \tag{1.4.5}$$

for σ-a.e. $x \in \partial_*\Omega$, which are, respectively, the (two-dimensional) harmonic boundary-to-domain double layer potential operator and the harmonic boundary-to-boundary double layer potential operator, whereas under the natural identification $\mathbb{R}^2 \equiv \mathbb{C}$ the operators $\mathcal{D}_{\widetilde{A}}$, $K_{\widetilde{A}}$ associated as in (1.3.18), (1.3.68) with Ω and the matrix $\widetilde{A}$ from (1.4.1) act on each function

$$f \in L^1\Big(\partial_*\Omega, \tfrac{\mathcal{H}^1(\zeta)}{1+|\zeta|}\Big), \tag{1.4.6}$$

according to

$$\mathcal{D}_{\widetilde{A}} f(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial_*\Omega} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \text{ for all } z \in \mathring{\Omega}, \tag{1.4.7}$$

$$K_{\widetilde{A}} f(z) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi\mathrm{i}} \int_{\partial_*\Omega \setminus \overline{B(z,\varepsilon)}} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \text{ for } \mathcal{H}^1\text{-a.e. } z \in \partial_*\Omega, \tag{1.4.8}$$

hence these are now the boundary-to-domain Cauchy integral operator and the boundary-to-boundary Cauchy integral operator, respectively. In this vein, we wish to recall that [68, Proposition 5.6.7] ensures that the limits in (1.4.5) and (1.4.8) do exist as indicated. If, however, Ω is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension 1) and has locally finite $\mathcal{H}^1$ measure (hence, in particular, if $\partial\Omega$ is a UR set), then the limits in (1.4.5) and (1.4.8) actually exist for $\mathcal{H}^1$-a.e. point in $\partial\Omega$.

Example 1.4.2 Consider next the higher-dimensional setting i.e., when $n \in \mathbb{N}$ with $n \geq 2$ is arbitrary. We are interested in the case of the Laplacian $L = \Delta$ in a nonempty, Lebesgue measurable, proper subset Ω of $\mathbb{R}^n$, having locally finite perimeter. As in the past, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Writing the Laplacian $\Delta = \partial_1^2 + \cdots + \partial_n^2$ as

$$\Delta = \sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k \quad \text{where} \quad A = \big(a_{jk}\big)_{1\leq j,k\leq n} \\ \text{has entries } a_{jk} := \delta_{jk} \text{ for each } j, k \in \{1, \dots, n\} \tag{1.4.9}$$

the recipes (1.3.18), (1.3.68) yield the standard boundary-to-domain and boundary-to-boundary harmonic double layer potential operators, acting on each function

$$f \in L^1\Big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big) \tag{1.4.10}$$

according to

$$\mathcal{D}_A f(x) = \frac{1}{\omega_{n-1}} \int_{\partial_*\Omega} \frac{\langle \nu(y), y - x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y) \quad \text{for all } x \in \mathring{\Omega}, \tag{1.4.11}$$

$$K_A f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial_*\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle \nu(y), y - x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y) \tag{1.4.12}$$

for σ-a.e. $x \in \partial_*\Omega$. On the other hand, adopting the Clifford algebra formalism recalled earlier in [68, §6.4] and expressing the Laplacian $\Delta = \partial_1^2 + \cdots + \partial_n^2$ as

$$\Delta = \sum_{j,k=1}^{n} \widetilde{a}_{jk}\partial_j\partial_k \quad \text{where} \quad \widetilde{A} = \big(\widetilde{a}_{jk}\big)_{1\leq j,k\leq n} \\ \text{has entries } \widetilde{a}_{jk} := -\mathbf{e}_j \odot \mathbf{e}_k \text{ for each } j, k \in \{1, \dots, n\} \tag{1.4.13}$$

now yields (in view of (1.3.22), (1.3.70), and the fact that $\partial_\nu^{A^\top} u = -(Du)\odot\nu$ for each Clifford algebra-valued function u) the integral operators acting on each function Clifford algebra-valued function

$$f \in L^1\Big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big) \otimes \mathcal{C}\ell_n \tag{1.4.14}$$

according to

$$\mathcal{D}_{\widetilde{A}} f(x) = \frac{1}{\omega_{n-1}} \int_{\partial_*\Omega} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\,\mathrm{d}\sigma(y) \quad \text{for all } x \in \mathring{\Omega}, \tag{1.4.15}$$

$$K_{\widetilde{A}} f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial_*\Omega\setminus\overline{B(x,\varepsilon)}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\,\mathrm{d}\sigma(y) \tag{1.4.16}$$

for σ-a.e. $x \in \partial_*\Omega$, i.e., the boundary-to-domain and boundary-to-boundary Cauchy-Clifford integral operators.

Once again, [68, Proposition 5.6.7] guarantees that the limits in (1.4.12) and (1.4.16) do exist as specified. This being said, if Ω is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (hence, in particular, if $\partial\Omega$ is a UR set), then the limits in (1.4.12) and (1.4.16) actually exist for σ-a.e. point in $\partial\Omega$.

Example 1.4.3 In the last part of Example 1.4.2, we could circumvent the direct use of the Clifford algebras by employing instead matrix formalism. To elaborate on this, fix $n \in \mathbb{N}$ with $n \geq 2$, and consider a family of real matrices $\{E_j\}_{1\leq j\leq n}$ satisfying

$$\begin{aligned} \big(E_j\big)^2 &= -I_{2^n\times 2^n} \ \text{ for each } \ j \in \{1,\dots,n\} \ \text{ and} \\ E_j E_k &= -E_k E_j \ \text{ for all } \ j,k \in \{1,\dots,n\} \ \text{ with } \ j \neq k. \end{aligned} \tag{1.4.17}$$

The existence of such a family has been established in [68, (6.4.14)] (with $m := n$). To proceed, define $M := 2^n$ and denote by $I_{M\times M}$ the $M \times M$ identity matrix. Consider the $M \times M$ second-order system in $\mathbb{R}^n$ defined as

$$L := \Delta \cdot I_{M\times M} \tag{1.4.18}$$

where $\Delta = \partial_1^2 + \cdots + \partial_n^2$ is the Laplacian in $\mathbb{R}^n$. For each $j,k \in \{1,\dots,n\}$ denote by $(a_{jk}^{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ the entries of the $M \times M$ matrix $-E_jE_k$, i.e.,

$$\begin{aligned} a_{jk}^{\alpha\beta} = -(E_jE_k)_{\alpha\beta} \ \text{ for each } \ j,k \in \{1,\dots,n\} \\ \text{and each } \ \alpha,\beta \in \{1,\dots,M\}. \end{aligned} \tag{1.4.19}$$

Then, with the summation convention over repeated indices in effect, (1.4.17) implies

$$\big(a_{jk}^{\alpha\beta}\partial_j\partial_k\big)_{1\leq\alpha,\beta\leq M} = -E_jE_k\partial_j\partial_k = -(E_j)^2\partial_j^2 = \Delta \cdot I_{M\times M}, \tag{1.4.20}$$

hence,

$$L = \big(a_{jk}^{\alpha\beta}\partial_j\partial_k\big)_{1\leq\alpha,\beta\leq M}. \tag{1.4.21}$$

Also, the fundamental solution E_L associated with the weakly elliptic system L as in [70, Theorem 1.4.2] is given by

$$E_L := E_\Delta \cdot I_{M\times M} \tag{1.4.22}$$

where E_Δ is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$, defined in (A.0.65). In particular, for each $x \in \mathbb{R}^n \setminus \{0\}$ we have

$$\partial_j(E_L)_{\gamma\beta}(x) = \frac{\delta_{\gamma\beta}}{\omega_{n-1}}\frac{x_j}{|x|^n} \ \text{ for } \ 1 \leq j \leq n \ \text{ and } \ 1 \leq \gamma,\beta \leq M. \tag{1.4.23}$$

Pressing on, suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ has $n \geq 2$) is a set of locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Finally, pick an arbitrary function

$$f \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M. \tag{1.4.24}$$

Then from (1.3.18), (1.4.19), and (1.4.23) we see that the boundary-to-domain double layer potential operator associated as in (1.3.18) with the set Ω and the system L written as in (1.4.21) acts on f according to

$$\mathcal{D}f(x) = \frac{1}{\omega_{n-1}} \int_{\partial_*\Omega} \sum_{j=1}^n E_j \frac{x_j - y_j}{|x-y|^n} \Big(\sum_{k=1}^n \nu_k(y) E_k\Big) f(y)\, d\sigma(y) \tag{1.4.25}$$

for every $x \in \mathring{\Omega}$. In terms of the boundary-to-domain Riesz transforms, $\mathcal{R}_j$ with $1 \leq j \leq n$, defined as in (A.0.188) (with $\Sigma := \partial\Omega$), formula (1.4.25) becomes

$$\mathcal{D}f = \frac{1}{2}\sum_{j=1}^n E_j \mathcal{R}_j \Big[\Big(\sum_{k=1}^n \nu_k E_k\Big) f\Big] \text{ in } \Omega. \tag{1.4.26}$$

Likewise, the boundary-to-boundary double layer potential operator associated as in (1.3.68) with the set Ω and the system L written as in (1.4.21) acts on any function f as in (1.4.24) according to

$$Kf = \frac{1}{2}\sum_{j=1}^n E_j R_j \Big[\Big(\sum_{k=1}^n \nu_k E_k\Big) f\Big] \tag{1.4.27}$$

where R_j for $1 \leq j \leq n$ are the boundary-to-boundary Riesz transforms defined as in (A.0.187) with $\Sigma := \partial_*\Omega$.

Example 1.4.4 Let us take yet another point of view on the manner in which boundary layer potentials may be associated with a given second-order weakly elliptic system. Suppose L is a given second-order, homogeneous, constant (complex) coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and denote by $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2]. Next, assume that

$$\widetilde{B} = \Big(\widetilde{b}_r^{\alpha\gamma}\Big)_{\substack{1\leq r\leq n\\ 1\leq\alpha\leq M\\ 1\leq\gamma\leq N}} \text{ and } B = \Big(b_s^{\gamma\beta}\Big)_{\substack{1\leq s\leq n\\ 1\leq\gamma\leq N\\ 1\leq\beta\leq M}} \tag{1.4.28}$$

are two coefficient tensors (of type $(1 \times n, M \times N)$ and $(n \times 1, N \times M)$, respectively), with complex entries, such that the associated first-order operators

$$\widetilde{D} := \Big(\widetilde{b}_r^{\alpha\gamma}\partial_r\Big)_{\substack{1\le\alpha\le M\\1\le\gamma\le N}} \text{ and } D := \Big(b_s^{\gamma\beta}\partial_s\Big)_{\substack{1\le\gamma\le N\\1\le\beta\le M}} \tag{1.4.29}$$

may be used to factor L as follows:

$$L = \widetilde{D}D. \tag{1.4.30}$$

In particular, this permits us to express

$$L = \Big(\widetilde{b}_r^{\alpha\gamma}b_s^{\gamma\beta}\partial_r\partial_s\Big)_{\substack{1\le\alpha\le M\\1\le\beta\le M}} \tag{1.4.31}$$

hence,

$$\begin{aligned} L = L_A \text{ where } A := A_{\widetilde{D},D} := \Big(a_{rs}^{\alpha\beta}\Big)_{\substack{1\le r,s\le n\\1\le\alpha,\beta\le M}} \\ \text{with each } a_{rs}^{\alpha\beta} := \widetilde{b}_r^{\alpha\gamma}b_s^{\gamma\beta}. \end{aligned} \tag{1.4.32}$$

Let us now consider a Lebesgue measurable nonempty proper set $\Omega\subset\mathbb{R}^n$ of locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω. Recall from [70, Convention 1.7.2] that whenever the function $u = (u_\beta)_{1\le\beta\le M}$ is of class $\mathscr{C}^1$ in a neighborhood of $\partial_*\Omega$, its conormal derivative associated with Ω and the coefficient tensor A from (1.4.32) is defined as

$$\begin{aligned} \partial_\nu^A u &= \Big(\nu_r a_{rs}^{\alpha\beta}(\partial_s u_\beta)\big|_{\partial_*\Omega}\Big)_{1\le\alpha\le M} = (-\mathrm{i})\Big(i\nu_r\widetilde{b}_r^{\alpha\gamma}b_s^{\gamma\beta}(\partial_s u_\beta)\big|_{\partial_*\Omega}\Big)_{1\le\alpha\le M} \\ &= (-\mathrm{i})\mathrm{Sym}(\widetilde{D};\nu)(Du)\big|_{\partial_*\Omega} \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega \end{aligned} \tag{1.4.33}$$

(see also [70, (1.7.42)] in this regard). Likewise, since

$$D^\top = -\Big(b_s^{\beta\gamma}\partial_s\Big)_{\substack{1\le\gamma\le N\\1\le\beta\le M}} \text{ and } \widetilde{D}^\top = -\Big(\widetilde{b}_r^{\gamma\alpha}\partial_r\Big)_{\substack{1\le\alpha\le M\\1\le\gamma\le N}} \tag{1.4.34}$$

it follows that

$$\begin{aligned} \partial_\nu^{A^\top} u &= \Big(\nu_r a_{sr}^{\beta\alpha}(\partial_s u_\beta)\big|_{\partial_*\Omega}\Big)_{1\le\alpha\le M} = (-\mathrm{i})\Big(i\nu_r\widetilde{b}_s^{\beta\gamma}b_r^{\gamma\alpha}(\partial_s u_\beta)\big|_{\partial_*\Omega}\Big)_{1\le\alpha\le M} \\ &= (-\mathrm{i})\mathrm{Sym}(D^\top;\nu)(\widetilde{D}^\top u)\big|_{\partial_*\Omega} \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \end{aligned} \tag{1.4.35}$$

In light of (1.3.22), (1.4.35), and [68, (1.7.17)] (as well as simple matrix formalism) we may express the action of the boundary-to-domain double layer operator (1.3.18) on any function f as in (1.3.17) at each point $x\in\mathring{\Omega}$ as

$$
\begin{aligned}
\mathcal{D}f(x) &= -\mathrm{i}\Big(\int_{\partial_*\Omega}\Big\langle \mathrm{Sym}\big(D^\top;\nu(y)\big)\widetilde{D}_y^\top\Big[\big(E_{L^\top}(x-y)\big)_{\bullet\gamma}\Big], f(y)\Big\rangle\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M}\\
&= \mathrm{i}\Big(\int_{\partial_*\Omega}\Big\langle \widetilde{D}_y^\top\Big[\big(E_{L^\top}(x-y)\big)_{\bullet\gamma}\Big], \mathrm{Sym}\big(D;\nu(y)\big)f(y)\Big\rangle\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M}\\
&= (-\mathrm{i})\int_{\partial_*\Omega}\big(\widetilde{D}^\top E_{L^\top}\big)^\top(x-y)\,\mathrm{Sym}\big(D;\nu(y)\big)f(y)\,\mathrm{d}\sigma(y),
\end{aligned}
\tag{1.4.36}
$$

where $E_{L^\top}$ is the fundamental solution associated with the transpose system $L^\top$ as in [70, Theorem 1.4.2], $\big(E_{L^\top}\big)_{\bullet\gamma}$ denotes the γ-th column of the matrix $E_{L^\top}$, and subscript y for $\widetilde{D}^\top$ indicates that said operator acts in the variable y. Also, in the last line of (1.4.36) $\widetilde{D}^\top E_{L^\top}$ is interpreted naturally, according to the ordinary multiplication of matrices. Bearing this in mind it is actually possible to eliminate the transposition(s) altogether in the last line of (1.4.36) and simply write, for each function f as in (1.3.17) and each point $x \in \mathring{\Omega}$,

$$
\mathcal{D}f(x) = (-\mathrm{i})\int_{\partial_*\Omega}\big(E_L\widetilde{D}\big)(x-y)\,\mathrm{Sym}\,\big(D;\nu(y)\big)f(y)\,\mathrm{d}\sigma(y),
\tag{1.4.37}
$$

with the convention that $E_L\widetilde{D}$ means ordinary multiplication of matrices (treating the product between a function $E_{\gamma\beta}$ and a partial derivative ∂_r as being $\partial_r E_{\gamma\beta}$).

Another, more direct, way of proving formula (1.4.36) is to start with (1.3.18) and then use (1.4.32), [70, (1.4.32)], (A.0.192), (1.4.34), to write

$$
\begin{aligned}
\mathcal{D}f(x) &= \Big(-\int_{\partial_*\Omega}\nu_s(y)a_{rs}^{\beta\alpha}\big(\partial_r(E_L)_{\gamma\beta}\big)(x-y)f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M}\\
&= \Big(-\int_{\partial_*\Omega}\nu_s(y)\widetilde{b}_r^{\beta\delta}\,b_s^{\delta\alpha}\big(\partial_r(E_{L^\top})_{\beta\gamma}\big)(x-y)f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M}\\
&= \Big(\int_{\partial_*\Omega}\big(-\widetilde{b}_r^{\beta\delta}\,\partial_r(E_{L^\top})_{\beta\gamma}\big)(x-y)\big(\nu_s(y)b_s^{\delta\alpha}f_\alpha(y)\big)\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M}\\
&= (-\mathrm{i})\Big(\int_{\partial_*\Omega}\big(\widetilde{D}^\top E_{L^\top}\big)_{\delta\gamma}(x-y)\Big(\mathrm{Sym}\big(D;\nu(y)\big)f(y)\Big)_\delta\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M}\\
&= (-\mathrm{i})\int_{\partial_*\Omega}\big(\widetilde{D}^\top E_{L^\top}\big)^\top(x-y)\,\mathrm{Sym}\big(D;\nu(y)\big)f(y)\,\mathrm{d}\sigma(y),
\end{aligned}
\tag{1.4.38}
$$

for each function f as in (1.3.17) and each point $x \in \mathring{\Omega}$.

In a similar fashion, we see from (1.3.70) and (1.4.35) that the action of the principal-value double layer potential operator (1.3.68) on any function f as in (1.3.17) may be recast as

$$
\begin{aligned}
&Kf(x)\\
&= -\mathrm{i}\Big(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \Big\langle \mathrm{Sym}\big(D^\top;\nu(y)\big)\widetilde{D}_y^\top\Big[\big(E_{L^\top}(x-y)\big)_{\bullet\gamma}\Big], f(y)\Big\rangle\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M}\\
&= \mathrm{i}\Big(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \Big\langle \widetilde{D}_y^\top\Big[\big(E_{L^\top}(x-y)\big)_{\bullet\gamma}\Big], \mathrm{Sym}\big(D;\nu(y)\big)f(y)\Big\rangle\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M}\\
&= (-\mathrm{i})\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \big(\widetilde{D}^\top E_{L^\top}\big)^\top(x-y)\,\mathrm{Sym}\big(D;\nu(y)\big)f(y)\,\mathrm{d}\sigma(y)\\
&= (-\mathrm{i})\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \big(E_L\widetilde{D}\big)(x-y)\,\mathrm{Sym}\,\big(D;\nu(y)\big)f(y)\,\mathrm{d}\sigma(y)
\end{aligned}
\tag{1.4.39}
$$

at σ-a.e. $x\in\partial_*\Omega$. From [68, Proposition 5.6.7] we know that the limit in (1.4.39) exists as indicated, and if Ω is actually a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (hence, in particular, if $\partial\Omega$ is a UR set) then the limit in (1.4.39) exists for σ-a.e. point in $\partial\Omega$.

Let us adopt the latter geometric setting, i.e., assume now that Ω is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (this is the case if, in particular, $\partial\Omega$ is a UR set). In such a setting, in view of (1.3.73) we may refashion the singular integral operator $K^{\#}$, originally defined in (1.3.72), as

$$
\begin{aligned}
&K^{\#}f(x)\\
&= -\mathrm{i}\Big(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \Big(\mathrm{Sym}\big(D^\top;\nu(x)\big)\widetilde{D}_x^\top\big[\big(E_{L^\top}(x-y)\big)_{\bullet\gamma}\big]\Big)_\alpha f_\gamma(y)\,\mathrm{d}\sigma(y)\Big)_{1\le\alpha\le M}\\
&= (-\mathrm{i})\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \mathrm{Sym}\big(D^\top;\nu(x)\big)\big(\widetilde{D}^\top E_{L^\top}\big)(x-y)f(y)\,\mathrm{d}\sigma(y)\\
&= (-\mathrm{i})\mathrm{Sym}\big(D^\top;\nu(x)\big)\Big(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big(\widetilde{D}^\top E_{L^\top}\big)(x-y)f(y)\,\mathrm{d}\sigma(y)\Big),
\end{aligned}
\tag{1.4.40}
$$

for each function f as in (1.3.71), at σ-a.e. point $x \in \partial_*\Omega$ (which is meaningful, according to [68, Proposition 5.6.7]).

Moving on, it is of interest to record a version of the integral representation formula from Theorem 1.3.3 emphasizing the role of factoring a given second-order system as a product of two first-order systems.

Corollary 1.4.5 *Let $\Omega \subseteq \mathbb{R}^n$, where*[10] *$n \geq 3$, be an open set with a lower Ahlfors regular boundary, and with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Let L be a homogeneous, weakly elliptic, second-order $M\times M$ system in $\mathbb{R}^n$, with complex constant coefficients. Suppose this may be factored as*

$$L = \widetilde{D}D \tag{1.4.41}$$

where $\widetilde{D}$ and D are two homogeneous, constant (complex) coefficient, first-order systems in $\mathbb{R}^n$.

Next, assume $u \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^M$ is a vector-valued function satisfying, for some aperture parameter $\kappa > 0$ and with all derivatives taken in the sense of distributions,

$$\begin{gathered} Lu \text{ belongs to } \Big[L^1\Big(\Omega, \frac{dy}{1+|y|^{n-2}}\Big)\Big]^M, \\ Du \text{ has locally integrable components in } \Omega, \\ u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ and } (Du)\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ exist } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega. \end{gathered} \tag{1.4.42}$$

In addition, assume the following integrability conditions hold:

$$\int_{\partial\Omega} \frac{(\mathcal{N}_\kappa u)(y)}{1+|y|^{n-1}}\, d\sigma(y) < \infty \text{ and } \int_{\partial\Omega} \frac{\big(\mathcal{N}_\kappa(Du)\big)(y)}{1+|y|^{n-2}}\, d\sigma(y) < \infty. \tag{1.4.43}$$

Finally, recall the Newtonian potential operator Π_Ω from (1.3.4), the single layer potential operator $\mathcal{S}$ from (1.3.6), the boundary-to-domain double layer potential operator $\mathcal{D}_{A_{\widetilde{D},D}}$ associated as in (1.3.18) with the set Ω and the coefficient tensor $A_{\widetilde{D},D}$ (defined as in (1.4.32) if $\widetilde{D}$, D are as in (1.4.29)), and the conormal derivative operator $\partial_\nu^{\widetilde{D},D}$ associate with the factorization (1.4.41) as in (A.0.185).

Then at $\mathcal{L}^n$-a.e. point $x \in \Omega$ one has

$$u(x) = \mathcal{D}_{A_{\widetilde{D},D}}\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x) - \mathcal{S}\big(\partial_\nu^{\widetilde{D},D}u\big)(x) + \Pi_\Omega(Lu)(x), \tag{1.4.44}$$

if either Ω is bounded, or $\partial\Omega$ is unbounded. In the case when Ω is an exterior domain, the same conclusion holds under the additional assumption that there exists $\lambda \in (1,\infty)$ such that

[10] see the last part in the statement for the case $n = 2$

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |u| \,\mathrm{d}\mathcal{L}^n = o(1) \ \text{ as } \ R \to \infty. \tag{1.4.45}$$

As a corollary, if in place of the first line in (1.4.42) *one now imposes the stronger condition that* $Lu = 0$ *in* Ω*, then* $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ *and at each point* $x \in \Omega$ *one has* (*with the same caveat when* Ω *is an exterior domain*)

$$\begin{aligned} u(x) &= \mathcal{D}_{A_{\widetilde{D},D}}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) - \mathcal{S}\big(\partial_\nu^{\widetilde{D},D} u\big)(x) \\ &= \int_{\partial_*\Omega} \big(\widetilde{D}^\top E_{L^\top}\big)^\top (x-y)\,(-\mathrm{i})\mathrm{Sym}\big(D;\nu(y)\big)\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)\,\mathrm{d}\sigma(y) \\ &\quad - \int_{\partial_*\Omega} E_L(x-y)(-\mathrm{i})\mathrm{Sym}\big(\widetilde{D};\nu(y)\big)\Big((Du)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(y)\,\mathrm{d}\sigma(y), \end{aligned} \tag{1.4.46}$$

where E_L *and* $E_{L^\top}$ *are, respectively, the fundamental solutions associated with* L *and* $L^\top$ *as in* [70, Theorem 1.4.2] (*and where* $\widetilde{D}^\top$ *acts on the columns of* $E_{L^\top}$).

Lastly, similar results are valid in the case when $n = 2$ *provided either*

$$\int_{S^1} \big[L(\xi)\big]^{-1} \,\mathrm{d}\mathcal{H}^1(\xi) = 0 \in \mathbb{C}^{M\times M}, \tag{1.4.47}$$

or otherwise one assumes that $\partial\Omega$ *is compact, replaces the first membership in* (1.4.42) *by*

$$\int_\Omega |(Lu)(y)| \ln(2+|y|)\,\mathrm{d}y < +\infty, \tag{1.4.48}$$

and, in the case when Ω *is an exterior domain, replaces* (1.4.45) *by*

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |u| \,\mathrm{d}\mathcal{L}^2 = o\big(\tfrac{1}{\ln R}\big) \ \text{ as } \ R \to \infty. \tag{1.4.49}$$

Proof All claims in the statement are clear from Theorem 1.3.3, definitions, (1.4.38), and elliptic regularity (cf. [68, (6.5.40) in Theorem 6.5.7]). □

Remark 1.4.6 *Consider the boundary layer representation formula* (1.4.46) (*in the context of Corollary 1.4.5*), *in the case when the weakly elliptic* $M \times M$ *system is factored as* $L = \widetilde{D}D$*, with* D*,* $\widetilde{D}$ *as in* (1.4.29) (*hence, the coefficient tensor* $A := A_{\widetilde{D},D}$ *used to represent* L *as* L_A *is as in* (1.4.32)), *and under the assumption that*

$$Du = 0 \ \text{ in } \ \Omega. \tag{1.4.50}$$

In such a scenario, as seen from (A.0.185), *the conormal derivative* $\partial_\nu^{A_{\widetilde{D},D}} u = \partial_\nu^{\widetilde{D},D} u$ *vanishes at* σ*-a.e. point on* $\partial_*\Omega$*. Also,* $Lu = \widetilde{D}(Du) = 0$ *in* Ω*. As such, the boundary layer representation formula* (1.4.46) *ultimately reduces to*

$$u(x) = \mathcal{D}_{A_{\widetilde{D},D}}\big(u\big|^{\rm n.t.}_{\partial\Omega}\big)(x) \tag{1.4.51}$$
$$= \int_{\partial_*\Omega} \big(\widetilde{D}^\top E_{L^\top}\big)^\top(x-y)\,(-{\rm i}){\rm Sym}\big(D;\nu(y)\big)\big(u\big|^{\rm n.t.}_{\partial\Omega}\big)(y)\,d\sigma(y)$$

for each point $x \in \Omega$, where $E_{L^\top}$ is the fundamental solution associated with the transpose system $L^\top$ as in [70, Theorem 1.4.2].

The most basic manifestation of this phenomenon is Cauchy's Reproducing Formula in complex analysis, allowing one to recover a holomorphic function from its boundary trace, via the boundary-to-domain Cauchy integral operator. See (1.4.71)-(1.4.73) *in Example 2 presented a little later below. In a nutshell, the conormal derivative associated with the factorization $\Delta = \overline{\partial}\partial$ of the Laplacian in $\mathbb{R}^2 \equiv \mathbb{C}$ of any function u which happens to be holomorphic in the open set $\Omega \subseteq \mathbb{R}^2$ always vanishes* (*since* $(-{\rm i}){\rm Sym}(\overline{\partial};\nu)(\partial u)\big|^{\rm n.t.}_{\partial\Omega} = 0$ *given that $\partial u = 0$ in Ω*) *and, as a result, the corresponding Green's Representation Formula for u regarded as a harmonic function in Ω becomes precisely Cauchy's Reproducing Formula for u viewed as a holomorphic function in Ω.*

A more general formulation of the result described in Remark 1.4.6 reads as follows:

Theorem 1.4.7 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, be an open set with a lower Ahlfors regular boundary, and such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$.*

Next, consider a homogeneous, first-order $N \times M$ system D with constant complex coefficients in $\mathbb{R}^n$ (*where $N, M \in \mathbb{N}$*) *which is injectively elliptic* (*cf.* [70, (1.3.18)]), *and suppose $\widetilde{D}$ is a homogeneous first-order $M \times N$ system with constant complex coefficients in $\mathbb{R}^n$ which complements D* (*i.e.,* [70, (1.3.21)] *holds*)*. In particular, $L := \widetilde{D}D$ is a weakly elliptic second-order $M \times M$ system in $\mathbb{R}^n$. Let $A_{\widetilde{D},D}$ be the coefficient tensor induced by the factorization $\widetilde{D}D$ of the system L, defined as in* (1.4.32)*. Also, let $\mathcal{D}$ be the boundary-to-domain double layer potential operator associated as in* (1.3.18) *with the set Ω and the coefficient tensor $A := A_{\widetilde{D},D}$.*

Finally, fix an aperture parameter $\kappa \in (0,\infty)$ and suppose $u : \Omega \to \mathbb{C}^M$ is a vector-valued function with Lebesgue measurable components satisfying[11]

$$\begin{gathered} \mathcal{N}_\kappa u \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big), \\ \Big(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)(x) \ \textit{exists for } \sigma\textit{-a.e. } x \in \partial_{\rm nta}\Omega, \\ \textit{and } Du \in \Big[L^1\Big(\Omega, \frac{dy}{1+|y|^{n-1}}\Big)\Big]^N. \end{gathered} \tag{1.4.52}$$

In the case when Ω is an exterior domain make the additional assumption that there exists $\lambda \in (1,\infty)$ such that

[11] with Du considered in the sense of distributions in Ω

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |u|\,\mathrm{d}\mathcal{L}^n = o(1) \ \ \textit{as} \ \ R \to \infty. \tag{1.4.53}$$

Then for any $\kappa' > 0$ *the nontangential trace* $u\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ *also exists at* σ*-a.e. point on* $\partial_{\mathrm{nta}}\Omega$ *and is actually independent of* κ'*. Moreover, with the dependence on the aperture parameter dropped, for* $\mathcal{L}^n$*-a.e.* $x \in \mathbb{R}^n \setminus \partial\Omega$ *one has*

$$\mathcal{D}\big(u\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(x) - \int_\Omega \big(\widetilde{D}^\top E_{L^\top}\big)^\top(x-y)(Du)(y)\,\mathrm{d}y = \begin{cases} u(x) \ \textit{if} \ x \in \Omega, \\ 0 \quad\ \ \textit{if} \ x \in \mathbb{R}^n \setminus \overline{\Omega}, \end{cases} \tag{1.4.54}$$

where $E_{L^\top}$ *is the matrix-valued fundamental solution associated with* $L^\top$ *as in* [70, Theorem 1.4.2]*, and where* $\widetilde{D}^\top$ *acting on the columns of* $E_{L^\top}$*.*

As a corollary, if the last property in (1.4.52) *is strengthened to*

$$Du = 0 \ \ \textit{in} \ \ \Omega, \tag{1.4.55}$$

then $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ *and* (1.4.54) *reduces to*

$$\mathcal{D}\big(u\big|_{\partial\Omega}^{\mathrm{n.t.}}\big) = \begin{cases} u \ \ \textit{in} \ \ \Omega, \\ 0 \ \ \textit{in} \ \ \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \tag{1.4.56}$$

For example, starting with a given second-order, weakly elliptic $M \times M$ system $L = \big(a_{rs}^{\alpha\beta}\partial_r\partial_s\big)_{1\le\alpha,\beta\le M}$ in $\mathbb{R}^n$, with constant (complex) coefficients, the factorization $L = \widetilde{D}D$ with $D, \widetilde{D}$ as in (1.4.67) makes formula (1.4.54) take the form

$$\begin{aligned} \mathcal{D}\big(u\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(x) + \Big(\int_\Omega a_{rs}^{\beta\gamma}(\partial_r E_{\alpha\beta})(x-y)(\partial_s u_\alpha)(y)\,\mathrm{d}y\Big)_{1\le\gamma\le M} \\ = \begin{cases} u(x) \ \ \text{if} \ \ x \in \Omega, \\ 0 \qquad \text{if} \ \ x \in \mathbb{R}^n \setminus \overline{\Omega}, \end{cases} \end{aligned} \tag{1.4.57}$$

where $E = (E_{\alpha\beta})_{1\le\alpha,,\beta\le M}$ is the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2], and $(u_\alpha)_{1\le\alpha\le M}$ are the scalar components of the vector-valued function u.

Let also emphasize the remarkable corollary that, in the context of Theorem 1.4.7, any function u satisfying

$$\begin{aligned} & u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Du = 0 \ \text{in} \ \Omega, \quad \mathcal{N}_\kappa u \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big), \\ & \text{the nontangential trace } u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega, \\ & \qquad \text{and if } \Omega \text{ is an exterior domain then (1.4.53) also holds,} \end{aligned} \tag{1.4.58}$$

is reproduced by $\mathcal{D}$ in the sense that (cf. (1.4.56))

$$u = \mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) \text{ in } \Omega. \tag{1.4.59}$$

Simply put, *any double layer reproduces something (specifically, the null-solutions of the second factor in the factorization of the original system that has produced said double layer in the first place).*

The proof of Theorem 1.4.7 is presented next.

Proof of Theorem 1.4.7 The first two properties in (1.4.52) together with [68, Corollary 8.9.9] imply that for any $\kappa' > 0$ the nontangential trace $u\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}$ exists at σ-a.e. point on $\partial_{\text{nta}}\Omega$ and is actually independent of the parameter κ'. Also, (1.4.54) is obtained by the equation in [70, (1.6.3)] and (1.4.36) (while bearing in mind [70, (1.6.1)]). Finally, if (1.4.55) holds then elliptic regularity implies that $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$, while (1.4.54) becomes (1.4.56).

Another way of justifying (1.4.56) is to note that if the equation (1.4.55) holds then $Lu = \widetilde{D}Du = 0$ in Ω, so the formula in question is a particular case of Theorem 1.3.3 (or Corollary 1.4.5) when either $n \geq 3$, or $n = 2$ and Ω is bounded, and of Theorem 1.3.4, when $n = 2$ and Ω is an exterior domain. □

For a given second-order system L, formulas (1.4.36)-(1.4.40) allow us to associate double layer potential operators with any factorization of L as in (1.4.30). This procedure also prefigures how one should associate double layer potential operators for a given second-order elliptic system in the manifold setting which is the composition, as in (1.4.30), of two first-order operators acting between vector bundles over said differential manifold; see [74], [75], and [76] in this regard. Indeed, a key advantage of working with a second-order system described through a (global) factorization as in (1.4.30) in place of a (local) description via a coefficient tensor A as in (1.4.32) is that the former (coordinate-free) formalism carries over to the setting of manifolds.

Of course, whenever the second-order system L is expressed as

$$L = \sum_{j=1}^{N} \widetilde{D}_j D_j \tag{1.4.60}$$

for some families of first-order differential operators $\widetilde{D}_j, D_j$ with $1 \leq j \leq N$, we may refashion (1.4.60) as the factorization

$$L = \widetilde{D}D \text{ where } \widetilde{D} := (\widetilde{D}_1, \dots, \widetilde{D}_N) \text{ and } D := \begin{pmatrix} D_1 \\ \vdots \\ D_N \end{pmatrix}. \tag{1.4.61}$$

Implementing (1.4.36)-(1.4.37) for the factorization (1.4.61) yields (retaining the earlier setting, and with the same algebraic conventions as before)

$$\mathcal{D}f(x) = \sum_{j=1}^{N} \int_{\partial_*\Omega} \big(\widetilde{D}_j^{\top} E_{L^{\top}}\big)^{\top}(x-y)(-\mathrm{i})\,\mathrm{Sym}\,\big(D_j;\nu(y)\big)f(y)\,\mathrm{d}\sigma(y)$$

$$= \sum_{j=1}^{N} \int_{\partial_*\Omega} \big(E_L \widetilde{D}_j\big)(x-y)(-\mathrm{i})\,\mathrm{Sym}\,\big(D_j;\nu(y)\big)f(y)\,\mathrm{d}\sigma(y) \tag{1.4.62}$$

for all $x \in \mathring{\Omega}$, while implementing (1.4.39) and (1.4.40) for the factorization (1.4.61) gives, at σ-a.e. point $x \in \partial_*\Omega$,

$$Kf(x) = \sum_{j=1}^{N} \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \big(\widetilde{D}_j^{\top} E_{L^{\top}}\big)^{\top}(x-y)(-\mathrm{i})\,\mathrm{Sym}\,\big(D_j;\nu(y)\big)f(y)\,\mathrm{d}\sigma(y)$$

$$= \sum_{j=1}^{N} \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \big(E_L \widetilde{D}_j\big)(x-y)(-\mathrm{i})\,\mathrm{Sym}\,\big(D_j;\nu(y)\big)f(y)\,\mathrm{d}\sigma(y) \tag{1.4.63}$$

and, respectively,

$$K^{\#}f(x) = \sum_{j=1}^{N} (-\mathrm{i})\mathrm{Sym}\,\big(D_j^{\top};\nu(x)\big)\Bigg(\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big(\widetilde{D}_j^{\top} E_{L^{\top}}\big)(x-y)f(y)\,\mathrm{d}\sigma(y)\Bigg). \tag{1.4.64}$$

It is instructive to provide concrete examples to the approach for defining boundary layer potentials in relation of a given second-order system proposed above.

Example 1.4.8 Suppose a second-order, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, with constant (complex) coefficients,

$$L = \big(a_{rs}^{\alpha\beta}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}, \tag{1.4.65}$$

has been given. Hence, with div understood as the differential operator taking any $w = (w_{s\beta})_{\substack{1\leq s\leq n\\ 1\leq\beta\leq M}}$ into $\mathrm{div}w := (\partial_s w_{s\beta})_{1\leq\beta\leq M}$, we have

$$L = L_A = \mathrm{div}A\nabla \;\text{ where }\; A = \Big(a_{rs}^{\alpha\beta}\Big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}. \tag{1.4.66}$$

Choosing

$$\begin{gathered} D := \nabla \text{ and } \widetilde{D} := \mathrm{div}A, \text{ i.e.,} \\ Du := \big(\partial_s u_\beta\big)_{\substack{1\leq s\leq n\\ 1\leq\beta\leq M}} \text{ if } u = (u_\beta)_{1\leq\beta\leq M} \text{ and} \\ \widetilde{D}w := \big(\partial_r a_{rs}^{\alpha\beta} w_{s\beta}\big)_{1\leq\alpha\leq M} \text{ if } w = (w_{s\beta})_{\substack{1\leq s\leq n\\ 1\leq\beta\leq M}} \end{gathered} \tag{1.4.67}$$

yields the factorization

$$L = \widetilde{D}D. \tag{1.4.68}$$

Also,

$$\begin{aligned} D^\top w &= \big(-\partial_s w_{s\beta}\big)_{1\le\beta\le M} \ \text{ if } w = (w_{s\beta})_{\substack{1\le s\le n\\ 1\le\beta\le M}}, \\ \widetilde{D}^\top u &= \big(-\partial_r a^{\alpha\beta}_{rs} u_\alpha\big)_{\substack{1\le s\le n\\ 1\le\beta\le M}} \ \text{ if } u = (u_\alpha)_{1\le\alpha\le M}, \\ \mathrm{Sym}(D^\top;\xi)w &= \big(-\mathrm{i}\xi_s w_{s\beta}\big)_{1\le\beta\le M} \\ &\text{if } w = (w_{s\beta})_{\substack{1\le s\le n\\ 1\le\beta\le M}} \text{ and } \xi = (\xi_s)_{1\le s\le n}. \end{aligned} \tag{1.4.69}$$

Next, fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^n$, having locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. In this context, whenever the function $u = (u_\alpha)_{1\le\alpha\le M}$ is of class $\mathscr{C}^1$ in a neighborhood of $\partial_*\Omega$, [70, Convention 1.7.2] tells us that at σ-a.e. point on $\partial_*\Omega$ we have

$$(-\mathrm{i})\mathrm{Sym}(D^\top;\nu)\big(\widetilde{D}^\top u\big)\big|_{\partial_*\Omega} = \Big(\nu_s a^{\alpha\beta}_{rs}(\partial_r u_\alpha)\big|_{\partial_*\Omega}\Big)_{1\le\beta\le M} = \partial^{A^\top}_\nu u. \tag{1.4.70}$$

Bearing this in mind it follows that the boundary layer potential operators associated as in formulas (1.4.36)-(1.4.40) with the factorization (1.4.68) of the second-order system L, with D and $\widetilde{D}$ as in (1.4.67), are precisely those considered earlier in (1.3.22), (1.3.68), (1.3.72). Incidentally, these identifications remain valid if instead of D, $\widetilde{D}$ as in (1.4.67) we take $D := A\nabla$ and $\widetilde{D} := \mathrm{div}$.

The bottom line is that the "old" point of view, of associating boundary layer potentials with a second-order system L as in (1.3.22), (1.3.68), (1.3.72), starting from the representation of L as L_A for some coefficient tensor A, may be subsumed into the "new" point of view, of associating boundary layer potentials with a second-order system L as in (1.4.36)-(1.4.40), starting from the factorization $L = \widetilde{D}D$ for some first-order systems $\widetilde{D}$, D.

Example 1.4.9 Work in $\mathbb{R}^2 \equiv \mathbb{C}$ and consider the factorization of the two-dimensional Laplacian $\Delta = \partial_x^2 + \partial_y^2$ given by

$$\Delta = \widetilde{D}D \ \text{ where } \ \widetilde{D} := \partial_x - \mathrm{i}\partial_y \ \text{ and } \ D := \partial_x + \mathrm{i}\partial_y. \tag{1.4.71}$$

Also, fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^2$, having locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^1\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \nu_2)$ the geometric measure theoretic outward unit normal to Ω, canonically identified with the complex-valued function $\nu = \nu_1 + \mathrm{i}\nu_2$ defined σ-a.e. on $\partial_*\Omega$. Then a simple direct computation reveals that the boundary layer potentials associated with Δ as in (1.4.36), (1.4.39) starting from the factorization given in (1.4.71) act on functions $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\Big)$ according to

$$\mathcal{D}f(z) = \frac{1}{2\pi\mathrm{i}}\int_{\partial_*\Omega}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \ \text{ for all } z \in \mathring{\Omega}, \tag{1.4.72}$$

and

$$Kf(z) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{\substack{y \in \partial_* \Omega \\ |x-y|>\varepsilon}} \frac{f(\zeta)}{\zeta - z} \, d\zeta \quad \text{for } \sigma\text{-a.e. } z \in \partial_* \Omega. \tag{1.4.73}$$

These are, of course, the boundary-to-domain and boundary-to-boundary versions of the Cauchy integral operator (cf. the discussion in [70, §1.1]).

On the other hand, if in place of (1.4.71) for the same two-dimensional Laplacian $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ we now consider the factorization

$$\Delta = \widetilde{D} D \quad \text{where } \widetilde{D} := (\partial_{x_1}, \partial_{x_2}) \text{ and } D := \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \end{pmatrix}, \tag{1.4.74}$$

then the double layer potential operators associated with the two-dimensional Laplacian as indicated in (1.4.36), (1.4.39) starting from the factorization (1.4.74) now act on functions $f \in L^1\Big(\partial_* \Omega, \frac{\sigma(x)}{1+|x|}\Big)$ according to

$$\mathcal{D} f(x) = \frac{1}{2\pi} \int_{\partial_* \Omega} \frac{\langle \nu(y), y - x \rangle}{|x - y|^2} f(y) \, d\sigma(y) \quad \text{for all } x \in \mathring{\Omega}, \tag{1.4.75}$$

and

$$Kf(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\substack{y \in \partial_* \Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(y), y - x \rangle}{|x - y|^2} f(y) \, d\sigma(y) \quad \text{for } \sigma\text{-a.e. } x \in \partial_* \Omega. \tag{1.4.76}$$

These are the boundary-to-domain and boundary-to-boundary versions of the two-dimensional harmonic double layer operator (compare with [70, (2.5.202), (2.5.203)] in the n-dimensional setting). From [68, Proposition 5.6.7] we know that the limits in (1.4.73) and (1.4.76) exist as indicated, but if Ω is actually a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (for example, if $\partial\Omega$ is a UR set) then said limits exist for σ-a.e. point in $\partial\Omega$.

In particular, he above discussion shows that in the two-dimensional setting both the Cauchy integral operator and the harmonic double layer arise in the same fashion, i.e., as boundary layer potential operators associated with two particular factorizations of the two-dimensional Laplacian.

From (1.4.58)-(1.4.59) we know that the boundary-to-domain double layer associated as in (1.4.36) with a factorization $\widetilde{D}D$ of a given weakly elliptic second-order system L reproduces null-solutions of the first-order system D (with proper nontangential boundary behavior). In the case of (1.4.71) we have chosen D to be (up to normalization) the Cauchy-Riemann operator, so its null-solutions are holomorphic functions in Ω. In turn, this yields the Cauchy reproducing formula for holomorphic functions with proper nontangential boundary behavior (see [70, (1.1.10)] in this regard). On the other hand, in the case of (1.4.74) we have taken D to be the gradient operator, so its null-solutions are locally constant functions in Ω. The corresponding reproducing formula for the double layer in this scenario is discussed in Proposition 1.3.6.

Example 1.4.10 Continue to work in $\mathbb{R}^2 \equiv \mathbb{C}$ and consider Bitsadze's operator

$$L_B := \partial_{\bar{z}}^2 \text{ where } \partial_{\bar{z}} := \tfrac{1}{2}\big(\partial_x + \mathrm{i}\partial_y\big) \tag{1.4.77}$$

(cf. [3], [4]). This is a weakly elliptic scalar operator and, up to an additive constant, the fundamental solution associated with L_B as in [70, Theorem 1.4.2] is (cf. [70, Proposition 1.4.34] and [70, (1.4.281)]) the distribution induced by the locally integrable function

$$E(z) = \frac{\bar{z}}{\pi z}, \qquad z \in \mathbb{C}. \tag{1.4.78}$$

Also, by design, we have a factorization

$$L_B = \widetilde{D}D \text{ where } \widetilde{D} = D = \tfrac{1}{2}\big(\partial_x + \mathrm{i}\partial_y\big). \tag{1.4.79}$$

For further use, note that

$$(L_B)^\top = L_B \text{ and } \widetilde{D}^\top = D^\top = -\tfrac{1}{2}\big(\partial_x + \mathrm{i}\partial_y\big). \tag{1.4.80}$$

Next, fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^2$, having locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and denote by $\nu = (\nu_1, \nu_2)$ the geometric measure theoretic outward unit normal to Ω, canonically identified with the complex-valued function $\nu = \nu_1 + \mathrm{i}\nu_2$ defined at σ-a.e. point on $\partial_*\Omega$. According to [70, Convention 1.7.2], for each function u of class $\mathscr{C}^1$ in a neighborhood of $\partial_*\Omega$, at σ-a.e. point on $\partial_*\Omega$ we have

$$(-\mathrm{i})\mathrm{Sym}(D^\top;\nu)\big(\widetilde{D}^\top u\big)\Big|_{\partial_*\Omega} = \tfrac{1}{2}\nu(\partial_{\bar{z}}u)\Big|_{\partial_*\Omega}. \tag{1.4.81}$$

In particular, since

$$(\partial_{\bar{z}}E)(z) = \frac{1}{\pi z} \text{ for each } z \in \mathbb{C} \setminus \{0\}, \tag{1.4.82}$$

it follows that for each $z \in \Omega$ and σ-a.e. $\zeta \in \partial_*\Omega$ we have

$$(-\mathrm{i})\mathrm{Sym}(D^\top;\nu(\zeta))\widetilde{D}_\zeta^\top[E(z-\zeta)] = -\frac{\nu(\zeta)}{2\pi(z-\zeta)}. \tag{1.4.83}$$

Finally, recall from (A.0.62) that

$$\nu(\zeta)\,\mathrm{d}\sigma(\zeta) = \frac{1}{\mathrm{i}}\mathrm{d}\zeta \text{ on } \partial_*\Omega. \tag{1.4.84}$$

Bearing these in mind, we then conclude that the boundary layer potential operators associated as in formulas (1.4.36), (1.4.39) with the factorization (1.4.79) of Bitsadze operator L_B from (1.4.77) act on functions $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\Big)$ according to

$$\mathcal{D}f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial_* \Omega} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \;\text{ for all }\; z \in \mathring{\Omega}, \tag{1.4.85}$$

and

$$Kf(z) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi \mathrm{i}} \int_{\substack{y \in \partial_* \Omega \\ |x-y|>\varepsilon}} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \;\text{ for } \sigma\text{-a.e. }\; z \in \partial_* \Omega. \tag{1.4.86}$$

Surprisingly, these are the boundary-to-domain and boundary-to-boundary versions of the Cauchy integral operator associated with Ω (compare with the discussion in Example 1.4.9). Let us also remark that [68, Proposition 5.6.7] guarantees the existence of the limit in (1.4.86), and if Ω is in fact a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension 1) and has locally finite $\mathcal{H}^1$ measure (in particular, if $\partial\Omega$ is a UR set) then said limit exists for σ-a.e. point in $\partial\Omega$.

Example 1.4.11 Work in the three-dimensional setting and denote by $\Delta = \sum_{j=1}^3 \partial_j^2$ the Laplacian in $\mathbb{R}^3$. Use this to define the second-order homogeneous constant coefficient system

$$L := \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix} = \Delta I_{3\times 3}. \tag{1.4.87}$$

In other words, L is simply the three-dimensional "vector Laplacian," which in particular means that $L^\top = L$. As is well known, the three-dimensional vector Laplacian may be written as

$$L = \nabla \mathrm{div} - \mathrm{curl}\,\mathrm{curl}, \tag{1.4.88}$$

hence

$$\begin{gathered} L = \sum_{j=1}^{2} \widetilde{D}_j D_j \;\text{ where} \\ \widetilde{D}_1 := \nabla, \quad D_1 := \mathrm{div}, \quad \widetilde{D}_2 := -\mathrm{curl}, \quad D_2 := \mathrm{curl}. \end{gathered} \tag{1.4.89}$$

As noted earlier in (1.4.60)-(1.4.61), we may recast (1.4.89) simply as

$$L = \widetilde{D} D \;\text{ where }\; \widetilde{D} := (\nabla, -\mathrm{curl}) \;\text{ and }\; D := \begin{pmatrix} \mathrm{div} \\ \mathrm{curl} \end{pmatrix}, \tag{1.4.90}$$

which we shall call the *Maxwell factorization of the vector Laplacian*. If E_Δ is as in (A.0.65) with $n = 3$, then

$$E_L = E_\Delta I_{3\times 3}. \tag{1.4.91}$$

Also,

$$\widetilde{D}^\top = \begin{pmatrix} \nabla^\top \\ -\mathrm{curl}^\top \end{pmatrix} = \begin{pmatrix} -\mathrm{div} \\ -\mathrm{curl} \end{pmatrix}. \tag{1.4.92}$$

Consequently, with the differential operators acting on columns,

$$\widetilde{D}^{\top} E_{L^{\top}} = \begin{pmatrix} -\mathrm{div} \\ -\mathrm{curl} \end{pmatrix} \begin{pmatrix} E_{\Delta} & 0 & 0 \\ 0 & E_{\Delta} & 0 \\ 0 & 0 & E_{\Delta} \end{pmatrix} = \begin{pmatrix} -\partial_1 E_{\Delta} & -\partial_2 E_{\Delta} & -\partial_3 E_{\Delta} \\ 0 & \partial_3 E_{\Delta} & -\partial_2 E_{\Delta} \\ -\partial_3 E_{\Delta} & 0 & \partial_1 E_{\Delta} \\ \partial_2 E_{\Delta} & -\partial_1 E_{\Delta} & 0 \end{pmatrix} \tag{1.4.93}$$

hence, further,

$$\big(\widetilde{D}^{\top} E_{L^{\top}}\big)^{\top} = \begin{pmatrix} -\partial_1 E_{\Delta} & 0 & -\partial_3 E_{\Delta} & \partial_2 E_{\Delta} \\ -\partial_2 E_{\Delta} & \partial_3 E_{\Delta} & 0 & -\partial_1 E_{\Delta} \\ -\partial_3 E_{\Delta} & -\partial_2 E_{\Delta} & \partial_1 E_{\Delta} & 0 \end{pmatrix}. \tag{1.4.94}$$

To proceed, fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^3$, having locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^2 \lfloor \partial\Omega$, and denote by $\nu = (\nu_1, \nu_2, \nu_3)$ the geometric measure theoretic outward unit normal to Ω. Having picked a vector-valued function $\vec{f} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^2}\Big)\Big]^3$ we therefore have

$$(-\mathrm{i})\mathrm{Sym}(D;\nu) = \begin{pmatrix} \nu \cdot \vec{f} \\ \nu \times \vec{f} \end{pmatrix}. \tag{1.4.95}$$

Then $(-\mathrm{i})\big(\widetilde{D}^{\top} E_{L^{\top}}\big)^{\top}\mathrm{Sym}(D;\nu)\vec{f}$ is identified, thanks to (1.4.93) and (1.4.95), as the vector

$$\begin{pmatrix} -(\partial_1 E_{\Delta})(\nu \cdot \vec{f}) - (\partial_3 E_{\Delta})(\nu \times \vec{f})_2 + (\partial_2 E_{\Delta})(\nu \times \vec{f})_3 \\ -(\partial_2 E_{\Delta})(\nu \cdot \vec{f}) + (\partial_3 E_{\Delta})(\nu \times \vec{f})_1 - (\partial_1 E_{\Delta})(\nu \times \vec{f})_3 \\ -(\partial_3 E_{\Delta})(\nu \cdot \vec{f}) - (\partial_2 E_{\Delta})(\nu \times \vec{f})_1 + (\partial_1 E_{\Delta})(\nu \times \vec{f})_2 \end{pmatrix}. \tag{1.4.96}$$

Ultimately, this analysis leads to the conclusion that the double layer potential operator associated with the three-dimensional vector-Laplacian (1.4.87) starting from the Maxwell factorization (1.4.90) takes the form

$$\mathcal{D}\vec{f}(x) = \int_{\partial_*\Omega} \Big\{(\nabla E_{\Delta})(x-y) \times \big(\nu \times \vec{f}\big)(y) - \big(\nu \cdot \vec{f}\big)(y)(\nabla E_{\Delta})(x-y)\Big\}\, \mathrm{d}\sigma(y) \tag{1.4.97}$$

at each point $x \in \mathring{\Omega}$. Likewise, the principal-value (or boundary-to-boundary) double layer potential operator associated with the three-dimensional vector-Laplacian (1.4.87) starting from the Maxwell factorization (1.4.90) is given by

$$K\vec{f}(x) = \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial_*\Omega \\ |x-y|>\varepsilon}} \Big\{(\nabla E_{\Delta})(x-y) \times \big(\nu \times \vec{f}\big)(y) - \big(\nu \cdot \vec{f}\big)(y)(\nabla E_{\Delta})(x-y)\Big\}\, \mathrm{d}\sigma(y) \tag{1.4.98}$$

at σ-a.e. point $x \in \partial_*\Omega$. Integral operators of this sort are relevant in the treatment of boundary value problems arising in the theory of electromagnetism; cf., e.g., [77]. The result recorded in [68, Proposition 5.6.7] guarantees the existence of the limit specified in (1.4.98), and tells us that if Ω is in fact a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (in particular, if $\partial\Omega$ is a UR set) then the limit in (1.4.98) actually exists for σ-a.e. point in $\partial\Omega$.

Lastly, we wish to remark that the boundary-to-domain double layer (1.4.97), which has been associated as in (1.4.36) with the Maxwell factorization of the vector Laplacian given in (1.4.90), reproduces the null-solutions of the first-order system D, i.e., divergence-free, curl-free, vector fields exhibiting proper nontangential boundary behavior (in the sense of (1.4.58)-(1.4.59)).

Example 1.4.12 Work in the Clifford algebra context and consider the factorization of the Laplacian $\Delta = \sum_{j=1}^{n} \partial_j^2$ in $\mathbb{R}^n$ given by

$$\Delta = \widetilde{D}D \text{ where } \widetilde{D} := D := \mathrm{i}\sum_{j=1}^{n} \mathbf{e}_j \partial_j. \tag{1.4.99}$$

To implement the recipes given in (1.4.36)-(1.4.39) for this factorization, we find it convenient to re-frame the discussion in the matrix formalism associated with the Clifford algebra $\mathcal{C}\ell_n$, as described in [68, (6.4.10)-(6.4.11)]. In such a scenario,

$$\begin{gathered}\text{identify } \mathbf{e}_j := E_j^n \in \mathbb{R}^{2^n \times 2^n} \text{ for each } j \in \{1, \dots, n\} \\ \text{and take } \odot \text{ to be the ordinary multiplication of matrices.}\end{gathered} \tag{1.4.100}$$

Hence, if we abbreviate $M := 2^n$, then

$$\begin{gathered}\mathcal{C}\ell_n \text{ is canonically identified with the sub-algebra} \\ \text{of } \mathbb{C}^{M\times M} \text{ generated by the matrices } \{E_j^n\}_{1\leq j\leq n}.\end{gathered} \tag{1.4.101}$$

Next, in the current language, the Dirac-type operator D from (1.4.99) becomes the first-order, homogeneous, constant (complex) coefficient $M \times M$ system

$$D = \mathrm{i}\sum_{j=1}^{n} E_j^n \partial_j. \tag{1.4.102}$$

Furthermore, the second-order, homogeneous, constant (real) coefficient weakly elliptic $M \times M$ system $L := \Delta I_{M\times M}$, where $I_{M\times M}$ denotes the $M \times M$ identity matrix, factors as $\widetilde{D}D$ with D as in (1.4.102) and $\widetilde{D} := D$. Then $L^\top = L$, so $E_{L^\top} = E_\Delta I_{M\times M}$ where E_Δ is the standard fundamental solution for the Laplacian (recalled in (A.0.65)). In addition, from [68, (6.4.16)] (cf. also [68, Lemma 6.4.1]) we see that

$$\widetilde{D}^\top = D^\top = D. \tag{1.4.103}$$

As a consequence of the aforementioned properties and [68, (6.4.16)], we may then compute

$$\begin{aligned}\big(\widetilde{D}^\top E_{L^\top}\big)^\top &= \big(DE_\Delta I_{M\times M}\big)^\top \\ &= \Big(i\sum_{j=1}^n(\partial_j E_\Delta)E_j^n\Big)^\top = (-\mathrm{i})\sum_{j=1}^n(\partial_j E_\Delta)E_j^n. \end{aligned} \tag{1.4.104}$$

To proceed, fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^n$, having locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Then

$$(-\mathrm{i})\mathrm{Sym}\big(D;\nu\big) = \mathrm{i}\sum_{j=1}^n \nu_j E_j^n \tag{1.4.105}$$

and (1.4.104) gives

$$\begin{aligned}\big(\widetilde{D}^\top E_{L^\top}\big)^\top(x-y) &= \frac{(-\mathrm{i})}{\omega_{n-1}}\sum_{j=1}^n \frac{x_j-y_j}{|x-y|^n}E_j^n \\ &\text{for all } x,y\in\mathbb{R}^n \text{ with } x\neq y.\end{aligned} \tag{1.4.106}$$

In view of the identities recorded in (1.4.105)-(1.4.106), and after eventually reverting to the standard Clifford algebra formalism, we then see that the boundary layer potentials associated with the Laplacian as in (1.4.36)-(1.4.39) starting from the factorization given in (1.4.99) act on arbitrary functions $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\otimes \mathcal{C}\ell_n$ according to

$$\mathcal{D}f(x) = \frac{1}{\omega_{n-1}}\int_{\partial_*\Omega}\frac{x-y}{|x-y|^n}\odot\nu(y)\odot f(y)\,\mathrm{d}\sigma(y) \ \text{ for all } x\in\mathring{\Omega}, \tag{1.4.107}$$

and

$$Kf(x) = \lim_{\varepsilon\to 0^+}\frac{1}{\omega_{n-1}}\int\limits_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}}\frac{x-y}{|x-y|^n}\odot\nu(y)\odot f(y)\,\mathrm{d}\sigma(y) \tag{1.4.108}$$

for σ-a.e. $x \in \partial_*\Omega$. These are the boundary-to-domain and boundary-to-boundary versions of the Cauchy-Clifford integral operator (cf. (A.0.53) and (A.0.54)). Once again, [68, Proposition 5.6.7] ensures the existence of the limit specified in (1.4.108), and tells us that if Ω is a Lebesgue measurable set whose topological boundary $\partial\Omega$ happens to be countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (in particular, if $\partial\Omega$ is a UR set) then the limit in (1.4.108) exists for σ-a.e. point in $\partial\Omega$.

Let us momentarily adopt the latter geometric setting, i.e., assume for the time being that Ω is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (this is the case if, in particular, $\partial\Omega$ is a UR set). In such a context, the principal-value integral operator $K^{\#}$ from (1.4.40) acts on each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big) \otimes \mathcal{C}\ell_n$ according to

$$K^{\#}f(x) = \lim_{\varepsilon\to 0^+} \frac{-1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu(x) \odot \frac{x-y}{|x-y|^n} \odot f(y)\,\mathrm{d}\sigma(y) \tag{1.4.109}$$

for σ-a.e. $x \in \partial_*\Omega$, which is the "transpose" Cauchy-Clifford operator (cf. (1.6.1)).

On the other hand, if for the same Laplacian $\Delta = \sum_{j=1}^n \partial_j^2$ we now consider the factorization

$$\Delta = \widetilde{D}D \text{ where } \widetilde{D} := (\partial_1, \dots, \partial_n) \text{ and } D := \begin{pmatrix} \partial_1 \\ \vdots \\ \partial_n \end{pmatrix} \tag{1.4.110}$$

in place of (1.4.99), then the boundary layer potential operators associated as in (1.4.36), (1.4.39) with some Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^n$ of locally finite perimeter and the n-dimensional Laplacian, starting from the factorization (1.4.110), now become

$$\mathcal{D}f(x) = \frac{1}{\omega_{n-1}} \int_{\partial_*\Omega} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y), \qquad \forall\, x \in \mathring{\Omega}, \tag{1.4.111}$$

and, respectively,

$$Kf(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y) \tag{1.4.112}$$

for σ-a.e. $x \in \partial_*\Omega$, i.e., the boundary-to-domain and boundary-to-boundary versions of the n-dimensional harmonic double layer operator (cf. [70, (2.5.202), (2.5.203)]).

Also, if Ω is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure, then the boundary layer potential operator associated with the n-dimensional Laplacian as in (1.4.40) starting from the factorization (1.4.110) now becomes

$$K^{\#}f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \frac{\langle \nu(x), x-y\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial_*\Omega, \tag{1.4.113}$$

which is the "transpose" principal-value harmonic double layer potential operator (cf. (A.0.102)).

In particular, this highlights the fact that both the Cauchy-Clifford integral operator and the harmonic double layer arise in the same fashion, i.e., as boundary layer potential operators associated with two particular factorizations of the Laplacian in $\mathbb{R}^n$.

Given that the factorization (1.4.99) of the Laplacian involves two copies of the standard Dirac operator in $\mathbb{R}^n$, from (1.4.58)-(1.4.59) we conclude that boundary-to-domain double layer (1.4.107) (associated as in (1.4.36) with said factorization) reproduces null-solutions of D, i.e., monogenic functions in Ω, with proper nontangential boundary behavior. See [70, (1.2.30)] in this regard. By way of contrast, in (1.4.110) we have taken D to be the gradient operator in $\mathbb{R}^n$, so its null-solutions are precisely locally constant functions in Ω. A suitable reproducing formula for the corresponding double layer (cf. (1.4.111)) in this scenario is provided in Proposition 1.3.6.

Example 1.4.13 There are a number of natural variants of Example 1.4.12. First, bring in the Clifford algebra $\mathcal{C}\ell_n$, but work in $\mathbb{R}^{n+1}$. Label the variables in $\mathbb{R}^{n+1}$ as $x_0, x_1, \dots, x_n$, and factor the Laplacian in $\mathbb{R}^{n+1}$, regarded as the $M \times M$ system $L := \Delta I_{2^n \times 2^n}$, in the following fashion:

$$\Delta = \widetilde{D} D \ \text{ where } \ \widetilde{D} := \partial_{x_0} - \sum_{j=1}^{n} \mathbf{e}_j \partial_{x_j} \ \text{ and } \ D := \partial_{x_0} + \sum_{j=1}^{n} \mathbf{e}_j \partial_{x_j}. \tag{1.4.114}$$

With this factorization, we can then associate double layer potentials as in (1.4.36) and (1.4.39). To be more specific, observe that $L^\top = L$, hence $E_{L^\top} = E_\Delta I_{M\times M}$ where E_Δ is the standard fundamental solution for the Laplacian in $\mathbb{R}^{n+1}$ (as in (A.0.65) with n replaced by $n+1$). Once again relying on [68, (6.4.16)] (or using [68, Lemma 6.4.1]), in place of (1.4.103) we now obtain

$$\widetilde{D}^\top = -\partial_{x_0} - \sum_{j=1}^{n} \mathbf{e}_j \partial_{x_j} = -D. \tag{1.4.115}$$

As a consequence of these properties and the fact that the transpose of $\mathbf{e}_j$ (identified with a matrix, as in (1.4.100)) is $-\mathbf{e}_j$ for each $j \in \{1, \dots, n\}$, we therefore arrive at

$$\big(\widetilde{D}^\top E_{L^\top}\big)^\top = -\big(D E_\Delta I_{M\times M}\big)^\top = -\partial_{x_0} E_\Delta + \sum_{j=1}^{n} \big(\partial_{x_j} E_\Delta\big)\mathbf{e}_j. \tag{1.4.116}$$

Thus, for all $x, y \in \mathbb{R}^n$ with $x \neq y$ we have

$$\big(\widetilde{D}^\top E_{L^\top}\big)^\top (x-y) = -\frac{1}{\omega_n} \frac{x_0 - y_0}{|x-y|^{n+1}} + \frac{1}{\omega_n} \sum_{j=1}^{n} \frac{x_j - y_j}{|x-y|^{n+1}} \mathbf{e}_j. \tag{1.4.117}$$

Next, fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^{n+1}$, of locally finite perimeter. If we abbreviate $\sigma := \mathcal{H}^n \lfloor \partial\Omega$ and denote by $\nu = (\nu_0, \nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω, then

$$(-\mathrm{i})\mathrm{Sym}\big(D;\nu\big)=\nu_0+\sum_{j=1}^{n}\nu_j\mathbf{e}_j\ \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{1.4.118}$$

The discussion above shows that the boundary layer potentials associated with the $(n+1)$-dimensional Laplacian as in (1.4.36)-(1.4.39) starting from the factorization given in (1.4.114) act on arbitrary functions $f\in L^1\Big(\partial_*\Omega,\frac{\sigma(y)}{1+|y|^n}\Big)\otimes C\ell_n$ according to

$$\mathcal{D}f(x)=\frac{1}{\omega_n}\int_{\partial_*\Omega}\frac{-(x_0-y_0)+\sum_{j=1}^{n}(x_j-y_j)\mathbf{e}_j}{|x-y|^{n+1}} \odot\Big(\nu_0(y)+\sum_{j=1}^{n}\nu_j(y)\mathbf{e}_j\Big)\odot f(y)\,\mathrm{d}\sigma(y) \tag{1.4.119}$$

at each point $x\in\mathring{\Omega}$, and, respectively,

$$Kf(x)=\lim_{\varepsilon\to 0^+}\frac{1}{\omega_n}\int_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}}\frac{-(x_0-y_0)+\sum_{j=1}^{n}(x_j-y_j)\mathbf{e}_j}{|x-y|^{n+1}} \odot\Big(\nu_0(y)+\sum_{j=1}^{n}\nu_j(y)\mathbf{e}_j\Big)\odot f(y)\,\mathrm{d}\sigma(y) \tag{1.4.120}$$

at σ-a.e. point $x\in\partial_*\Omega$ (see [68, Proposition 5.6.7] for the existence of the limit in (1.4.120)). These should be compared with the Cauchy-Clifford integral operators from (A.0.53) and (A.0.54). It may be verified without difficulty that, corresponding to $n=1$ (when there is only one imaginary unit $\mathbf{e}_1$, which we identify with $\mathrm{i}=\sqrt{-1}$), the operators (1.4.119)-(1.4.120) reduce precisely to the boundary-to-domain and the boundary-to-boundary Cauchy integral operators in the plane, from (1.4.85)-(1.4.86).

Second, denote by $\mathbb{H}$ the skew field of quaternions. Work in $\mathbb{R}^4$, where we label variables by x_0, x_1, x_2, x_3. Then, if $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard anticommuting imaginary units in $\mathbb{H}$, we may factor the Laplacian in $\mathbb{R}^4$ as

$$\begin{aligned}\Delta=\widetilde{D}D\ \text{ where }\ \widetilde{D}&:=\partial_{x_0}-\mathbf{i}\partial_{x_1}-\mathbf{j}\partial_{x_2}-\mathbf{k}\partial_{x_3}\\ \text{and }\ D&:=\partial_{x_0}+\mathbf{i}\partial_{x_1}+\mathbf{j}\partial_{x_2}+\mathbf{k}\partial_{x_3}.\end{aligned} \tag{1.4.121}$$

Once again, with this factorization we may associate double layer potentials as in (1.4.36) and (1.4.39). To implement this, much as in the past it is convenient to identify the multiplicative unit 1 in $\mathbb{H}$ with $I_{4\times 4}$, and regard $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as antisymmetric 4×4 matrices, namely

$$\mathbf{i} \equiv \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{j} \equiv \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{k} \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.4.122}$$

Then

$$\widetilde{D}^\top = -\partial_{x_0} - \mathbf{i}\partial_{x_1} - \mathbf{j}\partial_{x_2} - \mathbf{k}\partial_{x_3} = -D. \tag{1.4.123}$$

Next, denote by E_Δ the standard fundamental solution for the Laplacian Δ in $\mathbb{R}^4$ (defined as in (A.0.65) with $n := 4$). Then $L := \Delta I_{4\times 4}$ factors as in (1.4.121), and $L^\top = L$, so $E_{L^\top} = E_\Delta I_{4\times 4}$. Consequently,

$$\begin{aligned} \big(\widetilde{D}^\top E_{L^\top}\big)^\top &= \Big(- \partial_{x_0} E_\Delta - (\partial_{x_1} E_\Delta)\mathbf{i} - (\partial_{x_2} E_\Delta)\mathbf{j} - (\partial_{x_3} E_\Delta)\mathbf{k}\Big)^\top \\ &= -\partial_{x_0} E_\Delta + (\partial_{x_1} E_\Delta)\mathbf{i} + (\partial_{x_2} E_\Delta)\mathbf{j} + (\partial_{x_3} E_\Delta)\mathbf{k} \end{aligned} \tag{1.4.124}$$

As such, for any pair of distinct points $x, y \in \mathbb{R}^4$ we have

$$\begin{aligned} &\big(\widetilde{D}^\top E_{L^\top}\big)^\top (x-y) \\ &= -\frac{1}{\omega_3}\frac{x_0-y_0}{|x-y|^4} + \frac{1}{\omega_3}\frac{x_1-y_1}{|x-y|^4}\mathbf{i} + \frac{1}{\omega_3}\frac{x_2-y_2}{|x-y|^4}\mathbf{j} + \frac{1}{\omega_3}\frac{x_3-y_3}{|x-y|^4}\mathbf{k}. \end{aligned} \tag{1.4.125}$$

Going further, fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^4$, of locally finite perimeter. If we abbreviate $\sigma := \mathcal{H}^3\lfloor\partial\Omega$ and denote by $\nu = (\nu_0, \nu_1, \nu_2, \nu_3)$ the geometric measure theoretic outward unit normal to Ω, then

$$(-\mathrm{i})\mathrm{Sym}\big(D;\nu\big) = \nu_0 + \nu_1\mathbf{i} + \nu_2\mathbf{j} + \nu_3\mathbf{k} \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{1.4.126}$$

These computations allow us to conclude that the boundary layer potentials associated with the 4-dimensional Laplacian as in (1.4.36)-(1.4.39) starting from the factorization given in (1.4.121) act on functions $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^3}\Big) \otimes \mathbb{H}$ according to[12]

$$\begin{aligned} \mathcal{D}f(x) = \frac{1}{\omega_3}\int\limits_{\partial_*\Omega} &\frac{-(x_0-y_0) + (x_1-y_1)\mathbf{i} + (x_2-y_2)\mathbf{j} + (x_3-y_3)\mathbf{k}}{|x-y|^4} \cdot \\ &\cdot \Big(\nu_0(y) + \nu_1(y)\mathbf{i} + \nu_2(y)\mathbf{j} + \nu_3(y)\mathbf{k}\Big) \cdot f(y)\,\mathrm{d}\sigma(y) \end{aligned} \tag{1.4.127}$$

at each point $x \in \mathring{\Omega}$, and, respectively,

[12] with "·" denoting the multiplication in the field of quaternions $\mathbb{H}$

$$Kf(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_3} \int\limits_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \frac{-(x_0-y_0)+(x_1-y_1)\mathbf{i}+(x_2-y_2)\mathbf{j}+(x_3-y_3)\mathbf{k}}{|x-y|^4} \cdot \Big(\nu_0(y)+\nu_1(y)\mathbf{i}+\nu_2(y)\mathbf{j}+\nu_3(y)\mathbf{k}\Big)\cdot f(y)\,\mathrm{d}\sigma(y) \tag{1.4.128}$$

at σ-a.e. point $x \in \partial_*\Omega$ (see [68, Proposition 5.6.7] for the existence of the limit in (1.4.128)). One can make the case that these should be labeled as the Cauchy-Hamilton integral operators associated with the set Ω.

Let us describe a more inclusive point of view, which also allows the consideration of Cayley's algebra of octonions. We briefly discuss a general algebraic construction. Assume $\mathcal{A} = (\mathcal{A}, +, \cdot)$ is a real, unital, associative algebra endowed with a linear involution $a \mapsto a^c$. Assume the latter to be a *conjugation*, meaning that $(ab)^c = b^c a^c$ for every $a, b \in \mathcal{A}$. In particular, if $1_{\mathcal{A}}$ is the multiplicative unit in $\mathcal{A}$, then $1^c_{\mathcal{A}} = 1_{\mathcal{A}}$. Define the *trace* of an element $a \in \mathcal{A}$ as $\mathrm{Tr}_{\mathcal{A}}(a) := \frac{1}{2}(a + a^c)$. Next, consider the functor $\mathcal{A} \mapsto \mathcal{K}(\mathcal{A}) = (\mathcal{A} \times \mathcal{A}, +, \cdot)$ where the addition in $\mathcal{K}(\mathcal{A})$ is done componentwise while the multiplication and conjugation are, respectively, given by

$$(a,b)\cdot(\alpha,\beta) := (a\alpha - \beta^c b, \beta a + b\alpha^c), \quad (a,b)^c := (a^c, -b). \tag{1.4.129}$$

Assuming the algebra $\mathcal{A}$ is as above, the following claims are straightforward:

(1) $\mathcal{K}(\mathcal{A})$ is a real, unital algebra, equipped with a conjugation.
(2) $(1_{\mathcal{A}}, 0)$ is the multiplicative unit in $\mathcal{K}(\mathcal{A})$. Also, if $(\mathbf{e}_j)_{j\in J}$ are imaginary units which anticommute in $\mathcal{A}$ then $(0, 1_{\mathcal{A}})$ together with $(\mathbf{e}_j, 0)$, $(0, \mathbf{e}_j)$, for $j \in J$, are imaginary units which anticommute in $\mathcal{K}(\mathcal{A})$.
(3) If $\mathcal{A}$ is commutative then $\mathcal{K}(\mathcal{A})$ is associative.
(4) If $\mathcal{A}$ is associative and if any element has a scalar trace (i.e., if $\mathrm{Tr}_{\mathcal{A}}(a) \in \mathbb{R}$ for any $a \in \mathcal{A}$), then $\mathcal{K}(\mathcal{A})$ is an alternative algebra, i.e.,

$$[x,y,z] := (xy)z - x(yz) \tag{1.4.130}$$

is trilinear alternate, and any element in $\mathcal{K}(\mathcal{A})$ has a scalar trace.
(5) If $\mathrm{Tr}_{\mathcal{A}}(ab) = \mathrm{Tr}_{\mathcal{A}}(ba)$ for any $a, b \in \mathcal{A}$, then $\mathrm{Tr}_{\mathcal{K}(\mathcal{A})}(xy) = \mathrm{Tr}_{\mathcal{K}(\mathcal{A})}(yx)$ for any $x, y \in \mathcal{K}(\mathcal{A})$.
(6) If $\mathcal{A}$ is associative and $\mathrm{Tr}_{\mathcal{A}}(ab) = \mathrm{Tr}_{\mathcal{A}}(ba)$ for any $a, b \in \mathcal{A}$, then for any $x, y, z \in \mathcal{K}(\mathcal{A})$ one has $\mathrm{Tr}_{\mathcal{K}(\mathcal{A})}((xy)z) = \mathrm{Tr}_{\mathcal{K}(\mathcal{A})}(y(zx))$ or, equivalently, $\mathrm{Tr}_{\mathcal{K}(\mathcal{A})}[x,y,z] = 0$.
(7) If $\mathcal{A}$ is a normed algebra and $aa^c = |a|^2$ for each $a \in \mathcal{A}$, then $\mathcal{K}(\mathcal{A})$ becomes a normed algebra with $|x|^2 = xx^c = x^c x = |a|^2 + |b|^2$ for every $x = (a,b)$ in $\mathcal{K}(\mathcal{A})$. In particular, the multiplicative inverse of any element $x \in \mathcal{K}(\mathcal{A}) \setminus \{0\}$ is $x^{-1} = x^c/|x|^2$.
(8) Assume that $\mathcal{A}$ is a normed algebra such that $aa^c = |a|^2$, and so that any element in $\mathcal{A}$ has a scalar trace. Then $\mathcal{K}(\mathcal{A})$ turns into a real Hilbert space with respect to the pairing

$$\langle x, y\rangle := \mathrm{Tr}_{\mathcal{K}(\mathcal{A})}(xy^c) = \tfrac{1}{2}(xy^c + yx^c). \tag{1.4.131}$$

(9) $\mathscr{K}(\mathbb{R}) = \mathbb{C}$, the complex numbers, $\mathscr{K}(\mathbb{C}) = \mathbb{H}$, the quaternions, and $\mathscr{K}(\mathbb{H}) = \mathbb{O}$, the Cayley algebra of octonions.

Specializing these general considerations to the case of the Cayley algebra of octonions shows that the following properties are true:

(i) If i, j, k are the standard anticommuting imaginary units in $\mathbb{H}$, then $\xi_1 := (i, 0)$, $\xi_2 := (j, 0)$, $\xi_3 := (k, 0)$, $\xi_4 := (0, 1)$, $\xi_5 := (0, i)$, $\xi_6 := (0, j)$, $\xi_7 := (0, k)$ are anticommuting imaginary units which, along with $\xi_0 := (1, 0)$, the multiplicative unit in $\mathbb{O}$, form a basis for $\mathbb{O}$. Accordingly, one can embed $\mathbb{R}^8 \hookrightarrow \mathbb{O}$ by identifying each point $x = (x_\alpha)_\alpha \in \mathbb{R}^8$ with the Cayley number $\sum x_\alpha \xi_\alpha \in \mathbb{O}$.

(ii) Equipped with the pairing (1.4.131), the Cayley algebra $\mathbb{O}$ becomes a real, eight-dimensional Hilbert space, with $\{\xi_\alpha\}_\alpha$ an orthonormal basis.

(iii) If $x = \sum x_\alpha \xi_\alpha \in \mathbb{O}$, with $x_\alpha \in \mathbb{R}$, then $x^c = x_0 - x_1\xi_1 - \cdots - x_7\xi_7$. In particular, $\mathrm{Tr}_{\mathbb{O}}(x) = x_0$.

(iv) For each $x, y \in \mathbb{O}$ one has the identities $x(xy) = x^2 y$, $(xy)y = xy^2$, and also $(xy)y^{-1} = x = y^{-1}(yx)$ provided $y \neq 0$.

In the setting of the Cayley algebra $\mathbb{O}$, we may then introduce a first-order differential operator of Dirac type, namely

$$D := \frac{\partial}{\partial x_0} + \sum_{\alpha=1}^{7} \xi_\alpha \frac{\partial}{\partial x_\alpha}, \tag{1.4.132}$$

which acts on $\mathbb{O}$-valued functions (defined in $\mathbb{R}^8$) in a natural fashion. If we also set

$$D^c := \frac{\partial}{\partial x_0} - \sum_{\alpha=1}^{7} \xi_\alpha \frac{\partial}{\partial x_\alpha}, \tag{1.4.133}$$

then the transpose of the operator (1.4.132) with respect to the pairing (1.4.131) is $D^\top = -D^c$. Moreover, the Laplacian in $\mathbb{R}^8$ factors as

$$\Delta = \widetilde{D}D \quad \text{where } D \text{ is as in (1.4.132), and where } \widetilde{D} := D^c. \tag{1.4.134}$$

The key observation in this regard is that

$$\langle x \cdot y, z \rangle = \langle y, x^c \cdot z \rangle, \qquad \forall x, y, z \in \mathbb{O}. \tag{1.4.135}$$

Indeed, by (1.4.131), this is equivalent to $\mathrm{Tr}_{\mathbb{O}}((xy)z^c) = \mathrm{Tr}_{\mathbb{O}}(y(z^c x))$ which is true, by virtue of item *(6)* above. With (1.4.135) in hand, it is then straightforward to justify (1.4.134).

Having established the factorization in (1.4.134), we may then associate double layer potentials as in (1.4.36) and (1.4.39). Fix example, suppose a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^8$, having locally finite perimeter has been given. Denote by $\nu = (\nu_1, \dots, \nu_8)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^7 \lfloor \partial\Omega$. Then the same type of argument as in (1.4.116)-(1.4.118) shows that the boundary layer potentials associated with the Laplacian as in (1.4.36)-(1.4.39) based on the factorization given in (1.4.134) act on arbitrary functions $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^7}\Big) \otimes \mathbb{O}$ according to

$$\mathcal{D}f(x) = \frac{1}{\omega_7}\int_{\partial_*\Omega} \frac{-(x-y)^c}{|x-y|^8} \cdot \big(\nu(y)\cdot f(y)\big)\, d\sigma(y) \text{ for all } x \in \mathring{\Omega}, \tag{1.4.136}$$

and, with the existence of the principal-value limit guaranteed by [68, Proposition 5.6.7],

$$Kf(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_7} \int_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \frac{-(x-y)^c}{|x-y|^8} \cdot \big(\nu(y)\cdot f(y)\big)\, d\sigma(y) \tag{1.4.137}$$

for σ-a.e. $x \in \partial_*\Omega$, where the geometric measure theoretic outward unit normal to Ω is identified with an octonion-valued function $\nu : \partial_*\Omega \to \mathbb{O}$, via $\nu = \sum_{\alpha=0}^{8} \xi_\alpha \nu_\alpha$. These may be regarded as the boundary-to-domain and the boundary-to-boundary Cauchy-Cayley integral operators associated with Ω.

In all these cases, there are natural reproducing formulas for the appropriate double layers, involving null-solutions of corresponding first-order operator D, as indicated in (1.4.58)-(1.4.59).

Example 1.4.14 Recall the "deformation tensor" acting on any given vector-valued distribution $u = (u_j)_{1\leq j\leq n}$ defined in an open subset of $\mathbb{R}^n$ according to

$$\mathrm{Def}\, u := \Big(\tfrac{1}{2}\big(\partial_k u_j + \partial_j u_k\big)\Big)_{1\leq j,k\leq n}. \tag{1.4.138}$$

Its (real) transpose is acting on matrix-valued distribution $w = (w_{jk})_{1\leq j,k\leq n}$ according to

$$\mathrm{Def}^\top w = \Big(-\tfrac{1}{2}\big(\partial_k w_{jk} + \partial_k w_{kj}\big)\Big)_{1\leq j\leq n} \tag{1.4.139}$$

(cf. [70, (1.3.58)]). Let us also bring in the Jacobian operator. Specifically, for each vector-valued distribution $u = (u_j)_{1\leq j\leq n}$ defined in an open subset of $\mathbb{R}^n$ we set

$$\nabla u := \big(\partial_k u_j\big)_{1\leq j,k\leq n}. \tag{1.4.140}$$

Its (real) transpose sends a matrix-valued distribution $w = (w_{jk})_{1\leq j,k\leq n}$ into

$$\nabla^\top w = \big(-\partial_k w_{jk}\big)_{1\leq j\leq n} \tag{1.4.141}$$

(cf. [70, (1.3.57)]). In relation to these operators, it has been noted in [70, (1.7.44)] that, having fixed some Lamé moduli $\mu, \lambda \in \mathbb{C}$ then for each $\zeta \in \mathbb{C}$ we may express the complex Lamé system

$$L_{\lambda,\mu} := \mu\Delta + (\lambda+\mu)\nabla\mathrm{div} \tag{1.4.142}$$

as

$$L_{\lambda,\mu} = \widetilde{D}_\zeta D \tag{1.4.143}$$

where

$$\widetilde{D}_\zeta := \Big(-2\zeta\,\mathrm{Def}^\top, (\lambda+\mu-\zeta)\nabla, -(\mu-\zeta)\nabla^\top \Big) \text{ and } D := \begin{pmatrix} \mathrm{Def} \\ \mathrm{div} \\ \nabla \end{pmatrix} \tag{1.4.144}$$

are homogeneous, constant coefficient, first-order systems in $\mathbb{R}^n$. For future use, let us observe that the (real) transpose of $\widetilde{D}_\zeta$ is

$$\widetilde{D}_\zeta^\top = \begin{pmatrix} -2\zeta\,\mathrm{Def} \\ -(\lambda+\mu-\zeta)\mathrm{div} \\ -(\mu-\zeta)\nabla \end{pmatrix}. \tag{1.4.145}$$

As in [70, (1.7.46)], we agree to abbreviate

$$A_\zeta := A_{\widetilde{D}_\zeta, D} \text{ for each } \zeta \in \mathbb{C}, \tag{1.4.146}$$

where $A_{\widetilde{D}_\zeta, D}$ is the coefficient tensor associated with the systems $\widetilde{D}_\zeta$, D as in [70, (1.7.39)].

To proceed, fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^n$, having locally finite perimeter. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. For each $\zeta \in \mathbb{C}$ recall the conormal derivative operator from [70, (1.7.48)]. According to [70, Convention 1.7.2], whenever u is a vector-valued function of class $\mathscr{C}^1$ in a neighborhood of $\partial_*\Omega$, at σ-a.e. point on $\partial_*\Omega$ we have

$$\begin{aligned} \partial_\nu^{A_\zeta} u &= (-\mathrm{i})\mathrm{Sym}(\widetilde{D}_\zeta; \nu)(Du) \\ &= 2\zeta\,(\mathrm{Def}\,u)\nu + (\lambda+\mu-\zeta)(\mathrm{div} u)\nu + (\mu-\zeta)\,(\nabla u)\nu \text{ on } \partial_*\Omega, \end{aligned} \tag{1.4.147}$$

for each $\zeta \in \mathbb{C}$. As noted earlier, $\partial_\nu^{A_\zeta}$ is a one-parameter family of conormal derivative operators, containing the traction conormal derivative operator [70, (1.7.49)] (to which (1.4.147) reduces when $\zeta = \mu$), and also the pseudo-stress conormal derivative operator defined when $3\mu + \lambda \neq 0$ as in [70, (1.7.52)], corresponding to the choice of ζ made in [70, (1.7.53)].

Going further, the goal is to explicitly identify the format of the double layer potential operator produced by the recipe in (1.4.36), when the latter is applied for the factorization of the Lamé system given in (1.4.143). To this end, in view of the last formula in (1.4.144) it follows that for each given vector-valued function $f \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n$ we have

$$(-\mathrm{i})\mathrm{Sym}(D;\nu)f=\begin{pmatrix}\frac{1}{2}\big(\nu\otimes f+f\otimes\nu\big)\\ \nu\cdot f\\ f\otimes\nu\end{pmatrix}. \tag{1.4.148}$$

For economy of space, the right-hand side in (1.4.148) has been written as a three-component "vector" whose first and last components are $n\times n$ ordinary matrices. In fact, this is a genuine column-vector with $2n^2+1$ components, with the convention that the $n\times n$ matrix $\frac{1}{2}\big(\nu\otimes f+f\otimes\nu\big)$ is actually displayed as a column-vector with n^2 components, plus a similar convention for the $n\times n$ matrix $f\otimes\nu$.

Pressing on, to ensure the weak ellipticity of $L_{\lambda,\mu}$ assume

$$\mu\neq 0\ \text{ and }\ \lambda+2\mu\neq 0 \tag{1.4.149}$$

(cf. [70, (1.3.9)]) and bring in the matrix-valued fundamental solution $E=E_{L_{\lambda,\mu}}$ of $L_{\mu,\lambda}$ from [70, Proposition 1.4.4]. Our job is to identify the integrand in the last line of (1.4.36), i.e.,

$$(-\mathrm{i})\big(\widetilde{D}_\zeta^\top E_{L^\top_{\lambda,\mu}}\big)^\top(x-y)\mathrm{Sym}\big(D;\nu(y)\big)f(y). \tag{1.4.150}$$

In this regard, first note that $E_{L^\top_{\lambda,\mu}}=E$, since $L_{\lambda,\mu}$ is symmetric, so from (1.4.145) we conclude that

$$\widetilde{D}_\zeta^\top E_{L^\top_{\lambda,\mu}}=\begin{pmatrix}-2\zeta\,\mathrm{Def}\,E_{\bullet 1} & -2\zeta\,\mathrm{Def}\,E_{\bullet 2} & \cdots & -2\zeta\,\mathrm{Def}\,E_{\bullet n}\\ -(\lambda+\mu-\zeta)\mathrm{div}E_{\bullet 1} & -(\lambda+\mu-\zeta)\mathrm{div}E_{\bullet 2} & \cdots & -(\lambda+\mu-\zeta)\mathrm{div}E_{\bullet n}\\ -(\mu-\zeta)\nabla E_{\bullet 1} & -(\mu-\zeta)\nabla E_{\bullet 2} & \cdots & -(\mu-\zeta)\nabla E_{\bullet n}\end{pmatrix} \tag{1.4.151}$$

where $E_{\bullet 1},\dots,E_{\bullet n}$ are the columns of E. To be concise, the right-hand side in (1.4.151) has been written as a $3\times n$ "matrix" whose first and third rows are $n\times n$ ordinary matrices. In reality, this is a genuine $(2n^2+1)\times n$ matrix with the convention that each $n\times n$ matrix $\mathrm{Def}\,E_{\bullet j}$ is actually displayed as a column-vector with n^2 components, plus a similar convention for each $n\times n$ matrix $\nabla E_{\bullet j}$.

At this stage, we may use (1.4.148) and (1.4.151) (keeping in mind their respective interpretations) to identify (1.4.150) as the vector

$$-\zeta(\mathrm{Def}\,E)\big(\nu\otimes f+f\otimes\nu\big)-(\lambda+\mu-\zeta)(\nu\cdot f)(\mathrm{div}E)-(\mu-\zeta)\,(\nabla E)(f\otimes\nu), \tag{1.4.152}$$

with the variables x,y suppressed. If for any two matrices $U=(U_{jk})_{1\leq j,k\leq n}$ and $V=(V_{jk})_{1\leq j,k\leq n}$ we define the scalar $U \circledS V:=\sum_{j,k=1}^n U_{jk}V_{jk}$ then, above,

$$\begin{gathered}(\mathrm{Def}\,E)\big(\nu\otimes f+f\otimes\nu\big)\ \text{ is the vector whose}\\ j\text{-th component is }\ (\mathrm{Def}\,E_{\bullet j})\circledS\big(\nu\otimes f+f\otimes\nu\big),\end{gathered} \tag{1.4.153}$$

while

$$\mathrm{div}E:=\big(\mathrm{div}E_{\bullet j}\big)_{1\leq j\leq n}, \tag{1.4.154}$$

and

$$\begin{aligned}&(\nabla E)(f\otimes\nu)\ \text{ is the vector whose}\\&j\text{-th component is }\ (\nabla E_{\bullet j})\circledS(f\otimes\nu).\end{aligned}\tag{1.4.155}$$

Ultimately, this analysis leads to the conclusion that the double layer potential operator associated as in (1.4.36) with the Lamé system (1.4.142) starting from the factorization (1.4.143)-(1.4.144) takes the form

$$\begin{aligned}\mathcal{D}_{A_\zeta}f(x)=\int_{\partial_*\Omega}\Big\{&-\zeta(\operatorname{Def}E)(x-y)\big(\nu\otimes f+f\otimes\nu\big)(y)\\&-(\lambda+\mu-\zeta)(\nu\cdot f)(y)(\operatorname{div}E)(x-y)\\&-(\mu-\zeta)\,(\nabla E)(x-y)(f\otimes\nu)(y)\Big\}\,d\sigma(y)\end{aligned}\tag{1.4.156}$$

at each point $x\in\mathring{\Omega}$, with the conventions in (1.4.153)-(1.4.155) in effect.

Likewise, the principal-value (or boundary-to-boundary) double layer potential operator associated as in (1.4.39) with the Lamé system (1.4.142) starting from the factorization (1.4.143)-(1.4.144) is given by

$$\begin{aligned}K_{A_\zeta}f(x)=\lim_{\varepsilon\to0^+}\int_{\substack{y\in\partial_*\Omega\\|x-y|>\varepsilon}}\Big\{&-\zeta(\operatorname{Def}E)(x-y)\big(\nu\otimes f+f\otimes\nu\big)(y)\\&-(\lambda+\mu-\zeta)(\nu\cdot f)(y)(\operatorname{div}E)(x-y)\\&-(\mu-\zeta)\,(\nabla E)(x-y)(f\otimes\nu)(y)\Big\}\,d\sigma(y)\end{aligned}\tag{1.4.157}$$

at σ-a.e. point $x\in\partial_*\Omega$, again with the conventions made in (1.4.153)-(1.4.155). The existence of the principal-value limit is guaranteed by [68, Proposition 5.6.7], which also ensures that if Ω is actually a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (hence, in particular, if $\partial\Omega$ is a UR set) then the limit in (1.4.157) exists for σ-a.e. point in $\partial\Omega$.

There are other natural versions of the above construction, and we briefly elaborate on one such variant. Again, having fixed some Lamé moduli $\mu,\lambda\in\mathbb{C}$, for each $\zeta\in\mathbb{C}$ we may factor the complex Lamé system $L_{\lambda,\mu}$ recalled in (1.4.142) as

$$L_{\lambda,\mu}=\widetilde{D}D_\zeta\tag{1.4.158}$$

for the homogeneous, constant coefficient, first-order systems in $\mathbb{R}^n$ given by[13]

$$\widetilde{D}:=\Big(\operatorname{Def}^\top,\nabla,\nabla^\top\Big)\ \text{ and }\ D_\zeta:=\begin{pmatrix}-2\zeta\operatorname{Def}\\(\lambda+\mu-\zeta)\operatorname{div}\\-(\mu-\zeta)\nabla\end{pmatrix}.\tag{1.4.159}$$

[13] compared with (1.4.144), these differ only in the placement of the scalar coefficients -2ζ, $\lambda+\mu-\zeta$, $-(\mu-\zeta)$, which are now all attached with the second operator

It is then apparent from definitions that this factorization produces the same conormal derivative as the one associated with the factorization in (1.4.143), i.e., $\partial_\nu^{\widetilde{D},D_\zeta} = \partial_\nu^{\widetilde{D}_\zeta,D}$ and, in fact, even at the level of coefficient tensors we have $A_{\widetilde{D},D_\zeta} = A_{\widetilde{D}_\zeta,D} = A_\zeta$. Consequently, the layer potentials associated as in (1.4.36), (1.4.39) with the Lamé system (1.4.142) now starting from the factorization (1.4.158)-(1.4.159) continue to be given by (1.4.156)-(1.4.157).

For each $\zeta \in \mathbb{C}$, the boundary-to-domain double layer $\mathcal{D}_{A_\zeta}$ from (1.4.156) has been associated as in (1.4.36) the factorization $\widetilde{D}D_\zeta$ of the Lamé system $L_{\lambda,\mu}$ given in (1.4.158). As such, $\mathcal{D}_{A_\zeta}$ reproduces null-solutions of the first-order system D_ζ exhibiting proper nontangential boundary behavior, in the sense of (1.4.58)-(1.4.59). We wish to elaborate on the nature of such null-solutions. As seen from (1.4.159), a vector-valued function $\vec{\psi} = (\psi_j)_{1\leq j\leq n}$ is a null-solutions of system D_ζ in an open set $\Omega \subseteq \mathbb{R}^n$ if

$$\zeta\, \mathrm{Def}\, \vec{\psi} = 0, \quad (\lambda + \mu - \zeta)\, \mathrm{div}\vec{\psi} = 0, \quad (\mu - \zeta)\nabla\vec{\psi} = 0, \ \text{ in } \Omega. \tag{1.4.160}$$

If $\zeta \neq 0$ then the first condition above implies

$$\partial_i\psi_j = -\partial_j\psi_i \ \text{ in } \Omega \ \text{ for each } i, j \in \{1, \dots, n\}. \tag{1.4.161}$$

In turn, this forces $\mathrm{div}\vec{\psi} = 0$ (so the second condition in (1.4.160) is automatically satisfied in this case), and for each $i, j, k \in \{1, \dots, n\}$ we may write

$$\begin{aligned} \partial_i\partial_j\psi_k &= \partial_i(\partial_j\psi_k) = -\partial_i(\partial_k\psi_j) = -\partial_k(\partial_i\psi_j) \\ &= \partial_k(\partial_j\psi_i) = \partial_j(\partial_k\psi_i) = -\partial_j(\partial_i\psi_k) = -\partial_i\partial_j\psi_k, \end{aligned} \tag{1.4.162}$$

thus $\partial_i\partial_j\psi_k = 0$ in Ω. Hence, locally there exist constants a_{jk} with $a_{kj} = -a_{jk}$ such that $\partial_j\psi_k = a_{jk}$. Ultimately this shows that locally (and also globally if Ω is connected), there exist an antisymmetric matrix $A \in \mathbb{C}^{n\times n}$ along with a vector $b \in \mathbb{C}^n$ such that $\vec{\psi}(x) = Ax + b$. Finally, the final condition in (1.4.160) is discarded if $\zeta = \mu$, while if $\zeta \neq \mu$ then said condition forces $\vec{\psi}$ to be a locally constant vector field in Ω.

Example 1.4.15 Consider the complex Lamé system $L_{\lambda,\mu}$ in $\mathbb{R}^n$ defined in (1.4.142) for Lamé moduli $\lambda, \mu \in \mathbb{C}$. This system may be written in infinitely many ways as $\big(a_{rs}^{\alpha\beta}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq n}$. For example, we may express $L_{\lambda,\mu}$ as L_{A_ζ}, the system associated as in (A.0.139) with the coefficient tensor $A_\zeta := A_{\widetilde{D}_\zeta,D}$ defined in (1.4.146) for each $\zeta \in \mathbb{C}$. Recall that $A_{\widetilde{D}_\zeta,D}$ is the coefficient tensor associated as in [70, (1.7.39)] with the systems $\widetilde{D}_\zeta$, D defined in (1.4.144).

In a first stage, we wish to identify the latter coefficient tensor explicitly. To accomplish this, define Υ to be the set consisting of elements γ of the form (γ', γ'') with $\gamma' \in \{1, 2, 3\}$ and the convention that $\gamma'' \in \{1, \dots, n\}^2$ if either $\gamma' = 1$ or $\gamma' = 3$ while γ'' is simply suppressed if $\gamma' = 2$. This permits us to identify

$$D = \begin{pmatrix} \mathrm{Def} \\ \mathrm{div} \\ \nabla \end{pmatrix} = \sum_{s=1}^{n} \big(b_s^{\gamma\beta}\big)_{\substack{\gamma\in\Upsilon \\ 1\leq\beta\leq n}} \cdot \partial_s \tag{1.4.163}$$

where for each $\gamma = (\gamma', \gamma'') \in \Upsilon$ and $\beta \in \{1, \dots, n\}$ we have set

$$b_s^{\gamma\beta} := \begin{cases} \frac{1}{2}\big(\delta_{\gamma''(\beta,s)} + \delta_{\gamma''(s,\beta)}\big) & \text{if } \gamma' = 1, \\ \delta_{s\beta} & \text{if } \gamma' = 2, \\ \delta_{\gamma''(\beta,s)} & \text{if } \gamma' = 3, \end{cases} \tag{1.4.164}$$

(with the Kronecker delta's involving either two pairs or two singletons). Similarly, we may express

$$\widetilde{D}_\zeta = \Big(-2\zeta\,\mathrm{Def}^\top, (\lambda+\mu-\zeta)\nabla, -(\mu-\zeta)\nabla^\top \Big) = \sum_{r=1}^{n} \big(\widetilde{b}_r^{\alpha\gamma}\big)_{\substack{1\leq\alpha\leq n \\ \gamma\in\Upsilon}} \cdot \partial_r \tag{1.4.165}$$

where for each $\alpha \in \{1, \dots, n\}$ and $\gamma = (\gamma', \gamma'') \in \Upsilon$

$$\widetilde{b}_r^{\alpha\gamma} := \begin{cases} \zeta\big(\delta_{\gamma''(\alpha,r)} + \delta_{\gamma''(r,\alpha)}\big) & \text{if } \gamma' = 1, \\ (\lambda+\mu-\zeta)\delta_{\alpha r} & \text{if } \gamma' = 2, \\ (\mu-\zeta)\delta_{\gamma''(\alpha,r)} & \text{if } \gamma' = 3. \end{cases} \tag{1.4.166}$$

Then, according to [70, (1.7.39)], we have

$$A_\zeta = \big(a_{rs}^{\alpha\beta}(\zeta)\big)_{\substack{1\leq\alpha,\beta\leq n \\ 1\leq r,s\leq n}} \tag{1.4.167}$$

where each entry in A_ζ is given by

$$\begin{aligned} a_{rs}^{\alpha\beta}(\zeta) &:= \sum_{\gamma\in\Upsilon} \widetilde{b}_r^{\alpha\gamma} b_s^{\gamma\beta} \\ &= \sum_{\gamma''\in\{1,\dots,n\}^2} \zeta\big(\delta_{\gamma''(\alpha,r)} + \delta_{\gamma''(r,\alpha)}\big)\tfrac{1}{2}\big(\delta_{\gamma''(\beta,s)} + \delta_{\gamma''(s,\beta)}\big) \\ &\quad + (\lambda+\mu-\zeta)\delta_{\alpha r}\delta_{s\beta} \\ &\quad + \sum_{\gamma''\in\{1,\dots,n\}^2} (\mu-\zeta)\delta_{\gamma''(\alpha,r)}\delta_{\gamma''(\beta,s)} \\ &= \zeta\big(\delta_{\alpha\beta}\delta_{rs} + \delta_{\alpha s}\delta_{\beta r}\big) + (\lambda+\mu-\zeta)\delta_{\alpha r}\delta_{s\beta} + (\mu-\zeta)\delta_{\alpha\beta}\delta_{rs} \\ &= \mu\delta_{\alpha\beta}\delta_{rs} + (\lambda+\mu-\zeta)\delta_{\alpha r}\delta_{s\beta} + \zeta\delta_{\alpha s}\delta_{\beta r}. \end{aligned} \tag{1.4.168}$$

In summary, the family of coefficients associated with the family of factorizations of the Lamé system (1.4.142) as in (1.4.143) is explicitly described as

$$A_\zeta = \big(a^{\alpha\beta}_{jk}(\zeta)\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq n}} \text{ defined for each } \zeta\in\mathbb{C} \text{ according to}$$
$$a^{\alpha\beta}_{jk}(\zeta) := \mu\delta_{jk}\delta_{\alpha\beta} + (\mu+\lambda-\zeta)\delta_{j\alpha}\delta_{k\beta} + \zeta\delta_{j\beta}\delta_{k\alpha}, \qquad (1.4.169)$$
$$\text{for all } \ j,k,\alpha,\beta\in\{1,\dots,n\}.$$

In particular,

$$L_{\lambda,\mu} = \Big(a^{\alpha\beta}_{jk}(\zeta)\partial_j\partial_k\Big)_{1\leq\alpha,\beta\leq n} = L_{A_\zeta} \text{ for each } \zeta\in\mathbb{C}. \qquad (1.4.170)$$

In relation to this family of writings, it is of interest to directly identity the format of the double layer potential operators associated as in (1.3.68), (1.3.18) for the complex Lamé system $L_{\mu,\lambda}$ with Lamé moduli $\mu,\lambda\in\mathbb{C}$ satisfying (1.4.149). For this system, the fundamental solution E of $L_{\mu,\lambda}$ from [70, Theorem 1.4.2] has the explicit form $E=(E_{jk})_{1\leq j,k\leq n}$, a matrix whose (j,k) entry is defined at each point in $\mathbb{R}^n\setminus\{0\}$ according to [70, (1.4.72)].

Let us now fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^n$, having locally finite perimeter. Denote by $\nu=(\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. In such a setting, with each choice of $\zeta\in\mathbb{C}$, associate a double layer potential operator K_{A_ζ} as in (1.3.68). A direct computation based on [70, (1.4.72)], (1.4.169), and (1.3.68) then shows that the integral kernel $\Theta^\zeta(x,y)$ of the principal-value double layer potential operator K_{A_ζ} is an $n\times n$ matrix whose (j,k) entry, $1\leq j,k\leq n$, is explicitly given by

$$\begin{aligned}\Theta^\zeta_{jk}(x,y) = &-C_1(\zeta)\frac{\delta_{jk}}{\omega_{n-1}}\frac{\langle x-y,\nu(y)\rangle}{|x-y|^n}\\ &-(1-C_1(\zeta))\frac{n}{\omega_{n-1}}\frac{\langle x-y,\nu(y)\rangle(x_j-y_j)(x_k-y_k)}{|x-y|^{n+2}}\\ &-C_2(\zeta)\frac{1}{\omega_{n-1}}\frac{(x_j-y_j)\nu_k(y)-(x_k-y_k)\nu_j(y)}{|x-y|^n}, \end{aligned}\qquad (1.4.171)$$

for σ-a.e. $x\in\partial_*\Omega$ and $y\in\partial_*\Omega$, where the constants $C_1(\zeta),C_2(\zeta)\in\mathbb{C}$ are defined as

$$C_1(\zeta) := \frac{\mu(3\mu+\lambda)-\zeta(\mu+\lambda)}{2\mu(2\mu+\lambda)}, \qquad C_2(\zeta) := \frac{\mu(\mu+\lambda)-\zeta(3\mu+\lambda)}{2\mu(2\mu+\lambda)}. \qquad (1.4.172)$$

Thus, with notation introduced in (A.0.10), for each $\zeta\in\mathbb{C}$ the integral kernel $\Theta^\zeta(x,y)$ of K_{A_ζ} may be recast as

$$\Theta^{\zeta}(x,y) = -C_1(\zeta)\frac{1}{\omega_{n-1}}\frac{\langle x-y, \nu(y)\rangle}{|x-y|^n} I_{n\times n}$$
$$- (1 - C_1(\zeta))\frac{n}{\omega_{n-1}}\frac{\langle x-y, \nu(y)\rangle (x-y)\otimes(x-y)}{|x-y|^{n+2}}$$
$$- C_2(\zeta)\frac{1}{\omega_{n-1}}\frac{(x-y)\otimes\nu(y) - \nu(y)\otimes(x-y)}{|x-y|^n}, \tag{1.4.173}$$

for σ-a.e. $x \in \partial\Omega$ and $y \in \partial_*\Omega$, where $I_{n\times n}$ is the $n\times n$ identity matrix. Consequently, for each given vector-valued function $\vec{f} \in \big[L^1\big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\big)\big]^n$ and for σ-a.e. point $x \in \partial_*\Omega$ we have

$$K_{A_\zeta}\vec{f}(x) = -\frac{C_1(\zeta)}{\omega_{n-1}}\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\|x-y|>\varepsilon}}\frac{\langle x-y, \nu(y)\rangle}{|x-y|^n}\vec{f}(y)\,\mathrm{d}\sigma(y) \tag{1.4.174}$$
$$-\frac{n(1-C_1(\zeta))}{\omega_{n-1}}\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\|x-y|>\varepsilon}}\frac{\langle x-y, \nu(y)\rangle\langle x-y, \vec{f}(y)\rangle}{|x-y|^{n+2}}(x-y)\,\mathrm{d}\sigma(y)$$
$$-\frac{C_2(\zeta)}{\omega_{n-1}}\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\|x-y|>\varepsilon}}\frac{\langle \nu(y), \vec{f}(y)\rangle(x-y) - \langle x-y, \vec{f}(y)\rangle\nu(y)}{|x-y|^n}\,\mathrm{d}\sigma(y)$$

where the constants $C_1(\zeta), C_2(\zeta) \in \mathbb{C}$ are associated with each $\zeta \in \mathbb{C}$ as in (1.4.172). This agrees with the principal-value double layer potential operator for the Lamé system defined in (1.4.157).

Likewise, the boundary-to-domain double layer potential operator $\mathcal{D}_{A_\zeta}$ acts on each vector-valued function $\vec{f} \in \big[L^1\big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\big)\big]^n$ according to

$$\mathcal{D}_{A_\zeta}\vec{f}(x) = -\frac{C_1(\zeta)}{\omega_{n-1}}\int\limits_{\partial_*\Omega}\frac{\langle x-y, \nu(y)\rangle}{|x-y|^n}\vec{f}(y)\,\mathrm{d}\sigma(y) \tag{1.4.175}$$
$$-\frac{n(1-C_1(\zeta))}{\omega_{n-1}}\int\limits_{\partial_*\Omega}\frac{\langle x-y, \nu(y)\rangle\langle x-y, \vec{f}(y)\rangle}{|x-y|^{n+2}}(x-y)\,\mathrm{d}\sigma(y)$$
$$-\frac{C_2(\zeta)}{\omega_{n-1}}\int\limits_{\partial_*\Omega}\frac{\langle \nu(y), \vec{f}(y)\rangle(x-y) - \langle x-y, \vec{f}(y)\rangle\nu(y)}{|x-y|^n}\,\mathrm{d}\sigma(y)$$

for each $x \in \mathring{\Omega}$. Reassuringly, this agrees with the (boundary-to-domain) double layer potential operator associated the Lamé system in (1.4.156).

Example 1.4.16 Consider the factorization of the Laplacian Δ in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ given by

$$\Delta = \widetilde{D}D \text{ where } \widetilde{D} := (2\partial_{z_1}, \ldots, 2\partial_{z_n}) \text{ and } D := \begin{pmatrix} 2\partial_{\bar{z}_1} \\ \vdots \\ 2\partial_{\bar{z}_n} \end{pmatrix}. \tag{1.4.176}$$

Above, for each $j \in \{1, \ldots, n\}$ we have set

$$\partial_{z_j} := \tfrac{1}{2}\big(\partial_{x_j} - \mathrm{i}\partial_{y_j}\big) \text{ and } \partial_{\bar{z}_j} := \tfrac{1}{2}\big(\partial_{x_j} + \mathrm{i}\partial_{y_j}\big), \text{ assuming the } j\text{-th complex variable } z_j \in \mathbb{C} \text{ is expressed as } x_j + \mathrm{i}y_j \text{ with } x_j, y_j \in \mathbb{R}^n. \tag{1.4.177}$$

Fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^{2n} \equiv \mathbb{C}^n$, having locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$, and identify the geometric measure theoretic outward unit normal

$$\nu = (\nu_1, \nu_2, \ldots, \nu_{2n-1}, \nu_{2n}) \in \mathbb{R}^{2n} \tag{1.4.178}$$

to Ω with the complex vector

$$\nu_{\mathbb{C}} := (\nu_1 + \mathrm{i}\nu_2, \ldots, \nu_{2n-1} + \mathrm{i}\nu_{2n}) \in \mathbb{C}^n. \tag{1.4.179}$$

A straightforward interpretation of definitions then shows that the boundary layer potentials associated with the Laplacian as in (1.4.36)-(1.4.39) starting from the factorization given in (1.4.176) act on arbitrary complex-valued functions $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\Big)$ according to

$$\mathcal{D}f(z) = \frac{1}{\omega_{2n-1}} \int_{\partial_*\Omega} \frac{\big\langle \nu_{\mathbb{C}}(\zeta), \zeta - z\big\rangle_{\mathbb{C}}}{|z-\zeta|^{2n}} f(\zeta)\,\mathrm{d}\sigma(\zeta) \text{ for all } z \in \mathring{\Omega}, \tag{1.4.180}$$

where $\langle u, w\rangle_{\mathbb{C}} := \sum_{j=1}^n u_j\overline{w}_j$ for each $u = (u_j)_j \in \mathbb{C}^n$ and $w = (w_j)_j \in \mathbb{C}^n$, is the Hermitian complex-pairing, and

$$Kf(z) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{2n-1}} \int\limits_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \frac{\big\langle \nu_{\mathbb{C}}(\zeta), \zeta - z\big\rangle_{\mathbb{C}}}{|z-\zeta|^{2n}} f(\zeta)\,\mathrm{d}\sigma(\zeta) \tag{1.4.181}$$

for σ-a.e. $z \in \partial_*\Omega$. The result in [68, Proposition 5.6.7] guarantees the existence of the above limit and also ensures that if Ω is in fact a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $2n-1$) and has locally finite $\mathcal{H}^{2n-1}$ measure (in particular, if $\partial\Omega$ is a UR set) then the limit in (1.4.181) actually exists for σ-a.e. point in $\partial\Omega$.

As we shall see later on, (1.4.180) and (1.4.181) are precisely the boundary-to-domain and boundary-to-boundary versions of the Bochner-Martinelli integral operator (cf. (7.5.2) and (7.5.26)).

Example 1.4.17 Fix two integers $n, m \in \mathbb{N}$ with $n \geq 2$, and let $(\mathbb{R}^n)^m$ be embedded in $\mathcal{A} := \oplus\mathcal{C}\ell_n$, the sum of m copies of $\mathcal{C}\ell_n$. That is, if $x \in (\mathbb{R}^n)^m = \mathbb{R}^{nm}$ we write $x = (x_1, \ldots, x_m)$ with each $x_j = \sum_{\alpha=1}^n x_j^\alpha \mathbf{e}_\alpha \in \mathcal{C}\ell_n$ for $1 \leq j \leq m$. In this setting, the

Dirac operator corresponding to the j-th copy of $\mathcal{C}\ell_n$ is

$$D_j = \sum_{\alpha=1}^{n} \mathbf{e}_\alpha \frac{\partial}{\partial x_j^\alpha}, \tag{1.4.182}$$

while the global Dirac operator reads $\mathbb{D} := (D_j)_{1 \leq j \leq m}$. It follows that $\mathbb{D}$ is real (i.e. $\overline{\mathbb{D}} = \mathbb{D}$) and we have the factorization

$$\Delta = \sum_{j=1}^{m} \Delta_j = -\mathbb{D}^* \mathbb{D} = -\mathbb{D}^\top \mathbb{D}. \tag{1.4.183}$$

Here Δ is the Laplacian in the whole space $\mathbb{R}^{nm}$ and Δ_j is the Laplacian in the j-th factor of the Cartesian product $(\mathbb{R}^n)^m$, for each $1 \leq j \leq m$. Call a smooth $\mathcal{C}\ell_n$-valued function u defined in an open subset of $\mathbb{R}^{nm}$ `separately monogenic` if $D_j u = 0$ for each $j \in \{1, \dots, m\}$. Note that this amounts to the requirement that $\mathbb{D}u = 0$ (in particular, a separately monogenic function is harmonic).

We may recast (1.4.183) as the factorization of $L := \Delta$, the Laplacian in the whole space $\mathbb{R}^{nm}$, given by

$$\Delta = \widetilde{D} D \ \text{ where } \ \widetilde{D} := (-D_1, \ \dots, -D_m) \ \text{ and } \ D := \begin{pmatrix} D_1 \\ \vdots \\ D_m \end{pmatrix}. \tag{1.4.184}$$

Recall the standard fundamental solution for the Laplacian in $(\mathbb{R}^n)^m$, i.e., the function defined for each $x \in (\mathbb{R}^n)^m \setminus \{0\}$ as

$$\mathbb{E}(x) := \begin{cases} \dfrac{1}{\omega_{nm-1}(2 - nm)} \dfrac{1}{|x|^{nm-2}} & \text{if } nm \geq 3, \\[2ex] \dfrac{1}{2\pi} \ln |x| & \text{if } nm = 2, \end{cases} \tag{1.4.185}$$

where ω_{nm-1} stands for the area of the unit sphere in $\mathbb{R}^{nm}$. Observe that

$$L^\top = L = \Delta \ \text{ and } \ \widetilde{D}^\top = \begin{pmatrix} -D_1 \\ \vdots \\ -D_m \end{pmatrix}. \tag{1.4.186}$$

From (1.4.185), (1.4.186), and [68, (6.4.16)] we may then compute

$$\big(\widetilde{D}^\top E_{L^\top}\big)^\top = \big(\widetilde{D}^\top \mathbb{E}\big)^\top = \begin{pmatrix} -D_1\mathbb{E} \\ \vdots \\ -D_m\mathbb{E} \end{pmatrix}^\top = \big(D_1\mathbb{E}, \dots, D_n\mathbb{E}\big). \tag{1.4.187}$$

To proceed, let us fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^{nm}$, having locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{nm-1}\lfloor\partial\Omega$ and denote the geometric measure theoretic outward unit normal to Ω by

$$\begin{aligned} &\nu = \big(\nu_1, \nu_2, \dots, \nu_{nm}\big) = (\nu_j)_{1\leq j\leq m} : \partial^*\Omega \to (\mathbb{R}^n)^m \hookrightarrow \mathcal{A} \\ &\quad \text{with } \nu_j := \textstyle\sum_{\alpha=1}^n \nu_j^\alpha \mathbf{e}_\alpha \text{ for each } j \in \{1, \dots, m\}. \end{aligned} \tag{1.4.188}$$

Then

$$(-\mathrm{i})\mathrm{Sym}\big(D; \nu\big) = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_m \end{pmatrix} \tag{1.4.189}$$

so the expression defined for each pair of distinct points $x \in \mathbb{R}^n$ and $y \in \partial^*\Omega$ as

$$\begin{aligned} \Gamma(x, y) &:= (-\mathrm{i})\big(\widetilde{D}^\top E_{L^\top}\big)^\top(x-y)\mathrm{Sym}\big(D; \nu(y)\big) \\ &= \sum_{j=1}^m (D_j\mathbb{E})(x-y) \odot \nu_j(y) = \frac{1}{\omega_{nm-1}} \sum_{j=1}^m \frac{x_j - y_j}{|x-y|^{nm}} \odot \nu_j(y) \\ &= \frac{1}{\omega_{nm-1}} \sum_{j=1}^m \sum_{\alpha=1}^n \frac{x_j^\alpha - y_j^\alpha}{|x-y|^{nm}} \mathbf{e}_\alpha \odot \nu_j(y) \\ &= \frac{1}{\omega_{nm-1}} \sum_{j=1}^m \sum_{\alpha=1}^n \sum_{\beta=1}^n \frac{x_j^\alpha - y_j^\alpha}{|x-y|^{nm}} \nu_j^\beta(y) \mathbf{e}_\alpha \odot \mathbf{e}_\beta \end{aligned} \tag{1.4.190}$$

may be though of as the natural Cauchy kernel in this setting. In particular, the boundary layer potential operators associated with the nm-dimensional Laplacian as in (1.4.36)-(1.4.40) starting from the factorization (1.4.184) now act on each function $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^{nm-1}}\Big) \otimes C\ell_n$ according to

$$\mathcal{D}f(x) := \int_{\partial_*\Omega} \Gamma(x, y) \odot f(y)\, \mathrm{d}\sigma(y) \text{ for each } x \in \mathring{\Omega}, \tag{1.4.191}$$

and, respectively,

$$Kf(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \Gamma(x, y) \odot f(y)\, \mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial_*\Omega. \tag{1.4.192}$$

These are the boundary-to-domain and boundary-to-boundary versions of the Cauchy-Clifford integral operator in Clifford analysis for functions of several variables. The result in [68, Proposition 5.6.7] ensures the existence of the limit in (1.4.192) and also guarantees that if Ω actually is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (hence, in particular, if $\partial\Omega$ is a UR set) then the limit in (1.4.192) exists for σ-a.e. point in $\partial\Omega$.

Let us point out that while $\Gamma(x, y)$ is harmonic in x, it is not separately monogenic in the variable x when $m > 1$. Thus, the boundary-to-domain Cauchy-Clifford operator (1.4.191) is separately monogenic if and only if $m = 1$. The fact that $\mathcal{D}$ reproduces separately monogenic functions (in this regard see Remark 1.4.6 and Theorem 1.4.7, as well as [60]) may then be viewed as a version of the classical Bochner-Martinelli formula in $\mathbb{C}^n$ (cf. the discussion in §7.3).

Example 1.4.18 We employ notation and terminology from §7.3 to explain how the higher-degree Bochner-Martinelli integral operator $\mathcal{B}_{\alpha,\beta}$ with $\alpha, \beta \in \{0, 1, \dots, n\}$, introduced in Definition 7.3.1, fits into the blueprint for producing integral operators of double layer type presented in (1.4.36). Specifically, from (7.3.15)-(7.3.16) we know that the complex Laplacian $\square := -\frac{1}{2}\Delta$ in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ may be factored as

$$\square = \widetilde{D}D \text{ with } \widetilde{D} := D := (\bar{\partial} + \vartheta). \tag{1.4.193}$$

and from (7.3.20)-(7.3.21) we know that the double form $\Gamma_{\alpha,\beta}(\zeta, z)$ of type $\big((\alpha, \beta), (\beta, \alpha)\big)$ defined in (7.3.19) is a (suitably normalized) fundamental solution for $\square$. For further reference, observe that

$$\widetilde{D}^{\top}_{\zeta}\Gamma_{\alpha,\beta}(\zeta, z) = \bar{\partial}_{\zeta}\Gamma_{\alpha,\beta}(\zeta, z) + \vartheta_{\zeta}\Gamma_{\alpha,\beta}(\zeta, z). \tag{1.4.194}$$

To proceed, fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^{2n} \equiv \mathbb{C}^n$, having locally finite perimeter. Denote by ν its geometric measure theoretic outward unit normal, and abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. In this context, pick a differential form

$$f \in L^1\Big(\partial_*\Omega, \tfrac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\Big) \otimes \Lambda^{\alpha,\beta}. \tag{1.4.195}$$

Then from (7.2.3) and (7.2.16) we know that, at σ-a.e. $\zeta \in \partial_*\Omega$,

$$\mathrm{Sym}(D; \nu(\zeta))f(\zeta) = \frac{\mathrm{i}}{2}\,\nu(\zeta)^{0,1} \wedge f(\zeta) - \frac{\mathrm{i}}{2}\,\nu(\zeta)^{1,0} \vee f(\zeta). \tag{1.4.196}$$

On account of (1.4.194), (1.4.196), and simple degree considerations, the general recipe in (1.4.36) presently yields the double layer operator acting on f from (1.4.195) according to

$$\begin{aligned}\mathcal{D}f(z) = &-\frac{1}{2}\int_{\partial_*\Omega}\big\langle \nu^{0,1}(\zeta) \wedge f(\zeta), \bar{\partial}_{\zeta}\Gamma_{\alpha,\beta}(\zeta, z)\big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\ &+\frac{1}{2}\int_{\partial_*\Omega}\big\langle \nu^{1,0}(\zeta) \vee f(\zeta), \vartheta_{\zeta}\Gamma_{\alpha,\beta}(\zeta, z)\big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta), \quad \forall z \in \mathring{\Omega},\end{aligned} \tag{1.4.197}$$

where the Hermitian inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ is taken in the sense of (7.3.6).

The higher-degree Bochner-Martinelli integral operator $\mathcal{B}_{\alpha,\beta}$ corresponds precisely to the first line of (1.4.197) (cf. Definition 7.3.1). In this vein, observe that under the additional assumption that

$$\begin{gathered} f \text{ is complex tangential (cf. Definition 7.1.7),} \\ \text{i.e., } \nu(\zeta)^{1,0} \vee f(\zeta) = 0 \text{ at } \sigma\text{-a.e. } \zeta \in \partial_*\Omega, \end{gathered} \tag{1.4.198}$$

the integral in the second line of (1.4.197) drops out. The bottom line is that

> the higher-degree Bochner-Martinelli integral operator $\mathcal{B}_{\alpha,\beta}$ from Definition 7.3.1 and the double layer operator $\mathcal{D}$ from (1.4.197), constructed according to the general recipe described in (1.4.36) for the factorization of $L := \Box$ given in (1.4.193), agree whenever acting of differential forms f as in (1.4.195) which also happen to be complex tangential (cf. (1.4.198)). (1.4.199)

Similar considerations apply to the principal-value (boundary-to-boundary) Bochner-Martinelli integral operator $B_{\alpha,\beta}$ from (7.3.68), vis-a-vis to the the principal-value double layer operator K constructed according to the general blueprint described in (1.4.39) for the factorization of the complex Laplacian $L := \Box$ given in (1.4.193).

Example 1.4.19 There are other factorizations of the Laplacian which yield boundary layer potential operators of interest. For example, in the context of differential forms, we have (recall that d, δ stand, respectively, for the exterior derivative operator and its formal transpose)

$$\Delta = \widetilde{D}D \text{ with } \widetilde{D} := \mathrm{i}(\mathrm{d}, \delta) \text{ and } D := \mathrm{i}\begin{pmatrix} \delta \\ \mathrm{d} \end{pmatrix}, \tag{1.4.200}$$

as well as

$$\Delta = \widetilde{D}D \text{ with } \widetilde{D} := D := \mathrm{i}(\mathrm{d} + \delta). \tag{1.4.201}$$

These factorizations of the Hodge-Laplacian $\Delta = -(\mathrm{d}\delta + \delta\mathrm{d})$ lead to boundary layer potentials of the sort discussed in [76] in the context of Riemannian manifolds, generalizing those considered earlier in Example 1.4.11.

Example 1.4.20 Given a second-order $M \times M$ system L in $\mathbb{R}^n$, there are many choices of a coefficient tensor A which allows us to represent L as L_A (the system associated with A as in (1.3.2)), and all these choices correspond to typically different double layer potential operators. In fact, this is the case even for scalar operators (i.e., when $M = 1$). To illustrate this phenomenon, take the basic case when $L = \Delta$, the Laplacian in $\mathbb{R}^n$. For any given antisymmetric matrix $B = (b_{jk})_{1 \le j,k \le n} \in \mathbb{C}^{n \times n}$ we may then write (with the summation convention over repeated indices in effect throughout)

$$\begin{gathered} \Delta = (\delta_{jk} + b_{jk})\partial_j\partial_k, \text{ i.e., } \Delta = L_A \text{ with } (M = 1 \text{ and}) \\ A := (a_{jk})_{1 \le j,k \le n} \in \mathbb{C}^{n \times n} \text{ where } a_{jk} := \delta_{jk} + b_{jk} \text{ for } 1 \le j, k \le n. \end{gathered} \tag{1.4.202}$$

To proceed, fix a Lebesgue measurable nonempty proper subset Ω of $\mathbb{R}^n$, having locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Recall from [70, Convention 1.7.2] that whenever the function u is of class $\mathscr{C}^1$ in a neighborhood of $\partial_*\Omega$, then at σ-a.e. point on $\partial_*\Omega$ we may write

$$\begin{aligned}\partial_\nu^{A^\top} u &= \nu_j a_{kj}(\partial_k u)\big|_{\partial_*\Omega} = \nu_j(\delta_{jk} - b_{jk})(\partial_k u)\big|_{\partial_*\Omega} \\ &= \big\langle \nu, (\nabla u)\big|_{\partial_*\Omega}\big\rangle - \nu_j b_{jk}(\partial_k u)\big|_{\partial_*\Omega} \\ &= \big\langle \nu, (\nabla u)\big|_{\partial_*\Omega}\big\rangle - \tfrac{1}{2} b_{jk}\Big\{\nu_j(\partial_k u)\big|_{\partial_*\Omega} - \nu_k(\partial_j u)\big|_{\partial_*\Omega}\Big\}. \end{aligned} \tag{1.4.203}$$

In view of this, if E_Δ is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$ (cf. (A.0.65)), then the double layer potential operators $\mathcal{D}$, K associated with A and Ω as in (1.3.18) and (1.3.68), respectively, act on functions $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ according to

$$\begin{aligned}\mathcal{D}f(x) &= \int_{\partial_*\Omega} \partial_{\nu(y)}^{A^\top}\big[E_\Delta(x-y)\big] f(y)\,\mathrm{d}\sigma(y) \\ &= \int_{\partial_*\Omega} \big\langle \nu(y), \nabla_y[E_\Delta(x-y)]\big\rangle f(y)\,\mathrm{d}\sigma(y) \\ &\quad - \frac{b_{jk}}{2}\int_{\partial_*\Omega} \partial_{\tau_{jk}(y)}\big[E_\Delta(x-y)\big] f(y)\,\mathrm{d}\sigma(y) \\ &= \frac{1}{\omega_{n-1}}\int_{\partial_*\Omega} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y) \\ &\quad - \frac{b_{jk}}{2\omega_{n-1}}\int_{\partial_*\Omega} \frac{\nu_j(y)(y_k-x_k) - \nu_k(y)(y_j-x_j)}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y) \end{aligned} \tag{1.4.204}$$

at every $x \in \mathring{\Omega}$, and

$$\begin{aligned}(Kf)(x) &= \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y) \\ &\quad - \frac{b_{jk}}{2\omega_{n-1}} \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \frac{\nu_j(y)(y_k-x_k) - \nu_k(y)(y_j-x_j)}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y), \end{aligned} \tag{1.4.205}$$

at σ-a.e. point $x \in \partial_*\Omega$. The existence of the limit in (1.4.205) is guaranteed by [68, Proposition 5.6.7]. Also, if Ω is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure, then the operator $K^\#$ associated with A and Ω as in (1.3.72) acts on functions $f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ at σ-a.e. point $x \in \partial_*\Omega$ according to

$$
\begin{aligned}
(K^{\#} f)(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(x), x-y\rangle}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y) \\
- \frac{b_{jk}}{2\omega_{n-1}} \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\nu_j(x)(x_k-y_k)-\nu_k(x)(x_j-y_j)}{|x-y|^n} f(y)\,\mathrm{d}\sigma(y),
\end{aligned}
\tag{1.4.206}
$$

with the existence of the limit once again ensured by [68, Proposition 5.6.7]. Ultimately, the conclusion is that

as is visible from (1.4.203)-(1.4.206), the actual choice of the coefficient matrix A in the writing of the Laplacian in (1.4.202) directly affects the format of the conormal derivative as well as the boundary layer potentials $\mathcal{D}$, K, and $K^{\#}$ associated with A. (1.4.207)

It is also of significance to note that, as the above discussion indicates, even in the case of the (scalar) Laplacian there exist infinitely many conormal derivatives, hence infinitely many Neumann Problems. For example, if Ω is an Ahlfors regular domain, $\kappa > 0$ is a fixed background aperture parameter, and $p \in (1,\infty)$ is a given integrability exponent, then each boundary value problem of the form

$$
\begin{cases}
u \in \mathscr{C}^\infty(\Omega), \quad \Delta u = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma), \\
\Big\langle \nu, (\nabla u)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \Big\rangle + \tfrac{1}{2} b_{jk} \Big\{ \nu_j (\partial_k u)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - \nu_k (\partial_j u)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \Big\} = f \in L^p(\partial\Omega,\sigma),
\end{cases}
\tag{1.4.208}
$$

is a legitimate Neumann Problem for the Laplacian in Ω, corresponding to the conormal derivative ∂_ν^A for the coefficient matrix A as in (1.4.202).

Of course, the same type of phenomenon occurs for a general second-order homogeneous weakly elliptic system L in $\mathbb{R}^n$. Specifically, if $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n \\ 1\le\alpha,\beta\le M}}$ is a coefficient tensor allowing us to represent $L = L_A$ as in (1.3.2), then for any other coefficient tensor

$$
\begin{aligned}
& B = \big(b_{rs}^{\alpha\beta}\big)_{\substack{1\le r,s\le n \\ 1\le\alpha,\beta\le M}} \text{ antisymmetric in the lower indices, i.e., satisfying} \\
& \quad b_{rs}^{\alpha\beta} = -b_{sr}^{\alpha\beta} \text{ for each } r,s \in \{1,\dots,n\} \text{ and } \alpha,\beta \in \{1,\dots,M\}
\end{aligned}
\tag{1.4.209}
$$

we have $L = L_{A+B}$ and yet the conormal derivative ∂_ν^A as well as the boundary layer potentials $\mathcal{D}_A$, K_A, $K_A^{\#}$ associated with A are typically different from their counterparts ∂_ν^{A+B}, $\mathcal{D}_{A+B}$, K_{A+B}, $K_{A+B}^{\#}$ associated with $A+B$. In view of this, it is remarkable that certain combinations involving the aforementioned operators are unaffected when the underlying coefficient tensor changes. This is made precise in the proposition below.

Proposition 1.4.21 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \geq 3$, be an open set with a lower Ahlfors regular boundary, and with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Denote by ν the geometric measure theoretic outward unit normal to Ω. Let L be a homogeneous, weakly elliptic, second-order $M \times M$ system in $\mathbb{R}^n$, with complex constant coefficients, and pick a coefficient tensor A with the property that $L = L_A$. Recall the single layer potential operator $\mathscr{S}$ from (1.3.6). Also, let $\mathcal{D}_A$ and ∂_ν^A be, respectively, the double layer potential operator and conormal derivative associated with A and Ω as in (1.3.18) and, respectively, (A.0.184). Finally, fix an aperture parameter $\kappa > 0$ and suppose $u \in \big[W^{1,1}_{\rm loc}(\Omega)\big]^M$ is a vector-valued function satisfying*

$$\begin{gathered}\textit{the nontangential pointwise limits}\\ u\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega},\ (\nabla u)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\ \textit{exist } \sigma\textit{-a.e. on } \partial_{\rm nta}\Omega, \textit{ and}\\ \int_{\partial\Omega}\frac{(\mathcal{N}_\kappa u)(y)}{1+|y|^{n-1}}\,d\sigma(y) < \infty \textit{ and } \int_{\partial\Omega}\frac{\big(\mathcal{N}_\kappa(\nabla u)\big)(y)}{1+|y|^{n-2}}\,d\sigma(y) < \infty.\end{gathered} \tag{1.4.210}$$

Then for any other coefficient tensor $B = \big(b^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ which is antisymmetric in the lower indices (cf. (1.4.209)) one has

$$\mathcal{D}_A\Big(u\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) - \mathscr{S}(\partial_\nu^A u) = \mathcal{D}_{A+B}\Big(u\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) - \mathscr{S}(\partial_\nu^{A+B} u) \textit{ in } \Omega. \tag{1.4.211}$$

Furthermore, a similar result is valid when $n = 2$ provided $\partial\Omega$ is compact.

Proof Denote by $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2] and write $(u_\beta)_{1\leq\beta\leq M}$ for the scalar components of the vector-valued function u. Also, fix an arbitrary index $\gamma \in \{1,\dots,M\}$, and pick an arbitrary point $x \in \Omega$. Assume first that either Ω is bounded, or $\partial\Omega$ is unbounded. In such a scenario, consider the vector field $\vec{F}_x = \big(F_s\big)_{1\leq s\leq n}$ with components (throughout, the summation convention over repeated indices is in effect)

$$F_s := -b^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-\cdot)u_\alpha - E_{\gamma\alpha}(x-\cdot)b^{\alpha\beta}_{sr}\partial_r u_\beta \tag{1.4.212}$$

defined at $\mathcal{L}^n$-a.e. point in Ω. Reasoning as in the proof of [70, Theorem 1.5.1] (while keeping in mind the antisymmetry property (1.4.209)) we see that

$$\begin{gathered}\vec{F}_x \in \big[L^1_{\rm loc}(\Omega,\mathcal{L}^n)\big]^n,\quad {\rm div}\vec{F}_x = 0 \text{ in } \mathcal{D}'(\Omega),\\ \mathcal{N}^{\Omega\setminus K}_\kappa \vec{F}_x \in L^1(\partial\Omega,\sigma) \text{ if } K := \overline{B\big(x, \tfrac{1}{2}\,{\rm dist}(x,\partial\Omega)\big)},\\ \text{and the trace } \vec{F}_x\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial_{\rm nta}\Omega.\end{gathered} \tag{1.4.213}$$

In fact, on account of (1.4.212), (1.3.18), (1.3.6), and (A.0.184), we have

$$\Big(\mathcal{D}_{A+B}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\Big)_\gamma(x)-\Big(\mathcal{D}_{A}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\Big)_\gamma(x) \tag{1.4.214}$$

$$-\big(\mathcal{S}(\partial_\nu^{A+B}u)\big)_\gamma(x)+\big(\mathcal{S}(\partial_\nu^{A}u)\big)_\gamma(x)=\int_{\partial_*\Omega}\nu\cdot\big(\vec{F}_x\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\,d\sigma.$$

Granted (1.4.213)-(1.4.214), we may invoke [68, Theorem 1.4.1] and conclude (based on Divergence Formula [68, (1.4.6)], as well as the arbitrariness of the index $\gamma\in\{1,\dots,M\}$ and the point $x\in\Omega$) that identity (1.4.211) holds when either Ω is bounded, or $\partial\Omega$ is unbounded. In the case when Ω is an exterior domain (a scenario in which $\partial\Omega$ is bounded) we run the same argument as before but, this time, replace u by ψu where $\psi\in\mathscr{C}^\infty_c(\mathbb{R}^n)$ is a function satisfying $\psi\equiv 1$ near $\partial\Omega$. Finally, the argument in the two-dimensional case is similar. □

1.5 Calderón-Zygmund Function Theory for Boundary Layer Potentials

While certain features of boundary layer potentials are visible straight from invoking generic results valid in the general setting of spaces of homogeneous type, other, more delicate (typically cancelation and/or differential calculus sensitive) properties require fully employing the resourcefulness of the algebraic/geometric ambient. For example, while properties such as the fact that the boundary-to-boundary single layer S maps L^p into L^{p^*} if $1<p<n-1$ and $p^*:=\big(\frac{1}{p}-\frac{1}{n-1}\big)^{-1}$ is directly implied by the Fractional Integration Theorem (which is valid in general spaces of homogeneous type). This being said, showing that S has a regularizing/smoothing effect of order one requires tools from differential calculus. Similar issues arise in connection to double layer potential operators on Lebesgue and Sobolev spaces.

In this section we shall focus precisely on such aspects. Our first theorem elaborates on the rich Calderón-Zygmund theory which may be developed in relation to the boundary layer potential operators introduced earlier, for a given weakly elliptic second-order system and an open set with a UR boundary. For related results for boundary layer potential operators associated with *higher-order* weakly elliptic systems the reader is referred to the work in [78] in the class of Lipschitz domains, and [40] in the class of uniformly rectifiable domains.

Theorem 1.5.1 *Assume $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\geq 2$) is an open set with the property that $\partial\Omega$ is a* UR *set. Abbreviate $\sigma_*:=\mathcal{H}^{n-1}\lfloor\partial_*\Omega$ and $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu=(\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, for some $M\in\mathbb{N}$, consider a coefficient tensor $A=\big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with complex entries, with the property that the $M\times M$ homogeneous second-order system $L=L_A$ associated with A in $\mathbb{R}^n$ as in* (1.3.2) *is weakly elliptic* (*in the sense of* [70, (1.3.3) in Definition 1.3.1]). *Finally, consider the boundary layer potentials $\mathcal{S}$, S, $\mathcal{D}$, K, $K^\#$ associated with A and Ω as in* (1.3.6), (1.3.62), (1.3.18), (1.3.68), *and* (1.3.72), *respectively.*

Then, in relation to these operators, the following properties hold.

(i) For each $p \in [1, \infty)$ and $\kappa > 0$ there exists a finite constant $C > 0$ with the property that for every $f \in \big[L^p(\partial_*\Omega, \sigma_*)\big]^M$ one has

$$\big\|\mathcal{N}_\kappa(\mathcal{D}f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[L^p(\partial_*\Omega,\sigma_*)]^M} \ \text{ if } 1 < p < \infty, \tag{1.5.1}$$

$$\big\|\mathcal{N}_\kappa(\mathcal{D}f)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \leq C\|f\|_{[L^1(\partial_*\Omega,\sigma_*)]^M} \ \text{ if } p = 1. \tag{1.5.2}$$

Also, for each exponent $p \in (1, \infty)$, each Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$, and each aperture parameter $\kappa > 0$, there exists a constant $C \in (0, \infty)$ with the property that for every function $f \in \big[L^p(\partial_*\Omega, w\sigma_*)\big]^M$ one has

$$\big\|\mathcal{N}_\kappa(\mathcal{D}f)\big\|_{L^p(\partial\Omega,w\sigma)} \leq C\|f\|_{[L^p(\partial_*\Omega,w\sigma_*)]^M}. \tag{1.5.3}$$

Furthermore, given any $p \in (1, \infty)$, $q \in (0, \infty]$, and $\kappa > 0$, there exists a finite constant $C > 0$ with the property that for every $f \in \big[L^{p,q}(\partial_*\Omega, \sigma_*)\big]^M$ one has

$$\big\|\mathcal{N}_\kappa(\mathcal{D}f)\big\|_{L^{p,q}(\partial\Omega,\sigma)} \leq C\|f\|_{[L^{p,q}(\partial_*\Omega,\sigma_*)]^M}. \tag{1.5.4}$$

Also, in view of work in [68, §8.5]*, similar estimates to* (1.5.1)-(1.5.4) *are true with the nontangential maximal operator replaced by the tangential maximal operator* (*associated as in* [68, Definition 8.5.1] *with a sufficiently large power; cf.* (A.0.146)).

In addition, whenever $1 < p < \infty$ and $p \leq q \leq \infty$, there exists a constant $C \in (0, \infty)$ such that for each $f \in \big[L^p(\partial_*\Omega, \sigma_*)\big]^M$ one has

$$\Big\|\delta_{\partial\Omega}^{1+\frac{n-1}{p}-\frac{n}{q}}\big|\nabla(\mathcal{D}f)\big|\Big\|_{L^q(\Omega,\mathcal{L}^n)} \leq C\|f\|_{[L^p(\partial_*\Omega,\sigma_*)]^M}. \tag{1.5.5}$$

Finally, if one also imposes the condition that $\mathcal{H}^{n-1}\big(\partial\Omega\setminus\partial_*\Omega\big) = 0$ then, as noted earlier in [69, (10.2.253)-(10.2.254)], there exists some constant $C \in (0, \infty)$ with the property that for each function $f \in \big[L^p(\partial\Omega, \sigma)\cap\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M$ with $p \in (1, \infty)$ and $\eta \in (0, 1)$ one has[14]

$$\sup_{x\in\Omega}\big|(\mathcal{D}f)(x)\big| \leq C\|f\|_{[H^{1,\infty}(\partial\Omega,\sigma)^*]^M}. \tag{1.5.6}$$

(ii) For each function f belonging to the weighted boundary Sobolev space $\big[L^1_1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big)\big]^M$ (cf. (A.0.131)*), each aperture parameter $\kappa \in (0, \infty)$, and each index $\ell \in \{1, \dots, n\}$, the pointwise nontangential boundary trace*

$$\big(\partial_\ell\mathcal{D}f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists (in } \mathbb{C}^M\text{) at } \sigma_*\text{-a.e. point on } \partial_*\Omega. \tag{1.5.7}$$

---

[14] informally, this may be thought as the limiting case $p = \infty$ of (1.5.1)

As a corollary of [69, (11.7.22)], *this is true whenever* $f \in \big[L_1^{p,q}(\partial_*\Omega, \sigma_*)\big]^M$ *with* $p, q \in [1, \infty)$.
Moreover, for each $p, q \in [1, \infty)$ *and* $\kappa > 0$ *there exists some finite constant* $C > 0$, *depending only on* $\partial\Omega$, A, n, κ, p, q, *such that for each function* $f \in \big[L_1^{p,q}(\partial_*\Omega, \sigma_*)\big]^M$ *one has*

$$\big\|\mathcal{N}_\kappa(\mathcal{D}f)\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\mathcal{D}f)\big\|_{L^q(\partial\Omega,\sigma)} \leq C\|f\|_{[L_1^{p,q}(\partial_*\Omega,\sigma_*)]^M}$$
$$\text{if } \ p \in (1,\infty) \ \text{ and } \ q \in (1,\infty), \tag{1.5.8}$$

plus similar estimates in the case when either $p = 1$ *or* $q = 1$, *this time with the corresponding* L^1*-norm in the left side replaced by the weak-*L^1 (*quasi-*)*norm. In addition,*

if $p \in (1, \infty)$ *and the weight* w *belongs to* $A_p(\partial\Omega, \sigma)$, *then there exists some constant* $C \in (0, \infty)$ *with the property that for each* f *in the weighted Sobolev space* $\big[L_1^p(\partial_*\Omega, w\sigma_*)\big]^M$ *the following estimate holds*
$$\big\|\mathcal{N}_\kappa(\mathcal{D}f)\big\|_{L^p(\partial\Omega,w\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\mathcal{D}f)\big\|_{L^p(\partial\Omega,w\sigma)} \leq C\|f\|_{[L_1^p(\partial_*\Omega,w\sigma_*)]^M}. \tag{1.5.9}$$

In addition, in view of work in [68, §8.5], *similar estimates to* (1.5.8)-(1.5.9) *are valid with the nontangential maximal operator replaced by the tangential maximal operator* (*associated as in* [68, Definition 8.5.1] *with a sufficiently large power; cf.* (A.0.146)).
Finally, if Ω *is also a uniform domain, then for each vector-valued function* $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[L_1^{p,q}(\partial_*\Omega, \sigma_*)\big]^M$ *with* $p \in (1, \infty)$ *and* $q \in (n-1, \infty)$ *one has*

$$\mathcal{D}f \in \big[\dot{\mathscr{C}}^\eta(\overline{\Omega})\big]^M \ \text{ where } \ \eta := 1 - (n-1)/q \in (0, 1), \tag{1.5.10}$$

and there exists a constant $C = C(\Omega, A, q) \in (0, \infty)$ *with the property that*

$$\begin{aligned}\big\|\mathcal{D}f\big\|_{[\dot{\mathscr{C}}^\eta(\overline{\Omega})]^M} &\leq C\sum_{\alpha=1}^{M}\sum_{j,k=1}^{n}\big\|\partial_{\tau_{jk}} f_\alpha\big\|_{L^q(\partial_*\Omega,\sigma_*)} \\ &\leq C\|f\|_{[L_1^{p,q}(\partial_*\Omega,\sigma_*)]^M}.\end{aligned} \tag{1.5.11}$$

(iii) Fix $p, p' \in (1, \infty)$ *with* $1/p + 1/p' = 1$. *Then the operator*

$$K^\# : \big[L^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^p(\partial_*\Omega, \sigma_*)\big]^M \tag{1.5.12}$$

is well defined, linear, and bounded. Moreover, corresponding to $p = 1$, (1.3.72) *induces a well-defined, linear, and bounded operator*

$$K^\# : \big[L^1(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{1,\infty}(\partial_*\Omega, \sigma_*)\big]^M, \tag{1.5.13}$$

and the transpose of $K^\#$ *in* (1.5.12) *is*

$$K : \big[L^{p'}(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[L^{p'}(\partial\Omega, \sigma)\big]^M. \tag{1.5.14}$$

Corresponding to $p' = 1$, (1.5.14) *induces a well-defined, linear, and bounded mapping*

$$K : \big[L^{1}(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[L^{1,\infty}(\partial\Omega, \sigma)\big]^M. \tag{1.5.15}$$

In fact, similar results are valid for the operators $K, K^\#$ *acting on Lorentz spaces and Muckenhoupt weighted Lebesgue spaces. Specifically, for each given* $p \in (1,\infty)$, $q \in (0,\infty]$, *and* $w \in A_p(\partial\Omega,\sigma)$, *the operators*

$$K : \big[L^{p,q}(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[L^{p,q}(\partial\Omega, \sigma)\big]^M, \tag{1.5.16}$$

$$K^\# : \big[L^{p,q}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{p,q}(\partial_*\Omega, \sigma_*)\big]^M, \tag{1.5.17}$$

$$K : \big[L^{p}(\partial_*\Omega, w\sigma_*)\big]^M \longrightarrow \big[L^{p}(\partial\Omega, w\sigma)\big]^M, \tag{1.5.18}$$

$$K^\# : \big[L^{p}(\partial\Omega, w\sigma)\big]^M \longrightarrow \big[L^{p}(\partial_*\Omega, w\sigma_*)\big]^M, \tag{1.5.19}$$

are well-defined, linear, and bounded. Finally, if $q > 1$ *and* $p', q' \in (1,\infty)$ *are such that* $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$, *then the (real) transpose of the operator* K *in* (1.5.16) *is the operator* $K^\#$ *in* (1.5.17) *with* p, q *replaced by* p', q'. *Also, the (real) transpose of the operator* K *in* (1.5.18) *is the operator* $K^\#$ *in* (1.5.19) *with* p *replaced by* p' *and* w *replaced by* $w^{1-p'}$.

(iv) *Having fixed some arbitrary aperture parameter* $\kappa \in (0,\infty)$, *for each given vector-valued function* $f \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\Big)\Big]^M$ *the following nontangential boundary trace formula holds:*

$$\mathcal{D} f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\tfrac{1}{2}I + K\big) f \ \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega, \tag{1.5.20}$$

where I *is the identity operator. In particular, the jump-formula* (1.5.20) *is valid for each function* f *in the following function spaces:*

- $f \in \big[L^p(\partial_*\Omega, \sigma_*)\big]^M$ *with* $p \in [1,\infty)$ (*cf.* [68, (7.7.106)]);
- $f \in \big[L^{p,q}(\partial_*\Omega, \sigma_*)\big]^M$ *with* $p \in (1,\infty)$ *and* $q \in (0,\infty]$ (*cf.* [68, (7.7.107)]);
- $f \in \big[L^p(\partial_*\Omega, w\sigma_*)\big]^M$ *with* $p \in (1,\infty)$ *and the weight* $w \in A_p(\partial\Omega,\sigma)$ (*cf.* [68, (7.7.104)]);
- $f \in \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ *with* $p \in (1,\infty)$ *and* $\lambda \in (0, n-1)$ (*cf.* [69, (6.2.25)]).

Moreover, as a consequence of Proposition 1.3.6 and (1.5.20) (*or, alternatively,* (1.3.50) *and* [70, (2.3.15)]), *it follows that* *if Ω has a compact boundary, then for each $\lambda \in \mathbb{C}^M$ one has*

$$K\lambda = \begin{cases} +\frac{1}{2}\lambda \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega, \text{ if } \Omega \text{ is bounded,} \\ -\frac{1}{2}\lambda \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega, \text{ if } \Omega \text{ is unbounded.} \end{cases} \tag{1.5.21}$$

Finally, from (1.3.44)*,* (1.5.20)*,* (1.3.67)*, and [68, Proposition 8.8.6] one sees that for each point $x_o \in \Omega$ and for each index $\beta \in \{1, \dots, M\}$ one has*

$$\big(\tfrac{1}{2}I + K\big)\Big(E_{\bullet\beta}(\cdot - x_o)\big|_{\partial\Omega}\Big) = S\Big(\partial_\nu^A\big[E_{\bullet\beta}(\cdot - x_o)\big]\Big) \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{1.5.22}$$

(v) Under the additional assumption that

$$\mathcal{H}^{n-1}\big(\partial_{\text{nta}}\Omega \setminus \partial_*\Omega\big) = 0, \tag{1.5.23}$$

(*which is automatically satisfied if, e.g., Ω is a* UR *domain to begin with*) *it follows that the operator*

$$K : \big[L_1^p(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[L_1^p(\partial_*\Omega, \sigma_*)\big]^M \tag{1.5.24}$$

is well defined, linear, and bounded for each $p \in (1, \infty)$. More generally, if (1.5.23) *is satisfied then the operator*

$$K : \big[L_1^{p,q}(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[L_1^{p,q}(\partial_*\Omega, \sigma_*)\big]^M \tag{1.5.25}$$

is well defined, linear, and bounded for each $p, q \in (1, \infty)$, and

$$K : \big[L_1^p(\partial_*\Omega, w\sigma_*)\big]^M \longrightarrow \big[L_1^p(\partial_*\Omega, w\sigma_*)\big]^M \tag{1.5.26}$$

is well defined, linear, and bounded for each $p \in (1, \infty)$ and $w \in A_p(\partial\Omega, \sigma)$.

(vi) Continue to retain the additional assumption made in (1.5.23)*. Then for each $p, q \in (1, \infty)$ it follows that $K^\#$, originally acting on functions from $\big[L^p(\partial_*\Omega, \sigma)\big]^M$* (*regarding them as being extended by zero to the entire topological boundary $\partial\Omega$, and then applying $K^\#$ in the sense of* (1.5.12))*, further extends uniquely to a linear and bounded operator from the negative boundary Sobolev space $\big[L_{-1}^p(\partial_*\Omega, \sigma_*)\big]^M$ into itself and, more generally, from the off-diagonal negative Sobolev space $\big[L_{-1}^{p,q}(\partial_*\Omega, \sigma_*)\big]^M$ into itself. Furthermore, if one adopts the same notation $K^\#$ for said extensions, then the transpose of* (1.5.24) *is*

$$K^\# : \big[L_{-1}^{p'}(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[L_{-1}^{p'}(\partial_*\Omega, \sigma_*)\big]^M, \tag{1.5.27}$$

while the transpose of (1.5.25) *is*

$$K^\# : \big[L_{-1}^{p',q'}(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[L_{-1}^{p'q'}(\partial_*\Omega, \sigma_*)\big]^M, \tag{1.5.28}$$

where $p', q' \in (1, \infty)$ are such that $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

(vii) *Temporarily strengthen the original hypotheses by assuming that Ω is actually a* UR *domain. Then given any f belonging to the weighted boundary Sobolev space $\big[L^1_1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$, at σ-a.e. point $x \in \partial\Omega$ one has*

$$\big(\partial^A_\nu(\mathcal{D}f)\big)(x) \tag{1.5.29}$$
$$= \Bigg(\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y)\big(\partial_{\tau_{js}} f_\alpha\big)(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\le\mu\le M}$$

where the conormal derivative $\partial^A_\nu(\mathcal{D}f)$ is considered as in (A.0.184)*. As a corollary of* [69, (11.7.22)]*, this is true whenever $f \in \big[L^{p,q}_1(\partial\Omega,\sigma)\big]^M$ with $p, q \in [1, \infty)$.*

Hence, in such a setting, the conormal derivative of the double layer induces an operator

$$f \mapsto \big(\partial^A_\nu \mathcal{D}\big) f := \partial^A_\nu(\mathcal{D}f) \tag{1.5.30}$$

which for any given exponents $p, q \in (1, \infty)$ is well defined, linear, and bounded in the context

$$\partial^A_\nu \mathcal{D} : \big[L^{p,q}_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^q(\partial\Omega,\sigma)\big]^M. \tag{1.5.31}$$

Finally, it is apparent from (1.5.29) *that $(\partial^A_\nu \mathcal{D})f$ does not jump across the boundary* (*in the sense that it has the same nontangential boundary trace when considered from $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$*).

(viii) *Continue to assume that Ω is actually a* UR *domain. Also, pick exponents $p, p', q, q' \in (1, \infty)$ satisfying $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Then for any $f \in \big[L^{p,q}_1(\partial\Omega,\sigma)\big]^M$ and $g \in \big[L^{q',p'}_1(\partial\Omega,\sigma)\big]^M$ one has*

$$\int_{\partial\Omega} \big\langle \big(\partial^A_\nu \mathcal{D}\big) f, g \big\rangle \,\mathrm{d}\sigma = \int_{\partial\Omega} \big\langle f, \big(\partial^{A^\top}_\nu \mathcal{D}_{A^\top}\big) g \big\rangle \,\mathrm{d}\sigma \tag{1.5.32}$$

where $\partial^{A^\top}_\nu$ along with $\mathcal{D}_{A^\top}$ and, ultimately $\partial^{A^\top}_\nu \mathcal{D}_{A^\top}$, are defined as before with A now replaced by $A^\top$. As a consequence, whenever Ω is a UR *domain, the operator* (1.5.31) *has a unique extension to a well-defined, linear, and bounded mapping*

$$\partial^A_\nu \mathcal{D} : \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{q,p}_{-1}(\partial\Omega,\sigma)\big]^M, \tag{1.5.33}$$

namely the (*real*) *transpose of the operator*

$$\partial^{A^\top}_\nu \mathcal{D}_{A^\top} : \big[L^{q',p'}_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{p'}(\partial\Omega,\sigma)\big]^M. \tag{1.5.34}$$

(ix) *Suppose* $p \in (1, n-1)$ *and* $p^* := \big(1/p - 1/(n-1)\big)^{-1}$. *Then the boundary-to-boundary single layer potential operator* S *induces well-defined, linear, and bounded mappings*

$$S : \big[L^p(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[L_1^{p^*,p}(\partial_*\Omega, \sigma_*)\big]^M, \tag{1.5.35}$$

$$S : \big[L_{-1}^{p,p^*}(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[L^{p^*}(\partial_*\Omega, \sigma_*)\big]^M, \tag{1.5.36}$$

which act in a compatible fashion with one another. Also, if prime is used to indicate the Hölder conjugate exponent, the (*real*) *transpose of* (1.5.35) *is the operator*

$$S_{A^\top} : \big[L_{-1}^{(p^*)',p'}(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[L^{p'}(\partial_*\Omega, \sigma_*)\big]^M, \tag{1.5.37}$$

where $S_{A^\top}$ *is associated with the coefficient tensor* $A^\top$ (*the real transpose of* A) *in the same manner* S *has been associated with the original* A.
In addition, the operator S *from* (1.5.36) *further extends to a linear and bounded mapping*

$$\begin{gathered} S : \big[L_{-1}^{p,q}(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[L^{p^*}(\partial_*\Omega, \sigma_*)\big]^M + \big[L^q(\partial_*\Omega, \sigma_*)\big]^M \\ \text{for each given integrability exponent } q \in (1, \infty). \end{gathered} \tag{1.5.38}$$

Moreover, corresponding to $n \geq 3$ *and* $p = n-1$, *the following operator is linear and bounded:*

$$S : \big[L^{n-1,1}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^\infty(\partial\Omega, \sigma)\big]^M. \tag{1.5.39}$$

Finally, if $\partial\Omega$ *is actually bounded then the same results are also true for the exponent* $p \in [n-1, \infty)$, *this time regarding* p^* *as an arbitrary index in* $(1, \infty)$ (*unrelated to* p).

(x) *Given* $p \in (1, n-1)$ *along with* $q \in (1, \infty)$, *extend the action of the boundary-to-domain single layer* $\mathscr{S}$, *originally defined in* (1.3.6), *to the off-diagonal negative Sobolev space* $\big[L_{-1}^{p,q}(\partial_*\Omega, \sigma_*)\big]^M$ *by setting* (*again, with prime used to indicate Hölder conjugation*)

$$\mathscr{S} f(x) := {}_{[L_1^{p',q'}(\partial_*\Omega,\sigma_*)]^M}\Big\langle E(x-\cdot)\big|_{\partial_*\Omega}, f\Big\rangle_{[L_{-1}^{p,q}(\partial_*\Omega,\sigma_*)]^M} \quad \forall x \in \Omega, \tag{1.5.40}$$

for each $f \in \big[L_{-1}^{p,q}(\partial_*\Omega, \sigma_*)\big]^M$. *Then this is meaningfully defined and agrees with* $\mathscr{S}$ *from* (1.3.6) *when acting on the smaller space* $\big[L^p(\partial_*\Omega, \sigma_*)\big]^M$. *Moreover,*

$$\mathscr{S} : \big[L_{-1}^{p,q}(\partial_*\Omega, \sigma_*)\big]^M \longrightarrow \big[\mathscr{C}^\infty(\Omega)\big]^M \tag{1.5.41}$$

is continuous (*when the space on the right is equipped with the Frechét topology of uniform convergence of partial derivatives of any order on compact sets*), *and*

$$\partial^\gamma\big(\mathscr{S}f\big)(x) = {}_{[L_1^{p',q'}(\partial_*\Omega,\sigma_*)]^M}\Big\langle (\partial^\gamma E)(x-\cdot)\big|_{\partial_*\Omega}, f\Big\rangle_{[L_{-1}^{p,q}(\partial_*\Omega,\sigma_*)]^M} \tag{1.5.42}$$

for each functional $f \in \big[L_{-1}^{p,q}(\partial_*\Omega,\sigma_*)\big]^M$, *each multi-index* $\gamma \in \mathbb{N}_0^n$, *and each point* $x \in \Omega$. *Moreover,*

$$L(\mathscr{S}f) = 0 \text{ in } \Omega, \text{ for each functional } f \in \big[L_{-1}^{p,q}(\partial_*\Omega,\sigma_*)\big]^M. \tag{1.5.43}$$

Also, given any $\kappa > 0$, *for each* $f \in \big[L_{-1}^{p,q}(\partial_*\Omega,\sigma_*)\big]^M$ (*hence, in particular, for each function* $f \in \big[L^p(\partial_*\Omega,\sigma_*)\big]^M$; *cf.* [69, (11.8.30)]) *the nontangential pointwise trace* $\mathscr{S}f\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ *exists at* σ_**-a.e. point on* $\partial_*\Omega$; *in fact, with* Sf *considered in the sense of* (1.5.38) *one has*

$$\mathscr{S}f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = Sf \text{ at } \sigma_*\text{-a.e. point on } \partial_*\Omega. \tag{1.5.44}$$

Furthermore, if $p \in (1, n-1)$ *and* $p^* := \big(1/p - 1/(n-1)\big)^{-1}$, *then for each* $q \in (1,\infty)$ *and* $\kappa > 0$ *one has, in a quantitative fashion,*

$$\begin{gathered}\mathcal{N}_\kappa(\mathscr{S}f) \in L^{p^*}(\partial\Omega,\sigma) + L^q(\partial\Omega,\sigma)\\ \text{for each } \ f \in \big[L_{-1}^{p,q}(\partial_*\Omega,\sigma_*)\big]^M.\end{gathered} \tag{1.5.45}$$

In particular, there exists a finite constant $C = C(\Omega, A, p, \kappa) > 0$ *such that*

$$\begin{gathered}\big\|\mathcal{N}_\kappa(\mathscr{S}f)\big\|_{L^{p^*}(\partial\Omega,\sigma)} \le C\|f\|_{[L_{-1}^{p,p^*}(\partial_*\Omega,\sigma_*)]^M}\\ \text{for each } \ f \in \big[L_{-1}^{p,p^*}(\partial_*\Omega,\sigma_*)\big]^M,\end{gathered} \tag{1.5.46}$$

and, as a consequence of this and [69, (11.8.30)],

$$\begin{gathered}\big\|\mathcal{N}_\kappa(\mathscr{S}f)\big\|_{L^{p^*}(\partial\Omega,\sigma)} \le C\|f\|_{[L^p(\partial_*\Omega,\sigma_*)]^M}\\ \text{for each } \ f \in \big[L^p(\partial_*\Omega,\sigma_*)\big]^M.\end{gathered} \tag{1.5.47}$$

In addition, for each $p \in [1, n-1)$ *and each* $\kappa > 0$ *there exists some constant* $C = C(\Omega, p, \kappa) \in (0,\infty)$ *such that, for each function* $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$,

$$\begin{gathered}\text{the nontangential trace } (\nabla\mathscr{S}f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma_*\text{-a.e. on } \partial_*\Omega,\\ \big\|\mathcal{N}_\kappa(\nabla\mathscr{S}f)\big\|_{L^p(\partial\Omega,\sigma)} \le C\|f\|_{[L^p(\partial\Omega,\sigma)]^M} \text{ if } p \in (1, n-1),\\ \big\|\mathcal{N}_\kappa(\nabla\mathscr{S}f)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \le C\|f\|_{[L^1(\partial\Omega,\sigma)]^M} \text{ provided } p = 1.\end{gathered} \tag{1.5.48}$$

Moreover, corresponding to $p = n-1$,

$$\begin{gathered}\text{if } n \ge 3 \text{ then } \mathscr{S}f \text{ is bounded in } \Omega, \text{ for each function } f\\ \text{belonging to the Lorentz space } \big[L^{n-1,1}(\partial\Omega,\sigma)\big]^M.\end{gathered} \tag{1.5.49}$$

Once again, in view of work in [68, §8.5]*, similar estimates to those in* (1.5.46)-(1.5.48) *are valid with the nontangential maximal operator replaced by the tangential maximal operator* (*associated as in* [68, Definition 8.5.1] *with a sufficiently large power;* (A.0.146)).

Finally, all results in this item are also true for $p \in [n-1, \infty)$*, now regarding* p^* *as an arbitrary index in* $(1, \infty)$ (*unrelated to* p)*, in either of the following cases: (1)* $n \geq 3$ *and* $\partial\Omega$ *is bounded, (2)* $n = 2$ *and* Ω *is bounded, (3)* $n = 2$*,* Ω *is an exterior domain, and the ordinary nontangential maximal operator is truncated.*

(xi) Define the following modified version of the boundary-to-domain single layer operator

$$
\begin{aligned}
&\mathscr{S}_{\rm mod} f(x) := \int_{\partial\Omega} \big\{E(x-y) - E_*(-y)\big\} f(y)\, \mathrm{d}\sigma(y) \ \textit{for each}\ \ x \in \Omega,\\
&\textit{for each}\ \ f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M, \ \ \textit{where}\ \ E_* := E \cdot \mathbf{1}_{\mathbb{R}^n \setminus B(0,1)}.
\end{aligned}
\tag{1.5.50}
$$

Then for each $f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ *the function* $\mathscr{S}_{\rm mod} f$ *is well defined, belongs to the space* $\big[\mathscr{C}^\infty(\Omega)\big]^M$*, and for each multi-index* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \geq 1$ *one has*

$$
\partial^\alpha(\mathscr{S}_{\rm mod} f)(x) = \int_{\partial\Omega} (\partial^\alpha E)(x-y) f(y)\, \mathrm{d}\sigma(y) \ \textit{for each}\ \ x \in \Omega. \tag{1.5.51}
$$

In particular, (1.5.51) *implies that*

$$
L\big(\mathscr{S}_{\rm mod} f\big) = 0 \ \textit{in}\ \Omega \ \ \textit{for each}\ f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M, \tag{1.5.52}
$$

while identity (1.5.51) *and* [70, Theorem 2.5.1] *guarantee that, for each index* $j \in \{1, \dots, n\}$*, each aperture parameter* $\kappa \in (0, \infty)$*, and each function* f *in the space* $\Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$*, the nontangential boundary trace*

$$
\partial_j(\mathscr{S}_{\rm mod} f)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \ \textit{exists at}\ \sigma\textit{-a.e. point on}\ \ \partial_*\Omega. \tag{1.5.53}
$$

In addition, as a consequence of (1.5.51)*,* [70, (2.4.8)]*, and* [70, Theorem 1.4.2]*, for each aperture parameter* $\kappa \in (0, \infty)$ *one has*

$$
\begin{aligned}
&\mathcal{N}_\kappa\big(\partial_j(\mathscr{S}_{\rm mod} f)\big) \in L^p_{\rm loc}(\partial\Omega, \sigma), \quad j \in \{1, \dots, n\},\\
&\textit{for all}\ \ f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\rm loc}(\partial\Omega, \sigma)\Big]^M \ \ \textit{with}\ \ p \in (1, \infty).
\end{aligned}
\tag{1.5.54}
$$

Also, given any $f = (f_\beta)_{1 \leq \beta \leq M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$*, with the piece of notation introduced in* [70, (2.9.53)] *it follows that for each* $r \in \{1, \dots, n\}$ *one has*

$$\partial_r(\mathscr{S}_{\rm mod} f)(x) = \Big((\partial_r \mathscr{S}_{\alpha\beta}) f_\beta(x)\Big)_{1\le\alpha\le M} \quad \text{for each } x \in \Omega. \tag{1.5.55}$$

In fact, if $n \geq 3$ *and* $f \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\big)\Big]^M$, *or if* $n = 2$ *and the function* $f \in \Big[L^1\big(\partial\Omega, \ln(2+|x|)\sigma(x)\big)\Big]^M$, *(hence, in particular, if* $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ *with* $p \in [1, n-1)$*), then there exists a constant* $C_f \in \mathbb{C}^M$ *with the property that*

$$\mathscr{S}_{\rm mod} f = \mathscr{S} f + C_f \text{ at each point in } \Omega. \tag{1.5.56}$$

If actually the function f *belongs to the space* $\big[L^p(\partial\Omega,\sigma)\big]^M$ *for an arbitrary integrability exponent* $p \in (1,\infty)$*, then for each aperture parameter* $\kappa \in (0,\infty)$ *there exists some constant* $C = C(\Omega, A, \kappa, p) \in (0,\infty)$ *with the property that*

$$\begin{aligned}
&\Big\|\operatorname{dist}(\cdot,\partial\Omega)^{(n-1)/p} \cdot \sum_{j=1}^n \big|\partial_j(\mathscr{S}_{\rm mod} f)\big|\Big\|_{L^\infty(\Omega,\mathcal{L}^n)} \\
&\qquad \le C \sum_{j=1}^n \big\|\mathcal{N}_\kappa(\partial_j(\mathscr{S}_{\rm mod} f))\big\|_{L^p(\partial\Omega,\sigma)} \le C\|f\|_{[L^p(\partial\Omega,\sigma)]^M}.
\end{aligned} \tag{1.5.57}$$

Furthermore, for each function $f \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ *(hence, in particular, if* $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ *with* $p \in [1,\infty)$*) the conormal derivative* $\partial_\nu^A \mathscr{S}_{\rm mod} f$ *may be meaningfully considered in the sense of* (A.0.184)*, and one has the jump-formula*

$$\partial_\nu^A \mathscr{S}_{\rm mod} f = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega, \tag{1.5.58}$$

where I *is the identity operator, and* $K^\#_{A^\top}$ *is the operator associated as in* (1.3.72) *with the coefficient tensor* $A^\top$*. Also, if* $n \geq 3$ *and* $f \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\big)\Big]^M$ *(hence, in particular, if* $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ *with* $p \in [1, n-1)$*), or if* $n = 2$ *and* $f \in \Big[L^1\big(\partial\Omega, \ln(2+|x|)\sigma(x)\big)\Big]^M$*, the conormal derivative* $\partial_\nu^A \mathscr{S} f$ *is meaningfully defined in the sense of* (A.0.184) *and satisfies the jump-formula*

$$\partial_\nu^A \mathscr{S} f = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{1.5.59}$$

In addition, for each integrability exponent $p \in (1,\infty)$ *there exists some constant* $C = C(\partial\Omega, A, p) \in (0,\infty)$ *such that*

$$\begin{aligned}
&\text{if } n \geq 3 \text{ and } f \in \big[L^{n-1}(\partial\Omega,\sigma)\big]^M \text{ then } |\nabla \mathscr{S}_{\rm mod} f|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n \\
&\text{is a Carleson measure in } \Omega \text{ with constant} \le C\|f\|^p_{[L^{n-1}(\partial\Omega,\sigma)]^M};
\end{aligned} \tag{1.5.60}$$

in particular, corresponding to $p = 2$, it follows that[15]

$$\text{if } n \geq 3 \text{ and } f \in \big[L^{n-1}(\partial\Omega,\sigma)\big]^M \text{ then } |\nabla \mathscr{S}_{\rm mod} f|^2 \operatorname{dist}(\cdot,\partial\Omega)\, d\mathcal{L}^n \text{ is a Carleson measure in } \Omega \text{ with constant } \leq C\|f\|^2_{[L^{n-1}(\partial\Omega,\sigma)]^M}. \tag{1.5.61}$$

Moreover,

under the additional assumption that Ω is a uniform domain, for each function $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ with $p \in (n-1,\infty)$ it follows that $\mathscr{S}_{\rm mod} f$ belongs to $\big[\dot{\mathscr{C}}^\eta(\overline{\Omega})\big]^M$ where $\eta := 1-(n-1)/p \in (0,1)$ and one has $\big\|\mathscr{S}_{\rm mod} f\big\|_{[\dot{\mathscr{C}}^\eta(\overline{\Omega})]^M} \leq C\|f\|_{[L^p(\partial\Omega,\sigma)]^M}$ for some finite positive constant $C = C(\Omega, A, p)$. (1.5.62)

If $\partial\Omega$ is also bounded, then for each integrability exponent $p \in (1,\infty)$ there exists $C = C(\partial\Omega, A, p) \in (0,\infty)$ with the property that for every $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ one has

$$\sum_{j=1}^{n} \big\|\mathcal{N}_\kappa(\partial_j \mathscr{S} f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[L^p(\partial\Omega,\sigma)]^M} \tag{1.5.63}$$

plus a similar estimate with the nontangential maximal operator replaced by the tangential maximal operator (associated as in [68, Definition 8.5.1] *with a sufficiently large power;* (A.0.146)), *and*

$$|\nabla \mathscr{S} f|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega \text{ with constant } \leq C\|f\|^p_{[L^{n-1}(\partial\Omega,\sigma)]^M} \text{ if } n \geq 3 \text{ and } f \in \big[L^{n-1}(\partial\Omega,\sigma)\big]^M. \tag{1.5.64}$$

In particular, corresponding to $p = 2$,

$$\text{if } n \geq 3 \text{ and } f \in \big[L^{n-1}(\partial\Omega,\sigma)\big]^M \text{ then } |\nabla \mathscr{S} f|^2 \operatorname{dist}(\cdot,\partial\Omega)\, d\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega \text{ with constant } \leq C\|f\|^2_{[L^{n-1}(\partial\Omega,\sigma)]^M}. \tag{1.5.65}$$

Furthermore, if $\partial\Omega$ is also bounded, then with $K^\#_{A^\top}$ denoting the operator (1.3.72) *associated with the coefficient tensor $A^\top$, for each function $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ with $p \in [1,\infty)$ one has*

$$\partial^A_\nu \mathscr{S} f = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{1.5.66}$$

When Ω is also a uniform domain with a compact boundary, then (1.5.62) *may be rephrased in terms of $\mathscr{S}$ simply as the statement that*

[15] it is natural to refer to $\big|\nabla \mathscr{S}_{\rm mod} f\big|^2 \operatorname{dist}(\cdot,\partial\Omega)\, d\mathcal{L}^n$ as the Littlewood-Paley measure associated with f via the operator $\mathscr{S}_{\rm mod}$

$$\mathscr{S} : \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{\mathscr{C}}^\eta(\overline{\Omega})\big]^M \text{ boundedly} \\ \text{if } p \in (n-1,\infty) \text{ and } \eta := 1-(n-1)/p \in (0,1). \tag{1.5.67}$$

If $n \geq 3$ then for each truncation parameter $\varepsilon > 0$ one has

$$\mathcal{N}_\kappa^\varepsilon(\mathscr{S}_{\rm mod} f) \in L^{\frac{n-1}{n-2},\infty}_{\rm loc}(\partial\Omega,\sigma) \text{ for each } f \in \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \tag{1.5.68}$$

and, as a consequence of this and [68, (6.2.36)],

$$\mathcal{N}_\kappa^\varepsilon(\mathscr{S}_{\rm mod} f) \in \bigcap_{0<p<\frac{n-1}{n-2}} L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for each } f \in \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M, \tag{1.5.69}$$

while

$$\text{if } n = 2 \text{ then } \mathcal{N}_\kappa^\varepsilon(\mathscr{S}_{\rm mod} f) \in \textstyle\bigcap_{0<p<\infty} L^p_{\rm loc}(\partial\Omega,\sigma) \\ \text{for each function } f \in \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|}\Big)\Big]^M. \tag{1.5.70}$$

Finally, given any truncation parameter $\varepsilon > 0$ and any integrability exponent $p \in (1,\infty)$, it follows that

$$\text{for each } f \in \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma)\Big]^M \\ \text{on has } \mathcal{N}_\kappa^\varepsilon(\mathscr{S}_{\rm mod} f) \in L^q_{\rm loc}(\partial\Omega,\sigma), \tag{1.5.71}$$

where q is any number in $(1,\infty)$ if $n = 2$, while if $n \geq 3$ then q is given by

$$q := \begin{cases} \big(\frac{1}{p} - \frac{1}{n-1}\big)^{-1} & \text{if } 1 < p < n-1, \\ \text{any number in } (1,\infty) & \text{if } p = n-1, \\ \infty & \text{if } n-1 < p < \infty. \end{cases} \tag{1.5.72}$$

(xii) In analogy with (1.5.50)*, define the following modified version of the boundary-to-boundary single layer operator*

$$S_{\rm mod} f(x) := \int_{\partial\Omega} \big\{E(x-y) - E_*(-y)\big\} f(y)\,{\rm d}\sigma(y) \\ \text{at } \sigma\text{-a.e. } x \in \partial\Omega, \quad \text{for each function} \tag{1.5.73}$$

$$f \in \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M, \text{ where } E_* := E \cdot \mathbf{1}_{\mathbb{R}^n \setminus B(0,1)}.$$

Then this operator is meaningfully defined, via an absolutely convergent integral, at σ-a.e. point in $\partial\Omega$, and

$$S_{\rm mod} : \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \longrightarrow \big[L^1_{\rm loc}(\partial\Omega,\sigma)\big]^M \tag{1.5.74}$$

as well as

$$S_{\text{mod}} : \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\text{loc}}(\partial\Omega, w\sigma)\Big]^M \longrightarrow \big[L^p_{\text{loc}}(\partial\Omega, w\sigma)\big]^M \quad \text{for each weight } w \in A_p(\partial\Omega,\sigma) \text{ with } p \in (1,\infty), \tag{1.5.75}$$

and

$$S_{\text{mod}} : \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1-\varepsilon}}\big)\Big]^M \longrightarrow \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n}}\big)\Big]^M \quad \text{for each } \varepsilon > 0, \tag{1.5.76}$$

are well-defined, linear, and continuous mappings. In addition, the following mapping is well defined, linear, and bounded

$$S_{\text{mod}} : \big[L^p(\partial\Omega, w\sigma)\big]^M \longrightarrow \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n}}\Big)\Big]^M \quad \text{for each weight } w \in A_p(\partial\Omega,\sigma) \text{ with } p \in (1,\infty), \tag{1.5.77}$$

and

given any weight $w \in A_p(\partial\Omega,\sigma)$ with exponent $p \in (1,\infty)$, it follows that for each sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \big[L^p(\partial\Omega, w\sigma)\big]^M$ which is weak-$*$ convergent to some function $f \in \big[L^p(\partial\Omega, w\sigma)\big]^M$ one has $\lim_{j\to\infty} S_{\text{mod}} f_j = S_{\text{mod}} f$ in $\big[(\text{Lip}_c(\partial\Omega))'\big]^M$. (1.5.78)

Also, with the modified boundary-to-domain single layer operator $\mathscr{S}_{\text{mod}}$ as in (1.5.50), for each aperture parameter $\kappa > 0$ and each $f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ one has

the nontangential boundary limit $\big(\mathscr{S}_{\text{mod}} f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists (in $\mathbb{C}^M$) at σ-a.e. point on $A_\kappa(\partial\Omega)$, (1.5.79)

and, in fact,

$$\Big(\big(\mathscr{S}_{\text{mod}} f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) = (S_{\text{mod}} f)(x) \text{ at } \sigma\text{-a.e. point } x \in A_\kappa(\partial\Omega), \quad \text{in particular, (cf. [68, Proposition 8.8.4]) at } \sigma\text{-a.e. point } x \in \partial_*\Omega. \tag{1.5.80}$$

Furthermore, if $n \geq 3$ and the function $f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-2}}\big)\Big]^M$ (thus, in particular, if f belongs to the space $\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p \in [1, n-1)$), or if $n = 2$ and $f \in \Big[L^1\big(\partial\Omega, \ln(2+|x|)\sigma(x)\big)\Big]^M$, then there exists some constant $C_f \in \mathbb{C}^M$ with the property that

$$S_{\text{mod}} f = Sf + C_f \text{ at } \sigma\text{-a.e. point } x \in \partial_*\Omega. \tag{1.5.81}$$

Finally, if for each pair of indices $j, k \in \{1, \dots, n\}$ one considers the vector-valued version of the integral operator (1.2.3) *corresponding to choosing b to be the fundamental solution E, i.e., if for each function $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\Big]^M$ one defines*

$$T^{\#}_{jk} f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \big\{\nu_j(x)(\partial_k E)(x-y) - \nu_k(x)(\partial_j E)(x-y)\big\} f(y)\,\mathrm{d}\sigma(y) \tag{1.5.82}$$

at σ-a.e. point $x \in \partial_*\Omega$, then

$$\begin{aligned} &\textit{for each } f \in \Big[L^p_{\mathrm{loc}}(\partial\Omega,\sigma) \cap L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M \textit{ with } p \in (1,\infty) \\ &\qquad \textit{one has } \partial_{\tau_{jk}}\big(S_{\mathrm{mod}} f\big) = T^{\#}_{jk} f \textit{ at } \sigma\textit{-a.e. point on } \partial_*\Omega. \end{aligned} \tag{1.5.83}$$

In particular, [68, Lemma 7.7.13] *guarantees that the formula in the second line of* (1.5.83) *holds whenever $f \in \big[L^p(\partial\Omega, w\sigma)\big]^M$ with $p \in (1,\infty)$ and $w \in A_p(\partial\Omega,\sigma)$. Also,* (1.5.83), (1.5.56), (1.5.80)*, and* [68, (8.8.45)] *ensure that*

$$\begin{aligned} &\textit{for each } f \in \big[L^p(\partial\Omega,\sigma)\big]^M \textit{ with } p \in (1, n-1) \\ &\textit{one has } \partial_{\tau_{jk}}\big(Sf\big) = T^{\#}_{jk} f \textit{ at } \sigma\textit{-a.e. point on } \partial_*\Omega. \end{aligned} \tag{1.5.84}$$

(xiii) Make the assumption that Ω is a UR *domain. Then the following operator identities hold:*

$$\begin{aligned} &\big(\tfrac12 I + K\big)\circ\big(-\tfrac12 I + K\big) = S \circ \big(\partial^A_\nu \mathcal{D}\big) \\ &\textit{on } \big[L^{p,q}_1(\partial\Omega,\sigma)\big]^M \textit{ with } p\in(1,\infty) \textit{ and } q \in (1,n-1), \\ &\textit{as well as on } \big[L^p(\partial\Omega,\sigma)\big]^M \textit{ with } p \in \big(\tfrac{n-1}{n-2},\infty\big), \end{aligned} \tag{1.5.85}$$

$$\begin{aligned} &\big(\tfrac12 I + K^{\#}_{A^\top}\big)\circ\big(-\tfrac12 I + K^{\#}_{A^\top}\big) = \big(\partial^A_\nu \mathcal{D}\big)\circ S \\ &\textit{on } \big[L^p(\partial\Omega,\sigma)\big]^M \textit{ with } p \in (1,n-1), \textit{ as well as} \\ &\textit{on } \big[L^{p,p^*}_{-1}(\partial\Omega,\sigma)\big]^M \textit{ with } p\in(1,n-1) \textit{ and } \tfrac{1}{p^*} = \tfrac1p - \tfrac{1}{n-1}, \end{aligned} \tag{1.5.86}$$

$$\begin{aligned} &S \circ K^{\#}_{A^\top} = K \circ S \\ &\textit{on } \big[L^p(\partial\Omega,\sigma)\big]^M \textit{ with } p \in (1,n-1), \textit{ as well as} \\ &\textit{on } \big[L^{p,p^*}_{-1}(\partial\Omega,\sigma)\big]^M \textit{ with } p\in(1,n-1) \textit{ and } \tfrac{1}{p^*} = \tfrac1p - \tfrac{1}{n-1}, \end{aligned} \tag{1.5.87}$$

$$K^{\#}_{A^{\top}} \circ \big(\partial^{A}_{\nu}\mathcal{D}\big) = \big(\partial^{A}_{\nu}\mathcal{D}\big) \circ K$$
$$\text{on } \big[L^{p,q}_{1}(\partial\Omega,\sigma)\big]^{M} \text{ with } p \in (1,\infty) \text{ and } q \in (1,n-1), \tag{1.5.88}$$
$$\text{as well as on } \big[L^{p}(\partial\Omega,\sigma)\big]^{M} \text{ with } p \in (1,\infty).$$

Also, for each function $f \in \big[L^p(\partial\Omega, w\sigma)\big]^M$ *with* $p \in (1,\infty)$ *and each Muckenhoupt weight* $w \in A_p(\partial\Omega,\sigma)$*, at* σ*-a.e. point* $x \in \partial\Omega$ *one has* (*with the operators* $T^{\#}_{jk}$, $1 \leq j,k \leq n$*, defined as in* (1.5.82))

$$\big(\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)f\Big)(x) \tag{1.5.89}$$
$$= \left(\lim_{\varepsilon\to 0^{+}} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y)\big(T^{\#}_{js} f\big)_{\alpha}(y)\,\mathrm{d}\sigma(y)\right)_{1\leq\mu\leq M}$$
$$= \left(\lim_{\varepsilon\to 0^{+}} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y)\partial_{\tau_{js}}\big(S_{\text{mod}} f\big)_{\alpha}(y)\,\mathrm{d}\sigma(y)\right)_{1\leq\mu\leq M}.$$

Moreover, if $\partial\Omega$ *is bounded then one may allow any* $p,q \in (1,\infty)$ *in* (1.5.85)-(1.5.88)*, this time taking* $p^{*} \in (1,\infty)$ *arbitrary* (*and unrelated to* p).

(xiv) Revert back to assuming that $\Omega \subseteq \mathbb{R}^n$ *is an open set with the property that* $\partial\Omega$ *is a* UR *set. Denote by* $\mathfrak{A}_{\text{WE}}(n,M)$ *the collection of all coefficient tensors* $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ *with complex entries, with the property that the* $M \times M$ *homogeneous second-order system* L_A *associated with* A *in* $\mathbb{R}^n$ *as in* (1.3.2) *is weakly elliptic. Fix* $j \in \{1,\dots,n\}$ *and denote by* $\partial_j S_A$ *the principal-value singular integral operator on* $\partial\Omega$ *whose kernel is* $(\partial_j E_A)(x-y)$*, where* E_A *is the fundamental solution canonically associated with* L_A *as in* [70, Theorem 1.4.2]. *More specifically, for every* $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *set*

$$(\partial_j S_A) f(x) := \lim_{\varepsilon\to 0^{+}} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} (\partial_j E_A)(x-y) f(y)\,\mathrm{d}\sigma(y) \tag{1.5.90}$$

at σ*-a.e. point* $x \in \partial\Omega$*. Finally, recall the pieces of notation introduced in* (A.0.23), (A.0.24)*, and* (A.0.171).
Then the following operator-valued assignments are continuous:

$$\mathfrak{A}_{\rm WE}(n,M) \ni A \longmapsto \partial_j S_A \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega,\sigma)\big]^M\Big) \text{ if } 1 < p < \infty, \tag{1.5.91}$$

$$\mathfrak{A}_{\rm WE}(n,M) \ni A \longmapsto \partial_j S_A \in \mathrm{Bd}\Big(\big[L^1(\partial\Omega,\sigma)\big]^M \to \big[L^{1,\infty}(\partial\Omega,\sigma)\big]^M\Big), \tag{1.5.92}$$

$$\begin{gathered}\mathfrak{A}_{\rm WE}(n,M) \ni A \mapsto \partial_j S_A \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega,w\sigma)\big]^M\Big), \\ \text{if } 1 < p < \infty \text{ and } w \in A_p(\partial\Omega,\sigma),\end{gathered} \tag{1.5.93}$$

$$\begin{gathered}\mathfrak{A}_{\rm WE}(n,M) \ni A \mapsto \partial_j S_A \in \mathrm{Bd}\Big(\big[L^{p,q}(\partial\Omega,\sigma)\big]^M\Big) \\ \text{if } 1 < p < \infty \text{ and } 0 < q \leq \infty,\end{gathered} \tag{1.5.94}$$

and

$$\begin{gathered}\mathfrak{A}_{\rm WE}(n,M) \ni A \mapsto \partial_j S_A \in \mathrm{Bd}\Big(\big[H^{p,q}(\partial\Omega,\sigma)\big]^M \to \big[L^{p,q}(\partial\Omega,\sigma)\big]^M\Big) \\ \text{if } p \in \big(\tfrac{n-1}{n},\infty\big) \text{ and } q \in (0,\infty].\end{gathered} \tag{1.5.95}$$

(xv) *Make the assumption that* Ω *is a* UR *domain. Then for each* $p \in (1,\infty)$ *the following operator-valued assignments are continuous:*

$$\mathfrak{A}_{\rm WE}(n,M) \ni A \mapsto K_A \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega,\sigma)\big]^M\Big), \tag{1.5.96}$$

$$\mathfrak{A}_{\rm WE}(n,M) \ni A \mapsto K_A \in \mathrm{Bd}\Big(\big[L^p_1(\partial\Omega,\sigma)\big]^M\Big), \tag{1.5.97}$$

$$\mathfrak{A}_{\rm WE}(n,M) \ni A \mapsto K^\#_A \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega,\sigma)\big]^M\Big), \tag{1.5.98}$$

$$\mathfrak{A}_{\rm WE}(n,M) \ni A \mapsto K^\#_A \in \mathrm{Bd}\Big(\big[L^p_{-1}(\partial\Omega,\sigma)\big]^M\Big), \tag{1.5.99}$$

$$\mathfrak{A}_{\rm WE}(n,M) \ni A \mapsto \partial^A_\nu \mathcal{D}_A \in \mathrm{Bd}\Big(\big[L^p_1(\partial\Omega,\sigma)\big]^M \to \big[L^p(\partial\Omega,\sigma)\big]^M\Big), \tag{1.5.100}$$

$$\mathfrak{A}_{\rm WE}(n,M) \ni A \mapsto \partial^A_\nu \mathcal{D}_A \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega,\sigma)\big]^M \to \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M\Big). \tag{1.5.101}$$

If $\partial\Omega$ *is compact, then for each* $p \in (1,\infty)$ *the following operator-valued assignments are also continuous:*

$$\mathfrak{A}_{\rm WE}(n,M) \ni A \mapsto S_A \in \mathrm{Bd}\Big(\big[L^p(\partial\Omega,\sigma)\big]^M \to \big[L^p_1(\partial\Omega,\sigma)\big]^M\Big), \tag{1.5.102}$$

$$\mathfrak{A}_{\rm WE}(n,M) \ni A \mapsto S_A \in \mathrm{Bd}\Big(\big[L^p_{-1}(\partial\Omega,\sigma)\big]^M \to \big[L^p(\partial\Omega,\sigma)\big]^M\Big). \tag{1.5.103}$$

Finally, similar results are valid on Lorentz spaces, off-diagonal Sobolev spaces, as well as Muckenhoupt weighted Lebesgue and Sobolev spaces.

Proof The reader is reminded that, throughout the proof, the summation convention over repeated indices is in effect.

Proof of claims in item (i): The nontangential maximal function estimates in (1.5.1)-(1.5.2) are consequences of (1.3.68), [70, Theorem 1.4.2], and [70, Theorem 2.4.1], while the nontangential maximal function estimate from (1.5.3) is implied by (1.3.68), [70, Theorem 1.4.2], and [70, (2.4.18)]. Next, (1.5.4) is seen from (1.3.68), [70, Theorem 1.4.2], and [70, (2.4.23)]. Finally, the estimate in (1.5.5) is a consequence of [68, (6.5.31)] used with $u := \nabla \mathcal{D} f$, $p := q$, $q := p$, $s := 1 - \frac{1}{p}$ (also bearing in mind (1.3.24) and [68, (6.5.40) in Theorem 6.5.7]), as well as [70, (2.4.34)] used with $\Sigma := \partial\Omega$.

Proof of claims in item (ii): Having fixed $f = (f_\alpha)_{1 \le \alpha \le M} \in \big[L^1_1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big)\big]^M$, Lemma 1.3.2 gives that for each index $\ell \in \{1, \dots, n\}$ we have, at every $x \in \Omega$,

$$\partial_\ell\big(\mathcal{D} f\big)(x) = \left(\int_{\partial_*\Omega} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x - y)(\partial_{\tau_{\ell s}} f_\alpha)(y)\, d\sigma_*(y)\right)_{1 \le \gamma \le M}. \tag{1.5.104}$$

Since each $\partial_{\tau_{\ell s}} f_\alpha$ belongs to $L^1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big)$, the claim in (1.5.7) follows from (1.5.104), [70, Theorem 1.4.2], and [70, Theorem 2.5.1]. Also, the claims in (1.5.8) and the subsequent comment are seen from (1.5.104), [70, Theorem 1.4.2], and [70, Theorem 2.4.1]. The claim made in (1.5.9) may be justified, bearing [69, (11.7.21)] in mind, based on [70, Theorem 1.4.2] and [70, (2.4.18)]. Finally, assume that Ω is also a uniform domain, and that $f = (f_\alpha)_{1 \le \alpha \le M} \in \big[L^{p,q}_1(\partial_*\Omega, \sigma_*)\big]^M$ with $p \in (1, \infty)$ and $q \in (n - 1, \infty)$. In such a scenario, the Hölder regularity result in (1.5.10) and the estimate in (1.5.11) are implied by (1.5.8) and [68, Corollary 8.6.8]. This takes care of item *(ii)*.

Proof of claims in item (iii): These are all consequences of (1.3.72), [70, Theorem 1.4.2], [70, Theorem 2.3.2], (1.3.68), and [70, (2.3.25), (2.3.44), (2.3.60), (2.3.61)].

Proof of claims in item (iv): Fix an index $\gamma \in \{1, \dots, M\}$ along with some aperture parameter $\kappa > 0$, and observe that if $f = (f_\alpha)_{1 \le \alpha \le M} \in \big[L^1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big)\big]^M$ then according to (1.3.18), [70, Theorem 1.4.2], [70, Theorem 2.5.1], and (1.3.68), at σ_*-a.e. $x \in \partial_*\Omega$ we may write

$$\begin{aligned}\Big(\big(\mathcal{D}f\big)_\gamma\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) &= -\frac{1}{2\mathrm{i}}\widehat{\partial_r E_{\gamma\beta}}\big(\nu(x)\big)a^{\beta\alpha}_{rs}\nu_s(x)f_\alpha(x)\\ &\quad - \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \nu_s(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)f_\alpha(y)\,\mathrm{d}\sigma_*(y)\\ &= -\frac{1}{2}\Big(L\big(\nu(x)\big)^{-1}\Big)_{\gamma\beta}a^{\beta\alpha}_{rs}\nu_r(x)\nu_s(x)f_\alpha(x) + (Kf)_\gamma(x)\\ &= \frac{1}{2}\Big(L\big(\nu(x)\big)^{-1}\Big)_{\gamma\beta}\Big(L\big(\nu(x)\big)\Big)_{\beta\alpha}f_\alpha(x) + (Kf)_\gamma(x)\\ &= \tfrac{1}{2}\delta_{\gamma\alpha}f_\alpha(x) + (Kf)_\gamma(x) = \tfrac{1}{2}f_\gamma(x) + (Kf)_\gamma(x). \end{aligned} \tag{1.5.105}$$

On account of the arbitrariness of γ, formula (1.5.105) proves (1.5.20).

Proof of claims in item (v): Make the additional assumption stipulated in (1.5.23). Then the claim that for every $p \in (1,\infty)$ the operator K is well defined and bounded in the context of (1.5.24) becomes a consequence of (1.5.20), (1.5.8), and [69, Proposition 11.3.2] (whose applicability in the present case requires (1.5.23)). The linearity of K is clear from (1.3.68). The more general cases pertaining to the action of K on off-diagonal Sobolev spaces as in (1.5.25), and the action of K on weighted Sobolev spaces as in (1.5.26) are dealt with in a similar fashion.

Proof of claims in item (vi): Having fixed $p, p' \in (1,\infty)$ such that $1/p + 1/p' = 1$, we start by considering the operator

$$\widetilde{K^\#} : \big[L^{p'}_{-1}(\partial_*\Omega,\sigma_*)\big]^M \longrightarrow \big[L^{p'}_{-1}(\partial_*\Omega,\sigma_*)\big]^M \text{ defined by } \big\langle \widetilde{K^\#}f, g\big\rangle := \big\langle f, Kg\big\rangle$$
$$\text{for all } f \in \big[L^{p'}_{-1}(\partial_*\Omega,\sigma_*)\big]^M = \Big(\big[L^{p}_{1}(\partial_*\Omega,\sigma_*)\big]^M\Big)^* \text{ and } g \in \big[L^{p}_{1}(\partial_*\Omega,\sigma_*)\big]^M \tag{1.5.106}$$

where, in the present context, the angled brackets $\langle\cdot,\cdot\rangle$ denote the duality pairing between the spaces $\Big(\big[L^{p}_{1}(\partial_*\Omega,\sigma_*)\big]^M\Big)^*$ and $\big[L^{p}_{1}(\partial_*\Omega,\sigma_*)\big]^M$. Thanks to (A.0.136) and (1.5.24) it follows that $\widetilde{K^\#}$ is a well-defined, linear and bounded operator. Bring in the mapping E, extending functions originally defined on $\partial_*\Omega$ by zero to the entire topological boundary $\partial\Omega$. Then, since the transpose of $K^\#$ in (1.5.12) is K in (1.5.14), we conclude from (1.5.106), and [69, (11.8.4), (11.8.5)] that $\widetilde{K^\#}$ is the unique extension of the composition $K^\# \circ E$ (where $K^\#$ is as in (1.5.27)) to a continuous operator from $\big[L^{p'}_{-1}(\partial_*\Omega,\sigma_*)\big]^M$ into itself. Re-denoting $\widetilde{K^\#}$ simply as $K^\#$ then yields the claims in item *(vi)* pertaining to the diagonal scale of boundary Sobolev spaces. Finally, analogous claims for the off-diagonal scale of boundary Sobolev spaces may be justified in a very similar fashion.

Proof of claims in item (vii): Fix some $f = (f_\alpha)_{1\le\alpha\le M} \in \Big[L^1_1\Big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\Big)\Big]^M$. Then the current item *(ii)* ensures that the conormal derivative $\partial^A_\nu(\mathcal{D}f)$ is meaningfully defined as indicated in (A.0.184). More specifically, for each $\mu \in \{1, \dots, M\}$ and $\ell \in \{1, \dots, n\}$, at each point σ-a.e. $x \in \partial\Omega$ we may write

$$\begin{aligned}
\big((\partial^A_\nu \mathcal{D})f\big)_\mu(x) &= \nu_i(x) a^{\mu\gamma}_{ij} \partial_j \big(\mathcal{D}f\big)_\gamma\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}(x) \\
&= \frac{1}{2\mathrm{i}} \widehat{\partial_r E_{\gamma\beta}}\big(\nu(x)\big) a^{\beta\alpha}_{rs} a^{\mu\gamma}_{ij} \nu_i(x)(\partial_{\tau_{js}} f_\alpha)(x) \\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{js}} f_\alpha)(y)\,\mathrm{d}\sigma(y) \\
&= \frac{1}{2}\Big(L\big(\nu(x)\big)^{-1}\Big)_{\gamma\beta} a^{\beta\alpha}_{rs} a^{\mu\gamma}_{ij} \nu_i(x)\nu_r(x)(\partial_{\tau_{js}} f_\alpha)(x) \\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{js}} f_\alpha)(y)\,\mathrm{d}\sigma(y),
\end{aligned} \tag{1.5.107}$$

thanks to (A.0.184), (1.5.104), [70, Theorem 2.5.1], and [70, (1.4.30)] (with $\gamma := \mathbf{e}_r$ and $\xi := \nu(x)$). As regards the jump-term in (1.5.107), based on [69, (11.4.8)] we may compute

$$\begin{aligned}
&\big(L(\nu)^{-1}\big)_{\gamma\beta} a^{\beta\alpha}_{rs} a^{\mu\gamma}_{ij} \nu_i \nu_r \partial_{\tau_{js}} f_\alpha \\
&\qquad = \big(L(\nu)^{-1}\big)_{\gamma\beta} a^{\beta\alpha}_{rs} a^{\mu\gamma}_{ij} \nu_i \nu_r \nu_j \big(\nabla_{\tan} f_\alpha\big)_s \\
&\qquad\quad - \big(L(\nu)^{-1}\big)_{\gamma\beta} a^{\beta\alpha}_{rs} a^{\mu\gamma}_{ij} \nu_i \nu_r \nu_s \big(\nabla_{\tan} f_\alpha\big)_j \\
&\qquad = -\big(L(\nu)^{-1}\big)_{\gamma\beta} \big(L(\nu)\big)_{\mu\gamma} a^{\beta\alpha}_{rs} \nu_r \big(\nabla_{\tan} f_\alpha\big)_s \\
&\qquad\quad + \big(L(\nu)^{-1}\big)_{\gamma\beta} \big(L(\nu)\big)_{\beta\alpha} a^{\mu\gamma}_{ij} \nu_i \big(\nabla_{\tan} f_\alpha\big)_j \\
&\qquad = -a^{\mu\alpha}_{rs} \nu_r \big(\nabla_{\tan} f_\alpha\big)_s + a^{\mu\alpha}_{ij} \nu_i \big(\nabla_{\tan} f_\alpha\big)_j = 0.
\end{aligned} \tag{1.5.108}$$

Collectively, (1.5.107) and (1.5.108) prove (1.5.29). The boundedness of the conormal derivative operator in (1.5.31) when $p, q \in (1, \infty)$ is then clear from (1.5.29) and [70, Theorem 2.3.2].

Proof of claims in item (viii): Continue to assume that Ω is a UR domain in $\mathbb{R}^n$. Also, assume $\Omega \neq \mathbb{R}^n$ since otherwise there is nothing to prove. Define $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$. Then item *(7)* in [68, Lemma 5.10.9] ensures that Ω_- is also a UR domain, whose topological and geometric measure theoretic boundaries agree with those of Ω, and whose geometric measure theoretic outward unit normal is $-\nu$ at σ-a.e. point on $\partial\Omega$.

To proceed, select $f \in \big[L^{p,q}_1(\partial\Omega,\sigma)\big]^M$ and $g \in \big[L^{q',p'}_1(\partial\Omega,\sigma)\big]^M$ where $p, p', q, q' \in (1,\infty)$ satisfy $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. In order to stress the dependence on the coefficient tensor, write $\mathcal{D}_A$ for the boundary-to-domain double layer associated with A as in (1.3.18). Also, let $\mathcal{D}_{A^\top}$ be the boundary-to-domain double layer associated with the transpose coefficient tensor $A^\top$ (much as $\mathcal{D}$ is associated with A in (1.3.18)). Finally, define $u^\pm := \mathcal{D}_A f$ and $w^\pm := \mathcal{D}_{A^\top} g$ in $\Omega_\pm$.

Then our earlier results in items *(i)*, *(ii)*, *(iv)*, show that the pairs (u^+, w^+) and (u^-, w^-) satisfy [70, (1.7.76)] and the version of [70, (1.7.77)] with the nontangential maximal function memberships replaced as in [70, (1.7.83)], as well as [70, (1.7.80)], relative to Ω_+ and to Ω_-. In addition, in the case when $\Omega_\pm$ is an exterior domain, it is clear from (1.3.18), (1.5.104), and [70, (1.4.24)] that the functions $u^\pm, w^\pm$ satisfy

$$\begin{aligned} u^\pm(x) = O(|x|^{1-n}) \text{ and } w^\pm(x) = O(|x|^{1-n}) \text{ as } |x| \to \infty, \\ (\nabla u^\pm)(x) = O(|x|^{-n}) \text{ and } (\nabla w^\pm)(x) = O(|x|^{-n}) \text{ as } |x| \to \infty, \end{aligned} \tag{1.5.109}$$

which further implies that condition [70, (1.7.79)] formulated for the exterior domain $\Omega_\pm$ holds in such a scenario. As such, we may invoke Green's formula [70, (1.7.81)] which, in light of (1.5.20) and (1.5.29), permits us to write

$$\begin{aligned} &\int_{\partial\Omega} \Big\langle \big(\partial^A_\nu \mathcal{D}_A\big) f, \big(\pm \tfrac{1}{2} I + K_{A^\top}\big) g \Big\rangle \, d\sigma \\ &\qquad = \int_{\partial\Omega} \Big\langle \big(\pm \tfrac{1}{2} I + K_A\big) f, \big(\partial^{A^\top}_\nu \mathcal{D}_{A^\top}\big) g \Big\rangle \, d\sigma, \end{aligned} \tag{1.5.110}$$

where K_A is the boundary-to-boundary double layer associated with the coefficient tensor A as in (1.3.68), and $K_{A^\top}$ is defined similarly with A replaced by $A^\top$. Subtracting the two versions of this equality then produces (1.5.32).

Proof of claims in items (ix)-(x): These are direct consequences of (1.3.6), (1.3.62), and Proposition 1.2.4, whose applicability in the present setting is ensured by item *(3)* in [70, Theorem 1.4.2]. We also wish to note that in the proof of the claim made in relation to (1.5.37) the first formula in [70, (1.4.32)] is also used. Next, the fact that if $n \geq 3$ then S in (1.5.39) is a linear and bounded operator is a consequence of [68, (7.8.11)] and [70, Theorem 1.4.2]. Finally, the claim in (1.5.49) may be justified much as in [68, (7.8.10)-(7.8.11)].

Proof of claims in item (xi): The claims in the first part of the statement are easily justified using the Mean Value Theorem, the estimates for the fundamental solution from [70, Theorem 1.4.2], and [68, Lemma 7.2.1]. In the case when $\partial\Omega$ is actually a UR set and $f \in [L^p(\partial\Omega,\sigma)]^M$ with $p \in (1,\infty)$, the second estimate in (1.5.57) is implied by (1.5.51) and [70, Theorem 2.4.1], while the first estimate in (1.5.57) is a consequence of [68, Lemma 8.6.6].

Turning now to the task of justifying the jump-formula recorded in (1.5.58), fix an arbitrary function $f = (f_\gamma)_{1\leq\gamma\leq M} \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$. Then based on (A.0.184), (1.5.55), and the jump-formula [70, (2.9.54)], for each $\alpha \in \{1,\dots,M\}$ we may write, at σ-a.e. point $x \in \partial_*\Omega$,

$$
\begin{aligned}
(\partial_\nu^A \mathscr{S}_{\rm mod} f)_\alpha(x) &= \nu_r(x) a_{rs}^{\alpha\beta} \big(\partial_s(\mathscr{S}_{\rm mod} f)_\beta\big)\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}}(x) \\
&= \nu_r(x) a_{rs}^{\alpha\beta} \Big((\partial_s \mathscr{S}_{\beta\gamma}) f_\gamma\Big)\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}}(x) \\
&= \frac{1}{2} a_{rs}^{\alpha\beta} \nu_r(x)\nu_s(x) b_{\beta\gamma}(x) f_\gamma(x) \\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_r(x) a_{rs}^{\alpha\beta} (\partial_s E_{\beta\gamma})(x-y) f_\gamma(y)\, {\rm d}\sigma(y) \\
&= -\frac{1}{2} f_\alpha(x) + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_r(x) a_{rs}^{\alpha\beta} (\partial_s E_{\beta\gamma})(x-y) f_\gamma(y)\, {\rm d}\sigma(y),
\end{aligned}
\tag{1.5.111}
$$

where the last equality makes use of [70, (2.9.52)]. From (1.3.72) and the fact that $\big(E_{L_A}\big)^\top = E_{L_{A^\top}}$ (cf. [70, (1.4.32)]) we see that the above principal-value integral is precisely $\big(K^\#_{A^\top} f\big)_\alpha(x)$. Altogether, we conclude that for each index $\alpha \in \{1,\dots,M\}$ we have $(\partial_\nu^A \mathscr{S}_{\rm mod} f)_\alpha(x) = -\frac{1}{2} f_\alpha(x) + \big(K^\#_{A^\top} f\big)_\alpha(x)$ at σ-a.e. $x \in \partial_*\Omega$. In view of the arbitrariness of α, this establishes (1.5.58). Then the jump-formula (1.5.59) follows on account of this and (1.5.56).

Next, the claim in (1.5.60) is readily seen from (1.5.51) and [70, (2.4.48)] used with $k := \nabla E$, $p := n-1$, and $\theta := p-1$. That the Hölder estimate claimed in (1.5.62) holds when Ω is a uniform domain whose boundary is a UR set is a consequence of (1.5.57) and [68, (5.11.78)]. In the case when $\partial\Omega$ is bounded, $\mathscr{S}_{\rm mod} f$ defined in (1.5.50) differs from $\mathscr{S} f$ defined in (1.3.6) by a constant (which depends on the function $f \in \big[L^1(\partial\Omega,\sigma)\big]^M$). Hence, in such a case, $\nabla\mathscr{S}_{\rm mod} f = \nabla\mathscr{S} f$ in Ω. In particular, if $\partial\Omega$ is bounded then the nontangential maximal operator estimate (1.5.57) and the jump-formula (1.5.58) become, respectively, (1.5.63) and (1.5.66).

As regards the vanishing Carleson measure property in (1.5.64), we first observe that (1.5.60) presently gives that

$$
\begin{aligned}
&|\nabla\mathscr{S} f|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, {\rm d}\mathcal{L}^n \text{ is a Carleson measure in } \Omega \text{ with constant} \\
&\le C\|f\|^p_{[L^{n-1}(\partial\Omega,\sigma)]^M} \text{ whenever } f \in \big[L^{n-1}(\partial\Omega,\sigma)\big]^M.
\end{aligned}
\tag{1.5.112}
$$

There remains to show that the aforementioned Carleson measure is actually vanishing. To this end, pick an exponent $q \in (n-1,\infty)$ and define $\eta := 1-(n-1)/q \in (0,1)$. Also, select two arbitrary functions, $f \in \big[L^{n-1}(\partial\Omega,\sigma)\big]^M$ and $g \in \big[L^q(\partial\Omega,\sigma)\big]^M$. Then the function g belongs to $\big[L^{n-1}(\partial\Omega,\sigma)\big]^M$ (since $\partial\Omega$ is bounded), and for each $r \in \big(0, 2\,{\rm diam}(\partial\Omega)\big)$ and $x \in \partial\Omega$ we may estimate

$$\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathscr{S}(f-g)\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{\frac{1}{p}}$$
$$\leq C\|f-g\|_{[L^{n-1}(\partial\Omega,\sigma)]^M}, \tag{1.5.113}$$

thanks to (1.5.112) written for $f-g$ in place of f. In addition, we have

$$\begin{aligned}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}&\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathscr{S}g\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{\frac{1}{p}}\\ &\leq C\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\operatorname{dist}(\cdot,\partial\Omega)^{p\eta-1}\,d\mathcal{L}^n\Big)^{\frac{1}{p}}\|g\|_{[L^q(\partial\Omega,\sigma)]^M}\\ &\leq Cr^{\eta}\|g\|_{[L^q(\partial\Omega,\sigma)]^M}.\end{aligned} \tag{1.5.114}$$

The first inequality above uses the version of (1.5.57) written for $\mathscr{S}$ in place of $\mathscr{S}_{\rm mod}$ (and for g in place of f), while the second inequality is based on [68, (8.6.101)] used with $\lambda := 1-p\eta$, $\alpha := 1$, $\beta := n-1$, and $E := B(x,r)\cap\Omega$. Together, (1.5.113) and (1.5.114) imply

$$\lim_{R\to 0^+}\sup_{\substack{x\in\partial\Omega\text{ and}\\ r\in(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathscr{S}f\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{\frac{1}{p}}$$
$$\leq C\|f-g\|_{[L^{n-1}(\partial\Omega,\sigma)]^M}, \tag{1.5.115}$$

for some constant $C\in(0,\infty)$ independent of f and g. Having established this, (1.5.64) now follows, on account of the arbitrariness of $g\in\big[L^q(\partial\Omega,\sigma)\big]^M$ and the density of $L^q(\partial\Omega,\sigma)$ in $L^{n-1}(\partial\Omega,\sigma)$.

To justify the claims made in (1.5.68) and (1.5.70), pick $f\in\big[L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ along with a truncation parameter $\varepsilon>0$. Also, fix an arbitrary point $x_0\in\partial\Omega$, and choose some number $r>1+4|x_0|+2\varepsilon(1+\kappa)$. Split

$$\begin{gathered} f=f_1+f_2 \text{ on } \partial\Omega, \text{ where}\\ f_1:=\mathbf{1}_{\partial\Omega\cap B(x_0,2r)}\cdot f \text{ and } f_2:=\mathbf{1}_{\partial\Omega\setminus B(x_0,2r)}\cdot f.\end{gathered} \tag{1.5.116}$$

Hence,

$$\mathscr{S}_{\rm mod}f=\mathscr{S}_{\rm mod}f_1+\mathscr{S}_{\rm mod}f_2 \text{ in } \Omega. \tag{1.5.117}$$

Observe that

$$\begin{gathered}|x-y|\leq|x-z|+|z-y|<(1+\kappa)\operatorname{dist}\big(z,\partial\Omega\big)+|z-y|\leq(2+\kappa)|z-y|,\\ \text{for each points } x,y\in\partial\Omega \text{ and each point } z\in\Gamma_\kappa(x).\end{gathered} \tag{1.5.118}$$

Based on (1.5.50), [70, (1.4.24)], and (1.5.118) we then conclude that there exists some constant $C=C(L,\kappa,r,x_0)\in(0,\infty)$ with the property that if $x\in\partial\Omega\cap B(x_0,r)$

and $z \in \Gamma_\kappa(x)$ then in the case in which $n \geq 3$ we have

$$\begin{aligned}\big(\mathscr{S}_{\rm mod} f_1\big)(z) &\leq \int_{\partial\Omega\cap B(x_0,2r)} |E(z-y)||f(y)|\,{\rm d}\sigma(y)\\&\quad+\int_{\partial\Omega\cap B(x_0,2r)} |E_*(-y)||f(y)|\,{\rm d}\sigma(y)\\&\leq C\int_{\partial\Omega\cap B(x_0,2r)}\frac{|f(y)|}{|z-y|^{n-2}}\,{\rm d}\sigma(y)+C\int_{\partial\Omega\cap B(x_0,2r)}|f(y)|\,{\rm d}\sigma(y)\\&\leq C\int_{\partial\Omega\cap B(x_0,2r)}\frac{|f(y)|}{|x-y|^{n-2}}\,{\rm d}\sigma(y)+C\|f\|_{[L^1(\partial\Omega,\frac{\sigma}{1+|\cdot|^{n-1}})]^M}. \end{aligned}\tag{1.5.119}$$

Taking the supremum over $z \in \Gamma_\kappa(x)$ this further implies

$$\Big(\mathcal{N}_\kappa\big(\mathscr{S}_{\rm mod} f_1\big)\Big)(x) \leq C\cdot\big(I_{\partial\Omega\cap B(x_0,2r),1}|f|\big)(x)+C\|f\|_{[L^1(\partial\Omega,\frac{\sigma}{1+|\cdot|^{n-1}})]^M} \tag{1.5.120}$$

for each $x \in \partial\Omega \cap B(x_0,r)$, where $I_{\partial\Omega\cap B(x_0,2r),1}$ is the fractional integral operator defined as in (A.0.100) with $E := \partial\Omega \cap B(x_0,2r)$, $d := n-1$, $\alpha := 1$, $\mu := \sigma$, and ρ the ordinary Euclidean distance. From [68, (7.8.9)] we see (keeping in mind that f is locally integrable) that, as a function of $x \in \partial\Omega \cap B(x_0,r)$, the right side of (1.5.120) belongs to $L^{\frac{n-1}{n-2},\infty}\big(\partial\Omega\cap B(x_0,r),\sigma\big)$. In concert with [68, (6.2.16)], [68, (8.2.28)], and (1.5.120) this implies

$$\mathcal{N}_\kappa\big(\mathscr{S}_{\rm mod} f_1\big) \in L^{\frac{n-1}{n-2},\infty}\big(\partial\Omega\cap B(x_0,r),\sigma\big) \ \text{ if } n \geq 3. \tag{1.5.121}$$

When $n = 2$, the nature of the singularity in the fundamental solution E changes (see [70, (1.4.24)]). To handle this case, assume that $x \in \partial\Omega \cap B(x_0,r)$ and $z \in \Gamma_\kappa(x)$ with ${\rm dist}\big(z,\partial\Omega\big) < \varepsilon$. Since for each $y \in \partial\Omega \cap B(x_0,2r)$ we have

$$\begin{aligned}|z-y| &\leq |z-x|+|x-x_0|+|x_0-y|\\&<(1+\kappa)\,{\rm dist}\big(z,\partial\Omega\big)+r+2r\\&<(1+\kappa)\varepsilon+3r,\end{aligned}\tag{1.5.122}$$

it follows from [70, (1.4.24)] and (1.5.118) that for each $\alpha \in (0,1)$ there exists a constant $C_\alpha \in (0,\infty)$, which depends on L, κ, ε, α, r, and x_0, with the property that if $n = 2$ then

$$\big|E(z-y)\big| \leq C\big(1+\big|\ln|z-y|\big|\big) \leq \frac{C_\alpha}{|z-y|^{1-\alpha}} \leq \frac{C_\alpha}{|x-y|^{1-\alpha}}. \tag{1.5.123}$$

Granted this, when $n = 2$ in place of (1.5.119) we now see that, for each $\alpha \in (0,1)$,

$$\big(\mathscr{S}_{\rm mod} f_1\big)(z) \leq C \int_{\partial\Omega\cap B(x_0,2r)} \frac{|f(y)|}{|x-y|^{1-\alpha}}\,{\rm d}\sigma(y) + C\|f\|_{[L^1(\partial\Omega,\frac{\sigma}{1+|\cdot|})]^M} \tag{1.5.124}$$

for each $z \in \Gamma_\kappa(x)$ with ${\rm dist}\big(z,\partial\Omega\big) < \varepsilon$, where $C = C(L,\kappa,\alpha,\varepsilon,r,x_0) \in (0,\infty)$. After taking the supremum over all $z \in \Gamma_\kappa(x)$ satisfying ${\rm dist}\big(z,\partial\Omega\big) < \varepsilon$, when $n = 2$ we arrive at

$$\Big(\mathcal{N}_\kappa^\varepsilon\big(\mathscr{S}_{\rm mod} f_1\big)\Big)(x) \leq C\cdot\big(I_{\partial\Omega\cap B(x_0,2r),\alpha}|f|\big)(x) + C\|f\|_{[L^1(\partial\Omega,\frac{\sigma}{1+|\cdot|})]^M} \tag{1.5.125}$$

for each $x \in \partial\Omega \cap B(x_0,r)$, where $I_{\partial\Omega\cap B(x_0,2r),\alpha}$ is the fractional integral operator of order $\alpha \in (0,1)$, defined as in (A.0.100) with $E := \partial\Omega \cap B(x_0,2r)$, $d := 1$, $\mu := \sigma$, and ρ the ordinary Euclidean distance. Since f is a locally integrable function, from [68, (7.8.9)] we then conclude that, as a function of the variable $x \in \partial\Omega \cap B(x_0,r)$, the right side of (1.5.125) belongs to $L^{\frac{1}{\alpha},\infty}\big(\partial\Omega\cap B(x_0,r),\sigma\big)$. Together with [68, (6.2.16)], [68, (8.2.28)], and (1.5.125) this implies that

$$\mathcal{N}_\kappa^\varepsilon\big(\mathscr{S}_{\rm mod} f_1\big) \in L^{\frac{1}{\alpha},\infty}\big(\partial\Omega\cap B(x_0,r),\sigma\big) \text{ for each } \alpha \in (0,1), \text{ if } n = 2. \tag{1.5.126}$$

In concert with the embedding in [68, (6.2.38)], this ultimately proves the following companion to (1.5.121):

$$\mathcal{N}_\kappa^\varepsilon\big(\mathscr{S}_{\rm mod} f_1\big) \in L^p\big(\partial\Omega\cap B(x_0,r),\sigma\big) \text{ for each } p \in (0,\infty), \text{ if } n = 2. \tag{1.5.127}$$

There remains to estimate the contribution from f_2, which we shall do working under the assumption that $n \geq 2$. To set the stage, continue to assume that the point $x \in \partial\Omega \cap B(x_0,r)$ and pick some $z \in \Gamma_\kappa(x)$ with ${\rm dist}(z,\partial\Omega) < \varepsilon$. Then

$$\begin{aligned} |z| &\leq |z-x| + |x-x_0| + |x_0| \\ &< (1+\kappa)\,{\rm dist}\big(z,\partial\Omega\big) + r + |x_0| \leq (1+\kappa)\varepsilon + r + |x_0|. \end{aligned} \tag{1.5.128}$$

In turn, (1.5.128) implies that for each $\xi \in [0,z]$ and each $y \in \partial\Omega \setminus B(x_0,2r)$ we have

$$\begin{aligned} \tfrac{3}{2}r + \tfrac{1}{4}|x_0-y| &\leq |x_0-y| \leq |y-\xi| + |\xi| + |x_0| \\ &\leq |y-\xi| + |z| + |x_0| \leq |y-\xi| + \varepsilon(1+\kappa) + 2|x_0| + r. \end{aligned} \tag{1.5.129}$$

This entails

$$\Big(\tfrac{1}{2}r - \varepsilon(1+\kappa) - 2|x_0|\Big) + \tfrac{1}{4}|x_0-y| \leq |y-\xi| \tag{1.5.130}$$

which, bearing in mind the original choice of r, ultimately leads to the conclusion that

$$\tfrac{1}{4}|x_0-y| \leq |y-\xi| \text{ for each } \xi \in [0,z]. \tag{1.5.131}$$

Let us also observe that, since $2r > 1 + |x_0|$, for each point $y \in \partial\Omega \setminus B(x_0, 2r)$ we have $|y| > 1$. Keeping this in mind, we now use (1.5.50), the Mean Value Theorem, (1.5.128), (1.5.131), and [70, (1.4.24)] in [70, Theorem 1.4.2] to estimate

$$\begin{aligned}\big(\mathscr{S}_{\text{mod}} f_2\big)(z) &\leq \int_{\partial\Omega\setminus B(x_0,2r)} |E(z-y) - E(-y)||f(y)|\,\mathrm{d}\sigma(y)\\ &\leq C\int_{\partial\Omega\setminus B(x_0,2r)} \frac{|f(y)|}{|x_0 - y|^{n-1}}\,\mathrm{d}\sigma(y) \leq C\int_{\partial\Omega\setminus B(x_0,2r)} \frac{|f(y)|}{1+|y|^{n-1}}\,\mathrm{d}\sigma(y)\\ &\leq C\|f\|_{[L^1(\partial\Omega, \frac{\sigma}{1+|\cdot|^{n-1}})]^M} \end{aligned} \tag{1.5.132}$$

for some $C = C(L, \kappa, r, x_0, \varepsilon) \in (0, \infty)$. After taking the supremum in (1.5.132) over all $z \in \Gamma_\kappa(x)$ with $\operatorname{dist}(z, \partial\Omega) < \varepsilon$ we therefore arrive at

$$\Big(\mathcal{N}_\kappa^\varepsilon\big(\mathscr{S}_{\text{mod}} f_2\big)\Big)(x) \leq C\|f\|_{[L^1(\partial\Omega, \frac{\sigma}{1+|\cdot|^{n-1}})]^M} \text{ for all } x \in \partial\Omega \cap B(x_0, r). \tag{1.5.133}$$

In view of [68, (8.2.28)], this goes to show that

$$\mathcal{N}_\kappa^\varepsilon\big(\mathscr{S}_{\text{mod}} f_2\big) \in L^\infty\big(\partial\Omega \cap B(x_0, r), \sigma\big). \tag{1.5.134}$$

At this stage, from (1.5.117), (1.5.121), (1.5.134), and [68, (6.2.16), (8.2.9), (8.2.28)] we conclude that (1.5.68) holds, while (1.5.70) is implied by (1.5.117), (1.5.127), and (1.5.134).

Finally, consider the claim made in (1.5.71)-(1.5.72). When $n = 2$ this is seen directly from (1.5.70), so we shall focus on the case when $n \geq 3$. In this scenario, we re-run the argument which, starting with the decomposition in (1.5.116), has led to (1.5.121). Since, thanks to (1.5.71), we now have $f_1 \in L^p(\partial\Omega, \sigma)$, the mapping properties of the fractional integration operator of order one (of the sort recorded in [68, (7.8.7) and (7.8.14)-(7.8.15)]) now give that

$$\mathcal{N}_\kappa\big(\mathscr{S}_{\text{mod}} f_1\big) \in L^q\big(\partial\Omega \cap B(x_0, r), \sigma\big) \text{ with } q \text{ as in (1.5.72)}. \tag{1.5.135}$$

With this in hand, the same argument as before then completes the proof of (1.5.71).

Proof of claims in item (xii): Fix $x_0 \in \partial\Omega$ along with $r \in (0, \infty)$. Given a function $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$, for each $x \in \partial\Omega \cap B(x_0, r)$ split

$$
\begin{aligned}
\int_{\partial\Omega} \big|E(x-y)-E_*(-y)\big||f(y)|\,\mathrm{d}\sigma(y)
&= \int_{\partial\Omega\setminus B(x,1)} \big|E(x-y)-E_*(-y)\big||f(y)|\,\mathrm{d}\sigma(y)\\
&\quad + \int_{\partial\Omega\cap B(x,1)} \big|E(x-y)-E_*(-y)\big||f(y)|\,\mathrm{d}\sigma(y)\\
&=: I(x)+II(x). \qquad (1.5.136)
\end{aligned}
$$

The same argument used to prove [70, (2.3.117)] presently gives that, for some constant $C_{x_0,r}\in(0,\infty)$, we have

$$
\big|E(x-y)-E_*(-y)\big| \leq \frac{C_{x_0,r}}{1+|y|^{n-1}},
\qquad \text{for all } x\in B(x_0,r) \text{ and all } y\in\mathbb{R}^n\setminus B(x,1). \qquad (1.5.137)
$$

Consequently, the integral defining $I(x)$ is absolutely convergent and, moreover, the assignment $x\mapsto I(x)$ is measurable and bounded for $x\in B(x_0,r)$. In addition, the version of the Fractional Integration Theorem recorded in [68, (7.8.9)] ensures that the integral defining $II(x)$ in (1.5.136) is absolutely convergent for σ-a.e. point $x\in\partial\Omega\cap B(x_0,r)$ and that, as a function of the variable $x\in B(x_0,r)$, the term $II(x)$ is also absolutely integrable. This analysis ultimately shows that the definition of $(S_{\text{mod}}f)(x)$ in (1.5.73) involves an absolutely convergent integral for σ-a.e. $x\in\partial\Omega$, and also that S_{mod} in (1.5.74) is a well-defined, linear, and continuous operator.

To show that S_{mod} is also a well-defined, linear, and continuous operator in the context of (1.5.75), we reason much as above, this time starting with $f\in\big[L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^{n-1}}\big)\cap L^p_{\text{loc}}(\partial\Omega,w\sigma)\big]^M$ for some $w\in A_p(\partial\Omega,\sigma)$, $1<p<\infty$. Then $g:=f\cdot\mathbf{1}_{\partial\Omega\cap B(x_0,r+1)}\in\big[L^p(\partial\Omega,w\sigma)\big]^M$ and for each $x\in\partial\Omega\cap B(x_0,r)$ we may estimate

$$
\begin{aligned}
II(x) &= \int_{\partial\Omega\cap B(x,1)} \big|E(x-y)-E_*(-y)\big||f(y)|\,\mathrm{d}\sigma(y)\\
&\leq C\int_{\partial\Omega\cap B(x,1)} \frac{|g(y)|}{|x-y|^{n-2}}\,\mathrm{d}\sigma(y)\\
&\quad + \int_{\partial\Omega\cap B(x_0,r+1)} \big|E_*(-y)\big||f(y)|\,\mathrm{d}\sigma(y). \qquad (1.5.138)
\end{aligned}
$$

Together with [68, Lemma 7.7.16] this implies that the assignment $x\mapsto II(x)$ belongs to $L^p_{\text{loc}}(\partial\Omega,w\sigma)$ from which we ultimately conclude that S_{mod} in (1.5.75) is indeed a well-defined, linear, and continuous operator. Finally, the claims about (1.5.76) follow from (1.2.83) and [70, Theorem 1.4.2].

Let us now deal with the claim in (1.5.78). To this end, fix $p \in (1, \infty)$ and consider a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Also, assume $\{f_j\}_{j\in\mathbb{N}} \subseteq \big[L^p(\partial\Omega, w\sigma)\big]^M$ is weak-$*$ convergent to some function $f \in \big[L^p(\partial\Omega, w\sigma)\big]^M$, i.e.,

$$\lim_{j\to\infty} \int_{\partial\Omega} \langle f_j, F\rangle \, d\sigma = \int_{\partial\Omega} \langle f, F\rangle \, d\sigma \quad \text{for each } F \in \big[L^{p'}(\partial\Omega, w'\sigma)\big]^M, \tag{1.5.139}$$

where $p' \in (1, \infty)$ denotes the Hölder conjugate exponent of p and w' denotes the dual weight of w, thus $w' := w^{1-p'} \in A_{p'}(\partial\Omega, \sigma)$. To proceed, pick an arbitrary $\mathbb{C}^M$-valued test function $\phi \in \big[\mathrm{Lip}_c(\partial\Omega)\big]^M$. Choose some reference point $x_0 \in \partial\Omega$ and select $R \in (0, \infty)$ large enough so that $\operatorname{supp}\phi \subseteq \partial\Omega \cap B(x_0, R)$. Then for each $j \in \mathbb{N}$ we may decompose

$$\big\langle S_{\mathrm{mod}} f_j, \phi\big\rangle = \mathrm{I}_j + \mathrm{II}_j + \mathrm{III}_j \tag{1.5.140}$$

where the pairing in the left-hand side is taken in the sense of distributions on $\partial\Omega$, and where

$$\mathrm{I}_j := \int_{\partial\Omega} \Big\langle f_j(y), \mathbf{1}_{\partial\Omega\setminus B(x_0,2R)}(y) \cdot \int_{\partial\Omega} \big\{E(x-y) - E_*(-y)\big\}^{\top} \phi(x)\, d\sigma(x)\Big\rangle\, d\sigma(y), \tag{1.5.141}$$

$$\mathrm{II}_j := -\Big\langle \int_{\partial\Omega} E_*(-y)\mathbf{1}_{\partial\Omega\cap B(x_0,2R)}(y) f_j(y)\, d\sigma(y), \int_{\partial\Omega} \phi(x)\, d\sigma(x)\Big\rangle, \tag{1.5.142}$$

and

$$\mathrm{III}_j := \int_{\partial\Omega} \Big\langle f_j(y), \mathbf{1}_{\partial\Omega\cap B(x_0,2R)}(y) \cdot \int_{\partial\Omega\cap B(x_0,R)} E(x-y)^{\top} \phi(x)\, d\sigma(x)\Big\rangle\, d\sigma(y). \tag{1.5.143}$$

If for each $y \in \partial\Omega$ we define

$$F_1(y) := \mathbf{1}_{\partial\Omega\setminus B(x_0,2R)}(y) \cdot \int_{\partial\Omega} \big\{E(x-y) - E_*(-y)\big\}^{\top} \phi(x)\, d\sigma(x), \tag{1.5.144}$$

it follows that there exists a constant $C_{x_0,R,\phi} \in (0, \infty)$ with the property that

$$|F_1(y)| \leq \frac{C_{x_0,R,\phi}}{1 + |y|^{n-1}} \quad \text{for each } y \in \partial\Omega. \tag{1.5.145}$$

Since F_1 is a $\mathbb{C}^M$-valued σ-measurable function, we may then estimate

$$\int_{\partial\Omega} |F_1(y)|^{p'} w'(y)\, d\sigma(y) \leq C_{x_0,R,\phi} \int_{\partial\Omega} \frac{w'(y)}{(1 + |y|^{n-1})^{p'}}\, d\sigma(y) < +\infty, \tag{1.5.146}$$

by (1.5.145) and [68, (7.7.101)]. Hence, $F_1 \in \big[L^{p'}(\partial\Omega, w'\sigma)\big]^M$, so (1.5.139) gives

$$
\begin{aligned}
&\lim_{j\to\infty} \mathrm{I}_j \qquad (1.5.147)\\
&= \lim_{j\to\infty} \int_{\partial\Omega} \langle f_j(y), F_1(y)\rangle \,\mathrm{d}\sigma(y) = \int_{\partial\Omega} \langle f(y), F_1(y)\rangle \,\mathrm{d}\sigma(y)\\
&= \int_{\partial\Omega} \Big\langle f(y), \mathbf{1}_{\partial\Omega\setminus B(x_0,2R)}(y)\cdot \int_{\partial\Omega} \big\{E(x-y)-E_*(-y)\big\}^{\top}\phi(x)\,\mathrm{d}\sigma(x)\Big\rangle \,\mathrm{d}\sigma(y).
\end{aligned}
$$

To compute the limit of II_j as $j\to\infty$, define

$$
F_2(y) := -E_*(-y)\mathbf{1}_{\partial\Omega\cap B(x_0,2R)}(y) \text{ for each point } y\in\partial\Omega. \qquad (1.5.148)
$$

We then have $F_2 \in \big[L^\infty_{\rm comp}(\partial\Omega,\sigma)\big]^{M\times M} \subseteq \big[L^{p'}(\partial\Omega,w'\sigma)\big]^{M\times M}$, so based on (1.5.139) we may compute

$$
\begin{aligned}
\lim_{j\to\infty} \mathrm{II}_j &= \lim_{j\to\infty} \Big\langle \int_{\partial\Omega} F_2(y) f_j(y)\,\mathrm{d}\sigma(y), \int_{\partial\Omega}\phi(x)\,\mathrm{d}\sigma(x)\Big\rangle\\
&= \Big\langle \int_{\partial\Omega} F_2(y) f(y)\,\mathrm{d}\sigma(y), \int_{\partial\Omega}\phi(x)\,\mathrm{d}\sigma(x)\Big\rangle\\
&= -\Big\langle \int_{\partial\Omega} E_*(-y)\mathbf{1}_{\partial\Omega\cap B(x_0,2R)}(y) f(y)\,\mathrm{d}\sigma(y), \int_{\partial\Omega}\phi(x)\,\mathrm{d}\sigma(x)\Big\rangle. \qquad (1.5.149)
\end{aligned}
$$

Finally, to handle the limit of III_j as $j\to\infty$, for each $y\in\partial\Omega$ we define

$$
F_3(y) := \mathbf{1}_{\partial\Omega\cap B(x_0,2R)}(y)\cdot \int_{\partial\Omega\cap B(x_0,R)} E(x-y)^{\top}\phi(x)\,\mathrm{d}\sigma(x), \qquad (1.5.150)
$$

then since for each $x\in\operatorname{supp}\phi\subseteq\partial\Omega\cap B(x_0,R)$ and each $y\in\partial\Omega\cap B(x_0,2R)$ we may estimate $|y-x|\le|y-x_0|+|x_0-x|<2R+R=3R$, it follows that

$$
|F_3(y)| \le C \int_{\substack{x\in\partial\Omega\\ |y-x|<3R}} \frac{|\phi(x)|}{|y-x|^{n-2}}\,\mathrm{d}\sigma(x) \text{ for each } y\in\partial\Omega. \qquad (1.5.151)
$$

Since F_3 is a $\mathbb{C}^M$-valued σ-measurable function, from [68, Lemma 7.7.16] and (1.5.151) we then conclude that $F_3\in\big[L^{p'}(\partial\Omega,w'\sigma)\big]^M$ which, in concert with (1.5.139), permits us to write

$$
\begin{aligned}
\lim_{j\to\infty} \mathrm{III}_j &= \lim_{j\to\infty} \int_{\partial\Omega} \langle f_j(y), F_3(y)\rangle\,\mathrm{d}\sigma(y) = \int_{\partial\Omega}\langle f(y), F_3(y)\rangle\,\mathrm{d}\sigma(y) \qquad (1.5.152)\\
&= \int_{\partial\Omega} \Big\langle f(y), \mathbf{1}_{\partial\Omega\cap B(x_0,2R)}(y)\cdot\int_{\partial\Omega\cap B(x_0,R)} E(x-y)^{\top}\phi(x)\,\mathrm{d}\sigma(x)\Big\rangle\,\mathrm{d}\sigma(y).
\end{aligned}
$$

At this stage, from (1.5.140), (1.5.147), (1.5.149), and (1.5.152) we see that

$$\lim_{j\to\infty} \langle S_{\mathrm{mod}} f_j, \phi \rangle = \langle S_{\mathrm{mod}} f, \phi \rangle. \tag{1.5.153}$$

In view of the arbitrariness of $\phi \in [\mathrm{Lip}_c(\partial\Omega)]^M$ we therefore conclude that $\lim_{j\to\infty} S_{\mathrm{mod}} f_j = S_{\mathrm{mod}} f$ in $[(\mathrm{Lip}_c(\partial\Omega))']^M$, finishing the justification of (1.5.78).

The fact that the operator (1.5.77) is well defined, linear, and bounded is a consequence of (1.5.73), Proposition 1.2.6, [70, Theorem 1.4.2], and the embedding in [68, (7.7.102)].

To prove (1.5.79)-(1.5.80), fix an arbitrary function $f \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$. Having fixed a point $x_0 \in \partial\Omega$ along with an arbitrary number $r \in (0,\infty)$, decompose $\mathscr{S}_{\mathrm{mod}} f = \mathscr{S}_{\mathrm{mod}} f_1 + \mathscr{S}_{\mathrm{mod}} f_2$ with f_1, f_2 as in (1.5.116). Note that since $\mathscr{S}_{\mathrm{mod}} f_2$ has a continuous extension to $B(x_0, r)$ we trivially have that the nontangential trace $\mathscr{S}_{\mathrm{mod}} f_2\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ exists at every point in $A_\kappa(\partial\Omega) \cap B(x_0, r)$. More precisely,

$$\begin{aligned}\big(\mathscr{S}_{\mathrm{mod}} f_2\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}(x) &= \int_{\partial\Omega} \big\{E(x-y) - E_*(-y)\big\} f_2(y)\, d\sigma(y) \\ &= \big(S_{\mathrm{mod}} f_2\big)(x) \text{ for each } x \in A_\kappa(\partial\Omega) \cap B(x_0, r).\end{aligned} \tag{1.5.154}$$

Next, since $f_1 \in \big[L^1(\partial\Omega, \sigma)\big]^M$, at σ-a.e. point $x \in A_\kappa(\partial\Omega)$ we may write

$$\begin{aligned}\big(\mathscr{S}_{\mathrm{mod}} f_1\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}(x) &= \big(\mathscr{S} f_1\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}(x) - \int_{\partial\Omega} E_*(-y) f_1(y)\, d\sigma(y) \\ &= \int_{\partial\Omega} E(x-y) f_1(y)\, d\sigma(y) - \int_{\partial\Omega} E_*(-y) f_1(y)\, d\sigma(y) \\ &= \big(S_{\mathrm{mod}} f_1\big)(x),\end{aligned} \tag{1.5.155}$$

making use of [70, Proposition 2.5.39]. In turn, from (1.5.117) and (1.5.154)-(1.5.155) we conclude that

$$\Big(\big(\mathscr{S}_{\mathrm{mod}} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) = (S_{\mathrm{mod}} f)(x) \text{ at } \sigma\text{-a.e. } x \in B(x_0, r) \cap A_\kappa(\partial\Omega). \tag{1.5.156}$$

Upon recalling that $r > 0$ has been arbitrarily chosen, this ultimately proves (1.5.80).

The claim made in relation to (1.5.81) is a consequence of (1.5.80) (and [68, Proposition 8.8.4]), (1.5.56), and (1.3.67).

Let us justify the claim made in (1.5.83). To this end, fix $j, k \in \{1, \dots, n\}$ and some function $f \in \Big[L^p_{\mathrm{loc}}(\partial\Omega, \sigma) \cap L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ with $p \in (1,\infty)$. Also, choose some aperture parameter $\kappa \in (0,\infty)$. Then we may write

$$\begin{aligned}
&\partial_{\tau_{jk}}\big(S_{\rm mod} f\big)(x)\\
&\quad= \partial_{\tau_{jk}}\Big(\big(\mathscr{S}_{\rm mod} f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)(x)\\
&\quad= \nu_j(x)\Big(\Big(\partial_k\big(\mathscr{S}_{\rm mod} f\big)\Big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)(x) - \nu_k(x)\Big(\Big(\partial_j\big(\mathscr{S}_{\rm mod} f\big)\Big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)(x)\\
&\quad= \nu_j(x)\Big\{\frac{1}{2{\rm i}}\widehat{\partial_k E}\big(\nu(x)\big)f(x) + \lim_{\varepsilon\to0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}(\partial_k E)(x-y)f(y)\,{\rm d}\sigma(y)\Big\}\\
&\qquad - \nu_k(x)\Big\{\frac{1}{2{\rm i}}\widehat{\partial_j E}\big(\nu(x)\big)f(x) + \lim_{\varepsilon\to0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}(\partial_j E)(x-y)f(y)\,{\rm d}\sigma(y)\Big\}\\
&\quad= \big(T^{\#}_{jk} f\big)(x) \ \text{ at } \sigma\text{-a.e. } x\in\partial_*\Omega. \qquad (1.5.157)
\end{aligned}$$

The first equality in (1.5.157) uses indentity (1.5.80) and [68, Proposition 8.8.4]. The second equality in (1.5.157) is a consequence of [69, Proposition 11.3.2] applied here with $u := \mathscr{S}_{\rm mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M$, whose applicability in the current setting is ensured by (1.5.69), (1.5.54), (1.5.79), [68, Proposition 8.8.4], (1.5.51), and [70, Theorem 2.5.1]. The third equality in (1.5.157) is implied by [70, (2.5.4)] (keeping in mind (1.5.51)). Finally, the last equality in (1.5.157) is seen from (1.5.82) upon observing that at σ-a.e. point $x \in \partial_*\Omega$ we have

$$\begin{aligned}
&\nu_j(x)\widehat{\partial_k E}\big(\nu(x)\big) - \nu_k(x)\widehat{\partial_j E}\big(\nu(x)\big)\\
&\qquad = {\rm i}\nu_j(x)\nu_k(x)\big[L\big(\nu(x)\big)\big]^{-1} - {\rm i}\nu_k(x)\nu_j(x)\big[L\big(\nu(x)\big)\big]^{-1} = 0, \qquad (1.5.158)
\end{aligned}$$

thanks to [70, Theorem 1.4.2]. This establishes (1.5.83).

Proof of claims in item (xiii): Throughout, work under the assumption that Ω is a UR domain. To deal with the claims in (1.5.85) assume first that either Ω is bounded, or $\partial\Omega$ is unbounded. Pick a function $f \in \big[L^{p,q}_1(\partial\Omega,\sigma)\big]^M$ with $p \in (1,\infty)$ and $q \in (1, n-1)$, then introduce $u := \mathcal{D} f$ in Ω. From (1.5.8), (1.5.7), (1.5.20), (1.3.24), and [68, Lemma 7.2.1], it follows that u satisfies the hypotheses of Theorem 1.3.3. In view of (1.5.20) and (1.3.24), the integral representation formula (1.3.35) presently becomes

$$\mathcal{D} f = \mathcal{D}\Big(\big(\tfrac12 I + K\big)f\Big) - \mathscr{S}\big(\partial^A_\nu \mathcal{D} f\big) \ \text{ in } \Omega. \qquad (1.5.159)$$

On account of (1.5.20) and (1.5.44), taking nontangential traces produces

$$\big(\tfrac12 I + K\big)f = \big(\tfrac12 I + K\big)\big(\tfrac12 I + K\big)f - S\big(\partial^A_\nu \mathcal{D} f\big) \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \qquad (1.5.160)$$

After some simple algebra (1.5.160) implies $\big(\frac{1}{2}I+K\big)\big(-\frac{1}{2}I+K\big)f = S\big(\partial_\nu^A \mathcal{D} f\big)$ at σ-a.e. point on $\partial\Omega$. When Ω is an exterior domain, the same argument as above with Ω replaced by $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ leads to the same conclusion. Here, item *(7)* in [68, Lemma 5.10.9] is relevant. Hence, in all cases,

$$\begin{gathered}\big(\tfrac{1}{2}I+K\big)\circ\big(-\tfrac{1}{2}I+K\big) = S\circ\big(\partial_\nu^A\mathcal{D}\big) \ \text{ on } \ \big[L_1^{p,q}(\partial\Omega,\sigma)\big]^M \\ \text{with } \ p\in(1,\infty) \ \text{ and } \ q\in(1,n-1).\end{gathered} \tag{1.5.161}$$

This takes care of the first claim in (1.5.85). The second claim in (1.5.85) corresponds to functions belonging to $\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p\in\big(\frac{n-1}{n-2},\infty\big)$. Having fixed such a p, select $q\in(1,n-1)$ with the property that $q^* = p$ (i.e., $1/p = 1/q - 1/(n-1)$). Then (1.5.161) ensures that we have $\big(\frac{1}{2}I+K\big)\circ\big(-\frac{1}{2}I+K\big) = S\circ\big(\partial_\nu^A\mathcal{D}\big)$ on $\big[L_1^{q^*,q}(\partial\Omega,\sigma)\big]^M$, while from [69, (11.1.66)] we know that the latter space embeds densely into $\big[L^p(\partial\Omega,\sigma)\big]^M$. Based on this and the continuity properties of the operators involved (established earlier in (1.5.33) with $p := q^*$, and (1.5.28) with $p := q$, as well as in the current item *(iii)*), the second claim in (1.5.85) follows. This finishes the proof of (1.5.85).

Turning attention to (1.5.86), we first make the assumption that either Ω is bounded, or $\partial\Omega$ is unbounded. Select $f\in\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p\in(1,n-1)$ and define $u := \mathscr{S}f$ in Ω. From (1.5.48), (1.3.67), (1.3.57), (1.3.8), and [68, Lemma 7.2.1], it follows that u satisfies the hypotheses of Theorem 1.3.3. Thanks to (1.5.44), (1.5.59), and (1.3.8), the integral representation formula (1.3.35) currently reads

$$\mathscr{S}f = \mathcal{D}(Sf) - \mathscr{S}\Big(\big(-\tfrac{1}{2}I+K^{\#}_{A^\top}\big)f\Big) \ \text{ in } \ \Omega. \tag{1.5.162}$$

In view of (1.5.59), taking the conormal derivative ∂_ν^A of both sides then shows that, at σ-a.e. point on $\partial\Omega$, we have

$$\big(-\tfrac{1}{2}I+K^{\#}_{A^\top}\big)f = \big(\partial_\nu^A\mathcal{D}\big)(Sf) - \big(-\tfrac{1}{2}I+K^{\#}_{A^\top}\big)\big(-\tfrac{1}{2}I+K^{\#}_{A^\top}\big)f. \tag{1.5.163}$$

This yields, after some simple algebra, $\big(\frac{1}{2}I+K^{\#}_{A^\top}\big)\big(-\frac{1}{2}I+K^{\#}_{A^\top}\big)f = \big(\partial_\nu^A\mathcal{D}\big)(Sf)$ at σ-a.e. point on $\partial\Omega$. If Ω is an exterior domain, we run the same argument as above with Ω replaced by $\Omega_- := \mathbb{R}^n\setminus\overline{\Omega}$ and, in view of item *(7)* in [68, Lemma 5.10.9], reach the same conclusion. This establishes that, in all cases, we have

$$\begin{gathered}\big(\tfrac{1}{2}I+K^{\#}_{A^\top}\big)\circ\big(-\tfrac{1}{2}I+K^{\#}_{A^\top}\big) = \big(\partial_\nu^A\mathcal{D}\big)\circ S \\ \text{on } \ \big[L^p(\partial\Omega,\sigma)\big]^M \ \text{ with } \ p\in(1,n-1).\end{gathered} \tag{1.5.164}$$

With this in hand, the second version in (1.5.86) may then be justified by a density argument based on [69, (11.8.30)] and continuity properties of the operators involved established earlier (cf. (1.5.36), (1.5.33), and (1.5.28)). This completes the proof of (1.5.86).

To justify the claims in (1.5.87), assume for now that either Ω is bounded, or $\partial\Omega$ is unbounded. Fix $f\in\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p\in(1,n-1)$ and recall the integral identity established in (1.5.162). Taking nontangential traces then yields, on account of (1.5.20) and (1.5.44),

$$Sf = \big(\tfrac{1}{2}I + K\big)(Sf) - S\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)f\Big) \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{1.5.165}$$

from which we conclude, after some simple algebra, that $K(Sf) = S(K^{\#}_{A^{\top}}f)$ at σ-a.e. point on $\partial\Omega$. In the case when Ω is an exterior domain, we run the same argument as above with Ω replaced by $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ and, thanks to item *(7)* in [68, Lemma 5.10.9], reach the same conclusion. This proves that, in all cases, we have

$$S \circ K^{\#}_{A^{\top}} = K \circ S \text{ on } \big[L^p(\partial\Omega, \sigma)\big]^M \text{ with } p \in (1, n-1). \tag{1.5.166}$$

Having established this, the second version in (1.5.87) then follows by a density argument based on [69, (11.8.30)] and continuity properties of the operators involved established earlier (cf. (1.5.36), (1.5.28), and the current item *(iii)*). This finishes the proof of (1.5.87).

Going further, consider the claims in (1.5.88). In a first stage, assume that either Ω is bounded, or $\partial\Omega$ is unbounded. Also, pick $f \in \big[L^{p,q}_1(\partial\Omega, \sigma)\big]^M$ with $p \in (1, \infty)$ and $q \in (1, n-1)$. Then, as in the past, (1.5.159) holds. By applying the conormal derivative operator ∂^A_ν to both sides of (1.5.159) we arrive at the conclusion that

$$\partial^A_\nu \mathcal{D}f = \big(\partial^A_\nu \mathcal{D}\big)\big(\tfrac{1}{2}I + K\big)f - \big(-\tfrac{1}{2}I + K^{\#}_{A^{\top}}\big)\big(\partial^A_\nu \mathcal{D}f\big) \tag{1.5.167}$$

at σ-a.e. point on $\partial\Omega$. Hence,

$$\begin{gathered} K^{\#}_{A^{\top}} \circ \big(\partial^A_\nu \mathcal{D}\big) = \big(\partial^A_\nu \mathcal{D}\big) \circ K \text{ on } \big[L^{p,q}_1(\partial\Omega, \sigma)\big]^M \\ \text{with } p \in (1, \infty) \text{ and } q \in (1, n-1). \end{gathered} \tag{1.5.168}$$

When Ω is an exterior domain, the above argument with Ω replaced by $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ produces the same conclusion (once again, item *(7)* in [68, Lemma 5.10.9] is relevant here). Having proved (1.5.168), the version of the identity in (1.5.88) for functions in $\big[L^p(\partial\Omega, \sigma)\big]^M$ with $p \in (1, \infty)$ follows from this and a density argument (making use of the continuity properties of the operators involved; cf. the current items *(iii)*, *(vi)*, and *(viii)*). This finishes the proof of (1.5.88).

Moving on, the first equality in (1.5.89) corresponding to $f \in \big[L^p(\partial\Omega, \sigma)\big]^M$ with $p \in (1, n-1)$ is a consequence of (1.5.86), (1.5.29), and (1.5.84). Extending this formula to the case when $p \in (1, \infty)$ is done via density, given that all operators involved are bounded on $\big[L^p(\partial\Omega, w\sigma)\big]^M$ (cf. [70, (2.3.56)]). Having established this, the second equality in (1.5.89) follows from the first, keeping in mind (1.5.83) and [68, (7.7.104)].

An alternative proof of the first equality in (1.5.89) goes as follows. Pick an arbitrary function $f \in \big[L^p(\partial\Omega, \sigma)\big]^M$ with $p \in (1, \infty)$, then consider $u := \mathscr{S}_{\text{mod}} f$ in Ω. Also, fix an arbitrary aperture parameter $\kappa \in (0, \infty)$. Results in the current item *(xi)* then imply

$$\begin{aligned} & u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma), \\ & \text{the boundary traces } u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}, (\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \\ & u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = S_{\text{mod}} f \text{ and } \partial_\nu^A u = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f. \end{aligned} \tag{1.5.169}$$

In the case when Ω is an exterior domain we also have

$$⨍_{B(0,2R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{1.5.170}$$

In concert with the last two equalities in (1.5.157) the first trace formula in the last line of (1.5.169) shows that for each $j, k \in \{1, \dots, n\}$ and $\alpha \in \{1, \dots, M\}$ we have

$$\begin{aligned} & \nu_j(\partial_k u_\alpha)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - \nu_k(\partial_j u_\alpha)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = (T^\#_{jk} f)_\alpha \\ & \text{at } \sigma\text{-a.e. point on } \partial\Omega, \end{aligned} \tag{1.5.171}$$

where the singular integral operator $T^\#_{jk}$ is defined as in (1.5.82). As before, employ the notation $E = (E_{\gamma\beta})_{1 \leq \gamma,\beta \leq M}$ for the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2]. For each $\ell \in \{1, \dots, n\}$ and each $\gamma \in \{1, \dots, M\}$ we may then rely on (1.5.169)-(1.5.171), [70, (1.5.230)], and (A.0.184) to write

$$\begin{aligned} (\partial_\ell u_\gamma)(x) = & \int_{\partial\Omega} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x - y)(T^\#_{\ell s} f)_\alpha(y) \, d\sigma(y) \\ & - \int_{\partial\Omega} (\partial_\ell E_{\gamma\alpha})(x - y)\Big(\big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f\Big)_\alpha(y) \, d\sigma(y) \end{aligned} \tag{1.5.172}$$

at each point $x \in \Omega$. To proceed, fix an arbitrary index $\mu \in \{1, \dots, M\}$. Multiply both sides of (1.5.172) by $a^{\mu\gamma}_{j\ell}$, then go nontangentially to the boundary, subsequently multiply by ν_j, and finally sum up over $j \in \{1, \dots, n\}$. On account of the second trace formula in the last line of (1.5.169) and [70, (1.5.230)], we therefore arrive at

$$\begin{aligned} & \Big(\big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f\Big)_\mu(x) \\ & \quad = \big(\partial_\nu^A u\big)_\mu(x) = \nu_j(x) a^{\mu\gamma}_{j\ell} (\partial_\ell u_\gamma)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) \\ & \quad = \nu_j(x) \bigg[\int_{\partial\Omega} a^{\mu\gamma}_{j\ell} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(\cdot - y)(T^\#_{\ell s} f)_\alpha(y) \, d\sigma(y)\bigg]\bigg|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) \\ & \qquad - \nu_j(x)\bigg[\int_{\partial\Omega} a^{\mu\gamma}_{j\ell} (\partial_\ell E_{\gamma\alpha})(\cdot - y)\Big(\big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f\Big)_\alpha(y) \, d\sigma(y)\bigg]\bigg|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) \end{aligned} \tag{1.5.173}$$

at σ-a.e. point $x \in \partial\Omega$. In view of [70, (2.5.4)], the term involving the first nontangential trace above may be explicitly identified as

$$\nu_j(x)\bigg[\int_{\partial\Omega} a^{\mu\gamma}_{j\ell}\, a^{\beta\alpha}_{rs}\,(\partial_r E_{\gamma\beta})(\cdot - y)(T^{\#}_{\ell s} f)_\alpha(y)\,\mathrm{d}\sigma(y)\bigg]\bigg|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}(x) \tag{1.5.174}$$

$$= \nu_j(x) a^{\mu\gamma}_{j\ell}\, a^{\beta\alpha}_{rs}\,\frac{1}{2\mathrm{i}}\widehat{\partial_r E_{\gamma\beta}}\big(\nu(x)\big)(T^{\#}_{\ell s} f)_\alpha(x)$$

$$+ \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_j(x) a^{\mu\gamma}_{j\ell}\, a^{\beta\alpha}_{rs}\,(\partial_r E_{\gamma\beta})(x-y)(T^{\#}_{\ell s} f)_\alpha(y)\,\mathrm{d}\sigma(y).$$

As regards the first term in the right-hand side of the equality in (1.5.174), at σ-a.e. point $x \in \partial\Omega$ we have

$$\nu_j(x) a^{\mu\gamma}_{j\ell}\, a^{\beta\alpha}_{rs}\,\frac{1}{2\mathrm{i}}\widehat{\partial_r E_{\gamma\beta}}\big(\nu(x)\big)(T^{\#}_{\ell s} f)_\alpha(x) \tag{1.5.175}$$

$$= \frac{1}{2}\,\nu_j(x) a^{\mu\gamma}_{j\ell}\, a^{\beta\alpha}_{rs}\,\nu_r(x)\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta}(T^{\#}_{\ell s} f)_\alpha(x)$$

$$= \lim_{\varepsilon\to 0^+}\frac{1}{2}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_j(x) a^{\mu\gamma}_{j\ell}\, a^{\beta\alpha}_{rs}\,\nu_r(x)\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta}\times$$

$$\times\Big\{\nu_\ell(x)(\partial_s E_{\alpha\delta})(x-y) - \nu_s(x)(\partial_\ell E_{\alpha\delta})(x-y)\Big\} f_\delta(y)\,\mathrm{d}\sigma(y),$$

thanks to [70, (1.4.30)] and (1.5.82). In view of (A.0.141), for each $s \in \{1,\dots,n\}$ and $\alpha \in \{1,\dots,M\}$ we have

$$\nu_j(x) a^{\mu\gamma}_{j\ell}\, a^{\beta\alpha}_{rs}\,\nu_r(x)\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta}\nu_\ell(x) = -a^{\beta\alpha}_{rs}\,\nu_r(x)\big[L\big(\nu(x)\big)\big]_{\mu\gamma}\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta}$$

$$= -a^{\beta\alpha}_{rs}\,\nu_r(x)\delta_{\mu\beta} = -a^{\mu\alpha}_{rs}\,\nu_r(x), \tag{1.5.176}$$

while for each $\ell \in \{1,\dots,n\}$ and $\alpha \in \{1,\dots,M\}$ we have

$$\nu_j(x) a^{\mu\gamma}_{j\ell}\, a^{\beta\alpha}_{rs}\,\nu_r(x)\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta}\nu_s(x) = -\nu_j(x) a^{\mu\gamma}_{j\ell}\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta}\big[L\big(\nu(x)\big)\big]_{\beta\alpha}$$

$$= -\nu_j(x) a^{\mu\gamma}_{j\ell}\,\delta_{\gamma\alpha} = -a^{\mu\alpha}_{j\ell}\,\nu_j(x). \tag{1.5.177}$$

Combining (1.5.175)-(1.5.177) leads to the conclusion that

$$
\begin{aligned}
&\nu_j(x)a^{\mu\gamma}_{j\ell}a^{\beta\alpha}_{rs}\frac{1}{2\mathrm{i}}\widehat{\partial_r E_{\gamma\beta}}\big(\nu(x)\big)(T^{\#}_{\ell s}f)_\alpha(x)\\
&\quad=\lim_{\varepsilon\to 0^+}\frac{-1}{2}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\Big\{a^{\mu\alpha}_{rs}\nu_r(x)(\partial_s E_{\alpha\delta})(x-y)\\
&\qquad\qquad -a^{\mu\alpha}_{j\ell}\nu_j(x)(\partial_\ell E_{\alpha\delta})(x-y)\Big\}f_\delta(y)\,\mathrm{d}\sigma(y)\\
&\quad=0\ \text{ at }\sigma\text{-a.e. point }x\in\partial\Omega.
\end{aligned}\tag{1.5.178}
$$

Going further, the term involving the second nontangential trace in (1.5.173) is equal to (cf. [70, (1.5.230)], (1.5.51), and (1.5.58)),

$$
\nu_j(x)\Bigg[\int_{\partial\Omega}a^{\mu\gamma}_{j\ell}(\partial_\ell E_{\gamma\alpha})(\cdot-y)\Big(\big(-\tfrac{1}{2}I+K^{\#}_{A^\top}\big)f\Big)_\alpha(y)\,\mathrm{d}\sigma(y)\Bigg]\Bigg|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}(x)\tag{1.5.179}
$$

$$
=\Big(\partial^A_\nu\mathscr{S}_{\mathrm{mod}}\big(-\tfrac{1}{2}I+K^{\#}_{A^\top}\big)f\Big)_\mu(x)=\Big(\big(-\tfrac{1}{2}I+K^{\#}_{A^\top}\big)\big(-\tfrac{1}{2}I+K^{\#}_{A^\top}\big)f\Big)_\mu(x)
$$

at σ-a.e. point $x\in\partial\Omega$. At this stage, from (1.5.173), (1.5.174), (1.5.178), and (1.5.179) we readily conclude that the first equality in (1.5.89) holds for each function $f\in\big[L^p(\partial\Omega,\sigma)\big]^M$ with $p\in(1,\infty)$. The more general case when the function $f\in\big[L^p(\partial\Omega,w\sigma)\big]^M$ for some $w\in A_p(\partial\Omega,\sigma)$ with $p\in(1,\infty)$ then follows from this and density, keeping in mind the continuity of all singular integral operators involved on Muckenhoupt weighted Lebesgue spaces.

Finally, that in (1.5.85)-(1.5.88) we may allow $p,q,p^*\in(1,\infty)$ unrelated and unrestricted when $\partial\Omega$ is bounded, follows from the fact that the Lebesgue scale is nested, the addenda to the Fractional Integration Theorem in [68, (7.8.14)-(7.8.15)], and keeping in mind that the weight functions in [70, (1.5.3)] now behave like constants.

Proof of claims in item (xiv): Let us establish the continuity of the operator-valued assignment in (1.5.91) for some fixed integrability exponent $p\in(1,\infty)$. Fix a weakly elliptic coefficient tensor A_0 and pick another coefficient tensor A_1 with $|A_0-A_1|$ small enough. Define

$$
A_t:=A_0+t(A_1-A_0)\ \text{ for each }\ t\in[0,1].\tag{1.5.180}
$$

Having fixed $j\in\{1,\dots,n\}$, let $\partial_j S_{A_t}$ be the principal-value singular integral operator on $\partial\Omega$ whose kernel is $(\partial_j E_{A_t})(x-y)$. Then $\partial_j S_{A_0}-\partial_j S_{A_1}$ is a principal-value singular integral operator on $\partial\Omega$ whose kernel is $k(x-y)$ where, for each $z\in\mathbb{R}^n\setminus\{0\}$,

$$
k(z):=-\int_0^1 k_t(z)\,\mathrm{d}t\tag{1.5.181}
$$

with

$$k_t(z) := \frac{\mathrm{d}}{\mathrm{d}t}(\partial_j E_{A_t})(z) \text{ for every } t \in [0,1]. \tag{1.5.182}$$

In relation to this, we claim that

$$k_t \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^{M\times M} \text{ for every } t \in [0,1], \tag{1.5.183}$$

$$k_t(-z) = -k_t(z) \text{ for every } z \in \mathbb{R}^n \setminus \{0\} \text{ and } t \in [0,1], \tag{1.5.184}$$

$$k_t(\lambda z) = \lambda^{1-n} k_t(z) \text{ for every } z \in \mathbb{R}^n \setminus \{0\},\ \lambda > 0, \text{ and } t \in [0,1], \tag{1.5.185}$$

for every $N \in \mathbb{N}$ there exists $C = C(N, A_0) \in (0,\infty)$ such that

$$\sup_{t\in[0,1]} \big\|k_t\big|_{S^{n-1}}\big\|_{\mathscr{C}^N(S^{n-1})} \leq C|A_0 - A_1|. \tag{1.5.186}$$

Let us assume for the time being that (1.5.183)-(1.5.186) hold and indicate how these may be used to finish the proof of the claim in item *(15)*. Concretely, (1.5.183)-(1.5.186) ensure that k from (1.5.181) satisfies the properties listed in [70, (2.3.3)]. As such, [70, Theorem 2.3.2] is applicable to the operator $\partial_j S_{A_0} - \partial_j S_{A_1}$. In particular, [70, (2.3.20)] permits us to estimate

$$\begin{aligned} \big\|\partial_j S_{A_0} - \partial_j S_{A_1}\big\|_{[L^p(\partial\Omega,\sigma)]^M \to [L^p(\partial\Omega,\sigma)]^M} &\leq C\big\|k\big|_{S^{n-1}}\big\|_{\mathscr{C}^N(S^{n-1})} \\ &\leq C|A_0 - A_1|, \end{aligned} \tag{1.5.187}$$

where $C \in (0,\infty)$ depends only on n, p, A_0, as well as the Ahlfors regularity and UR constants of $\partial\Omega$. Of course, once this has been established, the continuity of the operator-valued assignment in (1.5.91) follows.

There remains to prove (1.5.183)-(1.5.186), a task to which we now turn. The key ingredient in this regard is having a more transparent formula for $k_t(z)$. To set the stage, write

$$\big(b_{rs}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}} := A_1 - A_0 \tag{1.5.188}$$

so that

$$\big(a_{rs}^{\alpha\beta}(t)\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}} := A_t = A_0 + t\cdot\big(b_{rs}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}} \text{ for each } t \in [0,1], \tag{1.5.189}$$

then for each $t \in [0,1]$ introduce

$$\Theta_t(\xi) := \frac{\mathrm{d}}{\mathrm{d}t}\Big[\big(a_{rs}^{\alpha\beta}(t)\xi_r\xi_s\big)_{1\leq\alpha,\beta\leq M}\Big]^{-1}, \qquad \forall\,\xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.5.190}$$

From (1.5.182) and (A.0.66) we then conclude that, for each $t \in [0,1]$ and each point $z \in \mathbb{R}^n \setminus \{0\}$,

$$k_t(z) = \frac{\partial}{\partial z_j}\left\{\frac{1}{4(2\pi \mathrm{i})^{n-1}}\Delta_z^{(n-1)/2}\int_{S^{n-1}} |\langle z,\xi\rangle|\Theta_t(\xi)\,\mathrm{d}\mathcal{H}^{n-1}(\xi)\right\} \tag{1.5.191}$$

if n is odd, and

$$k_t(z) = \frac{\partial}{\partial z_j}\left\{\frac{-1}{(2\pi \mathrm{i})^n}\Delta_z^{(n-2)/2}\int_{S^{n-1}}\big(\ln|\langle z,\xi\rangle|\big)\Theta_t(\xi)\,\mathrm{d}\mathcal{H}^{n-1}(\xi)\right\} \tag{1.5.192}$$

if n is even. To analyze (1.5.191)-(1.5.192), let us take a closer look at $\Theta_t(\xi)$ in (1.5.190). Pick an arbitrary number $t_o \in (0,1)$ and, for $\xi \in \mathbb{R}^n \setminus \{0\}$ fixed, compute

$$\begin{aligned}
&\Theta_{t_o}(\xi) \qquad\qquad (1.5.193)\\
&= \lim_{t\to 0}\frac{\left[\big(a_{rs}^{\alpha\beta}(t_o+t)\xi_r\xi_s\big)_{1\leq\alpha,\beta\leq M}\right]^{-1} - \left[\big(a_{rs}^{\alpha\beta}(t_o)\xi_r\xi_s\big)_{1\leq\alpha,\beta\leq M}\right]^{-1}}{t}\\
&= -\left[\big(a_{rs}^{\alpha\beta}(t_o)\xi_r\xi_s\big)_{1\leq\alpha,\beta\leq M}\right]^{-1}\times\\
&\quad\times\left(\lim_{t\to 0}\frac{\big(a_{rs}^{\alpha\beta}(t_o+t)\xi_r\xi_s\big)_{1\leq\alpha,\beta\leq M} - \big(a_{rs}^{\alpha\beta}(t_o)\xi_r\xi_s\big)_{1\leq\alpha,\beta\leq M}}{t}\right)\times\\
&\quad\times\left[\big(a_{rs}^{\alpha\beta}(t_o)\xi_r\xi_s\big)_{1\leq\alpha,\beta\leq M}\right]^{-1}\\
&= -\left[\big(a_{rs}^{\alpha\beta}(t_o)\xi_r\xi_s\big)_{1\leq\alpha,\beta\leq M}\right]^{-1}\big(b_{rs}^{\alpha\beta}\xi_r\xi_s\big)_{1\leq\alpha,\beta\leq M}\left[\big(a_{rs}^{\alpha\beta}(t_o)\xi_r\xi_s\big)_{1\leq\alpha,\beta\leq M}\right]^{-1}.
\end{aligned}$$

In turn, from (1.5.193) we deduce that

$$\Theta_t \in \big[\mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})\big]^{M\times M} \text{ for every } t\in[0,1], \tag{1.5.194}$$

$$\Theta_t(\xi) \text{ is even in } \xi\in\mathbb{R}^n\setminus\{0\} \text{ for every } t\in[0,1], \tag{1.5.195}$$

$$\Theta_t(\lambda\xi) = \lambda^{-2}\Theta_t(\xi) \text{ for every } \xi\in\mathbb{R}^n\setminus\{0\},\ \lambda\in\mathbb{C}\setminus\{0\}, \text{ and } t\in[0,1], \tag{1.5.196}$$

$$\sup_{0\leq t\leq 1}|\Theta_t(\xi)| \leq C|\xi|^{-2}|A_0 - A_1| \text{ for every } \xi\in\mathbb{R}^n\setminus\{0\}, \tag{1.5.197}$$

where $C\in(0,\infty)$ depends only on n, M, and A_0 (recall that A_1 is taken sufficiently close to A_0 to begin with). Hence $\Theta_t(\xi)$ behaves, in a quantitative fashion, uniformly in t, like the inverse $[L(\xi)]^{-1}$ appearing in (A.0.66). Bearing this in mind, we run the same type of argument as in the proof of [66, Theorem 11.1, pp. 393-395] which, starting with (A.0.66), establishes items *(1)*, *(3)*, *(4)* in [70, Theorem 1.4.2] (dealing, respectively, with smoothness, parity, homogeneity, and estimates for derivatives of the fundamental solution E), but for the function k_t from (1.5.191)-(1.5.192) in place of E. In the current case, this produces (1.5.183)-(1.5.186) and this finishes the proof of the claim made in relation to (1.5.91). The remaining claims in item *(xiv)* of the current theorem are dealt with analogously, making use of [70, (2.3.21), (2.3.53), (2.3.57)].

Proof of claims in item (xv): All desired conclusions follow from the continuity results established in item *(xiv)*, (1.3.68), (1.3.72), (1.5.29), (1.3.62), and the current item *(viii)* (cf. also Proposition 1.5.6 a little further below).

The proof of Theorem 1.5.1 is therefore complete. □

Theorem 1.5.1 is a powerful tool, which may be used with great effect for a variety of purposes. Below, we discuss a number of such applications. First, in Proposition 1.5.2 we shall employ Theorem 1.5.1 to shed further light on the structure of our brand of boundary Sobolev spaces on uniformly rectifiable sets.

Proposition 1.5.2 *Pick an integer $n \in \mathbb{N}$ with $n \geq 2$ and assume $\Omega \subsetneq \mathbb{R}^n$ is a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, then introduce $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$. Also, fix an integrability exponent $p \in (1,\infty)$ along with an aperture parameter $\kappa \in (0,\infty)$. In this setting, consider the class of functions $u^\pm : \Omega_\pm \to \mathbb{C}$ satisfying*

$$\begin{gathered} u^\pm \in \mathscr{C}^\infty(\Omega_\pm), \\ \mathcal{N}_\kappa u^\pm \in L^p(\partial\Omega,\sigma),\ \ \mathcal{N}_\kappa(\nabla u^\pm) \in L^p(\partial\Omega,\sigma), \\ u^\pm\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \textit{and}\ \big(\nabla u^\pm\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\ \textit{exist}\ \sigma\textit{-a.e. on}\ \partial\Omega. \end{gathered} \tag{1.5.198}$$

Then the L^p-based Sobolev space of order one on $\partial\Omega$ may be described as

$$L^p_1(\partial\Omega,\sigma) = \Big\{u^+\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} - u^-\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ :\ u^\pm \ \textit{are as in}\ (1.5.198)\Big\}. \tag{1.5.199}$$

Proof From item *(7)* in [68, Lemma 5.10.9] we know that Ω_- is also a UR domain, whose topological boundary agrees with that of $\Omega_+ = \Omega$. Granted this, the right-to-left inclusion in (1.5.199) becomes a consequence of [69, Proposition 11.3.2].

To prove the left-to-right inclusion in (1.5.199), pick $f \in L^p_1(\partial\Omega,\sigma)$ arbitrary and define

$$u^\pm(x) := \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \frac{\langle\nu(y), y-x\rangle}{|x-y|^n} f(y)\,d\sigma(y) \ \text{ for } x \in \Omega_\pm, \tag{1.5.200}$$

where ν is the geometric measure theoretic outward unit normal to Ω. Also, recall the principal-value harmonic double layer K associated with Ω as in [70, (2.5.203)]. From [70, Theorem 2.4.1] we deduce that

$$\big\|\mathcal{N}_\kappa u^\pm\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p(\partial\Omega,\sigma)}, \tag{1.5.201}$$

while from [70, (2.5.84)] we conclude that

$$u^\pm\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\pm\tfrac{1}{2}I + K\big)f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{1.5.202}$$

In particular,

$$u^+\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} - u^-\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{1.5.203}$$

Since from Theorem 1.5.1 we also know that

$$\begin{aligned}&\text{there exist } \nabla u^{\pm}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ at } \sigma\text{-a.e. point on } \partial\Omega,\\ &\text{and} \quad \big\|\mathcal{N}_\kappa(\nabla u^{\pm})\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p_1(\partial\Omega,\sigma)},\end{aligned} \tag{1.5.204}$$

we see that the functions $u^{\pm}$ are as in (1.5.198), so the desired conclusion follows.□

Next, we rely on Theorem 1.5.1 to show that the point-spectra of K and $K^{\#}_{A^{\top}}$ (acting on Lebesgue and Sobolev spaces on the boundary of a given UR domain) are actually closely related.

Proposition 1.5.3 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a* UR *domain with compact boundary and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. For some $M \in \mathbb{N}$, consider a coefficient tensor A with complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in* (1.3.2) *is weakly elliptic (in the sense of* [70, (1.3.3) in Definition 1.3.1]*). Recall the principal-value double layer K from* (1.3.68)*, and let $K^{\#}_{A^{\top}}$ be associated with $A^{\top}$ and Ω as in* (1.3.72)*. Finally, fix $p \in (1,\infty)$ along with $\lambda \in \mathbb{C} \setminus \{\pm\frac{1}{2}\}$.*

Then λ is an eigenvalue of K acting on $\big[L^p_1(\partial\Omega,\sigma)\big]^M$ (respectively, on $\big[L^p(\partial\Omega,\sigma)\big]^M$) if and only if λ is an eigenvalue of $K^{\#}_{A^{\top}}$ acting on $\big[L^p(\partial\Omega,\sigma)\big]^M$ (respectively, on $\big[L^p_{-1}(\partial\Omega,\sigma)\big]^M$).

Proof Assume first that $\lambda \in \mathbb{C}\setminus\{\pm\frac{1}{2}\}$ is an eigenvalue of K acting on $\big[L^p_1(\partial\Omega,\sigma)\big]^M$. Thus, there exists $f \in \big[L^p_1(\partial\Omega,\sigma)\big]^M$, $f \neq 0$, such that $Kf = \lambda f$. Then, if $\partial^A_\nu\mathcal{D}$ is as in item *(vii)* of Theorem 1.5.1, the function $g := \partial^A_\nu\mathcal{D}f$ belongs to $\big[L^p(\partial\Omega,\sigma)\big]^M$ (cf. (1.5.31)) and item *(xiii)* in Theorem 1.5.1 (cf. (1.5.88)) permits us to write

$$\begin{aligned}K^{\#}_{A^{\top}}g &= K^{\#}_{A^{\top}}\big(\partial^A_\nu\mathcal{D}\big)f = \big(\partial^A_\nu\mathcal{D}\big)(Kf)\\ &= \big(\partial^A_\nu\mathcal{D}\big)(\lambda f) = \lambda\big(\partial^A_\nu\mathcal{D}\big)f = \lambda g.\end{aligned} \tag{1.5.205}$$

Also, if S is the boundary-to-boundary single layer potential operator associated with A and Ω as in (1.3.62), based on item *(xiii)* in Theorem 1.5.1 (cf. (1.5.85)) we may write

$$Sg = S\big(\partial^A_\nu\mathcal{D}\big)f = \big(\tfrac{1}{2}I + K\big)\big(-\tfrac{1}{2}I + K\big)f = \big(\lambda^2 - \tfrac{1}{4}\big)f \neq 0, \tag{1.5.206}$$

since $\lambda^2 \neq \frac{1}{4}$ and $f \neq 0$. In turn, this forces $g \neq 0$ which ultimately proves that λ is an eigenvalue for $K^{\#}_{A^{\top}}$ on $\big[L^p(\partial\Omega,\sigma)\big]^M$. The case when $\lambda \in \mathbb{C} \setminus \{\pm\frac{1}{2}\}$ is an eigenvalue of K acting on $\big[L^p(\partial\Omega,\sigma)\big]^M$ is treated in a completely similar fashion, bearing in mind (1.5.33) and the fact that the operator identities in (1.5.88) and (1.5.85) continue to be valid on $\big[L^p(\partial\Omega,\sigma)\big]^M$.

Going further, assume now that $\lambda \in \mathbb{C} \setminus \{\pm\frac{1}{2}\}$ is an eigenvalue of $K^{\#}_{A^{\top}}$ acting on $\big[L^p(\partial\Omega,\sigma)\big]^M$. Thus, there exists some $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$, $f \neq 0$, with the property that $K^{\#}_{A^{\top}}f = \lambda f$. Then $g := Sf$ belongs to the Sobolev space $\big[L^p_1(\partial\Omega,\sigma)\big]^M$ (cf.

item *(ix)* in Theorem 1.5.1) and item *(xiii)* in Theorem 1.5.1 (cf. (1.5.87)) permits us to write

$$Kg = K(Sf) = S\big(K^{\#}_{A^\top} f\big) = S(\lambda f) = \lambda S f = \lambda g. \tag{1.5.207}$$

In addition, thanks to item *(xiii)* in Theorem 1.5.1 (cf. (1.5.86)) we have

$$\big(\partial^A_\nu \mathcal{D}\big)g = \big(\partial^A_\nu \mathcal{D}\big)(Sf) = \big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f = \big(\lambda^2 - \tfrac{1}{4}\big)f \neq 0, \tag{1.5.208}$$

since $\lambda^2 \neq \frac{1}{4}$ and $f \neq 0$. Consequently, $g \neq 0$ which ultimately proves that λ is an eigenvalue for K on $\big[L^p_1(\partial\Omega, \sigma)\big]^M$. Finally, the case when $\lambda \in \mathbb{C} \setminus \{\pm\frac{1}{2}\}$ is an eigenvalue of $K^{\#}_{A^\top}$ acting on $\big[L^p_{-1}(\partial\Omega, \sigma)\big]^M$ is dealt with in a completely analogous fashion, once again based on item *(ix)* in Theorem 1.5.1 and the operator identities in (1.5.87), (1.5.86). □

When specialized to the case $L = \Delta$, the Laplacian in $\mathbb{R}^n$, formula (1.5.89) from Theorem 1.5.1 acquires a particularly pleasant format, as described in the proposition below.

Proposition 1.5.4 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. In this setting, bring in the "standard" principal-value harmonic double layer defined for each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ and σ-a.e. point $x \in \partial\Omega$ according to*

$$K_\Delta f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(y), y - x\rangle}{|x-y|^n} f(y)\, d\sigma(y), \tag{1.5.209}$$

together with its formal transpose, i.e., the singular integral operator acting on each given function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ as

$$K^{\#}_\Delta f(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(x), x - y\rangle}{|x-y|^n} f(y)\, d\sigma(y), \tag{1.5.210}$$

at σ-a.e. $x \in \partial\Omega$. Also, consider the modified version of the boundary-to-boundary harmonic single layer operator (compare with (1.5.73))

$$\begin{gathered} S_{\Delta,\mathrm{mod}} f(x) := \int_{\partial\Omega} \big\{E_\Delta(x-y) - E_\Delta(-y)\cdot \mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}(-y)\big\} f(y)\, d\sigma(y) \\ \text{for each } f \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big) \text{ and } \sigma\text{-a.e. point } x \in \partial\Omega. \end{gathered} \tag{1.5.211}$$

Additionally, for each pair of indices $j,k \in \{1,\dots,n\}$ consider the operators[16] *defined for each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ and σ-a.e. point $x \in \partial\Omega$ as*

$$T_{jk}f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \big\{\nu_j(y)(\partial_k E_\Delta)(x-y) - \nu_k(y)(\partial_j E_\Delta)(x-y)\big\} f(y)\,\mathrm{d}\sigma(y) \tag{1.5.212}$$

and (compare with (1.5.82))

$$T^\#_{jk}f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \big\{\nu_j(x)(\partial_k E_\Delta)(x-y) - \nu_k(x)(\partial_j E_\Delta)(x-y)\big\} f(y)\,\mathrm{d}\sigma(y), \tag{1.5.213}$$

where E_Δ is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$ recalled in (1.1.1). Finally, fix some integrability exponent $p \in (1,\infty)$ and pick some Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$. Then

$$\big(\tfrac{1}{2}I + K_\Delta\big)\big(-\tfrac{1}{2}I + K_\Delta\big) = \frac{1}{2}\sum_{j,k=1}^{n}(T_{jk})^2 \ \text{ on } \ L^p(\partial\Omega, w\sigma), \tag{1.5.214}$$

and

$$\begin{aligned}\big(\tfrac{1}{2}I + K^\#_\Delta\big)\big(-\tfrac{1}{2}I + K^\#_\Delta\big) &= \frac{1}{2}\sum_{j,k=1}^{n}(T^\#_{jk})^2 \\ &= \frac{1}{2}\sum_{j,k=1}^{n}\big(\partial_{\tau_{jk}} S_{\Delta,\mathrm{mod}}\big)^2 \ \text{ on } \ L^p(\partial\Omega, w\sigma).\end{aligned} \tag{1.5.215}$$

Proof Select an arbitrary function $f \in L^p(\partial\Omega, w\sigma)$. Then formula (1.5.89) permits us to write (with $a^{\alpha\beta}_{rs} := \delta_{rs}$, with $M := 1$, and with the Greek letter suppressed)

[16] these are the versions of the singular integral operators from (1.2.2)-(1.2.3) corresponding to choosing b to be the standard fundamental solution E_Δ for the Laplacian in $\mathbb{R}^n$

$$
\begin{aligned}
&\big(\tfrac{1}{2}I + K^{\#}_{\Delta}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{\Delta}\big)f\Big)(x)\\
&\quad = \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x)\delta_{ij}\delta_{rs}(\partial_r E_\Delta)(x-y)\big(T^{\#}_{js}f\big)(y)\,\mathrm{d}\sigma(y)\\
&\quad = \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_j(x)(\partial_s E_\Delta)(x-y)\big(T^{\#}_{js}f\big)(y)\,\mathrm{d}\sigma(y)\\
&\quad = \frac{1}{2}\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \Big\{\nu_j(x)(\partial_s E_\Delta)(x-y)\\
&\qquad\qquad\qquad - \nu_s(x)(\partial_j E_\Delta)(x-y)\Big\}\big(T^{\#}_{js}f\big)(y)\,\mathrm{d}\sigma(y)\\
&\quad = \frac{1}{2}\Big(T^{\#}_{js}\big(T^{\#}_{js}f\big)\Big)(x) \ \text{ at } \sigma\text{-a.e. point } x\in\partial\Omega,
\end{aligned}
\tag{1.5.216}
$$

with the summation convention over repeated indices in effect. In the third equality above we have used the fact that $T^{\#}_{js}$ is antisymmetric in j and s, while the final equality is seen from (1.5.213).

From (1.5.216) we then conclude that the first equality in (1.5.215) holds. The second equality in (1.5.215) is a consequence of (1.5.83) (plus the subsequent comment) and definitions. Finally, formula (1.5.214) is implied by the first equality in (1.5.215) and duality. □

Moving on, our next theorem elaborates on the properties of acoustic boundary layer potentials from [70, §6.4], now considered in open sets with uniformly rectifiable boundaries.

Theorem 1.5.5 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\geq 2$) is an open set with the property that $\partial\Omega$ is a compact* UR *set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu = (\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, for some $M\in\mathbb{N}$, consider a complex coefficient tensor $A := \big(a^{JI}_{rs}\big)_{\substack{1\leq J,I\leq M\\ 1\leq r,s\leq n}}$ which allows writing the vector Laplacian as $\Delta = \mathrm{div}A\nabla$, i.e.,*

$$
\Delta I_{M\times M} = \Big(\sum_{r,s=1}^{n} a^{JI}_{rs}\partial_r\partial_s\Big)_{1\leq J,I\leq M}
\tag{1.5.217}
$$

where $I_{M\times M}$ is the $M\times M$ identity matrix. Finally, fix a wave number $k\in(0,\infty)$ and recall from (A.0.75) the radiating fundamental solution Φ_k of the Helmholtz operator $\Delta + k^2$. In this setting, for each vector-valued function $f = (f_J)_{1\leq J\leq M}$ in $\big[L^1(\partial_*\Omega,\sigma)\big]^M$ define the boundary-to-domain acoustic double layer potential operator at each $x\in\Omega$ as

$$\mathcal{D}f(x) := \Big(- \int_{\partial_*\Omega} \nu_s(y) a^{IJ}_{rs} (\partial_r \Phi_k)(x-y) f_J(y)\, \mathrm{d}\sigma(y) \Big)_{1 \le I \le M}, \tag{1.5.218}$$

and its principal-value version, the boundary-to-boundary acoustic double layer potential operator defined at σ-a.e. $x \in \partial\Omega$ as

$$Kf(x) := \Big(- \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial_*\Omega \\ |x-y|>\varepsilon}} \nu_s(y) a^{IJ}_{rs} (\partial_r \Phi_k)(x-y) f_J(y)\, \mathrm{d}\sigma(y) \Big)_{1 \le I \le M}. \tag{1.5.219}$$

Next, for each $f = (f_I)_{1 \le I \le M} \in \big[L^1(\partial\Omega, \sigma)\big]^M$ define the "transpose" boundary-to-boundary acoustic double layer potential operator at σ-a.e. $x \in \partial_*\Omega$ as

$$K^{\#}f(x) := \Big(\lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial_*\Omega \\ |x-y|>\varepsilon}} \nu_s(x) a^{IJ}_{rs} (\partial_r \Phi_k)(x-y) f_I(y)\, \mathrm{d}\sigma(y) \Big)_{1 \le J \le M}, \tag{1.5.220}$$

as well as the boundary-to-domain acoustic single layer potential operator

$$\mathcal{S}f(x) := \Big(\int_{\partial\Omega} \Phi_k(x-y) f_I(y)\, \mathrm{d}\sigma(y) \Big)_{1 \le I \le M} \quad \textit{for all } x \in \Omega, \tag{1.5.221}$$

and its boundary-to-boundary version

$$Sf(x) := \Big(\int_{\partial\Omega} \Phi_k(x-y) f_I(y)\, \mathrm{d}\sigma(y) \Big)_{1 \le I \le M} \quad \textit{for } \sigma\textit{-a.e. } x \in \partial\Omega. \tag{1.5.222}$$

Then for each aperture parameter $\kappa \in (0,\infty)$ and for each truncation parameter $\rho \in (0,\infty)$ the following properties hold.

(i) For every function $f \in \big[L^1(\partial_*\Omega, \sigma)\big]^M$ one has

$$\mathcal{D}f \in \big[\mathscr{C}^\infty(\Omega)\big]^M \ \textit{ and } \ (\Delta + k^2)\mathcal{D}f = 0 \ \textit{ in } \ \Omega. \tag{1.5.223}$$

Moreover, if Ω is an exterior domain then $\mathcal{D}f$ radiates at infinity (cf. [70, Definition 6.1.1]).
Also, for each $p \in (1,\infty)$ there exists some constant $C \in (0,\infty)$ such that for every $f \in \big[L^p(\partial_*\Omega, \sigma)\big]^M$ one has

$$\big\| \mathcal{N}^{\rho}_{\kappa}(\mathcal{D}f) \big\|_{L^p(\partial\Omega,\sigma)} \le C \|f\|_{[L^p(\partial_*\Omega,\sigma)]^M}. \tag{1.5.224}$$

(ii) For each function f belonging to the boundary Sobolev space $\big[L^1_1(\partial_*\Omega, \sigma)\big]^M$ and each index $\ell \in \{1, \dots, n\}$ the pointwise nontangential boundary trace

$$\big(\partial_\ell \mathcal{D}f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \textit{ exists (in } \mathbb{C}^M\textit{) at } \sigma\textit{-a.e. point on } \partial_*\Omega. \tag{1.5.225}$$

Moreover, for each $p \in (1,\infty)$ *there exists some finite constant* $C > 0$ *such that for each function* $f \in \big[L^p_1(\partial_*\Omega,\sigma)\big]^M$ *one has*

$$\big\|\mathcal{N}^{\rho}_{\kappa}(\mathcal{D}f)\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}^{\rho}_{\kappa}(\nabla\mathcal{D}f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[L^p_1(\partial_*\Omega,\sigma)]^M}. \tag{1.5.226}$$

(iii) Fix $p, p' \in (1,\infty)$ *with* $1/p + 1/p' = 1$. *Then the operators*

$$K : \big[L^p(\partial_*\Omega,\sigma)\big]^M \longrightarrow \big[L^p(\partial\Omega,\sigma)\big]^M, \tag{1.5.227}$$

$$K^{\#} : \big[L^{p'}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{p'}(\partial_*\Omega,\sigma)\big]^M, \tag{1.5.228}$$

are well-defined, linear, bounded, and transpose to one another.

(iv) If I *denotes the identity operator, then for each function* $f \in \big[L^1(\partial_*\Omega,\sigma)\big]^M$ *the following nontangential boundary trace formula holds:*

$$\mathcal{D}f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K\big)f \ \ \text{at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{1.5.229}$$

(v) Strengthen the original hypotheses on the underlying set by assuming that additional assumption that Ω *is a* UR *domain; in particular,* $\mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial_*\Omega\big) = 0$. *Also, fix* $p, p' \in (1,\infty)$ *with* $1/p + 1/p' = 1$. *Then the operators*

$$K : \big[L^p_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^p_1(\partial\Omega,\sigma)\big]^M, \tag{1.5.230}$$

$$K^{\#} : \big[L^{p'}_{-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{p'}_{-1}(\partial\Omega,\sigma)\big]^M, \tag{1.5.231}$$

are well-defined, linear, bounded, and transpose to one another.

(vii) Continue to assume that Ω *is a* UR *domain and recall the conormal derivative operator associated with the coefficient tensor* A *and the domain* Ω *as in* [70, (6.4.4)]. *Also, fix some* $p \in (1,\infty)$. *Then the conormal derivative of the double layer induces a well-defined, linear, and bounded operator in the context*

$$\big[L^p(\partial\Omega,\sigma)\big]^M \ni f \longmapsto \big(\partial^A_\nu\mathcal{D}\big)f := \partial^A_\nu(\mathcal{D}f) \in \big[L^p(\partial\Omega,\sigma)\big]^M. \tag{1.5.232}$$

(viii) Continue to assume that Ω *is actually a* UR *domain. Also, pick two integrability exponents* $p, p' \in (1,\infty)$ *satisfying* $1/p + 1/p' = 1$. *Then for any two given vector-valued functions* $f \in \big[L^p_1(\partial\Omega,\sigma)\big]^M$ *and* $g \in \big[L^{p'}_1(\partial\Omega,\sigma)\big]^M$ *one has*

$$\int_{\partial\Omega}\big\langle\big(\partial^A_\nu\mathcal{D}\big)f, g\big\rangle\,\mathrm{d}\sigma = \int_{\partial\Omega}\big\langle f, \big(\partial^{A^\top}_\nu\mathcal{D}_{A^\top}\big)g\big\rangle\,\mathrm{d}\sigma \tag{1.5.233}$$

where $\partial^{A^\top}_\nu$ *along with* $\mathcal{D}_{A^\top}$ *and, ultimately* $\partial^{A^\top}_\nu\mathcal{D}_{A^\top}$, *are defined as before with* A *now replaced by* $A^\top$. *As a consequence, whenever* Ω *is a* UR *domain, the operator* (1.5.31) *has a unique extension to a well-defined, linear, and bounded mapping*

$$\partial_\nu^A \mathcal{D} : \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M, \tag{1.5.234}$$

namely the (*real*) *transpose of the operator*

$$\partial_\nu^{A^\top} \mathcal{D}_{A^\top} : \big[L_1^{p'}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{p'}(\partial\Omega,\sigma)\big]^M. \tag{1.5.235}$$

(ix) Once again assume Ω *is a* UR *domain. Fix* $p, p' \in (1,\infty)$ *with* $1/p + 1/p' = 1$. *Then the boundary-to-boundary single layer potential operator* S *induces well-defined, linear, bounded, and compatible fashion with one another mappings*

$$S : \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L_1^p(\partial\Omega,\sigma)\big]^M, \tag{1.5.236}$$

$$S : \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^p(\partial\Omega,\sigma)\big]^M, \tag{1.5.237}$$

which are also transpose to one another.

(x) Assume Ω *is a* UR *domain and fix* $p, p' \in (1,\infty)$ *satisfying* $1/p+1/p' = 1$. *Extend the action of the boundary-to-boundary single layer* $\mathscr{S}$, *originally defined in* (1.5.221), *to each* $f = (f_I)_{1\leq I\leq M}$ *belonging to the negative Sobolev space* $\big[L^p_{-1}(\partial\Omega,\sigma)\big]^M$ *by setting*

$$\begin{gathered}\big(\mathscr{S} f\big)_I(x) := {}_{[L_1^{p'}(\partial\Omega,\sigma)]^M}\Big\langle \Phi_k(x-\cdot)\big|_{\partial\Omega}, f_I\Big\rangle_{[L^p_{-1}(\partial\Omega,\sigma)]^M} \\ \text{for every } x \in \Omega \text{ and every } I \in \{1,\dots,M\}.\end{gathered} \tag{1.5.238}$$

Then this operator is meaningfully defined and agrees with $\mathscr{S}$ *from* (1.5.221) *when acting on the smaller space* $\big[L^p(\partial\Omega,\sigma)\big]^M$. *Moreover,*

$$\mathscr{S} : \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathscr{C}^\infty(\Omega)\big]^M \tag{1.5.239}$$

is continuous (*when the space on the right is equipped with the Frechét topology of uniform convergence of partial derivatives of any order on compact sets*), *and*

$$\partial^\gamma\big(\mathscr{S} f\big)_I(x) = {}_{[L_1^{p'}(\partial\Omega,\sigma)]^M}\Big\langle (\partial^\gamma\Phi_k)(x-\cdot)\big|_{\partial\Omega}, f_I\Big\rangle_{[L^p_{-1}(\partial\Omega,\sigma)]^M} \tag{1.5.240}$$

for each $f = (f_I)_{1\leq I\leq M} \in \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M$, *each index* $I \in \{1,\dots,M\}$, *each multi-index* $\gamma \in \mathbb{N}_0^n$, *and each point* $x \in \Omega$. *Moreover,*

$$(\Delta + k^2)(\mathscr{S} f) = 0 \ \text{ in } \Omega, \text{ for each functional } f \in \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M, \tag{1.5.241}$$

and if Ω *is an exterior domain then* $\mathscr{S} f$ *also radiates at infinity* (*cf.* [70, Definition 6.1.1]). *In addition, for each* $f \in \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M$ *the nontangential pointwise trace* $\mathscr{S} f\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ *exists at* σ*-a.e. point on* $\partial\Omega$; *in fact, with* Sf *considered in the sense of* (1.5.237) *one has*

$$\mathscr{S}f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = Sf \ \textit{at } \sigma\textit{-a.e. point on } \partial\Omega. \tag{1.5.242}$$

Furthermore, there exists a finite constant $C > 0$ such that

$$\begin{aligned}&\textit{the nontangential trace } (\nabla\mathscr{S}f)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \textit{ exists } \sigma\textit{-a.e. on } \partial\Omega \textit{ and}\\ &\big\|\mathcal{N}_\kappa^\rho(\mathscr{S}f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[L^p_{-1}(\partial\Omega,\sigma)]^M} \textit{ for all } f \in \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M,\\ &\big\|\mathcal{N}_\kappa^\rho(\nabla\mathscr{S}f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[L^p(\partial\Omega,\sigma)]^M} \textit{ for all } f \in \big[L^p(\partial\Omega,\sigma)\big]^M\end{aligned} \tag{1.5.243}$$

Finally, for each $f \in \big[L^1(\partial\Omega,\sigma)\big]^M$ one has the jump-formula

$$\partial_\nu^A\mathscr{S}f = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)f \ \textit{at } \sigma\textit{-a.e. point on } \partial\Omega, \tag{1.5.244}$$

where I is the identity operator, and $K^\#_{A^\top}$ is the operator associated with the coefficient tensor $A^\top$ and the set Ω as in (1.5.220).

(xi) Suppose that in fact Ω is a UR *domain. For each pair of indices $j,\ell \in \{1,\dots,n\}$ and each function $f = (f_I)_{1\leq I\leq M} \in \big[L^1(\partial\Omega,\sigma)\big]^M$ define*

$$T^\#_{j\ell}f(x) := \Bigg(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big\{\nu_j(x)(\partial_\ell\Phi_k)(x-y) - \nu_\ell(x)(\partial_j\Phi_k)(x-y)\big\} f_I(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\leq I\leq M} \tag{1.5.245}$$

at σ-a.e. point $x \in \partial\Omega$. Then

$$\begin{aligned}&\textit{for each } f \in \big[L^1(\partial\Omega,\sigma)\big]^M \textit{ and } j,\ell \in \{1,\dots,n\}\\ &\textit{one has } \partial_{\tau_{j\ell}}(Sf) = T^\#_{j\ell}f \ \textit{at } \sigma\textit{-a.e. point on } \partial\Omega.\end{aligned} \tag{1.5.246}$$

(xii) Make the assumption that Ω is a UR *domain. Then for each $p \in (1,\infty)$ the following operator identities hold:*

$$\big(\tfrac{1}{2}I + K\big)\circ\big(-\tfrac{1}{2}I + K\big) = S\circ\big(\partial_\nu^A\mathscr{D}\big) \ \textit{on } \big[L(\partial\Omega,\sigma)\big]^M, \tag{1.5.247}$$

$$\big(\tfrac{1}{2}I + K^\#_{A^\top}\big)\circ\big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) = \big(\partial_\nu^A\mathscr{D}\big)\circ S \ \textit{on } \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M, \tag{1.5.248}$$

$$S\circ K^\#_{A^\top} = K\circ S \ \textit{on } \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M, \tag{1.5.249}$$

$$K^\#_{A^\top}\circ\big(\partial_\nu^A\mathscr{D}\big) = \big(\partial_\nu^A\mathscr{D}\big)\circ K \ \textit{on } \big[L^p(\partial\Omega,\sigma)\big]^M. \tag{1.5.250}$$

Proof This is a consequence of [70, Lemma 6.3.1] and Theorem 1.5.5 (together with its proof) used for the weakly elliptic system $L := \Delta I_{M\times M}$ in $\mathbb{R}^n$; other results from [70, Chapter 6] also play a role. The key observation is that under the identification $E \equiv \big(\Phi_k \delta_{IJ}\big)_{1\leq I,J\leq M}$ the operators (1.3.18), (1.3.68), (1.3.72), (1.3.6), and (1.3.62) correspond precisely to (1.5.218), (1.5.219), (1.5.220), (1.5.221), and (1.5.222), respectively. □

Our next result elaborates on the specific manner in which tangential derivatives commute with the action of the boundary-to-boundary double layer potential operator on functions belonging to boundary Sobolev spaces.

Proposition 1.5.6 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Let L be a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$ and let $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ be the matrix-valued fundamental solution associated with L as in* [70, Theorem 1.4.2]. *Also, pick a coefficient tensor $A = \Big(a^{\alpha\beta}_{rs}\Big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$ for which $L_A = L$ and bring in K, the boundary-to-boundary double layer potential operator associated with Ω and A as in* (1.3.68). *In addition, for each $j, k \in \{1, \dots, n\}$ define the singular integral operator U_{jk} acting on each given function $g = (g_{\alpha s})_{\substack{1\leq\alpha\leq M\\ 1\leq s\leq n}}$ with entries belonging to $L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ as $U_{jk}g = (U_{jk}g)_{1\leq\gamma\leq M}$ where, for every $\gamma \in \{1, \dots, M\}$ and for σ-a.e. point $x \in \partial\Omega$,*

$$
\begin{aligned}
(U_{jk}g)_\gamma(x) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.5.251)\\
:= -\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_k(x)-\nu_k(y)]\nu_j(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)g_{\alpha s}(y)\,\mathrm{d}\sigma(y)\\
+\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_j(x)-\nu_j(y)]\nu_k(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)g_{\alpha s}(y)\,\mathrm{d}\sigma(y)\\
+\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_k(y)-\nu_k(x)]\nu_s(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)g_{\alpha j}(y)\,\mathrm{d}\sigma(y)\\
-\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_j(y)-\nu_j(x)]\nu_s(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)g_{\alpha k}(y)\,\mathrm{d}\sigma(y).
\end{aligned}
$$

Finally, fix some integrability exponent $p \in (1, \infty)$ and consider a function

$$f = (f_\alpha)_{1\le\alpha\le M} \in \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma)\Big]^M \ \textit{such that}$$
$$\partial_{\tau_{jk}} f_\alpha \ \textit{belongs to}\ L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma) \tag{1.5.252}$$
$$\textit{for all}\ j,k\in\{1,\dots,n\}\ \textit{and}\ \alpha\in\{1,\dots,M\}.$$

Then for each $j,k\in\{1,\dots,n\}$ one has

$$\partial_{\tau_{jk}}(Kf) = K(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f), \tag{1.5.253}$$

where $\nabla_{\tan} f$ is regarded here as the $M\times n$ matrix-valued function whose (α,s) entry is the s-th component of the tangential gradient $\nabla_{\tan} f_\alpha$ (cf. (A.0.78)).

As an example, consider the case when $M := 1$ and $L := \Delta$, the Laplace operator in $\mathbb{R}^n$. Denote by K_Δ the standard boundary-to-boundary harmonic double layer operator associated with Ω, and recall the boundary-to-boundary Riesz transforms R_i, $1\le i\le n$, on $\partial\Omega$. Also, for each $j\in\{1,\dots,n\}$, denote by M_{ν_j} the operator of pointwise multiplication by ν_j, the j-th scalar component of ν. In this particular setting, Proposition 1.5.6 amounts to saying that for any scalar function with the property that there exists some $p\in(1,\infty)$ such that

$$f\in L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\cap L^p_{\rm loc}(\partial\Omega,\sigma)\ \text{ and}$$
$$\partial_{\tau_{jk}} f\in L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\cap L^p_{\rm loc}(\partial\Omega,\sigma)\ \text{ for all }\ j,k\in\{1,\dots,n\} \tag{1.5.254}$$

it follows that for each pair of indices $j,k\in\{1,\dots,n\}$ one has

$$\begin{aligned}
\partial_{\tau_{jk}}(Kf) &= K(\partial_{\tau_{jk}} f)\\
&\quad + \tfrac12\big[R_i, M_{\nu_k}\big]\big(\nu_j(\nabla_{\tan} f)_i\big) - \tfrac12\big[R_i, M_{\nu_j}\big]\big(\nu_k(\nabla_{\tan} f)_i\big)\\
&\quad - \tfrac12\big[R_i, M_{\nu_k}\big]\big(\nu_i(\nabla_{\tan} f)_j\big) + \tfrac12\big[R_i, M_{\nu_j}\big]\big(\nu_i(\nabla_{\tan} f)_k\big)\\
&= K(\partial_{\tau_{jk}} f) + \tfrac12\big[R_i, M_{\nu_j}\big]\big(\partial_{\tau_{ik}} f\big) - \tfrac12\big[R_i, M_{\nu_k}\big]\big(\partial_{\tau_{ij}} f\big),
\end{aligned} \tag{1.5.255}$$

where the last equality comes from [69, (11.4.8)]. In summary, tangential partial derivatives commute with the principal-value harmonic double layer, modulo commutators between Riesz transforms and operators of pointwise multiplication by the scalar components of the geometric measure theoretic outward unit normal.

Proof of Proposition 1.5.6 Fix $j,k\in\{1,\dots,n\}$ along with $\gamma\in\{1,\dots,M\}$ and recall the definition of the boundary-to-domain double layer potential operator from (1.3.18), as well as the definition of the operator introduced in [70, (2.9.53)]. Making use of (1.3.28) and [69, (11.4.8)], for each $x\in\Omega$ we may then write

$$
\begin{aligned}
\partial_j(\mathcal{D}f)_\gamma(x) &= \int_{\partial\Omega} \nu_j(y) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_s(y)\,\mathrm{d}\sigma(y) \\
&\quad - \int_{\partial\Omega} \nu_s(y) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_j(y)\,\mathrm{d}\sigma(y) \\
&= a^{\beta\alpha}_{rs}(\partial_r \mathcal{S}_{\gamma\beta})\big(\nu_j(\nabla_{\tan} f_\alpha)_s\big)(x) + \big(\mathcal{D}((\nabla_{\tan} f)^j)\big)_\gamma(x), \qquad (1.5.256)
\end{aligned}
$$

where $(\nabla_{\tan} f)^j$ is the $\mathbb{C}^M$-valued function whose α-th component is $(\nabla_{\tan} f_\alpha)_j$. Fix an aperture parameter $\kappa > 0$. From Lemma 1.3.2, [70, (2.4.8)], and [69, Proposition 11.3.2], we see that formula [69, (11.3.26)] holds for $u := \mathcal{D}f$. Based on this, (1.5.256), and the jump formulas in (1.5.20) and in [70, Corollary 2.9.5], at σ-a.e. point $x \in \partial\Omega$ we may then write (bearing in mind [70, (2.9.52)])

$$
\begin{aligned}
&\partial_{\tau_{jk}}(Kf)_\gamma(x) \qquad\qquad (1.5.257)\\
&= \partial_{\tau_{jk}}(\tfrac{1}{2}f + Kf)_\gamma(x) - \tfrac{1}{2}\partial_{\tau_{jk}} f_\gamma(x)\\
&= \nu_j(\partial_k \mathcal{D} f)_\gamma\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) - \nu_k(\partial_j \mathcal{D} f)_\gamma\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) - \tfrac{1}{2}\partial_{\tau_{jk}} f_\gamma(x)\\
&= -\tfrac{1}{2}\nu_j(x)\nu_k(x)\nu_r(x) a_{rs}^{\beta\alpha} b_{\gamma\beta}(x)(\nabla_{\tan} f_\alpha)_s(x)\\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_j(x)\nu_k(y) a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_s(y)\,\mathrm{d}\sigma(y)\\
&\quad + \tfrac{1}{2}\nu_j(x)(\nabla_{\tan} f_\gamma)_k(x) + \nu_j(x)\Big(K\big(\nabla_{\tan} f\big)^k\Big)_\gamma(x)\\
&\quad + \tfrac{1}{2}\nu_j(x)\nu_k(x)\nu_r(x) a_{rs}^{\beta\alpha} b_{\gamma\beta}(x)(\nabla_{\tan} f_\alpha)_s(x)\\
&\quad - \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_k(x)\nu_j(y) a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_s(y)\,\mathrm{d}\sigma(y)\\
&\quad - \tfrac{1}{2}\nu_k(x)(\nabla_{\tan} f_\gamma)_j(x) - \nu_k(x)\Big(K\big(\nabla_{\tan} f\big)^j\Big)_\gamma(x)\\
&\quad - \tfrac{1}{2}\partial_{\tau_{jk}} f_\gamma(x)\\
&= -\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_k(x)\nu_j(y) a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_s(y)\,\mathrm{d}\sigma(y)\\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_j(x)\nu_k(y) a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_s(y)\,\mathrm{d}\sigma(y)\\
&\quad + \Big(\big[K, M_{\nu_k}\big]\big(\nabla_{\tan} f\big)^j\Big)_\gamma(x) - \Big(\big[K, M_{\nu_j}\big]\big(\nabla_{\tan} f\big)^k\Big)_\gamma(x)\\
&\quad + \Big(K\big(\nu_j(\nabla_{\tan} f)^k - \nu_k(\nabla_{\tan} f)^j\big)\Big)_\gamma(x), \qquad\qquad (1.5.258)
\end{aligned}
$$

where for each $\ell \in \{1,\dots,n\}$ we have denoted by M_{ν_ℓ} the operator of pointwise multiplication by ν_ℓ, and $\big[K, M_{\nu_\ell}\big] := KM_{\nu_\ell} - M_{\nu_\ell}K$ is the commutator of K with M_{ν_ℓ}. Thanks to [69, (11.4.8)], the last line of (1.5.257) may be recast as

$$
\Big(K\big(\nu_j(\nabla_{\tan} f)^k - \nu_k(\nabla_{\tan} f)^j\big)\Big)_\gamma(x) = \big(K(\partial_{\tau_{jk}} f)\big)_\gamma(x). \qquad (1.5.259)
$$

Also, simple algebra gives

$$
\begin{aligned}
&- \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_k(x)\nu_j(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_s(y)\,\mathrm{d}\sigma(y)\\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_j(x)\nu_k(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_s(y)\,\mathrm{d}\sigma(y)\\
&= - \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_k(x)-\nu_k(y)]\nu_j(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_s(y)\,\mathrm{d}\sigma(y)\\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} [\nu_j(x)-\nu_j(y)]\nu_k(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_s(y)\,\mathrm{d}\sigma(y).
\end{aligned}
\tag{1.5.260}
$$

Having made these observations, there remains to note that if $\nabla_{\tan} f$ is regarded as a matrix-valued function whose (α, s) entry is the s-th component of $\nabla_{\tan} f_\alpha$, then (1.5.257) may be rewritten as claimed in (1.5.253). □

Our final result in this section describes the manner in which the double layer potential operators associated with a given weakly elliptic second-order $M \times M$ system L in $\mathbb{R}^n$ and a set of locally finite perimeter $\Omega \subseteq \mathbb{R}^n$ transform if one performs a linear change of variables in the Euclidean space via $W : \mathbb{R}^n \to \mathbb{R}^n$ for some non-singular matrix $W \in \mathbb{R}^{n\times n}$, or if L is multiplied (to the left, or to the right) by a non-singular matrix $C \in \mathbb{C}^{M\times M}$.

Proposition 1.5.7 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is an open set of locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, for some $M \in \mathbb{N}$, consider a coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in (1.3.2) is weakly elliptic (in the sense that $\det[L(\xi)] \neq 0$ for each $\xi \in \mathbb{R}^n \setminus \{0\}$).*

Given a non-singular matrix $W \in \mathbb{R}^{n\times n}$, consider the system $L \circ W$, formally defined as

$$
L \circ W := \Big(a^{\alpha\beta}_{jk}(W\nabla)_j(W\nabla)_k\Big)_{1\leq\alpha,\beta\leq M}. \tag{1.5.261}
$$

From [70, Proposition 1.4.3] *it is known that, with notation introduced in* [70, (1.4.56)-(1.4.57)],

$$
\widetilde{A} := W^\top \circ A \circ W \in \mathfrak{A}_{L\circ W} \ \text{ and } \ L \circ W \ \text{ is weakly elliptic}. \tag{1.5.262}
$$

Also, from [39] *it is know that the set $\widetilde{\Omega} := W^\top\Omega$ has the same (geometric measure theoretic) nature as Ω. Denote by $\widetilde{\nu}$ its geometric measure theoretic outward unit normal, and set $\widetilde{\sigma} := \mathcal{H}^{n-1}\lfloor\partial\widetilde{\Omega}$. In particular, it makes sense to consider* *the (boundary-to-domain and boundary-to-boundary) double layer potential operators*[17] *associated with the system* $L \circ W$ *and the set* $\widetilde{\Omega}$ *as in* (1.3.18) *and* (1.3.68), *respectively.*

Then, for these operators the following formulas hold:

$$\mathcal{D}_{\widetilde{A},\widetilde{\Omega}} g = \big(\mathcal{D}_{A,\Omega}(g \circ W^{\top})\big) \circ (W^{\top})^{-1} \text{ in } \widetilde{\Omega}, \tag{1.5.263}$$

and

$$K_{\widetilde{A},\widetilde{\Omega}} g = \big(K_{A,\Omega}(g \circ W^{\top})\big) \circ (W^{\top})^{-1} \text{ at } \widetilde{\sigma}\text{-a.e. point on } \partial\widetilde{\Omega}, \tag{1.5.264}$$

for each function $g \in \Big[L^1\Big(\partial_*\widetilde{\Omega}, \frac{\widetilde{\sigma}(x)}{1+|x|^{n-1}}\Big)\Big]^M$.

Furthermore, given any non-singular matrix $C = (c_{\beta\gamma})_{1\leq\beta,\gamma\leq M} \in \mathbb{C}^{M\times M}$, *the systems* LC *and* CL *(interpreted in the sense of multiplication of* $M \times M$ *matrices) are weakly elliptic, and if one defines*

$$A \diamond C := \big(a_{jk}^{\alpha\beta} c_{\beta\gamma}\big)_{\substack{1\leq\alpha,\gamma\leq M,\\ 1\leq j,k\leq n}} \qquad C \diamond A := \big(c_{\gamma\alpha} a_{jk}^{\alpha\beta}\big)_{\substack{1\leq\gamma,\beta\leq M,\\ 1\leq j,k\leq n}} \tag{1.5.265}$$

then

$$A \diamond C \in \mathfrak{A}_{LC}, \qquad C \diamond A \in \mathfrak{A}_{CL}, \tag{1.5.266}$$

and one has

$$\mathcal{D}_{A\diamond C,\Omega} = C^{-1} \circ \mathcal{D}_{A,\Omega} \circ C, \qquad \mathcal{D}_{C\diamond A,\Omega} = \mathcal{D}_{A,\Omega}, \tag{1.5.267}$$

as well as

$$K_{A\diamond C,\Omega} = C^{-1} \circ K_{A,\Omega} \circ C, \qquad K_{C\diamond A,\Omega} = K_{A,\Omega}. \tag{1.5.268}$$

Proof Formula (1.5.263) is seen from (1.3.18), (1.5.262), [70, (1.4.56)-(1.4.57)], the "surface-to-surface" change of variable formula [69, (11.6.4)] (used for the diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ given by $F(x) := W^{\top}x$ for each $x \in \mathbb{R}^n$), formula [69, (11.6.1)] describing the geometric measure theoretic outward unit normal $\widetilde{\nu}$ to $\widetilde{\Omega}$, item *(iii)* in [70, Proposition 1.4.3] (describing the "canonical" fundamental solution of $L \circ W$), and the Chain Rule. Formula (1.5.264) is proved in a similar fashion, now starting from (1.3.68) in place of (1.3.18). Finally, formulas (1.5.267)-(1.5.268) may be justified directly from (1.5.265), (1.3.18), and (1.3.68). □

[17] For the sake of clarity, we shall use both the coefficient tensor and the underlying set as subscripts in the notation for the aforementioned operators

1.6 Cauchy and Cauchy-Clifford Operators on Lebesgue and Sobolev Spaces

From the discussion in (1.4.13)-(1.4.16) we know that the boundary-to-domain and boundary-to-boundary Cauchy-Clifford integral operators may be regarded as double layer operators associated with the Laplacian. In view of this, and granted the availability of the rich Calderón-Zygmund theory developed in relation to layer potential operators associated with arbitrary weakly elliptic second-order systems, a host of results may instantaneously be deduced for the aforementioned Cauchy-Clifford integral operators on Lebesgue and Sobolev spaces defined on UR sets.

Alternatively, the results which are now available from Theorem 1.5.1 and Proposition 1.2.1 also allow for a quick and painless treatment of the Cauchy-Clifford integral operator on boundary Sobolev spaces. Since this latter approach reveals more about the algebraic structure of the Cauchy-Clifford integral operators, we present it below, in the proof of the following proposition:

Proposition 1.6.1 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is an open set with the property that $\partial\Omega$ is a* UR *set. Abbreviate $\sigma_* := \mathcal{H}^{n-1}\lfloor\partial_*\Omega$ and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, denote by ν the geometric measure theoretic outward unit normal to Ω. Recall the Cauchy-Clifford integral operators $\mathcal{C}$, $\mathfrak{C}$ associated with Ω as in (A.0.53) and (A.0.54), respectively. In addition, define the action of the "transpose" Cauchy-Clifford singular integral operator on any given Clifford algebra-valued function $f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big) \otimes \mathcal{C}\ell_n$*

$$\mathfrak{C}^{\#} f(x) := \lim_{\varepsilon\to 0^+} \frac{-1}{\omega_{n-1}} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu(x) \odot \frac{x-y}{|x-y|^n} \odot f(y)\, \mathrm{d}\sigma(y) \tag{1.6.1}$$

for σ-a.e. $x \in \partial_*\Omega$. Then the following statements are true.

(i) For each function $f \in L^{p,q}_1(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n$ with exponents $p, q \in [1, \infty)$, index $\ell \in \{1, \dots, n\}$, and aperture parameter $\kappa \in (0, \infty)$, the pointwise nontangential boundary trace

$$\big(\partial_\ell \mathcal{C} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exists (in } \mathcal{C}\ell_n\text{) at } \sigma_*\text{-a.e. point on } \partial_*\Omega, \tag{1.6.2}$$

and there exists some finite constant $C > 0$ depending only on $\partial\Omega$, n, p, q, and κ such that

$$\begin{gathered}\big\|\mathcal{N}_\kappa(\mathcal{C}f)\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\mathcal{C}f)\big\|_{L^q(\partial\Omega,\sigma)} \leq C\|f\|_{L^{p,q}_1(\partial_*\Omega,\sigma_*)\otimes\mathcal{C}\ell_n}\\ \text{if } \ p \in (1,\infty) \ \text{ and } \ q \in (1,\infty),\end{gathered} \tag{1.6.3}$$

plus similar estimates in the case when either $p = 1$ or $q = 1$, this time with the corresponding L^1-norm in the left side replaced by the weak-L^1 (quasi-)norm.

(ii) Fix $p, p' \in (1, \infty)$ with $1/p + 1/p' = 1$. Then the operator

$$\mathfrak{C}^{\#} : L^p(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \longrightarrow L^p(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n \tag{1.6.4}$$

is well defined, linear, and bounded. Also, corresponding to $p = 1$,

$$\mathfrak{C}^{\#} : L^1(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \longrightarrow L^{1,\infty}(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n \tag{1.6.5}$$

is well defined, linear, and bounded. Finally, the transpose of $\mathfrak{C}^{\#}$ *in* (1.6.4) *is*

$$\mathfrak{C} : L^{p'}(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n \longrightarrow L^{p'}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n. \tag{1.6.6}$$

(iii) Make the additional assumption that

$$\mathcal{H}^{n-1}\big(\partial_{\text{nta}}\Omega \setminus \partial_*\Omega\big) = 0. \tag{1.6.7}$$

Then for each $p \in (1, \infty)$ *one has*

$$\big(\mathfrak{C}^{\#}\big)^2 = \tfrac{1}{4}I \ \text{ on } \ L^p(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n. \tag{1.6.8}$$

Also, the operator

$$\mathfrak{C} : L^{p,q}_1(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n \longrightarrow L^{p,q}_1(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n \tag{1.6.9}$$

is well defined, linear, and bounded for each $p, q \in (1, \infty)$. *In particular, the operator*

$$\mathfrak{C} : L^p_1(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n \longrightarrow L^p_1(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n \tag{1.6.10}$$

is well defined, linear, and bounded for each $p \in (1, \infty)$.

(iv) Retain the additional assumption made in (1.6.7). *Then for each* $p, q \in (1, \infty)$ *it follows that the mapping* $\mathfrak{C}^{\#}$ *from* (1.6.4) *extends uniquely to a linear and bounded operator from the negative off-diagonal boundary Sobolev space* $L^{p,q}_{-1}(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n$ *into itself. Furthermore, if one retains the same notation* $\mathfrak{C}^{\#}$ *for this extension, the transpose of* (1.6.9) *is*

$$\mathfrak{C}^{\#} : L^{p',q'}_{-1}(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n \longrightarrow L^{p',q'}_{-1}(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n, \tag{1.6.11}$$

where $p', q' \in (1, \infty)$ *are such that* $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$. *In particular, the transpose of* (1.6.10) *is*

$$\mathfrak{C}^{\#} : L^{p'}_{-1}(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n \longrightarrow L^{p'}_{-1}(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n. \tag{1.6.12}$$

As a corollary of [70, (2.5.324)], item *(ii)* in Proposition 1.6.1, density, and duality, we also see that if $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, is a UR domain and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ then

$$\begin{aligned}&\mathfrak{C}^{\#} \text{ is bounded on } L^p(\partial\Omega, w)\otimes C\ell_n \text{ and}\\&\text{we have } \big(\mathfrak{C}^{\#}\big)^2 = \tfrac{1}{4}I \text{ on } L^p(\partial\Omega, w)\otimes C\ell_n\\&\text{whenever } p\in(1,\infty) \text{ and } w\in A_p(\partial\Omega,\sigma).\end{aligned}\tag{1.6.13}$$

This may also be seen from [70, Corollary 2.5.33] and duality.

Proof of Proposition 1.6.1 Upon recalling from [70, (2.5.311)] that

$$\begin{aligned}&\mathcal{C}f = \mathcal{D}f + \frac{1}{2}\cdot\sum_{j,k=1}^{n}\mathbf{e}_j\odot\mathbf{e}_k\odot\mathcal{R}_{jk}f \text{ in } \Omega\\&\text{for each function } f\in L^1\Big(\partial_*\Omega, \tfrac{\sigma(y)}{1+|y|^{n-1}}\Big)\otimes C\ell_n,\end{aligned}\tag{1.6.14}$$

where $\mathcal{D}$ is the harmonic double layer (cf. [70, Definition 2.5.17]) and the operators $\mathcal{R}_{jk}$ are as in (A.0.189), the claims in item *(i)* are direct consequences of part *(ii)* in Theorem 1.5.1 and part *(i)* in Proposition 1.2.1. Next, the fact that the operator $\mathfrak{C}^{\#}$ is well defined, linear, and bounded both in the context of (1.6.4) and in the case of (1.6.5) is guaranteed by [70, Theorem 2.3.2]. To proceed, fix $p'\in(1,\infty)$ is such that $1/p+1/p'=1$ and recall from [70, (2.5.312)] that

$$\mathfrak{C} = K + \frac{1}{2}\cdot\sum_{j,k=1}^{n}\mathbf{e}_j\odot\mathbf{e}_k\odot R_{jk} \text{ on } L^p(\partial_*\Omega,\sigma_*)\otimes C\ell_n,\tag{1.6.15}$$

where K is the principal-value harmonic double layer (cf. [70, (2.5.203)]) and the operators R_{jk} have been defined in (A.0.190). In relation to the latter family of operators, for each $j,k\in\{1,\dots,n\}$ and $f\in L^p(\partial\Omega,\sigma)\otimes C\ell_n$ let us also define (with $\nu_1,\dots,\nu_n$ denoting the components of ν)

$$\begin{aligned}R^{\#}_{jk}f(x) &:= \lim_{\varepsilon\to0^+}\int_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}}\Big[\nu_j(x)(\partial_k E_\Delta)(y-x)-\nu_k(x)(\partial_j E_\Delta)(y-x)\Big]f(y)\,\mathrm{d}\sigma(y),\\&= \lim_{\varepsilon\to0^+}\frac{-1}{\omega_{n-1}}\int_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}}\frac{\nu_j(x)(x_k-y_k)-\nu_k(x)(x_j-y_j)}{|x-y|^n}f(y)\,\mathrm{d}\sigma(y)\end{aligned}\tag{1.6.16}$$

for σ_*-a.e. $x\in\partial_*\Omega$. Then, on the one hand, from (1.6.15), (1.6.16), part *(iii)* in Theorem 1.5.1 (used with $L:=\Delta$), part *(ii)* in Proposition 1.2.1 (used with $b:=E_\Delta$ which is even and positive homogeneous of degree $2-n$ if $n\geq3$; the case $n=2$ is a minor variation), and [68, Lemma 6.4.1] it follows that the transpose of (1.6.15) is the operator

$$K^{\#} - \frac{1}{2} \cdot \sum_{j,k=1}^{n} \mathbf{e}_k \odot \mathbf{e}_j \odot R^{\#}_{jk}$$

$$= K^{\#} + \frac{1}{2} \cdot \sum_{j,k=1}^{n} \mathbf{e}_j \odot \mathbf{e}_k \odot R^{\#}_{jk} \ \text{ on } \ L^{p'}(\partial\Omega, \sigma) \otimes C\ell_n, \tag{1.6.17}$$

where $K^{\#}$ is defined as in (1.3.72) corresponding to $L := \Delta$, i.e.,

$$K^{\#} f(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\omega_{n-1}} \int_{\substack{y \in \partial\Omega \\ |x-y| > \varepsilon}} \frac{\langle \nu(x), x - y \rangle}{|x - y|^n} f(y) \, d\sigma(y) \tag{1.6.18}$$

for σ_*-a.e. $x \in \partial_*\Omega$. On the other hand, starting with (1.6.1) and reasoning much as in [70, (2.5.310)] (based on [68, (6.4.2)-(6.4.3)]) we see that

$$\mathfrak{C}^{\#} = K^{\#} + \frac{1}{2} \cdot \sum_{j,k=1}^{n} \mathbf{e}_j \odot \mathbf{e}_k \odot R^{\#}_{jk}. \tag{1.6.19}$$

From these, the last claim in item *(ii)* then readily follows. Finally, in view of (1.6.15), (1.6.19), and [70, (2.5.324)], the claims in items *(iii)-(iv)* are implied by parts *(v)-(vi)* in Theorem 1.5.1, parts *(iii)-(iv)* in Proposition 1.2.1, and [70, Proposition 2.5.32] (bearing in mind the current item *(ii)*). □

The next corollary contains a couple of useful identities involving Cauchy-Clifford singular integral operators and Riesz transforms on the boundary of a given UR domain.

Corollary 1.6.2 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, be a* UR *domain. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, let M_ν be the operator of pointwise Clifford algebra multiplication from the left by $\nu \equiv \nu_1 \mathbf{e}_1 + \cdots + \nu_n \mathbf{e}_n$. Then*

$$\mathfrak{C}^{\#} = M_\nu \mathfrak{C} M_\nu \ \text{ on } \ L^1\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^{n-1}}\Big) \otimes C\ell_n \tag{1.6.20}$$

and for each $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$ one has

$$\mathfrak{C}\mathfrak{C}^{\#} = -\frac{1}{4} \sum_{j=1}^{n} R_j^2 + \frac{1}{4} \sum_{1 \leq j < k \leq n} \mathbf{e}_j \odot \mathbf{e}_k [R_j, R_k] \ \text{ on } \ L^p(\partial\Omega, w) \otimes C\ell_n \tag{1.6.21}$$

where $(R_j)_{1 \leq j \leq n}$ are the Riesz transforms associated with $\Sigma := \partial\Omega$ as in (A.0.187).

Proof Formula (1.6.20) is a consequence of (A.0.54), (1.6.1), and the fact that at σ-a.e. point on $\partial\Omega$ we have $\nu \odot \nu = -1$ (cf. [68, (5.6.21), (6.4.1)], (A.0.178), and the fact that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$). Next observe from (A.0.53) and [68, (6.4.1)] that

$$\mathfrak{C}M_\nu = -\frac{1}{2}\sum_{j=1}^{n}\mathbf{e}_j R_j \ \text{ on } L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\otimes \mathcal{C}\ell_n. \tag{1.6.22}$$

Fix $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$. From (1.6.20), (1.6.22), and [68, (7.7.106)] we then deduce that, as operators on $L^p(\partial\Omega, w)\otimes \mathcal{C}\ell_n$,

$$\begin{aligned}
\mathfrak{C}\mathfrak{C}^{\#} &= \mathfrak{C}\big(M_\nu \mathfrak{C} M_\nu\big) = \big(\mathfrak{C}M_\nu\big)\big(\mathfrak{C}M_\nu\big) = \Big(-\frac{1}{2}\sum_{j=1}^{n}\mathbf{e}_j R_j\Big)\Big(-\frac{1}{2}\sum_{k=1}^{n}\mathbf{e}_k R_k\Big)\\
&= \frac{1}{4}\sum_{j,k=1}^{n}\mathbf{e}_j \odot \mathbf{e}_k R_j R_k = -\frac{1}{4}\sum_{j=1}^{n} R_j^2 + \frac{1}{4}\sum_{1\le j\neq k\le n}\mathbf{e}_j \odot \mathbf{e}_k R_j R_k\\
&= -\frac{1}{4}\sum_{j=1}^{n} R_j^2 + \frac{1}{4}\sum_{1\le j<k\le n}\mathbf{e}_j \odot \mathbf{e}_k (R_j R_k - R_k R_j).
\end{aligned} \tag{1.6.23}$$

Based on (1.6.23) and (1.6.20), we may therefore conclude that (1.6.21) holds. □

Special geometries lead to specialized formulas, as described in the next lemma.

Lemma 1.6.3 *If Ω is either a ball or a half-space in $\mathbb{R}^n$, then its outward unit normal ν satisfies*

$$-\nu(x)\odot(x-y) = (x-y)\odot\nu(y) \ \textit{for each } \ x, y \in \partial\Omega. \tag{1.6.24}$$

Proof Suppose Ω is a ball in $\mathbb{R}^n$. Via a translation and a dilation, there is no loss of generality in assuming that $\Omega = B(0, 1)$. Then $\nu(x) = x$ for each $x \in \partial\Omega$, hence

$$\begin{aligned}
\nu(x)\odot(x-y) + (x-y)\odot\nu(x) &= x\odot(x-y) + (x-y)\odot y\\
&= -1 - x\odot y + x\odot y - (-1)\\
&= 0 \ \text{ for each } \ x, y \in \partial\Omega,
\end{aligned} \tag{1.6.25}$$

which establishes the formula claimed in (1.6.24) in this case. Finally, if for some $\xi \in S^{n-1}$ and $x_0 \in \mathbb{R}^n$ we have $\Omega = \big\{x \in \mathbb{R}^n : \langle x - x_0, \xi\rangle < 0\big\}$, then $\nu = \xi$ on $\partial\Omega$. Keeping this in mind, [68, (6.4.6)] allows to write

$$\begin{aligned}
\nu(x)\odot(x-y) + (x-y)\odot\nu(x) &= \xi\odot(x-y) + (x-y)\odot\xi\\
&= -2\langle \xi, x-y\rangle\\
&= 0 \ \text{ for each } \ x, y \in \partial\Omega,
\end{aligned} \tag{1.6.26}$$

proving (1.6.24) in this case. □

As a result of Lemma 1.6.3, the Cauchy-Clifford singular integral agrees with its (formal) transpose on spheres and hyperplanes.

Lemma 1.6.4 *If Ω is either a ball or a half-space in $\mathbb{R}^n$, then $\mathfrak{C} = \mathfrak{C}^\#$ as operators acting on the space $L^1\Big(\partial\Omega, \frac{\mathcal{H}^{n-1}(x)}{1+|x|^{n-1}}\Big) \otimes C\ell_n$.*

Proof This is a direct consequence of Lemma 1.6.3 and definitions (cf. (A.0.54), (1.6.1)). □

In terms of Riesz transforms, the result in Lemma 1.6.4 yields the identities described in the next corollary.

Corollary 1.6.5 *Suppose Ω is either a ball or a half-space in $\mathbb{R}^n$, where $n \geq 2$, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Let $(R_j)_{1\leq j\leq n}$ are the Riesz transforms associated with $\Sigma := \partial\Omega$ as in (A.0.187). Also, fix some $w \in A_p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$. Then, as operators on $L^p(\partial\Omega, w)$,*

$$\sum_{j=1}^{n} R_j^2 = -I \ \text{ and } \ R_j R_k = R_k R_j \ \text{ for each } \ j, k \in \{1, \ldots, n\}. \tag{1.6.27}$$

Proof From (1.6.21), Lemma 1.6.4, and [70, (2.5.324)] we obtain

$$\frac{1}{4}I = -\frac{1}{4}\sum_{j=1}^{n} R_j^2 + \frac{1}{4}\sum_{1\leq j<k\leq n} \mathbf{e}_j \odot \mathbf{e}_k [R_j, R_k] \ \text{ on } \ L^p(\partial\Omega, w) \otimes C\ell_n. \tag{1.6.28}$$

Then (1.6.27) follows by restricting ourselves to scalar-valued functions in $L^p(\partial\Omega, w)$ and considering the vector and bi-vectors components in this identity separately. □

Here is another way of tying up Riesz transforms with single and double layers for the (vector) Laplacian; this is going to be of relevance to us later on.

Proposition 1.6.6 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, be a set of locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \ldots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Bring in the modified version $\mathscr{S}_{\text{mod}}$ of the single layer operator associated with the Laplacian Δ and the set Ω as in (1.5.50). Also, recall boundary-to-domain Riesz transforms, $\mathcal{R}_j$ indexed by $j \in \{1, \ldots, n\}$, defined as in (A.0.188) (with $\Sigma := \partial\Omega$). Next, consider a family of $2^n \times 2^n$ real matrices $\{E_j\}_{1\leq j\leq n}$ satisfying*[18]

$$\begin{aligned} \big(E_j\big)^2 &= -I_{2^n\times 2^n} \ \text{ for each } \ j \in \{1, \ldots, n\} \ \text{ and} \\ E_j E_k &= -E_k E_j \ \text{ for all } \ j, k \in \{1, \ldots, n\} \ \text{ with } \ j \neq k, \end{aligned} \tag{1.6.29}$$

and define first-order $2^n \times 2^n$ system

$$D := \sum_{j=1}^{n} E_j \partial_j. \tag{1.6.30}$$

[18] The existence of such a family has been established in [68, (6.4.14)] (with $m := n$)

Then for each function $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^{2^n}$ *one has*[19]

$$2D\mathscr{S}_{\rm mod} f = \sum_{j=1}^{n} E_j \mathcal{R}_j f \ \ \textit{in} \ \ \mathring{\Omega}. \tag{1.6.31}$$

Also, if $\mathcal{D}$ *is the boundary-to-domain double layer potential operator associated as in* (1.3.18) *with the set* Ω *and the vector Laplacian written as in* (1.4.21)*, i.e., if for each function*

$$f \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^{2^n} \tag{1.6.32}$$

and every $x \in \mathring{\Omega}$ *one defines*

$$\mathcal{D}f(x) = \frac{1}{\omega_{n-1}} \int_{\partial_*\Omega} \sum_{j=1}^{n} E_j \frac{x_j - y_j}{|x-y|^n} \Big(\sum_{k=1}^{n} \nu_k(y) E_k\Big) f(y)\, \mathrm{d}\sigma(y), \tag{1.6.33}$$

then for each f *as in* (1.6.32) *one has*

$$\mathcal{D}f = D\mathscr{S}_{\rm mod}\Big[\Big(\sum_{j=1}^{n} \nu_j E_j\Big) f\Big] \ \ \textit{in} \ \ \mathring{\Omega}. \tag{1.6.34}$$

Proof The identity claimed in (1.6.31) is seen from (1.5.50), (1.5.51), (A.0.65), (A.0.188), and (1.6.30). In turn, formula (1.6.34) is implied by (1.6.33), (A.0.188), and (1.6.31). □

It is natural to augment the picture emerging from Proposition 1.6.1 by discussing separately the classical Cauchy integral operator on UR sets in the complex plane, as the two-dimensional context accounts for a number of specialized properties (compared with the higher-dimensional setting considered earlier).

Proposition 1.6.7 *Suppose* $\Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$ *is an open set with the property that* $\partial\Omega$ *is a* UR *set. Abbreviate* $\sigma_* := \mathcal{H}^1 \lfloor \partial_*\Omega$ *and* $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$*, then denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. In this setting, consider the following Cauchy-type integral operators.*

First, for each complex-valued function $f \in L^1\Big(\partial_*\Omega, \frac{\sigma_*(\zeta)}{1+|\zeta|}\Big)$ *define (with* $\mathrm{d}\zeta$ *as in* (A.0.62)*)*

$$\mathscr{C}f(z) := \frac{1}{2\pi \mathrm{i}} \int_{\partial_*\Omega} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \ \ \textit{for each} \ \ z \in \Omega, \tag{1.6.35}$$

and its principal-value version

$$Cf(z) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi \mathrm{i}} \int_{\substack{\zeta\in\partial_*\Omega \\ |z-\zeta|>\varepsilon}} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta \ \ \textit{at } \sigma\textit{-a.e.} \ \ z \in \partial\Omega. \tag{1.6.36}$$

[19] with $\mathscr{S}_{\rm mod}$ acting on f componentwise

Second, for each complex-valued function $f \in L^1\Big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\Big)$ *set*

$$C^{\#} f(z) := -\nu(z)\Bigg(\lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int\limits_{\substack{\zeta\in\partial\Omega \\ |z-\zeta|>\varepsilon}} \frac{f(\zeta)}{\zeta - z}\, d\sigma(\zeta)\Bigg) \text{ at } \sigma_*\text{-a.e. } z \in \partial_*\Omega, \tag{1.6.37}$$

and make the convention that $C^{\#}$ *also acts in the same fashion on functions from* $L^1\Big(\partial_*\Omega, \frac{\sigma_*(\zeta)}{1+|\zeta|}\Big)$, *regarded them as defined on the entire topological boundary* $\partial\Omega$ *after being extended by zero outside* $\partial_*\Omega$. *In particular, this convention* (1.6.36), (1.6.36), *and* (A.0.62) *imply* (*with* $\overline{\nu}$ *denoting the complex conjugate of* ν) *that*

$$\begin{aligned} &C^{\#} f \;=\; -\nu\, C(\overline{\nu} f) \text{ at } \sigma_*\text{-a.e. point in } \partial_*\Omega, \text{ for each} \\ &\text{complex-valued function } f \text{ from } L^1\Big(\partial_*\Omega, \tfrac{\sigma_*(\zeta)}{1+|\zeta|}\Big). \end{aligned} \tag{1.6.38}$$

Recall the Cauchy-Riemann operator together with its conjugate,

$$\begin{gathered} \partial_{\bar z} := \tfrac{1}{2}\big(\partial_{z_1} - \tfrac{1}{i}\partial_{z_2}\big) \text{ and } \partial_z := \tfrac{1}{2}\big(\partial_{z_1} + \tfrac{1}{i}\partial_{z_2}\big) \\ \text{where } z = z_1 + iz_2 \in \mathbb{R} + i\mathbb{R} = \mathbb{C}, \end{gathered} \tag{1.6.39}$$

and consider the tangential derivative operator

$$\partial_\tau := \partial_{\tau_{12}} \tag{1.6.40}$$

where the latter is defined as in (A.0.183) *with* $j := 1$ *and* $k = 2$, *i.e., abbreviate*

$$\partial_\tau := \nu_1\partial_{\zeta_2} - \nu_2\partial_{\zeta_1} \text{ where } \nu = \nu_1 + i\nu_2 \text{ and } \zeta = \zeta_1 + i\zeta_2. \tag{1.6.41}$$

Finally, fix an aperture parameter $\kappa \in (0, \infty)$.

Then the following statements are true.

(i) *For each* $f \in L^1\Big(\partial_*\Omega, \frac{\sigma_*(\zeta)}{1+|\zeta|}\Big)$ *the function* $\mathscr{C} f$ *is smooth in* Ω *and satisfies*

$$\begin{aligned} &\partial_{\bar z}(\mathscr{C} f) = 0 \text{ at each point belonging to the set } \Omega, \text{ and} \\ &\mathscr{C} f\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = (\tfrac{1}{2}I + C)f \text{ at } \sigma_*\text{-a.e. point in } \partial_*\Omega. \end{aligned} \tag{1.6.42}$$

Also, for each $p \in [1, \infty)$ *there exists* $C \in (0, \infty)$ *depending only on* $\partial\Omega$, p, *and* κ *such that for each function* $f \in L^p(\partial_*\Omega, \sigma_*)$ *one has*

$$\big\|\mathcal{N}_\kappa(\mathscr{C} f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p(\partial_*\Omega,\sigma_*)} \quad \text{if } p > 1, \tag{1.6.43}$$

$$\big\|\mathcal{N}_\kappa(\mathscr{C} f)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \leq C\|f\|_{L^1(\partial_*\Omega,\sigma_*)} \quad \text{if } p = 1. \tag{1.6.44}$$

Finally, for each $f \in L^1\Big(\partial_*\Omega, \frac{\sigma_*(\zeta)}{1+|\zeta|}\Big)$ *one has*

$$2\partial_z \mathscr{S}_{\rm mod} f = -\mathscr{C}(\overline{\nu} f) \ \ in \ \ \Omega, \tag{1.6.45}$$

where $\mathscr{S}_{\rm mod}$ *is the modified harmonic single layer potential operator associated with* Ω *as in* (1.5.50).

(ii) For each function $f \in L^p_1(\partial_*\Omega, \sigma_*)$ *with* $p \in [1, \infty)$ *one has*

$$\partial_z(\mathscr{C} f) = (-\mathrm{i})\mathscr{C}(\overline{\nu}\, \partial_\tau f) \ \ at\ each\ point\ in\ \ \Omega. \tag{1.6.46}$$

As a consequence, for each function $f \in L^p_1(\partial_*\Omega, \sigma_*)$ *with* $p \in [1, \infty)$ *the pointwise nontangential boundary trace* $\big[\partial_z(\mathscr{C} f)\big]\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ *exists (in* $\mathbb{C}$*) at* σ_**-a.e. point on* $\partial_*\Omega$*, and there exists some finite constant* $C > 0$ *depending only on* $\partial\Omega$*,* p*, and* κ *such that*

$$\big\|\mathcal{N}_\kappa(\nabla \mathscr{C} f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|\partial_\tau f\|_{L^p(\partial_*\Omega,\sigma_*)} \quad \ if\ p > 1, \tag{1.6.47}$$

$$\big\|\mathcal{N}_\kappa(\nabla \mathscr{C} f)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \leq C\|\partial_\tau f\|_{L^1(\partial_*\Omega,\sigma_*)} \quad \ if\ p = 1. \tag{1.6.48}$$

(iii) Suppose $p, p' \in (1, \infty)$ *are such that* $1/p + 1/p' = 1$*. Then the operators*

$$C : L^p(\partial\Omega, \sigma) \longrightarrow L^p(\partial_*\Omega, \sigma_*), \tag{1.6.49}$$

$$C : L^1(\partial\Omega, \sigma) \longrightarrow L^{1,\infty}(\partial_*\Omega, \sigma_*), \tag{1.6.50}$$

are well-defined, linear, and bounded. In addition,

$$\text{the transpose of } C \text{ in (1.6.49) is } C^\top = C^\#, \tag{1.6.51}$$

where $C^\#$ *is the linear and bounded mapping*

$$C^\# : L^{p'}(\partial_*\Omega, \sigma_*) \longrightarrow L^{p'}(\partial\Omega, \sigma). \tag{1.6.52}$$

Finally, corresponding to $p = 1$*, the operator*

$$C^\# : L^1(\partial_*\Omega, \sigma_*) \longrightarrow L^{1,\infty}(\partial\Omega, \sigma) \tag{1.6.53}$$

is also well-defined, linear, and bounded.

(iv) Make the additional assumption that

$$\mathcal{H}^1\big(\partial_{\rm nta}\Omega \setminus \partial_*\Omega\big) = 0 \tag{1.6.54}$$

(which is automatically satisfied if Ω *is a* UR *domain to begin with). Then the operator*

$$C : L^p_1(\partial_*\Omega, \sigma_*) \longrightarrow L^p_1(\partial_*\Omega, \sigma_*) \tag{1.6.55}$$

is well defined, linear, and bounded for each $p \in (1, \infty)$*.*

(v) Retain the additional assumption made in (1.6.54)*. Then for each* $p \in (1,\infty)$ *it follows that* $C^{\#}$ *in* (1.6.52) *extends uniquely to a linear and bounded operator from the negative boundary Sobolev space* $L^p_{-1}(\partial_*\Omega, \sigma_*)$ *into itself. Furthermore, if one retains the same notation* $C^{\#}$ *for this extension, the transpose of* (1.6.55) *is*

$$C^{\#} : L^{p'}_{-1}(\partial_*\Omega, \sigma_*) \longrightarrow L^{p'}_{-1}(\partial_*\Omega, \sigma_*) \tag{1.6.56}$$

where $p' \in (1,\infty)$ *is such that* $1/p + 1/p' = 1$.

(vi) Once again, retain the additional assumption made in (1.6.54)*. Then for each* $p \in (1,\infty)$ *one has*

$$\partial_\tau \circ C + C^{\#} \circ \partial_\tau = 0 \ \text{ on } \ L^p_1(\partial_*\Omega, \sigma_*). \tag{1.6.57}$$

In particular, as a consequence of this and (1.6.38)*,*

$$\begin{aligned}&\partial_\tau(Cf) = \nu\, C(\overline{\nu}\, \partial_\tau f) \text{ at } \sigma_*\text{-a.e. point in } \partial_*\Omega, \text{ for}\\ &\text{each function } f \text{ belonging to } L^p_1(\partial_*\Omega, \sigma_*).\end{aligned} \tag{1.6.58}$$

Proof For each $f \in L^1\Big(\partial_*\Omega, \frac{\sigma_*(\zeta)}{1+|\zeta|}\Big)$, it follows from (1.6.35) that the function $\mathscr{C} f$ is holomorphic in Ω. The jump-formula in (1.6.42) is a consequence of item *(iv)* in Theorem 1.5.1 (in view of (1.4.72)-(1.4.73) in Example 1.4.9). The estimates in (1.6.43)-(1.6.44) are direct consequence of item *(i)* in Theorem 1.5.1 (again, bearing in mind the identification in (1.4.72)).

As far as (1.6.45) is concerned, for any $f \in L^1\Big(\partial_*\Omega, \frac{\sigma_*(\zeta)}{1+|\zeta|}\Big)$ (regarded as a function defined on the entire topological boundary, extending it by zero on $\partial\Omega \setminus \partial_*\Omega$ we may use (1.5.51) to write

$$\begin{aligned}2\partial_z(\mathscr{S}_{\rm mod} f)(z) &= \frac{1}{2\pi}\int_{\partial_*\Omega} \frac{\overline{z-\zeta}}{|z-\zeta|^2} f(\zeta)\, \mathrm{d}\sigma(\zeta)\\ &= \frac{1}{2\pi}\int_{\partial_*\Omega} \frac{1}{z-\zeta} f(\zeta)\, \mathrm{d}\sigma(\zeta) = -\mathscr{C}(\overline{\nu} f) \ \text{ in } \ \Omega.\end{aligned} \tag{1.6.59}$$

This takes care of all claims in the current item *(i)*.

To justify the first claim in item *(ii)*, fix an arbitrary point $z \in \Omega$. Since for each $\zeta \in \mathbb{C} \setminus \{z\}$ we have $\partial_{\bar\zeta}\big[\frac{1}{\zeta - z}\big] = 0$, at σ_*-a.e. $\zeta \in \partial_*\Omega$ we may then write

$$\begin{aligned}\nu(\zeta)\partial_\zeta\Big[\frac{1}{\zeta-z}\Big] &= \nu_1(\zeta)\partial_\zeta\Big[\frac{1}{\zeta-z}\Big] + \mathrm{i}\nu_2(\zeta)\partial_\zeta\Big[\frac{1}{\zeta-z}\Big]\\ &= \nu_1(\zeta)\big(\partial_\zeta - \partial_{\bar\zeta}\big)\Big[\frac{1}{\zeta-z}\Big] + \mathrm{i}\nu_2(\zeta)\big(\partial_\zeta + \partial_{\bar\zeta}\big)\Big[\frac{1}{\zeta-z}\Big]\\ &= (-\mathrm{i})\big(\nu_1(\zeta)\partial_{\zeta_2} - \nu_2(\zeta)\partial_{\zeta_1}\big)\Big[\frac{1}{\zeta-z}\Big].\end{aligned} \tag{1.6.60}$$

Consequently, given any $f \in L^p_1(\partial_*\Omega, \sigma_*)$ with $p \in [1, \infty)$, based on (1.6.35), (1.6.60), (1.6.41), and (A.0.62) we may compute

$$\begin{aligned}
\partial_z(\mathscr{C}f)(z) &= \frac{1}{2\pi}\int_{\partial_*\Omega} f(\zeta)\nu(\zeta)\partial_z\Big[\frac{1}{\zeta - z}\Big]\,d\sigma_*(\zeta)\\
&= -\frac{1}{2\pi}\int_{\partial_*\Omega} f(\zeta)\nu(\zeta)\partial_\zeta\Big[\frac{1}{\zeta - z}\Big]\,d\sigma_*(\zeta)\\
&= \frac{i}{2\pi}\int_{\partial_*\Omega} f(\zeta)\partial_{\tau(\zeta)}\Big[\frac{1}{\zeta - z}\Big]\,d\sigma_*(\zeta)\\
&= \frac{1}{2\pi i}\int_{\partial_*\Omega} (\partial_\tau f)(\zeta)\frac{1}{\zeta - z}\,d\sigma_*(\zeta)\\
&= (-i)\mathscr{C}(\overline{\nu}\,\partial_\tau f)(z) \ \text{ for each } \ z \in \Omega,
\end{aligned} \tag{1.6.61}$$

where the fourth equality uses the integration by parts formula on the boundary from [69, Lemma 11.1.7]. This finishes the proof of (1.6.46). With this in hand, all other claims in item *(ii)* follow with the help of what we have proved already in item *(i)*.

Moving on to item *(iii)*, the fact that the operators (1.6.49), (1.6.50), (1.6.52), (1.6.53) are all well-defined, linear, and bounded is a consequence of [70, Theorem 2.3.2] and (A.0.62). To prove the claim made in (1.6.51), for each integrability exponent $p \in (1, \infty)$, consider the extension by zero operator

$$\begin{gathered}
E : L^p(\partial_*\Omega, \sigma_*) \longrightarrow L^p(\partial\Omega, \sigma) \ \text{ given by}\\
Ef := \begin{cases} f \ \text{ on } \ \partial_*\Omega, \\ 0 \ \text{ on } \ \partial\Omega \setminus \partial_*\Omega, \end{cases} \quad \text{for each } \ f \in L^p(\partial_*\Omega, \sigma_*),
\end{gathered} \tag{1.6.62}$$

along with the operator of pointwise multiplication by the normal

$$\begin{gathered}
M : L^p(\partial\Omega, \sigma) \to L^p(\partial_*\Omega, \sigma_*),\\
Mf := \nu f \ \text{ for each } \ f \in L^p(\partial\Omega, \sigma).
\end{gathered} \tag{1.6.63}$$

Clearly, these are both linear and bounded operators. Also, a moment's reflection shows that

> the (real) transpose of the operator M in (1.6.63) is the composition $E \circ M \circ E : L^{p'}(\partial_*\Omega, \sigma_*) \longrightarrow L^{p'}(\partial\Omega, \sigma)$, where $p' \in (1, \infty)$ is the Hölder conjugate exponent of p. (1.6.64)

Next, bring in the principal-value singular integral operator

$$T : L^p(\partial\Omega, \sigma) \to L^p(\partial\Omega, \sigma) \tag{1.6.65}$$

acting on each $f \in L^p(\partial\Omega, \sigma)$ according to

$$Tf(z) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int\limits_{\substack{\zeta\in\partial\Omega \\ |z-\zeta|>\varepsilon}} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\sigma(\zeta) \text{ at } \sigma\text{-a.e. } z \in \partial\Omega. \tag{1.6.66}$$

From [70, Theorem 2.3.2] (cf. [70, (2.3.15), (2.3.18)]) we know that T is well defined, linear, and bounded. Moreover, [70, (2.3.25)] implies that

the (real) transpose of the singular integral operator (1.6.65) is the operator $-T : L^{p'}(\partial\Omega, \sigma) \longrightarrow L^{p'}(\partial\Omega, \sigma)$ where, once again, $p' \in (1, \infty)$ denotes the Hölder conjugate exponent of p. (1.6.67)

Upon noting that the Cauchy singular integral operator defined in (1.6.36) in the context $C : L^p(\partial_*\Omega, \sigma_*) \to L^p(\partial\Omega, \sigma)$ may be expressed as $C = T \circ E \circ M \circ E$, we may invoke (1.6.64) and (1.6.67) to conclude that its (real) transpose is the operator $C^\top : L^{p'}(\partial_*\Omega, \sigma_*) \to L^{p'}(\partial\Omega, \sigma)$, given by $C^\top = -M \circ T = C^\#$. This establishes (1.6.51) and finishes the treatment of item *(iii)*.

Moving on, the claims in the current items *(iv)*-*(v)* are direct consequences of items *(v)*-*(vi)* in [70, Theorem 2.3.2] (in view of the identification made in (1.4.73) in Example 1.4.9). At this stage, there remains to deal with item *(vi)*. To this end, fix some $f \in L^p_1(\partial_*\Omega, \sigma_*)$ with $p \in [1, \infty)$. From [69, Proposition 11.3.2] applied to the function $u := \mathscr{C} f$ in Ω we see that the nontangential boundary trace $u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ belongs to the boundary Sobolev space $L^p_1(\partial_*\Omega, \sigma_*)$ and

$$\partial_\tau\Big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) = \nu_1\Big((\partial_{z_2}u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) - \nu_2\Big((\partial_{z_1}u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) \text{ at } \sigma_*\text{-a.e. point on } \partial_*\Omega. \tag{1.6.68}$$

On the other hand, making use of the fact that the function u is holomorphic in Ω, we have $\partial_{z_2}u = \mathrm{i}\big(\partial_z - \partial_{\bar z}\big)u = \mathrm{i}\partial_z u$ and $\partial_{z_1}u = \big(\partial_z + \partial_{\bar z}\big)u = \partial_z u$. When used back in (1.6.68) this permits us to conclude that

$$\partial_\tau\Big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) = \mathrm{i}\nu\Big((\partial_z u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) \text{ at } \sigma_*\text{-a.e. point on } \partial_*\Omega. \tag{1.6.69}$$

In turn, by combining the jump-formula in (1.6.42) with (1.6.69), (1.6.46), and (1.6.38), we arrive at

$$\begin{aligned}
\tfrac12\partial_\tau f + \partial_\tau(Cf) &= \partial_\tau\Big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) = \mathrm{i}\nu\Big((\partial_z u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) \\
&= \nu\Big(\big(\mathscr{C}(\overline{\nu}\,\partial_\tau f)\big)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) = \tfrac12\partial_\tau f + \nu\, C(\overline{\nu}\,\partial_\tau f) \\
&= \tfrac12\partial_\tau f - C^\#(\partial_\tau f) \text{ at } \sigma_*\text{-a.e. point on } \partial_*\Omega.
\end{aligned} \tag{1.6.70}$$

From this, (1.6.57) readily follows, completing the proof of Proposition 1.6.7. □

A remarkable intertwining identity in the two-dimensional setting, whose proof involves Proposition 1.6.7, is discussed below (see also Propositions 2.3.13-2.3.14 in this regard).

Corollary 1.6.8 *Let $\Omega \subseteq \mathbb{R}^2$ be a* UR *domain. Set $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. In this setting, recall the two-dimensional the principal-value harmonic double layer acting on each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|}\big)$ according to* (*cf.* [70, (2.5.203)])

$$Kf(x) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^2} f(y)\, d\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{1.6.71}$$

along with the two-dimensional "transpose" principal-value harmonic double layer (*cf.* (1.6.18))

$$K^{\#} f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(x), x-y\rangle}{|x-y|^2} f(y)\, d\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{1.6.72}$$

Finally, fix an integrability exponent $p \in (1,\infty)$.

Then, with the tangential derivative operator

$$\partial_\tau := \partial_{\tau_{12}} \tag{1.6.73}$$

defined as in (A.0.183) *with $j := 1$ and $k := 2$, one has*

$$\partial_\tau \circ K = -K^{\#} \circ \partial_\tau \text{ on } L^p_1(\partial\Omega,\sigma). \tag{1.6.74}$$

Also, with $\partial_\nu \mathcal{D}$ denoting the normal derivative of the harmonic double layer (*cf.* (1.5.29)) *and with the principal-value singular integral operator R defined for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|}\big)$ and σ-a.e. point $x \in \partial\Omega$ as*

$$Rf(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \Big[\nu_1(y)(\partial_2 E_\Delta)(y-x) - \nu_2(y)(\partial_1 E_\Delta)(y-x)\Big] f(y)\, d\sigma(y), \tag{1.6.75}$$

where E_Δ is the standard fundamental solution for the two-dimensional Laplacian, one has

$$\partial_\nu \mathcal{D} = -\partial_\tau \circ R \text{ on } L^p_1(\partial\Omega,\sigma). \tag{1.6.76}$$

An inspection of the proof given below shows that

$$\text{formulas (1.6.74), (1.6.76) are actually true for the more inclusive off-diagonal Sobolev space } L^{p,q}_1(\partial\Omega,\sigma) \text{ with } p,q \in (1,\infty) \text{ arbitrary.} \tag{1.6.77}$$

Under stronger assumptions on the set Ω, formulas (1.6.74), (1.6.76) remain valid on yet even larger spaces. Specifically, via density (cf. [69, (11.8.4)]) and [69, Proposition 11.8.5], if $\Omega \subseteq \mathbb{R}^2$ is an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular then the operator identities in (1.6.74)-(1.6.76) are actually valid in the (larger) Lebesgue space $L^p(\partial\Omega, \sigma)$.

Proof of Corollary 1.6.8 Recall the singular integral operator R introduced in (1.6.75). For each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|}\big)$ let us also define

$$R^{\#} f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \Big[\nu_1(x)(\partial_2 E_\Delta)(y-x) - \nu_2(x)(\partial_1 E_\Delta)(y-x)\Big] f(y)\,\mathrm{d}\sigma(y), \tag{1.6.78}$$

at σ-a.e. point $x \in \partial\Omega$, where (ν_1, ν_2) are the scalar components of the normal vector ν and E_Δ is the standard fundamental solution for the (two-dimensional) Laplacian (cf. (1.1.1)). Then from (1.6.36), (1.6.71), and (1.6.75), on the one hand, and from (1.6.37), (1.6.72), and (1.6.78), on the other hand, we see that

$$C = K - iR \;\text{ and }\; C^{\#} = K^{\#} + \mathrm{i}R^{\#} \;\text{ on }\; L^p(\partial\Omega, \sigma). \tag{1.6.79}$$

Moreover, from (1.2.13) we conclude that for each $f \in L^p_1(\partial\Omega, \sigma)$ we have

$$\partial_\tau(Rf) = R^{\#}(\partial_\tau f) \;\text{ at } \sigma\text{-a.e. point on }\; \partial\Omega. \tag{1.6.80}$$

Then (1.6.74) follows by combining (1.6.57), (1.6.79), and (1.6.80).

Let us now turn our attention to (1.6.76). Given any $f \in L^p(\partial\Omega, \sigma)$, at each $x \in \Omega$ define

$$\mathcal{R} f(x) := \int\limits_{\partial\Omega} \Big[\nu_1(y)(\partial_2 E_\Delta)(y-x) - \nu_2(y)(\partial_1 E_\Delta)(y-x)\Big] f(y)\,\mathrm{d}\sigma(y). \tag{1.6.81}$$

Fix an aperture parameter $\kappa > 0$. From Proposition 1.2.1 it follows that

$$\begin{gathered} \mathcal{N}_\kappa(\mathcal{R}f) \in L^p(\partial\Omega, \sigma) \;\text{ and }\; \mathcal{R}f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = Rf \;\text{ for each }\; f \in L^p(\partial\Omega, \sigma), \\ \mathcal{N}_\kappa(\nabla\mathcal{R}f) \in L^p(\partial\Omega, \sigma) \;\text{ for each }\; f \in L^p_1(\partial\Omega, \sigma). \end{gathered} \tag{1.6.82}$$

Also, since for each $f \in L^p(\partial\Omega, \sigma)$ the function $\mathscr{C} f$ is holomorphic in Ω and

$$\mathscr{C} f = \mathcal{D} f - \mathrm{i}\mathcal{R} f \;\text{ in }\; \Omega, \tag{1.6.83}$$

it follows that $\mathcal{D} f$ and $-\mathcal{R} f$ are conjugate harmonic functions in Ω. That is, for each $f \in L^p(\partial\Omega, \sigma)$ we have

$$\partial_1 \mathcal{D} f + \partial_2 \mathcal{R} f = 0 \;\text{ and }\; \partial_2 \mathcal{D} f - \partial_1 \mathcal{R} f = 0 \;\text{ in }\; \Omega. \tag{1.6.84}$$

As a consequence of (1.6.84) and [69, Proposition 11.3.2], for each $f \in L^p_1(\partial\Omega, \sigma)$ we may then write

$$
\begin{aligned}
\partial_\nu \mathcal{D} f &= \nu \cdot \Big[\nabla(\mathcal{D} f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big] = \nu_1 \Big[\partial_1(\mathcal{D} f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big] + \nu_2 \Big[\partial_2(\mathcal{D} f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big] \\
&= -\nu_1 \Big[\partial_2(\mathcal{R} f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big] + \nu_2 \Big[\partial_1(\mathcal{R} f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big] \\
&= -\partial_\tau \Big[(\mathcal{R} f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big] = -\partial_\tau (R f), \qquad (1.6.85)
\end{aligned}
$$

proving (1.6.76). □

It is also of interest to augment the spectral results from Proposition 1.5.3 with the following symmetry property (with respect to the origin) for the point-spectra of the principal-value harmonic double layer acting on Lebesgue and Sobolev spaces on the boundary of two-dimensional UR domains.

Corollary 1.6.9 *Let $\Omega \subseteq \mathbb{R}^2$ be an arbitrary* UR *domain and set $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$. In this setting, recall the principal-value harmonic double layer K and its* (*real*) *transpose $K^\#$ from* (1.6.71)-(1.6.72)*. Then for each exponent $p \in (1, \infty)$ and each complex number $\lambda \in \mathbb{C} \setminus \{\pm\frac{1}{2}\}$ the following statements are true.*

(1) If λ is an eigenvalue of K acting on $L^p(\partial\Omega, \sigma)$ (respectively, on $L^p_1(\partial\Omega, \sigma)$) then $-\lambda$ is an eigenvalue of K acting on $L^p(\partial\Omega, \sigma)$ (respectively, on $L^p_1(\partial\Omega, \sigma)$).

(2) If λ is an eigenvalue of $K^\#$ acting on the space $L^p(\partial\Omega, \sigma)$ (respectively, on $L^p_{-1}(\partial\Omega, \sigma)$) then $-\lambda$ is an eigenvalue of $K^\#$ acting on the space $L^p(\partial\Omega, \sigma)$ (respectively, on $L^p_{-1}(\partial\Omega, \sigma)$).

Proof Suppose $\lambda \in \mathbb{C} \setminus \{\pm\frac{1}{2}\}$ is an eigenvalue of K acting on $L^p(\partial\Omega, \sigma)$. Hence, there exists $f \in L^p(\partial\Omega, \sigma)$, which is not zero σ-a.e. on $\partial\Omega$, with the property that $Kf = \lambda f$. Then the function $g := Rf$ (where R is the singular integral operator introduced in (1.6.75)) belongs to the Lebesgue space $L^p(\partial\Omega, \sigma)$ and, thanks to the first equality in [70, (2.5.334)], satisfies

$$
Kg = K(Rf) = -R(Kf) = -R(\lambda f) = -\lambda R f = -\lambda g. \qquad (1.6.86)
$$

As such, we may conclude that $-\lambda$ is an eigenvalue of K acting on $L^p(\partial\Omega, \sigma)$ as soon as we check that the function g is not zero σ-a.e. on $\partial\Omega$. To see that this is the case, we make use of the second equality in [70, (2.5.334)] to write

$$
Rg = R^2 f = \big(\tfrac{1}{2}I + K\big)\big(-\tfrac{1}{2}I + K\big) f = \big(\lambda^2 - \tfrac{1}{4}\big) f \neq 0, \qquad (1.6.87)
$$

given that $\lambda^2 \neq \frac{1}{4}$ and $f \neq 0$. This goes to show that, indeed, $g \neq 0$. The above argument also works for K acting on the Sobolev space $L^p_1(\partial\Omega, \sigma)$ since the operator identities in [70, (2.5.334)] are valid $L^p_1(\partial\Omega, \sigma)$ as well (as may be seen from (1.5.24) and (1.2.11)). The treatment of item *(1)* is therefore complete.

Finally, the claims in item *(2)* may be handled in an analogous fashion, this time making use of the operator identities (with $R^\#$ originally defined as in (1.6.78))

$$
\begin{gathered}
K^\# R^\# + R^\# K^\# = 0 \ \text{ and } \ \big(\tfrac{1}{2}I + K^\#\big)\big(-\tfrac{1}{2}I + K^\#\big) = \big(R^\#\big)^2 \\
\text{both on } \ L^p(\partial\Omega, \sigma) \ \text{ and on } \ L^p_{-1}(\partial\Omega, \sigma),
\end{gathered} \qquad (1.6.88)
$$

which follow from [70, (2.5.334)] and transposition (bearing in mind item *(vi)* in Theorem 1.5.1, and item *(iv)* in Proposition 1.2.1). □

For future endeavors, we find it useful to derive a version of the last formula from (1.6.88) featuring the composition between the tangential derivative operator and the modified harmonic single layer operator. This is made precise in the lemma below.

Lemma 1.6.10 *Let* $\Omega \subseteq \mathbb{R}^2$ *be an arbitrary* UR *domain and set* $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$*. In this setting, recall the transpose double layer* $K^{\#}$ *from* (1.6.72) *and bring in the two-dimensional modified boundary-to-boundary single layer operator* (*cf.* (1.5.73))

$$\begin{gathered} S_{\rm mod} f(x) := \int_{\partial\Omega} \big\{E(x-y) - E_*(-y)\big\} f(y)\,d\sigma(y) \ \textit{ at } \sigma\textit{-a.e. } x \in \partial\Omega, \\ \textit{for each } f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|}\big), \ \textit{ where } E_* := (2\pi)^{-1} \ln \cdot \mathbf{1}_{\mathbb{R}^2 \setminus B(0,1)}. \end{gathered} \tag{1.6.89}$$

Finally, consider the tangential derivative operator

$$\partial_\tau := \partial_{\tau_{12}} \tag{1.6.90}$$

defined as in (A.0.183) *with* $j := 1$ *and* $k := 2$*, and fix an integrability exponent* $p \in (1, \infty)$*. Then, with* I *denoting the identity,*

$$\begin{gathered} \big(\tfrac{1}{2}I + K^{\#}\big)\big(-\tfrac{1}{2}I + K^{\#}\big) = \big(\partial_\tau S_{\rm mod}\big)^2 \\ \textit{as operators on } L^p(\partial\Omega, \sigma). \end{gathered} \tag{1.6.91}$$

Proof From (1.6.78) and (1.5.82) we see that $R^{\#}$ coincides with $T^{\#}_{12}$, the singular integral operator defined as in (1.5.82) corresponding to $j = 1$, $k = 2$, $M = 1$, $n = 2$, $L = \Delta$, and $E := (2\pi)^{-1} \ln$. Bearing this in mind, we then see from (1.5.83) that

$$\partial_\tau \big(S_{\rm mod} f\big) = R^{\#} f \ \text{ for each } f \in L^p(\partial\Omega, \sigma). \tag{1.6.92}$$

With this in hand, the operator identity claimed in (1.6.91) follows from (1.6.88). □

1.7 Kernels and Images of Boundary Layer Potentials

With an eye toward the goal of eventually inverting boundary layer potentials (something that is relevant in the context of solving boundary value problems, a topic treated later on, in Volume IV; cf. [71]), one of the main goals in this section is to study the kernels and images of singular integral operators of boundary layer type.

To set the stage, the reader is reminded (cf. [70, Definition 1.3.2]) that a coefficient tensor with complex entries $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n \\ 1\leq\alpha,\beta\leq M}}$ is said to be `Legendre-Hadamard elliptic` provided there exists a real number $\kappa > 0$ such that the following condition is satisfied:

$$\operatorname{Re}\left[a_{rs}^{\alpha\beta}\xi_r\xi_s\overline{\eta_\alpha}\eta_\beta\right] \geq \kappa|\xi|^2|\eta|^2 \text{ for all}$$
$$\xi = (\xi_r)_{1\leq r\leq n} \in \mathbb{R}^n \text{ and } \eta = (\eta_\alpha)_{1\leq\alpha\leq M} \in \mathbb{C}^M. \tag{1.7.1}$$

A basic feature of a Legendre-Hadamard elliptic coefficient tensor $A \in \mathfrak{A}(n, M)$, which may be easily checked based on repeated applications of Plancherel's theorem and (1.7.1), is the fact that there exists a constant $c = c(A, n) \in (0, \infty)$ with the property that

$$\operatorname{Re}\int_{\mathbb{R}^n}\left\langle A\nabla u, \overline{\nabla u}\right\rangle d\mathcal{L}^n \geq c\int_{\mathbb{R}^n}|\nabla u|^2\,d\mathcal{L}^n, \qquad \forall\, u \in \left[W^{1,2}(\mathbb{R}^n)\right]^M. \tag{1.7.2}$$

Next, we make the following definition.

Definition 1.7.1 *Fix $n, M \in \mathbb{N}$, and consider a coefficient tensor $A = \left(a_{rs}^{\alpha\beta}\right)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with constant complex entries. Call A* `compliant` *provided A is positive semi-definite, in the sense that*

$$\operatorname{Re}\left\langle A\zeta, \overline{\zeta}\right\rangle = \operatorname{Re}\left(a_{rs}^{\alpha\beta}\zeta_s^\beta\overline{\zeta_r^\alpha}\right) \geq 0, \quad \forall\zeta = (\zeta_r^\alpha)_{\substack{1\leq r\leq n\\ 1\leq\alpha\leq M}} \in \mathbb{C}^{n\times M}, \tag{1.7.3}$$

and A has the property that for each $\zeta = (\zeta_r^\alpha)_{\substack{1\leq r\leq n\\ 1\leq\alpha\leq M}} \in \mathbb{C}^{n\times M}$ one has

$$\operatorname{Re}\left\langle A\zeta, \overline{\zeta}\right\rangle = 0 \Longleftrightarrow A\zeta = 0. \tag{1.7.4}$$

In relation to Definition 1.7.1, it is worth noting that

$$\text{any weakly elliptic compliant coefficient tensor} \atop \text{is Legendre-Hadamard elliptic.} \tag{1.7.5}$$

To justify (1.7.5), suppose $A = \left(a_{rs}^{\alpha\beta}\right)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ is a weakly elliptic compliant coefficient tensor, and fix an arbitrary vector $\xi = (\xi_r)_{1\leq r\leq n} \in \mathbb{R}^n \setminus \{0\}$. The fact that A is assumed to be weakly elliptic implies that

$$\text{the } M\times M \text{ matrix } \left(a_{rs}^{\alpha\beta}\xi_r\xi_s\right)_{1\leq\alpha,\beta\leq M} \text{ is invertible.} \tag{1.7.6}$$

Suppose now $\eta = (\eta_\alpha)_{1\leq\alpha\leq M} \in \mathbb{C}^M$ is such that

$$\operatorname{Re}\left[a_{rs}^{\alpha\beta}\xi_r\xi_s\overline{\eta_\alpha}\eta_\beta\right] = 0. \tag{1.7.7}$$

If we define

$$\zeta := (\xi_r\eta_\alpha)_{\substack{1\leq r\leq n\\ 1\leq\alpha\leq M}} \in \mathbb{C}^{n\times M} \tag{1.7.8}$$

then formula (1.7.7) may be recast simply as $\operatorname{Re}\left\langle A\zeta, \overline{\zeta}\right\rangle = 0$. In view of (1.7.4), this forces $0 = A\zeta = \left(a_{rs}^{\alpha\beta}\xi_s\eta_\beta\right)_{\substack{1\leq r\leq n\\ 1\leq\alpha\leq M}}$, hence $\left(a_{rs}^{\alpha\beta}\xi_r\xi_s\eta_\beta\right)_{1\leq\alpha\leq M} = 0 \in \mathbb{C}^M$. Bearing in mind (1.7.6), this allows us to conclude that $\eta = 0 \in \mathbb{C}^M$. Thus, (1.7.7) can only happen when $\eta = 0 \in \mathbb{C}^M$. The fact that A is positive semi-definite also implies (writing (1.7.3) for ζ as in (1.7.8)) that

$$\operatorname{Re}\left[a_{rs}^{\alpha\beta}\xi_r\xi_s\overline{\eta_\alpha}\eta_\beta\right] \geq 0 \text{ for each } \eta = (\eta_\alpha)_{1\leq\alpha\leq M} \in \mathbb{C}^M. \tag{1.7.9}$$

In concert with the homogeneity of the bilinear form associated with A, these properties ultimately show that there exists some $c \in (0,\infty)$ with the property that (1.7.1) holds. This finishes the proof of (1.7.5).

In the theorem below we establish injectivity properties for the boundary-to-boundary versions of the single and double layer potential operators.

Theorem 1.7.2 *Fix $n \in \mathbb{N}$ with $n \geq 2$. Suppose $\Omega \subseteq \mathbb{R}^n$ is an arbitrary* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Let A be a weakly elliptic coefficient tensor with complex entries, and consider the boundary layer potentials S, K, $K^\#$ associated with A and Ω as in* (1.3.62), (1.3.68), (1.3.72)*. Then the following properties hold.*

(1) Suppose A is actually Legendre-Hadamard elliptic (*cf.* (1.7.1))*. If $n \geq 3$ the boundary-to-boundary single layer operator S is injective in the context*

$$S : \left[L^{2(n-1)/n}(\partial\Omega,\sigma)\right]^M \longrightarrow \left[L^{2(n-1)/(n-2)}(\partial\Omega,\sigma)\right]^M \tag{1.7.10}$$

(*cf.* (1.3.63))*, while if $n = 2$ and $\partial\Omega$ is compact then*

$$\begin{gathered} \textit{if } f \in \left[L^p(\partial\Omega,\sigma)\right]^M \textit{ with } p \in (1,\infty) \textit{ and } \int_{\partial\Omega} f\,d\sigma = 0 \textit{ then} \\ Sf \textit{ constant on } \partial\Omega \textit{ forces } f \textit{ to vanish on } \partial\Omega. \end{gathered} \tag{1.7.11}$$

(2) Suppose A is positive semi-definite and Legendre-Hadamard elliptic and fix a complex number $z \in \mathbb{C} \setminus [-\frac{1}{2}, \frac{1}{2}]$. In addition, assume that either A is complex symmetric[20]*, or z is real. Then*

$$\begin{gathered} zI + K^\# : \left[L^{2(n-1)/n}(\partial\Omega,\sigma)\right]^M \longrightarrow \left[L^{2(n-1)/n}(\partial\Omega,\sigma)\right]^M \\ \textit{is an injective operator if } n \geq 3, \end{gathered} \tag{1.7.12}$$

and the operator

$$\begin{gathered} zI + K^\# : \left[L^p(\partial\Omega,\sigma)\right]^M \longrightarrow \left[L^p(\partial\Omega,\sigma)\right]^M \textit{ is injective} \\ \textit{for every } p \in (1,\infty) \textit{ if } \partial\Omega \textit{ is compact and } n = 2. \end{gathered} \tag{1.7.13}$$

(3) Given $p, p' \in (1,\infty)$ with $1/p + 1/p' = 1$ along with $z \in \mathbb{C}$, consider the space

$$\begin{aligned} L^p_{A,z}(\partial\Omega,\sigma) := \Big\{ f \in \left[L^p(\partial\Omega,\sigma)\right]^M : \int_{\partial\Omega} \langle f, g\rangle\,d\sigma = 0 \textit{ for each} \\ g \in \left[L^{p'}(\partial\Omega,\sigma)\right]^M \textit{ with } (zI + K_A)g = 0\Big\}. \end{aligned} \tag{1.7.14}$$

[20] i.e., Hermitian self-adjoint

Then for each $p \in (1, \infty)$ and $z \in \mathbb{C}$ one has

$$\overline{\mathrm{Im}\Big(zI + K^{\#}; \big[L^p(\partial\Omega, \sigma)\big]^M\Big)} = L^p_{A,z}(\partial\Omega, \sigma), \tag{1.7.15}$$

where the "bar" in the left-hand side indicates closure taken in $\big[L^p(\partial\Omega, \sigma)\big]^M$. Moreover,

$$\begin{gathered} \pm\tfrac{1}{2}I + K^{\#} : L^{2(n-1)/n}_{A^*,\pm 1/2}(\partial\Omega, \sigma) \longrightarrow \big[L^{2(n-1)/n}(\partial\Omega, \sigma)\big]^M \text{ injectively} \\ \text{if } n \geq 3 \text{ and } A^\top \text{ is assumed to be compliant (cf. Definition 1.7.1),} \end{gathered} \tag{1.7.16}$$

which, in concert with (1.7.15), shows that

$$\begin{gathered} \pm\tfrac{1}{2}I + K^{\#} : L^{2(n-1)/n}_{A,\pm 1/2}(\partial\Omega, \sigma) \longrightarrow L^{2(n-1)/n}_{A,\pm 1/2}(\partial\Omega, \sigma) \text{ injectively if } n \geq 3, \\ A \text{ is also assumed to be complex symmetric, and } A^\top \text{ is compliant.} \end{gathered} \tag{1.7.17}$$

(4) Assume A is positive definite, Ω is bounded, and $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. Then the operator

$$\begin{gathered} \tfrac{1}{2}I + K^{\#} : \big[L^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^p(\partial\Omega, \sigma)\big]^M \text{ is injective} \\ \text{if either } n \geq 3 \text{ and } 2(n-1)/n \leq p < \infty, \text{ or } n = 2 \text{ and } 1 < p < \infty. \end{gathered} \tag{1.7.18}$$

(5) Suppose the coefficient tensor A is complex symmetric and positive definite. Also, strengthen the hypotheses on the underlying domain by assuming that Ω is a bounded UR *domain satisfying a two-sided local John condition and such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. Then for each $z \in \mathbb{C} \setminus [-\frac{1}{2}, \frac{1}{2}]$ it follows that*

$$\begin{gathered} zI + K : \big[L^{2(n-1)/n}_1(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{2(n-1)/n}_1(\partial\Omega, \sigma)\big]^M \\ \text{is an injective operator if } n \geq 3, \end{gathered} \tag{1.7.19}$$

and the operator

$$\begin{gathered} zI + K : \big[L^p_1(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^p_1(\partial\Omega, \sigma)\big]^M \text{ is injective} \\ \text{for every } p \in (1, \infty) \text{ if } n = 2. \end{gathered} \tag{1.7.20}$$

(6) Assume the coefficient tensor A is positive definite. Also, suppose Ω is a bounded UR *domain satisfying a two-sided local John condition and such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. Then*

$$\begin{gathered} \tfrac{1}{2}I + K : \big[L^{2(n-1)/n}_1(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{2(n-1)/n}_1(\partial\Omega, \sigma)\big]^M \\ \text{is an injective operator if } n \geq 3, \end{gathered} \tag{1.7.21}$$

and the operator

$$\begin{gathered}\tfrac{1}{2}I+K:\big[L^p_1(\partial\Omega,\sigma)\big]^M\longrightarrow\big[L^p_1(\partial\Omega,\sigma)\big]^M\ \textit{is injective}\\ \textit{for every } p\in(1,\infty)\ \textit{if } n=2.\end{gathered}\tag{1.7.22}$$

Proof Throughout, we denote by $\nu=(\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω, and define $\Omega_+:=\Omega$ as well as $\Omega_-:=\mathbb{R}^n\setminus\overline{\Omega}$. [68, Lemma 5.10.9] then ensures that Ω_- is also a UR domain, whose topological boundary coincides with that of Ω, whose geometric measure theoretic boundary agrees with that of Ω, and whose geometric measure theoretic outward unit normal is the vector $-\nu$ at σ-a.e. point on $\partial\Omega$. Also, we shall let $\big(a^{\alpha\beta}_{rs}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$ be the entries of the coefficient tensor A, and denote by $E=(E_{\gamma\beta})_{1\le\gamma,\beta\le M}$ be the matrix-valued fundamental solution associated with the system

$$L:=\big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\le\alpha,\beta\le M}\tag{1.7.23}$$

(i.e., $L:=L_A=\operatorname{div}A\nabla$) as in [70, Theorem 1.4.2]. Finally, pick an aperture parameter $\kappa\in(0,\infty)$.

<u>*Proof of (1)*</u>: First, assume that $n\ge 3$ and pick a function $f\in\big[L^{2(n-1)/n}(\partial\Omega,\sigma)\big]^M$ satisfying $Sf=0$ on $\partial\Omega$. If we define

$$u(x):=\int_{\partial\Omega}E(x-y)f(y)\,d\sigma(y)\ \text{ for each }\ x\in\mathbb{R}^n\setminus\partial\Omega,\tag{1.7.24}$$

then [70, Proposition 2.5.40] implies that, when regarded as being defined $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$, the function u satisfies

$$\begin{gathered}u\in\big[L^{2n/(n-1)}(\mathbb{R}^n,\mathcal{L}^n)\big]^M,\quad \nabla u\in\big[L^2(\mathbb{R}^n,\mathcal{L}^n)\big]^{nM},\\ \text{and }\ u\in\big[W^{1,2}_{\rm loc}(\mathbb{R}^n)\big]^M.\end{gathered}\tag{1.7.25}$$

To proceed, introduce

$$u^\pm:=u\big|_{\Omega_\pm}.\tag{1.7.26}$$

From item *(x)* in Theorem 1.5.1 we know that

$$u^\pm\in\big[\mathscr{C}^\infty(\Omega_\pm)\big]^M,\qquad L_Au^\pm=0\ \text{ in }\ \Omega_\pm,\tag{1.7.27}$$

while (1.7.24), (1.5.47), (1.5.47), (1.3.67), (1.5.59), and (1.5.48) yield

$$\begin{gathered}\mathcal{N}_\kappa u^\pm\in L^{2(n-1)/(n-2)}(\partial\Omega,\sigma),\ \ \mathcal{N}_\kappa(\nabla u^\pm)\in L^{2(n-1)/n}(\partial\Omega,\sigma),\\ u^\pm\big|^{\kappa-\text{n.t.}}_{\partial\Omega}=Sf,\qquad \partial^A_\nu u^\pm=\big(\mp\tfrac{1}{2}I+K^\#_{A^\top}\big)f,\\ \text{and }\ \nabla u^\pm\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\ \text{ exist at }\sigma\text{-a.e. point on }\partial\Omega.\end{gathered}\tag{1.7.28}$$

In addition, thanks to (1.7.24) and [70, Theorem 1.4.2], it follows that for each $k\in\mathbb{N}_0$ we have

$$(\nabla^k u)(x) = O(|x|^{2-n-k}) \text{ as } |x| \to \infty \text{ if } \partial\Omega \text{ compact.} \tag{1.7.29}$$

Granted the properties listed in (1.7.27)-(1.7.29), we may invoke [70, Corollary 1.7.14] (presently used with $p := 2(n-1)/n$, $q := 2(n-1)/(n-2)$, $a := n-1$, $b := n-2$) and, bearing in mind that the outward unit normal for Ω_- is $-\nu$, conclude that

$$\int_{\Omega_\pm} \big\langle A\nabla u^\pm, \overline{\nabla u^\pm}\,\big\rangle \, d\mathcal{L}^n = \int_{\partial\Omega} \big\langle \big(-\tfrac{1}{2}I \pm K^\#_{A^\top}\big) f, \overline{Sf}\,\big\rangle \, d\sigma = 0 \tag{1.7.30}$$

since, by assumption, $Sf = 0$ on $\partial\Omega$. Collectively, from (1.7.25)-(1.7.30) we see that

$$\langle A\nabla u, \overline{\nabla u}\,\rangle \in L^1(\mathbb{R}^n, \mathcal{L}^n) \text{ and } \int_{\mathbb{R}^n} \big\langle A\nabla u, \overline{\nabla u}\,\big\rangle \, d\mathcal{L}^n = 0. \tag{1.7.31}$$

To proceed, pick a real-valued function $\varphi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ which is identically one in a neighborhood of the origin. If for each $j \in \mathbb{N}$ we set $\varphi_j(x) := \varphi(x/j)$ for every $x \in \mathbb{R}^n$, then

$$\begin{aligned} \lim_{j\to\infty} \int_{\mathbb{R}^n} & \big\langle A\nabla(\varphi_j u), \overline{\nabla(\varphi_j u)}\,\big\rangle \, d\mathcal{L}^n \\ &= \lim_{j\to\infty} \int_{\mathbb{R}^n} \varphi_j^2 \big\langle A\nabla u, \overline{\nabla u}\,\big\rangle \, d\mathcal{L}^n \\ &\quad + \lim_{j\to\infty} \int_{\mathbb{R}^n} O\big(|\varphi_j||\nabla\varphi_j||u||\nabla u| + |\nabla\varphi_j|^2|u|^2\big) \, d\mathcal{L}^n. \end{aligned} \tag{1.7.32}$$

Note that since $|x| \approx j$ on the support of $\nabla\varphi_j$, uniformly in $j \in \mathbb{N}$, based on (1.7.25) and (the trilinear version of) Hölder's inequality we may estimate

$$\begin{aligned} \int_{\mathbb{R}^n} |\varphi_j||\nabla\varphi_j||u||\nabla u| \, d\mathcal{L}^n &\leq Cj^{-1} \int_{\substack{x\in\mathbb{R}^n \\ |x|\approx j}} |u||\nabla u| \, d\mathcal{L}^n \\ &\leq Cj^{-1/2} \|u\|_{[L^{2n/(n-1)}(\mathbb{R}^n,\mathcal{L}^n)]^M} \|\nabla u\|_{[L^2(\mathbb{R}^n,\mathcal{L}^n)]^{nM}} \\ &= O(j^{-1/2}) \text{ as } j \to \infty, \end{aligned} \tag{1.7.33}$$

and

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla\varphi_j|^2|u|^2 \, d\mathcal{L}^n &\leq Cj^{-2} \int_{\substack{x\in\mathbb{R}^n \\ |x|\approx j}} |u|^2 \, d\mathcal{L}^n \\ &\leq Cj^{-1} \|u\|^2_{[L^{2n/(n-1)}(\mathbb{R}^n,\mathcal{L}^n)]^M} \\ &= O(j^{-1}) \text{ as } j \to \infty. \end{aligned} \tag{1.7.34}$$

Thanks to (1.7.31), (1.7.33), (1.7.34), and Lebesgue's Dominated Convergence Theorem we may therefore conclude from (1.7.32) that

$$\lim_{j\to\infty}\int_{\mathbb{R}^n}\left\langle A\nabla(\varphi_j u),\overline{\nabla(\varphi_j u)}\right\rangle \mathrm{d}\mathcal{L}^n = 0. \tag{1.7.35}$$

Since the last membership in (1.7.25) ensures that $\varphi_j u \in \left[W^{1,2}(\mathbb{R}^n)\right]^M$ for each $j\in\mathbb{N}$, the coercivity estimate (1.7.2) then gives, for some $c\in(0,\infty)$,

$$\begin{aligned} 0 &= \lim_{j\to\infty}\left\{\mathrm{Re}\int_{\mathbb{R}^n}\left\langle A\nabla(\varphi_j u),\overline{\nabla(\varphi_j u)}\right\rangle \mathrm{d}\mathcal{L}^n\right\} \\ &\geq c\lim_{j\to\infty}\int_{\mathbb{R}^n}|\nabla(\varphi_j u)|^2\,\mathrm{d}\mathcal{L}^n \\ &= c\lim_{j\to\infty}\int_{\mathbb{R}^n}\varphi_j^2|\nabla u|^2\,\mathrm{d}\mathcal{L}^n \\ &\quad+\lim_{j\to\infty}\int_{\mathbb{R}^n}O\big(|\varphi_j||\nabla\varphi_j||u||\nabla u|+|\nabla\varphi_j|^2|u|^2\big)\,\mathrm{d}\mathcal{L}^n \\ &= c\int_{\mathbb{R}^n}|\nabla u|^2\,\mathrm{d}\mathcal{L}^n, \end{aligned} \tag{1.7.36}$$

where the last equality makes use of (1.7.33), (1.7.34), (1.7.31), and Lebesgue's Dominated Convergence Theorem. This ultimately proves that

$$\nabla u = 0 \ \text{ in } \ \mathbb{R}^n. \tag{1.7.37}$$

In concert, the jump-formula (1.5.59), (A.0.184), and (1.7.37) establish

$$\big(\mp\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f = \partial^A_\nu u^{\pm} = 0 \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{1.7.38}$$

From (1.7.37) we finally conclude that

$$f = \partial^A_\nu u^- - \partial^A_\nu u^+ = 0, \tag{1.7.39}$$

which goes to show that the boundary-to-boundary single layer operator S is injective in the context of (1.7.10) if $n\geq 3$.

Let us now consider the case when $n=2$ and $\partial\Omega$ is compact. To fix ideas, assume Ω_+ is bounded (and Ω_- is an exterior domain). Fix $p\in(1,\infty)$. The goal is to prove (1.7.11). To this end, pick a function $f\in\left[L^p(\partial\Omega,\sigma)\right]^M$ with $\int_{\partial\Omega} f\,\mathrm{d}\sigma = 0$ and satisfying $Sf=c$, a constant on $\partial\Omega$. Define $u:\mathbb{R}^n\setminus\partial\Omega\to\mathbb{C}^M$ as in (1.7.24). In the present case, the effect of the extra cancelation assumption on f is the (improved) decay property

$$(\nabla^k u)(x) = O(|x|^{-1-k}) \ \text{ as } \ |x|\to\infty, \ \text{ for each } \ k\in\mathbb{N}_0. \tag{1.7.40}$$

See Lemma 1.3.1. Also, [70, Proposition 2.5.41] implies that, when regarded as being defined $\mathcal{L}^2$-a.e. in $\mathbb{R}^2$, the function u satisfies

$$u \in \bigcap_{2<q<\infty} \big[L^q(\mathbb{R}^2, \mathcal{L}^2)\big]^M \text{ and } \nabla u \in \bigcap_{1<q\leq 2p} \big[L^q(\mathbb{R}^2, \mathcal{L}^2)\big]^{2M}. \tag{1.7.41}$$

As in the past, introduce $u^\pm := u\big|_{\Omega_\pm}$ so

$$u^\pm \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M \text{ and } L_A u^\pm = 0 \text{ in } \Omega_\pm. \tag{1.7.42}$$

Also, from (1.7.24), (1.7.40), the fact that $\partial\Omega$ is compact, item *(x)* in Theorem 1.5.1, (1.3.67), (1.5.59), (1.5.48), and assumptions yield

$$\begin{gathered}\mathcal{N}_\kappa u^\pm \in \textstyle\bigcap_{0<q<\infty} L^q(\partial\Omega,\sigma),\ \ \mathcal{N}_\kappa(\nabla u^\pm) \in L^p(\partial\Omega,\sigma),\\ u^\pm\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = c, \qquad \partial_\nu^A u^\pm = \big(\mp\tfrac{1}{2}I + K^\#_{A^\top}\big)f,\\ \text{and } \nabla u^\pm\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exist at } \sigma\text{-a.e. point on } \partial\Omega.\end{gathered} \tag{1.7.43}$$

Thanks to these properties, [70, Corollary 1.7.14] applies (for the current p, and with $a := 2$, $b := 1$) and gives that, on the one hand,

$$\int_{\Omega_\pm} \big\langle A\nabla u^\pm, \overline{\nabla u^\pm}\big\rangle \, d\mathcal{L}^2 = \int_{\partial\Omega} \big\langle \big(-\tfrac{1}{2}I \pm K^\#_{A^\top}\big)f, \overline{c}\big\rangle \, d\sigma. \tag{1.7.44}$$

On the other hand, since Ω_+ is bounded,

$$\int_{\partial\Omega} \big\langle \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)f, \overline{c}\big\rangle \, d\sigma = \int_{\partial\Omega} \big\langle f, \big(-\tfrac{1}{2}I + K_{A^\top}\big)\overline{c}\big\rangle \, d\sigma = 0 \tag{1.7.45}$$

thanks to the duality result from item *(iii)* of Theorem 1.5.1 and (1.5.21) (written with $A^\top$ in place of A). In a similar fashion,

$$\begin{aligned}\int_{\partial\Omega} \big\langle \big(-\tfrac{1}{2}I - K^\#_{A^\top}\big)f, \overline{c}\big\rangle \, d\sigma &= \int_{\partial\Omega} \big\langle f, \big(-\tfrac{1}{2}I - K_{A^\top}\big)\overline{c}\big\rangle \, d\sigma = \int_{\partial\Omega} \langle f, -\overline{c}\,\rangle \, d\sigma \\ &= -\Big\langle \int_{\partial\Omega} f \, d\sigma, \overline{c}\Big\rangle = 0,\end{aligned} \tag{1.7.46}$$

with the last equality provided by the vanishing moment condition for f. Collectively, (1.7.44)-(1.7.46) imply that

$$\int_{\Omega_\pm} \big\langle A\nabla u^\pm, \overline{\nabla u^\pm}\big\rangle \, d\mathcal{L}^2 = 0. \tag{1.7.47}$$

Together with (1.7.41), this permits us to re-run the same argument which, starting with (1.7.31), has produced (1.7.39) to once again conclude that $f = 0$.

Proof of (2): First, suppose $n \geq 3$. Work under the assumption that A is positive semi-definite, complex symmetric, as well as Legendre-Hadamard elliptic, and that $z \in \mathbb{C} \setminus [-\frac{1}{2}, \frac{1}{2}]$ is arbitrary. Consider a function $f \in \big[L^{2(n-1)/n}(\partial\Omega, \sigma)\big]^M$ such that $(zI + K^{\#})f = 0$. If we define

$$u(x) := \int_{\partial\Omega} E^{\top}(x-y) f(y)\, \mathrm{d}\sigma(y) \ \text{ for each } \ x \in \mathbb{R}^n \setminus \partial\Omega, \tag{1.7.48}$$

then [70, Proposition 2.5.40] implies

$$\begin{aligned} u \in \big[L^{2n/(n-1)}(\mathbb{R}^n, \mathcal{L}^n)\big]^M, \quad \nabla u \in \big[L^2(\mathbb{R}^n, \mathcal{L}^n)\big]^{nM}, \\ \text{and } \ u \in \big[W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)\big]^M. \end{aligned} \tag{1.7.49}$$

Next, we invoke [70, Corollary 1.7.14] which, granted the current assumptions, applies to $u\big|_{\Omega_\pm}$ (and the system $L_{A^\top}$). On account of (1.3.67) and (1.5.59) (presently used with A replaced by $A^\top$) we may therefore write

$$\begin{aligned} 0 &= \int_{\partial\Omega} \big\langle (zI + K^{\#})f, \overline{S_{A^\top} f}\,\big\rangle\, \mathrm{d}\sigma \\ &= \int_{\partial\Omega} \Big\langle (-z + \tfrac{1}{2})(-\tfrac{1}{2}I + K^{\#})f + (z + \tfrac{1}{2})(\tfrac{1}{2}I + K^{\#})f, \overline{S_{A^\top} f}\Big\rangle\, \mathrm{d}\sigma \\ &= \big(-z + \tfrac{1}{2}\big) \int_{\Omega_+} \big\langle A^\top \nabla u, \overline{\nabla u}\,\big\rangle\, \mathrm{d}\mathcal{L}^n + \big(-z - \tfrac{1}{2}\big) \int_{\Omega_-} \big\langle A^\top \nabla u, \overline{\nabla u}\,\big\rangle\, \mathrm{d}\mathcal{L}^n. \end{aligned} \tag{1.7.50}$$

At this point, bring in the elementary fact that (easily justified by inspecting the real and imaginary parts)

$$\left.\begin{array}{l} a_\pm \in [0, \infty), \quad z \in \mathbb{C} \setminus \big[-\frac{1}{2}, \frac{1}{2}\big] \\ \big(-z + \frac{1}{2}\big)a_+ + \big(-z - \frac{1}{2}\big)a_- = 0 \end{array}\right\} \Longrightarrow a_+ = a_- = 0. \tag{1.7.51}$$

Recall that we are now assuming that A is positive semi-definite and complex symmetric. Thus, $A^\top$ is also complex symmetric, and A^* is positive semi-definite. Thanks to these properties and the second membership in (1.7.49) we have

$$a_\pm := \int_{\Omega_\pm} \big\langle A^\top \nabla u, \overline{\nabla u}\,\big\rangle\, \mathrm{d}\mathcal{L}^n \in [0, \infty). \tag{1.7.52}$$

Collectively, (1.7.50), (1.7.51), and (1.7.52) then permit us to conclude that

$$\int_{\Omega_+} \big\langle A^\top \nabla u, \overline{\nabla u}\,\big\rangle\, \mathrm{d}\mathcal{L}^n = 0 \ \text{ and } \ \int_{\Omega_-} \big\langle A^\top \nabla u, \overline{\nabla u}\,\big\rangle\, \mathrm{d}\mathcal{L}^n = 0, \tag{1.7.53}$$

hence, ultimately,

$$\int_{\mathbb{R}^n} \big\langle A^\top \nabla u, \overline{\nabla u}\,\big\rangle\, \mathrm{d}\mathcal{L}^n = 0. \tag{1.7.54}$$

Granted (1.7.54), the same reasoning based on the Legendre-Hadamard ellipticity of A (which also entails that $A^\top$ is Legendre-Hadamard elliptic) which, starting with (1.7.31) has produced (1.7.37), continues to work in the present setting (thanks to (1.7.48)). Much as before, from this we deduce that $\nabla u = 0$ in $\mathbb{R}^n$. Denoting $u^\pm := u\big|_{\Omega_\pm}$ and availing ourselves of the jump-formulas

$$\partial_\nu^{A^\top} u^\pm = \big(\mp \tfrac{1}{2} I + K^\#\big) f \text{ at } \sigma\text{-a.e. point on } \partial\Omega \tag{1.7.55}$$

(themselves seen from (1.5.59) in $\Omega_\pm$ and with $A^\top$ in place of A), we eventually arrive at the conclusion that $f = \partial_\nu^{A^\top} u^- - \partial_\nu^{A^\top} u^+ = 0$. The argument so far proves that if $n \geq 3$ then the operator $zI + K^\#$ is injective in the context of (1.7.12) provided A is positive semi-definite, complex symmetric, as well as Legendre-Hadamard elliptic, and $z \in \mathbb{C} \setminus \big[-\tfrac{1}{2}, \tfrac{1}{2}\big]$ is arbitrary.

Continue to assume that $n \geq 3$, but suppose now that $z \in \mathbb{R} \setminus \big[-\tfrac{1}{2}, \tfrac{1}{2}\big]$ while A is only positive semi-definite and Legendre-Hadamard elliptic. This time, in lieu of (1.7.51) we use

$$\left.\begin{array}{r} a_\pm \in \mathbb{C}, \quad \operatorname{Re} a_\pm \in [0,\infty), \quad z \in \mathbb{R} \setminus \big[-\tfrac{1}{2}, \tfrac{1}{2}\big] \\ \text{and } \big(-z + \tfrac{1}{2}\big) a_+ + \big(-z - \tfrac{1}{2}\big) a_- = 0 \end{array}\right\} \Longrightarrow \operatorname{Re} a_+ = \operatorname{Re} a_- = 0, \tag{1.7.56}$$

which is itself a consequence of (1.7.51). When applied to

$$a_\pm := \int_{\Omega_\pm} \big\langle A^\top \nabla u, \overline{\nabla u}\,\big\rangle \, d\mathcal{L}^n \in \mathbb{C}, \tag{1.7.57}$$

in place of (1.7.54) we now arrive at

$$\operatorname{Re} \int_{\mathbb{R}^n} \big\langle A^\top \nabla u, \overline{\nabla u}\,\big\rangle \, d\mathcal{L}^n = 0. \tag{1.7.58}$$

This is nonetheless just as effective in the context of (1.7.36), so we may complete the proof of the injectivity of the operator $zI + K^\#$ in the context of (1.7.12) much as before.

As an intermediate step, we next propose to show that

$$\begin{array}{l} \text{if } n \geq 2,\ \partial\Omega \text{ is compact, } z \in \mathbb{C} \text{ such that } z \neq -\tfrac{1}{2} \text{ if } \Omega \text{ is bounded and} \\ z \neq \tfrac{1}{2} \text{ if } \Omega \text{ is unbounded, and } f \in \big[L^p(\partial\Omega,\sigma)\big]^M \text{ with } p \in (1,\infty) \text{ is} \\ \text{such that } (zI + K^\#) f = 0, \text{ then necessarily } \int_{\partial\Omega} f \, d\sigma = 0. \end{array} \tag{1.7.59}$$

To see that this is the case, observe first that [68, Lemma 5.10.10] implies that either Ω_+ is bounded, or Ω_- is bounded. To fix ideas, assume Ω_+ is bounded. In such a scenario, applying Green's formula [70, (1.7.81)] (with $A^\top$ playing the role of A) to the functions $u := \mathscr{S}_{A^\top} f$ and $w := \lambda$ in Ω, for some arbitrary constant $\lambda \in \mathbb{C}^M$, yields

$$\Big\langle \int_{\partial\Omega} \big(-\tfrac{1}{2}I + K^{\#}\big) f \, \mathrm{d}\sigma, \lambda \Big\rangle = \int_{\partial\Omega} \big\langle \partial_\nu^{A^\top} u, w\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \big\rangle \, \mathrm{d}\sigma$$

$$= \int_{\partial\Omega} \big\langle u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \partial_\nu^{A} w \big\rangle \, \mathrm{d}\sigma = 0, \tag{1.7.60}$$

since $\partial_\nu^A w = 0$ at σ-a.e. point on $\partial\Omega$. From this and the arbitrariness of $\lambda \in \mathbb{C}^M$ we conclude that $\int_{\partial\Omega}(-\frac{1}{2}I + K^{\#})f \, \mathrm{d}\sigma = 0$. Upon noting that $(-\frac{1}{2}I + K^{\#})f = (-\frac{1}{2} - z)f$ and recalling that $z \neq -\frac{1}{2}$, we conclude that $\int_{\partial\Omega} f \, \mathrm{d}\sigma = 0$, as wanted. The case when Ω_- is bounded is dealt with similarly (applying Green's formula [70, (1.7.81)] in Ω_-, and using the fact that $z \neq \frac{1}{2}$), and this finishes the proof of (1.7.59).

Returning to the principal subject of discussion, we now observe that the very same argument used to establish the injectivity of (1.7.12) in the case $n \geq 3$ may be adapted to show that if $\partial\Omega$ is compact and $n = 2$ then the operator $zI + K^{\#}$ is injective on the space $\big[L^p(\partial\Omega, \sigma)\big]^M$ for every $p \in (1, \infty)$. Indeed, in such a case if $f \in \big[L^p(\partial\Omega, \sigma)\big]^M$ satisfies $(zI + K^{\#})f = 0$ then (1.7.59) guarantees that $\int_{\partial\Omega} f \, \mathrm{d}\sigma = 0$. In turn, this cancelation property on f translates into better-than-expected decay properties for the function u defined in (1.7.48), namely

$$(\nabla^k u)(x) = O(|x|^{-1-k}) \ \text{ as } \ |x| \to \infty, \ \text{ for each } \ k \in \mathbb{N}_0. \tag{1.7.61}$$

Granted this, the previous argument (used to deal with (1.7.12) in the case $n \geq 3$) goes through, and shows that (1.7.13) is injective.

<u>*Proof of (3)*</u>: That (1.7.15) holds for each $p \in (1, \infty)$ and $z \in \mathbb{C}$ follows from item *(iii)* in Theorem 1.5.1 and [69, (2.1.49)]. Assume next that $n \geq 3$ and that $A^\top$ is compliant. First, fix a function $f \in L^{2(n-1)/n}_{A^*,1/2}(\partial\Omega, \sigma) \subseteq \big[L^{2(n-1)/n}(\partial\Omega, \sigma)\big]^M$ with the property that $\big(\frac{1}{2}I + K^{\#}\big)f = 0$. Set

$$u^{\pm} := \mathscr{S}_{A^\top} f \ \text{ in } \ \Omega_{\pm}. \tag{1.7.62}$$

Then (1.5.66) (written for $A^\top$ in place of A) gives

$$\partial_\nu^{A^\top} u^- = \big(\tfrac{1}{2}I + K^{\#}\big)f = 0 \ \text{ on } \ \partial\Omega. \tag{1.7.63}$$

Next consider the function

$$g := u^-\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = S_{A^\top} f \in \big[L^{2(n-1)/(n-2)}(\partial\Omega, \sigma)\big]^M, \tag{1.7.64}$$

with the membership implied by (1.3.63) (used with $p := 2(n-1)/n$). As a consequence of (1.7.62), (1.7.63), (1.7.64), and (1.7.27)-(1.7.28) (written for $A^\top$ in place of A), the integral representation formula from Theorem 1.3.3 (again, written for $A^\top$ in place of A) presently gives

$$u^- = \mathscr{D}^-_{A^\top} g \ \text{ in } \ \Omega_- \tag{1.7.65}$$

where $\mathcal{D}^-_{A^\top}$ is the boundary-to-domain double layer operator associated with $A^\top$ and Ω_- as in (1.3.18). In view of (1.5.20) and the fact that the geometric measure theoretic outward unit normal for Ω_- is $-\nu$, going nontangentially to the boundary then yields $\big(\frac{1}{2}I + K_{A^\top}\big)g = 0$. After taking complex conjugates this proves

$$\big(\tfrac{1}{2}I + K_{A^*}\big)\overline{g} = 0. \tag{1.7.66}$$

By also keeping in mind that $f \in L^{2(n-1)/n}_{A^*,1/2}(\partial\Omega, \sigma)$, $g = S_{A^\top} f$, and $f = \big(\frac{1}{2}I - K^{\#}\big)f$ we may write

$$\begin{aligned} 0 &= \int_{\partial\Omega} \langle f, \overline{g}\rangle \, d\sigma = \int_{\partial\Omega} \Big\langle \big(\tfrac{1}{2}I - K^{\#}\big)f, \overline{S_{A^\top} f} \Big\rangle \, d\sigma \\ &= -\int_{\Omega_+} \Big\langle A^\top \nabla u^+, \overline{\nabla u^+} \Big\rangle \, d\mathcal{L}^n, \end{aligned} \tag{1.7.67}$$

where the last equality comes from [70, Corollary 1.7.14] (using $A^\top$ in place of A, and bearing in mind (1.7.62)). Consequently,

$$\int_{\Omega_+} \mathrm{Re}\, \Big\langle A^\top \nabla u^+, \overline{\nabla u^+} \Big\rangle \, d\mathcal{L}^n = 0 \tag{1.7.68}$$

and, given that $A^\top$ is positive semi-definite (as part of $A^\top$ being compliant; cf. Definition 1.7.1), this forces $\mathrm{Re}\, \Big\langle A^\top \nabla u^+, \overline{\nabla u^+} \Big\rangle = 0$ in Ω_+. From this and the fact that $A^\top$ is compliant, we conclude that

$$A^\top \nabla u^+ = 0 \ \text{ in } \Omega_+ \tag{1.7.69}$$

which, after going nontangentially to the boundary and taking the dot product with ν, further implies

$$\partial_\nu^{A^\top} u^+ = 0 \ \text{ on } \partial\Omega. \tag{1.7.70}$$

Collectively, (1.7.63), (1.7.70), (1.7.62), and (1.5.66) prove that

$$f = \partial_\nu^{A^\top} u^- - \partial_\nu^{A^\top} u^+ = 0 \ \text{ on } \partial\Omega. \tag{1.7.71}$$

Hence, $f = 0$ which ultimately shows that the operator $\frac{1}{2}I + K^{\#}$ is indeed injective when acting from the space $L^{2(n-1)/n}_{A^*,1/2}(\partial\Omega, \sigma)$. Finally, the fact that $-\frac{1}{2}I + K^{\#}$ is also injective when acting from the space $L^{2(n-1)/n}_{A^*,-1/2}(\partial\Omega, \sigma)$ is established in a similar fashion.

Proof of (4): Work under the assumption that the coefficient tensor A is positive definite, that the set Ω is bounded, and that the set $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$ is connected. Also, suppose $\big(\frac{1}{2}I + K^{\#}\big)f = 0$ for some function f such that $f \in \big[L^{2(n-1)/n}(\partial\Omega, \sigma)\big]^M$ if $n \geq 3$ and $f \in \big[L^p(\partial\Omega, \sigma)\big]^M$ with $p \in (1, \infty)$ arbitrary if $n = 2$. In particular, (1.7.59) implies $\int_{\partial\Omega} f \, d\sigma = 0$. Define $u^\pm := \mathcal{S}_{A^\top} f$ in $\Omega_\pm$. The aforementioned cancelation property of f ensures (cf. Lemma 1.3.1) that, in all dimensions $n \geq 2$,

$$(\nabla^k u^-)(x) = O(|x|^{-n+1-k}) \text{ as } |x| \to \infty, \text{ for each } k \in \mathbb{N}_0. \tag{1.7.72}$$

[70, Corollary 1.7.14] (with $A^\top$ in place of A) then gives

$$\int_{\Omega_-} \big\langle A^\top \nabla u^-, \overline{\nabla u^-} \big\rangle \, d\mathcal{L}^n = \int_{\partial\Omega} \big\langle \big(-\tfrac{1}{2}I - K^\#\big) f, \overline{S_{A^\top} f} \big\rangle \, d\sigma = 0, \tag{1.7.73}$$

given that, as part of the present assumptions, we have $\big(-\frac{1}{2}I - K^\#\big) f = 0$ on $\partial\Omega$. In particular,

$$\int_{\Omega_-} \operatorname{Re} \big\langle A^\top \nabla u^-, \overline{\nabla u^-} \big\rangle \, d\mathcal{L}^n = 0 \tag{1.7.74}$$

and, upon noting that $A^\top$ is also positive definite (since A is so), this ultimately implies

$$\nabla u^- = 0 \text{ in } \Omega_-. \tag{1.7.75}$$

Given that Ω_- is connected, this implies that u^- is constant in Ω_-. Since Ω_- is an exterior domain and u^- decays at infinity (as seen from (1.7.72)), and we conclude that

$$u^- = 0 \text{ in } \Omega_-. \tag{1.7.76}$$

As a consequence,

$$S_{A^\top} f = u^-\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = 0 \text{ on } \partial\Omega. \tag{1.7.77}$$

As such, we may write

$$0 = \int_{\partial\Omega} \big\langle \big(-\tfrac{1}{2}I + K^\#\big) f, \overline{S_{A^\top} f} \big\rangle \, d\sigma = \int_{\Omega_+} \big\langle A^\top \nabla u^+, \overline{\nabla u^+} \big\rangle \, d\mathcal{L}^n, \tag{1.7.78}$$

where the last equality comes from [70, Corollary 1.7.14]. Consequently,

$$\int_{\Omega_+} \operatorname{Re} \big\langle A^\top \nabla u^+, \overline{\nabla u^+} \big\rangle \, d\mathcal{L}^n = 0. \tag{1.7.79}$$

Given that $A^\top$ is positive definite (since A is so), this forces

$$\nabla u^+ = 0 \text{ in } \Omega_+ \tag{1.7.80}$$

which further implies

$$\partial_\nu^{A^\top} u^+ = 0 \text{ on } \partial\Omega. \tag{1.7.81}$$

Together with (1.7.76) this finally yields

$$f = \partial_\nu^{A^\top} u^- - \partial_\nu^{A^\top} u^+ = 0 \text{ on } \partial\Omega, \tag{1.7.82}$$

ultimately proving that the operator $\frac{1}{2}I + K^{\#}$ is indeed injective when acting on $\big[L^{2(n-1)/n}(\partial\Omega, \sigma)\big]^M$. From this, the desired conclusion follows, keeping in mind that $\partial\Omega$ is compact.

Proof of (5): In this portion of the proof we shall work under the assumption that the given coefficient tensor A is complex symmetric and positive definite. Also, we shall assume that Ω is a bounded UR domain satisfying a two-sided local John condition and such that Ω_- is connected. First, consider the case when $n \geq 3$. Pick an arbitrary complex number $z \in \mathbb{C} \setminus [-\frac{1}{2}, \frac{1}{2}]$ and suppose the vector-valued function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[L_1^{2(n-1)/n}(\partial\Omega, \sigma)\big]^M$ satisfies $(zI + K)f = 0$. Given the current assumptions, we may invoke [69, (11.5.201)] to conclude that

$$\big[L_1^{2(n-1)/n}(\partial\Omega, \sigma)\big]^M \hookrightarrow \big[L^{2(n-1)/(n-2)}(\partial\Omega, \sigma)\big]^M. \tag{1.7.83}$$

For each $x \in \mathbb{R}^n \setminus \partial\Omega$ let us now define

$$u(x) := \left(- \int_{\partial\Omega} \nu_s(y) a_{rs}^{\beta\alpha} (\partial_r E_{\gamma\beta})(x - y) f_\alpha(y)\, d\sigma(y) \right)_{1\leq\gamma\leq M}, \tag{1.7.84}$$

and also introduce

$$u_\pm := u\big|_{\Omega_\pm} \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M. \tag{1.7.85}$$

Thanks to assumptions, (1.7.83), and Theorem 1.5.1 we have

$$\begin{aligned} &\mathcal{N}_\kappa u_\pm \in L^{2(n-1)/(n-2)}(\partial\Omega, \sigma), \quad \mathcal{N}_\kappa(\nabla u_\pm) \in L^{2(n-1)/n}(\partial\Omega, \sigma), \\ &u_\pm\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = (\pm\tfrac{1}{2}I + K)f, \quad \partial_\nu^A u_\pm = \partial_\nu^A \mathcal{D} f, \text{ and } L_A u_\pm = 0 \text{ in } \Omega_\pm. \end{aligned} \tag{1.7.86}$$

In addition, from (1.7.84) and [70, (1.4.24)] we see that

$$(\nabla^k u_-)(x) = O(|x|^{1-n-k}) \text{ as } |x| \to \infty, \text{ for each } k \in \mathbb{N}_0. \tag{1.7.87}$$

Together, (1.7.86), (1.7.87), and [68, (8.6.51) in Proposition 8.6.3] further imply

$$u_\pm \in \big[L^{2n/(n-2)}(\Omega_\pm, \mathcal{L}^n)\big]^M \text{ and } \nabla u_\pm \in \big[L^2(\Omega_\pm, \mathcal{L}^n)\big]^{nM}. \tag{1.7.88}$$

Also, [70, Corollary 1.7.14] applies to $u_\pm$ in $\Omega_\pm$ and, on account of (1.7.86), yields

$$\begin{aligned} 0 &= \int_{\partial\Omega} \big\langle (zI + K)f, \overline{\partial_\nu^A \mathcal{D} f} \big\rangle\, d\sigma \\ &= \int_{\partial\Omega} \Big\langle (z + \tfrac{1}{2})(\tfrac{1}{2}I + K)f + (-z + \tfrac{1}{2})(-\tfrac{1}{2}I + K)f, \overline{\partial_\nu^A \mathcal{D} f} \Big\rangle\, d\sigma \\ &= \big(z + \tfrac{1}{2}\big) \int_{\Omega_+} \big\langle A^*\nabla u_+, \overline{\nabla u_+} \big\rangle\, d\mathcal{L}^n + \big(z - \tfrac{1}{2}\big) \int_{\Omega_-} \big\langle A^*\nabla u_-, \overline{\nabla u_-} \big\rangle\, d\mathcal{L}^n. \end{aligned} \tag{1.7.89}$$

Given that A is complex symmetric and positive definite, (1.7.88) ensures that

$$a_\pm := \int_{\Omega_\pm} \big\langle A^*\nabla u_\pm, \overline{\nabla u_\pm}\big\rangle \, d\mathcal{L}^n \in [0,\infty). \tag{1.7.90}$$

From (1.7.89), (1.7.90), and (1.7.51) (used with $-z$ in place of z) we conclude that

$$\int_{\Omega_+} \big\langle A^*\nabla u_+, \overline{\nabla u_+}\big\rangle \, d\mathcal{L}^n = 0 \text{ and } \int_{\Omega_-} \big\langle A^*\nabla u_-, \overline{\nabla u_-}\big\rangle \, d\mathcal{L}^n = 0. \tag{1.7.91}$$

Since $A^* = A$ is positive definite, this eventually forces

$$\nabla u_\pm = 0 \text{ in } \Omega_\pm. \tag{1.7.92}$$

Since Ω_- is connected, this implies that $u_- \equiv 0$ in Ω_- (since u_- vanishes at infinity; cf. (1.7.87)). In concert with (1.5.20), this further gives that

$$f = u_+\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - u_-\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = u_+\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ on } \partial\Omega. \tag{1.7.93}$$

Recall that we are presently assuming Ω_+ to be bounded. Writing (1.3.35) for u_+ in Ω_+ and taking into account (1.7.92) also yields

$$u_+ = \mathcal{D}\big(u_+\big|_{\partial\Omega}^{\mathrm{n.t.}}\big) \text{ in } \Omega_+. \tag{1.7.94}$$

From (1.7.93), (1.7.94), and (1.5.20) we then conclude that, on the one hand,

$$f = (\tfrac{1}{2}I + K)f, \text{ hence } Kf = \tfrac{1}{2}f. \tag{1.7.95}$$

On the other hand, $Kf = -zf$ with $z \neq -\frac{1}{2}$, so we ultimately conclude that $f = 0$. This proves that for every given spectral parameter $z \in \mathbb{C} \setminus \big[-\frac{1}{2}, \frac{1}{2}\big]$ the operator $zI + K$ is injective in the context of (1.7.19) if $n \geq 3$. Finally, the case of (1.7.20) in the two-dimensional setting is dealt with in an absolutely analogous fashion.

Proof of (6): Assume A is positive definite, Ω_+ is a bounded UR domain satisfying a two-sided local John condition, and Ω_- is connected. Let us first treat the case when $n \geq 3$. Suppose the function $f \in \big[L_1^{2(n-1)/n}(\partial\Omega,\sigma)\big]^M$ is such that $(\frac{1}{2}I + K)f = 0$ and define $u_\pm$ as in (1.7.85). Then (1.7.86)-(1.7.88) continue to hold. Also, writing (1.7.89) with $z := 1/2$ presently gives

$$0 = \int_{\partial\Omega} \big\langle (\tfrac{1}{2}I + K)f, \overline{\partial_\nu^A \mathcal{D}f}\big\rangle \, d\sigma = \int_{\Omega_+} \big\langle A^*\nabla u_+, \overline{\nabla u_+}\big\rangle \, d\mathcal{L}^n. \tag{1.7.96}$$

Since A^* is positive definite, this forces $\nabla u_+ = 0$ in Ω_+. In view of item *(vii)* of Theorem 1.5.1, this allows us to write

$$\partial_\nu^A u_- = \partial_\nu^A u_+ = 0 \text{ on } \partial\Omega. \tag{1.7.97}$$

Bearing this in mind and writing (1.7.89) for $z := -1/2$ then yields

$$0 = \int_{\partial\Omega} \big\langle (-\tfrac{1}{2}I + K)f, \overline{\partial_\nu^A \mathcal{D} f}\,\big\rangle \,\mathrm{d}\sigma = -\int_{\Omega_-} \big\langle A^* \nabla u_-, \overline{\nabla u_-}\,\big\rangle \,\mathrm{d}\mathcal{L}^n. \tag{1.7.98}$$

Given that A^* is positive definite, Ω_- is connected, and u_- vanishes at infinity, this implies that $u_- \equiv 0$ in Ω_-. Consequently,

$$(-\tfrac{1}{2}I + K)f = u_-\big|_{\partial\Omega}^{^{\kappa-\mathrm{n.t.}}} = 0 \text{ on } \partial\Omega, \tag{1.7.99}$$

hence

$$f = (\tfrac{1}{2}I + K)f - (-\tfrac{1}{2}I + K)f = 0 \text{ on } \partial\Omega. \tag{1.7.100}$$

Lastly, the case of (1.7.22) is dealt with in an absolutely analogous fashion. □

Via transposition, Theorem 1.7.2 yields the following results pertaining to the density of ranges of single and double layer potential operators.

Corollary 1.7.3 *Suppose $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is an arbitrary* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let A be a coefficient tensor with complex entries which is Legendre-Hadamard elliptic* (*cf.* (1.7.1))*, and consider the boundary layer potentials S, K, $K^\#$ associated with A and Ω as in* (1.3.62)*,* (1.3.68)*,* (1.3.72)*. Then the following claims are valid.*

(1) The boundary-to-boundary single layer operator S has dense range in the context

$$S : \Big[L_{-1}^{\frac{2(n-1)}{n}, \frac{2(n-1)}{n-2}}(\partial\Omega, \sigma)\Big]^M \longrightarrow \Big[L^{\frac{2(n-1)}{n-2}}(\partial\Omega, \sigma)\Big]^M \tag{1.7.101}$$

(*cf.* (1.5.36)) *if $n \geq 3$, and in the context*

$$S : \big[L_{-1}^{p,q}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^q(\partial\Omega, \sigma)\big]^M / \mathbb{C}^M \tag{1.7.102}$$

for every $p, q \in (1, \infty)$ and $n = 2$.

(2) Suppose A is also positive semi-definite, and fix some $z \in \mathbb{C} \setminus [-\frac{1}{2}, \frac{1}{2}]$. If either A is complex symmetric, or z is real, then the operator

$$zI + K : \big[L^{2(n-1)/(n-2)}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{2(n-1)/(n-2)}(\partial\Omega, \sigma)\big]^M \tag{1.7.103}$$

has dense range if $n \geq 3$, and

$$zI + K : \big[L^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^p(\partial\Omega, \sigma)\big]^M \tag{1.7.104}$$

has dense range for every $p \in (1, \infty)$ if $\partial\Omega$ is compact and $n = 2$.

(3) Assume A is positive definite, Ω is bounded, and $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. Then the operator

$$\begin{gathered} \tfrac{1}{2}I + K : \big[L^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^p(\partial\Omega, \sigma)\big]^M \text{ has dense range} \\ \text{if either } n \geq 3 \text{ and } 1 < p \leq 2(n-1)/(n-2), \\ \text{or } n = 2 \text{ and } 1 < p < \infty. \end{gathered} \tag{1.7.105}$$

(4) *Suppose the coefficient tensor A is complex symmetric and positive definite. Also, strengthen the hypotheses on the underlying domain by assuming that* Ω *is a bounded* UR *domain satisfying a two-sided local John condition and such that* $\mathbb{R}^n \setminus \overline{\Omega}$ *is connected. Then for any* $z \in \mathbb{C} \setminus [-\frac{1}{2}, \frac{1}{2}]$ *the operator*

$$zI + K^{\#} : \big[L^{2(n-1)/(n-2)}_{-1}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{2(n-1)/(n-2)}_{-1}(\partial\Omega, \sigma)\big]^M \tag{1.7.106}$$

(*cf.* (1.5.27)) *has dense range if* $n \geq 3$*, and*

$$zI + K^{\#} : \big[L^{p}_{-1}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{p}_{-1}(\partial\Omega, \sigma)\big]^M \tag{1.7.107}$$

has dense range for every $p \in (1, \infty)$ *if* $n = 2$.

(5) *Assume the coefficient tensor A is positive definite. Also, suppose* Ω *is a bounded* UR *domain satisfying a two-sided local John condition and such that* $\mathbb{R}^n \setminus \overline{\Omega}$ *is connected. Then the operator*

$$\begin{array}{c} \tfrac{1}{2}I + K^{\#} : \big[L^{2(n-1)/(n-2)}_{-1}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{2(n-1)/(n-2)}_{-1}(\partial\Omega, \sigma)\big]^M \\ \textit{has dense range if } n \geq 3, \end{array} \tag{1.7.108}$$

and the operator

$$\begin{array}{c} \tfrac{1}{2}I + K^{\#} : \big[L^{p}_{-1}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{p}_{-1}(\partial\Omega, \sigma)\big]^M \ \textit{has dense range} \\ \textit{for every } p \in (1, \infty) \textit{ if } n = 2. \end{array} \tag{1.7.109}$$

Proof All desired conclusions follow from Theorem 1.7.2 via transposition, with the help of items *(iii)*, *(vi)*, and *(ix)* in Theorem 1.5.1, as well as (A.0.136) and (A.0.137). □

The boundary layer potential operators studied in Theorem 1.5.1 turn out to be intimately connected with boundary value problems, even in the very general geometric setting considered in our next proposition. To facilitate its statement, we agree to denote by $\mathrm{Im}\big(T : X \to Y\big)$ the image of a linear operator $T : X \to Y$.

Proposition 1.7.4 *Suppose* $\Omega \subseteq \mathbb{R}^n$*, where* $n \geq 3$*, is a* UR *domain with compact boundary, and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$*. With the summation convention over repeated indices in effect, let*

$$L = \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M} \tag{1.7.110}$$

be a homogeneous, weakly elliptic, second-order $M \times M$ *system in* $\mathbb{R}^n$ (*for some* $M \in \mathbb{N}$)*, with complex constant coefficients, and recall the principal-value double layer* K *and the boundary single layer* S *associated, as in* (1.3.68) *and* (1.3.62)*, respectively, with the writing of the system* L *given in* (1.7.110)*. Having fixed an integrability exponent* $p \in (1, \infty)$ *along with an aperture parameter* $\kappa > 0$*, define the* L^p_1 *Regularity Problem for the system* L *in* Ω *as*

$$\begin{cases} u \in \left[\mathscr{C}^\infty(\Omega)\right]^M, \ Lu = 0 \ \textit{in} \ \Omega, \\ \mathcal{N}_\kappa u, \ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \ \textit{ at } \sigma\textit{-a.e. point on } \partial\Omega, \end{cases} \qquad \textit{where } f \in \left[L^p_1(\partial\Omega, \sigma)\right]^M, \tag{1.7.111}$$

with the understanding that, in the case when Ω *is an exterior domain, one also imposes the decay condition*

$$u(x) = O(|x|^{2-n}) \ \textit{as} \ |x| \to \infty. \tag{1.7.112}$$

Then the following statements are equivalent:

(a) The L^p_1 *Regularity Problem for the system* L *in* Ω *is solvable (i.e., for each boundary datum* $f \in \left[L^p_1(\partial\Omega, \sigma)\right]^M$ *one can find a function* u *as in (1.7.111)).*
(b) The following operator is surjective:

$$\begin{aligned} Q : \left[L^p_1(\partial\Omega, \sigma)\right]^M \oplus \left[L^p(\partial\Omega, \sigma)\right]^M &\longrightarrow \left[L^p_1(\partial\Omega, \sigma)\right]^M \\ \textit{given by } Q(g, h) := \left(\tfrac{1}{2}I + K\right)g + Sh \ &\textit{for every} \\ g \in \left[L^p_1(\partial\Omega, \sigma)\right]^M \ \textit{and} \ h \in \left[L^p(\partial\Omega, \sigma)\right]^M&. \end{aligned} \tag{1.7.113}$$

(c) One has

$$\begin{aligned} &\mathrm{Im}\Big(\tfrac{1}{2}I - K : \left[L^p_1(\partial\Omega, \sigma)\right]^M \to \left[L^p_1(\partial\Omega, \sigma)\right]^M\Big) \\ &\qquad\qquad \subseteq \mathrm{Im}\Big(S : \left[L^p(\partial\Omega, \sigma)\right]^M \to \left[L^p_1(\partial\Omega, \sigma)\right]^M\Big). \end{aligned} \tag{1.7.114}$$

Proof Let $A = \left(a^{\alpha\beta}_{rs}\right)_{\substack{1\le r,s\le n \\ 1\le\alpha,\beta\le M}}$ be the coefficient tensor used in the representation of the system L in (1.7.110) and recall the conormal derivative operator ∂^A_ν, with respect to A and Ω, from (A.0.184). Assume first that the L^p_1 Regularity Problem for the system L in Ω is solvable. Pick some function $f \in \left[L^p_1(\partial\Omega, \sigma)\right]^M$ arbitrary and let u solve (1.7.111) for this boundary datum. From the Fatou-type result established in [70, Theorem 3.3.4] we know that u has the additional property that

$$(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{exists at } \sigma\text{-a.e. point on } \partial\Omega. \tag{1.7.115}$$

Granted this, [70, Lemma 1.7.3] and [70, (1.7.15)] guarantee that the function $h := -\partial^A_\nu u$ is well defined and belongs to $\left[L^p(\partial\Omega, \sigma)\right]^M$. Also, since (1.7.112) implies [70, (1.5.5)] given that $n \geq 3$, the integral representation formula (1.3.35) from Theorem 1.3.3 presently yields

$$u = \mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) - \mathcal{S}\big(\partial^A_\nu u\big) = \mathcal{D}f + \mathcal{S}h \ \text{ in } \Omega. \tag{1.7.116}$$

In turn, from (1.7.116), (1.5.20), (1.5.44), and the last condition in (1.7.111), we deduce that

$$f = \big(\tfrac{1}{2}I + K\big)f + Sh \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{1.7.117}$$

or, equivalently, $\big(\frac{1}{2}I - K\big)f = Sh$. This ultimately goes to show that the inclusion in (1.7.114) holds, hence *(a)* ⇒ *(c)*.

To prove the implication *(c)* ⇒ *(b)*, observe that if an arbitrary $f \in \big[L^p_1(\partial\Omega,\sigma)\big]^M$ has been given, then (1.7.114) guarantees the existence of some $h \in \big[L^p(\partial\Omega,\sigma)\big]^M$ with the property that $\big(\frac{1}{2}I - K\big)f = Sh$. In view of (1.7.113), this may be further recast simply as $Q(f,h) = f$ which proves that the operator Q is indeed surjective.

There remains to show that *(b)* ⇒ *(a)*. To this end, start with some arbitrary $f \in \big[L^p_1(\partial\Omega,\sigma)\big]^M$ then use the surjectivity of the operator Q from (1.7.113) to conclude that there exist two functions, $g \in \big[L^p_1(\partial\Omega,\sigma)\big]^M$ and $h \in \big[L^p(\partial\Omega,\sigma)\big]^M$, such that $\big(\frac{1}{2}I + K\big)g + Sh = f$. Then, thanks to (1.3.8), (1.3.24), Theorem 1.5.1 (cf. (1.5.1), (1.5.20)), and [69, (11.8.4)], the function $u := \mathcal{D}g + \mathcal{S}h$ in Ω solves the L^p_1 Regularity Problem (1.7.111). The proof of Proposition 1.7.4 is now complete. □

We next elaborate on the fact that, given a weakly elliptic second-order system L, the solvability of the Regularity Problems for L on either side of a compact UR surface is actually equivalent to the surjectivity of the single layer operator associated with L on said surface.

Corollary 1.7.5 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \geq 3$, be a bounded* UR *domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Define $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$. Also, fix an integrability exponent $p \in (1,\infty)$ along with an aperture parameter $\kappa \in (0,\infty)$. Finally, for some $M \in \mathbb{N}$, consider a homogeneous, weakly elliptic, second-order $M \times M$ system L in $\mathbb{R}^n$ with constant (complex) coefficients, and recall the boundary-to-boundary single layer operator S associated with L and Ω as in (1.3.62).*

Then the following statements are equivalent:

(i) The L^p_1 Regularity Problems for the system L in $\Omega_\pm$ are solvable.

(ii) The operator $S : \big[L^p(\partial\Omega,\sigma)\big]^M \to \big[L^p_1(\partial\Omega,\sigma)\big]^M$ is surjective.

Proof This is a consequence of Proposition 1.7.4 (applied both to Ω_+ and Ω_-; here item *(7)* in [68, Lemma 5.10.9] is also relevant), upon observing that

$$\begin{aligned}\mathrm{Im}\Big(\tfrac{1}{2}I \mp K : \big[L^p_1(\partial\Omega,\sigma)\big]^M \to \big[L^p_1(\partial\Omega,\sigma)\big]^M\Big)& \\ \subseteq \mathrm{Im}\Big(S : \big[L^p(\partial\Omega,\sigma)\big]^M \to \big[L^p_1(\partial\Omega,\sigma)\big]^M\Big)&\end{aligned} \tag{1.7.118}$$

if and only if $S : \big[L^p(\partial\Omega,\sigma)\big]^M \to \big[L^p_1(\partial\Omega,\sigma)\big]^M$ is surjective. □

We conclude this section by proving a uniqueness result for the mixed boundary value problem formulated in a very general context.

Corollary 1.7.6 *Suppose $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, is an open set with a lower Ahlfors regular boundary and with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure.*

Hence, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. For some $M \in \mathbb{N}$, assume A is a complex coefficient tensor of type $(n \times n, M \times M)$ which is positive definite, in the sense that there exists some real number $c > 0$ such that

$$\operatorname{Re}\big\langle A\zeta, \overline{\zeta}\big\rangle = \operatorname{Re}\Big(a^{\alpha\beta}_{rs}\zeta^{\beta}_{s}\overline{\zeta^{\alpha}_{r}}\Big) \geq c|\zeta|^2, \quad \forall\,\zeta = (\zeta^{\alpha}_{r})_{\substack{1\leq r\leq n\\ 1\leq\alpha\leq M}} \in \mathbb{C}^{n\times M}. \tag{1.7.119}$$

Denote by L_A the homogeneous constant (complex) coefficient second-order $M \times M$ system in $\mathbb{R}^n$ associated with A as in (A.0.139). Finally, fix some aperture $\kappa > 0$ and exponents $p, q \in [1, \infty]$.

In this context, suppose u is a complex vector-valued function satisfying

$$\begin{cases} u \in \big[W^{1,1}_{\mathrm{loc}}(\Omega)\big]^M, \quad Lu = 0 \ \textit{in} \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma), \ \ \mathcal{N}_\kappa u \in L^q(\partial\Omega, \sigma), \\ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \textit{and} \ (\nabla u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \textit{exist} \ \sigma\textit{-a.e. on} \ \partial_{\mathrm{nta}}\Omega, \end{cases} \tag{1.7.120}$$

and (with the dependence on κ dropped)

$$\begin{gathered} \textit{at } \sigma\textit{-a.e. point } x \in \partial_*\Omega \textit{ one has that} \\ \textit{either } \ \big(u\big|^{\mathrm{n.t.}}_{\partial\Omega}\big)(x) = 0, \ \ \textit{or} \ \ \big(\partial^A_\nu u\big)(x) = 0. \end{gathered} \tag{1.7.121}$$

Then, in any of the following scenarios:

(i) $\partial\Omega$ is unbounded, $p = 2(n-1)/n$, and $q = 2(n-1)/(n-2)$;
(ii) Ω is bounded, $p \geq 2(n-1)/n$, and $q \geq p/(p-1)$;
(iii) Ω is an exterior domain, $p \geq 2(n-1)/n$, $q \geq p/(p-1)$, and there exist two real numbers, $a > n/2$ and $b > n-1-a$, such that

$$(\nabla u)(x) = O(|x|^{-a}) \ \textit{and} \ u(x) = O(|x|^{-b}) \ \textit{as} \ |x| \to \infty, \tag{1.7.122}$$

it follows that

$$\textit{the function } u \textit{ is locally constant in } \Omega. \tag{1.7.123}$$

Proof From [70, Corollary 1.7.14] we know that formula [70, (1.7.162)] holds in the present setting. Bearing in mind that $L_A u = 0$ and that the boundary term vanishes, thanks of (1.7.121), this implies that $\big\langle A\nabla u, \overline{\nabla u}\big\rangle = 0$ at $\mathcal{L}^n$-a.e. point in Ω. In view of positive definiteness property (1.7.119), this forces ∇u to vanish at $\mathcal{L}^n$-a.e. point in Ω. Hence, ultimately, u is locally constant in Ω. □

1.8 Modified Boundary Layer Potential Operators

As far as the boundary-to-domain version of the double layer potential operator associated with a given $M \times M$ weakly elliptic system and an open set $\Omega \subseteq \mathbb{R}^n$ with an Ahlfors regular boundary is concerned, we can go one step further and adapt the original definition from (1.3.17)-(1.3.18) as to allow functions from the space $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ (where, as usual, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$), which is *strictly bigger* than the space $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ considered in (1.3.17). One appealing virtue of such an extension is that this larger weighted Lebesgue space contains a multitude of other useful spaces, such as all Hölder functions as well as functions of bounded mean oscillations, on $\partial\Omega$.

Before stating our first main result in this regard, the reader is reminded that if $\Omega \subseteq \mathbb{R}^n$ is an open set with an Ahlfors regular boundary, a Carleson measure μ in Ω is said to be `vanishing` provided

$$\lim_{R\to 0^+}\Bigg\{\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}} \frac{\mu\big(B(x,r)\cap\Omega\big)}{\mathcal{H}^{n-1}\big(B(x,r)\cap\partial\Omega\big)}\Bigg\} = 0. \tag{1.8.1}$$

Definition 1.8.1 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set with an unbounded Ahlfors regular boundary. A Carleson measure μ in Ω* (*cf.* [70, (2.4.1)]) *is said to be* `super vanishing` *provided the following three conditions are satisfied:*

$$\lim_{R\to\infty}\Bigg\{\sup_{\substack{x\in\partial\Omega,\ r>0\\ B(x,r)\subseteq\mathbb{R}^n\setminus B(0,R)}} \frac{\mu\big(B(x,r)\cap\Omega\big)}{\mathcal{H}^{n-1}\big(B(x,r)\cap\partial\Omega\big)}\Bigg\} = 0, \tag{1.8.2}$$

$$\lim_{R\to 0^+}\Bigg\{\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}} \frac{\mu\big(B(x,r)\cap\Omega\big)}{\mathcal{H}^{n-1}\big(B(x,r)\cap\partial\Omega\big)}\Bigg\} = 0, \tag{1.8.3}$$

$$\lim_{R\to\infty}\Bigg\{\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(R,\infty)}} \frac{\mu\big(B(x,r)\cap\Omega\big)}{\mathcal{H}^{n-1}\big(B(x,r)\cap\partial\Omega\big)}\Bigg\} = 0. \tag{1.8.4}$$

We are now ready to state the theorem alluded to above.

Theorem 1.8.2 *Let $\Omega \subseteq \mathbb{R}^n$* (*where $n \in \mathbb{N}$, $n \geq 2$*) *be an open set with the property that $\partial\Omega$ is an Ahlfors regular set; in particular, Ω is a set of locally finite perimeter. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let $L = \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ be a homogeneous, weakly elliptic, constant* (*complex*) *coefficient, second-order $M \times M$ system in $\mathbb{R}^n$* (*for some $M \in \mathbb{N}$*).

In this setting, consider the following modified version of the double layer operator in (1.3.18) *acting on each function*

$$f = (f_\alpha)_{1\le\alpha\le M} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M \tag{1.8.5}$$

according to (recall that the summation convention over repeated indices is presently in effect):

$$\begin{aligned}&\big(\mathcal{D}_{\text{mod}} f\big)(x) \qquad (1.8.6)\\ &:= \Big(-\int_{\partial_*\Omega} \nu_s(y) a^{\beta\alpha}_{rs}\big\{(\partial_r E_{\gamma\beta})(x-y) - k_1^{(r\gamma\beta)}(-y)\big\} f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M}\end{aligned}$$

at each point $x \in \Omega$*, where* $k_1^{(r\gamma\beta)} := (\partial_r E_{\gamma\beta}) \cdot \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(0,1)}}$ *for every* $r \in \{1,\dots,n\}$*,* $\gamma,\beta \in \{1,\dots,M\}$*, and* $E = (E_{\gamma\beta})_{1\le\gamma,\beta\le M}$ *is the matrix-valued fundamental solution associated with L as in* [70, Theorem 1.4.2]*. Then the following properties hold.*

(1) The operator $\mathcal{D}_{\text{mod}}$ *is meaningfully defined, and satisfies*

$$\begin{aligned}&\mathcal{D}_{\text{mod}} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M \text{ and } L(\mathcal{D}_{\text{mod}} f) = 0 \text{ in } \Omega,\\ &\text{for each } f \in \Big[L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M.\end{aligned} \tag{1.8.7}$$

In addition, the operator $\mathcal{D}_{\text{mod}}$ *is compatible with* $\mathcal{D}$ *from* (1.3.18)*, in the sense that for each function f belonging to the smaller space* $\big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *(hence, in particular, for each function* $f \in \big[L^p(\partial_*\Omega,\sigma)\big]^M$ *with* $p \in [1,\infty)$*) the difference*

$$C_f := \mathcal{D}_{\text{mod}} f - \mathcal{D} f \text{ is a constant (belonging to } \mathbb{C}^M) \text{ in } \Omega. \tag{1.8.8}$$

As a consequence,

$$\nabla\mathcal{D}_{\text{mod}} f = \nabla\mathcal{D} f \text{ in } \Omega \text{ for each } f \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M. \tag{1.8.9}$$

Moreover,

$$\begin{aligned}&\mathcal{D}_{\text{mod}} \text{ maps constant } (\mathbb{C}^M\text{-valued}) \text{ functions on } \partial_*\Omega\\ &\text{into constant } (\mathbb{C}^M\text{-valued}) \text{ functions in } \Omega.\end{aligned} \tag{1.8.10}$$

In addition, at each point $x \in \Omega$ *one may express*

$$\begin{aligned}&\partial^\mu\big(\mathcal{D}_{\text{mod}} f\big)(x) = \Big(-\int_{\partial_*\Omega} \nu_s(y) a^{\beta\alpha}_{rs} (\partial^\mu \partial_r E_{\gamma\beta})(x-y) f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\le\gamma\le M}\\ &\text{for each } \mu \in \mathbb{N}_0^n \text{ with } |\mu| > 0 \text{ and } f = (f_\alpha)_{1\le\alpha\le M} \in \Big[L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^M.\end{aligned} \tag{1.8.11}$$

Finally, given any function

$$f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M$$
$$\text{with the property that } \ \partial_{\tau_{jk}} f_\alpha \in L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \tag{1.8.12}$$
$$\text{for all } j,k \in \{1,\dots,n\} \text{ and } \alpha \in \{1,\dots,M\},$$

it follows that for each index $\ell \in \{1,\dots,n\}$ *and each point* $x \in \Omega$ *one has*

$$\partial_\ell\big(\mathcal{D}_{\text{mod}} f\big)(x) = \left(\int_{\partial_*\Omega} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} f_\alpha)(y)\,\mathrm{d}\sigma(y)\right)_{1\leq\gamma\leq M}. \tag{1.8.13}$$

(2) For each $\eta \in (0,1)$ *there exists a constant* $C \in (0,\infty)$ *with the property that*

$$\sup_{x\in\Omega}\Big\{\operatorname{dist}(x,\partial\Omega)^{1-\eta}\big|\nabla\big(\mathcal{D}_{\text{mod}} f\big)(x)\big|\Big\} \leq C\|f\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M} \tag{1.8.14}$$

for every function $f \in \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M$. *Moreover,*

$$\begin{array}{l}\textit{if } \Omega \subseteq \mathbb{R}^n \textit{ is a uniform domain with the property that } \partial\Omega \textit{ is an Ahlfors} \\ \textit{regular set then } \mathcal{D}_{\text{mod}} : \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M \to \big[\dot{\mathscr{C}}^\eta\big(\overline{\Omega}\big)\big]^M \textit{ is a well-defined,} \\ \textit{linear, and bounded operator for each given exponent } \eta \in (0,1),\end{array} \tag{1.8.15}$$

whereas

$$\begin{array}{l}\textit{if } \Omega \subseteq \mathbb{R}^n \textit{ is an } \mathrm{NTA} \textit{ domain with an upper Ahlfors regular boundary} \\ \textit{then } \mathcal{D}_{\text{mod}} : \big[\dot{\mathscr{C}}^\eta_{\text{van}}(\partial\Omega)\big]^M \to \big[\dot{\mathscr{C}}^\eta_{\text{van}}\big(\overline{\Omega}\big)\big]^M \textit{ is a well-defined, linear, and} \\ \textit{bounded operator for each given exponent } \eta \in (0,1),\end{array} \tag{1.8.16}$$

where the homogeneous vanishing Hölder spaces intervening above are defined as in (A.0.48) (*with* $\Sigma := \partial\Omega$ *and* $\Sigma := \overline{\Omega}$, *respectively*). *Also, for each* $\eta \in (0,1)$ *and each* $p \in (1,\infty)$ *there exists some* $C \in (0,\infty)$ *with the property that for each function* $f \in \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M$ *one has*

$$\sup_{\substack{x\in\partial\Omega\\ r\in(0,\infty)}}\left(\frac{1}{r^{n-1+\eta p}}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{D}_{\text{mod}} f\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{1/p} \leq C\|f\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M} \tag{1.8.17}$$

and

$$\lim_{R\to 0^+}\left\{\sup_{\substack{x\in\partial\Omega\\ r\in(0,R)}}\left(\frac{1}{r^{n-1+\eta p}}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{D}_{\rm mod}f\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\right)^{1/p}\right\}$$
$$\leq C\operatorname{dist}\big(f,\big[\dot{\mathscr{C}}^{\eta}_{\rm van}(\partial\Omega)\big]^M\big), \tag{1.8.18}$$

where the distance is measured in the space $\Big(\big[\dot{\mathscr{C}}^{\eta}(\partial\Omega)\big]^M,\|\cdot\|_{[\dot{\mathscr{C}}^{\eta}(\partial\Omega)]^M}\Big)$. *As a corollary, if the function f actually belongs to the homogeneous vanishing Hölder space* $\big[\dot{\mathscr{C}}^{\eta}_{\rm van}(\partial\Omega)\big]^M$ *for some* $\eta\in(0,1)$, *then for each* $p\in(1,\infty)$ *one has*

$$\lim_{R\to 0^+}\left\{\sup_{\substack{x\in\partial\Omega\\ r\in(0,R)}}\left(\frac{1}{r^{n-1+\eta p}}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{D}_{\rm mod}f\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\right)^{1/p}\right\}=0. \tag{1.8.19}$$

(3) *Strengthen the original geometric hypotheses by assuming that $\partial\Omega$ is actually a* UR *set. Also, fix an aperture parameter* $\kappa\in(0,\infty)$. *Then, as a consequence of* (1.8.13) *and* [70, Theorem 2.5.1], *the nontangential boundary trace*

$$\begin{gathered}\big(\partial_\ell\mathcal{D}_{\rm mod}f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\ \text{ exists (in } \mathbb{C}^M) \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega,\\ \text{for each function } f \text{ as in (1.8.12) and each index } \ell\in\{1,\dots,n\}.\end{gathered} \tag{1.8.20}$$

Another corollary of (1.8.13) *and* [70, (2.4.8)] (*also keeping in mind* [70, Theorem 1.4.2]) *is the fact that for each* $\varepsilon>0$ *and each* $p\in(1,\infty)$

$$\begin{gathered}\mathcal{N}^{\varepsilon}_{\kappa}\big(\nabla(\mathcal{D}_{\rm mod}f)\big)\in L^p_{\rm loc}(\partial\Omega,\sigma)\ \text{ for each function}\\ f=(f_\alpha)_{1\leq\alpha\leq M}\in\left[L^1\Big(\partial_*\Omega,\frac{\sigma(x)}{1+|x|^n}\Big)\right]^M \text{ such that}\\ \partial_{\tau_{jk}}f_\alpha\in L^1\Big(\partial_*\Omega,\frac{\sigma(x)}{1+|x|^{n-1}}\Big)\cap L^p_{\rm loc}(\partial_*\Omega,\sigma)\\ \text{for all } j,k\in\{1,\dots,n\} \text{ and all } \alpha\in\{1,\dots,M\}.\end{gathered} \tag{1.8.21}$$

In addition, as seen from (1.8.6), [70, (2.5.32)], *and* [70, Theorem 1.4.2], *for each truncation parameter* $\varepsilon\in(0,\infty)$ *one has*

$$\begin{gathered}\mathcal{N}^{\varepsilon}_{\kappa}(\mathcal{D}_{\rm mod}f)\in L^p_{\rm loc}(\partial\Omega,\sigma)\ \text{ for each function}\\ f\in\Big[L^1\big(\partial_*\Omega,\tfrac{\sigma(x)}{1+|x|^n}\big)\cap L^p_{\rm loc}(\partial_*\Omega,\sigma)\Big]^M \text{ with } p\in(1,\infty).\end{gathered} \tag{1.8.22}$$

Furthermore, for each function f as in (1.8.12) *one has*

$$\big(\partial_\nu^A(\mathcal{D}_{\rm mod} f)\big)(x) \tag{1.8.23}$$
$$= \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y)\big(\partial_{\tau_{js}} f_\alpha\big)(y)\, d\sigma(y)\Bigg)_{1\le\mu\le M}$$

where the conormal derivative is considered as in (A.0.184).

Next, let $K_{\rm mod}$ *be the modified boundary-to-boundary double layer potential operator acting on each function* $f = (f_\alpha)_{1\le\alpha\le M} \in \Big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\Big]^M$ *at* σ*-a.e.* $x \in \partial\Omega$ *according to*

$$K_{\rm mod} f(x) \tag{1.8.24}$$
$$:= \Bigg(-\lim_{\varepsilon\to 0^+} \int\limits_{\partial_*\Omega} \nu_s(y) a^{\beta\alpha}_{rs} \big\{k^{(r\gamma\beta)}_\varepsilon(x-y) - k^{(r\gamma\beta)}_1(-y)\big\} f_\alpha(y)\, d\sigma(y)\Bigg)_{1\le\gamma\le M}$$

where

$$k^{(r\gamma\beta)}_\varepsilon := (\partial_r E_{\gamma\beta}) \cdot \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,\varepsilon)}} \text{ for each } \varepsilon > 0. \tag{1.8.25}$$

Then, as seen from definitions and [68, Proposition 5.6.7]*, the operator* $K_{\rm mod}$ *is compatible with* K (*acting on functions from* $\Big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ *as in* (1.3.68)) *in the sense that*

> *for each function* $f \in \Big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$ (*hence, in particular, for each function* $f \in \big[L^p(\partial_*\Omega,\sigma)\big]^M$ *with* $p \in [1,\infty)$) *the difference* $c_f := K_{\rm mod} f - Kf$ *is a constant* (*belonging to* $\mathbb{C}^M$) *on* $\partial\Omega$ *and satisfies* $|c_f| \le C_{\partial\Omega,L} \cdot \|f\|_{[L^1(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}})]^M}$ *for some finite* $C_{\partial\Omega,L} > 0$. (1.8.26)

Furthermore, the following jump-formula holds:

$$\begin{gathered}\big(\mathcal{D}_{\rm mod} f\big)\Big|^{\kappa-\rm n.t.}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{\rm mod}\big) f \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega, \\ \text{for each given function } f \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M,\end{gathered} \tag{1.8.27}$$

where, as usual, I *is the identity operator. As a consequence of* (1.8.27) *and* (1.8.10)*,*

> *if* $\partial\Omega$ *is a* UR *set then the operator* $K_{\rm mod}$ *maps constant* ($\mathbb{C}^M$*-valued*) *functions on* $\partial_*\Omega$ *into constant* ($\mathbb{C}^M$*-valued*) *functions on* $\partial_*\Omega$. (1.8.28)

(4) *Continue to work under the additional assumption that* $\partial\Omega$ *is a* UR *set. Then for each integrability exponent* $p \in (1,\infty)$ *there exists a constant* $C \in (0,\infty)$

with the property that for each function $f \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$ *the measure* $\big|\nabla\big(\mathcal{D}_{\rm mod}f\big)\big|^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n$ *is Carleson in* Ω*, in the quantitative sense that*

$$\sup_{x\in\partial\Omega,\,r>0}\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{D}_{\rm mod}f\big)\big|^p\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n \leq C\|f\|^p_{[\dot{\mathrm{BMO}}(\partial\Omega,\sigma)]^M} \tag{1.8.29}$$

(*with the piece of notation introduced in* (A.0.19)). *In particular, corresponding to* $p=2$*, it follows that*[21]

$$\begin{gathered}\big|\nabla\big(\mathcal{D}_{\rm mod}f\big)\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n \text{ is a Carleson measure in } \Omega,\\ \text{for each function } f\in\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M.\end{gathered} \tag{1.8.30}$$

Moreover, for each $p\in(1,\infty)$ *there exists a constant* $C\in(0,\infty)$ *with the property that for each function* $f\in\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$ *one has*

$$\lim_{R\to 0^+}\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{D}_{\rm mod}f\big)\big|^p\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}} \leq C\,\mathrm{dist}\big(f,[\mathrm{VMO}(\partial\Omega,\sigma)]^M\big) \tag{1.8.31}$$

where the distance in the right-hand side is considered in $[\mathrm{BMO}(\partial\Omega,\sigma)]^M$*. In particular,*

$$\begin{gathered}\big|\nabla\big(\mathcal{D}_{\rm mod}f\big)\big|^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\\ \text{is a vanishing Carleson measure in } \Omega\\ \text{for each } f\in\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M \text{ and each } p\in(1,\infty)\end{gathered} \tag{1.8.32}$$

and, corresponding to $p=2$*,*

$$\begin{gathered}\big|\nabla\big(\mathcal{D}_{\rm mod}f\big)\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega,\\ \text{for each function } f\in\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M.\end{gathered} \tag{1.8.33}$$

Furthermore, for each $p\in(1,\infty)$ *there exists a constant* $C\in(0,\infty)$*, which depends only on n, p, L, and the* UR *constants of* $\partial\Omega$*, with the property that for each function* $f\in\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$ *one has*

[21] it is natural to refer to $\big|\nabla\mathcal{D}_{\rm mod}f\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n$ as the Littlewood-Paley measure associated with f via the operator $\mathcal{D}_{\rm mod}$

$$\max\Bigg\{\lim_{R\to\infty}\sup_{\substack{x\in\partial\Omega,\ r>0\\ B(x,r)\subseteq\mathbb{R}^n\setminus B(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times$$
$$\times\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{D}_{\rm mod}f\big)\big|^p\,{\rm dist}\,(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{1/p},$$
$$\lim_{R\to 0^+}\sup_{x\in\partial\Omega,\ r\in(0,R)}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times$$
$$\times\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{D}_{\rm mod}f\big)\big|^p\,{\rm dist}\,(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{1/p},$$
$$\lim_{R\to\infty}\sup_{x\in\partial\Omega,\ r>R}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times$$
$$\times\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{D}_{\rm mod}f\big)\big|^p\,{\rm dist}\,(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{1/p}\Bigg\}$$
$$\leq C\,{\rm dist}\Big(f,\ \big[{\rm CMO}(\partial\Omega,\sigma)\big]^M\Big), \tag{1.8.34}$$

where the distance in the right-hand side is considered in $[{\rm BMO}(\partial\Omega,\sigma)]^M$ (*recall that* ${\rm CMO}(\partial\Omega,\sigma)$ *has been introduced in* (A.0.52)). *As a consequence of* (1.8.34) *and Definition 1.8.1,*

$$\big|\nabla\big(\mathcal{D}_{\rm mod}f\big)\big|^p\,{\rm dist}\,(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\ \text{ is a super vanishing Carleson measure in } \Omega,$$
$$\text{for each function } f\in\big[{\rm CMO}(\partial\Omega,\sigma)\big]^M \text{ and each } p\in(1,\infty). \tag{1.8.35}$$

(5) If $\partial\Omega$ *is bounded, then all properties listed in items (1)-(4) are valid for the ordinary double layer operator* $\mathcal{D}$, *as originally defined in* (1.3.18), *in place of its modified version* $\mathcal{D}_{\rm mod}$. *In particular, if* $\partial\Omega$ *is a compact* UR *set then for each* $p\in(1,\infty)$ *there exists a constant* $C\in(0,\infty)$ *with the property that for each function* $f\in\big[{\rm BMO}(\partial\Omega,\sigma)\big]^M$ *one has*

$$\lim_{R\to 0^+}\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{D}f\big)\big|^p{\rm dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{\frac{1}{p}}$$
$$\leq C\,{\rm dist}\big(f,[{\rm VMO}(\partial\Omega,\sigma)]^M\big) \tag{1.8.36}$$

where the distance in the right-hand side is considered in $[{\rm BMO}(\partial\Omega,\sigma)]^M$. *In particular, if* $\partial\Omega$ *is a compact* UR *set then*

$$\big|\nabla\big(\mathcal{D}f\big)\big|^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega, \text{ for each function } f\in\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M \text{ and each } p\in(1,\infty) \tag{1.8.37}$$

and, corresponding to $p=2$, it follows that[22]

$$\big|\nabla\big(\mathcal{D}f\big)\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega, \text{ for each function } f\in\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M \tag{1.8.38}$$

(6) *Strengthen the original geometric hypotheses by assuming now that Ω is a* UR *domain. Then for each $p\in(1,\infty)$ there exists a constant $C\in(0,\infty)$, which depends only on n, p, A, and the* UR *constants of $\partial\Omega$, with the property that for each function*

$$f\in\Big[L^1\Big(\partial\Omega,\,\frac{\sigma(x)}{1+|x|^n}\Big)\cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma)\Big]^M \tag{1.8.39}$$

each point $x_o\in\partial\Omega$, and each radius $r\in(0,\infty)$ one has

$$\begin{aligned}\Big(&⨍_{\Delta(x_o,r)}\Big|K_{\mathrm{mod}}f-⨍_{\Delta(x_o,r)}K_{\mathrm{mod}}f\,\mathrm{d}\sigma\Big|^p\,d\sigma\Big)^{1/p} \\ &\leq C\int_1^\infty\Big(⨍_{\Delta(x_o,\lambda r)}\Big|f-⨍_{\Delta(x_o,\lambda r)}f\,d\sigma\Big|^p\,\mathrm{d}\sigma\Big)^{1/p}\frac{\mathrm{d}\lambda}{\lambda^2},\end{aligned} \tag{1.8.40}$$

where $\Delta(x_o,R):=B(x_o,R)\cap\partial\Omega$ for each $R\in(0,\infty)$.
As a consequence of (1.8.40) *and the definition of the Fefferman-Stein sharp maximal operator* (*cf.* (A.0.195))*, it follows that for each $p\in(1,\infty)$ there exists some constant $C\in(0,\infty)$ such that the following pointwise inequality holds:*

$$\big(K_{\mathrm{mod}}f\big)^{\#}_p\leq C\cdot f^{\#}_p \text{ on } \partial\Omega, \text{ for every function } f\in\Big[L^1\Big(\partial\Omega,\,\tfrac{\sigma(x)}{1+|x|^n}\Big)\cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma)\Big]^M. \tag{1.8.41}$$

Also, if $p,p'\in(1,\infty)$ are such that $1/p+1/p'=1$ then given any functions

$$f\in\Big[L^p_{\mathrm{loc}}(\partial\Omega,\sigma)\cap L^1\Big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\Big)\Big]^M \text{ together with } g\in\big[L^{p'}_{\mathrm{comp}}(\partial\Omega,\sigma)\big]^M \text{ satisfying } \int_{\partial\Omega}g\,\mathrm{d}\sigma=0\in\mathbb{C}^M, \tag{1.8.42}$$

if follows that

[22] it is natural to refer to $\big|\nabla\mathcal{D}f\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n$ as the Littlewood-Paley measure associated with f via the operator $\mathcal{D}$

$$\int_{\partial\Omega} |K_{\text{mod}} f||g| \, d\sigma < +\infty, \quad \int_{\partial\Omega} |f||K^{\#} g| \, d\sigma < +\infty, \\ \textit{and} \quad \int_{\partial\Omega} \langle K_{\text{mod}} f, g\rangle \, d\sigma = \int_{\partial\Omega} \langle f, K^{\#} g\rangle \, d\sigma. \tag{1.8.43}$$

Finally, for each $p \in (1, \infty)$ there exists $C \in (0, \infty)$, which depends only on n, p, A, and the UR *constants of $\partial\Omega$, with the property that for each function $f \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\Big]^M$, each point $x_o \in \partial\Omega$, and each radius $r \in (0, \infty)$ one has*

$$\left(\frac{1}{\sigma\big(\Delta(x_o, r)\big)} \int_{B(x_o,r)\cap\Omega} \big|\nabla\big(\mathcal{D}_{\text{mod}} f\big)(x)\big|^p \operatorname{dist}(x, \partial\Omega)^{p-1} \, d\mathcal{L}^n(x)\right)^{1/p} \tag{1.8.44}$$

$$\leq C \int_1^\infty \left(⨍_{\Delta(x_o,\lambda r)} \Big| f - ⨍_{\Delta(x_o,\lambda r)} f \, d\sigma \Big|^p \, d\sigma \right)^{1/p} \frac{d\lambda}{\lambda^2}$$

where $\Delta(x_o, R) := B(x_o, R) \cap \partial\Omega$ for each $R \in (0, \infty)$.

As a corollary of (1.8.44) *and the definition of the Fefferman-Stein sharp maximal operator* (*cf.* (A.0.195))*, it follows that for each $p \in (1, \infty)$ there exists a constant $C \in (0, \infty)$ such that the pointwise inequality*

$$\sup_{r>0} \left(\frac{1}{\sigma\big(\Delta(x, r)\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{D}_{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1} \, d\mathcal{L}^n\right)^{1/p} \leq C \cdot f_p^{\#}(x) \tag{1.8.45}$$

holds for each point $x \in \partial\Omega$ and each function $f \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\Big]^M$.

Before presenting the proof of this theorem we make a number of comments.

<u>`Comment 1:`</u> Similar results to those presented in Theorem 1.8.2 are valid for modified boundary-to-domain "tangential" integral operators $\mathcal{T}_{jk}^{\text{mod}}$ with $j, k \in \{1, \ldots, n\}$, of the sort described in Proposition 1.8.16. Then, given any $p \in (1, \infty)$, there exists a constant $C \in (0, \infty)$ such that for each function $f \in \text{BMO}(\partial\Omega, \sigma)$ we have

$$\lim_{R\to 0^+} \sup_{\substack{x\in\partial\Omega \text{ and} \\ r\in(0,R)}} \left(\frac{1}{\sigma\big(B(x, r) \cap \partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}_{jk}^{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1} \, d\mathcal{L}^n\right)^{\frac{1}{p}}$$

$$\leq C \operatorname{dist}\big(f, \text{VMO}(\partial\Omega, \sigma)\big), \tag{1.8.46}$$

where the distance in the right-hand side is considered in the space $\text{BMO}(\partial\Omega, \sigma)$. As a consequence,

$$\big|\nabla\big(\mathcal{T}_{jk}^{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1} \, d\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega, \\ \text{for each function } f \in \text{VMO}(\partial\Omega, \sigma) \text{ and each } p \in (1, \infty). \tag{1.8.47}$$

In addition, for each $p \in (1,\infty)$ there exists a constant $C \in (0,\infty)$ with the property that for each function $f \in \mathrm{BMO}(\partial\Omega,\sigma)$ we have

$$\max\Big\{ \lim_{R\to\infty} \sup_{\substack{x\in\partial\Omega,\, r>0\\ B(x,r)\subseteq\mathbb{R}^n\setminus B(0,R)}} \Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times$$
$$\times\int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}^{\rm mod}_{jk} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{1/p},$$
$$\lim_{R\to 0^+} \sup_{x\in\partial\Omega,\, r\in(0,R)} \Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times$$
$$\times\int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}^{\rm mod}_{jk} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{1/p},$$
$$\lim_{R\to\infty} \sup_{x\in\partial\Omega,\, r>R} \Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times$$
$$\times\int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}^{\rm mod}_{jk} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{1/p}\Big\}$$
$$\leq C \operatorname{dist}\big(f\,,\,\mathrm{CMO}(\partial\Omega,\sigma)\big), \tag{1.8.48}$$

where the distance in the right-hand side is considered in the space $\mathrm{BMO}(\partial\Omega,\sigma)$ (recall that $\mathrm{CMO}(\partial\Omega,\sigma)$ has been introduced in (A.0.52)). As a corollary of (1.8.48) and Definition 1.8.1,

$$\begin{gathered}\big|\nabla\big(\mathcal{T}^{\rm mod}_{jk} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n \text{ is a super vanishing Carleson measure in } \Omega,\\ \text{for each function } f \in \mathrm{CMO}(\partial\Omega,\sigma) \text{ and each } p\in(1,\infty).\end{gathered} \tag{1.8.49}$$

Comment 2: Analogous properties to those discussed in Theorem 1.8.2 are also valid for the modified versions of the boundary-to-domain Cauchy-Clifford integral operator $\mathcal{C}_{\rm mod}$ defined in (1.8.238) in Corollary 1.8.24, hence further augmenting the results in [70, Proposition 2.5.29] and Proposition 1.6.1. This may be seen from [70, (2.5.311)], Theorem 1.8.2, and Comment 1 above.

Comment 3: Let $\mathcal{D}_{\Delta,\rm mod}$ be the modified boundary-to-domain harmonic double layer operator associated with Ω and $L := \Delta$ as in (1.8.6), i.e.,

$$\mathcal{D}_{\Delta,\rm mod} f(x) := \frac{1}{\omega_{n-1}}\int_{\partial_*\Omega}\Big\{\frac{\langle\nu(y),y-x\rangle}{|x-y|^n} - \frac{\langle\nu(y),y\rangle}{|y|^n}\cdot\mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,1)}}(y)\Big\} f(y)\,d\sigma(y) \tag{1.8.50}$$

for each $f \in L^1\big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^n}\big)$ and each $x \in \Omega$. If E_Δ denotes the standard fundamental solution for the Laplacian in $\mathbb{R}^n$ (cf. (A.0.65)), then formula (1.8.13) presently becomes (with the summation convention in effect)

$$\partial_\ell\big(\mathcal{D}_{\Delta,\mathrm{mod}} f\big)(x) = \int_{\partial_*\Omega} (\partial_j E_\Delta)(x-y)(\partial_{\tau_{\ell j}} f)(y)\,\mathrm{d}\sigma(y) \tag{1.8.51}$$

for each index $\ell \in \{1,\dots,n\}$, each point $x \in \Omega$, and each function

$$\begin{aligned} f \in L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \ \text{with the property that} \\ \partial_{\tau_{jk}} f \in L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \ \text{for all } j,k \in \{1,\dots,n\}. \end{aligned} \tag{1.8.52}$$

This may be further streamlined if we bring in $\mathscr{S}_{\Delta,\mathrm{mod}}$, the modified boundary-to-domain harmonic single layer operator[23] associated with Ω and $L := \Delta$ as in (1.5.50), i.e.,

$$\begin{aligned} \mathscr{S}_{\Delta,\mathrm{mod}} f(x) := \int_{\partial\Omega} \big\{E_\Delta(x-y) - E_\Delta(-y)\cdot \mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}(-y)\big\} f(y)\,\mathrm{d}\sigma(y) \\ \text{for each } f \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big) \text{ and each point } x \in \Omega. \end{aligned} \tag{1.8.53}$$

Then, thanks to (1.5.51), we may recast (1.8.51) simply as (again with the summation convention in effect)

$$\partial_\ell \mathcal{D}_{\Delta,\mathrm{mod}} f = \partial_j \mathscr{S}_{\Delta,\mathrm{mod}}(\partial_{\tau_{\ell j}} f) \ \text{ in } \Omega, \tag{1.8.54}$$

for each index $\ell \in \{1,\dots,n\}$ and each function f as in (1.8.52).

Comment 4: From [70, Proposition 2.3.3] we see that there are a number of alternative characterizations of the boundary-to-boundary double layer potential operator K_{mod}, originally introduced in (1.8.24). For one thing, given any function $f = (f_\alpha)_{1\le\alpha\le M} \in \big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ along with any surface ball $\Delta \subseteq \partial\Omega$, there exists a constant $C_{f,\Delta} \in \mathbb{R}$ with the property that for σ-a.e. point $x \in \Delta$ we have

$$\begin{aligned} (K_{\mathrm{mod}} f)(x) &= K\big(f\mathbf{1}_\Delta\big)(x) + C_{f,\Delta} \\ &+ \bigg(\int_{\partial_*\Omega\setminus\Delta} \nu_s(y) a^{\beta\alpha}_{rs}\big\{(\partial_r E_{\gamma\beta})(x-y) - (\partial_r E_{\gamma\beta})(x_\Delta - y)\big\} f_\alpha(y)\,\mathrm{d}\sigma(y)\bigg)_{1\le\gamma\le M} \end{aligned} \tag{1.8.55}$$

where $x_\Delta \in \partial\Omega$ is the center of the surface ball Δ.

For another thing, having fixed an arbitrary reference point $x_0 \in \partial\Omega$ then for any given function $f = (f_\alpha)_{1\le\alpha\le M} \in \big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ it follows that the limit

[23] recall the convention that when the single layer operator is applied to functions originally defined only on $\partial_*\Omega$, these functions are regarded as being extended by zero outside $\partial_*\Omega$, to the entire $\partial\Omega$

$$(\widetilde{K}_{\text{mod}} f)(x) := \lim_{j\to\infty} \Big\{ K\big(f\mathbf{1}_{B(x_0,j)\cap\partial\Omega}\big)(x) - \Big(\int_{\substack{y\in\partial_*\Omega \\ 1\le|x_0-y|<j}} \nu_s(y) a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x_0-y) f_\alpha(y)\, d\sigma(y)\Big)_{1\le\gamma\le M} \Big\} \tag{1.8.56}$$

exists at σ-a.e. point $x \in \partial\Omega$, and there exists a constant $\widetilde{C}_{f,x_0} \in \mathbb{R}$ with the property that

$$(\widetilde{K}_{\text{mod}} f)(x) = (K_{\text{mod}} f)(x) + \widetilde{C}_{f,x_0} \text{ at } \sigma\text{-a.e. } x \in \partial\Omega. \tag{1.8.57}$$

We now turn to the proof of Theorem 1.8.2.

Proof of Theorem 1.8.2 All claims in item *(1)* with the exception of (1.8.10) and (1.8.12)-(1.8.13) are direct consequences of definitions, [70, Corollary 2.5.3], and [70, Theorem 1.4.2]. As far as the claim made in (1.8.10) is concerned, having fixed two arbitrary points $x_0, x_1 \in \Omega$ along with some constant $\lambda \in \mathbb{C}^M$, we need to show that $(\mathcal{D}_{\text{mod}}\lambda)(x_0) - (\mathcal{D}_{\text{mod}}\lambda)(x_1) = 0$. A moment's reflection shows that this is further equivalent to proving that for each fixed $\alpha, \gamma \in \{1, \dots, M\}$ we have

$$\int_{\partial_*\Omega} \nu_s(y) a^{\beta\alpha}_{rs} \big\{(\partial_r E_{\gamma\beta})(x_0-y) - (\partial_r E_{\gamma\beta})(x_1-y)\big\}\, d\sigma(y) = 0. \tag{1.8.58}$$

To this end, consider the vector field $\vec{F} := (F_s)_{1\le s\le n}$ whose components are defined at $\mathcal{L}^n$-a.e. point $y \in \Omega$ by

$$F_s(y) := a^{\beta\alpha}_{rs} \big\{(\partial_r E_{\gamma\beta})(x_0-y) - (\partial_r E_{\gamma\beta})(x_1-y)\big\}, \qquad 1 \le s \le n. \tag{1.8.59}$$

Hence,

$$\vec{F} \in \big[L^1_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^n. \tag{1.8.60}$$

Also, a direct computation which also uses [70, (1.4.33)] gives that, with the divergence taken in the sense of distributions in Ω,

$$\begin{aligned} \operatorname{div}\vec{F} &= \partial_s F_s = -a^{\beta\alpha}_{rs} \partial_{y_r}\partial_{y_s}\big[E_{\gamma\beta}(x_0-\cdot)\big] + a^{\beta\alpha}_{rs} \partial_{y_r}\partial_{y_s}\big[E_{\gamma\beta}(x_1-\cdot)\big] \\ &= \delta_{\alpha\gamma}\delta_{x_0} - \delta_{\alpha\gamma}\delta_{x_1} \in \mathscr{E}'(\Omega) \end{aligned} \tag{1.8.61}$$

where, generally speaking, δ_x is the Dirac distribution in Ω with mass at $x \in \Omega$.

Fix a compact set $K \subset \Omega$ whose interior contains both x_0 and x_1. Then the same argument which has established [70, (2.3.117)] presently gives that there exists a constant $C_K \in (0, \infty)$ such that

$$|\vec{F}(y)| \le \frac{C_K}{1+|y|^n}, \qquad \forall y \in \Omega \setminus K. \tag{1.8.62}$$

Granted this, [68, Lemma 8.3.7] shows that there exists $C = C(K, \kappa, n) \in (0, \infty)$ such that

$$\left(\mathcal{N}_\kappa^{\Omega\setminus K}\vec{F}\right)(y) \le \frac{C}{1+|y|^n} \text{ for each } y \in \partial\Omega. \tag{1.8.63}$$

In turn, from (1.8.63), [68, (8.2.26)], and [68, Lemma 7.2.1], we conclude that

$$\mathcal{N}_\kappa^{\Omega\setminus K}\vec{F} \in L^1(\partial\Omega, \sigma). \tag{1.8.64}$$

In addition, since $\vec{F}$ is continuous in a neighborhood of $\partial\Omega$ it follows that the pointwise nontangential boundary trace

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at every point in } \partial_{\text{nta}}\Omega, \tag{1.8.65}$$

and at σ-a.e. $y \in \partial_*\Omega$ we have

$$\nu(y)\cdot\left(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\right)(y) = \nu_s(y)a_{rs}^{\beta\alpha}\big\{(\partial_r E_{\gamma\beta})(x_0-y)-(\partial_r E_{\gamma\beta})(x_1-y)\big\}. \tag{1.8.66}$$

Finally, we note that in the case when Ω is unbounded we have $|\vec{F}(y)| = O(|y|^{-n})$ for $y \in \Omega$ with $|y| \to \infty$. Hence, in such a scenario,

$$\int_{[B(0,2R)\setminus B(0,R)]\cap\Omega} |y\cdot\vec{F}(y)|\,\mathrm{d}\mathcal{L}^n(y) = O(R) \text{ as } R\to\infty. \tag{1.8.67}$$

At this stage, we may write

$$\begin{aligned}\int_{\partial_*\Omega} &\nu_s(y)a_{rs}^{\beta\alpha}\big\{(\partial_r E_{\gamma\beta})(x_0-y)-(\partial_r E_{\gamma\beta})(x_1-y)\big\}\,\mathrm{d}\sigma(y)\\ &= \int_{\partial_*\Omega}\nu(y)\cdot\left(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\right)(y)\,\mathrm{d}\sigma(y) = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F},1\big)_{\mathscr{C}_b^\infty(\Omega)}\\ &= \delta_{\alpha\gamma}-\delta_{\alpha\gamma} = 0.\end{aligned} \tag{1.8.68}$$

Above, the first equality comes from (1.8.66), the second equality is formula [68, (1.4.6)] (keeping in mind that the hypotheses of [68, Theorem 1.4.1] are satisfied, thanks to (1.8.60), (1.8.61), (1.8.65), (1.8.67)), the third equality is seen from (1.8.61), and the final equality is obvious. In turn, (1.8.68) establishes (1.8.58). This finishes the proof of (1.8.10).

Finally, the identity claimed in (1.8.13) for any function f as in (1.8.12) is established staring from (1.8.11), then reasoning as in (1.3.31) based on the integration by parts formula on the boundary from [69, Lemma 11.1.7]. The justification of the claims in item *(1)* is now complete.

Moving on, the first claim in item *(2)*, pertaining to (1.8.14), follows from (a vector-valued version of) [70, Lemma 2.1.2], used with $\mathcal{Q} := \nabla\mathcal{D}_{\text{mod}}$, bearing in mind that the constant C_2 defined in [70, (2.1.18)] presently vanishes, thanks to (1.8.10) (upon also noting that $\text{dist}(x,\partial\Omega) \le \text{dist}(x,\partial_*\Omega)$ for each $x \in \Omega$). Then (1.8.15) follows by combining (1.8.14) with [68, (5.11.78)].

As regards (1.8.16), suppose Ω is as stated there, and fix an arbitrary function $f \in \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M$ with $\eta \in (0,1)$. From [68, Proposition 5.9.16] we see that Ω is an Ahlfors regular domain. In view of the present goal, there is no loss of generality in assuming that Ω is also a uniform domain. More specifically, if $\partial\Omega$ is unbounded then [68, Lemma 5.11.3] tells us that Ω is connected which, in concert with item *(4)* in [68, Proposition 5.11.14], implies that Ω is a uniform domain. If $\partial\Omega$ is bounded, then [68, Lemma 5.11.3] tells us that Ω has finitely many connected components separated by a strictly positive distance from one another, and item *(4)* in [68, Proposition 5.11.14] ensures that each such component is a uniform domain. Then the conclusion we seek may be obtained by writing $\mathcal{D}_{\rm mod}$ as a sum of modified double layer potential operator corresponding to each connected component and by piecing together information about each individual piece.

In summary, there is no loss of generality in assuming that Ω is both an Ahlfors regular domain and a uniform domain. In particular, both (1.8.14) and (1.8.15) are presently valid. In particular, the latter implies that for some constant $C \in (0,\infty)$ independent of f we have

$$\mathcal{D}_{\rm mod} f \in \big[\dot{\mathscr{C}}^\eta(\overline{\Omega})\big]^M \text{ and } \big\|\mathcal{D}_{\rm mod} f\big\|_{[\dot{\mathscr{C}}^\eta(\overline{\Omega})]^M} \leq C\|f\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M}. \tag{1.8.69}$$

Let us now show that the operator $\mathcal{D}_{\rm mod}$ maps an arbitrary $f \in \big[\dot{\mathscr{C}}^\eta_{\rm van}(\partial\Omega)\big]^M$ into the space (cf. (A.0.48))

$$\big[\dot{\mathscr{C}}^\eta_{\rm van}(\overline{\Omega})\big]^M = \Big\{u \in \big[\dot{\mathscr{C}}^\eta(\overline{\Omega})\big]^M : \lim_{r\to 0^+}\Big(\sup_{x\in\overline{\Omega}} \|u\|_{\dot{\mathscr{C}}^\eta(B(x,r)\cap\overline{\Omega})}\Big) = 0\Big\}. \tag{1.8.70}$$

To this end, fix some arbitrary threshold $\varepsilon > 0$. Also, fix some $\beta \in (\eta, 1)$, and invoke [69, Theorem 3.2.2] to find a function

$$g \in \big[\dot{\mathscr{C}}^\eta(\partial\Omega) \cap \dot{\mathscr{C}}^\beta(\partial\Omega)\big]^M \text{ with } \|f - g\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M} < \varepsilon. \tag{1.8.71}$$

From (1.8.69) we know that $u := \mathcal{D}_{\rm mod} f$ belongs to $\big[\dot{\mathscr{C}}^\eta(\overline{\Omega})\big]^M$. Making use of notation introduced in the formulation of the result recorded in [68, Proposition 5.11.15], for each $x \in \partial\Omega$ and $r \in (0,R)$ we may estimate

$$
\begin{aligned}
&\big\|\mathcal{D}_{\mathrm{mod}}g\big\|_{[\dot{\mathscr{C}}^{\eta}(B(x,r)\cap\overline{\Omega})]^M}\\
&\quad= \big\|\mathcal{D}_{\mathrm{mod}}g\big\|_{[\dot{\mathscr{C}}^{\eta}(B(x,r)\cap\Omega)]^M} \leq C\big\|\mathcal{D}_{\mathrm{mod}}g\big\|_{[\dot{\mathscr{C}}^{\eta}(\Omega_{x,r})]^M}\\
&\quad\leq C\sup_{\Omega_{x,r}}\Big\{\big|\nabla(\mathcal{D}_{\mathrm{mod}}g)\big|\cdot\operatorname{dist}(\cdot,\partial\Omega_{x,r})^{1-\eta}\Big\}\\
&\quad= C\sup_{\Omega_{x,r}}\Big\{\big|\nabla(\mathcal{D}_{\mathrm{mod}}g)\big|\cdot\operatorname{dist}(\cdot,\partial\Omega_{x,r})^{1-\beta}\cdot\operatorname{dist}(\cdot,\partial\Omega_{x,r})^{\beta-\eta}\Big\}\\
&\quad\leq Cr^{\beta-\eta}\cdot\sup_{\Omega_{x,r}}\Big\{\big|\nabla(\mathcal{D}_{\mathrm{mod}}g)\big|\cdot\operatorname{dist}(\cdot,\partial\Omega_{x,r})^{1-\beta}\Big\}\\
&\quad\leq Cr^{\beta-\eta}\cdot\sup_{\Omega_{x,r}}\Big\{\big|\nabla(\mathcal{D}_{\mathrm{mod}}g)\big|\cdot\operatorname{dist}(\cdot,\partial\Omega)^{1-\beta}\Big\}\\
&\quad\leq Cr^{\beta-\eta}\cdot\sup_{\Omega}\Big\{\big|\nabla(\mathcal{D}_{\mathrm{mod}}g)\big|\cdot\operatorname{dist}(\cdot,\partial\Omega)^{1-\beta}\Big\}\\
&\quad\leq Cr^{\beta-\eta}\cdot\|g\|_{[\dot{\mathscr{C}}^{\beta}(\partial\Omega)]^M},
\end{aligned}
\tag{1.8.72}
$$

for some constant $C\in(0,\infty)$ independent of x, r, and g. The first inequality in (1.8.72) is simply a consequence of the monotonicity of Hölder semi-norm with respect to the domain, and [68, Proposition 5.11.15]. The second inequality in (1.8.72) is implied by [68, (5.11.78)] written with Ω replaced by $\Omega_{x,r}$. That the latter set is actually a uniform domain (as required in [68, (5.11.78)]) is seen from [68, Proposition 5.11.15] and [68, (5.11.65)]. The third inequality in (1.8.72) is a consequence of the fact that $\beta-\eta>0$ and $\operatorname{diam}\Omega_{x,r}\leq Cr$, which imply $\operatorname{dist}(\cdot,\partial\Omega_{x,r})^{\beta-\eta}\leq Cr^{\beta-\eta}$ at each point in $\Omega_{x,r}$. The fourth inequality in (1.8.72) follows from the monotonicity of the distance-to-the-boundary function (cf. [68, Lemma 8.1.1] and [68, Proposition 5.11.15]), bearing in mind that $1-\beta>0$. The fifth inequality in (1.8.72) merely uses the fact that $\Omega_{x,r}$ is a subset of Ω, while the final inequality in (1.8.72) comes from (1.8.14) written for g in place of f and β in place of η.

Having established (1.8.72), we may now conclude that there exists a constant $C\in(0,\infty)$ with the property that

$$
\sup_{x\in\partial\Omega}\big\|\mathcal{D}_{\mathrm{mod}}g\big\|_{[\dot{\mathscr{C}}^{\eta}(B(x,r)\cap\Omega)]^M}\leq Cr^{\beta-\eta}\cdot\|g\|_{[\dot{\mathscr{C}}^{\beta}(\partial\Omega)]^M}\quad\text{for each } r\in(0,R). \tag{1.8.73}
$$

For each fixed $r\in(0,R)$ we may now estimate

$$
\begin{aligned}
&\sup_{x\in\partial\Omega}\|u\|_{[\dot{\mathscr{C}}^{\eta}(B(x,r)\cap\Omega)]^M}\\
&\quad\leq \sup_{x\in\partial\Omega}\big\|\mathcal{D}_{\rm mod}g\big\|_{[\dot{\mathscr{C}}^{\eta}(B(x,r)\cap\Omega)]^M}+\sup_{x\in\partial\Omega}\big\|\mathcal{D}_{\rm mod}(f-g)\big\|_{[\dot{\mathscr{C}}^{\eta}(B(x,r)\cap\Omega)]^M}\\
&\quad\leq \sup_{x\in\partial\Omega}\big\|\mathcal{D}_{\rm mod}g\big\|_{[\dot{\mathscr{C}}^{\eta}(B(x,r)\cap\Omega)]^M}+\big\|\mathcal{D}_{\rm mod}(f-g)\big\|_{[\dot{\mathscr{C}}^{\eta}(\overline{\Omega})]^M}\\
&\quad\leq Cr^{\beta-\eta}\|g\|_{[\dot{\mathscr{C}}^{\beta}(\partial\Omega)]^M}+C\|f-g\|_{[\dot{\mathscr{C}}^{\eta}(\partial\Omega)]^M}\\
&\quad\leq Cr^{\beta-\eta}\|g\|_{[\dot{\mathscr{C}}^{\beta}(\partial\Omega)]^M}+C\varepsilon. \qquad (1.8.74)
\end{aligned}
$$

Above, we have used the triangle inequality in the first step, the monotonicity of Hölder semi-norm with respect to the domain in the second step, (1.8.73) and (1.8.14) (written for $f-g$ in place of f) in the third step, and (1.8.71) in the final step. In view of the arbitrariness of $\varepsilon>0$, from (1.8.74) we then readily conclude that

$$
\lim_{r\to 0^+}\Big(\sup_{x\in\partial\Omega}\|u\|_{[\dot{\mathscr{C}}^{\eta}(B(x,r)\cap\Omega)]^M}\Big)=0. \qquad (1.8.75)
$$

Next, for each fixed $r\in(0,R/5)$, let us estimate the Hölder semi-norm $\|u\|_{[\dot{\mathscr{C}}^{\eta}(B(x,r)\cap\Omega)]^M}$ in the situation when $x\in\overline{\Omega}\setminus\partial\Omega=\Omega$. Consider first the case when $\operatorname{dist}(x,\partial\Omega)\geq 4r$. Then $B(x,r)\subseteq\Omega$ and for each $y,z\in B(x,r)$ we may use the Mean Value Theorem to write

$$
\begin{aligned}
\frac{|u(y)-u(z)|}{|y-z|^{\eta}}&\leq|y-z|^{1-\eta}\cdot\sup_{\xi\in[y,z]}|(\nabla u)(\xi)|\\
&\leq(2r)^{1-\eta}\cdot\sup_{\xi\in[y,z]}|\nabla(\mathcal{D}_{\rm mod}g)(\xi)|\\
&\quad+(2r)^{1-\eta}\cdot\sup_{\xi\in[y,z]}|\nabla(\mathcal{D}_{\rm mod}(f-g))(\xi)|\\
&\leq Cr^{1-\eta}\|g\|_{[\dot{\mathscr{C}}^{\beta}(\partial\Omega)]^M}\cdot\sup_{\xi\in[y,z]}\operatorname{dist}(\xi,\partial\Omega)^{\beta-1}\\
&\quad+Cr^{1-\eta}\|f-g\|_{[\dot{\mathscr{C}}^{\eta}(\partial\Omega)]^M}\cdot\sup_{\xi\in[y,z]}\operatorname{dist}(\xi,\partial\Omega)^{\eta-1}\\
&\leq Cr^{\beta-\eta}\|g\|_{[\dot{\mathscr{C}}^{\beta}(\partial\Omega)]^M}+C\varepsilon, \qquad (1.8.76)
\end{aligned}
$$

where we have also relied on (1.8.14) (twice, first with g in place of f and β in place of η, and, second, with $f-g$ in place of f), as well as on (1.8.71), plus the fact that the distance from any point $\xi\in B(x,r)$ to $\partial\Omega$ is at least $3r$, while $\beta-1<0$ and $\eta-1<0$. Bearing in mind that $\varepsilon>0$ is arbitrary and that $\beta-\eta>0$, from (1.8.76) we then deduce that

$$\lim_{r\to 0^+}\Big(\sup_{\substack{x\in\Omega\text{ with}\\ \operatorname{dist}(x,\partial\Omega)\geq 4r}}\|u\|_{[\dot{\mathscr{C}}^\eta(B(x,r)\cap\Omega)]^M}\Big)=0. \tag{1.8.77}$$

At this stage, there remains to treat the case when the point $x\in\Omega$ satisfies $\operatorname{dist}(x,\partial\Omega)<4r$. Then there exists some point $x_*\in\partial\Omega$ with the property that $|x-x_*|<4r$, which further implies that we have $B(x,r)\cap\Omega\subseteq B(x_*,5r)\cap\Omega$. Consequently,

$$\begin{aligned}\|u\|_{[\dot{\mathscr{C}}^\eta(B(x,r)\cap\Omega)]^M}&\leq \|u\|_{[\dot{\mathscr{C}}^\eta(B(x_*,5r)\cap\Omega)]^M}\\ &\leq \sup_{z\in\partial\Omega}\|u\|_{[\dot{\mathscr{C}}^\eta(B(z,5r)\cap\Omega)]^M}.\end{aligned} \tag{1.8.78}$$

As such,

$$\begin{aligned}&\lim_{r\to 0^+}\Big(\sup_{\substack{x\in\Omega\text{ with}\\ \operatorname{dist}(x,\partial\Omega)<4r}}\|u\|_{[\dot{\mathscr{C}}^\eta(B(x,r)\cap\Omega)]^M}\Big)\\ &\qquad\leq \lim_{r\to 0^+}\Big(\sup_{z\in\partial\Omega}\|u\|_{[\dot{\mathscr{C}}^\eta(B(z,5r)\cap\Omega)]^M}\Big)=0,\end{aligned} \tag{1.8.79}$$

with the last equality coming from (1.8.73). Together, (1.8.73), (1.8.77), and (1.8.79) prove that for each $f\in\big[\dot{\mathscr{C}}^\eta_{\text{van}}(\partial\Omega)\big]^M$ the function $u=\mathcal{D}_{\text{mod}}f$ belongs to the space described in (1.8.70). The justification of (1.8.16) is therefore complete.

Going further, (1.8.17) may be proved by relying on (1.8.14) and the fact that there exists $C\in(0,\infty)$ such that

$$\sup_{x\in\partial\Omega}\int_{B(x,r)\cap\Omega}\operatorname{dist}(\cdot,\partial\Omega)^{\eta p-1}\,d\mathcal{L}^n\leq Cr^{n-1+\eta p}\quad\text{for each }r>0, \tag{1.8.80}$$

itself a consequence of item *(i)* in [68, Proposition 8.7.1] (see [68, (8.7.3)], presently used with $y:=x$, $N:=0$, and $\alpha:=1-\eta p<1$).

Let us now show that (1.8.19) holds when $f\in\big[\dot{\mathscr{C}}^\eta_{\text{van}}(\partial\Omega)\big]^M$. To this end, pick an arbitrary $\varepsilon>0$. Also, choose some $\beta\in(\eta,1)$ and select g as in (1.8.71). Then, on the one hand,

$$\sup_{\substack{x\in\partial\Omega\\ r\in(0,\infty)}}\bigg(\frac{1}{r^{n-1+\eta p}}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{D}_{\text{mod}}(f-g)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\bigg)^{1/p}\leq C\varepsilon, \tag{1.8.81}$$

by (1.8.17) and (1.8.71). On the other hand,

$$\sup_{\substack{x\in\partial\Omega\\ r\in(0,\infty)}}\left(\frac{1}{r^{n-1+\eta p}}\int_{B(x,r)\cap\Omega}\left|\nabla\mathcal{D}_{\rm mod}g\right|^p\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{1/p}$$

$$\leq\left(\sup_{\substack{x\in\partial\Omega\\ r\in(0,\infty)}}\frac{1}{r^{n-1+\eta p}}\int_{B(x,r)\cap\Omega}\mathrm{dist}(\cdot,\partial\Omega)^{\beta p-1}\,\mathrm{d}\mathcal{L}^n\right)^{1/p}\cdot\|g\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M}$$

$$\leq Cr^{\beta-\eta}\|g\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M},\tag{1.8.82}$$

by (1.8.17) used with g in place of f and with β in place of η, and by (1.8.80) used with β in place of η. Collectively, (1.8.81) and (1.8.82) then imply (1.8.19), in view of the arbitrariness of $\varepsilon > 0$. Finally, (1.8.18) is a consequence of (1.8.17) and (1.8.19). This takes care of item *(2)*.

To proceed, strengthen the hypotheses on Ω by assuming that $\partial\Omega$ is a UR set. Then the jump-formula (1.8.27) follows from definitions, [70, Corollary 2.5.3], and [70, Theorem 1.4.2]. With the exception of (1.8.23), all other claims in item *(3)* are justified as explained in the very statement of the theorem. To justify (1.8.23), select a function $f = (f_\alpha)_{1\leq\alpha\leq M}$ as in (1.8.12). From what we have proved so far in item *(3)* it follows that the conormal derivative $\partial^A_\nu(\mathcal{D}_{\rm mod}f)$ may be meaningfully defined, as in (A.0.184). Specifically, at σ-a.e. point $x\in\partial_*\Omega$ we have

$$\big(\partial^A_\nu(\mathcal{D}_{\rm mod}f)\big)(x)=\Big(\nu_i(x)a^{\mu\gamma}_{ij}\partial_j(\mathcal{D}_{\rm mod}f)_\gamma\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}(x)\Big)_{1\leq\mu\leq M}\tag{1.8.83}$$

$$=\left(\lim_{\varepsilon\to0^+}\int_{\substack{y\in\partial_*\Omega\\|x-y|>\varepsilon}}\nu_i(x)a^{\mu\gamma}_{ij}a^{\beta\alpha}_{rs}(\partial_rE_{\gamma\beta})(x-y)\big(\partial_{\tau_{js}}f_\alpha\big)(y)\,\mathrm{d}\sigma(y)\right)_{1\leq\mu\leq M}$$

where the first equality comes from (A.0.184) and the second equality is a consequence of (1.8.13) and the jump-formula from [70, Theorem 2.5.1] (reasoning as in (1.5.107)-(1.5.108)). This establishes (1.8.23).

As regards item *(4)*, the first order of business is to prove the estimate stated in (1.8.29). However, having established (1.8.10), the argument proceeds very much as in the case of the proof of [70, Corollary 2.4.2]. Specifically, pick an arbitrary function $f\in\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$. Having also fixed a point $x_o\in\partial\Omega$ and a scale $r\in\big(0,2\,\mathrm{diam}(\partial\Omega)\big)$, the same argument which has produced [70, (2.4.136)] (based on the estimate in [70, (2.4.34)] and the decay of the integral kernel of $\nabla\mathcal{D}_{\rm mod}$) now gives

$$\int_{B(x_o,r)\cap\Omega}\Big|\nabla\big(\mathcal{D}_{\rm mod}f\big)(x)\Big|^p\mathrm{dist}(x,\partial\Omega)^{p-1}\,\mathrm{d}x\tag{1.8.84}$$

$$\leq C\sigma\big(B(x_o,r)\cap\partial\Omega\big)f^\#_p(x_o)^p+C\sigma\big(B(x_o,r)\cap\partial\Omega\big)f^\#_1(x_o)^p,$$

where the L^q-based Fefferman-Stein maximal function $f_q^{\#}$, with $q \in [1,\infty)$, has been defined in (A.0.195). Granted this estimate, the version of (1.8.29) with the supremum taken in the regime $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ follows on account of [68, (7.4.111)]. Finally, the case when Ω is an exterior domain and $r \geq 2\,\mathrm{diam}(\partial\Omega)$ is handled much as in [70, (2.4.142), (2.4.143)]. Going further, consider the claim made in (1.8.31). To this end, pick $p \in (1,\infty)$ and select an arbitrary function $f \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$. Also, fix some exponent $\eta \in (0,1)$ and choose an arbitrary function

$$g \in \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M \cap \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M. \tag{1.8.85}$$

Then for each $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ and $x \in \partial\Omega$ we may estimate

$$\begin{aligned}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{D}_{\mathrm{mod}}(f-g)\big)\big|^p \mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}}\\ \leq C\|f-g\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M},\end{aligned} \tag{1.8.86}$$

thanks to (1.8.29) written $f-g$ in place of f. In addition, we have

$$\begin{aligned}&\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{D}_{\mathrm{mod}}g\big)\big|^p \mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}}\\ &\quad\leq C\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\mathrm{dist}(\cdot,\partial\Omega)^{p\eta-1}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}}\|g\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M}\\ &\quad\leq Cr^\eta\|g\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M}.\end{aligned} \tag{1.8.87}$$

Indeed, the first inequality above uses (1.8.14) (written for g in place of f), while the second inequality is based on [68, (8.6.101)] used with $\lambda := 1-p\eta$, $\alpha := 1$, $\beta := n-1$, $E := B(x,r)\cap\Omega$, and [68, (8.1.17)]. Collectively, (1.8.86) and (1.8.87) imply that

$$\begin{aligned}\lim_{R\to 0^+}\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{D}_{\mathrm{mod}}f\big)\big|^p \mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}}\\ \leq C\|f-g\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M},\end{aligned} \tag{1.8.88}$$

for some constant $C \in (0,\infty)$ independent of f and g. With this in hand, (1.8.31) follows on account of [69, Theorem 3.1.3] and the ability to choose g arbitrary as in (1.8.85). In turn, (1.8.31) readily implies (1.8.32)-(1.8.33). The estimate claimed in (1.8.34) is a particular case of a more general result established later in Theorem 5.1.22. This takes care of item *(4)*.

Next, the first claim in item *(5)* is clear from (1.8.8), while the remaining properties follow from what we have proved in item *(4)*, [69, (4.6.16)], and (1.8.8). As far as item *(6)* is concerned, the estimates claimed in (1.8.40) and (1.8.44) are special cases of more general results proved later in Theorem 5.1.15 and Theorem 5.1.8, respectively, while the claim formulated in (1.8.43) is a consequence of [70, (2.3.36)], (1.8.24), and (1.3.72). This completes the proof of Theorem 1.8.2. □

We continue by making three comments pertaining to Theorem 1.8.2. First, it is clear from (1.8.6) that for each function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\Big]^M$, each index $j \in \{1,\dots,n\}$, and each point $x \in \Omega$, we have

$$\partial_j\big(\mathcal{D}_{\text{mod}} f\big)(x) = \left(-\int_{\partial_*\Omega} \nu_s(y) a^{\beta\alpha}_{rs} (\partial_j\partial_r E_{\gamma\beta})(x-y) f_\alpha(y)\, \mathrm{d}\sigma(y) \right)_{1\leq\gamma\leq M}. \tag{1.8.89}$$

In particular, this formula is valid when the function f belongs to the space $\big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M$ for some $\eta \in (0,1)$, or to the space $\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$.

Our second comment further sheds light on the claims in item *(2)* of Theorem 1.8.2 in the case when Ω is an Ahlfors regular domain. Specifically, fix $f \in \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M$ with $\eta \in (0,1)$, along with some $j \in \{1,\dots,n\}$. Also, fix some point $x \in \Omega$. It follows from (1.8.89) that we may express $\partial_j\big(\mathcal{D}_{\text{mod}} f\big)(x)$ as

$$\partial_j\big(\mathcal{D}_{\text{mod}} f\big)(x) = \left(\int_{\partial\Omega} m^{\alpha\gamma}_{j,x}(y) f_\alpha(y)\, \mathrm{d}\sigma(y) \right)_{1\leq\gamma\leq M} \tag{1.8.90}$$

where, for each $\alpha,\gamma \in \{1,\dots,M\}$, we have set

$$m^{\alpha\gamma}_{j,x}(y) := -\nu_s(y) a^{\beta\alpha}_{rs} (\partial_j\partial_r E_{\gamma\beta})(x-y) \text{ for } \sigma\text{-a.e. } y \in \partial\Omega. \tag{1.8.91}$$

In particular, $m^{\alpha\gamma}_{j,x}$ is σ-measurable on $\partial\Omega$ and [70, Theorem 1.4.2] yields the estimate

$$\big|m^{\alpha\gamma}_{j,x}(y)\big| \leq C|x-y|^{-n} \text{ for } \sigma\text{-a.e. } y \in \partial\Omega. \tag{1.8.92}$$

Pick now

$$p := \tfrac{n-1}{n-1+\eta} \in \big(\tfrac{n-1}{n}, 1\big) \tag{1.8.93}$$

and note that this entails $\eta = (n-1)\big(\frac{1}{p}-1\big)$. Also, choose some $q \in (1,\infty)$ then select

$$d \in \Big((n-1)\big(\tfrac{q}{p}-1\big), nq-(n-1)\Big) \text{ and set } \theta := \tfrac{n-1}{d}\big(\tfrac{q}{p}-1\big) \in (0,1). \tag{1.8.94}$$

A direct computation based on (1.8.92) and [68, Lemma 7.2.1] then shows that

$$\big\|m^{\alpha\gamma}_{j,x}\big\|_{L^q(\partial\Omega,\sigma)} \leq C\,\mathrm{dist}(x,\partial\Omega)^{(n-1-nq)/q} \tag{1.8.95}$$

and, if $x_o \in \partial\Omega$ is such that $\mathrm{dist}(x,\partial\Omega) = |x-x_o|$,

$$\big\|m^{\alpha\gamma}_{j,x}\big\|_{L^q(\partial\Omega,\,|\cdot-x_o|^d\,\sigma)} \leq C\,\mathrm{dist}(x,\partial\Omega)^{(d+n-1-nq)/q}. \tag{1.8.96}$$

Collectively, (1.8.95)-(1.8.96) and (1.8.93)-(1.8.94) then permit us to estimate

$$\begin{aligned}\|m^{\alpha\gamma}_{j,x}\|^{1-\theta}_{L^q(\partial\Omega,\sigma)}\cdot\|m^{\alpha\gamma}_{j,x}\|^{\theta}_{L^q(\partial\Omega,\,|\cdot-x_o|^d\,\sigma)}&\\ &\le C\,\mathrm{dist}(x,\partial\Omega)^{(1-\theta)(n-1-nq)/q+\theta(d+n-1-nq)/q}\\ &=C\,\mathrm{dist}(x,\partial\Omega)^{\eta-1}.\end{aligned}\tag{1.8.97}$$

In particular, $m^{\alpha\gamma}_{j,x}\in L^q\big(\partial\Omega,(1+|\cdot-x_o|^d)\sigma\big)$. Moreover, if $\mathbf{e}_\alpha$ denotes the constant $\mathbb{C}^M$-valued function defined on $\partial\Omega$ whose α-th component is 1 and all others are zero, then (1.8.10) implies

$$\int_{\partial\Omega} m^{\alpha\gamma}_{j,x}(y)\,d\sigma(y)=\partial_j\big(\mathcal{D}_{\text{mod}}\mathbf{e}_\alpha\big)_\gamma(x)=0.\tag{1.8.98}$$

Granted these properties, [69, Corollary 4.5.3] applies and gives that

$$\begin{aligned}&m^{\alpha\gamma}_{j,x}\text{ is a multiple of a }H^p\text{-molecule on }\partial\Omega\text{ (cf. [69, Definition 4.5.1]),}\\ &\text{and we have }\|m^{\alpha\gamma}_{j,x}\|_{H^p(\partial\Omega,\sigma)}\le C\cdot\mathrm{dist}(x,\partial\Omega)^{\eta-1}\text{ for some finite con-}\\ &\text{stant }C>0\text{, independent of the given point }x.\end{aligned}\tag{1.8.99}$$

In turn, from (1.8.90), (1.8.99), and [69, Proposition 4.8.7] (which is currently applicable since any H^p-molecule on $\partial\Omega$ is also a multiple of a H^1-molecule on $\partial\Omega$), we deduce that, with $\langle\cdot,\cdot\rangle$ denoting the duality bracket between functions satisfying a homogeneous Hölder condition of order η on $\partial\Omega$, modulo constants, and the Hardy space H^p on $\partial\Omega$ (cf. [69, Theorem 4.6.1]), we have

$$\partial_j\big(\mathcal{D}_{\text{mod}}f\big)(x)=\begin{cases}\Big(\big\langle[f_\alpha],m^{\alpha\gamma}_{j,x}\big\rangle\Big)_{1\le\gamma\le M} & \text{if }\partial\Omega\text{ is unbounded,}\\ \Big(\big\langle f_\alpha,m^{\alpha\gamma}_{j,x}\big\rangle\Big)_{1\le\gamma\le M} & \text{if }\partial\Omega\text{ is bounded.}\end{cases}\tag{1.8.100}$$

In concert with [69, (4.6.9)], (1.8.99), and [69, (4.8.46)] (which we invoke in the case when $\partial\Omega$ is bounded, bearing (1.8.98) in mind), this ultimately allows us to estimate

$$\begin{aligned}\big|\nabla\big(\mathcal{D}_{\text{mod}}f\big)(x)\big|&\le\sum_{j=1}^{n}\sum_{\gamma=1}^{M}\|f_\alpha\|_{\dot{\mathscr{C}}^\eta(\partial\Omega)}\|m^{\alpha\gamma}_{j,x}\|_{H^p(\partial\Omega,\sigma)}\\ &\le C\cdot\mathrm{dist}(x,\partial\Omega)^{\eta-1}\|f\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M}.\end{aligned}\tag{1.8.101}$$

Since $x\in\Omega$ is arbitrary and $C\in(0,\infty)$ is independent of x and f, this gives an alternative proof of (1.8.14) (in the case when Ω is an Ahlfors regular domain).

Our third comment in relation to Theorem 1.8.2 is that all results there have natural counterparts for the Cauchy-Clifford integral operator $\mathcal{C}$, $\mathfrak{C}$ associated with Ω as in (A.0.53) and (A.0.54), respectively.

The fact that the normal derivative of the classical harmonic double layer potential has no jump across the boundary is, in a smooth setting, occasionally referred to as the

Lyapunov-Tauber theorem (cf., e.g., [31]). In the case of the Laplacian, the formula for the normal derivative of the double layer takes a particularly aesthetic form. This is indicated in the next proposition, which also serves as a natural justification of the aforementioned Lyapunov-Tauber phenomenon in a very general geometric and analytic setting.

Proposition 1.8.3 *Let* $\Omega \subseteq \mathbb{R}^n$ *(where* $n \in \mathbb{N}$, $n \geq 2$*) be a* UR *domain. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$*, and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Recall that* $\mathcal{D}_{\Delta,\mathrm{mod}}$ *is the modified boundary-to-domain harmonic double layer operator associated with* Ω *as in* (1.8.50)*, and that* $S_{\Delta,\mathrm{mod}}$ *is the modified boundary-to-boundary harmonic single layer operator from* (1.5.211)*. Then for each function*

$$
\begin{aligned}
&f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \text{ such that, for some } p \in (1,\infty),\\
&\partial_{\tau_{jk}} f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma) \text{ for all } j,k \in \{1,\dots,n\},
\end{aligned}
\tag{1.8.102}
$$

the following formula holds:

$$
\partial_\nu \mathcal{D}_{\Delta,\mathrm{mod}} f = \sum_{1\leq j<k\leq n} \partial_{\tau_{jk}}\big[S_{\Delta,\mathrm{mod}}\big(\partial_{\tau_{jk}} f\big)\big] \text{ in } \Omega. \tag{1.8.103}
$$

As a corollary, for each

$$
\begin{aligned}
&f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \text{ such that, for some } p \in (1,\infty),\\
&\partial_{\tau_{jk}} f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma) \text{ for all } j,k \in \{1,\dots,n\},
\end{aligned}
\tag{1.8.104}
$$

one has

$$
\partial_\nu \mathcal{D}_{\Delta} f = \sum_{1\leq j<k\leq n} \partial_{\tau_{jk}}\big[S_{\Delta,\mathrm{mod}}\big(\partial_{\tau_{jk}} f\big)\big] \text{ in } \Omega, \tag{1.8.105}
$$

where $\mathcal{D}_\Delta$ *is the ordinary boundary-to-domain harmonic double layer potential operator associated with the set* Ω*.*

Proof Denote by $\nu_1,\dots,\nu_n$ the scalar components of ν and select an aperture parameter $\kappa > 0$. Having fixed an arbitrary function f as in (1.8.102) we may write

$$
\begin{aligned}
\partial_\nu \mathcal{D}_{\Delta,\mathrm{mod}} f &= \sum_{j=1}^{n} \nu_j \big(\partial_j \mathcal{D}_{\Delta,\mathrm{mod}} f\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \sum_{j,k=1}^{n} \nu_j \big(\partial_k \mathcal{S}_{\Delta,\mathrm{mod}}(\partial_{\tau_{jk}} f)\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \\
&= \frac{1}{2}\sum_{j,k=1}^{n} \nu_j \big(\partial_k \mathcal{S}_{\Delta,\mathrm{mod}}(\partial_{\tau_{jk}} f)\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - \frac{1}{2}\sum_{j,k=1}^{n} \nu_k \big(\partial_j \mathcal{S}_{\Delta,\mathrm{mod}}(\partial_{\tau_{jk}} f)\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \\
&= \frac{1}{2}\sum_{j,k=1}^{n} \partial_{\tau_{jk}} \Big[\Big(\mathcal{S}_{\Delta,\mathrm{mod}}\big(\partial_{\tau_{jk}} f\big)\Big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big] = \frac{1}{2}\sum_{j,k=1}^{n} \partial_{\tau_{jk}} \Big[S_{\Delta,\mathrm{mod}}\big(\partial_{\tau_{jk}} f\big)\Big] \\
&= \sum_{1\le j<k\le n} \partial_{\tau_{jk}} \Big[S_{\Delta,\mathrm{mod}}\big(\partial_{\tau_{jk}} f\big)\Big]. \qquad (1.8.106)
\end{aligned}
$$

Above, the first equality is simply the definition of the normal derivative (cf. (A.0.184) for the Laplacian in its standard writing). The second equality comes from (1.8.54). The third equality is a consequence of the fact that $\partial_{\tau_{jk}}$ is antisymmetric in j and k. The fourth equality is implied by [69, (11.3.26)] written for the function $u := \mathcal{S}_{\Delta,\mathrm{mod}}\big(\partial_{\tau_{jk}} f\big)$, which satisfies the hypotheses of [69, Proposition 11.3.2]. Specifically, the first condition in [69, Proposition 11.3.2, (11.3.24)] is ensured by (1.5.69)-(1.5.70), while the second condition in [69, Proposition 11.3.2, (11.3.24)] is guaranteed by (1.5.54). Also, the existence of the nontangential traces in [69, Proposition 11.3.2, (11.3.25)] is seen from (1.5.80) and (1.5.53). Next, the fifth equality in (1.8.106) comes from (1.5.80), while the final equality in (1.8.106) is once again a consequence of the antisymmetry of $\partial_{\tau_{jk}}$ in j and k. This finishes the proof of (1.8.102).

Finally, (1.8.105) is a consequence of (1.8.102) and (1.8.8). □

Moving on, it is both useful and informative to provide concrete examples of modified double layer potential operators for which the results in Theorem 1.8.2 apply.

Example 1.8.4 Work in $\mathbb{R}^2 \equiv \mathbb{C}$ and consider the factorization of the two-dimensional Laplacian $\Delta = \partial_x^2 + \partial_y^2$ given by

$$
\Delta = \widetilde{D}D \text{ where } \widetilde{D} := \partial_x - \mathrm{i}\partial_y \text{ and } D := \partial_x + \mathrm{i}\partial_y. \qquad (1.8.107)
$$

Let $A_{\widetilde{D},D}$ be the coefficient tensor induced by this factorization of the Laplacian, defined as in (1.4.32) with $\widetilde{D}, D$ as in (1.8.107). Also, let $\Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$ be an open set with an Ahlfors regular boundary. Then a direct computation reveals that the modified boundary-to-domain double layer operator $\mathcal{D}_{\mathrm{mod}}$ associated as in (1.8.6) with the set Ω and the coefficient tensor $A := A_{\widetilde{D},D}$ acts on functions $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^2}\Big)$ according to

$$
\mathcal{D}_{\mathrm{mod}} f(z) = \frac{1}{2\pi\mathrm{i}} \int_{\partial_*\Omega} \Big\{\frac{1}{\zeta - z} - \frac{1}{\zeta}\mathbf{1}_{\mathbb{C}\setminus\overline{B(0,1)}}(\zeta)\Big\} f(\zeta)\,\mathrm{d}\zeta, \quad \forall z \in \Omega. \qquad (1.8.108)
$$

Also, the modified boundary-to-boundary double layer operator K_{mod} associated as in (1.8.24)-(1.8.25) with the set Ω and the coefficient tensor $A := A_{\widetilde{D},D}$ as above, turns out to act on each function $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^2}\Big)$ according to

$$K_{\text{mod}} f(z) = \lim_{\varepsilon\to 0^+} \frac{1}{2\pi \mathrm{i}} \int_{\partial_*\Omega} \Big\{ \frac{1}{\zeta - z} \mathbf{1}_{\mathbb{C}\setminus \overline{B(z,\varepsilon)}}(\zeta) - \frac{1}{\zeta} \mathbf{1}_{\mathbb{C}\setminus\overline{B(0,1)}}(\zeta) \Big\} f(\zeta)\, \mathrm{d}\zeta \quad (1.8.109)$$

at σ-a.e. $z \in \partial\Omega$. Naturally, these may be thought of as the modified versions of the standard boundary-to-domain and boundary-to-boundary Cauchy integral operators (cf. the discussion in [70, §1.1]). For these operators, the results in Theorem 1.8.2 apply.

Example 1.8.5 Work in the Clifford algebra context and consider the factorization of the Laplacian $\Delta = \sum_{j=1}^n \partial_j^2$ in $\mathbb{R}^n$ given by

$$\Delta = \widetilde{D} D \text{ where } \widetilde{D} := D := \mathrm{i} \sum_{j=1}^n \mathbf{e}_j \partial_j. \quad (1.8.110)$$

Let $A_{\widetilde{D},D}$ be the coefficient tensor induced by this factorization of the Laplacian, defined as in (1.4.32) with $\widetilde{D}$, D as in (1.8.110). In addition, let $\Omega \subseteq \mathbb{R}^n$ be an open set with an Ahlfors regular boundary. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. Then a direct computation shows that the modified boundary-to-domain double layer operator $\mathcal{D}_{\text{mod}}$ associated as in (1.8.6) with the set Ω and the coefficient tensor $A := A_{\widetilde{D},D}$ acts on functions $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^n}\Big) \otimes \mathcal{C}\ell_n$ according to

$$\mathcal{D}_{\text{mod}} f(x) = \frac{1}{\omega_{n-1}} \int_{\partial_*\Omega} \Big\{ \frac{x-y}{|x-y|^n} - \frac{-y}{|-y|^n} \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,1)}}(y) \Big\} \odot \nu(y) \odot f(y)\, \mathrm{d}\sigma(y) \quad (1.8.111)$$

at each point $x \in \Omega$. Moreover, the modified boundary-to-domain double layer operator K_{mod} associated as in (1.8.24)-(1.8.25) with the set Ω and the coefficient tensor $A := A_{\widetilde{D},D}$ as above, acts on each given function $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^n}\Big) \otimes \mathcal{C}\ell_n$ according to

$$\begin{aligned} K_{\text{mod}} f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial_*\Omega} \Big\{ & \frac{x-y}{|x-y|^n} \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(x,\varepsilon)}}(y) \\ & - \frac{-y}{|-y|^n} \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,1)}}(y) \Big\} \odot \nu(y) \odot f(y)\, \mathrm{d}\sigma(y) \end{aligned} \quad (1.8.112)$$

at σ-a.e. point $x \in \partial\Omega$. These may be naturally regarded as the modified versions of the standard boundary-to-domain and boundary-to-boundary Cauchy-Clifford integral operators (cf. (A.0.53) and (A.0.54)). For these operators, the results described in Theorem 1.8.2 are valid.

Example 1.8.6 Consider the factorization of the Laplacian Δ in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ given by

$$\Delta = \widetilde{D}D \text{ where } \widetilde{D} := (2\partial_{z_1}, \ldots, 2\partial_{z_n}) \text{ and } D := \begin{pmatrix} 2\partial_{\bar{z}_1} \\ \vdots \\ 2\partial_{\bar{z}_n} \end{pmatrix} \tag{1.8.113}$$

where, as in (1.4.177), we set $\partial_{z_j} := \frac{1}{2}\big(\partial_{x_j} - \mathrm{i}\partial_{y_j}\big)$ and $\partial_{\bar{z}_j} := \frac{1}{2}\big(\partial_{x_j} + \mathrm{i}\partial_{y_j}\big)$ for each index $j \in \{1, \ldots, n\}$. To proceed, let $A_{\widetilde{D},D}$ be the coefficient tensor induced by this factorization of the Laplacian, defined as in (1.4.32) with $\widetilde{D}$, D as in (1.8.110). Also, fix an open set $\Omega \subseteq \mathbb{R}^{2n}$ with an Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ and identify the geometric measure theoretic outward unit normal $\nu = (\nu_1, \nu_2, \ldots, \nu_{2n-1}, \nu_{2n}) \in \mathbb{R}^{2n}$ to Ω with the complex vector $\nu_{\mathbb{C}} := (\nu_1 + \mathrm{i}\nu_2, \ldots, \nu_{2n-1} + \mathrm{i}\nu_{2n}) \in \mathbb{C}^n$. A straightforward interpretation of definitions then shows that the modified boundary-to-domain double layer operator $\mathcal{D}_{\text{mod}}$ associated as in (1.8.6) with the set Ω and the coefficient tensor $A := A_{\widetilde{D},D}$ acts on arbitrary complex-valued functions $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n}}\Big)$ according to

$$\mathcal{D}_{\text{mod}} f(z) = \frac{1}{\omega_{2n-1}} \int_{\partial_*\Omega} \left\{ \frac{\big\langle \nu_{\mathbb{C}}(\zeta), \zeta - z\big\rangle_{\mathbb{C}}}{|z-\zeta|^{2n}} - \frac{\big\langle \nu_{\mathbb{C}}(\zeta), \zeta\big\rangle_{\mathbb{C}}}{|\zeta|^{2n}} \mathbf{1}_{\mathbb{C}^n \setminus \overline{B(0,1)}}(\zeta) \right\} f(\zeta)\, \mathrm{d}\sigma(\zeta) \tag{1.8.114}$$

at each point $z \in \Omega$, where $\langle u, w\rangle_{\mathbb{C}} := \sum_{j=1}^n u_j \overline{w}_j$ for each $u = (u_j)_j \in \mathbb{C}^n$ and $w = (w_j)_j \in \mathbb{C}^n$, is the Hermitian complex-pairing. Furthermore, the modified boundary-to-domain double layer operator K_{mod} associated as in (1.8.24)-(1.8.25) with the set Ω and the coefficient tensor $A := A_{\widetilde{D},D}$ as above, acts on each given complex-valued function $f \in L^1\Big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n}}\Big)$ according to

$$\begin{aligned} K_{\text{mod}} f(z) = \lim_{\varepsilon \to 0^+} \frac{1}{\omega_{2n-1}} \int_{\partial_*\Omega} \Bigg\{ & \frac{\big\langle \nu_{\mathbb{C}}(\zeta), \zeta - z\big\rangle_{\mathbb{C}}}{|z-\zeta|^{2n}} \mathbf{1}_{\mathbb{C}^n \setminus \overline{B(z,\varepsilon)}}(\zeta) \\ & - \frac{\big\langle \nu_{\mathbb{C}}(\zeta), \zeta\big\rangle_{\mathbb{C}}}{|\zeta|^{2n}} \mathbf{1}_{\mathbb{C}^n \setminus \overline{B(0,1)}}(\zeta) \Bigg\} f(\zeta)\, \mathrm{d}\sigma(\zeta) \end{aligned} \tag{1.8.115}$$

at σ-a.e. point $z \in \partial\Omega$. In a natural fashion, these may be thought of as the modified versions of the standard boundary-to-domain and boundary-to-boundary Bochner-Martinelli integral operators (cf. (7.5.2) and (7.5.26)). For these operators, the results in Theorem 1.8.2 therefore apply.

Example 1.8.7 There are other factorizations of the Laplacian which yield boundary layer potential operators of interest. For example, from (7.3.15)-(7.3.16) we know that the complex Laplacian $\square := -\frac{1}{2}\Delta$ in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ may be factored as

$$\square = \widetilde{D}D \ \text{ with } \ \widetilde{D} := D := (\bar{\partial} + \vartheta). \tag{1.8.116}$$

Also, in the context of differential forms, the Hodge-Laplacian $\Delta = -(\mathrm{d}\delta + \delta\mathrm{d})$ may be factored as (recall that d, δ stand, respectively, for the exterior derivative operator and its formal transpose)

$$\Delta = \widetilde{D}D \ \text{ with } \ \widetilde{D} := \mathrm{i}(\mathrm{d}, \delta) \ \text{ and } \ D := \mathrm{i}\begin{pmatrix} \delta \\ \mathrm{d} \end{pmatrix}, \tag{1.8.117}$$

as well as

$$\Delta = \widetilde{D}D \ \text{ with } \ \widetilde{D} := D := \mathrm{i}(\mathrm{d} + \delta). \tag{1.8.118}$$

All these factorizations naturally lead to modified boundary layer potentials for which the results in Theorem 1.8.2 are valid.

On to a new topic, in view of (1.8.26) and [69, (11.4.5)], the following result is a generalization of Proposition 1.5.6.

Proposition 1.8.8 *Assume $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider a homogeneous, second-order, constant complex coefficient, weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, and pick a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$ for which $L_A = L$. Let K be the boundary-to-boundary double layer potential operator associated with Ω and A as in (1.3.68), and bring in its modified version K_{mod} from (1.8.24). Finally, recall the family of singular integral operators U_{jk} with $j, k \in \{1, \dots, n\}$ defined in (1.5.251) and fix some integrability exponent $p \in (1, \infty)$.*

Then for each function

$$\begin{gathered} f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\mathrm{loc}}(\partial\Omega, \sigma)\Big]^M \ \textit{such that} \\ \partial_{\tau_{jk}} f_\alpha \ \textit{belongs to} \ L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\mathrm{loc}}(\partial\Omega, \sigma) \\ \textit{for all } \ j, k \in \{1, \dots, n\} \ \textit{and} \ \alpha \in \{1, \dots, M\}, \end{gathered} \tag{1.8.119}$$

and each pair of indices $j, k \in \{1, \dots, n\}$ one has

$$\partial_{\tau_{jk}}\big(K_{\text{mod}} f\big) = K(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f) \tag{1.8.120}$$

where, as in the case of (1.5.253), $\nabla_{\tan} f$ is regarded as the $M \times n$ matrix-valued function whose (α, s) entry is the s-th component of the tangential gradient $\nabla_{\tan} f_\alpha$ (cf. (A.0.78)).

In particular, Proposition 1.8.8 is applicable to all modified boundary-to-boundary double layer potential operator K_{mod} given in Examples 1.8.4-1.8.7.

Proof of Proposition 1.8.8 Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, fix an aperture parameter $\kappa > 0$, and pick $j, k \in \{1, \dots, n\}$ along with $\gamma \in \{1, \dots, M\}$. Given a function $f = (f_\alpha)_{1\leq\alpha\leq M}$ as in (1.8.119), we may then write

$$\partial_{\tau_{jk}}\big(K_{\rm mod} f\big)_\gamma = \partial_{\tau_{jk}}\big(\tfrac{1}{2}f + K_{\rm mod} f\big)_\gamma - \tfrac{1}{2}\partial_{\tau_{jk}} f_\gamma \tag{1.8.121}$$
$$= \nu_j\big(\partial_k(\mathcal{D}_{\rm mod} f)_\gamma\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} - \nu_k\big(\partial_j(\mathcal{D}_{\rm mod} f)_\gamma\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} - \tfrac{1}{2}\partial_{\tau_{jk}} f_\gamma$$

where the second equality is justified by invoking [69, Proposition 11.3.2] with $u := \mathcal{D}_{\rm mod} f$ and $p := 1$ (its present applicability is ensured by (1.8.20), (1.8.27), and (1.8.22)).

Next, recall the definition of the boundary-to-domain double layer potential operator from (1.3.18), as well as the definition of the operator introduced in [70, (2.9.53)]. Then making use of (1.8.13) and [69, (11.4.8)], for each $x \in \Omega$ we may express

$$\begin{aligned}\partial_j(\mathcal{D}_{\rm mod} f)_\gamma(x) &= \int_{\partial\Omega} \nu_j(y) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_s(y)\,{\rm d}\sigma(y)\\ &\quad - \int_{\partial\Omega} \nu_s(y) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)(\nabla_{\tan} f_\alpha)_j(y)\,{\rm d}\sigma(y)\\ &= a^{\beta\alpha}_{rs}(\partial_r \mathcal{S}_{\gamma\beta})\big(\nu_j(\nabla_{\tan} f_\alpha)_s\big)(x) + \big(\mathcal{D}((\nabla_{\tan} f)^j)\big)_\gamma(x),\end{aligned} \tag{1.8.122}$$

where $(\nabla_{\tan} f)^j$ is the $\mathbb{C}^M$-valued function whose α-th component is $(\nabla_{\tan} f_\alpha)_j$. Granted (1.8.121) and (1.8.122), the same type of argument as in (1.5.257)-(1.5.260) proves (1.8.120). □

Going further, the reader is reminded that *homogeneous* Sobolev space of order one have been introduced in [69, Definition 11.5.3] on suitable Ahlfors regular sets (cf. (A.0.127)-(A.0.128)).

Theorem 1.8.9 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set with the property that $\partial\Omega$ is a* UR *set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider a homogeneous, weakly elliptic, second-order $M \times M$ system L in $\mathbb{R}^n$, with complex constant coefficients, and recall the modified boundary-to-boundary single layer operator $S_{\rm mod}$ associated with L and Ω as in (1.5.73). Also, pick a coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$ for which $L_A = L$, and bring in the operator $K^\#_{A^\top}$ associated as in (1.3.72) with the coefficient tensor $A^\top$. Finally, fix an integrability exponent $p \in (1,\infty)$. Then the operator*

$$S_{\rm mod} : \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M \tag{1.8.123}$$

is well defined, linear, and bounded, when the target space is endowed with the induced by semi-norm (A.0.128). In addition,

$$\begin{aligned}&\big[S_{\rm mod}\big] : \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^M \ \text{ defined as}\\ &\big[S_{\rm mod}\big] f := \big[S_{\rm mod} f\big] \in \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^M, \quad \forall f \in \big[L^p(\partial\Omega,\sigma)\big]^M\end{aligned} \tag{1.8.124}$$

is also a well-defined, linear, and bounded operator, when the quotient space is endowed with the natural semi-norm[24] *introduced in* [69, (11.5.138)].

Furthermore, with $\mathscr{S}_{\rm mod}$ *denoting the modified version of the single layer operator acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *as in* (1.5.50)*, for each aperture parameter* $\kappa > 0$ *there exists some constant* $C = C(\Omega, L, n, p, \kappa) \in (0, \infty)$ *with the property that for each function* $f \in \big[L^p(\partial\Omega, \sigma)\big]^M$ *and each truncation parameter* $\varepsilon \in (0, \infty)$ *one has:*

$$
\begin{gathered}
\mathscr{S}_{\rm mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathscr{S}_{\rm mod} f\big) = 0 \ \textit{in} \ \Omega, \\
\mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big) \in L^p(\partial\Omega, \sigma), \ \big\|\mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[L^p(\partial\Omega,\sigma)]^M}, \\
\mathcal{N}_\kappa^\varepsilon(\mathscr{S}_{\rm mod} f) \in L^q_{\rm loc}(\partial\Omega, \sigma) \ \textit{for each} \ q \in \big(0, \tfrac{n-1}{n-2}\big), \\
\nabla(\mathscr{S}_{\rm mod} f)\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \ \textit{exists at } \sigma\textit{-a.e. point on} \ \partial_*\Omega, \\
\partial_\nu^A \mathscr{S}_{\rm mod} f = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f \ \textit{ at } \sigma\textit{-a.e. point on} \ \partial_*\Omega, \\
\textit{and} \ \Big(\big(\mathscr{S}_{\rm mod} f\big)\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\Big)(x) = (S_{\rm mod} f)(x) \ \textit{at } \sigma\textit{-a.e. point} \ x \in A_\kappa(\partial\Omega), \\
\textit{in particular } (\textit{cf.} \ [68, \text{Proposition 8.8.4}]), \textit{ at } \sigma\textit{-a.e. point } x \in \partial_*\Omega.
\end{gathered}
\tag{1.8.125}
$$

Proof The claims pertaining to the operator (1.8.123) are justified based on Proposition 1.2.6, [70, Theorem 1.4.2], (1.5.83), [68, (7.7.106)], and the fact that the singular integral operators defined in (1.5.82) induce bounded mappings (cf. [70, (2.3.20)] and [70, Theorem 1.4.2])

$$
T^\#_{jk} : \big[L^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^p(\partial_*\Omega, \sigma)\big]^M, \qquad \forall\, j, k \in \{1, \dots, n\}. \tag{1.8.126}
$$

Having dealt with (1.8.123), the claims regarding (1.8.124) readily follow. The properties in the first two lines of (1.8.125) are seen from item *(xi)* of Theorem 1.5.1 and [68, (7.7.106)]. The membership in the third line of (1.8.125) is a consequence of [68, (7.7.106)] and (1.5.69)-(1.5.70). The existence of the nontangential boundary trace in the fourth line of (1.8.125) comes from (1.5.53). Next, the jump-formula in the fifth line of (1.8.125) comes from (1.5.58), bearing [68, (7.7.106)] in mind. Finally, the boundary trace formula claimed in the last two lines of (1.8.125) is a consequence of (1.5.80) and [68, (7.7.106)]. □

On boundaries of planar chord-arc domains, the invertibility properties of boundary layer potentials for the Laplacian and related to one another in the manner described in the proposition below (see also Corollary 2.3.15 in this regard).

[24] recall from [69, Proposition 11.5.14] that said semi-norm is actually a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

Proposition 1.8.10 *Let $\Omega \subseteq \mathbb{R}^2$ be a chord-arc domain with unbounded boundary and abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$. Recall the principal-value harmonic double layer K and its (real) transpose $K^\#$ from (1.6.71)-(1.6.72). Also, bring in the two-dimensional modified boundary-to-boundary single layer operator $S_{\rm mod}$ from (1.6.89). Finally, fix $p, p' \in (1, \infty)$ with $1/p + 1/p' = 1$. Then*

the operators $\pm\frac{1}{2}I + K$ are isomorphisms on $L^p(\partial\Omega, \sigma)$, if and only if the operators $\pm\frac{1}{2}I + K^\#$ are isomorphisms on $L^{p'}(\partial\Omega, \sigma)$, if and only if $\big[S_{\rm mod}\big] : L^{p'}(\partial\Omega, \sigma) \to \dot{L}_1^{p'}(\partial\Omega, \sigma)/\sim$ (defined as in (1.8.124)) is an isomorphism. (1.8.127)

Proof The first equivalence in (1.8.127) follows by duality (cf. item *(iii)* in Theorem 1.5.1), while the second equivalence in (1.8.127) is a consequence of (1.8.124), (1.6.91), and [69, Proposition 11.5.15]. □

Next we take a look at injectivity properties of the modified boundary-to-boundary single layer operator in the context of (1.5.77), when the underlying domain is a *half-space*.

Proposition 1.8.11 *Consider a homogeneous, complex constant coefficient, second-order $M \times M$ system L in $\mathbb{R}^n$ (where $M, n \in \mathbb{N}$ with $n \geq 2$), satisfying the Legendre-Hadamard (strong) ellipticity condition* [70, (1.3.4) in Definition 1.3.2]. *Given a half-space $\Omega \subseteq \mathbb{R}^n$, abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and consider the modified boundary-to-boundary single layer operator $S_{\rm mod}$ associated with L and Ω as in (1.5.73). Finally, fix an integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$. Then for each function $f \in \big[L^p(\partial\Omega, w\sigma)\big]^M$ one has*

$$S_{\rm mod} f \text{ is a constant on } \partial\Omega \text{ if and only if } f = 0. \tag{1.8.128}$$

Proof Making a translation and a rotation, there is no loss of generality in assuming that $\Omega = \mathbb{R}^n_+$. Assume this is the case and consider a function $f \in \big[L^p(\partial\Omega, w\sigma)\big]^M$ with the property that $S_{\rm mod} f$ is constant on $\partial\Omega$. Also, fix an aperture parameter $\kappa \in (0, \infty)$ and some arbitrary truncation parameter $\varepsilon > 0$. If we define $\Omega_\pm := \mathbb{R}^n_\pm$ and set $u_\pm := \mathscr{S}_{\rm mod} f$ in $\Omega_\pm$ then the current assumptions, item *(xi)* in Theorem 1.5.1, and [68, Proposition 8.4.9] imply that

$$\begin{cases} u_\pm \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M, \quad Lu_\pm = 0 \text{ in } \Omega_\pm, \\ \mathcal{N}_\kappa(\nabla u_\pm) \in L^p(\partial\Omega, w\sigma), \ \mathcal{N}_\kappa^\varepsilon u_\pm \in L^p_{\rm loc}(\partial\Omega, w\sigma), \\ u_\pm\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}}, \ (\nabla u_\pm)\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \\ \text{and } u_\pm\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} = c \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \end{cases} \tag{1.8.129}$$

for some constant $c \in \mathbb{C}^M$. Introduce $w_j^\pm := \partial_j u_\pm$ in $\Omega_\pm$ with $1 \leq j \leq n-1$. Then for each $j \in \{1, \dots, n-1\}$ we have

$$\begin{cases} w_j^\pm \in \big[\mathscr{C}^\infty(\Omega_\pm)\big]^M, \quad Lw_j^\pm = 0 \text{ in } \Omega_\pm, \\ \mathcal{N}_\kappa w_j^\pm \in L^p(\partial\Omega, w\sigma), \text{ and} \\ w_j^\pm\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = 0 \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \end{cases} \tag{1.8.130}$$

To justify the property in the last line of (1.8.130) write (with $\nu = (0,\dots,0,-1) \in \mathbb{R}^n$ denoting the outward unit normal to Ω)

$$\begin{aligned} w_j^\pm\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} &= (\partial_j u_\pm)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = -\nu_n(\partial_j u_\pm)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \\ &= \nu_j(\partial_n u_\pm)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} - \nu_n(\partial_j u_\pm)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \\ &= \partial_{\tau_{jn}}\Big(u_\pm\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) = 0 \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \end{aligned} \tag{1.8.131}$$

with the final equality a consequence of (1.8.129) and [69, Proposition 11.3.2]. Since the system L satisfies the Legendre-Hadamard (strong) ellipticity condition, from (1.8.130) and the uniqueness result for the weighted Dirichlet Problem for L in the half-space proved in [61] we conclude that $w_j^\pm = 0$ in $\Omega_\pm$ for each $j \in \{1,\dots,n-1\}$. Hence,

$$\partial_j u_\pm = 0 \text{ in } \Omega_\pm \text{ for each } j \in \{1,\dots,n-1\}. \tag{1.8.132}$$

Pick now

$$A = \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\le j,k\le n \\ 1\le\alpha,\beta\le M}} \in \mathfrak{A}_L \tag{1.8.133}$$

and define

$$A_{jk} = \big(a_{jk}^{\alpha\beta}\big)_{1\le\alpha,\beta\le M} \in \mathbb{C}^{M\times M} \text{ for each } j,k \in \{1,\dots,n\}. \tag{1.8.134}$$

Thus, $L = A_{jk}\partial_j\partial_k$ and from the first line in (1.8.129) and (1.8.132) we conclude that, on the one hand,

$$0 = Lu_\pm = A_{jk}\partial_j\partial_k u_\pm = A_{nn}\partial_n\partial_n u_\pm \text{ in } \Omega_\pm. \tag{1.8.135}$$

On the other hand, the fact that L is weakly elliptic (itself a consequence of the Legendre-Hadamard strong ellipticity condition [70, (1.3.4) in Definition 1.3.2]) implies that its $M \times M$ symbol (or characteristic) matrix

$$L(\xi) = -\xi_j\xi_k A_{jk}, \qquad \xi = (\xi_j)_{1\le j\le n} \in \mathbb{R}^n, \tag{1.8.136}$$

satisfies

$$\det\big[L(\xi)\big] \neq 0, \qquad \forall\,\xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.8.137}$$

In particular, choosing $\xi := \mathbf{e}_n = (0,\dots,0,1) \in \mathbb{R}^n$ shows that $A_{nn} = -L(\mathbf{e}_n)$ is an invertible $M \times M$ matrix. Keeping this in mind we then conclude from (1.8.135) that

$$\partial_n \partial_n u_\pm \ \text{ in } \ \Omega_\pm. \tag{1.8.138}$$

In concert, (1.8.132) and (1.8.138) prove that $\nabla(\partial_n u_\pm) = 0$ in $\Omega_\pm$, hence $\partial_n u_\pm$ is a constant in $\Omega_\pm$, say

$$\partial_n u_\pm = c_\pm \in \mathbb{C}^M \ \text{ in } \ \Omega_\pm. \tag{1.8.139}$$

Consequently, (1.8.132), (1.8.139), and the fact that $\nu = (0, \dots, 0, -1) \in \mathbb{R}^n$ entail

$$\begin{aligned} \partial_\nu^A u_\pm &= \nu_j A_{jk} (\partial_k u_\pm)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} := \pm \nu_n A_{nn} (\partial_n u_\pm)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \\ &= \pm A_{nn} c_\pm \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \end{aligned} \tag{1.8.140}$$

Since, as seen from (1.5.58) (keeping [68, (7.7.104)] in mind), $f = -\partial_\nu^A u_+ - \partial_\nu^A u_-$ at σ-a.e. point on $\partial\Omega$, we deduce from (1.8.140) that f is a constant function on $\partial\Omega$. In view of the fact that $f \in \big[L^p(\partial\Omega, w\sigma)\big]^M$ this ultimately forces $f = 0$, as desired. This establishes the right-pointing implication in (1.8.128) and the opposite implication is trivial. □

Moving on, we take on the task of establishing basic functional analytic properties of the modified boundary-to-domain double layer potential operator acting on homogeneous Sobolev spaces defined on the boundary of a UR domain.

Theorem 1.8.12 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a* UR *domain. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. In addition, for some $M \in \mathbb{N}$, let $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1 \leq r,s \leq n \\ 1 \leq \alpha,\beta \leq M}}$ be a complex coefficient tensor with the property that*

$$L := \big(a_{rs}^{\alpha\beta} \partial_r \partial_s\big)_{1 \leq \alpha,\beta \leq M} \tag{1.8.141}$$

(i.e., $L := L_A$) is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Also, let $E = (E_{\gamma\beta})_{1 \leq \gamma,\beta \leq M}$ be the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2]. *In this setting, recall the modified version of the double layer operator $\mathcal{D}_{\text{mod}}$ acting on functions from $\Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M$ as in* (1.8.6)*. Finally, fix some integrability exponent $p \in (1, \infty)$ along with an aperture parameter $\kappa \in (0, \infty)$.*

Then there exists some constant $C = C(\Omega, A, n, p, \kappa) \in (0, \infty)$ with the property that for each function $f \in \big[\dot{L}_1^p(\partial\Omega, \sigma)\big]^M$ it follows that

$$\begin{gathered} \mathcal{D}_{\text{mod}} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathcal{D}_{\text{mod}} f\big) = 0 \ \text{ in } \ \Omega, \\ \big(\mathcal{D}_{\text{mod}} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ and } \ \big(\nabla \mathcal{D}_{\text{mod}} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exist at } \sigma\text{-a.e. point on } \partial\Omega, \\ \mathcal{N}_\kappa\big(\nabla \mathcal{D}_{\text{mod}} f\big) \in L^p(\partial\Omega, \sigma) \ \text{ and } \ \big\|\mathcal{N}_\kappa\big(\nabla \mathcal{D}_{\text{mod}} f\big)\big\|_{L^p(\partial\Omega,\sigma)} \leq C \|f\|_{[\dot{L}_1^p(\partial\Omega,\sigma)]^M}. \end{gathered} \tag{1.8.142}$$

In fact, for each function $f \in \big[\dot{L}_1^p(\partial\Omega, \sigma)\big]^M$ one has

$$\big(\mathcal{D}_{\rm mod} f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{\rm mod}\big) f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{1.8.143}$$

where I is the identity operator on $\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$, and $K_{\rm mod}$ is the modified boundary-to-boundary double layer potential operator from (1.8.24)-(1.8.25). *Also,*

$$\begin{array}{l}\text{if } p > n-1 \text{ then the operator } \mathcal{D}_{\rm mod} : \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M \to \big[\dot{\mathscr{C}}^\eta(\overline{\Omega})\big]^M \text{ is}\\ \text{well defined, linear, and bounded, with } \eta := 1 - \frac{n-1}{p} \in (0,1), \text{ provided}\\ \text{either } \Omega \subseteq \mathbb{R}^n \text{ is an open set satisfying a two-sided local John condi-}\\ \text{tion and whose boundary is Ahlfors regular, or } \Omega \text{ is simultaneously a}\\ \text{uniform domain and a UR domain in } \mathbb{R}^n.\end{array} \tag{1.8.144}$$

Moreover, given any function $f = (f_\alpha)_{1\leq\alpha\leq M}$ belonging to the homogeneous boundary Sobolev space $\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$, at σ-a.e. point $x \in \partial\Omega$ one has

$$\begin{aligned}&\big(\partial^A_\nu(\mathcal{D}_{\rm mod} f)\big)(x) \\ &= \Bigg(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y)\big(\partial_{\tau_{js}} f_\alpha\big)(y)\, d\sigma(y)\Bigg)_{1\leq\mu\leq M}\end{aligned} \tag{1.8.145}$$

where the conormal derivative is considered as in (A.0.184).

Furthermore, the operator

$$\begin{array}{l}\partial^A_\nu \mathcal{D}_{\rm mod} : \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^p(\partial\Omega,\sigma)\big]^M \ \text{ defined as}\\ \big(\partial^A_\nu \mathcal{D}_{\rm mod}\big) f := \partial^A_\nu(\mathcal{D}_{\rm mod} f) \text{ for each } f \in \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M\end{array} \tag{1.8.146}$$

is well defined, linear, and bounded, when the domain space is equipped with the semi-norm induced by (A.0.128). *In addition, for each $q \in [1,\infty)$, the operator* (1.8.146) *is an extension of the assignment* (*cf.* [69, (11.5.58)] *and* (1.5.30)-(1.5.31))

$$\big[L^{q,p}_1(\partial\Omega,\sigma)\big]^M \ni f \longmapsto \big(\partial^A_\nu \mathcal{D}\big) f := \partial^A_\nu(\mathcal{D} f) \in \big[L^p(\partial\Omega,\sigma)\big]^M. \tag{1.8.147}$$

Finally,

$$\begin{array}{l}\big[\partial^A_\nu \mathcal{D}_{\rm mod}\big] : \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[L^p(\partial\Omega,\sigma)\big]^M \ \text{ defined as}\\ \big[\partial^A_\nu \mathcal{D}_{\rm mod}\big][f] := \partial^A_\nu(\mathcal{D}_{\rm mod} f) \text{ for each } f \in \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M\end{array} \tag{1.8.148}$$

is a well-defined, linear, and bounded operator, when the quotient space is equipped with the natural semi-norm[25] *introduced in* [69, (11.5.138)].

[25] [69, Proposition 11.5.14] tells us that this semi-norm is fact a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

Proof For each $f \in \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$ the jump-formula (1.8.143) is seen from (1.8.27) (keeping in mind (A.0.127)). The claims in (1.8.142) are consequences of (A.0.127), (1.8.7), (1.8.27), (1.8.20), (1.8.13), [70, (2.4.9)], and [70, Theorem 1.4.2]. Also, the claim in (1.8.144) is a consequence of (1.8.142) and [68, Corollary 8.6.8] in the case when Ω is simultaneously a uniform domain and a UR domain, and is seen from [69, (11.5.60), (11.5.168), (11.5.201)] together with (1.8.15) in the case when $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular (in which scenario, Ω is known to be a UR domain; see [68, (5.11.27)]).

Going further, formula (1.8.145) is a direct consequence of (1.8.83) and [69, Definition 11.5.3] (cf. (A.0.127)-(A.0.128)). In turn, having established (1.8.145), the claims made in relation to (1.8.146) follow with the help of [70, Theorem 2.3.2] and [70, Theorem 1.4.2]. Also, the fact that the operator (1.8.146) is an extension of the assignment (1.8.147) is clear from [69, (11.5.58)], (1.5.30)-(1.5.31), and (1.5.29).

Finally, the claims pertaining to (1.8.148) are consequences of what we have proved so far and (1.8.10), which implies that $\partial^A_\nu \mathcal{D}_{\text{mod}}$ annihilates constants. □

Remark 1.8.13 *The results in Theorem 1.8.12 are applicable to all modified boundary-to-boundary double layer potential operators $\mathcal{D}_{\text{mod}}$, K_{mod} described in Examples 1.8.4-1.8.7.*

Moreover, the operator $\partial^A_\nu \mathcal{D}_{\text{mod}}$ from (1.8.146) *vanishes identically when $\mathcal{D}_{\text{mod}}$ is as in* (1.8.108) (*see the last part in Remark 1.4.6*). *The same the peculiarity* (*i.e., that the operator $\partial^A_\nu \mathcal{D}_{\text{mod}}$ from vanishes identically*) *is present when $\mathcal{D}_{\text{mod}}$ is as in* (1.8.111) (*see* (1.4.33)).

Our next result deals with the modified boundary-to-boundary double layer potential operator K_{mod} associated with a second-order weakly elliptic $M \times M$ system L and a domain $\Omega \subseteq \mathbb{R}^n$ satisfying suitable assumptions. The aim here is to identify geometric settings in which the homogeneous Sobolev space $\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$ is invariant under the action of K_{mod}.

Theorem 1.8.14 *Let $\Omega \subseteq \mathbb{R}^n$* (*where $n \in \mathbb{N}$ satisfies $n \geq 2$*) *be a* UR *domain, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let $L = \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ be a homogeneous, weakly elliptic, constant* (*complex*) *coefficient, second-order $M \times M$ system in $\mathbb{R}^n$* (*for some $M \in \mathbb{N}$*). *In this context, consider the modified boundary-to-boundary double layer potential operator K_{mod} from* (1.8.24)-(1.8.25). *Finally, select an integrability exponent $p \in (1,\infty)$. Then the following statements are true:*

(1) If Ω is also assumed to satisfy a local John condition, then the operator

$$K_{\text{mod}} : \big[\dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\text{loc}}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\text{loc}}(\partial\Omega,\sigma)\big]^M \tag{1.8.149}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.128). *As a consequence of this and* (1.8.28),

$$
\begin{aligned}
\big[K_{\text{mod}}\big] : \Big[\Big(\dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\text{loc}}(\partial\Omega,\sigma)\Big)\Big/\sim\Big]^M \\
\longrightarrow \Big[\Big(\dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\text{loc}}(\partial\Omega,\sigma)\Big)\Big/\sim\Big]^M
\end{aligned}
\tag{1.8.150}
$$

defined as

$$
\begin{aligned}
&\big[K_{\text{mod}}\big][f] := \big[K_{\text{mod}} f\big] \in \Big[\Big(\dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\text{loc}}(\partial\Omega,\sigma)\Big)\Big/\sim\Big]^M \\
&\quad \text{for each function } f \in \big[\dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\text{loc}}(\partial\Omega,\sigma)\big]^M
\end{aligned}
\tag{1.8.151}
$$

is also a well-defined linear and bounded operator, when the quotient spaces are equipped with the natural semi-norm introduced in [69, (11.5.138)]. *In addition, with U_{jk} for $j,k \in \{1,\dots,n\}$ denoting the family of singular integral operators defined in* (1.5.251)*, one has*

$$
\begin{aligned}
&\partial_{\tau_{jk}}\big(K_{\text{mod}} f\big) = K(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f) \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega \\
&\text{for each } f \in \big[\dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\text{loc}}(\partial\Omega,\sigma)\big]^M \text{ and each } j,k \in \{1,\dots,n\}.
\end{aligned}
\tag{1.8.152}
$$

(2) *Impose the stronger assumption that Ω is an* NTA *domain with an Ahlfors regular boundary*[26]*. Then the operator*

$$
K_{\text{mod}} : \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M
\tag{1.8.153}
$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.128). *As a corollary of* (1.8.153) *and* (1.8.28)*, the following is a well-defined linear and bounded*[27] *operator:*

$$
\begin{aligned}
&\big[K_{\text{mod}}\big] : \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^M \ \text{ defined as} \\
&\big[K_{\text{mod}}\big][f] := \big[K_{\text{mod}} f\big] \in \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^M, \quad \forall\, f \in \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M.
\end{aligned}
\tag{1.8.154}
$$

(3) *Strengthen the original hypotheses by now assuming that Ω is an open set in $\mathbb{R}^n$ satisfying a two-sided local John condition and whose boundary is Ahlfors regular*[28]*. Then, if U_{jk} with $j,k \in \{1,\dots,n\}$ is the family of singular integral operators defined in* (1.5.251)*, one has*

$$
\begin{aligned}
&\partial_{\tau_{jk}}\big(K_{\text{mod}} f\big) = K(\partial_{\tau_{jk}} f) + U_{jk}(\nabla_{\tan} f) \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega \\
&\quad \text{for each } f \in \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M \text{ and each } j,k \in \{1,\dots,n\}.
\end{aligned}
\tag{1.8.155}
$$

---

[26] a scenario in which Ω is known to be a UR domain; cf. [68, (5.11.5)]

[27] if $\partial\Omega$ is unbounded then the semi-norm introduced in [69, (11.5.138)] endowing the quotient spaces in (1.8.154) is actually a genuine norm and the homogeneous Sobolev spaces of order one, modulo constants, becomes Banach spaces; see [69, Proposition 11.5.14]

[28] in which scenario, Ω is known to be a UR domain; see [68, (5.11.27)]

It has been shown in [103] that

$$\text{if } \Omega \subseteq \mathbb{R}^n \text{ is an open set satisfying a two-sided corkscrew condition and whose boundary is Ahlfors regular, then } \Omega \text{ satisfies the local John condition if and only if } \Omega \text{ satisfies the local Harnack chain condition.} \tag{1.8.156}$$

As a consequence of (1.8.156) and the fact that the local John condition implies the corkscrew condition we then obtain the following result:

$$\text{if } \Omega \subseteq \mathbb{R}^n \text{ is an open set whose boundary is Ahlfors regular, then } \Omega \text{ satisfies a two-sided local John condition if and only if } \Omega \text{ is a two-sided NTA domain.} \tag{1.8.157}$$

In view of this, we conclude that if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular then the operator K_{mod} is still well defined, linear, bounded in the context of (1.8.153), and (1.8.154) continues to be a well-defined linear and bounded operator.

Let us now present the proof of Theorem 1.8.14.

Proof of Theorem 1.8.14 Pick an aperture parameter $\kappa \in (0, \infty)$. First, work under the assumption that $\Omega \subseteq \mathbb{R}^n$ is a UR domain satisfying a local John condition. Consider an arbitrary function

$$f \in \big[\dot{L}^p_1(\partial\Omega, \sigma) \cap L^p_{\text{loc}}(\partial\Omega, \sigma)\big]^M \tag{1.8.158}$$

and define $u := \mathcal{D}_{\text{mod}} f$ in Ω. Then $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ (cf. (1.8.7)), and the jump-formula (1.8.143) gives

$$u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_{\text{mod}}\big) f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{1.8.159}$$

From (1.8.142) we also know that

$$\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma) \ \text{ and } \ \big\|\mathcal{N}_\kappa(\nabla u)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[\dot{L}^p_1(\partial\Omega,\sigma)]^M} \tag{1.8.160}$$

for some constant $C \in (0, \infty)$ independent of f. Additionally, from (1.8.22) (whose applicability in the present setting is guaranteed by (1.8.158) and the current geometric assumptions on Ω) we know that for each truncation parameter $\varepsilon \in (0, \infty)$ we have

$$\mathcal{N}^\varepsilon_\kappa u \in L^p_{\text{loc}}(\partial\Omega, \sigma). \tag{1.8.161}$$

In view of these properties, we may reason as in the second half of the proof of [69, Proposition 11.5.12] and obtain that

$$u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ belongs to the space } \big[\dot{L}^p_1(\partial\Omega, \sigma) \cap L^p_{\text{loc}}(\partial\Omega, \sigma)\big]^M \text{ and } \Big\|u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big\|_{[\dot{L}^p_1(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[\dot{L}^p_1(\partial\Omega,\sigma)]^M} \tag{1.8.162}$$

for some constant $C \in (0, \infty)$ independent of f. In concert with (1.8.159), this ultimately shows that

$$\begin{aligned} &K_{\rm mod} f \ \text{ belongs to the space } \big[\dot{L}^p_1(\partial\Omega, \sigma) \cap L^p_{\rm loc}(\partial\Omega, \sigma)\big]^M \\ &\qquad \text{and } \big\|K_{\rm mod} f\big\|_{[\dot{L}^p_1(\partial\Omega,\sigma)]^M} \le C\|f\|_{[\dot{L}^p_1(\partial\Omega,\sigma)]^M}, \end{aligned} \tag{1.8.163}$$

from which the claims pertaining to (1.8.149) follow. In turn, the claims regarding the operator (1.8.154) are readily seen from what we have just proved, (1.8.28), and definitions. Finally, formula (1.8.155) is implied by Proposition 1.8.8 (bearing in mind (1.8.158) and [69, Definition 11.5.3]; cf. (A.0.127)-(A.0.128)). This takes care of item *(1)*.

To deal with the claims in item *(2)*, in place of the original geometric assumptions let us now assume that $\Omega \subseteq \mathbb{R}^n$ is an NTA domain with an upper Ahlfors regular boundary. These hypotheses then guarantee (cf. [68, (5.10.24)]) that Ω is a UR domain. Pick an arbitrary function $f \in \big[\dot{L}^p_1(\partial\Omega, \sigma)\big]^M$, and define $u := \mathcal{D}_{\rm mod} f$ in Ω. Then u continues to enjoy the same properties as before, up to (and including) (1.8.160). Granted these properties, we may invoke [69, Proposition 11.5.12] to conclude that

$$\begin{aligned} &K_{\rm mod} f \ \text{ belongs to the space } \big[\dot{L}^p_1(\partial\Omega, \sigma)\big]^M \\ &\text{and } \big\|K_{\rm mod} f\big\|_{[\dot{L}^p_1(\partial\Omega,\sigma)]^M} \le C\|f\|_{[\dot{L}^p_1(\partial\Omega,\sigma)]^M}, \end{aligned} \tag{1.8.164}$$

for some constant $C \in (0, \infty)$ independent of f. From this point on, the proof proceeds as before.

As regards item *(3)*, work under the stronger assumptions that Ω is an open set in $\mathbb{R}^n$ satisfying a two-sided local John condition and whose boundary is Ahlfors regular (as noted in [68, (5.11.27)], these hypotheses guarantee that Ω is a UR domain). Then the desired result is a direct consequence of what we have proved in item *(1)* and [69, Lemma 11.5.4]. □

Remark 1.8.15 *The results in Theorem 1.8.14 are applicable to all modified boundary-to-boundary double layer potential operators $K_{\rm mod}$ described in Examples 1.8.4-1.8.7.*

The following result augments Proposition 1.2.1, through the consideration of homogeneous Sobolev spaces. More general results of this type are contained in item *(9)* of Theorem 5.1.1.

Proposition 1.8.16 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \ge 2$) be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, consider a complex-valued function $b \in L^1_{\rm loc}(\mathbb{R}^n, \mathcal{L}^n)$ such that $b\big|_{\mathbb{R}^n\setminus\{0\}} \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ where $N = N(n) \in \mathbb{N}$ is a sufficiently large number, and with the property that ∇b is odd and positive homogeneous of degree $1-n$ in $\mathbb{R}^n \setminus \{0\}$. For each pair of indices $j, k \in \{1, \dots, n\}$ introduce the modified* *boundary-to-domain "tangential" integral operator* (*compare with* (1.2.1)) *acting on each function* $f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ *according to*

$$\mathcal{T}_{jk}^{\rm mod} f(x) := \int_{\partial\Omega} \Big\{ \nu_j(y)\big[(\partial_k b)(x-y) - \nu_k(y)(\partial_j b)(x-y)\big] - \big[(\partial_k b)(-y) - \nu_k(y)(\partial_j b)(-y)\big] \cdot \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(0,1)}}(y) \Big\} f(y)\, {\rm d}\sigma(y) \tag{1.8.165}$$

at each $x \in \Omega$. *Finally, fix an aperture parameter* $\kappa > 0$ *and pick an integrability exponent* $p \in (1,\infty)$.

Then there exists some constant $C = C(\Omega, b, n, p, \kappa) \in (0,\infty)$ *with the property that for each function* $f \in \dot{L}^p_1(\partial\Omega,\sigma)$ *it follows that*

$$\begin{gathered} \mathcal{T}_{jk}^{\rm mod} f \in \mathscr{C}^{N-1}(\Omega), \quad \big\|\mathcal{N}_\kappa\big(\nabla \mathcal{T}_{jk}^{\rm mod} f\big)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{\dot{L}^p_1(\partial\Omega,\sigma)}, \\ \big(\mathcal{T}_{jk}^{\rm mod} f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ and } \big(\nabla \mathcal{T}_{jk}^{\rm mod} f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ exist at } \sigma\text{-a.e. point on } \partial\Omega. \end{gathered} \tag{1.8.166}$$

In fact, for each function $f \in \dot{L}^p_1(\partial\Omega,\sigma)$ *one has*

$$(\mathcal{T}_{jk}^{\rm mod} f)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = T_{jk}^{\rm mod} f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{1.8.167}$$

where $T_{jk}^{\rm mod}$ *is the modified version of the operator* T_{jk} *from* (1.2.2). *Specifically, for each function*

$$f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \tag{1.8.168}$$

one defines

$$T_{jk}^{\rm mod} f(x) := \lim_{\varepsilon\to 0^+} \int_{\partial\Omega} \Big\{ \nu_j(y)\big(B_{k,\varepsilon}(x-y) - B_{k,1}(-y)\big) - \nu_k(y)\big(B_{j,\varepsilon}(x-y) - B_{j,1}(-y)\big) \Big\} f(y)\, {\rm d}\sigma(y) \tag{1.8.169}$$

at σ*-a.e. point* $x \in \partial\Omega$, *where*

$$B_{\ell,\varepsilon} := (\partial_\ell b)\cdot \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,\varepsilon)}} \text{ for each } \ell \in \{1,\dots,n\} \text{ and } \varepsilon > 0. \tag{1.8.170}$$

Under the additional assumption that Ω *satisfies a local John condition, the operator*

$$T_{jk}^{\rm mod} : \dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\rm loc}(\partial\Omega,\sigma) \longrightarrow \dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\rm loc}(\partial\Omega,\sigma) \tag{1.8.171}$$

is well defined, linear, and bounded, when the above spaces are equipped with the semi-norm (A.0.128).

Finally, if the given set Ω *is actually an* NTA *domain with an Ahlfors regular boundary*[29] *then the operator*

$$T_{jk}^{\text{mod}} : \dot{L}_1^p(\partial\Omega, \sigma) \longrightarrow \dot{L}_1^p(\partial\Omega, \sigma) \tag{1.8.172}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.128).

Proof The properties listed in (1.8.166) are justified by arguing as in the proof of (1.8.142). With (1.8.166) in hand, the fact that (1.8.171)-(1.8.172) are well-defined, linear, and bounded operators is then established much as in the proof of (1.8.149) and (1.8.153), respectively. □

The boundary-to-domain version of the modified double layer potential operator, along with the boundary-to-domain version of the modified single layer potential operator, play a basic role in the formulation of the following fundamental integral representation result.

Theorem 1.8.17 *Let* $\Omega \subseteq \mathbb{R}^n$ (*where* $n \in \mathbb{N}$, $n \geq 2$) *be an open set with the property that* $\partial\Omega$ *is an Ahlfors regular set; in particular,* Ω *is a set of locally finite perimeter. Denote by* ν *the geometric measure theoretic outward unit normal to* Ω *and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Also, for some* $M \in \mathbb{N}$, *let* $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ *be a complex coefficient tensor with the property that*

$$L := \big(a_{rs}^{\alpha\beta}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M} \tag{1.8.173}$$

(*i.e.,* $L := L_A$) *is a weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$. *In this setting, recall the modified version of the double layer operator* $\mathcal{D}_{\text{mod}}$ *acting on functions from* $\big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ *as in* (1.8.6), *and the modified version of the single layer operator* $\mathcal{S}_{\text{mod}}$ *acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *as in* (1.5.50). *Finally, fix an aperture parameter* $\kappa \in (0, \infty)$, *a truncation parameter* $\varepsilon \in (0, \infty)$, *and consider a function* $u : \Omega \to \mathbb{C}^M$ *satisfying*

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa^\varepsilon u \in L^1_{\text{loc}}(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega \text{ and } u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in \Big[L^1\Big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big)\Big]^M, \\ (\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big). \end{gathered} \tag{1.8.174}$$

In the case when Ω *is an exterior domain make the additional assumption that there exists* $\lambda \in (1, \infty)$ *such that*

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{1.8.175}$$

[29] in particular, this is the case if Ω is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular; cf. (1.8.157)

Then the conormal derivative $\partial_\nu^A u$, extended to the entire topological boundary by setting it to be zero outside $\partial_*\Omega$, belongs to $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ and there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω with the property that

$$u = \mathcal{D}_{\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) - \mathcal{S}_{\text{mod}}\big(\partial_\nu^A u\big) + c_u \ \text{ in } \ \Omega. \tag{1.8.176}$$

In addition, if $\mathcal{D}_{\text{mod}}$, $\mathcal{S}_{\text{mod}}$ are now regarded as operators mapping into functions defined in $\mathbb{R}^n \setminus \overline{\Omega}$, then

$$\mathcal{D}_{\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) - \mathcal{S}_{\text{mod}}\big(\partial_\nu^A u\big) \ \text{ is a locally constant function in } \mathbb{R}^n \setminus \overline{\Omega}. \tag{1.8.177}$$

Moreover, if in place of the second line in (1.8.174) *one assumes that*

$$u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega \ \text{ and } \ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in \Big[L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M, \tag{1.8.178}$$

then in place of (1.8.176) *one may conclude that* (*again, for some $\mathbb{C}^M$-valued locally constant function c_u in Ω, and with the same caveat in the case when Ω is an exterior domain*)

$$u = \mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) - \mathcal{S}_{\text{mod}}\big(\partial_\nu^A u\big) + c_u \ \text{ in } \ \Omega, \tag{1.8.179}$$

where $\mathcal{D}$ is the "ordinary" double layer operator, associated with Ω and A as in (1.3.18).

Also, if the last line in (1.8.174) *is strengthened to*

$$(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega \ \text{ and } \ \mathcal{N}_\kappa(\nabla u) \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\Big) \tag{1.8.180}$$

(*with the weight* $(1+|x|^{n-2})^{-1}$ *replaced by* $\ln(2+|x|)$ *if* $n = 2$) *then in place of* (1.8.176) *one may now conclude* (*once more, for some $\mathbb{C}^M$-valued locally constant function c_u in Ω, and with the imposition of the decay condition* (1.8.175) *when Ω is an exterior domain*) *that*

$$u = \mathcal{D}_{\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) - \mathcal{S}\big(\partial_\nu^A u\big) + c_u \ \text{ in } \ \Omega, \tag{1.8.181}$$

where $\mathcal{S}$ is the "ordinary" single layer operator, associated with Ω and L as in (1.3.6).

Finally, if in place of the second line in (1.8.174) *one now assumes* (1.8.178), *and in place of the last line in* (1.8.174) *one now assumes* (1.8.180), *then one concludes* (*with the same notation and conventions as before*) *that*

$$u = \mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) - \mathcal{S}\big(\partial_\nu^A u\big) + c_u \ \text{ in } \ \Omega. \tag{1.8.182}$$

It is of significance to observe that the locally function c_u appearing in the statement of Theorem 1.8.17 is actually a genuine constant (vector in $\mathbb{C}^M$) when $\partial\Omega$ is compact. Indeed, when Ω is bounded this follows from Theorem 1.3.3, (6.2.175), and (1.5.56), while if Ω is an exterior domain this is implied by [70, Theorem 1.5.7] (since (1.8.176) shows that u is bounded at infinity).

In addition, the locally constant function c_u intervening in the statement of Theorem 1.8.17 is in fact a genuine constant (vector in $\mathbb{C}^M$) whenever $\partial\Omega$ is unbounded and $\mathcal{N}_\kappa u \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$. To justify this, fix two arbitrary distinct points $x_0, x_1 \in \Omega$ along with an arbitrary index $\gamma \in \{1,\dots,n\}$. The idea is to employ the Divergence Theorem in the version recorded in [68, Theorem 1.4.1] for the vector field $\vec{F} = (F_s)_{1\leq s\leq n}$ with components given for each $s \in \{1,\dots,n\}$ by

$$
\begin{aligned}
F_s(y) := a^{\beta\alpha}_{rs}\Big[(\partial_r E_{\gamma\beta}(x_0 - y) - (\partial_r E_{\gamma\beta}(x_1 - y)\Big]u_\alpha(y)&\\
+\Big[E_{\gamma\alpha}(x_0 - y) - E_{\gamma\alpha}(x_1 - y)\Big]a^{\alpha\beta}_{sr}(\partial_r u_\beta)(y)&
\end{aligned}
\tag{1.8.183}
$$

at $\mathcal{L}^n$-a.e. $y \in \Omega$. This gives that the γ-th component of the vector $c_u(x_0) - c_u(x_1)$ is zero, from which the desired conclusion readily follows.

Let us turn now to the proof of Theorem 1.8.17.

Proof of Theorem 1.8.17 Work under the assumption that $u = (u_\beta)_{1\leq\beta\leq M}$ is as in (1.8.174). Since the current hypotheses imply $\mathcal{N}^\varepsilon_\kappa u,\ \mathcal{N}^\varepsilon_\kappa(\nabla u) \in L^1_{\rm loc}(\partial\Omega,\sigma)$, from (1.8.174) and [69, Proposition 11.3.2] we then conclude that the function $f := u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ considered on $\partial_*\Omega$ (cf. [68, (8.8.52)]) belongs to $\big[L^1_{1,{\rm loc}}(\partial_*\Omega,\sigma)\big]^M$ and satisfies

$$
\begin{gathered}
\partial_{\tau_{jk}} f = \nu_j\Big((\partial_k u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) - \nu_k\Big((\partial_j u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)\\
\text{at } \sigma\text{-a.e. point on } \partial_*\Omega, \text{ for each } j,k \in \{1,\dots,n\},
\end{gathered}
\tag{1.8.184}
$$

where $(\nu_1,\dots,\nu_n)$ are the scalar components of the geometric measure theoretic outward unit normal ν to Ω. In concert with (1.8.174) this also entails

$$
\begin{gathered}
f \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M \text{ and } \partial_{\tau_{jk}} f \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M\\
\text{for each pair of indices } j,k \in \{1,\dots,n\}.
\end{gathered}
\tag{1.8.185}
$$

To proceed, abbreviate (cf. (A.0.184))

$$
g := \partial^A_\nu u = \Big(\nu_r\big(a^{\alpha\beta}_{rs}\partial_s u_\beta\big)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)_{1\leq\alpha\leq M} \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega,
\tag{1.8.186}
$$

and note that (1.8.174) together with [68, (8.8.52), (8.9.8), (8.9.44)] ensure that

$$
g \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M.
\tag{1.8.187}
$$

Going further, define

$$
w := \mathcal{D}_{\rm mod} f - \mathcal{S}_{\rm mod}\, g \in \big[\mathscr{C}^\infty(\Omega)\big]^M,
\tag{1.8.188}
$$

and denote by $(w_\gamma)_{1\leq\gamma\leq M}$ the scalar components of w. Also, write $(f_\alpha)_{1\leq\alpha\leq M}$ and $(g_\alpha)_{1\leq\alpha\leq M}$ for the scalar components of f and g, respectively. Finally, denote by $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2]. Then for each index $\ell \in \{1,\dots,n\}$, each index $\gamma \in \{1,\dots,M\}$, and each point $x \in \Omega$ we may compute

$$
\begin{aligned}
(\partial_\ell w_\gamma)(x) &= \partial_\ell\big(\mathcal{D}_{\text{mod}} f\big)_\gamma(x) - \partial_\ell\big(\mathcal{S}_{\text{mod}} g\big)_\gamma(x) \\
&= \int_{\partial_*\Omega} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} f_\alpha)(y)\,\mathrm{d}\sigma(y) \\
&\quad - \int_{\partial_*\Omega} (\partial_\ell E_{\gamma\alpha})(x-y) g_\alpha(y)\,\mathrm{d}\sigma(y) \\
&= \int_{\partial_*\Omega} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)\times \\
&\qquad \times \Big\{\nu_\ell(y)\big((\partial_s u_\alpha)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(y) - \nu_s(y)\big((\partial_\ell u_\alpha)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(y)\Big\}\,\mathrm{d}\sigma(y) \\
&\quad - \int_{\partial_*\Omega} (\partial_\ell E_{\gamma\alpha})(x-y)\nu_r(y) a_{rs}^{\alpha\beta}\big((\partial_s u_\beta)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(y)\,\mathrm{d}\sigma(y) \\
&= (\partial_\ell u_\gamma)(x),
\end{aligned}
\tag{1.8.189}
$$

where the first equality above comes from (1.8.188), the second equality uses (1.8.12)-(1.8.13), (1.8.185), (1.5.51), (1.8.187), the third equality utilizes (1.8.184), and the final equality is provided by [70, (1.5.230)] (bearing in mind the properties in (1.8.174)). From (1.8.189) we then conclude that $\nabla w = \nabla u$ in Ω, which goes to show that the difference $c_u := u - w$ is a $\mathbb{C}^M$-valued locally constant function in Ω. The proof of (1.8.176) is therefore complete. The claim in (1.8.177) is established similarly, now making use of [70, (1.5.232)].

Next, that (1.8.179) holds under the assumptions made in (1.8.178) in place of the second line in (1.8.174) is implied by (1.8.176) and (1.8.8). Finally, (1.8.181) and (1.8.182) are dealt with similarly, now also taking into account (1.5.56). □

Here is a version of Theorem 1.8.17 in which no size conditions are explicitly imposed on the nontangential boundary trace of the function in question.

Corollary 1.8.18 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an Ahlfors regular domain satisfying a local John condition. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. For some $M \in \mathbb{N}$, let $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a complex coefficient tensor with the property that*

$$
L := \big(a_{rs}^{\alpha\beta}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M} \tag{1.8.190}
$$

(i.e., $L := L_A$) is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. In this setting, recall the modified version of the double layer operator $\mathcal{D}_{\text{mod}}$ acting on functions from

$\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ *as in* (1.8.6), *and the modified version of the single layer operator* $\mathscr{S}_{\rm mod}$ *acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *as in* (1.5.50). *Finally, fix an aperture parameter* $\kappa \in (0,\infty)$, *a truncation parameter* $\varepsilon \in (0,\infty)$, *and consider a function* $u : \Omega \to \mathbb{C}^M$ *satisfying:*

$$\begin{aligned} & u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{ in } \ \Omega, \\ & \mathcal{N}_\kappa^\varepsilon u \in L^q_{\rm loc}(\partial\Omega,\sigma) \ \text{ for some } \ q \in [1,\infty], \\ & \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma) \ \text{ for some } \ p \in (1,\infty), \\ & u\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \ \text{ and } \ (\nabla u)\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \ \text{ exist } \sigma\text{-a.e. on } \partial\Omega. \end{aligned} \tag{1.8.191}$$

In the case when Ω *is an exterior domain make the additional assumption that there exists* $\lambda \in (1,\infty)$ *such that*

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \ \text{ as } \ R \to \infty. \tag{1.8.192}$$

Then

$$\begin{aligned} & u\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \ \text{ belongs to } \ \big[\dot{L}^p_1(\partial\Omega,\sigma) \cap L^{q,p}_{\rm loc}(\partial\Omega,\sigma)\big]^M, \\ & \partial^A_\nu u \ \text{ belongs to } \ \big[L^p(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{1.8.193}$$

and there exists some $\mathbb{C}^M$*-valued locally constant function* c_u *in* Ω *with the property that*

$$u = \mathcal{D}_{\rm mod}\Big(u\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\Big) - \mathscr{S}_{\rm mod}\big(\partial^A_\nu u\big) + c_u \ \text{ in } \ \Omega. \tag{1.8.194}$$

In addition, if $\mathcal{D}_{\rm mod}$, $\mathscr{S}_{\rm mod}$ *are now regarded as operators mapping into functions defined in* $\mathbb{R}^n \setminus \overline{\Omega}$, *then*

$$\mathcal{D}_{\rm mod}\Big(u\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\Big) - \mathscr{S}_{\rm mod}\big(\partial^A_\nu u\big) \ \text{ is a locally constant function in } \mathbb{R}^n \setminus \overline{\Omega}. \tag{1.8.195}$$

Proof Granted the current geometric assumptions, [69, Proposition 11.5.13] applies and guarantees that the first property claimed in (1.8.193) holds. That the second property claimed in (1.8.193) is a consequence of (1.8.191) and [70, Lemma 1.7.3]. Granted these, we then see that all hypotheses of Theorem 1.8.17 are currently satisfied, so (1.8.194) is implied by (1.8.176), and (1.8.195) is implied by (1.8.177).□

We continue by presenting a version of the integral representation formula from Theorem 1.8.17 in an NTA domain with an Ahlfors regular boundary, for a null-solution of a weakly elliptic system such that the nontangential maximal operator of its gradient belongs to a Lebesgue space (a feature, ensuring that the nontangential boundary trace of said function belongs a homogeneous Sobolev space). A remarkable aspect of this result is the lack of explicit demands on the nontangential maximal operator of the function itself.

Theorem 1.8.19 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an* NTA *domain such that $\partial\Omega$ is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix an aperture parameter $\kappa \in (0, \infty)$. Next, let $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ (where $M \in \mathbb{N}$) be a complex coefficient tensor with the property that*

$$L := \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M} \tag{1.8.196}$$

(i.e., $L := L_A$) is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Recall the modified version of the double layer operator $\mathcal{D}_{\rm mod}$ acting on functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ as in (1.8.6), and the modified version of the single layer operator $\mathcal{S}_{\rm mod}$ acting on functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ as in (1.5.50). Finally, consider a function $u : \Omega \to \mathbb{C}^M$ satisfying

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma). \\ \text{for some integrability exponent } p \in (1, \infty), \end{gathered} \tag{1.8.197}$$

and, in the case when Ω is an exterior domain, make the additional assumption that there exists $\lambda \in (1, \infty)$ such that

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |\nabla u| \, \mathrm{d}\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{1.8.198}$$

Then

$$\begin{gathered} u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \text{ belongs to } \big[\dot{L}^p_1(\partial\Omega, \sigma) \cap L^p_{1,\mathrm{loc}}(\partial\Omega, \sigma)\big]^M, \\ (\nabla u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \text{ and } \partial^A_\nu u \text{ belongs to } \big[L^p(\partial\Omega, \sigma)\big]^M, \end{gathered} \tag{1.8.199}$$

and there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω with the property that

$$u = \mathcal{D}_{\rm mod}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) - \mathcal{S}_{\rm mod}\big(\partial^A_\nu u\big) + c_u \text{ in } \Omega. \tag{1.8.200}$$

In addition, if $\mathcal{D}_{\rm mod}$, $\mathcal{S}_{\rm mod}$ are now regarded as operators mapping into functions defined in $\mathbb{R}^n \setminus \overline{\Omega}$, then

$$\mathcal{D}_{\rm mod}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) - \mathcal{S}_{\rm mod}\big(\partial^A_\nu u\big) \text{ is a locally constant function in } \mathbb{R}^n \setminus \overline{\Omega}. \tag{1.8.201}$$

Moreover, if $p \in (1, n-1)$ (which implicitly forces $n \geq 3$) then there exists a constant $c \in \mathbb{C}^M$ such that

$$\begin{gathered} u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - c \in \big[L^{p^*}(\partial\Omega, \sigma)\big]^M \text{ and } \mathcal{N}_\kappa(u - c) \in L^{p^*}(\partial\Omega, \sigma) \\ \text{where } p^* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1} \in (1, \infty). \end{gathered} \tag{1.8.202}$$

As a consequence, if $p \in (1, n-1)$ (*again, this implicitly forces* $n \geq 3$) *and* p^* *is as above then*

$$u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in \left[L^{p^*}_{\text{loc}}(\partial\Omega, \sigma) \cap L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \right]^M \tag{1.8.203}$$

and

$$\mathcal{N}_\kappa u \in L^{p^*}_{\text{loc}}(\partial\Omega, \sigma) \cap L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big). \tag{1.8.204}$$

Proof From the current assumptions and [69, Proposition 11.5.12] we see that $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$ and belongs to $\big[\dot{L}^p_1(\partial\Omega, \sigma) \cap L^p_{1,\text{loc}}(\partial\Omega, \sigma)\big]^M$. From [68, Proposition 8.4.9] we also know that there exists some small $\varepsilon > 0$ such that

$$\mathcal{N}^\varepsilon_\kappa u \in L^p_{\text{loc}}(\partial\Omega, \sigma). \tag{1.8.205}$$

Next, the present hypotheses on Ω ensure (cf. [68, (5.10.24)]) that $\partial\Omega$ is a UR set and Ω satisfies a two-sided corkscrew condition. In particular, Ω is a UR domain. Keeping this in mind, the Fatou-type result from [70, Theorem 3.3.4] guarantees the nontangential boundary trace $(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists (in $\mathbb{C}^{n\cdot M}$) at σ-a.e. point on $\partial\Omega$. In particular, $\partial^A_\nu u$ is well defined and belongs to $\big[L^p(\partial\Omega, \sigma)\big]^M$ (cf. [70, Lemma 1.7.3]). Hence, all conditions in (1.8.174) are satisfied, and this permits us to invoke Theorem 1.8.17 to conclude that (1.8.200) and (1.8.201) hold.

For the remainder of the proof assume that $p \in (1, n-1)$ (hence necessarily $n \geq 3$), and set $p^* := \big(\frac{1}{p} - \frac{1}{n-1}\big)^{-1}$. Collectively, (1.8.205), [68, (8.9.8)], and the last condition in (1.8.197) imply that the function g defined as in [69, (11.5.71)] belongs to $L^p(\partial\Omega, \sigma)$. Having established this, from [68, (5.11.28)] and [69, Lemmas 11.5.8, 11.5.9] we infer the existence of a constant $c \in \mathbb{C}^M$ such that the function

$$u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} - c \ \text{ belongs to } \ \big[L^{p^*}(\partial\Omega, \sigma)\big]^M. \tag{1.8.206}$$

In particular,

$$u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in \big[L^{p^*}(\partial\Omega, \sigma)\big]^M + \mathbb{C}^M \hookrightarrow \left[L^{p^*}_{\text{loc}}(\partial\Omega, \sigma) \cap L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \right]^M. \tag{1.8.207}$$

Also, with $c \in \mathbb{C}^M$ as above we may rely on (1.8.206), (1.8.8), and (1.8.10) to write

$$\mathcal{D}_{\text{mod}}\Big(u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) = \mathcal{D}\Big(u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} - c\Big) + c^{(1)} \ \text{ in } \Omega. \tag{1.8.208}$$

for some constant $c^{(1)} \in \mathbb{C}^M$. To proceed, recall from the second line in (1.8.199) that $\partial^A_\nu u$ belongs to the space $\big[L^p(\partial\Omega, \sigma)\big]^M$ with $p \in (1, n-1)$. Based on this and (1.5.56) we then conclude that there exists some constant $c^{(2)} \in \mathbb{C}^M$ such that

$$\mathcal{S}_{\text{mod}}(\partial^A_\nu u) = \mathcal{S}(\partial^A_\nu u) + c^{(2)} \ \text{ in } \Omega. \tag{1.8.209}$$

Collectively, (1.8.208), (1.8.209), (1.8.200) imply that there exists a $\mathbb{C}^M$-valued locally constant function $\widetilde{c}_u$ in Ω (namely $\widetilde{c}_u := c^{(1)} - c^{(2)}$) with the property that

$$u = \mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - c\big) - \mathcal{S}\big(\partial_\nu^A u\big) + \widetilde{c}_u \ \text{ in } \Omega. \tag{1.8.210}$$

We next distinguish two cases. Assume first that $\partial\Omega$ is unbounded. Then [68, Lemma 5.11.3] gives that Ω is connected so $\widetilde{c}_u$ in (1.8.210) is actually a genuine constant, say $c_u \equiv c_o \in \mathbb{C}^M$. Keeping this in mind, (1.8.210) implies

$$\mathcal{N}_\kappa(u - c_o) \leq \mathcal{N}_\kappa\Big(\mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - c\big)\Big) + \mathcal{N}_\kappa\Big(\mathcal{S}\big(\partial_\nu^A u\big)\Big) \tag{1.8.211}$$

at each point on $\partial\Omega$. From (1.5.1), (2.2.39), and [68, (3.6.27)] we know that

$$\mathcal{N}_\kappa\Big(\mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - c\big)\Big) \in L^{p^*}(\partial\Omega, \sigma) \ \text{ and } \ \mathcal{N}_\kappa\Big(\mathcal{S}\big(\partial_\nu^A u\big)\Big) \in L^{p^*}(\partial\Omega, \sigma). \tag{1.8.212}$$

By combining (1.8.211), (1.8.212), and [68, (8.2.28)] we arrive at the conclusion that

$$\mathcal{N}_\kappa(u - c_o) \in L^{p^*}(\partial\Omega, \sigma). \tag{1.8.213}$$

In view of [68, (8.9.8)], this entails $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - c_o \in \big[L^{p^*}(\partial\Omega, \sigma)\big]^M$ which, in concert with (1.8.206) and the fact that we are presently assuming $\partial\Omega$ to be unbounded, forces $c_o = c$. Having established this, (1.8.202) follows from (1.8.206) and (1.8.213). In turn, (1.8.202) readily implies (1.8.203) and (1.8.204). Finally, consider the case when $\partial\Omega$ is bounded. Much as before, from (1.8.210) and (1.8.212) we see that

$$\mathcal{N}_\kappa(u - \widetilde{c}_u) \in L^{p^*}(\partial\Omega, \sigma). \tag{1.8.214}$$

Since in the current scenario [68, Lemma 5.11.3] implies that Ω has finitely many connected components, the range of the locally constant $\mathbb{C}^M$-valued function $\widetilde{c}_u$ is finite. Keeping in mind that now $\partial\Omega$ has finite measure, (1.8.214) and (1.8.206) imply that $\mathcal{N}_\kappa u \in L^{p^*}(\partial\Omega, \sigma)$ and $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in \big[L^{p^*}(\partial\Omega, \sigma)\big]^M$. Thus, (1.8.202) presently holds with $c := 0$, a scenario in which (1.8.203) and (1.8.204) readily follow from this. □

A result of the same flavor as the integral representation formula from Theorem 1.8.17, but for null-solutions of an injectively elliptic first-order system, is presented in the corollary below. This is akin a higher-dimensional Cauchy integral representation formula, in a very broad geometric setting and under very general analytic assumptions on the function involved.

Corollary 1.8.20 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, be an open set with an Ahlfors regular boundary and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider a homogeneous first-order $N \times M$ system D with constant complex coefficients in $\mathbb{R}^n$ (where $N, M \in \mathbb{N}$) which is injectively elliptic (cf. [70, (1.3.18)]), and suppose $\widetilde{D}$ is a homogeneous first-order $M \times N$ system with constant complex coefficients in $\mathbb{R}^n$ which complements D (i.e., [70, (1.3.21)] holds). In particular, $L := \widetilde{D}D$ is a weakly elliptic second-order $M \times M$ system in $\mathbb{R}^n$. Let $A_{\widetilde{D},D}$ be the coefficient tensor induced by the factorization $\widetilde{D}D$ of* *the system L, defined as in (1.4.32), and bring in the modified version of the double layer operator $\mathcal{D}_{\text{mod}}$ acting on functions from $\big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ as in (1.8.6) for the coefficient tensor $A := A_{\widetilde{D},D}$. Finally, fix an aperture parameter $\kappa \in (0,\infty)$ and suppose $u : \Omega \to \mathbb{C}^M$ is a vector-valued function satisfying*

$$
\begin{aligned}
&u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Du = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa^\varepsilon u \in L^1_{\text{loc}}(\partial\Omega,\sigma),\\
&u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega \text{ and } u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M,\\
&(\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big).
\end{aligned}
\tag{1.8.215}
$$

In the case when Ω is an exterior domain make the additional assumption that there exists $\lambda \in (1,\infty)$ such that

$$
\fint_{B(0,\lambda R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{1.8.216}
$$

Then there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω with the property that

$$
u = \mathcal{D}_{\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) + c_u \text{ in } \Omega. \tag{1.8.217}
$$

In addition, if $\mathcal{D}_{\text{mod}}$ is now regarded as an operator mapping into functions defined in $\mathbb{R}^n \setminus \overline{\Omega}$, then

$$
\mathcal{D}_{\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) \text{ is a locally constant function in } \mathbb{R}^n \setminus \overline{\Omega}. \tag{1.8.218}
$$

Furthermore, if in place of the second line in (1.8.215) one now assumes that

$$
u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega \text{ and } u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in \Big[L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M, \tag{1.8.219}
$$

then in place of (1.8.217) one may conclude that (again, for some $\mathbb{C}^M$-valued locally constant function c_u in Ω, and with the same caveat in the case when Ω is an exterior domain)

$$
u = \mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) + c_u \text{ in } \Omega, \tag{1.8.220}
$$

where $\mathcal{D}$ is the "ordinary" double layer operator, associated with the set Ω and the coefficient tensor $A := A_{\widetilde{D},D}$ as in (1.3.18).

Proof Upon observing that $Lu = \widetilde{D}Du = 0$ in Ω, we may invoke Theorem 1.8.17 for the coefficient tensor $A := A_{\widetilde{D},D}$. Since, as seen from (1.4.33), the conormal derivative $\partial_\nu^A u$ vanishes at σ-a.e. point on $\partial_*\Omega$ (given that we are presently assuming that $Du = 0$ in Ω), it follows that (1.8.176) reduces precisely to (1.8.217). Likewise, (1.8.177) turns into (1.8.218), while (1.8.179) now becomes (1.8.220). □

Here is a version of Corollary 1.8.20 for null-solutions of an injectively elliptic first-order system having the nontangential maximal functions of their gradients in a Lebesgue space.

Corollary 1.8.21 *Let* $\Omega \subseteq \mathbb{R}^n$ (*where* $n \in \mathbb{N}$, $n \geq 2$) *be an* NTA *domain such that* $\partial\Omega$ *is an Ahlfors regular set. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and fix an aperture parameter* $\kappa \in (0,\infty)$ *along with an integrability exponent* $p \in (1,\infty)$. *Next, consider a homogeneous first-order* $N \times M$ *system* D *with constant complex coefficients in* $\mathbb{R}^n$ (*where* $N, M \in \mathbb{N}$) *which is injectively elliptic* (*cf.* [70, (1.3.18)]), *and suppose* $\widetilde{D}$ *is a homogeneous first-order* $M \times N$ *system with constant complex coefficients in* $\mathbb{R}^n$ *which complements* D (*i.e.,* [70, (1.3.21)] *holds*). *In particular,* $L := \widetilde{D}D$ *is a weakly elliptic second-order* $M \times M$ *system in* $\mathbb{R}^n$. *Let* $A_{\widetilde{D},D}$ *be the coefficient tensor induced by the factorization* $\widetilde{D}D$ *of the system* L, *defined as in* (1.4.32), *and bring in the modified version of the double layer operator* $\mathcal{D}_{\text{mod}}$ *acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ *as in* (1.8.6) *for the coefficient tensor* $A := A_{\widetilde{D},D}$. *Finally, consider a function* $u : \Omega \to \mathbb{C}^M$ *satisfying*

$$u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Du = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma). \tag{1.8.221}$$

In the case when Ω *is an exterior domain make the additional assumption that there exists* $\lambda \in (1,\infty)$ *such that*

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{1.8.222}$$

Then

$$\begin{gathered} u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \text{ belongs to } \big[\dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{1,\text{loc}}(\partial\Omega,\sigma)\big]^M, \\ (\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega \text{ and belongs to } \big[L^p(\partial\Omega,\sigma)\big]^{M\cdot n}, \end{gathered} \tag{1.8.223}$$

and there exists some $\mathbb{C}^M$*-valued locally constant function* c_u *in* Ω *with the property that*

$$u = \mathcal{D}_{\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) + c_u \text{ in } \Omega. \tag{1.8.224}$$

In addition, if $\mathcal{D}_{\text{mod}}$ *is now regarded as an operator mapping into functions defined in* $\mathbb{R}^n \setminus \overline{\Omega}$, *then*

$$\mathcal{D}_{\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) \text{ is a locally constant function in } \mathbb{R}^n \setminus \overline{\Omega}. \tag{1.8.225}$$

Proof This follows from Corollary 1.8.20, much as the first main claim in Theorem 1.8.19 was proved by relying on Theorem 1.8.17. □

It is of interest to specialize the integral representation formulas deduced above to the two-dimensional setting. First, Corollary 1.8.20 used with the Cauchy-Riemann operator in the complex plane yields the following result:

Corollary 1.8.22 *Let $\Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$ be an open set with the property that $\partial\Omega$ is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and consider the following modified version of the boundary-to-domain Cauchy integral operator acting on each function*

$$f \in L^1\Big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^2}\Big) \tag{1.8.226}$$

according to

$$\big(\mathscr{C}_{\rm mod} f\big)(z) := \frac{1}{2\pi \mathrm{i}} \int_{\partial_*\Omega} \Big\{\frac{1}{\zeta - z} - \frac{1}{\zeta} \mathbf{1}_{\mathbb{C}\setminus \overline{B(0,1)}}(\zeta)\Big\} f(\zeta)\, \mathrm{d}\zeta \;\; \text{at each } \; z \in \Omega. \tag{1.8.227}$$

Also, consider a holomorphic function $u : \Omega \to \mathbb{C}$ satisfying

$$\begin{gathered} \mathcal{N}_\kappa^\varepsilon u \in L^1_{\rm loc}(\partial\Omega, \sigma) \text{ for some } \kappa \in (0,\infty) \text{ and } \varepsilon \in (0,\infty), \\ u\big|^{\kappa-\rm n.t.}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial_{\rm nta}\Omega \text{ and } u\big|^{\kappa-\rm n.t.}_{\partial\Omega} \in L^1\Big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^2}\Big), \\ (\nabla u)\big|^{\kappa-\rm n.t.}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial_{\rm nta}\Omega, \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^1\Big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\Big). \end{gathered} \tag{1.8.228}$$

In the case when Ω is an exterior domain make the additional assumption that there exists $\lambda \in (1,\infty)$ such that

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |\nabla u|\, \mathrm{d}\mathcal{L}^2 = o(1) \;\text{ as }\; R \to \infty. \tag{1.8.229}$$

Then there exists some complex-valued locally constant function c_u in Ω with the property that

$$u = \mathscr{C}_{\rm mod}\big(u\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big) + c_u \;\text{ in }\; \Omega. \tag{1.8.230}$$

In addition, if $\mathscr{C}_{\rm mod}$ is now regarded as an operator mapping into functions defined in $\mathbb{C} \setminus \overline{\Omega}$, then

$$\mathscr{C}_{\rm mod}\big(u\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big) \text{ is a locally constant function in } \mathbb{C}\setminus\overline{\Omega}. \tag{1.8.231}$$

Moreover, if in place of the second line in (1.8.228) *one now assumes that*

$$u\big|^{\kappa-\rm n.t.}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial_{\rm nta}\Omega \text{ and } u\big|^{\kappa-\rm n.t.}_{\partial\Omega} \in L^1\big(\partial_*\Omega, \tfrac{\sigma(\zeta)}{1+|\zeta|}\big), \tag{1.8.232}$$

then in place of (1.8.230) *one may conclude that* (*again, for some $\mathbb{C}^M$-valued locally constant function c_u in Ω, and with the same caveat in the case when Ω is an exterior domain*)

$$u = \mathscr{C}\big(u\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big) + c_u \;\text{ in }\; \Omega, \tag{1.8.233}$$

where $\mathscr{C}$ is the "ordinary" boundary-to-domain Cauchy integral operator (*associated with Ω as in* (1.6.35)).

Proof In view of the identification made in the first part of Example 1.4.9, this becomes a direct consequence of Corollary 1.8.20 used with $D := \partial_x + i\partial_y$ which, up to normalization, is the Cauchy-Riemann operator in the complex plane (cf. (1.6.39)). □

Second, Theorem 1.8.19 specialized to the Cauchy-Riemann operator in the complex plane yields the following result:

Corollary 1.8.23 *Let $\Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$ be an* NTA *domain such that $\partial\Omega$ is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and recall the modified version of the boundary-to-domain Cauchy integral operator $\mathscr{C}_{\rm mod}$ acting on functions (1.8.226) as in (1.8.227). Fix an aperture parameter $\kappa \in (0,\infty)$ along with an integrability exponent $p \in (1,\infty)$ and consider a holomorphic function $u : \Omega \to \mathbb{C}$ satisfying*[30] *$\mathcal{N}_\kappa(u') \in L^p(\partial\Omega, \sigma)$. In the case when Ω is an exterior domain make the additional assumption that there exists $\lambda \in (1,\infty)$ such that*

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |u'| \, d\mathcal{L}^2 = o(1) \ \ as \ \ R \to \infty. \tag{1.8.234}$$

Then

$$\begin{aligned} &u\Big|_{\partial\Omega}^{\kappa-\rm n.t.} \ \ \text{exists } \sigma\text{-a.e. on } \partial\Omega \text{ and belongs to } \dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{1,\rm loc}(\partial\Omega,\sigma), \\ &\quad u'\Big|_{\partial\Omega}^{\kappa-\rm n.t.} \ \ \text{exists } \sigma\text{-a.e. on } \partial\Omega \text{ and belongs to } \big[L^p(\partial\Omega,\sigma)\big]^2, \end{aligned} \tag{1.8.235}$$

and there exists some complex-valued locally constant function c_u in Ω with the property that

$$u = \mathscr{C}_{\rm mod}\Big(u\Big|_{\partial\Omega}^{\kappa-\rm n.t.}\Big) + c_u \ \ in \ \ \Omega. \tag{1.8.236}$$

In addition, if $\mathscr{C}_{\rm mod}$ is now regarded as an operator mapping into functions defined in $\mathbb{C} \setminus \overline{\Omega}$, then

$$\mathscr{C}_{\rm mod}\Big(u\Big|_{\partial\Omega}^{\kappa-\rm n.t.}\Big) \ \ \text{is a locally constant function in } \ \mathbb{C} \setminus \overline{\Omega}. \tag{1.8.237}$$

Proof This is justified by reasoning much as in the proof of Theorem 1.8.19 (used with D the Cauchy-Riemann operator in the complex plane), now making use of Corollary 1.8.22 in place of Theorem 1.8.17 (and keeping in mind that $|\nabla u| = |u'|$).□

In the same vein, it is also of interest to specialize our general integral representation formulas established earlier for injectively elliptic first-order systems to the case when of the classical Dirac operator. First, Corollary 1.8.20 implies the following result:

Corollary 1.8.24 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, be an open set with an Ahlfors regular boundary. Denote by ν the geometric measure theoretic outward unit normal to*

[30] with 'prime' denoting the ordinary complex derivative

Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Define the modified version of the boundary-to-domain Cauchy-Clifford operator $C_{\rm mod}$ acting on functions $f \in L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\otimes C\ell_n$ according to

$$C_{\rm mod} f(x) := \frac{1}{\omega_{n-1}}\int\limits_{\partial_*\Omega}\left\{\frac{x-y}{|x-y|^n} + \frac{y}{|y|^n}\mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,1)}}(y)\right\}\odot\nu(y)\odot f(y)\,{\rm d}\sigma(y), \tag{1.8.238}$$

at each point $x \in \Omega$. Finally, fix an aperture parameter $\kappa \in (0,\infty)$ and recall the classical Dirac operator D from (A.0.55)*. In this setting, suppose $u : \Omega \to C\ell_n$ is a Clifford algebra-valued function satisfying*

$$\begin{gathered} u \in \mathscr{C}^\infty(\Omega)\otimes C\ell_n, \quad Du = 0 \ \text{ in } \Omega, \quad \mathcal{N}_\kappa^\varepsilon u \in L^1_{\rm loc}(\partial\Omega,\sigma), \\ u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \ \text{exists } \sigma\text{-a.e. on } \partial_{\rm nta}\Omega \text{ and } u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \in L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\otimes C\ell_n, \\ (\nabla u)\big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \ \text{exists } \sigma\text{-a.e. on } \partial_{\rm nta}\Omega \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big). \end{gathered} \tag{1.8.239}$$

In the case when Ω is an exterior domain make the additional assumption that there exists $\lambda \in (1,\infty)$ such that

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |\nabla u|\,{\rm d}\mathcal{L}^n = o(1) \ \text{ as } R \to \infty. \tag{1.8.240}$$

Then there exists some Clifford algebra-valued locally constant function c_u in Ω with the property that

$$u = C_{\rm mod}\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\big) + c_u \ \text{ in } \Omega. \tag{1.8.241}$$

In addition, if $C_{\rm mod}$ is now regarded as an operator mapping into functions defined in $\mathbb{R}^n\setminus\overline{\Omega}$, then

$$C_{\rm mod}\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\big) \ \text{is a locally constant function in } \mathbb{R}^n\setminus\overline{\Omega}. \tag{1.8.242}$$

Furthermore, if in place of the second line in (1.8.239) *one now assumes that*

$$u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \ \text{exists } \sigma\text{-a.e. on } \partial_{\rm nta}\Omega \text{ and } u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \in L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\otimes C\ell_n, \tag{1.8.243}$$

then in place of (1.8.241) *one may conclude that* (*again, for some Clifford algebra-valued locally constant function c_u in Ω, and with the same caveat in the case when Ω is an exterior domain*)

$$u = C\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\big) + c_u \ \text{ in } \Omega, \tag{1.8.244}$$

where C is the "ordinary" Cauchy-Clifford operator, associated with the set Ω as in (A.0.53).

Proof This is a direct consequence of Corollary 1.8.20 used when both $\widetilde{D}$ and D are the classical Dirac operator (A.0.55). □

Lastly, Corollary 1.8.21 written for the Dirac operator produces the following result:

Corollary 1.8.25 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an* NTA *domain such that $\partial\Omega$ is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an aperture parameter $\kappa \in (0, \infty)$ along with an integrability exponent $p \in (1, \infty)$. Bring in the modified version of the boundary-to-domain Cauchy-Clifford operator $\mathcal{C}_{\text{mod}}$ acting on functions $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big) \otimes \mathcal{C}\ell_n$ as in (1.8.238), and recall the classical Dirac operator D from (A.0.55). Finally. consider a Clifford algebra-valued function $u : \Omega \to \mathcal{C}\ell_n$ satisfying*

$$u \in \mathscr{C}^\infty(\Omega) \otimes \mathcal{C}\ell_n, \quad Du = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma). \tag{1.8.245}$$

In the case when Ω is an exterior domain make the additional assumption that there exists $\lambda \in (1, \infty)$ such that

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{1.8.246}$$

Then

$$\begin{gathered} \text{the nontangential trace } u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ belongs to } \big(\dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{1,\text{loc}}(\partial\Omega,\sigma)\big) \otimes \mathcal{C}\ell_n, \\ (\nabla u)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega \text{ and belongs to } \big[L^p(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n\big]^n, \end{gathered} \tag{1.8.247}$$

and there exists some Clifford algebra-valued locally constant function c_u in Ω with the property that

$$u = \mathcal{C}_{\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) + c_u \text{ in } \Omega. \tag{1.8.248}$$

In addition, if $\mathcal{C}_{\text{mod}}$ is now regarded as an operator mapping into functions defined in $\mathbb{R}^n \setminus \overline{\Omega}$, then

$$\mathcal{C}_{\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) \text{ is a locally constant function in } \mathbb{R}^n \setminus \overline{\Omega}. \tag{1.8.249}$$

Proof This follows from Corollary 1.8.21 specialized to the case when both $\widetilde{D}$ and D are the classical Dirac operator (A.0.55). □

Singular integrals on rough surfaces do not constitute an algebra of operators. This being said, there are some remarkable composition identities involving the modified boundary-to-boundary double layer, its transpose version, the modified boundary-to-boundary single layer, and the conormal derivative of the modified double layer. These are made precise in Theorem 1.8.26 stated below (which should be compared with item *(xiii)* of Theorem 1.5.1).

Theorem 1.8.26 *Fix $n \in \mathbb{N}$ with $n \geq 2$ and assume $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. For some $M \in \mathbb{N}$, let $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a complex coefficient tensor with the property that*

$$L := \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M} \tag{1.8.250}$$

(i.e., $L := L_A$) is a weakly elliptic $M\times M$ system in $\mathbb{R}^n$. Having fixed some integrability exponent $p \in (1,\infty)$, recall the operators S_{mod} from (1.8.123), together with $\partial^A_\nu\mathcal{D}_{\text{mod}}$ from (1.8.146), and K_{mod} from (1.8.24). Finally, let $K^{\#}_{A^\top}$ be the operator associated with the coefficient tensor $A^\top$ and the set Ω as in (1.3.72). Then the following statements are true.

(1) For each function $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$, at σ-a.e. point on $\partial\Omega$ one has

$$\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big) = \big(\partial^A_\nu\mathcal{D}_{\text{mod}}\big)\big(S_{\text{mod}}f\big) \tag{1.8.251}$$

and there exists c_f, which is the nontangential trace on $\partial\Omega$ of some $\mathbb{C}^M$-valued locally constant function in Ω, with the property that

$$S_{\text{mod}}\big(K^{\#}_{A^\top}f\big) = K_{\text{mod}}\big(S_{\text{mod}}f\big) + c_f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{1.8.252}$$

(2) Impose the additional hypothesis that Ω satisfies a local John condition, and recall K_{mod} from (1.8.149). Then for each $f \in \big[\dot{L}^p_1(\partial\Omega,\sigma)\cap L^p_{\text{loc}}(\partial\Omega,\sigma)\big]^M$ one has

$$K^{\#}_{A^\top}\big(\partial^A_\nu\mathcal{D}_{\text{mod}}\big)f = \big(\partial^A_\nu\mathcal{D}_{\text{mod}}\big)\big(K_{\text{mod}}f\big) \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{1.8.253}$$

and there exists c_f, which is the nontangential trace on $\partial\Omega$ of some $\mathbb{C}^M$-valued locally constant function in Ω, with the property that at σ-a.e. point on $\partial\Omega$ one has

$$\big(\tfrac{1}{2}I + K_{\text{mod}}\big)\Big(\big(-\tfrac{1}{2}I + K_{\text{mod}}\big)f\Big) = S_{\text{mod}}\Big(\big(\partial^A_\nu\mathcal{D}_{\text{mod}}\big)f\Big) + c_f. \tag{1.8.254}$$

(3) If the original assumptions on the underlying domain are strengthened by now asking that Ω is actually an NTA *domain with an Ahlfors regular boundary*[31] *then, with the operator K_{mod} as in (1.8.153), both formula (1.8.253) as well as formula (1.8.254) actually hold for each function $f \in \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$.*

Proof The identity claimed in (1.8.251) is a consequence of (1.5.89), the fact that the operator (1.8.123) is well defined, linear, and bounded, plus (1.8.145)-(1.8.146).

To prove (1.8.252), start with $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ and define $u := \mathcal{S}_{\text{mod}}f$ in Ω. Having fixed an aperture parameter $\kappa \in (0,\infty)$ along with a truncation parameter

[31] in particular, this is the case if Ω is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular; cf. (1.8.157)

$\varepsilon \in (0, \infty)$, we conclude from (1.8.125) that u satisfies:

$$
\begin{gathered}
u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma), \\
\mathcal{N}_\kappa^\varepsilon u \in L^q_{\rm loc}(\partial\Omega, \sigma) \text{ for each } q \in \big(0, \tfrac{n-1}{n-2}\big), \\
(\nabla u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \\
\partial^A_\nu u = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \\
\text{and } u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = S_{\rm mod} f \text{ at } \sigma\text{-a.e. point on } \partial\Omega.
\end{gathered}
\tag{1.8.255}
$$

In addition, if Ω is an exterior domain then from (1.5.51) and [70, (1.4.24)] we see that

$$
⨍_{B(0,2R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{1.8.256}
$$

Granted these properties (and also keeping in mind (1.8.123), (A.0.127), and [68, (7.7.106)]), we may invoke Theorem 1.8.17 and conclude from (1.8.176) that there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω with the property that

$$
u = \mathcal{D}_{\rm mod}\big(S_{\rm mod} f\big) - \mathcal{S}_{\rm mod}\big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)f + c_u \text{ in } \Omega. \tag{1.8.257}
$$

Taking nontangential boundary traces in (1.8.257) and denoting $c_f := c_u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ then yields

$$
S_{\rm mod} f = \big(\tfrac{1}{2}I + K_{\rm mod}\big)\big(S_{\rm mod} f\big) - S_{\rm mod}\big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) + c_f, \tag{1.8.258}
$$

on account of (1.8.255), (1.8.123), (1.8.143), (1.5.12), and the last property in (1.8.125). Having established (1.8.258), simple algebra then yields (1.8.252). This takes care of *(1)*.

Consider next the first claim made in item *(2)*. To set the stage, bring in the matrix-valued fundamental solution $E = (E_{\gamma\beta})_{1\le\gamma,\beta\le M}$ associated with L as in [70, Theorem 1.4.2]. Also, fix an arbitrary function

$$
f \in \big[\dot{L}^p_1(\partial\Omega, \sigma) \cap L^p_{\rm loc}(\partial\Omega, \sigma)\big]^M \tag{1.8.259}
$$

and select an arbitrary index $\mu \in \{1, \dots, M\}$. In a first stage, our goal is to show that

$$
\begin{aligned}
&\Big(K^\#_{A^\top}\big(\partial^A_\nu \mathcal{D}_{\rm mod}\big)f\Big)_\mu(x) \\
&\quad = \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_i(x) a^{\mu\gamma}_{ij} a^{\beta\alpha}_{rs} (\partial_r E_{\gamma\beta})(x-y)\Big(\partial_{\tau_{js}}\big(K_{\rm mod} f\big)_\alpha\Big)(y)\, d\sigma(y)
\end{aligned}
\tag{1.8.260}
$$

at σ-a.e. point $x \in \partial\Omega$. To this end, define $u := \mathcal{D}_{\rm mod} f$ in Ω, and pick an aperture parameter $\kappa \in (0, \infty)$ along with a truncation parameter $\varepsilon \in (0, \infty)$. Then (1.8.142), (1.8.22), and (1.8.143) ensure that the following properties are satisfied:

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \qquad Lu = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa^\varepsilon u \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma), \qquad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma), \\ \text{the boundary traces } u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}},\ (\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\tfrac{1}{2}I + K_{\mathrm{mod}}\big)f \text{ and } \partial_\nu^A u = \big(\partial_\nu^A \mathcal{D}_{\mathrm{mod}}\big)f. \end{gathered} \tag{1.8.261}$$

In addition, [70, Theorem 1.4.2] implies that if Ω is an exterior domain we have

$$⨍_{B(0,2R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{1.8.262}$$

Let also note here that, as seen from (1.8.121) and (1.8.261), for each $\ell, s \in \{1, \dots, n\}$ and each $\alpha \in \{1, \dots, M\}$ we have

$$\nu_\ell\big((\partial_s u_\alpha)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) - \nu_s\big((\partial_\ell u_\alpha)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) = \partial_{\tau_{\ell s}}\big((\tfrac{1}{2}I + K_{\mathrm{mod}})f\big)_\alpha. \tag{1.8.263}$$

For each $\ell \in \{1, \dots, n\}$ and each $\gamma \in \{1, \dots, M\}$ we may then rely on (1.8.261)-(1.8.263), [70, (1.5.230)], and (A.0.184) to write

$$\begin{aligned} (\partial_\ell u_\gamma)(x) = &\int_{\partial\Omega} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)\partial_{\tau_{\ell s}}\big((\tfrac{1}{2}I + K_{\mathrm{mod}})f\big)_\alpha(y)\, d\sigma(y) \\ &- \int_{\partial\Omega} (\partial_\ell E_{\gamma\alpha})(x-y)\Big(\big(\partial_\nu^A \mathcal{D}_{\mathrm{mod}}\big)f\Big)_\alpha(y)\, d\sigma(y) \end{aligned} \tag{1.8.264}$$

at each point $x \in \Omega$. Next, fix an arbitrary index $\mu \in \{1, \dots, M\}$. Multiply both sides of (1.8.264) by $a_{j\ell}^{\mu\gamma}$, then go nontangentially to the boundary, subsequently multiply by ν_j, and finally sum up over $j \in \{1, \dots, n\}$. On account of [70, (1.5.230)], we therefore arrive at

$$\begin{aligned} &\Big(\big(\partial_\nu^A \mathcal{D}_{\mathrm{mod}}\big)f\Big)_\mu(x) \\ &= \nu_j(x) a_{j\ell}^{\mu\gamma}(\partial_\ell u_\gamma)(x) \\ &= \nu_j(x)\bigg[\int_{\partial\Omega} a_{j\ell}^{\mu\gamma} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(\cdot - y)\partial_{\tau_{\ell s}}\big((\tfrac{1}{2}I + K_{\mathrm{mod}})f\big)_\alpha(y)\, d\sigma(y)\bigg]\bigg|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x) \\ &\quad - \nu_j(x)\bigg[\int_{\partial\Omega} a_{j\ell}^{\mu\gamma}(\partial_\ell E_{\gamma\alpha})(\cdot - y)\Big(\big(\partial_\nu^A \mathcal{D}_{\mathrm{mod}}\big)f\Big)_\alpha(y)\, d\sigma(y)\bigg]\bigg|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x) \end{aligned} \tag{1.8.265}$$

at σ-a.e. point $x \in \partial\Omega$. The term involving the second nontangential trace in above is equal to (cf. [70, (1.5.230)], (1.5.51), and (1.5.58)),

$$\nu_j(x)\Big[\int_{\partial\Omega} a^{\mu\gamma}_{j\ell}(\partial_\ell E_{\gamma\alpha})(\cdot - y)\Big(\big(\partial^A_\nu \mathcal{D}_{\rm mod}\big)f\Big)_\alpha(y)\,{\rm d}\sigma(y)\Big]\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}(x) \tag{1.8.266}$$
$$= \Big(\partial^A_\nu \mathcal{S}_{\rm mod}\big(\partial^A_\nu \mathcal{D}_{\rm mod}\big)f\Big)_\mu(x) = \Big(\big(-\tfrac12 I + K^{\#}_{A^\top}\big)\big(\partial^A_\nu \mathcal{D}_{\rm mod}\big)f\Big)_\mu(x)$$

at σ-a.e. point $x \in \partial\Omega$.

In view of [70, (2.5.4)], the term involving the first nontangential trace in (1.8.265) may be explicitly identified as

$$\nu_j(x)\Big[\int_{\partial\Omega} a^{\mu\gamma}_{j\ell} a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(\cdot - y)\partial_{\tau_{\ell s}}\big((\tfrac12 I + K_{\rm mod})f\big)_\alpha(y)\,{\rm d}\sigma(y)\Big]\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}(x)$$
$$= \nu_j(x)a^{\mu\gamma}_{j\ell} a^{\beta\alpha}_{rs}\frac{1}{2{\rm i}}\widehat{\partial_r E_{\gamma\beta}}\big(\nu(x)\big)\partial_{\tau_{\ell s}}\big((\tfrac12 I + K_{\rm mod})f\big)_\alpha(x) \tag{1.8.267}$$
$$+ \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} a^{\mu\gamma}_{j\ell} a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\partial_{\tau_{\ell s}}\big((\tfrac12 I + K_{\rm mod})f\big)_\alpha(y)\,{\rm d}\sigma(y)$$

at σ-a.e. point $x \in \partial\Omega$. Bearing in mind [70, (1.4.30)] and (1.8.263), at σ-a.e. point $x \in \partial\Omega$ we may recast the first term in the right-hand side of (1.8.267) as

$$\nu_j(x)a^{\mu\gamma}_{j\ell} a^{\beta\alpha}_{rs}\frac{1}{2{\rm i}}\widehat{\partial_r E_{\gamma\beta}}\big(\nu(x)\big)\partial_{\tau_{\ell s}}\big((\tfrac12 I + K_{\rm mod})f\big)_\alpha(x) \tag{1.8.268}$$
$$= \frac12\,\nu_j(x)a^{\mu\gamma}_{j\ell} a^{\beta\alpha}_{rs}\nu_r(x)\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta}\times$$
$$\times\Big\{\nu_\ell(x)\big((\partial_s u_\alpha)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(x) - \nu_s(x)\big((\partial_\ell u_\alpha)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(x)\Big\}.$$

Recall from (1.5.176)-(1.5.177) that for each $s \in \{1,\dots,n\}$ and $\alpha \in \{1,\dots,M\}$ we have

$$\nu_j(x)a^{\mu\gamma}_{j\ell} a^{\beta\alpha}_{rs}\nu_r(x)\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta}\nu_\ell(x) = -a^{\mu\alpha}_{rs}\nu_r(x), \tag{1.8.269}$$

while for each $\ell \in \{1,\dots,n\}$ and $\alpha \in \{1,\dots,M\}$ we have

$$\nu_j(x)a^{\mu\gamma}_{j\ell} a^{\beta\alpha}_{rs}\nu_r(x)\big[L\big(\nu(x)\big)\big]^{-1}_{\gamma\beta}\nu_s(x) = -a^{\mu\alpha}_{j\ell}\nu_j(x). \tag{1.8.270}$$

From (1.8.268)-(1.8.270) we then see that

$$\nu_j(x)a^{\mu\gamma}_{j\ell}a^{\beta\alpha}_{rs}\frac{1}{2\mathrm{i}}\widehat{\partial_r E_{\gamma\beta}}\big(\nu(x)\big)\partial_{\tau_{\ell s}}\big((\tfrac12 I+K_{\rm mod})f\big)_\alpha(x)$$
$$= -\frac{1}{2}\Big\{a^{\mu\alpha}_{rs}\nu_r(x)\big((\partial_s u_\alpha)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(x) - a^{\mu\alpha}_{j\ell}\nu_j(x)\big((\partial_\ell u_\alpha)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(x)\Big\}.$$
$$= 0 \text{ at } \sigma\text{-a.e. point } x\in\partial\Omega. \tag{1.8.271}$$

Taking into account (1.8.271), (1.8.145), and (1.8.153), it follows that (1.8.267) becomes

$$\nu_j(x)\Bigg[\int_{\partial\Omega}a^{\mu\gamma}_{j\ell}a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(\cdot-y)\partial_{\tau_{\ell s}}\big((\tfrac12 I+K_{\rm mod})f\big)_\alpha(y)\,{\rm d}\sigma(y)\Bigg]\Bigg|^{\kappa-{\rm n.t.}}_{\partial\Omega}(x)$$
$$= \frac12\Big(\big(\partial^A_\nu \mathcal{D}_{\rm mod}\big)f\Big)_\mu(x)$$
$$+\lim_{\varepsilon\to0^+}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} a^{\mu\gamma}_{j\ell}a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\partial_{\tau_{\ell s}}\big(K_{\rm mod}f\big)_\alpha(y)\,{\rm d}\sigma(y) \tag{1.8.272}$$

at σ-a.e. point $x\in\partial\Omega$. Together, (1.8.265), (1.8.266), and (1.8.272) readily prove (1.8.260) (after some natural re-labeling). This completes the proof of (1.8.260) under the current assumptions. Finally, (1.8.253) is a consequence of (1.8.260), (1.8.145)-(1.8.146), and the fact that the space $\big[\dot{L}^p_1(\partial\Omega,\sigma)\cap L^p_{\rm loc}(\partial\Omega,\sigma)\big]^M$ is p-resently invariant under the action of $K_{\rm mod}$ (see Theorem 1.8.14).

There is yet another, more direct, proof of (1.8.253) in item *(2)* we would like to discuss. Specifically, granted the properties noted in (1.8.261)-(1.8.262), we may invoke Corollary 1.8.18 and conclude from (1.8.194) that there exists a $\mathbb{C}^M$-valued locally constant function c_u in Ω with the property that

$$u = \mathcal{D}_{\rm mod}\big(\tfrac12 I+K_{\rm mod}\big)f - \mathscr{S}_{\rm mod}\big(\partial^A_\nu\mathcal{D}_{\rm mod}\big)f + c_u \text{ in } \Omega. \tag{1.8.273}$$

Applying ∂^A_ν to both sides then yields, on account of (1.8.261), (1.8.146), the jump-formula in the fifth line of (1.8.125), and the fact that $\partial^A_\nu c_u = 0$ (as may be seen from (A.0.184)),

$$\big(\partial^A_\nu\mathcal{D}_{\rm mod}\big)f = \big(\partial^A_\nu\mathcal{D}_{\rm mod}\big)\big(\tfrac12 I+K_{\rm mod}\big)f - \big(-\tfrac12 I+K^\#_{A^\top}\big)\big(\partial^A_\nu\mathcal{D}_{\rm mod}\big)f, \tag{1.8.274}$$

from which (1.8.253) now readily follows after canceling like-terms.

To justify (1.8.254), start with f as in (1.8.259) and once again define $u := \mathcal{D}_{\rm mod}f$ in Ω. Then (1.8.273) continues to hold. If we introduce $c_f := c_u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$, then going nontangentially to the boundary in (1.8.273) yields

$$\big(\tfrac12 I+K_{\rm mod}\big)f = \big(\tfrac12 I+K_{\rm mod}\big)\big(\tfrac12 I+K_{\rm mod}\big)f - S_{\rm mod}\big(\partial^A_\nu\mathcal{D}_{\rm mod}\big)f + c_f, \tag{1.8.275}$$

thanks to the first formula in the fourth line of (1.8.261), the jump-formula (1.8.27) (bearing in mind (A.0.127) and the fact that the operator $K_{\rm mod}$ leaves the space $\big[\dot{L}^p_1(\partial\Omega,\sigma)\cap L^p_{\rm loc}(\partial\Omega,\sigma)\big]^M$ invariant; see Theorem 1.8.14), and the last property in (1.8.125) (also keeping in mind (1.8.146)). With (1.8.275) in hand, the identity claimed in (1.8.254) follows after some simple algebra. This completes the treatment of item *(2)*.

To deal with item *(3)*, work under the assumption that Ω is an NTA domain with an Ahlfors regular boundary. The proof of (1.8.253) for an arbitrary function $f\in\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$ proceeds along similar lines to the argument in the either of the two proofs of this identity provided in item *(2)*. As far as the first proof is concerned, given any $f\in\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$, we once again define $u:=\mathcal{D}_{\rm mod}f$ in Ω. This time, (1.8.142) and (1.8.143) imply (compare with (1.8.261))

$$\begin{gathered} u\in\big[\mathscr{C}^\infty(\Omega)\big]^M,\quad Lu=0\ \text{ in }\ \Omega,\quad \mathcal{N}_\kappa(\nabla u)\in L^p(\partial\Omega,\sigma),\\ \text{the boundary traces }\ u\big|^{\kappa-\text{n.t.}}_{\partial\Omega},\ (\nabla u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\ \text{ exist }\sigma\text{-a.e. on }\partial\Omega,\\ u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}=\big(\tfrac12 I+K_{\rm mod}\big)f\ \text{ and }\ \partial^A_\nu u=\big(\partial^A_\nu\mathcal{D}_{\rm mod}\big)f. \end{gathered}\tag{1.8.276}$$

In a first stage, the goal is to establish (1.8.260). For this we argue as before, the main difference is the justification of (1.8.263). Granted the current assumptions, we may invoke [68, Proposition 8.4.9] and conclude from [68, (8.4.109)] and (1.8.261) that there exists a truncation parameter $\varepsilon\in(0,\infty)$ such that

$$\mathcal{N}^\varepsilon_\kappa u\in L^p_{\rm loc}(\partial\Omega,\sigma).\tag{1.8.277}$$

In turn, (1.8.277) and (1.8.261) permit us to call upon [69, Proposition 11.3.2] to conclude from [69, (11.3.26)] and (1.8.276) that

$$\begin{aligned} \nu_\ell\big((\partial_s u_\alpha)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)-\nu_s\big((\partial_\ell u_\alpha)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)&\\ =\partial_{\tau_{\ell s}}\Big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)&=\partial_{\tau_{\ell s}}\big((\tfrac12 I+K_{\rm mod})f\big)_\alpha, \end{aligned}\tag{1.8.278}$$

at σ-a.e. point on $\partial\Omega$. Thus, (1.8.263) continues to hold under the current geometric assumptions, and from this point on the proof of (1.8.260) is completed as before. At this stage, (1.8.253) becomes a consequence of (1.8.260), (1.8.145)-(1.8.146), and the fact that the homogeneous Sobolev space $\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$ is presently invariant under the action of $K_{\rm mod}$ (see Theorem 1.8.14).

Let us also note that formula (1.8.253) may also be justified for each function $f\in\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$ much as in the second proof given in item *(2)*, now relying on the integral representation formula (1.8.200) from Theorem 1.8.19 for the function $u:=\mathcal{D}_{\rm mod}f$, then taking conormal derivatives.

Finally, we may also establish the veracity of formula (1.8.254) for each $f\in\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$ by once again employing the integral representation formula

(1.8.200) from Theorem 1.8.19 for the function $u := \mathcal{D}_{\text{mod}} f$, then going nontangentially to the boundary. □

Remark 1.8.27 *It is of interest to consider the operator identities described in Theorem 1.8.12 in the case when they involve modified boundary-to-boundary double layer potential operators of the sort described in Examples 1.8.4-1.8.5. Indeed, in such scenarios, the operator $\partial_\nu^A \mathcal{D}_{\text{mod}}$ vanishes identically (as noted in the last part in Remark 1.4.6 and in (1.4.33)). As such, the identity recorded in (1.8.254) simplifies in each of these settings. For example, re-branding the modified boundary-to-boundary Cauchy-Clifford integral operator (1.8.112) as $\mathfrak{C}_{\text{mod}}$, we conclude from (1.8.254) that for each $f \in \dot{L}^p_1(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ with $p \in (1, \infty)$ there exists c_f, which is the nontangential trace on $\partial\Omega$ of some $\mathcal{C}\ell_n$-valued locally constant function in Ω, with the property that*

$$\big(\tfrac{1}{2}I + \mathfrak{C}_{\text{mod}}\big)\Big(\big(-\tfrac{1}{2}I + \mathfrak{C}_{\text{mod}}\big) f\Big) = c_f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{1.8.279}$$

or, equivalently,

$$\mathfrak{C}_{\text{mod}}\big(\mathfrak{C}_{\text{mod}} f\big) = \tfrac{1}{4} f + c_f \text{ on } \partial\Omega. \tag{1.8.280}$$

In particular, if $\partial\Omega$ is connected and we let $\big[\mathfrak{C}_{\text{mod}}\big]$ be the operator $[f] \mapsto \big[\mathfrak{C}_{\text{mod}} f\big]$ (with brackets denoting equivalence classes modulo constants), we therefore obtain

$$\big[\mathfrak{C}_{\text{mod}}\big]^2 = \tfrac{1}{4} I \text{ on } \big[\dot{L}^p_1(\partial\Omega, \sigma)/\sim\big] \otimes \mathcal{C}\ell_n \text{ with } p \in (1, \infty). \tag{1.8.281}$$

See also (2.1.186)-(2.1.189) for related identities.

In the same setting as above, (1.8.252) yields (reasoning in a similar fashion) that for each given function $f \in \dot{L}^p_1(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ with $p \in (1, \infty)$ there exists some constant c_f, which is the nontangential trace on $\partial\Omega$ of some $\mathcal{C}\ell_n$-valued locally constant function in Ω, such that

$$S_{\text{mod}}\big(\mathfrak{C}^{\#} f\big) = \mathfrak{C}_{\text{mod}}\big(S_{\text{mod}} f\big) + c_f. \tag{1.8.282}$$

Of course, similar results are valid for the Cauchy operator in the complex plane (a scenario in which $n = 2$, and we identify $\mathbb{R}^2 \equiv \mathbb{C}$). For example, re-branding the modified boundary-to-boundary Cauchy integral operator (1.8.109) as C_{mod}, much as above we conclude from (1.8.254) that if $\partial\Omega$ is connected we have

$$\big[C_{\text{mod}}\big]^2 = \tfrac{1}{4} I \text{ on } \dot{L}^p_1(\partial\Omega, \sigma)/\sim \text{ with } p \in (1, \infty), \tag{1.8.283}$$

where $\big[C_{\text{mod}}\big]$ denotes the operator $[f] \mapsto \big[C_{\text{mod}} f\big]$ (again, with brackets denoting equivalence classes modulo constants).

Here is an application to the theory of Hardy spaces in which some of the above considerations play a role.

Proposition 1.8.28 *Let $\Omega \subseteq \mathbb{C} \equiv \mathbb{R}^2$ be a two-sided* NTA *domain such that $\partial\Omega$ is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and fix an integrability exponent $p \in (1, \infty)$ along with some aperture parameter $\kappa > 0$. Next, set*

$$\Omega_+ := \Omega, \qquad \Omega_- := \mathbb{C} \setminus \overline{\Omega}, \tag{1.8.284}$$

and consider the "positive/negative" homogeneous Hardy spaces with regularity on $\partial\Omega$*, defined as*

$$\dot{\mathcal{H}}^p_{1,\pm}(\partial\Omega,\sigma) := \Big\{ u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} : u \text{ holomorphic in } \Omega_\pm \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma)\Big\}. \tag{1.8.285}$$

Then, with $\sim$ *standing for equivalence modulo constants,*

$$\begin{gathered}\dot{\mathcal{H}}^p_{1,\pm}(\partial\Omega,\sigma)/\sim \;\text{ are closed subspaces of } \dot{L}^p_1(\partial\Omega,\sigma)/\sim \;\text{ and}\\ \dot{L}^p_1(\partial\Omega,\sigma)/\sim \, = \Big(\dot{\mathcal{H}}^p_{1,+}(\partial\Omega,\sigma)/\sim\Big) \oplus \Big(\dot{\mathcal{H}}^p_{1,-}(\partial\Omega,\sigma)/\sim\Big), \text{ direct sum.}\end{gathered} \tag{1.8.286}$$

In fact, if C_{mod} *is the modified boundary-to-boundary Cauchy singular integral operator acting on each function* $f \in L^1\Big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^2}\Big)$ *according to*

$$C_{\mathrm{mod}} f(z) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \Big\{ \frac{1}{\zeta - z} \mathbf{1}_{\mathbb{C}\setminus\overline{B(z,\varepsilon)}}(\zeta) - \frac{1}{\zeta} \mathbf{1}_{\mathbb{C}\setminus\overline{B(0,1)}}(\zeta) \Big\} f(\zeta)\, \mathrm{d}\zeta \tag{1.8.287}$$

at σ*-a.e.* $z \in \partial\Omega$ *(see (1.8.109)), and if* $\big[C_{\mathrm{mod}}\big]$ *denotes the operator* $[f] \mapsto \big[C_{\mathrm{mod}} f\big]$ *(with brackets denoting equivalence classes modulo constants), then*

$$\begin{aligned}\dot{\mathcal{H}}^p_{1,\pm}(\partial\Omega,\sigma)/\sim \, &= \mathrm{Ker}\Big(\mp \tfrac{1}{2} I + \big[C_{\mathrm{mod}}\big];\ \dot{L}^p_1(\partial\Omega,\sigma)/\sim\Big)\\ &= \mathrm{Im}\Big(\pm \tfrac{1}{2} I + \big[C_{\mathrm{mod}}\big];\ \dot{L}^p_1(\partial\Omega,\sigma)/\sim\Big).\end{aligned} \tag{1.8.288}$$

Proof For starters, observe that the present assumptions imply (see the last part in [68, Lemma 5.11.3]) that

$$\Omega_\pm \text{ are connected sets.} \tag{1.8.289}$$

In addition, from (1.8.285) and [69, Proposition 11.5.12] we see that $\dot{\mathcal{H}}^p_{1,\pm}(\partial\Omega,\sigma)$ are well-defined subspaces of $\dot{L}^p_1(\partial\Omega,\sigma)$. To proceed, from Example 1.8.4 and Remark 1.8.13 we conclude that for each $f \in \dot{L}^p_1(\partial\Omega,\sigma)$ the functions

$$u_\pm := \mathscr{C}_{\mathrm{mod}} f \ \text{ in } \Omega_\pm \tag{1.8.290}$$

are holomorphic and satisfy $\mathcal{N}_\kappa(\nabla u_\pm) \in L^p(\partial\Omega,\sigma)$ as well as

$$u_\pm\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2} I \pm C_{\mathrm{mod}}\big) f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{1.8.291}$$

In view of (1.8.285), we therefore have

$$\big(\tfrac{1}{2}I \pm C_{\text{mod}}\big)f \in \dot{\mathcal{H}}^p_{1,\pm}(\partial\Omega, \sigma), \tag{1.8.292}$$

and since $f = \big(\frac{1}{2}I + C_{\text{mod}}\big)f + \big(\frac{1}{2}I - C_{\text{mod}}\big)f$, we obtain

$$\dot{L}^p_1(\partial\Omega, \sigma) = \dot{\mathcal{H}}^p_{1,+}(\partial\Omega, \sigma) + \dot{\mathcal{H}}^p_{1,-}(\partial\Omega, \sigma), \tag{1.8.293}$$

hence also

$$\dot{L}^p_1(\partial\Omega, \sigma)/\sim = \Big(\dot{\mathcal{H}}^p_{1,+}(\partial\Omega, \sigma)/\sim\Big) + \Big(\dot{\mathcal{H}}^p_{1,-}(\partial\Omega, \sigma)/\sim\Big). \tag{1.8.294}$$

Suppose now $f \in \dot{\mathcal{H}}^p_{1,+}(\partial\Omega, \sigma) \cap \dot{\mathcal{H}}^p_{1,-}(\partial\Omega, \sigma)$. By definition, this implies the existence of two holomorphic functions $w_\pm$ in $\Omega_\pm$ with $\mathcal{N}_\kappa(\nabla w_\pm) \in L^p(\partial\Omega, \sigma)$ and satisfying $w_\pm\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = f$. Then (1.8.236) implies (bearing in mind (1.8.289)) that there exists a constant $c \in \mathbb{C}$ such that

$$w_- = \mathscr{C}_{\text{mod}}\Big(w_-\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) + c = \mathscr{C}_{\text{mod}}\Big(w_+\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) + c \ \text{ in } \Omega_- \tag{1.8.295}$$

which, according to (1.8.237), means that w_- is a constant function in Ω_- (again, here (1.8.289) is relevant). In a very similar fashion, we also obtain that w_+ is a constant function in Ω_+. Thus, $\mathscr{C}_{\text{mod}}f$ are constant functions in $\Omega_\pm$ which then forces $f = \big(\frac{1}{2}I + C_{\text{mod}}\big)f + \big(\frac{1}{2}I - C_{\text{mod}}\big)f$ to be itself a constant function on $\partial\Omega$. In concert with (1.8.294), this ultimately yields the direct sum decomposition in the second line of (1.8.286).

There is more one can extract from the above considerations. Specifically, recall (from the definition in (1.8.285)) that for any $f \in \dot{\mathcal{H}}^p_{1,+}(\partial\Omega, \sigma)$ there exists a holomorphic function u in Ω_+ with $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma)$ and such that $f = u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$. Then (1.8.236) implies (keeping in mind (1.8.289)) that there exists a constant $c \in \mathbb{C}$ such that

$$u = \mathscr{C}_{\text{mod}}\Big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) + c = \mathscr{C}_{\text{mod}}f + c \ \text{ in } \Omega_+. \tag{1.8.296}$$

After going nontangentially to the boundary we arrive at

$$f = u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + C_{\text{mod}}\big)f + c. \tag{1.8.297}$$

Hence

$$\big[C_{\text{mod}}\big][f] = \tfrac{1}{2}[f] \ \text{ for each } \ [f] \in \dot{\mathcal{H}}^p_{1,+}(\partial\Omega, \sigma)/\sim \tag{1.8.298}$$

and, analogously,

$$\big[C_{\text{mod}}\big][f] = -\tfrac{1}{2}[f] \ \text{ for each } \ [f] \in \dot{\mathcal{H}}^p_{1,-}(\partial\Omega, \sigma)/\sim\ . \tag{1.8.299}$$

These prove the first equality in (1.8.288) from which we also deduce that $\dot{\mathcal{H}}^p_{1,\pm}(\partial\Omega, \sigma)/\sim$ are closed subspaces of $\dot{L}^p_1(\partial\Omega, \sigma)/\sim$. Finally, the second equality in (1.8.288) is a consequence of (1.8.283). □

In closing we note that similar results are valid for homogeneous Hardy spaces with regularity in the higher-dimensional setting, now employing Clifford algebra-valued null-solutions of the Dirac operator in $\mathbb{R}^n$ in place of holomorphic functions in the complex plane.

Chapter 2
Layer Potential Operators on Hardy, BMO, VMO, and Hölder Spaces

The reader is reminded that Hölder, BMO, VMO, and Hardy spaces in the general setting of spaces of homogeneous type and on Ahlfors regular sets have been discussed at length in [68, §§ 7.3-7.4] and [69, Chapters 3-4]. This chapter is primarily focused on layer potential operators acting on Hardy, BMO, VMO, and Hölder spaces defined on boundaries of UR domains. A key aspect in this analysis is that a special algebraic structure is required of the integral kernel for a singular integral operator to map either of these spaces into itself, and the brand of Divergence Theorem produced in [68] plays a crucial role in ensuring this is indeed the case. In order to be more specific, let us pick a UR set $\Sigma \subseteq \mathbb{R}^n$ and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Then all garden-variety Calderón-Zygmund convolution-type singular integral operators T (of the sort considered in [70, Chapter 2]) are well defined and bounded in the context $T : H^p(\Sigma, \sigma) \to L^p(\Sigma, \sigma)$ for $\frac{n-1}{n} < p \leq 1$. This being said, they hopelessly fail to map the Hardy space $H^p(\Sigma, \sigma)$ into *itself* for $\frac{n-1}{n} < p \leq 1$. Well, a long time ago (in the late 1970's to be more precise; cf. [16]) R. Coifman and G. Weiss have taught us that when regarding Σ as a space of homogeneous type, i.e., equipped with the Euclidean metric and the doubling measure σ, for a linear and bounded integral operator $T : L^2(\Sigma, \sigma) \to L^2(\Sigma, \sigma)$ associated with an integral kernel satisfying "standard" size and regularity properties to actually map the Hardy space $H^1(\Sigma, \sigma)$ boundedly into itself it is necessary and sufficient that

$$m := Ta \text{ is a molecule on } \Sigma, \text{ for each atom } a \text{ on } \Sigma \tag{2.0.1}$$

(in a quantitative fashion). The delicate aspect is that, by design, a molecule m on Σ is supposed to satisfy the vanishing moment condition $\int_\Sigma m\, d\sigma = 0$, so we would need

$$\int_\Sigma Ta\, d\sigma = 0 \text{ for each atom } a \text{ on } \Sigma. \tag{2.0.2}$$

Alas, plain convolution-type operators (i.e., integral operators with integral kernel $k(x - y)$, where k is a smooth, odd, and positive homogeneous of degree $1 - n$ function in $\mathbb{R}^n \setminus \{0\}$) are really dull, as in general they lack any type of cancelation properties. This deficiency is predicated by the inability of the integral kernel k

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to "see the underlying surface" Σ, to which this is completely unrelated. This is why operators like the transpose harmonic double layer $K^{\#}_{\Delta}$ (from (A.0.102)) are decisively different. For example, corresponding to the case when $\Sigma := \partial\Omega$ for some UR domain $\Omega \subseteq \mathbb{R}^n$, we may verify (2.0.2) for $K^{\#}_{\Delta}$ playing the role of T by writing, for each atom a on $\partial\Omega$,

$$\int_{\partial\Omega} K^{\#}_{\Delta} a \, d\sigma = \int_{\partial\Omega} \big(-\tfrac{1}{2} I + K^{\#}_{\Delta} \big) a \, d\sigma$$
$$= \int_{\partial\Omega} \nu \cdot \Big(\vec{F}\big|^{\rm n.t.}_{\partial\Omega} \Big) \, d\sigma = \int_{\partial\Omega} {\rm div} \vec{F} \, d\mathcal{L}^n = 0, \tag{2.0.3}$$

where ν is the geometric measure theoretic outward unit normal to Ω, and the vector field $\vec{F}$ is given by $\nabla \mathscr{S} a$ (with $\mathscr{S}$ denoting the boundary-to-domain single layer potential operator for the Laplacian in Ω). The first equality uses the vanishing moment condition of the atom itself, the second equality is a particular case of the jump-formula (1.5.58), the third equality comes from an application of the Divergence Formula from [68, Theorem 1.2.1], and the last equality is a consequence of the fact that $\vec{F}$ is actually divergence-free in Ω (since ${\rm div} \nabla \mathscr{S} a = \Delta \mathscr{S} a = 0$).

This is the line of attack we adopt for proving mapping properties for transpose double layers associated with weakly elliptic second-order systems in UR domains. Once the action of the these double layers on the Hardy scale has been established, one can deal with similar issues on BMO, VMO, and Hölder spaces via duality. Much of the work in this chapter elaborates on this program.

2.1 Double Layer Potential Operators on Hardy, BMO, VMO, and Hölder Spaces

Consider the principal-value (p.v.) singular integral operator $K^{\#}$ from (1.3.72) in the case when the set $\Omega \subseteq \mathbb{R}^n$ is a UR domain and $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. In light of item *(6)* of [70, Theorem 2.3.2] (while also bearing in mind the qualities of the fundamental solution highlighted in [70, Theorem 1.4.2]) it follows that $K^{\#}$, originally acting on Lebesgue spaces on $\partial\Omega$ (as in (1.5.12)), extends uniquely to a linear and bounded operator

$$K^{\#} : \big[H^p(\partial\Omega, \sigma) \big]^M \longrightarrow \big[L^p(\partial\Omega, \sigma) \big]^M, \qquad p \in \big(\tfrac{n-1}{n}, \infty \big). \tag{2.1.1}$$

Specifically, given any $f = (f_\gamma)_{1 \leq \gamma \leq M} \in \big[H^p(\partial\Omega, \sigma) \big]^M$, the operator $K^{\#}$ acts on f as in (1.3.72) in the case when $p \in (1, \infty)$ and, if $p \in \big(\frac{n-1}{n}, 1 \big]$, the action of $K^{\#}$ on f is defined at σ-a.e. $x \in \partial\Omega$ according to

$$K^{\#} f(x) \tag{2.1.2}$$
$$= \left(\nu_s(x) a_{rs}^{\beta\alpha} \cdot \lim_{m\to\infty} \sum_{j=1}^{m} \lambda_{\gamma,j} \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} (\partial_r E_{\gamma\beta})(x-y) a_{\gamma,j}(y)\, \mathrm{d}\sigma(y) \right)_{1\leq\alpha\leq M}$$

where, for each $\gamma \in \{1, \dots, n\}$, we have relied on [69, Theorem 4.4.1] to expand $f_\gamma = \sum_{j=1}^{\infty} \lambda_{\gamma,j} a_{\gamma,j}$ (with convergence in $H^p(\partial\Omega, \sigma)$), with $\{\lambda_{\gamma,j}\}_{j\in\mathbb{N}} \in \ell^p$ and each $a_{\gamma,j}$ a (p, q)-atom on $\partial\Omega$ for some fixed background exponent $q \in [1, \infty]$ with $q > p$.

In fact, we may use real interpolation (as in [69, Theorem 4.3.1] and [68, (6.2.48)]) to conclude from (2.1.1) that $K^{\#}$ also induces a linear and bounded operator

$$K^{\#} : \left[H^{p,q}(\partial\Omega, \sigma)\right]^M \longrightarrow \left[L^{p,q}(\partial\Omega, \sigma)\right]^M, \quad \text{for } p \in \left(\tfrac{n-1}{n}, \infty\right) \text{ and } q \in (0, \infty]. \tag{2.1.3}$$

The main point of our first theorem in this section is that, due to its specific algebraic nature, the operator $K^{\#}$ actually maps Hardy spaces into Hardy spaces in a linear and bounded fashion.

Theorem 2.1.1 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). In this context, consider the boundary layer potential operator $K^{\#}$ associated with the system L and the set Ω as in* (1.3.72).

Then the operator $K^{\#}$, originally acting on Lebesgue spaces on $\partial\Omega$ (as in (1.5.12)*), extends uniquely to a linear and bounded mapping*

$$K^{\#} : \left[H^p(\partial\Omega, \sigma)\right]^M \longrightarrow \left[H^p(\partial\Omega, \sigma)\right]^M, \qquad p \in \left(\tfrac{n-1}{n}, 1\right]. \tag{2.1.4}$$

Moreover, various choices of the exponent p yield operators which are compatible with one another.

Ultimately, as a consequence of (2.1.4) *and* (1.5.12)*, one has a (unique) family of operators*

$$K^{\#} : \left[H^p(\partial\Omega, \sigma)\right]^M \longrightarrow \left[H^p(\partial\Omega, \sigma)\right]^M, \qquad p \in \left(\tfrac{n-1}{n}, \infty\right), \tag{2.1.5}$$

which are well-defined, linear, continuous, compatible with one another, and which agree with (1.3.72) *when $p \in (1, \infty)$. In addition,*

> *if $p \in \left(\tfrac{n-1}{n}, \infty\right)$, the composition to the left of $K^{\#}$ from* (2.1.5) *with the L^p-filtering operator $\mathfrak{H} : \left[H^p(\partial\Omega, \sigma)\right]^M \to \left[L^p(\partial\Omega, \sigma)\right]^M$ (cf.* [69, (4.9.2)]*) is $K^{\#} : \left[H^p(\partial\Omega, \sigma)\right]^M \to \left[L^p(\partial\Omega, \sigma)\right]^M$ defined as in* (2.1.1)-(2.1.2). (2.1.6)

Finally, the operator $K^{\#}$ *in the context of* (2.1.5) *depends in a continuous fashion on the underlying coefficient tensor* (*in the sense of item (xv) in Theorem 1.5.1*).

Proof Fix an arbitrary exponent $p \in \big(\frac{n-1}{n}, 1\big]$. The gist of the strategy is to show that $K^{\#}$ maps vector-valued atoms into a fixed multiple of vector-valued molecules for the vector Hardy space in question. Concretely, pick $q \in (1, \infty)$ and consider an arbitrary $\mathbb{C}^M$-valued (p, q)-atom on $\partial\Omega$. Recall from [69, (4.4.167)-(4.4.168)] that this means that $a : \partial\Omega \to \mathbb{C}^M$ is some σ-measurable function with the property that there exist a point $x_o \in \partial\Omega$ and some number $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ such that

$$\begin{aligned} &\mathrm{supp}\, a \subseteq B(x_o, r) \cap \partial\Omega, \\ &\|a\|_{[L^q(\partial\Omega,\sigma)]^M} \leq \sigma\big(B(x_o, r) \cap \partial\Omega\big)^{1/q-1/p}, \\ &\int_{\partial\Omega} a \, \mathrm{d}\sigma = 0 \in \mathbb{C}^M. \end{aligned} \tag{2.1.7}$$

In addition, in the case when $\partial\Omega$ is compact, we agree that each vector in $\mathbb{C}^M$ of norm $\leq \sigma(\partial\Omega)^{-1/p}$ is also considered, by definition, to be a $\mathbb{C}^M$-valued (p, q)-atom on $\partial\Omega$. The claim that we make is that

> the function $m := K^{\#}a$ is a fixed multiple of a $\mathbb{C}^M$-valued (p, q, ε)-molecule on $\partial\Omega$ (i.e., a $\mathbb{C}^M$-valued function whose scalar components are as in [69, Definition 4.5.1]), with $\varepsilon := \frac{1}{n-1}$. (2.1.8)

To prove this, first note that according to Theorem 1.5.1 the function m is meaningfully defined and belongs to the space $\big[L^q(\partial\Omega, \sigma)\big]^M$. In fact, thanks to item *(iii)* in Theorem 1.5.1 and (2.1.7), we have

$$\begin{aligned} \|m\|_{[L^q(\partial\Omega,\sigma)]^M} &= \|K^{\#}a\|_{[L^q(\partial\Omega,\sigma)]^M} \leq C\|a\|_{[L^q(\partial\Omega,\sigma)]^M} \\ &\leq C\sigma\big(B(x_o, r) \cap \partial\Omega\big)^{1/q-1/p}, \end{aligned} \tag{2.1.9}$$

for some finite constant $C = C(\partial\Omega, L, q) > 0$ independent of the atom.

To study the decay of m, observe that based on (2.1.7), the Mean Value Theorem, and [70, Theorem 1.4.2], we may estimate

$$\begin{aligned} |m(x)| &\leq C \int_{B(x_o,r)\cap\partial\Omega} |(\nabla E)(x - y) - (\nabla E)(x - x_o)||a(y)| \, \mathrm{d}\sigma(y) \\ &\leq C \frac{r}{|x - x_o|^n} \sigma\big(B(x_o, r) \cap \partial\Omega\big)^{1-1/p} \end{aligned} \tag{2.1.10}$$

at every $x \in \partial\Omega \setminus B(x_o, 2r)$. To proceed, for each $k \in \mathbb{N}$ define the boundary annulus

$$\widetilde{A}_k(x_o, r) := \big[B(x_o, 2^{k+1}r) \setminus B(x_o, 2^k r)\big] \cap \partial\Omega. \tag{2.1.11}$$

We may then rely on (2.1.10) and the Ahlfors regularity of $\partial\Omega$ to obtain to that, for each $k \in \mathbb{N}$,

$$\Big(\int_{\widetilde{A}_k(x_o,r)}|m|^q\,\mathrm{d}\sigma\Big)^{1/q}$$
$$\leq C\frac{r}{(2^k r)^n}\sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1-1/p}\sigma\big(B(x_o,2^{k+1}r)\cap\partial\Omega\big)^{1/q}$$
$$\leq C2^{k(n-1)[1/q-1-1/(n-1)]}\sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1/q-1/p} \tag{2.1.12}$$

for some finite constant $C=C(\partial\Omega,L)>0$ independent of the atom.

At this stage, the claim in (2.1.8) follows as soon as we check that

$$\int_{\partial\Omega} m\,\mathrm{d}\sigma = 0. \tag{2.1.13}$$

To this end, recall the matrix-valued fundamental solution $E=(E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ associated with L as in [70, Theorem 1.4.2]. For any given indices $\alpha,\beta\in\{1,\dots,M\}$ and $j\in\{1,\dots,n\}$ then introduce two integral operators acting on each scalar-valued function $f\in L^{p_o}(\partial\Omega,\sigma)$ with $p_o\in(1,\infty)$ according to

$$\mathcal{Q}^j_{\alpha\beta}f(x):=\int_{\partial\Omega}(\partial_j E_{\alpha\beta})(x-y)f(y)\,\mathrm{d}\sigma(y)\ \text{ for each }\ x\in\Omega, \tag{2.1.14}$$

and, respectively,

$$Q^j_{\alpha\beta}f(x):=\lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}(\partial_j E_{\alpha\beta})(x-y)f(y)\,\mathrm{d}\sigma(y)\ \text{ for }\sigma\text{-a.e. } x\in\partial\Omega. \tag{2.1.15}$$

Fix a background parameter $\kappa>0$ and pick $f\in L^{p_o}(\partial\Omega,\sigma)$ with $p_o\in(1,\infty)$. Also, select $\alpha,\beta\in\{1,\dots,M\}$ and $j\in\{1,\dots,n\}$. Then [70, Theorem 2.3.2] implies that the limit defining $Q^j_{\alpha\beta}f$ in (2.1.15) exists for σ-a.e. $x\in\partial\Omega$. Also, [70, Theorem 2.5.1] and [70, (1.4.30)] ensure that at σ-a.e. point $x\in\partial\Omega$ we have the jump-formula

$$\big(\mathcal{Q}^j_{\alpha\beta}f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}(x)=\frac{1}{2\mathrm{i}}\widehat{\partial_j E_{\alpha\beta}}\big(\nu(x)\big)f(x)+(Q^j_{\alpha\beta}f)(x)$$
$$=\frac{1}{2}\,b_{\alpha\beta}(x)\nu_j(x)f(x)+(Q^j_{\alpha\beta}f)(x), \tag{2.1.16}$$

where $\nu=(\nu_1,\dots,\nu_n)$ is the geometric measure theoretic outward unit normal to Ω, and

$$\begin{array}{l} b_{\alpha\beta}(x)\ \text{ is the }\ (\alpha,\beta)\text{-entry in the }\ M\times M\ \text{ matrix}\\ L(\nu(x))^{-1}=\Big[\Big(-a^{\gamma\delta}_{rs}\nu_r(x)\nu_s(x)\Big)_{1\leq\gamma,\delta\leq M}\Big]^{-1}\end{array} \tag{2.1.17}$$

(with the summation convention over repeated indices in effect). Pushing forth, fix an arbitrary index $\alpha\in\{1,\dots,M\}$ and bring in the vector field

$$\vec{F} = (F_s)_{1\leq s\leq n} \text{ with } F_s := a^{\beta\alpha}_{rs} Q^r_{\gamma\beta} a_\gamma \text{ in } \Omega, \text{ for each } s \in \{1,\dots,n\}, \tag{2.1.18}$$

where a_γ is the γ-th component of the $\mathbb{C}^M$-valued function a. Then, by design,

$$\vec{F} \in \big[\mathscr{C}^\infty(\Omega)\big]^n \tag{2.1.19}$$

and, thanks to (2.1.18), (2.1.14), [70, (1.4.33)], we have

$$\begin{aligned}(\mathrm{div}\vec{F})(x) &= a^{\beta\alpha}_{rs}(\partial_s Q^r_{\gamma\beta} a_\gamma)(x)\\ &= \int_{\partial\Omega} a^{\beta\alpha}_{rs}(\partial_s\partial_r E_{\gamma\beta})(x-y)a_\gamma(y)\,\mathrm{d}\sigma(y)\\ &= 0 \text{ for each } x \in \Omega.\end{aligned} \tag{2.1.20}$$

Since for each $s \in \{1,\dots,n\}$ the jump-formula (2.1.16) implies

$$\begin{aligned}F_s\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}(x) &= a^{\beta\alpha}_{rs}(Q^r_{\gamma\beta}a_\gamma)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}(x)\\ &= \frac{1}{2}b_{\gamma\beta}(x)a^{\beta\alpha}_{rs}\nu_r(x)a_\gamma(x) + a^{\beta\alpha}_{rs}(Q^r_{\gamma\beta}a_\gamma)(x)\end{aligned} \tag{2.1.21}$$

at σ-a.e. point $x \in \partial\Omega$, it follows that

$$\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.1.22}$$

Formula (2.1.21) also permits us to compute at σ-a.e. point $x \in \partial\Omega$

$$\begin{aligned}\nu(x)\cdot\Big(\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) &= \nu_s(x)\Big(F_s\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x)\\ &= \frac{1}{2}b_{\gamma\beta}(x)a^{\beta\alpha}_{rs}\nu_r(x)\nu_s(x)a_\gamma(x) + \nu_s(x)a^{\beta\alpha}_{rs}(Q^r_{\gamma\beta}a_\gamma)(x)\\ &= -\frac{1}{2}\delta_{\gamma\alpha}a_\gamma(x) + \int_{\partial\Omega}\nu_s(x)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)a_\gamma(y)\,\mathrm{d}\sigma(y)\\ &= -\frac{1}{2}a_\alpha(x) + (K^\# a)_\alpha(x) = -\frac{1}{2}a_\alpha(x) + m_\alpha(x),\end{aligned} \tag{2.1.23}$$

upon recalling (2.1.17), (2.1.15), (1.3.72), and the definition of m in (2.1.8). Since, as is apparent from (2.1.7), the components of the function $a : \partial\Omega \to \mathbb{C}^M$ are multiples of $(1,q)$-atoms on $\partial\Omega$, we may invoke [70, (2.4.14)] with $p = 1$ in order to conclude that

$$\mathcal{N}_\kappa\vec{F} \in L^1(\partial\Omega,\sigma). \tag{2.1.24}$$

Finally, the vanishing moment property of the atom (cf. the last line in (2.1.7)) together with (2.1.18), (2.1.14), and [70, (1.4.24)], imply that in the case when Ω is an exterior domain we have $\vec{F}(x) = O(|x|^{-n})$ as $x \in \Omega$ satisfies $|x| \to \infty$; hence, the pointwise decay property [68, (1.2.9)] is satisfied. (2.1.25)

Collectively, (2.1.19), (2.1.20), (2.1.22), (2.1.24), (2.1.25) guarantee the validity of the Divergence Formula [68, (1.2.2)] which, in light of (2.1.20) and (2.1.23), presently gives

$$0 = \int_{\Omega} \operatorname{div}\vec{F}\, d\mathcal{L}^n = \int_{\partial\Omega} \nu \cdot \big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\, d\sigma$$
$$= -\frac{1}{2}\int_{\partial\Omega} a_\alpha \, d\sigma + \int_{\partial\Omega} m_\alpha \, d\sigma = \int_{\partial\Omega} m_\alpha \, d\sigma, \tag{2.1.26}$$

bearing in mind the vanishing moment property in (2.1.7). Having established this, we then conclude that $\int_{\partial\Omega} m_\alpha \, d\sigma = 0$ for each index $\alpha \in \{1, \dots, M\}$, finishing the proof of (2.1.13).

Let us record our progress: from (2.1.9), (2.1.12), and (2.1.13) we deduce that, whenever a is as in (2.1.7), the components of the function $m := K^{\#}a : \partial\Omega \to \mathbb{C}^M$ are fixed multiples (i.e., the multiplicative constant is independent of the atom a) of molecules for the Hardy space $H^p(\partial\Omega, \sigma)$, in the sense of [69, Definition 4.5.1]. Granted this, it follows from [69, (4.5.6)] that

$$m \in \big[H^p(\partial\Omega, \sigma)\big]^M \text{ and } \|m\|_{[H^p(\partial\Omega,\sigma)]^M} \leq C(\partial\Omega, L, p) \in (0, \infty). \tag{2.1.27}$$

Also, in the case when $\partial\Omega$ is compact and the (p, q)-atom a is $\mathbb{C}^M$-valued constant function on $\partial\Omega$, of absolute value $\leq \sigma(\partial\Omega)^{-1/p}$, it follows from item *(ii-i)* of Theorem 1.5.1 that the fucntion $m := K^{\#}a$ belongs to $\big[L^2(\partial\Omega, \sigma)\big]^M$ and satisfies $\|m\|_{[L^2(\partial\Omega,\sigma)]^M} \leq C(\partial\Omega, L, p) \in (0, \infty)$. In view of this and the fact that $\big[L^2(\partial\Omega, \sigma)\big]^M$ presently embeds continuously into $\big[H^p(\partial\Omega, \sigma)\big]^M$ (cf. [69, (4.2.13)]), we see that the conclusions in (2.1.27) hold in this case as well.

Having established (2.1.27) in all circumstances, we may now invoke [69, Theorem 4.4.7] (whose applicability in the present setting makes use of (1.5.12)) to conclude that, indeed, the mapping $K^{\#}$, originally considered as in (1.5.12), extends uniquely to a linear and bounded operator from the Hardy space $\big[H^p(\partial\Omega, \sigma)\big]^M$ into itself. Finally, that various choices of $p \in \big(\frac{n-1}{n}, 1\big]$ in (2.1.4) yield operators which are compatible with one another may now be seen with the help of [69, Theorem 4.4.3].

Let us now deal with the claim made in (2.1.6). In this regard, fix some exponent $p \in \big(\frac{n-1}{n}, \infty\big)$ and observe that both the composition to the left of $K^{\#}$ from (2.1.5) with the L^p-filtering operator $\mathfrak{H} : \big[H^p(\partial\Omega, \sigma)\big]^M \to \big[L^p(\partial\Omega, \sigma)\big]^M$ and the operator $K^{\#}$ from (2.1.1) are well-defined continuous mappings from $\big[H^p(\partial\Omega, \sigma)\big]^M$ into $\big[L^p(\partial\Omega, \sigma)\big]^M$. Fix an exponent $q \in (1, \infty)$ and recall from [69, (4.4.114)] that the space $\big[H^p(\partial\Omega, \sigma) \cap L^q(\partial\Omega, \sigma)\big]^M$ is dense in $\big[H^p(\partial\Omega, \sigma)\big]^M$. Given that, thanks to (1.5.12) and [69, (4.9.3)], said operators agree on $\big[H^p(\partial\Omega, \sigma) \cap L^q(\partial\Omega, \sigma)\big]^M$, the desired conclusion follows.

To conclude the proof of Theorem 2.1.1 there remains to justify the very last claim in the statement. First, from (2.1.27) and the fact that the intervening constant $C(\partial\Omega, L, p) \in (0,\infty)$ depends in a bounded fashion on the coefficient tensor A (used to write L), we conclude that the operator $K^{\#}$ in the context of (2.1.5) depends in a bounded manner on the underlying coefficient tensor. Granted this, the desired conclusion follows from (1.5.96) and an interpolation inequality (of the sort discussed a little later in (2.1.37)). □

Recall the Lorentz-based Hardy spaces from [69, Definition 4.2.3] (cf. (A.0.81)).

Corollary 2.1.2 *Retain the setting of Theorem 2.1.1. Then for each* $p \in \big(\tfrac{n-1}{n}, \infty\big)$ *and* $q \in (0,\infty]$, *the operator* $K^{\#}$, *originally acting on Lebesgue spaces on* $\partial\Omega$ (*as in* (1.5.12)), *induces a linear and bounded mapping*

$$K^{\#} : \big[H^{p,q}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[H^{p,q}(\partial\Omega,\sigma)\big]^M \tag{2.1.28}$$

whose action continues to be compatible with that of $K^{\#}$ *in* (2.1.5). *Also,*

> *for each* $p \in \big(\tfrac{n-1}{n}, \infty\big)$ *and* $q \in (0,\infty]$, *the composition to the left of* $K^{\#}$ *from* (2.1.28) *with the operator* $\mathfrak{H}$ *mapping* $\big[H^{p,q}(\partial\Omega,\sigma)\big]^M$ *into* $\big[L^{p,q}(\partial\Omega,\sigma)\big]^M$ (*cf.* [69, (4.9.5) in Theorem 4.9.1]) *is the transpose double layer potential operator* $K^{\#} : \big[H^{p,q}(\partial\Omega,\sigma)\big]^M \to \big[L^{p,q}(\partial\Omega,\sigma)\big]^M$ *from* (2.1.3). (2.1.29)

Moreover,

$$K^{\#} : \big[L^{1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[H^{1,\infty}(\partial\Omega,\sigma)\big]^M \tag{2.1.30}$$

is a well-defined, linear and bounded mapping, and

> *the composition to the left of* $K^{\#}$ *from* (2.1.30) *with the filtering operator* $\mathfrak{H} : \big[H^{1,\infty}(\partial\Omega,\sigma)\big]^M \to \big[L^{1,\infty}(\partial\Omega,\sigma)\big]^M$ (*cf.* [69, (4.9.5) in Theorem 4.9.1]) *is the principal value operator* $K^{\#} : \big[L^{1}(\partial\Omega,\sigma)\big]^M \to \big[L^{1,\infty}(\partial\Omega,\sigma)\big]^M$ *from* (1.5.13). (2.1.31)

Proof The claims about (2.1.28) are seen from Theorem 2.1.1, [69, (4.3.3)], and real interpolation. In particular, (2.1.28) with $p = 1$ and $q = \infty$ gives that $K^{\#}$ maps $\big[H^{1,\infty}(\partial\Omega,\sigma)\big]^M$ boundedly into itself. Granted this, the fact that $K^{\#}$ in (2.1.30) is a well-defined, linear and bounded mapping follows on account of [69, (4.2.28)] (used with $\Sigma := \partial\Omega$).

To prove (2.1.29), fix $p \in \big(\tfrac{n-1}{n}, \infty\big)$ and $q \in (0,\infty]$, then select $p_0, p_1 \in \big(\tfrac{n-1}{n}, \infty\big)$ such that $p_0 < p < p_1$. Since the operators (2.1.5) act in a compatible fashion with one another, we may extend

$$K^{\#} : \big[H^{p_0}(\partial\Omega,\sigma) + H^{p_1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[H^{p_0}(\partial\Omega,\sigma) + H^{p_1}(\partial\Omega,\sigma)\big]^M \tag{2.1.32}$$

by setting $K^{\#}(f_0 + f_1) := K^{\#} f_0 + K^{\#} f_1$ for each $f_j \in \big[H^{p_j}(\partial\Omega, \sigma)\big]^M$ with $j \in \{0, 1\}$. Likewise, starting from the fact that the operators (2.1.1) act coherently, we may once again naturally extend

$$K^{\#} : \big[H^{p_0}(\partial\Omega, \sigma) + H^{p_1}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{p_0}(\partial\Omega, \sigma) + L^{p_1}(\partial\Omega, \sigma)\big]^M \tag{2.1.33}$$

Given that [69, Theorem 4.9.1] also allows us to canonically extend

$$\mathfrak{H} : \big[H^{p_0}(\partial\Omega, \sigma) + H^{p_1}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{p_0}(\partial\Omega, \sigma) + L^{p_1}(\partial\Omega, \sigma)\big]^M, \tag{2.1.34}$$

we may invoke (2.1.6) to conclude that the composition to the left of $K^{\#}$ from (2.1.32) with $\mathfrak{H}$ from (2.1.34) is the operator $K^{\#}$ from (2.1.33). With this in hand, and recalling from [69, (1.3.41), (4.3.3)] that

$$H^{p,q}(\partial\Omega, \sigma) \subseteq H^{p_0}(\partial\Omega, \sigma) + H^{p_1}(\partial\Omega, \sigma), \tag{2.1.35}$$

the conclusion in (2.1.29) follows.

At this stage in the proof there remains to deal with the claim in (2.1.31). To this end, observe first that both the composition to the left of the operator $K^{\#}$ from (2.1.30) with the filtering operator $\mathfrak{H} : \big[H^{1,\infty}(\partial\Omega, \sigma)\big]^M \to \big[L^{1,\infty}(\partial\Omega, \sigma)\big]^M$ and the principal-value singular integral operator $K^{\#} : \big[L^1(\partial\Omega, \sigma)\big]^M \to \big[L^{1,\infty}(\partial\Omega, \sigma)\big]^M$ from (1.5.13) are well-defined continuous mappings from the space $\big[L^1(\partial\Omega, \sigma)\big]^M$ into $\big[L^{1,\infty}(\partial\Omega, \sigma)\big]^M$. Fix $q \in (1, \infty)$. Since $\big[L^1(\partial\Omega, \sigma) \cap L^q(\partial\Omega, \sigma)\big]^M$ is dense in $\big[L^1(\partial\Omega, \sigma)\big]^M$ and said operators agree on $\big[L^1(\partial\Omega, \sigma) \cap L^q(\partial\Omega, \sigma)\big]^M$ (thanks to (1.5.12) and [69, (4.9.3)]), the desired conclusion follows. □

To sate our next result, recall the piece of notation introduced in (A.0.171).

Proposition 2.1.3 *Retain assumptions made in Theorem 2.1.1 on the set $\Omega \subseteq \mathbb{R}^n$ and the $M \times M$ system L. In addition, suppose*

$$\tfrac{n-1}{n} < p_o < p \le 1 < q_o < \infty, \quad 0 < q < \infty, \tag{2.1.36}$$

and consider $\theta \in (0, 1)$ such that $1/p = (1-\theta)/p_o + \theta/q_o$. Then there exists a finite constant $C > 0$, depending only on $\partial\Omega$, n, p_o, p, q_o, q, with the property that the norm of the operator $K^{\#}$ in (2.1.28) may be estimated as

$$\|K^{\#}\|_{\mathrm{Bd}([H^{p,q}(\partial\Omega,\sigma)]^M)} \le C \|K^{\#}\|^{1-\theta}_{\mathrm{Bd}([H^{p_o}(\partial\Omega,\sigma)]^M)} \cdot \|K^{\#}\|^{\theta}_{\mathrm{Bd}([L^{q_o}(\partial\Omega,\sigma)]^M)}. \tag{2.1.37}$$

Proof This is a consequence of Theorem 2.1.1, [69, (4.3.3)], the interpolation estimate from [69, Proposition 1.3.7, (1.3.64)], and (A.0.171). Attention should be paid to the fact that [69, (1.3.64)] works when the target spaces are quasi-normed lattices of functions (which is not the case for Hardy spaces H^p with $p \le 1$). One remedy is to apply [69, (1.3.64)] to the sub-linear operator $Tf := (K^{\#} f)^{\sharp}_{\gamma}$ (i.e., the Fefferman-Stein grand maximal function of $K^{\#} f$; cf. [69, (4.1.6)]) which now takes values in Lebesgue spaces. Alternatively, we may take advantage of the fact that $K^{\#}$

is a linear operator and use the version of [69, Proposition 1.3.7] discussed in [68, Remark 1.3.8].

In the special case when $q = p$ (a scenario in which $H^{p,q}(\partial\Omega, \sigma)$ simply becomes $H^p(\partial\Omega, \sigma)$; cf. [69, (4.2.25)]), a direct proof of the estimate recorded in (2.1.37) is as follows. Pick some arbitrary $\mathbb{C}^M$-valued (p, q_o)-atom a on $\partial\Omega$. Consider first the case when said atom has vanishing moment, i.e., $a \in \big[L^{q_o}_{\rm comp}(\partial\Omega, \sigma)\big]^M$ satisfies $\int_{\partial\Omega} a\,d\sigma = 0$, and there exist a point $x_o \in \partial\Omega$ along with some radius $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ such that $\mathrm{supp}\, a \subseteq B(x_o, r) \cap \partial\Omega$ and

$$\|a\|_{[L^{q_o}(\partial\Omega,\sigma)]^M} \leq \sigma\big(B(x_o, r) \cap \partial\Omega\big)^{1/q_o - 1/p}. \tag{2.1.38}$$

In particular, $\sigma\big(B(x_o, r) \cap \partial\Omega\big)^{1/p - 1/p_o} \cdot a$ is a $\mathbb{C}^M$-valued (p_o, q_o)-atom on $\partial\Omega$. Writing that its quasi-norm in $[H^{p_o}(\partial\Omega, \sigma)]^M$ is bounded by a constant which depends only on the environment then yields

$$\|a\|_{[H^{p_o}(\partial\Omega,\sigma)]^M} \leq C \cdot \sigma\big(B(x_o, r) \cap \partial\Omega\big)^{1/p_o - 1/p}. \tag{2.1.39}$$

Bearing in mind the choice of θ, from (2.1.38)-(2.1.39) we conclude that

$$\|a\|^{1-\theta}_{[H^{p_o}(\partial\Omega,\sigma)]^M} \cdot \|a\|^{\theta}_{[L^{q_o}(\partial\Omega,\sigma)]^M} \leq C, \tag{2.1.40}$$

for some finite constant $C > 0$ independent of the atom. Since the above discussion guarantees that $a \in [H^{p_o}(\partial\Omega, \sigma)]^M \cap [L^{q_o}(\partial\Omega, \sigma)]^M$, it follows from Theorem 2.1.1 and (1.5.12) that $K^{\#}a$ also belongs to $[H^{p_o}(\partial\Omega, \sigma)]^M \cap [L^{q_o}(\partial\Omega, \sigma)]^M$. Based on this, [69, Proposition 4.2.2], (A.0.171), and (2.1.40), we may then estimate

$$\begin{aligned}
\|K^{\#}a\|_{[H^p(\partial\Omega,\sigma)]^M} &\leq C\|K^{\#}a\|^{1-\theta}_{[H^{p_o}(\partial\Omega,\sigma)]^M}\|K^{\#}a\|^{\theta}_{[L^{q_o}(\partial\Omega,\sigma)]^M} \\
&\leq C\|K^{\#}\|^{1-\theta}_{\mathrm{Bd}([H^{p_o}(\partial\Omega,\sigma)]^M)} \cdot \|K^{\#}\|^{\theta}_{\mathrm{Bd}([L^{q_o}(\partial\Omega,\sigma)]^M)} \times \\
&\qquad \times \|a\|^{1-\theta}_{[H^{p_o}(\partial\Omega,\sigma)]^M} \cdot \|a\|^{\theta}_{[L^{q_o}(\partial\Omega,\sigma)]^M} \\
&\leq C\|K^{\#}\|^{1-\theta}_{\mathrm{Bd}([H^{p_o}(\partial\Omega,\sigma)]^M)} \cdot \|K^{\#}\|^{\theta}_{\mathrm{Bd}([L^{q_o}(\partial\Omega,\sigma)]^M)}.
\end{aligned} \tag{2.1.41}$$

The same type of estimate is valid when $\partial\Omega$ is bounded and the atom a is constant. With this in hand, [69, Theorem 4.4.7] applies and yields (2.1.37) in the case when $q = p$. □

Pressing on, we discuss the action of the integral operators $T^{\#}_{jk}$ defined in (1.2.3) on the scale of Lorentz-based Hardy spaces on $\partial\Omega$.

Theorem 2.1.4 *Let* $\Omega \subseteq \mathbb{R}^n$ *(where* $n \in \mathbb{N}$, $n \geq 2$*) be a* UR *domain and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Consider a complex-valued function* $b \in L^1_{\rm loc}(\mathbb{R}^n, \mathcal{L}^n)$ *with the property* $b\big|_{\mathbb{R}^n\setminus\{0\}} \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ *for some sufficiently large number* $N = N(n) \in \mathbb{N}$, *and such that* ∇b *is odd and positive homogeneous of degree* $1-n$ *in* $\mathbb{R}^n \setminus \{0\}$. *Finally, fix* $j, k \in \{1, \dots, n\}$ *and recall the integral operator* $T^{\#}_{jk}$ *defined as in* (1.2.3).

Then for each $p \in \big(\frac{n-1}{n}, 1\big]$ *the operator* $T^{\#}_{jk}$ *from* (1.2.3)*, originally acting on Lebesgue spaces on* $\partial\Omega$ (*as in item (ii) of Proposition 1.2.1*)*, extends uniquely to a linear and bounded mapping*

$$T^{\#}_{jk} : H^p(\partial\Omega,\sigma) \longrightarrow H^p(\partial\Omega,\sigma), \tag{2.1.42}$$

and there exists a constant $C = C(\Omega, p) \in (0,\infty)$ *with the property that*

$$\big\|T^{\#}_{jk}\big\|_{H^p(\partial\Omega,\sigma)\to H^p(\partial\Omega,\sigma)} \leq C \cdot \sum_{|\alpha|\leq N-1} \sup_{S^{n-1}} \big|\partial^\alpha(\nabla b)\big|. \tag{2.1.43}$$

As a consequence of (2.1.42)*, item (ii) of Proposition 1.2.1,* [69, (4.3.3)]*, and real interpolation, for each* $p \in \big(\frac{n-1}{n}, \infty\big)$ *and* $q \in (0,\infty]$ *the operator*

$$T^{\#}_{jk} : H^{p,q}(\partial\Omega,\sigma) \longrightarrow H^{p,q}(\partial\Omega,\sigma) \tag{2.1.44}$$

is well defined, linear, bounded, and satisfies a similar norm estimate as above.

Proof The strategy, similar to that used in the proof of Theorem 2.1.1, is to show that if $a : \partial\Omega \to \mathbb{C}$ is an arbitrary (p, q)-atom, where $q \in (1,\infty)$ is a fixed exponent, then $m := T^{\#}_{jk} a$ is a fixed multiple of a molecule for the Hardy space $H^p(\partial\Omega,\sigma)$ (cf. [69, Definition 4.5.1]). Specifically, assuming that $x_o \in \partial\Omega$ and $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ are such that

$$\begin{gathered}\operatorname{supp} a \subseteq B(x_o,r)\cap\partial\Omega, \quad \|a\|_{L^q(\partial\Omega,\sigma)} \leq \sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1/q-1/p}, \\ \text{and} \int_{\partial\Omega} a\,\mathrm{d}\sigma = 0,\end{gathered} \tag{2.1.45}$$

the goal is to show that there exists some finite constant $C = C(\partial\Omega, n, b, p, q) > 0$ independent of the atom in question with the property that

$$\begin{gathered}\|m\|_{L^q(\partial\Omega,\sigma)} \leq C\sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1/q-1/p} \text{ and, for each } k \in \mathbb{N}, \\ \Big(\int_{\widetilde{A}_k(x_o,r)} |m|^q\,\mathrm{d}\sigma\Big)^{1/q} \leq C 2^{k(n-1)[1/q-1-1/(n-1)]}\sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1/q-1/p},\end{gathered} \tag{2.1.46}$$

where

$$\widetilde{A}_k(x_o,r) := \big[B(x_o,2^{k+1}r)\setminus B(x_o,2^k r)\big]\cap\partial\Omega, \qquad \forall k\in\mathbb{N}, \tag{2.1.47}$$

as well as

$$\int_{\partial\Omega} m\,\mathrm{d}\sigma = 0. \tag{2.1.48}$$

The estimates in (2.1.46) may be established by closely mimicking the arguments used in the proof (2.1.9) and (2.1.12) (here, item *(ii)* in Proposition 1.2.1 plays a role).

To justify the vanishing moment condition claimed in (2.1.48), for each index $\ell \in \{1, \dots, n\}$ we introduce two integral operators acting on scalar-valued functions $f \in L^{p_o}(\partial\Omega, \sigma)$ with $p_o \in (1, \infty)$ according to

$$\mathscr{B}_\ell f(x) := \int_{\partial\Omega} (\partial_\ell b)(x-y) f(y)\, d\sigma(y) \ \text{ for each } \ x \in \Omega, \tag{2.1.49}$$

and, respectively,

$$B_\ell f(x) := \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} (\partial_\ell b)(x-y) f(y)\, d\sigma(y) \ \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{2.1.50}$$

To proceed, select some background parameter $\kappa > 0$ and fix an arbitrary function $f \in L^{p_o}(\partial\Omega, \sigma)$ with $p_o \in (1, \infty)$. In addition, pick an integer $\ell \in \{1, \dots, n\}$. [70, Theorem 2.3.2] then ensures that the limit defining $B_\ell f$ in (2.1.50) exists for σ-a.e. point $x \in \partial\Omega$. Also, [70, Theorem 2.5.1] implies that at σ-a.e. point $x \in \partial\Omega$ we have the jump-formula

$$\big(\mathscr{B}_\ell f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) = \frac{1}{2\mathrm{i}} \widehat{\partial_\ell b}\big(\nu(x)\big) f(x) + (B_\ell f)(x), \tag{2.1.51}$$

where $\nu = (\nu_1, \dots, \nu_n)$ is the geometric measure theoretic outward unit normal to Ω. Let us now define the vector field

$$\vec{F} := (\mathscr{B}_k a)\mathbf{e}_j - (\mathscr{B}_j a)\mathbf{e}_k \in \big[\mathscr{C}^\infty(\Omega)\big]^n. \tag{2.1.52}$$

Thanks to (2.1.52) and (2.1.49) we then have

$$\begin{aligned}(\mathrm{div}\vec{F})(x) &= \partial_j(\mathscr{B}_k a)(x) - \partial_k(\mathscr{B}_j a)(x) \\ &= \int_{\partial\Omega} (\partial_j \partial_k b)(x-y) a(y)\, d\sigma(y) - \int_{\partial\Omega} (\partial_k \partial_j b)(x-y) a(y)\, d\sigma(y) \\ &= 0 \ \text{ for each } \ x \in \Omega. \end{aligned} \tag{2.1.53}$$

Furthermore, the jump-formula (2.1.51) implies

$$\begin{aligned}\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) &= (\mathscr{B}_k a)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x)\mathbf{e}_j - (\mathscr{B}_j a)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x)\mathbf{e}_k \\ &= \frac{1}{2\mathrm{i}} \widehat{\partial_k b}\big(\nu(x)\big) a(x)\mathbf{e}_j + (B_k a)(x)\mathbf{e}_j \\ &\quad - \frac{1}{2\mathrm{i}} \widehat{\partial_j b}\big(\nu(x)\big) a(x)\mathbf{e}_k - (B_j a)(x)\mathbf{e}_k \end{aligned} \tag{2.1.54}$$

at σ-a.e. point $x \in \partial\Omega$. Consequently, at σ-a.e. point $x \in \partial\Omega$ we have

$$\begin{aligned}\nu(x)\cdot\Big(\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) &= \frac{1}{2\mathrm{i}}\widehat{\partial_k b}\big(\nu(x)\big)\nu_j(x)a(x) + \nu_j(x)(B_k a)(x)\\ &\quad - \frac{1}{2\mathrm{i}}\widehat{\partial_j b}\big(\nu(x)\big)\nu_k(x)a(x) - \nu_k(x)(B_j a)(x)\\ &= \int_{\partial\Omega}\big\{\nu_j(x)(\partial_k b)(x-y) - \nu_k(x)(\partial_j b)(x-y)\big\}a(y)\,\mathrm{d}\sigma(y)\\ &= (T^{\#}_{jk}a)(x) = m(x), \end{aligned} \tag{2.1.55}$$

by also relying on [70, (2.5.19)], (2.1.50), and (1.2.3). Upon noting from (2.1.45) that the function $a : \partial\Omega \to \mathbb{C}$ is a multiple of an $(1, q)$-atom on $\partial\Omega$, we may use [70, (2.4.14)] with $p = 1$ to conclude that

$$\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma). \tag{2.1.56}$$

Finally, the vanishing moment property of the atom (cf. the last property in (2.1.45)) together with (2.1.52) and (2.1.49) imply that

$$\text{in the case when } \Omega \text{ is an exterior domain we have } \vec{F}(x) = O(|x|^{-n}) \text{ as } x \in \Omega \text{ satisfies } |x| \to \infty\text{; thus, the decay condition [68, (1.2.9)] is presently satisfied.} \tag{2.1.57}$$

Together, (2.1.52), (2.1.53), (2.1.54), (2.1.56), and (2.1.57) ensure the validity of the Divergence Formula [68, (1.2.2)]. In light of (2.1.53) and (2.1.55) this currently permits us to write

$$0 = \int_\Omega \operatorname{div}\vec{F}\,\mathrm{d}\mathcal{L}^n = \int_{\partial\Omega}\nu\cdot\big(\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma = \int_{\partial\Omega} m\,\mathrm{d}\sigma, \tag{2.1.58}$$

finishing the proof of (2.1.48). In summary, whenever a is as in (2.1.45), the function $m := T^{\#}_{jk}a$ is a fixed multiple of a molecule for the Hardy space $H^p(\partial\Omega, \sigma)$, in the sense of [69, Definition 4.5.1]. Having established this, it follows from [69, (4.5.6)] that there exists some finite constant $C > 0$ independent of the atom a such that

$$m \in H^p(\partial\Omega, \sigma) \text{ and } \|m\|_{H^p(\partial\Omega,\sigma)} \le C. \tag{2.1.59}$$

At this stage, there remains to consider the case when $\partial\Omega$ is compact and the atom a is of the form $\sigma(\partial\Omega)^{-1/p}$. In this scenario, item *(ii)* of Proposition 1.2.1 implies that the function $m := T^{\#}_{jk}a$ belongs to the space $L^2(\partial\Omega, \sigma)$ and satisfies $\|m\|_{L^2(\partial\Omega,\sigma)} \le C(\partial\Omega, b, p) \in (0, \infty)$. Bearing in mind that, in the current setting, $L^2(\partial\Omega, \sigma)$ embeds continuously into $H^p(\partial\Omega, \sigma)$ (cf. [69, (4.2.13)]), we see that the conclusions in (2.1.59) are valid in this case as well. Thus, (2.1.59) is true in all circumstances. Granted this, we may once again rely on [69, Theorem 4.4.7] (keeping (1.2.7) in mind) to conclude that, as claimed, the mapping $T^{\#}_{jk}$, originally considered as in (1.2.3), extends uniquely to a linear and bounded operator from the Hardy space $H^p(\partial\Omega, \sigma)$ into itself. □

A corollary of Theorems 2.1.1-2.1.4 worth mentioning at this stage pertains to the transpose Cauchy-Clifford singular integral operator $\mathfrak{C}^\#$ from (1.6.1). To put matters in perspective, recall that the principal-value singular integral operator $\mathfrak{C}^\#$ from (1.6.1), when $\Omega \subseteq \mathbb{R}^n$ is a UR domain and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, may be expressed at σ-a.e. $x \in \partial\Omega$ as

$$\mathfrak{C}^\# f(x) = -\nu(x) \odot \left(\lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot f(y)\, \mathrm{d}\sigma(y) \right). \tag{2.1.60}$$

As such, item *(6)* of [70, Theorem 2.3.2] guarantees that $\mathfrak{C}^\#$, originally acting on Lebesgue spaces on $\partial\Omega$ (as in (1.6.4)), extends uniquely to a linear and bounded operator

$$\mathfrak{C}^\# : H^p(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow L^p(\partial\Omega, \sigma) \otimes C\ell_n, \qquad p \in \left(\tfrac{n-1}{n}, \infty\right). \tag{2.1.61}$$

We may further interpolate (based on [69, Theorem 4.3.1] and [68, (6.2.48)]) to obtain a linear and bounded operator

$$\begin{gathered} \mathfrak{C}^\# : H^{p,q}(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow L^{p,q}(\partial\Omega, \sigma) \otimes C\ell_n, \\ \text{for } p \in \left(\tfrac{n-1}{n}, \infty\right) \text{ and } q \in (0, \infty]. \end{gathered} \tag{2.1.62}$$

This being said, due to its special algebraic nature, the operator $\mathfrak{C}^\#$ turns out to map Hardy spaces (respectively, Lorentz-based Hardy spaces) into themselves in a linear and bounded fashion. This, along with other related properties, are discussed in the theorem below.

Theorem 2.1.5 *Let $\Omega \subseteq \mathbb{R}^n$ be a* UR *domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Then $\mathfrak{C}^\#$, originally considered on Lebesgue spaces as in item (ii) of Proposition 1.6.1, extends to a linear and bounded operator in the context*

$$\mathfrak{C}^\# : H^p(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow H^p(\partial\Omega, \sigma) \otimes C\ell_n, \qquad \forall p \in \left(\tfrac{n-1}{n}, \infty\right) \tag{2.1.63}$$

and the operators $\mathfrak{C}^\#$ corresponding to various values of $p \in \left(\tfrac{n-1}{n}, \infty\right)$ are compatible with one another. Also,

for each $p \in \left(\tfrac{n-1}{n}, \infty\right)$, the composition to the left of $\mathfrak{C}^\#$ from (2.1.63) with the L^p-filtering operator $\mathfrak{H} : H^p(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow L^p(\partial\Omega, \sigma) \otimes C\ell_n$ (cf. [69, (4.9.2)]) is $\mathfrak{C}^\# : H^p(\partial\Omega, \sigma) \otimes C\ell_n \to L^p(\partial\Omega, \sigma) \otimes C\ell_n$ from (2.1.61). (2.1.64)

In addition,

$$\left(\mathfrak{C}^\#\right)^2 = \tfrac{1}{4} I \text{ on } H^p(\partial\Omega, \sigma) \otimes C\ell_n \text{ with } p \in \left(\tfrac{n-1}{n}, \infty\right). \tag{2.1.65}$$

More generally, for each $p \in \left(\frac{n-1}{n}, \infty\right)$ *and* $q \in (0, \infty]$ *the operator*

$$\mathfrak{C}^{\#} : H^{p,q}(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow H^{p,q}(\partial\Omega, \sigma) \otimes C\ell_n \tag{2.1.66}$$

is well defined, linear bounded, satisfies

$$\left(\mathfrak{C}^{\#}\right)^2 = \tfrac{1}{4} I \ \text{ on } \ H^{p,q}(\partial\Omega, \sigma) \otimes C\ell_n, \tag{2.1.67}$$

and

for each given exponents $p \in \left(\frac{n-1}{n}, \infty\right)$ *and* $q \in (0, \infty]$, *the composition to the left of the operator* $\mathfrak{C}^{\#}$ *defined in* (2.1.66) *with the filtering operator* $\mathfrak{H} : H^{p,q}(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow L^{p,q}(\partial\Omega, \sigma) \otimes C\ell_n$ (*cf.* [69, (4.9.5) in Theorem 4.9.1]) *is the operator* $\mathfrak{C}^{\#} : H^{p,q}(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow L^{p,q}(\partial\Omega, \sigma) \otimes C\ell_n$ *considered in* (2.1.62).
(2.1.68)

Finally,

$$\mathfrak{C}^{\#} : L^1(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow H^{1,\infty}(\partial\Omega, \sigma) \otimes C\ell_n \tag{2.1.69}$$

is a well-defined, linear and bounded mapping, and

the composition to the left of the operator $\mathfrak{C}^{\#}$ *from* (2.1.69) *with the filtering operator* $\mathfrak{H} : H^{1,\infty}(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow L^{1,\infty}(\partial\Omega, \sigma) \otimes C\ell_n$ (*cf.* [69, (4.9.5) in Theorem 4.9.1]) *is the operator* $\mathfrak{C}^{\#} : L^1(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow L^{1,\infty}(\partial\Omega, \sigma) \otimes C\ell_n$ *considered in the principal-value sense* (*cf.* (1.6.5)).
(2.1.70)

Before presenting the proof of this theorem we wish to note that since (2.1.67) implies

$$\left(\mathfrak{C}^{\#}\right)^2 = \tfrac{1}{4} I \ \text{ on } \ H^{1,\infty}(\partial\Omega, \sigma) \otimes C\ell_n, \tag{2.1.71}$$

and since [69, (4.2.28)] gives

$$L^1(\partial\Omega, \sigma) \otimes C\ell_n \hookrightarrow H^{1,\infty}(\partial\Omega, \sigma) \otimes C\ell_n, \tag{2.1.72}$$

we conclude that

$$\mathfrak{C}^{\#}\left(\mathfrak{C}^{\#} f\right) = \tfrac{1}{4} f \ \text{ for each } \ f \in L^1(\partial\Omega, \sigma) \otimes C\ell_n, \tag{2.1.73}$$

where $\mathfrak{C}^{\#} f$ is considered in the sense of (2.1.69), and the subsequent action of $\mathfrak{C}^{\#}$ on this distribution is taken in the sense of (2.1.66) with $p = 1$ and $q = \infty$.

Proof of Theorem 2.1.5 The claims made in regard to (2.1.63) follow from formula (1.6.19), elaborating on the nature of the components of $\mathfrak{C}^{\#}$, and Theorems 2.1.1-2.1.4. That the operators $\mathfrak{C}^{\#}$ in (2.1.5) corresponding to various values of

$p \in \big(\frac{n-1}{n}, \infty\big)$ are compatible with one another is a consequence of [69, Theorem 4.4.3] and (1.6.1). Going further, that (2.1.66) is a well-defined linear and bounded operator is implied by (2.1.63), item *(ii)* of Proposition 1.6.1, [69, (4.3.3)], and real interpolation. Next, recall from [70, Proposition 2.5.32] that $\mathfrak{C}^2 = \frac{1}{4}I$ on $L^q(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ with $q \in (1, \infty)$. In concert with item *(ii)* in Proposition 1.6.1 this gives

$$\big(\mathfrak{C}^{\#}\big)^2 = \tfrac{1}{4}I \text{ on } L^q(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \text{ with } q \in (1, \infty). \tag{2.1.74}$$

Fix now $p \in \big(\frac{n-1}{n}, 1\big]$ along with $q \in (1, \infty)$. Then (2.1.74) implies that we have $\big(\mathfrak{C}^{\#}\big)^2 = \frac{1}{4}I$ on the space $H^{p,q}_{\text{fin}}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$. Granted this, from [69, (4.4.114)] and (2.1.63) we then conclude that

$$\big(\mathfrak{C}^{\#}\big)^2 = \tfrac{1}{4}I \text{ on } H^p(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \text{ with } p \in \big(\tfrac{n-1}{n}, 1\big]. \tag{2.1.75}$$

Combining (2.1.74) and (2.1.75) we see that (2.1.65) holds.

Next, (2.1.64) is justified by reasoning as in the proof of (2.1.6), while (2.1.68) is established much as (2.1.29). Taking $p = 1$ and $q = \infty$ in (2.1.66) gives that $\mathfrak{C}^{\#}$ maps $H^{1,\infty}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ boundedly into itself. Having established this, we may now conclude that $\mathfrak{C}^{\#}$ in (2.1.69) is a well-defined, linear and bounded mapping on account of [69, (4.2.28)] (used here with $\Sigma := \partial\Omega$). Also, the claim in (2.1.70) may be proved along the lines of (2.1.31).

Finally, there remains to prove (2.1.67). Suppose $\frac{n-1}{n} < p_0 < p_1 < \infty$. From the first part of the proof we know that $\big(\mathfrak{C}^{\#}\big)^2$ acts like $\frac{1}{4}I$ both on $H^{p_0}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ and on $H^{p_1}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$. Since $\mathfrak{C}^{\#}$ on $H^{p_0}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ is compatible with $\mathfrak{C}^{\#}$ on $H^{p_1}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$, we conclude that, on the one hand, $\big(\mathfrak{C}^{\#}\big)^2$ acts like $\frac{1}{4}$ on $H^{p_0}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n + H^{p_1}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$. On the other hand, from [69, (1.3.41), (4.3.3)] we see that

$$H^{p,q}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \subseteq H^{p_0}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n + H^{p_1}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \tag{2.1.76}$$

whenever $p \in (p_0, p_1)$ and $q \in (0, \infty]$. Thus, ultimately, we obtain $\big(\mathfrak{C}^{\#}\big)^2 = \frac{1}{4}I$ on $H^{p,q}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ whenever $p_0 < p < p_1$ and $0 < q \leq \infty$. Given that p_0, p_1 have been arbitrarily chosen in $\big(\frac{n-1}{n}, \infty\big)$, this finishes the proof of (2.1.67). □

The Clifford algebra formalism allows us to consider the Riesz transforms bundled together, into a single entity, we call the boundary-to-domain `Clifford-Riesz transform`. Specifically, given any Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ then define the action of the Clifford-Riesz transform on functions $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes \mathcal{C}\ell_n$ as

$$\mathcal{R}_{\mathcal{C}\ell} f(x) := \frac{2}{\omega_{n-1}} \int_{\partial\Omega} \frac{x-y}{|x-y|^n} \odot f(y)\, d\sigma(y), \quad \forall x \in \Omega. \tag{2.1.77}$$

At least formally, the scalar components of the Clifford-Riesz transform agree, up to normalization, with the first-order partial derivatives of the single layer operator. The study of the Clifford-Riesz transform in arbitrary UR domains is taken up next.

Theorem 2.1.6 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In relation to the Clifford-Riesz transform (2.1.77) the following properties are true.*

(a) With D denoting the Dirac operator (A.0.55), one has

$$D\big(\mathcal{R}_{C\ell} f\big) = 0 \text{ in } \Omega, \quad \forall f \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big) \otimes C\ell_n. \tag{2.1.78}$$

(b) For each integrability exponent $p \in (1,\infty)$ and each aperture parameter $\kappa > 0$ one has[1]

$$\mathcal{R}_{C\ell} : L^p(\partial\Omega,\sigma) \otimes C\ell_n \longrightarrow N^p_\kappa(\Omega;\sigma) \otimes C\ell_n \quad \text{linearly and boundedly.} \tag{2.1.79}$$

Hence, for each given exponent $p \in (1,\infty)$ and $\kappa \in (0,\infty)$ there exists a constant $C = C(\Omega, p, \kappa) \in (0,\infty)$ such that for each $f \in L^p(\partial\Omega,\sigma) \otimes C\ell_n$ one has

$$\big\|\mathcal{N}_\kappa\big(\mathcal{R}_{C\ell} f\big)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p(\partial\Omega,\sigma)\otimes C\ell_n} \tag{2.1.80}$$

Moreover, corresponding to $p = 1$, there exists a constant $C = C(\Omega,\kappa) \in (0,\infty)$ such that

$$\big\|\mathcal{N}_\kappa\big(\mathcal{R}_{C\ell} f\big)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \leq C\|f\|_{L^1(\partial\Omega,\sigma)\otimes C\ell_n} \tag{2.1.81}$$

for each $f \in L^1(\partial\Omega,\sigma) \otimes C\ell_n$. Finally, for each $f \in L^p(\partial\Omega,\sigma) \otimes C\ell_n$ with $p \in [1,\infty)$ and $\kappa > 0$ one has

$$\nu \odot \big(\mathcal{R}_{C\ell} f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(I - 2\mathfrak{C}^{\#}\big) f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{2.1.82}$$

where $\mathfrak{C}^{\#}$ is the principal-value singular integral operator from (1.6.1).

(c) Given any integrability exponent $p \in \big(\frac{n-1}{n}, 1\big]$ and aperture parameter $\kappa > 0$, the Clifford-Riesz transform (2.1.79) extends uniquely to a linear and bounded operator

$$\mathcal{R}_{C\ell} : H^p(\partial\Omega,\sigma) \otimes C\ell_n \longrightarrow N^p_\kappa(\Omega;\sigma) \otimes C\ell_n. \tag{2.1.83}$$

Specifically, if for each $j \in \{1,\dots,n\}$ one introduces

$$\Phi_j(x) := \frac{1}{\omega_{n-1}} \frac{x_j}{|x|^n}, \qquad \forall x = (x_1,\dots,x_n) \in \mathbb{R}^n \setminus \{0\}, \tag{2.1.84}$$

then said extension of (2.1.79) acts according to[2]

---

[1] with $N^p_\kappa(\Omega;\sigma)$ defined as in [68, Proposition 8.3.5]

[2] with duality brackets $\langle\cdot,\cdot\rangle$ in the sense of [69, Theorem 4.6.1]

$$\begin{gathered}\big(\mathcal{R}_{C\ell} f\big)(x) = 2\sum_J \sum_{j=1}^n \big\langle \big[\Phi_j(x-\cdot)\big|_{\partial\Omega}\big], f_J\big\rangle \mathbf{e}_j \odot e_J \\ \text{for each } f = \textstyle\sum_J f_J e_J \in H^p(\partial\Omega,\sigma)\otimes C\ell_n \text{ and } x\in\Omega,\end{gathered} \tag{2.1.85}$$

in the case when $\partial\Omega$ is unbounded, and a similar formula in the case when $\partial\Omega$ is bounded (omitting taking the equivalence class, modulo constants, of $\Phi_j(x-\cdot)\big|_{\partial\Omega}$ this time). Hence, given any $p\in\big(\frac{n-1}{n},\infty\big)$ and $\kappa\in(0,\infty)$, the Clifford-Riesz transform $\mathcal{R}_{C\ell}$ (defined as in (2.1.77), (2.1.85)) satisfies

$$\big\|\mathcal{N}_\kappa\big(\mathcal{R}_{C\ell} f\big)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{H^p(\partial\Omega,\sigma)\otimes C\ell_n} \tag{2.1.86}$$

for each $f\in H^p(\partial\Omega,\sigma)\otimes C\ell_n$, for some constant $C=C(\Omega,p,\kappa)\in(0,\infty)$ independent of f. In addition,

$$D\big(\mathcal{R}_{C\ell} f\big) = 0 \text{ in } \Omega, \quad \forall f\in H^p(\partial\Omega,\sigma)\otimes C\ell_n. \tag{2.1.87}$$

(d) For each $f\in H^p(\partial\Omega,\sigma)\otimes C\ell_n$ with $p\in\big(\frac{n-1}{n},\infty\big)$, the Clifford-Riesz transform $\mathcal{R}_{C\ell}$ (defined as in (2.1.77), (2.1.85)) satisfies

$$\nu \circledbullet \big(\mathcal{R}_{C\ell} f\big) = \big(I-2\mathfrak{C}^\#\big)f \text{ in } H^p(\partial\Omega,\sigma)\otimes C\ell_n, \tag{2.1.88}$$

where the Clifford bullet product is defined as in (A.0.167), and where $\mathfrak{C}^\#$ is the "transpose" Cauchy-Clifford operator from (2.1.63).

Proof The claim in item *(a)* is a straightforward consequence of definitions, bearing in mind that the function $\mathbb{R}^n\setminus\{0\}\ni x\mapsto x/|x|^n\in C\ell_n$ is a null-solution of D. As regards item *(b)*, the claims in (2.1.79)-(2.1.81) are clear from [70, Theorem 2.4.1], while the claim in (2.1.82) is implied by (2.1.77), [70, Theorem 2.5.1], and (1.6.1). Alternatively, we may rely on (A.0.53), [70, (2.5.309)], and the observation that, with ν denoting the geometric measure theoretic outward unit normal to Ω, we have

$$\mathcal{R}_{C\ell} f = -2\mathcal{C}(\nu\odot f) \text{ in } \Omega, \text{ for each } f\in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big)\otimes C\ell_n. \tag{2.1.89}$$

Moving on to item *(c)*, the claims pertaining to (2.1.83)-(2.1.86) are consequences of definitions and [70, Theorem 2.4.1], while (2.1.87) follows with the help of Lemma 2.2.1.

To deal with the claim in item *(d)*, assume first that $p\in\big(\frac{n-1}{n},1\big]$. Thanks to item *(c)* and the discussion in [69, Example 10.2.14], the mapping

$$H^p(\partial\Omega,\sigma)\otimes C\ell_n \ni f \longmapsto \nu\circledbullet\big(\mathcal{R}_{C\ell} f\big)\in H^p(\partial\Omega,\sigma)\otimes C\ell_n \tag{2.1.90}$$

is well defined, linear and bounded. From (2.1.63) we know that $\mathfrak{C}^\#$ is also a well-defined, linear and bounded operator from $H^p(\partial\Omega,\sigma)\otimes C\ell_n$ into itself. As such, the claim in (2.1.88) follows as soon as we show that these two operators agree on a dense subspace of $H^p(\partial\Omega,\sigma)\otimes C\ell_n$. To this end, pick $q\in(1,\infty)$ and consider $f\in H^{p,q}_{\text{fin}}(\partial\Omega,\sigma)\otimes C\ell_n$. Then from [69, (10.2.104)] and (2.1.82) we see that for each fixed aperture parameter $\kappa>0$ we have

$$\nu \circledbullet (\mathcal{R}_{C\ell} f) = \nu \odot (\mathcal{R}_{C\ell} f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = (I - 2\mathfrak{C}^{\#}) f. \tag{2.1.91}$$

On account of [69, (4.4.114)] and the fact that the principal-value singular integral operator $\mathfrak{C}^{\#}$ from (1.6.1) acts in a compatible fashion with $\mathfrak{C}^{\#}$ in (2.1.63), the desired conclusion now follows. Finally, that the same conclusion holds when $p \in (1, \infty)$ is seen from (2.1.91) which remains valid when $f \in L^p(\partial\Omega, \sigma) \otimes C\ell_n$. □

There are many well-documented instances when the space L^∞ is inadequate and BMO turns out to be the correct substitute. For example, the Riesz transforms in the Euclidean space are not bounded on L^∞ but (their modified versions) are bounded on BMO, and the latter is the smallest space containing L^∞ with this property. We wish to show that the principal-value double layer operator associated with a given weakly elliptic system in a UR domain may be suitably extended to the space of functions of bounded mean oscillations in a manner that renders that map continuous from said space into itself. This is made precise in our next theorem.

Theorem 2.1.7 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and define $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. Also, let $L = \big(a^{\alpha\beta}_{rs} \partial_r \partial_s\big)_{1 \leq \alpha,\beta \leq M}$ be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). In this context, consider the boundary layer potential operators K and $K^{\#}$, associated with the system L and the set Ω as in (1.3.68) and (1.3.72), respectively. Finally, recall the modified boundary-to-boundary double layer potential operator K_{mod} from (1.8.24)-(1.8.25).*

Then the operators

$$K_{\text{mod}} : \big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M \tag{2.1.92}$$

and

$$\begin{gathered}\big[K_{\text{mod}}\big] : \big[\mathrm{BMO}(\partial\Omega, \sigma)/\sim\big]^M \longrightarrow \big[\mathrm{BMO}(\partial\Omega, \sigma)/\sim\big]^M \ \text{ defined as} \\ \big[K_{\text{mod}}\big][f] := \big[K_{\text{mod}} f\big] \ \text{ for each function } f \in \big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M\end{gathered} \tag{2.1.93}$$

are well-defined, linear, and bounded. Moreover, in the case when $\partial\Omega$ is bounded, a scenario in which one has

$$\big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M \subseteq \bigcap_{0<p<\infty} \big[L^p(\partial\Omega, \sigma)\big]^M, \tag{2.1.94}$$

the operator K acting on the Lebesgue scale $\big[L^p(\partial\Omega, \sigma)\big]^M$ with $p \in (1, \infty)$ has $\big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M$ as an invariant subspace, and its restriction

$$K : \big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M \tag{2.1.95}$$

is a well-defined, linear and bounded operator. Finally, *the (real) transpose of the operator $K^{\#}$ from (2.1.4) with $p = 1$ is, respectively, $\big[K_{\text{mod}}\big]$ from (2.1.93) if $\partial\Omega$ is unbounded, and K from (2.1.95) if $\partial\Omega$ is bounded,* (2.1.96)

that is, for each $f \in \big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M$ and $g \in \big[H^1(\partial\Omega, \sigma)\big]^M$ one has (with duality brackets as in [69, Theorem 4.6.1])

$$\big\langle \big[K_{\text{mod}} f\big], g\big\rangle = \big\langle [f], K^{\#} g\big\rangle \text{ if } \partial\Omega \text{ is unbounded,} \tag{2.1.97}$$

$$\big\langle K f, g\big\rangle = \big\langle f, K^{\#} g\big\rangle \text{ if } \partial\Omega \text{ is bounded.} \tag{2.1.98}$$

Proof Recall from [68, (7.4.118)] that

$$\big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M \subseteq \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^n}\Big)\Big]^M. \tag{2.1.99}$$

If we now pick an arbitrary function $f = (f_\alpha)_{1 \le \alpha \le M} \in \big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M$, from (2.1.99), [70, (2.3.35)], (1.8.24), and [68, (7.4.105)] we see that

$$K_{\text{mod}} f \in \bigcap_{1 \le p < \infty} \big[L^p_{\text{loc}}(\partial\Omega, \sigma)\big]^M. \tag{2.1.100}$$

Let us now consider an arbitrary $\mathbb{C}^M$-valued $(1, \infty)$-atom $a = (a_\gamma)_{1 \le \gamma \le M}$ on $\partial\Omega$ with vanishing moment[3]. Concretely, a is $\mathbb{C}^M$-valued, σ-measurable function defined on $\partial\Omega$, with the property that there exist $x_o \in \partial\Omega$ and $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ such that

$$\begin{gathered} \operatorname{supp} a \subseteq B(x_o, r) \cap \partial\Omega, \quad \|a\|_{[L^\infty(\partial\Omega,\sigma)]^M} \le \sigma\big(B(x_o, r) \cap \partial\Omega\big)^{-1}, \\ \text{and } \int_{\partial\Omega} a \, d\sigma = 0 \in \mathbb{C}^M. \end{gathered} \tag{2.1.101}$$

From (2.1.8) and [69, (4.5.33)] we know that

$$\int_{\partial\Omega} |f||K^{\#} a| \, d\sigma < +\infty, \tag{2.1.102}$$

and we make the claim that we also have $\int_{\partial\Omega} |K_{\text{mod}} f||a| \, d\sigma < +\infty$ as well as

$$\int_{\partial\Omega} \langle K_{\text{mod}} f, a\rangle \, d\sigma = \int_{\partial\Omega} \langle f, K^{\#} a\rangle \, d\sigma. \tag{2.1.103}$$

Indeed, this is a direct consequence of (1.8.43). A direct proof of (2.1.103) goes as follows. First, bearing (2.1.102) in mind, we write (using the summation convention over repeated indices, as in the past)

[3] this is always the case if $\partial\Omega$ is unbounded

$$\int_{\partial\Omega} f_\alpha (K^\# a)_\alpha \, d\sigma = \int_{\partial\Omega\cap B(x_o,2r)} f_\alpha (K^\# a)_\alpha \, d\sigma + \int_{\partial\Omega\setminus B(x_o,2r)} f_\alpha (K^\# a)_\alpha \, d\sigma$$
$$=: \mathrm{I} + \mathrm{II}, \tag{2.1.104}$$

with the last equality defining I and II. In relation to these, note that on account of item *(iii)* in Theorem 1.5.1 and [68, (7.4.105)] we may re-write I as

$$\mathrm{I} = \int_{\partial\Omega} \big(f \cdot \mathbf{1}_{\partial\Omega\cap B(x_o,2r)}\big)_\alpha (K^\# a)_\alpha \, d\sigma = \int_{\partial\Omega} \Big(K\big(f \cdot \mathbf{1}_{\partial\Omega\cap B(x_o,2r)}\big)\Big)_\gamma a_\gamma \, d\sigma$$
$$= \int_{\partial\Omega} \Big(K_{\text{mod}}\big(f \cdot \mathbf{1}_{\partial\Omega\cap B(x_o,2r)}\big)\Big)_\gamma a_\gamma \, d\sigma \tag{2.1.105}$$

since $K\big(f \cdot \mathbf{1}_{\partial\Omega\cap B(x_o,2r)}\big)$ differs from $K_{\text{mod}}\big(f \cdot \mathbf{1}_{\partial\Omega\cap B(x_o,2r)}\big)$ by a constant (from $\mathbb{C}^M$) on $\partial\Omega$, thanks to [70, (2.3.34)] and the fact that the atom has integral zero (cf. the last property in (2.1.101)). Also,

$$
\begin{aligned}
\mathrm{II} &= \int_{\partial\Omega\setminus B(x_o,2r)} f_\alpha(x)\bigg(\int_{\partial\Omega} \nu_s(x) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y) a_\gamma(y)\,\mathrm{d}\sigma(y)\bigg)\,\mathrm{d}\sigma(x)\\
&= \int_{\partial\Omega\setminus B(x_o,2r)} f_\alpha(x)\bigg(\int_{\partial\Omega} \nu_s(x) a^{\beta\alpha}_{rs} k^{(r\gamma\beta)}_{\varepsilon}(x-y) a_\gamma(y)\,\mathrm{d}\sigma(y)\bigg)\,\mathrm{d}\sigma(x)\\
&= \int_{\partial\Omega\setminus B(x_o,2r)} f_\alpha(x)\bigg(\int_{\partial\Omega} \nu_s(x) a^{\beta\alpha}_{rs} \big\{k^{(r\gamma\beta)}_{\varepsilon}(x-y)\\
&\qquad\qquad + k^{(r\gamma\beta)}_{1}(-x)\big\} a_\gamma(y)\,\mathrm{d}\sigma(y)\bigg)\,\mathrm{d}\sigma(x)\\
&= \int_{\partial\Omega\setminus B(x_o,2r)} f_\alpha(x)\bigg(\int_{\partial\Omega\cap B(x_o,r)} \nu_s(x) a^{\beta\alpha}_{rs} \big\{k^{(r\gamma\beta)}_{\varepsilon}(x-y)\\
&\qquad\qquad + k^{(r\gamma\beta)}_{1}(-x)\big\} a_\gamma(y)\,\mathrm{d}\sigma(y)\bigg)\,\mathrm{d}\sigma(x)\\
&= \int_{\partial\Omega\cap B(x_o,r)} \bigg(\int_{\partial\Omega\setminus B(x_o,2r)} \nu_s(x) a^{\beta\alpha}_{rs} \big\{k^{(r\gamma\beta)}_{\varepsilon}(x-y)\\
&\qquad\qquad + k^{(r\gamma\beta)}_{1}(-x)\big\} f_\alpha(x)\,\mathrm{d}\sigma(x)\bigg) a_\gamma(y)\,\mathrm{d}\sigma(y)\\
&= \int_{\partial\Omega} \lim_{\varepsilon\to 0^+} \bigg(\int_{\partial\Omega\setminus B(x_o,2r)} \nu_s(x) a^{\beta\alpha}_{rs} \big\{k^{(r\gamma\beta)}_{\varepsilon}(x-y)\\
&\qquad\qquad + k^{(r\gamma\beta)}_{1}(-x)\big\} f_\alpha(x)\,\mathrm{d}\sigma(x)\bigg) a_\gamma(y)\,\mathrm{d}\sigma(y)\\
&= \int_{\partial\Omega} \Big(K_{\mathrm{mod}}\big(f\cdot \mathbf{1}_{\partial\Omega\setminus B(x_o,2r)}\big)\Big)_\gamma(y)\; a_\gamma(y)\,\mathrm{d}\sigma(y).
\end{aligned}
\tag{2.1.106}
$$

The first equality in (2.1.106) is implied by (1.3.72) bearing in mind that, thanks to the first property in (2.1.101), the variables x, y are uniformly separated. The second equality in (2.1.106) uses (1.8.25) and is valid for each choice $\varepsilon \in (0, r)$. The third equality in (2.1.106) is a consequence of the cancelation property of the atom (cf. the last property in (2.1.101)), while the fourth equality in (2.1.106) is seen from the first property in (2.1.101). The fifth equality in (2.1.106) follows from Fubini's Theorem whose applicability is presently ensured by the fact that the double integral is absolutely convergent, thanks to the properties listed in the first line of

(2.1.101), the estimate in [70, (2.3.117)] (with a constant which stays bounded for x in a compact subset of $\partial\Omega$), and (2.1.99). The sixth equality in (2.1.106) uses the fact that the inner integral is actually independent of $\varepsilon \in (0, r)$, and also the support condition for the atom. Finally, the last equality in (2.1.106) is seen from (1.8.24). At this stage, from (2.1.104)-(2.1.106) we conclude that

$$\int_{\partial\Omega} f_\alpha (K^\# a)_\alpha \, d\sigma = \int_{\partial\Omega} \big(K_{\text{mod}} f\big)_\gamma a_\gamma \, d\sigma, \tag{2.1.107}$$

from which (2.1.103) follows.

Let us now suppose that $\partial\Omega$ is unbounded. Then, on the one hand, based on (5.1.232) and [69, Proposition 4.8.6] (whose applicability with $g := K^\# a$ is ensured by (2.1.4) and (2.1.102)), for each $\mathbb{C}^M$-valued $(1,\infty)$-atom $a = (a_\gamma)_{1\leq\gamma\leq M}$ on $\partial\Omega$ we may write

$$\int_{\partial\Omega} \big(K_{\text{mod}} f\big)_\gamma a_\gamma \, d\sigma = \big\langle [f], K^\# a \big\rangle \tag{2.1.108}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket between the John-Nirenberg space of functions of bounded mean oscillations on $\partial\Omega$, modulo constants, and the Hardy space H^1 on Σ (cf. [69, Theorem 4.6.1]). In concert with [69, (4.6.9)], this permits us to estimate

$$\begin{aligned} \Big| \int_{\partial\Omega} \big(K_{\text{mod}} f\big)_\gamma a_\gamma \, d\sigma \Big| &= \big| \big\langle [f], K^\# a \big\rangle \big| \\ &\leq C \|f\|_{[\text{BMO}(\partial\Omega,\sigma)]^M} \cdot \|K^\# a\|_{[H^1(\partial\Omega,\sigma)]^M} \\ &\leq C \|f\|_{[\text{BMO}(\partial\Omega,\sigma)]^M}, \end{aligned} \tag{2.1.109}$$

where the last inequality is based on (2.1.4) and [69, (4.5.5)-(4.5.6)]. On the other hand, from (2.1.100), (A.0.20), [68, Proposition 7.4.12], and (2.1.101) we see that

$$\begin{aligned} &\big\|K_{\text{mod}} f\big\|_{[\text{BMO}(\partial\Omega,\sigma)]^M} \\ &\quad \leq C \cdot \sup \Big\{ \Big| \int_{\partial\Omega} \big(K_{\text{mod}} f\big)_\gamma a_\gamma \, d\sigma \Big| : \ a = (a_\gamma)_{1\leq\gamma\leq M} \\ &\qquad\qquad \text{is a } \mathbb{C}^M\text{-valued } (1,\infty)\text{-atom on } \partial\Omega \Big\}. \end{aligned} \tag{2.1.110}$$

Together, (2.1.109) and (2.1.110) give

$$\big\|K_{\text{mod}} f\big\|_{[\text{BMO}(\partial\Omega,\sigma)]^M} \leq C \|f\|_{[\text{BMO}(\partial\Omega,\sigma)]^M}, \tag{2.1.111}$$

for some $C \in (0,\infty)$ independent of f. Hence, the operator K_{mod} is a well-defined, linear, bounded mapping from $\big[\text{BMO}(\partial\Omega,\sigma)\big]^M$ into itself when $\partial\Omega$ is unbounded.

In view of the format of the norm on $\widetilde{\mathrm{BMO}}(\partial\Omega,\sigma)$ (cf. [68, (7.4.95)]) and (1.8.28), we then also conclude that the mapping in (2.1.93) is well defined, linear, and bounded, when $\partial\Omega$ is unbounded. With these properties in hand, proving (2.1.97) comes down (thanks to the last property in [69, (4.4.114)] and the continuity of $K^{\#}$ on the Hardy scale) to showing that

$$\big\langle [K_{\mathrm{mod}} f], a\big\rangle = \big\langle [f], K^{\#} a\big\rangle \tag{2.1.112}$$

for each $f \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$ and each $\mathbb{C}^M$-valued $(1,\infty)$-atom a on $\partial\Omega$. This, however, is clear from (2.1.108) and the duality result from [69, Theorem 4.6.1, (4.6.8)] (bearing in mind that we already know that $K_{\mathrm{mod}} f \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$).

At this stage, all claims pertaining to (2.1.92)-(2.1.93) have been justified when $\partial\Omega$ is an unbounded set. As such, there remains to treat the case when $\partial\Omega$ is a bounded set. In such a scenario, (2.1.94) holds thanks to [68, (7.4.105)]. As a consequence of this and [70, (2.3.34)], for each function $f = (f_\alpha)_{1\le\alpha\le M} \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$ and each $a = (a_\gamma)_{1\le\gamma\le M}$ as in (2.1.101) we have

$$\int_{\partial\Omega} f_\alpha (K^{\#} a)_\alpha \, d\sigma = \int_{\partial\Omega} (Kf)_\gamma a_\gamma \, d\sigma. \tag{2.1.113}$$

Then the same argument which, starting with (5.1.232), has produced (5.1.239) presently gives

$$\|Kf\|_{[\dot{\mathrm{BMO}}(\partial\Omega,\sigma)]^M} \le C\|f\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M} \text{ for each } f \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M. \tag{2.1.114}$$

From this, (1.5.12), and [69, (4.6.18)] we then conclude that K is a well-defined, linear, and bounded operator in the context of (2.1.95).

Next, (2.1.94) and [70, (2.3.34)] currently give that $\big[K_{\mathrm{mod}} f\big] = [Kf]$ for each $f \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$. With this in hand, it follows from (2.1.95) that the mapping in (2.1.93) is well defined, linear, and bounded. Granted this, for each $f \in \mathrm{BMO}(\partial\Omega,\sigma)$ we may estimate, bearing in mind (A.0.20), the fact that $C_f := K_{\mathrm{mod}} f - Kf$ is a constant as in [70, (2.3.34)], and [69, (4.6.18)]:

$$\begin{aligned}
\big\|K_{\mathrm{mod}} f\big\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M} &\le \big\|K_{\mathrm{mod}} f - Kf\big\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M} + \|Kf\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \\
&\le \sigma(\partial\Omega)|C_f| + C\|f\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M} \\
&\le C\|f\|_{[L^1(\partial\Omega,\sigma)]^M} + C\|f\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M} \\
&\le C\|f\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M}.
\end{aligned} \tag{2.1.115}$$

This completes the proof of the fact that the mapping (2.1.92) is well defined and bounded. Next, for each $f = (f_\alpha)_{1\le\alpha\le M} \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$ and each $\mathbb{C}^M$-valued $(1,\infty)$-atom $a = (a_\gamma)_{1\le\gamma\le M}$ on $\partial\Omega$ we may write

$$\big\langle Kf, a\big\rangle = \int_{\partial\Omega} (Kf)_\gamma a_\gamma \, d\sigma = \int_{\partial\Omega} f_\alpha (K^{\#} a)_\alpha \, d\sigma = \big\langle f, K^{\#} a\big\rangle \tag{2.1.116}$$

by [69, Proposition 4.8.6] (keeping in mind (2.1.94) and (1.5.12)), (1.2.7), and (2.1.4). This proves (2.1.98) and completes the proof of Theorem 2.1.7. □

Remark 2.1.8 *The results in Theorem 2.1.7 are applicable to all boundary-to-boundary double layer potential operators K, $K^{\#}$ from Examples 1.4.9-1.4.20 and their modified versions $K_{\rm mod}$ from Examples 1.8.4-1.8.7.*

In particular, Theorem 2.1.7 permits us to verify the integral identities claimed below.

Corollary 2.1.9 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ satisfies $n \geq 2$) is a* UR *domain, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, suppose L is a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$), and consider the boundary layer potential operator $K^{\#}$ associated with the system L and the set Ω as in* (1.3.72).

Then for each function $f \in \big[H^1(\partial\Omega, \sigma)\big]^M$ one has

$$f \in \big[L^1(\partial\Omega, \sigma)\big]^M, \qquad K^{\#} f \in \big[L^1(\partial\Omega, \sigma)\big]^M, \tag{2.1.117}$$

and, with all integrals involved absolutely convergent,

$$\int_{\partial\Omega} K^{\#} f \, d\sigma = \begin{cases} +\dfrac{1}{2}\displaystyle\int_{\partial\Omega} f \, d\sigma & \text{if } \Omega \text{ is bounded,} \\ -\dfrac{1}{2}\displaystyle\int_{\partial\Omega} f \, d\sigma & \text{if } \Omega \text{ is unbounded and } \partial\Omega \text{ is bounded,} \\ \quad 0 & \text{if } \partial\Omega \text{ is unbounded.} \end{cases} \tag{2.1.118}$$

Proof Pick $f \in \big[H^1(\partial\Omega, \sigma)\big]^M$ arbitrary. Then the claims in (2.1.117) are immediate consequences of [69, (4.2.10)] and (2.1.4) (used with $p := 1$). Next, the branch of (2.1.118) corresponding to the case when $\partial\Omega$ is unbounded is seen from Theorem 2.1.1 and [69, (4.4.9)]. Suppose now that $\partial\Omega$ is bounded. Then for each $\lambda \in \mathbb{C}^M$ we may write

$$\begin{aligned} \Big\langle \int_{\partial\Omega} K^{\#} f \, d\sigma \,,\, \lambda \Big\rangle &= \int_{\partial\Omega} \langle K^{\#} f, \lambda \rangle \, d\sigma = {}_{[H^1(\partial\Omega,\sigma)]^M}\big\langle K^{\#} f, \lambda \big\rangle_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M} \\ &= {}_{[H^1(\partial\Omega,\sigma)]^M}\big\langle f, K\lambda \big\rangle_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M} \\ &= {}_{[H^1(\partial\Omega,\sigma)]^M}\big\langle f, \pm\tfrac{1}{2}\lambda \big\rangle_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M} \\ &= \pm\frac{1}{2}\Big\langle \int_{\partial\Omega} f \, d\sigma \,,\, \lambda \Big\rangle \end{aligned} \tag{2.1.119}$$

where the $\pm$ signs correspond to Ω being bounded and unbounded, respectively. Indeed, the first equality in (2.1.119) is trivial, the second one is a consequence of Theorem 2.1.1 and [69, Proposition 4.8.6], the third one comes from (2.1.98), the fourth one is guaranteed by (1.5.21), and the fifth one is once again implied by [69, Proposition 4.8.6]. Having proved (2.1.119), on account of the arbitrariness of $\lambda \in \mathbb{C}^M$ we then conclude that the branches of (2.1.118) corresponding to the case when $\partial\Omega$ is bounded are valid as well. □

A companion result to Theorem 2.1.7 is Theorem 2.1.10 below, to the effect that the action of the principal-value double layer operator associated with a given weakly elliptic system in a UR domain may also be extended, in a linear and bounded fashion, to the entire scale of Hölder spaces on the boundary. See also Proposition 2.1.12 for a result of similar flavor.

Theorem 2.1.10 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). In this context, consider the boundary layer potential operators K and $K^{\#}$, associated with the system L and the set Ω as in (1.3.68) and (1.3.72), respectively. In addition, bring back the operator K_{mod} defined in (1.8.24). Lastly, consider*

$$\eta \in (0,1) \ \text{and define} \ p := \tfrac{n-1}{n-1+\eta} \in \big(\tfrac{n-1}{n}, 1\big). \tag{2.1.120}$$

or, equivalently, start with

$$p \in \big(\tfrac{n-1}{n}, 1\big) \ \text{and define} \ \eta := (n-1)\big(\tfrac{1}{p} - 1\big) \in (0,1). \tag{2.1.121}$$

Then the operators

$$K_{\text{mod}} : \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M \longrightarrow \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M \tag{2.1.122}$$

and

$$\begin{aligned}&\big[K_{\text{mod}}\big] : \big[\dot{\mathscr{C}}^\eta(\partial\Omega)/\sim\big]^M \longrightarrow \big[\dot{\mathscr{C}}^\eta(\partial\Omega)/\sim\big]^M \ \text{defined as}\\ &\big[K_{\text{mod}}\big][f] := \big[K_{\text{mod}} f\big] \ \text{for each function} \ f \in \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M\end{aligned} \tag{2.1.123}$$

are well defined, linear, and bounded. Moreover, in the case when $\partial\Omega$ is bounded, a scenario in which one has

$$\big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M = \big[\mathscr{C}^\eta(\partial\Omega)\big]^M \subseteq \bigcap_{0<q\leq\infty} \big[L^q(\partial\Omega,\sigma)\big]^M, \tag{2.1.124}$$

the operator K acting on the Lebesgue scale $\big[L^q(\partial\Omega,\sigma)\big]^M$ with $q \in (1,\infty)$ has $\big[\mathscr{C}^\eta(\partial\Omega)\big]^M$ as an invariant subspace, and its restriction

$$K : \big[\mathscr{C}^\eta(\partial\Omega)\big]^M \longrightarrow \big[\mathscr{C}^\eta(\partial\Omega)\big]^M \tag{2.1.125}$$

is a well-defined, linear and bounded operator. Finally, *the (real) transpose of the operator* $K^{\#}$ *from* (2.1.4) *with* $p \in \left(\frac{n-1}{n}, 1\right)$ *is, respectively,* $\left[K_{\text{mod}}\right]$ *from* (2.1.123) *if* $\partial\Omega$ *is unbounded, and* K *from* (2.1.125) *if* $\partial\Omega$ *is bounded,* (2.1.126)

that is, for each $f \in \left[\dot{\mathscr{C}}^{\eta}(\partial\Omega)\right]^M$ *and* $g \in \left[H^p(\partial\Omega,\sigma)\right]^M$ *one has (with duality brackets as in* [69, Theorem 4.6.1])

$$\left\langle\left[K_{\text{mod}} f\right], g\right\rangle = \left\langle [f], K^{\#} g\right\rangle \ \textit{if } \partial\Omega \textit{ is unbounded,} \tag{2.1.127}$$

$$\left\langle K f, g\right\rangle = \left\langle f, K^{\#} g\right\rangle \ \textit{if } \partial\Omega \textit{ is bounded.} \tag{2.1.128}$$

Proof All results are largely dealt with as their counterparts treated in Theorem 2.1.7. More specifically, from [68, (7.4.119)] we know that

$$\dot{\mathscr{C}}^{\eta}(\partial\Omega) \subseteq L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big). \tag{2.1.129}$$

Then for any function $f = (f_\alpha)_{1\le\alpha\le M} \in \left[\dot{\mathscr{C}}^{\eta}(\partial\Omega)\right]^M$, from (2.1.129), [70, (2.3.35)], and (1.8.24) we see that

$$K_{\text{mod}} f \in \bigcap_{1\le q<\infty} \left[L^q_{\text{loc}}(\partial\Omega,\sigma)\right]^M. \tag{2.1.130}$$

Also, from (2.1.8) and [69, Lemma 4.5.5] we know that

$$\int_{\partial\Omega} |f||K^{\#} a| \, d\sigma < +\infty. \tag{2.1.131}$$

With this in hand, the same argument that has produced (2.1.107) currently gives

$$\int_{\partial\Omega} f_\alpha (K^{\#} a)_\alpha \, d\sigma = \int_{\partial\Omega} \big(K_{\text{mod}} f\big)_\gamma a_\gamma \, d\sigma \tag{2.1.132}$$

for each $\mathbb{C}^M$-valued (p,∞)-atom $a = (a_\gamma)_{1\le\gamma\le M}$ on $\partial\Omega$ with vanishing moment. Granted this, we may reason as in (2.1.108)-(2.1.111), now relying on [69, Proposition 4.8.7] in place of [69, Proposition 4.8.6], and [68, Proposition 7.4.8] in place of [68, Proposition 7.4.12], to conclude that there exists $C \in (0,\infty)$ such that

$$\left\|K_{\text{mod}} f\right\|_{[\dot{\mathscr{C}}^{\eta}(\partial\Omega)]^M} \le C\|f\|_{[\dot{\mathscr{C}}^{\eta}(\partial\Omega)]^M} \ \text{ for each } f \in \left[\dot{\mathscr{C}}^{\eta}(\partial\Omega)\right]^M. \tag{2.1.133}$$

From this and (1.8.28), the claims concerning the operators (2.1.122)-(2.1.123) follow. Once these have been established, proving (2.1.127) reduces (in view of the last property in [69, (4.4.114)]) to checking that

$$\begin{gathered}\left\langle\left[K_{\text{mod}} f\right], a\right\rangle = \left\langle [f], K^{\#} a\right\rangle \ \text{ for each} \\ \text{function } f \in \left[\dot{\mathscr{C}}^{\eta}(\partial\Omega)\right]^M \text{ and each } (p,\infty)\text{-atom } a \text{ on } \partial\Omega.\end{gathered} \tag{2.1.134}$$

Since we already know that $K_{\text{mod}} f \in \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M$, from the duality result in [69, Theorem 4.6.1, (4.6.8)] we conclude that

$$\big\langle \big[K_{\text{mod}} f\big], a \big\rangle = \int_{\partial\Omega} (K_{\text{mod}} f)_\gamma a_\gamma \, \mathrm{d}\sigma. \tag{2.1.135}$$

Observe that since a is a (p,∞)-atom, hence also a multiple of some $(1,\infty)$-atom, from (2.1.4) we have

$$K^\# a \in \big[H^1(\partial\Omega,\sigma)\big]^M \cap \big[H^p(\partial\Omega,\sigma)\big]^M. \tag{2.1.136}$$

Thanks to (2.1.131) and (2.1.136), [69, Proposition 4.8.7] applies and gives

$$\big\langle [f], K^\# a \big\rangle = \int_{\partial\Omega} f_\alpha (K^\# a)_\alpha \, \mathrm{d}\sigma. \tag{2.1.137}$$

Now (2.1.134) follows from (2.1.135), (2.1.137), and (2.1.132).

In the case when $\partial\Omega$ is bounded, starting with the fact that (2.1.113) holds for each $f \in \big[\mathscr{C}^\eta(\partial\Omega)\big]^M$ and each $\mathbb{C}^M$-valued (p,∞)-atom $a = (a_\gamma)_{1\leq\gamma\leq M}$ on $\partial\Omega$, the same argument which has proved (2.1.133), outlined above, now gives

$$\|Kf\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M} \leq C\|f\|_{[\mathscr{C}^\eta(\partial\Omega)]^M} \text{ for each } f \in \big[\mathscr{C}^\eta(\partial\Omega)\big]^M. \tag{2.1.138}$$

In addition, for each function $f \in \big[\mathscr{C}^\eta(\partial\Omega)\big]^M$ and σ-a.e. point $x \in \partial\Omega$ we may use (1.5.21) and [68, (7.2.5)] (with $X := \partial\Omega$, $r := 2\,\mathrm{diam}(\partial\Omega)$, $d := n-1$, and $\delta := \eta$) to estimate

$$\begin{aligned} |(Kf)(x)| &\leq \big|\big(K(f - f(x))\big)(x)\big| + \tfrac{1}{2}|f(x)| \\ &\leq C\|f\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M} \int_{\partial\Omega} \frac{\mathrm{d}\sigma(y)}{|x-y|^{n-1-\eta}} + \tfrac{1}{2}|f(x)| \\ &\leq C\big[\mathrm{diam}(\partial\Omega)\big]^\eta \|f\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M} + \tfrac{1}{2}\sup_{\partial\Omega}|f|, \end{aligned} \tag{2.1.139}$$

for some $C \in (0,\infty)$ which depends only on L, n, and the lower ADR constant of Ω. In view of this and the fact that, as seen from (2.1.138), the operator K maps $\big[\mathscr{C}^\eta(\partial\Omega)\big]^M$ into continuous functions on $\partial\Omega$, we conclude that there exists some constant $C \in (0,\infty)$ with the property that

$$\sup_{x\in\partial\Omega} |(Kf)(x)| \leq C\|f\|_{[\mathscr{C}^\eta(\partial\Omega)]^M} \text{ for each } f \in \big[\mathscr{C}^\eta(\partial\Omega)\big]^M. \tag{2.1.140}$$

Ultimately, (2.1.140) and (2.1.138) imply

$$\|Kf\|_{[\mathscr{C}^\eta(\partial\Omega)]^M} \leq C\|f\|_{[\mathscr{C}^\eta(\partial\Omega)]^M} \text{ for each } f \in \big[\mathscr{C}^\eta(\partial\Omega)\big]^M. \tag{2.1.141}$$

This proves that the operator K is well defined and bounded in the context of (2.1.125). Finally, (2.1.128) is justified as before, based on density (cf. [69, (4.4.114)]) and the fact that (2.1.116) continues to hold for each function $f \in \big[\mathscr{C}^\eta(\partial\Omega)\big]^M$ and each $\mathbb{C}^M$-valued (p, ∞)-atom a on $\partial\Omega$ (cf. (2.1.125), the bounded set version of the duality result from [69, Theorem 4.6.1, (4.6.8)], item *(iii)* in Theorem 1.5.1, and [69, Corollary 4.8.11], keeping in mind (2.1.8)). The proof of Theorem 2.1.10 is therefore complete. □

Similar results as in the first part of Theorem 2.1.10 are also valid on our scale of vanishing Hölder spaces, introduced in [69, §3.2]. A formal statement is contained in our next corollary.

Corollary 2.1.11 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). In this context, consider the boundary-to-boundary layer potential operator K and its modified version K_{mod} associated with the system L and the set Ω in as in (1.3.68) and (1.8.24), respectively. Finally, fix an arbitrary exponent*

$$\eta \in (0, 1) \tag{2.1.142}$$

and recall the (homogeneous and inhomogeneous) vanishing Hölder spaces $\dot{\mathscr{C}}^\eta_{\text{van}}(\partial\Omega)$ and $\mathscr{C}^\eta_{\text{van}}(\partial\Omega)$ defined as in (A.0.48) and (A.0.49), respectively, with $\Sigma := \partial\Omega$.

Then the operators

$$K_{\text{mod}} : \big[\dot{\mathscr{C}}^\eta_{\text{van}}(\partial\Omega)\big]^M \longrightarrow \big[\dot{\mathscr{C}}^\eta_{\text{van}}(\partial\Omega)\big]^M \tag{2.1.143}$$

and

$$\begin{aligned} &\big[K_{\text{mod}}\big] : \big[\dot{\mathscr{C}}^\eta_{\text{van}}(\partial\Omega)/\sim\big]^M \longrightarrow \big[\dot{\mathscr{C}}^\eta_{\text{van}}(\partial\Omega)/\sim\big]^M \text{ defined as} \\ &\big[K_{\text{mod}}\big][f] := \big[K_{\text{mod}} f\big] \text{ for each function } f \in \big[\dot{\mathscr{C}}^\eta_{\text{van}}(\partial\Omega)\big]^M \end{aligned} \tag{2.1.144}$$

are well defined, linear, and bounded. Moreover, in the case when $\partial\Omega$ is bounded, a scenario in which one has

$$\big[\dot{\mathscr{C}}^\eta_{\text{van}}(\partial\Omega)\big]^M = \big[\mathscr{C}^\eta_{\text{van}}(\partial\Omega)\big]^M \subseteq \big[\mathscr{C}^\eta(\partial\Omega)\big]^M, \tag{2.1.145}$$

the operator K acting on the ordinary Hölder space $\big[\mathscr{C}^\eta(\partial\Omega)\big]^M$ (as discussed in Theorem 2.1.10) has $\big[\mathscr{C}^\eta_{\text{van}}(\partial\Omega)\big]^M$ as an invariant subspace, and its restriction

$$K : \big[\mathscr{C}^\eta_{\text{van}}(\partial\Omega)\big]^M \longrightarrow \big[\mathscr{C}^\eta_{\text{van}}(\partial\Omega)\big]^M \tag{2.1.146}$$

is a well-defined, linear and bounded operator.

Proof All results are consequences of Theorem 2.1.10, the definitions made in (A.0.48), and (A.0.49), as well as the density result from [69, Theorem 3.2.2]. □

In Theorem 2.1.10 we have established that the principal-value double layer potential operators is a well-defined linear and bounded operator on the scale of Hölder spaces considered on boundaries of UR domains. Remarkably, such a result continues to be true when the uniform rectifiability assumption for the boundary is replaced by a mere upper Ahlfors regularity demand. This is made precise in the proposition below.

Proposition 2.1.12 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ satisfies $n \geq 2$) be a Lebesgue measurable set with the property that $\partial\Omega$ is bounded and $\partial_*\Omega$ is upper Ahlfors regular (in particular,* [68, (5.6.35)] *guarantees that Ω is a set of locally finite perimeter). Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, for some $M \in \mathbb{N}$, consider a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in* (1.3.2) *is weakly elliptic (in the sense of* [70, (1.3.3) in Definition 1.3.1]*). Finally, let $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ be the matrix-valued fundamental solution associated with L as in* [70, Theorem 1.4.2]. *Then the following statements are true.*

(i) For each function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[\mathscr{C}^\eta(\partial_*\Omega)\big]^M$ with $\eta \in (0, 1)$ the limit

$$
Kf(x) := \Bigg(-\lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \nu_s(y)a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)f_\alpha(y)\,d\sigma(y)\Bigg)_{1\leq\gamma\leq M} \tag{2.1.147}
$$

exists at σ-a.e. point $x \in \partial_*\Omega$. In fact, at σ-a.e. point $x \in \partial_*\Omega$ the principal-value double layer potential defined in (2.1.147) *may be expressed as*

$$
\begin{aligned}
Kf(x) = \Bigg(-\int_{\partial_*\Omega} \nu_s(y)a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)\big(f_\alpha(y)-f_\alpha(x)\big)\,d\sigma(y)\Bigg)_{1\leq\gamma\leq M}& \\
+\begin{cases} +\frac{1}{2}f(x) & \text{if } \Omega \text{ is bounded,}\\ -\frac{1}{2}f(x) & \text{if } \Omega \text{ is unbounded.}\end{cases}&
\end{aligned} \tag{2.1.148}
$$

(ii) If for each $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[\mathscr{C}^\eta(\partial_*\Omega)\big]^M$ with $\eta \in (0, 1)$ one redefines Kf on a σ-nullset contained in $\partial_*\Omega$ as being the function in (2.1.148) *at each point $x \in \partial_*\Omega$, the assignment $f \mapsto Kf$ thus defined induces a well-defined, linear, and bounded operator*

$$
K : \big[\mathscr{C}^\eta(\partial_*\Omega)\big]^M \longrightarrow \big[\mathscr{C}^\eta(\partial_*\Omega)\big]^M \tag{2.1.149}
$$

for each $\eta \in (0, 1)$, with the property that for each $\lambda \in \mathbb{C}^M$ one has

$$K\lambda \equiv \begin{cases} +\frac{1}{2}\lambda \ \text{ on } \partial_*\Omega, \text{ if } \Omega \text{ is bounded,} \\ -\frac{1}{2}\lambda \ \text{ on } \partial_*\Omega, \text{ if } \Omega \text{ is unbounded.} \end{cases} \tag{2.1.150}$$

(iii) Let $\mathcal{D}$ be the boundary-to-domain double layer operators associated with the set Ω and the coefficient tensor A as in (1.3.18). Also, fix an arbitrary aperture parameter $\kappa > 0$. Then, under the additional assumption that the set Ω is open, the following jump-formula (where I denotes the identity operator) is valid:

$$\begin{aligned} &\text{for each } f \in \big[\mathscr{C}^\eta(\partial_*\Omega)\big]^M \text{ with } \eta \in (0,1) \text{ one has} \\ &\big(\mathcal{D}f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K\big)f \ \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \end{aligned} \tag{2.1.151}$$

Proof For each $\alpha, \gamma \in \{1, \dots, M\}$ define (with the summation convention over repeated indices in effect) $\vec{k}_{\alpha\gamma} := -\big(d^{\beta\alpha}_{rs}\partial_r E_{\gamma\beta}\big)_{1\leq s\leq n} \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^n$ which is an odd, positive homogeneous of degree $1-n$, divergence-free vector field (cf. [70, Theorem 1.4.2]). Thanks to [70, (1.4.25)], we also have

$$\begin{aligned} \vartheta_{\alpha\gamma} &:= \int_{S^{n-1}} \big\langle \omega, \vec{k}_{\alpha\gamma}(\omega)\big\rangle \, d\mathcal{H}^{n-1}(\omega) \\ &= -\int_{S^{n-1}} \omega_s d^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(\omega) \, d\mathcal{H}^{n-1}(\omega) = -\delta_{\alpha\gamma}. \end{aligned} \tag{2.1.152}$$

If $\mathcal{Z}_{\alpha\gamma}$ is the integral operator associated as in [70, (2.5.157)] with $\vec{k}_{\alpha\gamma}$, then it becomes apparent from (1.3.18) that for each $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[L^1(\partial_*\Omega, \sigma)\big]^M$ we have

$$\mathcal{D}f = \big(\mathcal{Z}_{\alpha\gamma} f_\alpha\big)_{1\leq\gamma\leq M} \ \text{ in } \Omega. \tag{2.1.153}$$

Also, if $Z_{\alpha\gamma}$ is the principal-value singular integral operator associated as in [70, (2.5.166)] with $\vec{k}_{\alpha\gamma}$, then from (2.1.147) it follows that for each vector-valued function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[L^1(\partial_*\Omega, \sigma)\big]^M$ we have

$$Kf = \big(Z_{\alpha\gamma} f_\alpha\big)_{1\leq\gamma\leq M} \ \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{2.1.154}$$

Granted (2.1.152)-(2.1.154), all desired claims follow from [70, Proposition 2.5.16]. □

Moving on, Theorem 2.1.7, Theorem 2.1.10, and [69, Theorem 3.1.3] make it possible to handle the (modified version of the) principal-value double layer potential operator on the Sarason space, of functions of vanishing mean oscillations, considered on the boundary of an arbitrary UR domain.

Corollary 2.1.13 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a* UR *domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$), and recall the boundary layer potential operator K, and its modified version K_{mod}, associated with the system L and the set Ω as in (1.3.68) and (1.8.24), respectively.*

Then the operator K_{mod} *acting on* $\big[\text{BMO}(\partial\Omega,\sigma)\big]^M$ (*cf. Theorem 2.1.7*) *has* $\big[\text{VMO}(\partial\Omega,\sigma)\big]^M$ *as an invariant subspace, hence its restriction*

$$K_{\text{mod}} : \big[\text{VMO}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\text{VMO}(\partial\Omega,\sigma)\big]^M \tag{2.1.155}$$

is a well-defined, linear and bounded operator. In addition, the following operator is also well defined, linear and bounded:

$$\begin{gathered}\big[K_{\text{mod}}\big] : \big[\text{VMO}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\text{VMO}(\partial\Omega,\sigma)/\sim\big]^M \ \textit{defined as}\\ \big[K_{\text{mod}}\big][f] := \big[K_{\text{mod}}f\big] \ \textit{for each function}\ f\in\big[\text{VMO}(\partial\Omega,\sigma)\big]^M.\end{gathered} \tag{2.1.156}$$

Finally, if $\partial\Omega$ *is compact then the operator* K *acting on* $\big[\text{BMO}(\partial\Omega,\sigma)\big]^M$ (*cf.* (2.1.95)) *has* $\big[\text{VMO}(\partial\Omega,\sigma)\big]^M$ *as an invariant subspace, hence its restriction*

$$K : \big[\text{VMO}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\text{VMO}(\partial\Omega,\sigma)\big]^M \tag{2.1.157}$$

is a well-defined, linear and bounded operator, and

$$\begin{gathered}\textit{the (real) transpose of the operator (2.1.157) is the operator } K^{\#} :\\ \big[H^1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[H^1(\partial\Omega,\sigma)\big]^M \ \textit{(cf. Theorem 2.1.1)}.\end{gathered} \tag{2.1.158}$$

Proof This is a consequence of Theorem 2.1.7, Theorem 2.1.10, [69, Theorem 3.1.3], [69, (3.1.50)], Theorem 2.1.1, (1.8.28), and [69, (4.6.22)-(4.6.23)]. □

We wish to note that the same template used in the proofs of Theorem 2.1.7, Theorem 2.1.10, Corollary 2.1.11, and Corollary 2.1.13 can handle other important classes of operators. To provide further concrete example, suppose $\Omega\subseteq\mathbb{R}^n$ is an UR domain and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix a complex-valued function $b\in L^1_{\text{loc}}(\mathbb{R}^n,\mathcal{L}^n)$ with $b\big|_{\mathbb{R}^n\setminus\{0\}}\in\mathscr{C}^N(\mathbb{R}^n\setminus\{0\})$ for some sufficiently large number $N=N(n)\in\mathbb{N}$, and such that ∇b is odd and positive homogeneous of degree $1-n$ in $\mathbb{R}^n\setminus\{0\}$. Assume first that $\partial\Omega$ is unbounded and bring in the operators T^{mod}_{jk} defined as in (1.8.168)-(1.8.170). Then for each $j,k\in\{1,\dots,n\}$ the operators T^{mod}_{jk} map constant functions on $\partial\Omega$ into constant functions on $\partial\Omega$ and, with brackets denoting equivalence classes modulo constants, the naturally induced maps

$$\begin{gathered}\big[T^{\text{mod}}_{jk}\big] : \widetilde{\text{BMO}}(\partial\Omega,\sigma)\longrightarrow\widetilde{\text{BMO}}(\partial\Omega,\sigma),\\ \text{defined as } \big[T^{\text{mod}}_{jk}\big][f] := \big[T^{\text{mod}}_{jk}f\big] \text{ for each}\\ f\in\text{BMO}(\partial\Omega,\sigma)\subset L^1\Big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\Big),\end{gathered} \tag{2.1.159}$$

and, for every $\eta\in(0,1)$,

$$\begin{aligned}&\big[T^{\rm mod}_{jk}\big] : \dot{\mathscr{C}}^\eta(\partial\Omega)/\sim \longrightarrow \dot{\mathscr{C}}^\eta(\partial\Omega)/\sim,\\ &\text{defined as } \big[T^{\rm mod}_{jk}\big][f] := \big[T^{\rm mod}_{jk} f\big] \text{ for each}\\ &f \in \dot{\mathscr{C}}^\eta(\partial\Omega) \subset L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big),\end{aligned} \tag{2.1.160}$$

are all well defined, linear, and bounded. Boundedness results similar to (2.1.160) are also valid on homogeneous vanishing Hölder spaces (see Corollary 2.1.11). In addition, with the duality brackets as in [69, Theorem 4.6.1], for each pair of integers $j, k \in \{1, \dots, n\}$ we have

$$\begin{aligned}&\big\langle \big[T^{\rm mod}_{jk} f\big], g\big\rangle = -\big\langle [f], T^{\#}_{jk} g\big\rangle \text{ for each}\\ &f \in \mathrm{BMO}(\partial\Omega,\sigma) \subset L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \text{ and } g \in H^1(\partial\Omega,\sigma),\end{aligned} \tag{2.1.161}$$

as well as

$$\begin{aligned}&\big\langle \big[T^{\rm mod}_{jk} f\big], g\big\rangle = -\big\langle [f], T^{\#}_{jk} g\big\rangle \text{ for each}\\ &f \in \dot{\mathscr{C}}^\eta(\partial\Omega) \subset L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \text{ and } g \in H^p(\partial\Omega,\sigma)\\ &\text{with } p \in \big(\tfrac{n-1}{n}, 1\big) \text{ and } \eta := (n-1)\big(\tfrac{1}{p}-1\big) \in (0,1),\end{aligned} \tag{2.1.162}$$

where the operator $T^{\#}_{jk}$ is as in (2.1.42). In fact, each $T^{\rm mod}_{jk}$ is also well defined, linear, and bounded on plain $\mathrm{BMO}(\partial\Omega,\sigma)$ (i.e., without modding out constants). Considered as such, said operator has $\mathrm{VMO}(\partial\Omega,\sigma)$ as an invariant subspace, and both

$$T^{\rm mod}_{jk} : \mathrm{VMO}(\partial\Omega,\sigma) \longrightarrow \mathrm{VMO}(\partial\Omega,\sigma), \tag{2.1.163}$$

together with

$$\begin{aligned}&\big[T^{\rm mod}_{jk}\big] : \widetilde{\mathrm{VMO}}(\partial\Omega,\sigma) \longrightarrow \widetilde{\mathrm{VMO}}(\partial\Omega,\sigma),\\ &\text{defined as } \big[T^{\rm mod}_{jk}\big][f] := \big[T^{\rm mod}_{jk} f\big] \text{ for each}\\ &f \in \mathrm{VMO}(\partial\Omega,\sigma) \subset L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big),\end{aligned} \tag{2.1.164}$$

are well-defined, linear, and bounded operators.

Moreover, in the case when $\partial\Omega$ is bounded, the principal-value singular integral operator T_{jk} from (1.2.2) has $\mathrm{BMO}(\partial\Omega,\sigma)$, $\mathscr{C}^\eta(\partial\Omega)$ and $\mathscr{C}^\eta_{\rm van}(\partial\Omega)$ with $\eta \in (0,1)$, as well as $\mathrm{VMO}(\partial\Omega,\sigma)$ as invariant subspaces, and its restrictions

$$T_{jk} : \mathrm{BMO}(\partial\Omega,\sigma) \longrightarrow \mathrm{BMO}(\partial\Omega,\sigma), \tag{2.1.165}$$

$$T_{jk} : \mathscr{C}^\eta(\partial\Omega) \longrightarrow \mathscr{C}^\eta(\partial\Omega) \text{ for each } \eta \in (0,1), \tag{2.1.166}$$

$$T_{jk} : \mathscr{C}^\eta(\partial\Omega)_{\rm van} \longrightarrow \mathscr{C}^\eta_{\rm van}(\partial\Omega) \text{ for each } \eta \in (0,1), \tag{2.1.167}$$

$$T_{jk} : \mathrm{VMO}(\partial\Omega,\sigma) \longrightarrow \mathrm{VMO}(\partial\Omega,\sigma), \tag{2.1.168}$$

are well-defined, linear, and bounded operators which satisfy

$$\langle T_{jk}f, g\rangle = -\langle f, T^{\#}_{jk}g\rangle, \quad \forall f \in \mathrm{BMO}(\partial\Omega, \sigma),\ \forall g \in H^1(\partial\Omega, \sigma), \tag{2.1.169}$$

$$\langle T_{jk}f, g\rangle = -\langle f, T^{\#}_{jk}g\rangle, \quad \forall f \in \mathscr{C}^{\eta}(\partial\Omega),\ \forall g \in H^p(\partial\Omega, \sigma), \tag{2.1.170}$$

if $p \in \big(\frac{n-1}{n}, 1\big)$ and $\eta := (n-1)\big(\frac{1}{p} - 1\big) \in (0, 1)$.

The following result, which complements Corollary 2.1.13, pertains to the action of the modified boundary-to-boundary double layer potential operator on the space CMO.

Corollary 2.1.14 *Let* $\Omega \subseteq \mathbb{R}^n$ (*where* $n \in \mathbb{N}$, $n \geq 2$) *be a* UR *domain with an unbounded boundary, and define* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Also, let* L *be a homogeneous, weakly elliptic, constant* (*complex*) *coefficient, second-order* $M \times M$ *system in* $\mathbb{R}^n$ (*for some* $M \in \mathbb{N}$). *In this context, consider the modified boundary-to-boundary double layer potential operator* K_{mod} *as in* (1.8.24)-(1.8.25), *and the corresponding transpose double layer potential operator* $K^{\#}$ *associated with the system* L *and the set* Ω *as in* (1.3.72). *Then the operators*

$$K_{\text{mod}} : \big[\mathrm{CMO}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathrm{CMO}(\partial\Omega, \sigma)\big]^M \tag{2.1.171}$$

and

$$\begin{gathered}\big[K_{\text{mod}}\big] : \big[\mathrm{CMO}(\partial\Omega, \sigma)/\sim\big]^M \longrightarrow \big[\mathrm{CMO}(\partial\Omega, \sigma)/\sim\big]^M \ \textit{defined as} \\ \big[K_{\text{mod}}\big][f] := \big[K_{\text{mod}}f\big] \ \textit{for each function}\ f \in \big[\mathrm{CMO}(\partial\Omega, \sigma)\big]^M\end{gathered} \tag{2.1.172}$$

are well-defined, linear, and bounded. Moreover,

under the identification [69, (4.6.17)] *the* (*real*) *transpose of the operator* K_{mod} *from* (2.1.171) *is the operator* $K^{\#}$ *from* (2.1.4) *with* $p = 1$, *while under the identification* [69, (4.6.21)] *the* (*real*) *transpose of the operator* $\big[K_{\text{mod}}\big]$ *from* (2.1.172) *is the operator* $K^{\#}$ *from* (2.1.4) *with* $p = 1$. (2.1.173)

Proof Given any $\varphi \in \big[\mathrm{Lip}_c(\partial\Omega)\big]^M$ we have $\varphi \in \big[\dot{\mathscr{C}}^{\eta}(\partial\Omega)\big]^M$ for each $\eta \in (0, 1)$. As such, Theorem 2.1.10 ensures that

$$K_{\text{mod}}\varphi \in \big[\dot{\mathscr{C}}^{\eta}(\partial\Omega)\big]^M \ \text{ for each } \eta \in (0, 1). \tag{2.1.174}$$

From (1.8.26), (1.3.68), and [70, Theorem 1.4.2] we also see that there exists a constant $C_\varphi \in \mathbb{C}^M$ with the property that

$$\big(K_{\text{mod}}\varphi\big)(x) = C_\varphi + O(|x|^{1-n}) \ \text{ as } \partial\Omega \ni x \to \infty. \tag{2.1.175}$$

Together, (2.1.174) and (2.1.175) prove that

$$\text{for each } \varphi \in \big[\mathrm{Lip}_c(\partial\Omega)\big]^M \text{ there exists some } C_\varphi \in \mathbb{C}^M \text{ such that } K_{\text{mod}}\varphi - C_\varphi \text{ is a continuous functions on } \partial\Omega \text{ which vanish at infinity.} \tag{2.1.176}$$

Based on this, (A.0.52), [69, (4.6.14)], and the fact that the operator K_{mod} is well defined, linear, and bounded in the context of (2.1.92), we then conclude that K_{mod} maps the space $\big[\mathrm{CMO}(\partial\Omega, \sigma)\big]^M$ linearly and boundedly into itself. This takes care of (2.1.171), and the claims about (2.1.172) follow on account of the first line in [69, (4.6.14)].

Finally, the claims in (2.1.173) are consequences of [69, (4.6.17), (4.6.21)] and (2.1.97). □

Remark 2.1.15 *The results in Theorem 2.1.10, Corollary 2.1.11, and Corollaries 2.1.13-2.1.14 are applicable to all boundary-to-boundary double layer potential operators K, $K^{\#}$ from Examples 1.4.9-1.4.20 and their modified versions K_{mod} from Examples 1.8.4-1.8.7.*

Let us further elaborate on Remark 2.1.8 and Remark 2.1.15 by describing the analogues of Theorem 2.1.7, Theorem 2.1.10, Corollary 2.1.11, and Corollary 2.1.14 for the Cauchy-Clifford singular integral operator $\mathfrak{C}$ defined in (A.0.54). These results further complement the picture emerging from [70, Proposition 2.5.29] and Proposition 1.6.1 in relation to this operator. Specifically, having fixed a UR domain $\Omega \subseteq \mathbb{R}^n$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Assume first that $\partial\Omega$ is unbounded. For each function

$$f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \otimes \mathcal{C}\ell_n \tag{2.1.177}$$

we define the modified boundary-to-boundary Cauchy-Clifford integral

$$\mathfrak{C}_{\text{mod}} f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \Big\{ \frac{x-y}{|x-y|^n} \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(x,\varepsilon)}}(y) + \frac{y}{|y|^n} \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(0,1)}}(y) \Big\} \odot \nu(y) \odot f(y)\, d\sigma(y) \tag{2.1.178}$$

at σ-a.e. $x \in \partial\Omega$. Then $\mathfrak{C}_{\text{mod}}$ maps constant ($\mathcal{C}\ell_n$-valued) functions on $\partial\Omega$ into constant ($\mathcal{C}\ell_n$-valued) functions on $\partial\Omega$ and induces well-defined, linear, and bounded mappings in the following contexts:

$$\begin{gathered} \big[\mathfrak{C}_{\text{mod}}\big] : \big(\mathrm{BMO}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n\big) / \sim \longrightarrow \big(\mathrm{BMO}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n\big) / \sim \\ \big[\mathfrak{C}_{\text{mod}}\big][f] := \big[\mathfrak{C}_{\text{mod}} f\big] \text{ for each } f \in \mathrm{BMO}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n, \end{gathered} \tag{2.1.179}$$

$$\begin{gathered} \big[\mathfrak{C}_{\text{mod}}\big] : \big(\mathrm{VMO}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n\big) / \sim \longrightarrow \big(\mathrm{VMO}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n\big) / \sim \\ \big[\mathfrak{C}_{\text{mod}}\big][f] := \big[\mathfrak{C}_{\text{mod}} f\big] \text{ for each } f \in \mathrm{VMO}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n, \end{gathered} \tag{2.1.180}$$

$$\begin{aligned}&\big[\mathfrak{C}_{\text{mod}}\big] : \big(\mathrm{CMO}(\partial\Omega,\sigma)\otimes C\ell_n\big)/\sim\longrightarrow\big(\mathrm{CMO}(\partial\Omega,\sigma)\otimes C\ell_n\big)/\sim\\&\big[\mathfrak{C}_{\text{mod}}\big][f]:=\big[\mathfrak{C}_{\text{mod}}f\big]\ \text{ for each }\ f\in\mathrm{CMO}(\partial\Omega,\sigma)\otimes C\ell_n,\end{aligned}\tag{2.1.181}$$

and, for every $\eta\in(0,1)$,

$$\begin{aligned}&\big[\mathfrak{C}_{\text{mod}}\big] : \big(\dot{\mathscr{C}}^{\eta}(\partial\Omega)\otimes C\ell_n\big)/\sim\longrightarrow\big(\dot{\mathscr{C}}^{\eta}(\partial\Omega)\otimes C\ell_n\big)/\sim\\&\big[\mathfrak{C}_{\text{mod}}\big][f]:=\big[\mathfrak{C}_{\text{mod}}f\big]\ \text{ for each }\ f\in\dot{\mathscr{C}}^{\eta}(\partial\Omega)\otimes C\ell_n,\end{aligned}\tag{2.1.182}$$

$$\begin{aligned}&\big[\mathfrak{C}_{\text{mod}}\big] : \big(\dot{\mathscr{C}}^{\eta}_{\text{van}}(\partial\Omega)\otimes C\ell_n\big)/\sim\longrightarrow\big(\dot{\mathscr{C}}^{\eta}_{\text{van}}(\partial\Omega)\otimes C\ell_n\big)/\sim\\&\big[\mathfrak{C}_{\text{mod}}\big][f]:=\big[\mathfrak{C}_{\text{mod}}f\big]\ \text{ for each }\ f\in\dot{\mathscr{C}}^{\eta}_{\text{van}}(\partial\Omega)\otimes C\ell_n,\end{aligned}\tag{2.1.183}$$

with brackets denoting equivalence classes modulo constants. Moreover, these operators are compatible (in a suitable sense) with the action of the principal-value Cauchy-Clifford singular integral operator $\mathfrak{C}$ on Lebesgue spaces. In addition, with the duality brackets as in [69, Theorem 4.6.1], from (2.1.97) we presently deduce that

$$\begin{aligned}&\big\langle\big[\mathfrak{C}_{\text{mod}}f\big],g\big\rangle=\big\langle[f],\mathfrak{C}^{\#}g\big\rangle\ \text{ for each}\\&f\in\mathrm{BMO}(\partial\Omega,\sigma)\otimes C\ell_n\subset L^1\Big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\Big)\otimes C\ell_n\\&\text{and }\ g\in H^1(\partial\Omega,\sigma)\otimes C\ell_n,\end{aligned}\tag{2.1.184}$$

while (2.1.127) currently implies

$$\begin{aligned}&\big\langle\big[\mathfrak{C}_{\text{mod}}f\big],g\big\rangle=\big\langle[f],\mathfrak{C}^{\#}g\big\rangle\ \text{ for each}\\&f\in\dot{\mathscr{C}}^{\eta}(\partial\Omega)\otimes C\ell_n\subset L^1\Big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\Big)\otimes C\ell_n\\&\text{and }\ g\in H^p(\partial\Omega,\sigma)\otimes C\ell_n\ \text{ with}\\&p\in\big(\tfrac{n-1}{n},1\big)\ \text{ and }\ \eta:=(n-1)\big(\tfrac{1}{p}-1\big)\in(0,1).\end{aligned}\tag{2.1.185}$$

In concert with (2.1.65) and the duality result from [69, Theorem 4.6.1], these further entail

$$\big[\mathfrak{C}_{\text{mod}}\big]^2=\tfrac14 I\ \text{ on }\ \big(\mathrm{BMO}(\partial\Omega,\sigma)\otimes C\ell_n\big)/\sim,\tag{2.1.186}$$

$$\big[\mathfrak{C}_{\text{mod}}\big]^2=\tfrac14 I\ \text{ on }\ \big(\mathrm{VMO}(\partial\Omega,\sigma)\otimes C\ell_n\big)/\sim,\tag{2.1.187}$$

$$\big[\mathfrak{C}_{\text{mod}}\big]^2=\tfrac14 I\ \text{ on }\ \big(\mathrm{CMO}(\partial\Omega,\sigma)\otimes C\ell_n\big)/\sim,\tag{2.1.188}$$

$$\big[\mathfrak{C}_{\text{mod}}\big]^2=\tfrac14 I\ \text{ on }\ \big(\dot{\mathscr{C}}^{\eta}(\partial\Omega)\otimes C\ell_n\big)/\sim\ \text{ with }\ \eta\in(0,1),\tag{2.1.189}$$

$$\big[\mathfrak{C}_{\text{mod}}\big]^2=\tfrac14 I\ \text{ on }\ \big(\dot{\mathscr{C}}^{\eta}_{\text{van}}(\partial\Omega)\otimes C\ell_n\big)/\sim\ \text{ with }\ \eta\in(0,1).\tag{2.1.190}$$

Similar results are also valid in the case when Ω has a bounded boundary. Specifically, if $\partial\Omega$ is a bounded set, then $\mathrm{BMO}(\partial\Omega,\sigma)\otimes C\ell_n$, $\mathrm{VMO}(\partial\Omega,\sigma)\otimes C\ell_n$, as well as $\mathscr{C}^{\eta}(\partial\Omega) \otimes C\ell_n$ and $\mathscr{C}^{\eta}_{\text{van}}(\partial\Omega) \otimes C\ell_n$ with $\eta \in (0,1)$ are all invariant subspaces of the standard Cauchy-Clifford singular integral operator $\mathfrak{C}$ acting on Lebesgue spaces (as in item *(ii)* of Proposition 1.6.1), the induced mappings

$$\mathfrak{C} : \text{BMO}(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow \text{BMO}(\partial\Omega, \sigma) \otimes C\ell_n, \tag{2.1.191}$$

$$\mathfrak{C} : \text{VMO}(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow \text{VMO}(\partial\Omega, \sigma) \otimes C\ell_n, \tag{2.1.192}$$

$$\mathfrak{C} : \mathscr{C}^{\eta}(\partial\Omega) \otimes C\ell_n \longrightarrow \mathscr{C}^{\eta}(\partial\Omega) \otimes C\ell_n \ \text{ with } \eta \in (0,1), \tag{2.1.193}$$

$$\mathfrak{C} : \mathscr{C}^{\eta}(\partial\Omega) \otimes C\ell_n \longrightarrow \mathscr{C}^{\eta}_{\text{van}}(\partial\Omega) \otimes C\ell_n \ \text{ with } \eta \in (0,1), \tag{2.1.194}$$

are well defined, linear, bounded,

$$\langle \mathfrak{C} f, g \rangle = \langle f, \mathfrak{C}^{\#} g \rangle \tag{2.1.195}$$

(with the duality brackets as in [69, Theorem 4.6.1]) when either

$$f \in \text{BMO}(\partial\Omega, \sigma) \otimes C\ell_n \ \text{ and } \ g \in H^1(\partial\Omega, \sigma) \otimes C\ell_n, \tag{2.1.196}$$

or

$$\begin{gathered} f \in \mathscr{C}^{\eta}(\partial\Omega) \otimes C\ell_n \ \text{ and } \ g \in H^p(\partial\Omega, \sigma) \otimes C\ell_n \\ \text{with } p \in \big(\tfrac{n-1}{n}, 1\big) \ \text{ and } \ \eta := (n-1)\big(\tfrac{1}{p} - 1\big) \in (0,1), \end{gathered} \tag{2.1.197}$$

and, finally,

$$\mathfrak{C}^2 = \tfrac{1}{4} I \ \text{ on } \ \text{BMO}(\partial\Omega, \sigma) \otimes C\ell_n, \tag{2.1.198}$$

$$\mathfrak{C}^2 = \tfrac{1}{4} I \ \text{ on } \ \text{VMO}(\partial\Omega, \sigma) \otimes C\ell_n, \tag{2.1.199}$$

$$\mathfrak{C}^2 = \tfrac{1}{4} I \ \text{ on } \ \mathscr{C}^{\eta}(\partial\Omega) \otimes C\ell_n \ \text{ with } \eta \in (0,1), \tag{2.1.200}$$

$$\mathfrak{C}^2 = \tfrac{1}{4} I \ \text{ on } \ \mathscr{C}^{\eta}_{\text{van}}(\partial\Omega) \otimes C\ell_n \ \text{ with } \eta \in (0,1). \tag{2.1.201}$$

2.2 Single Layer Operators Acting from Hardy Spaces

As a prelude to the treatment of certain classes of integral operators (of single layer type) on boundary Hardy spaces later on in this section, in the next lemma we look at the issue of "differentiation under the duality pairing" which ultimately decides the regularity of the integral operators in question.

Lemma 2.2.1 *Consider a closed Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$ and let $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix a function $b \in \mathscr{C}^{N+1}\big(\mathbb{R}^n \setminus \{0\}\big)$ for some $N \in \mathbb{N}_0$, with the property that for each multi-index $\alpha \in \mathbb{N}_0^n$ of length $|\alpha| \leq N$ and each $\varepsilon > 0$ one has*

$$\sup_{x \in \mathbb{R}^n \setminus B(0,\varepsilon)} |(\partial^{\alpha} b)(x)| < +\infty. \tag{2.2.1}$$

Then for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and each $x \in \mathbb{R}^n \setminus \Sigma$ one has

$$(\partial^\alpha b)(x - \cdot)\big|_\Sigma \in \bigcap_{0<\eta<1} \mathscr{C}^\eta(\Sigma) \subset L^\infty(\Sigma, \sigma) \subset \mathrm{BMO}(\Sigma, \sigma). \tag{2.2.2}$$

Also, if for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and each $f \in H^p(\Sigma, \sigma)$ with $p \in \big(\frac{n-1}{n}, 1\big]$ one considers the function

$$(\mathcal{T}_\alpha f)(x) := \begin{cases} \big\langle (\partial^\alpha b)(x - \cdot), f \big\rangle & \text{if } \Sigma \text{ is bounded,} \\ \big\langle [(\partial^\alpha b)(x - \cdot)], f \big\rangle & \text{if } \Sigma \text{ is unbounded,} \end{cases} \qquad \forall x \in \mathbb{R}^n \setminus \Sigma, \tag{2.2.3}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality bracket on Σ described in [69, Theorem 4.6.1], *then $\mathcal{T}_\alpha f$ is well defined, and satisfies*

$$\mathcal{T}_\alpha f \in \mathscr{C}^N(\mathbb{R}^n \setminus \Sigma) \text{ and } \partial^\beta(\mathcal{T}_\alpha f) = \mathcal{T}_{\alpha+\beta} f \\ \text{for each } \beta \in \mathbb{N}_0^n \text{ with } |\beta| \leq N - |\alpha|. \tag{2.2.4}$$

Proof Pick $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and an arbitrary point $x \in \mathbb{R}^n \setminus \Sigma$. Using (2.2.1) with $\varepsilon \in \big(0, \mathrm{dist}(x, \Sigma)\big)$ then proves that the function $(\partial^\alpha b)(x - \cdot)$ is both bounded and Lipschitz in $\mathbb{R}^n \setminus \overline{B(0, \varepsilon)}$. Since our choice of ε ensures that Σ is contained in this set, and since both boundedness and Lipschitzianity are hereditary properties, we conclude that $(\partial^\alpha b)(x - \cdot)\big|_\Sigma$ is both bounded and Lipschitz on Σ. With this in hand, the membership in (2.2.2) follows with the help of [68, (7.4.105)] and [68, (7.3.25)].

Having proved (2.2.2), from the duality result recorded in [69, Theorem 4.6.1] we may now conclude that $\mathcal{T}_\alpha f$ is well defined in $\mathbb{R}^n \setminus \Sigma$ for each given $f \in H^p(\Sigma, \sigma)$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$. Next, fix an arbitrary $f \in H^p(\Sigma, \sigma)$ and suppose until mentioned otherwise that Σ is unbounded. According to [69, Theorem 4.4.1], there exist a numerical sequence $\{\lambda_j\}_{j \in \mathbb{N}} \in \ell^p(\mathbb{N})$ along with a sequence $\{a_j\}_{j \in \mathbb{N}}$ of $(p, 2)$-atoms on Σ such that

$$f = \lim_{m \to \infty} f_m \text{ in } H^p(\Sigma, \sigma) \text{ where, for each } m \in \mathbb{N}, \\ f_m := \sum_{j=1}^m \lambda_j a_j \in L^2_{\mathrm{comp}}(\Sigma, \sigma). \tag{2.2.5}$$

If for each $m \in \mathbb{N}$ we now set

$$u_m(x) := \big\langle [b(x - \cdot)], f_m \rangle, \qquad \forall x \in \mathbb{R}^n \setminus \Sigma, \tag{2.2.6}$$

then the duality result from [69, Theorem 4.6.1, (4.6.8)] permits us to express

$$
\begin{aligned}
u_m(x) &= \sum_{j=1}^{m} \lambda_j \int_{\Sigma} b(x-y) a_j(y)\, \mathrm{d}\sigma(y) \\
&= \int_{\Sigma} b(x-y) f_m(y)\, \mathrm{d}\sigma(y), \qquad \forall x \in \mathbb{R}^n \setminus \Sigma. \qquad (2.2.7)
\end{aligned}
$$

It is then clear from this representation that for each $m \in \mathbb{N}$ we have

$$
\begin{aligned}
& u_m \in \mathscr{C}^N(\mathbb{R}^n \setminus \Sigma) \text{ and, for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq N \text{ and } x \in \mathbb{R}^n \setminus \Sigma, \\
& (\partial^\alpha u_m)(x) = \int_{\Sigma} (\partial^\alpha b)(x-y) f_m(y)\, \mathrm{d}\sigma(y) = \big\langle [(\partial^\alpha b)(x-\cdot)], f_m \big\rangle,
\end{aligned} \qquad (2.2.8)
$$

where the last equality relies on (2.2.2) and the duality result from [69, Theorem 4.6.1, (4.6.8)]. If for each given $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ and each given $x \in \mathbb{R}^n \setminus \Sigma$ we introduce

$$
C_b^\alpha(x) := \begin{cases} \|(\partial^\alpha b)(x-\cdot)\|_{\dot{\mathscr{C}}^{(n-1)(1/p-1)}(\Sigma)} & \text{if } p \in \big(\tfrac{n-1}{n}, 1\big), \\ \|(\partial^\alpha b)(x-\cdot)\|_{\mathrm{BMO}(\Sigma,\sigma)} & \text{if } p = 1, \end{cases} \qquad (2.2.9)
$$

then based on (2.2.3), (2.2.8), [69, (4.6.9)], and (2.2.9) we may estimate

$$
\begin{aligned}
\big|(\mathcal{T}_\alpha f)(x) - (\partial^\alpha u_m)(x)\big| &= \Big|\big\langle [(\partial^\alpha b)(x-\cdot)], f \big\rangle - \big\langle [(\partial^\alpha b)(x-\cdot)], f_m \big\rangle\Big| \\
&= \Big|\big\langle [(\partial^\alpha b)(x-\cdot)], f - f_m \big\rangle\Big| \\
&\leq C_b^\alpha(x) \|f - f_m\|_{H^p(\Sigma,\sigma)}, \qquad (2.2.10)
\end{aligned}
$$

for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$, $m \in \mathbb{N}$, and $x \in \mathbb{R}^n \setminus \Sigma$. An inspection of the argument used to justify (2.2.2) reveals that, for each $x_o \in \mathbb{R}^n \setminus \Sigma$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$ fixed, the quantity (2.2.9) satisfies

$$
\sup_{x \in B(x_o, r)} C_b^\alpha(x) < +\infty \text{ if } 0 < r < \operatorname{dist}(x_o, \Sigma). \qquad (2.2.11)
$$

In concert with (2.2.5) and (2.2.10), this proves that

$$
\begin{aligned}
&\text{for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq N, \text{ the sequence } \big\{\partial^\alpha u_m\big\}_m \text{ converges} \\
&\text{to the function } \mathcal{T}_\alpha f \text{ uniformly on compact subsets of } \mathbb{R}^n \setminus \Sigma.
\end{aligned} \qquad (2.2.12)
$$

From this and the differentiation theorem for sequences of functions we then readily conclude that (2.2.4) holds in the case when Σ is unbounded. Finally, the case when Σ is bounded is treated, *mutatis mutandis*, identically. □

We continue by considering integral operators of single layer type and studying the manner in which their weak tangential derivatives act on inhomogeneous Hardy-based Sobolev spaces.

Proposition 2.2.2 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. For some sufficiently large $N = N(n) \in \mathbb{N}$, let $b \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ be a complex-valued function which is even and positive homogeneous of degree $2-n$. Consider the integral operator acting on each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\big)$ (hence, in particular, for each $f \in L^p(\partial\Omega, \sigma)$ with $p \in (1, n-1)$) according to*

$$\mathscr{B} f(x) := \int_{\partial\Omega} b(x-y) f(y)\, \mathrm{d}\sigma(y) \ \textit{for each}\ x \in \Omega. \tag{2.2.13}$$

Then, for each $p \in \big(\frac{n-1}{n}, n-1\big)$ and $q \in (0, \infty]$, the operator (2.2.13) extends to a continuous linear mapping

$$\mathscr{B} : H^{p,q}(\partial\Omega, \sigma) \longrightarrow \mathscr{C}^N(\Omega) \tag{2.2.14}$$

which, for each $j, k \in \{1, \dots, n\}$ and each $f \in H^{p,q}(\partial\Omega, \sigma)$, satisfies

$$\dot{\partial}_{\tau_{jk}}\big(\mathscr{B} f\big) = T^{\#}_{jk} f \ \textit{ in }\ H^{p,q}(\partial\Omega, \sigma), \tag{2.2.15}$$

where the operator $T^{\#}_{jk}$ is as in (2.1.44).

Let us also consider $\mathscr{B}_{\text{mod}}$, the modified version of (2.2.13) acting on each function f belonging to the larger space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ (hence, in particular, for each $f \in L^p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$) according to

$$\mathscr{B}_{\text{mod}} f(x) := \int_{\partial\Omega} \Big\{ b(x-y) - b(-y)\mathbf{1}_{\mathbb{R}^n \setminus B(0,1)}(y) \Big\} f(y)\, \mathrm{d}\sigma(y) \tag{2.2.16}$$

at every $x \in \Omega$. Then for each function $f \in L^{p,q}(\partial\Omega, \sigma)$, with exponents $p \in (1, \infty)$, $q \in (0, \infty]$, and each $j, k \in \{1, \dots, n\}$ one has

$$\dot{\partial}_{\tau_{jk}}\big(\mathscr{B}_{\text{mod}} f\big) = T^{\#}_{jk} f \ \textit{ at } \sigma\textit{-a.e. point on }\ \partial\Omega, \tag{2.2.17}$$

where the operator $T^{\#}_{jk}$ is now interpreted as in item (ii) of Proposition 1.2.1.

Before presenting the proof of Proposition 2.2.2 we wish to comment on some of its immediate ramifications. To set the stage, let $\Omega \subseteq \mathbb{R}^n$ be a UR domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Suppose $b \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ (for some sufficiently large $N \in \mathbb{N}$) is a complex-valued function which is even and positive homogeneous of degree $2-n$. Also, pick two arbitrary indices $j, k \in \{1, \dots, n\}$ and recall the integral operator $\mathcal{T}_{jk}$ associated with Ω and b as in (1.2.1). Finally, consider $\mathscr{B}$ as in (2.2.14). As consequence of [69, Lemma 11.10.3] and [69, Corollary 11.1.8], we then see that for each distribution f belonging to the Hardy-based inhomogeneous Sobolev space $H^{q,p}_1(\partial\Omega, \sigma)$ (cf. [69, Definition 11.10.6]) with $p \in \big(\frac{n-1}{n}, n-1\big)$ and $q \in (1, \infty)$ we have

$$\mathcal{T}_{jk} f = \mathscr{B}\big(\partial_{\tau_{jk}} f\big) \ \text{ in } \Omega. \tag{2.2.18}$$

In concert, (2.2.18) and (2.2.15) further imply that for every $j, k, r, s \in \{1, \dots, n\}$ we have (compare with (1.2.13))

$$\begin{aligned} &\dot{\partial}_{\tau_{rs}}\big(\mathcal{T}_{jk} f\big) = T^{\#}_{rs}\big(\partial_{\tau_{jk}} f\big) \text{ in } H^p(\partial\Omega, \sigma), \text{ for each} \\ &f \in H^{q,p}_1(\partial\Omega, \sigma) \text{ with } p \in \big(\tfrac{n-1}{n}, n-1\big) \text{ and } q \in (1, \infty). \end{aligned} \tag{2.2.19}$$

Here is the proof of Proposition 2.2.2.

Proof of Proposition 2.2.2 The extension of the operator $\mathscr{B}$, originally considered as in (2.2.13), to a continuous linear mapping in the context of (2.2.14) may be justified based on Lemma 2.2.1 and real-interpolation (cf. [69, Theorem 4.3.1]); see also [70, (2.4.24)] in this regard. As such, given any aperture parameter $\kappa > 0$ along with $p \in \big(\frac{n-1}{n}, n-1\big)$ and $q \in (0, \infty]$, we conclude from [70, (2.4.27)] that there exists a constant $C \in (0, \infty)$ with the property that

$$\big\|\mathcal{N}_\kappa(\nabla \mathscr{B} f)\big\|_{L^{p,q}(\partial\Omega,\sigma)} \leq C\|f\|_{H^{p,q}(\partial\Omega,\sigma)} \text{ for each } f \in H^{p,q}(\partial\Omega, \sigma). \tag{2.2.20}$$

To proceed, fix two indices $j, k \in \{1, \dots, n\}$. Granted (2.2.20), the result described in [69, Example 10.2.2] applies and [69, (10.2.13)] gives that

$$\begin{aligned} &\text{for each } f \in H^{p,q}(\partial\Omega, \sigma) \text{ with } p \in \big(\tfrac{n-1}{n}, n-1\big) \text{ and } q \in (0, \infty], \\ &\text{the distribution } \dot{\partial}_{\tau_{jk}}(\mathscr{B} f) \text{ belongs to the Lorentz-based Hardy space} \\ &H^{p,q}(\partial\Omega, \sigma) \text{ and there exists a finite constant } C > 0, \text{ depending only} \\ &\text{on } \Omega, n, b, p, q, \text{ such that } \big\|\dot{\partial}_{\tau_{jk}}(\mathscr{B} f)\big\|_{H^{p,q}(\partial\Omega,\sigma)} \leq C\|f\|_{H^{p,q}(\partial\Omega,\sigma)}. \end{aligned} \tag{2.2.21}$$

In view of the fact that the operator $T^{\#}_{jk}$ is also well-defined, linear, and bounded on $H^{p,q}(\partial\Omega, \sigma)$, in order to conclude (2.2.15) it suffices to show that

$$\dot{\partial}_{\tau_{jk}}\big(\mathscr{B} f\big) = T^{\#}_{jk} f \text{ for each } f \in L^p(\partial\Omega, \sigma) \text{ with } p \in (1, n-1). \tag{2.2.22}$$

Indeed, via density this identity continues to hold for any $f \in H^p(\partial\Omega, \sigma)$ with $p \in \big(\frac{n-1}{n}, n-1\big)$ from which (2.2.15) is then deduced using real-interpolation (cf. [69, Theorem 4.3.1]).

To justify (2.2.22), fix an arbitrary function $f \in L^p(\partial\Omega, \sigma)$ with $p \in (1, n-1)$. Based on [69, Example 10.2.10] (see the equality in [69, (10.2.92)], in particular) and [70, (2.4.9)] we conclude that $\dot{\partial}_{\tau_{jk}}\big(\mathscr{B} f\big)$ belongs to $L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ and, for some background aperture parameter $\kappa > 0$,

$$\dot{\partial}_{\tau_{jk}}\big(\mathscr{B} f\big) = \nu_j(\partial_k \mathscr{B} f)\big|^{\kappa-\text{n.t.}}_{\partial\Omega} - \nu_k(\partial_j \mathscr{B} f)\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{2.2.23}$$

where $\nu = (\nu_1, \dots, \nu_n)$ stands for the geometric measure theoretic outward unit normal to Ω. In concert with the jump-formula described in (2.1.49)-(2.1.51) and the identity proved in (1.2.18), this permits us to compute

$$
\begin{aligned}
&\overset{\bullet}{\partial}_{\tau_{jk}}\big(\mathscr{B}f\big)(x)\\
&\quad= \frac{1}{2\mathrm{i}}\,\nu_j(x)\widehat{\partial_k b}\big(\nu(x)\big)f(x) + \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_j(x)(\partial_k b)(x-y)f(y)\,\mathrm{d}\sigma(y)\\
&\qquad - \frac{1}{2\mathrm{i}}\,\nu_k(x)\widehat{\partial_j b}\big(\nu(x)\big)f(x) - \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_k(x)(\partial_j b)(x-y)f(y)\,\mathrm{d}\sigma(y)\\
&\quad= \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big\{\nu_j(x)(\partial_k b)(x-y) - \nu_k(x)(\partial_j b)(x-y)\big\}f(y)\,\mathrm{d}\sigma(y)\\
&\quad= T^{\#}_{jk}f(x) \text{ at } \sigma\text{-a.e. point } x\in\partial\Omega,
\end{aligned}
\tag{2.2.24}
$$

with the last equality coming from (1.2.3). Hence, (2.2.22) is established and, with it, the proof of (2.2.15) is complete.

Turning our attention to (2.2.17), first observe that for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ and each index $\ell \in \{1,\dots,n\}$ we have

$$
\partial_\ell\big(\mathscr{B}_{\mathrm{mod}}f\big)(x) = \int_{\partial\Omega}(\partial_\ell b)(x-y)f(y)\,\mathrm{d}\sigma(y) \text{ for all } x\in\Omega. \tag{2.2.25}
$$

In concert with [70, Theorem 2.4.1], this proves that for each $p \in (1,\infty)$ and $q \in (0,\infty]$ there exists a constant $C \in (0,\infty)$ with the property that

$$
\big\|\mathcal{N}_\kappa\big(\nabla(\mathscr{B}_{\mathrm{mod}}f)\big)\big\|_{L^{p,q}(\partial\Omega,\sigma)} \leq C\|f\|_{L^{p,q}(\partial\Omega,\sigma)} \tag{2.2.26}
$$

for all $f \in L^{p,q}(\partial\Omega,\sigma)$. Going further, pick two indices $j,k \in \{1,\dots,n\}$. Having shown (2.2.26), we may once again invoke [69, Example 10.2.2] to conclude that

> for each function $f \in L^{p,q}(\partial\Omega,\sigma)$ with integrability exponents $p \in (1,\infty)$ and $q \in (0,\infty]$, the distribution $\overset{\bullet}{\partial}_{\tau_{jk}}\big(\mathscr{B}_{\mathrm{mod}}f\big)$ belongs to the Lorentz space $L^{p,q}(\partial\Omega,\sigma)$ and there exists some finite constant $C > 0$, depending only on Ω, n, b, p, q, with the property that $\big\|\overset{\bullet}{\partial}_{\tau_{jk}}\big(\mathscr{B}_{\mathrm{mod}}f\big)\big\|_{L^{p,q}(\partial\Omega,\sigma)} \leq C\|f\|_{L^{p,q}(\partial\Omega,\sigma)}$. (2.2.27)

Given that the operator $T^{\#}_{jk}$ is also well-defined, linear, and bounded on $L^{p,q}(\partial\Omega,\sigma)$ (cf. item *(ii)* in Proposition 1.2.1), the formula claimed in (2.2.17) follows via density and embeddings (cf. [68, (6.2.51)-(6.2.52)]) as soon as we show that

$$
\overset{\bullet}{\partial}_{\tau_{jk}}\big(\mathscr{B}_{\mathrm{mod}}f\big) = T^{\#}_{jk}f \text{ for each } f\in L^p(\partial\Omega,\sigma) \text{ with } p\in(1,n-1). \tag{2.2.28}
$$

However, (2.2.28) is implied by the fact that, as seen from [69, Example 10.2.2], the operator $\overset{\bullet}{\partial}_{\tau_{jk}}$ annihilates constants and the observation that, as seen from definitions, for each function f belonging to the space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ (in particular, for each function $f \in L^p(\partial\Omega, \sigma)$ with exponent $p \in [1, \infty)$) the difference $C_f := \mathscr{B}_{\rm mod} f - \mathscr{B} f$ is a constant in Ω. (2.2.29)

This completes the proof of (2.2.17). □

One of the main points of the theorem below is that the action of the boundary-to-domain version of the single layer, originally defined as in (1.3.6) on Lebesgue space L^p with $p \in [1, n-1)$, may be extended in a natural fashion to the scale of Hardy spaces H^p for p in the range $\big(\frac{n-1}{n}, n-1\big)$ if $n \geq 3$. Moreover, there is a natural version of said single layer in the two-dimensional setting, acting on the scale of Hardy spaces H^p with $p \in \big(\frac{1}{2}, 1\big]$.

Theorem 2.2.3 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is an open set with an Ahlfors regular boundary $\partial\Omega$. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit to Ω. For some $M \in \mathbb{N}$, consider a coefficient tensor $A = \Big(a^{\alpha\beta}_{rs}\Big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with complex entries, with the property that the $M\times M$ second-order system (as always, the summation convention over repeated indices is in effect)*

$$L_A := \Big(a^{\alpha\beta}_{rs}\partial_r\partial_s\Big)_{1\leq\alpha,\beta\leq M} \tag{2.2.30}$$

is weakly elliptic. Also, denote by $A^\top$ the transpose of the coefficient tensor A (as defined in (1.3.3)). Finally, let $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ be the fundamental solution associated with the system L_A as in [70, Theorem 1.4.2].

In this setting, for each $p \in \big(\frac{n-1}{n}, 1\big]$ define the boundary-to-domain single layer potential operator $\mathscr{S}$ acting on each $f = (f_\beta)_{1\leq\beta\leq M} \in \big[H^p(\partial\Omega, \sigma)\big]^M$ according to

$$\mathscr{S}f(x) := \begin{cases} \Big(\big\langle E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta\big\rangle\Big)_{1\leq\alpha\leq M} & \text{if } \partial\Omega \text{ is bounded,}\\[2ex] \Big(\big\langle [E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}], f_\beta\big\rangle\Big)_{1\leq\alpha\leq M} & \text{if } \partial\Omega \text{ is unbounded,}\end{cases} \tag{2.2.31}$$

at every $x \in \Omega$, where $\langle\cdot,\cdot\rangle$ stands for the duality bracket on $\partial\Omega$, described in [69, Theorem 4.6.1] *(with $\Sigma := \partial\Omega$). Then the following properties hold.*

(1) For each $p \in \big(\frac{n-1}{n}, 1\big]$, the single layer induces a well-defined, linear operator in the context

$$\mathscr{S} : \big[H^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathscr{C}^\infty(\Omega)\big]^M \tag{2.2.32}$$

which is also continuous when $\big[\mathscr{C}^\infty(\Omega)\big]^M$ is equipped with the Frechét topology of uniform convergence of partial derivatives on compact sets. In additions, the operators in (2.2.32) corresponding to various values of $p \in \big(\frac{n-1}{n}, 1\big]$ are compatible with one another.
Moreover, for each $f = (f_\beta)_{1\leq\beta\leq M} \in [H^p(\partial\Omega, \sigma)]^M$ with integrability exponent $p \in \big(\frac{n-1}{n}, 1\big]$ one has

$$L_A\big(\mathscr{S}f\big) = 0 \ \textit{in} \ \Omega, \tag{2.2.33}$$

and, for each multi-index $\gamma \in \mathbb{N}_0^n$ *and each point* $x \in \Omega$,

$$\partial^\gamma\big(\mathscr{S}f\big)(x) = \begin{cases} \Big(\big\langle(\partial^\gamma E_{\alpha\beta})(x-\cdot)\big|_{\partial\Omega}, f_\beta\big\rangle\Big)_{1\le\alpha\le M} & \textit{if } \partial\Omega \textit{ is bounded,} \\ \Big(\big\langle[(\partial^\gamma E_{\alpha\beta})(x-\cdot)\big|_{\partial\Omega}], f_\beta\big\rangle\Big)_{1\le\alpha\le M} & \textit{if } \partial\Omega \textit{ is unbounded.} \end{cases} \tag{2.2.34}$$

In addition, for each $f \in [H^p(\partial\Omega,\sigma)]^M$ *with* $p \in \big(\frac{n-1}{n}, 1\big]$ *and each* $\gamma \in \mathbb{N}_0^n$ *one has*

$$\begin{gathered} \partial^\gamma\big(\mathscr{S}f\big)(x) = O(|x|^{2-n-|\gamma|}) \ \textit{as} \ |x| \to \infty, \\ \textit{if } \Omega \textit{ is an exterior domain, and either } n \ge 3 \textit{ or } |\gamma| > 0. \end{gathered} \tag{2.2.35}$$

Finally,

> *if* $n = 2$ *then for each* $f \in \big[H^1(\partial\Omega,\sigma)\big]^M$ *the function* $\mathscr{S}f \in \big[\mathscr{C}^\infty(\Omega)\big]^M$, *originally defined for each* $x \in \Omega$ *as in* (2.2.31), *extends* (*via the same formula*) *to a continuous function in* $\mathbb{R}^2$, *and if* Ω *is not an exterior domain it is bounded by* $C\|f\|_{[H^1(\partial\Omega,\sigma)]^M}$, *for some* $C = C(\Omega, A) \in (0,\infty)$. (2.2.36)

(2) For each $p \in \big(\frac{n-1}{n}, 1\big]$, *the operator* $\mathscr{S}$ *from* (2.2.31)-(2.2.32) *satisfies*

$$\begin{gathered} \mathscr{S}f(x) = \int_{\partial\Omega} E(x-y)f(y)\,d\sigma(y) \ \textit{for each} \ x \in \Omega, \ \textit{whenever} \\ f \in \big[H^p(\partial\Omega,\sigma)\big]^M \cap \big[L^q(\partial\Omega,\sigma)\big]^M \textit{ with } q \in (1, n-1). \end{gathered} \tag{2.2.37}$$

Hence, the operator $\mathscr{S}$ *from* (2.2.31)-(2.2.32) *is compatible with the single layer operator introduced earlier in* (1.3.6). *In particular, the latter operator may be regarded as a natural extension of the former. Together, they give rise to a well-defined, linear and bounded mapping*

$$\mathscr{S} : \big[H^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad p \in \begin{cases} \big(\frac{n-1}{n}, n-1\big) & \textit{if } n \ge 3, \\ \big(\frac{1}{2}, 1\big] & \textit{if } n = 2, \end{cases} \tag{2.2.38}$$

defined as in (2.2.31)-(2.2.32) *whenever* $p \in \big(\frac{n-1}{n}, 1\big]$, *and defined as in* (1.3.6) *whenever* $n \ge 3$ *and* $p \in \big(1, n-1\big)$. *The operator* $\mathscr{S}$ *in* (2.2.38) *has the property that for each integrability exponent* $p \in \big(\frac{n-1}{n}, n-1\big)$ *and each aperture parameter* $\kappa \in (0,\infty)$ *there exists some constant* $C = C(\Omega, A, \kappa, p) \in (0,\infty)$ *such that for each* $f \in \big[H^p(\partial\Omega,\sigma)\big]^M$ *one has*

$$\text{if either } n \geq 3, \text{ or } n = 2 \text{ and } \Omega \text{ is not an exterior domain, then}$$
$$\big\|\mathcal{N}_\kappa(\mathscr{S} f)\big\|_{L^{p^*}(\partial\Omega,\sigma)} \leq C\|f\|_{[H^p(\partial\Omega,\sigma)]^M} \text{ with } p^* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1}, \tag{2.2.39}$$

and

$$\text{if } n = 2 \text{ and } \Omega \text{ is an exterior domain then for each } R > 0$$
$$\big\|\mathcal{N}_\kappa^{\Omega\cap B(0,R)}(\mathscr{S} f)\big\|_{L^{p^*}(\partial\Omega,\sigma)} \leq C_R\|f\|_{[H^p(\partial\Omega,\sigma)]^M} \text{ with } p^* := \big(\tfrac{1}{p} - 1\big)^{-1}. \tag{2.2.40}$$

Moreover, in the two-dimensional case one may allow $p = 1$ in (2.2.39)-(2.2.40) (*in which scenario one has $p^* = \infty$), that is,*

$$\text{if } n = 2 \text{ and } \Omega \text{ is not an exterior domain, then}$$
$$\big\|\mathcal{N}_\kappa(\mathscr{S} f)\big\|_{L^\infty(\partial\Omega,\sigma)} \leq C\|f\|_{[H^1(\partial\Omega,\sigma)]^M} \text{ for all } f \in \big[H^1(\partial\Omega,\sigma)\big]^M, \tag{2.2.41}$$

and

$$\text{if } n = 2 \text{ and } \Omega \text{ is an exterior domain, then for each } R > 0$$
$$\big\|\mathcal{N}_\kappa^{\Omega\cap B(0,R)}(\mathscr{S} f)\big\|_{L^\infty(\partial\Omega,\sigma)} \leq C_R\|f\|_{[H^1(\partial\Omega,\sigma)]^M} \text{ for all } f \in \big[H^1(\partial\Omega,\sigma)\big]^M. \tag{2.2.42}$$

Finally, when either $n \geq 3$ or $\partial\Omega$ is bounded, the single layer operator from (1.3.6) *corresponding to $p = 1$, i.e.,*

$$\mathscr{S} : \big[L^1(\partial\Omega,\sigma)\big]^M \to \big[\mathscr{C}^\infty(\Omega)\big]^M \text{ mapping each } f \in \big[L^1(\partial\Omega,\sigma)\big]^M$$
$$\text{into } \mathscr{S} f(x) := \int_{\partial\Omega} E(x-y) f(y)\, d\sigma(y) \text{ for every } x \in \Omega, \tag{2.2.43}$$

extends the operator $\mathscr{S}$ from (2.2.38) *corresponding to $p = 1$. Moreover, for each given $p \in \big(\frac{n-1}{n}, 1\big]$, the operator $\mathscr{S}$ in* (2.2.43) *agrees with the operator $\mathscr{S}$ from* (2.2.38) *on the subspace $\big[H^p(\partial\Omega,\sigma)\big]^M \cap \big[H^1(\partial\Omega,\sigma)\big]^M$ of $\big[L^1(\partial\Omega,\sigma)\big]^M$.*

(3) Assume $\partial\Omega$ is a UR *set, and pick an aperture parameter $\kappa > 0$. Then for each integrability exponent $p \in \big(\frac{n-1}{n}, n-1\big)$ if $n \geq 3$, or $p \in \big(\frac{1}{2}, 1\big]$ if $n = 2$, there exists a constant $C = C(\Omega, A, \kappa, p) \in (0,\infty)$ with the property that for every $f \in \big[H^p(\partial\Omega,\sigma)\big]^M$ one has*

$$\sum_{j=1}^n \big\|\mathcal{N}_\kappa(\partial_j \mathscr{S} f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[H^p(\partial\Omega,\sigma)]^M}. \tag{2.2.44}$$

(4) *Suppose $\partial\Omega$ is a* UR *set. Then, if $f \in [H^p(\partial\Omega,\sigma)]^M$ with $p \in \big(\frac{n-1}{n},1\big]$, the weak conormal derivative $\dot{\partial}^A_\nu \mathscr{S} f$, taken in the sense of* [69, Definition 10.2.18], *is meaningfully defined in the Hardy space $[H^p(\partial\Omega,\sigma)]^M$ and satisfies, for a constant $C = C(\Omega, A, p) \in (0,\infty)$ independent of f,*

$$\big\|\dot{\partial}^A_\nu \mathscr{S} f\big\|_{[H^p(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[H^p(\partial\Omega,\sigma)]^M}. \tag{2.2.45}$$

In addition, the weak conormal derivative $\dot{\partial}^A_\nu \mathscr{S} f$, considered in the sense of [69, Definition 10.2.18], *is also meaningfully defined for every $f \in [L^1(\partial\Omega,\sigma)]^M$ and satisfies, for some $C = C(\Omega, A) \in (0,\infty)$ independent of f,*

$$\big\|\dot{\partial}^A_\nu \mathscr{S} f\big\|_{[H^{1,\infty}(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[L^1(\partial\Omega,\sigma)]^M}. \tag{2.2.46}$$

If in fact Ω is a UR *domain, then for every $f \in [H^p(\partial\Omega,\sigma)]^M$ with $p \in \big(\frac{n-1}{n},1\big]$ one actually has the jump-formula*

$$\dot{\partial}^A_\nu \mathscr{S} f = \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f \ \text{ in } \big[H^p(\partial\Omega,\sigma)\big]^M, \tag{2.2.47}$$

where I is the identity operator, and $K^{\#}_{A^\top}$ is the operator associated as in (2.1.4) *with the transpose coefficient tensor $A^\top$ (in place of A).*

(5) *Fix $\kappa > 0$ along with $q \in (0,\infty]$ and $p \in \big(\frac{n-1}{n}, n-1\big)$. Then the single layer operator* (2.2.38) *further extends to a linear and continuous mapping*

$$\mathscr{S} : \big[H^{p,q}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathscr{C}^\infty(\Omega)\big]^M \tag{2.2.48}$$

with the property that there exists some constant $C = C(\Omega, A, \kappa, p, q) \in (0,\infty)$ such that for each $f \in \big[H^{p,q}(\partial\Omega,\sigma)\big]^M$ one has

$$\sum_{j=1}^n \big\|\mathcal{N}_\kappa(\partial_j \mathscr{S} f)\big\|_{L^{p,q}(\partial\Omega,\sigma)} \leq C\|f\|_{[H^{p,q}(\partial\Omega,\sigma)]^M}, \tag{2.2.49}$$

as well as

$$\begin{gathered}\text{if } n = 2 \text{ and } \Omega \text{ is not an exterior domain, then}\\ \big\|\mathcal{N}_\kappa(\mathscr{S} f)\big\|_{L^{p^*,q}(\partial\Omega,\sigma)} \leq C\|f\|_{[H^{p,q}(\partial\Omega,\sigma)]^M} \ \text{ with } \ p^* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1},\end{gathered} \tag{2.2.50}$$

and

$$\begin{gathered}\text{if } n = 2 \text{ and } \Omega \text{ is an exterior domain, then for each } R > 0\\ \big\|\mathcal{N}_\kappa^{\Omega\cap B(0,R)}(\mathscr{S} f)\big\|_{L^{p^*,q}(\partial\Omega,\sigma)} \leq C_R\|f\|_{[H^{p,q}(\partial\Omega,\sigma)]^M}\\ \text{with } \ p^* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1}.\end{gathered} \tag{2.2.51}$$

In particular, (2.2.50)-(2.2.51) *with* $q := p$ *give the following sharper versions of* (2.2.39)-(2.2.40)*:*

$$\text{if } n = 2 \text{ and } \Omega \text{ is not an exterior domain, then}$$
$$\big\|\mathcal{N}_\kappa(\mathscr{S} f)\big\|_{L^{p^*,p}(\partial\Omega,\sigma)} \leq C\|f\|_{[H^p(\partial\Omega,\sigma)]^M} \text{ with } p^* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1}, \tag{2.2.52}$$

and, respectively,

$$\text{if } n = 2 \text{ and } \Omega \text{ is an exterior domain, then for each } R > 0$$
$$\big\|\mathcal{N}_\kappa^{\Omega\cap B(0,R)}(\mathscr{S} f)\big\|_{L^{p^*,p}(\partial\Omega,\sigma)} \leq C_R\|f\|_{[H^p(\partial\Omega,\sigma)]^M} \text{ with } p^* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1}, \tag{2.2.53}$$

Also, whenever $f = (f_\beta)_{1\leq\beta\leq M} \in \big[H^{p,q}(\partial\Omega,\sigma)\big]^M$ *has compact support* (*as a distribution, which is automatically the case if* $\partial\Omega$ *is compact*) *and the function* $\psi \in \mathrm{Lip}_c(\partial\Omega)$ *is identically one near* supp f *one has*

$$\begin{aligned}&\big(\mathscr{S} f\big)_\alpha(x) = {}_{\mathrm{Lip}_c(\partial\Omega)}\Big\langle \psi E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta\Big\rangle_{(\mathrm{Lip}_c(\partial\Omega))'}\\ &\text{for each } x\in\Omega \text{ and each } \alpha\in\{1,\dots,M\}.\end{aligned} \tag{2.2.54}$$

Moreover, if in fact Ω *is a* UR *domain, then for every* $f \in [H^{p,q}(\partial\Omega,\sigma)]^M$ *one has the jump-formulas*

$$\begin{aligned}\dot{\partial}^A_\nu \mathscr{S} f &= \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f \text{ in } \big[H^{p,q}(\partial\Omega,\sigma)\big]^M \text{ if } p \leq 1,\\ \partial^A_\nu \mathscr{S} f &= \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f \text{ in } \big[L^{p,q}(\partial\Omega,\sigma)\big]^M \text{ if } p > 1,\end{aligned} \tag{2.2.55}$$

as well as

$$\begin{aligned}&\Big(\nu_r a^{\alpha\beta}_{rs}\big[\partial_s(\mathscr{S} f)_\beta\big]\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)_{1\leq\alpha\leq M} = -\tfrac{1}{2}\mathfrak{H} f + K^\#_{A^\top} f \text{ at } \sigma\text{-a.e. point on } \partial\Omega,\\ &\text{with } \mathfrak{H} \text{ as in [69, Theorem 4.9.1, (4.9.5)] and } K^\#_{A^\top} \text{ now regarded}\\ &\text{as a mapping from } \big[H^{p,q}(\partial\Omega,\sigma)\big]^M \text{ into } \big[L^{p,q}(\partial\Omega,\sigma)\big]^M \text{ (cf. (2.1.3))}.\end{aligned} \tag{2.2.56}$$

(6) Assume Ω *is a* UR *domain, and consider a factorization of the original system of the form*

$$L = \widetilde{D} D \tag{2.2.57}$$

where $\widetilde{D}$ *is a homogeneous, constant* (*complex*) *coefficient, first-order* $M \times N$ *system in* $\mathbb{R}^n$*, and* D *is a homogeneous, constant* (*complex*) *coefficient, first-order* $N \times M$ *system in* $\mathbb{R}^n$*, say*

$$\widetilde{D} = \Big(\sum_{j=1}^{n} \widetilde{b}_j^{\alpha\gamma} \partial_j \Big)_{\substack{1 \le \alpha \le M \\ 1 \le \gamma \le N}} \text{ and } D = \Big(\sum_{k=1}^{n} b_k^{\gamma\beta} \partial_k \Big)_{\substack{1 \le \gamma \le N \\ 1 \le \beta \le M}}. \tag{2.2.58}$$

Also, define the coefficient tensor (with the summation convention over repeated indices in effect)

$$A_{\widetilde{D},D} := \big(a_{jk}^{\alpha\beta} \big)_{\substack{1 \le j,k \le n \\ 1 \le \alpha,\beta \le M}} \text{ where each } a_{jk}^{\alpha\beta} := \widetilde{b}_j^{\alpha\gamma} b_k^{\gamma\beta}. \tag{2.2.59}$$

Then for every $\kappa > 0$ *and every* $f \in [H^{p,q}(\partial\Omega, \sigma)]^M$ *with* $p \in \big(\frac{n-1}{n}, \infty\big)$ *and* $q \in (0, \infty]$ *one has the jump-formulas*

$$(-\mathrm{i})\mathrm{Sym}\big(\widetilde{D}; \nu\big) \bullet (D\mathscr{S} f) = \Big(-\tfrac{1}{2} I + K^{\#}_{A^{\top}_{\widetilde{D},D}} \Big) f \text{ if } p \le 1, \tag{2.2.60}$$

$$(-\mathrm{i})\mathrm{Sym}\big(\widetilde{D}; \nu\big)(D\mathscr{S}_{\text{mod}} f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \Big(-\tfrac{1}{2} I + K^{\#}_{A^{\top}_{\widetilde{D},D}} \Big) f \text{ if } p > 1, \tag{2.2.61}$$

where the distribution in the left-hand side of (2.2.60) *is defined as in* [69, Proposition 10.2.11], *and where* $K^{\#}_{A^{\top}_{\widetilde{D},D}}$ *is the transpose double layer associated with the domain* Ω *and the transpose coefficient tensor* $\big(A_{\widetilde{D},D}\big)^{\top}$.

(7) Continue to assume that Ω *is a* UR *domain, and fix two arbitrary indices* $j, k \in \{1, \dots, n\}$. *Then for each* $p \in \big(\frac{n-1}{n}, \infty\big)$ *and* $q \in (0, \infty]$ *the operator* $T^{\#}_{jk}$ *from* (1.5.82) *extends to a linear and bounded mapping*

$$T^{\#}_{jk} : \big[H^{p,q}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[H^{p,q}(\partial\Omega, \sigma)\big]^M \tag{2.2.62}$$

and, with this interpretation,

$$\begin{gathered} \text{for each } f \in \big[H^{p,q}(\partial\Omega, \sigma)\big]^M \text{ with } \big(\tfrac{n-1}{n}, n-1\big) \text{ and } q \in (0, \infty] \\ \text{one has } \dot{\partial}_{\tau_{jk}} \big(\mathscr{S} f\big) = T^{\#}_{jk} f \text{ in } \big[H^{p,q}(\partial\Omega, \sigma)\big]^M. \end{gathered} \tag{2.2.63}$$

Also,

$$\begin{gathered} \text{if } n = 2 \text{ then for each } f \in \big[H^{1}(\partial\Omega, \sigma)\big]^M \text{ one has} \\ \dot{\partial}_{\tau_{jk}} \big(\mathscr{S} f\big) = T^{\#}_{jk} f \text{ in } \big[H^{1}(\partial\Omega, \sigma)\big]^M. \end{gathered} \tag{2.2.64}$$

Finally, for each function $f \in \big[L^{p,q}(\partial\Omega, \sigma)\big]^M$ *with* $p \in (1, \infty)$ *and* $q \in (0, \infty]$ *one has*

$$\dot{\partial}_{\tau_{jk}} \big(\mathscr{S}_{\text{mod}} f\big) = T^{\#}_{jk} f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{2.2.65}$$

where $T^{\#}_{jk}$ *is now interpreted as a bounded operator on* $\big[L^{p,q}(\partial\Omega, \sigma)\big]^M$.

It is also worth noting that, as an inspection of the proof of Theorem 2.2.3 reveals, all results in items *(1)-(2)*, with the exception of (2.2.33) and (2.2.36), are of purely real variable nature, as they utilize only generic size and regularity properties of the fundamental solution E. As such, analogous results are valid for the more general class of operators in which E is replaced by a kernel function $b \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying (2.2.1) for each multi-index $\alpha \in \mathbb{N}_0^n$, as well as

$$\begin{gathered} |(\partial^\alpha b)(x)| \leq C|x|^{-(n-2+|\alpha|)} \text{ for each } x \in \mathbb{R}^n \setminus \{0\} \\ \text{and each multi-index } \alpha \in \mathbb{N}_0^n \text{ of length } |\alpha| \leq 1. \end{gathered} \tag{2.2.66}$$

We now turn to the task of giving the proof of Theorem 2.2.3.

Proof of Theorem 2.2.3 That the operator in (2.2.32) is well defined, linear, continuous, and that (2.2.34) holds, follow from Lemma 2.2.1 and [70, Theorem 1.4.2] when $n \geq 3$, and from [70, (1.4.22)-(1.4.23)], [69, Lemma 4.8.2], and [70, (1.4.44)] when $n = 2$. In turn, (2.2.34) readily implies (2.2.33) (cf. [70, (1.4.21)]). Next, that the operators in (2.2.32) corresponding to various values of p in $\big(\frac{n-1}{n}, 1\big]$ are compatible with one another is a consequence of (2.2.31) and [69, Theorem 4.4.3].

To justify the claim made in (2.2.35), work under the assumption that Ω is an exterior domain and fix $f = (f_\beta)_{1 \leq \beta \leq M} \in [H^p(\partial\Omega, \sigma)]^M$ where $p \in \big(\frac{n-1}{n}, 1\big]$ along with $\gamma \in \mathbb{N}_0^n$. Then from (2.2.34) we know that for each $\alpha \in \{1, \dots, M\}$ we have

$$\partial^\gamma \big(\mathscr{S} f\big)_\alpha(x) = {}_{(H^p(\partial\Omega,\sigma))^*}\Big\langle (\partial^\gamma E_{\alpha\beta})(x - \cdot)\Big|_{\partial\Omega}, f_\beta \Big\rangle_{H^p(\partial\Omega,\sigma)} \tag{2.2.67}$$

at each point $x \in \Omega$. Suppose first that $p < 1$ and set $\eta := (n-1)\big(\frac{1}{p} - 1\big) \in (0, 1)$. In concert with the duality result from [69, Theorem 4.6.1] this implies that, for each point $x \in \Omega$,

$$\Big|\partial^\gamma \big(\mathscr{S} f\big)(x)\Big| \leq C \Big\|(\partial^\gamma E)(x - \cdot)\Big\|_{[\mathscr{C}^\eta(\partial\Omega)]^{M\times M}} \|f\|_{[H^p(\partial\Omega,\sigma)]^M}. \tag{2.2.68}$$

To estimate the Hölder norm above, fix some $R \in (0, \infty)$ large enough so that $\partial\Omega \subseteq B(0, R)$ and make use of [68, (7.3.25)] and [70, Theorem 1.4.2] to write

$$\begin{aligned} &\Big\|(\partial^\gamma E)(x - \cdot)\Big\|_{[\mathscr{C}^\eta(\partial\Omega)]^{M\times M}} \\ &\qquad \leq C\Big(\sup_{y \in \partial\Omega} \Big|(\partial^\gamma E)(x - y)\Big| + \Big\|(\partial^\gamma E)(x - \cdot)\Big\|_{[\mathrm{Lip}(\partial\Omega)]^{M\times M}} \Big) \\ &\qquad \leq C\Big(\sup_{|y| \leq R} \Big|(\partial^\gamma E)(x - y)\Big| + \sup_{|y| \leq R} \Big|(\nabla\partial^\gamma E)(x - y)\Big| \Big) \\ &\qquad = O(|x|^{2-n-|\gamma|}) \text{ as } |x| \to \infty, \end{aligned} \tag{2.2.69}$$

assuming that either $n \geq 3$, or $|\gamma| > 0$. The case when $p = 1$ is dealt with similarly, now estimating

$$\begin{aligned}\left|\partial^\gamma(\mathscr{S}f)(x)\right| &\leq C\left\|(\partial^\gamma E)(x-\cdot)\right\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^{M\times M}}\|f\|_{[H^1(\partial\Omega,\sigma)]^M}\\ &\leq C\sup_{y\in\partial\Omega}\left|(\partial^\gamma E)(x-y)\right|\|f\|_{[H^1(\partial\Omega,\sigma)]^M}\\ &= O(|x|^{2-n-|\gamma|})\ \text{ as }\ |x|\to\infty,\end{aligned} \tag{2.2.70}$$

again, assuming that either $n \geq 3$, or $|\gamma| > 0$. The proof of (2.2.35) is therefore complete.

Finally, the claim made in (2.2.36) is seen from [70, (1.4.22)-(1.4.23)] and [69, Lemma 4.8.2] (since a similar result as in [69, Lemma 4.8.2] is valid with the logarithm replaced by the function Φ from [70, (1.4.23)]). This takes care of the claims in item *(1)*.

Turning attention to item *(2)*. In concert with [69, Proposition 4.8.6] if $p = 1$, and with [69, Proposition 4.8.7] (together with [69, Proposition 4.2.2]) if $p \in \left(\frac{n-1}{n}, 1\right)$, the estimate in [70, (2.5.558)] shows that the operator $\mathscr{S}$ from (2.2.31)-(2.2.32) satisfies the compatibility condition in (2.2.37). In particular, this justifies retaining the same notation for the operator in (2.2.38).

Consider next the task of proving the estimate claimed in (2.2.39), working first under the assumption that $n \geq 3$. In view of [68, (3.6.27)], the range $p \in (1, n-1)$ is covered by (1.3.57). In fact, using notation introduced in (A.0.168) we may rephrase (1.3.57) as the statement that

$$\begin{gathered}\mathscr{S} : \left[L^q(\partial\Omega,\sigma)\right]^M \longrightarrow N_\kappa^{q^*}(\Omega;\sigma) \text{ is a well-defined, bounded}\\ \text{operator if } q \in (1, n-1) \text{ and } q^* := \big(1/q - 1/(n-1)\big)^{-1}.\end{gathered} \tag{2.2.71}$$

To proceed, consider the case when $p \in \left(\frac{n-1}{n}, 1\right]$. Fix some $q \in (1, n-1)$ and consider an arbitrary $\mathbb{C}^M$-valued (p,q)-atom a on $\partial\Omega$. Hence, there exist $x_o \in \partial\Omega$ along with $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ such that

$$\begin{gathered}\mathrm{supp}\, a \subseteq B(x_o, r)\cap\partial\Omega,\\ \|a\|_{[L^q(\partial\Omega,\sigma)]^M} \leq \sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1/q-1/p},\\ \text{and } \int_{\partial\Omega} a\, d\sigma = 0.\end{gathered} \tag{2.2.72}$$

If $\partial\Omega$ is bounded then, by definition, constant functions of absolute value less than or equal to $\sigma(\partial\Omega)^{-1/p}$ are also considered to be (p,q)-atoms. Then, based on Hölder's inequality, the Ahlfors regularity of $\partial\Omega$, (2.2.39), and (2.2.72), we may estimate

$$\begin{aligned}
&\int_{B(x_o,2r)\cap\partial\Omega} \big|\mathcal{N}_\kappa(\mathcal{S}a)\big|^{p^*}\,\mathrm{d}\sigma \\
&\qquad \le \Big(\int_{B(x_o,2r)\cap\partial\Omega} \big|\mathcal{N}_\kappa(\mathcal{S}a)\big|^{q^*}\,\mathrm{d}\sigma\Big)^{p^*/q^*}\cdot\sigma\big(B(x_o,2r)\cap\partial\Omega\big)^{1-p^*/q^*} \\
&\qquad \le C\cdot r^{(n-1)(1-p^*/q^*)}\cdot\big\|\mathcal{N}_\kappa(\mathcal{S}a)\big\|^{p^*}_{L^{q^*}(\partial\Omega,\sigma)} \\
&\qquad \le C\cdot r^{(n-1)(1-p^*/q^*)}\cdot\|a\|^{p^*}_{[L^q(\partial\Omega,\sigma)]^M} \\
&\qquad \le C\cdot r^{(n-1)(1-p^*/q^*)}\cdot r^{p^*(n-1)(1/q-1/p)} = C, \qquad (2.2.73)
\end{aligned}$$

for some constant $C \in (0,\infty)$ independent of the atom in question. Next, pick an arbitrary point $x \in \partial\Omega \setminus B(x_o, 2r)$, along with some arbitrary $z \in \Gamma_\kappa(x)$ and $y \in \partial\Omega \cap B(x_o, r)$. Then the Mean Value Theorem, the estimates for E available from [70, Theorem 1.4.2], and [68, (8.1.8)] imply that

$$|E(z-y) - E(z-x_o)| \le Cr|z-x_o|^{-(n-1)} \le Cr|x-x_o|^{-(n-1)} \qquad (2.2.74)$$

for some constant $C \in (0,\infty)$ independent of x, x_o, z, r. Since a belongs to $\big[L^q(\partial\Omega,\sigma)\big]^M$, has bounded support and integrates to zero, (2.2.37) permits us to write

$$\mathcal{S}a(z) = \int_{\partial\Omega}\big\{E(z-y) - E(z-x_o)\big\}a(y)\,\mathrm{d}\sigma(y). \qquad (2.2.75)$$

Collectively, (2.2.75), (2.2.74), and (2.2.72) allow us to estimate (bearing in mind that $\partial\Omega$ is Ahlfors regular)

$$\begin{aligned}
\big|\mathcal{S}a(z)\big| &\le Cr|x-x_o|^{-(n-1)}\int_{\partial\Omega\cap B(x_o,r)}|a(y)|\,\mathrm{d}\sigma(y) \\
&\le Cr|x-x_o|^{-(n-1)}\cdot\|a\|_{[L^q(\partial\Omega,\sigma)]^M}\cdot\sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1-1/q} \\
&\le C\frac{r^{1+(n-1)(1-1/p)}}{|x-x_o|^{n-1}}. \qquad (2.2.76)
\end{aligned}$$

By taking the supremum of both sides of (2.2.76) in $z \in \Gamma_\kappa(x)$ we therefore arrive at

$$\big(\mathcal{N}_\kappa(\mathcal{S}a)\big)(x) \le C\frac{r^{1+(n-1)(1-1/p)}}{|x-x_o|^{n-1}} \quad \text{for all } x \in \partial\Omega\setminus B(x_o,2r). \qquad (2.2.77)$$

On account of (2.2.77) and [68, (7.2.5)] we then obtain (bearing in mind that $p^*(n-1) > n-1$)

$$\begin{aligned}\int_{\partial\Omega\setminus B(x_o,2r)} \big|\mathcal{N}_\kappa(\mathscr{S}a)\big|^{p^*}\,\mathrm{d}\sigma &\leq C\int_{\partial\Omega\setminus B(x_o,2r)} \frac{r^{p^*+p^*(n-1)(1-1/p)}}{|x-x_o|^{(n-1)p^*}}\,\mathrm{d}\sigma\\ &\leq C\cdot r^{p^*+p^*(n-1)(1-1/p)}\cdot r^{(n-1)-p^*(n-1)}\\ &= C, \end{aligned}\tag{2.2.78}$$

for some constant $C\in(0,\infty)$ independent of the atom a. At this stage, combining (2.2.73) with (2.2.78) shows that for each (p,q)-atom a with zero integral on $\partial\Omega$ we have

$$\big\|\mathcal{N}_\kappa(\mathscr{S}a)\big\|_{L^{p^*}(\partial\Omega,\sigma)}\leq C \tag{2.2.79}$$

for some constant $C\in(0,\infty)$ independent of a. Recall that when $\partial\Omega$ is bounded, constant functions of absolute value $\leq\sigma(\partial\Omega)^{-1/p}$ are also considered, by definition, to be (p,q)-atoms. In such a scenario, (2.2.79) continues to hold, by virtue of (1.3.59), since we are currently assuming $n\geq 3$. Thus, (2.2.79) is true for all (p,q)-atoms a on $\partial\Omega$ in such a case.

Granted (2.2.71) and (2.2.79), we may now invoke [69, Theorem 4.4.7] with (X,τ) and $(Y,\|\cdot\|)$ taken to be, respectively, $N_\kappa^{q^*}(\Omega;\sigma)$ and $N_\kappa^{p^*}(\Omega;\sigma)$ (both of which are Banach spaces; cf. [68, Proposition 8.3.5]). In light of the remark made in [69, (4.4.146)] (presently used with $\theta:=1$ and $s:=1$) this guarantees that the operator $\mathscr{S}$ from (2.2.71) extends uniquely, in the manner described in [69, (4.4.144)], to a linear and bounded operator

$$\widetilde{\mathscr{S}}:\big[H^p(\partial\Omega,\sigma)\big]^M\longrightarrow N_\kappa^{p^*}(\Omega;\sigma). \tag{2.2.80}$$

To complete the proof of (2.2.39) in the case when $n\geq 3$ there remains to show that

$$\widetilde{\mathscr{S}}f=\mathscr{S}f\ \text{ for every }\ f\in\big[H^p(\partial\Omega,\sigma)\big]^M\ \text{ with }\ p\in\big(\tfrac{n-1}{n},1\big]. \tag{2.2.81}$$

With this finality in mind, pick an arbitrary distribution $f\in\big[H^p(\partial\Omega,\sigma)\big]^M$ with integrability exponent $p\in\big(\frac{n-1}{n},1\big]$ and, for some fixed exponent $q\in(1,n-1)$, consider a numerical sequence $\{\lambda_i\}_{i\in\mathbb{N}}\in\ell^p(\mathbb{N})$ along with a sequence $\{a_i\}_{i\in\mathbb{N}}$ of $\mathbb{C}^M$-valued (p,q)-atoms on $\partial\Omega$ with the property that $f_j:=\sum_{i=1}^j\lambda_i a_i$ converges to f in $\big[H^p(\partial\Omega,\sigma)\big]^M$ as $j\to\infty$. Then [69, (4.4.144)] implies that $\widetilde{\mathscr{S}}f=\lim\limits_{j\to\infty}\mathscr{S}f_j$ in $N_\kappa^{p^*}(\Omega;\sigma)$. From this and [68, (8.3.33)] we further conclude that $\mathscr{S}f_j$ converges to $\widetilde{\mathscr{S}}f$ pointwise in Ω as $j\to\infty$. As such, for each $x\in\Omega$ we may write, assuming that $\partial\Omega$ is unbounded,

$$\begin{aligned}\widetilde{\mathscr{S}}f(x)&=\lim_{j\to\infty}\mathscr{S}f_j(x)=\lim_{j\to\infty}\big\langle[E(x-\cdot)],f_j\big\rangle\\ &=\big\langle[E(x-\cdot)],f\big\rangle=\mathscr{S}f(x),\end{aligned}\tag{2.2.82}$$

where we have also used (2.2.31) in the second and fourth equalities. Hence, for each $x \in \Omega$ we have $\widetilde{\mathscr{S}} f(x) = \mathscr{S} f(x)$, and the same conclusion holds when $\partial\Omega$ is bounded (via a similar argument). The proof of (2.2.39) is therefore complete, in the case when $n \geq 3$.

To deal with the two-dimensional setting, first observe that (2.2.41)-(2.2.42) are implied by (2.2.36) and [68, (8.2.28)]. Next, assume $n = 2$, fix $p \in \big(\frac{1}{2}, 1\big)$, and set $p^* := \big(\frac{1}{p} - 1\big)^{-1}$. We largely reason as before. The main difference is that if a is as in (2.2.72) (this time, with $q \in [1, \infty]$ arbitrary), then in place of (2.2.73) we now estimate (again, for some constant $C \in (0, \infty)$ independent of the atom in question)

$$\begin{aligned}\int_{B(x_o,2r)\cap\partial\Omega} \big|\mathcal{N}_\kappa(\mathscr{S}a)\big|^{p^*}\,d\sigma &\leq \big\|\mathcal{N}_\kappa(\mathscr{S}a)\big\|^{p^*}_{L^\infty(\partial\Omega,\sigma)} \cdot \sigma\big(B(x_o, 2r)\cap\partial\Omega\big)\\ &\leq C\cdot r\cdot \|a\|^{p^*}_{[H^1(\partial\Omega,\sigma)]^M}\\ &\leq C\cdot r\cdot \Big(\sigma\big(B(x_o,r)\cap\partial\Omega\big)\Big)^{p^*(1-1/p)} \leq C, \end{aligned}\tag{2.2.83}$$

thanks to (2.2.41), the fact that [69, (4.4.6)] and [69, Theorem 4.4.1] imply

$$\|a\|_{[H^1(\partial\Omega,\sigma)]^M} \leq C\Big(\sigma\big(B(x_o,r)\cap\partial\Omega\big)\Big)^{1-1/p}, \tag{2.2.84}$$

the Ahlfors regularity of $\partial\Omega$, and the definition of p^*. The argument in (2.2.74)-(2.2.78) goes through and, as before, we conclude that (2.2.79) holds if either Ω is bounded, or $\partial\Omega$ is unbounded. With this in hand, the same type of reasoning as in the end-game of the case $n \geq 3$ establishes (2.2.39) when $n = 2$ and either Ω is bounded, or $\partial\Omega$ is unbounded.

Finally, when $n = 2$ and Ω is an exterior domain, (2.2.79) is true provided a truncated nontangential maximal operator is used (see (1.3.60)). This establishes (2.2.40).

Going further, the compatibility between (2.2.43) and $\mathscr{S}$ from (2.2.38) with $p = 1$ is a consequence of [69, Lemma 4.6.5]. Finally, the fact that for each $p \in \big(\frac{n-1}{n}, 1\big]$ the operator $\mathscr{S}$ in (2.2.43) agrees with the operator $\mathscr{S}$ from (2.2.38) when considered on $\big[H^p(\partial\Omega,\sigma)\cap H^1(\partial\Omega,\sigma)\big]^M$ is a consequence of [69, Lemma 4.6.6]. This concludes the treatment of item *(2)*.

As regards item *(3)*, if $\partial\Omega$ is a UR set then the nontangential estimate in (2.2.44) is implied by (2.2.34), [70, (2.4.14)], [70, (2.4.9)] (keeping in mind [68, (3.6.27)]), and [70, Theorem 1.4.2].

Next, let us deal with the claims in item *(4)*. For now, assume that $\partial\Omega$ is a UR set. First, having fixed an arbitrary $f \in \big[H^p(\partial\Omega,\sigma)\big]^M$ with $p \in \big(\frac{n-1}{n}, 1\big]$ along with some background aperture parameter $\kappa > 0$, the proof so far guarantees that the function $u := \mathscr{S} f$ satisfies

$$u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L_A u = 0 \text{ in } \Omega, \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma). \tag{2.2.85}$$

In fact, there exists a constant $C = C(\Omega, A, \kappa, p) \in (0,\infty)$ independent of f such that

$$\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[H^p(\partial\Omega,\sigma)]^M}. \tag{2.2.86}$$

Granted these, [69, Theorem 10.2.24] applies and gives $\overset{\bullet}{\partial}{}^A_\nu \mathscr{S} f \in \big[H^p(\partial\Omega,\sigma)\big]^M$ and

$$\big\|\overset{\bullet}{\partial}{}^A_\nu \mathscr{S} f\big\|_{[H^p(\partial\Omega,\sigma)]^M} \leq C\big\|\mathcal{N}_\kappa(\nabla \mathscr{S} f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[H^p(\partial\Omega,\sigma)]^M}, \tag{2.2.87}$$

proving (2.2.45). In particular, we have a well-defined, linear, and bounded assignment

$$\big[H^p(\partial\Omega,\sigma)\big]^M \ni f \longmapsto \overset{\bullet}{\partial}{}^A_\nu \mathscr{S} f \in \big[H^p(\partial\Omega,\sigma)\big]^M. \tag{2.2.88}$$

Estimate in (2.2.46) is established in a similar manner, making use of [69, Theorem 10.2.24] and the estimate in the last line in (1.5.48).

Going further, strengthen the hypotheses by assuming that Ω is a UR domain in $\mathbb{R}^n$. In this setting, fix $p \in \big(\frac{n-1}{n}, 1\big]$ along with some $q \in (1, n-1)$. Suppose first that the function f is an arbitrary $\mathbb{C}^M$-valued (p,q)-atom on $\partial\Omega$. Then f is a multiple of a $\mathbb{C}^M$-valued $(1,q)$-atom, so f also belongs to the space $\big[H^1(\partial\Omega,\sigma)\big]^M$. If we define $u := \mathscr{S} f$ then, in light of (2.2.86), the latter membership ensures that $\mathcal{N}_\kappa(\nabla u)$ belongs to $L^1(\partial\Omega,\sigma)$. In addition, from (2.2.34) and [69, Proposition 4.8.7] it follows that for each $j \in \{1,\dots,n\}$ and each $\alpha \in \{1,\dots,M\}$ we have

$$(\partial_j u_\alpha)(x) = \int_{\partial\Omega} (\partial_j E_{\alpha\beta})(x-y) f_\beta(y)\, d\sigma(y), \qquad \forall\, x \in \Omega. \tag{2.2.89}$$

Then [70, Theorem 2.5.1] applies and gives that, for every $j \in \{1,\dots,n\}$ and $\alpha \in \{1,\dots,M\}$, the nontangential trace

$$(\partial_j u_\alpha)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega. \tag{2.2.90}$$

At this stage, item *(3)* in [69, Theorem 10.2.24] applies and the pointwise formula for the weak conormal derivative $\overset{\bullet}{\partial}{}^A_\nu u \in \big[H^1(\partial\Omega,\sigma)\big]^M \subset \big[L^1(\partial\Omega,\sigma)\big]^M$ given in [69, (10.2.186)] then permits us to express

$$\begin{aligned} \overset{\bullet}{\partial}{}^A_\nu u &= \Big(\nu_r a^{\alpha\beta}_{rs}\big(\partial_s(\mathscr{S} f)_\beta\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)_{1\leq\alpha\leq M} \\ &= \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \end{aligned} \tag{2.2.91}$$

where the last equality is a consequence of (2.2.37), (1.5.59), and the fact that, as a $\mathbb{C}^M$-valued (p,q)-atom, the function f belongs to $\big[L^q(\partial\Omega,\sigma)\big]^M$ (recall that $q \in (1, n-1)$). Let us record our progress. From (2.2.45) and (2.2.91) it follows that the assignment

$$\big[H^p(\partial\Omega,\sigma)\big]^M \ni f \longmapsto \overset{\bullet}{\partial}{}^A_\nu \mathscr{S} f \in \big[H^p(\partial\Omega,\sigma)\big]^M \tag{2.2.92}$$

is well defined, linear, continuous, and agrees with the operator $-\frac{1}{2}I + K^{\#}_{A^\top}$ when acting on arbitrary $\mathbb{C}^M$-valued (p, q)-atoms on $\partial\Omega$. Since from Theorem 2.1.1 we know that $-\frac{1}{2}I + K^{\#}_{A^\top}$ is also a well-defined linear and bounded operator on $\big[H^p(\partial\Omega, \sigma)\big]^M$, formula (2.2.47) now follows via a standard density argument (relying on [69, (4.4.114)]).

Next, with the exception of (2.2.54) and (2.2.56), the claims in item *(5)* are consequences of the properties established so far, along with [69, Theorem 4.3.1], (1.5.59), and [69, Proposition 1.3.7] (with the manner in which $\mathscr{S}$ acts in the context of (2.2.48) determined by [69, (1.3.41), (4.3.3)], and (2.2.31)). To justify (2.2.54), we reason much as in [70, (2.4.93), (2.4.94)]. Specifically, fix a compactly supported distribution $f = (f_\beta)_{1\le\beta\le M} \in \big[H^{p,q}(\partial\Omega, \sigma)\big]^M$ along with $\psi \in \mathrm{Lip}_c(\partial\Omega)$ which is identically one near supp f. Also, pick $p_0, p_1 \in \big(\frac{n-1}{n}, n-1\big)$ such that $p_0 < p < p_1$ and decompose f as $f^{(0)} + f^{(1)}$ where $f^{(i)} = (f^{(i)}_\beta)_{1\le\beta\le M} \in \big[H^{p_i}(\partial\Omega, \sigma)\big]^M$ for $i \in \{0, 1\}$. Next, choose a function $\theta \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ satisfying $\theta \equiv 1$ near the origin in $\mathbb{R}^n$, then for each $R > 0$ set $\theta_R(x) := \theta(x/R)$ for every $x \in \mathbb{R}^n$. Finally, fix a point $x \in \Omega$ along with an index $\alpha \in \{1, \dots, M\}$. We may then rely on [68, (7.3.17)] (with $\alpha := 1$) to conclude that

$$\lim_{R\to\infty} \theta_R E_{\alpha\bullet}(x-\cdot)\big|_{\partial\Omega} = E_{\alpha\bullet}(x-\cdot)\big|_{\partial\Omega} \text{ in } \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M \text{ for each } \eta \in (0, 1). \quad (2.2.93)$$

Together with (2.2.31) and [69, Lemma 4.6.4], this permits us to write

$$\big(\mathscr{S} f^{(i)}\big)_\alpha(x) = \lim_{R\to\infty} {}_{\mathrm{Lip}_c(\partial\Omega)}\Big\langle \theta_R E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f^{(i)}_\beta \Big\rangle_{(\mathrm{Lip}_c(\partial\Omega))'} \text{ for } i \in \{0, 1\}. \quad (2.2.94)$$

Summing up the two formulas in (2.2.94) then yields

$$\begin{aligned}\big(\mathscr{S} f\big)_\alpha(x) &= \lim_{R\to\infty} {}_{\mathrm{Lip}_c(\partial\Omega)}\Big\langle \theta_R E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta \Big\rangle_{(\mathrm{Lip}_c(\partial\Omega))'} \\ &= \lim_{R\to\infty} {}_{\mathrm{Lip}_c(\partial\Omega)}\Big\langle \theta_R \psi E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta \Big\rangle_{(\mathrm{Lip}_c(\partial\Omega))'} \\ &= {}_{\mathrm{Lip}_c(\partial\Omega)}\Big\langle \psi E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta \Big\rangle_{(\mathrm{Lip}_c(\partial\Omega))'} \end{aligned} \quad (2.2.95)$$

which proves (2.2.54). The jump-formula in (2.2.56) may be justified based on [70, (2.5.65)], upon noting that using the notation introduced in (A.0.141) for each $\alpha, \gamma \in \{1, \dots, M\}$ we may compute, based on [70, (1.4.30)] (used with $\gamma := \mathbf{e}_s$ and $\xi := \nu(x)$),

$$\begin{aligned}\frac{1}{2\mathrm{i}}\,\nu_r(x)a^{\alpha\beta}_{rs}\widehat{(\partial_s E_{\beta\gamma})}\big(\nu(x)\big) &= \frac{1}{2\mathrm{i}}\,\nu_r(x)a^{\alpha\beta}_{rs}\Big[\mathrm{i}\nu_s(x)\Big(\big[L(\nu(x))\big]^{-1}\Big)_{\beta\gamma}\Big]\\ &= -\frac{1}{2}\Big(L(\nu(x))\Big)_{\alpha\beta}\Big(\big[L(\nu(x))\big]^{-1}\Big)_{\beta\gamma}\\ &= -\frac{1}{2}\delta_{\gamma\alpha}\ \text{ for } \sigma\text{-a.e. } x\in\partial_*\Omega. \end{aligned}\tag{2.2.96}$$

Going further, the claims in item *(6)* are consequences of the current item *(5)*, [69, Proposition 10.2.21], and (1.5.55). As regards item *(7)*, the fact that $T^{\#}_{jk}$, originally defined as in (1.5.82), extends to a linear and bounded mapping in the context of (2.2.62) is a direct consequence of Theorem 2.1.4 (keeping in mind [70, Theorem 1.4.2]; cf. (2.1.44)), whereas the formula claimed in (2.2.63) is seen directly from Proposition 2.2.2 when $n\geq 3$ (cf. (2.2.15)), and from a slight variant of it when $n=2$, in which (2.2.22) is now tested for $f\in H^1(\partial\Omega,\sigma)$ (as a byproduct, the latter also justifies (2.2.64)). Finally, (2.2.65) is a direct consequence of the last part in Proposition 2.2.2 (bearing in mind [70, Theorem 1.4.2]; cf. (2.2.17)) when $n\geq 3$, and from a slight variant of it when $n=2$ (in which (2.2.28) is now tested for $f\in H^1(\partial\Omega,\sigma)$). This completes the proof of Theorem 2.2.3. □

We now present the version of [69, Corollary 10.2.28] regarding the (uniform) membership to the weak Hardy space $H^{1,\infty}$ of conormal and tangential derivatives of the fundamental solution of a weakly elliptic homogeneous constant complex coefficient elliptic second-order system when the singularity is located on the boundary of the domain. Such a version, which is going to be useful in the proof of Proposition 2.2.5 stated a little later, involves the notion of principal-value distribution on a given UR set (cf. [69, Proposition 11.9.1]).

Corollary 2.2.4 *Suppose $\Omega\subseteq\mathbb{R}^n$, where $n\in\mathbb{N}$ satisfies $n\geq 2$, is a* UR *domain. Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu=(\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, for some $M\in\mathbb{N}$, let $A=\Big(a^{\alpha\beta}_{rs}\Big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a constant (complex) coefficient tensor with the property that the associated homogeneous second-order $M\times M$ system $L_A:=\big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ (with the summation convention over repeated indices in effect) is weakly elliptic in $\mathbb{R}^n$. Finally, denote by $E=(E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ the matrix-valued fundamental solution associated with L_A as in* [70, Theorem 1.4.2]. *Then*

$$\begin{aligned}&\textit{the distribution } \Big(\mathrm{P.V.}\big(\nu_r a^{\alpha\gamma}_{rs}(\partial_s E_{\gamma\beta})(x_o-\cdot)\big|_{\partial\Omega}\big)\Big)_{1\leq\alpha,\beta\leq M}\ \textit{belongs to}\\ &\textit{the weak Hardy space } \big[H^{1,\infty}(\partial\Omega,\sigma)\big]^{M\times M}\ \textit{for } \sigma\textit{-a.e. point } x_o\in\partial\Omega,\end{aligned}\tag{2.2.97}$$

in a uniform fashion, i.e., there exists $C\in(0,\infty)$ such that

$$\sum_{\alpha,\beta=1}^{M} \Big\| \mathrm{P.V.}\big(\nu_r a_{rs}^{\alpha\gamma}(\partial_s E_{\gamma\beta})(x_o - \cdot)\big|_{\partial\Omega}\big)\Big\|_{H^{1,\infty}(\partial\Omega,\sigma)} \leq C \text{ for } \sigma\text{-a.e. } x_o \in \partial\Omega. \tag{2.2.98}$$

Moreover, for any $\alpha, \beta \in \{1, \dots, M\}$ *and* $j, k \in \{1, \dots, n\}$*, one has*

$$\mathrm{P.V.}\big(\nu_j(\partial_k E_{\alpha\beta})(x_o - \cdot)\big|_{\partial\Omega}\big) - \mathrm{P.V.}\big(\nu_k(\partial_j E_{\alpha\beta})(x_o - \cdot)\big|_{\partial\Omega}\big) \in H^{1,\infty}(\partial\Omega,\sigma)$$
$$\text{for } \sigma\text{-a.e. point } x_o \in \partial\Omega, \text{ in a uniform fashion.} \tag{2.2.99}$$

In addition, for every index $\beta \in \{1, \dots, M\}$ *and* σ*-a.e. point* $x_o \in \partial\Omega$ *one has*

$$\dot{\partial}_\nu^A\big[E_{\bullet\beta}(\cdot - x_o)\big] = -\tfrac{1}{2}\delta_{x_o}\mathbf{e}_\beta - \Big(\mathrm{P.V.}\big(\nu_r a_{rs}^{\alpha\gamma}(\partial_s E_{\gamma\beta})(x_o - \cdot)\big|_{\partial\Omega}\big)\Big)_{1\leq\alpha\leq M}$$
$$\text{as distributions in } \big[H^{1,\infty}(\partial\Omega,\sigma)\big]^M. \tag{2.2.100}$$

Finally, for each fixed $\beta \in \{1, \dots, M\}$ *the difference*

$$\dot{\partial}_\nu^A\big[E_{\bullet\beta}(\cdot - x_0)\big] - \dot{\partial}_\nu^A\big[E_{\bullet\beta}(\cdot - x_1)\big]$$
$$\text{belongs to } \textstyle\bigcap_{\frac{n-1}{n}<p<\infty} \big[H^p(\partial\Omega,\sigma)\big]^M \text{ for } \sigma\text{-a.e. } x_0, x_1 \in \partial\Omega, \tag{2.2.101}$$

and for each fixed $\alpha, \beta \in \{1, \dots, M\}$ *the difference*

$$\mathrm{P.V.}\big(\nu_j(\partial_k E_{\alpha\beta})(x_0 - \cdot)\big|_{\partial\Omega}\big) - \mathrm{P.V.}\big(\nu_k(\partial_j E_{\alpha\beta})(x_0 - \cdot)\big|_{\partial\Omega}\big)$$
$$- \mathrm{P.V.}\big(\nu_j(\partial_k E_{\alpha\beta})(x_1 - \cdot)\big|_{\partial\Omega}\big) + \mathrm{P.V.}\big(\nu_k(\partial_j E_{\alpha\beta})(x_1 - \cdot)\big|_{\partial\Omega}\big)$$

$$\text{belongs to } \bigcap_{\frac{n-1}{n}<p<\infty} H^p(\partial\Omega,\sigma) \text{ for } \sigma\text{-a.e. } x_0, x_1 \in \partial\Omega, \tag{2.2.102}$$

though the uniformity of the membership is now lost.

Proof All claims except (2.2.100) are seen from [69, (4.2.17)], [69, (4.2.34)], and [69, Proposition 11.9.2] by specializing $\vec{k}$ as in [69, (10.2.241), (10.2.242)]. Finally, (2.2.100) follows from (A.0.182) and [69, (11.9.30)] (used for $\vec{F}$ as in [69, (11.9.18)] and with $\vec{k}$ as in [69, (10.2.241)]), bearing in mind [70, (1.4.25)] and the fact that ∇E is odd. □

We are now ready to discuss the action of single and double layer potential operators in the context of Theorem 2.2.3 (cf. (2.2.54)) and Corollary 2.1.2 (cf. (2.1.28)) on certain concrete distributions belonging to the scale of Lorentz-based Hardy spaces.

Proposition 2.2.5 *Let* $\Omega \subseteq \mathbb{R}^n$ *(where* $n \in \mathbb{N}$*,* $n \geq 2$*) be a* UR *domain. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* $\nu = (\nu_1, \dots, \nu_n)$ *the geometric measure theoretic outward unit normal to* Ω*. Let* L *be a homogeneous, weakly elliptic, constant* *(complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). Denote by $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ the matrix-valued fundamental solution associated with L as in* [70, Theorem 1.4.2]. *Finally, pick a constant (complex) coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with the property that $L = L_A$. Then the following claims are true.*

(1) If $\mathscr{S}$ is the boundary-to-domain single layer potential operator (associated with the system L and the set Ω) acting on the scale of Lorentz-based Hardy spaces as in (2.2.48)*, then for every index $\beta \in \{1, \dots, M\}$ and σ-a.e. point $x_o \in \partial\Omega$ one has*

$$\mathscr{S}(\delta_{x_o}\mathbf{e}_\beta) = E_{\bullet\beta}(\cdot - x_o) \text{ in } \Omega. \tag{2.2.103}$$

(2) If $K^\#$ is the boundary layer potential operator (associated with the coefficient tensor A and the set Ω) acting on the scale of Lorentz-based Hardy spaces as in Corollary 2.1.2, then for every index $\beta \in \{1, \dots, M\}$ and σ-a.e. point $x_o \in \partial\Omega$ one has

$$\begin{aligned} K^\#(\delta_{x_o}\mathbf{e}_\beta) &= -\Big(\mathrm{P.V.}\big(\nu_r a_{sr}^{\gamma\alpha}(\partial_s E_{\beta\gamma})(x_o - \cdot)\big|_{\partial\Omega}\big)\Big)_{1\leq\alpha\leq M} \\ &= \dot{\partial}_\nu^{A^\top}\big[(E_{L^\top})_{\bullet\beta}(\cdot - x_o)\big] + \tfrac{1}{2}\delta_{x_o}\mathbf{e}_\beta \in \big[H^{1,\infty}(\partial\Omega,\sigma)\big]^M. \end{aligned} \tag{2.2.104}$$

In particular, for every index $\beta \in \{1, \dots, M\}$ and σ-a.e. point $x_o \in \partial\Omega$ one has

$$\big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)(\delta_{x_o}\mathbf{e}_\beta) = \dot{\partial}_\nu^A\big[E_{\bullet\beta}(\cdot - x_o)\big] \text{ in } \big[H^{1,\infty}(\partial\Omega,\sigma)\big]^M. \tag{2.2.105}$$

(3) If $K^\#$ is now regarded as a mapping from $\big[H^{1,\infty}(\partial\Omega,\sigma)\big]^M$ into $\big[L^{1,\infty}(\partial\Omega,\sigma)\big]^M$ (cf. (2.1.3)*), then for every index $\beta \in \{1, \dots, M\}$ and σ-a.e. point $x_o \in \partial\Omega$ one has*

$$K^\#(\delta_{x_o}\mathbf{e}_\beta) = -\Big(\nu_r a_{sr}^{\gamma\alpha}(\partial_s E_{\beta\gamma})(x_o - \cdot)\big|_{\partial\Omega}\Big)_{1\leq\alpha\leq M} \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.2.106}$$

(4) If $\mathfrak{H}$ is the filtering operator, considered as in [69, Theorem 4.9.1, (4.9.5)] *with $p := 1$ and $q := \infty$, then for every indices $\alpha, \beta \in \{1, \dots, M\}$ and σ-a.e. point $x_o \in \partial\Omega$ one has*

$$\begin{gathered} \mathfrak{H}\Big(\mathrm{P.V.}\big(\nu_r a_{sr}^{\gamma\alpha}(\partial_s E_{\beta\gamma})(x_o - \cdot)\big|_{\partial\Omega}\big)\Big) = \nu_r a_{sr}^{\gamma\alpha}(\partial_s E_{\beta\gamma})(x_o - \cdot)\big|_{\partial\Omega} \\ \text{at } \sigma\text{-a.e. point on } \partial\Omega. \end{gathered} \tag{2.2.107}$$

Proof Fix $\beta \in \{1, \dots, M\}$. From [69, Example 4.2.4] we know that for each point $x_o \in \partial\Omega$ the distribution $\delta_{x_o}\mathbf{e}_\beta$ belongs to $\big[H^{1,\infty}(\partial\Omega,\sigma)\big]^M$. Since $\delta_{x_o}\mathbf{e}_\beta$ has also compact support, (2.2.54) implies (2.2.103). Next, the first jump-formula in (2.2.55) presently gives

$$\dot{\partial}_\nu^A\mathscr{S}(\delta_{x_o}\mathbf{e}_\beta) = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)(\delta_{x_o}\mathbf{e}_\beta) \text{ in } \big[H^{1,\infty}(\partial\Omega,\sigma)\big]^M. \tag{2.2.108}$$

On the one hand, from (2.2.100) and (2.2.103) we see that for σ-a.e. $x_o \in \partial\Omega$ we have

$$\overset{\bullet}{\partial}{}^A_\nu \mathscr{S}(\delta_{x_o}\mathbf{e}_\beta) = -\tfrac{1}{2}\delta_{x_o}\mathbf{e}_\beta - \Big(\text{P.V.}\big(\nu_r a^{\alpha\gamma}_{rs}(\partial_s E_{\gamma\beta})(x_o - \cdot)\big|_{\partial\Omega}\big)\Big)_{1\leq\alpha\leq M} \quad \text{as distributions in } \big[\big(\text{Lip}_c(\partial\Omega)\big)'\big]^M. \tag{2.2.109}$$

Comparing now (2.2.108) with (2.2.109) yields the first equality in (2.2.104) after slight adjustments in notation (taking into account [70, (1.7.2)] and the first formula in [70, (1.4.32)]). The second equality in (2.2.104) is seen from (2.2.100) written with $A^\top$ in place of A.

Going further, the claim made in item *(3)* is justified by writing the jump-formula (2.2.56) with $A^\top$ in place of A and for the choice $f := \delta_{x_o}\mathbf{e}_\beta \in \big[H^{1,\infty}(\partial\Omega,\sigma)\big]^M$ (cf. [69, (4.2.34)]), then invoking [69, (4.9.8)] and (2.2.54). Finally, the claim in item *(4)* is seen by applying $\mathfrak{H}$ to (2.2.104), then taking into account (2.1.29) and (2.2.106).□

We shall temporarily digress for the purpose of further elaborating on the scope of Proposition 2.2.5. To set stage, recall from Theorem 2.1.5 that the boundary-to-boundary transpose Cauchy-Clifford operator $\mathfrak{C}^\#$ is well defined, linear, and bounded on the scale of Lorentz-based Hardy spaces considered on the boundary of a UR domain $\Omega \subseteq \mathbb{R}^n$ (cf. (2.1.66)). In this regard, it is remarkable that for σ-a.e. point $x_o \in \partial\Omega$ we have

$$\mathfrak{C}^\# \delta_{x_o} = \text{P.V.}\Big(\nu \odot \Phi(x_o - \cdot)\big|_{\partial\Omega}\Big) \in H^{1,\infty}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \tag{2.2.110}$$

where ν is the geometric measure theoretic outward unit normal to Ω, the measure $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and

$$\Phi(x) := \frac{1}{\omega_{n-1}}\frac{x}{|x|^n}, \qquad \forall\, x \in \mathbb{R}^n \setminus \{0\}. \tag{2.2.111}$$

For example, in the particular case when we take $\Omega := \mathbb{R}^n_+$, the upper half-space in $\mathbb{R}^n$, formula (2.2.110) implies (as may be seen after unraveling notation) that $R_j\delta_0 = \frac{2}{\omega_{n-1}}\text{P.V.}\frac{x_j}{|x|^n}$, where R_j is j-th Riesz transform in $\mathbb{R}^{n-1}$ and δ_0 denotes the Dirac distribution with mass at the origin in $\mathbb{R}^{n-1}$. Hence, it is natural to think of (2.2.110) as a generalization of this classical fact in Harmonic Analysis.

Formula (2.2.110) is proved by computing the Clifford bullet product $\nu \bullet \big(\mathcal{R}_{\mathcal{C}\ell}\delta_{x_o}\big)$ in two ways. Specifically, first we employ (2.1.88) to conclude that for each $x_o \in \partial\Omega$ we have

$$\nu \bullet \big(\mathcal{R}_{\mathcal{C}\ell}\delta_{x_o}\big) = \big(I - 2\mathfrak{C}^\#\big)\delta_{x_o} \text{ in } H^{1,\infty}(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n. \tag{2.2.112}$$

Second, starting from the observation that for each $x_o \in \partial\Omega$ and each $x \in \Omega$ we have

$$\big(\mathcal{R}_{\mathcal{C}\ell}\delta_{x_o}\big)(x) = 2\Phi(x - x_o), \tag{2.2.113}$$

we may use the very definition of the Clifford bullet product given in (A.0.167) to compute (along the lines of the argument which has eventually produced [69, (11.9.30)])

$$\nu \bullet \Phi(\cdot - x_o) = \tfrac{1}{2}\delta_{x_o} - \mathrm{P.V.}\Big(\nu \odot \Phi(x_o - \cdot)\big|_{\partial\Omega}\Big) \tag{2.2.114}$$

as distributions on $\partial\Omega$, for σ-a.e. point $x_o \in \partial\Omega$. Comparing (2.2.112) with (2.2.114) (while keeping in mind (2.2.113)) yields (2.2.110). To close, we remark that if $\mathfrak{H}$ is the filtering operator considered as in [69, Theorem 4.9.1, (4.9.5)] (with $p := 1$ and $q := \infty$) then for σ-a.e. point $x_o \in \partial\Omega$ we have (compare with (2.2.107))

$$\mathfrak{H}\Big(\mathrm{P.V.}\big(\nu \odot \Phi(x_o - \cdot)\big|_{\partial\Omega}\big)\Big) = \nu \odot \Phi(x_o - \cdot)\big|_{\partial\Omega} \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.2.115}$$

Returning now to the task of studying the action of the single layer potential operator on Hardy spaces, the philosophy emerging from our theorem below is that the boundary-to-boundary version of the single layer operator acts naturally from the *entire* scale of boundary Hardy spaces, i.e., H^p with $\frac{n-1}{n} < p < \infty$, if $n \geq 3$. When $n = 2$, the boundary-to-boundary single layer acts naturally from Hardy spaces H^p with $\frac{1}{2} < p < 1$.

Theorem 2.2.6 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ satisfies*[4] *$n \geq 3$, be an open set with the property that $\partial\Omega$ is an Ahlfors regular set, and denote $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. For some $M \in \mathbb{N}$, consider a coefficient tensor A of type $(n \times n, M \times M)$ as in* (1.3.1) *with the property that the $M \times M$ second-order system L_A defined as in* (2.2.30) *is weakly elliptic and denote by E the fundamental solution associated with the system L_A as in* [70, Theorem 1.4.2].

In this setting, recall the definition of the boundary-to-boundary single layer operator S from (1.3.62). *In relation to this operator, the following statements are true.*

(1) For each $p \in \big(\frac{n-1}{n}, 1\big]$, the operator S from (1.3.63) *further extends, in a unique fashion, to a linear and bounded operator*[5]

$$\begin{gathered} S : \big[H^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{p^*}(\partial\Omega, \sigma)\big]^M \\ \text{where } p^* := \Big(\tfrac{1}{p} - \tfrac{1}{n-1}\Big)^{-1} \in \big(1, \tfrac{n-1}{n-2}\big]. \end{gathered} \tag{2.2.116}$$

Furthermore, various choices of p yield operators which are compatible with one another.

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[4] The two-dimensional case is dealt with separately, in item *(8)*

[5] When $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is a compact Ahlfors regular set, (2.2.116) may be seen from [69, Proposition 11.11.5] and item *(ix)* in Theorem 1.5.1.

Finally, for each given $p \in \big(\frac{n-1}{n}, 1\big]$, *the operator* S *in* (2.2.116) *agrees with the operator* S *from* (1.3.64) *on the subspace* $\big[H^p(\partial\Omega,\sigma)\big]^M \cap \big[H^1(\partial\Omega,\sigma)\big]^M$ *of* $\big[L^1(\partial\Omega,\sigma)\big]^M$.

(2) *Suppose* $r \in (n-1,\infty)$ *and define* $\eta := 1-(n-1)/r \in (0,1)$. *Then the operator* S *from* (1.3.63) *further extends, in a unique fashion, to a linear and bounded mapping*

$$\big[L^r(\partial\Omega,\sigma)\big]^M \ni f \longmapsto \begin{cases} Sf \in \big[\mathscr{C}^\eta(\partial\Omega)\big]^M & \text{if } \partial\Omega \text{ is bounded,} \\ [Sf] \in \big[\dot{\mathscr{C}}^\eta(\partial\Omega)/\sim\big]^M & \text{if } \partial\Omega \text{ is unbounded.} \end{cases} \tag{2.2.117}$$

Moreover, if

$$\begin{aligned} p \in \big(\tfrac{n-1}{n}, 1\big), \quad \eta := (n-1)\big(\tfrac{1}{p}-1\big) \in (0,1), \\ p^* := \Big(\tfrac{1}{p} - \tfrac{1}{n-1}\Big)^{-1} \in \big(1, \tfrac{n-1}{n-2}\big), \end{aligned} \tag{2.2.118}$$

and

$$(p^*)' := \big(1 - \tfrac{1}{p^*}\big)^{-1} \in (n-1,\infty) \tag{2.2.119}$$

denotes the Hölder conjugate exponent of p^*, *then for each distribution* f *in the space* $\big[H^p(\partial\Omega,\sigma)\big]^M$ *and each function* $g \in \big[L^{(p^*)'}(\partial\Omega,\sigma)\big]^M$ *one has*

$$\int_{\partial\Omega} \langle Sf, g\rangle \, d\sigma = \begin{cases} {}_{[H^p(\partial\Omega,\sigma)]^M}\big\langle f, [S_{L^\top} g]\big\rangle_{[\dot{\mathscr{C}}^\eta(\partial\Omega)/\sim]^M} & \text{if } \partial\Omega \text{ is unbounded,} \\ {}_{[H^p(\partial\Omega,\sigma)]^M}\big\langle f, S_{L^\top} g\big\rangle_{[\mathscr{C}^\eta(\partial\Omega)]^M} & \text{if } \partial\Omega \text{ is bounded,} \end{cases} \tag{2.2.120}$$

where $S_{L^\top}$ *is the single layer potential operator associated with* $L^\top$, *the* (*real*) *transpose of* L, *and the brackets in the right side indicate duality in the sense of* [69, Theorem 4.6.1].

(3) *The operator* S *from* (1.3.63) *also extends, in a unique fashion, to a linear and bounded mapping*

$$\big[L^{n-1}(\partial\Omega,\sigma)\big]^M \ni f \mapsto \begin{cases} Sf \in \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M & \text{if } \partial\Omega \text{ is bounded,} \\ [Sf] \in \big[\widetilde{\mathrm{VMO}}(\partial\Omega,\sigma)\big]^M & \text{if } \partial\Omega \text{ is unbounded.} \end{cases} \tag{2.2.121}$$

In particular,

$$\big[L^{n-1}(\partial\Omega,\sigma)\big]^M \ni f \mapsto \begin{cases} Sf \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M & \text{if } \partial\Omega \text{ is bounded,} \\ [Sf] \in \big[\widetilde{\mathrm{BMO}}(\partial\Omega,\sigma)\big]^M & \text{if } \partial\Omega \text{ is unbounded,} \end{cases} \tag{2.2.122}$$

is also a well-defined, linear, and continuous operator. Furthermore, for each given pair of functions, $f \in \big[H^1(\partial\Omega,\sigma)\big]^M$ *and* $g \in \big[L^{n-1}(\partial\Omega,\sigma)\big]^M$*, one has*

$$\int_{\partial\Omega} \langle Sf, g\rangle \,\mathrm{d}\sigma \tag{2.2.123}$$
$$= \begin{cases} {}_{[H^1(\partial\Omega,\sigma)]^M}\big\langle f, [S_{L^\top} g]\big\rangle_{[\mathrm{BMO}(\partial\Omega,\sigma)/\sim]^M} & \text{if } \partial\Omega \text{ is unbounded,} \\ {}_{[H^1(\partial\Omega,\sigma)]^M}\big\langle f, S_{L^\top} g\big\rangle_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M} & \text{if } \partial\Omega \text{ is bounded,} \end{cases}$$

where $S_{L^\top}$ *is the single layer potential operator associated with* $L^\top$*, the* (*real*) *transpose of L, and the brackets in the right side indicate duality in the sense of* [69, Theorem 4.6.1].

(4) The operators from items (1)-(3) act in a coherent fashion with one another, as well as with the operator S from (1.3.63)*. As such, they may be glued together to create an operator acting from the global Hardy scale* $\big[H^p(\partial\Omega,\sigma)\big]^M$ *with* $p \in \big(\frac{n-1}{n}, \infty\big)$ *as follows:*

$$S : \big[H^p(\partial\Omega,\sigma)\big]^M \to \begin{cases} \big[L^{p^*}(\partial\Omega,\sigma)\big]^M & \text{if } p \in \big(\frac{n-1}{n}, n-1\big),\ \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n-1}, \\ \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M & \text{if } p = n-1 \text{ and } \partial\Omega \text{ is bounded,} \\ \big[\widetilde{\mathrm{BMO}}(\partial\Omega,\sigma)\big]^M & \text{if } p = n-1 \text{ and } \partial\Omega \text{ is unbounded,} \\ \big[\mathscr{C}^\eta(\partial\Omega)\big]^M & \text{if } p \in (n-1,\infty),\ \eta = 1 - \frac{n-1}{p}, \\ & \text{and } \partial\Omega \text{ is bounded,} \\ \big[\dot{\mathscr{C}}^\eta(\partial\Omega)/\sim\big]^M & \text{if } p \in (n-1,\infty),\ \eta = 1 - \frac{n-1}{p}, \\ & \text{and } \partial\Omega \text{ is unbounded.} \end{cases} \tag{2.2.124}$$

See also (2.3.12) *and* (4.3.43) *in this regard.*

(5) In addition to the boundary-to-boundary single layer S from (2.2.124)*, recall its modified version* S_{mod} *from* (1.5.73)*. Then for each function* $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ *with* $1 < p < \infty$ *one has*

$$Sf = \begin{cases} S_{\mathrm{mod}} f + C_f & \text{if } p \in (1, n-1) \text{ or } \partial\Omega \text{ is bounded,} \\ \big[S_{\mathrm{mod}} f\big] & \text{if } p \in [n-1,\infty) \text{ and } \partial\Omega \text{ is unbounded,} \end{cases} \tag{2.2.125}$$

for some constant $C_f \in \mathbb{C}^M$ *which depends on* f.

(6) Given an exponent $p \in \big(\frac{n-1}{n}, n-1\big)$*, recall the boundary-to-domain single layer operator* $\mathscr{S}$ *from* (2.2.38)*. Also, fix some arbitrary aperture parameter* $\kappa \in (0,\infty)$*. Then for each* $f \in \big[H^p(\partial\Omega,\sigma)\big]^M$ *it follows that*

the nontangential boundary limit $(\mathscr{S}f)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ *exists (in* $\mathbb{C}^M$*) at* σ*-a.e. point on* $A_\kappa(\partial\Omega)$ *(hence also at* σ*-a.e. point on* $\partial_*\Omega$*; cf.* [68, (8.8.45)]), (2.2.126)

and, with S *denoting the boundary-to-boundary single layer operator from* (2.2.124)*, one has*

$$\Big((\mathscr{S}f)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = (Sf)(x) \ \text{at } \sigma\text{-a.e. point } x \in A_\kappa(\partial\Omega) \quad \text{(hence also at } \sigma\text{-a.e. point } x \in \partial_*\Omega\text{; cf. [68, (8.8.45)]).} \tag{2.2.127}$$

(7) For each $p \in \big(\frac{n-1}{n}, n-1\big)$ *and* $q \in (0,\infty]$*, the operator* S *from* (2.2.124) *induces a well-defined, linear and bounded mapping*

$$S : \big[H^{p,q}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{p^*,q}(\partial\Omega,\sigma)\big]^M \quad \text{where } p^* := \Big(\frac{1}{p} - \frac{1}{n-1}\Big)^{-1} \in (1,\infty). \tag{2.2.128}$$

In particular,

$$S : \big[H^{p}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{p^*,p}(\partial\Omega,\sigma)\big]^M \ \text{boundedly if } p \in \big(\tfrac{n-1}{n}, n-1\big) \text{ and } p^* := \Big(\frac{1}{p} - \frac{1}{n-1}\Big)^{-1}. \tag{2.2.129}$$

Moreover, having fixed an arbitrary aperture parameter $\kappa > 0$*, for each* $f \in \big[H^{p,q}(\partial\Omega,\sigma)\big]^M$ *with* $p \in \big(\frac{n-1}{n}, n-1\big)$ *and* $q \in (0,\infty]$ *it follows that the nontangential boundary limit* $(\mathscr{S}f)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ *exists (in* $\mathbb{C}^M$*) at* σ*-a.e. point on* $A_\kappa(\partial\Omega)$*, and, with* S *denoting the boundary-to-boundary single layer operator from* (2.2.128)*, the boundary trace formula* (2.2.127) *continues to hold.*

(8) Assume $n = 2$*. Consider the operator*

$$S : \big[H^{1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{\infty}(\partial\Omega,\sigma) \cap \mathscr{C}^0(\partial\Omega)\big]^M \tag{2.2.130}$$

acting on each $f = (f_\alpha)_{1\le\alpha\le M} \in \big[H^{1}(\partial\Omega,\sigma)\big]^M$ *according to*

$$Sf(x) := \begin{cases} \Big(\big\langle E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta\big\rangle\Big)_{1\le\alpha\le M} & \text{if } \partial\Omega \text{ is bounded,} \\ \Big(\big\langle [E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}], f_\beta\big\rangle\Big)_{1\le\alpha\le M} & \text{if } \partial\Omega \text{ is unbounded,} \end{cases} \tag{2.2.131}$$

at every point $x \in \partial\Omega$ *(where* $\langle\cdot,\cdot\rangle$ *stands for the duality bracket on* $\partial\Omega$*, described in* [69, Theorem 4.6.1]*, used with* $\Sigma := \partial\Omega$*). Then this is a well-defined, linear, continuous mapping. Moreover, for each distribution* $f \in \big[H^{1}(\partial\Omega,\sigma)\big]^M$ *one has* $(\mathscr{S}f)\big|_{\partial\Omega} = Sf$*, where* $\mathscr{S}f \in \big[\mathscr{C}^0(\overline{\Omega})\big]^M$ *is considered as in* (2.2.36)*. In* *particular, for each aperture parameter* $\kappa > 0$ *and each* $f \in \big[H^1(\partial\Omega, \sigma)\big]^M$ *one has*

$$\begin{gathered}(\mathscr{S} f)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = Sf \ \text{ at } \sigma\text{-a.e. point on } A_\kappa(\partial\Omega) \\ (\text{hence also at } \sigma\text{-a.e. point on } \partial_*\Omega;\ cf.\ [68, (8.8.45)]).\end{gathered} \tag{2.2.132}$$

Also, for each $p \in \big(\tfrac{1}{2}, 1\big)$ *the operator* (2.2.130)-(2.2.131) *extends uniquely to a linear and bounded mapping*

$$S : \big[H^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{p^*}(\partial\Omega, \sigma)\big]^M \ \text{ where } \ p^* := \big(\tfrac{1}{p} - 1\big)^{-1}, \tag{2.2.133}$$

which continues to satisfy (2.2.132) *for each* $\kappa > 0$ *and each* $f \in \big[H^p(\partial\Omega, \sigma)\big]^M$. *In addition, via real interpolation* (*cf.* [69, (4.3.3)] *and* [68, (6.2.48)]), *from* (2.2.133) *one further obtains that*

$$\begin{gathered}S : \big[H^{p,q}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{p^*,q}(\partial\Omega, \sigma)\big]^M \ \text{ boundedly} \\ \text{if } p \in \big(\tfrac{1}{2}, 1\big),\ q \in (0, \infty],\ \text{ and } p^* := \big(\tfrac{1}{p} - 1\big)^{-1},\end{gathered} \tag{2.2.134}$$

hence, in particular,

$$\begin{gathered}S : \big[H^{p}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{p^*,p}(\partial\Omega, \sigma)\big]^M \ \text{ boundedly} \\ \text{whenever } p \in \big(\tfrac{1}{2}, 1\big) \ \text{ and } \ p^* := \big(\tfrac{1}{p} - 1\big)^{-1},\end{gathered} \tag{2.2.135}$$

and (2.2.132) *remains true for each* $f \in \big[H^{p,q}(\partial\Omega, \sigma)\big]^M$ *with* $p \in \big(\tfrac{1}{2}, 1\big)$ *and* $q \in (0, \infty]$.
Finally, if

$$\begin{gathered}\text{the set } \partial\Omega \text{ is bounded,} \\ p \in \big(\tfrac{1}{2}, 1\big), \quad \eta := \tfrac{1}{p} - 1 \in (0, 1), \\ p^* := \big(\tfrac{1}{p} - 1\big)^{-1} \in (1, \infty), \quad (p^*)' := \big(1 - \tfrac{1}{p^*}\big)^{-1} \in (1, \infty)\end{gathered} \tag{2.2.136}$$

then the operator S *from* (1.3.63) *further extends, in a unique fashion, to a linear and bounded mapping*

$$S : \big[L^{(p^*)'}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathscr{C}^\eta(\partial\Omega)\big]^M \tag{2.2.137}$$

and for each distribution $f \in \big[H^p(\partial\Omega, \sigma)\big]^M$ *and each function* g *in the space* $\big[L^{(p^*)'}(\partial\Omega, \sigma)\big]^M$ *one has*

$$\int_{\partial\Omega} \langle Sf, g\rangle \, d\sigma = {}_{[H^p(\partial\Omega,\sigma)]^M}\big\langle f, S_{L^\top} g\big\rangle_{[\mathscr{C}^\eta(\partial\Omega)]^M} \tag{2.2.138}$$

where $S_{L^\top}$ is the single layer potential operator associated with $L^\top$, the (real) transpose of L, and the brackets in the right side indicate duality in the sense of [69, Theorem 4.6.1].

An inspection of the proof of Theorem 2.2.6 reveals that all results in items *(1)-(7)* are of a purely real variable nature, making use of only generic size and regularity properties of the fundamental solution E. Consequently, analogous results are valid for more general classes of operators in which E is replaced by a kernel function enjoying similar size and regularity properties.

We wish to remark that formula (2.2.127) is optimal, in the sense that $A_\kappa(\partial\Omega)$ is the largest subset of $\partial\Omega$ where it is meaningful to consider the nontangential trace $\mathscr{S}f\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$. Let us also note that, if $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ denotes the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2], then since $\delta_{x_o}\mathbf{e}_\beta \in \big[H^{1,\infty}(\partial\Omega,\sigma)\big]^M$ for each $\beta\in\{1,\dots,M\}$ (cf. [69, (4.2.34)]) the last claim in item *(7)* and (2.2.103) imply that for each $\beta\in\{1,\dots,M\}$ and each $x_o\in\partial\Omega$ we have

$$S\big(\delta_{x_o}\mathbf{e}_\beta\big)(x) = \big(E_{\alpha\beta}(x-x_o)\big)_{1\leq\alpha\leq M} \text{ for } \sigma\text{-a.e. } x\in\partial_{\text{nta}}\Omega. \tag{2.2.139}$$

We now turn to the proof of Theorem 2.2.6.

Proof of Theorem 2.2.6 To prove the claims in item *(1)* fix an exponent $p\in\big(\frac{n-1}{n},1\big]$ along with some $q\in(1,n-1)$ and set $q^* := \big(\frac{1}{q}-\frac{1}{n-1}\big)^{-1}$. Also, consider an arbitrary $\mathbb{C}^M$-valued (p,q)-atom a on $\partial\Omega$. Thus, there exist a point $x_o\in\partial\Omega$ and some radius $r\in\big(0,\operatorname{diam}(\partial\Omega)\big)$ such that the properties in (2.2.72) hold. In addition, in the case when $\partial\Omega$ is bounded, constant functions of absolute value $\leq\sigma(\partial\Omega)^{-1/p}$ are also considered to be (p,q)-atoms. Making use of Hölder's inequality, the doubling property of σ, (1.3.63), and (2.2.72), we may then write

$$\begin{aligned}
&\int_{B(x_o,2r)\cap\partial\Omega}|Sa|^{p^*}\,\mathrm{d}\sigma\\
&\qquad\leq\Big(\int_{B(x_o,2r)\cap\partial\Omega}|Sa|^{q^*}\,\mathrm{d}\sigma\Big)^{p^*/q^*}\cdot\sigma\big(B(x_o,2r)\cap\partial\Omega\big)^{1-p^*/q^*}\\
&\qquad\leq\|Sa\|^{p^*}_{[L^{q^*}(\partial\Omega,\sigma)]^M}\cdot\sigma\big(B(x_o,2r)\cap\partial\Omega\big)^{1-p^*/q^*}\\
&\qquad\leq C\cdot\|a\|^{p^*}_{[L^q(\partial\Omega,\sigma)]^M}\cdot\sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1-p^*/q^*}=C,
\end{aligned}\tag{2.2.140}$$

for some constant $C\in(0,\infty)$ independent of the given atom. Going further, pick an arbitrary point $x\in\partial\Omega\setminus B(x_o,2r)$, along with some arbitrary $y\in\partial\Omega\cap B(x_o,r)$. Then the Mean Value Theorem and the estimates for E from [70, Theorem 1.4.2] imply

$$|E(x-y)-E(x-x_o)|\leq Cr|x-x_o|^{-(n-1)} \tag{2.2.141}$$

for some constant $C \in (0, \infty)$ independent of x, x_o, r. Bearing in mind that a belongs to $\big[L^q(\partial\Omega, \sigma)\big]^M$, has bounded support and integrates to zero, (1.3.62) allows us to express

$$(\mathcal{S}a)(x) = \int_{\partial\Omega} \big\{E(x - y) - E(x - x_o)\big\} a(y)\, \mathrm{d}\sigma(y). \tag{2.2.142}$$

In turn, (2.2.75), (2.2.74), and (2.2.72) permit us to estimate (upon recalling that $\partial\Omega$ is Ahlfors regular)

$$\begin{aligned} |(\mathcal{S}a)(x)| &\le Cr|x - x_o|^{-(n-1)} \int_{\partial\Omega\cap B(x_o,r)} |a(y)|\, \mathrm{d}\sigma(y) \\ &\le Cr|x - x_o|^{-(n-1)} \cdot \|a\|_{[L^q(\partial\Omega,\sigma)]^M} \cdot \sigma\big(B(x_o, r) \cap \partial\Omega\big)^{1-1/q} \\ &\le C\frac{r^{1+(n-1)(1-1/p)}}{|x - x_o|^{n-1}}. \end{aligned} \tag{2.2.143}$$

Given that $p^*(n - 1) > n - 1$, from (2.2.143) and [68, (7.2.5)] we then obtain

$$\begin{aligned} \int_{\partial\Omega\setminus B(x_o,2r)} |\mathcal{S}a|^{p^*}\, \mathrm{d}\sigma &\le C \int_{\partial\Omega\setminus B(x_o,2r)} \frac{r^{p^*+p^*(n-1)(1-1/p)}}{|x - x_o|^{(n-1)p^*}}\, \mathrm{d}\sigma(x) \\ &\le C \cdot r^{p^*+p^*(n-1)(1-1/p)} \cdot r^{(n-1)-p^*(n-1)} = C, \end{aligned} \tag{2.2.144}$$

for some constant $C \in (0, \infty)$ independent of the atom a. Finally, combining (2.2.73) with (2.2.78) proves that for each $\mathbb{C}^M$-valued (p, q)-atom a with zero integral on $\partial\Omega$ we have

$$\|\mathcal{S}a\|_{[L^{p^*}(\partial\Omega,\sigma)]^M} \le C \tag{2.2.145}$$

for some constant $C \in (0, \infty)$ independent of a. Recall that when $\partial\Omega$ is bounded, constant functions of absolute value less than or equal to $\sigma(\partial\Omega)^{-1/p}$ are also considered to be (p, q)-atoms. In such a case, (2.2.145) continues to hold thanks to (1.3.66). Hence, (2.2.145) is true for all $\mathbb{C}^M$-valued (p, q)-atoms a on $\partial\Omega$.

Now that (1.3.63) and (2.2.145) have been established, the plan is now to invoke [69, Theorem 4.4.7] with $(X, \tau) := L^{q^*}(\partial\Omega, \sigma)$ and $(Y, \|\cdot\|) := L^{p^*}(\partial\Omega, \sigma)$. In view of the remark made in [69, (4.4.146)] (presently used with $\theta := 1$ and $s := 1$) this abstract extension result guarantees that $\mathcal{S}$ from (1.3.63) extends uniquely, in the manner described in [69, (4.4.144)], to a linear and bounded operator as in (2.2.116). That various choices of p in the interval $\big(\frac{n-1}{n}, 1\big]$ yield operators in (2.2.116) which are compatible with one another may be justified with the help of the simultaneous convergence result proved in [69, Theorem 4.4.3].

To justify the final claim in item *(1)*, fix some $q \in (1, n - 1)$ and pick an arbitrary function $f \in \big[H^p(\partial\Omega, \sigma)\big]^M \cap \big[H^1(\partial\Omega, \sigma)\big]^M$. Then [69, Theorem 4.4.3] ensures the existence of a sequence

$$\{f_N\}_{N\in\mathbb{N}} \subseteq \big[L^q_{\rm comp}(\partial\Omega,\sigma)\big]^M \cap \big[H^p(\partial\Omega,\sigma)\big]^M \cap \big[H^1(\partial\Omega,\sigma)\big]^M$$
$$\text{so that } \lim_{N\to\infty} f_N = f \text{ both in } \big[H^p(\partial\Omega,\sigma)\big]^M \text{ and } \big[H^1(\partial\Omega,\sigma)\big]^M, \qquad (2.2.146)$$
$$\text{hence also in the space } \big[L^1(\partial\Omega,\sigma)\big]^M.$$

To clarify notation, let us temporarily write S_{H^p} for the operator S in (2.2.116), and write S_{L^1} for the operator S in (1.3.64). From what we have proved already it follows that, for each $N \in \mathbb{N}$, we have

$$S_{H^p} f_N(x) = \int_{\partial\Omega} E(x-y) f_N(y)\,{\rm d}\sigma(y) = S_{L^1} f_N(x) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \quad (2.2.147)$$

Based on (2.2.146), the boundedness of (2.2.116), the boundedness of (1.3.64), and [68, Lemma 6.2.6], we may then write

$$\lim_{N\to\infty} S_{L^1} f_N = S_{L^1} f \text{ in } \big[L^{(n-1)/(n-2),\infty}(\partial\Omega,\sigma)\big]^M, \qquad (2.2.148)$$
$$\text{hence also in } \big[L^1_{\rm loc}(\partial\Omega,\sigma)\big]^M,$$

and

$$\lim_{N\to\infty} S_{H^p} f_N = S_{H^p} f \text{ in } \big[L^{p^*}(\partial\Omega,\sigma)\big]^M, \qquad (2.2.149)$$
$$\text{hence also in } \big[L^1_{\rm loc}(\partial\Omega,\sigma)\big]^M.$$

Finally, from (2.2.147), (2.2.148), (2.2.149) we conclude that $S_{L^1} f = S_{H^p} f$ at σ-a.e. point on $\partial\Omega$, hence the desired conclusion follows. This finishes the treatment of the claims made in item *(1)*.

On to item *(2)*, given $r \in (n-1,\infty)$ define

$$p := \Big(\frac{n}{n-1} - \frac{1}{r}\Big)^{-1} \in \Big(\frac{n-1}{n}, 1\Big) \text{ and } \eta := 1 - \frac{n-1}{r} \in (0,1), \qquad (2.2.150)$$

then consider $p^* \in (1,\infty)$ such that $1/p^* = 1/p - 1/(n-1)$. These choices entail

$$(n-1)\Big(\frac{1}{p} - 1\Big) = \eta \text{ and } (p^*)' = r. \qquad (2.2.151)$$

Let $S_{A^\top}$ be the boundary-to-boundary single layer associated with transpose coefficient tensor $A^\top$ from (1.3.3) in the same manner S has been associated with A in (1.3.62). From the current item *(1)* we know that

$$S_{A^\top} : \big[H^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{p^*}(\partial\Omega,\sigma)\big]^M \qquad (2.2.152)$$

is a well-defined linear and bounded operator. Thanks to (2.2.151) and the duality result recoded in [69, Theorem 4.6.1], its transpose is

$$\big(S_{A^\top}\big)^\top : \big[L^r(\partial\Omega,\sigma)\big]^M \longrightarrow \begin{cases} \big[\mathscr{C}^\eta(\partial\Omega)\big]^M & \text{if } \partial\Omega \text{ is bounded,} \\ \big[\dot{\mathscr{C}}^\eta(\partial\Omega)/\sim\big]^M & \text{if } \partial\Omega \text{ is unbounded.} \end{cases} \qquad (2.2.153)$$

We claim that this is compatible the operator S from (1.3.63). To see that this is the case, suppose first that $\partial\Omega$ is unbounded. In such a scenario, pick an arbitrary $f \in \big[L^r(\partial\Omega,\sigma)\big]^M \cap \big[L^q(\partial\Omega,\sigma)\big]^M$ with $q \in (1, n-1)$, and select a representative g for $\big(S_{A^\top}\big)^\top f$, i.e., select a function $g \in \dot{\mathscr{C}}^\eta(\partial\Omega)$ such that $\big(S_{A^\top}\big)^\top f = [g]$. Finally, let $q^* \in (1,\infty)$ satisfy $1/q^* = 1/q - 1/(n-1)$, and denote by $(q^*)' \in (1, n-1)$ the Hölder conjugate exponent of q^*. Then for each $\mathbb{C}^M$-valued $\big(p,(q^*)'\big)$-atom a on $\partial\Omega$ we may write

$$\int_{\partial\Omega}\langle g,a\rangle\,\mathrm{d}\sigma = \big\langle [g],a\big\rangle = \big\langle \big(S_{A^\top}\big)^\top f, a\big\rangle$$
$$= \int_{\partial\Omega}\langle f, S_{A^\top}a\rangle\,\mathrm{d}\sigma = \int_{\partial\Omega}\langle Sf,a\rangle\,\mathrm{d}\sigma, \tag{2.2.154}$$

where $S_{A^\top}a$ is considered in the sense of (1.3.62), i.e.,

$$\big(S_{A^\top}a\big)(x) = \int_{\partial\Omega} E_{A^\top}(x-y)f(y)\,\mathrm{d}\sigma(y) \ \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{2.2.155}$$

The last equality in (2.2.154) is a consequence of Fubini's Theorem, whose applicability is ensured by the fact that if I_1 is the fractional integral operator of order 1 on $\partial\Omega$ (cf. (A.0.100)) then [70, (1.4.24)], Hölder's inequality, and [68, (7.8.7)] imply

$$\int_{\partial\Omega}\int_{\partial\Omega}|E_{A^\top}(x-y)||f(x)||a(y)|\,\mathrm{d}\sigma(x)\,\mathrm{d}\sigma(y)$$
$$\leq C\int_{\partial\Omega}(I_1|f|)(y)|a(y)|\,\mathrm{d}\sigma(y)$$
$$\leq C\big\|I_1|f|\big\|_{L^{q^*}(\partial\Omega,\sigma)}\|a\|_{[L^{(q^*)'}(\partial\Omega,\sigma)]^M}$$
$$\leq C\|f\|_{[L^q(\partial\Omega,\sigma)]^M}\|a\|_{[L^{(q^*)'}(\partial\Omega,\sigma)]^M} < +\infty. \tag{2.2.156}$$

This finishes the proof of (2.2.154). In turn, from (2.2.154) and [69, Lemma 4.6.9] we conclude that $[Sf] = [g] = \big(S_{A^\top}\big)^\top f$, where Sf is computed in the sense of (1.3.62), Thus, the operator S from (1.3.63) further extends to a linear and bounded operator as in (2.2.117). Thanks to [68, (3.1.14)], such an extension is necessarily unique. Dealing with the case when $\partial\Omega$ is bounded is very similar, and this completes the treatment of the first claim in item *(2)*.

Finally, the claim in (2.2.120) may be justified by first considering the case when f is a $(p,2)$-atom and when $g \in L^{(p^*)'}(\partial\Omega,\sigma) \cap L^2(\partial\Omega,\sigma)$. In such a scenario, the desired conclusion is seen from [69, Lemma 4.6.6], (2.2.117) used for the exponent $r := (p^*)' \in (n-1,\infty)$, and (1.3.62). Having established this, we may then invoke [69, (4.4.172)], the density of $L^{(p^*)'}(\partial\Omega,\sigma) \cap L^2(\partial\Omega,\sigma)$ in $L^{(p^*)'}(\partial\Omega,\sigma)$, and continuity of both sides of (2.2.120) with respect to f and g (cf. (2.2.116), (2.2.117), and the duality result from [69, Theorem 4.6.1]) to conclude that (2.2.120) in full generality.

Turning to item *(3)*, we run the same type of argument used in the treatment of item *(2)* in which we now take $r := n-1$. Such a choice leads to considering $p = 1$ in (2.2.150) and the role of the Hölder spaces is now played by BMO. We therefore arrive at the conclusion that S from (1.3.63) extends, in a unique fashion, to a linear and bounded operator as in (2.2.122). In fact, this conclusion may be further refined. Specifically, pick an integrability exponent $r \in (n-1, \infty)$ and, given any $f \in \big[L^{n-1}(\partial\Omega, \sigma)\big]^M$, select a sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \big[L^r(\partial\Omega, \sigma)\big]^M$ with the property that $f_j \to f$ in $\big[L^r(\partial\Omega, \sigma)\big]^M$ as $j \to \infty$. Then, if we define $\eta := 1 - (n-1)/r \in (0,1)$, from item *(2)* and (2.2.122) it follows that the sequence $\{Sf_j\}_{j\in\mathbb{N}} \subseteq \big[\mathscr{C}^\eta(\partial\Omega)\big]^M$ converges to Sf in $\big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M$ if $\partial\Omega$ is bounded, and the sequence $\big\{[Sf_j]\big\}_{j\in\mathbb{N}} \subseteq \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^M$ converges to $[Sf]$ in $\big[\widetilde{\mathrm{BMO}}(\partial\Omega, \sigma)\big]^M$ if $\partial\Omega$ is unbounded. On account of [69, (3.1.50)] and [69, Theorem 3.1.3] we then conclude that actually $Sf \in \big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M$, finishing the proof of the first claim made in item *(3)*.

To deal with the claim in (2.2.123), first consider the case when f is a $(1,2)$-atom and when $g \in L^{n-1}(\partial\Omega, \sigma) \cap L^2(\partial\Omega, \sigma)$. In such a setting, the desired conclusion is seen from [69, Lemma 4.6.5], (2.2.122), and (1.3.62). Having treated this special case, we may then rely on [69, (4.4.172)], the density of $L^{n-1}(\partial\Omega, \sigma) \cap L^2(\partial\Omega, \sigma)$ in $L^{n-1}(\partial\Omega, \sigma)$, and continuity of both sides of (2.2.123) with respect to f and g (cf. (2.2.116), (2.2.122), and the duality result from [69, Theorem 4.6.1]) to conclude that (2.2.123) holds as stated.

As regards item *(4)*, since we already know that the operators from items *(1)*-*(3)* are compatible with S from (1.3.63), we only have to show that said operators act in a coherent fashion with one another. The latter property is, however, a consequence of the former and the simultaneous convergence results from [69, Theorem 4.4.3] and [68, (3.1.14)].

Moving on to item *(5)*, assume first that either $p \in (1, n-1)$ or $\partial\Omega$ is bounded, and fix some function $f \in \big[L^p(\partial\Omega, \sigma)\big]^M$. Then (1.5.73) and (1.3.62) imply (that all integrals involved are absolutely convergent is ensured by the Fractional Integration Theorem, [70, (1.4.24)], and [68, Lemma 7.2.1])

$$
\begin{aligned}
(S_{\mathrm{mod}} f)(x) &= \int_{\partial\Omega} E(x-y) f(y)\, d\sigma(y) - \int_{\partial\Omega} E_*(-y) f(y)\, d\sigma(y) \\
&= (Sf)(x) - C_f \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \qquad (2.2.157)
\end{aligned}
$$

with $C_f := \int_{\partial\Omega} E_*(-y) f(y)\, d\sigma(y) \in \mathbb{C}^M$. Consider next the case when $\partial\Omega$ is unbounded and $p \in [n-1, \infty)$. Fix some integrability exponent $q \in (1, n-1)$ and, having picked an arbitrary function $f \in \big[L^p(\partial\Omega, \sigma)\big]^M$, choose a sequence of functions $\{f_j\}_{j\in\mathbb{N}} \subseteq \big[L^p(\partial\Omega, \sigma)\big]^M \cap \big[L^q(\partial\Omega, \sigma)\big]^M$ which converges to f in the space $\big[L^p(\partial\Omega, \sigma)\big]^M$. Write S_{L^p} and S_{L^q} for the versions of the single layer operator from (2.2.124) acting from the spaces $\big[L^p(\partial\Omega, \sigma)\big]^M$ and $\big[L^q(\partial\Omega, \sigma)\big]^M$, respectively. Then, with convergence in $\big[L^1_{\mathrm{loc}}(\partial\Omega, \sigma)/\sim\big]^M$, we have

$$\big[S_{\rm mod} f\big] = \lim_{j\to\infty} \big[S_{\rm mod} f_j\big] = \lim_{j\to\infty} \big[S_{L^q} f_j\big] = \lim_{j\to\infty} S_{L^p} f_j, \tag{2.2.158}$$

where the first equality is implied by (1.5.74), bearing in mind that $\big[L^p(\partial\Omega,\sigma)\big]^M$ embeds continuously into $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$, the second equality is a consequence of (2.2.157) written for f_j in place of f, and the third equality is guaranteed by the first claim in the current item *(4)*. In turn, from (2.2.158) we conclude that, after eventually passing to a subsequence, $S_{L^p} f_{j_k}$ converges to $\big[S_{\rm mod} f\big]$ pointwise σ-a.e. on $\partial\Omega$ as $k\to\infty$. On the other hand, the current item *(4)* gives that $\lim_{k\to\infty} S_{L^p} f_{j_k} = Sf$ in $\big[\widetilde{\rm BMO}(\partial\Omega,\sigma)\big]^M$ or $\big[\dot{\mathscr{C}}^\eta(\partial\Omega)/\sim\big]^M$, depending on whether $p=n-1$ or $p>n-1$ (in which case $\eta = 1-(n-1)/p$) which goes to show that $S_{L^p} f_{j_k}$ has a subsequence which converges to Sf pointwise σ-a.e. on $\partial\Omega$. Altogether, this argument shows that $Sf = \big[S_{\rm mod} f\big]$ at σ-a.e. point in $\partial\Omega$, completing the treatment of item *(5)*.

Consider next the claims in item *(6)*. The case when $p\in(1,n-1)$ is covered by (1.3.67), so there remains to deal with situation when $p\in\big(\frac{n-1}{n},1\big]$. Fix such an exponent p along with an aperture parameter $\kappa>0$. The idea is to use [68, Proposition 6.2.11] for the following choices. First, take $\mathscr{X} := \Omega\cup A_\kappa(\partial\Omega)$ presently endowed with the topology inherited from the ambient Euclidean space, and $X := A_\kappa(\partial\Omega)$. Hence, $\mathscr{X}\setminus X = \Omega$. Second, we take $\mu := \mathcal{H}^{n-1}\lfloor A_\kappa(\partial\Omega)$ which is a locally finite complete Borel-regular measure on X by virtue of [68, Lemma 3.6.4] (the fact that the hypotheses in [68, (3.6.25)] are presently satisfied is seen from [68, (8.8.5)] and the upper Ahlfors regularity of $\partial\Omega$). If for each $x\in X = A_\kappa(\partial\Omega)$ we take $\Gamma(x) := \Gamma_\kappa(x)$, then condition [68, (6.2.71)] is satisfied thanks to (A.0.2). Next, we take $Y := \big[H^p(\partial\Omega,\sigma)\big]^M$ which is a quasi-Banach space, and consider the operator $\mathcal{T}$ mapping vectors $f\in Y$ into $\mathbb{C}^M$-valued continuous functions defined in $\mathscr{X}\setminus X = \Omega$ according to $\mathcal{T}f := \mathscr{S}f$ (cf. (2.2.32)). Since $\mathscr{S}$ is linear, [68, (6.2.72)] holds. Going further, fix some $q\in(1,n-1)$ and take $\mathcal{V} := \big[H^{p,q}_{\rm fin}(\partial\Omega,\sigma)\big]^M$. From [69, (4.4.114)] we know that $\mathcal{V}$ is a dense linear subspace of Y. Finally, take the integrability exponent p appearing in the statement of [68, Proposition 6.2.11] to presently be p^*. These choices imply that the associated maximal operator (cf. [68, (6.2.73)]) is

$$(T_\star f)(x) = \mathcal{N}_\kappa(\mathscr{S}f)(x), \quad \forall f\in\big[H^p(\partial\Omega,\sigma)\big]^M, \quad \forall x\in A_\kappa(\partial\Omega). \tag{2.2.159}$$

Since, according to item *(3)* in Theorem 2.2.3, this maximal operator maps $\big[H^p(\partial\Omega,\sigma)\big]^M$ boundedly into the Lorentz space $L^{p^*,\infty}\big(A_\kappa(\partial\Omega),\mathcal{H}^{n-1}\lfloor A_\kappa(\partial\Omega)\big)$, it follows that hypothesis [68, (6.2.74)] is satisfied. Finally, that for every $f\in\mathcal{V}$ the limit in [68, (6.2.75)] exists becomes a consequence of (1.3.67) (bearing in mind [69, (4.4.114)] and our choice of q).

At this stage, the application of [68, Proposition 6.2.11] in the manner just described guarantees that

$$\begin{aligned}&\text{for each } f \in \big[H^p(\partial\Omega,\sigma)\big]^M \text{ with } p \in \big(\tfrac{n-1}{n},1\big] \text{ the nontangential}\\&\text{limit } (\mathscr{S}f)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \mathcal{H}^{n-1}\text{-a.e. point in } A_\kappa(\partial\Omega), \text{ and we have}\\&\Big|(\mathscr{S}f)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big| \leq \mathcal{N}_\kappa(\mathscr{S}f) \text{ at } \mathcal{H}^{n-1}\text{-a.e. point in } A_\kappa(\partial\Omega).\end{aligned} \tag{2.2.160}$$

Having proved this, [68, Corollary 8.9.7] (used with $\mu := \mathcal{H}^{n-1}\lfloor\partial\Omega$) additionally gives that

$$\begin{aligned}&\text{for each } f \in \big[H^p(\partial\Omega,\sigma)\big]^M \text{ with } p \in \big(\tfrac{n-1}{n},1\big] \text{ the function defined}\\&\mathcal{H}^{n-1}\text{-a.e. as } A_\kappa(\partial\Omega) \ni x \longmapsto \Big((\mathscr{S}f)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) \text{ is } \mathcal{H}^{n-1}\text{-measurable.}\end{aligned} \tag{2.2.161}$$

From (2.2.160), (2.2.161), and item *(3)* in Theorem 2.2.3 we then conclude that for each $p \in \big(\frac{n-1}{n},1\big]$ there exists $C \in (0,\infty)$ with the property that

$$\Big\|(\mathscr{S}f)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big\|_{[L^{p^*}(A_\kappa(\partial\Omega),\mathcal{H}^{n-1}\lfloor A_\kappa(\partial\Omega))]^M} \leq C\|f\|_{[H^p(\partial\Omega,\sigma)]^M}, \tag{2.2.162}$$

for every $f \in \big[H^p(\partial\Omega,\sigma)\big]^M$. Hence,

$$\begin{aligned}\big[H^p(\partial\Omega,\sigma)\big]^M \ni f \longmapsto (\mathscr{S}f)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in \Big[L^{p^*}\big(A_\kappa(\partial\Omega),\mathcal{H}^{n-1}\lfloor A_\kappa(\partial\Omega)\big)\Big]^M\\ \text{is a well-defined, linear and bounded operator.}\end{aligned} \tag{2.2.163}$$

On the other hand, item *(1)* of the current theorem implies that S also induces a well-defined, linear and bounded operator in the same functional analytic context. Moreover, as seen from item *(2)* in Theorem 2.2.3, item *(2)* of the current theorem, (1.3.6), (1.3.62), and (1.3.67), when acting on $\big[H^{p,q}_{\rm fin}(\partial\Omega,\sigma)\big]^M$, these two operators yield functions which agree $\mathcal{H}^{n-1}$-a.e. on $A_\kappa(\partial\Omega)$. Since the operators in questions are continuous and $\big[H^{p,q}_{\rm fin}(\partial\Omega,\sigma)\big]^M$ is dense in $\big[H^p(\partial\Omega,\sigma)\big]^M$, we ultimately conclude that (2.2.127) holds for each $f \in \big[H^p(\partial\Omega,\sigma)\big]^M$.

As regards the claims in item *(7)*, the boundedness of the boundary-to-boundary single layer operator in the context of (2.2.128) is a consequence of the current item *(4)*, [69, Theorem 4.3.1], [68, (6.2.48)], and real interpolation. Finally, that for each $f \in \big[H^{p,q}(\partial\Omega,\sigma)\big]^M$ the nontangential boundary limit $(\mathscr{S}f)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists σ-a.e. on $A_\kappa(\partial\Omega)$, and that the boundary trace formula (2.2.127) continues to hold in this case, may be justified by reasoning as in the proof of (2.2.126)-(2.2.127).

Let us now deal with item *(8)*, working in the two-dimensional setting. For starters, all claims up to, and including (2.2.132), made in relation to the operator (2.2.130)-(2.2.131) are direct consequence of (2.2.36). To proceed, fix an exponent $p \in \big(\frac{1}{2},1\big)$ and set $p^* := \big(\frac{1}{p}-1\big)^{-1}$. Also, pick some $\mathbb{C}^M$-valued $(p,2)$-atom a on $\partial\Omega$. Retaining earlier notation, in place of (2.2.140) we now have

$$\int_{B(x_o,2r)\cap\partial\Omega} |Sa|^{p^*}\, \mathrm{d}\sigma \leq \big\|Sa\big\|^{p^*}_{[L^\infty(\partial\Omega,\sigma)]^M} \cdot \sigma\big(B(x_o,2r)\cap\partial\Omega\big)$$
$$\leq Cr \cdot \|a\|^{p^*}_{[H^1(\partial\Omega,\sigma)]^M} \leq C, \tag{2.2.164}$$

thanks to the boundedness of the operator (2.2.130), the Ahlfors regularity of $\partial\Omega$, the estimate recorded in (2.2.84), and the definition of p^*. Above, $C \in (0,\infty)$ is a constant independent of the atom in question. Since the argument in (2.2.141)-(2.2.144) continues to work when $n = 2$, we ultimately conclude that we presently still have (2.2.145) for all $\mathbb{C}^M$-valued $(p,2)$-atoms a on $\partial\Omega$. With this in hand, we may now invoke [69, Theorem 4.4.7] with

$$q := 1,\ (X,\tau) := L^\infty(\partial\Omega,\sigma),\ \text{and}\ (Y,\|\cdot\|) := L^{p^*}(\partial\Omega,\sigma). \tag{2.2.165}$$

Bearing in mind [69, (4.4.146)] (presently used with $\theta := 1$ and $s := 1$) this guarantees that S from (2.2.130)-(2.2.131) extends uniquely, in the manner described in [69, (4.4.144)], to a linear and bounded operator in the context of (2.2.133).

Next, the fact that the single layer in the context of (2.2.133) continues to satisfy (2.2.132) for each $\kappa > 0$ and each $f \in \big[H^p(\partial\Omega,\sigma)\big]^M$ may be justified arguing much as in the proof of the claims made in the current item *(6)*, now making use of (2.2.133). The very last claim in item *(8)*, concerning the validity of (2.2.132) for each $f \in \big[H^{p,q}(\partial\Omega,\sigma)\big]^M$ with $p \in \big(\frac{1}{2},1\big)$ and $q \in (0,\infty]$, is seen from what we have just proved and [69, (1.3.41), (4.3.3)].

Finally, under the assumptions in (2.2.136) the claims in (2.2.136)-(2.2.138) may be justified by reasoning much as in the proofs of (2.2.117) and (2.2.120), making use of (2.2.133). This finishes the proof of Theorem 2.2.6. □

In the last part of this section, we wish to augment the result established in Theorem 1.8.19 by now working in a setting which places the (weak) conormal derivative of the function in question in a Hardy space H^p with $p \leq 1$. Specifically, we have the basic integral representation formula stated in following theorem[6]:

Theorem 2.2.7 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an* NTA *domain with the property that $\partial\Omega$ is an Ahlfors regular set. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, for some $M \in \mathbb{N}$, let L be a weakly elliptic homogeneous second-order $M \times M$ system in $\mathbb{R}^n$, with constant complex coefficients. Next, fix an aperture parameter $\kappa \in (0,\infty)$ and consider a function $u : \Omega \to \mathbb{C}^M$ satisfying*

$$u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{in}\ \Omega, \ \text{and}$$
$$\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma) \ \text{for some}\ p \in \big(\tfrac{n-1}{n},1\big]. \tag{2.2.166}$$

Then the following statements are true.

[6] for a related result, see Theorem 2.3.16

(a) *The nontangential boundary trace* $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ *exists at* σ*-a.e. point on* $\partial\Omega$ *and, as a function, this trace belongs to the weighted Lebesgue space* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$.

(b) *If either* $n \geq 3$, *or* $n = 2$ *and* $p \in \big(\frac{1}{2}, 1\big)$, *it follows that there exists some constant* $c \in \mathbb{C}^M$ *with the property that* $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - c$ *belongs to the space* $\big[L^{p^*}(\partial\Omega, \sigma)\big]^M$, *where* $p^* := \big(\frac{1}{p} - \frac{1}{n-1}\big)^{-1} \in (1, \infty)$. *In particular,* $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in \big[L^{p^*}_{\mathrm{loc}}(\partial\Omega, \sigma)\big]^M$ *in this scenario. Finally, in the remaining case, i.e., when* $n = 2$ *and* $p = 1$, *the nontangential boundary trace* $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ *belongs to the John-Nirenberg space* $\big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M$.

(c) *In the case when* Ω *is an exterior domain make the additional assumption that there exists* $\lambda \in (1, \infty)$ *such that*

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |\nabla u| \, \mathrm{d}\mathcal{L}^n = o(1) \ \text{ as } \ R \to \infty. \tag{2.2.167}$$

Then for each complex coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ *with the property that* $L = L_A$ *the weak conormal derivative* $\dot{\partial}_\nu^A u$ *belongs to the Hardy space* $\big[H^p(\partial\Omega, \sigma)\big]^M$ *and there exists some* $\mathbb{C}^M$*-valued locally constant function* c_u *in* Ω *with the property that*

$$u = \mathcal{D}_{\mathrm{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) - \mathcal{S}\big(\dot{\partial}_\nu^A u\big) + c_u \ \text{ in } \ \Omega, \tag{2.2.168}$$

where $\mathcal{D}_{\mathrm{mod}}$ *is the modified double layer potential operator associated with the coefficient tensor* A (*acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ *as in* (1.8.6)), *and where the single layer potential operator acts on distributions in the Hardy space* $\big[H^p(\partial\Omega, \sigma)\big]^M$ *as in* (2.2.32).

(d) *Assume that either* $n \geq 3$, *or* $n = 2$ *and* $p \in \big(\frac{1}{2}, 1\big)$ *and* Ω *not an exterior domain. Then there exists a constant* $c \in \mathbb{C}^M$ *such that*

$$\mathcal{N}_\kappa(u - c) \in L^{p^*}(\partial\Omega, \sigma) \ \text{ where } \ p^* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1} \in (1, \infty), \tag{2.2.169}$$

and, as a consequence, in this case one has

$$\mathcal{N}_\kappa u \in L^{p^*}_{\mathrm{loc}}(\partial\Omega, \sigma) \cap L^1\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^n}\Big). \tag{2.2.170}$$

Also, in the case when $n = 2$, $p \in \big(\frac{1}{2}, 1\big)$, *and* Ω *is an exterior domain, the same results in* (2.2.169)-(2.2.170) *remain true provided a truncated nontangential maximal operator, say* $\mathcal{N}_\kappa^{\Omega\cap B(0,R)}$ *with* $R \in (0, \infty)$, *is employed in place of* $\mathcal{N}_\kappa = \mathcal{N}_\kappa^\Omega$.
Finally, in the case when $n = 2$ *and* $p = 1$ *one has*

$$\mathcal{N}_\kappa^\varepsilon u \in L^q_{\mathrm{loc}}(\partial\Omega, \sigma) \textit{ for each } \varepsilon > 0 \textit{ and each } q \in (0, \infty). \tag{2.2.171}$$

Through the consideration of the sub-unital integrability exponents, Theorem 2.2.7 complements the result in Theorem 1.8.19 (which deals with the super-unital range). An integral representation similar in spirit to (2.2.168) is also established in Theorem 2.3.16 under weaker geometric hypotheses on the underlying domain Ω but for stronger analytical assumptions on the function u. See also Corollary 2.3.17 for a version of (2.2.168) in which we are allowed to take $c_u = 0$.

Proof of Theorem 2.2.7 For starters, [68, Proposition 8.9.22] and [68, (5.2.4)] ensure that the nontangential trace

$$u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \tag{2.2.172}$$

and the function

$$g := \begin{cases} \mathcal{N}_\kappa(\nabla u) \text{ if } \partial\Omega \text{ is unbounded,} \\ \Big|u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big| + \mathcal{N}_\kappa(\nabla u) \text{ if } \partial\Omega \text{ is bounded,} \end{cases} \tag{2.2.173}$$

belongs to $L^p(\partial\Omega, \sigma)$. Granted this, from [68, (5.11.28)] and [69, Lemma 11.5.8] we see that there exists a constant $C \in (0, \infty)$ independent of u with the property that

$$\begin{gathered} \Big|\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) - \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)\Big| \leq C|x - y| \cdot \big[g(x) + g(y)\big] \\ \text{for } \sigma\text{-a.e. points } x, y \in \partial\Omega. \end{gathered} \tag{2.2.174}$$

Then [69, Lemma 11.5.9] applies with $f := u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ and $\Sigma := \partial\Omega$. If either $n \geq 3$ or $n = 2$ and $p \in \big(\frac{1}{2}, 1\big)$, this guarantees the existence of some constant $c \in \mathbb{C}^M$ such that the function $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - c$ belongs to $\big[L^{p^*}(\partial\Omega, \sigma)\big]^M$. On account of this, we may also conclude that

$$u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in \big[L^{p^*}(\partial\Omega, \sigma)\big]^M + \mathbb{C}^M \hookrightarrow \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^n}\Big)\Big]^M, \tag{2.2.175}$$

in this scenario. In the remaining case, i.e., when $n = 2$ and $p = 1$, from item *(2)* in [69, Lemma 11.5.9] and [68, (7.4.118)] (used here with $p = 1$, $d = 1$, and $\varepsilon = 1$) we see that

$$u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in \big[\mathrm{BMO}(\partial\Omega, \sigma)\big]^M \hookrightarrow \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^2}\Big)\Big]^M. \tag{2.2.176}$$

Altogether, the argument so far takes care of the claims made in items *(a)-(b)*.

Going further, from the second line in (2.2.166) and [68, (8.6.51)] we see (also bearing in mind (A.0.105)) that

$$\nabla u \in \big[L^{np/(n-1)}_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)\big]^{M\cdot n}. \tag{2.2.177}$$

In addition, from [68, Proposition 8.4.9] we deduce that

$$\mathcal{N}_\kappa^\rho u \in L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for each } \rho>0. \tag{2.2.178}$$

In concert with [68, (8.1.18), (8.6.51)], this also implies

$$u \in \big[L^{np/(n-1)}_{\rm bdd}(\Omega,\mathcal{L}^n)\big]^M. \tag{2.2.179}$$

Ultimately, from (2.2.177)-(2.2.179) and (A.0.208) we conclude that

$$u \in \big[W^{1,\,np/(n-1)}_{\rm bdd}(\Omega)\big]^M. \tag{2.2.180}$$

For later use, let us also note here that (2.2.172), (2.2.178), [68, Corollary 8.9.9], [68, (8.8.52)], and [68, (5.2.4)] guarantee that

$$u\big|^{\widetilde{\kappa}-{\rm n.t.}}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial\Omega \text{ for each } \widetilde{\kappa}\in(0,\infty), \text{ and this trace is actually independent of the aperture parameter.} \tag{2.2.181}$$

Henceforth, in the case when Ω is an exterior domain, make the additional assumption that (2.2.167) holds for some $\lambda \in (1,\infty)$. To proceed, pick a complex coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\le r,s\le n\\ 1\le\alpha,\beta\le M}}$ with the property that $L = L_A$. Then, granted the present hypotheses, [70, Theorem 3.3.1] yields a number of conclusions. First, for any $\ell, s \in \{1,\dots,n\}$ we have

$$\partial_{\tau_{\ell s}} u \in \big[H^p(\partial\Omega,\sigma)\big]^M \text{ and } \partial^A_\nu u \in \big[H^p(\partial\Omega,\sigma)\big]^M. \tag{2.2.182}$$

Second, if $E = (E_{\gamma\beta})_{1\le\gamma,\beta\le M}$ is the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2] and if $(u_\gamma)_{1\le\gamma\le M}$ are the scalar components of the function u, then with the duality pairings understood in the sense of [69, Theorem 4.6.1] with $\Sigma := \partial\Omega$ and the summation convention over repeated indices in effect it follows that for each $\ell \in \{1,\dots,n\}$ and each $\gamma \in \{1,\dots,M\}$ we have

$$\begin{aligned}(\partial_\ell u_\gamma)(x) &= a^{\beta\alpha}_{rs}\Big\langle \big[(\partial_r E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}\big], \partial_{\tau_{\ell s}} u_\alpha \Big\rangle \\ &\quad - \Big\langle \big[(\partial_\ell E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}\big], \big(\partial^A_\nu u\big)_\beta \Big\rangle, \quad \text{for all } x\in\Omega,\end{aligned} \tag{2.2.183}$$

assuming $\partial\Omega$ is unbounded. A similar formula holds when $\partial\Omega$ is bounded, this time omitting taking equivalence classes of functions modulo constants in the duality pairings in (2.2.183). The idea is to further transform the first duality pairing in (2.2.183). We shall do so while henceforth assuming that $\partial\Omega$ is unbounded, since a very similar (and slightly simpler) argument works in the case when $\partial\Omega$ is bounded as well.

To get started, fix $\ell, s, r \in \{1,\dots,n\}$ along with $\alpha,\beta,\gamma \in \{1,\dots,M\}$. Also, pick $x \in \Omega$ then choose some $\varepsilon \in \big(0, \tfrac12\,{\rm dist}(x,\partial\Omega)\big)$. Consider a scalar-valued function $\eta \in \mathscr{C}^\infty(\mathbb{R}^n)$ satisfying $\eta = 0$ on $B(0,1)$, $\eta = 1$ on $\mathbb{R}^n \setminus B(0,2)$, and define

$\eta_\varepsilon(y) := \eta\big((y-x)/\varepsilon\big)$ for every $y \in \mathbb{R}^n$. This ensures that

$$\eta_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n), \quad \eta_\varepsilon \equiv 1 \text{ on } \mathbb{R}^n \setminus B(x, 2\varepsilon), \quad \eta_\varepsilon \equiv 0 \text{ on } B(x,\varepsilon). \tag{2.2.184}$$

As a consequence, we have that the function $E_{\gamma\beta}(x-\cdot)\eta_\varepsilon$ belongs to $\mathscr{C}^\infty(\mathbb{R}^n)$ and coincides with $E_{\gamma\beta}(x-\cdot)$ near $\partial\Omega$. Next, choose a function $\theta \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ satisfying $0 \le \theta \le 1$, $\theta \equiv 1$ on $B(0,1)$, $\theta \equiv 0$ on $\mathbb{R}^n \setminus B(0,2)$ and, for each $R > 0$, define $\theta_R(y) := \theta(y/R)$ for every $y \in \mathbb{R}^n$. The general weak-$*$ convergence results from [69, Lemma 4.8.4] and [69, Lemma 4.8.1] (also bearing in mind the trivial bounded embedding $L^\infty(\partial\Omega,\sigma) \hookrightarrow \mathrm{BMO}(\partial\Omega,\sigma)$) imply that we have

$$\begin{aligned}
\Big[(\partial_r E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}\Big] &= \Big[\big((\partial_r E_{\gamma\beta})(x-\cdot)\eta_\varepsilon\big)\big|_{\partial\Omega}\Big] \\
&= \lim_{R\to\infty} \Big[\big(\theta_R(\partial_r E_{\gamma\beta})(x-\cdot)\eta_\varepsilon\big)\big|_{\partial\Omega}\Big] \text{ weak-}* \text{ in } \big(H^p(\partial\Omega,\sigma)\big)^* \\
&= \begin{cases} \dot{\mathscr{C}}^{(n-1)(1/p-1)}(\partial\Omega)/\sim & \text{if } p<1, \\ \widetilde{\mathrm{BMO}}(\partial\Omega,\sigma) & \text{if } p=1. \end{cases}
\end{aligned} \tag{2.2.185}$$

Consequently,

$$\Big\langle\Big[(\partial_r E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}\Big], \partial_{\tau_{\ell s}} u_\alpha\Big\rangle = \lim_{R\to\infty}\Big\langle\Big[\big(\theta_R(\partial_\ell E_{\gamma\beta})(x-\cdot)\eta_\varepsilon\big)\big|_{\partial\Omega}\Big], \partial_{\tau_{\ell s}} u_\alpha\Big\rangle. \tag{2.2.186}$$

Let us momentarily abbreviate

$$\Psi := \theta_R(\partial_r E_{\gamma\beta})(x-\cdot)\eta_\varepsilon. \tag{2.2.187}$$

Then $\Psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ and for each $R>0$ we have

$$\begin{aligned}
&\Big\langle\Big[\big(\theta_R(\partial_r E_{\gamma\beta})(x-\cdot)\eta_\varepsilon\big)\big|_{\partial\Omega}\Big], \partial_{\tau_{\ell s}} u_\alpha\Big\rangle \\
&\qquad = {}_{\mathrm{Lip}_c(\partial\Omega)}\Big\langle\Big[\big(\theta_R(\partial_r E_{\gamma\beta})(x-\cdot)\eta_\varepsilon\big)\big|_{\partial\Omega}\Big], \partial_{\tau_{\ell s}} u_\alpha\Big\rangle_{(\mathrm{Lip}_c(\partial\Omega))'} \\
&\qquad = \int_\Omega \big\{(\partial_s u_\alpha)(\partial_\ell \Psi) - (\partial_\ell u_\alpha)(\partial_s \Psi)\big\}\,d\mathcal{L}^n,
\end{aligned} \tag{2.2.188}$$

thanks to [69, Lemma 4.6.4] and [68, Example 4.2.4] (cf. (A.0.175)-(A.0.176) in the Glossary). At this stage, we may invoke [68, Proposition 2.8.17] for the vector field

$$\vec{F} := u_\alpha(\partial_\ell\Psi)\mathbf{e}_s - u_\alpha(\partial_s\Psi)\mathbf{e}_\ell \tag{2.2.189}$$

and with p replaced by $np/(n-1) \in (1, \infty)$, its applicability in the present context being ensured by (2.2.180) and (2.2.181). From this and the independence of the nontangential boundary trace on the aperture parameter (cf. (2.2.181)) we may then conclude that

$$\Big\langle \Big[\big(\theta_R (\partial_r E_{\gamma\beta})(x-\cdot)\eta_\varepsilon\big)\Big|_{\partial\Omega} \Big], \dot{\partial}_{\tau_{\ell s}} u_\alpha \Big\rangle$$

$$= \int_{\partial\Omega} \big(u_\alpha\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) \Big\{ \nu_s (\partial_\ell \Psi)\big|_{\partial\Omega} - \nu_\ell (\partial_s \Psi)\big|_{\partial\Omega} \Big\}\, d\sigma. \tag{2.2.190}$$

Upon recalling the definition of Ψ from (2.2.187) and bearing in mind that $\eta_\varepsilon \equiv 1$ near $\partial\Omega$ while $\nabla\eta_\varepsilon$ is supported away from $\partial\Omega$ (as may be seen from (2.2.184)), we may then conclude from (2.2.186) and (2.2.190) that

$$a^{\beta\alpha}_{rs} \Big\langle \big[(\partial_r E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega} \big], \dot{\partial}_{\tau_{\ell s}} u_\alpha \Big\rangle = \lim_{R\to\infty} \mathrm{I}_R - \lim_{R\to\infty} \mathrm{II}_R \tag{2.2.191}$$

where, for each $R > 0$, we have set

$$\mathrm{I}_R := \int_{\partial\Omega} a^{\beta\alpha}_{rs} \big(u_\alpha\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) \Big\{ \nu_s (\partial_\ell \theta_R)(\partial_r E_{\gamma\beta})(x-\cdot) - \nu_\ell (\partial_s \theta_R)(\partial_r E_{\gamma\beta})(x-\cdot) \Big\}\, d\sigma, \tag{2.2.192}$$

and

$$\mathrm{II}_R := \int_{\partial\Omega} a^{\beta\alpha}_{rs} \big(u_\alpha\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) \Big\{ \nu_s \theta_R (\partial_\ell \partial_r E_{\gamma\beta})(x-\cdot) - \nu_\ell \theta_R (\partial_s \partial_r E_{\gamma\beta})(x-\cdot) \Big\}\, d\sigma. \tag{2.2.193}$$

Since for each fixed $x \in \Omega$, each $y \in \partial\Omega$, and each $R > 0$ we have

$$|(\nabla\theta_R)(y)| \leq C R^{-1} \mathbf{1}_{|y|\approx R} \text{ and } |(\nabla E)(x-y)| \leq \frac{C_x}{1+|y|^{n-1}}, \tag{2.2.194}$$

it follows that

$$\begin{aligned} \big|\mathrm{I}_R\big| &\leq C(x, A, \partial\Omega) \int_{\partial\Omega} \big|\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(y)\big| \big(R^{-1}\mathbf{1}_{|y|\approx R}\big) \frac{1}{1+|y|^{n-1}}\, d\sigma(y) \\ &\leq C(x, A, \partial\Omega) \int_{\substack{y\in\partial\Omega \\ |y|\approx R}} \frac{\big|\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(y)}{1+|y|^n}\, d\sigma(y). \end{aligned} \tag{2.2.195}$$

In view of the fact that $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\Big]^M$ (cf. (2.2.175)-(2.2.176)), Lebesgue's Dominated Convergence Theorem gives

$$\lim_{R\to\infty}\int\limits_{\substack{y\in\partial\Omega\\ |y|\approx R}}\frac{\big|\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)\big|}{1+|y|^n}\,d\sigma(y)=0. \tag{2.2.196}$$

From (2.2.195) and (2.2.196) we then conclude that

$$\lim_{R\to\infty}\mathrm{I}_R=0. \tag{2.2.197}$$

Next, for each $R>0$ decompose

$$\mathrm{II}_R=\mathrm{II}_R^{(a)}+\mathrm{II}_R^{(b)} \tag{2.2.198}$$

where

$$\mathrm{II}_R^{(a)}:=\int_{\partial\Omega}a_{rs}^{\beta\alpha}\big(u_\alpha\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)\nu_s\theta_R(\partial_\ell\partial_r E_{\gamma\beta})(x-\cdot)\,d\sigma, \tag{2.2.199}$$

and

$$\mathrm{II}_R^{(b)}:=-\int_{\partial\Omega}a_{rs}^{\beta\alpha}\big(u_\alpha\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)\nu_\ell\theta_R(\partial_s\partial_r E_{\gamma\beta})(x-\cdot)\,d\sigma. \tag{2.2.200}$$

Upon noting that $a_{rs}^{\beta\alpha}(\partial_s\partial_r E_{\gamma\beta})(x-\cdot)=0$ on $\partial\Omega$ (as seen from [70, (1.4.33)]), we deduce that

$$\mathrm{II}_R^{(b)}=0\ \text{ for each }\ R>0. \tag{2.2.201}$$

Also, since for each $y\in\partial\Omega$ we have

$$\lim_{R\to\infty}\theta_R(y)=1,\quad |\theta_R(y)|\le C,\ \text{ and }\ |(\nabla^2E)(x-y)|\le\frac{C_x}{1+|y|^n}, \tag{2.2.202}$$

Lebesgue's Dominated Convergence Theorem presently applies, keeping in mind that, as noted in (2.2.175)-(2.2.176), we have $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\in\big[L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\big)\big]^M$. This permits us to compute

$$\begin{aligned}\lim_{R\to\infty}\mathrm{II}_R^{(a)}&=\lim_{R\to\infty}\int_{\partial\Omega}\nu_s(y)a_{rs}^{\beta\alpha}\theta_R(y)(\partial_\ell\partial_r E_{\gamma\beta})(x-y)\big(u_\alpha\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)\,d\sigma(y)\\&=\int_{\partial\Omega}\nu_s(y)a_{rs}^{\beta\alpha}(\partial_\ell\partial_r E_{\gamma\beta})(x-y)\big(u_\alpha\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)\,d\sigma(y)\\&=-\partial_\ell\Big[\mathcal{D}_{\mathrm{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)\Big]_\gamma(x),\end{aligned} \tag{2.2.203}$$

with the last equality coming from (1.8.11) (again, keeping in mind (2.2.175)-(2.2.176)).

Collectively, (2.2.191), (2.2.197), (2.2.198), (2.2.201), and (2.2.203) then imply that the first term in the right-hand side of (2.2.183) may be recast as

$$d_{rs}^{\beta\alpha}\Big\langle\big[(\partial_r E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}\big],\,\overset{\bullet}{\partial}_{\tau_{\ell s}}u_\alpha\Big\rangle=\partial_\ell\Big[\mathcal{D}_{\rm mod}\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\big)\Big]_\gamma(x).\tag{2.2.204}$$

From (2.2.34) we also see that the second term in the right-hand side of (2.2.183) may be expressed as

$$-\Big\langle\big[(\partial_\ell E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}\big],\,\big(\overset{\bullet}{\partial}{}^A_\nu u\big)_\beta\Big\rangle=-\partial_\ell\big[\mathcal{S}(\overset{\bullet}{\partial}{}^A_\nu u)\big]_\gamma(x).\tag{2.2.205}$$

Altogether, from (2.2.183), (2.2.204), and (2.2.205) we conclude that

$$(\partial_\ell u_\gamma)(x)=\partial_\ell\Big[\mathcal{D}_{\rm mod}\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\big)\Big]_\gamma(x)-\partial_\ell\big[\mathcal{S}(\overset{\bullet}{\partial}{}^A_\nu u)\big]_\gamma(x),\qquad\forall\,x\in\Omega.\tag{2.2.206}$$

In view of the arbitrariness of $\ell\in\{1,\dots,n\}$ and each $\gamma\in\{1,\dots,M\}$, this further implies that there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω such that (2.2.168) holds. This concludes the treatment of item *(c)*.

As regards item *(d)*, work under the assumption that either $n\geq 3$, or $n=2$ and $p\in\big(\frac{1}{2},1\big)$. Then, as we have seen in item *(b)*, there exists $c\in\mathbb{C}^M$ such that $u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}-c$ belongs to $\big[L^{p^*}(\partial\Omega,\sigma)\big]^M$. Use (2.2.168) to write

$$u=\mathcal{D}_{\rm mod}\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}-c\big)-\mathcal{S}\big(\overset{\bullet}{\partial}{}^A_\nu u\big)+c_u^{(1)}\ \text{ in }\ \Omega,\tag{2.2.207}$$

where $c_u^{(1)}:=c_u+\mathcal{D}_{\rm mod}c$ is a locally constant $\mathbb{C}^M$-valued function in Ω (as may be seen from (1.8.10)). Since $u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}-c\in\big[L^{p^*}(\partial\Omega,\sigma)\big]^M$ with $p^*\in(1,\infty)$ in the current case, we may invoke (1.8.8) to conclude that the difference

$$\begin{gathered}c_u^{(2)}:=\mathcal{D}_{\rm mod}\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}-c\big)-\mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}-c\big)\\ \text{is a constant (belonging to }\mathbb{C}^M\text{) in }\Omega.\end{gathered}\tag{2.2.208}$$

In particular, $\widetilde{c}_u:=c_u^{(1)}+c_u^{(2)}$ is a locally constant $\mathbb{C}^M$-valued functions in Ω and (2.2.207)-(2.2.208) entail

$$u=\mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}-c\big)-\mathcal{S}\big(\overset{\bullet}{\partial}{}^A_\nu u\big)+\widetilde{c}_u\ \text{ in }\ \Omega.\tag{2.2.209}$$

In turn, this implies that for each Lebesgue measurable set $E\subseteq\Omega$ we have

$$\mathcal{N}_\kappa^E(u-\widetilde{c}_u)\leq\mathcal{N}_\kappa^E\Big(\mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}-c\big)\Big)+\mathcal{N}_\kappa^E\Big(\mathcal{S}\big(\overset{\bullet}{\partial}{}^A_\nu u\big)\Big)\tag{2.2.210}$$

at each point on $\partial\Omega$. Recall from (1.5.1) and (2.2.39), (2.2.40) that we have

$$\mathcal{N}_\kappa\Big(\mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}-c\big)\Big)\in L^{p^*}(\partial\Omega,\sigma),\tag{2.2.211}$$

as well as

$$\begin{gathered}\mathcal{N}_\kappa\Big(\mathscr{S}\big(\dot{\partial}^A_\nu u\big)\Big) \in L^{p^*}(\partial\Omega,\sigma)\\ \text{if either } n \geq 3, \text{ or } n = 2 \text{ and } \Omega \text{ is not an exterior domain,}\end{gathered} \tag{2.2.212}$$

and

$$\begin{gathered}\mathcal{N}_\kappa^{\Omega\cap B(0,R)}\Big(\mathscr{S}\big(\dot{\partial}^A_\nu u\big)\Big) \in L^{p^*}(\partial\Omega,\sigma)\\ \text{for each } R \in (0,\infty) \text{ if } n = 2 \text{ and } \Omega \text{ is an exterior domain.}\end{gathered} \tag{2.2.213}$$

Then from (2.2.210), (2.2.211)-(2.2.213), and [68, (8.2.28)] we conclude that there exists a locally constant $\mathbb{C}^M$-valued function $\widetilde{c}_u$ in Ω such that

$$\begin{gathered}\mathcal{N}_\kappa(u - \widetilde{c}_u) \in L^{p^*}(\partial\Omega,\sigma)\\ \text{if either } n \geq 3, \text{ or } n = 2 \text{ and } \Omega \text{ is not an exterior domain,}\end{gathered} \tag{2.2.214}$$

while

$$\begin{gathered}\mathcal{N}_\kappa^{\Omega\cap B(0,R)}(u - \widetilde{c}_u) \in L^{p^*}(\partial\Omega,\sigma)\\ \text{for each } R \in (0,\infty) \text{ if } n = 2 \text{ and } \Omega \text{ is an exterior domain.}\end{gathered} \tag{2.2.215}$$

If $\partial\Omega$ is unbounded then [68, Lemma 5.11.3] gives that Ω is connected so $\widetilde{c}_u$ is actually a genuine constant in $\mathbb{C}^M$, which establishes (2.2.169) in this case. If $\partial\Omega$ is bounded then [68, Lemma 5.11.3] tells us that Ω has finitely many connected components, so the range of the locally constant $\mathbb{C}^M$-valued function $\widetilde{c}_u$ is finite. Since in this case $\partial\Omega$ has finite measure, (2.2.214) implies that $\mathcal{N}_\kappa u \in L^{p^*}(\partial\Omega,\sigma)$ if either $n \geq 3$, or $n = 2$ and Ω is not an exterior domain, so (2.2.169) presently holds with $c := 0$. In turn, (2.2.170) is a consequence of (2.2.169). Also, (2.2.215) implies that if $n = 2$ and Ω is an exterior domain then $\mathcal{N}_\kappa^{\Omega\cap B(0,R)} u \in L^{p^*}(\partial\Omega,\sigma)$ for each $R \in (0,\infty)$, so the desired conclusions in this case follow as well.

Finally, that (2.2.171) holds in the case when $n = 2$ and $p = 1$ is seen from (2.2.168), the fact that we now have $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$, the last embedding in [68, (7.4.105)], [68, (7.4.118)] used with $d := 1$ and $\varepsilon := 1$, (1.8.22) used with $n = 2$ and q in place of p, (2.2.41)-(2.2.42), and the fact that (as already observed above) any locally constant function in Ω has finite range. □

We conclude with a result in the spirit of [69, Theorem 11.10.10] in which no explicit assumptions are made on the nontangential boundary trace and the nontangential maximal operator of the function itself.

Theorem 2.2.8 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an* NTA *domain whose boundary is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix an aperture parameter $\kappa \in (0,\infty)$. Also, suppose $u : \Omega \to \mathbb{C}$ is a function satisfying*

$$\begin{gathered}u \text{ belongs to } \mathscr{C}^1(\Omega) \text{ and}\\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma) \text{ for some } p \in \big(\tfrac{n-1}{n},\infty\big).\end{gathered} \tag{2.2.216}$$

Then the nontangential boundary trace $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ *exists at* σ*-a.e. point on* $\partial\Omega$ *and is independent of the aperture parameter* κ*. As a function,* $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ *belongs to the homogeneous Hardy-based Sobolev space* $\dot{H}^p_1(\partial\Omega,\sigma)$*, and satisfies*

$$\partial_{\tau_{jk}}\Big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big) = \dot{\partial}_{\tau_{jk}} u \in H^p(\partial\Omega,\sigma) \text{ for all } j,k\in\{1,\dots,n\}, \tag{2.2.217}$$

as well as

$$\Big\|u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\|_{\dot{H}^p_1(\partial\Omega,\sigma)} \leq C\big\|\mathcal{N}_\kappa(\nabla u)\big\|_{L^p(\partial\Omega,\sigma)} \tag{2.2.218}$$

for some constant $C = C(\Omega,n,\kappa,p)\in(0,\infty)$ *independent of* u*.*

It is noteworthy that Theorem 2.2.8 may be used in concert with [70, Theorem 3.3.1] and item *(i)* in Theorem 2.3.1 to give an alternative proof of Theorem 2.2.7.

Proof of Theorem 2.2.8 The case when $p\in(1,\infty)$ is a consequence of [70, Lemma 2.5.9], [69, (4.2.9), (11.10.30)], [69, Proposition 10.2.11], and [68, Proposition 8.4.9], so we shall restrict attention for the rest of the proof to the case when

$$p\in\big(\tfrac{n-1}{n},\,1\big]. \tag{2.2.219}$$

First, [68, Proposition 8.9.22] and [68, (5.2.4)] guarantee that the nontangential trace

$$u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \tag{2.2.220}$$

and that

$$g := \begin{cases} \mathcal{N}_\kappa(\nabla u) & \text{if } \partial\Omega \text{ is unbounded,} \\ \Big|u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big| + \mathcal{N}_\kappa(\nabla u) & \text{if } \partial\Omega \text{ is bounded,} \end{cases} \tag{2.2.221}$$

is a well-defined function belonging to $L^p(\partial\Omega,\sigma)$. As such, [68, (5.11.28)] and [69, Lemma 11.5.8] show that there exists a constant $C\in(0,\infty)$ independent of u with the property that

$$\begin{gathered}\Big|\Big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) - \Big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(y)\Big| \leq C|x-y|\cdot\big[g(x)+g(y)\big] \\ \text{for } \sigma\text{-a.e. points } x,y\in\partial\Omega.\end{gathered} \tag{2.2.222}$$

Granted this, [69, Lemma 11.5.9] applies with $f := u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ and $\Sigma := \partial\Omega$. When either $n\geq 3$, or $n=2$ and $p\in\big(\tfrac12,1\big)$, this implies the existence of some constant $c\in\mathbb{C}$ such that $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - c$ is a function belonging to the space $L^{p^*}(\partial\Omega,\sigma)$, where $p^* := \big(\tfrac1p - \tfrac{1}{n-1}\big)^{-1}\in(1,\infty)$. With this in hand, we then conclude that in this scenario we have

$$u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in L^{p^*}(\partial\Omega,\sigma)+\mathbb{C} \hookrightarrow L^1\Big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\Big). \tag{2.2.223}$$

In the remaining case, i.e., when $n = 2$ and $p = 1$, from item *(2)* in [69, Lemma 11.5.9] and [68, (7.4.118)] (presently used with $p := 1$, $d := 1$, and $\varepsilon := 1$) we see that

$$u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in \mathrm{BMO}(\partial\Omega, \sigma) \hookrightarrow L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^2}\Big). \tag{2.2.224}$$

The argument so far shows that, in all cases,

$$u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \subseteq L^1_{\mathrm{loc}}(\partial\Omega, \sigma). \tag{2.2.225}$$

Next, from the second line in (2.2.216) and [68, (8.6.51) in Proposition 8.6.3] we see (also keeping in mind (A.0.105)) that

$$\nabla u \in \big[L^{np/(n-1)}_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)\big]^n. \tag{2.2.226}$$

In addition, from [68, Proposition 8.4.9] we deduce that

$$\mathcal{N}^{\rho}_{\kappa} u \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma) \text{ for each } \rho > 0. \tag{2.2.227}$$

In concert with [68, (8.6.51) in Proposition 8.6.3] and [68, (8.1.18)], this also implies

$$u \in L^{np/(n-1)}_{\mathrm{bdd}}(\Omega, \mathcal{L}^n). \tag{2.2.228}$$

Together, (2.2.226), (2.2.228), and (A.0.208) permit us to conclude that

$$u \in W^{1,np/(n-1)}_{\mathrm{bdd}}(\Omega). \tag{2.2.229}$$

Let us also record here that, as seen from (2.2.220), (2.2.227), [68, Corollary 8.9.9], and [68, (5.2.4), (8.8.52)], the nontangential trace

$$\begin{array}{l} u\Big|_{\partial\Omega}^{\widetilde{\kappa}-\mathrm{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega \text{ for each } \widetilde{\kappa} \in (0,\infty), \text{ and this} \\ \text{trace is actually independent of the aperture parameter.} \end{array} \tag{2.2.230}$$

Fix now an arbitrary test function $\psi \in \mathrm{Lip}_c(\partial\Omega)$, and consider a function Ψ satisfying

$$\begin{array}{l} \Psi \in \mathrm{Lip}(\overline{\Omega}),\ \Psi\big|_{\partial\Omega} = \psi, \text{ and } \Psi \equiv 0 \\ \text{outside of some compact subset of } \overline{\Omega}. \end{array} \tag{2.2.231}$$

Let us also fix a pair of indices $j, k \in \{1, \dots, n\}$. We claim that

$$\begin{array}{c} \text{there exists some sequence } \{\varphi_\ell\}_{\ell\in\mathbb{N}} \subseteq \mathscr{C}^\infty_c(\mathbb{R}^n) \text{ such that} \\ \operatorname{supp}\varphi_\ell \subseteq K \text{ for all } \ell \in \mathbb{N} \text{ and some compact set } K \subseteq \mathbb{R}^n, \\ \varphi_\ell \longrightarrow \Psi \text{ uniformly on compact sets in } \mathbb{R}^n \text{ as } \ell \to \infty, \\ (\nabla\varphi_\ell)\big|_\Omega \longrightarrow (\nabla\Psi)\big|_\Omega \text{ weak-}* \text{ in } \big[L^\infty(\Omega, \mathcal{L}^n)\big]^n \text{ as } \ell \to \infty, \\ \partial_{\tau_{jk}}\varphi_\ell \text{ converges to } \partial_{\tau_{jk}}\psi \text{ weak-}* \text{ in } L^\infty(\partial\Omega, \sigma) \text{ as } \ell \to \infty. \end{array} \tag{2.2.232}$$

Indeed, such a sequence is constructed as in [69, Remark 11.1.13] (with $\partial_*\Omega$ replaced by $\partial\Omega$, f replaced by ψ, and F replaced by Ψ) by mollifying Ψ. The property in the third line of (2.2.232) comes from [69, Lemma 11.1.12], and the property in the last line of (2.2.232) is proved as in [69, (11.3.13)].

To proceed, for each $\ell \in \mathbb{N}$ define the vector field

$$\vec{F}_\ell := u(\partial_k\varphi_\ell)\mathbf{e}_j - u(\partial_j\varphi_\ell)\mathbf{e}_k \in \big[\mathscr{C}^1(\Omega)\big]^n. \tag{2.2.233}$$

From (2.2.233) and (2.2.231) we then see that

$$\begin{gathered}\vec{F}_\ell \;\text{ belongs to the space }\; \big[W^{1,np/(n-1)}(\Omega)\big]^n,\\ \text{and vanishes outside of }\; K\cap\Omega,\end{gathered} \tag{2.2.234}$$

while (2.2.230) further implies that

$$\begin{gathered}\vec{F}_\ell\big|_{\partial\Omega}^{\widetilde{\kappa}-\mathrm{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega \text{ for each } \widetilde{\kappa}\in(0,\infty)\text{, and this trace}\\ \text{is actually independent of the aperture parameter.}\end{gathered} \tag{2.2.235}$$

Let us also observe that, thanks to (2.2.233) and (2.2.220), at σ-a.e. point on $\partial\Omega$ we have

$$\begin{gathered}\vec{F}_\ell\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(\partial_k\varphi_\ell)\big|_{\partial\Omega}\mathbf{e}_j - \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(\partial_j\varphi_\ell)\big|_{\partial\Omega}\mathbf{e}_k,\\ \text{and }\operatorname{div}\vec{F}_\ell = (\partial_j u)(\partial_k\varphi_\ell) - (\partial_k u)(\partial_j\varphi_\ell).\end{gathered} \tag{2.2.236}$$

Observe from (2.2.225) that

$$\begin{gathered}u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \;\text{ belongs to }\; L^1_{\mathrm{loc}}(\partial\Omega,\sigma),\;\text{ hence}\\ \partial_{\tau_{jk}}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) \in \big(\mathrm{Lip}_c(\partial\Omega)\big)' \;\text{ by [69, Definition 11.2.1].}\end{gathered} \tag{2.2.237}$$

Also, from (2.2.216) and [69, Example 10.2.2] we see that

$$\begin{gathered}\dot{\partial}_{\tau_{jk}}u \;\text{ is a well-defined distribution in }\; H^p(\partial\Omega,\sigma)\\ \text{satisfying }\; \big\|\dot{\partial}_{\tau_{jk}}u\big\|_{H^p(\partial\Omega,\sigma)} \le C\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,\sigma)},\end{gathered} \tag{2.2.238}$$

for some constant $C\in(0,\infty)$ independent of u.

Denote by $\nu = (\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω. Bearing in mind the memberships from (2.2.237) and (2.2.238), we are now prepared to compute (with all pointy brackets denoting distributional pairings on $\partial\Omega$):

$$
\begin{aligned}
\Big\langle \partial_{\tau_{jk}}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big), \psi\Big\rangle &= -\int_{\partial\Omega}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\partial_{\tau_{jk}}\psi\,\mathrm{d}\sigma \\
&= -\lim_{\ell\to\infty}\int_{\partial\Omega}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\partial_{\tau_{jk}}\varphi_\ell\,\mathrm{d}\sigma \\
&= -\lim_{\ell\to\infty}\int_{\partial\Omega}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\big\{\nu_j(\partial_k\varphi_\ell)-\nu_k(\partial_j\varphi_\ell)\big\}\,\mathrm{d}\sigma \\
&= -\lim_{\ell\to\infty}\int_{\partial\Omega}\nu\cdot\big(\vec{F}_\ell\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma \\
&= -\lim_{\ell\to\infty}\int_{\Omega}\operatorname{div}\vec{F}_\ell\,\mathrm{d}\mathcal{L}^n \\
&= -\lim_{\ell\to\infty}\int_{\Omega}\big\{(\partial_j u)(\partial_k\varphi_\ell)-(\partial_k u)(\partial_j\varphi_\ell)\big\}\,\mathrm{d}\mathcal{L}^n \\
&= -\int_{\Omega}\big\{(\partial_j u)(\partial_k\Psi_\ell)-(\partial_k u)(\partial_j\Psi_\ell)\big\}\,\mathrm{d}\mathcal{L}^n \\
&= \big\langle \dot{\partial}_{\tau_{jk}}u, \psi\big\rangle.
\end{aligned}
\tag{2.2.239}
$$

Above, the first equality comes from [69, Definition 11.2.1], while the second equality is based on (2.2.232) and (2.2.225). The third equality in (2.2.239) is implied by (A.0.183), and the fourth equality in (2.2.239) is a consequence of the property recorded in the first line of (2.2.236). The fifth equality in (2.2.239) is a consequence of [68, Proposition 2.8.17], whose applicability in the current setting is guaranteed by the assumptions on Ω, together with (2.2.234) and the fact that $(np)/(n-1) > 1$ (see (2.2.219)), as well as (2.2.235). The sixth equality in (2.2.239) is seen from (2.2.236), while the seventh equality in (2.2.239) is deduced from (2.2.232) and (2.2.229) (again, bearing in mind that $(np)/(n-1) > 1$ thanks to (2.2.219)). The final equality in (2.2.239) follows from (2.2.231) and [69, Example 10.2.2].

Having established (2.2.239), in view of the arbitrariness of the test function $\psi \in \mathrm{Lip}_c(\partial\Omega)$ we conclude that

$$
\partial_{\tau_{jk}}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) = \dot{\partial}_{\tau_{jk}}u \ \text{ in } \big(\mathrm{Lip}_c(\partial\Omega)\big)'. \tag{2.2.240}
$$

From this and (2.2.238) we may now conclude that (2.2.217) holds. Finally, the fact that $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ belongs to the homogeneous Hardy-based Sobolev space $\dot{H}^p_1(\partial\Omega,\sigma)$ and satisfies (2.2.218) then becomes a consequence of (2.2.217), (2.2.238), (2.2.237), and [69, Definition 11.10.5] (cf. (A.0.89)-(A.0.90)). □

2.3 Integral Operators of Layer Potential Type on Hardy-Based Sobolev Spaces and BMO_{-1}

In the theorem below we shall augment Theorem 1.5.1 and Theorem 2.2.6, by including results involving the scale of Hardy-based Sobolev spaces and BMO_{-1}.

Theorem 2.3.1 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. For $M \in \mathbb{N}$ given, consider a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in (1.3.2) is weakly elliptic (in the sense of* [70, (1.3.3) in Definition 1.3.1]*). Finally, fix an aperture parameter $\kappa \in (0, \infty)$ along with an exponent*

$$p \in \big(\tfrac{n-1}{n}, 1\big]. \tag{2.3.1}$$

Then the following properties hold.

(i) Recall the modified boundary-to-domain double layer potential operator $\mathcal{D}_{\text{mod}}$ acting on functions from the space $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ as in (1.8.6). Then for each $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[\dot{H}_1^p(\partial\Omega, \sigma)\big]^M$, $\gamma \in \{1, \dots, M\}$, $\ell \in \{1, \dots, n\}$, and $x \in \Omega$ one has (using the summation convention)

$$\partial_\ell\big(\mathcal{D}_{\text{mod}} f\big)_\gamma(x) = \begin{cases} -a_{rs}^{\beta\alpha}\Big\langle(\partial_r E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}, \partial_{\tau_{\ell s}} f_\alpha\Big\rangle & \text{if } \partial\Omega \text{ bounded,} \\ -a_{rs}^{\beta\alpha}\Big\langle\big[(\partial_r E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}\big], \partial_{\tau_{\ell s}} f_\alpha\Big\rangle & \text{if } \partial\Omega \text{ unbounded,} \end{cases} \tag{2.3.2}$$

where the pairings in (2.3.2) are understood in the sense of [69, Theorem 4.6.1] *(keeping in mind that each tangential derivative $\partial_{\tau_{\ell s}} f_\alpha$ belongs to the Hardy space $H^p(\partial\Omega, \sigma)$). As a consequence of (2.3.2),* [70, Theorem 1.4.2]*, and* [70, Corollary 2.5.4]*,*

$$\text{for each } f \in \big[\dot{H}_1^p(\partial\Omega, \sigma)\big]^M, \text{ the nontangential boundary trace } \big(\nabla\mathcal{D}_{\text{mod}} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists (in } \mathbb{C}^{M\cdot n}) \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.3.3}$$

Also, there exists some finite constant $C > 0$, depending only on $\partial\Omega$, A, n, κ, and p, such that for each $f \in \big[\dot{H}_1^p(\partial\Omega, \sigma)\big]^M$ one has

$$\big\|\mathcal{N}_\kappa(\nabla\mathcal{D}_{\text{mod}} f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[\dot{H}_1^p(\partial\Omega,\sigma)]^M}. \tag{2.3.4}$$

(ii) The following operator is well defined, linear, and bounded:

$$\begin{aligned} &\dot{\partial}_\nu^A\mathcal{D}_{\text{mod}} : \big[\dot{H}_1^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[H^p(\partial\Omega, \sigma)\big]^M \text{ defined as} \\ &\big(\dot{\partial}_\nu^A\mathcal{D}_{\text{mod}}\big)f := \dot{\partial}_\nu^A(\mathcal{D}_{\text{mod}} f) \text{ for each } f \in \big[\dot{H}_1^p(\partial\Omega, \sigma)\big]^M, \end{aligned} \tag{2.3.5}$$

where the weak conormal derivative is considered in the sense of [69, Definition 10.2.18]. *In turn, the operator* (2.3.5) *induces a bounded linear mapping*

$$\dot{\partial}^A_\nu \mathcal{D} : \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[H^p(\partial\Omega,\sigma)\big]^M \text{ for each } q \in [1,\infty). \tag{2.3.6}$$

The operators $\dot{\partial}^A_\nu \mathcal{D}$ *thus considered for various values of* $p \in \big(\frac{n-1}{n}, 1\big]$ *and* $q \in [1,\infty)$ *are compatible with another. In addition,*

having fixed integrability exponents $p_0, q_0 \in (1,\infty)$ *and* $q \in [1,\infty)$*, it follows that for each* $f \in \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M \cap \big[L^{p_0,q_0}_1(\partial\Omega,\sigma)\big]^M$ *the function* $\partial^A_\nu \mathcal{D} f \in \big[L^{q_0}(\partial\Omega,\sigma)\big]^M$ *with* $\partial^A_\nu \mathcal{D}$ *as in* (1.5.31) *induces the same distribution on* $\partial\Omega$ *as* $\dot{\partial}^A_\nu \mathcal{D} f \in \big[H^p(\partial\Omega,\sigma)\big]^M$ *with* $\dot{\partial}^A_\nu \mathcal{D}$ *as in* (2.3.6). (2.3.7)

Moreover,

if $\Omega \subseteq \mathbb{R}^n$ *is an open set satisfying a two-sided local John condition whose boundary is compact and Ahlfors regular and* $q \in (1,\infty)$*, then the mapping* $\dot{\partial}^A_\nu \mathcal{D} : \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M \to \big[H^p(\partial\Omega,\sigma)\big]^M \hookrightarrow \big[\mathrm{Lip}\,(\partial\Omega)'\big]^M$ *induced by the operator* (2.3.6) *composed with the inclusion* [69, (4.2.8)] *is compatible with* $\partial^A_\nu \mathcal{D} : \big[L^q(\partial\Omega,\sigma)\big]^M \to \big[L^q_{-1}(\partial\Omega,\sigma)\big]^M \hookrightarrow \big[\mathrm{Lip}\,(\partial\Omega)'\big]^M$*, the mapping induced by the operator* (1.5.33) *composed with the inclusion* [69, (11.8.15)].
(2.3.8)

(iii) The boundary-to-boundary double layer potential operator K *from* (1.5.24) *extends to a linear and bounded mapping*

$$K : \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M \text{ for each } q \in (1,\infty). \tag{2.3.9}$$

(iv) The boundary-to-boundary single layer potential operator from Theorem 2.2.6 *induces a linear and bounded mapping*

$$S : \big[H^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^M. \tag{2.3.10}$$

Also, if $T^{\#}_{jk}$ *is the operator in* (2.1.42) *corresponding to* $b := E$ *(the matrix-valued fundamental solution associated with* L *as in* [70, Theorem 1.4.2]) *then for each* $j,k \in \{1,\dots,n\}$ *one has*

$$\partial_{\tau_{jk}}(Sf) = T^{\#}_{jk} f \text{ for each } f \in \big[H^p(\partial\Omega,\sigma)\big]^M. \tag{2.3.11}$$

Finally, the operator S *from* (2.2.116) *induces a linear and bounded mapping*

$$S : \big[H^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[H^{p^*,p}_1(\partial\Omega,\sigma)\big]^M \quad \text{where } p^* := \Big(\frac{1}{p} - \frac{1}{n-1}\Big)^{-1} \in \Big(1, \frac{n-1}{n-2}\Big]. \tag{2.3.12}$$

(v) *Assume $n \geq 3$. Then the (real) transpose of the operator (2.3.12) corresponding to $p := 1$, i.e.,*

$$S : \left[H^1(\partial\Omega, \sigma)\right]^M \longrightarrow \left[H_1^{\frac{n-1}{n-2},1}(\partial\Omega, \sigma)\right]^M \tag{2.3.13}$$

is the linear and bounded mapping

$$S^\top : \left[\mathrm{BMO}_{-1}(\partial\Omega, \sigma)\right]^M \longrightarrow \begin{cases} \left[\mathrm{BMO}(\partial\Omega, \sigma)\right]^M & \text{if } \partial\Omega \text{ is bounded,} \\ \left[\mathrm{BMO}(\partial\Omega, \sigma)/\sim\right]^M & \text{if } \partial\Omega \text{ is unbounded.} \end{cases} \tag{2.3.14}$$

Also, the (real) transpose of the operator (2.3.9) corresponding to $p := 1$ and $q := \frac{n-1}{n-2}$, i.e.,

$$K : \left[H_1^{\frac{n-1}{n-2},1}(\partial\Omega, \sigma)\right]^M \longrightarrow \left[H_1^{\frac{n-1}{n-2},1}(\partial\Omega, \sigma)\right]^M, \tag{2.3.15}$$

is the linear and bounded mapping

$$K^\top : \left[\mathrm{BMO}_{-1}(\partial\Omega, \sigma)\right]^M \longrightarrow \left[\mathrm{BMO}_{-1}(\partial\Omega, \sigma)\right]^M. \tag{2.3.16}$$

This mapping is compatible with $K^\#$, in the sense that

$$\begin{aligned} &K^\top f = K^\# f \ \text{ as functionals in } \left[\mathrm{BMO}_{-1}(\partial\Omega, \sigma)\right]^M \\ &\text{for each } f \in \left[L^{n-1}(\partial\Omega, \sigma)\right]^M \hookrightarrow \left[\mathrm{BMO}_{-1}(\partial\Omega, \sigma)\right]^M. \end{aligned} \tag{2.3.17}$$

In addition, $\left[\mathrm{VMO}_{-1}(\partial\Omega, \sigma)\right]^M$ is an invariant subspace of $K^\top$ in (2.3.16), hence

$$K^\top : \left[\mathrm{VMO}_{-1}(\partial\Omega, \sigma)\right]^M \longrightarrow \left[\mathrm{VMO}_{-1}(\partial\Omega, \sigma)\right]^M \tag{2.3.18}$$

is a well-defined, linear, and bounded mapping.

(vi) *Continue to assume $n \geq 3$ and, this time, also make the additional assumption that $\partial\Omega$ is compact. Then the boundary-to-domain single layer potential operator associated with L and Ω induces a well-defined linear mapping in the context*

$$\mathscr{S} : \left[\mathrm{BMO}_{-1}(\partial\Omega, \sigma)\right]^M \longrightarrow \left[\mathscr{C}^\infty(\Omega)\right]^M \tag{2.3.19}$$

if for each $f = (f_\beta)_{1 \leq \beta \leq M} \in \left[\mathrm{BMO}_{-1}(\partial\Omega, \sigma)\right]^M$ one sets

$$\mathscr{S} f(x) := \Big(\big\langle E_{\alpha\beta}(x - \cdot)\big|_{\partial\Omega}, f_\beta\big\rangle\Big)_{1 \leq \alpha \leq M} \ \text{ for all } x \in \Omega. \tag{2.3.20}$$

In formula (2.3.20) above, $E = (E_{\alpha\beta})_{1 \leq \alpha,\beta \leq M}$ is the matrix-valued fundamental solution associated with the system L as in [70, Theorem 1.4.2]*, and*

$\langle\cdot,\cdot\rangle$ *stands for the duality bracket between the space* $H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)$ *and the space* $\mathrm{BMO}_{-1}(\partial\Omega,\sigma)=\Big(H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)\Big)^*$ (*cf.* [69, Definition 11.10.9]; *see also* (A.0.32)). *In addition, for each* $f\in\big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$ *one has*

$$L_A\big(\mathscr{S}f\big)=0 \ \text{ in } \ \Omega, \tag{2.3.21}$$

and for each $p\in(1,\infty)$ *there exists some constant* $C=C(\Omega,L,p)\in(0,\infty)$ *such that*

$$\begin{array}{l}|\nabla\mathscr{S}f|^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n \text{ is a Carleson measure in}\\ \text{the set } \Omega \text{ with constant} \leq C\|f\|^p_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}.\end{array} \tag{2.3.22}$$

In particular, corresponding to $p=2$, *it follows that*[7]

$$\begin{array}{l}|\nabla\mathscr{S}f|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,d\mathcal{L}^n \text{ is a Carleson measure in } \Omega \text{ with constant}\\ \leq C\|f\|^2_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}, \text{ for each } f\in\big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M.\end{array} \tag{2.3.23}$$

Also, for each given $f\in\big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$, *the boundary trace*

$$\begin{array}{l}\mathscr{S}f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \text{ and, in fact,}\\ \Big(\mathscr{S}f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x)=\big((S_{L^\top})^\top f\big)(x) \text{ at } \sigma\text{-a.e. point } x\in\partial\Omega,\end{array} \tag{2.3.24}$$

where $S_{L^\top}$ *is the boundary-to-boundary single layer potential operator associated with* $L^\top$, *considered* (*cf.* (2.3.13)) *as a mapping from* $\big[H^1(\partial\Omega,\sigma)\big]^M$ *into* $\big[H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)\big]^M$, *and* $(S_{L^\top})^\top$ *stands for its* (*real*) *transpose* (*cf.* (2.3.14)). *Finally, for each* $f\in\big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$ *one has*

$$\lim_{R\to0^+}\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathscr{S}f\big)\big|^p\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{\frac{1}{p}}$$
$$\leq C\,\mathrm{dist}\big(f,[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)]^M\big) \tag{2.3.25}$$

where the distance in the right-hand side is considered in $[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M$. *As a consequence,*

$$\begin{array}{l}|\nabla\mathscr{S}f|^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n \text{ is a vanishing Carleson measure}\\ \text{in the set } \Omega \text{ for each } f\in\big[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)\big]^M \text{ and } p\in(1,\infty),\end{array} \tag{2.3.26}$$

hence, corresponding to $p=2$,

$$\begin{array}{l}|\nabla\mathscr{S}f|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,d\mathcal{L}^n \text{ is a vanishing Carleson}\\ \text{measure in } \Omega \text{ for each } f\in\big[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)\big]^M.\end{array} \tag{2.3.27}$$

[7] it is natural to think of $\big|\nabla\mathscr{S}f\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,d\mathcal{L}^n$ as the Littlewood-Paley measure associated with f via the operator $\mathscr{S}$

(vii) Strengthen the original hypotheses by assuming that $n \geq 3$ and Ω is an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. Then $S^\top$ from (2.3.14) is compatible with $S_{L^\top}$, the single layer potential operator associated with $L^\top$ and Ω as in (1.3.62), in the sense that

$$\begin{aligned} &S^\top f = S_{L^\top} f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \text{ for each} \\ &f \in \big[L^{n-1}(\partial\Omega,\sigma)\big]^M \hookrightarrow \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M. \end{aligned} \tag{2.3.28}$$

Moreover, the restriction of the operator (2.3.14) to $\big[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)\big]^M$ induces a well-defined, linear, and bounded mapping

$$S^\top : \big[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M. \tag{2.3.29}$$

Parenthetically, we wish to note that (2.2.139), [69, (4.2.17)], and (2.3.10) offer an alternative proof of the result recorded in [69, (11.10.66)], corresponding to (special, matrix-valued case) $b := E$ and $n \geq 3$.

Proof of Theorem 2.3.1 Fix some $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^M$. Also, pick some $\gamma \in \{1,\dots,M\}$, $\ell \in \{1,\dots,n\}$, and select an arbitrary point $x \in \Omega$. Since $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^n}\Big)\Big]^M$, it follows that $\mathcal{D}_{\rm mod} f$ is meaningfully defined, and we may use (1.8.11) together with the first two lines in (1.3.31) to write (using the summation convention over repeated indices)

$$\partial_\ell\big(\mathcal{D}_{\rm mod} f\big)_\gamma(x) = -\int_{\partial\Omega} a^{\beta\alpha}_{rs}\, \partial_{\tau_{\ell s}(y)}[(\partial_r E_{\gamma\beta})(x-y)] f_\alpha(y)\,\mathrm{d}\sigma(y). \tag{2.3.30}$$

To proceed, choose $\varepsilon \in \big(0, \mathrm{dist}(x,\partial\Omega)\big)$, set $\mathcal{U}_\varepsilon := \big\{y \in \mathbb{R}^n : \mathrm{dist}(y,\partial\Omega) < \varepsilon\big\}$ and fix an index $\beta \in \{1,\dots,M\}$ along with $r \in \{1,\dots,n\}$. Then the function $\varphi(y) := (\partial_r E_{\gamma\beta})(x-y)$ for each $y \in \mathcal{U}_\varepsilon$, satisfies

$$\begin{aligned} &\varphi \in \mathscr{C}^1(\mathcal{U}_\varepsilon) \ \text{ and there exists } C \in (0,\infty) \text{ such that} \\ &|\varphi(y)| \leq \frac{C}{1+|y|^{n-1}} \ \text{ and } \ |(\nabla\varphi)(y)| \leq \frac{C}{1+|y|^n} \ \text{ for all } \ y \in \mathcal{U}_\varepsilon. \end{aligned} \tag{2.3.31}$$

Then (2.3.31) ensures the applicability of [69, Lemma 11.10.4] for the current choice of φ which, for each $s \in \{1,\dots,n\}$, allows us to write

$$\begin{aligned} &\int_{\partial\Omega} \partial_{\tau_{\ell s}(y)}[(\partial_r E_{\gamma\beta})(x-y)] f_\alpha(y)\,\mathrm{d}\sigma(y) \\ &\qquad = \begin{cases} -\Big\langle (\partial_r E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}, \partial_{\tau_{\ell s}} f_\alpha \Big\rangle & \text{if } \partial\Omega \text{ bounded,} \\ -\Big\langle \big[(\partial_r E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}\big], \partial_{\tau_{\ell s}} f_\alpha \Big\rangle & \text{if } \partial\Omega \text{ unbounded,} \end{cases} \end{aligned} \tag{2.3.32}$$

where the pairings in (2.3.32) are understood in the sense of [69, Theorem 4.6.1] (keeping in mind that $\partial_{\tau_{\ell s}} f_\alpha \in H^p(\partial\Omega, \sigma)$). Collectively, (2.3.30) and (2.3.32) prove (2.3.2). Since the pairings in the right-hand side of (2.3.2) are of the form [70, (2.4.16)] (corresponding to the kernel $k := \partial_r E_{\gamma\beta}$) and since $\partial_{\tau_{\ell s}} f_\alpha \in H^p(\partial\Omega, \sigma)$, we may apply [70, (2.4.14)] in [70, Theorem 2.4.1] to conclude that

$$\big\|\mathcal{N}_\kappa(\partial_\ell \mathcal{D}_{\text{mod}} f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C \sum_{\alpha=1}^{M} \sum_{s=1}^{n} \|\partial_{\tau_{\ell s}} f_\alpha\|_{H^p(\partial\Omega,\sigma)} \tag{2.3.33}$$

for some finite $C > 0$ independent of f. This establishes (2.3.4) and finishes the treatment of the claims made in item *(i)*.

Moving on to item *(ii)*, the claims about the operator (2.3.5) are clear from (A.0.89), (1.3.24), (2.3.4), and [69, Theorem 10.2.24]. Collectively, these imply that for each $f \in \big[\dot{H}^p_1(\partial\Omega, \sigma)\big]^M$, the function $u := \mathcal{D} f$ belongs to $\big[\mathscr{C}^\infty(\Omega)\big]^M$, and satisfies $L_A u = 0$ in Ω as well as $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma)$. As such, the weak conormal derivative $\dot{\partial}^A_\nu u$, considered in the sense of [69, Definition 10.2.18], satisfies

$$\dot{\partial}^A_\nu u \in \big[H^p(\partial\Omega, \sigma)\big]^M \text{ and } \big\|\dot{\partial}^A_\nu u\big\|_{[H^p(\partial\Omega,\sigma)]^M} \leq C\big\|\mathcal{N}_\kappa(\nabla u)\big\|_{L^p(\partial\Omega,\sigma)} \tag{2.3.34}$$

for some constant $C = C(\Omega, A, \kappa, p) \in (0, \infty)$. The estimate in (2.3.34) in combination with (2.3.4) then yields

$$\big\|\dot{\partial}^A_\nu(\mathcal{D}_{\text{mod}} f)\big\|_{[H^p(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M}, \tag{2.3.35}$$

as desired.

Next, the fact that the operator (2.3.6) is well defined, linear, and bounded is seen from what we have proved so far and (1.8.8). Also, from (A.0.181)-(A.0.182) and (A.0.165)-(A.0.166) we see that for each $f \in \big[H^{q,p}_1(\partial\Omega, \sigma)\big]^M$ and $\Phi \in \mathrm{Lip}_c(\mathbb{R}^n)$ we have

$$_{[\mathrm{Lip}_c(\partial\Omega)']^M}\Big\langle \dot{\partial}^A_\nu \mathcal{D} f, \Phi\big|_{\partial\Omega}\Big\rangle_{[\mathrm{Lip}_c(\partial\Omega)]^M} = \int_\Omega \big\langle A\nabla(\mathcal{D} f), \nabla\Phi\big\rangle \, d\mathcal{L}^n. \tag{2.3.36}$$

From this it is clear that the mappings induced by $\dot{\partial}^A_\nu \mathcal{D}$ in the context of (2.3.6) corresponding to various values of $p \in \big(\frac{n-1}{n}, 1\big]$ and $q \in [1, \infty)$ are compatible with another.

Let us now check the compatibility claim made in (2.3.7). With this goal in mind, fix an arbitrary function $f \in \big[H^{q,p}_1(\partial\Omega, \sigma)\big]^M \cap \big[L^{p_0,q_0}_1(\partial\Omega, \sigma)\big]^M$ for some $p_0, q_0 \in (1, \infty)$. Then from [69, Definitions 11.1.2 and 11.10.6] it follows that $\partial_{\tau_{jk}} f$ belongs to $\big[H^p(\partial\Omega, \sigma)\big]^M \cap \big[L^{q_0}(\partial\Omega, \sigma)\big]^M$ for each indices $j, k \in \{1, \ldots, n\}$. Since $L^{q_0}(\partial\Omega, \sigma) = H^{q_0}(\partial\Omega, \sigma)$ (recall that $q_0 > 1$), [69, Proposition 4.2.2] implies $\partial_{\tau_{jk}} f \in \big[H^1(\partial\Omega, \sigma)\big]^M$ for each $j, k \in \{1, \ldots, n\}$. Hence $f \in \big[H^{q,1}_1(\partial\Omega, \sigma)\big]^M$, Since the mappings induced by $\dot{\partial}^A_\nu \mathcal{D}$ in the context of (2.3.6) are compatible with one another, henceforth we shall consider $\dot{\partial}^A_\nu \mathcal{D} f$ regarding f as a function in

$\left[H_1^{q,1}(\partial\Omega,\sigma)\right]^M$. Then item *(i)* in the current theorem (applied with $p = 1$) together with (1.8.8) give

$$\mathcal{N}_\kappa(\nabla\mathcal{D}f) \in L^1(\partial\Omega,\sigma). \tag{2.3.37}$$

Moreover, [69, (4.2.10)] guarantees that $\partial_{\tau_{jk}} f$ belongs to $\left[L^1(\partial\Omega,\sigma)\right]^M$ for each $j,k \in \{1,\dots,n\}$, thus $f \in \left[L_1^{q,1}(\partial\Omega,\sigma)\right]^M$. As such, we may apply item *(ii)* in Theorem 1.5.1 (with $p := q$, $q := 1$, and keeping in mind that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ in our setting; cf. the definition of a UR domain from [68, Definition 5.10.6]) to obtain that

$$(\nabla\mathcal{D}f)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.3.38}$$

Having established (2.3.37) and (2.3.38), we may invoke item *(3)* in [69, Theorem 10.2.24] to conclude that $\dot{\partial}_\nu^A(\mathcal{D}f)$, considered in the sense of [69, Definition 10.2.18], actually belongs to $\left[L^1(\partial\Omega,\sigma)\right]^{n\cdot M}$ and has the pointwise formula

$$\dot{\partial}_\nu^A(\mathcal{D}f) = \Big(\nu_r a_{rs}^{\alpha\beta}\big(\partial_s u_\beta\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)_{1\leq\alpha\leq M} \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.3.39}$$

The desired conclusion now follows upon observing that the right-hand side of (2.3.39) is actually $(\partial_\nu^A\mathcal{D})f$, with the operator $\partial_\nu^A\mathcal{D}$ as in (1.5.31) (see item *(vii)* in Theorem 1.5.1).

Let us now justify the compatibility claim made in (2.3.8). To this end, work under the assumption that $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. Also, fix $q \in (1,\infty)$ and pick $f \in \left[H_1^{q,p}(\partial\Omega,\sigma)\right]^M \hookrightarrow \left[L^q(\partial\Omega,\sigma)\right]^M$ arbitrary. The goal is to show that

$$\dot{\partial}_\nu^A\mathcal{D}f = \partial_\nu^A\mathcal{D}f \text{ in } \left[\mathrm{Lip}\,(\partial\Omega)'\right]^M, \tag{2.3.40}$$

where we have considered $\dot{\partial}_\nu^A\mathcal{D}f \in \left[H^p(\partial\Omega,\sigma)\right]^M \hookrightarrow \left[\mathrm{Lip}\,(\partial\Omega)'\right]^M$ via (2.3.6) and [69, (4.2.8)], while $\partial_\nu^A\mathcal{D}f \in \left[L_{-1}^q(\partial\Omega,\sigma)\right]^M \hookrightarrow \left[\mathrm{Lip}\,(\partial\Omega)'\right]^M$ via (1.5.33) and [69, (11.8.15)]. Pick an arbitrary vector-valued test function $\phi \in \left[\mathrm{Lip}\,(\partial\Omega)\right]^M$, with the aim of proving that

$${}_{[\mathrm{Lip}\,(\partial\Omega)']^M}\big\langle\dot{\partial}_\nu^A\mathcal{D}f,\phi\big\rangle_{[\mathrm{Lip}\,(\partial\Omega)]^M} = {}_{[\mathrm{Lip}\,(\partial\Omega)']^M}\big\langle\partial_\nu^A\mathcal{D}f,\phi\big\rangle_{[\mathrm{Lip}\,(\partial\Omega)]^M}. \tag{2.3.41}$$

To this end, we distinguish two cases. First, suppose $p \in \left(\frac{n-1}{n},1\right)$ and define

$$\eta := (n-1)\left(\tfrac{1}{p}-1\right) \in (0,1). \tag{2.3.42}$$

Bring in the sequence $\{\varphi_\ell\}_{\ell\in\mathbb{N}} \subseteq \left[\mathscr{C}_c^\infty(\mathbb{R}^n)\right]^M$ associated with the scalar components of the function $f := \phi \in \left[L_1^\infty(\partial\Omega,\sigma)\right]^M$ as in [69, Remark 11.1.13]. Then [69, (11.1.92)-(11.1.94)] guarantee that

$$\text{there exists a compact set } K \subset \mathbb{R}^n \text{ such that } \operatorname{supp}\varphi_\ell \subseteq K \text{ for all } \ell \in \mathbb{N}, \tag{2.3.43}$$

and the functions $\psi_\ell := \varphi_\ell\big|_{\partial\Omega}$, for each $\ell \in \mathbb{N}$, satisfy the following properties:

$$\lim_{\ell\to\infty}\Big\{\sup_{x\in\partial\Omega}\big|\psi_\ell(x)-\phi(x)\big|\Big\}=0, \tag{2.3.44}$$

$$\sup_{\ell\in\mathbb{N}}\|\psi_\ell\|_{[\mathscr{C}^\alpha(\partial\Omega)]^M}<+\infty \text{ for each } \alpha\in(0,1), \tag{2.3.45}$$

and

$$\begin{gathered}\partial_{\tau_{jk}}\psi_\ell\longrightarrow\partial_{\tau_{jk}}\phi \text{ weak-}* \text{ in } \big[L^\infty(\partial\Omega,\sigma)\big]^M \text{ as } \ell\to\infty,\\ \text{for each pair of indices } j,k\in\{1,\dots,n\}.\end{gathered} \tag{2.3.46}$$

As a consequence of (2.3.44), (2.3.45), and [69, (4.8.29) in Lemma 4.8.4] we also have

$$\lim_{\ell\to\infty}\psi_\ell=\phi \text{ weak-}* \text{ in } \big[\mathscr{C}^\eta(\partial\Omega)\big]^M, \tag{2.3.47}$$

whereas (2.3.44) implies

$$\lim_{\ell\to\infty}\psi_\ell=\varphi \text{ in } \big[L^\infty(\partial\Omega,\sigma)\big]^M, \text{ hence also in } \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M. \tag{2.3.48}$$

Finally, as a consequence of (2.3.43), (2.3.44), (2.3.46), and [69, Lemma 11.8.3] we have

$$\lim_{\ell\to\infty}\psi_\ell=\phi \text{ weak-}* \text{ in } \big[L^q_1(\partial\Omega,\sigma)\big]^M. \tag{2.3.49}$$

To get going, write

$$\begin{aligned}{}_{[\mathrm{Lip}(\partial\Omega)']^M}\big\langle\dot{\partial}^A_\nu\mathcal{D}f,\phi\big\rangle_{[\mathrm{Lip}(\partial\Omega)]^M}&={}_{[H^p(\partial\Omega,\sigma)]^M}\big\langle\dot{\partial}^A_\nu\mathcal{D}f,\phi\big\rangle_{[\mathscr{C}^\eta(\partial\Omega)]^M}\\&=\lim_{\ell\to\infty}{}_{[H^p(\partial\Omega,\sigma)]^M}\big\langle\dot{\partial}^A_\nu\mathcal{D}f,\psi_\ell\big\rangle_{[\mathscr{C}^\eta(\partial\Omega)]^M}\\&=\lim_{\ell\to\infty}{}_{[\mathrm{Lip}(\partial\Omega)']^M}\big\langle\dot{\partial}^A_\nu\mathcal{D}f,\psi_\ell\big\rangle_{[\mathrm{Lip}(\partial\Omega)]^M},\end{aligned} \tag{2.3.50}$$

with the first and last equalities coming from [69, (4.6.31)] (bearing in mind (2.3.6)), and the second equality implied by (2.3.47).

Let us now fix $\ell\in\mathbb{N}$. From (A.0.181)-(A.0.182) and (A.0.165)-(A.0.166) we see that

$$_{[\mathrm{Lip}(\partial\Omega)']^M}\big\langle\dot{\partial}^A_\nu\mathcal{D}f,\psi_\ell\big\rangle_{[\mathrm{Lip}_c(\partial\Omega)]^M}=\int_\Omega\big\langle A\nabla(\mathcal{D}f),\nabla\varphi_\ell\big\rangle\,\mathrm{d}\mathcal{L}^n. \tag{2.3.51}$$

Also,

$$\begin{aligned}\int_\Omega\big\langle A\nabla(\mathcal{D}f),\nabla\varphi_\ell\big\rangle\,\mathrm{d}\mathcal{L}^n&=\int_\Omega\big\langle A^\top\nabla\varphi_\ell,\nabla(\mathcal{D}f)\big\rangle\,\mathrm{d}\mathcal{L}^n\\&=\int_{\partial\Omega}\big\langle\partial^{A^\top}_\nu\varphi_\ell,\big(\tfrac12 I+K\big)f\big\rangle\,\mathrm{d}\sigma-\int_\Omega\big\langle L^\top\varphi_\ell,\mathcal{D}f\big\rangle\,\mathrm{d}\mathcal{L}^n\end{aligned} \tag{2.3.52}$$

by the Green type formula [70, (1.7.121)], used with $A^\top$ in place of A, with $u := \varphi_\ell\big|_\Omega$ and $w := \mathcal{D}f$. The fact that $f \in \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M$ implies, in view of (1.5.1), (2.3.4), and (1.8.8),

$$\mathcal{N}_\kappa(\mathcal{D}f) \in L^q(\partial\Omega,\sigma) \ \text{ and } \ \mathcal{N}_\kappa(\nabla\mathcal{D}f) \in L^p(\partial\Omega,\sigma). \tag{2.3.53}$$

In concert with [68, (8.6.51)] and the fact that $p > \frac{n-1}{n}$, the latter membership further entails (with K as in (2.3.43))

$$\nabla\mathcal{D}f \in \big[L^{np/(n-1)}(\Omega\cap K,\mathcal{L}^n)\big]^{nM} \hookrightarrow \big[L^1(\Omega\cap K,\mathcal{L}^n)\big]^{nM}. \tag{2.3.54}$$

Also, from (1.5.20) we know that for our choice of w we have $w\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\frac12 I + K\big)f$ at σ-a.e. point on $\partial\Omega$, for any aperture parameter $\kappa \in (0,\infty)$. These properties ensure that the hypotheses of [70, Theorem 1.7.12] are presently satisfied, so (2.3.52) is justified. Combining (2.3.51)-(2.3.52) leads to the conclusion that

$$\begin{aligned} {}_{[\mathrm{Lip}(\partial\Omega)']^M}\big\langle \dot{\partial}^A_\nu \mathcal{D}f, \psi_\ell\big\rangle_{[\mathrm{Lip}_c(\partial\Omega)]^M} &= \int_{\partial\Omega} \big\langle \partial^{A^\top}_\nu \varphi_\ell, \big(\tfrac12 I + K\big)f\big\rangle\,\mathrm{d}\sigma \\ &\quad - \int_\Omega \big\langle L^\top\varphi_\ell, \mathcal{D}f\big\rangle\,\mathrm{d}\mathcal{L}^n. \end{aligned} \tag{2.3.55}$$

Going further, continue to keep $\ell \in \mathbb{N}$ arbitrary and fixed. We make the claim that for each function $g \in \big[L^q(\partial\Omega,\sigma)\big]^M$ we have

$$\begin{aligned} \int_{\partial\Omega} \big\langle \partial^{A^\top}_\nu \varphi_\ell, \big(\tfrac12 I + K\big)g\big\rangle\,\mathrm{d}\sigma - \int_\Omega \big\langle L^\top\varphi_\ell, \mathcal{D}g\big\rangle\,\mathrm{d}\mathcal{L}^n. \\ = \int_{\partial\Omega} \big\langle g, \partial^{A^\top}_\nu \mathcal{D}_{A^\top}\psi_\ell\big\rangle\,\mathrm{d}\sigma. \end{aligned} \tag{2.3.56}$$

The justification of this claim rests on two observations. The first observation we make is that both sides of (2.3.56) depend linearly and continuously on the function $g \in \big[L^q(\partial\Omega,\sigma)\big]^M$. Indeed, this is apparent from definitions, the continuity of K on the space $\big[L^q(\partial\Omega,\sigma)\big]^M$ (cf. item *(iii)* in Theorem 1.5.1), the fact that (1.5.1) and [68, (8.6.51)] imply the continuity of

$$\mathcal{D} : \big[L^q(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^1(\Omega\cap K,\mathcal{L}^n)\big]^M, \tag{2.3.57}$$

and the fact that $\partial^{A^\top}_\nu \mathcal{D}_{A^\top}\psi_\ell \in \big[L^{q'}(\partial\Omega,\sigma)\big]^M$ where $q' := \big(1-\frac1q\big)^{-1} \in (1,\infty)$ (cf. [69, (11.1.81)] and (1.5.31)).

The second observation we wish to make in relation to (2.3.56) is that said formula is true whenever $g \in \big[L^q_1(\partial\Omega,\sigma)\big]^M$. To see this, note that since we have the inclusion $\big[L^q_1(\partial\Omega,\sigma)\big]^M \subseteq \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M$ (as is apparent from [68, (3.6.27)], [69, (4.2.21)], and definitions), the same argument that led to formula (2.3.55) presently shows that, for each $g \in \big[L^q_1(\partial\Omega,\sigma)\big]^M$, we have

$$\int_{\partial\Omega} \big\langle \partial^{A^\top}_\nu \varphi_\ell, \big(\tfrac{1}{2}I + K\big)g\big\rangle \, d\sigma - \int_\Omega \big\langle L^\top \varphi_\ell, \mathcal{D}g\big\rangle \, d\mathcal{L}^n$$
$$= {}_{[\mathrm{Lip}(\partial\Omega)']^M}\big\langle \dot{\partial}^A_\nu \mathcal{D}g, \psi_\ell\big\rangle_{[\mathrm{Lip}_c(\partial\Omega)]^M}. \tag{2.3.58}$$

Knowing that $g \in \big[L^q_1(\partial\Omega,\sigma)\big]^M$ allows us to replace $\dot{\partial}^A_\nu \mathcal{D}g$ by $\partial^A_\nu \mathcal{D}g$ (see the compatibility result in (2.3.7), here used with $p_0 = q_0 = q$). Hence,

$$\int_{\partial\Omega} \big\langle \partial^{A^\top}_\nu \varphi_\ell, \big(\tfrac{1}{2}I + K\big)g\big\rangle \, d\sigma - \int_\Omega \big\langle L^\top \varphi_\ell, \mathcal{D}g\big\rangle \, d\mathcal{L}^n$$
$$= {}_{[\mathrm{Lip}(\partial\Omega)']^M}\big\langle \partial^A_\nu \mathcal{D}g, \psi_\ell\big\rangle_{[\mathrm{Lip}_c(\partial\Omega)]^M} = \int_{\partial\Omega} \big\langle \partial^A_\nu \mathcal{D}g, \psi_\ell\big\rangle \, d\sigma$$
$$= \int_{\partial\Omega} \big\langle g, \partial^{A^\top}_\nu \mathcal{D}_{A^\top}\psi_\ell\big\rangle \, d\sigma, \tag{2.3.59}$$

where the second equality above is a consequence of [68, (4.1.47)], and the final equality in (2.3.59) is provided by (1.5.32) (bearing in mind that the function $\psi_\ell \in \big[L^{q'}_1(\partial\Omega,\sigma)\big]^M$; cf. [69, (11.1.81)]). Formula (2.3.59) justifies (2.3.56) in the case when g belongs to the space $\big[L^q_1(\partial\Omega,\sigma)\big]^M$. Recall from [69, (11.1.66)] that $\big[L^q_1(\partial\Omega,\sigma)\big]^M \hookrightarrow \big[L^q(\partial\Omega,\sigma)\big]^M$ densely. In view of this and the fact that, as already noted, both sides of (2.3.56) depend linearly and continuously on the function $g \in \big[L^q(\partial\Omega,\sigma)\big]^M$, we may ultimately conclude that (2.3.56) holds for each function $g \in \big[L^q(\partial\Omega,\sigma)\big]^M$.

Having established (2.3.56) for arbitrary functions $g \in \big[L^q(\partial\Omega,\sigma)\big]^M$, write (2.3.56) for $g := f$ then combine the resulting formula with (2.3.55) to deduce that, for each fixed $\ell \in \mathbb{N}$, we have

$${}_{[\mathrm{Lip}(\partial\Omega)']^M}\big\langle \dot{\partial}^A_\nu \mathcal{D}f, \psi_\ell\big\rangle_{[\mathrm{Lip}_c(\partial\Omega)]^M} = \int_{\partial\Omega} \big\langle f, \partial^{A^\top}_\nu \mathcal{D}_{A^\top}\psi_\ell\big\rangle \, d\sigma. \tag{2.3.60}$$

If we now pass to limit, as $\ell \to \infty$, and also bring into the mix (2.3.50), we arrive at the conclusion that, on the one hand,

$${}_{[\mathrm{Lip}\,(\partial\Omega)']^M}\big\langle \dot{\partial}^A_\nu \mathcal{D}f, \phi\big\rangle_{[\mathrm{Lip}\,(\partial\Omega)]^M} = \lim_{\ell\to\infty} \int_{\partial\Omega} \big\langle f, \partial^{A^\top}_\nu \mathcal{D}_{A^\top}\psi_\ell\big\rangle \, d\sigma. \tag{2.3.61}$$

On the other hand, for each fixed $\ell \in \mathbb{N}$, we may rely on (1.5.33)-(1.5.34) to write (viewing f simply as a function in $\big[L^q(\partial\Omega,\sigma)\big]^M$, and ψ_ℓ as a function in $\big[L^{q'}_1(\partial\Omega,\sigma)\big]^M$)

$$\int_{\partial\Omega} \big\langle f, \partial^{A^\top}_\nu \mathcal{D}_{A^\top}\psi_\ell\big\rangle \, d\sigma = {}_{[L^q_{-1}(\partial\Omega,\sigma)]^M}\big\langle \partial^A_\nu \mathcal{D}f, \psi_\ell\big\rangle_{[L^{q'}_1(\partial\Omega,\sigma)]^M} \tag{2.3.62}$$

where $\partial_\nu^A \mathcal{D}$ acts as in (1.5.33) (with $p := q$). Together, (2.3.61), (2.3.62), (2.3.49), and [69, (11.8.16)] then permit us to write

$$
\begin{aligned}
{}_{[\mathrm{Lip}(\partial\Omega)']^M}\big\langle \dot{\partial}_\nu^A \mathcal{D} f, \phi \big\rangle_{[\mathrm{Lip}(\partial\Omega)]^M} &= \lim_{\ell\to\infty} {}_{[L^q_{-1}(\partial\Omega,\sigma)]^M}\big\langle \partial_\nu^A \mathcal{D} f, \psi_\ell \big\rangle_{[L^{q'}_1(\partial\Omega,\sigma)]^M} \\
&= {}_{[L^q_{-1}(\partial\Omega,\sigma)]^M}\big\langle \partial_\nu^A \mathcal{D} f, \phi \big\rangle_{[L^{q'}_1(\partial\Omega,\sigma)]^M} \\
&= {}_{[\mathrm{Lip}(\partial\Omega)']^M}\big\langle \partial_\nu^A \mathcal{D} f, \phi \big\rangle_{[\mathrm{Lip}(\partial\Omega)]^M}. \qquad (2.3.63)
\end{aligned}
$$

This finishes the proof of (2.3.41), hence (2.3.40) is established in the case when $p < 1$. Finally, the case when $p = 1$ follows along very similar lines, now using the H^1-BMO duality in place of the H^p-$\mathscr{C}^\eta$ duality (cf. [69, Lemma 4.6.4]), and (2.3.48). Hence, (2.3.8) is justified for any p as in (2.3.1). This completes the proof of item *(ii)*.

Next, assume $p \in \big(\frac{n-1}{n}, 1\big]$ and $q \in (1, \infty)$ and take $f \in \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M$. By the current item *(i)* and (1.8.9) we have

$$
\begin{aligned}
&\mathcal{N}_\kappa\big(\nabla \mathcal{D} f\big) \in L^p(\partial\Omega,\sigma) \ \text{ and} \\
&\big\|\mathcal{N}_\kappa(\nabla \mathcal{D} f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M}.
\end{aligned} \qquad (2.3.64)
$$

Moreover,

$$
\mathcal{N}_\kappa(\mathcal{D} f) \in L^q(\partial\Omega,\sigma) \hookrightarrow L^1_{\mathrm{loc}}(\partial\Omega,\sigma) \ \text{ and} \qquad (2.3.65)
$$

$$
(\mathcal{D} f)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K\big)f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \qquad (2.3.66)
$$

thanks to item *(i)* in Theorem 1.5.1 (with $p := q$), and item *(iv)* in Theorem 1.5.1 (bearing in mind that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ in the current setting). Given that $\mathcal{D} f \in [\mathscr{C}^\infty(\Omega)]^M$, the properties listed in (2.3.64)-(2.3.66) ensure the applicability of [69, Theorem 11.10.10] which presently yields

$$
\partial_{\tau_{jk}}(Kf) \in H^p(\partial\Omega,\sigma) \ \text{ for each } j,k \in \{1,\dots,n\}, \qquad (2.3.67)
$$

and

$$
\sum_{j,k=1}^n \big\|\partial_{\tau_{jk}}(Kf)\big\|_{H^p(\partial\Omega,\sigma)} \leq C\|f\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M}. \qquad (2.3.68)
$$

Since from [70, (2.3.18)] we also know that K maps $\big[L^q(\partial\Omega,\sigma)\big]^M$ boundedly into itself, based on (2.3.67)-(2.3.68), (2.3.4), and [69, Definition 11.10.6] we ultimately conclude that

$$
\begin{aligned}
&Kf \in \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M \ \text{ and} \\
&\|Kf\|_{[H^{q,p}_1(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[H^{q,p}_1(\partial\Omega,\sigma)]^M}.
\end{aligned} \qquad (2.3.69)
$$

Hence, the claim made in item *(iii)* follows.

Turning to the claim made in item *(iv)*, define $p^* := \big(\frac{1}{p} - \frac{1}{n-1}\big)^{-1} \in (1, \infty]$ and fix $j, k \in \{1, \dots, n\}$ arbitrary. From Theorem 2.2.6 we know that

$$S : \big[H^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^{p^*}(\partial\Omega, \sigma)\big]^M \tag{2.3.70}$$

is well defined, linear, and bounded. In concert with [69, (11.2.6)] this implies that

$$\partial_{\tau_{jk}} S : \big[H^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\big(\mathrm{Lip}_c(\partial\Omega)\big)'\big]^M \text{ continuously.} \tag{2.3.71}$$

Since from Theorem 2.1.4 (cf. (2.1.42)) and [69, (4.2.8)] we also have

$$T^{\#}_{jk} : \big[H^p(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\big(\mathrm{Lip}_c(\partial\Omega)\big)'\big]^M \text{ continuously,} \tag{2.3.72}$$

in order to conclude that (2.3.11) holds it suffices to show (cf. [69, Theorem 4.4.1]) that for each (p, ∞)-atom a for $\big[H^p(\partial\Omega, \sigma)\big]^M$ we have

$$\partial_{\tau_{jk}}(Sa) = T^{\#}_{jk} a \text{ in } \big[\big(\mathrm{Lip}_c(\partial\Omega)\big)'\big]^M. \tag{2.3.73}$$

To this end, consider an arbitrary test function $\phi \in \big[\mathrm{Lip}_c(\partial\Omega)\big]^M$. With $S_{A^\top}$ being the single layer potential associated with the coefficient tensor $A^\top$, the real transpose of A, in the same manner S has been associated with the original A, we may then write

$$\begin{aligned}
&{}_{[(\mathrm{Lip}_c(\partial\Omega))']^M}\big\langle \partial_{\tau_{jk}}(Sa), \phi\big\rangle_{[\mathrm{Lip}_c(\partial\Omega)]^M} \\
&\qquad = -{}_{[(\mathrm{Lip}_c(\partial\Omega))']^M}\big\langle Sa, \partial_{\tau_{jk}}\phi\big\rangle_{[\mathrm{Lip}_c(\partial\Omega)]^M} \\
&\qquad = -\int_{\partial\Omega} \langle Sa, \partial_{\tau_{jk}}\phi\rangle \, d\sigma = -\int_{\partial\Omega} \langle a, S_{A^\top}(\partial_{\tau_{jk}}\phi)\rangle \, d\sigma \\
&\qquad = -\int_{\partial\Omega} \langle a, T_{jk}\phi\rangle \, d\sigma = \int_{\partial\Omega} \langle T^{\#}_{jk} a, \phi\rangle \, d\sigma \\
&\qquad = {}_{[(\mathrm{Lip}_c(\partial\Omega))']^M}\big\langle T^{\#}_{jk} a, \phi\big\rangle_{[\mathrm{Lip}_c(\partial\Omega)]^M}
\end{aligned} \tag{2.3.74}$$

thanks to (2.3.70), [69, Definition 11.2.1], Fubini's Theorem, a slight variant of (1.2.29), item *(ii)* in Proposition 1.2.1, and [68, Proposition 4.1.4]. This establishes (2.3.73), finishing the proof of (2.3.11).

In turn, from (2.3.70), the fact that

$$\bigcup_{1 \le q \le \infty} L^q(\partial\Omega, \sigma) \subseteq L^1\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^n}\Big), \tag{2.3.75}$$

(2.3.11), (2.1.42), and [69, Definition 11.10.5] (cf. (A.0.89)-(A.0.90)) we conclude that the boundary-to-boundary single layer induces a well-defined, linear, and bounded mapping in the context of (2.3.10). Finally, the claim about (2.3.12) follows from what we have just proved, (2.3.70), and [69, Definition 11.10.6]. This finishes the proof of item *(iv)*.

That $S^{\top}$, the (real) transpose of (2.3.12) corresponding to $p = 1$, is a well-defined, linear, and bounded mapping in the context of (2.3.14) follows from [69, (1.2.67)], the duality result recorded in [69, Theorem 4.6.1], and (A.0.32). Likewise, that (2.3.16) is a well-defined linear and bounded operator follows from (2.3.9) (used with $p := 1$ and $q := \frac{n-1}{n-2}$) and (A.0.32). To show that the operator $K^{\top}$, considered in the context of (2.3.16), is compatible with $K^{\#}$ acting on the space $\big[L^{n-1}(\partial\Omega,\sigma)\big]^M$, pick an arbitrary function $f \in \big[L^{n-1}(\partial\Omega,\sigma)\big]^M$. Hence also $f \in \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$ (cf. [69, (11.10.52)]) and for each $g \in \big[H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)\big]^M$ we may write

$$\begin{aligned}
&{}_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}\big\langle K^{\top} f, g\big\rangle_{[H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)]^M} \\
&\qquad = {}_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}\big\langle f, Kg\big\rangle_{[H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)]^M} \\
&\qquad = {}_{[L^{n-1}(\partial\Omega,\sigma)]^M}\big\langle f, Kg\big\rangle_{[L^{\frac{n-1}{n-2}}(\partial\Omega,\sigma)]^M} \\
&\qquad = {}_{[L^{n-1}(\partial\Omega,\sigma)]^M}\big\langle K^{\#} f, g\big\rangle_{[L^{\frac{n-1}{n-2}}(\partial\Omega,\sigma)]^M} \\
&\qquad = {}_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}\big\langle K^{\#} f, g\big\rangle_{[H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)]^M}.
\end{aligned} \tag{2.3.76}$$

The first equality above is implied by the fact that $K^{\top}$ is the transpose of (2.3.15). The second equality in (2.3.76) comes from the compatibility property recorded in [69, (11.10.55)]. The third equality in (2.3.76) is a consequence of item *(iii)* in Theorem 1.5.1, while the fourth equality in (2.3.76) is once again seen from [69, (11.10.55)]. With (2.3.76) in hand, (2.3.17) follows.

Next, that $K^{\top}$ from (2.3.16) has $\big[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)\big]^M$ as an invariant subspace is seen from the boundedness of (2.3.16), (2.3.17), the fact that $K^{\#}$ maps $\big[L^{n-1}(\partial\Omega,\sigma)\big]^M$ into itself (cf. item *(iii)* in Theorem 1.5.1), (A.0.207), and [69, Lemma 1.2.20].

Consider next the task of addressing the claims made in item *(vi)*. Throughout, work under the assumption that $n \geq 3$ and that $\partial\Omega$ is bounded. To get started, fix some arbitrary functional $f = (f_\beta)_{1\leq\beta\leq M} \in \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$. From [69, Proposition 11.10.7] and (A.0.32) we know that for each $\beta \in \{1,\dots,M\}$ there exist $f_0^{(\beta)} \in L^{n-1}(\partial\Omega,\sigma)$ and $f_{jk}^{(\beta)} \in \mathrm{BMO}(\partial\Omega,\sigma)$ with $1 \leq j < k \leq n$ satisfying (for some constant $C \in (0,\infty)$ independent of f)

$$\|f_0^{(\beta)}\|_{L^{n-1}(\partial\Omega,\sigma)} + \sum_{1\leq j<k\leq n} \|f_{jk}^{(\beta)}\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \leq C\|f_\beta\|_{\mathrm{BMO}_{-1}(\partial\Omega,\sigma)} \tag{2.3.77}$$

and, for every function $g \in H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)$,

$$
\begin{aligned}
&{}_{\mathrm{BMO}_{-1}(\partial\Omega,\sigma)}\big\langle f_\beta, g\big\rangle_{H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)} \\
&\qquad = \int_{\partial\Omega} f_0^{(\beta)} g \,\mathrm{d}\sigma + \sum_{1\leq j<k\leq n} {}_{\mathrm{BMO}(\partial\Omega,\sigma)}\big\langle f_{jk}^{(\beta)}, \partial_{\tau_{jk}} g\big\rangle_{H^1(\partial\Omega,\sigma)}.
\end{aligned}
\tag{2.3.78}
$$

Having picked $x \in \Omega$ and $\alpha \in \{1, \dots, M\}$, we may then use (2.3.78) to write

$$
\begin{aligned}
\big(\mathscr{S} f\big)_\alpha(x) &= {}_{\mathrm{BMO}_{-1}(\partial\Omega,\sigma)}\Big\langle f_\beta, E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}\Big\rangle_{H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)} \\
&= \int_{\partial\Omega} E_{\alpha\beta}(x-y) f_0^{(\beta)}(y)\,\mathrm{d}\sigma(y) \\
&\quad + \sum_{1\leq j<k\leq n} {}_{\mathrm{BMO}(\partial\Omega,\sigma)}\Big\langle f_{jk}^{(\beta)}, \partial_{\tau_{jk}}\big[E_{\alpha\beta}(x-\cdot)\big]\Big\rangle_{H^1(\partial\Omega,\sigma)} \\
&= \int_{\partial\Omega} E_{\alpha\beta}(x-y) f_0^{(\beta)}(y)\,\mathrm{d}\sigma(y) \\
&\quad + \sum_{1\leq j<k\leq n} \int_{\partial\Omega} \partial_{\tau_{jk}(y)}\big[E_{\alpha\beta}(x-y)\big] f_{jk}^{(\beta)}(y)\,\mathrm{d}\sigma(y),
\end{aligned}
\tag{2.3.79}
$$

where the last equality is based on [69, Lemma 4.6.5] (whose applicability in the present setting is guaranteed by [68, (7.4.106)] and [70, Theorem 1.4.2]). Going further, for any two pairs of indices, $j, k \in \{1, \dots, n\}$ and $\alpha, \beta \in \{1, \dots, M\}$, introduce the integral operators acting on each function $\phi \in L^1(\partial\Omega\,\sigma)$ according to

$$
\mathcal{T}_{jk}^{\alpha\beta}\phi(x) := \int_{\partial\Omega} \big\{\nu_j(y)(\partial_k E_{\alpha\beta})(x-y) - \nu_k(y)(\partial_j E_{\alpha\beta})(x-y)\big\}\phi(y)\,\mathrm{d}\sigma(y)
\tag{2.3.80}
$$

at each $x \in \Omega$, where $\nu = (\nu_1, \dots, \nu_n)$ is the geometric measure theoretic outward unit normal to Ω. In this notation, and with $f_0 := (f_0^{(\beta)})_{1\leq\beta\leq M}$, the formula recorded in (2.3.79) may be recast (bearing in mind [68, (7.4.106)]) simply as

$$
\mathscr{S} f = \mathscr{S} f_0 - \sum_{1\leq j<k\leq n} \big(\mathcal{T}_{jk}^{\alpha\beta} f_{jk}^{(\beta)}\big)_{1\leq\alpha\leq M} \ \text{ in } \Omega.
\tag{2.3.81}
$$

It is then clear from (2.3.81) that (2.3.19) is a well-defined linear mapping, and that (2.3.21) holds. In addition, from (2.3.81), (1.5.60), (1.5.56), [70, Corollary 2.4.2], and (2.3.77) we see that (2.3.22) is true. From (2.3.81), (1.5.44), (1.2.4), and [68, (7.4.106)] we also conclude that $\mathscr{S} f\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$ and, in fact, at σ-a.e. point on $\partial\Omega$ we have

$$
\mathscr{S} f\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = S f_0 - \sum_{1\leq j<k\leq n} \big(T_{jk}^{\alpha\beta} f_{jk}^{(\beta)}\big)_{1\leq\alpha\leq M}
\tag{2.3.82}
$$

where, for each $j, k \in \{1, \dots, n\}$ and $\alpha, \beta \in \{1, \dots, M\}$,

$$T_{jk}^{\alpha\beta}\phi(x) \tag{2.3.83}$$
$$:= \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \big\{\nu_j(y)(\partial_k E_{\alpha\beta})(x-y) - \nu_k(y)(\partial_j E_{\alpha\beta})(x-y)\big\}\phi(y)\,d\sigma(y),$$

for every function $\phi \in L^1(\partial\Omega, \sigma)$ and σ-a.e. point $x \in \partial\Omega$. Having established this, for each test function $\psi = (\psi_\alpha)_{1\leq\alpha\leq M} \in \big[\mathrm{Lip}(\partial\Omega)\big]^M$ we may then write, thanks to (1.3.62) and [70, (2.3.25)],

$$\int_{\partial\Omega} \Big\langle \mathscr{S} f\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \psi\Big\rangle\,d\sigma$$
$$= \int_{\partial\Omega} \big\langle S f_0, \psi\big\rangle\,d\sigma - \sum_{1\leq j<k\leq n} \int_{\partial\Omega} \big(T_{jk}^{\alpha\beta} f_{jk}^{(\beta)}\big)\psi_\alpha\,d\sigma$$
$$= \int_{\partial\Omega} \big\langle f_0, S_{L^\top}\psi\big\rangle\,d\sigma + \sum_{1\leq j<k\leq n} \int_{\partial\Omega} f_{jk}^{(\beta)}\big(\big(T_{jk}^{\alpha\beta}\big)^{\#}\psi_\alpha\big)\,d\sigma \tag{2.3.84}$$

where for each $j, k \in \{1, \dots, n\}$ and $\alpha, \beta \in \{1, \dots, M\}$ we have denoted by $T_{jk}^{\alpha\beta}$ the integral operator acting on each function $\varphi \in L^1(\partial\Omega, \sigma)$ according to

$$\big(T_{jk}^{\alpha\beta}\big)^{\#}\varphi(x) \tag{2.3.85}$$
$$:= \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \big\{\nu_j(x)(\partial_k E_{\alpha\beta})(x-y) - \nu_k(x)(\partial_j E_{\alpha\beta})(x-y)\big\}\varphi(y)\,d\sigma(y),$$

at σ-a.e. point $x \in \partial\Omega$. In relation to this operator we wish to observe that, as seen from (1.5.83) and the first formula in [70, (1.4.32)], for each $j, k \in \{1, \dots, n\}$ and each $\beta \in \{1, \dots, M\}$ we have

$$\partial_{\tau_{jk}}\big(S_{L^\top}\psi\big)_\beta = \big(T_{jk}^{\alpha\beta}\big)^{\#}\psi_\alpha \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.3.86}$$

Together, (2.3.84) and (2.3.86) prove that

$$\begin{aligned}\int_{\partial\Omega}\Big\langle \mathscr{S}f\big|^{\kappa-\text{n.t.}}_{\partial\Omega},\psi\Big\rangle\,\mathrm{d}\sigma &= \int_{\partial\Omega}\big\langle f_0, S_{L^\top}\psi\big\rangle\,\mathrm{d}\sigma\\ &\quad+\sum_{1\le j<k\le n}\int_{\partial\Omega} f^{(\beta)}_{jk}\,\partial_{\tau_{jk}}\big(S_{L^\top}\psi\big)_\beta\,\mathrm{d}\sigma\\ &=\int_{\partial\Omega} f^{(\beta)}_0\big(S_{L^\top}\psi\big)_\beta\,\mathrm{d}\sigma\\ &\quad+\sum_{1\le j<k\le n}{}_{\mathrm{BMO}(\partial\Omega,\sigma)}\Big\langle f^{(\beta)}_{jk},\partial_{\tau_{jk}}\big(S_{L^\top}\psi\big)_\beta\Big\rangle_{H^1(\partial\Omega,\sigma)}\\ &={}_{\mathrm{BMO}_{-1}(\partial\Omega,\sigma)}\Big\langle f_\beta,\big(S_{L^\top}\psi\big)_\beta\Big\rangle_{H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)}\\ &={}_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}\Big\langle f,S_{L^\top}\psi\Big\rangle_{[H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)]^M}\\ &={}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M}\Big\langle (S_{L^\top})^\top f,\psi\Big\rangle_{[H^1(\partial\Omega,\sigma)]^M}\\ &=\int_{\partial\Omega}\Big\langle (S_{L^\top})^\top f,\psi\Big\rangle\,\mathrm{d}\sigma, \end{aligned}\tag{2.3.87}$$

where the second equality uses [69, Lemma 4.6.5] (whose applicability in the present setting is ensured by [68, (7.4.106)] and [70, Theorem 1.4.2]). In view of [68, (3.7.23)], we may now conclude from (2.3.87) that $\mathscr{S}f\big|^{\kappa-\text{n.t.}}_{\partial\Omega}=(S_{L^\top})^\top f$ at σ-a.e. point on $\partial\Omega$. This finishes the proof of (2.3.24).

To complete the treatment of the claims in item *(vi)* there remains to justify (2.3.25). To this end, pick $p\in(1,\infty)$ and select an arbitrary $f\in\big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$. Also, choose an arbitrary function $g\in\big[L^{n-1}(\partial\Omega,\sigma)\big]^M$. In particular, we have $g\in\big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$ by [69, (11.10.52)]. Then for each $r\in\big(0,2\,\mathrm{diam}(\partial\Omega)\big)$ and $x\in\partial\Omega$ we may estimate

$$\begin{aligned}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathscr{S}(f-g)\big)\big|^p\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac1p}&\\ \le C\|f-g\|_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M},&\end{aligned}\tag{2.3.88}$$

thanks to (2.3.22) (written for $f-g$ in place of f). In addition, from (1.5.64) we know that

$$\lim_{R\to0^+}\sup_{\substack{x\in\partial\Omega\text{ and}\\ r\in(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathscr{S}g\big)\big|^p\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac1p}=0.\tag{2.3.89}$$

Collectively, (2.3.88) and (2.3.89) imply

$$\lim_{R\to 0^+} \sup_{\substack{x\in\partial\Omega \text{ and} \\ r\in(0,R)}} \left(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathscr{S}f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n \right)^{\frac{1}{p}}$$
$$\leq C\|f-g\|_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}, \tag{2.3.90}$$

for some constant $C \in (0,\infty)$ independent of f and g. With this in hand, (2.3.25) follows on account of [69, (11.10.53)] and the arbitrariness of $g \in \big[L^{n-1}(\partial\Omega,\sigma)\big]^M$.

To deal with the claims in item *(vii)*, assume $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 3$, is an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. To justify the identity in (2.3.28), pick an arbitrary function $f \in \big[L^{n-1}(\partial\Omega,\sigma)\big]^M \hookrightarrow \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$ (cf. [69, (11.10.52)]). Fix $q,q' \in (1,\infty)$ with $1/q + 1/q' = 1$ and recall that a fractional integration operator of order 1 maps $\big[L^{n-1}(\partial\Omega,\sigma)\big]^M$ boundedly into $\big[L^{q'}(\partial\Omega,\sigma)\big]^M$. For each $\mathbb{C}^M$-valued $(q,1)$-atom g on $\partial\Omega$ we may then write

$$\begin{aligned}
\int_{\partial\Omega} \langle S^\top f, g\rangle\,d\sigma &= {}_{[\mathrm{BMO}(\partial\Omega,\sigma)]^M}\big\langle S^\top f, g\big\rangle_{[H^1(\partial\Omega,\sigma)]^M} \\
&= {}_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}\big\langle f, Sg\big\rangle_{[H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)]^M} \\
&= {}_{[L^{n-1}(\partial\Omega,\sigma)]^M}\big\langle f, Sg\big\rangle_{[L^{\frac{n-1}{n-2}}(\partial\Omega,\sigma)]^M} \\
&= \int_{\partial\Omega} \langle f, Sg\rangle\,d\sigma = \int_{\partial\Omega} \langle S_{L^\top} f, g\rangle\,d\sigma.
\end{aligned} \tag{2.3.91}$$

The first equality above is implied [69, Lemma 4.6.5] (bearing in mind [68, (7.4.106)]). The second equality in (2.3.91) uses the fact that $S^\top$ is the transpose of S in (2.3.13). The third equality in (2.3.91) comes from the compatibility property recorded in [69, (11.10.55)]. For the fourth equality in (2.3.91), we have used the fact that the duality between $L^{n-1}(\partial\Omega,\sigma)$ and $L^{\frac{n-1}{n-2}}(\partial\Omega,\sigma)$ is given by the integral pairing. Finally, the fifth equality in (2.3.91) is a consequence of (1.3.62), Fubini's Theorem, and the first formula in [70, (1.4.32)]. Once (2.3.91) has been established, (2.3.28) follows with the help of [69, Lemma 4.6.9].

Finally, that the restriction of the operator (2.3.14) to $\big[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)\big]^M$ induces a well-defined, linear, and bounded mapping in the context described in (2.3.29), is seen from (2.3.14), (2.3.28), the fact that $S_{L^\top}$ maps $\big[L^{n-1}(\partial\Omega,\sigma)\big]^M$ into $\big[L_1^{n-1}(\partial\Omega,\sigma)\big]^M$ (as seen from item *(ix)* in Theorem 1.5.1), [69, (11.5.203)], and [69, Lemma 1.2.20]. This completes the treatment of item *(vii)*, and finishes the proof of Theorem 2.3.1. □

For further use, we note here that Theorem 2.3.1 implies the following result.

Corollary 2.3.2 *Suppose* $\Omega \subseteq \mathbb{R}^n$ *(where* $n \in \mathbb{N}$ *satisfies* $n \geq 2$*) is a* UR *domain. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. For* $M \in \mathbb{N}$*, consider a coefficient tensor* $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n \\ 1\leq\alpha,\beta\leq M}}$ *with* *complex entries, with the property that the $M\times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in* (1.3.2) *is weakly elliptic* (*in the sense of* [70], (1.3.3) in Definition 1.3.1]). *Finally, fix an aperture parameter $\kappa \in (0,\infty)$ along with an exponent $p \in \big(\frac{n-1}{n}, 1\big]$. Then the following properties hold.*

(i) *Recall the modified boundary-to-domain double layer potential operator $\mathcal{D}_{\text{mod}}$ acting on functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ as in* (1.8.6). *Then the operator*

$$\begin{gathered}\big[\dot{\partial}^A_\nu \mathcal{D}_{\text{mod}}\big] : \big[\dot{H}^p_1(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[H^p(\partial\Omega,\sigma)\big]^M \text{ defined as} \\ \big[\dot{\partial}^A_\nu \mathcal{D}_{\text{mod}}\big][f] := \dot{\partial}^A_\nu(\mathcal{D}_{\text{mod}} f) \text{ for each } f \in \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^M,\end{gathered} \tag{2.3.92}$$

is well defined, linear, and bounded, when the quotient space is equipped with the semi-quasinorm[8] *introduced in* (A.0.92).

(ii) *Recall the boundary-to-boundary single layer potential operator S from Theorem* 2.2.6. *Then the operator*

$$\begin{gathered}[S] : \big[H^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{H}^p_1(\partial\Omega,\sigma)/\sim\big]^M \text{ defined as} \\ [S]f := [Sf] \in \big[\dot{H}^p_1(\partial\Omega,\sigma)/\sim\big]^M, \quad \forall f \in \big[H^p(\partial\Omega,\sigma)\big]^M.\end{gathered} \tag{2.3.93}$$

is well defined, linear, and bounded, when the quotient space is equipped with the semi-quasinorm introduced in (A.0.92).

Proof All claims are direct consequences of Theorem 2.3.1 and definitions. □

Up to additive constants, functions in homogeneous Hardy-based Sobolev spaces actually belong to a Lebesgue space corresponding to a sharp embedding exponent. This is made precise in the theorem below.

Theorem 2.3.3 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) be an open set satisfying a two-sided local John condition with an Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also,*

$$\text{fix } p \in \big(\tfrac{n-1}{n}, n-1\big) \text{ and define } p_* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1} \in (1,\infty). \tag{2.3.94}$$

Then for each function $f \in \dot{H}^p_1(\partial\Omega,\sigma)$ one may find some number $c = c(f) \in \mathbb{C}$ such that

$$f - c \text{ belongs to the space } L^{p_*}(\partial\Omega,\sigma), \tag{2.3.95}$$

and, in addition,

[8] if in fact $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then Proposition 2.3.8 guarantees that said semi-quasinorm becomes a genuine quasinorm, making $\dot{H}^p_1(\partial\Omega,\sigma)/\sim$ a quasi-Banach space

if $\partial\Omega$ is unbounded, there exists $C = C(\Omega, n, p) \in (0, \infty)$ so that

$$\|f - c\|_{L^{p_*}(\partial\Omega,\sigma)} \leq C \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{H^p(\partial\Omega,\sigma)} = C\|f\|_{\dot{H}^p_1(\partial\Omega,\sigma)}. \tag{2.3.96}$$

In fact, with $\mathbb{C}$ canonically identified with the space of constant functions on $\partial\Omega$, one has

$$\dot{H}^p_1(\partial\Omega, \sigma) = \mathbb{C} + H^{p_*,p}_1(\partial\Omega, \sigma). \tag{2.3.97}$$

As a consequence,

$$\dot{H}^p_1(\partial\Omega, \sigma) \subseteq L^{p_*}_{\mathrm{loc}}(\partial\Omega, \sigma), \tag{2.3.98}$$

and

$$\begin{gathered}\text{if } \partial\Omega \text{ is unbounded then for each } f \in \dot{H}^p_1(\partial\Omega, \sigma) \text{ one has}\\ \|f\|_{\dot{H}^p_1(\partial\Omega,\sigma)} = 0 \Longleftrightarrow f \text{ is a constant function on } \partial\Omega.\end{gathered} \tag{2.3.99}$$

Lastly, one has

$$\begin{gathered}H^{1,p}_1(\partial\Omega, \sigma) = \dot{H}^p_1(\partial\Omega, \sigma) = H^{p_*,p}_1(\partial\Omega, \sigma)\\ \text{as sets, provided } \partial\Omega \text{ is bounded,}\end{gathered} \tag{2.3.100}$$

as well as

$$\begin{gathered}H^{1,p}_1(\partial\Omega, \sigma) \hookrightarrow L^{p_*}(\partial\Omega, \sigma)\\ \text{continuously and densely, if } \partial\Omega \text{ is bounded,}\end{gathered} \tag{2.3.101}$$

and

$$\begin{gathered}\text{if } \partial\Omega \text{ is bounded, there exists a constant } C \in (1, \infty) \text{ such that}\\ C^{-1}\|f\|_{H^{p_*,p}_1(\partial\Omega,\sigma)} \leq \|f\|_{H^{1,p}_1(\partial\Omega,\sigma)} \leq C\|f\|_{H^{p_*,p}_1(\partial\Omega,\sigma)}\\ \text{for all } f \in H^{1,p}_1(\partial\Omega, \sigma) = \dot{H}^p_1(\partial\Omega, \sigma) = H^{p_*,p}_1(\partial\Omega, \sigma).\end{gathered} \tag{2.3.102}$$

Note that if $p \in \big(\frac{n-1}{n}, 1\big)$ then $p_* := \big(\frac{1}{p} - \frac{1}{n-1}\big)^{-1} \in \big(1, \frac{n-1}{n-2}\big)$. That the latter range is optimal as far as the conclusion in (2.3.95) is concerned, may be seen from [69, (11.10.66)].

Proof of Theorem 2.3.3 Consider first the case when $p \in (1, n-1)$. Then from [69, (11.10.30)], item *(1)* in [69, Theorem 11.5.16], [69, (11.5.102)], [69, (11.1.27)], and [68, Corollary 3.7.3] we see that all claims made in the statement of the theorem are true. We are therefore left with consider the case when either

$$n \geq 3 \text{ and } p \in \big(\tfrac{n-1}{n}, 1\big], \tag{2.3.103}$$

or

$$n = 2 \text{ and } p \in \big(\tfrac{1}{2}, 1\big). \tag{2.3.104}$$

We shall simultaneously consider both scenarios described in (2.3.103)-(2.3.104). The key claim we make is that there exists some constant $C = C(\Omega, n, p) \in (0, \infty)$ such that for each given $f \in \dot{H}^p_1(\partial\Omega, \sigma)$ we may find some non-negative function $g \in L^p(\partial\Omega, \sigma)$ with the property that

$$|f(x) - f(y)| \leq |x - y| \cdot \big[g(x) + g(y)\big] \text{ for } \sigma\text{-a.e. } x, y \in \partial\Omega, \tag{2.3.105}$$

and for which

$$\|g\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{\dot{H}^p_1(\partial\Omega,\sigma)} \text{ if } \partial\Omega \text{ is unbounded.} \tag{2.3.106}$$

To justify this claim, we argue largely as in the proof of [69, Lemma 11.5.10]. For starters, define $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$. From the current hypotheses and [68, (5.11.27)] together with item *(7)* in [68, Lemma 5.10.9] we then conclude that both Ω_+ and Ω_- are UR domains satisfying a local John condition, and with $\partial\Omega_+ = \partial\Omega = \partial\Omega_-$. As in the proof of [69, Lemma 11.5.10], denote by K_{mod} the modified principal-value harmonic double layer operator associated with Ω and $L := \Delta$ as in (1.8.24). Fix an arbitrary function $f \in \dot{H}^p_1(\partial\Omega, \sigma)$, so

$$\begin{gathered} f \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \subseteq L^1_{\text{loc}}(\partial\Omega, \sigma) \text{ satisfying} \\ \partial_{\tau_{jk}} f \in H^p(\partial\Omega, \sigma) \text{ for all } j, k \in \{1, \dots, n\} \end{gathered} \tag{2.3.107}$$

(see (A.0.89)), and decompose

$$f = f_+ - f_- \text{ where } f_\pm := \big(\pm \tfrac{1}{2}I + K_{\text{mod}}\big)f, \tag{2.3.108}$$

To proceed, define $u_\pm := \pm\mathcal{D}^\pm_{\text{mod}} f$ in $\Omega_\pm$, where $\mathcal{D}^\pm_{\text{mod}}$ are the modified boundary-to-domain harmonic double layer operators associated with $\Omega_\pm$ and $L := \Delta$ as in (1.8.6). Then from Theorem 1.8.2, and Theorem 2.3.1 we see that $u_\pm : \Omega_\pm \to \mathbb{C}$ are well-defined (harmonic) functions belonging to $\mathscr{C}^\infty(\Omega_\pm)$ which, for any given aperture parameter $\kappa > 0$, satisfy

$$\begin{gathered} u_\pm\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\pm \tfrac{1}{2}I + K_{\text{mod}}\big)f = f_\pm \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \\ \mathcal{N}_\kappa(\nabla u_\pm) \text{ belongs to the space } L^p(\partial\Omega, \sigma), \text{ and} \\ \big\|\mathcal{N}_\kappa(\nabla u_\pm)\big\|_{L^p(\partial\Omega,\sigma)} \leq C \sum_{j,k=1}^n \big\|\partial_{\tau_{jk}} f\big\|_{H^p(\partial\Omega,\sigma)} = C\|f\|_{\dot{H}^p_1(\partial\Omega,\sigma)}, \end{gathered} \tag{2.3.109}$$

for some constant $C = C(\Omega, n, \kappa, p) \in (0, \infty)$. At this point, the discussion branches out, depending on whether $\partial\Omega$ is unbounded, or bounded. For now, assume that $\partial\Omega$ is unbounded, a scenario in which [69, Lemma 11.5.8] implies

$$\begin{gathered} |f_\pm(x) - f_\pm(y)| \leq |x - y| \cdot \big[g_\pm(x) + g_\pm(y)\big] \text{ for } \sigma\text{-a.e. } x, y \in \partial\Omega, \\ \text{where } g_\pm := C \cdot \mathcal{N}_\kappa(\nabla u_\pm). \end{gathered} \tag{2.3.110}$$

If we now define $g := g_+ + g_-$ then the claims in (2.3.105)-(2.3.106) in the case when $\partial\Omega$ is unbounded are seen from (2.3.109), and (2.3.110).

Moving on, consider the scenario in which (2.3.103) holds and $\partial\Omega$ is bounded. Define $f_\pm$ and $u_\pm$ as before. These are functions which continue to enjoy the properties described in (2.3.108)-(2.3.109). What changes is that now [69, Lemma 11.5.8] gives

$$\begin{gathered} |f_\pm(x) - f_\pm(y)| \leq |x-y| \cdot \big[g_\pm(x) + g_\pm(y)\big] \text{ for } \sigma\text{-a.e. } x, y \in \partial\Omega, \\ \text{where } g_\pm := C\Big(\big|u_\pm\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big| + \mathcal{N}_\kappa(\nabla u_\pm)\Big). \end{gathered} \tag{2.3.111}$$

If we define $g := g_+ + g_-$ then (2.3.105) holds, but proving the membership of g to $L^p(\partial\Omega, \sigma)$ requires more work than before (due to the presence of the nontangential boundary traces in the formula for $g_\pm$ in (2.3.111)). Specifically, we first note that

$$u_\pm\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f_\pm \in L^{1,\infty}(\partial\Omega, \sigma) \subseteq \bigcap_{0<q<1} L^q(\partial\Omega, \sigma), \tag{2.3.112}$$

thanks to the formula in the first line of (2.3.109), the mapping property recorded in [70, (2.3.33)], the membership in the first line of [69, (11.5.105)], the fact that $\partial\Omega$ is bounded, and the embedding in [68, (6.2.38)]. In turn, from (2.3.112), the definitions of $g_\pm$ from (2.3.111), and the membership in the second line of (2.3.109) we then see that

$$g_\pm \in L^p(\partial\Omega, \sigma) + \bigcap_{0<q<1} L^q(\partial\Omega, \sigma), \tag{2.3.113}$$

hence

$$g = g_+ + g_- \in L^p(\partial\Omega, \sigma) + \bigcap_{0<q<1} L^q(\partial\Omega, \sigma). \tag{2.3.114}$$

If $p < 1$, this puts g in $L^p(\partial\Omega, \sigma)$, and we are done. If $p = 1$, we arrive at the conclusion that

$$g \in \bigcap_{0<q<1} L^q(\partial\Omega, \sigma). \tag{2.3.115}$$

As noted earlier, the function g satisfies (2.3.105). This permits us to invoke [69, Lemma 11.5.9] and deduce that

$$\begin{gathered} \text{whenever } q \in \big(\tfrac{n-1}{n}, 1\big) \text{ and } q_* := \big(\tfrac{1}{q} - \tfrac{1}{n-1}\big)^{-1} \text{ then} \\ q_* \in \big(1, \tfrac{n-1}{n-2}\big) \text{ and } f \text{ belongs to the space } L^{q_*}(\partial\Omega, \sigma). \end{gathered} \tag{2.3.116}$$

Finally, from (2.3.116) and [70, (2.3.35)] we infer that

$$u_\pm\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\pm\tfrac{1}{2}I + K_{\text{mod}}\big)f \in L^{q_*}(\partial\Omega, \sigma), \tag{2.3.117}$$

which, in light of [69, (11.5.118)] and the definitions in [69, (11.5.113)], implies

$$g \in L^{q_*}(\partial\Omega, \sigma) + L^1(\partial\Omega, \sigma) = L^1(\partial\Omega, \sigma), \tag{2.3.118}$$

since $\partial\Omega$ is presently assumed to be bounded. This establishes (2.3.105) for some non-negative function $g \in L^p(\partial\Omega, \sigma)$ in all cases described in (2.3.103)-(2.3.104).

Having proved (2.3.105)-(2.3.106), we may rely on item *(1)* in [69, Lemma 11.5.9] and conclude that for each $f \in \dot{H}^p_1(\partial\Omega, \sigma)$ there exists $c = c(f) \in \mathbb{C}$ such that (2.3.95) and (2.3.96) are true. In turn, from (2.3.95) and (A.0.89) we see that (2.3.97) holds.

If $\partial\Omega$ is bounded, the first equality in (2.3.100) is implied by [69, (11.10.41)], while the second equality in (2.3.100) is a consequence of (2.3.97) (bearing in mind that $\mathbb{C} \subseteq H^{p_*,p}_1(\partial\Omega, \sigma)$ now).

Next, consider the task of proving (2.3.101). Henceforth, assume that $\partial\Omega$ is bounded. Then [69, (11.10.41)] and (2.3.98) imply that

$$H^{1,p}_1(\partial\Omega, \sigma) \subseteq L^{p_*}(\partial\Omega, \sigma) \text{ as sets.} \tag{2.3.119}$$

Consequently, the identity operator

$$H^{1,p}_1(\partial\Omega, \sigma) \ni f \longmapsto f \in L^{p_*}(\partial\Omega, \sigma) \tag{2.3.120}$$

is a well-defined linear mapping. We claim that its graph is closed in the product topology on $H^{1,p}_1(\partial\Omega, \sigma) \times L^{p_*}(\partial\Omega, \sigma)$. To justify this claim, suppose

$$\begin{gathered} \{f_j\}_{j\in\mathbb{N}} \subseteq H^{1,p}_1(\partial\Omega, \sigma) \text{ and } (g, h) \in H^{1,p}_1(\partial\Omega, \sigma) \times L^{p_*}(\partial\Omega, \sigma) \\ \text{are such that } \lim_{j\to\infty} f_j = g \text{ in } H^{1,p}_1(\partial\Omega, \sigma) \text{ and } \lim_{j\to\infty} f_j = h \text{ in } L^{p_*}(\partial\Omega, \sigma). \end{gathered} \tag{2.3.121}$$

We then conclude from (A.0.94) that $\lim_{j\to\infty} f_j = g$ in $L^1(\partial\Omega, \sigma)$. Given that we also have $\lim_{j\to\infty} f_j = h$ in $L^1(\partial\Omega, \sigma)$ (since $\partial\Omega$ is bounded and $p_* > 1$), it follows that $g = h$. Thus, (g, h) belongs to the graph (2.3.120), proving that this graph is closed. Since $H^{1,p}_1(\partial\Omega, \sigma)$ and $L^{p_*}(\partial\Omega, \sigma)$ are quasi-Banach spaces, the version of the Closed Graph Theorem for quasi-Banach spaces proved in [73, Corollary 6.78, p. 442] implies that (2.3.120) is a bounded mapping. This establishes the continuity of the embedding in (2.3.101). The fact that this embedding also has dense range is a consequence of [69, (11.10.39)] and [68, (3.7.22)]. Finally, the first inequality in (2.3.102) is implied by (2.3.101), while the second inequality in (2.3.102) follows from definitions and the fact that $L^{p_*}(\partial\Omega, \sigma) \hookrightarrow L^1(\partial\Omega, \sigma)$ continuously. □

A version of Theorem 2.3.3 corresponding to the forbidden values $n = 2$ and $p = 1$ is presented below.

Proposition 2.3.4 *Let $\Omega \subseteq \mathbb{R}^2$ be an open set satisfying a two-sided local John condition whose boundary is an Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$. Then*

$$\dot{H}^1_1(\partial\Omega, \sigma) \subseteq \mathrm{VMO}(\partial\Omega, \sigma) \subseteq \mathrm{BMO}(\partial\Omega, \sigma) \tag{2.3.122}$$

and the inclusion $\dot{H}^1_1(\partial\Omega, \sigma) \hookrightarrow \mathrm{BMO}(\partial\Omega, \sigma)$ is actually continuous if $\partial\Omega$ is unbounded. More specifically,

$$\begin{gathered} \textit{if } \partial\Omega \textit{ is unbounded, there exists } C = C(\Omega) \in (0,\infty) \textit{ such that} \\ \|f\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \leq C\|f\|_{\dot{H}^1_1(\partial\Omega,\sigma)} \textit{ for all } f \in \dot{H}^1_1(\partial\Omega,\sigma). \end{gathered} \tag{2.3.123}$$

As a consequence of (2.3.123) *and* [68, (7.4.93)],

$$\begin{gathered} \textit{if } \partial\Omega \textit{ is unbounded then for each } f \in \dot{H}^1_1(\partial\Omega,\sigma) \textit{ one has} \\ \|f\|_{\dot{H}^1_1(\partial\Omega,\sigma)} = 0 \iff f \textit{ is a constant function on } \partial\Omega. \end{gathered} \tag{2.3.124}$$

Finally,

$$\begin{gathered} H^{1,1}_1(\partial\Omega,\sigma) \hookrightarrow \mathrm{VMO}(\partial\Omega,\sigma) \\ \textit{continuously and densely, if } \partial\Omega \textit{ is bounded,} \end{gathered} \tag{2.3.125}$$

hence, in particular,

$$\begin{gathered} \textit{if } \partial\Omega \textit{ is bounded, there exists some constant } C \in (0,\infty) \textit{ such that} \\ \|f\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \leq C\Big(\|f\|_{L^1(\partial\Omega,\sigma)} + \|f\|_{\dot{H}^1_1(\partial\Omega,\sigma)}\Big) = C\|f\|_{H^{1,1}_1(\partial\Omega,\sigma)} \\ \textit{for all functions } f \in \dot{H}^1_1(\partial\Omega,\sigma) = H^{1,1}_1(\partial\Omega,\sigma) \ (\textit{cf. } [69, (11.10.41)]). \end{gathered} \tag{2.3.126}$$

Proof Reasoning as in the proof of Theorem 2.3.3 with $n = 2$ and $p = 1$, we see that there exists some constant $C = C(\Omega) \in (0,\infty)$ such that for each $f \in \dot{H}^1_1(\partial\Omega,\sigma)$ it is possible to find some non-negative function $g \in L^1(\partial\Omega,\sigma)$ with the property that

$$|f(x) - f(y)| \leq |x-y| \cdot \big[g(x) + g(y)\big] \text{ for } \sigma\text{-a.e. } x, y \in \partial\Omega, \tag{2.3.127}$$

and for which

$$\|g\|_{L^1(\partial\Omega,\sigma)} \leq C\|f\|_{\dot{H}^1_1(\partial\Omega,\sigma)} \text{ if } \partial\Omega \text{ is unbounded.} \tag{2.3.128}$$

Granted these, we may invoke item *(2)* in [69, Lemma 11.5.9] (with $n = 2$) to conclude that f belongs to $\mathrm{VMO}(\partial\Omega,\sigma)$ and that (2.3.123) holds.

For the remainder of the proof assume that $\partial\Omega$ is bounded. Then [69, (11.10.41)] and (2.3.122) imply that

$$H^{1,1}_1(\partial\Omega,\sigma) \subseteq \mathrm{VMO}(\partial\Omega,\sigma) \text{ as sets.} \tag{2.3.129}$$

In particular, the identity operator

$$H^{1,1}_1(\partial\Omega,\sigma) \ni f \longmapsto f \in \mathrm{VMO}(\partial\Omega,\sigma) \tag{2.3.130}$$

is a well-defined linear mapping. Also, we claim that its graph is closed in the product topology on $H^{1,1}_1(\partial\Omega,\sigma) \times \mathrm{VMO}(\partial\Omega,\sigma)$. Indeed, if

$$\{f_j\}_{j\in\mathbb{N}} \subseteq H_1^{1,1}(\partial\Omega,\sigma) \text{ and } (g,h) \in H_1^{1,1}(\partial\Omega,\sigma)\times \mathrm{VMO}(\partial\Omega,\sigma)$$
$$\text{are such that } \lim_{j\to\infty} f_j = g \text{ in } H_1^{1,1}(\partial\Omega,\sigma) \text{ and } \lim_{j\to\infty} f_j = h \text{ in } \mathrm{BMO}(\partial\Omega,\sigma) \tag{2.3.131}$$

we conclude from (A.0.94) and [68, (7.4.105)] that $\lim_{j\to\infty} f_j = g$ in $L^1(\partial\Omega,\sigma)$ and $\lim_{j\to\infty} f_j = h$ in $L^1(\partial\Omega,\sigma)$. Hence, $g = h$. Consequently, (g,h) belongs to the graph (2.3.130), proving that said graph is closed. Given that both $H_1^{1,1}(\partial\Omega,\sigma)$ and $\mathrm{VMO}(\partial\Omega,\sigma)$ are Banach spaces, the Closed Graph Theorem implies that (2.3.130) is a bounded mapping. This establishes the continuity of the embedding in (2.3.125), That this embedding also has dense range is implied by [69, (3.1.51), (11.10.39)]. Finally, (2.3.126) is seen from (2.3.125). □

Remark 2.3.5 *Assume $\Omega \subseteq \mathbb{R}^2$ is an open set satisfying a two-sided local John condition, with a compact Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^1\lfloor\partial\Omega$. Then Proposition 2.3.4,* [69, Definition 11.10.6]*, and* [68, (7.4.105)] *imply that*

$$H_1^{1,1}(\partial\Omega,\sigma) = H_1^{q,1}(\partial\Omega,\sigma) \text{ for each } q \in [1,\infty). \tag{2.3.132}$$

In such a setting, we then proceed to define (*compare with* [69, Definition 11.10.9]*; see also* (A.0.32))

$$\mathrm{BMO}_{-1}(\partial\Omega,\sigma) := \Big(H_1^{1,1}(\partial\Omega,\sigma)\Big)^* = \Big(H_1^{q,1}(\partial\Omega,\sigma)\Big)^* \tag{2.3.133}$$

for each $q \in [1,\infty)$.

Also,

$$\begin{gathered}\text{for an arbitrary integrability exponent } q \in (1,\infty) \text{ we define}\\ \mathrm{VMO}_{-1}(\partial\Omega,\sigma) := \text{ the closure of } L^q(\partial\Omega,\sigma) \text{ in } \mathrm{BMO}_{-1}(\partial\Omega,\sigma).\end{gathered} \tag{2.3.134}$$

In relation to this, first we note that (2.3.133) *and* [69, (11.10.47)] *imply we have a well-defined continuous embedding*

$$L^q(\partial\Omega,\sigma) \hookrightarrow \mathrm{BMO}_{-1}(\partial\Omega,\sigma) \text{ for each } q \in (1,\infty) \tag{2.3.135}$$

which, in turn, makes the definition of $\mathrm{VMO}_{-1}(\partial\Omega,\sigma)$ given above meaningful. Also, the fact that $L^{q_1}(\partial\Omega,\sigma) \hookrightarrow L^{q_0}(\partial\Omega,\sigma)$ continuously and densely whenever $1 < q_0 < q_1 < \infty$ makes the definition of $\mathrm{VMO}_{-1}(\partial\Omega,\sigma)$ unambiguous.

Retain the setting of Remark 2.3.5. Then, as is apparent from definitions,

$$\begin{gathered}\mathrm{VMO}_{-1}(\partial\Omega,\sigma) \text{ is a closed subspace of } \mathrm{BMO}_{-1}(\partial\Omega,\sigma)\text{, and the space}\\ L^q(\partial\Omega,\sigma) \text{ embeds continuously and densely into } \mathrm{VMO}_{-1}(\partial\Omega,\sigma) \text{ for}\\ \text{each } q \in (1,\infty).\end{gathered} \tag{2.3.136}$$

In fact, since $\big\{\phi|_{\partial\Omega} : \phi \in \mathscr{C}^\infty(\mathbb{R}^2)\big\}$ is dense in any $L^q(\partial\Omega,\sigma)$ with $q \in (1,\infty)$ (cf. [68, (3.7.22)]), we conclude from (2.3.136) that the space $\{\phi|_{\partial\Omega} : \phi \in \mathscr{C}^\infty(\mathbb{R}^2)\}$ is dense in $\mathrm{VMO}_{-1}(\partial\Omega,\sigma)$; as a consequence, $\mathrm{VMO}_{-1}(\partial\Omega,\sigma)$ is the closure of $\mathrm{Lip}(\partial\Omega)$ in the space $\mathrm{BMO}_{-1}(\partial\Omega,\sigma)$. (2.3.137)

Finally, we wish to observe that, [69, (11.10.49)], (2.3.133), and [69, (1.2.25)] imply that the following compatibility result is valid for each $q, q' \in (1,\infty)$ satisfying $1/q + 1/q' = 1$:

$$\begin{aligned}\int_{\partial\Omega} fg\, d\sigma = {}_{L^{q'}(\partial\Omega,\sigma)}\big\langle f, g\big\rangle_{L^q(\partial\Omega,\sigma)} &= {}_{\mathrm{BMO}_{-1}(\partial\Omega,\sigma)}\big\langle f, g\big\rangle_{H_1^{q,1}(\partial\Omega,\sigma)} \\ \text{for all } f \in L^{q'}(\partial\Omega,\sigma) &\hookrightarrow \mathrm{BMO}_{-1}(\partial\Omega,\sigma) \\ \text{and all } g \in H_1^{q,1}(\partial\Omega,\sigma) &\hookrightarrow L^q(\partial\Omega,\sigma).\end{aligned} \tag{2.3.138}$$

We continue by presenting the versions of [69, Proposition 11.11.5] and [69, Proposition 11.11.6] corresponding to the forbidden values $n = 2$ and $p = 1$.

Proposition 2.3.6 *Let $\Omega \subseteq \mathbb{R}^2$ be an open set satisfying a two-sided local John condition, with a compact Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^1\lfloor\partial\Omega$. Then*

$$H^1(\partial\Omega,\sigma) \hookrightarrow \big(H_1^{1,1}(\partial\Omega,\sigma)\big)^* = \mathrm{BMO}_{-1}(\partial\Omega,\sigma) \ \textit{continuously}, \tag{2.3.139}$$

and

$$\begin{aligned}{}_{H^1(\partial\Omega,\sigma)}\big\langle f, g\big\rangle_{\mathrm{BMO}(\partial\Omega,\sigma)} &= {}_{\mathrm{BMO}_{-1}(\partial\Omega,\sigma)}\big\langle f, g\big\rangle_{H_1^{1,1}(\partial\Omega,\sigma)} \\ \textit{for every } f \in H^1(\partial\Omega,\sigma) &\hookrightarrow \mathrm{BMO}_{-1}(\partial\Omega,\sigma) \\ \textit{and } g \in H_1^{1,1}(\partial\Omega,\sigma) &\hookrightarrow \mathrm{VMO}(\partial\Omega,\sigma) \hookrightarrow \mathrm{BMO}(\partial\Omega,\sigma),\end{aligned} \tag{2.3.140}$$

where the above embeddings are provided by (2.3.139), (2.3.125)*, and* [69, (3.1.2)].

Proof That we have the continuous embedding claimed in (2.3.139) follows from (2.3.125), [69, (4.6.22)], and [69, Lemma 1.2.1]. A direct proof, which also gives the compatibility of the pairings claimed in (2.3.140) goes as follows. First, observe that there exists a constant $C \in (0,\infty)$ with the property that for each $\Lambda \in \big(\mathrm{VMO}(\partial\Omega,\sigma)\big)^*$ we have

$$\sup\Big\{|\Lambda f| : f \in H_1^{1,1}(\partial\Omega,\sigma),\ \|f\|_{H_1^{1,1}(\partial\Omega,\sigma)} \leq 1\Big\} \leq C\big\|\Lambda\big\|_{(\mathrm{VMO}(\partial\Omega,\sigma))^*}. \tag{2.3.141}$$

Indeed, this is a direct consequence of the continuity of the embedding in (2.3.125). We next consider

$$\begin{aligned}\Phi : \big(\mathrm{VMO}(\partial\Omega,\sigma)\big)^* &\longrightarrow \big(H_1^{1,1}(\partial\Omega,\sigma)\big)^* \ \text{defined as} \\ \big(\mathrm{VMO}(\partial\Omega,\sigma)\big)^* \ni \Lambda &\mapsto \Phi(\Lambda) := \Lambda\big|_{H_1^{1,1}(\partial\Omega,\sigma)} \in \big(H_1^{1,1}(\partial\Omega,\sigma)\big)^*.\end{aligned} \tag{2.3.142}$$

The fact that this is a well defined linear operator is then seen from (2.3.141) which actually implies that for each $\Lambda \in \big(\mathrm{VMO}(\partial\Omega,\sigma)\big)^*$ we have

$$\begin{aligned}&\Lambda\big|_{H^{1,1}_1(\partial\Omega,\sigma)} \text{ belongs to } \big(H^{1,1}_1(\partial\Omega,\sigma)\big)^* \text{ and}\\&\Big\|\Lambda\big|_{H^{1,1}_1(\partial\Omega,\sigma)}\Big\|_{(H^{1,1}_1(\partial\Omega,\sigma))^*} \leq C\|\Lambda\|_{(\mathrm{VMO}(\partial\Omega,\sigma))^*}.\end{aligned} \tag{2.3.143}$$

In fact, (2.3.143) also implies that Φ is continuous in the context of (2.3.142). Going further, we claim that Φ from (2.3.142) is injective. To this end, assume the functional $\Lambda \in \big(\mathrm{VMO}(\partial\Omega,\sigma)\big)^*$ is such that $\Phi(\Lambda) = 0$ in $\big(H^{1,1}_1(\partial\Omega,\sigma)\big)^*$. This forces $\Lambda f = 0$ for each function $f \in H^{1,1}_1(\partial\Omega,\sigma)$. In particular, $\Lambda f = 0$ for each $f \in \mathrm{Lip}(\partial\Omega)$, thanks to [69, (11.10.39)]. In concert with [69, (3.1.51)], this ultimately forces $\Lambda f = 0$ for each $f \in \mathrm{VMO}(\partial\Omega,\sigma)$, hence $\Lambda = 0$ as a functional in $\big(\mathrm{VMO}(\partial\Omega,\sigma)\big)^*$. This proves that Φ is indeed injective in the context of (2.3.142).

Next, with any given function $f \in H^1(\partial\Omega,\sigma)$, let us associate the functional $\Lambda_f \in \big(\mathrm{VMO}(\partial\Omega,\sigma)\big)^*$ by setting

$$\Lambda_f(\phi) := {}_{H^1(\partial\Omega,\sigma)}\big\langle f,\phi\big\rangle_{\mathrm{BMO}(\partial\Omega,\sigma)} \quad \text{for each } \phi \in \mathrm{VMO}(\partial\Omega,\sigma). \tag{2.3.144}$$

According to [69, (4.6.22)-(4.6.23)],

$$\begin{aligned}&\text{the assignment } H^1(\partial\Omega,\sigma) \ni f \longmapsto \Lambda_f \in \big(\mathrm{VMO}(\partial\Omega,\sigma)\big)^* \text{ is}\\&\text{the concrete embodiment of the implicit identification in the}\\&\text{formula } H^1(\partial\Omega,\sigma) = \big(\mathrm{VMO}(\partial\Omega,\sigma)\big)^*.\end{aligned} \tag{2.3.145}$$

Then

$$\begin{aligned}&\text{the inclusion in (2.3.139) is given by}\\&H^1(\partial\Omega,\sigma) \ni f \mapsto \Phi(\Lambda_f) \in \big(H^{1,1}_1(\partial\Omega,\sigma)\big)^*.\end{aligned} \tag{2.3.146}$$

Granted (2.3.145) and the properties of Φ established so far, it follows that the map in (2.3.146) is well defined, linear, continuous, and injective. Thus, (2.3.139) holds.

As far as (2.3.140) is concerned, pick two arbitrary functions, $f \in H^1(\partial\Omega,\sigma)$ and $g \in H^{1,1}_1(\partial\Omega,\sigma)$. Then, since $g \in H^{1,1}_1(\partial\Omega,\sigma) \hookrightarrow \mathrm{VMO}(\partial\Omega,\sigma)$ (cf. (2.3.125)), we may use (2.3.146), (2.3.142), and (2.3.144) to write

$$\begin{aligned}{}_{(H^{1,1}_1(\partial\Omega,\sigma))^*}\big\langle f,g\big\rangle_{H^{1,1}_1(\partial\Omega,\sigma)} &= {}_{(H^{1,1}_1(\partial\Omega,\sigma))^*}\big\langle \Phi(\Lambda_f),g\big\rangle_{H^{1,1}_1(\partial\Omega,\sigma)}\\&= {}_{(H^{1,1}_1(\partial\Omega,\sigma))^*}\Big\langle \Lambda_f\big|_{H^{1,1}_1(\partial\Omega,\sigma)},g\Big\rangle_{H^{1,1}_1(\partial\Omega,\sigma)}\\&= \Lambda_f(g) = {}_{H^1(\partial\Omega,\sigma)}\big\langle f,g\big\rangle_{\mathrm{BMO}(\partial\Omega,\sigma)}\end{aligned} \tag{2.3.147}$$

as wanted. □

We next study the action of the boundary-to-domain single layer potential operator on the negative Sobolev space BMO_{-1} in the two-dimensional setting.

Theorem 2.3.7 *Assume $\Omega \subseteq \mathbb{R}^2$ is an open set satisfying a two-sided local John condition whose boundary is a compact Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^1\lfloor\partial\Omega$. Let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M\times M$ system in $\mathbb{R}^2$ (for some $M \in \mathbb{N}$), which is weakly elliptic (in the sense of*

[70, (1.3.3) in Definition 1.3.1]) *Finally, denote by* $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ *the matrix-valued fundamental solution associated with the system* L *as in* [70, Theorem 1.4.2].

Then the boundary-to-domain single layer potential operator associated with L *and* Ω *as in* (1.3.6) *induces a well-defined linear mapping in the context*

$$\mathscr{S} : \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathscr{C}^\infty(\Omega)\big]^M \tag{2.3.148}$$

if for each $f = (f_\beta)_{1\leq\beta\leq M} \in \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$ *one sets*

$$\mathscr{S}f(x) := \Big(\big\langle E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta\big\rangle\Big)_{1\leq\alpha\leq M} \text{ for all } x\in\Omega, \tag{2.3.149}$$

where the pairing $\langle\cdot,\cdot\rangle$ *above stands for the duality bracket between the space* $H^{1,1}_1(\partial\Omega,\sigma)$ *and the space* $\mathrm{BMO}_{-1}(\partial\Omega,\sigma) = \Big(H^{1,1}_1(\partial\Omega,\sigma)\Big)^*$ (*cf.* (2.3.133)). *In addition, for each* $f \in \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$ *one has*

$$L\big(\mathscr{S}f\big) = 0 \text{ in } \Omega, \tag{2.3.150}$$

and for each $p\in(1,\infty)$ *there exists some constant* $C = C(\Omega,L,p)\in(0,\infty)$ *such that*

$$\begin{array}{l} |\nabla\mathscr{S}f|^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathscr{L}^2 \text{ is a Carleson measure in} \\ \text{the set } \Omega \text{ with constant} \leq C\|f\|^p_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}. \end{array} \tag{2.3.151}$$

In particular, corresponding to $p = 2$, *it follows that*[9]

$$\begin{array}{l} |\nabla\mathscr{S}f|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,d\mathscr{L}^2 \text{ is a Carleson measure in } \Omega \text{ with constant} \\ \leq C\|f\|^2_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}, \text{ for each functional } f\in\big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M. \end{array} \tag{2.3.152}$$

Also, for each given $f \in \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$, *the boundary trace*

$$\begin{array}{c} \mathscr{S}f\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \text{ and, in fact,} \\ \Big(\mathscr{S}f\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) = \big((S_{L^\top})^\top f\big)(x) \text{ at } \sigma\text{-a.e. point } x\in\partial\Omega, \end{array} \tag{2.3.153}$$

where $S_{L^\top}$ *is the boundary-to-boundary single layer potential operator associated with* $L^\top$, *considered as a mapping from* $\big[H^1(\partial\Omega,\sigma)\big]^M$ *into* $\big[H^{q,1}_1(\partial\Omega,\sigma)\big]^M$ *with* $q\in(1,\infty)$ (*cf.* (2.3.13)), *and* $(S_{L^\top})^\top$, *mapping*

$$\Big(\big[H^{q,1}_1(\partial\Omega,\sigma)\big]^M\Big)^* = \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M \tag{2.3.154}$$

into

$$\Big(\big[H^1(\partial\Omega,\sigma)\big]^M\Big)^* = \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M, \tag{2.3.155}$$

[9] it is natural to regard $\big|\nabla\mathscr{S}f\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,d\mathscr{L}^2$ as the Littlewood-Paley measure associated with f via the operator $\mathscr{S}$

stands for its (real) transpose (cf. the duality result from [69, Theorem 4.6.1] *and Remark 2.3.5).*

Finally, for each $f \in \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$ *one has*

$$\lim_{R\to 0^+}\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathscr{S}f\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathscr{L}^2\Big)^{\frac{1}{p}}$$
$$\leq C\operatorname{dist}\big(f,[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)]^M\big) \tag{2.3.156}$$

where the distance in the right-hand side is considered in $[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M$. *As a consequence,*

$$|\nabla\mathscr{S}f|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathscr{L}^2 \text{ is a vanishing Carleson measure in } \Omega \text{ for each } f\in\big[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)\big]^M \text{ and } p\in(1,\infty), \tag{2.3.157}$$

hence, corresponding to $p=2$,

$$|\nabla\mathscr{S}f|^2\operatorname{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathscr{L}^2 \text{ is a vanishing Carleson measure in } \Omega \text{ for each } f\in\big[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)\big]^M. \tag{2.3.158}$$

Proof Throughout, fix $q,q'\in(1,\infty)$ with $1/q+1/q'=1$. Then [69, Proposition 11.10.7] and Remark 2.3.5 guarantee the existence of a constant $C\in(0,\infty)$ with the property that for each functional

$$f=(f_\beta)_{1\leq\beta\leq M}\in\big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M=\Big(\big[H^{q,1}_1(\partial\Omega,\sigma)\big]^M\Big)^* \tag{2.3.159}$$

we can find two families of functions, $f_0^{(\beta)}\in L^{q'}(\partial\Omega,\sigma)$ and $f_{12}^{(\beta)}\in\mathrm{BMO}(\partial\Omega,\sigma)$ with $\beta\in\{1,\dots,M\}$, satisfying

$$\sum_{\beta=1}^M\Big\{\big\|f_0^{(\beta)}\big\|_{L^{q'}(\partial\Omega,\sigma)}+\big\|f_{12}^{(\beta)}\big\|_{\mathrm{BMO}(\partial\Omega,\sigma)}\Big\}\leq C\|f\|_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M} \tag{2.3.160}$$

as well as

$$\begin{aligned}&{}_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}\big\langle f,g\big\rangle_{[H^{q,1}_1(\partial\Omega,\sigma)]^M}\\&\qquad=\int_{\partial\Omega}f_0^{(\beta)}g_\beta\,\mathrm{d}\sigma+{}_{\mathrm{BMO}(\partial\Omega,\sigma)}\big\langle f_{12}^{(\beta)},\partial_{\tau_{12}}g_\beta\big\rangle_{H^1(\partial\Omega,\sigma)}\end{aligned} \tag{2.3.161}$$

for every function $g=(g_\beta)_{1\leq\beta\leq M}\in\big[H^{q,1}_1(\partial\Omega,\sigma)\big]^M$.

Let us abbreviate $f_0:=(f_0^{(\beta)})_{1\leq\beta\leq M}\in\big[L^{q'}(\partial\Omega,\sigma)\big]^M$. Also, for each pair of indeces $\alpha,\beta\in\{1,\dots,M\}$ denote by $\mathcal{T}_{12}^{\alpha\beta}$ the integral operator defined as in (2.3.80) with $j:=1$ and $k:=2$. Then the same argument which has produced (2.3.81) currently gives

$$\mathscr{S} f = \mathscr{S} f_0 - \big(\mathcal{T}_{12}^{\alpha\beta} f_{12}^{(\beta)}\big)_{1 \le \alpha \le M} \text{ in } \Omega. \tag{2.3.162}$$

From (2.3.162) it is then clear that (2.3.148) is a well-defined linear mapping and that (2.3.150) holds. In addition, the same type of argument as in (2.3.82)-(2.3.87) leads to the conclusion that the boundary trace formula (2.3.153) is true.

Next, the claims in (2.3.151) follow from (2.3.162), [70, Corollary 2.4.2], and (2.3.160), as soon as we show that:

$$\begin{array}{l} \text{if } p, q \in (1,\infty) \text{ and } h \in \big[L^q(\partial\Omega,\sigma)\big]^M \text{ then} \\ |\nabla \mathscr{S} h|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^2 \text{ is a Carleson measure} \\ \text{in } \Omega \text{ with constant} \le C\|h\|^p_{[L^q(\partial\Omega,\sigma)]^M}. \end{array} \tag{2.3.163}$$

To justify (2.3.163), fix $h \in \big[L^q(\partial\Omega,\sigma)\big]^M$ and $p \in (1,\infty)$. Set $u := \mathscr{S} h$ in Ω. Pick $R \in (0,\infty)$ large enough so that $\big\{x \in \mathbb{R}^2 : \operatorname{dist}(x,\partial\Omega) \le 2 \operatorname{diam} \partial\Omega\big\}$ is contained in $B(0,R)$. We claim that

$$\begin{array}{l} |\nabla u|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1} \in L^2(\Omega_R,\mathcal{L}^2) \text{ where } \Omega_R := \Omega \cap B(0,R) \text{ and} \\ \Big\| |\nabla u|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\Big\|_{L^2(\Omega_R,\mathcal{L}^2)} \le C\|h\|^p_{[L^q(\partial\Omega,\sigma)]^M}. \end{array} \tag{2.3.164}$$

Assuming this claim for the time being, for each $x \in \partial\Omega$ and $r \in \big(0, 2 \operatorname{diam} \partial\Omega\big)$ we may estimate

$$\begin{aligned} &\int_{\Omega \cap B(x,r)} |\nabla \mathscr{S} h|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^2 \\ &\qquad \le \Big(\int_{\Omega\cap B(x,r)} |\nabla u|^{2p} \operatorname{dist}(\cdot,\partial\Omega)^{2(p-1)}\,\mathrm{d}\mathcal{L}^2\Big)^{1/2} \Big(\mathcal{L}^2\big(\Omega \cap B(x,r)\big)\Big)^{1/2} \\ &\qquad \le \Big\| |\nabla u|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\Big\|_{L^2(\Omega_R,\mathcal{L}^2)} \cdot \Big(\mathcal{L}^2\big(B(x,r)\big)\Big)^{1/2} \\ &\qquad \le C r \|h\|^p_{[L^q(\partial\Omega,\sigma)]^M}, \end{aligned} \tag{2.3.165}$$

from which (2.3.163) follows. There remains to prove the claim made in (2.3.164). To this end, observe that (1.5.56) and (1.5.57) presently give (for some fixed aperture parameter $\kappa > 0$)

$$\begin{aligned} \Big\| \operatorname{dist}(\cdot,\partial\Omega)^{1/q} \cdot \big|\nabla u\big| \Big\|_{L^\infty(\Omega,\mathcal{L}^2)} &\le C \big\|\mathcal{N}_\kappa(\nabla u)\big\|_{L^q(\partial\Omega,\sigma)} \\ &\le C\|h\|_{[L^q(\partial\Omega,\sigma)]^M}. \end{aligned} \tag{2.3.166}$$

In concert with [68, (8.6.51)] this permits us to conclude that

$$\Big\| \operatorname{dist}(\cdot,\partial\Omega)^{1/q} \cdot \big|\nabla u\big| \Big\|_{L^\infty(\Omega_R,\mathcal{L}^2)} \le C\|h\|_{[L^q(\partial\Omega,\sigma)]^M}, \tag{2.3.167}$$

and

$$\Big\||\nabla u|\Big\|_{L^{2q}(\Omega_R,\mathcal{L}^2)} \leq C\|h\|_{[L^q(\partial\Omega,\sigma)]^M}. \tag{2.3.168}$$

If $q \leq p$, use (2.3.167)-(2.3.168) to write

$$\begin{aligned}
\Big\||\nabla u|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\Big\|^2_{L^2(\Omega_R,\mathcal{L}^2)} &= \int_{\Omega_R} |\nabla u|^{2p} \operatorname{dist}(\cdot,\partial\Omega)^{2(p-1)}\,\mathrm{d}\mathcal{L}^2 \\
&= \int_{\Omega_R} |\nabla u|^{2q}\Big(|\nabla u||\operatorname{dist}(\cdot,\partial\Omega)^{1/q}\Big)^{2(p-q)}\operatorname{dist}(\cdot,\partial\Omega)^{2p(1-1/q)}\,\mathrm{d}\mathcal{L}^2 \\
&\leq (2R)^{2p(1-1/q)}\Big\|\operatorname{dist}(\cdot,\partial\Omega)^{1/q}\cdot\big|\nabla u\big|\Big\|^{2(p-q)}_{L^\infty(\Omega_R,\mathcal{L}^2)}\Big\||\nabla u|\Big\|^{2q}_{L^{2q}(\Omega_R,\mathcal{L}^2)} \\
&\leq C(2R)^{2p(1-1/q)}\|h\|^{2p}_{[L^q(\partial\Omega,\sigma)]^M},
\end{aligned} \tag{2.3.169}$$

from which (2.3.164) follows. If $p < q$, use Hölder's inequality and (2.3.168) to write

$$\begin{aligned}
\Big\||\nabla u|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\Big\|^2_{L^2(\Omega_R,\mathcal{L}^2)} &= \int_{\Omega_R} |\nabla u|^{2p} \operatorname{dist}(\cdot,\partial\Omega)^{2(p-1)}\,\mathrm{d}\mathcal{L}^2 \\
&\leq \Big(\int_{\Omega_R} |\nabla u|^{2q}\operatorname{dist}(\cdot,\partial\Omega)^{2q(p-1)/p}\,\mathrm{d}\mathcal{L}^2\Big)^{p/q}\cdot\Big(\mathcal{L}^2(\Omega_R)\Big)^{1-p/q} \\
&\leq C(2R)^{2(p-1)}R^{2(1-p/q)}\Big\||\nabla u|\Big\|^{2p}_{L^{2q}(\Omega_R,\mathcal{L}^2)} \\
&\leq CR^{2p(1-1/q)}\|h\|^{2p}_{[L^q(\partial\Omega,\sigma)]^M},
\end{aligned} \tag{2.3.170}$$

from which (2.3.164) once again follows. This completes the proof of (2.3.151).

Finally, let us justify (2.3.156). To this end, pick some $f \in \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$ and select $p \in (1,\infty)$. Also, choose an arbitrary function $g \in \big[L^p(\partial\Omega,\sigma)\big]^M$. In particular, $g \in \big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^M$ by (2.3.135). Then for each $r \in \big(0, 2\operatorname{diam}(\partial\Omega)\big)$ and $x \in \partial\Omega$ we may estimate

$$\begin{aligned}
\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathscr{S}(f-g)\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^2\Big)^{\frac{1}{p}} \\
\leq C\|f-g\|_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M},
\end{aligned} \tag{2.3.171}$$

thanks to (2.3.151) (written for $f-g$ in place of the function f). In addition, with $\eta := 1-1/p \in (0,1)$ we have

$$\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathscr{S}g\big)\big|^{p}\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^2\Big)^{\frac{1}{p}}$$
$$\leq C\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\operatorname{dist}(\cdot,\partial\Omega)^{p\eta-1}\,d\mathcal{L}^2\Big)^{\frac{1}{p}}\|g\|_{[L^p(\partial\Omega,\sigma)]^M}$$
$$\leq Cr^{\eta}\|g\|_{[L^p(\partial\Omega,\sigma)]^M}. \tag{2.3.172}$$

The first inequality above uses the version of (1.5.57) written for $\mathscr{S}$ in place of $\mathscr{S}_{\text{mod}}$ (and for g in place of f), while the second inequality is based on [68, (8.6.101)] used with $n := 2$, $\lambda := 1 - p\eta$, $\alpha := 1$, $\beta := 1$, $E := B(x,r)\cap\Omega$, and [68, (8.1.17)]. Together, (2.3.171) and (2.3.172) imply

$$\lim_{R\to 0^+}\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathscr{S}f\big)\big|^{p}\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^2\Big)^{\frac{1}{p}}$$
$$\leq C\|f-g\|_{[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)]^M}, \tag{2.3.173}$$

for some constant $C\in(0,\infty)$ independent of f and g. With this in hand, (2.3.156) follows on account of the arbitrariness of the function g in $\big[L^p(\partial\Omega,\sigma)\big]^M$ and the fact that the latter space is dense in $[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)]^M$ (see (2.3.136)). □

In a favorable geometric setting, the homogeneous Hardy-based Sobolev space $\dot{H}^p_1$ (modulo constants) becomes a quasi-Banach space when equipped with the quasi-norm assigning to each function the sum of the H^p quasi-norms of its tangential derivatives. This is made precise in our next proposition.

Proposition 2.3.8 *Assume $\Omega\subseteq\mathbb{R}^n$, where $n\in\mathbb{N}$ with $n\geq 2$, is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and pick some exponent $p\in\big(\frac{n-1}{n},1\big]$. Let $\dot{H}^p_1(\partial\Omega,\sigma)/\sim$ denote the quotient space of classes $[\cdot]$ of equivalence modulo constants of functions in $\dot{H}^p_1(\partial\Omega,\sigma)$, equipped with the semi-quasinorm (A.0.92). Then (A.0.92) is a genuine quasinorm on $\dot{H}^p_1(\partial\Omega,\sigma)/\sim$, and $\dot{H}^p_1(\partial\Omega,\sigma)/\sim$ is a quasi-Banach space when equipped with the quasinorm (A.0.92).*

Proof That the semi-quasinorm (A.0.92) is actually a quasinorm on the space $\dot{H}^p_1(\partial\Omega,\sigma)/\sim$ is readily seen from (2.3.99) if $n\geq 3$ and (2.3.124) if $n=2$. To show that $\dot{H}^p_1(\partial\Omega,\sigma)/\sim$ is complete when equipped with the quasinorm (A.0.92), let $\{f_\alpha\}_{\alpha\in\mathbb{N}}\subseteq\dot{H}^p_1(\partial\Omega,\sigma)$ be such that $\big\{[f_\alpha]\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in the quotient space $\dot{H}^p_1(\partial\Omega,\sigma)/\sim$. Then for each fixed $j,k\in\{1,\dots,n\}$ it follows that $\big\{\partial_{\tau_{jk}}f_\alpha\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $H^p(\partial\Omega,\sigma)$. As the latter space is complete, it follows that there exists $g_{jk}\in H^p(\partial\Omega,\sigma)$ such that

$$\partial_{\tau_{jk}}f_\alpha\longrightarrow g_{jk}\ \text{ in } H^p(\partial\Omega,\sigma)\ \text{ as }\alpha\to\infty. \tag{2.3.174}$$

To proceed, we shall consider two cases.

`Case I:` *Assume that either $n \geq 3$, or $n = 2$ and we have $p \in \big(\tfrac{1}{2}, 1\big)$.* Define the exponent $p_* := \big(\frac{1}{p} - \frac{1}{n-1}\big)^{-1}$ and note that, thanks to the working assumptions, $p_* \in (1, \infty)$. The current assumptions also allow us to invoke Theorem 2.3.3 which guarantees the existence of some constant $C = C(\Omega, n, p) \in (0, \infty)$ with the property that for each function $f \in \dot{H}^p_1(\partial\Omega, \sigma)$ we may find $c(f) \in \mathbb{C}$ such that $f - c(f)$ belongs to the space $L^{p_*}(\partial\Omega, \sigma)$ and

$$\|f - c(f)\|_{L^{p_*}(\partial\Omega,\sigma)} \leq C \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{H^p(\partial\Omega,\sigma)} = C\|f\|_{\dot{H}^p_1(\partial\Omega,\sigma)}. \tag{2.3.175}$$

Since $\partial\Omega$ is unbounded, it follows that $c(f)$ is uniquely determined by f. In particular, the assignment $\dot{H}^p_1(\partial\Omega, \sigma) \ni f \mapsto c(f) \in \mathbb{C}$ is linear. Hence, if we set

$$\widetilde{f_\alpha} := f_\alpha - c(f_\alpha) \text{ for each } \alpha \in \mathbb{N}, \tag{2.3.176}$$

it follows that

$$\begin{aligned}
\|\widetilde{f_\alpha} - \widetilde{f_\beta}\|_{L^{p_*}(\partial\Omega,\sigma)} &= \big\|f_\alpha - c(f_\alpha) - \big(f_\beta - c(f_\beta)\big)\big\|_{L^{p_*}(\partial\Omega,\sigma)} \\
&= \big\|(f_\alpha - f_\beta) - c(f_\alpha - f_\beta)\big\|_{L^{p_*}(\partial\Omega,\sigma)} \\
&\leq C\|f_\alpha - f_\beta\|_{\dot{H}^p_1(\partial\Omega,\sigma)} = C\big\|[f_\alpha - f_\beta]\big\|_{\dot{H}^p_1(\partial\Omega,\sigma)/\sim} \\
&= C\big\|[f_\alpha] - [f_\beta]\big\|_{\dot{H}^p_1(\partial\Omega,\sigma)/\sim} \text{ for each } \alpha, \beta \in \mathbb{N}.
\end{aligned} \tag{2.3.177}$$

In view of the fact that $\big\{[f_\alpha]\big\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $\dot{H}^p_1(\partial\Omega, \sigma)\big/\sim$, this implies that $\{\widetilde{f_\alpha}\}_{\alpha\in\mathbb{N}}$ is a Cauchy sequence in $L^{p_*}(\partial\Omega, \sigma)$. In particular, there exists $f \in L^{p_*}(\partial\Omega, \sigma)$ such that

$$\widetilde{f_\alpha} \longrightarrow f \text{ in } L^{p_*}(\partial\Omega, \sigma) \text{ as } \alpha \to \infty. \tag{2.3.178}$$

In addition, since $\partial_{\tau_{jk}} \widetilde{f_\alpha} = \partial_{\tau_{jk}} f_\alpha$ in $\big(\mathrm{Lip}_c(\partial\Omega)\big)'$ for each $j, k \in \{1, \dots, n\}$ and $\alpha \in \mathbb{N}$, from (2.3.174) we conclude that for each $j, k \in \{1, \dots, n\}$ we have

$$\partial_{\tau_{jk}} \widetilde{f_\alpha} \longrightarrow g_{jk} \text{ in } H^p(\partial\Omega, \sigma) \text{ as } \alpha \to \infty. \tag{2.3.179}$$

Consequently, for each $j, k \in \{1, \dots, n\}$ and each test function $\varphi \in \mathrm{Lip}_c(\partial\Omega)$ we may write

$$
\begin{aligned}
{}_{(\mathrm{Lip}_c(\partial\Omega))'}\big\langle \partial_{\tau_{jk}} f, \varphi \big\rangle_{\mathrm{Lip}_c(\partial\Omega)} &= -\int_{\partial\Omega} f(\partial_{\tau_{jk}}\varphi)\,\mathrm{d}\sigma \\
&= -\lim_{\alpha\to\infty}\int_{\partial\Omega} \widetilde{f_\alpha}(\partial_{\tau_{jk}}\varphi)\,\mathrm{d}\sigma \\
&= \lim_{\alpha\to\infty} {}_{(\mathrm{Lip}_c(\partial\Omega))'}\big\langle \partial_{\tau_{jk}} \widetilde{f_\alpha}, \varphi \big\rangle_{\mathrm{Lip}_c(\partial\Omega)} \\
&= \lim_{\alpha\to\infty} {}_{H^p(\partial\Omega)}\big\langle \partial_{\tau_{jk}} \widetilde{f_\alpha}, [\varphi] \big\rangle_{(H^p(\partial\Omega))^*} \\
&= {}_{H^p(\partial\Omega)}\big\langle g_{jk}, [\varphi] \big\rangle_{(H^p(\partial\Omega))^*} \\
&= {}_{(\mathrm{Lip}_c(\partial\Omega))'}\big\langle g_{jk}, \varphi \big\rangle_{\mathrm{Lip}_c(\partial\Omega)},
\end{aligned}
\tag{2.3.180}
$$

thanks to [69, Definition 11.2.1], (2.3.178), the duality result recorded in [69, Theorem 4.6.1], [69, Lemma 4.6.4], and (2.3.179). In turn, this implies that

$$
\partial_{\tau_{jk}} f = g_{jk} \in H^p(\partial\Omega,\sigma) \text{ for each } j,k\in\{1,\dots,n\}. \tag{2.3.181}
$$

Upon recalling that $f \in L^{p_*}(\partial\Omega,\sigma) \subseteq L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)$, we then conclude that $f \in \dot{H}^p_1(\partial\Omega,\sigma)$.

Finally, from (2.3.174), (2.3.181), and (A.0.92) we deduce that the sequence $\{[f_\alpha]\}_{\alpha\in\mathbb{N}}$ converges to $[f]$, the class of f, in the quotient space $\dot{H}^p_1(\partial\Omega,\sigma)/\sim$. This proves that, in Case I, the space $\dot{H}^p_1(\partial\Omega,\sigma)/\sim$ is indeed complete when equipped with the quasinorm (A.0.92).

`Case II:` *Assume* $n = 2$ *and* $p = 1$. From Proposition 2.3.4 and the fact that $\mathrm{BMO}(\partial\Omega,\sigma)$ is complete, it follows that there exists some $f \in \mathrm{BMO}(\partial\Omega,\sigma)$ such that

$$
f_\alpha \longrightarrow f \text{ in } \mathrm{BMO}(\partial\Omega,\sigma) \text{ as } \alpha\to\infty. \tag{2.3.182}
$$

In view of [68, (7.4.118)], we then have $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^2}\Big)$ and $f_\alpha \longrightarrow f$ in $L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$ as $\alpha\to\infty$. Granted these, then the same type of argument as before shows

$$
\partial_{\tau_{jk}} f = g_{jk} \in H^1(\partial\Omega,\sigma) \text{ for each } j,k\in\{1,\dots,2\}, \tag{2.3.183}
$$

where the functions $g_{jk} \in H^1(\partial\Omega,\sigma)$ are as in (2.3.174) with $p = 1$. Hence, f belongs to $\dot{H}^1_1(\partial\Omega,\sigma)$ and the sequence $\{[f_\alpha]\}_{\alpha\in\mathbb{N}}$ converges to $[f]$, the class of f, in the quotient space $\dot{H}^1_1(\partial\Omega,\sigma)/\sim$. The final conclusion is that, in the current case, the space $\dot{H}^1_1(\partial\Omega,\sigma)/\sim$ is complete when equipped with the quasinorm (A.0.92). □

Next we study the mapping properties of the modified boundary-to-boundary double layer potential operator acting on the scale of homogeneous Hardy-based Sobolev space $\dot{H}^p_1$ (modulo constants).

Theorem 2.3.9 *Fix $n, M \in \mathbb{N}$ with $n \geq 2$. Assume Ω is an* NTA *domain with an Ahlfors regular boundary*[10] *and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, consider a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ having complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in* (1.3.2) *is weakly elliptic* (*in the sense of* [70, (1.3.3) in Definition 1.3.1]). *Lastly, fix an aperture parameter $\kappa \in (0, \infty)$ along with an exponent $p \in \big(\frac{n-1}{n}, 1\big]$.*

Then the modified boundary-to-boundary double layer potential operator $K_{\rm mod}$, originally defined as in (1.8.24)*, induces a linear and bounded mapping*

$$K_{\rm mod} : \big[\dot{H}^p_1(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\dot{H}^p_1(\partial\Omega, \sigma)\big]^M. \tag{2.3.184}$$

As a consequence of this and (1.8.28)*, the operator*

$$\begin{gathered}\big[K_{\rm mod}\big] : \big[\dot{H}^p_1(\partial\Omega, \sigma)/\sim\big]^M \longrightarrow \big[\dot{H}^p_1(\partial\Omega, \sigma)/\sim\big]^M \ \textit{defined as}\\ \big[K_{\rm mod}\big][f] := \big[K_{\rm mod} f\big] \in \big[\dot{H}^p_1(\partial\Omega, \sigma)/\sim\big]^M, \quad \forall f \in \big[\dot{H}^p_1(\partial\Omega, \sigma)\big]^M,\end{gathered} \tag{2.3.185}$$

is well defined, linear, and bounded, when all quotient spaces are equipped with the semi-quasinorm[11] *introduced in* (A.0.92).

Proof Pick $f \in \big[\dot{H}^p_1(\partial\Omega, \sigma)\big]^M$ and set $u := \mathcal{D}_{\rm mod} f$ in Ω. From (A.0.89), (1.8.7), and item *(i)* in Theorem 2.3.1 we know that $u : \Omega \to \mathbb{C}^M$ is a well-defined function which, for any aperture parameter $\kappa > 0$, satisfies

$$\begin{gathered}u \text{ belongs to } \big[\mathscr{C}^\infty(\Omega)\big]^M \text{ and } Lu = 0 \text{ in } \Omega,\\ u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{\rm mod}\big)f \text{ at } \sigma\text{-a.e. point on } \partial\Omega,\\ \mathcal{N}_\kappa(\nabla u) \text{ belongs to the space } L^p(\partial\Omega, \sigma),\\ \text{and } \big\|\mathcal{N}_\kappa(\nabla u)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M},\end{gathered} \tag{2.3.186}$$

for some constant $C = C(\Omega, n, \kappa, p, L) \in (0, \infty)$. In particular, Theorem 2.2.8 applies and gives that the nontangential boundary trace $u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ belongs to $\big[\dot{H}^p_1(\partial\Omega, \sigma)\big]^M$, and (2.2.218) holds. In turn, this implies that $K_{\rm mod} f$ belongs to $\big[\dot{H}^p_1(\partial\Omega, \sigma)\big]^M$ and, for some constant $C = C(\Omega, n, \kappa, p, L) \in (0, \infty)$ independent of f, we may write

[10] in particular, this is the case if Ω is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular; cf. (1.8.157)

[11] Proposition 2.3.8 tells us that if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then said semi-quasinorm is actually a genuine quasinorm, and $\dot{H}^p_1(\partial\Omega, \sigma)/\sim$ becomes a quasi-Banach space

$$
\begin{aligned}
\|K_{\rm mod} f\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M} &\leq C\big\|\big(\tfrac{1}{2}I + K_{\rm mod}\big)f\,\big\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M} + C\|f\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M} \\
&\leq C\Big\|u\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M} + C\|f\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M} \\
&\leq C\big\|\mathcal{N}_\kappa(\nabla u)\big\|_{L^p(\partial\Omega,\sigma)} + C\|f\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M} \\
&\leq C\|f\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M}.
\end{aligned}
\tag{2.3.187}
$$

Thus, $K_{\rm mod}$ induces a well-defined, linear, and bounded mapping in the context of (2.3.184). The claims pertaining to (2.3.185) readily follow from this. □

Remark 2.3.10 *The results in Theorem 2.3.9 apply to all modified boundary-to-boundary double layer potential operators* $K_{\rm mod}$ *described in Examples 1.8.4-1.8.7.*

We are now in a position to augment Theorem 1.8.26 with composition identities involving the modified boundary-to-boundary double layer, its transpose version, the modified boundary-to-boundary single layer, and the conormal derivative of the modified double layer, now considered on homogeneous Hardy-based Sobolev spaces (for related results, see also Theorem 2.3.18).

Theorem 2.3.11 *Assume* $\Omega \subseteq \mathbb{R}^n$ *(where* $n \in \mathbb{N}$, $n \geq 2$*) is an* NTA *domain whose boundary is an Ahlfors regular set. Denote by* ν *the geometric measure theoretic outward unit normal to* Ω *and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$*. Next, for some* $M \in \mathbb{N}$*, let* $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ *be a complex coefficient tensor with the property that*

$$
L := \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}
\tag{2.3.188}
$$

(i.e., $L := L_A$*) is a weakly elliptic* $M \times M$ *system in* $\mathbb{R}^n$*. Having fixed an exponent*

$$
p \in \big(\tfrac{n-1}{n}, 1\big],
\tag{2.3.189}
$$

recall the operators S *from (2.3.10),* $\dot{\partial}^A_\nu \mathcal{D}_{\rm mod}$ *from (2.3.5), and* $K_{\rm mod}$ *from (2.3.184). Finally, let* $K^{\#}_{A^\top}$ *be the operator associated with the coefficient tensor* $A^\top$ *and the set* Ω *as in (2.1.4). Then the following statements are true.*

(1) For each $f \in \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^M$ *there exists* c_f*, which is the nontangential trace on* $\partial\Omega$ *of some* $\mathbb{C}^M$*-valued locally constant function in* Ω*, with the property that at* σ*-a.e. point on* $\partial\Omega$ *one has*

$$
\big(\tfrac{1}{2}I + K_{\rm mod}\big)\Big(\big(-\tfrac{1}{2}I + K_{\rm mod}\big)f\Big) = S\Big(\big(\dot{\partial}^A_\nu \mathcal{D}_{\rm mod}\big)f\Big) + c_f.
\tag{2.3.190}
$$

(2) For each function $f \in \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^M$ *one has*

$$
\big(\dot{\partial}^A_\nu \mathcal{D}_{\rm mod}\big)\big(K_{\rm mod} f\big) = K^{\#}_{A^\top}\big(\dot{\partial}^A_\nu \mathcal{D}_{\rm mod}\big)f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega.
\tag{2.3.191}
$$

(3) For each $f \in [H^p(\partial\Omega, \sigma)]^M$ *there exists* c_f*, which is the nontangential trace on* $\partial\Omega$ *of some* $\mathbb{C}^M$*-valued locally constant function in* Ω*, with the property that*

$$S\big(K^{\#}_{A^\top} f\big) = K_{\text{mod}}\big(Sf\big) + c_f \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.3.192}$$

(4) For each $f \in [H^p(\partial\Omega, \sigma)]^M$*, at* σ*-a.e. point on* $\partial\Omega$ *one has*

$$\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big) = \big(\dot{\partial}^A_\nu \mathcal{D}_{\text{mod}}\big)\big(Sf\big). \tag{2.3.193}$$

Proof The argument largely proceeds as in the proof of Theorem 1.8.26, now substituting the integral representation formulas (1.8.176) from Theorem 1.8.17, (1.8.194) from Corollary 1.8.18, and (1.8.200) from Theorem 1.8.19, with the integral representation formula (2.2.168) from Theorem 2.2.7, and keeping in mind the mapping properties for layer potentials from Theorem 2.1.1, Theorem 2.3.1, Theorem 2.3.9, Theorem 2.2.3, and Theorem 2.2.6.

Turning to specifics, observe first that the present hypotheses imply (cf. [68, (5.10.33)]) that Ω is a UR domain. Fix an aperture parameter $\kappa \in (0, \infty)$. To justify the claim made in item *(1)*, pick some arbitrary function $f \in [\dot{H}^p_1(\partial\Omega, \sigma)]^M$ and define $u := \mathcal{D}_{\text{mod}} f$ in Ω. In view of (A.0.89), we see that

$$f \in L^1\Big[\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^n}\Big)\Big]^M \tag{2.3.194}$$

which goes to show that u is a well-defined function (cf. (1.8.5)-(1.8.6)). In addition, (1.8.7), (1.8.27), (2.3.3), (2.3.4), and (2.3.5) guarantee that

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma), \\ \text{the boundary traces } u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}, \ (\nabla u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \\ u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{\text{mod}}\big)f \in \big[\dot{H}^p_1(\partial\Omega, \sigma)\big]^M, \text{ and} \\ \dot{\partial}^A_\nu u = \big(\dot{\partial}^A_\nu \mathcal{D}_{\text{mod}}\big)f \in \big[H^p(\partial\Omega, \sigma)\big]^M. \end{gathered} \tag{2.3.195}$$

Moreover, [70, Theorem 1.4.2] implies that if Ω is an exterior domain we have

$$\fint_{B(0,2R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{2.3.196}$$

Granted these properties, Theorem 2.2.7 applies, and according to the integral representation formula (2.2.168) we may write

$$\begin{aligned} u &= \mathcal{D}_{\text{mod}}\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big) - \mathscr{S}\big(\dot{\partial}^A_\nu u\big) + c_u \\ &= \mathcal{D}_{\text{mod}}\Big(\big(\tfrac{1}{2}I + K_{\text{mod}}\big)f\Big) - \mathscr{S}\Big(\big(\dot{\partial}^A_\nu \mathcal{D}_{\text{mod}}\big)f\Big) + c_u \text{ in } \Omega, \end{aligned} \tag{2.3.197}$$

for some $\mathbb{C}^M$-valued locally constant function c_u in Ω. The idea is to now take non-tangential boundary traces in (2.3.197). With $c_f := c_u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$, on account of (1.8.27), (2.2.127) (if $n \geq 3$), and (2.2.132) (plus the comment right after (2.2.133), if $n = 2$), we obtain

$$\big(\tfrac{1}{2}I + K_{\rm mod}\big)f = \big(\tfrac{1}{2}I + K_{\rm mod}\big)\Big(\big(\tfrac{1}{2}I + K_{\rm mod}\big)f\Big) - S\Big(\big(\dot{\partial}_\nu^A \mathcal{D}_{\rm mod}\big)f\Big) + c_f \tag{2.3.198}$$

at σ-a.e. point on $\partial\Omega$. With this in hand, (2.3.190) readily follows, after simple algebra. This takes care of the claim made in item *(1)*.

Taking weak conormal derivatives in (2.3.197) leads to

$$\begin{aligned}\big(\dot{\partial}_\nu^A \mathcal{D}_{\rm mod}\big)f &= \big(\dot{\partial}_\nu^A \mathcal{D}_{\rm mod}\big)\Big(\big(\tfrac{1}{2}I + K_{\rm mod}\big)f\Big) - \dot{\partial}_\nu^A \mathscr{S}\Big(\big(\dot{\partial}_\nu^A \mathcal{D}_{\rm mod}\big)f\Big) + \dot{\partial}_\nu^A c_u \\ &= \big(\dot{\partial}_\nu^A \mathcal{D}_{\rm mod}\big)\Big(\big(\tfrac{1}{2}I + K_{\rm mod}\big)f\Big) - \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)\Big(\big(\dot{\partial}_\nu^A \mathcal{D}_{\rm mod}\big)f\Big),\end{aligned} \tag{2.3.199}$$

thanks to (2.3.5), (2.2.47), and the fact that $\dot{\partial}_\nu^A c_u = 0$ (as may be seen from [69, Definition 10.2.18]). Having established (2.3.199), the claim made in (2.3.191) readily follows after some simple algebra. The treatment of item *(2)* is therefore complete.

Consider next the claims made in items *(3)-(4)*. To set the stage, pick an arbitrary distribution $f \in \big[H^p(\partial\Omega,\sigma)\big]^M$ and define $u := \mathscr{S}f$ in Ω. Then Theorem 2.2.3 ensures that u is a well-defined function in Ω satisfying:

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma), \\ \text{the boundary traces } u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}},\ (\nabla u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \\ u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = Sf \in \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^M, \\ \dot{\partial}_\nu^A u = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)f \in \big[H^p(\partial\Omega,\sigma)\big]^M, \\ \text{and } (\nabla u)(x) = O(|x|^{1-n}) \text{ as } |x| \to \infty. \end{gathered} \tag{2.3.200}$$

See (2.2.32), (2.2.33), (2.2.35), (2.2.44), (2.2.47), (2.2.127) (if $n \geq 3$), and (2.2.132) (plus the comment right after (2.2.133), if $n = 2$). We should also point out that the memberships in lines 3-4 above come from (2.3.10) and (2.1.4), respectively. In particular, in the case when Ω is an exterior domain, the last property listed in (2.3.200) entails

$$⨍_{B(0,2R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{2.3.201}$$

In turn, these properties permit us to invoke Theorem 2.2.7, and the integral representation formula (2.2.168) presently gives that there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω such that

$$
\begin{aligned}
u &= \mathcal{D}_{\text{mod}}\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) - \mathcal{S}\big(\partial_\nu^A u\big) + c_u \\
&= \mathcal{D}_{\text{mod}}(Sf) - \mathcal{S}\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big) \text{ in } \Omega.
\end{aligned} \tag{2.3.202}
$$

To proceed, define $c_f := c_u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$. After taking nontangential boundary traces in (2.3.202) we arrive at

$$
Sf = \big(\tfrac{1}{2}I + K_{\text{mod}}\big)(Sf) - S\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big) + c_f \tag{2.3.203}
$$

at σ-a.e. point on $\partial\Omega$, thanks to (2.3.200), (2.2.127) (if $n \geq 3$), and (2.2.132) (plus the comment right after (2.2.133), if $n = 2$), also bearing in mind the memberships $\big(-\frac{1}{2}I + K^{\#}_{A^\top}\big)f \in \big[H^p(\partial\Omega,\sigma)\big]^M$ (see Theorem 2.1.1) and $Sf \in \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^M$ (see (2.3.10)). Having proved (2.3.203), the claim in (1.8.252) follows after simple algebra. This takes care of item *(3)*.

To justify (2.3.193), observe that taking weak conormal derivatives in (2.3.202) leads to

$$
\begin{aligned}
\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f &= \dot{\partial}_\nu^A u = \dot{\partial}_\nu^A \mathcal{D}_{\text{mod}}(Sf) - \dot{\partial}_\nu^A \mathcal{S}\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big) + \dot{\partial}_\nu^A c_u \\
&= \dot{\partial}_\nu^A \mathcal{D}_{\text{mod}}(Sf) - \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big),
\end{aligned} \tag{2.3.204}
$$

by virtue of (2.3.200), (2.3.5), (2.2.47), and the fact that $\dot{\partial}_\nu^A c_u = 0$ (as may be seen from [69, Definition 10.2.18]). At this stage, (2.3.193) follows from (2.3.204) after canceling like-terms. □

Remark 2.3.12 *From Remark 1.8.27, Theorem 2.3.11 (cf. (2.3.190)), and Theorem 2.3.9 we conclude that the modified boundary-to-boundary Cauchy-Clifford integral operator $\mathfrak{C}_{\text{mod}}$ (cf. (1.8.112)) is well defined, linear and bounded in the context*

$$
\mathfrak{C}_{\text{mod}} : \dot{H}^p_1(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \longrightarrow \dot{H}^p_1(\partial\Omega,\sigma) \otimes \mathcal{C}\ell_n \text{ for } p \in \big(\tfrac{n-1}{n},\infty\big), \tag{2.3.205}
$$

and, if $\partial\Omega$ is connected and we let $\big[\mathfrak{C}_{\text{mod}}\big]$ be the operator $[f] \mapsto \big[\mathfrak{C}_{\text{mod}} f\big]$ (with brackets denoting equivalence classes modulo constants), we have

$$
\big[\mathfrak{C}_{\text{mod}}\big]^2 = \tfrac{1}{4}I \text{ on } \big[\dot{H}^p_1(\partial\Omega,\sigma)/\sim\big] \otimes \mathcal{C}\ell_n \text{ with } p \in \big(\tfrac{n-1}{n},\infty\big). \tag{2.3.206}
$$

Below we extend the scope of the intertwining identity proved in Corollary 1.6.8.

Proposition 2.3.13 *Let $\Omega \subseteq \mathbb{R}^2$ be an open set satisfying a two-sided local John condition and having an Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^1\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Let $K_{\Delta,\text{mod}}$ be the modified boundary-to-boundary harmonic double layer operator associated with Ω and $L := \Delta$ as in (1.8.24), i.e.,*

$$K_{\Delta,\mathrm{mod}} f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{2\pi} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \Big\{ \frac{\langle \nu(y), y-x\rangle}{|x-y|^2} \cdot \mathbf{1}_{\mathbb{R}^2\setminus\overline{B(x,\varepsilon)}}(y) - \frac{\langle \nu(y), y\rangle}{|y|^2} \cdot \mathbf{1}_{\mathbb{R}^2\setminus\overline{B(0,1)}}(y) \Big\} f(y)\, \mathrm{d}\sigma(y) \tag{2.3.207}$$

for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(y)}{1+|y|^2}\big)$ *and for* σ*-a.e. point* $x \in \partial\Omega$*. In addition, recall from Theorem 2.1.1 that the* (*two-dimensional*) *"transpose" principal-value harmonic double layer* $K^\#$ *from* (1.6.72) *induces a bounded mapping on* $H^p(\partial\Omega, \sigma)$ *for each* $p \in \big(\frac{1}{2}, 1\big]$*.*

Then, with the tangential derivative operator

$$\partial_\tau := \partial_{\tau_{12}} \tag{2.3.208}$$

defined as in (A.0.183) *with* $j := 1$ *and* $k := 2$*, one has*

$$\partial_\tau \circ K_{\Delta,\mathrm{mod}} = -K^\# \circ \partial_\tau \ \text{ on } \dot{H}^p_1(\partial\Omega, \sigma) \text{ with } p \in \big(\tfrac{1}{2}, 1\big]. \tag{2.3.209}$$

Proof For starters, we note that the hypotheses on the underlying set imply that Ω is a UR domain (see [68, (5.10.33)]). Let $\mathcal{D}_{\Delta,\mathrm{mod}}$ be the modified boundary-to-domain harmonic double layer operator associated, in the plane, with Ω and $L := \Delta$ as in (1.8.6), i.e.,

$$\mathcal{D}_{\Delta,\mathrm{mod}} f(x) := \frac{1}{2\pi} \int_{\partial\Omega} \Big\{ \frac{\langle \nu(y), y-x\rangle}{|x-y|^2} - \frac{\langle \nu(y), y\rangle}{|y|^2} \cdot \mathbf{1}_{\mathbb{R}^2\setminus\overline{B(0,1)}}(y) \Big\} f(y)\, \mathrm{d}\sigma(y) \tag{2.3.210}$$

for each $f \in L^1\big(\partial\Omega, \frac{\sigma(y)}{1+|y|^2}\big)$ and each $x \in \Omega$. With $E_\Delta(x) := (2\pi)^{-1} \ln|x|$ for $x \in \mathbb{R}^2 \setminus \{0\}$ denoting the standard fundamental solution for the Laplacian in the two-dimensional setting, let us also define the following modified version of the boundary-to-domain operator (1.6.81):

$$\mathcal{R}_{\mathrm{mod}} f(x) := \int_{\partial\Omega} \Big\{ \Big[\nu_1(y)(\partial_2 E_\Delta)(y-x) - \nu_2(y)(\partial_1 E_\Delta)(y-x) \Big] - \Big[\nu_1(y)(\partial_2 E_\Delta)(y) - \nu_2(y)(\partial_1 E_\Delta)(y) \Big] \cdot \mathbf{1}_{\mathbb{R}^2\setminus\overline{B(0,1)}}(y) \Big\} f(y)\, \mathrm{d}\sigma(y) \tag{2.3.211}$$

for each $f \in L^1\big(\partial\Omega, \frac{\sigma(y)}{1+|y|^2}\big)$ and each $x \in \Omega$. Finally, recall the modified version of the boundary-to-domain Cauchy integral operator $\mathscr{C}_{\mathrm{mod}}$ acting on functions (1.8.226) as in (1.8.227). Since for each real-valued function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^2}\big)$ we have

$$\begin{aligned} &\mathscr{C}_{\text{mod}} f \ \text{ is a holomorphic function in } \ \Omega, \\ &\text{and } \mathscr{C}_{\text{mod}} f = \mathcal{D}_{\Delta,\text{mod}} f - i\mathcal{R}_{\text{mod}} f \ \text{ in } \ \Omega, \end{aligned} \tag{2.3.212}$$

it follows that $\mathcal{D}_{\Delta,\text{mod}} f$ and $-\mathcal{R}_{\text{mod}} f$ are conjugate harmonic functions in Ω. In turn, this implies that

$$\begin{aligned} &\text{for each (complex-valued) function } \ f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^2}\big) \ \text{ we have} \\ &\mathcal{D}_{\Delta,\text{mod}} f,\ \mathcal{R}_{\text{mod}} f \in \mathscr{C}^\infty(\Omega), \quad \Delta\mathcal{D}_{\Delta,\text{mod}} f = \Delta\mathcal{R}_{\text{mod}} f = 0 \ \text{ in } \ \Omega, \\ &\partial_1\mathcal{D}_{\Delta,\text{mod}} f + \partial_2\mathcal{R}_{\text{mod}} f = 0 \ \text{ and } \ \partial_2\mathcal{D}_{\Delta,\text{mod}} f - \partial_1\mathcal{R}_{\text{mod}} f = 0 \ \text{ in } \ \Omega. \end{aligned} \tag{2.3.213}$$

Also, from Theorem 1.8.12 and Proposition 1.8.16 we know that

$$\begin{aligned} &\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{\Delta,\text{mod}} f\big) \in L^p(\partial\Omega,\sigma) \ \text{ and } \ \mathcal{N}_\kappa\big(\nabla\mathcal{R}_{\text{mod}} f\big) \in L^p(\partial\Omega,\sigma) \\ &\qquad \text{for each function } \ f \in \dot{H}^p_1(\partial\Omega,\sigma). \end{aligned} \tag{2.3.214}$$

To proceed, recall that the tangential derivative operator $\partial_\tau := \partial_{\tau_{12}}$ is defined as in (A.0.183) with $j := 1$ and $k := 2$. Consider the boundary-to-domain single layer potential operator $\mathscr{S}$ associated with Ω and $L := \Delta$ as in (2.2.31). Finally, recall the duality brackets from [69, Theorem 4.6.1]. Then for each $f \in \dot{H}^p_1(\partial\Omega,\sigma)$ with $p \in \big(\tfrac12, 1\big]$ and each $\ell \in \{1,2\}$ we may write

$$\begin{aligned} -\partial_\ell\mathcal{R}_{\text{mod}} f(x) &= \int_{\partial\Omega} \Big\{\nu_1(y)(\partial_\ell\partial_2 E_\Delta)(y-x) - \nu_2(y)(\partial_\ell\partial_1 E_\Delta)(y-x)\Big\} f(y)\,d\sigma(y) \\ &= \int_{\partial\Omega} \partial_{\tau(y)}\big\{(\partial_\ell E_\Delta)(y-x)\big\} f(y)\,d\sigma(y) \\ &= \begin{cases} \big\langle (\partial_\ell E_\Delta)(x-\cdot)\big|_{\partial\Omega}, \partial_\tau f\big\rangle & \text{if } \partial\Omega \text{ is bounded,} \\[2ex] \big\langle \big[(\partial_\ell E_\Delta)(x-\cdot)\big|_{\partial\Omega}\big], \partial_\tau f\big\rangle & \text{if } \partial\Omega \text{ is unbounded,} \end{cases} \\ &= \partial_\ell\mathscr{S}(\partial_\tau f)(x) \ \text{ for each } \ x \in \Omega, \end{aligned} \tag{2.3.215}$$

thanks to (2.3.211), (A.0.183), [69, Lemma 11.10.4], and (2.2.34). Thus,

$$\begin{aligned} &\nabla\mathcal{R}_{\text{mod}} f = -\nabla\mathscr{S}(\partial_\tau f) \ \text{ in } \ \Omega, \\ &\text{for each } \ f \in \dot{H}^p_1(\partial\Omega,\sigma) \ \text{ with } \ p \in \big(\tfrac12, 1\big]. \end{aligned} \tag{2.3.216}$$

Consider now an arbitrary $f \in \dot{H}^p_1(\partial\Omega,\sigma)$ with $p \in \big(\tfrac12, 1\big]$. Then f belongs to $L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^2}\big)$ and has $\partial_\tau f \in H^p(\partial\Omega,\sigma)$. Granted the current geometric assumptions on Ω, (2.3.98) (when $\tfrac12 < p < 1$), as well as (2.3.122) and [68, (7.4.105)] (when $p = 1$), allow us to conclude that

$$f \in \bigcup_{1<q<\infty} L^q_{\rm loc}(\partial\Omega, \sigma). \tag{2.3.217}$$

In concert with [70, (2.5.32)], this further implies that each truncation parameter $\varepsilon \in (0, \infty)$ we have

$$\mathcal{N}^{\varepsilon}_{\kappa}(\mathcal{D}_{\Delta,\rm mod} f) \in \bigcup_{1<q<\infty} L^q_{\rm loc}(\partial\Omega, \sigma) \subseteq L^1_{\rm loc}(\partial\Omega, \sigma). \tag{2.3.218}$$

Finally, we note from (2.3.214) and [68, (8.6.51)] that

$$\nabla(\mathcal{D}_{\Delta,\rm mod} f) \in \big[L^1_{\rm bdd}(\Omega, \mathcal{L}^2)\big]^2, \tag{2.3.219}$$

and recall from (1.8.27) that we have the jump-formula

$$\big(\mathcal{D}_{\Delta,\rm mod} f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{\Delta,\rm mod}\big) f \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.3.220}$$

Keeping these properties in mind, we may write

$$\begin{aligned}
\big(-\tfrac{1}{2}I + K^{\#}\big)(\partial_\tau f) &= \overset{\bullet}{\partial}_\nu \mathcal{S}(\partial_\tau f) = \nu \bullet \nabla \mathcal{S}(\partial_\tau f) \\
&= -\nu \bullet \nabla \mathcal{R}_{\rm mod} f = -\nu \bullet \Big(\partial_2 \mathcal{D}_{\Delta,\rm mod} f, -\partial_1 \mathcal{D}_{\Delta,\rm mod} f\Big) \\
&= \overset{\bullet}{\partial}_\tau \big(\mathcal{D}_{\Delta,\rm mod} f\big) = -\partial_\tau \big(\mathcal{D}_{\Delta,\rm mod} f\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big) \\
&= -\partial_\tau \big[\big(\tfrac{1}{2}I + K_{\Delta,\rm mod}\big) f\big] \text{ in } H^p(\partial\Omega, \sigma).
\end{aligned} \tag{2.3.221}$$

Above, the first equality is a consequence of (2.2.47), the second equality comes from the definition of the weak normal derivative (cf. (A.0.181)-(A.0.182)), the third equality has been proved in (2.3.216), the fourth equality is contained in (2.3.213), the fifth equality is seen from [69, Example 10.2.2], the sixth equality is provided by [69, Proposition 11.3.1] (whose applicability in the present setting is ensured by (2.3.218), (2.3.219), and (2.3.220)), and the final equality in (2.3.221) uses (2.3.220).

At this stage, the conclusion claimed in (2.3.209) is readily seen from (2.3.221). □

Our next proposition contains intertwining identities in the two-dimensional setting, in the spirit of Corollary 1.6.8 and Proposition 2.3.13.

Proposition 2.3.14 *Let $\Omega \subseteq \mathbb{R}^2$ be an open set satisfying a two-sided local John condition and whose boundary is an Ahlfors regular set. Introduce $\sigma := \mathcal{H}^1\lfloor\partial\Omega$ and abbreviate*

$$\partial_\tau := \partial_{\tau_{12}} \tag{2.3.222}$$

with the latter tangential derivative operator on $\partial\Omega$ defined as in (A.0.183) with $j := 1$ and $k := 2$. Lastly, bring back the modified boundary-to-boundary harmonic double layer operator $K_{\Delta,\rm mod}$ from (2.3.207), and recall the "transpose" principal-value harmonic double layer $K^{\#}$ from (1.6.72). Then one has

$$\partial_\tau \circ K_{\Delta,\mathrm{mod}} = -K^{\#} \circ \partial_\tau \ \ on \ \ \dot{L}^p_1(\partial\Omega,\sigma) \ \ with \ \ p \in (1,\infty). \tag{2.3.223}$$

In particular, for each $p \in (1,\infty)$ *one has*

$$\begin{gathered} \partial_\tau \circ \big[K_{\Delta,\mathrm{mod}}\big] = -K^{\#} \circ \partial_\tau \\ \text{as operators from } \dot{L}^p_1(\partial\Omega,\sigma)\big/\sim \ \text{ into } \ L^p(\partial\Omega,\sigma), \end{gathered} \tag{2.3.224}$$

where

$$\begin{gathered} \partial_\tau : \dot{L}^p_1(\partial\Omega,\sigma)\big/\sim \longrightarrow L^p(\partial\Omega,\sigma) \ \text{ is defined as} \\ \partial_\tau[f] := \partial_\tau f \ \text{ for each } \ f \in \dot{L}^p_1(\partial\Omega,\sigma), \end{gathered} \tag{2.3.225}$$

and (*compare with* (1.8.154))

$$\begin{gathered} \big[K_{\Delta,\mathrm{mod}}\big] : \dot{L}^p_1(\partial\Omega,\sigma)\big/\sim \longrightarrow \dot{L}^p_1(\partial\Omega,\sigma)\big/\sim \ \text{ is defined as} \\ \big[K_{\Delta,\mathrm{mod}}\big][f] := \big[K_{\Delta,\mathrm{mod}} f\big] \in \dot{L}^p_1(\partial\Omega,\sigma)\big/\sim \ \text{ for each } \ f \in \dot{L}^p_1(\partial\Omega,\sigma). \end{gathered} \tag{2.3.226}$$

Finally, as a consequence of (2.3.223) *and* (2.3.209), *one also has*

$$\partial_\tau \circ K_{\Delta,\mathrm{mod}} = -K^{\#} \circ \partial_\tau \ \ on \ \ \dot{H}^p_1(\partial\Omega,\sigma) \ \ with \ \ p \in \big(\tfrac{1}{2},\infty\big). \tag{2.3.227}$$

Proof Fix some aperture parameter $\kappa \in (0,\infty)$, and pick an arbitrary function $f \in \dot{L}^p_1(\partial\Omega,\sigma)$ with $1 < p < \infty$. Then from Theorem 1.8.12 we know that

$$\begin{gathered} \mathcal{D}_{\Delta,\mathrm{mod}} f \in \mathscr{C}^\infty(\Omega), \quad \mathcal{N}_\kappa\big(\nabla \mathcal{D}_{\Delta,\mathrm{mod}} f\big) \in L^p(\partial\Omega,\sigma) \ \text{ and} \\ \big(\mathcal{D}_{\Delta,\mathrm{mod}} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{\Delta,\mathrm{mod}}\big) f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \end{gathered} \tag{2.3.228}$$

and also

$$\big(\nabla \mathcal{D}_{\Delta,\mathrm{mod}} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \text{ exist at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.3.229}$$

In addition, from [69, (11.5.59)] and (1.8.22) we see that

$$\mathcal{N}^{\varepsilon}_\kappa(\mathcal{D}_{\Delta,\mathrm{mod}} f) \in L^p_{\mathrm{loc}}(\partial\Omega,\sigma) \ \text{ for each } \ \varepsilon > 0. \tag{2.3.230}$$

Then, if E_Δ is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$ (cf. (A.0.65)), at σ-a.e. point $x \in \partial\Omega$ we may compute

$$
\begin{aligned}
\partial_\tau\big[\big(\tfrac{1}{2}I + K_{\Delta,\mathrm{mod}}\big)f\big](x) &= \partial_\tau\big(\mathcal{D}_{\Delta,\mathrm{mod}} f\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) \\
&= \nu_1(x)\Big[\big(\partial_2(\mathcal{D}_{\Delta,\mathrm{mod}} f)\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big](x) - \nu_2(x)\Big[\big(\partial_1(\mathcal{D}_{\Delta,\mathrm{mod}} f)\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big](x) \\
&= \frac{1}{2}\nu_1(x)^2(\partial_\tau f)(x) - \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_1(x)(\partial_2 E_\Delta)(x-y)(\partial_\tau f)(y)\,\mathrm{d}\sigma(y) \\
&\quad + \frac{1}{2}\nu_2(x)^2(\partial_\tau f)(x) - \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu_2(x)(\partial_1 E_\Delta)(x-y)(\partial_\tau f)(y)\,\mathrm{d}\sigma(y) \\
&= \tfrac{1}{2}(\partial_\tau f)(x) - K^{\#}(\partial_\tau f)(x).
\end{aligned}
\tag{2.3.231}
$$

The first equality above is a consequence of the second line in (2.3.228). The second equality above is implied by [69, Proposition 11.3.2], whose present applicability is guaranteed by (2.3.228)-(2.3.230). The third equality above may be justified with the help of (1.8.51) and the jump-formula [70, (2.5.341) in Proposition 2.5.35].

The final equality in (2.3.231) is implied by the definition of $K^{\#}$ (cf. (1.6.72)) and the fact that $\nu_1^2 + \nu_2^2 = 1$ at σ-a.e. point on $\partial\Omega$. At this stage, (2.3.223) follows readily from (2.3.231), and everything else is a direct consequence of this identity and definitions.

In closing, we wish to note that an alternative proof of (2.3.223) is obtain by combining the version of the formula (1.6.74) recorded in (1.6.77) together with [69, (11.5.161)], and keeping in mind (1.8.28) plus item *(5)* in [69, Lemma 11.1.3]. □

Availing ourselves of the intertwining formulas established in Proposition 2.3.14 we may further augment Proposition 1.8.10 as follows:

Corollary 2.3.15 *Let $\Omega \subseteq \mathbb{R}^2$ be a chord-arc domain with unbounded boundary and abbreviate $\sigma := \mathcal{H}^1\lfloor\partial\Omega$. Recall the transpose harmonic double layer $K^{\#}$ from* (1.6.72)*, and bring in the modified boundary-to-boundary harmonic double layer operator $\big[K_{\Delta,\mathrm{mod}}\big]$ acting as in* (2.3.226)*. Then for each $p \in (1,\infty)$ and each $z \in \mathbb{C}$ it follows that*

$$
\begin{aligned}
&\text{the operator } zI + K^{\#} \text{ is an isomorphism on } L^p(\partial\Omega,\sigma) \text{ if and only if} \\
&\text{the operator } -zI + \big[K_{\Delta,\mathrm{mod}}\big] \text{ is an isomorphism on } \dot{L}^p_1(\partial\Omega,\sigma)\big/\sim.
\end{aligned}
\tag{2.3.232}
$$

Proof This is a direct consequence of the intertwining formula (2.3.224) and the isomorphism proved in [69, Proposition 11.5.15]. □

We next take up the task of establishing the analogue of the integral representation formula (1.3.35) from Theorem 1.3.3 for functions u satisfying, among other things, $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma)$ with $p \in \big(\frac{n-1}{n}, n-1\big)$. The chief novel difficulty is that now $\mathcal{N}_\kappa(\nabla u)$ may not even be locally integrable, so the hypotheses of Theorem 1.3.3 are clearly violated. Related results are contained in Theorem 2.2.7 and Theorem 1.8.19.

Before stating our new result, the reader is reminded that the maximal operator $\mathfrak{P}$ has been defined in (A.0.173).

Theorem 2.3.16 *Suppose $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ satisfies*[12] *$n \geq 3$, is an Ahlfors regular domain. Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, for some $M \in \mathbb{N}$, consider a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in* (1.3.2) *is weakly elliptic* (*in the sense of* [70, (1.3.3) in Definition 1.3.1]). *In this setting, consider the potential operators $\mathcal{D}$, $\mathscr{S}$, Π_Ω, associated with A and Ω as in* (1.3.18), (1.3.6), (1.3.4). *Suppose $u \in \big[W^{1,1}_{\rm loc}(\Omega)\big]^M$ is a vector-valued function with the property that, for some aperture parameter $\kappa \in (0,\infty)$, satisfies*

$$\begin{aligned} &\text{the trace } u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \\ &\int_{\partial\Omega} \frac{(\mathcal{N}_\kappa u)(y)}{1+|y|^{n-1-\delta}}\,{\rm d}\sigma(y) < +\infty \text{ for some } \delta > 0, \\ &\text{and } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma) \text{ for some } p \in \big(\tfrac{n-1}{n}, n-1\big). \end{aligned} \tag{2.3.233}$$

In addition, with Lu considered in the sense of distributions in Ω, assume that

$$Lu \in \big[L^1_{\rm loc}(\Omega,\mathcal{L}^n)\big]^M \text{ and } \mathfrak{P}(Lu) \in L^p(\partial\Omega,\sigma). \tag{2.3.234}$$

Finally, fix a point $x \in \mathbb{R}^n \setminus \partial\Omega$ with the property that

$$\int_\Omega \frac{|(Lu)(y)|}{|x-y|^{n-2}}\,{\rm d}y < +\infty. \tag{2.3.235}$$

Then the weak conormal derivative $\dot{\partial}^A_\nu u$, defined as in [69, Theorem 10.2.24, (10.2.148)], *belongs to the Hardy space $\big[H^p(\partial\Omega,\sigma)\big]^M$ and one has* (*with the single layer operator acting on distributions in the Hardy space as in* (2.2.31))

$$\mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\big)(x) - \mathscr{S}(\dot{\partial}^A_\nu u)(x) + \Pi_\Omega(Lu)(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \overline{\Omega}. \end{cases} \tag{2.3.236}$$

if either $\partial\Omega$ is unbounded, or Ω is bounded. In the case when Ω is an exterior domain, the same conclusion is valid under the additional assumption that there exists $\lambda \in (1,\infty)$ such that

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |u|\,{\rm d}\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{2.3.237}$$

In fact, a similar result is valid when $n = 2$ provided either

[12] see the last claim in the statement for $n = 2$

$$\int_{S^1} \big[L(\xi)\big]^{-1}\, d\mathcal{H}^1(\xi) = 0 \in \mathbb{C}^{M\times M}, \tag{2.3.238}$$

or otherwise one assumes that $\partial\Omega$ is compact, replaces (2.3.235) *by*

$$\int_\Omega |(Lu)(y)|\big(1 + \big|\ln|x-y|\big|\big)\, dy < +\infty, \tag{2.3.239}$$

and, in the case when Ω is an exterior domain, replaces (2.3.237) *by*

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |u|\, d\mathcal{L}^2 = o\big(\tfrac{1}{\ln R}\big) \ \text{as} \ R \to \infty. \tag{2.3.240}$$

We note that when $n = 2$ and Ω is an exterior domain, (2.3.240) is implied by the weaker decay condition (2.3.237), provided (2.3.234)-(2.3.235) are replaced by $Lu = 0$ in Ω. Indeed, this is seen from [70, (1.5.205), (1.5.213)]. We also wish to note that [68, Lemma 7.2.1] and Hölder's inequality imply that

the integrability condition from the second line of (2.3.233) is automatically satisfied if $\mathcal{N}_\kappa u \in L^q(\partial\Omega, \sigma)$ for some $q \in [1, \infty)$. (2.3.241)

Proof of Theorem 2.3.16 Assume first that $n \geq 3$. For starters, (2.3.233) and [68, Lemma 8.3.1] imply that

$$\text{the function } u \text{ is locally Lipschitz in } \Omega. \tag{2.3.242}$$

In addition, from the first two lines in (2.3.233), [68, (8.9.8)], [68, Proposition 8.8.4], [68, Corollary 8.9.6], and the fact that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ we conclude that

$$u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\Big]^M. \tag{2.3.243}$$

Moreover, from [69, (10.2.182)] and (2.3.233)-(2.3.234) we know that

$$\dot{\partial}^A_\nu u \in \big[H^p(\partial\Omega, \sigma)\big]^M. \tag{2.3.244}$$

We find it convenient to divide the remaining portion of the proof into two cases, depending on the size of p.

`Case I:` *Suppose $p \in \big(\frac{n-1}{n}, 1\big]$.* Assume for now that $\partial\Omega$ is unbounded, and fix a point $x \in \Omega$ for which (2.3.235) holds. Also, select an arbitrary index $\alpha \in \{1, \dots, M\}$. Consider a scalar-valued function $\eta \in \mathscr{C}^\infty(\mathbb{R}^n)$ with the property that $\eta = 0$ on $B(0,1)$ and $\eta = 1$ on $\mathbb{R}^n \setminus B(0,2)$. For each number $\varepsilon \in \big(0, \frac{1}{2}\operatorname{dist}(x, \partial\Omega)\big)$ define $\eta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by setting

$$\eta_\varepsilon(y) := \eta\Big(\frac{y-x}{\varepsilon}\Big) \ \text{ for every } \ y \in \mathbb{R}^n. \tag{2.3.245}$$

Then $\eta_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n)$ is a bounded function satisfying

$$\lim_{\varepsilon\to 0^+}\eta_\varepsilon(y)=1 \text{ for every } y\in\mathbb{R}^n\setminus\{x\}, \tag{2.3.246}$$

and there exists a constant $C\in(0,\infty)$ such that for each $\varepsilon\in\big(0,\tfrac12\operatorname{dist}(x,\partial\Omega)\big)$ we have

$$\begin{gathered}1-\eta_\varepsilon\in\mathscr{C}^\infty_c(\Omega),\quad \eta_\varepsilon\equiv 0 \text{ on } B(x,\varepsilon),\quad \operatorname{supp}(\nabla\eta_\varepsilon)\subseteq\overline{B(x,2\varepsilon)}\setminus B(x,\varepsilon),\\ \text{and } |(\nabla^j\eta_\varepsilon)(y)|\leq C\varepsilon^{-j} \text{ for every } j\in\mathbb{N}_0 \text{ and every } y\in\mathbb{R}^n.\end{gathered} \tag{2.3.247}$$

In particular, for each $\beta\in\{1,\dots,M\}$ the function $E_{\alpha\beta}(x-\cdot)\eta_\varepsilon$ belongs to $\mathscr{C}^\infty(\mathbb{R}^n)$ and coincides with $E_{\alpha\beta}(x-\cdot)$ near $\partial\Omega$.

To proceed, fix $\varepsilon\in\big(0,\tfrac12\operatorname{dist}(x,\partial\Omega)\big)$ and observe that according to (2.2.31) we have (with the summation convention over repeated indices in effect)

$$\begin{aligned}\big(\mathscr{S}\dot{\partial}^A_\nu u\big)_\alpha(x)&=\Big\langle\big[E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}\big],(\dot{\partial}^A_\nu u)_\beta\Big\rangle\\ &=\Big\langle\big[\big(E_{\alpha\beta}(x-\cdot)\eta_\varepsilon\big)\big|_{\partial\Omega}\big],(\dot{\partial}^A_\nu u)_\beta\Big\rangle\end{aligned} \tag{2.3.248}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality bracket on $\partial\Omega$, described in [69, Theorem 4.6.1] (with $\Sigma:=\partial\Omega$). Let us also fix some number $\lambda\in(1,\infty)$ and select a function $\theta\in\mathscr{C}^\infty_c(\mathbb{R}^n)$ satisfying $0\leq\theta\leq 1$, $\theta\equiv 1$ on $B(0,1)$, $\theta\equiv 0$ on $\mathbb{R}^n\setminus B(0,\lambda)$ and, for each $R>0$, define $\theta_R(x):=\theta(x/R)$ for every $x\in\mathbb{R}^n$. For each $\beta\in\{1,\dots,M\}$ we then have

$$\begin{gathered}\lim_{R\to\infty}\Big[\big(\theta_R E_{\alpha\beta}(x-\cdot)\eta_\varepsilon\big)\big|_{\partial\Omega}\Big]=\big[\big(E_{\alpha\beta}(x-\cdot)\eta_\varepsilon\big)\big|_{\partial\Omega}\big]\\ \text{weak-}*\text{ in }\begin{cases}\dot{\mathscr{C}}^{(n-1)(1/p-1)}(\partial\Omega)\big/\sim & \text{if } p<1,\\ \widetilde{\mathrm{BMO}}(\partial\Omega,\sigma) & \text{if } p=1.\end{cases}\end{gathered} \tag{2.3.249}$$

This is implied by the general weak-$*$ convergence results established in [69, Lemma 4.8.4] and, respectively, [69, Lemma 4.8.1] (in the latter case also bearing in mind the trivial bounded embedding $L^\infty(\partial\Omega,\sigma)\hookrightarrow\mathrm{BMO}(\partial\Omega,\sigma)$). At this stage, from (2.3.248), (2.3.249), (2.3.244), the duality result from [69, Theorem 4.6.1], and [69, Lemma 4.6.4] we conclude that

$$\big(\mathscr{S}\dot{\partial}^A_\nu u\big)_\alpha(x)=\lim_{R\to\infty}{}_{(\mathrm{Lip}_c(\partial\Omega))'}\Big\langle(\dot{\partial}^A_\nu u)_\beta,\big(\theta_R E_{\alpha\beta}(x-\cdot)\eta_\varepsilon\big)\big|_{\partial\Omega}\Big\rangle_{\mathrm{Lip}_c(\partial\Omega)}. \tag{2.3.250}$$

Pressing on, recall from (A.0.181)-(A.0.182) that the weak conormal derivative of u associated with the coefficient tensor A is defined as the distribution

$$\begin{gathered}\dot{\partial}^A_\nu u:=\big(\nu\bullet\vec{F}^\gamma\big)_{1\leq\gamma\leq M}\text{ in }\Big[\big(\mathrm{Lip}_c(\partial\Omega)\big)'\Big]^M\text{ where}\\ \vec{F}^\gamma:=(A\nabla u)_\gamma=\big(a^{\gamma\beta}_{rs}\partial_s u_\beta\big)_{1\leq r\leq n}\text{ for each }\gamma\in\{1,\dots,M\}.\end{gathered} \tag{2.3.251}$$

Based on (2.3.250), (2.3.251), and [68, Proposition 4.2.3] we obtain

$$\begin{aligned}
\big(\mathscr{S}\dot{\partial}^A_\nu u\big)_\alpha(x) &= \lim_{R\to\infty}\int_\Omega (\mathrm{div}\vec{F}^\gamma)\theta_R E_{\alpha\gamma}(x-\cdot)\eta_\varepsilon\,\mathrm{d}\mathcal{L}^n \\
&\quad + \lim_{R\to\infty}\int_\Omega \big\langle \vec{F}^\gamma, \nabla\big(\theta_R E_{\alpha\gamma}(x-\cdot)\eta_\varepsilon\big)\big\rangle\,\mathrm{d}\mathcal{L}^n \\
&= \lim_{R\to\infty}\int_\Omega \big\langle Lu\,,\theta_R E_{\alpha\bullet}(x-\cdot)\eta_\varepsilon\big\rangle\,\mathrm{d}\mathcal{L}^n \\
&\quad + \lim_{R\to\infty}\int_\Omega a^{\gamma\beta}_{rs}(\partial_s u_\beta)\partial_r\big(\theta_R E_{\alpha\gamma}(x-\cdot)\eta_\varepsilon\big)\,\mathrm{d}\mathcal{L}^n. \qquad (2.3.252)
\end{aligned}$$

Thanks to (2.3.235), [70, (1.4.24)], and Lebesgue's Dominated Convergence Theorem,

$$\lim_{\varepsilon\to 0^+}\lim_{R\to\infty}\int_\Omega \big\langle Lu, \theta_R E_{\alpha\bullet}(x-\cdot)\eta_\varepsilon\big\rangle\,\mathrm{d}\mathcal{L}^n = \big(\Pi_\Omega(Lu)\big)_\alpha(x). \qquad (2.3.253)$$

Together, (2.3.252) and (2.3.253) permit us to conclude that

$$\begin{aligned}
&\big(\mathscr{S}\dot{\partial}^A_\nu u\big)_\alpha(x) - \big(\Pi_\Omega(Lu)\big)_\alpha(x) \qquad (2.3.254)\\
&\qquad = \lim_{\varepsilon\to 0^+}\lim_{R\to\infty}\int_\Omega (\partial_s u_\beta)a^{\gamma\beta}_{rs}\partial_r\big(\theta_R E_{\alpha\gamma}(x-\cdot)\eta_\varepsilon\big)\,\mathrm{d}\mathcal{L}^n.
\end{aligned}$$

Recall that $\theta_R E(x-\cdot)\eta_\varepsilon \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ and that $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma)$ with $p \in \big(\tfrac{n-1}{n}, 1\big]$ implies, via [68, (8.6.51)], that $\nabla u \in \big[L^1_{\mathrm{bdd}}(\Omega,\mathcal{L}^n)\big]^{M\cdot n}$. Also, the integrability condition in the second line of (2.3.233) implies that $\mathcal{N}_\kappa u \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$. Bearing these in mind and relying on the integration by parts formula established in [68, Theorem 1.7.1], we may further transform this formula into

$$\begin{aligned}
&\big(\mathscr{S}\dot{\partial}^A_\nu u\big)_\alpha(x) - \big(\Pi_\Omega(Lu)\big)_\alpha(x) \\
&\qquad = -\lim_{\varepsilon\to 0^+}\lim_{R\to\infty}\int_\Omega u_\beta a^{\gamma\beta}_{rs}\partial_r\partial_s\big(\theta_R E_{\alpha\gamma}(x-\cdot)\eta_\varepsilon\big)\,\mathrm{d}\mathcal{L}^n \\
&\qquad\quad + \lim_{R\to\infty}\int_{\partial\Omega} \nu_s\big(u_\beta\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)a^{\gamma\beta}_{rs}\partial_r\big(\theta_R E_{\alpha\gamma}(x-\cdot)\big)\,\mathrm{d}\sigma \\
&\qquad =: \lim_{\varepsilon\to 0^+}\mathrm{I}_\varepsilon + \mathrm{II}. \qquad (2.3.255)
\end{aligned}$$

Using the product rule for the derivative ∂_r yields

$$\begin{aligned}\mathrm{II} &= \lim_{R\to\infty}\int_{\partial\Omega}\nu_s\big(u_\beta\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)a_{rs}^{\gamma\beta}(\partial_r\theta_R)E_{\alpha\gamma}(x-\cdot)\,\mathrm{d}\sigma\\ &\quad-\lim_{R\to\infty}\int_{\partial\Omega}\nu_s\big(u_\beta\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)a_{rs}^{\gamma\beta}\theta_R(\partial_r E_{\alpha\gamma})(x-\cdot)\,\mathrm{d}\sigma\\ &=: \mathrm{II}^{(1)}-\mathrm{II}^{(2)}.\end{aligned} \tag{2.3.256}$$

Based on [70, (1.4.24)] plus the fact that for every $j\in\mathbb{N}$ we have

$$\sup_{\mathbb{R}^n}|\nabla^j\theta_R|\leq CR^{-j}\ \text{ and }\ \mathrm{supp}(\nabla^j\theta_R)\subseteq\overline{B(0,\lambda R)}\setminus B(0,R)\ \text{ for each } R>0, \tag{2.3.257}$$

we may estimate

$$\begin{aligned}&\Big|\int_{\partial\Omega}\nu_s\big(u_\beta\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)a_{rs}^{\gamma\beta}(\partial_r\theta_R)E_{\alpha\gamma}(x-\cdot)\,\mathrm{d}\sigma\Big|\\ &\qquad\leq C\int_{\partial\Omega\cap[B(0,\lambda R)\setminus B(0,R)]}\frac{\big|\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)\big|}{1+|y|^{n-1}}\,\mathrm{d}\sigma(y)\\ &\qquad= o(1)\ \text{ as }\ R\to\infty,\end{aligned} \tag{2.3.258}$$

where the last equality is a consequence of Lebesgue's Dominated Convergence Theorem, whose applicability is ensured by (2.3.243). Thus,

$$\mathrm{II}^{(1)}=0. \tag{2.3.259}$$

In view of (1.3.18), from (2.3.243) and Lebesgue's Dominated Convergence Theorem we also conclude that

$$\mathrm{II}^{(2)}=\int_{\partial\Omega}\nu_s\big(u_\beta\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)a_{rs}^{\gamma\beta}(\partial_r E_{\alpha\gamma})(x-\cdot)\,\mathrm{d}\sigma=-\Big(\mathcal{D}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)\Big)_\alpha(x). \tag{2.3.260}$$

Going further, recall that $a_{rs}^{\gamma\beta}(\partial_r\partial_s E_{\alpha\gamma})(x-\cdot)=0$ in $\mathbb{R}^n\setminus\{x\}$ (cf. [70, (1.4.33)]) for each index $\beta\in\{1,\dots,M\}$. We split I_ε from (2.3.255) as

$$\mathrm{I}_\varepsilon=\mathrm{I}^{(1)}+\mathrm{I}^{(2)}+\mathrm{I}^{(3)}+\mathrm{I}_\varepsilon^{(4)}+\mathrm{I}_\varepsilon^{(5)}+\mathrm{I}_\varepsilon^{(6)} \tag{2.3.261}$$

where

$$\mathrm{I}^{(1)} := \lim_{R\to\infty} \int_\Omega u_\beta a^{\gamma\beta}_{rs} (\partial_r \theta_R)(\partial_s E_{\alpha\gamma})(x-\cdot)\,\mathrm{d}\mathcal{L}^n,$$

$$\mathrm{I}^{(2)} := \lim_{R\to\infty} \int_\Omega u_\beta a^{\gamma\beta}_{rs} (\partial_s \theta_R)(\partial_r E_{\alpha\gamma})(x-\cdot)\,\mathrm{d}\mathcal{L}^n,$$

$$\mathrm{I}^{(3)} := -\lim_{R\to\infty} \int_\Omega u_\beta a^{\gamma\beta}_{rs} (\partial_r \partial_s \theta_R) E_{\alpha\gamma}(x-\cdot)\,\mathrm{d}\mathcal{L}^n,$$

$$\mathrm{I}^{(4)}_\varepsilon := \lim_{R\to\infty} \int_\Omega u_\beta a^{\gamma\beta}_{rs} \theta_R (\partial_r E_{\alpha\gamma})(x-\cdot)\partial_s \eta_\varepsilon\,\mathrm{d}\mathcal{L}^n,$$

$$\mathrm{I}^{(5)}_\varepsilon := \lim_{R\to\infty} \int_\Omega u_\beta a^{\gamma\beta}_{rs} \theta_R (\partial_s E_{\alpha\gamma})(x-\cdot)\partial_r \eta_\varepsilon\,\mathrm{d}\mathcal{L}^n,$$

$$\mathrm{I}^{(6)}_\varepsilon := -\lim_{R\to\infty} \int_\Omega u_\beta a^{\gamma\beta}_{rs} \theta_R E_{\alpha\gamma}(x-\cdot)\partial_r \partial_s \eta_\varepsilon\,\mathrm{d}\mathcal{L}^n. \tag{2.3.262}$$

Note that since we are presently assuming $\partial\Omega$ to be unbounded, estimate [68, (8.6.51)] is applicable with $E_R := \Omega \cap \big[B(0,\lambda R)\setminus B(0,R)\big]$ and $p := (n-1)/n$, and this gives

$$\|u\|_{L^1(E_R,\mathcal{L}^n)} \le C\|\mathcal{N}_\kappa u\|_{L^{(n-1)/n}(\pi_\kappa(E_R),\sigma)}, \qquad \forall R > 0. \tag{2.3.263}$$

On the other hand, based on Hölder's inequality, [68, (8.1.17)], and the upper-Ahlfors regularity of $\partial\Omega$ we may estimate

$$\begin{aligned}
&\|\mathcal{N}_\kappa u\|_{L^{(n-1)/n}(\pi_\kappa(E_R),\sigma)} \\
&\quad = \Big[\int_{\pi_\kappa(E_R)} (\mathcal{N}_\kappa u)(y)^{\frac{n-1}{n}}\,\mathrm{d}\sigma(y)\Big]^{\frac{n}{n-1}} \\
&\quad = \left[\int_{\pi_\kappa(E_R)} \Big(\frac{(\mathcal{N}_\kappa u)(y)}{1+|y|^{n-1-\delta}}\Big)^{\frac{n-1}{n}} \cdot \big(1+|y|^{n-1-\delta}\big)^{\frac{n-1}{n}}\,\mathrm{d}\sigma(y)\right]^{\frac{n}{n-1}} \\
&\quad \le \Big(\int_{\pi_\kappa(E_R)} \frac{(\mathcal{N}_\kappa u)(y)}{1+|y|^{n-1-\delta}}\,\mathrm{d}\sigma(y)\Big) \cdot \Big(\int_{\pi_\kappa(E_R)} \big(1+|y|^{n-1-\delta}\big)^{n-1}\,\mathrm{d}\sigma(y)\Big)^{\frac{1}{n-1}} \\
&\quad \le CR^{n-\delta}\Big(\int_{\partial\Omega} \frac{(\mathcal{N}_\kappa u)(y)}{1+|y|^{n-1-\delta}}\,\mathrm{d}\sigma(y)\Big).
\end{aligned} \tag{2.3.264}$$

Collectively, (2.3.262), (2.3.257), [70, (1.4.24)], (2.3.263), (2.3.264), and (2.3.233) imply

$$
\begin{aligned}
\big|\mathrm{I}^{(1)}\big| &\leq C \limsup_{R\to\infty} \Big\{ R^{-n} \int_{\Omega\cap[B(0,\lambda R)\setminus B(0,R)]} |u|\,\mathrm{d}\mathcal{L}^n \Big\} \\
&= C \limsup_{R\to\infty} \Big\{ R^{-n} \|u\|_{L^1(E_R,\mathcal{L}^n)} \Big\} \\
&\leq C\Big(\int_{\partial\Omega} \frac{(\mathcal{N}_\kappa u)(y)}{1+|y|^{n-1-\delta}}\,\mathrm{d}\sigma(y) \Big) \cdot \limsup_{R\to\infty} R^{-\delta} = 0
\end{aligned}
\tag{2.3.265}
$$

hence

$$
\mathrm{I}^{(1)} = 0. \tag{2.3.266}
$$

Moreover, in a completely analogous manner we also have

$$
\mathrm{I}^{(2)} = 0 \ \text{ and } \ \mathrm{I}^{(3)} = 0. \tag{2.3.267}
$$

Next, we decompose

$$
\mathrm{I}^{(4)}_\varepsilon = \int_\Omega u_\beta(y) a^{\gamma\beta}_{rs} (\partial_r E_{\alpha\gamma})(x-y)(\partial_s \eta_\varepsilon)(y)\,\mathrm{d}y = \mathrm{I}^{(4a)}_\varepsilon + \mathrm{I}^{(4b)}_\varepsilon \tag{2.3.268}
$$

where

$$
\begin{aligned}
\mathrm{I}^{(4a)}_\varepsilon &:= \int_\Omega \big(u_\beta(y) - u_\beta(x)\big) a^{\gamma\beta}_{rs} (\partial_r E_{\alpha\gamma})(x-y)(\partial_s \eta_\varepsilon)(y)\,\mathrm{d}y, \\
\mathrm{I}^{(4b)}_\varepsilon &:= u_\beta(x) \int_\Omega a^{\gamma\beta}_{rs} (\partial_r E_{\alpha\gamma})(x-y)(\partial_s \eta_\varepsilon)(y)\,\mathrm{d}y.
\end{aligned}
\tag{2.3.269}
$$

In relation to $\mathrm{I}^{(4a)}_\varepsilon$ observe that (2.3.242) permits us to estimate

$$
\limsup_{\varepsilon\to 0^+} \big|\mathrm{I}^{(4a)}_\varepsilon\big| \leq C \cdot \limsup_{\varepsilon\to 0^+} \fint_{B(x,2\varepsilon)} \big|u(y) - u(x)\big|\,\mathrm{d}y = 0. \tag{2.3.270}
$$

Also, [70, (1.4.33)] eventually allows us to conclude that for each $\beta \in \{1,\dots,M\}$ we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+}\int_\Omega a^{\gamma\beta}_{rs}(\partial_r E_{\alpha\gamma})(x-y)(\partial_s\eta_\varepsilon)(y)\,dy\\
&\qquad = \lim_{\varepsilon\to 0^+}\int_\Omega a^{\gamma\beta}_{rs}(\partial_r E_{\alpha\gamma})(x-y)\partial_s(\eta_\varepsilon-1)(y)\,dy\\
&\qquad = -\lim_{\varepsilon\to 0^+}{}_{\mathcal{D}'(\Omega)}\Big\langle a^{\gamma\beta}_{rs}\partial_r\big[E_{\alpha\gamma}(x-\cdot)\big],\partial_s(\eta_\varepsilon-1)\Big\rangle_{\mathcal{D}(\Omega)}\\
&\qquad = \lim_{\varepsilon\to 0^+}{}_{\mathcal{D}'(\Omega)}\Big\langle a^{\gamma\beta}_{rs}\partial_s\partial_r\big[E_{\alpha\gamma}(x-\cdot)\big],\eta_\varepsilon-1\Big\rangle_{\mathcal{D}(\Omega)}\\
&\qquad = \lim_{\varepsilon\to 0^+}\delta_{\alpha\beta}(\eta_\varepsilon-1)(x)=-\delta_{\alpha\beta}. \qquad (2.3.271)
\end{aligned}
$$

By combining (2.3.268)-(2.3.271) we then arrive at the conclusion that

$$\lim_{\varepsilon\to 0^+}\mathrm{I}^{(4)}_\varepsilon=-u_\alpha(x). \qquad (2.3.272)$$

In a completely similar fashion, we also have

$$\lim_{\varepsilon\to 0^+}\mathrm{I}^{(5)}_\varepsilon=-u_\alpha(x)\ \text{ and }\ \lim_{\varepsilon\to 0^+}\mathrm{I}^{(6)}_\varepsilon=u_\alpha(x). \qquad (2.3.273)$$

Bringing together (2.3.255), (2.3.256), (2.3.259), (2.3.260), (2.3.261), (2.3.266), (2.3.267), (2.3.272), and (2.3.273) now yields the formula

$$\big(\mathscr{S}\dot{\partial}^A_\nu u\big)_\alpha(x)-\big(\Pi_\Omega(Lu)\big)_\alpha(x)=\Big(\mathcal{D}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\Big)_\alpha(x)-u_\alpha(x). \qquad (2.3.274)$$

Given that $\alpha\in\{1,\dots,M\}$ is arbitrary, this establishes (2.3.236) in the case when $\partial\Omega$ is unbounded and $x\in\Omega$.

In the case when $\partial\Omega$ is unbounded and $x\in\mathbb{R}^n\setminus\overline{\Omega}$, we reason in a completely similar manner. Keeping in mind that this time we actually have $\eta_\varepsilon\equiv 1$ in Ω, it is clear from (2.3.262) that we now have $\mathrm{I}^{(4)}_\varepsilon=\mathrm{I}^{(5)}_\varepsilon=\mathrm{I}^{(6)}_\varepsilon=0$. Ultimately, this explains why the right-hand side of (2.3.236) is zero in this case. Next, the case when Ω is bounded (and $x\in\mathbb{R}^n\setminus\partial\Omega$) is treated, *mutatis mutandis*, identically, since (2.3.263) remains valid in such a scenario.

To complete the treatment in Case I, there remains to consider the case when Ω is an exterior domain, when estimate (2.3.263) may fail to hold with $C\in(0,\infty)$ independent of R (see the nature of the constant involved in [68, (8.6.51) in Proposition 8.6.3]). However, in such a situation the decay condition (2.3.237) turns out to be a good substitute, as this once again implies (as in the first line of (2.3.265)) that (2.3.266)-(2.3.267) hold.

`Case II:` *Suppose* $p\in(1,n-1)$. The argument proceeds largely as before, with some natural alterations. First, (2.3.244) becomes

$$\dot{\partial}^A_\nu u\in\big[L^p(\partial\Omega,\sigma)\big]^M, \qquad (2.3.275)$$

and in place of (2.3.248) we now have

$$\big(\mathscr{S}\dot{\partial}^A_\nu u\big)_\alpha(x) = \int_{\partial\Omega} E_{\alpha\beta}(x-\cdot)\eta_\varepsilon(\dot{\partial}^A_\nu u)_\beta \,\mathrm{d}\sigma. \tag{2.3.276}$$

The integral in (2.3.276) is absolutely convergent since if p' is the Hölder conjugate exponent of p, then the membership $p \in (1, n-1)$ implies $\int_{\partial\Omega} |E(x-\cdot)|^{p'}\,\mathrm{d}\sigma < +\infty$ thanks to [70, (1.4.24)] and [68, (7.2.5)]. This observation also permits us to write

$$\big(\mathscr{S}\dot{\partial}^A_\nu u\big)_\alpha(x) = \lim_{R\to\infty}\int_{\partial\Omega} \theta_R E_{\alpha\beta}(x-\cdot)\eta_\varepsilon(\dot{\partial}^A_\nu u)_\beta \,\mathrm{d}\sigma, \tag{2.3.277}$$

which shows that (2.3.250) is valid in the present case as well. From this point on, the proof proceeds as before, and the desired conclusion follows.

In fact, the same argument as above works when $n = 2$ provided either (2.3.238) holds, or otherwise we assume that $\partial\Omega$ is compact, replace (2.3.235) by (2.3.239) and, in the case when Ω is an exterior domain, replace [70, (1.5.5)] by (2.3.240) (see [70, (1.4.24)]). This completes the proof of Theorem 2.3.16. □

A consequence of Theorem 2.3.16 worth stating as a stand-alone result is the integral representation formula in the corollary below.

Corollary 2.3.17 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an* NTA *domain with the property that $\partial\Omega$ is a compact Ahlfors regular set. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, for some $M \in \mathbb{N}$, let L be a weakly elliptic homogeneous second-order $M \times M$ system in $\mathbb{R}^n$, with constant complex coefficients. Next, fix an aperture parameter $\kappa \in (0,\infty)$ and consider a function $u : \Omega \to \mathbb{C}^M$ satisfying*

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{ in } \Omega, \quad \text{and} \\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma) \ \text{for some } \ p \in \big(\tfrac{n-1}{n}, n-1\big). \end{gathered} \tag{2.3.278}$$

Finally, in the case when Ω is an exterior domain make the additional assumption that there exists $\lambda \in (1,\infty)$ such that

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |u|\,\mathrm{d}\mathcal{L}^n = o(1) \ \text{ as } \ R \to \infty. \tag{2.3.279}$$

Then for each complex coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with the property that $L = L_A$ the weak conormal derivative $\dot{\partial}^A_\nu u$ belongs to the Hardy space $\big[H^p(\partial\Omega,\sigma)\big]^M$, the nontangential boundary trace $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ exists at σ-a.e. point on $\partial\Omega$ and, as a function, belongs to $\big[L^{p^*}(\partial\Omega,\sigma)\big]^M$ where $p^* := \big(\frac{1}{p}-\frac{1}{n-1}\big)^{-1} \in (1,\infty)$. With these interpretations, the following integral representation formula holds:

$$u = \mathcal{D}\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) - \mathscr{S}\big(\dot{\partial}^A_\nu u\big) \ \text{ in } \Omega, \tag{2.3.280}$$

where $\mathcal{D}$ is the double layer potential operator associated with the coefficient tensor A and the domain Ω as in (1.3.18), and where the single layer potential operator acts on distributions in the Hardy space $\big[H^p(\partial\Omega,\sigma)\big]^M$ as in (2.2.31), (1.3.6).

Proof Recall that any NTA domain with an Ahlfors regular boundary is an Ahlfors regular domain (cf. (A.0.1) and [68, (5.2.4)]). Since $\partial\Omega$ is bounded, Theorem 2.2.7 and Theorem 1.8.19 enure that the nontangential boundary trace $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$, and $\mathcal{N}_\kappa u \in L^{p^*}(\partial\Omega,\sigma)$. In particular, all conditions in (2.3.233) are satisfied. Granted this, (2.3.280) follows from Theorem 2.3.16 (see also the first comment following its statement). □

Having established in Theorem 2.3.1 mapping properties for boundary layer potentials involving the Hardy-based Sobolev spaces $H^{q,p}_1(\partial\Omega,\sigma)$ with $p \in \big(\frac{n-1}{n},1\big]$ and $q \in [1,\infty]$ from [69, Definition 11.10.6], the goal of our next theorem in this section is to extend the operator identities from item *(xiii)* in Theorem 1.5.1 to the latter scale (for related results, see also Theorem 2.3.11).

Theorem 2.3.18 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a* UR *domain. In the case when $n = 2$ make the additional assumption that $\partial\Omega$ is compact. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, for some $M \in \mathbb{N}$, consider a coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with complex entries, with the property that the $M\times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in (1.3.2) is weakly elliptic (in the sense of* [70, (1.3.3) in Definition 1.3.1]). *In this setting, consider the boundary layer potential operators S, $\dot{\partial}^A_\nu\mathcal{D}$, K, associated with A and Ω as in Theorem 2.3.1, and let $K^{\#}_{A^\top}$ be the operator associated with $A^\top$ and Ω as in Theorem 2.1.1. Then for any two exponents, $q \in (1,\infty)$ and*

$$p \in \big(\tfrac{n-1}{n},1\big] \cap \big(\tfrac{n-1}{n},n-1\big) = \begin{cases} \big(\frac{n-1}{n},1\big] & \text{if } n \geq 3,\\ \big(\frac{1}{2},1\big) & \text{if } n = 2,\end{cases} \tag{2.3.281}$$

the following operator identities hold:

$$\big(\tfrac{1}{2}I + K\big)\circ\big(-\tfrac{1}{2}I + K\big) = S\circ\big(\dot{\partial}^A_\nu\mathcal{D}\big) \ \text{ on } \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M, \tag{2.3.282}$$

$$\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\circ\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big) = \big(\dot{\partial}^A_\nu\mathcal{D}\big)\circ S \ \text{ on } \big[H^{p}(\partial\Omega,\sigma)\big]^M, \tag{2.3.283}$$

$$S\circ K^{\#}_{A^\top} = K\circ S \ \text{ on } \big[H^{p}(\partial\Omega,\sigma)\big]^M, \tag{2.3.284}$$

$$K^{\#}_{A^\top}\circ\big(\dot{\partial}^A_\nu\mathcal{D}\big) = \big(\dot{\partial}^A_\nu\mathcal{D}\big)\circ K \ \text{ on } \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M. \tag{2.3.285}$$

Henceforth, strengthen the original geometric assumptions, now demanding that Ω is an open set in $\mathbb{R}^n$ satisfying a two-sided local John condition with an Ahlfors regular boundary. In the two-dimensional setting, also assume that $\partial\Omega$ is compact. Then for each $f \in \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^M$ there exists c_f, which is the nontangential trace *on $\partial\Omega$ of some $\mathbb{C}^M$-valued locally constant function in Ω, with the property that at σ-a.e. point on $\partial\Omega$ one has*

$$\big(\tfrac{1}{2}I + K_{\text{mod}}\big)\Big(\big(-\tfrac{1}{2}I + K_{\text{mod}}\big)f\Big) = S\Big(\big(\dot{\partial}^A_\nu \mathcal{D}_{\text{mod}}\big)f\Big) + c_f, \tag{2.3.286}$$

with S as in (2.3.10), $\dot{\partial}^A_\nu \mathcal{D}_{\text{mod}}$ as in (2.3.5), and K_{mod} as in (2.3.184). In addition, for each function $f \in \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^M$ one has

$$\big(\dot{\partial}^A_\nu \mathcal{D}_{\text{mod}}\big)\big(K_{\text{mod}} f\big) = K^{\#}_{A^\top}\big(\dot{\partial}^A_\nu \mathcal{D}_{\text{mod}}\big)f \ \ \textit{at } \sigma\textit{-a.e. point on } \partial\Omega. \tag{2.3.287}$$

Proof To prove (2.3.282), pick a function $f \in \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^M$ and define $u := \mathcal{D}f$ in Ω. From (2.3.4), (1.5.1), (1.5.20), (1.3.24), and (2.3.241), it follows that u satisfies the hypotheses of Theorem 2.3.16 (including (2.3.237) in the case when Ω is an exterior domain). In view of (1.5.20) and (1.3.24), the integral representation formula (2.3.236) presently becomes

$$\mathcal{D}f = \mathcal{D}\Big(\big(\tfrac{1}{2}I + K\big)f\Big) - \mathscr{S}\big((\dot{\partial}^A_\nu \mathcal{D})f\big) \ \text{ in } \Omega. \tag{2.3.288}$$

Upon recalling (1.5.20), (2.3.6), (2.2.127) (also keeping in mind (A.0.1)), and taking nontangential traces then yields

$$\big(\tfrac{1}{2}I + K\big)f = \big(\tfrac{1}{2}I + K\big)\big(\tfrac{1}{2}I + K\big)f - S\big((\dot{\partial}^A_\nu \mathcal{D})f\big) \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.3.289}$$

After some simple algebra this proves (2.3.282).

Moving on, select $f \in \big[H^p(\partial\Omega,\sigma)\big]^M$ and define $u := \mathscr{S}f$ in Ω. From (2.2.39)-(2.2.41), (2.2.44), (2.2.127), (2.2.132), (1.3.8), and (2.3.241), it follows that u satisfies the hypotheses of Theorem 2.3.16 in the case when either Ω is bounded or $\partial\Omega$ is unbounded. Thanks to (2.2.127) (also keeping in mind (A.0.1)), (2.2.33), (2.2.47), and (2.1.4), the integral representation formula (2.3.236) currently reads

$$\mathscr{S}f = \mathcal{D}(Sf) - \mathscr{S}\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big) \ \text{ in } \Omega. \tag{2.3.290}$$

Upon recalling (2.2.47), (2.3.12), and (2.3.6), taking the weak conormal derivative $\dot{\partial}^A_\nu$ of both sides of (2.3.290) then shows that

$$\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f = \big(\dot{\partial}^A_\nu \mathcal{D}\big)(Sf) - \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f \tag{2.3.291}$$

in $\big[H^p(\partial\Omega,\sigma)\big]^M$. After some simple algebra, this establishes (2.3.283) in the case when either Ω is bounded or $\partial\Omega$ is unbounded. Also, if staring with (2.3.290) we now take nontangential traces then, on account of (1.5.20) and (2.2.127) (also bearing in mind (A.0.1)), we arrive at

$$Sf = \big(\tfrac{1}{2}I + K\big)(Sf) - S\Big(\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f\Big) \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{2.3.292}$$

from which (2.3.284) follows in the case when either Ω is bounded or $\partial\Omega$ is unbounded, after some simple algebra. Finally, when Ω is an exterior domain, we may prove (2.3.283) and (2.3.284) in a similar fashion with Ω replaced by $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$.

Consider (2.3.285). Given any $f \in \big[H_1^{q,p}(\partial\Omega,\sigma)\big]^M$, write (2.3.288) then apply the weak conormal derivative operator $\dot{\partial}_\nu^A$ to both sides to obtain

$$\big(\dot{\partial}_\nu^A \mathcal{D}\big) f = \big(\dot{\partial}_\nu^A \mathcal{D}\big)\big(\tfrac{1}{2}I + K\big) f - \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big)\big((\dot{\partial}_\nu^A \mathcal{D}) f\big), \tag{2.3.293}$$

on account of (2.3.6) and (2.2.47). From this, (2.3.285) readily follows. This takes care of all claims in the first half of the statement of the theorem.

To deal with the claims in the second half of the statement, assume now that Ω is an open set in $\mathbb{R}^n$ satisfying a two-sided local John condition whose boundary is Ahlfors regular. In the two-dimensional setting make the additional assumption that $\partial\Omega$ is compact. Finally, suppose p is an in (2.3.281). The latter condition enures that $p^* := \big(\frac{1}{p} - \frac{1}{n-1}\big)^{-1}$ is a well-defined number, belonging to $(1,\infty)$. Fix an arbitrary function $f \in \big[\dot{H}_1^p(\partial\Omega,\sigma)\big]^M$. From Theorem 2.3.3 we know that there exists $c_0 \in \mathbb{C}^M$ with the property that

$$\widetilde{f} := f - c_0 \in \big[L^{p^*}(\partial\Omega,\sigma)\big]^M. \tag{2.3.294}$$

Define

$$u := \mathcal{D}\widetilde{f} \text{ in } \Omega, \tag{2.3.295}$$

and note that, by virtue of (2.3.294) and (1.8.10), we may recast this as

$$u = \mathcal{D}_{\text{mod}} f + c_1 \text{ for some constant } c_1 \in \mathbb{C}^M. \tag{2.3.296}$$

As a consequence of (2.3.294)-(2.3.296), (1.3.24), items *(i)* and *(iv)* in Theorem 1.5.1, as well as items *(i)* and *(iv)* in Theorem 2.3.1, for any aperture parameter $\kappa > 0$ we have

$$\begin{gathered} u \text{ belongs to } \big[\mathscr{C}^\infty(\Omega)\big]^M \text{ and } Lu = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa(\nabla u) \text{ belongs to the space } L^p(\partial\Omega,\sigma), \\ \mathcal{N}_\kappa u \in L^{p^*}(\partial\Omega,\sigma) \subseteq \bigcup_{\delta>0} L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1-\delta}}\Big), \\ \big(\tfrac{1}{2}I + K\big)\widetilde{f} = u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_{\text{mod}}\big) f + c_1, \\ \text{and } \dot{\partial}_\nu^A u = (\dot{\partial}_\nu^A \mathcal{D}_{\text{mod}}) f. \end{gathered} \tag{2.3.297}$$

In the case when Ω is an exterior domain, we also have

$$\fint_{B(0,2R)\setminus B(0,R)} |u| \, \mathrm{d}\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{2.3.298}$$

Granted these properties, Theorem 2.3.16 applies, and from (2.3.236) and (2.3.297) we see that

$$u = \mathcal{D}\Big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big) - \mathcal{S}(\dot{\partial}_\nu^A u) \ \text{ in } \ \Omega. \tag{2.3.299}$$

Note that for some constants $c_2, c_3, c_4 \in \mathbb{C}^M$ we may write

$$\begin{aligned}\mathcal{D}\Big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big) &= \mathcal{D}\Big(\big(\tfrac{1}{2}I + K\big)\widetilde{f}\Big) = \mathcal{D}_{\mathrm{mod}}\Big(\big(\tfrac{1}{2}I + K\big)\widetilde{f}\Big) + c_2 \\ &= \mathcal{D}_{\mathrm{mod}}\Big(\big(\tfrac{1}{2}I + K_{\mathrm{mod}}\big)\widetilde{f}\Big) + c_3 = \mathcal{D}_{\mathrm{mod}}\Big(\big(\tfrac{1}{2}I + K_{\mathrm{mod}}\big)f\Big) + c_4 \ \text{ in } \ \Omega,\end{aligned} \tag{2.3.300}$$

thanks to (1.8.10), (1.8.28), and (2.3.297). After combining (2.3.299) with (2.3.300) and the last property in (2.3.297) we arrive at

$$u = \mathcal{D}_{\mathrm{mod}}\Big(\big(\tfrac{1}{2}I + K_{\mathrm{mod}}\big)f\Big) - \mathcal{S}(\dot{\partial}_\nu^A \mathcal{D}_{\mathrm{mod}})f + c_4 \ \text{ in } \ \Omega. \tag{2.3.301}$$

Going nontangentially to the boundary in (2.3.301) and making use of (2.3.297), (2.3.184), (A.0.89), (1.8.27), (2.3.5), and (2.2.126), further yields

$$\big(\tfrac{1}{2}I + K_{\mathrm{mod}}\big)f + c_1 = \big(\tfrac{1}{2}I + K_{\mathrm{mod}}\big)\Big(\big(\tfrac{1}{2}I + K_{\mathrm{mod}}\big)f\Big) - S(\dot{\partial}_\nu^A \mathcal{D}_{\mathrm{mod}})f + c_4, \tag{2.3.302}$$

from which (2.3.286) follows with $c_f := c_1 - c_4 \in \mathbb{C}^M$. Applying the weak conormal derivative $\dot{\partial}_\nu^A$ to all terms in (2.3.301) and relying on (2.3.297), (2.3.184), (2.3.5), and (2.2.47) produces

$$(\dot{\partial}_\nu^A \mathcal{D}_{\mathrm{mod}})f = (\dot{\partial}_\nu^A \mathcal{D}_{\mathrm{mod}})\Big(\big(\tfrac{1}{2}I + K_{\mathrm{mod}}\big)f\Big) - \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)(\dot{\partial}_\nu^A \mathcal{D}_{\mathrm{mod}})f, \tag{2.3.303}$$

which readily justifies (2.3.287). □

Chapter 3
Layer Potential Operators on Calderón, Morrey-Campanato, and Morrey Spaces

The reader is reminded that Morrey-Campanato spaces, Morrey spaces, and their pre-duals on Ahlfors regular sets in $\mathbb{R}^n$ have been discussed at length in [69, Chapter 6].

The main aim in this chapter is to study singular integral operators of boundary layer type on Calderón spaces (cf. §3.1), Morrey-Campanato spaces (cf. §3.2), and Morrey spaces (cf. §3.3), on domains with uniformly rectifiable boundaries.

In §3.1 we introduce Calderón spaces (aka C_p^α-spaces) on Ahlfors regular sets, by requiring the membership of a "fractional" Fefferman-Stein sharp maximal function (obtained by modifying the standard version, used in connection with the John-Nirenberg space BMO, as indicated in (3.1.1)) to a Lebesgue space. See (3.1.10) and (3.1.14) in this regard. In the entire Euclidean space, such Calderón spaces have been studied at length in [23], [14], [6], [80], [94] and the references included there. The main novel aspect of our analysis is the consideration of the action of singular integral operators of boundary layer type on Calderón spaces defined on uniformly rectifiable sets. Our main result in this regard is Theorem 3.1.1, whose proof makes use of cancelation properties specific to (modified) double layer operators which, in turn, are a consequence of the Divergence Theorem.

Boundary-to-boundary double layer potential operators are studied on Morrey-Campanato spaces in §3.2. The starting point is establishing the boundedness result described in Theorem 3.2.1 for the transpose double layer $K^\#$ associated with a given weakly elliptic system on the boundary of an arbitrary UR domain, acting on the pre-dual of the Morrey-Campanato space introduced in [69, Chapter 6]. The proof of Theorem 3.2.1 makes essential use of the atomic/molecular theory developed in [69, Chapter 6] in relation to these pre-dual spaces. In the process of proving that $K^\#$ maps atoms to molecules, the full force of our Divergence Theorem from [68, §1.2] is required. With this in hand, a duality argument then gives the boundedness of double layer potential operators K on Morrey-Campanato spaces; see Theorem 3.2.2 for details.

Finally, in §3.3 we take up the task of studying the action of singular integral operators of layer potential type on Morrey spaces and their pre-duals, considered on boundaries of uniformly rectifiable domains. At its core, the approach we adopt in this endeavor rests on two basic aspects. First, in earlier work special care has been

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taken to establish integral identities and operator identities in a sufficiently general functional analytic setting (i.e., involving spaces of functions in which Morrey spaces embed naturally). See [70, Theorem 1.7.10], Proposition 1.2.1, Proposition 1.2.2, Lemma 1.3.2, along with the jump-formulas (1.5.20) and (1.5.58) in Theorem 1.5.1, in this regard. Second, in [70, Theorem 2.6.1], [70, Proposition 2.6.2], and [70, Proposition 2.6.3] we have proved norm estimates in Morrey spaces and their pre-duals for generic convolution-type singular integral operators. In turn, these garden-variety singular integral operators serve as building blocks for the algebraically more sophisticated boundary layer potential operators we presently consider. In concert, these two aspects work with great efficiency and yield a satisfactory theory for boundary layer potential operators acting on Morrey spaces and their pre-duals, considered on uniformly rectifiable sets.

More specifically, in Theorem 3.3.1 we prove boundedness properties, jump-relations, and nontangential maximal function estimates for boundary layer potential operators on Morrey and block spaces in arbitrary UR domains. In Theorem 3.3.2 we further extend the scope of this theorem by now allowing our layer potential operators to act on Morrey-based Sobolev spaces. Any boundary-to-domain version of the double layer potential operator associated with a given weakly elliptic system and UR domain with compact boundary is shown to satisfy certain fractional Carleson measure estimates when acting on Morrey spaces, of the sort established in Theorem 3.3.3. Next, Theorem 3.3.5 deals with the modified single layer potential operator mapping Morrey and block spaces into Morrey-based and, respectively, block-based Sobolev spaces. Mapping properties for modified boundary-to-domain double layer potential operators involving Morrey-based and block-based Sobolev spaces are subsequently treated in Theorem 3.3.6. In Theorem 3.3.8 we then establish mapping properties for modified boundary-to-boundary double layer potential operators acting on the scales of Morrey-based and block-based Sobolev spaces.

Moving on, in Theorem 3.3.10 we prove a very useful integral representation formula for null-solutions of weakly elliptic second-order systems in NTA domains with Ahlfors regular boundaries with the property that the nontangential maximal operator of their gradients belong to Morrey and block spaces. Finally, this is then used in Theorem 3.3.12 to derive operator identities akin to those established in Theorem 1.8.26, now involving Morrey-based and block-based homogeneous Sobolev spaces.

3.1 Boundary Layer Potentials on Calderón Spaces

We begin by introducing two maximal operators, then associate with them what we shall call Calderón spaces (aka C_p^α-spaces) on uniformly rectifiable sets, with the ultimate goal of study singular integral operators in such a setting. Specifically, let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Having fixed an exponent $q \in [1, \infty)$ along with a power $\eta \in \mathbb{R}$, we consider the following "fractional" version of the Fefferman-Stein sharp maximal operator (A.0.195), acting on each $f \in L^1_{\rm loc}(\Sigma,\sigma)$ as

$$f^{\#}_{q,\eta}(x) := \sup_{R>0}\left\{R^{-\eta}\Big(\fint_{\Sigma\cap B(x,R)}\big|f(y)-f_{\Delta(x,R)}\big|^q\,d\sigma(y)\Big)^{1/q}\right\},\ \ \forall\,x\in\Sigma, \tag{3.1.1}$$

where $f_{\Delta(x,R)} := \fint_{\Sigma\cap B(x,R)} f\,d\sigma$ for each $R>0$ and $x\in\Sigma$. This turns out to be a σ-measurable function[1] and, obviously,

$$(f-c)^{\#}_{q,\eta} = f^{\#}_{q,\eta}\ \text{ for each }\ c\in\mathbb{C}. \tag{3.1.2}$$

For each $f \in L^1_{\rm loc}(\Sigma,\sigma)$ let us also set

$$\widetilde{f^{\#}_{q,\eta}}(x) := \sup_{R>0}\left\{R^{-\eta}\Big(\fint_{\Sigma\cap B(x,R)}|f(y)-f(x)|^q\,d\sigma(y)\Big)^{1/q}\right\},\ \ \forall\,x\in\Sigma. \tag{3.1.3}$$

Since for any two points $x,y\in\Sigma$ and any $R>0$ we have

$$\begin{aligned}\big|f(y)-f_{\Delta(x,R)}\big| &\le |f(y)-f(x)| + \Big|f(x)-\fint_{\Sigma\cap B(x,R)} f\,d\sigma\Big|\\ &\le |f(y)-f(x)| + \fint_{\Sigma\cap B(x,R)}|f(x)-f(z)|\,d\sigma(z),\end{aligned} \tag{3.1.4}$$

it follows that

$$\begin{aligned}\fint_{\Sigma\cap B(x,R)}\big|f(y)-f_{\Delta(x,R)}\big|^q\,d\sigma(y) &\le C\fint_{\Sigma\cap B(x,R)}|f(y)-f(x)|^q\,d\sigma(y)\\ &\quad+\fint_{\Sigma\cap B(x,R)}|f(x)-f(z)|^q\,d\sigma(z)\\ &= C\fint_{\Sigma\cap B(x,R)}|f(y)-f(x)|^q\,d\sigma(y)\end{aligned} \tag{3.1.5}$$

hence, ultimately,

$$f^{\#}_{q,\eta}(x) \le C\cdot\widetilde{f^{\#}_{q,\eta}}(x)\ \text{ at each }\ x\in\Sigma. \tag{3.1.6}$$

In the opposite direction, we claim that if $\eta>0$ then there exists $C\in(0,\infty)$ with the property that for each function $f\in L^1_{\rm loc}(\Sigma,\sigma)$ we have

$$\widetilde{f^{\#}_{q,\eta}}(x) \le C\cdot f^{\#}_{q,\eta}(x)\ \text{ at }\sigma\text{-a.e. }\ x\in\Sigma. \tag{3.1.7}$$

Indeed, we shall show that (3.1.7) holds at each Lebesgue point $x\in\Sigma$ for f. To see that this is the case, fix such a point and, given the nature of the conclusion we

[1] the same argument used in the proofs of [68, Theorems 6.3.3 and 7.6.1] applies

seek, observe that there is no loss of generality in assuming that $f^{\#}_{q,\eta}(x) < +\infty$. In turn, this ensures that actually $f \in L^q_{\rm loc}(\Sigma, \sigma)$. To proceed, pick a scale $R > 0$. The estimate in [68, (7.4.121)] written with $2^{-j-1}R$ in place of r then gives that for each $j \in \mathbb{N}$ we have

$$\begin{aligned}\Big(\fint_{\Sigma\cap B(x,R)} \big|f(y) - f_{\Delta(x,2^{-j-1}R)}\big|^q \, d\sigma(y)\Big)^{1/q} \\ \leq C \sum_{k=0}^{j} \Big(\fint_{\Sigma\cap B(x,2^{-k}R)} \big|f(y) - f_{\Delta(x,2^{-k}R)}\big|^q \, d\sigma(y)\Big)^{1/q} \\ \leq C \sum_{k=0}^{\infty} (2^{-k}R)^\eta f^{\#}_{q,\eta}(x) = CR^\eta f^{\#}_{q,\eta}(x),\end{aligned} \tag{3.1.8}$$

for some finite $C > 0$ (since $\eta > 0$). After passing to limit as $j \to \infty$ we therefore arrive at

$$\Big(\fint_{\Sigma\cap B(x,R)} |f(y) - f(x)|^q \, d\sigma(y)\Big)^{1/q} \leq CR^\eta f^{\#}_{q,\eta}(x), \tag{3.1.9}$$

on account of Lebesgue's Differentiation Theorem (cf. [68, Proposition 7.4.4] whose applicability in the present setting is guaranteed by [68, Lemma 3.6.4]) and Lebesgue's Dominated Convergence Theorem (which uses the membership of f to $L^q_{\rm loc}(\Sigma, \sigma)$). At this stage, (3.1.7) follows from (3.1.9) after dividing by R^η and taking the supremum over $R > 0$.

With Σ, σ as above, for each $p \in [1, \infty]$, $q \in [1, \infty)$, and $\eta \in \mathbb{R}$ we define

$$\dot{C}^p_{q,\eta}(\Sigma, \sigma) := \Big\{f \in L^1_{\rm loc}(\Sigma, \sigma) : \ f^{\#}_{q,\eta} \in L^p(\Sigma, \sigma)\Big\} \tag{3.1.10}$$

equipped with the semi-norm

$$\|f\|_{\dot{C}^p_{q,\eta}(\Sigma,\sigma)} := \big\|f^{\#}_{q,\eta}\big\|_{L^p(\Sigma,\sigma)}, \qquad \forall f \in \dot{C}^p_{q,\eta}(\Sigma, \sigma). \tag{3.1.11}$$

Note that if we set

$$\big\|\,[f]\,\big\|_{\dot{C}^p_{q,\eta}(\Sigma,\sigma)/\sim} := \|f\|_{\dot{C}^p_{q,\eta}(\Sigma,\sigma)} \ \text{ for each } \ f \in L^1_{\rm loc}(\Sigma, \sigma), \tag{3.1.12}$$

then $\big\|\,[\cdot]\,\big\|_{\dot{C}^p_{q,\eta}(\Sigma,\sigma)/\sim}$ becomes a genuine norm on the quotient space

$$\dot{C}^p_{q,\eta}(\Sigma, \sigma)/\sim\, := \big\{[f] : \ f \in \dot{C}^p_{q,\eta}(\Sigma, \sigma)\big\}. \tag{3.1.13}$$

Let us also introduce

$$\begin{aligned} C^p_{q,\eta}(\Sigma,\sigma) &:= L^p(\Sigma,\sigma) \cap \dot{C}^p_{q,\eta}(\Sigma,\sigma) \\ &= \Big\{ f \in L^p(\Sigma,\sigma) : \ f^\#_{q,\eta} \in L^p(\Sigma,\sigma) \Big\} \end{aligned} \tag{3.1.14}$$

and equip this space with the norm

$$\|f\|_{C^p_{q,\eta}(\Sigma,\sigma)} := \|f\|_{L^p(\Sigma,\sigma)} + \big\|f^\#_{q,\eta}\big\|_{L^p(\Sigma,\sigma)}, \quad \forall f \in C^p_{q,\eta}(\Sigma,\sigma). \tag{3.1.15}$$

Since the maximal functions from (3.1.3) and (3.1.1) have been first introduced in the Euclidean setting in [8] and [10], respectively, we shall refer to (3.1.10) and (3.1.14) as (homogeneous and inhomogeneous) `Calderón spaces`. A wealth of information concerning the Euclidean version of these Calderón spaces may be found in [23], [14], [6], [80], [94] and the references therein. The novelty here is the consideration of the action of singular integral operators on Calderón spaces defined on uniformly rectifiable sets (cf. Theorem 3.1.1 below in this regard).

In relation to these scales of spaces we wish to make two comments. First, since [68, (7.4.115)] (used with $X := \Sigma$, $\mu := \sigma$, $d := n-1$, $\varepsilon := 1$, and $p := 1$) gives that for each function $f \in L^1_{\rm loc}(\Sigma,\sigma)$, each point $x_0 \in \Sigma$, and each scale $r \in (0,\infty)$, we have

$$\begin{aligned} &\int_\Sigma \frac{\big|f(x) - f_{\Delta(x_0,r)}\big|}{\big[r + |x-x_0|\big]^n}\, d\sigma(x) \\ &\qquad \le \frac{C}{r}\int_1^\infty \Big(⨍_{\Sigma\cap B(x_0,\lambda r)} \big|f(x) - f_{\Delta(x_0,\lambda r)}\big|^q\, d\sigma(x)\Big)^{1/q} \frac{d\lambda}{\lambda^2} \\ &\qquad \le \frac{C}{r^{1-\eta}}\Big(\int_1^\infty \frac{d\lambda}{\lambda^{2-\eta}}\Big) f^\#_{q,\eta}(x_0), \end{aligned} \tag{3.1.16}$$

it follows that

$$C^p_{q,\eta}(\Sigma,\sigma) \subset \dot{C}^p_{q,\eta}(\Sigma,\sigma) \subset L^1\Big(\Sigma, \frac{\sigma(x)}{1+|x|^n}\Big) \text{ whenever } \eta < 1. \tag{3.1.17}$$

Our second comment is that, as is visible from [68, Proposition 7.4.9] (whose present applicability is ensured by [68, Lemma 3.6.4] used with $s := n-1$), (A.0.195), and [68, (7.4.113)], corresponding to the case when $p = \infty$ we have

$$\dot{C}^\infty_{q,\eta}(\Sigma,\sigma) = \begin{cases} \dot{\mathscr{C}}^\eta(\Sigma) & \text{if } \eta > 0, \\ \mathrm{BMO}(\Sigma,\sigma) & \text{if } \eta = 0. \end{cases} \tag{3.1.18}$$

The action of the boundary-to-boundary version of double layer potential operators associated with a given homogeneous, weakly elliptic, constant coefficient, second-order system on uniformly rectifiable sets has been studied on Hölder spaces in Theorem 2.1.10 and on the John-Nirenberg space of functions of bounded mean oscillations in Theorem 2.1.7. We wish to augment this body of results by considering below mapping properties for said double layer potential operators on Calderón spaces with $p < \infty$.

Theorem 3.1.1 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. In addition, let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). In this setting, let K be a principal-value double layer operator associated with Ω and L as in (1.3.68), and recall its modified version K_{mod} introduced in (1.8.24)-(1.8.25). Also, fix $q \in (1,\infty)$ and $\eta \in (0,1)$.*

Then there exists a constant $C \in (0,\infty)$ with the property that, with the maximal operator defined as in (3.1.1) (with $\Sigma := \partial\Omega$), one has

$$\big(K_{\text{mod}} f\big)^{\#}_{q,\eta}(x) \leq C\, f^{\#}_{q,\eta}(x) \ \text{ at } \sigma\text{-a.e. } x \in \partial\Omega, \tag{3.1.19}$$

for every function $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M$. As a consequence, given any $p \in [1,\infty)$, it follows that the operator

$$\begin{gathered} [K_{\text{mod}}] : \big[\dot{C}^p_{q,\eta}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{C}^p_{q,\eta}(\partial\Omega,\sigma)/\sim\big]^M \\ \text{defined by } [K_{\text{mod}}][f] := \big[K_{\text{mod}} f\big] \text{ for each } f \in \big[\dot{C}^p_{q,\eta}(\partial\Omega,\sigma)\big]^M \end{gathered} \tag{3.1.20}$$

is well defined, linear, and bounded (considering the space $\big[\dot{C}^p_{q,\eta}(\partial\Omega,\sigma)/\sim\big]^M$ equipped with the norm defined as in (3.1.12)). Moreover,

$$\big(K f\big)^{\#}_{q,\eta}(x) \leq C\, f^{\#}_{q,\eta}(x) \ \text{ at } \sigma\text{-a.e. } x \in \partial\Omega, \tag{3.1.21}$$

for every function $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ with $p \in [1,\infty)$. Finally, whenever one has $p \in (1,\infty)$ it follows that the operator

$$K : \big[C^p_{q,\eta}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[C^p_{q,\eta}(\partial\Omega,\sigma)\big]^M \tag{3.1.22}$$

is well defined, linear, and bounded (considering the space $\big[C^p_{q,\eta}(\partial\Omega,\sigma)\big]^M$ equipped with the norm defined as in (3.1.15)).

As is apparent from the proof given below, similar results hold for double layer potential operators acting on Lorentz-based Calderón spaces, defined in a similar fashion to (3.1.10), (3.1.14), this time demanding the membership of the maximal function to Lorentz spaces in lieu of Lebesgue spaces.

Proof of Theorem 3.1.1 Select a function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M$ and pick $x \in \partial\Omega$ at which (3.1.7) holds (with $\Sigma := \partial\Omega$). If for each $R > 0$ we define

$$g_R(y) := \big(f(y) - f(x)\big)\mathbf{1}_{\partial\Omega\cap B(x,2R)}(y), \qquad \forall\, y \in \partial\Omega, \tag{3.1.23}$$

then Hölder's inequality and (3.1.7) (keeping in mind (3.1.3)) permit us to estimate

$$\|g_R\|_{[L^1(\partial\Omega,\sigma)]^M} \le CR^{n-1}\Big(\fint_{\Sigma\cap B(x,2R)}|f(y)-f(x)|^q\,\mathrm{d}\sigma(y)\Big)^{1/q} \le CR^{n-1+\eta}f^{\#}_{q,\eta}(x), \qquad \forall R>0. \tag{3.1.24}$$

As a consequence, for each fixed scale $R>0$ we have

$$\begin{aligned}\int_{\partial\Omega\setminus B(x,2R)}\frac{|f(y)-f(x)|}{|y-x|^n}\,\mathrm{d}\sigma(y) &= \sum_{j=1}^{\infty}\int_{\partial\Omega\cap[B(x,2^{j+1}R)\setminus B(x,2^jR)]}\frac{|f(y)-f(x)|}{|y-x|^n}\,\mathrm{d}\sigma(y)\\ &\le \sum_{j=1}^{\infty}\big(2^jR\big)^{-n}\big\|g_{2^jR}\big\|_{[L^1(\partial\Omega,\sigma)]^M}\\ &\le \sum_{j=1}^{\infty}\big(2^jR\big)^{-n}\big(2^jR\big)^{n-1+\eta}f^{\#}_{q,\eta}(x)\\ &= CR^{\eta-1}f^{\#}_{q,\eta}(x),\end{aligned} \tag{3.1.25}$$

for some constant $C>0$, independent of f and x, which is finite since $\eta<1$. In a similar fashion, for each $R>0$ we may estimate

$$\begin{aligned}\int_{\partial\Omega\cap B(x,2R)}\frac{|f(y)-f(x)|}{|y-x|^{n-1}}\,\mathrm{d}\sigma(y) &= \sum_{j=0}^{\infty}\int_{\partial\Omega\cap[B(x,2^{-j+1}R)\setminus B(x,2^{-j}R)]}\frac{|f(y)-f(x)|}{|y-x|^{n-1}}\,\mathrm{d}\sigma(y)\\ &\le \sum_{j=0}^{\infty}\big(2^{-j}R\big)^{-(n-1)}\big\|g_{2^{-j}R}\big\|_{[L^1(\partial\Omega,\sigma)]^M}\\ &\le \sum_{j=0}^{\infty}\big(2^{-j}R\big)^{-(n-1)}\big(2^{-j}R\big)^{n-1+\eta}f^{\#}_{q,\eta}(x)\\ &= CR^{\eta}f^{\#}_{q,\eta}(x),\end{aligned} \tag{3.1.26}$$

for some constant $C>0$, independent of f and x, which is finite since $\eta>0$.

Going further, from (1.8.28) we see that for each $z\in\partial\Omega$ we have

$$\big(K_{\text{mod}}f\big)(z)-\big(K_{\text{mod}}f\big)(x)=\Big(K_{\text{mod}}\big(f-f(x)\big)\Big)(z)-\Big(K_{\text{mod}}\big(f-f(x)\big)\Big)(x). \tag{3.1.27}$$

Write L explicitly as $\big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\le\alpha,\beta\le M}$. If $\nu=(\nu_1,\dots,\nu_n)$ denotes the geometric measure theoretic outward unit normal to Ω, then for each fixed $\gamma\in\{1,\dots,M\}$ and $z\in B(x,R)\cap\partial\Omega$ we may then employ (3.1.27) and (1.8.24) to express (with $k^{(r\gamma\beta)}_{\varepsilon}$ as in (1.8.25))

$$\begin{aligned}
&\big(K_{\rm mod}f\big)_\gamma(z)-\big(K_{\rm mod}f\big)_\gamma(x)\\
&=\lim_{\varepsilon\to 0^+}\int_{\partial\Omega}\nu_s(y)a^{\beta\alpha}_{rs}\big\{k^{(r\gamma\beta)}_{\varepsilon}(x-y)-k^{(r\gamma\beta)}_{\varepsilon}(z-y)\big\}\big(f_\alpha(y)-f_\alpha(x)\big)\,{\rm d}\sigma(y)\\
&=\int_{\partial\Omega\setminus B(x,2R)}\nu_s(y)a^{\beta\alpha}_{rs}\big\{(\partial_r E_{\gamma\beta})(x-y)-(\partial_r E_{\gamma\beta})(z-y)\big\}\big(f_\alpha(y)-f_\alpha(x)\big)\,{\rm d}\sigma(y)\\
&\quad+\lim_{\varepsilon\to 0^+}\int_{\partial\Omega\cap B(x,2R)}\nu_s(y)a^{\beta\alpha}_{rs}k^{(r\gamma\beta)}_{\varepsilon}(x-y)\big(f_\alpha(y)-f_\alpha(x)\big)\,{\rm d}\sigma(y)+\big(Kg_R\big)(z).
\end{aligned}\tag{3.1.28}$$

On account of (3.1.28), the Mean Value Theorem, [70, (1.4.24)], (3.1.25), and (3.1.25), for each point $z\in B(x,R)\cap\partial\Omega$ we may then estimate

$$\begin{aligned}
&\big|\big(K_{\rm mod}f\big)(z)-\big(K_{\rm mod}f\big)(x)\big|\\
&\qquad\le\big|\big(Kg_R\big)(z)\big|+C|z-x|\cdot\int_{\partial\Omega\setminus B(x,2R)}\frac{|f(y)-f(x)|}{|x-y|^n}\,{\rm d}\sigma(y)\\
&\qquad\quad+C\int_{\partial\Omega\cap B(x,2R)}\frac{|f(y)-f(x)|}{|x-y|^{n-1}}\,{\rm d}\sigma(y)\\
&\qquad\le\big|\big(Kg_R\big)(z)\big|+CR^\eta f^{\#}_{q,\eta}(x).
\end{aligned}\tag{3.1.29}$$

In turn, after raising the most extreme sides to the q-th power and integrating over $\partial\Omega\cap B(x,R)$ with respect to σ in the variable z, this permits us to write

$$\begin{aligned}
&\Big(⨏_{\partial\Omega\cap B(x,R)}\big|\big(K_{\rm mod}f\big)(z)-\big(K_{\rm mod}f\big)(x)\big|^q\,{\rm d}\sigma(z)\Big)^{1/q}\\
&\qquad\le CR^{-(n-1)/q}\big\|Kg_R\big\|_{[L^q(\partial\Omega,\sigma)]^M}+CR^\eta f^{\#}_{q,\eta}(x)\\
&\qquad\le CR^{-(n-1)/q}\big\|g_R\big\|_{[L^q(\partial\Omega,\sigma)]^M}+CR^\eta f^{\#}_{q,\eta}(x)\\
&\qquad\le CR^\eta f^{\#}_{q,\eta}(x),
\end{aligned}\tag{3.1.30}$$

thanks to boundedness of K on $\big[L^q(\partial\Omega,\sigma)\big]^M$ (cf. Theorem 1.5.1), (3.1.23), and (3.1.7).

At this stage, (3.1.19) follows from (3.1.30) and (3.1.6), bearing in mind that σ-a.e. $x \in \partial\Omega$ is a Lebesgue point for f (cf. [68, Proposition 7.4.4] and [68, Lemma 3.6.4]). As far as the claim pertaining to the operator in (3.1.20) is concerned, for each given $f \in \dot{\mathscr{C}}^p_{q,\eta}(\partial\Omega, \sigma)$ we may write

$$\begin{aligned}
\big\|[K_{\text{mod}}][f]\big\|_{[\dot{\mathscr{C}}^p_{q,\eta}(\partial\Omega,\sigma)/\sim]^M} &= \big\|[K_{\text{mod}} f]\big\|_{[\dot{\mathscr{C}}^p_{q,\eta}(\partial\Omega,\sigma)/\sim]^M} \\
&= \big\|K_{\text{mod}} f\big\|_{[\dot{\mathscr{C}}^p_{q,\eta}(\partial\Omega,\sigma)]^M} = \big\|(K_{\text{mod}} f)^\#_{q,\eta}\big\|_{L^p(\partial\Omega,\sigma)} \\
&\leq C\|f^\#_{q,\eta}\|_{L^p(\partial\Omega,\sigma)} = C\|f\|_{[\dot{\mathscr{C}}^p_{q,\eta}(\partial\Omega,\sigma)]^M} \\
&= C\big\|[f]\big\|_{[\dot{\mathscr{C}}^p_{q,\eta}(\partial\Omega,\sigma)/\sim]^M} \qquad (3.1.31)
\end{aligned}$$

thanks to (3.1.20), (3.1.11), (3.1.19), and (3.1.12). The desired conclusions follow from this. Next, the claim in (3.1.21) becomes a consequence of (3.1.19), (1.8.24), (1.3.68), and (3.1.2). Finally, that the operator K in (3.1.22) is well defined, linear, and bounded whenever $p \in (1, \infty)$ is seen from (3.1.21), (3.1.14)-(3.1.15), and the boundedness of K on $\big[L^p(\partial\Omega, \sigma)\big]^M$ (cf. Theorem 1.5.1). □

3.2 Boundary Layer Potentials on Morrey-Campanato Spaces and Their Pre-Duals

In this section we turn our attention to singular integral operators on Morrey-Campanato spaces, as well as their pre-duals, on uniformly rectifiable sets. We begin by establishing the boundedness result described in the theorem below for the transpose (principal-value) double layers, associated with a given weakly elliptic system on the boundary of an arbitrary UR domain, acting on the pre-dual space of the Morrey-Campanato space introduced in [69, Chapter 6].

Theorem 3.2.1 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). In this context, consider the boundary layer potential operator $K^\#$ associated with the system L and the set Ω as in (1.3.72). Finally, select $q \in (1, \infty)$ and $\lambda \in (0, n-1)$.*

Then the operator

$$K^\# : \big[L^r(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L^r(\partial\Omega, \sigma)\big]^M \text{ with } r := \frac{q(n-1)}{n-1+\lambda(q-1)} \qquad (3.2.1)$$

(cf. (1.5.12)), has $\big[\mathscr{H}^{q,\lambda}(\partial\Omega, \sigma)\big]^M$ as an invariant subspace (cf. [69, (6.1.22)]), and

$$K^{\#} : \Big(\big[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma) \big]^M , \|\cdot\|_{[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)]^M} \Big) \longrightarrow \Big(\big[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma) \big]^M , \|\cdot\|_{[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)]^M} \Big) \tag{3.2.2}$$

is a linear and bounded mapping.

Proof Assume first that $\partial\Omega$ is unbounded and let $a \in \big[L^q(\partial\Omega,\sigma)\big]^M$ be an arbitrary $\mathbb{C}^M$-valued $\mathscr{H}^{q,\lambda}$-atom on $\partial\Omega$. Then (cf. [69, (6.1.15)]) there exist $x_o \in \partial\Omega$ and $R \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ such that

$$\operatorname{supp} a \subseteq B(x_o,R) \cap \partial\Omega, \quad \|a\|_{[L^q(\partial\Omega,\sigma)]^M} \leq R^{\lambda(\frac{1}{q}-1)}, \quad \int_{\partial\Omega} a\, d\sigma = 0. \tag{3.2.3}$$

Contrasting the above properties to [69, (4.4.2)] shows that $R^{(\lambda-n+1)(1-1/q)}a$ is an ordinary $\mathbb{C}^M$-valued $(1,q)$-atom (i.e., an L^q-normalized atom for the Hardy space H^1 on $\partial\Omega$). Consequently, (2.1.8) gives that

the function $m := R^{(\lambda-n+1)(1-1/q)}K^{\#}a$ is a fixed multiple of a $\mathbb{C}^M$-valued $(1,q,\varepsilon)$-molecule on $\partial\Omega$, in the sense considered in [69, Definition 4.5.1] for the choice $\varepsilon := 1/(n-1)$. (3.2.4)

Using the boundedness of $K^{\#}$ on Lebesgue spaces (cf. (1.5.12)) and the normalization of the atom in (3.2.3), we estimate

$$\int_{\partial\Omega} |K^{\#}a|^q \, d\sigma \leq C \int_{\partial\Omega} |a|^q \, d\sigma \leq C R^{\lambda(1-q)}, \tag{3.2.5}$$

for some constant $C \in (0,\infty)$ independent of the $\mathscr{H}^{q,\lambda}$-atom a. Also, if we choose (for $\varepsilon := 1/(n-1)$, as above)

$$\theta \in \Big((n-1)(q-1),\, (n-1)[(1+\varepsilon)q-1] \Big), \tag{3.2.6}$$

then by relying on (3.2.4) and [69, (4.5.1)] we may write (using the piece of notation introduced in [69, (4.5.2)] with $\Sigma := \partial\Omega$)

$$
\begin{aligned}
\int_{\partial\Omega} &|(K^{\#}a)(x)|^q |x-x_o|^\theta \,\mathrm{d}\sigma(x) \\
&= R^{(n-1-\lambda)(q-1)} \int_{\partial\Omega} |m(x)|^q |x-x_o|^\theta \,\mathrm{d}\sigma(x) \\
&\leq CR^{(n-1-\lambda)(q-1)} \sum_{k=0}^{\infty} (2^k R)^\theta \int_{A_k(x_o,R)} |m|^q \,\mathrm{d}\sigma \\
&\leq CR^{(n-1-\lambda)(q-1)+\theta} \sum_{k=0}^{\infty} 2^{k(n-1)[1-(1+\varepsilon)q]+k\theta} \sigma\big(B(x_o,R)\cap\partial\Omega\big)^{1-q} \\
&\leq CR^{(n-1-\lambda)(q-1)+\theta} \Big(\sum_{k=0}^{\infty} 2^{k\{\theta-(n-1)[(1+\varepsilon)q-1]\}}\Big) R^{-(n-1)(q-1)} \\
&= CR^{\lambda(1-q)+\theta},
\end{aligned}
\tag{3.2.7}
$$

for some $C \in (0,\infty)$ independent of the $\mathscr{H}^{q,\lambda}$-atom a. In addition, from (3.2.4) and (2.1.13) we see that

$$
\int_{\partial\Omega} K^{\#}a \,\mathrm{d}\sigma = 0. \tag{3.2.8}
$$

Collectively, (3.2.5), (3.2.7), (3.2.8), and [69, (6.1.37)] prove that

$$
\text{the function } K^{\#}a \text{ is a fixed multiple of a } \mathbb{C}^M\text{-valued } \mathscr{H}^{q,\lambda,\theta}\text{-molecule on } \partial\Omega, \text{ for } \theta \text{ chosen as in (3.2.6).} \tag{3.2.9}
$$

In turn, from (3.2.9) and [69, (6.1.38)] we conclude that there exists a constant $C \in (0,\infty)$ independent of the $\mathscr{H}^{q,\lambda}$-atom a with the property that

$$
K^{\#}a \in \big[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)\big]^M \ \text{ and } \ \|K^{\#}a\|_{[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)]^M} \leq C. \tag{3.2.10}
$$

Suppose now that some function $f \in \big[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ has been given. Then there exist some numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$ along with a sequence $\{a_j\}_{j\in\mathbb{N}}$ of $\mathbb{C}^M$-valued $\mathscr{H}^{q,\lambda}$-atoms on $\partial\Omega$ such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ in the sense of distributions on $\partial\Omega$. Thanks to (A.0.85)-[69, (6.1.22)], the series $\sum_{j=1}^{\infty} \lambda_j a_j$ actually converges to f in $\big[L^r(\partial\Omega,\sigma)\big]^M$ with $r := \frac{q(n-1)}{n-1+\lambda(q-1)}$. Also, (3.2.10) readily implies that the sequence $\big\{K^{\#}\big(\sum_{j=1}^{N} \lambda_j a_j\big)\big\}_{N\in\mathbb{N}}$ is Cauchy, hence convergent, in the Banach space $\Big(\big[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)\big]^M, \|\cdot\|_{[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)]^M}\Big)$. In light of [69, (6.1.22)], this latter convergence takes place in $\big[L^r(\partial\Omega,\sigma)\big]^M$ as well. Since $K^{\#}$ is continuous on $\big[L^r(\partial\Omega,\sigma)\big]^M$, it follows that $\big\{K^{\#}\big(\sum_{j=1}^{N} \lambda_j a_j\big)\big\}_{N\in\mathbb{N}}$ converges to $K^{\#}f$ in the space $\big[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$. Bearing (3.2.10) in mind, this argument ultimately proves that

$$K^{\#} f \in \big[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)\big]^M \text{ and } \|K^{\#} f\|_{[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)]^M}, \tag{3.2.11}$$

from which all desired conclusions follow in the case when $\partial\Omega$ is unbounded.

Finally, in the case when $\partial\Omega$ is bounded we reason similarly, except that now we also have to consider the action of $K^{\#}$ on an atom with constant components. Since on account of (1.5.12) and [69, (6.1.24)] the resulting function belongs to $\big[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$, all desired conclusions follow in this case as well. □

In turn, Theorem 3.2.1 is a key ingredient in the proof of the fact that the principal-value double layer potential operators associated with a given weakly elliptic system on the boundary of an arbitrary UR domain act in a natural fashion on Morrey-Campanato spaces (both homogeneous and inhomogeneous), as indicated in our next theorem.

Theorem 3.2.2 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$), and consider a principal-value double layer potential operator K associated with the system L and the set Ω as in* (1.3.68)*. Finally, select $p \in (1,\infty)$ and $\lambda \in (0, n-1)$.*

Then the operator

$$K : \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^p(\partial\Omega,\sigma)\big]^M \tag{3.2.12}$$

(which is well defined, linear and bounded; cf. Theorem 1.5.1), has the inhomogeneous Morrey-Campanato space $\big[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (cf. (A.0.119)*) as an invariant subspace, and*

$$K : \Big(\big[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \|\cdot\|_{[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)]^M}\Big) \longrightarrow \Big(\big[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \|\cdot\|_{[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)]^M}\Big) \tag{3.2.13}$$

is a linear and bounded mapping.

Moreover, if K_{mod} is the modified version of the principal-value double layer operator introduced in (1.8.24)-(1.8.25)*, the assignment*

$$[K_{\text{mod}}] : \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M$$
$$[K_{\text{mod}}][f] := [K_{\text{mod}} f] \text{ for each } f \in \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^M \tag{3.2.14}$$

is well defined, linear and bounded, assuming each quotient space above equipped with the norm $\|[\cdot]\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M}$ (cf. [69, (6.1.9)]*). In particular,*

$K_{\rm mod} : \left[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\right]^M \to \left[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\right]^M$ *is well defined, linear, and there exists some constant* $C \in (0,\infty)$ *with the property that* $\|K_{\rm mod} f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}$ *for each function* $f \in \left[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\right]^M$. (3.2.15)

Finally, assuming that $q \in (1,\infty)$ *satisfies* $1/p + 1/q = 1$,

the (real) transpose of the operator $K^\#$ *from* (3.2.2) *is, respectively,* $\left[K_{\rm mod}\right]$ *from* (3.2.14) *if* $\partial\Omega$ *is unbounded, and* K *from* (3.2.13) *if* $\partial\Omega$ *is bounded,* (3.2.16)

that is, for each $f \in \left[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\right]^M$ *and* $g \in \left[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)\right]^M$ *one has*

$$\left\langle \left[K_{\rm mod} f\right], g\right\rangle = \left\langle [f], K^\# g\right\rangle \text{ if } \partial\Omega \text{ is unbounded,} \tag{3.2.17}$$

$$\left\langle K f, g\right\rangle = \left\langle f, K^\# g\right\rangle \text{ if } \partial\Omega \text{ is bounded,} \tag{3.2.18}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality bracket between the Morrey-Campanato space and its pre-dual* (*cf.* [69, (6.1.25)]).

Proof To fix ideas, assume $\partial\Omega$ is unbounded (the case when $\partial\Omega$ is bounded is very similar). Recall the operator $K^\#$ associated with the given system L and the set Ω as in (1.3.72). Let $q \in (1,\infty)$ be the exponent satisfying $1/p + 1/q = 1$ and consider consider the operator

$$\begin{gathered} \widetilde{K} : \left[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim\right]^M \longrightarrow \left[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim\right]^M \\ \text{defined by } \left\langle \widetilde{K}[f], g\right\rangle := \left\langle [f], K^\# g\right\rangle \text{ for every} \\ f \in \left[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\right]^M \text{ and } g \in \left[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)\right]^M \end{gathered} \tag{3.2.19}$$

where the angled brackets $\langle\cdot,\cdot\rangle$ stand for the duality pairing between the homogeneous Morrey-Campanato space, modulo constants, and its pre-dual (cf. [69, (6.1.25)]). Thanks to Theorem 3.2.1 and [69, (6.1.25)] it follows that $\widetilde{K}$ is a well-defined, linear and bounded operator. This is going to be of significance shortly.

For now, fix an arbitrary $\mathbb{C}^M$-valued $\mathscr{H}^{q,\lambda}$-atom $a = (a_\alpha)_{1\leq\alpha\leq M}$ on $\partial\Omega$. Also, pick an arbitrary function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \left[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\right]^M$. Since from [69, (6.1.3), (6.1.14)] (presently used with $\Sigma := \partial\Omega$) we know that

$$f \in \left[L^p_{\rm loc}(\partial\Omega,\sigma)\right]^M \cap \left[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\right]^M, \tag{3.2.20}$$

we may invoke [70, (2.3.35)] (bearing in mind (1.8.24)) to conclude that

$$K_{\rm mod} f \in \left[L^p_{\rm loc}(\partial\Omega,\sigma)\right]^M. \tag{3.2.21}$$

We also have $\int_{\partial\Omega} |f||K^\# a|\,d\sigma < +\infty$, thanks to (3.2.9) and [69, Lemma 6.1.3] (used with $\Sigma := \partial\Omega$). Granted this and (3.2.20), we may reason as in (2.1.104)-(2.1.106) to conclude that (cf. (2.1.107))

$$\int_{\partial\Omega} f_\alpha (K^\# a)_\alpha \, d\sigma = \int_{\partial\Omega} \big(K_{\mathrm{mod}} f\big)_\alpha a_\alpha \, d\sigma, \tag{3.2.22}$$

using the summation convention over repeated indexes. As such, we may write

$$\int_{\partial\Omega} (K_{\mathrm{mod}} f)_\alpha a_\alpha \, d\sigma = \int_{\partial\Omega} f_\alpha (K^\# a)_\alpha \, d\sigma = \big\langle [f], K^\# a \big\rangle = \big\langle \widetilde{K}[f], a \big\rangle, \tag{3.2.23}$$

where the second equality above is a consequence of (3.2.9) and [69, Lemma 6.1.3] (again, used with $\Sigma := \partial\Omega$), while the third equality in (3.2.23) comes from (3.2.19). If we now pick a function $h = (h_\alpha)_{1\leq\alpha\leq M} \in \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ such that $[h] = \widetilde{K}[f]$, then (3.2.23), [69, Lemma 6.1.3], and [69, (6.1.88)] permit us to write

$$\int_{\partial\Omega} (K_{\mathrm{mod}} f)_\alpha a_\alpha \, d\sigma = \big\langle [h], a \big\rangle = \int_{\partial\Omega} h_\alpha a_\alpha \, d\sigma. \tag{3.2.24}$$

On account of this, the arbitrariness of the $\mathscr{H}^{q,\lambda}$-atom a, the fact that any such atom is a multiple of an ordinary $(1,q)$-atom (for the scale of Hardy spaces on $\partial\Omega$), [69, Lemma 4.6.9], and the fact that for each $\alpha \in \{1,\dots,M\}$ both $(K_{\mathrm{mod}} f)_\alpha$ and h_α belong to $L^p_{\mathrm{loc}}(\partial\Omega,\sigma)$, we finally conclude that

$$[K_{\mathrm{mod}} f] = [h] = \widetilde{K}[f]. \tag{3.2.25}$$

In turn, based on (3.2.25), the boundedness of the operator (3.2.19), and [69, (6.1.9)], we may estimate

$$\begin{aligned}\big\| [K_{\mathrm{mod}} f] \big\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M} &= \big\|\widetilde{K}[f] \big\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M} \\ &\leq C \big\| [f] \big\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M},\end{aligned} \tag{3.2.26}$$

for some constant $C \in (0,\infty)$ independent of the function $f \in \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. This proves that the assignment (3.2.14) is indeed well defined, linear and bounded. The version of the claims in (3.2.16)-(3.2.18) corresponding to the case when $\partial\Omega$ is unbounded is also implicit in what we have proved so far.

In the case when $f \in \big[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)\big]^M = \big[L^p(\partial\Omega,\sigma) \cap \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, we know from [70, (2.3.34)] that the difference $K_{\mathrm{mod}} f - Kf$ is a constant on $\partial\Omega$. On account of this, [69, (6.1.8), (6.1.9)], and (3.2.26) we may then compute

$$\begin{aligned}\|Kf\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M} &= \|K_{\mathrm{mod}} f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M} = \big\| [K_{\mathrm{mod}} f] \big\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M} \\ &\leq C \big\| [f] \big\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim]^M} = C\|f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}.\end{aligned} \tag{3.2.27}$$

Since the operator (3.2.12) is also bounded, from (3.2.27) and (A.0.120) we ultimately obtain

$$\|Kf\|_{[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)]^M}, \tag{3.2.28}$$

for some $C \in (0,\infty)$ independent of $f \in \big[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. With this in hand, all desired conclusions in the statement of the theorem about the operator K (including the version of the claims in (3.2.16)-(3.2.18) when $\partial\Omega$ is bounded) now readily follow. □

It is also of interest to study the action of the boundary-to-domain double layer potential operators associated with a given weakly elliptic system in an arbitrary UR domain on homogeneous and inhomogeneous Morrey-Campanato spaces. This topic is addressed in the theorem below.

Theorem 3.2.3 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$) and recall the boundary-to-domain double layer operator $\mathcal{D}$ associated with L and Ω as in* (1.3.18)*, as well as its modified version $\mathcal{D}_{\rm mod}$ defined in* (1.8.6)*. Also, fix an integrability exponent $p \in (1,\infty)$ along with a parameter $\lambda \in (0, n-1)$.*

Then there exists a constant $C \in (0,\infty)$ with the property that

$$\sup_{x\in\Omega}\Big\{\operatorname{dist}(x,\partial\Omega)^{1+\frac{n-1-\lambda}{p}}\big|\nabla\big(\mathcal{D}_{\rm mod}f\big)(x)\big|\Big\} \leq C\|f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M} \tag{3.2.29}$$

for each function $f \in \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. In particular, there exists some constant $C \in (0,\infty)$ such that

$$\begin{aligned}\sup_{x\in\Omega}\Big\{\operatorname{dist}(x,\partial\Omega)^{1+\frac{n-1-\lambda}{p}}\big|\nabla\big(\mathcal{D}f\big)(x)\big|\Big\} &\leq C\|f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}\\ &\leq C\|f\|_{[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)]^M}\end{aligned} \tag{3.2.30}$$

for each function $f \in \big[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (see also [70, (2.6.5)]*).*

Proof Having fixed an arbitrary point $x \in \Omega$, recall from (1.8.90)-(1.8.92) that for each given function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\Big]^M$ we may express

$$\partial_j\big(\mathcal{D}_{\rm mod}f\big)(x) = \Big(\int_{\partial\Omega} m^{\alpha\gamma}_{j,x}(y)f_\alpha(y)\,d\sigma(y)\Big)_{1\leq\gamma\leq M}, \quad 1 \leq j \leq n, \tag{3.2.31}$$

for some family of σ-measurable functions $m^{\alpha\gamma}_{j,x}$ defined on $\partial\Omega$ which satisfy, for some constant $C \in (0,\infty)$ independent of x,

$$\big|m^{\alpha\gamma}_{j,x}(y)\big| \leq C|x-y|^{-n} \text{ at } \sigma\text{-a.e. } y \in \partial\Omega, \tag{3.2.32}$$

as well as (cf. (1.8.98))

$$\int_{\partial\Omega} m^{\alpha\gamma}_{j,x}(y)\,d\sigma(y) = 0. \tag{3.2.33}$$

To proceed, let $q \in (1,\infty)$ be such that $1/p + 1/q = 1$, define $\eta := 1 + (n-1-\lambda)/p$, and pick a number θ such that $(n-1)(q-1) < \theta < nq - n + 1$ (which is always possible). Also, let $x_o \in \partial\Omega$ be such that $|x - x_o| = \operatorname{dist}(x, \partial\Omega)$ and abbreviate $R := \operatorname{dist}(x, \partial\Omega)$. Finally, fix $j \in \{1, \dots, n\}$ along with $\alpha, \gamma \in \{1, \dots, M\}$. Then the function $R^\eta \cdot m^{\alpha\gamma}_{j,x}$ is σ-measurable and there exists some constant $C \in (0,\infty)$, independent of x, with the property that

$$
\begin{aligned}
\int_{\partial\Omega} & \big|R^\eta \cdot m^{\alpha\gamma}_{j,x}(y)\big|^q \, \mathrm{d}\sigma(y) \\
&\leq C R^{\eta q} \int_{\partial\Omega} \frac{\mathrm{d}\sigma(y)}{|x-y|^{qn}} \\
&= C R^{\eta q} \int_{B(x_o,2R)\cap\partial\Omega} \frac{\mathrm{d}\sigma(y)}{|x-y|^{qn}} + C R^{\eta q} \int_{\partial\Omega\setminus B(x_o,2R)} \frac{\mathrm{d}\sigma(y)}{|x-y|^{qn}} \\
&\leq C R^{\eta q} R^{-nq} \sigma\big(B(x_o,2R)\cap\partial\Omega\big) + C R^{\eta q} \int_{\partial\Omega\setminus B(x_o,2R)} \frac{\mathrm{d}\sigma(y)}{|y-x_o|^{qn}} \\
&\leq C R^{\eta q} R^{n-1-nq} = C R^{\lambda(1-q)}.
\end{aligned}
\tag{3.2.34}
$$

Indeed, the first inequality above is a consequence of (3.2.32), the last inequality is implied by the upper Ahlfors regularity of $\partial\Omega$ together with [68, Lemma 7.2.1] (bearing in mind that $nq > n-1$; cf. [68, (7.2.5)]), and the final equality is implied by the choice of η and q. In a similar fashion, since $qn - \theta > n - 1$ we may estimate

$$
\begin{aligned}
\int_{\partial\Omega} & \big|R^\eta \cdot m^{\alpha\gamma}_{j,x}(y)\big|^q |y - x_o|^\theta \, \mathrm{d}\sigma(y) \\
&\leq C R^{\eta q} \int_{\partial\Omega} \frac{|y-x_o|^\theta}{|x-y|^{qn}} \, \mathrm{d}\sigma(y) \\
&= C R^{\eta q} \int_{B(x_o,2R)\cap\partial\Omega} \frac{|y-x_o|^\theta}{|x-y|^{qn}} \, \mathrm{d}\sigma(y) \\
&\quad + C R^{\eta q} \int_{\partial\Omega\setminus B(x_o,2R)} \frac{|y-x_o|^\theta}{|x-y|^{qn}} \, \mathrm{d}\sigma(y) \\
&\leq C R^{\eta q} R^{\theta - nq} \sigma\big(B(x_o,2R)\cap\partial\Omega\big) + C R^{\eta q} \int_{\partial\Omega\setminus B(x_o,2R)} \frac{\mathrm{d}\sigma(y)}{|y-x_o|^{qn-\theta}} \\
&\leq C R^{\eta q} R^{n-1-nq+\theta} = C R^{\lambda(1-q)+\theta},
\end{aligned}
\tag{3.2.35}
$$

for some constant $C \in (0,\infty)$ independent of x. Collectively, (3.2.33), (3.2.34), (3.2.35) prove that

$$
\text{each } R^\eta \cdot m^{\alpha\gamma}_{j,x} \text{ is a fixed multiple of some } \mathscr{H}^{q,\lambda,\theta}\text{-molecule on the set } \partial\Omega \text{ (in the sense described in [69, (6.1.37)]).} \tag{3.2.36}
$$

In turn, from (3.2.36) and [69, (6.1.88)] we conclude that there exists a constant $C \in (0, \infty)$ independent of x with the property that

$$\|R^\eta \cdot m^{\alpha\gamma}_{j,x}\|_{\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)} \leq C \tag{3.2.37}$$

Given any function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega, \sigma)\big]^M$, for each $j \in \{1, \dots, n\}$ and $\gamma \in \{1, \dots, M\}$ we may now estimate (bearing in mind [69, (6.1.14)])

$$\begin{aligned}
\operatorname{dist}(x, \partial\Omega)^{1+\frac{n-1-\lambda}{p}} \big|\partial_j \big(\mathcal{D}_{\text{mod}} f\big)_\gamma(x)\big| &= R^\eta \big|\partial_j \big(\mathcal{D}_{\text{mod}} f\big)_\gamma(x)\big| \\
&= \Big|\int_{\partial\Omega} R^\eta \cdot m^{\alpha\gamma}_{j,x}(y) f_\alpha(y)\, d\sigma(y)\Big| \\
&= \begin{cases} \big|\big\langle [f_\alpha], R^\eta \cdot m^{\alpha\gamma}_{j,x}\big\rangle\big| & \text{if } \partial\Omega \text{ is unbounded} \\ \big|\big\langle f_\alpha, R^\eta \cdot m^{\alpha\gamma}_{j,x}\big\rangle\big| & \text{if } \partial\Omega \text{ is bounded} \end{cases} \\
&\leq C\|f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M} \|R^\eta \cdot m^{\alpha\gamma}_{j,x}\|_{\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)} \\
&\leq C\|f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}.
\end{aligned} \tag{3.2.38}$$

In (3.2.38), the first equality is implied the formulas for R and η, the second equality uses (3.2.31), the third equality is a consequence of (3.2.36) and [69, Lemma 6.1.3], the first inequality is seen from [69, (6.1.9), (6.1.29)] in the case when $\partial\Omega$ is unbounded as well as their counterparts in the case when $\partial\Omega$ is bounded, and the final inequality comes from (3.2.37).

In view of the arbitrariness of $x \in \Omega$, $j \in \{1, \dots, n\}$, and $\gamma \in \{1, \dots, M\}$, the estimate claimed in (3.2.29) now readily follows from (3.2.38). Finally, (3.2.30) is a consequence of (3.2.29) and (1.8.9) (bearing in mind [68, (7.7.106)]). □

The (modified version of the) double layer potential operator associated with a weakly elliptic system in a UR domain acting on the Morrey-Campanato spaces also satisfies a fractional Carleson measure estimate of the sort described in our next theorem (see also Example 5.1.13 for a more general result of this flavor).

Theorem 3.2.4 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. For some $M \in \mathbb{N}$, suppose L is an $M \times M$ homogeneous, weakly elliptic, constant (complex) coefficient, second-order system in $\mathbb{R}^n$, and recall the modified boundary-to-domain double layer operator $\mathcal{D}_{\text{mod}}$ associated with L and Ω as in* (1.8.6)*. Finally, fix $p \in (1, \infty)$ and $\lambda \in (0, n-1)$.*

Then there exists a constant $C \in (0, \infty)$ with the property that the fractional Carleson measure estimate

$$\sup_{x\in\partial\Omega \text{ and } r>0}\left\{r^{-\lambda}\int_{B(x,r)\cap\Omega}\left|\nabla\big(\mathcal{D}_{\text{mod}}f\big)(y)\right|^p \operatorname{dist}(y,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n(y)\right\}^{\frac{1}{p}} \leq C\|f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M} \tag{3.2.39}$$

holds for each function f belonging to the homogeneous Morrey-Campanato space $\big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$.

As is apparent from an inspection of the proof of Theorem 3.2.4, the same type of fractional Carleson measure estimate holds for other types of singular integral operators which exhibit similar size and cancellation properties as the family of double layers associated with weakly elliptic systems. For example, this is the case for the operators Q_{ij} defined in [70, (2.4.115)] hence, in particular, for the operators $\nabla\mathcal{R}_{jk}$ considered in [70, (2.5.266)]. Similar fractional Carleson measure estimates for the gradient of the Cauchy-Clifford operator $\nabla\mathcal{C}$ from [70, (2.5.298)] may also be obtained, thanks to the discussion in Example 1.4.12, as a particular case of Theorem 3.2.4.

Proof of Theorem 3.2.4 Fix an arbitrary function $f\in\big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. Also, fix a point $x_o\in\partial\Omega$ and for each $R>0$ abbreviate

$$\Delta_R := B(x_o,R)\cap\partial\Omega,\ \ T(\Delta_R) := B(x_o,R)\cap\Omega,\ \text{ and }\ f_{\Delta_R} := \fint_{\Delta_R} f\,\mathrm{d}\sigma. \tag{3.2.40}$$

Then (A.0.118) ensures that for each $R\in\big(0,2\operatorname{diam}(\partial\Omega)\big)$ we have

$$\fint_{\Delta_R}\left|f-f_{\Delta_R}\right|^p\mathrm{d}\sigma \leq CR^{-(n-1-\lambda)}\|f\|^p_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}. \tag{3.2.41}$$

In addition, pick a scale $r\in\big(0,2\operatorname{diam}(\partial\Omega)\big)$ and consider a cutoff function η in $\mathbb{R}^n$ satisfying

$$\begin{gathered}\eta\in\mathscr{C}^\infty_c(\mathbb{R}^n),\quad 0\leq\eta\leq 1,\quad \eta\equiv 1 \text{ on } B(x_o,2r),\\ \eta\equiv 0 \text{ outside } B(x_o,4r), \text{ and}\\ |\partial^\alpha\eta(x)|\leq C_\alpha r^{-|\alpha|},\quad \forall x\in\mathbb{R}^n \text{ and } \forall\alpha\in\mathbb{N}_0^n.\end{gathered} \tag{3.2.42}$$

We then proceed to split

$$f=\eta(f-f_{\Delta_{4r}})+(1-\eta)(f-f_{\Delta_{4r}})+f_{\Delta_{4r}}. \tag{3.2.43}$$

From (3.2.43) and the property of $\mathcal{D}_{\text{mod}}$ recorded in (1.8.10) we see that

$$\nabla\big(\mathcal{D}_{\text{mod}}f\big)=\nabla\Big(\mathcal{D}_{\text{mod}}\big(\eta(f-f_{\Delta_{4r}})\big)\Big)+\nabla\Big(\mathcal{D}_{\text{mod}}\big((1-\eta)(f-f_{\Delta_{4r}})\big)\Big). \tag{3.2.44}$$

Thus,

$$
\begin{aligned}
\int_{T(\Delta_r)} \Big|\nabla\big(\mathcal{D}_{\text{mod}} f\big)(x)\Big|^p \operatorname{dist}(x,\partial\Omega)^{p-1}\,\mathrm{d}x \\
&\leq C\int_{T(\Delta_r)} \Big|\nabla\Big(\mathcal{D}_{\text{mod}}\big(\eta(f-f_{\Delta_{4r}})\big)\Big)(x)\Big|^p \operatorname{dist}(x,\partial\Omega)^{p-1}\,\mathrm{d}x \\
&\quad + C\int_{T(\Delta_r)} \Big|\nabla\Big(\mathcal{D}_{\text{mod}}\big((1-\eta)(f-f_{\Delta_{4r}})\big)\Big)(x)\Big|^p \operatorname{dist}(x,\partial\Omega)^{p-1}\,\mathrm{d}x \\
&=: \mathrm{I}+\mathrm{II}. \qquad (3.2.45)
\end{aligned}
$$

We shall show that

$$
\mathrm{I} \leq C\|f\|^p_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M} \cdot r^\lambda \ \text{ and } \ \mathrm{II} \leq C\|f\|^p_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M} \cdot r^\lambda. \qquad (3.2.46)
$$

To justify the first inequality in (3.2.46), with $\mathcal{D}$ denoting a boundary-to-domain double layer operator associated with L and Ω as in (1.3.18), we write

$$
\begin{aligned}
\mathrm{I} &\leq C\int_{\Omega} \Big|\nabla\Big(\mathcal{D}_{\text{mod}}\big(\eta(f-f_{\Delta_{4r}})\big)\Big)(x)\Big|^p \operatorname{dist}(x,\partial\Omega)^{p-1}\,\mathrm{d}x \\
&= C\int_{\Omega} \Big|\nabla\Big(\mathcal{D}\big(\eta(f-f_{\Delta_{4r}})\big)\Big)(x)\Big|^p \operatorname{dist}(x,\partial\Omega)^{p-1}\,\mathrm{d}x \\
&\leq C\int_{\partial\Omega} \big|\eta(f-f_{\Delta_{4r}})\big|^p\,\mathrm{d}\sigma \leq C\int_{\Delta_{4r}} \big|f-f_{\Delta_{4r}}\big|^p\,\mathrm{d}\sigma \\
&= Cr^{n-1} \fint_{\Delta_{4r}} \big|f-f_{\Delta_{4r}}\big|^p\,\mathrm{d}\sigma \leq Cr^\lambda \|f\|^p_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}. \qquad (3.2.47)
\end{aligned}
$$

Above, the first inequality follows from the definition of I in (3.2.45), the subsequent equality is justified by (1.8.9), [69, (6.1.3)], and [68, (7.7.106)], the second inequality is implied by [70, (2.4.34)], (1.3.18), and [70, Theorem 1.4.2], the third inequality is clear from the support properties of the function η introduced in (3.2.42), the subsequent equality is obvious, and the fourth inequality comes from (3.2.41).

To justify the second inequality in (3.2.46), we first observe that for each $x \in T(\Delta_r)$ we have

$$
\Big|\nabla\Big(\mathcal{D}_{\text{mod}}\big((1-\eta)(f-f_{\Delta_{4r}})\big)\Big)(x)\Big| \leq C\int_{\partial\Omega\setminus\Delta_{2r}} \frac{|f(y)-f_{\Delta_{4r}}|}{|y-x_o|^n}\,\mathrm{d}\sigma(y), \qquad (3.2.48)
$$

based on (1.8.11), the properties of the fundamental solution E from [70, Theorem 1.4.2] and of the function η from [70, (2.4.132)], and the fact that since $x \in T(\Delta_r)$ we have $|x-y| \approx |x_o-y|$ uniformly for $y \in \partial\Omega \setminus \Delta_{2r}$. In turn, from the first inequality in [68, (7.4.115)] (used with $X = \partial\Omega$, $\mu := \sigma$, $p := 1$, $d := n-1$, $\varepsilon := 1$, $q := p$, and with the letter t used in place of λ) we obtain

$$
\begin{aligned}
\int_{\partial\Omega\setminus\Delta_{2r}} \frac{|f(y)-f_{\Delta_{4r}}|}{|y-x_o|^n}\,d\sigma(y)
&\le C\int_{\partial\Omega} \frac{|f(y)-f_{\Delta_{4r}}|}{\big[r+|y-x_o|\big]^n}\,d\sigma(y)\\
&\le \frac{C}{r}\int_1^\infty \Big(\fint_{\Delta(x_o,\,tr)} \big|f(x)-f_{\Delta(x_o,\,tr)}\big|^p\,d\sigma(x)\Big)^{1/p}\frac{dt}{t^2}\\
&\le C\|f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}\cdot r^{-1}\Big(\int_1^\infty (tr)^{-(n-1-\lambda)/p}\,\frac{dt}{t^2}\Big)\\
&= C\|f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}\cdot r^{-1-(n-1-\lambda)/p}.
\end{aligned}
\tag{3.2.49}
$$

In the third inequality above we have used (3.2.41), and the final inequality takes into account the fact that $\int_1^\infty \frac{dt}{t^{2+(n-1-\lambda)/p}} < +\infty$. By combining (3.2.48) and (3.2.49) we obtain

$$
\begin{aligned}
\mathrm{II} &\le C\|f\|^p_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}\cdot r^{-p-(n-1-\lambda)}\int_{T(\Delta_r)} \operatorname{dist}(x,\partial\Omega)^{p-1}\,dx\\
&\le C\|f\|^p_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}\cdot r^{-p-(n-1-\lambda)}\cdot r^{p-1}\cdot r^n\\
&= C\|f\|^p_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}\cdot r^{\lambda}.
\end{aligned}
\tag{3.2.50}
$$

This establishes the second inequality in (3.2.46). At this stage, the version of (3.2.39) with the supremum taken in the regime $0 < r < 2\,\mathrm{diam}(\partial\Omega)$ follows from (3.2.45) and (3.2.46). Finally, the version of (3.2.39) with the supremum taken in the regime $r \ge 2\,\mathrm{diam}(\partial\Omega)$ is trivially implied by what we have just proved (since in this case we have $B(x,r)\cap\Omega = \Omega$ for each $x \in \partial\Omega$). □

3.3 Boundary Layer Potential Operators on Morrey Spaces and Their Pre-Duals

The results in this section deal with singular integral operators of boundary layer type, acting on Morrey spaces and their pre-duals, considered on boundaries of uniformly rectifiable domains. In a nutshell, the approach we adopt in the treatment of singular integral operators of boundary layer type rests on two basic aspects. First, in earlier work special care has been taken to establish *integral formulas* and *operator identities* (such as those in [70, Theorem 1.7.10], Proposition 1.2.1, Proposition 1.2.2, Lemma 1.3.2, along with the jump-formulas (1.5.20) and (1.5.58) in Theorem 1.5.1, etc.) in an inclusive enough functional analytic setting which, in particular, allows the consideration of Morrey spaces and their pre-duals. Second, in [70, Theorem 2.6.1], [70, Proposition 2.6.2], and [70, Proposition 2.6.3] we have proved *norm estimates* in Morrey spaces and their pre-duals for generic, garden-variety singular integral operators, which in turn serve as building blocks for the algebraically more sophisticated boundary layer potential operators we wish to consider. Together, these two aspects work with great efficiency to yield very satisfactory theory for boundary layer potential operators acting on Morrey spaces and their pre-duals, considered on uniformly rectifiable sets. Here is the theorem which backs up these heuristic considerations.

Theorem 3.3.1 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a* UR *domain. Define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). Write $L = L_A$ for a choice of a coefficient tensor A and consider the layer potential operators K, $\mathcal{D}$, $\mathcal{S}$, S, $\mathcal{S}_{\text{mod}}$, and $K^{\#}$ associated with the coefficient tensor A and the set Ω as in* (1.3.68)*,* (1.3.18)*,* (1.3.6)*,* (1.3.62)*,* (1.5.50)*, and* (1.3.72)*, respectively. Finally, select $p, q \in (1,\infty)$ such that $1/p + 1/q = 1$, along with $\lambda \in (0, n-1)$, and some aperture parameter $\kappa > 0$.*

Then the following statements are true.

(i) The operators

$$K, K^{\#} : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \tag{3.3.1}$$

are well-defined, linear, and bounded. In addition, for each multi-index $\alpha \in \mathbb{N}_0^n$ there exists $C \in (0,\infty)$, depending only on the Ahlfors regularity constants of $\partial\Omega$, L, n, p, λ, and α, with the property that

$$\sup_{x\in\Omega}\Big\{\operatorname{dist}(x,\partial\Omega)^{|\alpha|+\frac{n-1-\lambda}{p}}\big|\partial^\alpha\big(\mathcal{D}f\big)(x)\big|\Big\} \leq C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \quad \text{for each function } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{3.3.2}$$

and, with $C \in (0,\infty)$ now depending on the UR *character of $\partial\Omega$, L, n, κ, p, and λ,*

$$\big\|\mathcal{N}_\kappa\big(\mathcal{D}f\big)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa\big(\nabla(\mathcal{S}_{\text{mod}}f)\big)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \quad \text{for each function } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{3.3.3}$$

In light of work in [68, §8.5]*, a similar estimate to* (3.3.3) *is true with the nontangential maximal operator replaced by the tangential maximal operator (associated as in* [68, Definition 8.5.1] *with a sufficiently large power;* (A.0.146)*). Furthermore, for each given function f in the Morrey space $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ the following nontangential boundary trace formulas hold (with I denoting the identity operator)*

$$\mathcal{D}f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K\big)f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{3.3.4}$$

and

$$\begin{aligned} &\nabla(\mathscr{S}_{\rm mod} f)\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \text{ and} \\ &\partial_\nu^A \mathscr{S}_{\rm mod} f = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \end{aligned} \tag{3.3.5}$$

where $K^\#_{A^\top}$ is the operator associated as in (1.3.72) *with the coefficient tensor* $A^\top$.

In addition, the operators $K, K^\#$ in the context of (3.3.1) *depend in a continuous fashion on the underlying coefficient tensor* (*in the sense of item (xv) in Theorem 1.5.1*).

Finally, similar results are valid with the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ replaced throughout by its version $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ (*defined as in* (A.0.149) *with $\Sigma := \partial\Omega$*).

(ii) Assume $n \geq 3$ and suppose

$$\lambda \in (0, n-2), \quad 1 < p < n-1-\lambda, \quad p_* := \Big(\frac{1}{p} - \frac{1}{n-1-\lambda}\Big)^{-1}. \tag{3.3.6}$$

Then the boundary-to-boundary single layer potential operator induces well-defined, linear, and bounded mappings in the following settings:

$$S : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M^{p_*,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{3.3.7}$$

$$S : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M_1^{p_*,p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{3.3.8}$$

Also, there exists a constant $C \in (0,\infty)$ such that

$$\begin{aligned} &\big\|\mathcal{N}_\kappa\big(\mathscr{S} f\big)\big\|_{M^{p_*,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \\ &\text{for each function } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{3.3.9}$$

plus a similar estimate in which the nontangential maximal operator is replaced by the tangential maximal operator (*associated as in* [68, Definition 8.5.1] *with a sufficiently large power;* (A.0.146)), *and*

$$\begin{aligned} &\text{for each } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \text{ the boundary trace} \\ &\mathscr{S} f\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \text{ and, in fact,} \\ &\Big(\mathscr{S} f\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\Big)(x) = (Sf)(x) \text{ at } \sigma\text{-a.e. point } x \in \partial\Omega. \end{aligned} \tag{3.3.10}$$

Moreover, similar results are valid for the pre-duals of Morrey spaces, as well as their own pre-duals. Specifically, the jump-formula in (3.3.10) *remains valid when the function f belongs the space $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ or to the space $\big[\mathcal{B}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, and there exists some constant $C \in (0,\infty)$ with the property that*

$$\max\Big\{\|Sf\|_{[M^{p_*,\lambda}(\partial\Omega,\sigma)]^M},\big\|\mathcal{N}_\kappa(\mathscr{S}f)\big\|_{M^{p_*,\lambda}(\partial\Omega,\sigma)},\|Sf\|_{[M_1^{p_*,p,\lambda}(\partial\Omega,\sigma)]^M}\Big\}$$
$$\leq C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M},\ \textit{for each}\ f\in\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{3.3.11}$$

and

$$\max\Big\{\|Sf\|_{[\mathcal{B}^{p_*,\lambda}(\partial\Omega,\sigma)]^M},\big\|\mathcal{N}_\kappa(\mathscr{S}f)\big\|_{\mathcal{B}^{p_*,\lambda}(\partial\Omega,\sigma)},\|Sf\|_{[\mathscr{H}_1^{p_*,p,\lambda}(\partial\Omega,\sigma)]^M}\Big\}$$
$$\leq C\|f\|_{[\mathcal{B}^{p,\lambda}(\partial\Omega,\sigma)]^M}\ \textit{for each}\ f\in\big[\mathcal{B}^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{3.3.12}$$

(iii) The operators

$$K,K^{\#}:\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M\longrightarrow\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M \tag{3.3.13}$$

are well-defined, linear, bounded, and depend in a continuous fashion on the underlying coefficient tensor (*in the sense of item (xv) in Theorem 1.5.1*). *In fact, the real transpose of K in* (3.3.13) (*respectively, $K^{\#}$ in* (3.3.13)) *is the operator $K^{\#}$ in* (3.3.1) (*respectively, K in* (3.3.1)). *Also, the real transpose of K acting on $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$* (*respectively, $K^{\#}$ acting on $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$*) *is the operator $K^{\#}$ in* (3.3.13) (*respectively, K in* (3.3.13)).
Moreover, there exists a constant $C\in(0,\infty)$ such that

$$\big\|\mathcal{N}_\kappa(\mathcal{D}f)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}+\big\|\mathcal{N}_\kappa\big(\nabla(\mathscr{S}_{\rm mod}f)\big)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\leq C\|f\|_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M}$$
$$\textit{for each function}\ f\in\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{3.3.14}$$

Thanks to work in [68, §8.5], *an estimate analogous to* (3.3.14) *holds if the nontangential maximal operator is replaced by the tangential maximal operator* (*associated as in* [68, Definition 8.5.1] *with a sufficiently large power;* (A.0.146)).
Finally, the jump-formulas (3.3.4)-(3.3.5) *remain valid for arbitrary vector-valued functions $f\in\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$.*

(iv) The following property holds:

$$\begin{array}{l}\textit{for each given sequence } \{f_j\}_{j\in\mathbb{N}}\subseteq\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \textit{ which is}\\ \textit{weak-}*\textit{ convergent to some function } f\in\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \textit{ one}\\ \textit{has } \lim\limits_{j\to\infty}S_{\rm mod}f_j=S_{\rm mod}f \textit{ in } \big[({\rm Lip}_c(\partial\Omega))'\big]^M.\end{array} \tag{3.3.15}$$

(v) Make the additional assumption that $\partial\Omega$ is compact. Then the boundary-to-boundary single layer potential operator induces well-defined, linear, and bounded mappings in the following settings:

$$S : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{3.3.16}$$

$$S : \big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{3.3.17}$$

$$S : \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{3.3.18}$$

Also, for each large $R \in (0,\infty)$ there exists a constant $C \in (0,\infty)$ such that

$$\begin{aligned} &\big\|\mathcal{N}_\kappa^{\Omega\cap B(0,R)}(\mathscr{S}f)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\mathscr{S}f)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \\ &\qquad\qquad \text{for each function } f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{3.3.19}$$

and

$$\begin{aligned} &\big\|\mathcal{N}_\kappa^{\Omega\cap B(0,R)}(\mathscr{S}f)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\mathscr{S}f)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M} \\ &\qquad\qquad \text{for each function } f \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{3.3.20}$$

with the convention that $\mathcal{N}_\kappa^{\Omega\cap B(0,R)}$ may be replaced by $\mathcal{N}_\kappa$ if $n \geq 3$. Finally, the boundary trace

$$\begin{aligned} &\mathscr{S}f\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \text{ and, in fact,} \\ &\Big(\mathscr{S}f\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = (Sf)(x) \text{ at } \sigma\text{-a.e. point } x \in \partial\Omega, \end{aligned} \tag{3.3.21}$$

for each function f belonging to either $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$, or $\big[\mathcal{B}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$.

(vi) Once again make the additional assumption that $\partial\Omega$ is compact. Then the boundary-to-domain single layer potential operator induces a well-defined linear mapping in the context

$$\mathscr{S} : \big[M_{-1}^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathscr{C}^\infty(\Omega)\big]^M \tag{3.3.22}$$

if for each $f = (f_\beta)_{1\leq\beta\leq M} \in \big[M_{-1}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ one sets

$$\mathscr{S}f(x) := \Big(\big\langle E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta\big\rangle\Big)_{1\leq\alpha\leq M} \text{ for all } x \in \Omega, \tag{3.3.23}$$

where $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ is the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2]*, and $\langle\cdot,\cdot\rangle$ stands for the duality bracket between the spaces $\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)$ and $M_{-1}^{p,\lambda}(\partial\Omega,\sigma)$ (cf.* (A.0.156)*). In addition, for each given $f \in \big[M_{-1}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ one has*

$$L_A(\mathscr{S}f) = 0 \text{ in } \Omega, \tag{3.3.24}$$

for each large $R > 0$ *there exists some constant* $C = C(\Omega, L, \kappa, p, \lambda, R) \in (0, \infty)$ *such that*

$$\left\| \mathcal{N}_\kappa^{\Omega\cap B(0,R)}(\mathcal{S} f) \right\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[M^{p,\lambda}_{-1}(\partial\Omega,\sigma)]^M} \tag{3.3.25}$$

(*with the convention that* $\mathcal{N}_\kappa^{\Omega\cap B(0,R)}$ *may be replaced by* $\mathcal{N}_\kappa$ *if* $n \geq 3$), *and the boundary trace*

$$\begin{gathered} \mathcal{S} f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \text{ and, in fact,} \\ \Big(\mathcal{S} f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = \big((S_{L^\top})^\top f\big)(x) \text{ at } \sigma\text{-a.e. point } x \in \partial\Omega, \end{gathered} \tag{3.3.26}$$

where $S_{L^\top}$ *is the boundary-to-boundary single layer potential operator associated with* $L^\top$*, acting from* $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ *into* $\big[\mathcal{B}^{q,\lambda}_1(\partial\Omega,\sigma)\big]^M$ (*cf.* (3.3.17)), *and* $(S_{L^\top})^\top$ *stands for its* (*real*) *transpose, i.e.,*

$$(S_{L^\top})^\top : \big[M^{p,\lambda}_{-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \tag{3.3.27}$$

(*see* [69, Definition 11.8.9] *or* (A.0.156), *and* [69, Proposition 6.2.8]).

(vii) Once more make the additional assumption that $\partial\Omega$ *is compact. Then the boundary-to-domain single layer potential operator induces a well-defined linear mapping in the context*

$$\mathcal{S} : \big[\mathcal{B}^{q,\lambda}_{-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathscr{C}^\infty(\Omega)\big]^M \tag{3.3.28}$$

with the convention that for each $f = (f_\beta)_{1\leq\beta\leq M} \in \big[\mathcal{B}^{q,\lambda}_{-1}(\partial\Omega,\sigma)\big]^M$ *one defines*

$$\mathcal{S} f(x) := \Big(\big\langle E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta\big\rangle\Big)_{1\leq\alpha\leq M} \text{ for all } x \in \Omega, \tag{3.3.29}$$

where $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ *is the matrix-valued fundamental solution associated with* L *as in* [70, Theorem 1.4.2], *and* $\langle\cdot,\cdot\rangle$ *stands for the duality bracket between* $\mathring{M}^{p,\lambda}_1(\partial\Omega,\sigma)$ *and* $\mathcal{B}^{q,\lambda}_{-1}(\partial\Omega,\sigma)$ (*cf.* (A.0.37)). *Moreover, for each given distribution* $f \in \big[\mathcal{B}^{q,\lambda}_{-1}(\partial\Omega,\sigma)\big]^M$ *one has*

$$L_A\big(\mathcal{S} f\big) = 0 \text{ in } \Omega, \tag{3.3.30}$$

for each large $R > 0$ *there exists some constant* $C = C(\Omega, L, \kappa, q, \lambda, R) \in (0, \infty)$ *such that*

$$\left\| \mathcal{N}_\kappa^{\Omega\cap B(0,R)}(\mathcal{S} f) \right\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[\mathcal{B}^{q,\lambda}_{-1}(\partial\Omega,\sigma)]^M} \tag{3.3.31}$$

(*with the convention that* $\mathcal{N}_\kappa^{\Omega\cap B(0,R)}$ *may be replaced by* $\mathcal{N}_\kappa$ *if* $n \geq 3$), *and the boundary trace*

$$\begin{aligned}&\mathscr{S}f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \text{ and, in fact,}\\ &\Big(\mathscr{S}f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = \big((S_{L^\top})^\top f\big)(x) \text{ at } \sigma\text{-a.e. point } x\in\partial\Omega,\end{aligned} \tag{3.3.32}$$

where $S_{L^\top}$ is the boundary-to-boundary single layer potential operator associated with $L^\top$, considered as a mapping from $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ into $\big[\mathring{M}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ (cf. (3.3.18)), and $(S_{L^\top})^\top$ stands for its (real) transpose, i.e.,

$$(S_{L^\top})^\top : \big[\mathscr{B}_{-1}^{q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathscr{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M \tag{3.3.33}$$

(*see* [69, Definition 11.8.9] *or* (A.0.37)*, and* [69, Proposition 6.2.16]).

(viii) Strengthen the hypotheses on Ω by now assuming that this is an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. Then the operator (3.3.27) *induces a well-defined, linear, bounded mapping*

$$(S_{L^\top})^\top : \big[\mathring{M}_{-1}^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{3.3.34}$$

and for each large $R>0$

$$\mathcal{N}_\kappa^{\Omega\cap B(0,R)}(\mathscr{S}f) \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) \text{ for each } f\in\big[\mathring{M}_{-1}^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \tag{3.3.35}$$

(*with the convention that $\mathcal{N}_\kappa^{\Omega\cap B(0,R)}$ may be replaced by $\mathcal{N}_\kappa$ if $n\geq 3$*)*, where $\mathring{M}_{-1}^{p,\lambda}(\partial\Omega,\sigma)$ is the space introduced in* [69, Definition 11.8.12]*; cf.* (A.0.149).

Proof The claims about the operators (3.3.1) are direct consequences of definitions, [70, (2.6.1)], and [69, (6.2.5)]. The fact that the operators $K, K^\#$ in the context of (3.3.1) depend in a continuous fashion on the underlying coefficient tensor is a consequence of [69, Proposition 6.2.12] and item *(xv)* in Theorem 1.5.1. Next, the estimate in (3.3.2) follows from [70, (2.6.5)] and [69, (6.2.5)] (in this regard, see also (3.2.29) and [69, (6.2.34)], bearing in mind (1.8.8) and [69, (6.2.25)]), while the estimate in (3.3.3) is implied by [70, (2.6.4)], [69, (6.2.5)], and (1.5.51). Also, the jump-formulas (3.3.4), (3.3.5) are direct consequences of (1.5.20), (1.5.53), (1.5.58), and [69, (6.2.25)]. The very last claim in item *(i)* is a direct consequence of the last property recorded in part *(1)* of [70, Theorem 2.6.1]. Next, all claims in item *(ii)* with the exception of those pertaining to S mapping into Sobolev-like spaces (such as (3.3.8), etc.) are direct consequences of [70, Propositions 2.6.2-2.6.3] (used with $\Sigma := \partial\Omega$ and $\alpha := 1$) and [70, Theorem 1.4.2] (the fact that $K, K^\#$ in (3.3.13) depend in a continuous fashion on the underlying coefficient tensor may be seen from the corresponding continuity result on $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ from item *(i)*, and the duality result from [69, Proposition 6.2.16]). To deal with (3.3.8), we begin by noting that from [68, (6.2.23)] (with $\tau := 1$ and $a := n-2$), (3.3.6), and [69, (6.2.7)] we have

$$\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \hookrightarrow \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\Big)\cap L^p_{\text{loc}}(\partial\Omega,\sigma)\Big]^M. \tag{3.3.36}$$

Granted this, we may invoke (1.2.30) which, together with [70, (2.6.1)], [70, Theorem 1.4.2], and [69, (6.2.3)], gives that for each $f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ and $j,k \in \{1,\dots,n\}$ we have

$$\begin{gathered}\partial_{\tau_{jk}}(Sf) \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \text{ and} \\ \big\|\partial_{\tau_{jk}}(Sf)\big\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}\end{gathered} \tag{3.3.37}$$

for some constant $C \in (0,\infty)$ independent of f. Together with (3.3.7) and [69, (11.7.14)], this proves that the operator S in (3.3.8) is well defined, linear and bounded. All other claims pertaining to S mapping into Sobolev-like spaces of the sort considered in the last portion of item *(ii)* are dealt with in a similar fashion. Next, the claims in item *(iii)* are consequences of part *(4)* in [70, Theorem 2.6.1] (keeping in mind [69, (6.2.74)]).

The claim in (3.3.15) is justified by reasoning much as in the proof of (1.5.78). Specifically, suppose the sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ is weak-$*$ convergent to some $f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. In view of [69, Proposition 6.2.8] this amounts to

$$\lim_{j\to\infty}\int_{\partial\Omega}\langle f_j, F\rangle\,d\sigma = \int_{\partial\Omega}\langle f, F\rangle\,d\sigma \text{ for each } F \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{3.3.38}$$

Pick an arbitrary $\mathbb{C}^M$-valued test function $\phi \in \big[\mathrm{Lip}_c(\partial\Omega)\big]^M$, then choose a reference point $x_0 \in \partial\Omega$ along with some radius $R \in (0,\infty)$ large enough to ensure that $\operatorname{supp}\phi \subseteq \partial\Omega \cap B(x_0,R)$. Then for each $j \in \mathbb{N}$ decompose $\big\langle S_{\text{mod}} f_j, \phi\big\rangle$ as in (1.5.140). Consider the function F_1 defined on $\partial\Omega$ as in (1.5.144). Since (1.5.145) holds, we conclude from [69, (6.2.245)] that $F_1 \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ given that $N := n-1$ satisfies $n-1 > \frac{\lambda(q-1)+n-1}{q}$ (recall that $\lambda < n-1$). As such, we conclude (much as in (1.5.147), keeping in mind (3.3.38)) that

$$\lim_{j\to\infty} \mathrm{I}_j = \int_{\partial\Omega}\big\langle f(y), F_1(y)\big\rangle\,d\sigma(y). \tag{3.3.39}$$

Next, thanks to [69, (6.2.69)] the matrix-valued function F_2 defined on $\partial\Omega$ as in (1.5.148) satisfies $F_2 \in \big[L^\infty_{\text{comp}}(\partial\Omega,\sigma)\big]^{M\times M} \subseteq \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^{M\times M}$. Consequently, the same type of argument as in (1.5.149) (now taking into account (3.3.38)) yields

$$\lim_{j\to\infty} \mathrm{II}_j = \Big\langle \int_{\partial\Omega} F_2(y)f(y)\,d\sigma(y), \int_{\partial\Omega}\phi(x)\,d\sigma(x)\Big\rangle. \tag{3.3.40}$$

There remains to handle the limit of III_j as $j \to \infty$. To this end, define F_3 as in (1.5.150) then use (1.5.151) and [68, (7.7.120)] to estimate

$$|F_3| \leq C\mathcal{M}_{\partial\Omega}\phi \text{ pointwise on } \partial\Omega, \tag{3.3.41}$$

where, as usual, $\mathcal{M}_{\partial\Omega}$ is the Hardy-Littlewood maximal operator on the set $\partial\Omega$. Since [69, (6.2.69)] also ensures that $\phi \in \big[L^\infty_{\text{comp}}(\partial\Omega,\sigma)\big]^M \subseteq \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$, [69, Corollary 6.2.11] guarantees that $\mathcal{M}_{\partial\Omega}\phi$ belongs to $\mathscr{B}^{q,\lambda}(\partial\Omega,\sigma)$. In view of this, we may then conclude from (3.3.41) and [69, (6.2.75)] that $F_3 \in \left[\mathscr{B}^{q,\lambda}(\partial\Omega,\sigma)\right]^M$. With this in hand, we deduce from (3.3.38) that

$$\lim_{j\to\infty} \mathrm{III}_j = \int_{\partial\Omega} \langle f(y), F_3(y)\rangle \, d\sigma(y). \tag{3.3.42}$$

In turn, from (1.5.140), (3.3.39), (3.3.40), and (3.3.42) we see that

$$\lim_{j\to\infty} \big\langle S_{\text{mod}} f_j, \phi\big\rangle = \big\langle S_{\text{mod}} f, \phi\big\rangle, \tag{3.3.43}$$

which ultimately proves that $\lim\limits_{j\to\infty} S_{\text{mod}} f_j = S_{\text{mod}} f$ in $\left[(\mathrm{Lip}_c(\partial\Omega))'\right]^M$, as wanted.

Moving on, all the claims made in item *(v)* of the theorem are consequences of what we have proved so far, [68, Lemma 7.7.16], [70, (2.5.548)], and [69, Corollaries 6.2.11, 6.2.13]. Consider next the claims made in item *(vi)*. Work under the assumption that $\partial\Omega$ is compact. To get started, fix some arbitrary $f = (f_\beta)_{1\le\beta\le M} \in \left[M^{p,\lambda}_{-1}(\partial\Omega,\sigma)\right]^M$. From [69, Proposition 11.8.10] we know that for each $\beta \in \{1,\dots,M\}$ there exist $f_0^{(\beta)}$, $f_{jk}^{(\beta)} \in M^{p,\lambda}(\partial\Omega,\sigma)$, with $1 \le j < k \le n$, satisfying

$$\|f_0^{(\beta)}\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \sum_{1\le j<k\le n} \|f_{jk}^{(\beta)}\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \le C\|f_\beta\|_{M^{p,\lambda}_{-1}(\partial\Omega,\sigma)} \tag{3.3.44}$$

and

$$\begin{aligned} {}_{M^{p,\lambda}_{-1}(\partial\Omega,\sigma)}\big\langle f_\beta, g\big\rangle_{\mathscr{B}^{q,\lambda}_1(\partial\Omega,\sigma)} &= \int_{\partial\Omega} \Big(f_0^{(\beta)} g + \sum_{1\le j<k\le n} f_{jk}^{(\beta)} \partial_{\tau_{jk}} g\Big)\, d\sigma \\ &\text{for every function } g \in \mathscr{B}^{q,\lambda}_1(\partial\Omega,\sigma). \end{aligned} \tag{3.3.45}$$

For each $x \in \Omega$ and $\alpha \in \{1,\dots,M\}$ we may then write

$$\begin{aligned} \big(\mathscr{S} f\big)_\alpha(x) &= {}_{M^{p,\lambda}_{-1}(\partial\Omega,\sigma)}\big\langle f_\beta, E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}\big\rangle_{\mathscr{B}^{q,\lambda}_1(\partial\Omega,\sigma)} \\ &= \int_{\partial\Omega} E_{\alpha\beta}(x-y) f_0^{(\beta)}(y)\, d\sigma(y) \\ &\quad + \sum_{1\le j<k\le n} \int_{\partial\Omega} \partial_{\tau_{jk}(y)}\big[E_{\alpha\beta}(x-y)\big] f_{jk}^{(\beta)}(y)\, d\sigma(y). \end{aligned} \tag{3.3.46}$$

To proceed, for any two pairs of indices, $j,k \in \{1,\dots,n\}$ and $\alpha,\beta \in \{1,\dots,M\}$, introduce the integral operators acting on each function $\phi \in L^1(\partial\Omega,\sigma)$ according to

$$\mathcal{T}_{jk}^{\alpha\beta}\phi(x) := \int_{\partial\Omega} \big\{\nu_j(y)(\partial_k E_{\alpha\beta})(x-y) - \nu_k(y)(\partial_j E_{\alpha\beta})(x-y)\big\}\phi(y)\, d\sigma(y) \tag{3.3.47}$$

at each $x \in \Omega$, where $\nu = (\nu_1, \dots, \nu_n)$ is the geometric measure theoretic outward unit normal to Ω. In terms of this piece of notation, and with $f_0 := (f_0^{(\beta)})_{1 \le \beta \le M}$, the formula recorded in (3.3.46) may be recast (bearing in mind [69, (6.2.25)]) simply as

$$\mathscr{S} f = \mathscr{S} f_0 - \sum_{1 \le j < k \le n} \big(\mathcal{T}_{jk}^{\alpha\beta} f_{jk}^{(\beta)} \big)_{1 \le \alpha \le M} \text{ in } \Omega. \tag{3.3.48}$$

It is then apparent from (3.3.48) that (3.3.22) is a well-defined linear mapping and that (3.3.24) holds. In addition, from (3.3.48), (3.3.19), [70, (2.6.4)], [69, (6.2.5)], and (3.3.44) we see that (3.3.25) is true. From (3.3.48), (3.3.21), (1.2.4), and [69, (6.2.25)] we also conclude that $\mathscr{S} f\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$ and, in fact, at σ-a.e. point on $\partial\Omega$ we have

$$\mathscr{S} f\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = S f_0 - \sum_{1 \le j < k \le n} \big(T_{jk}^{\alpha\beta} f_{jk}^{(\beta)} \big)_{1 \le \alpha \le M} \tag{3.3.49}$$

where, for each $j, k \in \{1, \dots, n\}$ and $\alpha, \beta \in \{1, \dots, M\}$,

$$\begin{aligned} &T_{jk}^{\alpha\beta} \phi(x) \\ &:= \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y| > \varepsilon}} \big\{ \nu_j(y)(\partial_k E_{\alpha\beta})(x-y) - \nu_k(y)(\partial_j E_{\alpha\beta})(x-y) \big\} \phi(y)\, d\sigma(y), \end{aligned} \tag{3.3.50}$$

for every function $\phi \in L^1(\partial\Omega, \sigma)$ and σ-a.e. point $x \in \partial\Omega$. Having established this, for each function $\psi = (\psi_\alpha)_{1 \le \alpha \le M} \in \big[\text{Lip}(\partial\Omega)\big]^M$ we may then write, thanks to (1.3.62) and [70, (2.6.15)],

$$\begin{aligned} &{}_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \Big\langle \mathscr{S} f\big|_{\partial\Omega}^{\kappa-\text{n.t.}}, \psi \Big\rangle_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M} \\ &\qquad = {}_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \big\langle S f_0, \psi \big\rangle_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M} \\ &\qquad\quad - \sum_{1 \le j < k \le n} {}_{M^{p,\lambda}(\partial\Omega,\sigma)} \big\langle T_{jk}^{\alpha\beta} f_{jk}^{(\beta)}, \psi_\alpha \big\rangle_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \\ &\qquad = {}_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \big\langle f_0, S_{L^\top} \psi \big\rangle_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M} \\ &\qquad\quad + \sum_{1 \le j < k \le n} {}_{M^{p,\lambda}(\partial\Omega,\sigma)} \big\langle f_{jk}^{(\beta)}, \big(T_{jk}^{\alpha\beta}\big)^{\#} \psi_\alpha \big\rangle_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \end{aligned} \tag{3.3.51}$$

where for each $j, k \in \{1, \dots, n\}$ and $\alpha, \beta \in \{1, \dots, M\}$ we have set

$$\big(T^{\alpha\beta}_{jk}\big)^{\#}\varphi(x) \tag{3.3.52}$$
$$:= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big\{\nu_j(x)(\partial_k E_{\alpha\beta})(x-y) - \nu_k(x)(\partial_j E_{\alpha\beta})(x-y)\big\}\varphi(y)\,d\sigma(y),$$

for every function $\varphi \in L^1(\partial\Omega,\sigma)$ and σ-a.e. point $x \in \partial\Omega$. In relation to this operator we wish to observe that, as seen from (1.5.83) and the first formula in [70, (1.4.32)], for each $j,k \in \{1,\dots,n\}$ and each $\beta \in \{1,\dots,M\}$ we have

$$\partial_{\tau_{jk}}\big(S_{L^\top}\psi\big)_\beta = \big(T^{\alpha\beta}_{jk}\big)^{\#}\psi_\alpha \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{3.3.53}$$

Together, (3.3.51) and (3.3.53) prove (bearing in mind the nature of the duality pairing involved; cf. [69, Proposition 6.2.8]) that

$$\begin{aligned}
&{}_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}\Big\langle \mathscr{S}f\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, \psi\Big\rangle_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M}\\
&\qquad = {}_{M^{p,\lambda}(\partial\Omega,\sigma)}\big\langle f^{(\beta)}_0, (S_{L^\top}\psi)_\beta\big\rangle_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\\
&\qquad\quad + \sum_{1\le j<k\le n} {}_{M^{p,\lambda}(\partial\Omega,\sigma)}\big\langle f^{(\beta)}_{jk}, \partial_{\tau_{jk}}\big(S_{L^\top}\psi\big)_\beta\big\rangle_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}\\
&\qquad = \int_{\partial\Omega}\Big(f^{(\beta)}_0\big(S_{L^\top}\psi\big)_\beta + \sum_{1\le j<k\le n} f^{(\beta)}_{jk}\partial_{\tau_{jk}}\big(S_{L^\top}\psi\big)_\beta\Big)\,d\sigma\\
&\qquad = {}_{M^{p,\lambda}_{-1}(\partial\Omega,\sigma)}\big\langle f_\beta, \big(S_{L^\top}\psi\big)_\beta\big\rangle_{\mathcal{B}^{q,\lambda}_1(\partial\Omega,\sigma)}\\
&\qquad = {}_{[M^{p,\lambda}_{-1}(\partial\Omega,\sigma)]^M}\big\langle f, S_{L^\top}\psi\big\rangle_{[\mathcal{B}^{q,\lambda}_1(\partial\Omega,\sigma)]^M}\\
&= {}_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}\big\langle (S_{L^\top})^\top f, \psi\big\rangle_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M}.
\end{aligned} \tag{3.3.54}$$

In view of [69, (6.2.70)] and [69, Proposition 6.2.8], we conclude from formula (3.3.54) that $\mathscr{S}f\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = (S_{L^\top})^\top f$ as functions in $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. This finishes the proof of (3.3.26). The treatment of the claims in item *(vi)* is therefore complete. In fact, all claims made in item *(vii)* are dealt with similarly, now making use of [69, Proposition 6.2.16] and [69, (6.2.20)].

Finally, there remains to deal with the claims made in item *(viii)*. Work under the assumption that Ω is an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. Also, define $s := \frac{p(n-1)}{n-1-\lambda} \in (p,\infty)$.

We claim that $(S_{L^\top})^\top$ from (3.3.27) is compatible with S, the single layer potential operator (associated with L and Ω) considered as in item *(ix)* of Theorem 1.5.1, in the sense that

$$(S_{L^\top})^\top f = Sf \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \text{ for each}$$
$$f \in \big[L^s_{-1}(\partial\Omega,\sigma)\big]^M \hookrightarrow \big[M^{p,\lambda}_{-1}(\partial\Omega,\sigma)\big]^M \text{ (cf. [69, (11.8.43)]).} \tag{3.3.55}$$

To justify (3.3.55), fix an arbitrary $f \in \big[L^s_{-1}(\partial\Omega,\sigma)\big]^M \hookrightarrow \big[M^{p,\lambda}_{-1}(\partial\Omega,\sigma)\big]^M$. Also, denote by $r := \frac{q(n-1)}{n-1+\lambda(q-1)} \in (1,q)$ the Hölder conjugate exponent of s. In particular, from [69, (11.7.27)] we know that $\mathcal{B}^{q,\lambda}_1(\partial\Omega,\sigma) \hookrightarrow L^r_1(\partial\Omega,\sigma)$. Then for each function $g \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ the embedding in [69, (6.2.71)] guarantees that $g \in \big[L^r(\partial\Omega,\sigma)\big]^M$, and we may write

$$\begin{aligned}
&{}_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}\big\langle (S_{L^\top})^\top f, g\big\rangle_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M}\\
&\qquad = {}_{[M^{p,\lambda}_{-1}(\partial\Omega,\sigma)]^M}\big\langle f, S_{L^\top} g\big\rangle_{[\mathcal{B}^{q,\lambda}_1(\partial\Omega,\sigma)]^M}\\
&\qquad = {}_{[L^s_{-1}(\partial\Omega,\sigma)]^M}\big\langle f, S_{L^\top} g\big\rangle_{[L^r_1(\partial\Omega,\sigma)]^M}\\
&\qquad = {}_{[L^s(\partial\Omega,\sigma)]^M}\big\langle Sf, g\big\rangle_{[L^r(\partial\Omega,\sigma)]^M}\\
&\qquad = \int_{\partial\Omega} \langle Sf, g\rangle \, d\sigma\\
&\qquad = {}_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}\big\langle Sf, g\big\rangle_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M}.
\end{aligned} \tag{3.3.56}$$

The first equality above is implied by the fact that $(S_{L^\top})^\top$ is the transpose of $S_{L^\top}$ acting from $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ into $\big[\mathcal{B}^{q,\lambda}_1(\partial\Omega,\sigma)\big]^M$ (cf. (3.3.18)). The second equality in (3.3.56) comes from the compatibility property recorded in [69, (11.8.44)]. The third equality in (3.3.56) is a consequence of item *(ix)* in Theorem 1.5.1, and [69, (11.5.202)]. For the fourth equality in (3.3.56) we have used the fact that duality between $L^s(\partial\Omega,\sigma)$ and $L^r(\partial\Omega,\sigma)$ is given by the integral pairing. Finally, the fifth equality in (3.3.56) is seen from [69, Proposition 6.2.8]. Once (3.3.56) has been established, (3.3.55) follows by invoking [69, (6.2.80)].

Since S maps $\big[L^s_{-1}(\partial\Omega,\sigma)\big]^M$ into $\big[L^s(\partial\Omega,\sigma)\big]^M$ (cf. item *(ix)* of Theorem 1.5.1), we conclude from [69, Definition 11.8.12] (see (A.0.149)), (3.3.55), (A.0.149), (3.3.27), and [69, Lemma 1.2.20] that actually $(S_{L^\top})^\top$ is a well-defined, linear, and bounded operator in the setting described in (3.3.34).

To justify (3.3.35), first assume that $n \geq 3$. We claim that the mapping

$$\big[M^{p,\lambda}_{-1}(\partial\Omega,\sigma)\big]^M \ni f \longmapsto \mathcal{N}_\kappa(\mathcal{S}f) \in M^{p,\lambda}(\partial\Omega,\sigma) \tag{3.3.57}$$

is continuous. To justify this claim, pick $f, g \in \big[M^{p,\lambda}_{-1}(\partial\Omega,\sigma)\big]^M$ arbitrary. Then (3.3.31) gives $\mathcal{N}_\kappa(\mathcal{S}f)$, $\mathcal{N}_\kappa(\mathcal{S}g) \in M^{p,\lambda}(\partial\Omega,\sigma)$. In particular,

$$\big(\mathcal{N}_\kappa(\mathcal{S}f)\big)(x) < +\infty \text{ and } \big(\mathcal{N}_\kappa(\mathcal{S}g)\big)(x) < +\infty \text{ for } \sigma\text{-a.e. point } x \in \partial\Omega. \tag{3.3.58}$$

Granted this, we may invoke [68, (8.2.11)] to write

$$\Big|\big(\mathcal{N}_\kappa(\mathscr{S}f)\big)(x)-\big(\mathcal{N}_\kappa(\mathscr{S}g)\big)(x)\Big|\leq\big(\mathcal{N}_\kappa(\mathscr{S}(f-g))\big)(x) \tag{3.3.59}$$

at σ-a.e. point $x\in\partial\Omega$. In concert with [69, (6.2.3)] and (3.3.25), this gives

$$\begin{aligned}\big\|\mathcal{N}_\kappa(\mathscr{S}f)-\mathcal{N}_\kappa(\mathscr{S}g)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}&\leq\big\|\mathcal{N}_\kappa(\mathscr{S}(f-g))\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}\\&\leq C\|f-g\|_{[M^{p,\lambda}_{-1}(\partial\Omega,\sigma)]^M}\end{aligned} \tag{3.3.60}$$

for some $C\in(0,\infty)$ independent of f,g. Finally, (3.3.60) readily implies that the mapping (3.3.57) is indeed continuous.

To proceed, fix an arbitrary $f\in\big[\mathring{M}^{p,\lambda}_{-1}(\partial\Omega,\sigma)\big]^M$. From [69, Definition 11.8.12] (cf. (A.0.149)) we know that there exists a sequence $\{f_j\}_{j\in\mathbb{N}}\subseteq\big[L^s_{-1}(\partial\Omega,\sigma)\big]^M$ which converges to f in $\big[M^{p,\lambda}_{-1}(\partial\Omega,\sigma)\big]^M$. As a result of this, (1.5.45), and the continuity of the mapping (3.3.57) we conclude that the sequence $\big\{\mathcal{N}_\kappa(\mathscr{S}f_j)\big\}_{j\in\mathbb{N}}$ is contained in $L^s(\partial\Omega,\sigma)$ and converges to $\mathcal{N}_\kappa(\mathscr{S}f)$ in $M^{p,\lambda}(\partial\Omega,\sigma)$. In view of (A.0.149), this places $\mathcal{N}_\kappa(\mathscr{S}f)$ in $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$, thus establishing (3.3.35). Finally, the case when $n=2$ and a truncated nontangential maximal operator is employed is dealt with very similarly. □

There are also natural results (of the sort discussed in Theorem 1.5.1 for the brand of boundary Sobolev spaces introduced in [69, Chapter 11]) involving layer potential operators acting on Morrey-based Sobolev spaces, as defined in (A.0.150)-(A.0.151), or even more generally, on off-diagonal Morrey-based Sobolev spaces, as defined in (A.0.152)-(A.0.153), and the related brands from (A.0.154), as well as on block-based Sobolev spaces, as defined in (A.0.33)-(A.0.34), and their off-diagonal versions from (A.0.35)-(A.0.36).

Theorem 3.3.2 *Assume $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\geq 2$) is a* UR *domain. Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Net, let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M\times M$ system in $\mathbb{R}^n$ (for some $M\in\mathbb{N}$). Write $L=L_A$ for a choice of a coefficient tensor A and consider the layer potential operators K, $\mathcal{D}$, $\mathscr{S}$, S, $\mathscr{S}_{\rm mod}$, and $K^\#$ associated with the coefficient tensor A and the set Ω as in (1.3.68), (1.3.18), (1.3.6), (1.3.62), (1.5.50), and (1.3.72), respectively. Finally, pick two arbitrary integrability exponents $p,q\in(1,\infty)$, along with a number $\lambda\in(0,n-1)$, and an aperture parameter $\kappa\in(0,\infty)$. Then the following statements are true.*

(i) For each f in the off-diagonal Morrey-based Sobolev space $\big[M^{p,q,\lambda}_1(\partial\Omega,\sigma)\big]^M$ it follows that

$$\begin{aligned}&\textit{the nontangential boundary trace } \big(\partial_\ell\mathcal{D}f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\textit{ exists}\\&\textit{(in } \mathbb{C}^M\textit{) at } \sigma\textit{-a.e. point on } \partial\Omega, \textit{ for each } \ell\in\{1,\dots,n\},\end{aligned} \tag{3.3.61}$$

and there exists some finite constant $C > 0$, depending only on the UR *character of $\partial\Omega$, L, n, κ, p, q, λ, such that*

$$\big\|\mathcal{N}_\kappa(\mathcal{D}f)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\mathcal{D}f)\big\|_{M^{q,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[M_1^{p,q,\lambda}(\partial\Omega,\sigma)]^M}. \tag{3.3.62}$$

In addition, the operator

$$K : \big[M_1^{p,q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M_1^{p,q,\lambda}(\partial\Omega,\sigma)\big]^M \tag{3.3.63}$$

is well defined, linear, bounded, and depends in a continuous fashion on the underlying coefficient tensor (*in the sense of item (xv) in Theorem 1.5.1*).

(ii) Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω, and choose a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ such that $L = L_A$. Then, given any function $f = (f_\alpha)_{1\leq\alpha\leq M}$ belonging to the off-diagonal Morrey-based Sobolev space $\big[M_1^{p,q,\lambda}(\partial\Omega,\sigma)\big]^M$, at σ-a.e. point $x \in \partial\Omega$ one has

$$\begin{aligned}&\big(\partial_\nu^A(\mathcal{D}f)\big)(x) \\ &= \left(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x)a_{ij}^{\mu\gamma}a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{\tau_{js}}f_\alpha\big)(y)\,\mathrm{d}\sigma(y)\right)_{1\leq\mu\leq M}\end{aligned} \tag{3.3.64}$$

where the conormal derivative $\partial_\nu^A(\mathcal{D}f)$ is originally defined as in (A.0.184)*. Moreover, the conormal derivative of the double layer*

$$f \mapsto \big(\partial_\nu^A\mathcal{D}\big)f := \partial_\nu^A(\mathcal{D}f) \tag{3.3.65}$$

induces a well-defined, linear, and bounded operator

$$\partial_\nu^A\mathcal{D} : \big[M_1^{p,q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M^{q,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{3.3.66}$$

Also, if the exponents $p', q' \in (1,\infty)$ are such $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, then for any two given vector-valued functions $f \in \big[M_1^{p,q,\lambda}(\partial\Omega,\sigma)\big]^M$ and $g \in \big[\mathcal{B}_1^{q',p',\lambda}(\partial\Omega,\sigma)\big]^M$ one has

$$\int_{\partial\Omega}\big\langle\big(\partial_\nu^A\mathcal{D}\big)f, g\big\rangle\,\mathrm{d}\sigma = \int_{\partial\Omega}\big\langle f, \big(\partial_\nu^{A^\top}\mathcal{D}_{A^\top}\big)g\big\rangle\,\mathrm{d}\sigma \tag{3.3.67}$$

where $\partial_\nu^{A^\top}$ along with $\mathcal{D}_{A^\top}$ and, ultimately $\partial_\nu^{A^\top}\mathcal{D}_{A^\top}$, are defined as before with A now replaced by $A^\top$.

(iii) Similar results to those in items (i)-(ii) are valid for functions in the space $\big[\mathring{M}_1^{p,q,\lambda}(\partial\Omega,\sigma)\big]^M$ defined in (A.0.154)*, as well as the space $\big[\mathcal{B}_1^{p,q,\lambda}(\partial\Omega,\sigma)\big]^M$*

defined in (A.0.35)-(A.0.36)*. Finally, the operator identities in item (xiii) of Theorem 1.5.1 also have natural counterparts on these scales of spaces.*

Proof Given an arbitrary function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[M_1^{p,q,\lambda}(\partial\Omega,\sigma)\big]^M$, from Lemma 1.3.2 and [69, (11.7.24)] we conclude that for each index $\ell \in \{1,\dots,n\}$ we have

$$\partial_\ell\big(\mathcal{D}f\big)(x) = \left(\int_{\partial\Omega} a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x-y)(\partial_{\tau_{\ell s}} f_\alpha)(y)\,\mathrm{d}\sigma(y)\right)_{1\leq\gamma\leq M}, \tag{3.3.68}$$

for all $x \in \Omega$. Granted this, [70, Theorem 2.5.1] then justifies the claim made in (3.3.61) (bearing in mind [69, (6.2.25), (11.7.14)]). Also, the estimate in (3.3.62) is implied by (3.3.68), [69, (11.7.14)], [70, (2.6.4)], and [70, Theorem 1.4.2]. Next, the fact that the operator K in (3.3.63) is well defined, linear, and bounded, follows from (3.3.1), [69, (11.3.26)] (used for $u := \mathcal{D}f$ with $f \in \big[M_1^{p,q,\lambda}(\partial\Omega,\sigma)\big]^M$; the fact that [69, Proposition 11.3.2] applies to this function is ensured by (3.3.62), (3.3.3), (3.3.4), (3.3.61), and [69, (6.2.25)]), and [69, (6.2.3)]. The fact that the operator K in the context of (3.3.63) depends continuously on the underlying coefficient tensor may be seen using the corresponding continuity result for K in (3.3.1) and Proposition 1.5.6. This takes care of item *(i)*.

As regards the claims in item *(ii)*, the validity of the formula (3.3.64) for each given function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[M_1^{p,q,\lambda}(\partial\Omega,\sigma)\big]^M$ is ensured by (1.5.29), bearing in mind the second embedding in [69, (11.7.24)]. In turn, from (3.3.64), [69, (11.7.14)], [70, (2.6.1)], [70, Theorem 1.4.2], and [69, (6.2.5)] we conclude that, indeed, the operator in (3.3.66) is well defined, linear, and bounded. Next, the integral identity claimed in (3.3.67) may be justified in the same manner as (1.5.32), bearing in mind that the main ingredients in the proof of the (1.5.32) (namely Green's formula [70, (1.7.81)], as well as the jump-formula (1.5.20), and (1.5.29)) are all valid for Morrey spaces and their pre-duals (cf. [70, (1.7.75)], (3.3.4), and (3.3.64)). Finally, the claims in item *(iii)* are dealt with in a very similar fashion. □

It turns out that the boundary-to-domain double layer potential operators associated with a given weakly elliptic system and UR domain with compact boundary satisfy certain fractional Carleson measure estimates when acting on Morrey spaces. This is made precise in the theorem below (see also Example 5.1.10 for a more general result of this flavor).

Theorem 3.3.3 *Let* $\Omega \subseteq \mathbb{R}^n$ (*where* $n \in \mathbb{N}$, $n \geq 2$) *be a* UR *domain and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$*. For some* $M \in \mathbb{N}$*, suppose* L *is a homogeneous, weakly elliptic, constant* (*complex*) *coefficient, second-order,* $M \times M$ *system in* $\mathbb{R}^n$*, and consider a boundary-to-domain double layer operator* $\mathcal{D}$ *associated with* L *and* Ω *as in* (1.3.18)*. Also, fix an integrability exponent* $p \in (1,\infty)$ *along with some parameter* $\lambda \in (0, n-1)$.

Then there exists a constant $C \in (0,\infty)$ *with the property that the fractional Carleson measure estimate*

$$\sup_{x\in\partial\Omega \text{ and } r>0}\left\{r^{-\lambda}\int_{B(x,r)\cap\Omega}\left|\nabla\big(\mathcal{D}f\big)(y)\right|^p \operatorname{dist}(y,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n(y)\right\}^{\frac{1}{p}}$$

$$\leq C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M} \tag{3.3.69}$$

holds for each function f in the Morrey space $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. In addition, there exists $C\in(0,\infty)$ with the property that for each function $f\in\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ one has

$$\lim_{R\to 0^+}\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}}\left(r^{-\lambda}\int_{B(x,r)\cap\Omega}\left|\nabla\big(\mathcal{D}f\big)\right|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}}$$

$$\leq C\operatorname{dist}\big(f,[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)]^M\big), \tag{3.3.70}$$

where the distance is measured in $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. In particular, for each function f belonging to the space $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ one has

$$\lim_{R\to 0^+}\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}}\left(r^{-\lambda}\int_{B(x,r)\cap\Omega}\left|\nabla\big(\mathcal{D}f\big)\right|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}}=0. \tag{3.3.71}$$

Proof The fractional Carleson measure estimate claimed in (3.3.69) is a consequence of Theorem 3.2.4, [69, (6.2.44)], (1.8.9), and [68, (7.7.106)] (see also [69, (6.2.49)] in the case when $\partial\Omega$ is bounded). Consider next the claim made in (3.3.70). To this end, fix some $\eta\in(0,1)$ and select an arbitrary function $f\in\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. Also, choose some $g\in\big[\mathrm{Lip}_c(\partial\Omega)\big]^M$. Then for each location $x\in\partial\Omega$ and each scale $r\in\big(0,2\operatorname{diam}(\partial\Omega)\big)$ we may rely on (3.3.69) to write

$$\left(r^{-\lambda}\int_{B(x,r)\cap\Omega}\left|\nabla\big(\mathcal{D}(f-g)\big)\right|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}}$$

$$\leq C\|f-g\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}. \tag{3.3.72}$$

Also, we have

$$\begin{aligned}&\left(r^{-\lambda}\int_{B(x,r)\cap\Omega}\left|\nabla\big(\mathcal{D}g\big)\right|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}}\\ &\quad\leq Cr^{\frac{n-1-\lambda}{p}}\left(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\left|\nabla\big(\mathcal{D}g\big)\right|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}}\\ &\quad\leq C\|g\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^M}\cdot r^{\eta+\frac{n-1-\lambda}{p}},\end{aligned} \tag{3.3.73}$$

where the last inequality comes from (1.8.87). Given that $\eta + (n-1-\lambda)/p > 0$, from (3.3.72) and (3.3.73) we then conclude that

$$\lim_{R\to 0^+} \sup_{\substack{x\in\partial\Omega \text{ and} \\ r\in(0,R)}} \Big(r^{-\lambda} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{D} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n \Big)^{\frac{1}{p}} \\ \leq C\|f-g\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}, \tag{3.3.74}$$

for some constant $C \in (0,\infty)$ independent of f and g. Having established this, (3.3.70) follows on account of [69, (6.2.20)] and the arbitrariness of the function $g \in \big[\mathrm{Lip}_c(\partial\Omega)\big]^M$. Finally, (3.3.70) readily implies (3.3.71). □

The results presented in this section about the double layer operators associated with arbitrary weakly elliptic systems are in effect for particular embodiments of this class of singular integral operators, such as those discussed in Examples 1-7, just prior to the statement of Proposition 1.4.21 in the build-up to Theorem 1.5.1. In particular, they apply to the Cauchy-Clifford integral operator. In relation to this, we wish to single out the following version of [70, Proposition 2.5.32] which opens the door for developing a rich theory of Hardy spaces in the context of Morrey spaces and their pre-duals.

Proposition 3.3.4 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ satisfies $n \geq 2$) is a* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, pick $p, q \in (1,\infty)$ such that $1/p + 1/q = 1$ and select $\lambda \in (0, n-1)$ arbitrary.*

Then the principal-value Cauchy-Clifford integral operator $\mathfrak{C}$ associated with Ω as in (A.0.54) induces well-defined linear and bounded mappings

$$\mathfrak{C} : M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \longrightarrow M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \tag{3.3.75}$$

$$\mathfrak{C} : \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \tag{3.3.76}$$

$$\mathfrak{C} : \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \longrightarrow \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \tag{3.3.77}$$

and has the property that

$$\mathfrak{C}^2 = \tfrac{1}{4} I \text{ on either of the spaces} \\ M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n,\ \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n,\ \text{or } \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n. \tag{3.3.78}$$

Likewise, the transpose Cauchy-Clifford integral operator $\mathfrak{C}^{\#}$ associated with Ω as in (1.6.1) induces well-defined linear and bounded mappings

$$\mathfrak{C}^{\#} : M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \longrightarrow M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \tag{3.3.79}$$

$$\mathfrak{C}^{\#} : \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \longrightarrow \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \tag{3.3.80}$$

$$\mathfrak{C}^{\#} : \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \longrightarrow \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \tag{3.3.81}$$

and has the property that

$$\begin{gathered}\big(\mathfrak{C}^{\#}\big)^2=\tfrac{1}{4}I \ \textit{ on either of the spaces} \\ M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n,\ \mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n,\ \textit{ or }\ \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n.\end{gathered} \tag{3.3.82}$$

Finally,

$$\textit{the (real) transpose of } \mathfrak{C} \textit{ from (3.3.76) is } \mathfrak{C}^{\#} \textit{ from (3.3.81)}, \tag{3.3.83}$$

$$\textit{the (real) transpose of } \mathfrak{C} \textit{ from (3.3.77) is } \mathfrak{C}^{\#} \textit{ from (3.3.79)}, \tag{3.3.84}$$

$$\textit{the (real) transpose of } \mathfrak{C}^{\#} \textit{ from (3.3.80) is } \mathfrak{C} \textit{ from (3.3.77)}, \tag{3.3.85}$$

$$\textit{the (real) transpose of } \mathfrak{C}^{\#} \textit{ from (3.3.81) is } \mathfrak{C} \textit{ from (3.3.75)}. \tag{3.3.86}$$

Proof Combining [70, Theorem 2.6.1] together with (A.0.54) and (1.6.1) yields the claims pertaining to the operators (3.3.75)-(3.3.77) and (3.3.79)-(3.3.81). Having established these boundedness properties, the claims in (3.3.83)-(3.3.86) then follow on account of [70, (2.6.15), (2.6.16)], (A.0.54), (1.6.1), and [68, Lemma 6.4.1].

To prove that $\mathfrak{C}^2=\frac{1}{4}I$ on $M^{p,\lambda}(\partial\Omega,\sigma)$, let $f\in M^{p,\lambda}(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n$ be arbitrary and define $u:=Cf$ in Ω where the boundary-to-domain Cauchy-Clifford operator C is as in (A.0.53). Also, pick an arbitrary aperture parameter $\kappa>0$. Based on (3.3.3) and (3.3.4) we conclude that

$$\begin{aligned}&u\in\mathscr{C}^\infty(\Omega)\otimes\mathcal{C}\ell_n, \quad Du=0 \text{ in } \Omega,\\ &u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}=(\tfrac{1}{2}I+\mathfrak{C})f \text{ at } \sigma\text{-a.e. point on } \partial\Omega,\\ &\mathcal{N}_\kappa u \text{ belongs to the Morrey space } M^{p,\lambda}(\partial\Omega,\sigma),\\ &u(x)=O(|x|^{1-n}) \text{ if } \Omega \text{ is an exterior domain.}\end{aligned} \tag{3.3.87}$$

In concert with [69, (6.2.25)], the next-to-last property above implies that

$$\int_{\partial\Omega}\frac{(\mathcal{N}_\kappa u)(x)}{1+|x|^{n-1}}\,d\sigma(x)<+\infty, \tag{3.3.88}$$

while the last property in (3.3.87) implies that, in the case when Ω is an exterior domain,

$$\int_{[B(0,2R)\setminus B(0,R)]\cap\Omega}|u(x)|\,d\mathcal{L}^n=o(R^n) \text{ as } R\to\infty. \tag{3.3.89}$$

With (3.3.87)-(3.3.89) in hand, we may rely on [70, Theorem 1.2.2] to conclude that

$$\begin{aligned}u(x)&=\frac{1}{\omega_{n-1}}\int_{\partial_*\Omega}\frac{x-y}{|x-y|^n}\odot\nu(y)\odot\Big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(y)\,d\sigma(y)\\ &=\big(C(\tfrac{1}{2}I+\mathfrak{C})f\big)(x) \text{ at each } x\in\Omega.\end{aligned} \tag{3.3.90}$$

Taking the nontangential boundary traces of the most extreme sides in (3.3.90) then yields (on account of (3.3.4))

$$(\tfrac{1}{2}I + \mathfrak{C})f = (\tfrac{1}{2}I + \mathfrak{C})f(\tfrac{1}{2}I + \mathfrak{C})f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{3.3.91}$$

This goes to show that, indeed, $\mathfrak{C}^2 = \frac{1}{4}I$ on $M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n$, hence also on $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n$ by [69, (6.2.15)] and (3.3.76). That we also have $\mathfrak{C}^2 = \frac{1}{4}I$ on $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n$ is established in a similar fashion, this time taking into account item *(iii)* of Theorem 3.3.1 and the embedding in [69, (6.2.230)]. This completes the proof of (3.3.78).

Finally, from (3.3.78) and (3.3.83)-(3.3.84) we see that $\big(\mathfrak{C}^{\#}\big)^2 = \frac{1}{4}I$ both on $M^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n$ and on $\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n$. The former also implies $\big(\mathfrak{C}^{\#}\big)^2 = \frac{1}{4}I$ on $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n$, thanks to (3.3.80) and [69, (6.2.15)]. □

We continue by discussing mapping properties of the modified single layer potential operator involving Morrey-based and block-based Sobolev spaces in arbitrary UR domains.

Theorem 3.3.5 *Let* $\Omega \subseteq \mathbb{R}^n$ *(where* $n \in \mathbb{N}$, $n \geq 2$*) be a* UR *domain, and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$*. Also, consider a homogeneous, weakly elliptic, second-order* $M\times M$ *system* L *in* $\mathbb{R}^n$*, with complex constant coefficients, and recall the modified boundary-to-boundary single layer operator* S_{mod} *associated with* L *and* Ω *as in* (1.5.73)*. Finally, fix some integrability exponents* $p, q \in (1,\infty)$ *along with some parameter* $\lambda \in (0, n-1)$*. Then the following properties are true.*

(1) The modified boundary-to-boundary single layer operator induces a mapping

$$S_{\text{mod}} : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M \tag{3.3.92}$$

which is well defined, linear, and bounded, when the target space is endowed with the semi-norm induced by (A.0.158)*.*

(2) The following operator,

$$\big[S_{\text{mod}}\big] : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M \ \textit{defined as}$$
$$\big[S_{\text{mod}}\big]f := \big[S_{\text{mod}}f\big] \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M, \quad \forall f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M. \tag{3.3.93}$$

is well defined, linear, and bounded, when the quotient space is equipped with the semi-norm[2] *introduced in* [69, (11.13.51)].

(3) With $\mathscr{S}_{\text{mod}}$ *denoting the modified version of the single layer operator acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *as in* (1.5.50)*, for each given aperture*

[2] Recall from [69, Proposition 11.13.10] that this semi-norm is fact a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

parameter $\kappa > 0$ *there exists some constant* $C = C(\Omega, L, n, p, \lambda, \kappa) \in (0, \infty)$ *with the property that for each given function* $f \in \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ *one has*

$$\begin{gathered}\mathscr{S}_{\rm mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathscr{S}_{\rm mod} f\big) = 0 \ \textit{in} \ \Omega, \\ \mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big) \in M^{p,\lambda}(\partial\Omega, \sigma), \\ \big\|\mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \le C\|f\|_{[M^{p,\lambda}(\partial\Omega,\sigma)]^M}, \\ \textit{and} \ \Big(\big(\mathscr{S}_{\rm mod} f\big)\Big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big)(x) = (S_{\rm mod} f)(x) \ \textit{at} \ \sigma\textit{-a.e. point} \ x \in A_\kappa(\partial\Omega) \\ \textit{hence, (cf. [68, Proposition 8.8.4] and (A.0.204)), at } \sigma\textit{-a.e. point } x \in \partial\Omega. \end{gathered} \tag{3.3.94}$$

(4) Similar properties to those described in items (1)-(3) are valid for block spaces (and block-based homogeneous Sobolev spaces) in place of Morrey spaces (and Morrey-based homogeneous Sobolev spaces). More specifically, the operator

$$S_{\rm mod} : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big]^M \tag{3.3.95}$$

is well defined, linear, and bounded, when the target space is endowed with the semi-norm induced by (A.0.40). Also,

$$\begin{gathered}\big[S_{\rm mod}\big] : \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)/\sim\big]^M \ \textit{defined as} \\ \big[S_{\rm mod}\big] f := \big[S_{\rm mod} f\big] \in \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)/\sim\big]^M, \quad \forall f \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \end{gathered} \tag{3.3.96}$$

is a well-defined linear and bounded operator, in the scenario in which the quotient space is endowed with the semi-norm[3] *introduced in* [69, (11.13.70)]. *Finally, for each* $\kappa > 0$ *there exists a constant* $C = C(\Omega, L, n, q, \lambda, \kappa) \in (0, \infty)$ *with the property that for each function* $f \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M$ *one has*

$$\begin{gathered}\mathscr{S}_{\rm mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathscr{S}_{\rm mod} f\big) = 0 \ \textit{in} \ \Omega, \\ \mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big) \in \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma), \\ \big\|\mathcal{N}_\kappa\big(\nabla \mathscr{S}_{\rm mod} f\big)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \le C\|f\|_{[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)]^M}, \\ \textit{and} \ \Big(\big(\mathscr{S}_{\rm mod} f\big)\Big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big)(x) = (S_{\rm mod} f)(x) \ \textit{at} \ \sigma\textit{-a.e. point} \ x \in \partial\Omega. \end{gathered} \tag{3.3.97}$$

(5) Analogous properties to those presented in items (1)-(3) above are also valid for vanishing Morrey spaces $\mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$ *(cf. (A.0.149)) and vanishing Morrey-based homogeneous Sobolev spaces* $\dot{\mathbb{M}}^{p,\lambda}_1(\partial\Omega, \sigma)$ *(cf.* [69, Definition 11.13.15],

[3] according to [69, Proposition 11.13.12], said semi-norm is actually a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

or (A.0.159)-(A.0.160)) *in place of Morrey spaces and Morrey-based homogeneous Sobolev spaces, respectively.*

Proof The claims regarding the operator (3.3.92) are seen from (A.0.157), (1.5.75) (with $w \equiv 1$), (1.5.76) (with $0 < \varepsilon < (n-1-\lambda)/p$), [68, (6.2.23)] (with $\tau = 1$ and $a = n-1-\varepsilon$), [69, (6.2.25)], (1.5.83), and the fact that the singular integral operators defined in (1.5.82) induce bounded mappings (cf. [70, (2.6.1)], [69, (6.2.5)], and [70, Theorem 1.4.2])

$$T^{\#}_{jk} : \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \qquad \forall\, j,k \in \{1,\dots,n\}. \tag{3.3.98}$$

Once we have dealt with (3.3.92), the claims regarding (3.3.93) readily follow. The properties in the first line of (3.3.94) are seen from (1.5.51), (1.5.52), and [69, (6.2.25)]. The properties in the second line of (3.3.94) are consequences of [69, (6.2.25)], (1.5.51), [70, Theorem 1.4.2], [70, (2.6.4)], and [69, (6.2.5)]. The boundary trace formula in (3.3.94) is implied by (1.5.80) and [69, (6.2.25)]. Next, the claims in item *(4)* are justified in an analogous fashion, now making use of the fact that

$$T^{\#}_{jk} : \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M, \qquad \forall\, j,k \in \{1,\dots,n\} \tag{3.3.99}$$

are well-defined, linear, and bounded operators (as may be seen from [70, (2.6.14)], [69, (6.2.74)], and [70, Theorem 1.4.2]). Finally, the claim in item *(5)* is seen by reasoning in a similar fashion, bearing in mind (A.0.159) and making use of the fact that

$$T^{\#}_{jk} : \big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \qquad \forall\, j,k \in \{1,\dots,n\} \tag{3.3.100}$$

are well-defined, linear, and bounded operators (cf. [70, Theorem 2.6.1] and [69, (6.2.17)]). □

Going further, we consider mapping properties of the modified boundary-to-domain double layer potential operators involving Morrey-based and block-based Sobolev spaces in arbitrary UR domains.

Theorem 3.3.6 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a* UR *domain. Denote by $\nu = (\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In addition, for some $M \in \mathbb{N}$, let $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a complex coefficient tensor with the property that*

$$L := \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M} \tag{3.3.101}$$

(i.e., $L := L_A$) is a weakly elliptic $M\times M$ system in $\mathbb{R}^n$. Also, let $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ be the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2]. *In this setting, recall the modified version of the double layer operator $\mathcal{D}_{\text{mod}}$ acting on functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ as in* (1.8.6)*. Finally, fix some integrability*

exponents $p, q \in (1, \infty)$ *along with a number* $\lambda \in (0, n-1)$, *and an aperture parameter* $\kappa \in (0, \infty)$. *Then the following statements are true.*

(1) *There exists some constant* $C = C(\Omega, A, n, p, \lambda, \kappa) \in (0, \infty)$ *with the property that for each function* $f \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)\big]^M$ *it follows that*

$$\begin{gathered}\mathcal{D}_{\rm mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathcal{D}_{\rm mod} f\big) = 0 \text{ in } \Omega,\\ \big(\mathcal{D}_{\rm mod} f\big)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ and } \big(\nabla\mathcal{D}_{\rm mod} f\big)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ exist at } \sigma\text{-a.e. point on } \partial\Omega,\\ \mathcal{N}_\kappa\big(\nabla\mathcal{D}_{\rm mod} f\big) \text{ belongs to } M^{p,\lambda}(\partial\Omega, \sigma) \text{ and}\\ \big\|\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{\rm mod} f\big)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)]^M}.\end{gathered} \tag{3.3.102}$$

In fact, for each function $f \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)\big]^M$ *one has*

$$(\mathcal{D}_{\rm mod} f)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + K_{\rm mod}\big)f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{3.3.103}$$

where I *is the identity operator on* $\big[\dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)\big]^M$, *and* $K_{\rm mod}$ *is the modified boundary-to-boundary double layer potential operator from* (1.8.24)-(1.8.25).

(2) *Given any function* $f = (f_\alpha)_{1\leq\alpha\leq M}$ *belonging to the Morrey-based homogeneous boundary Sobolev space* $\big[\dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)\big]^M$, *at* σ*-a.e. point* $x \in \partial\Omega$ *one has*

$$\begin{aligned}&\big(\partial^A_\nu(\mathcal{D}_{\rm mod} f)\big)(x) \\ &= \Bigg(\lim_{\varepsilon\to0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_i(x)a^{\mu\gamma}_{ij}a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)\big(\partial_{\tau_{js}} f_\alpha\big)(y)\,{\rm d}\sigma(y)\Bigg)_{1\leq\mu\leq M}\end{aligned} \tag{3.3.104}$$

where the conormal derivative is considered as in (A.0.184).

(3) *The operator*

$$\begin{gathered}\partial^A_\nu\mathcal{D}_{\rm mod} : \big[\dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \text{ defined as}\\ \big(\partial^A_\nu\mathcal{D}_{\rm mod}\big)f := \partial^A_\nu(\mathcal{D}_{\rm mod} f) \text{ for each } f \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)\big]^M\end{gathered} \tag{3.3.105}$$

is well defined, linear, and bounded, when the domain space is equipped with the semi-norm induced by (A.0.158). *In addition,*

$$\begin{gathered}\big[\partial^A_\nu\mathcal{D}_{\rm mod}\big] : \big[\dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)/\sim\big]^M \longrightarrow \big[M^{p,\lambda}(\partial\Omega, \sigma)\big]^M \text{ defined as}\\ \big[\partial^A_\nu\mathcal{D}_{\rm mod}\big][f] := \partial^A_\nu(\mathcal{D}_{\rm mod} f) \text{ for each } f \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)\big]^M,\end{gathered} \tag{3.3.106}$$

is a well-defined linear and bounded operator, when the quotient space is equipped with the semi-norm[4] *introduced in* [69, (11.13.51)].

(4) *Similar properties to those described in items (1)-(3) are valid for block spaces* (*and block-based homogeneous Sobolev spaces*) *in place of Morrey spaces* (*and Morrey-based homogeneous Sobolev spaces*). *Concretely, there exists a constant* $C = C(\Omega, A, n, q, \lambda, \kappa) \in (0, \infty)$ *with the property that for each function* $f \in \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big]^M$ *one has*

$$
\begin{aligned}
&\mathcal{D}_{\rm mod} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad L\big(\mathcal{D}_{\rm mod} f\big) = 0 \text{ in } \Omega,\\
&\big(\mathcal{D}_{\rm mod} f\big)\big|^{\kappa-\rm n.t.}_{\partial\Omega} = \big(\tfrac12 I + K_{\rm mod}\big) f \text{ at } \sigma\text{-a.e. point on } \partial\Omega,\\
&\big(\nabla\mathcal{D}_{\rm mod} f\big)\big|^{\kappa-\rm n.t.}_{\partial\Omega} \text{ exist at } \sigma\text{-a.e. point on } \partial\Omega,\\
&\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{\rm mod} f\big) \text{ belongs to } \mathcal{B}^{q,\lambda}(\partial\Omega, \sigma) \text{ and}\\
&\big\|\mathcal{N}_\kappa\big(\nabla\mathcal{D}_{\rm mod} f\big)\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)]^M}.
\end{aligned}
\tag{3.3.107}
$$

Also, formula (3.3.104) *remains true for* $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big]^M$ *and the operator*

$$
\begin{aligned}
&\partial^A_\nu \mathcal{D}_{\rm mod} : \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \text{ defined as}\\
&(\partial^A_\nu \mathcal{D}_{\rm mod}) f := \partial^A_\nu(\mathcal{D}_{\rm mod} f) \text{ for each } f \in \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big]^M
\end{aligned}
\tag{3.3.108}
$$

is well defined, linear, and bounded, when the domain space is equipped with the semi-norm induced by [69, (11.13.66)]. *Finally,*

$$
\begin{aligned}
&\big[\partial^A_\nu \mathcal{D}_{\rm mod}\big] : \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)/\sim\big]^M \longrightarrow \big[\mathcal{B}^{q,\lambda}(\partial\Omega, \sigma)\big]^M \text{ defined as}\\
&\big[\partial^A_\nu \mathcal{D}_{\rm mod}\big][f] := \partial^A_\nu(\mathcal{D}_{\rm mod} f) \text{ for each } f \in \big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega, \sigma)\big]^M
\end{aligned}
\tag{3.3.109}
$$

is a well-defined linear and bounded operator, assuming the quotient space is endowed with the semi-norm[5] *introduced in* [69, (11.13.70)].

(5) *Analogous properties to those presented in items (1)-(3) above are also valid for vanishing Morrey-based homogeneous Sobolev spaces* $\mathring{\mathbb{M}}^{p,\lambda}_1(\partial\Omega, \sigma)$ (*cf.* [69, Definition 11.13.15], *or* (A.0.159)-(A.0.160)) *in place of Morrey-based homogeneous Sobolev spaces.*

---

[4] recall from [69, Proposition 11.13.10] that said semi-norm is in fact a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

[5] from [69, Proposition 11.13.10] we know that this semi-norm is actually a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

Proof That the jump-formula (3.3.103) is true for each $f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ follows from (1.8.27) (bearing in mind (A.0.157)). The claims in (3.3.102) are consequences of (A.0.157), (1.8.7), (1.8.27), (1.8.20), (1.8.13), [70, Theorem 2.6.1], and [70, Theorem 1.4.2].

Also, (3.3.104) is a consequence of (1.8.83) and [69, Definition 11.13.1] (cf. (A.0.157)-(A.0.158)). Having established (3.3.104), the claims made in relation to (3.3.105) follow with the help of [70, Theorem 2.6.1] and [70, Theorem 1.4.2]. The claim pertaining to (3.3.106) is a consequence of what we have proved so far and (1.8.10), which implies that $\partial_\nu^A \mathcal{D}_{\rm mod}$ annihilates constants. Finally, the claims in items *(4)* and *(5)* are justified in a similar fashion. □

Remark 3.3.7 *The results in Theorem 3.3.6 are applicable to all modified boundary-to-boundary double layer potential operators $\mathcal{D}_{\rm mod}$, $K_{\rm mod}$ described in Examples 1.8.4-1.8.7. In addition, the operator $\partial_\nu^A \mathcal{D}_{\rm mod}$ vanishes identically in the context of* (3.3.105) *and* (3.3.108) *when $\mathcal{D}_{\rm mod}$ is as in* (1.8.108) (*see the last part in Remark 1.4.6*)*. The same the peculiarity* (*i.e., that $\partial_\nu^A \mathcal{D}_{\rm mod}$ from vanishes identically*) *is present when $\mathcal{D}_{\rm mod}$ is as in* (1.8.111) (*see* (1.4.33)).

The next item on the current agenda is to establish mapping properties for the modified boundary-to-boundary double layer potential operator acting on the scales of Morrey-based and block-based Sobolev spaces in the geometric setting described in the theorem below.

Theorem 3.3.8 *Let $\Omega \subseteq \mathbb{R}^n$* (*where $n \in \mathbb{N}$, $n \geq 2$*) *be a* UR *domain satisfying a local John condition, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let $L = \big(a_{rs}^{\alpha\beta}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ be a homogeneous, weakly elliptic, constant* (*complex*) *coefficient, second-order $M \times M$ system in $\mathbb{R}^n$* (*for some integer $M \in \mathbb{N}$*)*. In this context, consider the modified boundary-to-boundary double layer potential operator $K_{\rm mod}$ from* (1.8.24)-(1.8.25)*. Finally, select some integrability exponents $p, q \in (1,\infty)$ along with a parameter $\lambda \in (0, n-1)$. Then the following statements are valid.*

(1) The modified boundary-to-boundary double layer potential operator induces a mapping

$$K_{\rm mod} : \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \tag{3.3.110}$$

which is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.158)*. As a corollary of* (3.3.110) *and* (1.8.28)*, the operator*

$$\big[K_{\rm mod}\big] : \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \text{ defined as}$$
$$\big[K_{\rm mod}\big][f] := \big[K_{\rm mod} f\big] \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M, \quad \forall f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \tag{3.3.111}$$

is well defined, linear, and bounded, assuming all quotient spaces are endowed with the semi-norm[6] *introduced in* [69, (11.13.51)].

(2) If U_{jk} *with* $j,k\in\{1,\dots,n\}$ *is the family of singular integral operators defined in* (1.5.251)*, then*

$$\begin{aligned}&\partial_{\tau_{jk}}\big(K_{\text{mod}}f\big)=K(\partial_{\tau_{jk}}f)+U_{jk}(\nabla_{\tan}f)\ \text{ at }\sigma\text{-a.e. point on }\partial\Omega\\&\text{for each } f\in\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \text{ and each } j,k\in\{1,\dots,n\}.\end{aligned}\tag{3.3.112}$$

(3) Similar properties to those described in items (1)-(2) are valid for block-based homogeneous Sobolev spaces in place Morrey-based homogeneous Sobolev spaces. Specifically,

$$K_{\text{mod}}:\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M\longrightarrow\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M\tag{3.3.113}$$

is a well-defined, linear, and bounded operator when the spaces involved are endowed with the semi-norm [69, (11.13.66)]. *Also,*

$$\begin{aligned}&\big[K_{\text{mod}}\big]:\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M\longrightarrow\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M\ \text{ defined as}\\&\big[K_{\text{mod}}\big][f]:=\big[K_{\text{mod}}f\big]\in\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M,\quad\forall f\in\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M\end{aligned}\tag{3.3.114}$$

is a well-defined linear and bounded mapping, when all quotient spaces are endowed with the semi-norm[7] *introduced in* [69, (11.13.70)]. *Finally,*

$$\begin{aligned}&\partial_{\tau_{jk}}\big(K_{\text{mod}}f\big)=K(\partial_{\tau_{jk}}f)+U_{jk}(\nabla_{\tan}f)\ \text{ at }\sigma\text{-a.e. point on }\partial\Omega\\&\text{for each } f\in\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \text{ and each } j,k\in\{1,\dots,n\}.\end{aligned}\tag{3.3.115}$$

(4) Analogous properties to those presented in items (1)-(2) above are also valid for vanishing Morrey-based homogeneous Sobolev spaces $\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)$ *(cf.* [69, Definition 11.13.15]*, or* (A.0.159)-(A.0.160)*) in place of Morrey-based homogeneous Sobolev spaces.*

Proof Fix an aperture parameter $\kappa\in(0,\infty)$ along with a truncation parameter $\varepsilon\in(0,\infty)$. Next, consider an arbitrary function

$$f\in\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M\subseteq\Big[L^1\Big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\Big)\cap L^p_{\text{loc}}(\partial\Omega,\sigma)\Big]^M\tag{3.3.116}$$

(cf. (A.0.157)), and define $u:=\mathcal{D}_{\text{mod}}f$ in Ω. Then from (3.3.116), (3.3.102), (1.8.22), and the jump-formula (3.3.103) we see that

[6] recall from [69, Proposition 11.13.10] that this semi-norm is fact a genuine norm if $\Omega\subseteq\mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

[7] according to [69, Proposition 11.13.12] said semi-norm is actually a true norm if $\Omega\subseteq\mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

$$\begin{aligned} & u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\tfrac{1}{2}I + K_{\mathrm{mod}}\big)f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \\ & \mathcal{N}_\kappa^\varepsilon u \text{ belongs to } L^p_{\mathrm{loc}}(\partial\Omega,\sigma), \quad \mathcal{N}_\kappa(\nabla u) \text{ belongs to } M^{p,\lambda}(\partial\Omega,\sigma), \\ & \text{and } \big\|\mathcal{N}_\kappa(\nabla u)\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M}, \end{aligned} \tag{3.3.117}$$

for some constant $C = C(\Omega, L, \kappa, p, \lambda) \in (0,\infty)$ independent of f. Granted these properties and the current geometric assumptions, we may invoke [69, Proposition 11.13.8] to conclude that

$$\begin{aligned} & u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ belongs to the space } \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \\ & \text{and } \Big\|u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\|_{[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M}. \end{aligned} \tag{3.3.118}$$

Then from (3.3.117) and (3.3.118) we deduce that

$$\begin{aligned} & K_{\mathrm{mod}} f \text{ belongs to the space } \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \\ & \text{and } \big\|K_{\mathrm{mod}} f\big\|_{[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)]^M}, \end{aligned} \tag{3.3.119}$$

from which the claims regarding (3.3.110) follow. Next, the claims concerning the operator (3.3.111) are readily seen from what we have just proved, (1.8.28), and definitions. Also, formula (3.3.112) is a consequence of Proposition 1.8.8 and [69, Definition 11.13.1] (cf. (A.0.157)-(A.0.158)). Finally, the claims in items *(3)*-*(4)* are established in a similar fashion, now making use of the trace results from [69, Proposition 11.13.14] (as well as the embedding in [69, (6.2.71)]), and [69, Proposition 11.13.17] (together with (5.1.53)), respectively. □

Remark 3.3.9 *The results in Theorem 3.3.8 are applicable to all modified boundary-to-boundary double layer potential operators K_{mod} described in Examples 1.8.4-1.8.7.*

Changing topics, we now turn to integral representation formulas involving null-solutions of weakly elliptic second-order systems in NTA domains with Ahlfors regular boundaries. The goal is to produce a version of Theorem 1.8.19 with Morrey and block spaces playing the role of the Lebesgue spaces in the formulation of said result.

Theorem 3.3.10 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an* NTA *domain such that $\partial\Omega$ is an Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Let $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n \\ 1\leq\alpha,\beta\leq M}}$ (where $M \in \mathbb{N}$) be a complex coefficient tensor with the property that*

$$L := \big(a_{rs}^{\alpha\beta}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M} \tag{3.3.120}$$

(i.e., $L := L_A$) is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Once again, recall the modified version of the double layer operator $\mathcal{D}_{\mathrm{mod}}$ acting on functions from

$\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ *as in* (1.8.6)*, and the modified version of the single layer operator* $\mathscr{S}_{\rm mod}$ *acting on functions from* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *as in* (1.5.50)*. Finally, fix an aperture parameter* $\kappa \in (0,\infty)$ *along with an integrability exponent* $p \in (1,\infty)$ *and a number* $\lambda \in (0, n-1)$*. In this setting, consider a function* $u : \Omega \to \mathbb{C}^M$ *satisfying*

$$u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{in} \ \Omega, \quad \mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega, \sigma). \tag{3.3.121}$$

In the case when Ω *is an exterior domain make the additional assumption that there exists* $\lambda \in (1,\infty)$ *such that*

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |\nabla u|\, \mathrm{d}\mathcal{L}^n = o(1) \ \text{as} \ R \to \infty. \tag{3.3.122}$$

Then

$$\begin{aligned} &u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \text{exists } \sigma\text{-a.e. on } \partial\Omega \text{ and belongs to } \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M, \\ &(\nabla u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \text{exists } \sigma\text{-a.e. on } \partial\Omega \text{ and } \partial^A_\nu u \text{ belongs to } \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{3.3.123}$$

and there exists some $\mathbb{C}^M$*-valued locally constant function* c_u *in* Ω *with the property that*

$$u = \mathscr{D}_{\rm mod}\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big) - \mathscr{S}_{\rm mod}\big(\partial^A_\nu u\big) + c_u \ \text{in} \ \Omega. \tag{3.3.124}$$

In particular, if in place of the last condition in (3.3.121) *one assumes that* $\mathcal{N}_\kappa(\nabla u)$ *belongs to* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ *(which is a subspace of* $M^{p,\lambda}(\partial\Omega,\sigma)$*; cf.* (A.0.149)*) then* $u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \in \big[\mathring{\dot{\mathbb{M}}}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$*, the conormal derivative* $\partial^A_\nu u$ *belongs to* $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$*, and the integral representation formula* (3.3.124) *continues to hold.*

Finally, if the last condition in (3.3.121) *is changed to* $\mathcal{N}_\kappa(\nabla u) \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)$ *for some exponent* $q \in (1,\infty)$ *then* $u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ *exists at* σ*-a.e. point on* $\partial\Omega$ *and belongs to* $\big[\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)\big]^M$*, the trace* $(\nabla u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ *exists at* σ*-a.e. point on* $\partial\Omega$*, the conormal derivative* $\partial^A_\nu u$ *belongs to* $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$*, and the integral representation formula* (3.3.124) *once again continues to hold.*

Proof The current assumptions and [69, Proposition 11.13.7] imply that $u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists at σ-a.e. point on $\partial\Omega$ and belongs to $\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$. From [69, (6.2.7)] and [68, Proposition 8.4.9] we also see that there exists some $\varepsilon > 0$ such that $\mathcal{N}^\varepsilon_\kappa u \in L^p_{\rm loc}(\partial\Omega,\sigma)$. The present hypotheses on Ω ensure (cf. [68, (5.10.24)]) that Ω is a UR domain. Bearing this in mind, the Fatou-type result from [70, Theorem 3.3.4] ensures that the nontangential boundary trace $(\nabla u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists (in $\mathbb{C}^{n\cdot M}$) at σ-a.e. point on $\partial\Omega$. As such, $\partial^A_\nu u$ is well defined and belongs to $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (cf. [70, (1.7.16)], [69, (6.2.3)], and [69, (6.2.5)]). Thus, all conditions in (1.8.174) are satisfied, and this permits us to invoke Theorem 1.8.17 to conclude that (3.3.124) holds. Next, the claims made in the scenario in which the last condition in (3.3.121) is strengthened to $\mathcal{N}_\kappa(\nabla u) \in \mathring{M}^{p,\lambda}(\partial\Omega, \sigma)$ are seen from what we have proved so far, (A.0.149), [69, Proposition 11.13.16], and [69, (6.2.17), (6.2.18)]. Finally, the claims in the very last portion of the statement of the theorem are justified in a similar fashion, bearing in mind [69, (6.2.71)]. □

We continue by presenting a variant of Corollary 1.8.18 in which the Lebesgue spaces are replaced by appropriate Morrey and block spaces. In contrast to Theorem 3.3.10, this is formulated in an Ahlfors regular domain satisfying a local John condition.

Theorem 3.3.11 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an Ahlfors regular domain satisfying a local John condition. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Let $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ (where $M \in \mathbb{N}$) be a complex coefficient tensor with the property that*

$$L := \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M} \tag{3.3.125}$$

(i.e., $L := L_A$) is a weakly elliptic $M \times M$ system in $\mathbb{R}^n$. In this setting, bring in the modified version of the double layer operator $\mathcal{D}_{\rm mod}$ acting on functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ as in (1.8.6), and the modified version of the single layer operator $\mathcal{S}_{\rm mod}$ acting on functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ as in (1.5.50). Finally, fix an aperture parameter $\kappa \in (0,\infty)$ along with a truncation parameter $\varepsilon \in (0,\infty)$, an integrability exponent $p \in (1,\infty)$, a number $\lambda \in (0, n-1)$, and consider a function $u : \Omega \to \mathbb{C}^M$ satisfying:

$$\begin{gathered} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \ \text{ in } \ \Omega, \\ u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \text{ and } \ (\nabla u)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \\ \mathcal{N}^{\varepsilon}_\kappa u \ \text{ belongs to the space } \ L^p_{\rm loc}(\partial\Omega,\sigma), \\ \mathcal{N}_\kappa(\nabla u) \ \text{ belongs to the space } \ M^{p,\lambda}(\partial\Omega,\sigma). \end{gathered} \tag{3.3.126}$$

In the case when Ω is an exterior domain make the additional assumption that there exists $\lambda \in (1,\infty)$ such that

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |\nabla u| \,\mathrm{d}\mathcal{L}^n = o(1) \ \text{ as } \ R \to \infty. \tag{3.3.127}$$

Then

$$\begin{gathered} u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \text{ belongs to } \ \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M, \\ \partial^A_\nu u \ \text{ belongs to } \ \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \end{gathered} \tag{3.3.128}$$

and there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω with the property that

$$u = \mathcal{D}_{\rm mod}\Big(u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) - \mathcal{S}_{\rm mod}\big(\partial^A_\nu u\big) + c_u \ \text{ in } \ \Omega. \tag{3.3.129}$$

In particular, if in place of the last condition in (3.3.126) *one assumes that* $\mathcal{N}_\kappa(\nabla u)$ *belongs to* $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ (*which is a subspace of* $M^{p,\lambda}(\partial\Omega,\sigma)$*; cf.* (A.0.149)) *then* $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in \big[\mathring{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *the conormal derivative* $\partial_\nu^A u$ *belongs to* $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$*, and the integral representation formula* (3.3.129) *continues to hold.*

Finally, if for some exponent $q \in (1,\infty)$ *the last two lines in* (3.3.126) *are changed to*

$$\begin{gathered}\mathcal{N}_\kappa^\varepsilon u \in L^{q_\lambda}_{\mathrm{loc}}(\partial\Omega,\sigma) \ \textit{with} \ q_\lambda := \tfrac{q(n-1)}{n-1+\lambda(q-1)} \in (1,q), \\ \textit{and} \ \mathcal{N}_\kappa(\nabla u) \ \textit{belongs to the space} \ \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma),\end{gathered} \tag{3.3.130}$$

then $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ *belongs to* $\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$*, the conormal derivative* $\partial_\nu^A u$ *belongs to* $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$*, and the integral representation formula* (3.3.129) *once again continues to hold.*

Proof In view of the current geometric assumptions, from (3.3.126), [69, Proposition 11.13.8], and (A.0.157) we conclude that

$$u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \hookrightarrow \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M. \tag{3.3.131}$$

In particular, the first membership in (3.3.128) holds. Also, (A.0.184), (3.3.126), [68, (8.9.8)], [68, Corollary 8.9.6], [69, (6.2.3)], and [69, (6.2.5)] we see that the second membership in (3.3.128) is true as well.

Going further, we note that [69, (6.2.25)] implies

$$M^{p,\lambda}(\partial\Omega,\sigma) \hookrightarrow L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big). \tag{3.3.132}$$

From (3.3.126), (3.3.127), (3.3.131), and (3.3.132) we then see that all hypotheses of Theorem 1.8.17 are presently satisfied. As such, (1.8.176) implies (3.3.129).

Let us now replace the last condition in (3.3.126) with the assumption that $\mathcal{N}_\kappa(\nabla u)$ belongs to $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$. Then [69, Proposition 11.13.17] guarantees that the function $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ belongs to the space $\big[\mathring{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. Also, (A.0.184), [68, (8.9.8)], [68, Corollary 8.9.6], and [69, (6.2.17), (6.2.18)] we see that the conormal derivative $\partial_\nu^A u$ now belongs to $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. Finally, since $\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)$ is a subspace of $M^{p,\lambda}(\partial\Omega,\sigma)$ (cf. (A.0.149)), from what we have proved earlier we conclude that the integral representation formula (3.3.129) continues to hold in this case.

As regards the claims in the last paragraph of the statement, fix some exponent $q \in (1,\infty)$ and replace the last two lines in (3.3.126) by (3.3.130). Then [69, Proposition 11.13.13] and (A.0.38) guarantee that

$$u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in \big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M \hookrightarrow \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M. \tag{3.3.133}$$

In addition, (A.0.184), the first two lines in (3.3.126), (3.3.130), [68, (8.9.8)], [68, Corollary 8.9.6], [69, (6.2.75)], and [69, (6.2.74)] ensure that the conormal derivative $\partial_\nu^A u$ now belongs to the space $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$. Next, we note that [69, (6.2.71)] and [68, (7.7.106)] imply

$$\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \hookrightarrow L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big). \tag{3.3.134}$$

Collectively, the first two lines in (3.3.126), (3.3.130), (3.3.127), (3.3.133), and (3.3.134) imply that all hypotheses of Theorem 1.8.17 are presently satisfied. As a result, (1.8.176) follows from (3.3.129). □

Together with Theorem 1.8.17, Theorem 3.3.11 is one of the main ingredients in the proof of certain operator identities akin to those established in Theorem 1.8.26, now involving Morrey-based and block-based homogeneous Sobolev spaces.

Theorem 3.3.12 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. For some $M \in \mathbb{N}$, let $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a complex coefficient tensor with the property that*

$$L := \big(a_{rs}^{\alpha\beta}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M} \tag{3.3.135}$$

(i.e., $L := L_A$) is a weakly elliptic $M\times M$ system in $\mathbb{R}^n$. Fix some integrability exponents $p, q \in (1,\infty)$ along with a parameter $\lambda \in (0, n-1)$, then recall the operators $S_{\rm mod}$ from Theorem 3.3.5, $\partial_\nu^A\mathcal{D}_{\rm mod}$ from Theorem 3.3.6, and associate the modified boundary-to-boundary double layer potential operator $K_{\rm mod}$ with Ω and A as in (1.8.24)-(1.8.25). Finally, let $K^\#_{A^\top}$ be the operator associated with the coefficient tensor $A^\top$ and the set Ω as in (1.3.72). Then the following statements are true.

(1) For each function f belonging to either $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (in particular, to $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$; cf. [69, (6.2.15)]*), or $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$, there exists c_f, which is the nontangential trace on $\partial\Omega$ of some $\mathbb{C}^M$-valued locally constant function in Ω, with the property that*

$$S_{\rm mod}\big(K^\#_{A^\top}f\big) = K_{\rm mod}\big(S_{\rm mod}f\big) + c_f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{3.3.136}$$

In particular, if $\partial\Omega$ is connected then

$$\begin{gathered}\big[S_{\rm mod}\big]K^\#_{A^\top} = \big[K_{\rm mod}\big]\big[S_{\rm mod}\big] \ \text{ acting from either}\\ \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \quad \big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \ \text{ or } \ \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M.\end{gathered} \tag{3.3.137}$$

(2) For each function f belonging to either $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (in particular, to $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$; cf. [69, (6.2.15)]*), or $\big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$, at σ-a.e. point on $\partial\Omega$ one has*

$$\big(\tfrac12 I + K^\#_{A^\top}\big)\Big(\big(-\tfrac12 I + K^\#_{A^\top}\big)f\Big) = \big(\partial_\nu^A\mathcal{D}_{\rm mod}\big)\big(S_{\rm mod}f\big). \tag{3.3.138}$$

In particular,

$$\begin{aligned}&\big(\tfrac{1}{2}I + K^{\#}_{A^\top}\big)\big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big) = \big[\partial^A_\nu \mathcal{D}_{\rm mod}\big]\big[S_{\rm mod}\big] \textit{ acting from either}\\ &\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \quad \big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \textit{ or } \big[\mathscr{B}^{q,\lambda}(\partial\Omega,\sigma)/\sim\big]^M.\end{aligned} \tag{3.3.139}$$

(3) Strengthen the original hypotheses by now also assuming that Ω satisfies a local John condition. Then for each function f belonging to either $\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$ (hence, in particular, to the space $\big[\dot{\mathbb{M}}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$; cf. [69, (11.13.87)]*), or $\big[\dot{\mathscr{B}}^{q,\lambda}_1(\partial\Omega,\sigma)\big]^M$, there exists c_f, which is the nontangential trace on $\partial\Omega$ of some $\mathbb{C}^M$-valued locally constant function in Ω, with the property that at σ-a.e. point on $\partial\Omega$ one has*

$$\big(\tfrac{1}{2}I + K_{\rm mod}\big)\Big(\big(-\tfrac{1}{2}I + K_{\rm mod}\big)f\Big) = S_{\rm mod}\Big(\big(\partial^A_\nu \mathcal{D}_{\rm mod}\big)f\Big) + c_f. \tag{3.3.140}$$

In particular, if $\partial\Omega$ is connected then

$$\begin{aligned}&\big(\tfrac{1}{2}I + \big[K_{\rm mod}\big]\big)\big(-\tfrac{1}{2}I + \big[K_{\rm mod}\big]\big) = \big[S_{\rm mod}\big]\big[\partial^A_\nu \mathcal{D}_{\rm mod}\big] \textit{ acting from either}\\ &\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M, \quad \big[\dot{\mathbb{M}}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M, \textit{ or } \big[\dot{\mathscr{B}}^{q,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M.\end{aligned} \tag{3.3.141}$$

(4) Continue to impose the additional assumption that Ω satisfies a local John condition. Then for each function f belonging to either $\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$ (in particular, to $\big[\dot{\mathbb{M}}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M$; cf. [69, (11.13.87)]*), or $\big[\dot{\mathscr{B}}^{q,\lambda}_1(\partial\Omega,\sigma)\big]^M$, one has*

$$\big(\partial^A_\nu \mathcal{D}_{\rm mod}\big)\big(K_{\rm mod} f\big) = K^{\#}_{A^\top}\big(\partial^A_\nu \mathcal{D}_{\rm mod}\big)f \textit{ at } \sigma\textit{-a.e. point on } \partial\Omega. \tag{3.3.142}$$

In particular,

$$\begin{aligned}&\big[\partial^A_\nu \mathcal{D}_{\rm mod}\big]\big[K_{\rm mod}\big] = K^{\#}_{A^\top}\big[\partial^A_\nu \mathcal{D}_{\rm mod}\big] \textit{ acting from either}\\ &\big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M, \quad \big[\dot{\mathbb{M}}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M, \textit{ or } \big[\dot{\mathscr{B}}^{q,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M.\end{aligned} \tag{3.3.143}$$

Proof To set the stage, fix an aperture parameter $\kappa \in (0,\infty)$ along with a truncation parameter $\varepsilon \in (0,\infty)$. Consider the claims made in item *(1)*. Having pick an arbitrary function

$$f \in \big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M \hookrightarrow \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma)\Big]^M, \tag{3.3.144}$$

(cf. [69, (6.2.25)]), define $u := \mathscr{S}_{\rm mod} f$ in Ω and note that

$$\begin{aligned}
&u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega,\\
&\mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega,\sigma), \quad \mathcal{N}_\kappa^{\varepsilon} u \in L^1_{\rm loc}(\partial\Omega,\sigma),\\
&(\nabla u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega,\\
&\partial^A_\nu u = \big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f \text{ at } \sigma\text{-a.e. point on } \partial\Omega,\\
&u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = S_{\rm mod} f \in \big[M^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M \hookrightarrow \big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\big]^M,
\end{aligned} \tag{3.3.145}$$

thanks to (3.3.144), (3.3.94), (1.5.71), (3.3.5), (3.3.92), and (A.0.157). Moreover, if Ω is an exterior domain then from (1.5.51) and [70, (1.4.24)] we see that

$$⨍_{B(0,2R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{3.3.146}$$

Granted these properties, we may invoke Theorem 1.8.17 and conclude from (1.8.176) that there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω with the property that

$$u = \mathcal{D}_{\rm mod}\big(S_{\rm mod} f\big) - \mathscr{S}_{\rm mod}\big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) f + c_u \text{ in } \Omega. \tag{3.3.147}$$

Taking nontangential boundary traces in (3.3.147) and denoting $c_f := c_u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ then yields

$$S_{\rm mod} f = \big(\tfrac{1}{2}I + K_{\rm mod}\big)\big(S_{\rm mod} f\big) - S_{\rm mod}\big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) + c_f, \tag{3.3.148}$$

on account of the last line in (3.3.145), (1.8.27), (3.3.1), and the last property in (3.3.94). With (3.3.148) in hand, simple algebra then establishes (3.3.136) for arbitrary functions f belonging to $\big[M^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (hence, in particular, for arbitrary functions in $\big[\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$; cf. [69, (6.2.15)]).

The proof of (3.3.136) for arbitrary functions $f \in \big[\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ is carried out along similar lines, making use of item *(4)* in Theorem 3.3.5 (and also keeping in mind [69, Definition 11.13.11], or (A.0.38)-(A.0.40), together with the embeddings in [69, (6.2.71)] and [68, (7.7.106)]). Finally, the claims in (3.3.137) is a direct consequence of (3.3.136). This takes care of *(1)*.

Moving on, justifying (3.3.138) proceeds as above, up to (3.3.147), at which point we apply conormal derivatives to all terms involved. Thanks to (1.5.58) and Theorem 3.3.6, this ultimately proves (3.3.138) (bearing in mind that $\partial^A_\nu c_u = 0$; cf. (A.0.184)). In turn, this readily implies (3.3.139). All claims in item *(2)* are therefore established.

To deal with items *(3)*-*(4)*, work under the stronger assumptions that Ω is a UR domain satisfying a local John condition. To get going, pick an arbitrary function

$$f \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M \hookrightarrow \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma)\Big]^M, \tag{3.3.149}$$

(see (A.0.157)), and define $u := \mathcal{D}_{\text{mod}} f$ in Ω. Then u is well defined (cf. item *(1)* in Theorem 1.8.2) and satisfies:

$$
\begin{gathered}
u \in \big[\mathscr{C}^\infty(\Omega)\big]^M, \quad Lu = 0 \text{ in } \Omega, \\
\mathcal{N}_\kappa(\nabla u) \in M^{p,\lambda}(\partial\Omega,\sigma), \quad \mathcal{N}_\kappa^{\varepsilon} u \in L^p_{\text{loc}}(\partial\Omega,\sigma), \\
(\nabla u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \\
\partial_\nu^A u = \big(\partial_\nu^A \mathcal{D}_{\text{mod}}\big) f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \\
u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + K_{\text{mod}}\big) f \in \big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M \hookrightarrow \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big)\Big]^M,
\end{gathered}
\tag{3.3.150}
$$

by virtue of (3.3.149), item *(1)* in Theorem 3.3.6, item *(1)* in Theorem 3.3.8, (1.8.22), and (1.8.20). In addition, when Ω is an exterior domain we see from (3.3.149), (1.8.11), and [70, (1.4.24)] that

$$
⨍_{B(0,2R)\setminus B(0,R)} |\nabla u| \, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{3.3.151}
$$

Once the aforementioned properties have been validated, Theorem 3.3.11 applies (see (3.3.126)) and (3.3.129) presently implies that there exists some $\mathbb{C}^M$-valued locally constant function c_u in Ω with the property that

$$
u = \mathcal{D}_{\text{mod}}\big(\tfrac{1}{2}I + K_{\text{mod}}\big) f - \mathscr{S}_{\text{mod}}\big(\partial_\nu^A \mathcal{D}_{\text{mod}}\big) f + c_u \text{ in } \Omega. \tag{3.3.152}
$$

If we now set $c_f := c_u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ and go nontangentially to the boundary in (3.3.152) we arrive at

$$
\big(\tfrac{1}{2}I + K_{\text{mod}}\big) f = \big(\tfrac{1}{2}I + K_{\text{mod}}\big)\big(\tfrac{1}{2}I + K_{\text{mod}}\big) f - S_{\text{mod}}\big(\partial_\nu^A \mathcal{D}_{\text{mod}}\big) + c_f, \tag{3.3.153}
$$

after taking into account (3.3.150), (3.3.110), (3.3.103), (3.3.105), and the last property in (3.3.94). In turn, (3.3.153) readily establishes (3.3.140) for arbitrary functions f belonging to $\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ (hence, in particular, for arbitrary functions in $\big[\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\big]^M$; cf. [69, (11.13.87)]).

The version of (3.3.140) for arbitrary functions $f \in \big[\dot{\mathscr{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\big]^M$ is dealt with analogously, now relying on item *(4)* in Theorem 3.3.5, item *(4)* in Theorem 3.3.6, item *(3)* in Theorem 3.3.8, and keeping in mind [69, Definition 11.13.11] (cf. (A.0.38)-(A.0.40)). Upon noting that (3.3.141) is a direct consequence of (3.3.140), the treatment of item *(3)* is complete.

Finally, the proof of (3.3.142) proceeds as above, up to the integral representation formula (3.3.152), at which stage we now take the conormal derivatives of all terms involved. In view of Theorem 3.3.5, Theorem 3.3.6, Theorem 3.3.8, and Theorem 3.3.1, this leads to (3.3.138) (once again, keeping in mind that $\partial_\nu^A c_u = 0$; cf. (A.0.184)). This takes care of the current item *(4)*, and finishes the proof of the theorem. □

Remark 3.3.13 *It is of particular interest to consider the operator identities presented in Theorem 3.3.12 when they involve the brand of modified boundary-to-boundary double layer potential operators described in Examples 1.8.4-1.8.5. This is because the operator* $\partial_\nu^A \mathcal{D}_{\rm mod}$ *vanishes identically in these scenarios* (*as noted in the last part in Remark 1.4.6 and in* (1.4.33)). *As a result, the identity recorded in* (3.3.140) *simplifies in each of these instances.*

For example, for the modified boundary-to-boundary Cauchy-Clifford integral $\mathfrak{C}_{\rm mod}$ *defined in* (2.1.178) *we obtain from* (3.3.140) *that for each function* f *belonging to either* $\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\otimes C\ell_n$ *or* $\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)\otimes C\ell_n$ *there exists* c_f, *which is the nontangential trace on* $\partial\Omega$ *of some* $C\ell_n$*-valued locally constant function in* Ω, *with the property that*

$$\big(\tfrac{1}{2}I+\mathfrak{C}_{\rm mod}\big)\Big(\big(-\tfrac{1}{2}I+\mathfrak{C}_{\rm mod}\big)f\Big)=c_f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{3.3.154}$$

Hence, $\mathfrak{C}_{\rm mod}\big(\mathfrak{C}_{\rm mod}f\big)=\frac{1}{4}f+c_f$ *on* $\partial\Omega$. *In particular, if* $\partial\Omega$ *is connected and we let* $\big[\mathfrak{C}_{\rm mod}\big]$ *be the operator* $[f]\mapsto\big[\mathfrak{C}_{\rm mod}f\big]$ (*with brackets denoting equivalence classes modulo constants*), *we therefore obtain*

$$\big[\mathfrak{C}_{\rm mod}\big]^2=\tfrac{1}{4}I \tag{3.3.155}$$

on $\big[\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\ \big]\otimes C\ell_n$ *and* $\big[\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)/\sim\ \big]\otimes C\ell_n$. *See also* (1.8.279) *and* (2.1.186)-(2.1.189) *for related identities.*

Chapter 4
Layer Potential Operators Acting from Boundary Besov and Triebel-Lizorkin Spaces

Besov and Triebel-Lizorkin spaces on Ahlfors regular subsets of $\mathbb{R}^n$ have been discussed at length in [69, Chapter 18]. The primary goal in this chapter is to explore the mapping properties of singular integral operators of layer potential type acting from (boundary) Besov and Triebel-Lizorkin spaces considered in very broad geometric settings. Given the nature of the spaces involved, the operators in question are expected to have certain inherent cancelation properties (see, e.g., the boundedness criterion for an operator T from [32, Theorem A, p. 129] requiring that $T(1) = 0$ in a suitable sense; cf. also [22, Theorem 4.26, p. 123] in this regard), which generic Calderón-Zygmund convolution type operators typically fail to satisfy. This explains the focus on singular integral operators of *layer potential type* adopted here.

From this perspective, we find it instructive to briefly elaborate on the crucial cancelation property "$K(1) = 0$" alluded to above, for a boundary-to-boundary double layer operator K associated with a given weakly elliptic second-order $M \times M$ system and a UR domain $\Omega \subseteq \mathbb{R}^n$ with compact boundary. Using the brand of "wavelets" introduced in [22, Definition 2.2, p. 28], this ultimately boils down to checking that

$$\int_{\partial\Omega} K^{\#}\psi \, d\sigma = 0 \text{ for each } \mathbb{C}^M\text{-valued wavelet } \psi \text{ on } \partial\Omega, \tag{4.0.1}$$

where $K^{\#}$ stands for the transpose double layer and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is the "surface measure" on the topological boundary of Ω. It turns out that the size and cancelation properties of such a vector-valued wavelet ψ on $\partial\Omega$ permit us to conclude that actually ψ belongs to the Hardy space $\big[H^1(\partial\Omega, \sigma)\big]^M$. This is the manner in which the present enterprise ties up with analysis from §2.1, since Corollary 2.1.9 then guarantees that (4.0.1) holds. For the Besov scale, this ultimately proves that in the setting described above any boundary-to-boundary double layer induces a well-defined linear and bounded mapping

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$$
\begin{aligned}
&K:\big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M,\\
&p\in\big(\tfrac{n-1}{n},\infty\big],\quad q\in(0,\infty],\quad (n-1)\big(\tfrac{1}{p}-1\big)_+<s<1.
\end{aligned}
\tag{4.0.2}
$$

The next step is to use duality in order to obtain boundedness results for the transpose double layer $K^\#$ on Besov spaces $\big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M$ in the range $1<p,q<\infty$ and $0<s<1$ (where duality works). We desire to further augment this range by allowing sub-unital exponents p,q (and negative smoothness). The idea is to start from the fact that $K^\#$ is known to be bounded on the Hardy scale $[H^p(\partial\Omega,\sigma)]^M$ with $p\in\big(\frac{n-1}{n},1\big]$ and then employ [69, Theorem 7.8.21] where the envelopes of Hardy spaces (considered in the sense discussed at length in the first part of [69, §7.8]) have been identified as being Besov spaces of the nature we currently desire. This typifies the flavor of §4.1, where the Triebel-Lizorkin scale and other singular integral operators of interest are also considered.

Changing gears, in §4.2 we consider mapping properties of boundary-to-domain layer potential operators acting from Besov spaces into some suitably weighted Sobolev spaces, in the context of open sets with compact Ahlfors regular boundaries. In this regard, we first establish mapping properties for certain boundary-to-domain integral operators acting from Besov spaces with a positive amount of smoothness into Lebesgue spaces suitably weighted in terms of the distance to the boundary. See Theorem 4.2.1 and Theorem 4.2.6, where we identify the key qualities of the integral operators in question which produce desirable results of this flavor. The mapping properties thus obtained are subsequently specialized to double layer and single layer operators, and some of the main results derived in this fashion are contained in Theorem 4.2.3 and Theorem 4.2.10. See also Theorem 4.2.5 where the boundary behavior of the double layer potential operator acting from boundary Besov spaces into weighted Sobolev spaces is discussed. A program of similar aims is carried out in §4.3, this time in relation to the action of boundary-to-domain layer potential operators acting from (boundary) Besov spaces into Besov and Triebel-Lizorkin spaces (defined in open sets).

In §4.4, the final section in this chapter, we derive a number of useful integral representation formulas of layer potential type in the context of Besov and Triebel-Lizorkin spaces. For example, in Theorem 4.4.1 we prove a basic integral representation formula in such a setting, to the effect that a function which is a null-solution of a weakly elliptic system may be expressed as the action of the (boundary-to-domain) double layer operator on the boundary trace of said function and the action of the (boundary-to-domain) single layer operator on the conormal derivative of the original function. Other results of similar flavor are subsequently derived (including some in relation to first-order systems). In turn, such integral representation formulas are used to conclude a number of remarkable properties for functions in Besov and Triebel-Lizorkin spaces (in open subsets of $\mathbb{R}^n$) which are null-solutions of a weakly elliptic system.

4.1 Boundary-to-Boundary Layer Potentials from Besov and Triebel-Lizorkin Spaces into Themselves

Recall the principal-value (p.v.) singular integral operator K associated with a given a weakly elliptic system L and a UR domain $\Omega \subseteq \mathbb{R}^n$ as in (1.3.68). It turns out that thanks to its specific algebraic nature, the operator K acts naturally on the scales of Besov and Triebel-Lizorkin spaces on $\partial\Omega$. Specifically, we have the following result.

Theorem 4.1.1 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ satisfies $n \geq 2$) is a* UR *domain with compact boundary and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). In this setting, consider the boundary layer potential operator K associated with the system L and the set Ω as in* (1.3.68).

Then the operator K, originally acting on Lebesgue spaces on $\partial\Omega$ (cf. (1.5.14)*), extends uniquely to linear and bounded mappings*

$$K : \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M,$$
$$p \in \big(\tfrac{n-1}{n},\infty\big], \quad q \in (0,\infty], \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \tag{4.1.1}$$

and

$$K : \big[F^{p,q}_s(\partial\Omega,\sigma)\big]^M \longrightarrow \big[F^{p,q}_s(\partial\Omega,\sigma)\big]^M,$$
$$p \in \big(\tfrac{n-1}{n},\infty\big), \quad q \in \big(\tfrac{n-1}{n},\infty\big], \quad (n-1)\big(\tfrac{1}{\min\{p,q\}}-1\big)_+ < s < 1. \tag{4.1.2}$$

Moreover, various choices of the exponents yield operators which are compatible with one another.

Proof Since $\partial\Omega$ is compact, from [69, Proposition 7.7.2], and [69, (7.1.55), (7.7.5)] we see that

$$\big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M \hookrightarrow \big[L^1(\partial\Omega,\sigma)\big]^M \text{ whenever}$$
$$p \in \big(\tfrac{n-1}{n},\infty\big], \quad q \in (0,\infty], \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \tag{4.1.3}$$

and

$$\big[F^{p,q}_s(\partial\Omega,\sigma)\big]^M \hookrightarrow \big[L^1(\partial\Omega,\sigma)\big]^M \text{ whenever}$$
$$p \in \big(\tfrac{n-1}{n},\infty\big), \quad q \in \big(\tfrac{n-1}{n},\infty\big], \quad (n-1)\big(\tfrac{1}{\min\{p,q\}}-1\big)_+ < s < 1. \tag{4.1.4}$$

As a consequence of these embeddings and [70, Theorem 2.3.2], it follows that K, originally defined as a principal-value singular integral operator on the Lebesgue space $\big[L^1(\partial\Omega,\sigma)\big]^M$ (cf. [70, (2.3.19)]), acts on a meaningful manner on functions belonging to the family of Besov and Triebel-Lizorkin spaces described in (4.1.1)-(4.1.2).

To see that actually K induces well-defined, linear, and bounded mappings in such settings, the idea is to check that the one-sided $T(1)$ theorem from [32, Theorem A, p. 129] (cf. also [33], [22, Theorem 4.26, p. 123], [56], and [21]), which guarantees the boundedness of singular integral operators on Besov and Triebel-Lizorkin spaces considered in the abstract setting of spaces of homogeneous type, is presently applicable. For starters, from (1.3.68) and Theorem 1.5.1 we see that K is a Calderón-Zygmund operator in the sense of [22, Definition 1.9, p. 14], whose kernel satisfies conditions (1.3)-(1.5) on p. 126 of [32] (and also conditions (1.10)-(1.11) in [22, Definition 1.8, p. 14]). In addition, as noted in the comment following [22, Definition 1.15, p. 19], the fact that K is bounded on the scale of Lebesgue spaces (cf. Theorem 1.5.1) guarantees that K satisfies the weak boundedness property (cf. [22, Definition 1.8, p. 14]). As such, matters are reduced to checking that $K(1) = 0$. The latter cancelation condition should be suitably understood. Indeed, according to the comment following [22, Theorem 4.27, p. 123] (cf. also item *(iii)* of Theorem 1.5.1), what this ultimately boils down to is having

$$\int_{\partial\Omega} K^{\#}\psi \, d\sigma = 0 \ \text{ for each } \mathbb{C}^M\text{-valued wavelet } \psi \text{ on } \partial\Omega. \tag{4.1.5}$$

The notion of wavelet referred to above is introduced in [22, Definition 2.2, p. 28]. An inspection of this definition (as well as [22, Definition 2.1, p. 27]) reveals that wavelets are test functions in the sense of [69, Definition 7.1.1] satisfying a vanishing moment condition. That is, $\psi : \partial\Omega \to \mathbb{C}^M$ has the property that there exist two numbers $\gamma \in (0,\infty)$, $\beta \in (0,1]$, along with a point $x_0 \in \partial\Omega$ and a radius $r \in (0,\infty)$ such that (for some constant $C \in (0,\infty)$)

$$|\psi(x)| \leq C \frac{r^{\gamma}}{\big(r + |x - x_0|\big)^{n-1+\gamma}} \ \text{ for each } \ x \in \partial\Omega, \tag{4.1.6}$$

$$\begin{aligned} &|\psi(x) - \psi(y)| \leq C \frac{r^{\gamma}|x-y|^{\beta}}{\big(r + |x - x_0|\big)^{n-1+\gamma+\beta}} \\ &\quad \text{whenever } x, y \in \partial\Omega \text{ are such that } |x-y| < (r + |x - x_0|)/4, \end{aligned} \tag{4.1.7}$$

and

$$\int_{\partial\Omega} \psi \, d\sigma = 0. \tag{4.1.8}$$

The claim that we make in relation to such a function ψ is that, irrespective of whether $\partial\Omega$ is bounded,

$$\psi \in \big[H^1(\partial\Omega, \sigma)\big]^M. \tag{4.1.9}$$

In turn, this may be justified by invoking [69, Corollary 4.5.3] with $p := 1$. Specifically, choosing

$$q \in (1, \infty) \text{ arbitrary and } d \in \big((n-1)(q-1), (n-1)(q-1) + q\gamma\big) \tag{4.1.10}$$

ensures (in light of (4.1.6) and [68, (7.2.5)]) that $\psi \in \big[L^q\big(\partial\Omega, (1 + |\cdot - x_0|^d)\sigma\big)\big]^M$. Granted this membership and the cancelation condition (4.1.8), [69, Corollary 4.5.3] applies and gives (4.1.9). Having proved (4.1.9), we may now rely on Corollary 2.1.9 together with (4.1.8) to conclude that we have $\int_{\partial\Omega} K^{\#}\psi \, d\sigma = 0$.

Having established (4.1.5), the door is open for employing the one-sided $T(1)$ theorem from [32, Theorem A, p. 129]. This gives that K is well defined, linear, and bounded both in the context of (4.1.1) and (4.1.2). Finally, that various versions of K in (4.1.1)-(4.1.2) are compatible with one another (as well as with other earlier versions of K considered so far) is clear from (4.1.3)-(4.1.4). □

Remark 4.1.2 *If in place of* [32, Theorem A, p. 129] *we use the one-sided $T(1)$ theorem from* [22, Theorem 4.26, p. 123] *then the same type of argument as in the proof of Theorem 4.1.1 gives that whenever $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain with unbounded boundary then K also acts naturally on homogeneous Besov and Triebel-Lizorkin spaces on the boundary, i.e.,*

$$\begin{gathered} K : \big[\dot{B}^{p,q}_s(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\dot{B}^{p,q}_s(\partial\Omega, \sigma)\big]^M, \\ p \in \big(\tfrac{n-1}{n}, \infty\big), \quad q \in (0, \infty), \quad (n-1)\big(\tfrac{1}{p} - 1\big)_+ < s < 1, \end{gathered} \tag{4.1.11}$$

and

$$\begin{gathered} K : \big[\dot{F}^{p,2}_s(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\dot{F}^{p,2}_s(\partial\Omega, \sigma)\big]^M, \\ p \in \big(\tfrac{n-1}{n}, \infty\big), \quad (n-1)\big(\tfrac{1}{\min\{p,2\}} - 1\big)_+ < s < 1. \end{gathered} \tag{4.1.12}$$

are well-defined, linear, bounded operators, which act in a compatible fashion with one another.

As commented at length on earlier occasions, there is a large variety of double layer operators associated with a given weakly elliptic system, and Theorem 4.1.1 is applicable to all such operators. A concrete case involves the boundary-to-boundary Cauchy-Clifford integral operator $\mathfrak{C}$ defined in (A.0.54). Indeed, as noted in Example 1.4.12 this is a special case of a double layer (associated with the Laplacian). In light of this, Theorem 4.1.1 implies that whenever $\Omega \subseteq \mathbb{R}^n$ is a UR domain with compact boundary and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ then

$$\begin{gathered} \mathfrak{C} : B^{p,q}_s(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow B^{p,q}_s(\partial\Omega, \sigma) \otimes C\ell_n, \\ p \in \big(\tfrac{n-1}{n}, \infty\big], \quad q \in (0, \infty], \quad (n-1)\big(\tfrac{1}{p} - 1\big)_+ < s < 1, \end{gathered} \tag{4.1.13}$$

and

$$\begin{gathered} \mathfrak{C} : F^{p,q}_s(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow F^{p,q}_s(\partial\Omega, \sigma) \otimes C\ell_n, \\ p \in \big(\tfrac{n-1}{n}, \infty\big), \quad q \in \big(\tfrac{n-1}{n}, \infty\big], \quad (n-1)\big(\tfrac{1}{\min\{p,q\}} - 1\big)_+ < s < 1, \end{gathered} \tag{4.1.14}$$

are well-defined, linear, and bounded operators. In addition, from [70, (2.5.332)], (4.1.13)-(4.1.14), and [69, Lemma 7.1.10] it follows that

$$\begin{gathered}\mathfrak{C}^2 = \tfrac{1}{4} I \text{ on } B^{p,q}_s(\partial\Omega,\sigma) \otimes C\ell_n \text{ when}\\ p \in \big(\tfrac{n-1}{n},\infty\big), \quad q \in (0,\infty], \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1,\end{gathered} \tag{4.1.15}$$

and

$$\begin{gathered}\mathfrak{C}^2 = \tfrac{1}{4} I \text{ on } F^{p,q}_s(\partial\Omega,\sigma) \otimes C\ell_n \text{ when}\\ p \in \big(\tfrac{n-1}{n},\infty\big), \quad q \in \big(\tfrac{n-1}{n},\infty\big], \quad (n-1)\big(\tfrac{1}{\min\{p,q\}}-1\big)_+ < s < 1,\end{gathered} \tag{4.1.16}$$

first when $q < \infty$, then the end-point $q = \infty$ is included as a result of this and loose embeddings (cf. items *(iii)*-*(iv)* in [69, Proposition 7.7.1]).

Mapping properties analogous to (4.1.13)-(4.1.14) also hold for the boundary-to-boundary version of the ordinary Cauchy operator in the complex plane, and for the boundary-to-boundary Bochner-Martinelli integral operator in the context of several complex variables (cf. Example 1.4.9 and Example 1.4.16). Finally, there are also results in the spirit of Remark 4.1.2 for all these operators.

Another type of singular integral operator which acts naturally on boundary Besov and Triebel-Lizorkin spaces with a positive amount of smoothness is discussed in the theorem below.

Theorem 4.1.3 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a* UR *domain with compact boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_j)_{1\leq j\leq n}$ the geometric measure theoretic outward unit normal to Ω. Next, let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n,\mathcal{L}^n)$ be a complex-valued function with the property that $b\big|_{\mathbb{R}^n\setminus\{0\}} \in \mathscr{C}^N(\mathbb{R}^n\setminus\{0\})$ where $N = N(n) \in \mathbb{N}$ is a sufficiently large number, and such that ∇b is odd and positive homogeneous of degree $1-n$ in $\mathbb{R}^n\setminus\{0\}$. Finally, for each pair of indices $j,k \in \{1,\dots,n\}$ introduce the integral operator acting on each function $f \in L^1(\partial\Omega,\sigma)$ according to*

$$T_{jk} f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big\{\nu_j(y)(\partial_k b)(x-y) - \nu_k(y)(\partial_j b)(x-y)\big\} f(y)\,\mathrm{d}\sigma(y), \tag{4.1.17}$$

for σ-a.e. $x \in \partial\Omega$. Then the operator T_{jk} extends uniquely to linear and bounded mappings

$$\begin{gathered}T_{jk} : B^{p,q}_s(\partial\Omega,\sigma) \longrightarrow B^{p,q}_s(\partial\Omega,\sigma),\\ p \in \big(\tfrac{n-1}{n},\infty\big), \quad q \in (0,\infty), \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1,\end{gathered} \tag{4.1.18}$$

and

$$
\begin{aligned}
&T_{jk} : F^{p,q}_s(\partial\Omega,\sigma) \longrightarrow F^{p,q}_s(\partial\Omega,\sigma),\\
&p,q \in \big(\tfrac{n-1}{n},\infty\big), \quad (n-1)\big(\tfrac{1}{\min\{p,q\}}-1\big)_+ < s < 1.
\end{aligned}
\tag{4.1.19}
$$

Furthermore, different choices of the exponents yield operators which are compatible with one another.

Proof Given any $\ell \in \{1,\dots,n\}$ define

$$
\begin{aligned}
&\mathscr{B}_\ell f(x) := \int_{\partial\Omega} (\partial_\ell b)(x-y) f(y)\,\mathrm{d}\sigma(y) \text{ at each } x\in\Omega,\\
&\qquad\text{for every function } f\in L^1(\partial\Omega,\sigma).
\end{aligned}
\tag{4.1.20}
$$

Also, fix $\kappa\in(0,\infty)$ along with $\ell\in\{1,\dots,n\}$. Then for each $f\in L^1(\partial\Omega,\sigma)$ we have

$$
\begin{aligned}
\Big(\mathscr{B}_\ell f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) &= \frac{1}{2\mathrm{i}}\widehat{\partial_\ell b}\big(\nu(x)\big) f(x)\\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} (\partial_\ell b)(x-y) f(y)\,\mathrm{d}\sigma(y)
\end{aligned}
\tag{4.1.21}
$$

at σ-a.e. point $x\in\partial\Omega$. For each $j,k\in\{1,\dots,n\}$ we may then rely on (4.1.21), (1.2.18), and (1.2.3) to compute

$$
\begin{aligned}
&\nu_j(x)\Big(\mathscr{B}_k f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) - \nu_k(x)\Big(\mathscr{B}_j f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x)\\
&\quad = \frac{1}{2\mathrm{i}}\nu_j(x)\widehat{\partial_k b}\big(\nu(x)\big) f(x) + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_j(x)(\partial_k b)(x-y) f(y)\,\mathrm{d}\sigma(y)\\
&\qquad - \frac{1}{2\mathrm{i}}\nu_k(x)\widehat{\partial_j b}\big(\nu(x)\big) f(x) - \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \nu_k(x)(\partial_j b)(x-y) f(y)\,\mathrm{d}\sigma(y)\\
&\quad = (T^{\#}_{jk} f)(x) \text{ at } \sigma\text{-a.e. point } x\in\partial\Omega.
\end{aligned}
\tag{4.1.22}
$$

To proceed, fix some $\psi\in H^1(\partial\Omega,\sigma)$ along with $j,k\in\{1,\dots,n\}$ and define the vector field

$$
\vec{F} := \big(\mathscr{B}_k\psi\big)\mathbf{e}_j - \big(\mathscr{B}_j\psi\big)\mathbf{e}_k \text{ in } \Omega. \tag{4.1.23}
$$

Then from (4.1.23), [70, (2.4.14)] (with $p := 1$), and (4.1.22) we see that

$$
\begin{aligned}
&\vec{F}\in\big[\mathscr{C}^1(\Omega)\big]^n, \quad \operatorname{div}\vec{F} = 0 \text{ in } \Omega, \quad \mathcal{N}_\kappa\vec{F}\in L^1(\partial\Omega,\sigma),\\
&\qquad \text{and } \vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = T^{\#}_{jk}\psi \text{ at } \sigma\text{-a.e. point } x\in\partial\Omega.
\end{aligned}
\tag{4.1.24}
$$

In the case when Ω is not an exterior domain [68, Theorem 1.2.1] applies and the Divergence Formula recorded in [68, (1.2.2)] presently gives

$$\int_{\partial\Omega} T^{\#}_{jk}\psi \, d\sigma = \int_{\partial\Omega} \nu \cdot \big(\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big) \, d\sigma = \int_{\Omega} \operatorname{div}\vec{F} \, d\mathcal{L}^n = 0, \tag{4.1.25}$$

hence, in this case,

$$\int_{\partial\Omega} T^{\#}_{jk}\psi \, d\sigma = 0. \tag{4.1.26}$$

When Ω is an exterior domain, then working with the bounded UR domain $\mathbb{R}^n \setminus \overline{\Omega}$ in place of Ω (cf. item *(7)* in [68, Lemma 5.10.9]) and once again arrive at the same conclusion as in (4.1.26).

Together with item *(ii)* in Proposition 1.2.1, the cancelation property established in (4.1.26) then permits us to invoke the one-sided $T(1)$ theorem [32, Theorem A, p. 129] (cf. also [22, Theorem 4.26, p. 123]) to conclude, much as in the proof of Theorem 4.1.1, that T_{jk} induces well-defined, linear, and continuous mappings (which act in a coherent fashion) in the context of both (4.1.18) and (4.1.19). □

We now proceed to study the action of the "transpose" double layer $K^{\#}$ (originally defined on Lebesgue spaces in (1.3.72)) on Besov and Triebel-Lizorkin scales defined on boundaries of UR domains. The staring point the following partial result obtained from Theorem 4.1.1 and duality. Given the format of [69, Proposition 7.6.1], for now we restricting ourselves to portions of the aforementioned scales consisting of Banach spaces.

Corollary 4.1.4 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain with compact boundary and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a homogeneous, weakly elliptic, constant (complex) coefficient, second-order $M \times M$ system in $\mathbb{R}^n$ (for some $M \in \mathbb{N}$). In this context, consider the boundary layer potential operator $K^{\#}$ associated with the system L and the set Ω as in* (1.3.72)*.*

Then the operator $K^{\#}$, originally acting on Lebesgue spaces on $\partial\Omega$ (cf. (1.5.12)*), extends uniquely to linear and bounded mappings*

$$\begin{gathered} K^{\#} : \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M, \\ p \in (1,\infty], \quad q \in (0,\infty], \quad s \in (0,1), \end{gathered} \tag{4.1.27}$$

and

$$\begin{gathered} K^{\#} : \big[F^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[F^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M, \\ p \in (1,\infty), \quad q \in (1,\infty], \quad s \in (0,1). \end{gathered} \tag{4.1.28}$$

In addition, various choices of the parameters p, q, s yield operators which are compatible with one another. Finally, if the exponents $p, q, p', q' \in (1,\infty)$ satisfy $1/p + 1/p' = 1 = 1/q + 1/q'$ and $s \in (0,1)$, then

$$
\begin{aligned}
&{}_{[B^{p,q}_{-s}(\partial\Omega,\sigma)]^M}\big\langle K^\# f, g\big\rangle_{[B^{p',q'}_{s}(\partial\Omega,\sigma)]^M} = {}_{[B^{p,q}_{-s}(\partial\Omega,\sigma)]^M}\big\langle f, Kg\big\rangle_{[B^{p',q'}_{s}(\partial\Omega,\sigma)]^M} \\
&\qquad \textit{for each } f \in \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M \textit{ and } g \in \big[B^{p',q'}_{s}(\partial\Omega,\sigma)\big]^M,
\end{aligned}
\tag{4.1.29}
$$

and

$$
\begin{aligned}
&{}_{[F^{p,q}_{-s}(\partial\Omega,\sigma)]^M}\big\langle K^\# f, g\big\rangle_{[F^{p',q'}_{s}(\partial\Omega,\sigma)]^M} = {}_{[F^{p,q}_{-s}(\partial\Omega,\sigma)]^M}\big\langle f, Kg\big\rangle_{[F^{p',q'}_{s}(\partial\Omega,\sigma)]^M} \\
&\qquad \textit{for each } f \in \big[F^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M \textit{ and } g \in \big[F^{p',q'}_{s}(\partial\Omega,\sigma)\big]^M.
\end{aligned}
\tag{4.1.30}
$$

Proof First, in the case when $q \in (1,\infty]$, the claimed mapping properties are consequence of Theorem 4.1.1 and [69, Proposition 7.6.1] (also bearing in mind [69, Lemma 7.1.10] and item *(iii)* in Theorem 1.5.1). Second, the extension to $q \in (0,\infty]$ in (4.1.27) is achieved by invoking the real interpolation result from [69, (7.4.2)]. Finally, (4.1.29)-(4.1.30) are seen from what we have proved so far, item *(iii)* in Theorem 1.5.1, [69, (7.1.62)], and Theorem 4.1.1. □

In our next theorem we further augment the results from Corollary 4.1.4 by now allowing sub-unital integrability exponents.

Theorem 4.1.5 *Let* $\Omega \subseteq \mathbb{R}^n$ (*where* $n \in \mathbb{N}$, $n \geq 2$) *be a* UR *domain with compact boundary and set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Consider a homogeneous, weakly elliptic, constant* (*complex*) *coefficient, second-order* $M \times M$ *system* L *in* $\mathbb{R}^n$ (*for some* $M \in \mathbb{N}$), *and bring in the boundary layer potential operator* $K^\#$ *associated with the system* L *and the set* Ω *as in* (1.3.72).

Then the operator $K^\#$, *originally considered as in Corollary 4.1.4, further extends, in a unique fashion, to linear and bounded mappings*

$$
\begin{aligned}
&K^\# : \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M \\
&\textit{provided } s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s},\infty\big], \quad q \in (0,\infty],
\end{aligned}
\tag{4.1.31}
$$

and

$$
\begin{aligned}
&K^\# : \big[F^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[F^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M, \\
&\textit{provided } s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s},\infty\big), \quad q \in \big(\tfrac{n-1}{n-s},\infty\big].
\end{aligned}
\tag{4.1.32}
$$

Moreover, various choices of the parameters p, q, s *yield operators which are compatible with one another.*

Proof Let us first observe that whenever s, p, q are as in (4.1.31) the lower bound on p implies $s + (n-1)\big(\frac{1}{p} - 1\big) < 1$, hence we may choose some exponent $\widetilde{p} \in (1,\infty)$ with the property that

$$
\widetilde{s} := s + (n-1)\Big(\frac{1}{p} - \frac{1}{\widetilde{p}}\Big) < 1. \tag{4.1.33}
$$

As such, we may invoke [69, Theorem 7.7.4] to conclude that, with $\widetilde{q} \in (1, \infty)$ arbitrary, we have a continuous embedding

$$\big[B^{p,q}_{-s}(\partial\Omega, \sigma)\big]^M \hookrightarrow \big[B^{\widetilde{p},\widetilde{q}}_{-\widetilde{s}}(\partial\Omega, \sigma)\big]^M. \tag{4.1.34}$$

Since from (4.1.27) in Corollary 4.1.4 we already know that $K^\#$ is well defined on the larger space $\big[B^{\widetilde{p},\widetilde{q}}_{-\widetilde{s}}(\partial\Omega, \sigma)\big]^M$ it follows that it is meaningful to consider $K^\# f$ whenever $f \in \big[B^{p,q}_{-s}(\partial\Omega, \sigma)\big]^M$ with s, p, q as in (4.1.31). In particular, $K^\#$ stays compatible with its earlier version considered in Corollary 4.1.4.

Suppose now that two arbitrary parameters $s^* \in (0, 1)$ and $p^* \in \big(\frac{n-1}{n-s^*}, 1\big]$ have been given. Define

$$p := \big(\tfrac{1}{p^*} + \tfrac{s^*}{n-1}\big)^{-1} \tag{4.1.35}$$

and note that this entails

$$\tfrac{n-1}{n} < p < p^* \leq 1 \text{ and } s^* = (n-1)\big(\tfrac{1}{p} - \tfrac{1}{p^*}\big). \tag{4.1.36}$$

Granted this, [69, (7.8.153)] applies and yields

$$\mathcal{E}_{p^*}\Big(\big[H^p(\partial\Omega, \sigma)\big]^M\Big) = \big[B^{p^*,p^*}_{-s^*}(\partial\Omega, \sigma)\big]^M. \tag{4.1.37}$$

In turn, from [69, (7.1.60)], (4.1.37), (2.1.4) in Theorem 2.1.1, and [69, (7.8.56) in Proposition 7.8.9] we conclude that

whenever $s^* \in (0, 1)$ and $p^* \in \big(\frac{n-1}{n-s^*}, 1\big]$, it follows that the operator $K^\# : \big[F^{p,2}_0(\partial\Omega, \sigma)\big]^M \to \big[F^{p,2}_0(\partial\Omega, \sigma)\big]^M$, with $p \in \big(\frac{n-1}{n}, 1\big)$ defined as in (4.1.35), extends (in a unique fashion) to a linear and bounded mapping $\widehat{K^\#} : \big[B^{p^*,p^*}_{-s^*}(\partial\Omega, \sigma)\big]^M \to \big[B^{p^*,p^*}_{-s^*}(\partial\Omega, \sigma)\big]^M$. (4.1.38)

From embeddings (cf. (4.1.34)) we also see that the various extensions $\widehat{K^\#}$ described in (4.1.38) act in a coherent fashion with one another, and they are also compatible with $K^\#$ from Corollary 4.1.4. For this reason, we agree to drop the "hat", and simply refer to $\widehat{K^\#}$ as $K^\#$.

Consider now $s \in (0, 1)$, $p \in \big(\frac{n-1}{n-s}, 1\big]$, and $q \in (0, \infty]$. Choose $s^*_0 \in (0, s)$ and $s^*_1 \in (s, 1)$ close enough to s so that

$$\max\big\{\tfrac{n-1}{n-s^*_0}, \tfrac{n-1}{n-s^*_1}\big\} < p, \tag{4.1.39}$$

then pick $\theta \in (0, 1)$ such that $s = (1-\theta)s^*_0 + \theta s^*_1$. Finally, define $p^* := p$. Then, according to the real interpolation result from [69, (7.4.2)] (presently used with $p := p^*$, $q_0 := p^*$, $q_1 := p^*$, $s_0 := -s^*_0$, $s_1 := -s^*_1$) we have

$$\Big(\big[B^{p^*,p^*}_{-s^*_0}(\partial\Omega, \sigma)\big]^M, \big[B^{p^*,p^*}_{-s^*_1}(\partial\Omega, \sigma)\big]^M\Big)_{\theta,q} = \big[B^{p,q}_{-s}(\partial\Omega, \sigma)\big]^M. \tag{4.1.40}$$

In concert, (4.1.38), [69, Proposition 1.3.7], and (4.1.40) then prove that the operator $K^{\#}$ is bounded in the context of

$$K^{\#} : \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^M, \\ s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s}, 1\big], \quad q \in (0,\infty]. \tag{4.1.41}$$

Another proof of the fact that the operator $K^{\#}$ is bounded in the context of (4.1.41) is as follows. Start with an arbitrary $f \in \big[B^{p,p}_{-s}(\partial\Omega,\sigma)\big]^M$, with $s \in (0,1)$ and $p \in \big(\frac{n-1}{n-s}, 1\big]$, then invoke [69, Corollary 7.2.9] to decompose it as

$$f = g + \sum_{j\in\mathbb{N}} \lambda_j a_j \text{ in } \big[B^{p,p}_{-s}(\partial\Omega,\sigma)\big]^M, \tag{4.1.42}$$

for some function

$$g \in \big[B^{p,p}_{-s}(\partial\Omega,\sigma)\big]^M \cap \Big(\bigcap_{1\leq p^*\leq\infty} \big[L^{p^*}(\partial\Omega,\sigma)\big]^M\Big) \tag{4.1.43}$$

satisfying

$$\|g\|_{[B^{p,p}_{-s}(\partial\Omega,\sigma)]^M} + \sup_{1\leq p^*\leq\infty} \|g\|_{[L^{p^*}(\partial\Omega,\sigma)]^M} \leq C\|f\|_{\big[B^{p,p}_{-s}(\partial\Omega,\sigma)\big]^M} \tag{4.1.44}$$

and for some family $\{a_j\}_{j\in\mathbb{N}}$ of η-smooth atoms of type $(p,-s)$ on $\partial\Omega$ along with some numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}} \in \ell^p$ satisfying

$$\Big(\sum_{j\in\mathbb{N}} |\lambda_j|^p\Big)^{1/p} \leq C\|f\|_{[B^{p,p}_{-s}(\partial\Omega,\sigma)]^M}. \tag{4.1.45}$$

From [69, (7.2.18)] and (2.1.8) we then conclude that, for each $j \in \mathbb{N}$,

the function $m_j := K^{\#}a_j$ is a fixed multiple of a $\mathbb{C}^M$-valued (p_o, q_o, ε)-molecule on $\partial\Omega$ (cf. [69, Definition 4.5.1]), if we take $p_o := \big(\frac{1}{p} + \frac{s}{n-1}\big)^{-1} \in \big(\frac{n-1}{n}, 1\big)$, $q_o \in [1,\infty]$, and $\varepsilon := \frac{1}{n-1}$. (4.1.46)

In turn, (4.1.46) and [69, (7.2.16)] imply that

for each $j \in \mathbb{N}$, the function $m_j := K^{\#}a_j$ is a fixed multiple of some $\mathbb{C}^M$-valued $(p, q_o, \varepsilon, 1-s)$-rough molecule (in the sense of [69, Definition 7.2.3]). (4.1.47)

Combining (4.1.47) with [69, Theorem 7.2.4] then proves (on account of (4.1.34) and (4.1.27) in Corollary 4.1.4) that

$$\begin{aligned}\Big\|K^{\#}\Big(\sum_{j\in\mathbb{N}}\lambda_j a_j\Big)\Big\|_{[B^{p,p}_{-s}(\partial\Omega,\sigma)]^M} &= \Big\|\sum_{j\in\mathbb{N}}\lambda_j K^{\#}a_j\Big\|_{[B^{p,p}_{-s}(\partial\Omega,\sigma)]^M}\\ &\le \Big(\sum_{j\in\mathbb{N}}|\lambda_j|^p\Big)^{1/p}\\ &\le C\|f\|_{[B^{p,p}_{-s}(\partial\Omega,\sigma)]^M},\end{aligned} \tag{4.1.48}$$

where the last inequality is provided by (4.1.45). From this, (4.1.42)-(4.1.44), (1.5.12), and [69, (7.7.8)] we then deduce that $K^{\#}$ maps $\big[B^{p,p}_{-s}(\partial\Omega,\sigma)\big]^M$ boundedly into itself whenever $s\in(0,1)$ and $p\in\big(\frac{n-1}{n-s},1\big]$. Via real interpolation (cf. [69, Theorem 7.4.1]) we then conclude that the operator in (4.1.41) is indeed well defined, linear, and bounded.

At this stage, the claims pertaining to (4.1.31) now become consequences of (4.1.27) and (4.1.41).

Alternatively, we could have established the boundedness of $K^{\#}$ in the context of (4.1.31) by relying on the one-sided $T(1)$ theorem from [32, Theorem B, p. 129]. More specifically, the latter result requires several hypotheses which guarantee the desired boundedness result. First, we need $K^{\#}$ to satisfy $T^*(1)=0$. This is a condition amounting to $K(1)=0$, which has been already checked in the course of the proof of Theorem 4.1.1. Second, we need $K^{\#}$ to satisfy the weak boundedness property (aka WBP). This follows from the fact that the operator in question is bounded on $\big[L^2(\partial\Omega,\sigma)\big]^M$ (cf. item *(iii)* in Theorem 1.5.1). Finally, $K^{\#}$ is required to have an integral kernel satisfying the estimates demanded in [32, (1.3), (1.4), and (1.6), p. 126], and this is clear from (1.3.72) and [70, Theorem 1.4.2] (bearing in mind that we are currently assuming $\partial\Omega$ to be compact).

In fact, the one-sided $T(1)$ theorem from [32, Theorem B, p. 129] also gives boundedness results on the scale of Triebel-Lizorkin spaces in the setting of spaces of homogeneous type, for a class of operators to which $K^{\#}$ belongs (as just noted above). When specialized to the present setting, this theorem then establishes the boundedness of $K^{\#}$ in the context of (4.1.32). □

Remark 4.1.6 Much as Theorem 4.1.1 has given (4.1.13)-(4.1.14), Theorem 4.1.5 applies to the "transpose" principal-value Cauchy-Clifford operator $\mathfrak{C}^{\#}$ (defined in (1.6.1)) and shows that, whenever $\Omega\subseteq\mathbb{R}^n$ is a UR domain with compact boundary and $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$, then

$$\begin{gathered}\mathfrak{C}^{\#}: B^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n\longrightarrow B^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n,\\ s\in(0,1),\quad p\in\big(\tfrac{n-1}{n-s},\infty\big],\quad q\in(0,\infty],\end{gathered} \tag{4.1.49}$$

and

$$\begin{gathered}\mathfrak{C}^{\#}: F^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n\longrightarrow F^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n,\\ s\in(0,1),\quad p\in\big(\tfrac{n-1}{n-s},\infty\big),\quad q\in\big(\tfrac{n-1}{n-s},\infty\big],\end{gathered} \tag{4.1.50}$$

are well-defined, linear, and bounded operators. Moreover, whenever the exponents $p, q, p', q' \in (1, \infty)$ satisfy $1/p + 1/p' = 1 = 1/q + 1/q'$ and $s \in (0, 1)$, then (compare with item *(ii)* in Proposition 1.6.1)

$$\begin{gathered} {}_{B^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n}\big\langle \mathfrak{C}^{\#} f, g\big\rangle_{B^{p',q'}_{s}(\partial\Omega,\sigma)\otimes C\ell_n} = {}_{B^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n}\big\langle f, \mathfrak{C} g\big\rangle_{B^{p',q'}_{s}(\partial\Omega,\sigma)\otimes C\ell_n} \\ \text{for each } f \in B^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n \text{ and } g \in B^{p',q'}_{s}(\partial\Omega,\sigma)\otimes C\ell_n, \end{gathered} \tag{4.1.51}$$

and

$$\begin{gathered} {}_{F^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n}\big\langle \mathfrak{C}^{\#} f, g\big\rangle_{F^{p',q'}_{s}(\partial\Omega,\sigma)\otimes C\ell_n} = {}_{F^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n}\big\langle f, \mathfrak{C} g\big\rangle_{F^{p',q'}_{s}(\partial\Omega,\sigma)\otimes C\ell_n} \\ \text{for each } f \in F^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n \text{ and } g \in F^{p',q'}_{s}(\partial\Omega,\sigma)\otimes C\ell_n. \end{gathered} \tag{4.1.52}$$

Finally, from (1.6.8), (4.1.49)-(4.1.50), and [69, Lemma 7.1.10] it follows that

$$\begin{gathered} (\mathfrak{C}^{\#})^2 = \tfrac{1}{4} I \text{ on } B^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n \text{ when} \\ s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s}, \infty\big), \quad q \in (0, \infty], \end{gathered} \tag{4.1.53}$$

and

$$\begin{gathered} (\mathfrak{C}^{\#})^2 = \tfrac{1}{4} I \text{ on } F^{p,q}_{-s}(\partial\Omega,\sigma)\otimes C\ell_n \text{ when} \\ s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s}, \infty\big), \quad q \in \big(\tfrac{n-1}{n-s}, \infty\big], \end{gathered} \tag{4.1.54}$$

first when $q < \infty$, then the end-point $q = \infty$ is included as a result of this and loose embeddings (cf. items *(iii)-(iv)* in [69, Proposition 7.7.1]).

Here is another category of singular integral operators which behave naturally on boundary Besov and Triebel-Lizorkin spaces with a negative amount of smoothness.

Theorem 4.1.7 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a* UR *domain with compact boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_j)_{1\leq j\leq n}$ the geometric measure theoretic outward unit normal to Ω. Next, let $b \in L^1_{\rm loc}(\mathbb{R}^n, \mathcal{L}^n)$ be a complex-valued function with the property that $b\big|_{\mathbb{R}^n\setminus\{0\}} \in \mathscr{C}^N(\mathbb{R}^n\setminus\{0\})$ where $N = N(n) \in \mathbb{N}$ is a sufficiently large number, and such that ∇b is odd and positive homogeneous of degree $1-n$ in $\mathbb{R}^n\setminus\{0\}$. Finally, for each given pair of indices $j, k \in \{1, \dots, n\}$ introduce the integral operator acting on each function $f \in L^1(\partial\Omega, \sigma)$ according to*

$$T^{\#}_{jk} f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \big\{\nu_j(x)(\partial_k b)(x-y) - \nu_k(x)(\partial_j b)(x-y)\big\} f(y)\,\mathrm{d}\sigma(y), \tag{4.1.55}$$

at σ-a.e. $x \in \partial\Omega$. Then the operator $T^{\#}_{jk}$ extends uniquely to linear and bounded mappings

$$T^{\#}_{jk} : B^{p,q}_{-s}(\partial\Omega, \sigma) \longrightarrow B^{p,q}_{-s}(\partial\Omega, \sigma),$$
$$s \in (0, 1), \quad p \in \left(\tfrac{n-1}{n-s}, \infty\right], \quad q \in (0, \infty], \tag{4.1.56}$$

and

$$T^{\#}_{jk} : F^{p,q}_{-s}(\partial\Omega, \sigma) \longrightarrow F^{p,q}_{-s}(\partial\Omega, \sigma),$$
$$s \in (0, 1), \quad p \in \left(\tfrac{n-1}{n-s}, \infty\right), \quad q \in \left(\tfrac{n-1}{n-s}, \infty\right]. \tag{4.1.57}$$

Also, various choices of the exponents yield operators which are compatible with one another. Finally, if $p, q, p', q' \in (1, \infty)$ *satisfy* $1/p + 1/p' = 1 = 1/q + 1/q'$ *and* $s \in (0, 1)$*, then*

$${}_{B^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle T^{\#}_{jk} f, g\big\rangle_{B^{p',q'}_{s}(\partial\Omega,\sigma)} = -\,{}_{B^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle f, T_{jk} g\big\rangle_{B^{p',q'}_{s}(\partial\Omega,\sigma)}$$
$$\textit{for each } f \in B^{p,q}_{-s}(\partial\Omega, \sigma) \textit{ and } g \in B^{p',q'}_{s}(\partial\Omega, \sigma), \tag{4.1.58}$$

and

$${}_{F^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle T^{\#}_{jk} f, g\big\rangle_{F^{p',q'}_{s}(\partial\Omega,\sigma)} = -\,{}_{F^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle f, T_{jk} g\big\rangle_{F^{p',q'}_{s}(\partial\Omega,\sigma)}$$
$$\textit{for each } f \in F^{p,q}_{-s}(\partial\Omega, \sigma) \textit{ and } g \in F^{p',q'}_{s}(\partial\Omega, \sigma). \tag{4.1.59}$$

Proof The claims about (4.1.56)-(4.1.57) may be justified with the help of Theorem 4.1.3, item *(ii)* in Proposition 1.2.1, and Theorem 2.1.4 much in the same way Theorem 4.1.5 has been proved making use of Theorem 4.1.1, item *(iii)* in Theorem 1.5.1, and Theorem 2.1.1. Finally, (4.1.58)-(4.1.59) are seen from what we have proved so far, Theorem 4.1.3, item *(ii)* in Proposition 1.2.1, and [69, (7.1.62)]. □

4.2 Boundary-to-Domain Layer Potentials from Besov Spaces into Weighted Sobolev Spaces

In this section we take up the task of studying mapping properties of boundary-to-domain layer potential operators acting from Besov spaces into weighted Sobolev spaces.

For starters, we shall prove mapping properties for certain boundary-to-domain integral operators acting from Besov spaces with a positive amount of smoothness into Lebesgue spaces suitably weighted in terms of the distance to the boundary. Below, we shall establish a general result of this nature which identifies the key qualities of said integral operator which produce a desirable result of this flavor. Subsequently, we shall specialize this to a number of concrete integral operators of practical interest. For now, the reader is reminded that, having fixed background parameter $\theta \in (0, 1)$, the solid maximal function $u_{\star,\theta}$ is associated with any given function u as in [68, §6.6]. Also, recall that for each number $a \in \mathbb{R}$ we use the abbreviation $(a)_+ := \max\{a, 0\}$.

Theorem 4.2.1 *Let Ω be an open set in $\mathbb{R}^n$, $n \geq 2$, whose boundary is an Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, consider a measurable function $q : \Omega \times \partial\Omega \to \mathbb{C}$ with the property that there exist $\varepsilon \in [0, 1)$, $k \in \mathbb{R}$, and some finite constant $c_o > 0$ such that*

$$|q(x, y)| \leq c_o \delta_{\partial\Omega}(x)^{-k} \Big(\frac{\delta_{\partial\Omega}(x)}{|x - y|}\Big)^{-\varepsilon} \frac{1}{|x - y|^n} \quad \text{for every point } x \in \Omega \text{ and } \sigma\text{-a.e. point } y \in \partial\Omega. \tag{4.2.1}$$

Associated with it, consider the integral operator acting on each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-\varepsilon}}\big)$ according to

$$\mathcal{Q}f(x) := \int_{\partial\Omega} q(x, y) f(y)\, d\sigma(y) \ \text{for} \ x \in \Omega, \tag{4.2.2}$$

and assume that

$$\mathcal{Q} \text{ annihilates constants.} \tag{4.2.3}$$

Then, if

$$\tfrac{n-1}{n-\varepsilon} < p \leq \infty \ \text{and} \ (n - 1)\big(\tfrac{1}{p} - 1\big)_+ < s < 1 - \varepsilon, \tag{4.2.4}$$

it follows that for each $\theta \in (0, 1)$ there exists some $C = C(\Omega, \varepsilon, k, c_o, p, s, \theta) > 0$ such that

$$\Big\| \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \big|\mathcal{Q}f\big|_{\star,\theta} \Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{B^{p,p}_s(\partial\Omega,\sigma)}, \quad \forall f \in B^{p,p}_s(\partial\Omega, \sigma), \tag{4.2.5}$$

with the understanding that when $p > 1$ the solid maximal function $\big|\mathcal{Q}f\big|_{\star,\theta}$ is replaced by $\big|\mathcal{Q}f\big|$.

A few remarks are in order. First, from (4.2.1) it follows that

$$|q(x, y)| \leq \frac{c_o \delta_{\partial\Omega}(x)^{-k-\varepsilon}}{|x - y|^{n-\varepsilon}} \ \text{for all} \ x \in \Omega \ \text{and} \ y \in \partial\Omega, \tag{4.2.6}$$

which, in concert with [68, Lemma 7.2.1], implies that

> the operator $\mathcal{Q}$ acts on functions from the weighted Lebesgue space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-\varepsilon}}\big)$ meaningfully (via an absolutely convergent integral). (4.2.7)

In particular, it follows that the action of $\mathcal{Q}$ on constants functions is meaningful, hence it makes sense to impose the demand in (4.2.3).

Second, from [69, Proposition 7.9.4], [69, (7.1.59)], and [68, (7.4.119)], we see that

$$B^{p,p}_s(\partial\Omega, \sigma) \hookrightarrow L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-\varepsilon}}\big) \quad \text{if } \varepsilon \in [0, 1) \text{ and } p, s \text{ are as in (4.2.4).} \tag{4.2.8}$$

Together with (4.2.7) this shows that $\mathcal{Q}$ acts in a meaningful fashion on the Besov spaces considered in Theorem 4.2.1.

Third, [68, (6.6.6) in Lemma 6.6.1] ensures that $\big|\mathcal{Q}f\big|_{\star,\theta}$ is a Lebesgue measurable function in Ω, which bounds $|\mathcal{Q}f\big|$ at $\mathcal{L}^n$-a.e. point in Ω. As a consequence of this and (4.2.5) we therefore have (with p, s as in (4.2.4))

$$\Big\|\delta_{\partial\Omega}^{k+1-\frac{1}{p}-s}\cdot\mathcal{Q}f\Big\|_{L^p(\Omega,\mathcal{L}^n)}\leq C\|f\|_{B^{p,p}_s(\partial\Omega,\sigma)},\quad\forall f\in B^{p,p}_s(\partial\Omega,\sigma).\tag{4.2.9}$$

Fourth, an inspection of the proof reveals that the same type of result is true in the case when the kernel q is matrix-valued, and the functions f are vector-valued (with the understanding that the definition of $\mathcal{Q}$ in (4.2.2) now takes into account the natural action of the matrix $q(x, y)$ on the vector $f(y)$).

We now turn to the task of presenting the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1 Throughout, fix $\theta\in(0,1)$. We shall show that whenever p, s are as in (4.2.4) there exists a constant $C\in(0,\infty)$ such that for each function $f\in B^{p,p}_s(\partial\Omega,\sigma)$ we have

$$\Big\|\delta_{\partial\Omega}^{k+1-\frac{1}{p}-s}\big|\mathcal{Q}f\big|_{\star,\theta}\Big\|_{L^p(\Omega,\mathcal{L}^n)}\leq C\|f\|_{B^{p,p}_s(\partial\Omega,\sigma)},\tag{4.2.10}$$

with the convention that when $p>1$ the solid maximal function $|\mathcal{Q}f\big|_{\star,\theta}$ is simply replaced by $|\mathcal{Q}f\big|$. Consider first the case $p=1$, in which scenario we shall prove that there exists some finite $C>0$ such that if $0<s<1-\varepsilon$ then

$$\Big\|\delta_{\partial\Omega}^{k-s}\big|\mathcal{Q}f\big|_{\star,\theta}\Big\|_{L^1(\Omega,\mathcal{L}^n)}\leq C\|f\|_{B^{1,1}_s(\partial\Omega,\sigma)},\quad\forall f\in B^{1,1}_s(\partial\Omega,\sigma).\tag{4.2.11}$$

To justify this, fix an arbitrary function $f\in B^{1,1}_s(\partial\Omega,\sigma)$. Thanks to (4.2.3) we may write

$$(\mathcal{Q}f)(x)=\int_{\partial\Omega}q(x,y)(f(y)-f(z))\,d\sigma(y)\ \text{ for all }\ x\in\Omega,\ \ z\in\partial\Omega.\tag{4.2.12}$$

Combining (4.2.12) with (4.2.1) further yields

$$\big|(\mathcal{Q}f)(x)\big|\leq c_o\delta_{\partial\Omega}(x)^{-k-\varepsilon}\int_{\partial\Omega}|x-y|^{-n+\varepsilon}|f(y)-f(z)|\,d\sigma(y),\tag{4.2.13}$$

for all $x\in\Omega$ and $z\in\partial\Omega$. Next, fix $c>1$ and for each $x\in\Omega$ define the set

$$E_{x,c}:=\big\{z\in\partial\Omega:\ |x-z|<c\delta_{\partial\Omega}(x)\big\}.\tag{4.2.14}$$

Also, consider $x^*\in\partial\Omega$ with the property that $|x-x^*|=\delta_{\partial\Omega}(x)$. Then for each $z\in E_{x,c}$ we have $|z-x^*|\leq|x-z|+|x-x^*|<(c+1)\delta_{\partial\Omega}(x)$. Moreover, if $0<\lambda<c-1$, then for every $z\in\partial\Omega$ satisfying $|z-x^*|<\lambda\delta_{\partial\Omega}(x)$ we may estimate $|x-z|\leq|x-x^*|+|x^*-z|<c\delta_{\partial\Omega}(x)$. As such,

$$B\big(x^*,\lambda\delta_{\partial\Omega}(x)\big)\cap\partial\Omega\subseteq E_{x,c}\subseteq B\big(x^*,(c+1)\delta_{\partial\Omega}(x)\big)\cap\partial\Omega.\tag{4.2.15}$$

It is also apparent from (4.2.14) that $E_{x,c}$ is a relatively open subset of $\partial\Omega$, hence σ-measurable. On account of this, (4.2.15), and the Ahlfors regularity of $\partial\Omega$, we conclude that

$$\sigma(E_{x,c}) \approx \delta_{\partial\Omega}(x)^{n-1}. \tag{4.2.16}$$

Having established this, we proceed to take the integral average with respect to σ over the set $E_{x,c}$ of both sides of (4.2.13). In doing so, and relying on (4.2.16), we arrive at the conclusion that, for each $x \in \Omega$,

$$\big|(\mathcal{Q}f)(x)\big| \leq C\delta_{\partial\Omega}(x)^{-n+1-k-\varepsilon} \int_{E_{x,c}} \int_{\partial\Omega} \frac{|f(y)-f(z)|}{|x-y|^{n-\varepsilon}}\, d\sigma(y)\, d\sigma(z). \tag{4.2.17}$$

Pressing on, pick an arbitrary point $x^0 \in \Omega$. Elementary considerations show that

$$x \in B\big(x^0, \theta \cdot \delta_{\partial\Omega}(x^0)\big) \Longrightarrow \begin{cases} (1-\theta)\delta_{\partial\Omega}(x^0) \leq \delta_{\partial\Omega}(x) \leq (1+\theta)\delta_{\partial\Omega}(x^0), \\ \text{and} \\ E_{x,c} \subseteq E_{x^0,c_0} \ \text{ where } \ c_0 := c(1+\theta)+\theta. \end{cases} \tag{4.2.18}$$

Furthermore,

$$x \in B\big(x^0, \theta \cdot \delta_{\partial\Omega}(x^0)\big) \Longrightarrow |x-y| \approx |x^0-y| \ \text{ uniformly for } \ y \in \partial\Omega. \tag{4.2.19}$$

Consequently, from (4.2.18)-(4.2.19) and (4.2.17) we deduce that for each point $x \in B\big(x^0, \theta \cdot \delta_{\partial\Omega}(x^0)\big)$ we have

$$\big|(\mathcal{Q}f)(x)\big| \leq C\delta_{\partial\Omega}(x^0)^{-n+1-k-\varepsilon} \int_{E_{x^0,c_0}} \int_{\partial\Omega} \frac{|f(y)-f(z)|}{|x^0-y|^{n-\varepsilon}}\, d\sigma(y)\, d\sigma(z). \tag{4.2.20}$$

Taking the supremum over $x \in B\big(x^0, \theta \cdot \delta_{\partial\Omega}(x^0)\big)$ then gives (keeping in mind (A.0.194))

$$\big|(\mathcal{Q}f)\big|_{\star,\theta}(x^0) \leq C\delta_{\partial\Omega}(x^0)^{-n+1-k-\varepsilon} \int_{E_{x^0,c_0}} \int_{\partial\Omega} \frac{|f(y)-f(z)|}{|x^0-y|^{n-\varepsilon}}\, d\sigma(y)\, d\sigma(z), \tag{4.2.21}$$

for every $x^0 \in \Omega$. Hence, if we re-denote x^0 by x, then multiply both sides of (4.2.21) by $\delta_{\partial\Omega}(x)^{k-s}$ and, finally, integrate the resulting expressions over Ω with respect to x, we obtain (after an application of Fubini's Theorem)

$$\int_{\Omega} \delta_{\partial\Omega}(x)^{k-s} \big| \mathcal{Q} f \big|_{\star,\theta}(x)\,\mathrm{d}x \tag{4.2.22}$$
$$\leq C \int_{\Omega} \delta_{\partial\Omega}(x)^{1-n-s-\varepsilon} \int_{E_{x,c_0}} \int_{\partial\Omega} \frac{|f(y) - f(z)|}{|x-y|^{n-\varepsilon}}\,\mathrm{d}\sigma(y)\,\mathrm{d}\sigma(z)\,\mathrm{d}x$$
$$= C \int_{\partial\Omega} \int_{\partial\Omega} |f(y) - f(z)| \Big(\int_{\Gamma_\kappa(z)} \frac{\delta_{\partial\Omega}(x)^{1-n-s-\varepsilon}}{|x-y|^{n-\varepsilon}}\,\mathrm{d}x \Big)\,\mathrm{d}\sigma(y)\,\mathrm{d}\sigma(z),$$

where $\kappa := c_0 - 1 > 0$ and

$$\Gamma_\kappa(z) := \big\{ x \in \Omega : \ |x - z| < (1+\kappa)\delta_{\partial\Omega}(x) \big\}. \tag{4.2.23}$$

At this point we invoke [68, Proposition 8.7.8] with $\Sigma := \partial\Omega$, $\beta := -1 + n + s + \varepsilon$, and $M := n - \varepsilon$ (note that, since $p = 1$ and $0 < s < 1 - \varepsilon$, these choices entail $\beta < n$ and $M > n - \beta$, as required by the hypotheses of said proposition). As a consequence of this result, the integral over $\Gamma_\kappa(z)$ from (4.2.22) is bounded by $C|z - y|^{-(n-1+s)}$. The latter used back in (4.2.22) then yields (4.2.11), upon recalling [69, (7.9.11)].

Now we turn our attention to the case when $p = \infty$ and $0 < s < 1 - \varepsilon$. The goal is to show that there exists a constant $C \in (0, \infty)$ with the property that for each $f \in B^{\infty,\infty}_s(\partial\Omega, \sigma)$ we have

$$\Big\| \delta^{k+1-s}_{\partial\Omega} \big| \mathcal{Q} f \big| \Big\|_{L^\infty(\Omega, \mathcal{L}^n)} \leq C \|f\|_{B^{\infty,\infty}_s(\partial\Omega,\sigma)}. \tag{4.2.24}$$

To this end, fix $f \in B^{\infty,\infty}_s(\partial\Omega, \sigma)$. Also, let $x \in \Omega$ be arbitrary and, again, denote by $x^* \in \partial\Omega$ a point with the property that $|x - x^*| = \delta_{\partial\Omega}(x)$. Then, by once more appealing to (4.2.3), we may estimate

$$\delta_{\partial\Omega}(x)^{k+1-s} \big| \mathcal{Q} f(x) \big| \leq \delta_{\partial\Omega}(x)^{k+1-s} \int_{\partial\Omega} |q(x,y)||f(y) - f(x^*)|\,\mathrm{d}\sigma(y). \tag{4.2.25}$$

Since $f \in B^{\infty,\infty}_s(\partial\Omega, \sigma) = \mathscr{C}^s(\partial\Omega)$ (recall [69, (7.1.59)]), we have

$$|f(y) - f(x^*)| \leq \|f\|_{B^{\infty,\infty}_s(\partial\Omega,\sigma)} |y - x^*|^s, \qquad \forall\, y \in \partial\Omega. \tag{4.2.26}$$

To proceed, set $r := |x - x^*| = \delta_{\partial\Omega}(x)$ and bound the right-hand side of (4.2.25) by $I_1 + I_2$, where I_1, I_2 correspond to taking $y \in B(x^*, 2r) \cap \partial\Omega$ and $y \in \partial\Omega \setminus B(x^*, 2r)$, respectively, in the integral in (4.2.25). Making use of (4.2.26), (4.2.1), and the fact that if $y \in B(x^*, 2r) \cap \partial\Omega$ then $|x - y| \geq r$ hence also $|y - x^*| \leq 2r \leq 2|x - y|$, we obtain

$$I_1 \leq C \|f\|_{B^{\infty,\infty}_s(\partial\Omega,\sigma)} r^{k+1-s} \int_{B(x^*,2r)\cap\partial\Omega} \frac{|y - x^*|^s}{r^{k+\varepsilon}|x-y|^{n-\varepsilon}}\,\mathrm{d}\sigma(y)$$
$$\leq C \|f\|_{B^{\infty,\infty}_s(\partial\Omega,\sigma)} r^{-n+1} \sigma\big(B(x^*, 2r) \cap \partial\Omega\big) \leq C \|f\|_{B^{\infty,\infty}_s(\partial\Omega,\sigma)}, \tag{4.2.27}$$

where in the last inequality in (4.2.27) we have used the Ahlfors regularity of $\partial\Omega$. Turning our attention to I_2, observe that if the point $y \in \partial\Omega \setminus B(x^*, 2r)$ then we have $|y - x^*| \le |x - y| + r \le |x - y| + \frac{1}{2}|y - x^*|$ which further implies $|y - x^*| \le 2|x - y|$. Thus, starting again with (4.2.26) and (4.2.1), we may now write

$$
\begin{aligned}
I_2 &\le C\|f\|_{B^{\infty,\infty}_s(\partial\Omega,\sigma)} r^{k+1-s} \int_{\partial\Omega\setminus B(x^*,2r)} \frac{|y - x^*|^s}{r^{k+\varepsilon}|x - y|^{n-\varepsilon}}\, d\sigma(y) \\
&\le C\|f\|_{B^{\infty,\infty}_s(\partial\Omega,\sigma)} r^{1-s-\varepsilon} \int_{\partial\Omega\setminus B(x^*,2r)} \frac{d\sigma(y)}{|y - x^*|^{n-s-\varepsilon}} \\
&\le C\|f\|_{B^{\infty,\infty}_s(\partial\Omega,\sigma)},
\end{aligned}
\tag{4.2.28}
$$

thanks to the first estimate in [68, (7.2.5)] (whose applicability is ensured by the fact that we presently have $n - s - \varepsilon > n - 1$). Now (4.2.24) follows by combining (4.2.27) and (4.2.28). This completes the treatment of the case when $p = \infty$ and $0 < s < 1 - \varepsilon$.

To treat the case when $1 < p < \infty$, the idea is to use what we have proved so far and interpolation. More precisely, assume $0 < s_0 < s_1 < 1 - \varepsilon$ and consider the family of linear operators

$$
L_z := \delta_{\partial\Omega}^{k+z-[(1-z)s_0+zs_1]} Q \ \text{ for } z \in \mathbb{C} \text{ with } 0 \le \operatorname{Re} z \le 1. \tag{4.2.29}
$$

Observe that L_z depends analytically on z and

$$
\operatorname{Re} z = 0 \Longrightarrow |L_z f| = \delta_{\partial\Omega}^{k-s_0}|Qf|, \tag{4.2.30}
$$

$$
\operatorname{Re} z = 1 \Longrightarrow |L_z f| = \delta_{\partial\Omega}^{k+1-s_1}|Qf|. \tag{4.2.31}
$$

Based on this observation, our results for $p = 1$ and $p = \infty$ (the former in concert with [68, (6.6.6) in Lemma 6.6.1]) lead to the conclusion that the operators

$$
L_z : B^{1,1}_{s_0}(\partial\Omega, \sigma) \longrightarrow L^1(\Omega, \mathcal{L}^n) \ \text{ for } \operatorname{Re} z = 0, \tag{4.2.32}
$$

$$
L_z : B^{\infty,\infty}_{s_1}(\partial\Omega, \sigma) \longrightarrow L^\infty(\Omega, \mathcal{L}^n) \ \text{ for } \operatorname{Re} z = 1, \tag{4.2.33}
$$

are well-defined, linear, and bounded. Granted these, the complex interpolation results from [69, Theorem 7.5.2] may be used in concert with Stein's interpolation theorem for analytic families of operators (see, e.g., [11, Theorem 3.4, pp. 151-152] for a versatile variant). This allows us to conclude that the operator

$$
\delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} Q : B^{p,p}_s(\partial\Omega, \sigma) \longrightarrow L^p(\Omega, \mathcal{L}^n) \tag{4.2.34}
$$

is well defined, linear, and bounded for each $s \in (0, 1 - \varepsilon)$ and $p \in [1, \infty]$.

Alternatively, we may reach the same conclusions regarding the operator (4.2.34) using the real method of interpolation. Specifically, given any $s \in (0, 1 - \varepsilon)$, from (4.2.11) and [68, Lemma 6.6.1] we know that the operator

$$\delta_{\partial\Omega}^{k+1-s} Q : B_s^{1,1}(\partial\Omega, \sigma) \longrightarrow L^1\big(\Omega, \delta_{\partial\Omega}^{-1}\mathcal{L}^n\big) \tag{4.2.35}$$

is well defined, linear, and bounded, whereas (4.2.24) implies that we also have a well-defined, linear, and bounded operator

$$\delta_{\partial\Omega}^{k+1-s} Q : B_s^{\infty,\infty}(\partial\Omega, \sigma) \longrightarrow L^\infty(\Omega, \mathcal{L}^n) = L^\infty\big(\Omega, \delta_{\partial\Omega}^{-1}\mathcal{L}^n\big) \tag{4.2.36}$$

(with the above equality a consequence of the fact that the measures $\mathcal{L}^n$ and $\delta_{\partial\Omega}^{-1}\mathcal{L}^n$ are mutually absolutely continuous in Ω). Then from (4.2.35)-(4.2.36), the real interpolation result from [69, (7.4.18)] (used with $\Sigma := \partial\Omega$, $p_0 := 1$, $p_1 := \infty$), and well-known real interpolation results for generic Lebesgue spaces (cf., e.g., [2, Theorem 5.2.1, p. 109]; see also [26] for useful results regarding the real interpolation of generic weighted Lebesgue spaces) we conclude that the operator

$$\delta_{\partial\Omega}^{k+1-s} Q : B_s^{p,p}(\partial\Omega, \sigma) \longrightarrow L^p\big(\Omega, \delta_{\partial\Omega}^{-1}\mathcal{L}^n\big) \tag{4.2.37}$$

is well defined, linear, and bounded whenever $1 \le p \le \infty$ and $0 < s < 1 - \varepsilon$. A moment's refection shows that this is equivalent to having the operator (4.2.34) well-defined, linear, and bounded for each $s \in (0, 1 - \varepsilon)$ and $p \in [1, \infty]$. In turn, this amounts to having the estimate claimed (4.2.10) valid in the range $1 \le p \le \infty$ and $0 < s < 1 - \varepsilon$ (with the agreement that when $p > 1$ the subscripts $\star, \theta$ are omitted). This finishes the proof of estimate (4.2.10) in the range $1 \le p \le \infty$ and $0 < s < 1-\varepsilon$ (again, with the understanding that when $p > 1$ the subscripts $\star, \theta$ are omitted).

We are left analyzing the case when $\frac{n-1}{n-\varepsilon} < p < 1$ and $(n-1)\big(\frac{1}{p}-1\big) < s < 1-\varepsilon$. In this scenario, we shall first prove that for each fixed $\eta \in (s, 1)$ there exists some finite constant $C > 0$ such that

$$\Big\| \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \big|Qa\big|_{\star,\theta} \Big\|_{L^p(\Omega, \mathcal{L}^n)} \le C \text{ for each smooth } \eta\text{-block } a \text{ of type } (p, s) \tag{4.2.38}$$

(cf. [69, Definition 7.2.1]). To this end, fix k and η as above, and assume that

$$\|a\|_{L^\infty(\partial\Omega,\sigma)} \le r^{s-\frac{n-1}{p}} \text{ for some } r > 0, \tag{4.2.39}$$

$$\operatorname{supp} a \subseteq B(x_a, r) \cap \partial\Omega \text{ for some } x_a \in \partial\Omega, \tag{4.2.40}$$

$$|a(x) - a(y)| \le r^{s-\frac{n-1}{p}-\eta}|x - y|^\eta \text{ for all } x, y \in \partial\Omega. \tag{4.2.41}$$

For $\tau \in \mathbb{R}$ to be specified later, consider the re-normalization

$$\widetilde{a}(x) := r^\tau a(x) \text{ for each } x \in \partial\Omega. \tag{4.2.42}$$

For now, assume that $\tau + s - (n-1)\big(\frac{1}{p} - 1\big) \in (0, 1)$, i.e.,

$$(n-1)\big(\tfrac{1}{p} - 1\big) - s < \tau < 1 + (n-1)\big(\tfrac{1}{p} - 1\big) - s. \tag{4.2.43}$$

Then, using the properties of a it is easy to check that $\widetilde{a}$ satisfies conditions (4.2.39)-(4.2.41) with s replaced by $\tau + s - (n-1)\big(\frac{1}{p} - 1\big)$ and with p replaced by 1. Thus, $\widetilde{a}$ is a smooth η-block of type $\big(1, \tau + s - (n-1)\big(\frac{1}{p} - 1\big)\big)$ if we also assume

$$\eta > \tau + s - (n-1)\big(\tfrac{1}{p} - 1\big). \tag{4.2.44}$$

Granted this, [69, Theorem 7.2.8] ensures that

$$\widetilde{a} \in B^{1,1}_{\tau+s-(n-1)(\frac{1}{p}-1)}(\partial\Omega, \sigma) \text{ and } \|\widetilde{a}\|_{B^{1,1}_{\tau+s-(n-1)(\frac{1}{p}-1)}(\partial\Omega,\sigma)} \leq C, \tag{4.2.45}$$

for some finite constant $C > 0$ independent of a. Under the additional assumption that

$$\tau < 1 - \varepsilon + (n-1)\big(\tfrac{1}{p} - 1\big) - s, \tag{4.2.46}$$

this allows us to use the bounds already proved for the operator $\mathcal{Q}$ corresponding to $p = 1$ in order to obtain that

$$\Big\|\delta_{\partial\Omega}^{k-\tau-s+(n-1)(\frac{1}{p}-1)}\big|\mathcal{Q}\widetilde{a}\big|_{\star,\theta}\Big\|_{L^1(\Omega,\mathcal{L}^n)} \leq C\|\widetilde{a}\|_{B^{1,1}_{\tau+s-(n-1)(\frac{1}{p}-1)}(\partial\Omega,\sigma)} \leq C. \tag{4.2.47}$$

Recall that $\theta \in (0,1)$ has been fixed at the beginning of the proof. Applying Hölder's inequality and using (4.2.47) we may then estimate

$$\begin{aligned}
&\int_{B(x_a,2(1-\theta)^{-1}r)\cap\Omega} \Big(\delta_{\partial\Omega}(x)^{k+1-\frac{1}{p}-s}\big|\mathcal{Q}a\big|_{\star,\theta}(x)\Big)^p \, \mathrm{d}x \\
&\quad = r^{-\tau p} \int_{B(x_a,2(1-\theta)^{-1}r)\cap\Omega} \delta_{\partial\Omega}(x)^{kp+p-1-sp}\big|\mathcal{Q}\widetilde{a}\big|^p_{\star,\theta}(x)\,\mathrm{d}x \\
&\quad \leq r^{-\tau p}\Big(\int_{B(x_a,2(1-\theta)^{-1}r)\cap\Omega} \delta_{\partial\Omega}(x)^{k-\tau-s+(n-1)\left(\frac{1}{p}-1\right)}\big|\mathcal{Q}\widetilde{a}\big|_{\star}(x)\,\mathrm{d}x\Big)^p \times \\
&\qquad\qquad \times \Big(\int_{B(x_a,2(1-\theta)^{-1}r)\cap\Omega} \delta_{\partial\Omega}(x)^{\frac{\tau p}{1-p}-n}\,\mathrm{d}x\Big)^{1-p} \\
&\quad \leq C\, r^{-\tau p}\Big(\int_{B(x_a,2(1-\theta)^{-1}r)\cap\Omega} \delta_{\partial\Omega}(x)^{\frac{\tau p}{1-p}-n}\,\mathrm{d}x\Big)^{1-p}.
\end{aligned} \tag{4.2.48}$$

If we further impose the condition

$$(n-1)\big(\tfrac{1}{p} - 1\big) < \tau, \tag{4.2.49}$$

we may then apply [68, (8.7.3)] with $\alpha := n - \frac{\tau p}{1-p}$ and $N := 0$ (also, with $\Sigma := \partial\Omega$, $y := x_a$, and r replaced by $2(1-\theta)^{-1}r$) to conclude that

$$\int_{B(x_a, 2(1-\theta)^{-1}r)\cap\Omega} \delta_{\partial\Omega}(x)^{\frac{\tau p}{1-p}-n}\,\mathrm{d}x \le Cr^{\frac{\tau p}{1-p}}. \tag{4.2.50}$$

The key observation now is that conditions (4.2.43), (4.2.46), (4.2.44), and (4.2.49) may be simultaneously satisfied for a suitable choice of τ, since $0 < s < 1-\varepsilon$ and $s < \eta < 1$. Having chosen such a number τ, a combination of (4.2.48), (4.2.50), and (4.2.47) yields

$$\int_{B(x_a, 2(1-\theta)^{-1}r)\cap\Omega} \Big(\delta_{\partial\Omega}(x)^{k+1-\frac{1}{p}-s}\big|Qa\big|_{\star,\theta}(x)\Big)^p\,\mathrm{d}x \le C, \tag{4.2.51}$$

for some finite constant $C > 0$ which is independent of the atom in question.

Next, we turn our attention to the contribution away from the support of the atom. Using (4.2.39), (4.2.40), and (4.2.1) we see that

$$\begin{aligned}\big|(Qa)(x)\big| &\le C\int_{B(x_a,r)\cap\partial\Omega}\frac{|a(y)|}{\delta_{\partial\Omega}(x)^{k+\varepsilon}|x-y|^{n-\varepsilon}}\,\mathrm{d}\sigma(y)\\ &\le \frac{C\,r^{s+(n-1)(1-\frac{1}{p})}}{\delta_{\partial\Omega}(x)^{k+\varepsilon}|x-x_a|^{n-\varepsilon}} \ \text{ if } x\in\Omega\setminus B(x_a,2r).\end{aligned} \tag{4.2.52}$$

To proceed, observe that

$$x^0\in\Omega\setminus B\big(x_a,2(1-\theta)^{-1}r\big) \text{ and } x\in B\big(x^0,\theta\cdot\delta_{\partial\Omega}(x^0)\big) \Longrightarrow |x-x_a|\ge 2r. \tag{4.2.53}$$

Indeed, the second membership forces $\delta_{\partial\Omega}(x) > (1-\theta)\delta_{\partial\Omega}(x^0)$. Hence, in the case in which we have $\delta_{\partial\Omega}(x^0) \ge 2(1-\theta)^{-1}r$ we may write

$$|x-x_a| \ge \delta_{\partial\Omega}(x) > (1-\theta)\delta_{\partial\Omega}(x^0) \ge 2r, \tag{4.2.54}$$

as wanted. On the other hand, if $\delta_{\partial\Omega}(x^0) < 2(1-\theta)^{-1}r$ then

$$\begin{aligned}|x-x_a| &\ge |x^0-x_a| - |x-x^0| \ge 2(1-\theta)^{-1}r - \theta\cdot\delta_{\partial\Omega}(x^0)\\ &> \tfrac{2}{1-\theta}r - \tfrac{2\theta}{1-\theta}r = 2r,\end{aligned} \tag{4.2.55}$$

finishing the justification of (4.2.53). In turn, from (4.2.53), (4.2.52), (A.0.194), and (4.2.18)-(4.2.19) we obtain

$$\big|Qa\big|_{\star,\theta}(x^0) \le \frac{C\,r^{s+(n-1)(1-\frac{1}{p})}}{\delta_{\partial\Omega}(x^0)^{k+\varepsilon}|x^0-x_a|^{n-\varepsilon}} \ \text{ if } x^0\in\Omega\setminus B\big(x_a,2(1-\theta)^{-1}r\big). \tag{4.2.56}$$

At this point we find it convenient to revert to denoting x^0 by x, and use (4.2.56) and [68, (8.7.5) in Proposition 8.7.1] with $\alpha := 1 + p(s-1+\varepsilon)$ and $N := p(n-\varepsilon)$ (note that $\alpha < 1$ since $s < 1-\varepsilon$, while $\alpha > n-N$ since $s > (n-1)(1/p-1)$) in order to conclude that

$$\int_{\Omega\setminus B(x_a,2(1-\theta)^{-1}r)} \delta_{\partial\Omega}(x)^{(k+1-\frac{1}{p}-s)p}\,\big|\mathcal{Q}a\big|^p_{\star,\theta}(x)\,\mathrm{d}x \leq C. \tag{4.2.57}$$

Now estimate (4.2.38) follows from (4.2.51) and (4.2.57).

Going further, fix $\frac{n-1}{n-\varepsilon} < p < 1$ along with $(n-1)\big(\frac{1}{p}-1\big) < s < 1-\varepsilon$ and consider some arbitrary $f \in B^{p,p}_s(\partial\Omega,\sigma)$. Then [69, Theorem 7.2.7] ensures the existence of a sequence $\{a_j\}_{j\in\mathbb{N}}$ of smooth η-blocks of type (p,s) on $\partial\Omega$ along with a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ belonging to ℓ^p (a space with which $b^{p,p}(\partial\Omega)$ identifies; cf. [69, (7.2.24) in Definition 7.2.5]) with the property that

$$\Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{1/p} \leq C\|f\|_{B^{p,p}_s(\partial\Omega,\sigma)} < +\infty, \tag{4.2.58}$$

for some constant $C \in (0,\infty)$ independent of f, and such that if

$$f_m := \sum_{j=1}^{m}\lambda_j a_j \ \text{ for each } \ m \in \mathbb{N} \tag{4.2.59}$$

then

$$f_m \longrightarrow f \ \text{ in } \ B^{p,p}_s(\partial\Omega,\sigma) \ \text{ as } \ m \to \infty. \tag{4.2.60}$$

Let us also observe that if we set

$$s_* := s-(n-1)\big(\tfrac{1}{p}-1\big) \in (0,s) \subseteq (0,1-\varepsilon), \tag{4.2.61}$$

then $0 < s_* < s < 1$ and $\frac{1}{p}-\frac{s}{n-1} = 1-\frac{s_*}{n-1}$. As such, [69, Theorem 7.7.4] (presently employed with $\Sigma := \partial\Omega$) gives

$$B^{p,p}_s(\partial\Omega,\sigma) \hookrightarrow B^{1,1}_{s_*}(\partial\Omega,\sigma) \ \text{ continuously.} \tag{4.2.62}$$

In particular, from (4.2.60) and (4.2.62) we conclude that

$$f_m \longrightarrow f \ \text{ in } \ B^{1,1}_{s_*}(\partial\Omega,\sigma) \ \text{ as } \ m \to \infty. \tag{4.2.63}$$

To proceed, abbreviate

$$F := \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s}\,(\mathcal{Q}f)_{\star,\theta} \ \text{ and} \tag{4.2.64}$$

$$F_m := \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s}\,(\mathcal{Q}f_m)_{\star,\theta} \ \text{ for each } \ m \in \mathbb{N}. \tag{4.2.65}$$

In relation to these functions we make several remarks. First, based on (4.2.65), (4.2.59), [68, (6.6.4), (6.6.9)], (4.2.38), and (4.2.58), for each $m \in \mathbb{N}$ we may estimate (bearing in mind that we are presently assuming $0 < p < 1$)

$$\|F_m\|^p_{L^p(\Omega,\mathcal{L}^n)} \leq \sum_{j=1}^{m} |\lambda_j|^p \Big\| \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \big|\mathcal{Q} a_j\big|_{\star,\theta} \Big\|^p_{L^p(\Omega,\mathcal{L}^n)}$$

$$\leq C \sum_{j=1}^{m} |\lambda_j|^p \leq C\|f\|^p_{B^{p,p}_s(\partial\Omega,\sigma)}, \tag{4.2.66}$$

for some constant $C \in (0,\infty)$ which is independent of f and m. Likewise, given any $m', m'' \in \mathbb{N}$ with $m' \leq m''$, we may use (4.2.65), [68, (6.6.4), (6.6.10)], (4.2.59), and (4.2.38) to estimate (again, mindful of the fact that $0 < p < 1$)

$$\big\|F_{m'} - F_{m''}\big\|^p_{L^p(\Omega,\mathcal{L}^n)} = \Big\| \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \Big| \big(\mathcal{Q} f_{m'}\big)_{\star,\theta} - \big(\mathcal{Q} f_{m''}\big)_{\star,\theta} \Big| \Big\|^p_{L^p(\Omega,\mathcal{L}^n)}$$

$$\leq \Big\| \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \big(\mathcal{Q}(f_{m'} - f_{m''})\big)_{\star,\theta} \Big\|^p_{L^p(\Omega,\mathcal{L}^n)}$$

$$\leq \sum_{j=m'}^{m''} |\lambda_j|^p \Big\| \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \big|\mathcal{Q} a_j\big|_{\star,\theta} \Big\|^p_{L^p(\Omega,\mathcal{L}^n)}$$

$$\leq C \sum_{j=m'}^{m''} |\lambda_j|^p, \tag{4.2.67}$$

for some constant $C \in (0,\infty)$ which is independent of f, m, m''. Finally, in a similar fashion, for each $m \in \mathbb{N}$ we may also employ (4.2.11) in order to write

$$\Big\| \delta_{\partial\Omega}^{-1+\frac{1}{p}} \cdot F - \delta_{\partial\Omega}^{-1+\frac{1}{p}} \cdot F_m \Big\|_{L^1(\Omega,\mathcal{L}^n)} = \Big\| \delta_{\partial\Omega}^{k-s} \Big| \big(\mathcal{Q} f\big)_{\star,\theta} - \big(\mathcal{Q} f_m\big)_{\star,\theta} \Big| \Big\|_{L^1(\Omega,\mathcal{L}^n)}$$

$$\leq \Big\| \delta_{\partial\Omega}^{k-s} \big(\mathcal{Q}(f - f_m)\big)_{\star,\theta} \Big\|_{L^1(\Omega,\mathcal{L}^n)}$$

$$\leq C\|f - f_m\|_{B^{1,1}_s(\partial\Omega,\sigma)}, \tag{4.2.68}$$

for some constant $C \in (0,\infty)$ which is independent of f, m.

Let us summarize our progress. From (4.2.66), (4.2.67), (4.2.68), (4.2.58), (4.2.60), and (4.2.63) we conclude that

$$F_m \in L^p(\Omega,\mathcal{L}^n) \text{ for all } m \in \mathbb{N} \text{ and } \sup_{m\in\mathbb{N}} \|F_m\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{B^{p,p}_s(\partial\Omega,\sigma)}, \tag{4.2.69}$$

$$\{F_m\}_{m\in\mathbb{N}} \text{ is a Cauchy sequence in } L^p(\Omega,\mathcal{L}^n), \text{ and} \tag{4.2.70}$$

$$\delta_{\partial\Omega}^{-1+\frac{1}{p}} \cdot F_m \longrightarrow \delta_{\partial\Omega}^{-1+\frac{1}{p}} \cdot F \text{ in } L^1(\Omega,\mathcal{L}^n) \text{ as } m \to \infty. \tag{4.2.71}$$

In turn, (4.2.70) implies that there exists some $\widetilde{F} \in L^p(\Omega, \mathcal{L}^n)$ such that $F_m \to \widetilde{F}$ in $L^p(\Omega, \mathcal{L}^n)$ as $m \to \infty$. Consequently, there exists a subsequence $\{F_{m_i}\}_{i\in\mathbb{N}}$ with the property that

$$F_{m_i} \to \widetilde{F} \text{ at } \mathcal{L}^n\text{-a.e. point in } \Omega \text{ as } i \to \infty. \tag{4.2.72}$$

Also, from (4.2.69) we see that

$$\|\widetilde{F}\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{B^{p,p}_s(\partial\Omega,\sigma)} \tag{4.2.73}$$

for some constant $C \in (0, \infty)$ which is independent of the function f. Finally, (4.2.71) guarantees that $\delta_{\partial\Omega}^{-1+\frac{1}{p}} \cdot F_{m_i} \to \delta_{\partial\Omega}^{-1+\frac{1}{p}} \cdot F$ in $L^1(\Omega, \mathcal{L}^n)$ as $i \to \infty$, hence there exists a sub-subsequence $\{F_{m_{i_j}}\}_{j\in\mathbb{N}}$ with the property that $\delta_{\partial\Omega}^{-1+\frac{1}{p}} \cdot F_{m_{i_j}} \to \delta_{\partial\Omega}^{-1+\frac{1}{p}} \cdot F$ at $\mathcal{L}^n$-a.e. point in Ω as $j \to \infty$. Since the latter further implies that $F_{m_{i_j}} \to F$ at $\mathcal{L}^n$-a.e. point in Ω as $j \to \infty$, we then conclude from this and (4.2.72) that $F = \widetilde{F}$ at $\mathcal{L}^n$-a.e. point in Ω. In concert with (4.2.73) and (4.2.64), this ultimately yields (4.2.10) when $\frac{n-1}{n-\varepsilon} < p < 1$ and $(n-1)\big(\frac{1}{p}-1\big) < s < 1-\varepsilon$. At this point, the estimate claimed in (4.2.10) has been established for the full range of indices p, s indicated in (4.2.4). The proof of Theorem 4.2.1 is therefore complete. □

Theorem 4.2.1 has many remarkable consequences and in a series of corollaries we single out several applications to double layer like integral operators.

Corollary 4.2.2 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set whose boundary is Ahlfors regular. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let $L = \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ be an $M\times M$ second-order, homogeneous, constant (complex) coefficient, weakly elliptic system in $\mathbb{R}^n$, and denote by $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ the matrix-valued fundamental solution associated with L as in* [70, Theorem 1.4.2]. *Finally, fix $\theta \in (0,1)$ and assume that*

$$\tfrac{n-1}{n} < p \leq \infty \text{ and } (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1. \tag{4.2.74}$$

Then the following conclusions are valid.

(a) If $\mathcal{D}$ is the double layer potential operator associated with L and Ω as in (1.3.18) then given any multi-index $\gamma \in \mathbb{N}_0^n$ with $|\gamma| > 0$ there exists some finite constant $C = C(\Omega, L, p, s, \theta, \gamma) > 0$ such that for each $f \in \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$ one has

$$\Big\|\delta_{\partial\Omega}^{|\gamma|-\frac{1}{p}-s}\big|\partial^\gamma(\mathcal{D}f)\big|_{\star,\theta}\Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{[B^{p,p}_s(\partial\Omega,\sigma)]^M} \tag{4.2.75}$$

provided either $p < \infty$, or $\partial\Omega$ is a bounded set.

(b) Make the additional assumption that $\partial\Omega$ is bounded. For each pair of indices $j,k \in \{1,\dots,n\}$ consider the integral operator $\mathcal{U}_{jk}$ acting on any $f \in \big[L^1(\partial_*\Omega,\sigma)\big]^M$ according to

$$\mathcal{U}_{jk}f(x) := \int_{\partial_*\Omega}\big\{\nu_j(y)(\partial_k E)(x-y) - \nu_k(y)(\partial_j E)(x-y)\big\}f(y)\,\mathrm{d}\sigma(y) \tag{4.2.76}$$

for all $x \in \Omega$. Then for each $\gamma \in \mathbb{N}_0^n$ with $|\gamma| > 0$ there exists some finite constant $C = C(\Omega, L, p, s, \theta, \gamma) > 0$ with the property that for each function $f \in \big[B_s^{p,p}(\partial\Omega,\sigma)\big]^M$ one has

$$\sum_{j,k=1}^{n} \Big\| \delta_{\partial\Omega}^{|\gamma|-\frac{1}{p}-s} \big|\partial^\gamma(\mathcal{U}_{jk} f)\big|_{\star,\theta} \Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C \|f\|_{[B_s^{p,p}(\partial\Omega,\sigma)]^M}. \tag{4.2.77}$$

(c) Continue to work under the additional assumption that $\partial\Omega$ is bounded. Then the following versions of (4.2.75) and (4.2.77) corresponding to the case when $|\gamma| = 0$ also hold: for each cutoff function $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ there exists a constant $C = C(\Omega, L, p, s, \theta, \psi) \in (0,\infty)$ with the property that for each given function $f \in \big[B_s^{p,p}(\partial\Omega,\sigma)\big]^M$ one has

$$\begin{aligned} &\Big\| \delta_{\partial\Omega}^{1-\frac{1}{p}-s} \psi \big|\mathcal{D} f\big| \Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C \|f\|_{[B_s^{p,p}(\partial\Omega,\sigma)]^M} \text{ and} \\ &\Big\| \delta_{\partial\Omega}^{1-\frac{1}{p}-s} \big|\mathcal{D} f\big|_{\star,\theta} \Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C \|f\|_{[B_s^{p,p}(\partial\Omega,\sigma)]^M} \text{ if } \Omega \text{ bounded,} \end{aligned} \tag{4.2.78}$$

as well as

$$\begin{aligned} &\sum_{j,k=1}^{n} \Big\| \delta_{\partial\Omega}^{1-\frac{1}{p}-s} \psi \big|\mathcal{U}_{jk} f\big| \Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C \|f\|_{[B_s^{p,p}(\partial\Omega,\sigma)]^M} \text{ and} \\ &\sum_{j,k=1}^{n} \Big\| \delta_{\partial\Omega}^{1-\frac{1}{p}-s} \big|\mathcal{U}_{jk} f\big|_{\star,\theta} \Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C \|f\|_{[B_s^{p,p}(\partial\Omega,\sigma)]^M} \text{ if } \Omega \text{ bounded.} \end{aligned} \tag{4.2.79}$$

Proof From [69, Theorem 7.7.4] and [69, (7.9.10)-(7.9.11)] we conclude that

$$\begin{aligned} &\text{if } \tfrac{n-1}{n} < p < \infty \text{ and } (n-1)\big(\tfrac{1}{p} - 1\big)_+ < s < 1 \\ &\text{there exist } p_* \in (1,\infty) \text{ and } s_* \in (0,1) \text{ such that} \\ &B_s^{p,p}(\partial\Omega,\sigma) \hookrightarrow B_{s_*}^{p_*,p_*}(\partial\Omega,\sigma) \hookrightarrow L^{p_*}(\partial\Omega,\sigma). \end{aligned} \tag{4.2.80}$$

Also, corresponding to $p = \infty$, the identification in [69, (7.1.59)] implies that

$$\begin{aligned} &\text{if } \partial\Omega \text{ is a compact set and } 0 < s < 1 \text{ then} \\ &B_s^{\infty,\infty}(\partial\Omega,\sigma) = \mathscr{C}^s(\partial\Omega) \hookrightarrow L^{p_*}(\partial\Omega,\sigma) \text{ for each } p_* \in (0,\infty]. \end{aligned} \tag{4.2.81}$$

From (4.2.80)-(4.2.81) and (1.3.24) we see that whenever p, s are as in (4.2.74) the double layer potential operator $\mathcal{D}$ is well defined on the Besov space $\big[B_s^{p,p}(\partial\Omega,\sigma)\big]^M$ and, in fact,

$$\begin{aligned} &\mathcal{D} f \in \big[\mathscr{C}^\infty(\Omega)\big]^M \text{ and } L(\mathcal{D} f) = 0 \text{ in } \Omega, \\ &\text{for each } f \in \big[B_s^{p,p}(\partial\Omega,\sigma)\big]^M. \end{aligned} \tag{4.2.82}$$

To proceed, pick an arbitrary multi-index $\gamma \in \mathbb{N}_0^n$ with $|\gamma| > 0$ and define the operator Q as in (4.2.2) for the matrix-valued integral kernel

$$q(x,y) := \Big(-\nu_s(y) d_{rs}^{\beta\alpha} (\partial^\gamma \partial_r E_{\mu\beta})(x-y) \mathbf{1}_{\partial_*\Omega}(y) \Big)_{1 \le \mu,\alpha \le M} \quad \text{for every } x \in \Omega \text{ and } \sigma\text{-a.e. } y \in \partial\Omega. \tag{4.2.83}$$

From (4.2.83) and [70, Theorem 1.4.2] we then see that there exists $C \in (0,\infty)$ with the property that

$$|q(x,y)| \le C |\partial^\gamma (\nabla E)(x-y)| \le \frac{C}{|x-y|^{n-1+|\gamma|}} \le C \frac{\delta_{\partial\Omega}(x)^{1-|\gamma|}}{|x-y|^n} \quad \text{for every } x \in \Omega \text{ and } \sigma\text{-a.e. } y \in \partial\Omega. \tag{4.2.84}$$

As such, estimate (4.2.1) is satisfied with $\varepsilon := 0$ and $k := |\gamma| - 1$. In addition, if $\mathcal{D}_{\text{mod}}$ is as in (1.8.6), then (1.8.11), (1.8.10), and the fact that $|\gamma| > 0$ permit us to write

$$Q\,\mathbf{e}_\ell = \partial^\gamma \big(\mathcal{D}_{\text{mod}} \mathbf{e}_\ell \big) = 0 \ \text{ in } \ \Omega \ \text{ for each } \ \ell \in \{1, \dots, n\}. \tag{4.2.85}$$

Hence, all hypotheses of Theorem 4.2.1 are satisfied by the current choice of Q. Upon noting that, as seen from (4.2.80)-(4.2.81) and (1.3.18),

$$Qf = \partial^\gamma (\mathcal{D} f) \ \text{ for each } \ f \in \big[B_s^{p,p}(\partial\Omega, \sigma) \big]^M \quad \text{provided either } p < \infty, \text{ or } \partial\Omega \text{ is a bounded set,} \tag{4.2.86}$$

we conclude from (4.2.5), (4.2.82), [68, (6.5.40) in Theorem 6.5.7], and [68, (6.6.91)] that (4.2.75) holds. This takes care of item *(a)*.

As regards the operators $\mathcal{U}_{jk}$ from (4.2.76), observe that if $\partial\Omega$ is bounded then given any $j, k \in \{1, \dots, n\}$ for each function $f \in \big[L_1^1(\partial_*\Omega, \sigma) \big]^M$ we may write

$$\mathcal{U}_{jk} f(x) = \int_{\partial_*\Omega} E(x-y)(\partial_{\tau_{jk}} f)(y)\, d\sigma(y), \qquad \forall x \in \Omega, \tag{4.2.87}$$

thanks to (A.0.183) and the boundary integration by parts formula [69, (11.1.62)]. In particular, the cancelation condition (4.2.3) is satisfied by $\partial^\gamma \mathcal{U}_{jk}$ for each $\gamma \in \mathbb{N}_0^n$. Since when $|\gamma| > 0$ the integral kernel of $\partial^\gamma \mathcal{U}_{jk}$ also satisfies (4.2.1) with $\varepsilon := 0$ and $k := |\gamma| - 1$, Theorem 4.2.1 applies and yields (4.2.77) (bearing in mind [68, (6.5.40) in Theorem 6.5.7], [68, (6.6.91)], and the fact that $L(\mathcal{U}_{jk} f) = 0$ in Ω for each function $f \in \big[B_s^{p,p}(\partial\Omega, \sigma) \big]^M$). This takes care of item *(b)*.

There remains to deal with the claims in item *(c)*. Again, work under the assumption that $\partial\Omega$ is bounded. Fix an arbitrary $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ and consider the following version of the double layer:

$$\widetilde{\mathcal{D}}f := \begin{cases} \psi\mathcal{D}f - \psi \fint_{\partial\Omega} f \, d\sigma & \text{if } \Omega \text{ is bounded,} \\ \psi\mathcal{D}f & \text{if } \Omega \text{ is unbounded.} \end{cases} \tag{4.2.88}$$

Since under the present assumptions we have the continuous embedding (cf. (4.2.80)-(4.2.81))

$$B^{p,p}_s(\partial\Omega, \sigma) \hookrightarrow L^1(\partial\Omega, \sigma), \tag{4.2.89}$$

it follows that the above definition is meaningful. Note that, thanks to Proposition 1.3.6, the operator $\widetilde{\mathcal{D}}$ annihilates constants. Also, since $\partial\Omega$ is bounded it follows that the integral kernel of $\widetilde{\mathcal{D}}$ satisfies (4.2.1) with $k := 0$ and $\varepsilon := 0$. Granted these properties we may invoke Theorem 4.2.1, and (4.2.5) (with $k := 0$) presently ensures (bearing in mind [68, (6.6.6)]) that there exists $C \in (0, \infty)$ with the property that for each $f \in \big[B^{p,p}_s(\partial\Omega, \sigma)\big]^M$ we have

$$\Big\| \delta_{\partial\Omega}^{1-\frac{1}{p}-s} \big| \widetilde{\mathcal{D}}f \big| \Big\|_{L^p(\Omega, \mathcal{L}^n)} \leq C \|f\|_{[B^{p,p}_s(\partial\Omega,\sigma)]^M}. \tag{4.2.90}$$

In turn, from (4.2.90), (4.2.88), (4.2.89), and item *(i)* in [68, Proposition 8.7.1], applied here with $\Sigma := \partial\Omega$, $r := 2\,\text{diam}(\Omega \cap \text{supp}\,\psi) + 2\,\text{dist}\big(\partial\Omega, \Omega \cap \text{supp}\,\psi\big)$, $\alpha := 1 - p(1-s)$, and $N := 0$, we then conclude

$$\Big\| \delta_{\partial\Omega}^{1-\frac{1}{p}-s} \psi \big| \mathcal{D}f \big| \Big\|_{L^p(\Omega, \mathcal{L}^n)} \leq C \|f\|_{[B^{p,p}_s(\partial\Omega,\sigma)]^M} \tag{4.2.91}$$

for every $f \in \big[B^{p,p}_s(\partial\Omega, \sigma)\big]^M$. This is the first inequality in (4.2.78). When Ω is bounded, the second estimate claimed in (4.2.78) is a consequence of the first (now taking the cutoff function ψ to be identically one near $\overline{\Omega}$), and also relying on (1.3.24), [68, (6.5.40) in Theorem 6.5.7], and [68, (6.6.91)].

Next, the first estimate claimed in (4.2.79) follows from Theorem 4.2.1, whose applicability is ensured by (4.2.87) and the observation that the integral kernel of $\psi\mathcal{U}_{jk}$ satisfies (4.2.1) with $\varepsilon := 0$ and $k := 0$. Finally, in the case when Ω is bounded, the second estimate in (4.2.79) is a consequence of the first estimate in (4.2.79), also keeping in mind [68, (6.5.40) in Theorem 6.5.7], [68, (6.6.91)], and the fact that we have $L(\mathcal{U}_{jk}f) = 0$ in Ω for each $f \in \big[B^{p,p}_s(\partial\Omega, \sigma)\big]^M$. □

Recall the agreement made in [69, Convention 8.3.7] (cf. also (A.0.217)), and the scale of weighted maximal Sobolev spaces from [69, Definition 8.6.1] (see also (A.0.215)-(A.0.216)).

Theorem 4.2.3 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set with a compact Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Let L be a second-order, homogeneous, constant (complex) coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and let $\mathcal{D}$ be the boundary-to-domain double layer potential operator associated with L and Ω as in (1.3.18). Finally, fix*

$$\tfrac{n-1}{n} < p < \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \\ \textit{and set } a := 1 - s - \tfrac{1}{p}. \tag{4.2.92}$$

Then, with [69, Convention 8.3.7] (*cf. also* (A.0.217)) *assumed throughout, the following operators are well defined, linear, and continuous:*

$$\mathcal{D} : \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \longrightarrow \big[W^{1,p}_a(\Omega)_{\rm bdd}\big]^M \textit{ and} \\ \mathcal{D} : \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \longrightarrow \big[W^{1,p}_{a,\star}(\Omega)\big]^M \cap \operatorname{Ker} L \textit{ if } \Omega \textit{ is bounded.} \tag{4.2.93}$$

Strengthen the original geometric hypotheses by assuming now that $\partial\Omega$ *is a* UR *set, and now restrict to* $p \in (1,\infty)$. *Then, corresponding to the limiting case* $s = 0$ *of* (4.2.93), *the following operators are well defined, linear, and continuous*

$$\mathcal{D} : \big[L^p(\partial_*\Omega,\sigma)\big]^M \longrightarrow \big[W^{1,p}_{1-\frac{1}{p}}(\Omega)_{\rm bdd}\big]^M \textit{ and} \\ \mathcal{D} : \big[L^p(\partial_*\Omega,\sigma)\big]^M \longrightarrow \big[W^{1,p}_{1-\frac{1}{p},\star}(\Omega)\big]^M \cap \operatorname{Ker} L \textit{ if } \Omega \textit{ is bounded,} \tag{4.2.94}$$

while corresponding to the limiting case $s = 1$ *the following operators are well defined, linear, and continuous:*

$$\mathcal{D} : \big[L^p_1(\partial_*\Omega,\sigma)\big]^M \longrightarrow \big[W^{2,p}_{1-\frac{1}{p}}(\Omega)_{\rm bdd}\big]^M \textit{ and} \\ \mathcal{D} : \big[L^p_1(\partial_*\Omega,\sigma)\big]^M \longrightarrow \big[W^{2,p}_{1-\frac{1}{p},\star}(\Omega)\big]^M \cap \operatorname{Ker} L \textit{ if } \Omega \textit{ is bounded.} \tag{4.2.95}$$

Finally, similar results are valid for the family of integral operators $\mathcal{U}_{jk}$ *defined as in* (4.2.76) *for each* $j, k \in \{1,\dots,n\}$.

Proof Combining (4.2.75) (with $|\gamma| = 1$) and the first estimate in (4.2.78). takes care of the first operator in (4.2.93). With this in hand, the claims about the second operator in (4.2.93) follow (assuming Ω is bounded) with help from (1.3.24), [68, (6.5.40) in Theorem 6.5.7], and [68, (6.6.91)].

Let us now work under the stronger assumption that $\partial\Omega$ is a UR set. Fix some $p \in (1,\infty)$. Then (1.5.5) with $q := p$ yields

$$\Big\|\delta_{\partial\Omega}^{1-\frac{1}{p}}\big|\nabla(\mathcal{D}f)\big|\Big\|_{L^p(\Omega,\mathcal{L}^n)} \le C\|f\|_{[L^p(\partial_*\Omega,\sigma)]^M}. \tag{4.2.96}$$

We need a similar estimate without the gradient. To this end, pick a cutoff function $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$. Also, recall from (1.3.18) that the integral kernel of the operator $\mathcal{D}$ is the matrix-valued function defined at each $x \in \Omega$ and σ-a.e. $y \in \partial\Omega$ by

$$k(x,y) := \Big(-\mathbf{1}_{\partial_*\Omega}(y)\nu_j(y)a^{\beta\alpha}_{rj}(\partial_r E_{\gamma\beta})(x-y)\Big)_{1\le\gamma,\alpha\le M} \tag{4.2.97}$$

where $L = \big(a^{\alpha\beta}_{rj}\partial_r\partial_j\big)_{1\le\alpha,\beta\le M}$ is the writing of the given system with respect to which the double layer operator $\mathcal{D}$ has been set up, $\nu = (\nu_j)_{1\le j\le n}$ is the geometric measure theoretic outward unit normal to Ω, and $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ is the matrix-valued fundamental solution associated with L as in [70, Theorem 1.4.2]. From (4.2.97) and the estimates in [70, Theorem 1.4.2] we see that for each $\varepsilon_* < 1$ there exists a constant $C \in (0,\infty)$ such that $|\psi(x)k(x,y)| \leq C|x-y|^{-(n-\varepsilon_*)}$ for all $x \in \Omega$ and σ-a.e. $y \in \partial\Omega$. As such, choosing $r > 0$ large enough so that $\Omega \cap \operatorname{supp}\psi \subseteq \{x \in \Omega : \delta_{\partial\Omega}(x) < r\}$ along with $\varepsilon_* \in (0,1)$, permits us to invoke [68, Proposition 8.7.10] (for the operator $T := \psi\mathcal{D}$) with $\beta := 1 - \frac{1}{p}$ to conclude that there exists a constant $C \in (0,\infty)$ such that for every $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ we have

$$\Big\|\delta_{\partial\Omega}^{1-\frac{1}{p}}\big|\psi\mathcal{D}f\big|\Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{[L^p(\partial\Omega,\sigma)]^M}. \tag{4.2.98}$$

In concert, (4.2.96) and (4.2.98) readily imply the claims about the first operator in (4.2.94). When Ω is bounded, the claims about the second operator in (4.2.94) are consequences of what we have just proved and [69, Lemma 8.6.2]. Likewise, the claims about the operators in (4.2.95) are consequences of (4.2.94), Lemma 1.3.2, item *(4)* in [70, Theorem 2.4.1] (cf. [70, (2.4.34)]), [70, Theorem 1.4.2], [68, (6.6.91)], (1.3.24), and [68, (6.5.40) in Theorem 6.5.7].

Finally, since the family of operators $\mathcal{U}_{jk}$, defined for $j,k \in \{1,\dots,\}$ as in (4.2.76), satisfy the same key analytical and algebraic properties that allowed us to treat the double layer $\mathcal{D}$ (cf. (4.2.77), (4.2.87), and the fact that $L(\mathcal{U}_{jk}f) = 0$ in Ω for each $f \in \big[L^1(\partial_*\Omega,\sigma)\big]^M$ in particular), similar mapping properties continue to hold for said family of operators. □

We have already commented that there is a large variety of double layer operators associated with a given weakly elliptic system, and Corollary 4.2.2 together with Theorem 4.2.3 apply to all such operators. A case in point is as follows. As noted in Example 1.4.12, the boundary-to-domain Cauchy-Clifford integral operator $\mathcal{C}$ (defined in (A.0.53)) is a particular example of a double layer (associated with the Laplacian). As such, Corollary 4.2.2 implies that whenever $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is an open set with an Ahlfors regular boundary, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and p, s are as in (4.2.74), then for each $\theta \in (0,1)$ and each $\gamma \in \mathbb{N}_0^n$ with $|\gamma| > 0$ there exists some finite constant $C = C(\Omega,p,s,\theta,\gamma) > 0$ such that

$$\Big\|\delta_{\partial\Omega}^{|\gamma|-\frac{1}{p}-s}\big|\partial^\gamma(\mathcal{C}f)\big|_{\star,\theta}\Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{B^{p,p}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n} \tag{4.2.99}$$

for each $f \in B^{p,p}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n$. In the same geometric setting, Theorem 4.2.3 further implies that

$$\begin{gathered}\mathcal{C} : B^{p,p}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \longrightarrow W^{1,p}_a(\Omega)_{\text{bdd}}\otimes\mathcal{C}\ell_n \\ \text{is a continuous operator if } \tfrac{n-1}{n} < p < \infty, \\ (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1 \text{ and } a := 1 - s - \tfrac{1}{p},\end{gathered} \tag{4.2.100}$$

and

$$\begin{aligned} &\mathcal{C} : B^{p,p}_s(\partial\Omega,\sigma)\otimes \mathcal{C}\ell_n \longrightarrow W^{1,p}_{a,\star}(\Omega)\otimes \mathcal{C}\ell_n \\ &\text{is a bounded operator whenever } \Omega \text{ is bounded,} \\ &\tfrac{n-1}{n} < p < \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \text{ and } a := 1-s-\tfrac{1}{p}. \end{aligned} \tag{4.2.101}$$

Similar considerations also apply to the boundary-to-domain version of the ordinary Cauchy operator in the complex plane, and to the boundary-to-domain Bochner-Martinelli integral operator in the context of several complex variables (cf. Example 1.4.9 and Example 1.4.16).

Theorem 4.2.1 is also directly applicable to the following brand of boundary-to-domain integral operators (with kernels of a purely real variable nature, lacking any direct link to partial differential equations).

Corollary 4.2.4 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) is an open set with the property that $\partial\Omega$ is an Ahlfors regular set; in particular, Ω is a set of locally finite perimeter. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Next, suppose $k \in \mathscr{C}^2(\mathbb{R}^n \setminus \{0\})$ is function which is positive homogeneous of degree $-N$ for some $N \geq n-1$, and for each $i, j \in \{1, \dots, n\}$ consider the integral operator $\mathcal{Q}_{ij}$ acting on functions $f \in L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ according to*

$$\mathcal{Q}_{ij} f(x) := \int_{\partial_*\Omega} \Big\{\nu_i(y)\partial_{y_j}\big[k(x-y)\big] - \nu_j(y)\partial_{y_i}\big[k(x-y)\big]\Big\} f(y)\,\mathrm{d}\sigma(y) \tag{4.2.102}$$

for each $x \in \Omega$. Finally, fix $\theta \in (0,1)$ and assume that

$$\tfrac{n-1}{n} < p \leq \infty \text{ and } (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1. \tag{4.2.103}$$

Then there exists some finite constant $C = C(\Omega, k, p, s, \theta, N, n) > 0$ with the property that for each $f \in B^{p,p}_s(\partial\Omega,\sigma)$ one has

$$\Big\| \delta_{\partial\Omega}^{N-n+2-\frac{1}{p}-s} \big(\mathcal{Q}_{ij} f\big)_{\star,\theta} \Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{B^{p,p}_s(\partial\Omega,\sigma)} \tag{4.2.104}$$

with the convention that when $p > 1$ the solid maximal function $\big(\mathcal{Q}_{ij} f\big)_{\star,\theta}$ is replaced by $\mathcal{Q}_{ij} f$. In particular,

$$\begin{aligned} &\text{if } N := n-1 \text{ and } a := 1-s-\tfrac{1}{p}, \text{ then the operator} \\ &\mathcal{Q}_{ij} : B^{p,p}_s(\partial\Omega,\sigma) \to L^p\big(\Omega, \delta_{\partial\Omega}^{ap}\mathcal{L}^n\big) \text{ is well defined,} \\ &\text{linear, and bounded.} \end{aligned} \tag{4.2.105}$$

Proof Upon recalling from [70, (2.4.120), (2.4.130)] that

$$\mathcal{Q}_{ij} 1 \equiv 0 \text{ in } \Omega, \tag{4.2.106}$$

the fact that the estimate in (4.2.104) holds follows from Theorem 4.2.1 presently used with $\varepsilon := 0$, $k := N - (n-1)$, and

$$q(x,y) := \Big\{\nu_i(y)\partial_{y_j}\big[k(x-y)\big] - \nu_j(y)\partial_{y_i}\big[k(x-y)\big]\Big\}\mathbf{1}_{\partial_*\Omega}(y) \tag{4.2.107}$$

for each $x \in \Omega$ and $y \in \partial\Omega$. In turn, (4.2.104) implies readily implies the claim in (4.2.105). □

Next, we discuss the boundary behavior of the double layer potential operator, acting from boundary Besov spaces into weighted Sobolev spaces in open sets.

Theorem 4.2.5 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an (ε,δ)-domain whose boundary is a compact* UR *set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and make the additional assumption that $\partial_*\Omega$ has full measure in $\partial\Omega$, i.e., $\sigma\big(\partial\Omega\setminus\partial_*\Omega\big) = 0$. Next, let L be a second-order, homogeneous, constant (complex) coefficient, weakly elliptic $M\times M$ system in $\mathbb{R}^n$, and let $\mathcal{D}$ be the double layer potential operator associated with the system L and the set Ω as in* (1.3.18). *Finally, pick an integrability exponent $p \in (1,\infty)$, a smoothness index $s \in (0,1)$, and introduce*

$$a := 1 - s - \tfrac{1}{p} \in \big(-\tfrac{1}{p}, 1 - \tfrac{1}{p}\big). \tag{4.2.108}$$

Then the double layer operator acting in the context of

$$\mathcal{D} : \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \longrightarrow \big[W^{1,p}_a(\Omega)_{\rm bdd}\big]^M \tag{4.2.109}$$

(*cf.* (4.2.93)), *satisfies the jump-formula*

$$\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathcal{D} = \tfrac{1}{2}I + K \ \text{ on } \ \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M, \tag{4.2.110}$$

where $\mathrm{Tr}_{\Omega\to\partial\Omega} : \big[W^{1,p}_a(\Omega)_{\rm bdd}\big]^M \to \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$ *is the boundary trace operator considered in* [69, Theorem 8.3.6] (*further extended as in Remark 2 following its statement*), *I denotes the identity operator on $\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$, and K is the boundary-to-boundary double layer operator acting on $\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$ as in* (4.1.1) *of Theorem 4.1.1.*

Proof Since the set $\partial\Omega$ is compact, [68, (5.11.35)] ensures that $\mathrm{rad}\,(\Omega) > 0$. Hence, all operators involved are well defined, linear, and bounded, thanks to Theorem 4.2.3, [69, Theorem 8.3.6], and Theorem 4.1.1. Fix a sufficiently large aperture parameter $\kappa > 0$ so that the conclusion in [69, Corollary 8.3.9] holds. Also, pick an arbitrary cutoff function $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ satisfying $\psi \equiv 1$ near $\partial\Omega$. Then for each given function $f \in \big[\mathrm{Lip}\,(\partial\Omega)\big]^M \subseteq \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$ we may write

$$\begin{aligned}\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathcal{D}\big)f &= \mathrm{Tr}_{\Omega\to\partial\Omega}\big(\psi\mathcal{D}f\big) = \big(\psi\mathcal{D}f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \\ &= (\tfrac{1}{2}I + K)f \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega,\end{aligned} \tag{4.2.111}$$

by virtue of [69, Corollary 8.3.9], item *(iv)* of Theorem 1.5.1, and the fact that $\psi \equiv 1$ near $\partial\Omega$. As a consequence of this, the operators $\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathcal{D}$ and $\frac{1}{2}I + K$ agree on $\big[\mathrm{Lip}\,(\partial\Omega)\big]^M$. Since said operators are continuous on $\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$ and since $\big[\mathrm{Lip}\,(\partial\Omega)\big]^M$ is a dense subset of $\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$ (cf. [69, Lemma 7.1.10]), we ultimately conclude that the jump-formula claimed in (4.2.110) holds for each function belonging to $\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$. □

Theorem 4.2.5 is applicable to the entire family of double layer operators associated with a given weakly elliptic system. This includes the boundary-to-domain Cauchy-Clifford integral operator $\mathcal{C}$ (which fits into such a setting as indicated in Example 1.4.12). In such a case, Theorem 4.2.5 gives that whenever $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\geq 2$) is an (ε,δ)-domain whose boundary is a compact UR set with the property that $\sigma\big(\partial\Omega\setminus\partial_*\Omega\big)=0$, $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$, $p\in(1,\infty)$, $s\in(0,1)$, and $a:=1-s-\frac{1}{p}$, then

$$\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\,\mathcal{C}=\tfrac{1}{2}I+\mathfrak{C}\ \text{ on }\ B^{p,p}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \tag{4.2.112}$$

where $\mathrm{Tr}_{\Omega\to\partial\Omega}:W^{1,p}_a(\Omega)_{\mathrm{bdd}}\otimes\mathcal{C}\ell_n\longrightarrow B^{p,p}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n$ is (the Clifford algebra version of) the boundary trace operator from [69, Theorem 8.3.6] (further extended as in Remark 2 following its statement), $\mathcal{C}$ is the boundary-to-domain version of the Cauchy-Clifford integral operator considered in the context of (4.2.100), I denotes the identity operator on $B^{p,p}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n$, and $\mathfrak{C}$ is the boundary-to-boundary Cauchy-Clifford integral operator acting on $B^{p,p}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n$ as in (4.1.13) (with $q=p$). Analogous considerations also apply to the boundary-to-domain version of the ordinary Cauchy operator in the complex plane, and to the boundary-to-domain Bochner-Martinelli integral operator in the context of several complex variables (cf. Example 1.4.9 and Example 1.4.16).

Moving on, we propose to study mapping properties of boundary-to-domain integral operators acting from Besov spaces with a negative amount of smoothness into suitably weighted Lebesgue spaces. Once again, the strategy is to establish a general result of this nature which identifies those basic features of said integral operator which are responsible for results of this flavor.

Theorem 4.2.6 *Let Ω be an open set in $\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\geq 2$) with a compact Ahlfors regular boundary, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider a function $r:\Omega\times\partial\Omega\to\mathbb{C}$ with the property that there exist $\varepsilon\in[0,1)$, $k\in\mathbb{R}$, and a finite constant $C_0>0$ such that*

$$|r(x,y)|\leq C_0\delta_{\partial\Omega}(x)^{-k}\Big(\frac{\delta_{\partial\Omega}(x)}{|x-y|}\Big)^{-\varepsilon}\frac{1}{|x-y|^{n-1}} \tag{4.2.113}$$
for every point $x\in\Omega$ and every point $y\in\partial\Omega$.

In addition, suppose there exists a constant $C_1\in(1,\infty)$ such that

$$|r(x,y)-r(x,z)|\leq C_0\delta_{\partial\Omega}(x)^{-k}\Big(\frac{\delta_{\partial\Omega}(x)}{|x-y|}\Big)^{-\varepsilon}\frac{|y-z|}{|x-y|^{n}}. \tag{4.2.114}$$
for every $x\in\Omega$ and every $y,z\in\partial\Omega$ with $|x-y|\geq C_1|y-z|$.

In this context, consider the integral operator acting on each function $f\in L^1(\partial\Omega,\sigma)$ according to

$$\mathcal{R}f(x) := \int_{\partial\Omega} r(x,y)f(y)\,\mathrm{d}\sigma(y), \qquad x \in \Omega. \tag{4.2.115}$$

Finally, select a function $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$*, and assume*

$$\tfrac{n-1}{n-\varepsilon} < p \leq \infty \text{ and } (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1-\varepsilon. \tag{4.2.116}$$

Then the action of the operator $\mathcal{R}$ *may be extended to* $B^{p,p}_{s-1}(\partial\Omega,\sigma)$ *and there exists a finite constant* $C = C(\Omega,\psi,\varepsilon,C_0,C_1,p,s) > 0$ *with the property that for every* $f \in B^{p,p}_{s-1}(\partial\Omega,\sigma)$ *one has*

$$\Big\|\delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \cdot \psi\mathcal{R}f\Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{B^{p,p}_{s-1}(\partial\Omega,\sigma)}. \tag{4.2.117}$$

Before presenting the proof of this theorem we wish to make two comments. The first comment pertains to the nature of the estimate demanded in (4.2.114). Specifically, assume Ω satisfies the following connectivity property:

there exists some constant $c \in (0,\infty)$ such that for each pair of points $y,z \in \partial\Omega$ there exists a rectifiable path $t \mapsto \gamma_{y,z}(t)$ joining y and z in Ω, parametrized with respect to its arc-length (hence $|\gamma'_{y,z}(t)| = 1$ for $\mathcal{H}^1$-a.e. t), and with the property that length$\,(\gamma_{y,z}) \leq c|y-z|$. (4.2.118)

For example, any open set with compact boundary and satisfying a local John condition has the aforementioned connectivity property. Also, assume that for each fixed $x \in \Omega$, we may extend $r(x,\cdot)$ to a function of class $\mathscr{C}^1$ in $\Omega \setminus \{x\}$ with the property that there exists $C \in (0,\infty)$ such that

$$|\nabla_y r(x,y)| \leq C\delta_{\partial\Omega}(x)^{-k}\Big(\frac{\delta_{\partial\Omega}(x)}{|x-y|}\Big)^{-\varepsilon}\frac{1}{|x-y|^n} \quad \text{for each } y \in \Omega\setminus\{x\}. \tag{4.2.119}$$

We then claim that (4.2.114) is presently satisfied. To see that this the case, fix $x \in \Omega$ and pick a constant $C_1 > 0$ much larger than c. Also, select a pair of arbitrary points $y,z \in \partial\Omega$ and set $L := \text{length}\,(\gamma_{y,z})$. Then, on account of (4.2.113) and (4.2.118), for each $x \in \Omega$ with $|x-y| \geq C_1|y-z|$ we may estimate

$$\begin{aligned} |r(x,y)-r(x,z)| &\leq \int_0^L |(\nabla_y r)(x,\gamma_{y,z}(t))|\,\mathrm{d}t \\ &\leq C\cdot L\cdot \sup_{0<t<L}\frac{\delta_{\partial\Omega}(x)^{-k-\varepsilon}}{|x-\gamma_{y,z}(t)|^{n-\varepsilon}} \\ &\leq C|y-z|\frac{\delta_{\partial\Omega}(x)^{-k-\varepsilon}}{|x-y|^{n-\varepsilon}}, \end{aligned} \tag{4.2.120}$$

where we have used the fact that $|x-\gamma_{y,z}(t)| \geq C|x-y|$ for every $t \in [0,L]$ if $|x-y| \geq C_1|y-z|$, and that $L \leq c|y-z|$ (cf. (4.2.118)).

The second comment is that the same type of result continues to hold in the case when the kernel r is matrix-valued, and the functions f are vector-valued (with the understanding that the definition of $\mathcal{R}$ in (4.2.115) now takes into account the natural action of the matrix $r(x, y)$ on the vector $f(y)$).

After this digression, we turn to the proof of Theorem 4.2.6.

Proof of Theorem 4.2.6 The proof of (4.2.117) consists of treating several special cases, then using interpolation to cover the whole range of indices. We begin by considering the situation when $p = \infty$. In this scenario, fix a bounded subset Ω_o of Ω. The crucial step is establishing that there exists $C \in (0, \infty)$ with the property that

$$r(x, \cdot) \in B^{1,1}_{1-s}(\partial\Omega, \sigma) \text{ and } \big\|r(x, \cdot)\big\|_{B^{1,1}_{1-s}(\partial\Omega,\sigma)} \leq C\delta_{\partial\Omega}(x)^{s-k-1} \\ \text{for every point } x \in \Omega_o. \tag{4.2.121}$$

To this end, fix $x \in \Omega_o$ and observe that, thanks to [69, (7.9.10)], the claims in (4.2.121) follow as soon as we prove that

$$\int_{\partial\Omega}\int_{\partial\Omega} \frac{|r(x, y) - r(x, z)|}{|y - z|^{n-s}} \, d\sigma(y) \, d\sigma(z) \leq C\delta_{\partial\Omega}(x)^{s-k-1} \tag{4.2.122}$$

and

$$\int_{\partial\Omega} |r(x, y)| \, d\sigma(y) \leq C\delta_{\partial\Omega}(x)^{s-k-1} \tag{4.2.123}$$

for some $C \in (0, \infty)$ independent of x. As regards (4.2.122), with the constant C_1 as in (4.2.114), for $z \in \partial\Omega$ arbitrary split the inner integral according to whether $|x - y| < C_1|y - z|$, or $|x - y| \geq C_1|y - z|$. The focus then becomes bounding $\int_{\partial\Omega} I_\ell(z) \, d\sigma(z)$ by $C\delta_{\partial\Omega}(x)^{s-k-1}$ for $\ell \in \{1, 2, 3\}$, where

$$I_1(z) := \int_{y\in\partial\Omega,\ |x-y|<C_1|y-z|} \frac{|r(x, y)|}{|y - z|^{n-s}} \, d\sigma(y), \tag{4.2.124}$$

$$I_2(z) := \int_{y\in\partial\Omega,\ |x-y|<C_1|y-z|} \frac{|r(x, z)|}{|y - z|^{n-s}} \, d\sigma(y), \tag{4.2.125}$$

$$I_3(z) := \int_{y\in\partial\Omega,\ |x-y|\geq C_1|y-z|} \frac{|r(x, y) - r(x, z)|}{|y - z|^{n-s}} \, d\sigma(y). \tag{4.2.126}$$

We start by making use of (4.2.113) to estimate $I_1(z)$ as follows

$$I_1(z) \leq C\delta_{\partial\Omega}(x)^{-k-\varepsilon} \int_{\substack{y\in\partial\Omega \\ |x-y|<C_1|y-z|}} \frac{d\sigma(y)}{|x - y|^{n-1-\varepsilon}|y - z|^{n-s}}, \tag{4.2.127}$$

which further implies that

$$
\begin{aligned}
\int_{\partial\Omega} & I_1(z)\,\mathrm{d}\sigma(z) \\
&\le C\delta_{\partial\Omega}(x)^{-k-\varepsilon}\int_{\partial\Omega}\frac{1}{|x-y|^{n-1-\varepsilon}}\Big(\int_{\partial\Omega\setminus B(y,|x-y|/C_1)}\frac{\mathrm{d}\sigma(z)}{|y-z|^{n-s}}\Big)\,\mathrm{d}\sigma(y) \\
&\le C\delta_{\partial\Omega}(x)^{-k-\varepsilon}\int_{\partial\Omega}\frac{1}{|x-y|^{n-1-\varepsilon}}|x-y|^{s-1}\,\mathrm{d}\sigma(y) \\
&\le C\delta_{\partial\Omega}(x)^{-k-\varepsilon}\delta_{\partial\Omega}(x)^{s-1+\varepsilon} = C\delta_{\partial\Omega}(x)^{s-k-1}.
\end{aligned}
\tag{4.2.128}
$$

The first inequality in (4.2.128) is due to Fubini's Theorem, while for the second and third inequalities we have used [68, (7.2.5)] and [68, (8.7.92)].

In fact, the same argument works to bound the integral $\int_{\partial\Omega} I_2(z)\,\mathrm{d}\sigma(z)$. Indeed, $|x-y| < C_1|y-z|$ implies $|x-z| < (1+C_1)|y-z|$ which allows us to enlarge the domain of integration so that, after using Fubini's Theorem, we then return to a similar expression to the one just handled above.

Consider next the task of estimating $\int_{\partial\Omega} I_3(z)\,\mathrm{d}\sigma(z)$. A combination of (4.2.114) with (4.2.126) produces

$$
I_3(z) \le C\delta_{\partial\Omega}(x)^{-k-\varepsilon}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|\ge C_1|y-z|}}\frac{\mathrm{d}\sigma(y)}{|x-y|^{n-\varepsilon}|y-z|^{n-1-s}}
\tag{4.2.129}
$$

which, in concert with [68, (7.2.5)] and [68, (8.7.92)], allows us to estimate

$$
\begin{aligned}
\int\limits_{\partial\Omega} & I_3(z)\,\mathrm{d}\sigma(z) \\
&\le C\delta_{\partial\Omega}(x)^{-k-\varepsilon}\int\limits_{\partial\Omega}\frac{1}{|x-y|^{n-\varepsilon}}\Big(\int\limits_{\partial\Omega\cap B(y,|x-y|/C_1)}\frac{\mathrm{d}\sigma(z)}{|y-z|^{n-1-s}}\Big)\,\mathrm{d}\sigma(y) \\
&\le C\delta_{\partial\Omega}(x)^{-k-\varepsilon}\int\limits_{\partial\Omega}\frac{\mathrm{d}\sigma(y)}{|x-y|^{n-\varepsilon-s}} \le C\delta_{\partial\Omega}(x)^{-k-\varepsilon}\delta_{\partial\Omega}(x)^{-1+s+\varepsilon} \\
&= C\delta_{\partial\Omega}(x)^{s-k-1}.
\end{aligned}
\tag{4.2.130}
$$

Collectively, the estimates for $\int_{\partial\Omega} I_\ell(z)\,\mathrm{d}\sigma(z)$ with $\ell = \{1,2,3\}$ yield the bound (4.2.122). This concludes the proof of (4.2.122).

Turning our attention to (4.2.123), we use (4.2.113) and [68, (8.7.92)] to estimate

$$
\begin{aligned}
\int\limits_{\partial\Omega}|r(x,y)|\,\mathrm{d}\sigma(y) &\le C_0\delta_{\partial\Omega}(x)^{-k-\varepsilon}\int\limits_{\partial\Omega}\frac{1}{|x-y|^{n-1-\varepsilon}}\,\mathrm{d}\sigma(y) \\
&\le C\delta_{\partial\Omega}(x)^{-k-\varepsilon} \le C\delta_{\partial\Omega}(x)^{s-k-1},
\end{aligned}
\tag{4.2.131}
$$

where the last inequality is based on the observation that our hypotheses imply $-\varepsilon - s + 1 > 0$ and that we have $\delta_{\partial\Omega}(x)$ stays bounded on the bounded set Ω_o. This establishes (4.2.123) and finishes the proof of (4.2.121).

Going further, recall from [69, Proposition 7.6.1] that

$$B^{\infty,\infty}_{s-1}(\partial\Omega,\sigma) = \big(B^{1,1}_{1-s}(\partial\Omega,\sigma)\big)^*. \tag{4.2.132}$$

On account of (4.2.132) and (4.2.121), we may then define the action of the operator $\mathcal{R}$ on each $f \in B^{\infty,\infty}_{s-1}(\partial\Omega,\sigma)$, regarded as a functional in $\big(B^{1,1}_{1-s}(\partial\Omega,\sigma)\big)^*$, according to

$$(\mathcal{R}f)(x) := {}_{B^{1,1}_{1-s}(\partial\Omega,\sigma)}\big\langle r(x,\cdot), f\big\rangle_{(B^{1,1}_{1-s}(\partial\Omega,\sigma))^*} \text{ for every } x \in \Omega. \tag{4.2.133}$$

This is compatible with the original definition of $\mathcal{R}$ and, in concert with (4.2.121), shows that there exists a constant $C \in (0,\infty)$ with the property that

$$\Big\|\delta^{k+1-s}_{\partial\Omega} \cdot \psi\mathcal{R}f\Big\|_{L^\infty(\Omega,\mathcal{L}^n)} \leq C\|f\|_{B^{\infty,\infty}_{s-1}(\partial\Omega,\sigma)} \text{ for all } f \in B^{\infty,\infty}_{s-1}(\partial\Omega,\sigma). \tag{4.2.134}$$

Consider next the case when $p = 1$ and $s \in (0, 1-\varepsilon)$. The first order of business is to extend the action of the operator $\mathcal{R}$ to the Besov space $B^{1,1}_{s-1}(\partial\Omega,\sigma)$. With this aim in mind, observe that (4.2.114) implies that for each fixed point $x \in \Omega$ the function $r(x,\cdot)$ is Lipschitz on $\partial\Omega$. Since the latter set is compact, it follows that, on the one hand,

$$r(x,\cdot) \in \mathscr{C}^{1-s}(\partial\Omega) \text{ for each } x \in \Omega. \tag{4.2.135}$$

On the other hand, [69, Proposition 7.6.1] and [69, (7.1.59)] ensure that

$$B^{1,1}_{s-1}(\partial\Omega,\sigma) \hookrightarrow \big(B^{1,1}_{s-1}(\partial\Omega,\sigma)\big)^{**} = \big(\mathscr{C}^{1-s}(\partial\Omega)\big)^*. \tag{4.2.136}$$

Based on (4.2.135)-(4.2.136) we may then naturally define the action of the operator $\mathcal{R}$ on each given $f \in B^{1,1}_{s-1}(\partial\Omega,\sigma)$ by setting

$$(\mathcal{R}f)(x) := {}_{\mathscr{C}^{1-s}(\partial\Omega)}\big\langle r(x,\cdot), f\big\rangle_{(\mathscr{C}^{1-s}(\partial\Omega))^*} \text{ for every } x \in \Omega. \tag{4.2.137}$$

Our next priority is to show that there exists some finite constant $C > 0$ with the property that

$$\Big\|\delta^{k-s}_{\partial\Omega} \cdot \psi\mathcal{R}f\Big\|_{L^1(\Omega,\mathcal{L}^n)} \leq C\|f\|_{(\mathscr{C}^{1-s}(\partial\Omega))^*} \text{ for each } f \in \big(\mathscr{C}^{1-s}(\partial\Omega)\big)^*. \tag{4.2.138}$$

In light of (4.2.136), this then proves (4.2.117) when $p = 1$ and $s \in (0, 1-\varepsilon)$.

As a preamble to the proof of (4.2.138) we first propose to show that for each fixed function $g \in L^\infty(\Omega,\mathcal{L}^n)$ with $\|g\|_{L^\infty(\Omega,\mathcal{L}^n)} \leq 1$ the estimate

$$\Big\|\int_\Omega \delta_{\partial\Omega}(x)^{k-s}\,\psi(x) r(x,\cdot) g(x)\,\mathrm{d}x\Big\|_{\mathscr{C}^{1-s}(\partial\Omega)} \leq C \tag{4.2.139}$$

holds for some finite constant $C > 0$ independent of g. With this goal in mind, observe that (4.2.139) is a consequence of the following two inequalities. First, we claim that for every $y, z \in \partial\Omega$ the inequality

$$\Big|\int_\Omega g(x)\delta_{\partial\Omega}(x)^{k-s}\psi(x)\big(r(x,y)-r(x,z)\big)\,\mathrm{d}x\Big| \leq C\,|y-z|^{1-s} \tag{4.2.140}$$

holds for some constant $C \in (0,\infty)$ independent of y, z. Second, we claim that there exists a finite constant $C > 0$ such that we have the inequality

$$\Big|\int_\Omega g(x)\delta_{\partial\Omega}(x)^{k-s}\psi(x)r(x,y)\,\mathrm{d}x\Big| \leq C, \quad \forall\, y \in \partial\Omega. \tag{4.2.141}$$

To prove (4.2.140), fix two arbitrary points $y, z \in \partial\Omega$ and recall (4.2.118). We can then bound the integral in the left-hand side of (4.2.140) by

$$\begin{aligned} &C\int_{\Omega\cap B(y,C_1|y-z|)} \delta_{\partial\Omega}(x)^{k-s}|r(x,y)|\,\mathrm{d}x \\ &+C\int_{\Omega\cap B(y,C_1|y-z|)} \delta_{\partial\Omega}(x)^{k-s}|r(x,z)|\,\mathrm{d}x \\ &+C\int_{\Omega\setminus B(y,C_1|y-z|)} \delta_{\partial\Omega}(x)^{k-s}|r(x,y)-r(x,z)|\,\mathrm{d}x \\ &=: \mathrm{I}+\mathrm{II}+\mathrm{III}. \end{aligned} \tag{4.2.142}$$

As far as I is concerned, note that

$$\mathrm{I} \leq C\int_{\Omega\cap B(y,C_1|y-z|)} \frac{\delta_{\partial\Omega}(x)^{-s-\varepsilon}}{|x-y|^{n-1-\varepsilon}}\,\mathrm{d}x \leq C|y-z|^{1-s}, \tag{4.2.143}$$

by (4.2.113) and by [68, (8.7.3)] applied with $r := C_1|y-z|$, $\alpha := s+\varepsilon$, and $N := n-1-\varepsilon$. Since $\Omega\cap B(y,C_1|y-z|) \subseteq \Omega\cap B\big(z,(1+C_1)|y-z|\big)$, a similar argument applies to II. Thus, we are left with estimating III. For this, we use (4.2.114) together with [68, (8.7.5)] applied with $r := C_1|y-z|$, $\alpha := s+\varepsilon$, and $N := n-\varepsilon$ (choices for which we have $n-N < \alpha < 1$, as required for the applicability of [68, (8.7.5)]) and obtain

$$\begin{aligned} \mathrm{III} &\leq C|y-z|\int_{\Omega\setminus B(y,C_1|y-z|)} \frac{\delta_{\partial\Omega}(x)^{-s-\varepsilon}}{|x-y|^{n-\varepsilon}}\,\mathrm{d}x \\ &\leq C|y-z|\cdot|y-z|^{-s} = C|y-z|^{1-s}, \end{aligned} \tag{4.2.144}$$

as desired. This finishes the proof of (4.2.140). As far as (4.2.141) is concerned, choose a finite number $R > \sup\{|a-b| : a \in \partial\Omega,\ b \in \Omega\cap\operatorname{supp}\psi\}$. Also, pick an arbitrary point $y \in \partial\Omega$. Then [68, (8.7.3)] used with $r := R$, $\alpha := s+\varepsilon < 1$, and $N := n-1-\varepsilon < n-\alpha$ gives

$$\Big|\int_\Omega g(x)\delta_{\partial\Omega}(x)^{k-s}\psi(x)r(x,y)\,\mathrm{d}x\Big| \leq C\int_{B(y,R)\setminus\partial\Omega}\frac{\delta_{\partial\Omega}(x)^{-s-\varepsilon}}{|x-y|^{n-1-\varepsilon}}\,\mathrm{d}x \leq C\,R^{1-s}. \tag{4.2.145}$$

This establishes the estimate claimed in (4.2.141), and completes the proof of (4.2.139).

Having justified (4.2.139), fix some $f \in \big(\mathscr{C}^{1-s}(\partial\Omega)\big)^*$ and for each function $g \in L^\infty_{\rm comp}(\Omega,\mathcal{L}^n)$ with $\|g\|_{L^\infty(\Omega,\mathcal{L}^n)} \leq 1$ use (4.2.137) together with (4.2.139) to write

$$\begin{aligned}
&\Big|\int_\Omega g(x)\delta_{\partial\Omega}(x)^{k-s}\,\psi(x)(\mathcal{R}f)(x)\,\mathrm{d}x\Big| \\
&\qquad = \Big|{}_{\mathscr{C}^{1-s}(\partial\Omega)}\Big\langle \int_\Omega \delta_{\partial\Omega}(x)^{k-s}\,\psi(x)r(x,\cdot)g(x)\,\mathrm{d}x, f\Big\rangle_{(\mathscr{C}^{1-s}(\partial\Omega))^*}\Big| \\
&\qquad \leq \Big\|\int_\Omega \delta_{\partial\Omega}(x)^{k-s}\,\psi(x)r(x,\cdot)g(x)\,\mathrm{d}x\Big\|_{\mathscr{C}^{1-s}(\partial\Omega)}\|f\|_{(\mathscr{C}^{1-s}(\partial\Omega))^*} \\
&\qquad \leq C\|f\|_{(\mathscr{C}^{1-s}(\partial\Omega))^*}.
\end{aligned} \tag{4.2.146}$$

On account of the arbitrariness of g, this ultimately establishes (4.2.138). Hence, the proof of (4.2.117) in the case when $p = 1$ and $s \in (0, 1-\varepsilon)$ is finished.

Moving on, we treat (4.2.117) in the case when $1 < p < \infty$ and $0 < s < 1-\varepsilon$. The idea is to rely on what we proved so far and invoke Stein's interpolation theorem for analytic families of operators. Specifically, suppose $0 < s_0 < s_1 < 1-\varepsilon$ and consider the family of linear operators

$$L_z f := \delta_{\partial\Omega}^{k+z-[(1-z)s_0+zs_1]}\,\psi\mathcal{R}f \ \text{ for } z\in\mathbb{C} \text{ with } 0 \leq \operatorname{Re} z \leq 1. \tag{4.2.147}$$

This family exhibits an analytical dependence on the parameter z, and satisfies

$$\operatorname{Re} z = 0 \Longrightarrow |L_z f| = \delta_{\partial\Omega}^{k-s_0}|\psi\mathcal{R}f|, \tag{4.2.148}$$

$$\operatorname{Re} z = 1 \Longrightarrow |L_z f| = \delta_{\partial\Omega}^{k+1-s_1}|\psi\mathcal{R}f|. \tag{4.2.149}$$

This observation and the results already proved for $p = 1$ and $p = \infty$ then ensure that the operators

$$L_z : B^{1,1}_{s_0-1}(\partial\Omega,\sigma) \longrightarrow L^1(\Omega,\mathcal{L}^n) \ \text{ for } \operatorname{Re} z = 0, \tag{4.2.150}$$

$$L_z : B^{\infty,\infty}_{s_1-1}(\partial\Omega,\sigma) \longrightarrow L^\infty(\Omega,\mathcal{L}^n) \ \text{ for } \operatorname{Re} z = 1, \tag{4.2.151}$$

are well-defined, linear, and bounded. Granted these, Stein's interpolation theorem for analytic families of operators applies and, when used in concert with our complex interpolation results from [69, Theorem 7.5.2], allows us to conclude that the operator

$$\delta_{\partial\Omega}^{k+1-\frac{1}{p}-s}\cdot\psi\mathcal{R} : B^{p,p}_{s-1}(\partial\Omega,\sigma) \longrightarrow L^p(\Omega,\mathcal{L}^n) \tag{4.2.152}$$

is well defined, linear, and bounded, for every $s \in (0, 1-\varepsilon)$ and $p \in [1, \infty]$. This finishes the proof of estimate (4.2.117) for the range of indices $s \in (0, 1-\varepsilon)$ and $p \in [1, \infty]$.

Alternatively, we may arrive at the same conclusions regarding the operator (4.2.152) using the real method of interpolation. Concretely, given any $s \in (0, 1-\varepsilon)$, from (4.2.136)-(4.2.138) we may conclude that the operator

$$\delta_{\partial\Omega}^{k+1-s} \cdot \psi\mathcal{R} : B^{1,1}_{s-1}(\partial\Omega, \sigma) \longrightarrow L^1\big(\Omega, \delta_{\partial\Omega}^{-1}\mathcal{L}^n\big) \tag{4.2.153}$$

is well defined, linear, and bounded, while from (4.2.132)-(4.2.134) we know that we have a well-defined, linear, and bounded operator

$$\delta_{\partial\Omega}^{k+1-s} \cdot \psi\mathcal{R} : B^{\infty,\infty}_{s-1}(\partial\Omega, \sigma) \longrightarrow L^\infty(\Omega, \mathcal{L}^n) = L^\infty\big(\Omega, \delta_{\partial\Omega}^{-1}\mathcal{L}^n\big) \tag{4.2.154}$$

(with the above equality a consequence of the fact that the measures $\mathcal{L}^n$ and $\delta_{\partial\Omega}^{-1}\mathcal{L}^n$ are mutually absolutely continuous in Ω). Then from (4.2.153)-(4.2.154), the real interpolation result from [69, (7.4.18)] (used with $\Sigma := \partial\Omega$, $p_0 := 1$, $p_1 := \infty$), and well-known real interpolation results for generic Lebesgue spaces (cf., e.g., [2, Theorem 5.2.1, p. 109]) we conclude that the operator

$$\delta_{\partial\Omega}^{k+1-s} \cdot \psi\mathcal{R} : B^{p,p}_{s-1}(\partial\Omega, \sigma) \longrightarrow L^p\big(\Omega, \delta_{\partial\Omega}^{-1}\mathcal{L}^n\big) \tag{4.2.155}$$

is well defined, linear, and bounded whenever $1 \leq p \leq \infty$ and $0 < s < 1-\varepsilon$. In turn, this is equivalent to having estimate (4.2.117) valid in the range $1 \leq p \leq \infty$ and $0 < s < 1-\varepsilon$.

At this stage, there remains to analyze the case when

$$\tfrac{n-1}{n-\varepsilon} < p < 1 \ \text{ and } \ (n-1)\big(\tfrac{1}{p} - 1\big) < s < 1-\varepsilon. \tag{4.2.156}$$

In this scenario, the plan is to eventually use the decomposition from [69, Theorem 7.2.7] of f into a linear combination of smooth blocks and smooth atoms. To be specific, fix $\eta \in (1-s, 1)$ and recall from [69, Definition 7.2.1] that an η-smooth atom of type $(p, s-1)$ is a function $a \in L^\infty(\partial\Omega, \sigma)$ with the property that there exist $r > 0$ and $x_a \in \partial\Omega$ such that

$$(1)\ \operatorname{supp} a \subseteq B(x_a, r) \cap \partial\Omega, \tag{4.2.157}$$

$$(2)\ \|a\|_{L^\infty(\partial\Omega,\sigma)} \leq r^{s-1-\frac{n-1}{p}}, \tag{4.2.158}$$

$$(3)\ |a(x) - a(y)| \leq r^{s-1-\frac{n-1}{p}-\eta} \cdot |x-y|^\eta \ \text{ for all } x, y \in \partial\Omega, \tag{4.2.159}$$

$$(4)\ \int_{\partial\Omega} a \, d\sigma = 0. \tag{4.2.160}$$

An η-smooth block enjoys similar properties, except that (4.2.160) is not necessarily satisfied. This being said, a key provision in [69, Theorem 7.2.7] is that all blocks involved in the decomposition of a given distribution $f \in B^{p,p}_{s-1}(\partial\Omega, \sigma)$ have uniformly large supports (say, surface balls of a common radius ρ).

In view of these considerations, the immediate goal is to show that for each fixed $\eta \in (1-s, 1)$ there exists some constant $C = C(\Omega, \psi, C_0, C_1, p, s) \in (0, \infty)$ such that

$$\Big\| \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \cdot \psi \mathcal{R} a \Big\|_{L^p(\Omega, \mathcal{L}^n)} \leq C \tag{4.2.161}$$

whenever a is either a smooth η-atom on $\partial\Omega$, or a smooth η-block of type $(p, s-1)$ with $r = \rho$, a fixed positive number which depends only on $\partial\Omega$. To justify this claim, first assume the function $a \in L^\infty(\partial\Omega, \sigma)$ is as in (4.2.157)-(4.2.160). Fix $x \in \Omega \setminus B(x_a, C_1 r)$ and, for each point $y \in \partial\Omega \cap B(x_a, r)$, use the vanishing moment condition of the atom together with estimate (4.2.114) and the Ahlfors regularity of $\partial\Omega$ to write

$$\begin{aligned} |\psi(x)| \big|(\mathcal{R}a)(x)\big| &= |\psi(x)| \Big| \int_{\partial\Omega \cap B(x_a, r)} \big(r(x, x_a) - r(x, y)\big) a(y) \, d\sigma(y) \Big| \\ &\leq C\, r^n \|a\|_{L^\infty(\partial\Omega, \sigma)} \frac{\delta_{\partial\Omega}(x)^{-k-\varepsilon}}{|x - x_a|^{n-\varepsilon}} \\ &\leq C\, r^{s+(n-1)(1-\frac{1}{p})} \frac{\delta_{\partial\Omega}(x)^{-k-\varepsilon}}{|x - x_a|^{n-\varepsilon}}. \end{aligned} \tag{4.2.162}$$

Consequently,

$$\begin{aligned} &\int_{\Omega \setminus B(x_a, C_1 r)} \delta_{\partial\Omega}(x)^{kp+p-1-ps} |\psi(x)|^p \big|(\mathcal{R}a)(x)\big|^p \, dx \\ &\qquad \leq C r^{ps+(n-1)(p-1)} \int_{\Omega \setminus B(x_a, C_1 r)} \frac{\delta_{\partial\Omega}(x)^{p-p\varepsilon-1-ps}}{|x - x_a|^{p(n-\varepsilon)}} \, dx. \end{aligned} \tag{4.2.163}$$

With $\alpha := 1 - p(1 - s - \varepsilon)$ and $N := p(n - \varepsilon)$, the conditions $1 > \alpha > n - N$ are satisfied, given that we assume (4.2.156). Thus, [68, Proposition 8.7.1] applies and allows us to further bound the right-hand side in (4.2.163) by $C r^{ps+(n-1)(p-1)} \cdot r^{n-(1-p(1-s-\varepsilon))-p(n-\varepsilon)} = C$. This proves that

$$\int_{\Omega \setminus B(x_a, C_1 r)} \delta_{\partial\Omega}(x)^{kp+p-1-ps} |\psi(x)|^p \big|(\mathcal{R}a)(x)\big|^p \, dx \leq C, \tag{4.2.164}$$

for some finite constant $C > 0$, depending only on Ω, ψ, p, s, C_o, C_1.

The treatment of the situation when (4.2.160) is dropped and, instead, it is assumed that $r = \rho$, where $\rho \in (0, \infty)$ is a fixed number depending only on $\partial\Omega$, is similar. In this case, the fact that x belongs to a bounded subset of Ω (namely $\Omega \cap \operatorname{supp} \psi$) compensates the fact that we no longer assume the vanishing moment condition (4.2.160). Specifically, the same type of argument that has produced (4.2.162) now gives (using (4.2.113), the fact that $|x-y| \geq \big(1-C_1^{-1}\big)|x-x_a|$ for each point $x \in \Omega \setminus B(x_a, C_1 r)$ and each $y \in B(x_a, r) \cap \partial\Omega$, and the Ahlfors regularity of $\partial\Omega$),

$$|\psi(x)|\big|(\mathcal{R}a)(x)\big| \tag{4.2.165}$$
$$\leq C\, r^{s+(n-1)(1-\frac{1}{p})}\delta_{\partial\Omega}(x)^{-k}\left(\frac{\delta_{\partial\Omega}(x)}{|x-x_a|}\right)^{-\varepsilon}\frac{1}{|x-x_a|^n}\Big(\frac{|x-x_a|}{r}\Big)|\psi(x)|,$$

for each $x \in \Omega \setminus B(x_a, C_1 r)$. Upon noticing that

$$\big(|x-x_a|/r\big)|\psi(x)| \leq \rho^{-1}\big(\sup_{\mathbb{R}^n}|\psi|\big)\sup\big\{|x-y| : \ x \in \operatorname{supp}\psi,\ y \in \partial\Omega\big\}$$
$$= C(\partial\Omega, \psi) < \infty, \tag{4.2.166}$$

we deduce that $\psi(x)(\mathcal{R}a)(x)$ satisfies a bound of the same nature as in (4.2.162). Thus, (4.2.164) holds in this case as well.

Let us now consider the contribution coming from integrating near the support of the function $a \in L^\infty(\partial\Omega, \sigma)$. The strategy is to rely on a rescaling argument in order to be able to use what we have already proved in the case $p = 1$. To do so, pick $\tau \in \mathbb{R}$ and introduce

$$\widetilde{a}(x) := r^\tau a(x) \ \text{ for } \ x \in \partial\Omega. \tag{4.2.167}$$

For now, assume that

$$(n-1)\big(\tfrac{1}{p}-1\big) - s < \tau < (n-1)\big(\tfrac{1}{p}-1\big) + 1 - s. \tag{4.2.168}$$

Based on the properties of a (i.e., (4.2.157)-(4.2.160), or (4.2.157)-(4.2.159)) it is not difficult to check that $\widetilde{a}$ is an η-smooth atom of type $\big(1, \tau+s-(n-1)(\frac{1}{p}-1)-1\big)$, or an η-smooth block of type $\big(1, \tau+s-(n-1)(\frac{1}{p}-1)-1\big)$, depending on the original nature of $a \in L^\infty(\partial\Omega, \sigma)$, provided

$$\eta > 1 - \tau - s + (n-1)\big(\tfrac{1}{p}-1\big). \tag{4.2.169}$$

Assuming this is the case, [69, Theorem 7.2.8] guarantees the existence of a constant $C \in (0, \infty)$, independent of a, such that

$$\widetilde{a} \in B^{1,1}_{\tau+s-(n-1)(\frac{1}{p}-1)-1}(\partial\Omega, \sigma) \ \text{ and } \ \|\widetilde{a}\|_{B^{1,1}_{\tau+s-(n-1)(\frac{1}{p}-1)-1}(\partial\Omega,\sigma)} \leq C \tag{4.2.170}$$

If we further require that

$$\tau < (n-1)\big(\tfrac{1}{p}-1\big) + 1 - \varepsilon - s, \tag{4.2.171}$$

we may use the bounds already proved for the operator $\mathcal{R}$ corresponding to $p = 1$ in order to obtain

$$\Big\|\delta_{\partial\Omega}^{k-\tau-s+(n-1)(\frac{1}{p}-1)} \cdot \psi\mathcal{R}\widetilde{a}\Big\|_{L^1(\Omega,\mathcal{L}^n)} \leq C\|\widetilde{a}\|_{B^{1,1}_{\tau+s-(n-1)(\frac{1}{p}-1)-1}(\partial\Omega,\sigma)} \leq C. \tag{4.2.172}$$

Hence, by first applying Hölder's inequality, then (4.2.172), and then [68, (8.7.3)] with $\alpha := n - \frac{\tau p}{1-p}$ and $N := 0$, we may further estimate

$$\begin{aligned}
&\int_{B(x_a,C_1 r)\cap\Omega} \Big(\delta_{\partial\Omega}(x)^{k+1-\frac{1}{p}-s}|\psi(x)|\big|\mathcal{R}a(x)\big|\Big)^p \,\mathrm{d}x \\
&\leq C\,r^{-\tau p}\Bigg(\int_{B(x_a,C_1 r)\cap\Omega} \delta_{\partial\Omega}(x)^{k-\tau-s+(n-1)(\frac{1}{p}-1)}|\psi(x)|\big|\mathcal{R}\widetilde{a}(x)\big|\,\mathrm{d}x\Bigg)^p \\
&\qquad\qquad \times\Bigg(\int_{B(x_a,C_1 r)\cap\Omega} \delta_{\partial\Omega}(x)^{\frac{\tau p}{1-p}-n}\,\mathrm{d}x\Bigg)^{1-p} \\
&\leq C\,r^{-\tau p}\Bigg(\int_{B(x_a,C_1 r)\cap\Omega} \delta_{\partial\Omega}(x)^{\frac{\tau p}{1-p}-n}\,\mathrm{d}x\Bigg)^{1-p} \leq C, \qquad (4.2.173)
\end{aligned}$$

provided

$$(n-1)\big(\tfrac{1}{p}-1\big) < \tau. \qquad (4.2.174)$$

Note that (4.2.174) is needed in order to ensure that the necessary condition $\alpha < 1$, under which [68, (8.7.3)] with $N = 0$ holds, is satisfied. Given the assumptions in (4.2.116) and bearing in mind that $1-s < \eta < 1$, it follows that it is possible to select τ simultaneously satisfying (4.2.168), (4.2.169), (4.2.171), and (4.2.174). For such a τ fixed, estimate (4.2.173) holds. Now (4.2.161) follows by combining (4.2.164) and (4.2.173), completing the proof of (4.2.161) in the case when s and p satisfy (4.2.156).

Moving on, we now consider the case when f is an arbitrary distribution in the Besov space $B^{p,p}_{s-1}(\partial\Omega,\sigma)$ with $\frac{n-1}{n-\varepsilon} < p < 1$ and $(n-1)\big(\frac{1}{p}-1\big) < s < 1-\varepsilon$. [69, Theorem 7.2.7] guarantees the existence of a sequence $\{a_j\}_{j\in\mathbb{N}}$ consisting of either η-smooth atoms of type $(p, s-1)$, or η-smooth blocks of type $(p, s-1)$ supported in surface balls of a common radius ρ, along with a numerical sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ belonging to ℓ^p (a space with which $b^{p,p}(\partial\Omega)$ naturally identifies; cf. [69, (7.2.24) in Definition 7.2.5]) satisfying

$$\Big(\sum_{j=1}^{\infty}|\lambda_j|^p\Big)^{1/p} \leq C\|f\|_{B^{p,p}_{s-1}(\partial\Omega,\sigma)} < +\infty, \qquad (4.2.175)$$

for some constant $C \in (0,\infty)$ independent of f, and such that if

$$f_m := \sum_{j=1}^{m}\lambda_j a_j \ \text{ for each } \ m\in\mathbb{N} \qquad (4.2.176)$$

then

$$f_m \longrightarrow f \text{ in } B^{p,p}_{s-1}(\partial\Omega,\sigma) \text{ as } m \to \infty. \tag{4.2.177}$$

Observe that if we set

$$s_* := s - (n-1)\big(\tfrac{1}{p} - 1\big) \in (0,s) \subseteq (0, 1-\varepsilon), \tag{4.2.178}$$

then $0 < s_* < s < 1$ and $\frac{1}{p} - \frac{s-1}{n-1} = 1 - \frac{s_*-1}{n-1}$. Granted these, [69, Theorem 7.7.4] (used with $\Sigma := \partial\Omega$) gives

$$B^{p,p}_{s-1}(\partial\Omega,\sigma) \hookrightarrow B^{1,1}_{s_*-1}(\partial\Omega,\sigma) \text{ continuously.} \tag{4.2.179}$$

As a consequence of (4.2.177) and (4.2.179) we conclude that

$$f_m \longrightarrow f \text{ in } B^{1,1}_{s_*-1}(\partial\Omega,\sigma) \text{ as } m \to \infty. \tag{4.2.180}$$

Let us now introduce

$$G := \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \cdot \psi\mathcal{R}f \text{ and } G_m := \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \cdot \psi\mathcal{R}f_m \text{ for each } m \in \mathbb{N}. \tag{4.2.181}$$

These functions enjoy several remarkable properties. First, as consequence of (4.2.181), (4.2.176), (4.2.161), and (4.2.175), for each $m \in \mathbb{N}$ we may estimate (mindful that we are presently assuming $0 < p < 1$)

$$\begin{aligned} \|G_m\|^p_{L^p(\Omega,\mathcal{L}^n)} &\leq \sum_{j=1}^m |\lambda_j|^p \Big\| \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \cdot \psi\mathcal{R}a_j \Big\|^p_{L^p(\Omega,\mathcal{L}^n)} \\ &\leq C \sum_{j=1}^m |\lambda_j|^p \leq C\|f\|^p_{B^{p,p}_{s-1}(\partial\Omega,\sigma)}, \end{aligned} \tag{4.2.182}$$

for some $C \in (0,\infty)$ independent of f and m. Similarly, given any $m', m'' \in \mathbb{N}$ with $m' \leq m''$, based on (4.2.181), (4.2.176), and (4.2.161) to estimate (again, keeping in mind that $0 < p < 1$)

$$\begin{aligned} \big\|G_{m'} - G_{m''}\big\|^p_{L^p(\Omega,\mathcal{L}^n)} &\leq \Big\| \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \cdot \psi\mathcal{R}(f_{m'} - f_{m''}) \Big\|^p_{L^p(\Omega,\mathcal{L}^n)} \\ &\leq \sum_{j=m'}^{m''} |\lambda_j|^p \Big\| \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \cdot \psi\mathcal{R}a_j \Big\|^p_{L^p(\Omega,\mathcal{L}^n)} \\ &\leq C \sum_{j=m'}^{m''} |\lambda_j|^p, \end{aligned} \tag{4.2.183}$$

for some constant $C \in (0,\infty)$ independent of f, m, m''. Lastly, for each $m \in \mathbb{N}$ we may employ (4.2.138) to write

$$\begin{aligned}\Big\|\delta_{\partial\Omega}^{-1+\frac{1}{p}}\cdot G-\delta_{\partial\Omega}^{-1+\frac{1}{p}}\cdot G_m\Big\|_{L^1(\Omega,\mathcal{L}^n)}&=\Big\|\delta_{\partial\Omega}^{k-s}\cdot\psi\mathcal{R}(f-f_m)\Big\|_{L^1(\Omega,\mathcal{L}^n)}\\&\leq C\|f-f_m\|_{B^{1,1}_{s-1}(\partial\Omega,\sigma)},\end{aligned}\tag{4.2.184}$$

for a constant $C\in(0,\infty)$ independent of f,m. Collectively, (4.2.182), (4.2.183), (4.2.184), (4.2.175), (4.2.177), and (4.2.180) permit us to conclude that

$$G_m\in L^p(\Omega,\mathcal{L}^n)\ \text{ for all }\ m\in\mathbb{N}\ \text{ and }\ \sup_{m\in\mathbb{N}}\|G_m\|_{L^p(\Omega,\mathcal{L}^n)}\leq C\|f\|_{B^{p,p}_{s-1}(\partial\Omega,\sigma)},\tag{4.2.185}$$

$$\{G_m\}_{m\in\mathbb{N}}\ \text{ is a Cauchy sequence in }\ L^p(\Omega,\mathcal{L}^n),\ \text{ and}\tag{4.2.186}$$

$$\delta_{\partial\Omega}^{-1+\frac{1}{p}}\cdot G_m\longrightarrow\delta_{\partial\Omega}^{-1+\frac{1}{p}}\cdot G\ \text{ in }\ L^1(\Omega,\mathcal{L}^n)\ \text{ as }\ m\to\infty.\tag{4.2.187}$$

Next, from (4.2.186) we see that there exists some function $\widetilde{G}\in L^p(\Omega,\mathcal{L}^n)$ such that $G_m\to\widetilde{G}$ in $L^p(\Omega,\mathcal{L}^n)$ as $m\to\infty$. As such, there exists a subsequence $\{G_{m_i}\}_{i\in\mathbb{N}}$ with the property that

$$G_{m_i}\to\widetilde{G}\ \text{ at }\mathcal{L}^n\text{-a.e. point in }\ \Omega\ \text{ as }\ i\to\infty.\tag{4.2.188}$$

Also, (4.2.185) implies that

$$\|\widetilde{G}\|_{L^p(\Omega,\mathcal{L}^n)}\leq C\|f\|_{B^{p,p}_{s-1}(\partial\Omega,\sigma)}\tag{4.2.189}$$

for some constant $C\in(0,\infty)$ which is independent of the function f. Finally, (4.2.187) ensures that $\delta_{\partial\Omega}^{-1+\frac{1}{p}}\cdot G_{m_i}\to\delta_{\partial\Omega}^{-1+\frac{1}{p}}\cdot G$ in $L^1(\Omega,\mathcal{L}^n)$ as $i\to\infty$, hence there exists a sub-subsequence $\{G_{m_{i_j}}\}_{j\in\mathbb{N}}$ with the property that $\delta_{\partial\Omega}^{-1+\frac{1}{p}}\cdot G_{m_{i_j}}\to\delta_{\partial\Omega}^{-1+\frac{1}{p}}\cdot G$ at $\mathcal{L}^n$-a.e. point in Ω as $j\to\infty$. Since the latter further implies that $G_{m_{i_j}}\to G$ at $\mathcal{L}^n$-a.e. point in Ω as $j\to\infty$, we then deduce from this and (4.2.188) that $G=\widetilde{G}$ at $\mathcal{L}^n$-a.e. point in Ω. Together with (4.2.73) and (4.2.181), this ultimately establishes (4.2.117) in the case when $\frac{n-1}{n-\varepsilon}<p<1$ and $(n-1)\big(\frac{1}{p}-1\big)<s<1-\varepsilon$. Hence, the estimate claimed in (4.2.117) has been now proved for the full range of indices p,s described in (4.2.116). This finishes the proof of Theorem 4.2.6. □

Theorem 4.2.6 has a host of remarkable applications, and in the next three corollaries we explore some of its concrete manifestations.

Corollary 4.2.7 *Suppose $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\geq 2$) is an open set with a compact Ahlfors regular boundary and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, assume $b\in\mathscr{C}^1(\mathbb{R}^n\setminus\{0\})$ is a function with the property that there exist $N_o,N\in\mathbb{R}$ and some $C\in(0,\infty)$ such that*

$$\begin{gathered}|b(z)|\leq C|z|^{-N_o}\ \textit{ and }\ |(\nabla b)(z)|\leq C|z|^{-N-1}\\ \textit{for each }\ z\in B(0,1)\setminus\{0\}.\end{gathered}\tag{4.2.190}$$

Define the integral operator acting on each function $f \in L^1(\partial\Omega, \sigma)$ *according to*

$$(\mathcal{B}f)(x) := \int_{\partial\Omega} b(x-y)f(y)\,\mathrm{d}\sigma(y), \qquad x \in \Omega. \tag{4.2.191}$$

Finally, select a function $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ *and assume* p, s, k *satisfy*

$$\begin{aligned} &\tfrac{n-1}{n} < p \leq \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \\ &k \geq \max\{N, N_o\} - n + 1 \ \text{ and } \ k > s - 1. \end{aligned} \tag{4.2.192}$$

Then $\mathcal{B}$ *may be extended to the Besov space* $B^{p,p}_{s-1}(\partial\Omega, \sigma)$ *and there exists a finite constant* $C > 0$*, which depends only on* $\Omega, b, \psi, N, N_o, p, s, k$*, with the property that for each* $f \in B^{p,p}_{s-1}(\partial\Omega, \sigma)$ *one has*

$$\Big\| \delta_{\partial\Omega}^{k+1-\frac{1}{p}-s} \cdot \psi\mathcal{B}f \Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{B^{p,p}_{s-1}(\partial\Omega,\sigma)}. \tag{4.2.193}$$

Proof Define

$$r(x,y) := b(x-y), \ \text{ for each } \ x \in \Omega \ \text{ and } \ y \in \partial\Omega, \tag{4.2.194}$$

and note that, thanks to (4.2.190) we have

$$|r(x,y)| \leq C|x-y|^{-N_o} \ \text{ for all } \ x \in \Omega, \ \ y \in \partial\Omega, \tag{4.2.195}$$

and also using the Mean Value Theorem,

$$\begin{aligned} &|r(x,y) - r(x,z)| \leq C|y-z||x-y|^{-N-1} \ \text{ for all } \ x \in \Omega, \ \ y, z \in \partial\Omega \\ &\qquad\qquad \text{with the property that } |x-y| \geq 2|y-z|. \end{aligned} \tag{4.2.196}$$

Let us next remark that, since Ω is assumed to be bounded,

$$\begin{aligned} &\text{for each } \alpha, \beta \in \mathbb{R} \text{ satisfying } \alpha \geq (\beta)_+ \text{ there exists a constant} \\ &C \in (0,\infty) \text{ with the property that } \delta_{\partial\Omega}(x)^\alpha \leq C|x-y|^\beta \text{ for} \\ &\text{each } x \in \Omega \text{ and each } y \in \partial\Omega. \end{aligned} \tag{4.2.197}$$

Set $\widetilde{N} := \max\{N, N_o\}$. Based on (4.2.192) and the observation that $0 < 1 - s < 1$, we may choose a number $\varepsilon \in [0,1)$ (which is smaller than, but very close to, $1-s$) satisfying

$$0 \leq \varepsilon < 1 - s \ \text{ and } \ k \geq \max\{\widetilde{N} - n + 1, -\varepsilon\}. \tag{4.2.198}$$

Together with the first line in (4.2.192), the upper bound for ε (in the double inequality above) ensures that

$$(n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1 - \varepsilon \ \text{ and } \ \tfrac{n-1}{n-\varepsilon} < p \leq \infty. \tag{4.2.199}$$

Also, since the last inequality in (4.2.198) implies

$$k + \varepsilon \geq \max\{\widetilde{N} - n + 1 + \varepsilon, 0\} = (\widetilde{N} - n + 1 + \varepsilon)_{+} \tag{4.2.200}$$

we may invoke the estimate in (4.2.197) (with $\alpha := k + \varepsilon$ and $\beta := \widetilde{N} - n + 1 + \varepsilon$) to conclude that there exists some finite constant $C > 0$ with the property that

$$\delta_{\partial\Omega}(x)^{k+\varepsilon} \leq C|x - y|^{\widetilde{N}-n+1+\varepsilon} \text{ for all } x \in \Omega \text{ and } y \in \partial\Omega. \tag{4.2.201}$$

In turn, from (4.2.201) and (4.2.195)-(4.2.196) it readily follows that the function (4.2.194) satisfies (4.2.113)-(4.2.114) for the present ε and k, with $C_1 := 2$. Since (4.2.192) is also satisfied (cf. (4.2.198)) by the current choice of ε, it follows that Theorem 4.2.6 applies and gives that $\mathscr{B}$ may be extended to the Besov space $B^{p,p}_{s-1}(\partial\Omega, \sigma)$ in such a way that (4.2.193) is satisfied. □

Next, we turn our attention to the boundary-to-domain single layer acting on Besov spaces. First, we introduce this operator property.

Proposition 4.2.8 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set with a compact Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a second-order, homogeneous, constant (complex) coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and recall the boundary-to-domain single layer potential operator $\mathscr{S}$ defined in relation to L and Ω as in (1.3.6), where $E = (E_{\alpha\beta})_{1\leq\alpha,\beta\leq M}$ is the fundamental solution associated with the system L as in* [70, Theorem 1.4.2]. *Finally, define*

$$\mathscr{S}f(x) := \Big({}_{\mathrm{Lip}\,(\partial\Omega)}\big\langle E_{\alpha\beta}(x - \cdot)\big|_{\partial\Omega}, f_\beta\big\rangle_{(\mathrm{Lip}\,(\partial\Omega))'}\Big)_{1\leq\alpha\leq M} \tag{4.2.202}$$
for each $f = (f_\beta)_{1\leq\beta\leq M} \in \big[(\mathrm{Lip}\,(\partial\Omega))'\big]^M$ *and each* $x \in \Omega$.

Then, if

$$\tfrac{n-1}{n} < p \leq \infty, \quad 0 < q \leq \infty, \quad (n-1)\big(\tfrac{1}{p} - 1\big)_{+} < s < 1, \tag{4.2.203}$$

the action of the single layer potential operator $\mathscr{S}$ may be naturally adapted to the Besov space $\big[B^{p,q}_{s-1}(\partial\Omega, \sigma)\big]^M$ in the following fashion. Since (4.2.203) entails $\frac{1}{p} - \frac{s}{n-1} \in \big(-\frac{s}{n-1}, 1\big)$, if $p < \infty$ it is possible to choose $p_* \in (1,\infty)$ and $s_* \in (0, s)$ such that $\frac{1}{p} - \frac{s}{n-1} = \frac{1}{p_*} - \frac{s_*}{n-1}$. According to [69, Theorem 7.7.4], *this guarantees that in this case for any $q_* \in (1,\infty)$ one has*

$$B^{p,q}_{s-1}(\partial\Omega, \sigma) \hookrightarrow B^{p_*,q_*}_{s_*-1}(\partial\Omega, \sigma). \tag{4.2.204}$$

When $p = \infty$, then [69, (7.7.6)] *may be invoked to ensure that (4.2.204) holds for any $p_*, q_* \in (1,\infty)$ and $s_* \in (0, s)$. Now, given any $f = (f_\beta)_{1\leq\beta\leq M} \in \big[B^{p,p}_{s-1}(\partial\Omega, \sigma)\big]^M$ define*

$$\mathscr{S}f(x) := \Big({}_{(B^{p_*,q_*}_{s_*-1}(\partial\Omega,\sigma))^*}\big\langle E_{\alpha\beta}(x - \cdot)\big|_{\partial\Omega}, f_\beta\big\rangle_{B^{p_*,q_*}_{s_*-1}(\partial\Omega,\sigma)}\Big)_{1\leq\alpha\leq M} \tag{4.2.205}$$

for each and each $x \in \Omega$, presently viewing each f_α in $B^{p_*,q_*}_{s_*-1}(\partial\Omega,\sigma)$ (via (4.2.204)) and considering each $E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}$ in $\big(B^{p_*,q_*}_{s_*-1}(\partial\Omega,\sigma)\big)^*$ via

$$E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega} \in \mathrm{Lip}\,(\partial\Omega) \subseteq B^{p'_*,q'_*}_{1-s_*}(\partial\Omega,\sigma) = \big(B^{p_*,q_*}_{s_*-1}(\partial\Omega,\sigma)\big)^* \tag{4.2.206}$$

where p'_*, q'_* are the Hölder conjugate exponent of p_*, q_* (cf. [69, Proposition 7.6.1]). *With this interpretation it follows that*

$$\mathscr{S} : \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathscr{C}^\infty(\Omega)\big]^M \tag{4.2.207}$$

is a well-defined linear operator which is also continuous when $\big[\mathscr{C}^\infty(\Omega)\big]^M$ is equipped with the Frechét topology of uniform convergence of partial derivatives on compact sets. Moreover, one may take derivatives of $\mathscr{S}f$ by differentiating under the duality pairing (4.2.205) in a natural fashion, i.e.,

$$\partial^\gamma(\mathscr{S}f)(x) = \Big({}_{(B^{p_*,q_*}_{s_*-1}(\partial\Omega,\sigma))^*}\big\langle(\partial^\gamma E_{\alpha\beta})(x-\cdot)\big|_{\partial\Omega}, f_\beta\big\rangle_{B^{p_*,q_*}_{s_*-1}(\partial\Omega,\sigma)}\Big)_{1\leq\alpha\leq M}$$
$$\text{for each multi-index } \gamma \in \mathbb{N}_0^n \text{ and each point } x \in \Omega, \tag{4.2.208}$$

and one has

$$L(\mathscr{S}f) = 0 \ \text{ in } \Omega \text{ for each } f \in \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M. \tag{4.2.209}$$

Finally,

if Ω also satisfies a two-sided local John condition then the single layer operator in the context of (4.2.207) is compatible with the single layer operator in the context of (1.5.41). (4.2.210)

Proof From [69, Proposition 7.6.2] we know that the definition in (4.2.205) is unambiguous. The fact that (4.2.207) is a well-defined, linear, continuous operators, and that (4.2.208) holds may be seen by reasoning as in the proof of Lemma 2.2.1 (in which (2.2.5) is now replaced by $f = \lim_{N\to\infty} P_N f$, with family of operators $\{P_N\}_{N\in\mathbb{N}}$ defined as in [69, (7.3.37)]). Also, (4.2.209) is implied by (4.2.208) and [70, Theorem 1.4.2].

To justify the compatibility claim in (4.2.210), work under the additional assumption that Ω also satisfies a two-sided local John condition. Then (4.2.204) and [69, (11.8.15), (11.11.12)] imply that we have the following continuous embeddings:

$$B^{p,q}_{s-1}(\partial\Omega,\sigma) \hookrightarrow B^{p_*,q_*}_{s_*-1}(\partial\Omega,\sigma) \hookrightarrow L^{p_*}_{-1}(\partial\Omega,\sigma) \hookrightarrow \big(\mathrm{Lip}\,(\partial\Omega)\big)'. \tag{4.2.211}$$

Fix $f = (f_\beta)_{1\leq\beta\leq M} \in \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M$. Then for each index $\alpha \in \{1,\dots,M\}$ and fixed point $x \in \Omega$ we may write

$$
\begin{aligned}
\big(\mathscr{S}f\big)_\alpha(x) &= {}_{(B^{p_*,q_*}_{s_*-1}(\partial\Omega,\sigma))^*}\big\langle E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta\big\rangle_{B^{p_*,q_*}_{s_*-1}(\partial\Omega,\sigma)} \\
&= {}_{\mathrm{Lip}(\partial\Omega)}\big\langle E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta\big\rangle_{(\mathrm{Lip}(\partial\Omega))'} \\
&= {}_{(L^{p_*}_{-1}(\partial\Omega,\sigma))^*}\big\langle E_{\alpha\beta}(x-\cdot)\big|_{\partial\Omega}, f_\beta\big\rangle_{L^{p_*}_{-1}(\partial\Omega,\sigma)}
\end{aligned}
\tag{4.2.212}
$$

thanks to (4.2.205), (4.2.206), [69, (7.6.9), (11.8.16)], and (A.0.136). This proves (4.2.210). □

We shall now establish the following result.

Corollary 4.2.9 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set with a compact Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, let L be a second-order, homogeneous, constant (complex) coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and recall the boundary-to-domain single layer potential operator $\mathscr{S}$ acting in the context of Besov spaces as in Proposition 4.2.8. Finally, assume that*

$$
\tfrac{n-1}{n} < p \leq \infty \ \ \textit{and}\ \ (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1. \tag{4.2.213}
$$

Then for each multi-index $\gamma \in \mathbb{N}_0^n$ and each number $k \in \mathbb{R}$ satisfying

$$
k \geq |\gamma| - 1 \ \ \textit{and}\ \ k > s-1 \tag{4.2.214}
$$

and each parameter $\theta \in (0,1)$ there exists a finite constant $C > 0$, which depends only on Ω, L, p, s, γ, k, θ, with the property that for each $f \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M$ one has

$$
\Big\|\delta_{\partial\Omega}^{k+1-\frac{1}{p}-s}\big|\partial^\gamma\mathscr{S}f\big|_{\star,\theta}\Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{[B^{p,p}_{s-1}(\partial\Omega,\sigma)]^M} \quad \textit{provided the set } \Omega \textit{ is bounded.} \tag{4.2.215}
$$

Moreover, for each cutoff function $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ there exists a finite constant $C > 0$, which depends only on Ω, L, p, s, γ, k, ψ, with the property that for each $f \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M$ one has

$$
\Big\|\delta_{\partial\Omega}^{k+1-\frac{1}{p}-s}\psi\big|\partial^\gamma\mathscr{S}f\big|\Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{[B^{p,p}_{s-1}(\partial\Omega,\sigma)]^M}. \tag{4.2.216}
$$

Proof Let E be the fundamental solution associated with L as in [70, Theorem 1.4.2]. The idea is to apply Corollary 4.2.7 with $b := \partial^\gamma E$. If either $n \geq 3$ or $|\gamma| > 0$, then the hypotheses in (4.2.190) and (4.2.192) are satisfied with $N_o := N := n-2+|\gamma|$ and the current p, s, k. In the remaining case, when $n = 2$ and $|\gamma| = 0$, then the hypotheses in (4.2.190) and (4.2.192) are satisfied with any $N_o \in (0,s)$, $N := 0$, and the current p, s, k. Granted these, (4.2.215) follows from the version of (4.2.193) for a bounded domain (in which case we may take the cutoff function $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ to be identically one near $\overline{\Omega}$), also bearing in mind (4.2.209), [68, (6.5.40) in Theorem 6.5.7], and [68, (6.6.91)]. Finally, (4.2.216) is implied by (4.2.193) as is. □

In turn, Corollary 4.2.9 is the main ingredient in the proof of the fact that the boundary-to-domain single layer operator maps Besov spaces into weighted maximal Sobolev spaces.

Theorem 4.2.10 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set with a compact Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, suppose L is a second-order, homogeneous, constant (complex) coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and recall the boundary-to-domain single layer potential operator $\mathscr{S}$, associated with L and Ω, acting in the context of Besov spaces as in Proposition 4.2.8. Finally, let*

$$\begin{gathered} \tfrac{n-1}{n} < p < \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \\ \text{and set } a := 1 - s - \tfrac{1}{p}. \end{gathered} \tag{4.2.217}$$

Then, with [69, Convention 8.3.7] (cf. also (A.0.217)) assumed throughout, the following operators are well defined, linear, and continuous:

$$\begin{gathered} \mathscr{S} : \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[W^{1,p}_{a}(\Omega)_{\rm bdd}\big]^M \ \text{ and} \\ \mathscr{S} : \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[W^{1,p}_{a,\star}(\Omega)\big]^M \cap \operatorname{Ker} L \ \text{ if } \Omega \text{ is bounded.} \end{gathered} \tag{4.2.218}$$

Furthermore, corresponding to the end-point case $p = \infty$,

$$\begin{gathered} \text{if } \Omega \text{ is also assumed to be a locally uniform domain,} \\ \text{then the operator } \mathscr{S} : \big[B^{\infty,\infty}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathscr{C}^{s}_{\rm bdd}(\Omega)\big]^M \\ \text{is well defined, linear, and continuous for each } s \in (0,1). \end{gathered} \tag{4.2.219}$$

Next, strengthen the original hypotheses on Ω by assuming that $\partial\Omega$ is actually a UR *set. Then, corresponding to the limiting case $s = 0$ of (4.2.218), the operators*

$$\begin{gathered} \mathscr{S} : \big[L^{p}_{-1}(\partial_*\Omega,\sigma)\big]^M \longrightarrow \big[W^{1,p}_{1-\frac{1}{p}}(\Omega)_{\rm bdd}\big]^M \ \text{ and} \\ \mathscr{S} : \big[L^{p}_{-1}(\partial_*\Omega,\sigma)\big]^M \longrightarrow \big[W^{1,p}_{1-\frac{1}{p},\star}(\Omega)\big]^M \cap \operatorname{Ker} L \ \text{ if } \Omega \text{ is bounded,} \end{gathered} \tag{4.2.220}$$

are well defined, linear, and continuous for each $p \in (1,\infty)$, while corresponding to the limiting case $s = 1$ of (4.2.218) the operators

$$\begin{gathered} \mathscr{S} : \big[L^{p}(\partial_*\Omega,\sigma)\big]^M \longrightarrow \big[W^{2,p}_{1-\frac{1}{p}}(\Omega)_{\rm bdd}\big]^M \ \text{ and} \\ \mathscr{S} : \big[L^{p}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[W^{2,p}_{1-\frac{1}{p},\star}(\Omega)\big]^M \cap \operatorname{Ker} L \ \text{ if } \Omega \text{ is bounded} \end{gathered} \tag{4.2.221}$$

are also well defined, linear, and continuous for each $p \in (1,\infty)$.

Proof Throughout, fix an arbitrary cutoff function $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$. Specializing (4.2.216) twice, first to the case when $|\gamma| = 0$ and $k = 0$, then to the case when $|\gamma| = 1$ and $k = 0$, implies that there exists some constant $C \in (0,\infty)$ with the property that

$$\left\|\delta_{\partial\Omega}^{a}\psi\big|\mathscr{S}f\big|\right\|_{L^p(\Omega,\mathcal{L}^n)}+\left\|\delta_{\partial\Omega}^{a}\psi\big|\nabla\mathscr{S}f\big|\right\|_{L^p(\Omega,\mathcal{L}^n)}\leq C\|f\|_{[B^{p,p}_{s-1}(\partial\Omega,\sigma)]^M} \tag{4.2.222}$$

for each $f\in\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M$. From this and (A.0.211) we conclude that the first operator in (4.2.218) is well defined, linear, and continuous. When Ω is bounded, fix some $\theta\in(0,1)$ and specialize (4.2.215) twice, first taking $|\gamma|=0$ and $k=0$, then taking $|\gamma|=1$ and $k=0$. Collectively, these imply that there exists $C\in(0,\infty)$ such that

$$\left\|\delta_{\partial\Omega}^{a}\big|\mathscr{S}f\big|_{\star,\theta}\right\|_{L^p(\Omega,\mathcal{L}^n)}+\left\|\delta_{\partial\Omega}^{a}\big|\nabla\mathscr{S}f\big|_{\star,\theta}\right\|_{L^p(\Omega,\mathcal{L}^n)}\leq C\|f\|_{[B^{p,p}_{s-1}(\partial\Omega,\sigma)]^M} \tag{4.2.223}$$

for each $f\in\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M$. Hence, the claims regarding the second operator in (4.2.218) follow from this in light of (A.0.211) and (4.2.209).

For the next segment in the proof, work under the additional assumption that Ω is a locally uniform domain. Define $\Omega_r:=\big\{x\in\Omega:\ \operatorname{dist}(x,\partial\Omega)\geq r\big\}$ for a sufficiently small threshold $r>0$, and fix $s\in(0,1)$. We shall use the version of (4.2.216) corresponding to $p=\infty$ when $|\gamma|=1$ and $k=0$. In concert with [68, (5.11.75)], this guarantees (bearing in mind that ψ is compactly supported) that there exists a constant $C=C(\Omega,L,s)\in(0,\infty)$ such that

$$\begin{aligned}\|\psi\mathscr{S}f\|_{[\mathscr{C}^s(\Omega)]^M}&\leq C\left\|\delta_{\partial\Omega}^{1-s}\big|\nabla(\psi\mathscr{S}f)\big|\right\|_{L^\infty(\Omega,\mathcal{L}^n)}+C\cdot\sup_{\Omega_r}\big|\psi\mathscr{S}f\big|\\&\leq C\|f\|_{[B^{\infty,\infty}_{s-1}(\partial\Omega,\sigma)]^M}\end{aligned} \tag{4.2.224}$$

for each function $f\in\big[B^{\infty,\infty}_{s-1}(\partial\Omega,\sigma)\big]^M$. Thus, the claim made in (4.2.219) is justified.

For the remainder of the proof strengthen make the assumption that $\partial\Omega$ is actually a UR set, and fix some $p\in(1,\infty)$. To set the stage, we make the claim that there exists some constant $C\in(0,\infty)$ with the property that for each $h\in\big[L^p(\partial\Omega,\sigma)\big]^M$ we have

$$\left\|\delta_{\partial\Omega}^{1-\frac{1}{p}}\psi\big|\partial^\gamma\mathscr{S}h\big|\right\|_{L^p(\Omega,\mathcal{L}^n)}\leq C\|h\|_{[L^p(\partial\Omega,\sigma)]^M}\ \text{ whenever } |\gamma|\leq 2. \tag{4.2.225}$$

Indeed, when $|\gamma|\leq 1$, use [69, (7.7.8)] to embed

$$L^p(\partial\Omega,\sigma)\hookrightarrow B^{p,p}_{s-1}(\partial\Omega,\sigma)\ \text{ for any }\ s\in(0,1). \tag{4.2.226}$$

Fix $s\in(0,1)$. Upon observing that $\delta_{\partial\Omega}^{1-\frac{1}{p}}\leq C\delta_{\partial\Omega}^{1-s-\frac{1}{p}}$ on $\Omega\cap\operatorname{supp}\psi$, we may invoke (4.2.216) to estimate

$$\begin{aligned}\left\|\delta_{\partial\Omega}^{1-\frac{1}{p}}\psi\big|\partial^\gamma\mathscr{S}h\big|\right\|_{L^p(\Omega,\mathcal{L}^n)}&\leq C\left\|\delta_{\partial\Omega}^{1-s-\frac{1}{p}}\psi\big|\partial^\gamma\mathscr{S}h\big|\right\|_{L^p(\Omega,\mathcal{L}^n)}\\&\leq C\|h\|_{[B^{p,p}_{s-1}(\partial\Omega,\sigma)]^M}\leq C\|h\|_{[L^p(\partial\Omega,\sigma)]^M},\end{aligned} \tag{4.2.227}$$

as wanted. Finally, if $|\gamma| = 2$, say $\gamma = \mathbf{e}_i + \mathbf{e}_j$ for $i, j \in \{1, \dots, n\}$, and if we set

$$\mathcal{T}_j h(x) := \int_{\partial\Omega} (\partial_j E)(x - y) h(y)\, d\sigma(y), \qquad \forall x \in \Omega, \tag{4.2.228}$$

then, thanks to [70, (2.4.34)], we have

$$\begin{aligned}
\Big\| \delta_{\partial\Omega}^{1-\frac{1}{p}} \psi \big| \partial^\gamma \mathscr{S} h \big| \Big\|_{L^p(\Omega, \mathcal{L}^n)} &\leq C \Big\| \delta_{\partial\Omega}^{1-\frac{1}{p}} \big| \partial^\gamma \mathscr{S} h \big| \Big\|_{L^p(\Omega, \mathcal{L}^n)} \\
&\leq C \Big(\int_\Omega |(\partial_i \mathcal{T}_j h)(x)|^p \operatorname{dist}(x, \partial\Omega)^{p-1}\, dx \Big)^{1/p} \\
&\leq C \|h\|_{[L^p(\partial\Omega, \sigma)]^M},
\end{aligned} \tag{4.2.229}$$

which finishes the proof of (4.2.225).

Turning to the claims in (4.2.220) in earnest, pick some $f \in \big[L^p_{-1}(\partial_*\Omega, \sigma)\big]^M$ and recall from (1.5.40) the manner in which $\mathscr{S}$ acts on f. Specifically, from [69, Proposition 11.8.2] we know that there exist $f_0,\ f_{jk} \in \big[L^p(\partial_*\Omega, \sigma)\big]^M$, for all $1 \leq j < k \leq n$, satisfying

$$\|f_0\|_{[L^p(\partial_*\Omega, \sigma)]^M} + \sum_{1 \leq j < k \leq n} \|f_{jk}\|_{[L^p(\partial_*\Omega, \sigma)]^M} \leq C \|f\|_{[L^p_{-1}(\partial_*\Omega, \sigma)]^M} \tag{4.2.230}$$

for some finite constant $C > 0$ independent of f and, with $1 < p' < \infty$ such that $1/p + 1/p' = 1$,

$$\begin{gathered}
{}_{[L^p_{-1}(\partial_*\Omega, \sigma)]^M} \big\langle f, g \big\rangle_{[L^{p'}_1(\partial_*\Omega, \sigma)]^M} = \int_{\partial_*\Omega} \Big(\langle f_0, g \rangle + \sum_{1 \leq j < k \leq n} \langle f_{jk}, \partial_{\tau_{jk}} g \rangle \Big)\, d\sigma \\
\text{for every function } g \in \big[L^{p'}_1(\partial_*\Omega, \sigma)\big]^M.
\end{gathered} \tag{4.2.231}$$

Then if $E = (E_{\alpha\beta})_{1 \leq \alpha, \beta \leq M}$ is the matrix-valued fundamental solution associated with L in $\mathbb{R}^n$ as in [70, Theorem 1.4.2], then from (1.5.40) and (4.2.231) we conclude that for each $x \in \Omega$ we have

$$\begin{aligned}
(\mathscr{S} f)(x) &= \int_{\partial_*\Omega} E(x - y) f_0(y)\, d\sigma(y) \\
&\quad + \sum_{1 \leq j < k \leq n} \int_{\partial_*\Omega} \partial_{\tau_{jk}(y)}[E(x - y)] f_{jk}(y)\, d\sigma(y) \\
&= (\mathscr{S} \widetilde{f_0})(x) + \sum_{1 \leq j < k \leq n} \Big\{ \partial_j \mathscr{S}(\nu_k \widetilde{f_{jk}})(x) - \partial_k \mathscr{S}(\nu_j \widetilde{f_{jk}})(x) \Big\},
\end{aligned} \tag{4.2.232}$$

where $\widetilde{f_0}$, $\widetilde{f_{jk}}$ are the extensions of the functions $f_0,\ f_{jk} \in \big[L^p(\partial_*\Omega, \sigma)\big]^M$ by zero outside $\partial_*\Omega$, to the entire $\partial\Omega$. Thanks to this representation formula, the estimate established in (4.2.225) (also used with ψ replaced by various derivatives of ψ, as needed), and (4.2.230), we then conclude that

$$\Big\|\delta_{\partial\Omega}^{1-\frac{1}{p}}|\psi\mathcal{S}f|\Big\|_{L^p(\Omega,\mathcal{L}^n)}+\Big\|\delta_{\partial\Omega}^{1-\frac{1}{p}}|\nabla(\psi\mathcal{S}f)|\Big\|_{L^p(\Omega,\mathcal{L}^n)}\leq C\|f\|_{[L^p_{-1}(\partial_*\Omega,\sigma)]^M},\quad(4.2.233)$$

for some constant $C\in(0,\infty)$ independent of f. This proves that the first operator in (4.2.220) is well defined, linear, and bounded. With this in hand, the claims about the second operator in (4.2.220) are seen to be true by virtue of [69, Lemma 8.6.2].

Finally, the fact that the first operator in (4.2.221) is well defined, linear, and bounded is seen directly from (4.2.225) and definitions (cf. (A.0.211)), while the claims about the second operator in (4.2.221) follow from this and [69, Lemma 8.6.2]. □

4.3 Boundary-to-Domain Layer Potentials from Besov Spaces into Besov and Triebel-Lizorkin Spaces

We are now in a position to elucidate the smoothing effect of the boundary-to-domain double layer potential operator, measured on Besov and Triebel-Lizorkin scales.

Theorem 4.3.1 *Suppose $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\geq 2$) is an (ε,δ)-domain with a compact Ahlfors regular boundary, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, suppose L is a second-order, homogeneous, constant (complex) coefficient, weakly elliptic $M\times M$ system in $\mathbb{R}^n$, and recall the boundary-to-domain double layer potential operator $\mathcal{D}$ associated with L and Ω as in* (1.3.18)*. Then, with* [69, Convention 8.3.7] *(cf. also* (A.0.217)*) assumed throughout, the following assertions are true.*

(1) The double layer operator $\mathcal{D}$ induces a well-defined, linear, and continuous mapping

$$\mathcal{D}:\big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M\longrightarrow\Big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M\ \textit{whenever}\quad \frac{n-1}{n}<p<\infty,\quad (n-1)\big(\tfrac{1}{p}-1\big)_+<s<1,\quad 0<q\leq\infty.\quad(4.3.1)$$

(2) The double layer operator $\mathcal{D}$ induces a well-defined, linear, and continuous mapping

$$\mathcal{D}:\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M\longrightarrow\Big[F^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M\ \textit{whenever}\quad \frac{n-1}{n}<p<\infty,\quad (n-1)\big(\tfrac{1}{p}-1\big)_+<s<1,\quad \frac{n}{n+s+1/p}<q\leq\infty.\quad(4.3.2)$$

(3) Corresponding to taking $s:=2-\frac{1}{p}$ and $q:=2$ in (4.3.2)*, the double layer operator $\mathcal{D}$ induces a well-defined, linear, and continuous mapping*

$$\mathcal{D} : \Big[B^{p,p}_{2-\frac{1}{p}}(\partial\Omega, \sigma)\Big]^M \longrightarrow \big[h^p_2(\Omega)_{\rm bdd}\big]^M \ \textit{with}\ \tfrac{n}{n+1} < p < 1, \tag{4.3.3}$$

where $h^p_2(\Omega)$ *is the local Hardy-based Sobolev space of order* 2 *in* Ω (*cf.* [69, (9.2.43)]).
Also, corresponding to having $s := 1 - \frac{1}{p}$ *and* $q := 2$ *in* (4.3.2)*, the double layer operator* $\mathcal{D}$ *induces a well-defined, linear, and continuous mapping*

$$\mathcal{D} : \Big[B^{p,p}_{1-\frac{1}{p}}(\partial\Omega, \sigma)\Big]^M \longrightarrow \big[W^{1,p}_{\rm bdd}(\Omega)\big]^M \ \textit{for each}\ p \in (1, \infty). \tag{4.3.4}$$

(*which is a particular case of Theorem* 4.2.3).

(4) Strengthen the original hypotheses on Ω *by assuming that* $\partial\Omega$ *is actually a* UR *set. Then, as a limiting case of* (4.3.1)-(4.3.2)*, formally corresponding to making* $s := 0$*, the operators*

$$\begin{gathered}\mathcal{D} : \big[L^p(\partial_*\Omega, \sigma)\big]^M \longrightarrow \Big[B^{p,q}_{\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M \ \textit{with} \\ 1 < p < \infty \ \textit{and}\ p \leq q \leq \infty,\end{gathered} \tag{4.3.5}$$

as well as

$$\begin{gathered}\mathcal{D} : \big[L^p(\partial_*\Omega, \sigma)\big]^M \longrightarrow \Big[F^{p,q}_{\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M \ \textit{with} \\ 1 < p < \infty \ \textit{and}\ \tfrac{n}{n+1/p} < q \leq \infty,\end{gathered} \tag{4.3.6}$$

are well defined, linear, and continuous. In particular, corresponding to the case when $p = q = 2$*, it follows that* (*recall* (A.0.86) *and* [69, (9.2.22)])

$$\mathcal{D} : \big[L^2(\partial_*\Omega, \sigma)\big]^M \longrightarrow \big[H^{1/2}_{\rm bdd}(\Omega)\big]^M \tag{4.3.7}$$

is a well-defined, linear, and continuous operator. Furthermore, as a limiting case of (4.3.1)-(4.3.2)*, formally corresponding to making* $s := 1$*, the operators*

$$\begin{gathered}\mathcal{D} : \big[L^p_1(\partial_*\Omega, \sigma)\big]^M \longrightarrow \Big[B^{p,q}_{1+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M \ \textit{with} \\ 1 < p < \infty \ \textit{and}\ p \leq q \leq \infty,\end{gathered} \tag{4.3.8}$$

as well as

$$\begin{gathered}\mathcal{D} : \big[L^p_1(\partial_*\Omega, \sigma)\big]^M \longrightarrow \Big[F^{p,q}_{1+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M \ \textit{with} \\ 1 < p < \infty \ \textit{and}\ \tfrac{n}{n+1+1/p} < q \leq \infty,\end{gathered} \tag{4.3.9}$$

are well-defined, linear, and continuous. In particular, corresponding to the case when $p = q = 2$*, it follows that* (*recall* (A.0.86) *and* [69, (9.2.22)])

$$\mathcal{D} : \big[L^2_1(\partial_*\Omega, \sigma)\big]^M \longrightarrow \big[H^{3/2}_{\rm bdd}(\Omega)\big]^M \tag{4.3.10}$$

is a well-defined, linear, and continuous operator.

(5) *Similar results to those recorded in items (1)-(4) above for the double layer* $\mathcal{D}$ *also hold for the integral operators* $\mathcal{U}_{jk}$*, defined for* $j, k \in \{1, \dots, \}$ *as in* (4.2.76).

Before presenting the proof of this result we wish to remark that whenever the set Ω is actually bounded, we have $B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\text{bdd}} = B^{p,q}_{s+\frac{1}{p}}(\Omega)$, $F^{p,q}_{s+\frac{1}{p}}(\Omega)$, $h^p_2(\Omega)_{\text{bdd}} = h^p_2(\Omega)$, $W^{1,p}_{\text{bdd}}(\Omega) = W^{1,p}(\Omega)$, etc., i.e., the subscript $_{\text{bdd}}$ may be omitted in all cases.

Here is the proof of Theorem 4.3.1.

Proof of Theorem 4.3.1 Assume first that

$$\tfrac{n-1}{n} < p < \infty, \quad (n-1)\big(\tfrac{1}{p} - 1\big)_+ < s < 1, \quad \tfrac{n}{n+s+1/p} < q \leq \infty. \tag{4.3.11}$$

The fact that $\frac{1}{p} - \frac{s}{n-1} < 1$, itself a consequence of the bounds for s in (4.3.11), implies that

$$\text{there exists } s_* \in (0, s) \text{ such that } \tfrac{1}{p} - \tfrac{s}{n-1} + \tfrac{s_*}{n-1} < 1. \tag{4.3.12}$$

Consequently,

$$p_* := \Big(\tfrac{1}{p} - \tfrac{s}{n-1} + \tfrac{s_*}{n-1}\Big)^{-1} \Longrightarrow p_* \in \big(\max\{1, p\}, \infty\big). \tag{4.3.13}$$

In particular, from (4.3.12)-(4.3.13), [69, Theorem 7.7.4], and [69, (7.9.10)-(7.9.11)] we conclude that we have the continuous embeddings

$$\big[B^{p,p}_s(\partial\Omega, \sigma)\big]^M \hookrightarrow \big[B^{p_*,p_*}_{s_*}(\partial\Omega, \sigma)\big]^M \hookrightarrow \big[L^{p_*}(\partial\Omega, \sigma)\big]^M. \tag{4.3.14}$$

To proceed, fix an arbitrary $f \in \big[B^{p,p}_s(\partial\Omega, \sigma)\big]^M$. Then the function $u := \mathcal{D}f$ is a smooth null-solution of the system L in Ω (cf. (1.3.24)). In addition, from (4.3.13)-(4.3.14), (1.5.1), and [68, (8.6.51)] we see that

$$u \in \big[L^p_{\text{bdd}}(\Omega, \mathcal{L}^n)\big]^M. \tag{4.3.15}$$

In concert with Theorem 4.2.3, the proof so far shows that if $a := 1 - s - \frac{1}{p}$ then

$$\begin{aligned} &u \in \big[L^p_{\text{bdd}}(\Omega, \mathcal{L}^n)\big]^M \cap \operatorname{Ker} L \text{ such that} \\ &\psi u \in \big[W^{1,p}_a(\Omega)\big]^M \text{ for each } \psi \in \mathscr{C}^\infty_c(\mathbb{R}^n), \end{aligned} \tag{4.3.16}$$

with quantitative control. Granted this, we may then conclude from [69, Corollary 9.2.32] (used with $k := 0$) that the double layer potential operator induces a well-defined, linear, and continuous mapping in the context of (4.3.2).

In turn, from (4.3.2) and real interpolation, based on [69, (7.4.4) in Theorem 7.4.1] and [69, Theorem 9.2.22], we conclude that the double layer potential operator also induces a well-defined, linear, and continuous mapping in the context of (4.3.1).

Next, the claim pertaining to (4.3.3) is seen by specializing the result in (4.3.2) to the case when $\frac{n}{n+1} < p < 1$, $s := 2 - \frac{1}{p}$, and $q := 2$, bearing in mind the identification result from [69, Proposition 9.2.7] (currently used with $k := 2$). Also, in the case when Ω is bounded, the assertions made in relation to (4.3.5)-(4.3.7) are clear from item *(11)* of [70, Theorem 2.4.1] (whose present applicability is ensured by (1.3.18) and [70, Theorem 1.4.2]). When Ω is an exterior domain, apply what we have just proved to the bounded set $\Omega_R := \Omega \cap B(0, R)$ for some sufficiently large $R > 0$, identifying function from $L^p(\partial_*\Omega, \sigma)$ with function in $L^p(\partial_*\Omega_R, \sigma_R)$ (where $\sigma_R := \mathcal{H}^{n-1}\lfloor\partial\Omega_R$) by extending them by zero from $\partial_*\Omega$ to $\partial_*\Omega_R$.

Alternatively, given any $f \in \big[L^p(\partial_*\Omega, \sigma)\big]^M$, the function $u := \mathcal{D}f$ is known from (4.2.94), (1.5.1), and [68, (8.6.51)] to satisfy

$$\begin{aligned} &u \in \big[L^p_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^M \cap \operatorname{Ker} L \ \text{ such that} \\ &\psi u \in \big[W^{1,p}_{1-\frac{1}{p}}(\Omega)\big]^M \ \text{ for each } \psi \in \mathscr{C}^\infty_c(\mathbb{R}^n), \end{aligned} \tag{4.3.17}$$

with quantitative control. Having established this, we may then conclude from [69, Corollary 9.2.32] (with $k := 0$, $s := 0$, and $a := 1 - \frac{1}{p}$) that the double layer potential operator induces a well-defined, linear, and continuous mapping both in the context of (4.3.6) and in the context of (4.3.5).

Going further, consider the claims concerning the double layer potential operator in (4.3.8)-(4.3.10). Given any $f \in \big[L^p_1(\partial_*\Omega, \sigma)\big]^M$ with $1 < p < \infty$, the function $u := \mathcal{D}f$ is a smooth null-solution of the system L in Ω (cf. (1.3.24)). Also, [68, (8.6.51)] together with (1.5.1) ensure that $u \in \big[L^p_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^M$. In concert with (4.2.95) from Theorem 4.2.3 the argument so far gives that

$$\begin{aligned} &u \in \big[L^p_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^M \cap \operatorname{Ker} L \ \text{ such that} \\ &\psi u \in \big[W^{2,p}_{1-\frac{1}{p}}(\Omega)\big]^M \ \text{ for each } \psi \in \mathscr{C}^\infty_c(\mathbb{R}^n), \end{aligned} \tag{4.3.18}$$

with quantitative control. With this in hand, [69, Corollary 9.2.32] permits us to conclude that the double layer potential operator induces a well-defined, linear, and continuous mapping in the context of (4.3.8)-(4.3.9). In turn, (4.3.10) is a consequence of this and [69, (9.2.22)].

Finally, since the operators $\mathcal{U}_{jk}$, defined for $j, k \in \{1, \dots, \}$ as in (4.2.76), satisfy the same key analytical and algebraic properties that allowed us to deduce the mapping properties for the double layer $\mathcal{D}$ recorded in items *(1)-(4)* (cf. (4.2.77), (4.2.87), and the fact that $L(\mathcal{U}_{jk}f) = 0$ in Ω for each $f \in \big[L^1(\partial_*\Omega, \sigma)\big]^M$ in particular), said mapping properties continue to hold for this family of operators. □

Theorem 4.3.1 specialized to the case of the boundary-to-domain Cauchy-Clifford integral operator $\mathcal{C}$ (regarded as a special case of a double layer associated, as explained in Example 1.4.12, with the Laplacian) yields the conclusion that whenever $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is an (ε, δ)-domain with a compact Ahlfors regular boundary, and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, then

$$\begin{gathered} C : B^{p,q}_s(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\text{bdd}} \otimes C\ell_n \text{ with} \\ \tfrac{n-1}{n} < p < \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \quad 0 < q \leq \infty, \end{gathered} \tag{4.3.19}$$

$$\begin{gathered} C : B^{p,p}_s(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega)_{\text{bdd}} \otimes C\ell_n \text{ with} \\ \tfrac{n-1}{n} < p < \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \quad \tfrac{n}{n+s+1/p} < q \leq \infty, \end{gathered} \tag{4.3.20}$$

and

$$C : B^{p,p}_{2-\frac{1}{p}}(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow h^p_2(\Omega)_{\text{bdd}} \otimes C\ell_n \text{ with } \tfrac{n}{n+1} < p < 1, \tag{4.3.21}$$

are all well-defined, linear, and continuous operators. Moreover, the analogues of the claims in item *(4)* of Theorem 4.3.1 (regarding the end-point cases $s = 0$ and $s = 1$ of (4.3.19)-(4.3.20)) hold for the the boundary-to-domain Cauchy-Clifford integral operator C as well. Finally, similar considerations also apply to the boundary-to-domain version of the ordinary Cauchy operator in the complex plane, and to the boundary-to-domain Bochner-Martinelli integral operator in the context of several complex variables (cf. Example 1.4.9 and Example 1.4.16).

We continue by discussing the boundary behavior of the (boundary-to-domain version of the) double layer potential operator acting from boundary Besov spaces into Besov and Triebel-Lizorkin spaces defined on domains.

Theorem 4.3.2 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an (ε, δ)-domain whose boundary is a compact* UR *set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and make the additional assumption that $\sigma\big(\partial\Omega \setminus \partial_*\Omega\big) = 0$. Finally, suppose L is a second-order, homogeneous, constant (complex) coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and recall the boundary-to-domain double layer potential operator $\mathcal{D}$ associated with L and Ω as in* (1.3.18)*. Then the following claims are true.*

(1) One has the jump-formula

$$\begin{gathered} \text{Tr}_{\Omega\to\partial\Omega} \circ \mathcal{D} = \tfrac{1}{2}I + K \;\; on \;\; \big[B^{p,q}_s(\partial\Omega, \sigma)\big]^M, \;\; whenever \\ \tfrac{n-1}{n} < p < \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \quad 0 < q \leq \infty, \end{gathered} \tag{4.3.22}$$

where $\mathcal{D} : \big[B^{p,q}_s(\partial\Omega, \sigma)\big]^M \to \big[B^{p,q}_{s+1/p}(\Omega)_{\text{bdd}}\big]^M$ is the double layer potential operator considered in (4.3.1)*, $\text{Tr}_{\Omega\to\partial\Omega} : \big[B^{p,q}_{s+1/p}(\Omega)_{\text{bdd}}\big]^M \to \big[B^{p,q}_s(\partial\Omega, \sigma)\big]^M$ is the boundary trace operator from* [69, (9.4.91) in item *(ii)* of Theorem 9.4.5] *(further extended as in Remark 3 following the statement of* [69, Theorem 9.4.5]*), I denotes the identity operator on $\big[B^{p,q}_s(\partial\Omega, \sigma)\big]^M$, and K is the boundary-to-boundary double layer operator acting on $\big[B^{p,q}_s(\partial\Omega, \sigma)\big]^M$ as in* (4.1.1) *of Theorem 4.1.1.*

(2) One has the jump-formula

$$\begin{gathered}\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathcal{D}=\tfrac{1}{2}I+K\ \ on\ \ \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M,\ \ whenever\\ \tfrac{n-1}{n}<p<\infty,\quad (n-1)\big(\tfrac{1}{p}-1\big)_+<s<1,\quad \tfrac{n}{n+s+1/p}<q\le\infty,\end{gathered}\tag{4.3.23}$$

where, this time, $\mathrm{Tr}_{\Omega\to\partial\Omega}:\big[F^{p,q}_{s+1/p}(\Omega)_{\mathrm{bdd}}\big]^M\to\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$ *is the boundary trace operator from* [69, (9.4.93) in item *(ii)* of Theorem 9.4.5] (*further extended as in Remark 3 following the statement of* [69, Theorem 9.4.5]), $\mathcal{D}:\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M\to\big[F^{p,q}_{s+1/p}(\Omega)_{\mathrm{bdd}}\big]^M$ *is the double layer potential operator considered in* (4.3.2), *I denotes the identity operator on* $\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$ *and, finally, K is the boundary-to-boundary double layer operator acting on* $\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$ *as in* (4.1.1) *of Theorem 4.1.1.*

We wish to note that, as a particular case of Theorem 4.3.2, similar results are valid for the Cauchy-Clifford integral operator, the ordinary Cauchy operator in the complex plane, and the Bochner-Martinelli integral operator in the context of several complex variables.

Proof of Theorem 4.3.2 Consider the claim made in item *(1)*. For starters, observe that all operators involved are well-defined, linear, and continuous, in the contexts specified there thanks to item *(1)* of Theorem 4.3.1, item *(ii)* of [69, Theorem 9.4.5], and Theorem 4.1.1. Fix a sufficiently large aperture parameter $\kappa>0$ as in [69, Corollary 8.3.9]. Then for each function f belonging to $\big[\mathrm{Lip}\,(\partial\Omega)\big]^M\subseteq\big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M$ we may rely on [69, Corollary 8.3.9] and item *(iv)* of Theorem 1.5.1 to write

$$\begin{aligned}\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathcal{D}\big)f&=\mathrm{Tr}_{\Omega\to\partial\Omega}\big(\mathcal{D}f\big)=\big(\mathcal{D}f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\\ &=\big(\tfrac{1}{2}I+K\big)f\ \text{ at }\sigma\text{-a.e. point on }\partial\Omega.\end{aligned}\tag{4.3.24}$$

Hence, the operators $\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathcal{D}$ and $\frac{1}{2}I+K$ agree on $\big[\mathrm{Lip}\,(\partial\Omega)\big]^M$. Given that said operators are continuous on $\big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M$ and given that $\big[\mathrm{Lip}\,(\partial\Omega)\big]^M$ is a dense subset of $\big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M$ (cf. [69, Lemma 7.1.10]), we ultimately conclude that the jump-formula claimed in (4.3.22) holds for each function belonging to $\big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M$. Finally, the proof of the jump-formula in the context described in item *(2)* is similar, relying on the same main ingredients as before, and this finishes the proof of the theorem. □

Next, we turn our attention to the smoothing properties of the boundary-to-domain single layer potential operator, measured on Besov and Triebel-Lizorkin scales.

Theorem 4.3.3 *Let $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\ge 2$) be an (ε,δ)-domain with a compact Ahlfors regular boundary, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, suppose L is a second-order, homogeneous, constant (complex) coefficient, weakly elliptic $M\times M$ system in $\mathbb{R}^n$, and recall the boundary-to-domain single layer potential operator $\mathscr{S}$, associated with L and Ω, acting in the context of Besov spaces as in*

Proposition 4.2.8. Then, with [69, Convention 8.3.7] (*cf. also* (A.0.217)) *assumed throughout, the following assertions are true.*

(1) The single layer operator $\mathscr{S}$ induces a well-defined, linear, and continuous mapping

$$\begin{gathered}\mathscr{S} : \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \Big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M \ \textit{whenever} \\ \tfrac{n-1}{n} < p < \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \quad 0 < q \leq \infty.\end{gathered} \tag{4.3.25}$$

(2) The single layer operator $\mathscr{S}$ induces a well-defined, linear, and continuous mapping

$$\begin{gathered}\mathscr{S} : \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \Big[F^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M \ \textit{whenever} \\ \tfrac{n-1}{n} < p < \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \quad \tfrac{n}{n+s+1/p} < q \leq \infty.\end{gathered} \tag{4.3.26}$$

(3) Corresponding to $s := 2 - \frac{1}{p}$ and $q := 2$ in (4.3.25)-(4.3.26)*, the single layer operator induces a well-defined, linear, and continuous mapping*

$$\mathscr{S} : \Big[B^{p,p}_{1-\frac{1}{p}}(\partial\Omega,\sigma)\Big]^M \longrightarrow \big[h^p_2(\Omega)_{\rm bdd}\big]^M \ \textit{with} \ \tfrac{n}{n+1} < p < 1, \tag{4.3.27}$$

where $h^p_2(\Omega)$ is the local Hardy-based Sobolev space of order 2 *in* Ω (*cf.* [69, (9.2.43)]).

(4) Strengthen the original hypotheses on Ω by assuming that $\partial\Omega$ is actually a UR *set. Then, as a limiting case of* (4.3.25)-(4.3.26)*, formally corresponding to making $s := 0$, the operators*

$$\begin{gathered}\mathscr{S} : \big[L^p_{-1}(\partial_*\Omega,\sigma)\big]^M \longrightarrow \Big[B^{p,q}_{\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M \ \textit{with} \\ 1 < p < \infty \ \textit{and} \ p \leq q \leq \infty,\end{gathered} \tag{4.3.28}$$

as well as

$$\begin{gathered}\mathscr{S} : \big[L^p_{-1}(\partial_*\Omega,\sigma)\big]^M \longrightarrow \Big[F^{p,q}_{\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M \ \textit{with} \\ 1 < p < \infty \ \textit{and} \ \tfrac{n}{n+1/p} < q \leq \infty,\end{gathered} \tag{4.3.29}$$

are well-defined, linear, and continuous. In particular, corresponding to the case when $p = q = 2$, it follows that (*recall* (A.0.86) *and* [69, (9.2.22)])

$$\mathscr{S} : \big[L^2_{-1}(\partial_*\Omega,\sigma)\big]^M \longrightarrow \big[H^{1/2}_{\rm bdd}(\Omega)\big]^M \tag{4.3.30}$$

is a well-defined, linear, and continuous operator. Furthermore, as a limiting case of (4.3.25)-(4.3.26)*, formally corresponding to making $s := 1$, the operators*

$$\mathscr{S} : \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \Big[B^{p,q}_{1+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M \ \textit{with} \\ 1 < p < \infty \ \textit{and} \ p \leq q \leq \infty, \tag{4.3.31}$$

as well as

$$\mathscr{S} : \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \Big[F^{p,q}_{1+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M \ \textit{with} \\ 1 < p < \infty \ \textit{and} \ \tfrac{n}{n+1+1/p} < q \leq \infty, \tag{4.3.32}$$

are well defined, linear, and continuous. In particular, corresponding to the case when $p = q = 2$*, it follows that* (*recall* (A.0.86) *and* [69, (9.2.22)])

$$\mathscr{S} : \big[L^2(\partial\Omega,\sigma)\big]^M \longrightarrow \big[H^{3/2}_{\rm bdd}(\Omega)\big]^M \tag{4.3.33}$$

is a well-defined, linear, and continuous operator.

Proof Consider first the case when the indices p, q, s are as in (4.3.26). Select some arbitrary function $f \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M$ and consider $u := \mathscr{S}f$ in Ω. From (4.2.216) written for $|\gamma| = 0$ and $k := s - 1 + \frac{1}{p}$ we see that, in a quantitative fashion,

$$u \in \big[L^p_{\rm bdd}(\Omega,\mathcal{L}^n)\big]^M. \tag{4.3.34}$$

If we now set $a := 1 - s - \frac{1}{p}$, from (4.3.34), (4.2.207), (4.2.209), and Theorem 4.2.10 we deduce that

$$u \in \big[L^p_{\rm bdd}(\Omega,\mathcal{L}^n)\big]^M \cap \operatorname{Ker} L \ \text{ such that} \\ \psi u \in \big[W^{1,p}_a(\Omega)\big]^M \ \text{ for each } \psi \in \mathscr{C}^\infty_c(\mathbb{R}^n), \tag{4.3.35}$$

with quantitative control. Having established this, we may then conclude from [69, Corollary 9.2.32] that the single layer potential operator induces a well-defined, linear, and continuous mapping in the context of (4.3.26). With this in hand, from (4.3.26) and real interpolation (using [69, (7.4.4) in Theorem 7.4.1] and [69, Theorem 9.2.22]) we conclude that the single layer potential operator also induces a well-defined, linear, and bounded mapping in the context of (4.3.25). Next, the claim regarding (4.3.27) is seen by specializing the result in (4.3.26) to the case when $\frac{n}{n+1} < p < 1$, $s := 2 - \frac{1}{p}$, and $q := 2$, keeping in mind the identification result from [69, Proposition 9.2.7] (used here with $k := 2$).

Turning our attention to item *(4)*, fix an arbitrary $f \in \big[L^p_{-1}(\partial_*\Omega,\sigma)\big]^M$ and define $u := \mathscr{S}f$ in Ω. From [68, (8.6.51)] and (1.5.45) (used with $q := p$) we see that $u \in \big[L^p_{\rm bdd}(\Omega,\mathcal{L}^n)\big]^M$ in a quantitative fashion. Combining this with (4.2.220) we then conclude that

$$u \in \big[L^p_{\rm bdd}(\Omega,\mathcal{L}^n)\big]^M \cap \operatorname{Ker} L \ \text{ such that} \\ \psi u \in \big[W^{1,p}_{1-\frac{1}{p}}(\Omega)\big]^M \ \text{ for each } \psi \in \mathscr{C}^\infty_c(\mathbb{R}^n), \tag{4.3.36}$$

with quantitative control. With this in hand, [69, Corollary 9.2.32] then gives that the single layer potential operator induces a well-defined, linear, and continuous mapping both in the context of (4.3.28) and in the context of (4.3.29).

Let us now analyze the single layer operator in the context of (4.3.32). Given $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ with $1 < p < \infty$, consider the function $u := \mathscr{S} f$ in Ω. Then from (4.3.34) and (4.2.221) we conclude that

$$
\begin{aligned}
&u \in \big[L^p_{\rm bdd}(\Omega,\mathcal{L}^n)\big]^M \cap \operatorname{Ker} L \ \text{ such that}\\
&\psi u \in \big[W^{2,p}_{1-\frac{1}{p}}(\Omega)\big]^M \ \text{ for each } \psi \in \mathscr{C}^\infty_c(\mathbb{R}^n),
\end{aligned}
\tag{4.3.37}
$$

with quantitative control. Once this has been established, we may invoke [69, Corollary 9.2.32] to conclude that the single layer potential operator induces a well-defined, linear, and continuous mapping both in the context of (4.3.31) and in the context of (4.3.32). □

Let us now consider the action of the boundary-to-boundary single layer potential operator acting on Besov scales.

Theorem 4.3.4 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set with a compact Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, suppose L is a second-order, homogeneous, constant (complex) coefficient, weakly elliptic $M \times M$ system in $\mathbb{R}^n$, and recall the boundary-to-boundary single layer potential operator S, initially associated with L and Ω as in (1.3.62). Then the following statements are valid.*

(1) The single layer operator S from (2.2.116) further extends, in a unique fashion, to a linear and bounded mapping

$$
\begin{aligned}
&S : \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{p^*,q}(\partial\Omega,\sigma)\big]^M\\
&\textit{whenever } \tfrac{n-1}{n} < p < 1, \quad 0 < q \leq \infty,\\
&(n-1)\big(\tfrac{1}{p}-1\big) < s < 1, \ \textit{and } p^* := \Big(\tfrac{1}{p} - \tfrac{s}{n-1}\Big)^{-1}.
\end{aligned}
\tag{4.3.38}
$$

Moreover, various choices of p, q, s, p^* as above yield operators which are compatible with one another. In particular, the single layer operator S from (2.2.116) induces well-defined linear and bounded mappings, which are compatible with each other, in the context

$$
\begin{aligned}
&S : \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{p^*}(\partial\Omega,\sigma)\big]^M \ \textit{whenever}\\
&\tfrac{n-1}{n} < p < 1, \quad (n-1)\big(\tfrac{1}{p}-1\big) < s < 1, \quad p^* := \Big(\tfrac{1}{p} - \tfrac{s}{n-1}\Big)^{-1}.
\end{aligned}
\tag{4.3.39}
$$

(2) If, in addition to the original hypotheses, Ω is also assumed to satisfy a two-sided local John condition, then the single layer operator S from (1.3.62) further extends, in a unique fashion, to a linear and bounded mapping

$$S : \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,q}_{s}(\partial\Omega,\sigma)\big]^M$$
$$\textit{whenever } 1 < p < \infty, \quad 0 < q \le \infty, \quad 0 < s < 1. \tag{4.3.40}$$

Once again, various choices of p, q, s as above yield operators which are compatible with one another. Moreover, if the exponents $p, q, p', q' \in (1,\infty)$ satisfy $1/p + 1/p' = 1 = 1/q + 1/q'$ and $s \in (0,1)$, then

$${}_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^M}\big\langle Sf, g\big\rangle_{[B^{p',q'}_{-s}(\partial\Omega,\sigma)]^M} = {}_{[B^{p,q}_{s-1}(\partial\Omega,\sigma)]^M}\big\langle f, S_{L^\top} g\big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^M}$$
$$\textit{for each } f \in \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \textit{ and } g \in \big[B^{p',q'}_{-s}(\partial\Omega,\sigma)\big]^M, \tag{4.3.41}$$

where $S_{L^\top}$ is associated with the system $L^\top$ (the real transpose of L) in the same manner S has been associated with the original system L.

(3) In addition to the original hypotheses, make also the assumption that Ω is an (ε,δ)-domain. Then the single layer operator S from (4.3.39) further extends, in a unique fashion, to a linear and bounded mapping

$$S : \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,q}_{s}(\partial\Omega,\sigma)\big]^M \textit{ whenever}$$
$$\tfrac{n-1}{n} < p < \infty, \quad 0 < q \le \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1. \tag{4.3.42}$$

Various choices of p, q, s as above yield operators which are compatible with one another. Also, as a consequence of (4.3.42) and [69, (7.7.54)], it follows that the single layer potential operator induces a well-defined, linear, and continuous mapping

$$S : H^p(\partial\Omega,\sigma) \longrightarrow B^{p^*,q^*}_{1-(n-1)(\frac{1}{p}-\frac{1}{p^*})}(\partial\Omega,\sigma) \textit{ whenever}$$
$$\tfrac{n-1}{n} < p < p^* \le 1 \textit{ and } 0 < q^* \le \infty. \tag{4.3.43}$$

Furthermore, one has the trace formula

$$\mathrm{Tr}_{\Omega\to\partial\Omega} \circ \mathscr{S} = S \textit{ on } \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \tag{4.3.44}$$

when

$\mathrm{Tr}_{\Omega\to\partial\Omega} : \big[B^{p,q}_{s+1/p}(\Omega)_{\rm bdd}\big]^M \to \big[B^{p,q}_{s}(\partial\Omega,\sigma)\big]^M$ *is the boundary trace operator from* [69, (9.4.91) in item *(ii)* of Theorem 9.4.5] (*further extended as indicated in Remark 3 following its statement*), $\mathscr{S} : \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \to \big[B^{p,q}_{s+1/p}(\Omega)_{\rm bdd}\big]^M$ *is the boundary-to-domain single layer from* (4.3.25), *and S is the boundary-to-boundary single layer potential operator from* (4.3.42), (4.3.45)

and also when

$$\begin{array}{l}\mathrm{Tr}_{\Omega\to\partial\Omega} : \big[F^{p,q}_{s+1/p}(\Omega)_{\rm bdd}\big]^M \to \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \textit{ is the boundary trace operator from } [69, (9.4.93) \text{ in item } \textit{(ii)} \text{ of Theorem 9.4.5}] \\ (\textit{and further extended as indicated in Remark 3 following its statement}), \mathscr{S} : \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \to \big[F^{p,q}_{s+1/p}(\Omega)_{\rm bdd}\big]^M \textit{ is the boundary-to-domain single layer from } (4.3.26), \textit{ and } S \textit{ is the boundary-to-boundary single layer potential operator mapping } \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \\ \textit{the space into } \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \textit{ as in } (4.3.42) \textit{ with } q := p.\end{array} \tag{4.3.46}$$

Finally, if $p \in (1,\infty)$, $q := p$, $s \in (0,1)$, *and* $a := 1 - s - \frac{1}{p}$, *the trace formula* (4.3.44) *also holds when assuming*

$$\begin{array}{l}\mathrm{Tr}_{\Omega\to\partial\Omega} : \big[W^{1,p}_a(\Omega)_{\rm bdd}\big]^M \to \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \textit{ is the boundary trace operator discussed in } [69, \text{Theorem 8.3.6}] \ (\textit{and further extended as indicated in Remark 2 following the statement of the theorem}), \mathscr{S} : \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \to \big[W^{1,p}_a(\Omega)_{\rm bdd}\big]^M \textit{ is the boundary-to-domain single layer from } (4.2.218), \textit{ and } S \textit{ is the boundary-to-boundary single layer potential operator mapping } \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \\ \textit{into } \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \textit{ as in } (4.3.42) \textit{ with } q := p.\end{array} \tag{4.3.47}$$

(4) If, in addition to the original hypotheses, Ω *is also assumed to be a locally uniform domain, then the single layer operator* S *from* (4.3.39) *induces a linear and bounded mapping*

$$S : \big[B^{\infty,\infty}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{\infty,\infty}_{s}(\partial\Omega,\sigma)\big]^M \textit{ for each } s \in (0,1). \tag{4.3.48}$$

Furthermore, this extension satisfies

$$\begin{array}{c} Sf = (\mathscr{S}f)\big|_{\partial\Omega} \textit{ for each } f \in \big[B^{\infty,\infty}_{s-1}(\partial\Omega,\sigma)\big]^M \textit{ where} \\ \mathscr{S}f \textit{ is regarded as a function locally in } \big[\mathscr{C}^s(\overline{\Omega})\big]^M \ (\textit{cf.}\ (4.2.219)).\end{array} \tag{4.3.49}$$

(5) If, in addition to the original hypotheses, Ω *is also assumed to an* (ε,δ)*-domain satisfying an exterior local John condition, then*

$$\begin{array}{l}\textit{the duality formula } (4.3.41) \textit{ remains valid if either } 1 < p < \infty, \\ 0 < q < \infty,\ 0 < s < 1, \textit{ or } p = 1,\ 0 < q \leq 1,\ 0 < s < 1 \ (\textit{with} \\ \textit{the convention made in } [69, (7.6.1)]).\end{array} \tag{4.3.50}$$

Proof Let p, s, p^* be as in (4.3.39) and define

$$q := \Big(\tfrac{1-s}{n-1} + \tfrac{1}{p}\Big)^{-1} \in \big(\tfrac{n-1}{n}, p\big). \tag{4.3.51}$$

In particular,

$$\tfrac{n-1}{n} < q < p \leq 1 \ \text{ and } \ s - 1 = -(n-1)\big(\tfrac{1}{q} - \tfrac{1}{p}\big). \tag{4.3.52}$$

Granted this, [69, (7.8.153)] applies and yields

$$\mathcal{E}_p\Big(\big[H^q(\partial\Omega,\sigma)\big]^M\Big) = \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M. \tag{4.3.53}$$

Also, [69, (7.8.35)] guarantees that

$$\mathcal{E}_p\Big(\big[L^{p^*}(\partial\Omega,\sigma)\big]^M\Big) = \big[L^{p^*}(\partial\Omega,\sigma)\big]^M. \tag{4.3.54}$$

From item *(1)* of Theorem 2.2.6 we know that the single layer operator S induces a linear and bounded mapping in the context of

$$S : \big[H^q(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^{p^*}(\partial\Omega,\sigma)\big]^M. \tag{4.3.55}$$

In turn, from (4.3.53), (4.3.54), (4.3.55), and [69, (7.8.56) in Proposition 7.8.9] we conclude that the single layer operator S from (4.3.55) extends (in a unique fashion) to a linear and bounded mapping

$$\widehat{S} : \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \to \big[L^{p^*}(\partial\Omega,\sigma)\big]^M. \tag{4.3.56}$$

Via embeddings it may be seen that the operators $\widehat{S}$ corresponding to various values of p, s, p^* as in (4.3.39) act in a coherent fashion with one another, and they are also compatible with S from Theorem 2.2.6. As such, we may drop the “hat”, and simply refer to $\widehat{S}$ from (4.3.56) simply as S. This establishes the claims made in relation to (4.3.39). With this in hand, the claims regarding (4.3.38) are then justified using the real interpolation results from [69, (7.4.2)] and [68, (6.2.48)].

Moving on, make the additional assumption that Ω satisfies a two-sided local John condition (cf. [68, Definition 5.11.7]; see also (A.0.104)). In particular, from [68, (5.10.24), (5.11.26)] we conclude that Ω is actually a UR domain. Thanks to this, the results in item *(ix)* of Theorem 1.5.1 imply that for each $p \in (1,\infty)$ the single layer potential operator S induces well-defined, linear, and bounded mappings

$$\begin{aligned} S &: \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^p_1(\partial\Omega,\sigma)\big]^M, \\ S &: \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^p(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{4.3.57}$$

which act in a compatible fashion with one another. Granted this, the claims pertaining to (4.3.40) in the current part *(2)* follow from our real interpolation results obtained in [69, Theorem 11.12.2] and [69, (11.12.60) in Corollary 11.12.3]. Finally, (4.3.41) is seen from what we have proved so far, item *(ix)* in Theorem 1.5.1, and [69, (7.1.62)].

To deal with the claims made in part *(3)* of the theorem, make the additional assumption that Ω is an (ε,δ)-domain and suppose

$$\tfrac{n-1}{n} < p < \infty, \quad 0 < q \leq \infty, \quad (n-1)\big(\tfrac{1}{p} - 1\big)_+ < s < 1. \tag{4.3.58}$$

Then the result in item *(1)* of Theorem 4.3.3 gives that the boundary-to-domain single layer potential operator $\mathscr{S}$ induces a well-defined, linear, and continuous mapping

$$\mathscr{S} : \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \Big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M. \tag{4.3.59}$$

Also, from item *(ii)* of [69, Theorem 9.4.5] we know that the (vector-valued) boundary trace operator

$$\mathrm{Tr}_{\Omega\to\partial\Omega} : \Big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^M \longrightarrow \big[B^{p,q}_{s}(\partial\Omega,\sigma)\big]^M \tag{4.3.60}$$

is well defined, linear, and continuous. Consequently, the composition of the operators in (4.3.59)-(4.3.60), i.e.,

$$\widetilde{S} := \mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathscr{S} : \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,q}_{s}(\partial\Omega,\sigma)\big]^M \tag{4.3.61}$$

is also a well-defined, linear, and continuous operator. The goal now is to show that $\widetilde{S}$ from (4.3.61) agrees with S when acting on "nice" functions, in the common domain.

To this end, fix a sufficiently large aperture parameter $\kappa > 0$ as in [69, Corollary 8.3.9]. Then for each function $f \in \big[\mathrm{Lip}\,(\partial\Omega)\big]^M$ we may rely on [69, Corollary 8.3.9], (1.3.67), [68, (8.8.69)] (also bearing in mind [68, Lemma 5.11.9)] and [68, (5.11.35)]) to write

$$\begin{aligned}\widetilde{S}f &= \big(\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathscr{S}\big)f = \mathrm{Tr}_{\Omega\to\partial\Omega}\big(\mathscr{S}f\big)\\ &= \big(\mathscr{S}f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = Sf \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega.\end{aligned} \tag{4.3.62}$$

Hence, the operators $\widetilde{S}$ and S agree on $\big[\mathrm{Lip}\,(\partial\Omega)\big]^M$. As a consequence of this and the boundedness of (4.3.61), for each $f \in \big[\mathrm{Lip}\,(\partial\Omega)\big]^M \subseteq \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M$ we have $Sf = \widetilde{S}f \in \big[B^{p,q}_{s}(\partial\Omega,\sigma)\big]^M$ and

$$\|Sf\|_{[B^{p,q}_s(\partial\Omega,\sigma)]^M} = \big\|\widetilde{S}f\big\|_{[B^{p,q}_s(\partial\Omega,\sigma)]^M} \leq C\|f\|_{[B^{p,q}_{s-1}(\partial\Omega,\sigma)]^M}, \tag{4.3.63}$$

for some constant $C \in (0,\infty)$ independent of the function f. Given that $\big[\mathrm{Lip}\,(\partial\Omega)\big]^M$ is a dense subset of $\big[B^{p,q}_{s}(\partial\Omega,\sigma)\big]^M$ (cf. [69, Lemma 7.1.10]), we ultimately conclude that the single layer operator S from (4.3.39) extends, in a unique fashion, to a linear and bounded mapping in the context of (4.3.42).

Going further, the fact that the trace formulas (4.3.44)-(4.3.47) are valid is implicit in the manner in which the single layer potential operator S has been defined in the context of (4.3.42) (see (4.3.62) and recall the boundedness properties of $\mathscr{S}$ from (4.3.25), (4.3.26), (4.2.218), together with the boundedness properties $\mathrm{Tr}_{\Omega\to\partial\Omega}$ from [69, (9.4.91) in item *(ii)* of Theorem 9.4.5] as well as [69, (9.4.93) in item *(ii)* of Theorem 9.4.5], and [69, (8.3.38)]).

Under the assumption that Ω is also a locally uniform domain, the claim made in item *(4)* is proved in a similar manner, making use of (4.2.219) and [69, (7.1.59)], now taking $\widetilde{S}$ to be the mapping

$$\big[B^{\infty,\infty}_{s-1}(\partial\Omega,\sigma)\big]^M \ni f \longmapsto (\mathscr{S}f)\big|_{\partial\Omega} \in \big[B^{\infty,\infty}_{s}(\partial\Omega,\sigma)\big]^M. \tag{4.3.64}$$

Finally, consider the claim made in item *(5)*, now assuming that Ω is also (ε,δ)-domain satisfying an exterior local John condition. Together, [68, Proposition 5.11.14] and [68, (5.11.28)] then ensure that Ω is also a locally uniform domain satisfying an interior local John condition. Granted these properties, the claim made in (4.3.50) may be justified based on the continuity/compatibility result just established, the duality formula (4.3.41), the density result from [69, Lemma 7.1.10], and the compatibility of pairings from [69, Proposition 7.6.2]. Specifically, assume that either $1 < p < \infty$, $0 < q < \infty$, $0 < s < 1$, or $p = 1$, $0 < q \leq 1$, $0 < s < 1$. Also, pick $s_* \in (s,1)$ and $p_* \in (1,p')$. Granted these choices, [69, Proposition 7.7.2] gives that

$$B^{p',q'}_{-s}(\partial\Omega,\sigma) \subseteq B^{p_*,p_*}_{-s_*}(\partial\Omega,\sigma). \tag{4.3.65}$$

Also, having fixed an arbitrary $f \in B^{p,q}_{s-1}(\partial\Omega,\sigma)$, [69, Lemma 7.1.10] tells us that there exists a sequence

$$\{f_j\}_{j\in\mathbb{N}} \subseteq \big[\mathrm{Lip}(\partial\Omega)\big]^M \text{ such that } \lim_{j\to\infty} f_j = f \text{ in } \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M. \tag{4.3.66}$$

In particular, (4.3.66) and the continuity of the operator (4.3.42) gives

$$\lim_{j\to\infty} Sf_j = Sf \text{ in } B^{p,q}_{s}(\partial\Omega,\sigma). \tag{4.3.67}$$

From (4.3.42) and (4.3.48), it follows that $S_{L^\top}g \in \big[B^{p',q'}_{1-s}(\partial\Omega,\sigma)\big]^M$ for each $g \in B^{p',q'}_{-s}(\partial\Omega,\sigma)$ it follows . Bearing this in mind we may then write

$$\begin{aligned}
&{}_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^M}\big\langle Sf, g\big\rangle_{[B^{p',q'}_{-s}(\partial\Omega,\sigma)]^M}\\
&\quad = \lim_{j\to\infty} {}_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^M}\big\langle Sf_j, g\big\rangle_{[B^{p',q'}_{-s}(\partial\Omega,\sigma)]^M}\\
&\quad = \lim_{j\to\infty} {}_{[B^{(p_*)',(p_*)'}_{s_*}(\partial\Omega,\sigma)]^M}\big\langle Sf_j, g\big\rangle_{[B^{p_*,q_*}_{-s_*}(\partial\Omega,\sigma)]^M}\\
&\quad = \lim_{j\to\infty} {}_{[B^{(p_*)',(p_*)'}_{s_*-1}(\partial\Omega,\sigma)]^M}\big\langle f_j, S_{L^\top}g\big\rangle_{[B^{p_*,q_*}_{1-s_*}(\partial\Omega,\sigma)]^M}\\
&\quad = \lim_{j\to\infty} {}_{[B^{p,q}_{s-1}(\partial\Omega,\sigma)]^M}\big\langle f_j, S_{L^\top}g\big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^M}\\
&\quad = {}_{[B^{p,q}_{s-1}(\partial\Omega,\sigma)]^M}\big\langle f, S_{L^\top}g\big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^M}.
\end{aligned} \tag{4.3.68}$$

Above, the first equality is implied by (4.3.67) and the continuity of the duality pairing (cf. item *(i)* in [69, Proposition 7.6.1]). The second equality in (4.3.68) uses (4.3.65), the fact that the membership of f_j to $\big[B^{(p_*)',(p_*)'}_{s_*-1}(\partial\Omega,\sigma)\big]^M$ implies $Sf_j \in \big[B^{(p_*)',(p_*)'}_{s_*}(\partial\Omega,\sigma)\big]^M$ (cf. (4.3.40)), and the compatibility of pairings from [69, Proposition 7.6.2]. The third equality in (4.3.68) comes from (4.3.41). The fourth equality in (4.3.68) is once again implied by the compatibility of pairings from [69, Proposition 7.6.2]. The final equality in (4.3.68) is a result of (4.3.66). This establishes (4.3.50), so the proof of Theorem 4.3.4 is complete. □

Our next theorem is concerned with the action of the conormal derivative of the double layer potential operator on boundary Besov spaces.

Theorem 4.3.5 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Next, for some fixed $M \in \mathbb{N}$, consider a coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in (1.3.2) is weakly elliptic (in the sense of* [70, (1.3.3) in Definition 1.3.1]). *Finally, fix*

$$1 < p < \infty, \quad 0 < q \leq \infty, \quad 0 < s < 1. \tag{4.3.69}$$

Then the operator $\partial^A_\nu\mathcal{D}$ from (1.5.31) extends, in a unique fashion, to a bounded linear mapping

$$\partial^A_\nu\mathcal{D} : \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \tag{4.3.70}$$

and various choices of p, q, s as in (4.3.69) yield operators which are compatible with one another. Moreover, if $s \in (0,1)$ and the exponents $p, q, p', q' \in (1,\infty)$ satisfy $1/p + 1/p' = 1 = 1/q + 1/q'$ then

$$\begin{aligned} &{}_{[B^{p,q}_{s-1}(\partial\Omega,\sigma)]^M}\big\langle \partial^A_\nu\mathcal{D}f, g\big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^M} \\ &\qquad = {}_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^M}\big\langle f, \partial^{A^\top}_\nu\mathcal{D}_{A^\top}g\big\rangle_{[B^{p',q'}_{-s}(\partial\Omega,\sigma)]^M} \\ &\text{for each } f \in \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M \text{ and } g \in \big[B^{p',q'}_{1-s}(\partial\Omega,\sigma)\big]^M, \end{aligned} \tag{4.3.71}$$

where $\partial^{A^\top}_\nu\mathcal{D}_{A^\top}$ is associated with $A^\top$ (the real transpose of A) in the same manner $\partial^A_\nu\mathcal{D}$ has been associated with the original coefficient tensor A.

Furthermore, under the additional assumption that Ω is an (ε,δ)-domain, one has the following compatibility result

$$\begin{aligned} &\partial^A_\nu\big(\mathcal{D}f, 0\big) = \partial^A_\nu\big(\psi\mathcal{D}f, L(\psi\mathcal{D}f)\big) = \big(\partial^A_\nu\mathcal{D}\big)f \\ &\text{for all } f \in \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M \text{ and each cutoff function} \\ &\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n) \text{ with } \psi \equiv 1 \text{ near } \partial\Omega, \end{aligned} \tag{4.3.72}$$

in each of the following scenarios:

(a) Assume $q := p$ *and, for each* $f \in \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$*, the conormal derivatives in* (4.3.72) *are considered as in* [69, Remark 8.5.4] *and* [69, Proposition 8.5.3] (*by viewing* $\mathscr{D}f$ *as a function in* $\big[W^{1,p}_a(\Omega)_{\text{bdd}}\big]^M$ *with* $a := 1 - s - \frac{1}{p}$*; cf. Theorem 4.2.3*)*, while* $\big(\partial^A_\nu \mathscr{D}\big)f$ *in the right-hand side of* (4.3.72) *is taken in the sense of* (4.3.70) *with* $q := p$*.*

(b) Assume $q \in (1,\infty)$ *and, for each* $f \in \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M$*, the conormal derivatives in* (4.3.72) *are considered as in* [69, Remark 9.5.3] *and* [69, Proposition 9.5.2] *with* $\mathbb{A} := B$ (*by viewing* $\mathscr{D}f$ *as a function in* $\big[B^{p,q}_{s+1/p}(\Omega)_{\text{bdd}}\big]^M$*; cf. item (1) in Theorem 4.3.1*)*, while* $\big(\partial^A_\nu \mathscr{D}\big)f$ *in the right-hand side of* (4.3.72) *is taken in the sense of* (4.3.70)*.*

(c) Assume $q := p$ *and, for each* $f \in \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$*, the conormal derivatives in* (4.3.72) *are considered as in* [69, Remark 9.5.3] *and* [69, Proposition 9.5.2] *with* $\mathbb{A} := F$ (*by viewing* $\mathscr{D}f$ *as a function in* $\big[F^{p,q_o}_{s+1/p}(\Omega)_{\text{bdd}}\big]^M$ *with* $1 < q_o < \infty$*; cf. item (2) in Theorem 4.3.1*)*, while* $\big(\partial^A_\nu \mathscr{D}\big)f$ *in the right-hand side of* (4.3.72) *is taken in the sense of* (4.3.70) *with* $q := p$*.*

Finally, retaining the additional assumption that Ω *is an* (ε,δ)*-domain, it follows that, given any* $p \in (1,\infty)$ *and* $s \in (0,1)$*, for each* $s_* \in (s,1)$ *the operator*

$$\partial^A_\nu \mathscr{D} : \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \tag{4.3.73}$$

maps $\big[B^{\infty,\infty}_{s_*}(\partial\Omega,\sigma)\big]^M$ *into* $\big[B^{\infty,\infty}_{s_*-1}(\partial\Omega,\sigma)\big]^M$ *and the map induced in this context, i.e.,*

$$\partial^A_\nu \mathscr{D} : \big[B^{\infty,\infty}_{s_*}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{\infty,\infty}_{s_*-1}(\partial\Omega,\sigma)\big]^M, \tag{4.3.74}$$

is well defined, linear, and bounded.

Proof From [68, (5.2.4), (5.10.24), (5.11.26)] and [68,] we see that Ω is actually a UR domain. Having established this, the results in items *(vii)*-*(viii)* of Theorem 1.5.1 imply that for each $p \in (1,\infty)$ the operator $\partial^A_\nu \mathscr{D}$ induces well-defined, linear, and bounded mappings

$$\begin{aligned}
\partial^A_\nu \mathscr{D} &: \big[L^p_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^p(\partial\Omega,\sigma)\big]^M, \\
\partial^A_\nu \mathscr{D} &: \big[L^p(\partial\Omega,\sigma)\big]^M \longrightarrow \big[L^p_{-1}(\partial\Omega,\sigma)\big]^M,
\end{aligned} \tag{4.3.75}$$

which are compatible with each other. Granted this, all claims pertaining to (4.3.70) follow from the real interpolation results from [69, Theorem 11.12.2] and [69, (11.12.60) in Corollary 11.12.3]. Furthermore, (4.3.71) is seen from what we have proved so far, item *(viii)* in Theorem 1.5.1, and [69, (7.1.62)].

To deal with the claims in the second part of the statement of the theorem, make the additional assumption that Ω is an (ε,δ)-domain. Consider first the situation when $f \in \big[L^p_1(\partial\Omega,\sigma)\big]^M$, and fix an aperture parameter $\kappa > 0$. From items *(i)-(ii)* in Theorem 1.5.1 we know that

$$\begin{aligned}&\mathcal{D}f\big|^{\kappa-\text{n.t.}}_{\partial\Omega},\ \big(\nabla\mathcal{D}f\big)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\ \text{ exist at } \sigma\text{-a.e. point on } \partial\Omega, \text{ and}\\ &\qquad\quad \mathcal{N}_\kappa(\mathcal{D}f),\ \mathcal{N}_\kappa(\nabla\mathcal{D}f)\ \text{ belong to }\ L^p(\partial\Omega,\sigma).\end{aligned} \tag{4.3.76}$$

Hence, if we now introduce $u = (u_\beta)_{1\leq\beta\leq M} := \mathcal{D}f$ in Ω, it is then meaningful to consider $\partial^A_\nu u$ in the sense of (A.0.184). In fact, according to (A.0.184) and item *(vii)* in Theorem 1.5.1 (cf. (1.5.29) and (1.5.30)) we have

$$\partial^A_\nu u = \big(\partial^A_\nu\mathcal{D}\big)f. \tag{4.3.77}$$

To proceed, let us now pick an arbitrary function $\varphi \in \big[\mathrm{Lip}\,(\partial\Omega)\big]^M$. It is a classical fact that this may be extended to some $\Phi = (\Phi_\alpha)_{1\leq\alpha\leq M} \in \big[\mathrm{Lip}_c(\mathbb{R}^n)\big]^M$. Also, pick a cutoff function $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ with $\psi \equiv 1$ near $\partial\Omega$. Observe that

$$\big\langle A\nabla(\psi u),\nabla\Phi\big\rangle = a^{\alpha\beta}_{jk}\partial_k(\psi u_\beta)\partial_j\Phi_\alpha\ \text{ belongs to }\ L^1(\Omega,\mathcal{L}^n), \tag{4.3.78}$$

thanks to (4.3.76) and [68, (8.6.51)]. Then, with $p' \in (1,\infty)$ denoting the Hölder conjugate exponent of p, we may write

$$\begin{aligned}\int_{\partial\Omega}&\big\langle\big(\partial^A_\nu\mathcal{D}\big)f,\varphi\big\rangle\,\mathrm{d}\sigma\\ &= \int_{\partial\Omega}\big\langle\partial^A_\nu u,\Phi\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big\rangle\,\mathrm{d}\sigma = \int_{\partial\Omega}\big\langle\partial^A_\nu(\psi u),\Phi\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big\rangle\,\mathrm{d}\sigma\\ &= a^{\alpha\beta}_{jk}\int_\Omega\partial_k(\psi u_\beta)\partial_j\Phi_\alpha\,\mathrm{d}\mathcal{L}^n + \int_\Omega\big\langle L(\psi u),\Phi\big\rangle\,\mathrm{d}\mathcal{L}^n\\ &= {}_{([B^{p',p'}_{1-s}(\partial\Omega,\sigma)]^M)^*}\Big\langle\partial^A_\nu\big(\psi u, L(\psi u)\big),\mathrm{Tr}_{\Omega\to\partial\Omega}\big(\Phi\big|_\Omega\big)\Big\rangle_{[B^{p',p'}_{1-s}(\partial\Omega,\sigma)]^M}\\ &= {}_{([B^{p',p'}_{1-s}(\partial\Omega,\sigma)]^M)^*}\Big\langle\partial^A_\nu\big(\psi\mathcal{D}f, L(\psi\mathcal{D}f)\big),\varphi\Big\rangle_{[B^{p',p'}_{1-s}(\partial\Omega,\sigma)]^M}.\end{aligned} \tag{4.3.79}$$

The first equality above comes from (4.3.77) and the fact that $\Phi\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \Phi\big|_{\partial\Omega} = \varphi$ at every point on $A_\kappa(\partial\Omega)$ (cf. [68, (8.9.10)]), hence at σ-a.e. point on $\partial\Omega$ (cf. [68, Corollary 8.8.9]). The second equality in (4.3.79) is implied by the Green type formula [70, (1.7.121)], whose present applicability is ensured by (4.3.78) and the fact that $L_A(\psi u) \in \big[\mathscr{C}^\infty_c(\Omega)\big]^M$ (cf. (1.3.24)). The third equality uses the generalized Green's formula from [69, (8.5.18)]. To see that the latter is applicable in the present context requires two observations. First, since the space $L^p_1(\partial\Omega,\sigma)$ is contained in $B^{p,p}_s(\partial\Omega,\sigma)$ (cf. [69, (11.11.12)]), Theorem 4.2.3 guarantees that $\psi u \in \big[W^{1,p}_a(\Omega)\big]^M$ with $a := 1 - s - \frac{1}{p}$. Second, the function $\Phi\big|_\Omega$ does belong to $\big[W^{1,p'}_{-a}(\Omega)\big]^M$, as may be seen from [69, (8.3.36)]. The final equality in (4.3.79) is a consequence of the definition of u and [69, (8.3.39)].

In turn, from (4.3.79) and [69, Lemma 7.1.10] we ultimately conclude that if $f \in \big[L_1^p(\partial\Omega,\sigma)\big]^M$ then $\big(\partial_\nu^A \mathcal{D}\big)f$, originally viewed as a function in $\big[L^p(\partial\Omega,\sigma)\big]^M$ (cf. (1.5.31)), induces a linear and continuous functional in $\Big(\big[B_{1-s}^{p',p'}(\partial\Omega,\sigma)\big]^M\Big)^*$ which actually coincides with the conormal derivative $\partial_\nu^A\big(\psi\mathcal{D}f, L(\psi\mathcal{D}f)\big)$. Since both assignments $f \mapsto \big(\partial_\nu^A\mathcal{D}\big)f$ and $f \mapsto \partial_\nu^A\big(\psi\mathcal{D}f, L(\psi\mathcal{D}f)\big)$ are continuous from $\big[B_s^{p,p}(\partial\Omega,\sigma)\big]^M$ into $\big[B_{s-1}^{p,p}(\partial\Omega,\sigma)\big]^M$ (as evident from (4.3.70), and [69, Proposition 8.5.3] in combination with Theorem 4.2.3), we finally conclude, on account of the density of $\big[L_1^p(\partial\Omega,\sigma)\big]^M$ into $\big[B_s^{p,p}(\partial\Omega,\sigma)\big]^M$ (cf. [69, Proposition 11.11.3]), that the identification claimed in (4.3.72) holds in the scenario described in item *(a)* of the theorem.

That (4.3.72) is also valid in the scenarios described in items *(b)*-*(c)* of the theorem is then proved in a similar fashion, making use of [69, Proposition 9.5.2] in place of [69, Proposition 8.5.3], and of Theorem 4.3.1 in place of Theorem 4.2.3.

To deal with the final claim in the statement of the theorem, retain the additional assumption that Ω is an (ε,δ)-domain. In view of the fact that the conormal derivative of the double layer $\partial_\nu^A\mathcal{D}$ does not jump across the boundary (cf. as seen from (1.5.29) and the fact that $\partial_\nu^A\mathcal{D}$ from (4.3.70) is the unique extension of (1.5.31)), there is no loss of generality in assuming that Ω is also bounded (cf. [68, Lemma 5.10.10]). To proceed, fix $p \in (1,\infty)$, $s \in (0,1)$, and $s_* \in (s,1)$. From (4.3.70) we know that the operator (4.3.73) is well defined, linear, and bounded. Also, item *(i)* in [69, Proposition 7.7.2] implies that $\big[B_{s_*}^{\infty,\infty}(\partial\Omega,\sigma)\big]^M$ is a subspace of $\big[B_s^{p,p}(\partial\Omega,\sigma)\big]^M$. Given any $f \in \big[B_{s_*}^{\infty,\infty}(\partial\Omega,\sigma)\big]^M$ set $u := \mathcal{D}f$ and use (4.3.72) and [69, (8.5.32)] to write

$$(\partial_\nu^A\mathcal{D})f = \partial_\nu^A(\mathcal{D}f,0) = \partial_\nu^A(u,0) = \partial_\nu^A u \in \big[B_{s_*-1}^{\infty,\infty}(\partial\Omega,\sigma)\big]^M, \tag{4.3.80}$$

where the conormal derivative $\partial_\nu^A u$ is defined as in [69, (8.5.29)-(8.5.30)]. This proves that the operator (4.3.73) maps $\big[B_{s_*}^{\infty,\infty}(\partial\Omega,\sigma)\big]^M$ into $\big[B_{s_*-1}^{\infty,\infty}(\partial\Omega,\sigma)\big]^M$. Finally, from (4.3.80), [69, (8.5.31)], (1.8.14), (1.8.9), and [69, (7.1.59)] we deduce that

$$\begin{aligned}
\big\|(\partial_\nu^A\mathcal{D})f\big\|_{[B_{s_*-1}^{\infty,\infty}(\partial\Omega,\sigma)]^M} &= \big\|\partial_\nu^A u\big\|_{[B_{s_*-1}^{\infty,\infty}(\partial\Omega,\sigma)]^M} \leq C\Big\|\delta_{\partial\Omega}^{1-s_*}\big|\nabla u\big|\Big\|_{L^\infty(\Omega,\mathcal{L}^n)} \\
&= C\Big\|\delta_{\partial\Omega}^{1-s_*}\big|\nabla(\mathcal{D}f)\big|\Big\|_{L^\infty(\Omega,\mathcal{L}^n)} \leq C\|f\|_{[\dot{\mathscr{C}}^{s_*}(\partial\Omega)]^M} \\
&\leq C\|f\|_{[B_{s_*}^{\infty,\infty}(\partial\Omega)]^M}.
\end{aligned} \tag{4.3.81}$$

This establishes the fact that the operator (4.3.73) induces a well defined, linear, and bounded mapping in the context of (4.3.74). □

The conormal derivative of the boundary-to-domain single layer potential operator acting on Besov spaces satisfies jump-formulas of the sort described in the theorem below.

Theorem 4.3.6 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an* NTA *domain with a compact Ahlfors regular boundary. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Next, for some* $M \in \mathbb{N}$*, consider a coefficient tensor* $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ *with complex entries, with the property that the* $M \times M$ *homogeneous second-order system* $L = L_A$ *associated with* A *in* $\mathbb{R}^n$ *as in* (1.3.2) *is weakly elliptic* (*in the sense of* [70, (1.3.3) in Definition 1.3.1]). *Recall the boundary-to-domain single layer potential operator* $\mathscr{S}$ (*associated with* L *and* Ω) *acting in the context of Besov spaces as in Proposition 4.2.8, and the boundary-to-boundary "transpose" double layer potential operator* $K^{\#}_{A^\top}$ (*associated with* $A^\top$ *and* Ω) *acting in the context of Besov spaces as in Theorem 4.1.5. Finally, fix*

$$1 < p < \infty, \quad 1 < q < \infty, \quad 0 < s < 1. \tag{4.3.82}$$

Then the jump-formula

$$\begin{aligned} &\partial_\nu^A\big(\mathscr{S}f, 0\big) = \partial_\nu^A\big(\psi\mathscr{S}f, \phi\big) = \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f \\ &\text{for all } f \in \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M \text{ and all } \psi \in \mathscr{C}^\infty_c(\mathbb{R}^n) \\ &\text{with } \psi \equiv 1 \text{ near } \partial\Omega, \text{ where } \phi := L(\psi\mathscr{S}f) \in \big[\mathscr{C}^\infty_c(\Omega)\big]^M, \end{aligned} \tag{4.3.83}$$

(*with* I *denoting the identity operator*) *holds in each of the following scenarios:*

(a) Assume $q := p$*, for each* $f \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M$ *consider the conormal derivatives in* (4.3.83) *as in* [69, Remark 8.5.4] *and* [69, Proposition 8.5.3] (*by regarding* $\mathscr{S}f$ *as a function in* $\big[W^{1,p}_a(\Omega)_{\rm bdd}\big]^M$ *with* $a := 1 - s - \frac{1}{p}$*; cf. Theorem 4.2.10*), *and interpret* $K^{\#}_{A^\top}$ *as in* (4.1.27) (*with* $A^\top$ *in place of* A*, with* $q := p$*, and with* s *replaced by* $1 - s$).

(b) For each $f \in \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M$*, the conormal derivatives in* (4.3.83) *are considered as in* [69, Remark 9.5.3] *and* [69, Proposition 9.5.2] *with* $\mathbb{A} := B$ (*by viewing* $\mathscr{S}f$ *in* $\big[B^{p,q}_{s+1/p}(\Omega)_{\rm bdd}\big]^M$*; cf. item (1) in Theorem 4.3.3*), *while* $K^{\#}_{A^\top}$ *in* (4.3.83) *is understood as in* (4.1.27) (*with* $A^\top$ *in place of* A*, and with* s *replaced by* $1 - s$).

(c) Assume $q := p$ *and, for each* $f \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M$*, the conormal derivatives in* (4.3.83) *are considered in the sense of* [69, Remark 9.5.3] *and* [69, Proposition 9.5.2] *with* $\mathbb{A} := F$ (*by viewing* $\mathscr{S}f$ *as a function in* $\big[F^{p,q_o}_{s+1/p}(\Omega)_{\rm bdd}\big]^M$ *with* $1 < q_o < \infty$*; cf. item (2) in Theorem 4.3.3*), *while* $K^{\#}_{A^\top}$ *in* (4.3.83) *is understood as in* (4.1.27) (*with* $A^\top$ *in place of* A*, with* $q := p$*, and with* s *replaced by* $1 - s$).

Proof Let us establish the jump-formula (4.3.83) in the scenario described in item *(a)* in the statement of the theorem. For starters, observe that [68, (5.10.24)] implies that Ω is actually a UR domain. Granted this, Corollary 4.1.4 applies and gives that

$$K^{\#}_{A^\top} : \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \text{ boundedly.} \tag{4.3.84}$$

In addition, having fixed an arbitrary cutoff function $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ with $\psi \equiv 1$ near $\partial\Omega$, from Theorem 4.2.10 and [69, Proposition 8.5.3] we also see that

$$\begin{gathered}\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \ni f \longmapsto \partial^A_\nu\big(\psi\mathscr{S}f,\phi_f\big) \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \\ \text{where } \phi_f := L(\psi\mathscr{S}f) \in \big[\mathscr{C}^\infty_c(\Omega)\big]^M,\end{gathered} \tag{4.3.85}$$

is a well-defined, linear, and bounded operator. On account of (4.3.84)-(4.3.85), the jump-formula claimed in (4.3.83) follows as soon as we show that the operators in (4.3.84)-(4.3.85) agree on a dense subset of $\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M$. By [69, Lemma 7.1.10], it therefore suffices to prove that

$$\partial^A_\nu\big(\psi\mathscr{S}f,\phi_f\big) = \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f \text{ for each } f \in \big[\mathrm{Lip}\,(\partial\Omega)\big]^M. \tag{4.3.86}$$

With this goal in mind, fix an arbitrary $f \in \big[\mathrm{Lip}\,(\partial\Omega)\big]^M$ and set

$$u = (u_\beta)_{1\le\beta\le M} := \psi\mathscr{S}f \in \big[\mathscr{C}^\infty(\Omega)\big]^M. \tag{4.3.87}$$

Also, fix an aperture parameter $\kappa > 0$. From (4.3.87) and item *(x)* in Theorem 1.5.1 we see that

$$\begin{gathered} u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ (\nabla u)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exist at } \sigma\text{-a.e. point on } \partial\Omega, \\ Lu = \phi_f \in \big[\mathscr{C}^\infty_c(\Omega)\big]^M \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma). \end{gathered} \tag{4.3.88}$$

It is then meaningful to consider $\partial^A_\nu u$ in the pointwise sense of (A.0.184). In fact, according to item *(xi)* in Theorem 1.5.1 we have

$$\partial^A_\nu u = \big(-\tfrac{1}{2}I + K^{\#}_{A^\top}\big)f. \tag{4.3.89}$$

Going further, let us now pick an arbitrary function $\varphi \in \big[\mathrm{Lip}\,(\partial\Omega)\big]^M$. As is well known, there exists some $\Phi = (\Phi_\alpha)_{1\le\alpha\le M} \in \big[\mathrm{Lip}_c(\mathbb{R}^n)\big]^M$ such that $\Phi\big|_{\partial\Omega} = \varphi$. Thanks to (4.3.88) and [68, (8.6.51)] we have

$$\big\langle A\nabla u, \nabla\Phi\big\rangle = a^{\alpha\beta}_{jk}(\partial_k u_\beta)(\partial_j\Phi_\alpha) \text{ belongs to } L^1(\Omega,\mathscr{L}^n), \tag{4.3.90}$$

Let $p' \in (1,\infty)$ denote the Hölder conjugate exponent of p. Then, with justifications to be provided momentarily, we may compute

$$\begin{aligned}
&{}_{([B^{p',p'}_{1-s}(\partial\Omega,\sigma)]^M)^*}\Big\langle\big(-\tfrac{1}{2}I+K^{\#}_{A^\top}\big)f,\varphi\Big\rangle_{[B^{p',p'}_{1-s}(\partial\Omega,\sigma)]^M} \\
&\quad=\int_{\partial\Omega}\big\langle\big(-\tfrac{1}{2}I+K^{\#}_{A^\top}\big)f,\varphi\big\rangle\,\mathrm{d}\sigma=\int_{\partial\Omega}\big\langle\partial^A_\nu u,\Phi\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big\rangle\,\mathrm{d}\sigma \\
&\quad=\int_\Omega\big\langle\phi_f,\Phi\big\rangle\,\mathrm{d}\mathcal{L}^n+a^{\alpha\beta}_{jk}\int_\Omega\partial_k u_\beta\partial_j\Phi_\alpha\,\mathrm{d}\mathcal{L}^n \\
&\quad={}_{([B^{p',p'}_{1-s}(\partial\Omega,\sigma)]^M)^*}\Big\langle\partial^A_\nu(u,\phi_f),\mathrm{Tr}_{\Omega\to\partial\Omega}\big(\Phi\big|_\Omega\big)\Big\rangle_{[B^{p',p'}_{1-s}(\partial\Omega,\sigma)]^M} \\
&\quad={}_{([B^{p',p'}_{1-s}(\partial\Omega,\sigma)]^M)^*}\Big\langle\partial^A_\nu(\psi\mathscr{S}f,\phi_f),\varphi\Big\rangle_{[B^{p',p'}_{1-s}(\partial\Omega,\sigma)]^M}.
\end{aligned}\tag{4.3.91}$$

Above, the first equality comes from (4.3.89) and the fact that $\Phi\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}=\Phi\big|_{\partial\Omega}=\varphi$ at every point on $A_\kappa(\partial\Omega)$ (cf. [68, (8.9.10)]), hence at σ-a.e. point on $\partial\Omega$ (cf. [68, Corollary 8.8.9]). In view of the fact that $Lu=\phi_f$ in Ω (cf. (4.3.88)), the second equality in (4.3.91) is implied by the Green type formula [70, (1.7.121)], whose present applicability is ensured by (4.3.90). The third equality in (4.3.91) uses the generalized Green's formula from [69, (8.5.18)], which is applicable in the present context. Indeed, since $\mathrm{Lip}\,(\partial\Omega)$ is contained in $B^{p,p}_s(\partial\Omega,\sigma)$ (cf. [69, (7.1.62)]), Theorem 4.2.10 ensures that $u\in\big[W^{1,p}_a(\Omega)\big]^M$ with $a:=1-s-\frac{1}{p}$. Also, [69, (8.3.36)] implies that $\Phi\big|_\Omega\in\big[W^{1,p'}_{-a}(\Omega)\big]^M$. The last equality in (4.3.91) is a consequence of the definition of u and [69, (8.3.39)]. Thus, (4.3.91) is fully justified.

At this stage, from the resulting identity in (4.3.91) and [69, Lemma 7.1.10] we may conclude that $\big(-\frac{1}{2}I+K^{\#}_{A^\top}\big)f$, originally viewed as a function in $\big[L^p(\partial\Omega,\sigma)\big]^M$ (cf. (1.5.12)), induces a linear and continuous functional in $\Big(\big[B^{p',p'}_{1-s}(\partial\Omega,\sigma)\big]^M\Big)^*$ which actually coincides with the conormal derivative $\partial^A_\nu(\psi\mathscr{S}f,\phi_f)$. This establishes (4.3.86) which, together with [69, (8.5.23)], proves the jump-formula (4.3.83) in the scenario described in item *(a)*.

Finally, that (4.3.83) is also valid in the scenarios described in items *(b)*-*(c)* of the theorem may be justified in a similar manner, now relying on [69, Remark 9.5.3] in place of [69, Remark 8.5.4], and of Theorem 4.3.3 in place of Theorem 4.2.10. □

4.4 Integral Representation Formulas of Layer Potential Type, and Consequences

The first main result in this section is a basic integral representation formula of a null-solution of a weakly elliptic system, which allows us to recover such a function from the action of the (boundary-to-domain) double layer operator on the boundary trace of said function and the action of the (boundary-to-domain) single layer operator on the conormal derivative of the original function.

Theorem 4.4.1 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, be an (ε, δ)-domain with a compact Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Next, for some $M \in \mathbb{N}$, consider a coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1 \leq r,s \leq n \\ 1 \leq \alpha,\beta \leq M}}$ with complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in* (1.3.2) *is weakly elliptic* (*in the sense of* [70, (1.3.3) in Definition 1.3.1]). *Finally, let $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ be a null-solution of the system L in Ω which, in the case when Ω is an exterior domain, satisfies*

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |u| \, \mathrm{d}\mathcal{L}^2 = o(1) \text{ as } R \to \infty, \tag{4.4.1}$$

for some $\lambda \in (1, \infty)$.

Then the integral representation formula

$$u = \mathcal{D}\big(\mathrm{Tr}_{\Omega \to \partial\Omega}\, u\big) - \mathcal{S}\big(\partial_\nu^A(u, 0)\big) \text{ in } \Omega \tag{4.4.2}$$

holds in any of the following scenarios:

(1) One assumes that

$$\begin{gathered} u \text{ belongs to the Besov space } \big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\mathrm{bdd}}\big]^M \text{ with} \\ 1 < p < \infty, \quad 0 < s < 1, \quad 1 < q < \infty, \end{gathered} \tag{4.4.3}$$

$\mathrm{Tr}_{\Omega \to \partial\Omega} : \big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\mathrm{bdd}}\big]^M \to \big[B^{p,q}_s(\partial\Omega, \sigma)\big]^M$ *is the boundary trace operator from* [69, (9.4.105)] (*thus,* $\mathrm{Tr}_{\Omega \to \partial\Omega}\, u$ *lies in the space* $\big[B^{p,q}_s(\partial\Omega, \sigma)\big]^M$), (4.4.4)

$\mathcal{D} : \big[B^{p,q}_s(\partial\Omega, \sigma)\big]^M \to \big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\mathrm{bdd}}\big]^M$ *is the boundary-to-domain double layer potential operator associated with A and Ω, which is well defined in such a context by item (1) in Theorem 4.3.1,* (4.4.5)

∂_ν^A *is the conormal derivative operator associated with A and Ω as in* [69, Remark 9.5.3] *with $A := B$* (*so* $\partial_\nu^A(u, 0) \in \big[B^{p,q}_{s-1}(\partial\Omega, \sigma)\big]^M$), (4.4.6)

and

$\mathcal{S} : \big[B^{p,q}_{s-1}(\partial\Omega, \sigma)\big]^M \to \big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\mathrm{bdd}}\big]^M$ *is the boundary-to-domain single layer operator* (*associated with A and Ω*) *as in* (4.3.25). (4.4.7)

(2) One assumes that

$$\begin{gathered}\textit{u belongs to the Triebel-Lizorkin space } \big[F^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\big]^M \textit{ with}\\ 1<p<\infty,\quad 0<s<1,\quad 1<q<\infty,\end{gathered}\tag{4.4.8}$$

$$\begin{gathered}\mathrm{Tr}_{\Omega\to\partial\Omega}:\big[F^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\big]^M\to\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \textit{ is the boundary trace}\\ \textit{operator from } [69, (9.4.106)]\ (\textit{hence } \mathrm{Tr}_{\Omega\to\partial\Omega}u\in\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M),\end{gathered}\tag{4.4.9}$$

$$\begin{gathered}\mathscr{D}:\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M\to\big[F^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\big]^M \textit{ is the boundary-to-domain}\\ \textit{double layer potential operator associated with } A \textit{ and } \Omega\textit{, which is well}\\ \textit{defined in such a context by item (2) in Theorem 4.3.1,}\end{gathered}\tag{4.4.10}$$

$$\begin{gathered}\partial^A_\nu \textit{ is the conormal derivative operator associated with } A \textit{ and } \Omega \textit{ as in}\\ [69, \text{Remark } 9.5.3] \textit{ with } A:=F\ (\textit{so } \partial^A_\nu(u,0)\in\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M),\end{gathered}\tag{4.4.11}$$

and

$$\begin{gathered}\mathscr{S}:\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M\to\big[F^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\big]^M \textit{ is the boundary-to-domain}\\ \textit{single layer operator (associated with } A \textit{ and } \Omega\textit{) as in (4.3.26).}\end{gathered}\tag{4.4.12}$$

(3) One also assumes that $\mathbb{R}^n\setminus\Omega$ *is n-thick, and that*

$$\begin{gathered}\textit{u belongs to the weighted Sobolev space } \big[W^{1,p}_a(\Omega)_{\rm bdd}\big]^M,\ \textit{with}\\ 1<p<\infty,\quad 0<s<1,\quad a:=1-s-\tfrac{1}{p},\end{gathered}\tag{4.4.13}$$

$$\begin{gathered}\mathrm{Tr}_{\Omega\to\partial\Omega}:\big[W^{1,p}_a(\Omega)_{\rm bdd}\big]^M\to\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \textit{ is the trace from}\\ [69, \text{Theorem } 8.3.6]\ (\textit{so that } \mathrm{Tr}_{\Omega\to\partial\Omega}u\in\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \textit{ by } [69,\\ (8.3.38)]),\end{gathered}\tag{4.4.14}$$

$$\begin{gathered}\mathscr{D}:\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M\to\big[W^{1,p}_a(\Omega)_{\rm bdd}\big]^M \textit{ is the boundary-to-domain}\\ \textit{double layer potential operator associated with } A \textit{ and } \Omega\textit{, which is well}\\ \textit{defined in the present context thanks to Theorem 4.2.3,}\end{gathered}\tag{4.4.15}$$

$$\partial_\nu^A \textit{ is the conormal derivative operator associated with } A \textit{ and } \Omega \textit{ as in } [69, \text{Remark 8.5.4}] \ (\textit{so, } \partial_\nu^A(u,0) \textit{ ends up in } \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M), \tag{4.4.16}$$

and

$$\mathscr{S} : \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \to \big[W^{1,p}_a(\Omega)_{\text{bdd}}\big]^M \textit{ is the boundary-to-domain single layer operator (associated with } A \textit{ and } \Omega) \textit{ as in (4.2.218).} \tag{4.4.17}$$

Furthermore, if Ω *is an exterior domain and, in place of* (4.4.1)*, one now only assumes*

$$\fint_{B(0,\lambda R)\setminus B(0,R)} |u|\,\mathrm{d}\mathcal{L}^2 = o(R) \ \textit{ as } \ R \to \infty, \tag{4.4.18}$$

for some $\lambda \in (1,\infty)$*, then in place of the integral representation formula* (4.4.2) *one now concludes that there exists* $c \in \mathbb{C}^M$ *such that*

$$u = \mathscr{D}\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u\big) - \mathscr{S}\big(\partial_\nu^A(u,0)\big) + c \ \textit{ in } \ \Omega, \tag{4.4.19}$$

in any of the scenarios described in items (1)-(3) above.

Proof Consider the scenario specified in item *(1)*. Fix a point $x \in \Omega$ and choose a scalar-valued function $\theta \in \mathscr{C}^\infty(\mathbb{R}^n)$ with the property that $\theta = 0$ on $B(0,1)$ and $\theta = 1$ on $\mathbb{R}^n \setminus B(0,2)$. For each

$$\varepsilon \in \Big(0, \min\big\{1, \tfrac{1}{2}\,\mathrm{dist}(x,\partial\Omega)\big\}\Big) \tag{4.4.20}$$

define $\theta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by setting

$$\theta_\varepsilon(y) := \theta\Big(\frac{y-x}{\varepsilon}\Big) \ \text{ for every } \ y \in \mathbb{R}^n. \tag{4.4.21}$$

Then $\theta_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n)$ is a bounded function satisfying

$$\lim_{\varepsilon\to 0^+} \theta_\varepsilon(y) = 1 \ \text{ for every } \ y \in \mathbb{R}^n \setminus \{x\}, \tag{4.4.22}$$

and there exists a constant $C \in (0,\infty)$ such that for each ε as in (4.4.20) we have

$$\begin{gathered} 1-\theta_\varepsilon \in \mathscr{C}^\infty_c(\Omega), \quad \theta_\varepsilon \equiv 0 \ \text{ on } \ B(x,\varepsilon), \\ \mathrm{supp}\,(\nabla\theta_\varepsilon) \subseteq \overline{B(x,2\varepsilon)} \setminus B(x,\varepsilon), \ \text{ and} \\ |(\nabla^j\theta_\varepsilon)(y)| \leq C\varepsilon^{-j} \ \text{ for every } \ j \in \mathbb{N}_0 \ \text{ and every } \ y \in \mathbb{R}^n. \end{gathered} \tag{4.4.23}$$

Also, pick a scalar-valued function $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ such that $\psi = 1$ on $B(0,1)$, and for each $R > 0$ define $\psi_R := \psi(\cdot/R)$. Henceforth, fix $R \in (0,\infty)$ sufficiently large, as to have

$$\partial\Omega \subseteq B(0,R) \ \text{ as well as } \ B(x,1) \subseteq B(0,R). \tag{4.4.24}$$

To proceed, bring in the matrix-valued fundamental solution $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ associated with the system L as in [70, Theorem 1.4.2]. Fix $\gamma \in \{1,\dots,M\}$ and observe that $E_{\gamma\beta}(x-\cdot)\theta_\varepsilon \in \mathscr{C}^\infty(\overline{\Omega})$ for each $\beta \in \{1,\dots,M\}$. Hence,

$$\begin{aligned} & w := \big(E_{\gamma\beta}(x-\cdot)\theta_\varepsilon\psi_R\big)_{1\leq\beta\leq M} \in \big[\mathscr{C}^\infty(\overline{\Omega})\big]^M \\ & \text{and } w \text{ vanishes outside of a bounded subset of } \Omega. \end{aligned} \tag{4.4.25}$$

As a consequence, $w \in \Big[B^{p',q'}_{1-s+\frac{1}{p'}}(\Omega)\Big]^M$ where p', q' are the Hölder conjugate exponents of p, q. Since the assumptions on p, q, s from (4.4.3) ensure that

$$B^{p,q}_{s+1/p}(\Omega) \hookrightarrow L^p(\Omega, \mathcal{L}^n) \hookrightarrow L^1(\Omega, \mathcal{L}^n), \tag{4.4.26}$$

(recall that Ω has finite measure), we my also define $g \in \Big[\big(B^{p,q}_{s+1/p}(\Omega)\big)^*\Big]^M$ by setting

$$\begin{aligned} & {}_{[(B^{p,q}_{s+1/p}(\Omega))^*]^M}\big\langle g, G\big\rangle_{[B^{p,q}_{s+1/p}(\Omega)]^M} := \int_\Omega (L^\top w)_\alpha G_\alpha \,\mathrm{d}\mathcal{L}^n \\ & \text{for each } G = (G_\alpha)_{1\leq\alpha\leq M} \in \Big[B^{p,q}_{s+\frac{1}{p}}(\Omega)\Big]^M. \end{aligned} \tag{4.4.27}$$

In particular, the function $L^\top w \in \big[\mathscr{C}^\infty(\overline{\Omega})\big]^M$ satisfies

$$L^\top w = g\lfloor_\Omega \text{ in } \big[\mathcal{D}'(\Omega)\big]^M. \tag{4.4.28}$$

We now invoke the version of the generalized "full" Green's formula [69, (9.5.20)] for Besov spaces, applied with u the function from the statement of the theorem (which is currently assumed to satisfy (4.4.3)), with $f := 0 \in \Big[\big(B^{p',q'}_{1-s+1/p'}(\Omega)\big)^*\Big]^M$, and with w and g as above. Specifically, having selected some cutoff function $\xi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ with $\xi \equiv 1$ near both $\partial\Omega$ and the support of w (playing the role of ψ in [69, (9.5.20)]), we obtain

$$\begin{aligned} & {}_{([B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^M)^*}\big\langle \partial^A_\nu(u,0), \mathrm{Tr}_{\Omega\to\partial\Omega}\, w\big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^M} \\ & \qquad - {}_{([B^{p,q}_{s}(\partial\Omega,\sigma)]^M)^*}\big\langle \partial^{A^\top}_\nu(w,g), \mathrm{Tr}_{\Omega\to\partial\Omega}\, u\big\rangle_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^M} \\ & \qquad = -{}_{[(B^{p,q}_{s+1/p}(\Omega))^*]^M}\big\langle g, \xi u\big\rangle_{[B^{p,q}_{s+1/p}(\Omega)]^M} \\ & \qquad = -\int_\Omega (L^\top w)_\alpha \xi u_\alpha \,\mathrm{d}\mathcal{L}^n = -\int_\Omega (L^\top w)_\alpha u_\alpha \,\mathrm{d}\mathcal{L}^n, \end{aligned} \tag{4.4.29}$$

where the last equality comes from (4.4.27). Note that thanks to (4.4.25) and Proposition 4.2.8 we have

$$
{}_{([B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^M)^*}\Big\langle \partial^A_\nu(u,0), \operatorname{Tr}_{\Omega\to\partial\Omega} w\Big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^M} = \Big(\mathscr{S}\big(\partial^A_\nu(u,0)\big)\Big)_\gamma(x). \tag{4.4.30}
$$

Also, if $(\nu_1,\dots,\nu_n)$ are the components of ν then

$$
\begin{aligned}
&{}_{([B^{p,q}_{s}(\partial\Omega,\sigma)]^M)^*}\Big\langle \partial^{A^\top}_\nu(w,g), \operatorname{Tr}_{\Omega\to\partial\Omega} u\Big\rangle_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^M} \\
&\qquad = \int_{\partial_*\Omega} \nu_j a^{\beta\alpha}_{kj}(\partial_k w_\beta)\big|_{\partial\Omega}\big(\operatorname{Tr}_{\Omega\to\partial\Omega} u\big)_\alpha \, d\sigma \\
&\qquad = -\int_{\partial_*\Omega} \nu_j a^{\beta\alpha}_{kj}(\partial_k E_{\gamma\beta})(x-\cdot)\big(\operatorname{Tr}_{\Omega\to\partial\Omega} u\big)_\alpha \, d\sigma \\
&\qquad = \Big(\mathcal{D}\big(\operatorname{Tr}_{\Omega\to\partial\Omega} u\big)\Big)_\gamma(x),
\end{aligned} \tag{4.4.31}
$$

by [69, Proposition 9.5.4], (4.4.25), and (1.3.18). Let us take a second look at the last integral in (4.4.29). For starters, use the product rule and support considerations (cf. (4.4.24) and (4.4.20)) to expand

$$
\begin{aligned}
-(L^\top w)_\alpha u_\alpha &= -a^{\beta\alpha}_{kj}(\partial_k\partial_j w_\beta)u_\alpha \\
&= a^{\beta\alpha}_{kj}(\partial_j E_{\gamma\beta})(x-\cdot)(\partial_k\theta_\varepsilon)u_\alpha + a^{\beta\alpha}_{kj}(\partial_j E_{\gamma\beta})(x-\cdot)(\partial_k\psi_R)u_\alpha \\
&\quad + a^{\beta\alpha}_{kj}(\partial_k E_{\gamma\beta})(x-\cdot)(\partial_j\theta_\varepsilon)u_\alpha + a^{\beta\alpha}_{kj}(\partial_k E_{\gamma\beta})(x-\cdot)(\partial_j\psi_R)u_\alpha \\
&\quad - a^{\beta\alpha}_{kj}E_{\gamma\beta}(x-\cdot)(\partial_j\partial_k\theta_\varepsilon)u_\alpha - a^{\beta\alpha}_{kj}E_{\gamma\beta}(x-\cdot)(\partial_j\partial_k\psi_R)u_\alpha \\
&\quad - a^{\beta\alpha}_{kj}(\partial_j\partial_k E_{\gamma\beta})(x-\cdot)\theta_\varepsilon\psi_R u_\alpha,
\end{aligned} \tag{4.4.32}
$$

and observe that the last term above vanishes in $\Omega\setminus\{x\}$ since for each $\alpha\in\{1,\dots,M\}$ we have

$$
a^{\beta\alpha}_{kj}(\partial_j\partial_k E_{\gamma\beta})(x-\cdot) = 0 \ \text{ in } \ \mathbb{R}^n\setminus\{x\}, \tag{4.4.33}
$$

by [70, (1.4.33)]. Thus,

$$
\begin{aligned}
-\int_\Omega (L^\top w)_\alpha u_\alpha \, d\mathcal{L}^n &= \mathrm{I}_\varepsilon + \mathrm{I}'_\varepsilon + \mathrm{II}_\varepsilon + \mathrm{II}''_\varepsilon + \mathrm{III}_\varepsilon + \mathrm{III}'_\varepsilon \\
&\quad + \mathrm{IV}_R + \mathrm{V}_R + \mathrm{VI}_R,
\end{aligned} \tag{4.4.34}
$$

where

$$
\begin{aligned}
\mathrm{I}_\varepsilon &:= \int_\Omega a^{\beta\alpha}_{kj}(\partial_j E_{\gamma\beta})(x-y)(\partial_k\theta_\varepsilon)(y)\big(u_\alpha(y)-u_\alpha(x)\big)\,\mathrm{d}y,\\
\mathrm{II}_\varepsilon &:= \int_\Omega a^{\beta\alpha}_{kj}(\partial_k E_{\gamma\beta})(x-y)(\partial_j\theta_\varepsilon)(y)\big(u_\alpha(y)-u_\alpha(x)\big)\,\mathrm{d}y, \qquad (4.4.35)\\
\mathrm{III}_\varepsilon &:= -\int_\Omega a^{\beta\alpha}_{kj} E_{\gamma\beta}(x-y)(\partial_j\partial_k\theta_\varepsilon)(y)\big(u_\alpha(y)-u_\alpha(x)\big)\,\mathrm{d}y,
\end{aligned}
$$

then

$$
\begin{aligned}
\mathrm{I}'_\varepsilon &:= \Big\{\int_\Omega a^{\beta\alpha}_{kj}(\partial_j E_{\gamma\beta})(x-y)(\partial_k\theta_\varepsilon)(y)\,\mathrm{d}y\Big\}u_\alpha(x),\\
\mathrm{II}'_\varepsilon &:= \Big\{\int_\Omega a^{\beta\alpha}_{kj}(\partial_k E_{\gamma\beta})(x-y)(\partial_j\theta_\varepsilon)(y)\,\mathrm{d}y\Big\}u_\alpha(x), \qquad (4.4.36)\\
\mathrm{III}'_\varepsilon &:= -\Big\{\int_\Omega a^{\beta\alpha}_{kj} E_{\gamma\beta}(x-y)(\partial_j\partial_k\theta_\varepsilon)(y)\,\mathrm{d}y\Big\}u_\alpha(x),
\end{aligned}
$$

and, finally,

$$
\begin{aligned}
\mathrm{IV}_R &:= \int_\Omega a^{\beta\alpha}_{kj}(\partial_j E_{\gamma\beta})(x-y)(\partial_k\psi_R)(y)u_\alpha(y)\,\mathrm{d}y,\\
\mathrm{V}_R &:= \int_\Omega a^{\beta\alpha}_{kj}(\partial_k E_{\gamma\beta})(x-y)(\partial_j\psi_R)(y)u_\alpha(y)\,\mathrm{d}y, \qquad (4.4.37)\\
\mathrm{VI}_R &:= -\int_\Omega a^{\beta\alpha}_{kj} E_{\gamma\beta}(x-y)(\partial_j\partial_k\psi_R)(y)u_\alpha(y)\,\mathrm{d}y.
\end{aligned}
$$

In relation to I_ε we note that since u is continuous at x we may estimate (using [70, (1.4.24)] and [70, (1.5.25)])

$$
\limsup_{\varepsilon\to 0^+}\big|\mathrm{I}_\varepsilon\big| \leq C\limsup_{\varepsilon\to 0^+} ⨍_{B(x,2\varepsilon)}\big|u_\alpha(y)-u_\alpha(x)\big|\,\mathrm{d}y = 0. \qquad (4.4.38)
$$

As regards the expression appearing in the curly brackets in I'_ε, by relying on (4.4.23) and [70, (1.4.33)] we may conclude that for each fixed index $\alpha \in \{1,\dots,M\}$ we have

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+} \int_\Omega a^{\beta\alpha}_{kj}(\partial_j E_{\gamma\beta})(x-y)(\partial_k\theta_\varepsilon)(y)\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0^+} \int_\Omega a^{\beta\alpha}_{kj}(\partial_j E_{\gamma\beta})(x-y)\partial_k(\theta_\varepsilon - 1)(y)\,\mathrm{d}y \\
&= -\lim_{\varepsilon\to 0^+} {}_{\mathcal{D}'(\Omega)}\Big\langle a^{\beta\alpha}_{kj}\partial_j\big[E_{\gamma\beta}(x-\cdot)\big], \partial_k(\theta_\varepsilon - 1)\Big\rangle_{\mathcal{D}(\Omega)} \\
&= \lim_{\varepsilon\to 0^+} {}_{\mathcal{D}'(\Omega)}\Big\langle a^{\beta\alpha}_{kj}\partial_k\partial_j\big[E_{\gamma\beta}(x-\cdot)\big], \theta_\varepsilon - 1\Big\rangle_{\mathcal{D}(\Omega)} \\
&= \delta_{\alpha\gamma}\lim_{\varepsilon\to 0^+} {}_{\mathcal{D}'(\Omega)}\big\langle \delta_x, \theta_\varepsilon - 1\big\rangle_{\mathcal{D}(\Omega)} \\
&= \delta_{\alpha\gamma}\lim_{\varepsilon\to 0^+}(\theta_\varepsilon - 1)(x) = -\delta_{\alpha\gamma}.
\end{aligned}
\tag{4.4.39}
$$

Consequently,

$$
\lim_{\varepsilon\to 0^+} \mathrm{I}'_\varepsilon = -u_\gamma(x). \tag{4.4.40}
$$

Likewise,

$$
\limsup_{\varepsilon\to 0^+} \big|\mathrm{II}_\varepsilon\big| = 0 \quad\text{and}\quad \lim_{\varepsilon\to 0^+} \mathrm{II}'_\varepsilon = -u_\gamma(x). \tag{4.4.41}
$$

In a very similar fashion,

$$
\lim_{\varepsilon\to 0^+} \mathrm{III}'_\varepsilon = u_\gamma(x) \tag{4.4.42}
$$

and, if $n \geq 3$,

$$
\limsup_{\varepsilon\to 0^+} \big|\mathrm{III}_\varepsilon\big| \leq C \limsup_{\varepsilon\to 0^+} ⨍_{B(x,2\varepsilon)} \big|u_\alpha(y) - u_\alpha(x)\big|\,\mathrm{d}y = 0. \tag{4.4.43}
$$

When $n = 2$ we only have $|E(x-y)| \leq C_0\big(1 + \big|\ln|x-y|\big|\big)$ for $y \in \mathbb{R}^n \setminus \{x\}$ (cf. [70, (1.4.24)]), but u is of class $\mathscr{C}^\infty$ in Ω (being a null-solution of the weakly elliptic system L; cf. [68, Theorem 6.5.7]), hence locally Lipschitz, so we may employ the Mean Value Theorem to estimate

$$
\begin{aligned}
\limsup_{\varepsilon\to 0^+} \big|\mathrm{III}_\varepsilon\big| &\leq C \limsup_{\varepsilon\to 0^+} \big(1 + \big|\ln\varepsilon\big|\big) ⨍_{B(x,2\varepsilon)} \big|u_\alpha(y) - u_\alpha(x)\big|\,\mathrm{d}y \\
&\leq C \limsup_{\varepsilon\to 0^+} \big(1 + \big|\ln\varepsilon\big|\big)\varepsilon = 0.
\end{aligned}
\tag{4.4.44}
$$

Finally, from definitions, (4.4.1), [70, (1.5.205), (1.5.213)], and [70, Theorem 1.4.2] we see that

$$
\lim_{R\to\infty} \mathrm{IV}_R = 0, \qquad \lim_{R\to\infty} \mathrm{V}_R = 0, \qquad \lim_{R\to\infty} \mathrm{VI}_R = 0. \tag{4.4.45}
$$

Collectively, (4.4.32)-(4.4.45) prove that

$$-\int_{\Omega}(L^{\top}w)_{\alpha}u_{\alpha}\,\mathrm{d}\mathcal{L}^{n}=-u_{\gamma}(x). \tag{4.4.46}$$

At this stage, from (4.4.29), (4.4.30), (4.4.31), and (4.4.46) we conclude that

$$u_{\gamma}(x)=\Big(\mathcal{D}\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\,u\big)\Big)_{\gamma}(x)-\Big(\mathcal{S}\big(\partial_{\nu}^{A}(u,0)\big)\Big)_{\gamma}(x). \tag{4.4.47}$$

In view of the arbitrariness of $\gamma\in\{1,\dots,M\}$ and $x\in\Omega$, this establishes the integral representation formula (4.4.2) in the scenario described in item *(1)* of the theorem.

Going further, that the integral representation formula (4.4.2) also holds in the scenarios described in items *(2)*-*(3)* in the statement of the theorem is established in a completely analogous fashion (for the situation in item *(3)*, the generalized "full" Green formula from [69, (8.5.26)] is now used, and [69, Proposition 8.5.6] is now employed in place of [69, Proposition 9.5.4]).

There remains to justify formula (4.4.19), working now under the assumption that Ω is an exterior domains and, in place of (4.4.1), we only have (4.4.18). To this end, fix a point $x_0\in\mathbb{R}^n\setminus\overline{\Omega}$ and, in lieu of (4.4.25), now consider

$$w:=\Big(\big(E_{\gamma\beta}(x-\cdot)-E_{\gamma\beta}(x_0-\cdot)\big)\theta_{\varepsilon}\psi_{R}\Big)_{1\leq\beta\leq M}\in\big[\mathscr{C}^{\infty}(\overline{\Omega})\big]^{M}. \tag{4.4.48}$$

The idea is to run the same argument as above for this choice of w. The manner in which the above alteration manifests itself in this process is via the following estimates (themselves consequences of [70, Theorem 1.4.2] and the Mean Value Theorem)

$$\begin{gathered}|\nabla\psi_R|\big|(\nabla E)(x-\cdot)-(\nabla E)(x_0-\cdot)\big|\leq CR^{-n-1}\ \text{ and}\\ |\nabla^2\psi_R|\big|E(x-\cdot)-E(x_0-\cdot)\big|\leq CR^{-n-1},\end{gathered} \tag{4.4.49}$$

which are valid for some $C=C(L,x,x_0)\in(0,\infty)$ independent of R. Together with (4.4.18), they permit us to justify (the corresponding version of) (4.4.45) in the present setting. Much as before, we arrive at the conclusion that (4.4.19) holds in any of the scenarios described in items *(1)*-*(3)* with $c:=(c_\gamma)_{1\leq\gamma\leq M}\in\mathbb{C}^M$ given by

$$\begin{aligned}c_{\gamma}:=&\int_{\partial\Omega}\nu_s(y)a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(x_0-y)\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\,u\big)_{\alpha}(y)\,\mathrm{d}\sigma(y)\\&+{}_{([B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^M)^*}\Big\langle\partial_\nu^A(u,0),E_{\gamma\,\bullet}(x_0-\cdot)\big|_{\partial\Omega}\Big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^M}\end{aligned} \tag{4.4.50}$$

for each $\gamma\in\{1,\dots,M\}$. The proof of Theorem 4.4.1 is therefore complete. □

The next goal is to establish operator identities involving the double and single layer potentials associated with a weakly elliptic system, similar in format to those from item *(xiii)* in Theorem 1.5.1, when considering said boundary layer potentials on Besov spaces.

Theorem 4.4.2 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set satisfying a two-sided local John condition and whose boundary is a compact Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Next, for some $M \in \mathbb{N}$, let $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a coefficient tensor with complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in (1.3.2) is weakly elliptic (in the sense of* [70, (1.3.3) in Definition 1.3.1]). *Consider the boundary layer potentials $K, K^\#, S, \partial_\nu^A \mathcal{D}$ canonically associated with A and Ω. Finally, let $K^\#_{A^\top}$ be associated with $A^\top$ (the real transpose of A) in the same manner $K^\#$ has been associated with the original coefficient tensor A. Then the following operator identities hold:*

$$\begin{gathered}\big(\tfrac{1}{2}I + K\big)\circ\big(-\tfrac{1}{2}I + K\big) = S\circ\big(\partial_\nu^A\mathcal{D}\big)\ \text{ on }\ \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M\\ \text{with } p\in(1,\infty),\ q\in(0,\infty],\ \text{and } s\in(0,1),\end{gathered}\tag{4.4.51}$$

$$\begin{gathered}\big(\tfrac{1}{2}I + K^\#_{A^\top}\big)\circ\big(-\tfrac{1}{2}I + K^\#_{A^\top}\big) = \big(\partial_\nu^A\mathcal{D}\big)\circ S\ \text{ on }\ \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M\\ \text{with } p\in(1,\infty),\ q\in(0,\infty],\ \text{and } s\in(0,1),\end{gathered}\tag{4.4.52}$$

$$\begin{gathered}S\circ K^\#_{A^\top} = K\circ S\ \text{ on }\ \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^M\\ \text{with } p\in(1,\infty),\ q\in(0,\infty],\ \text{and } s\in(0,1),\end{gathered}\tag{4.4.53}$$

$$\begin{gathered}K^\#_{A^\top}\circ\big(\partial_\nu^A\mathcal{D}\big) = \big(\partial_\nu^A\mathcal{D}\big)\circ K\ \text{ on }\ \big[B^{p,q}_{s}(\partial\Omega,\sigma)\big]^M\\ \text{with } p\in(1,\infty),\ q\in(0,\infty],\ \text{and } s\in(0,1).\end{gathered}\tag{4.4.54}$$

Moreover,

under the additional assumption that Ω is a uniform domain, the operator identities in (4.4.51)-(4.4.54) *also hold $p = q = \infty$ and $s \in (0,1)$.* (4.4.55)

Proof Without loss of generality (otherwise working with $\mathbb{R}^n \setminus \overline{\Omega}$ in place of Ω; cf. [68, Lemma 5.10.9] and [68, Lemma 5.10.10]) we may assume that Ω is bounded. The action of the boundary-to-boundary single layer potential operator and the conormal derivative of the double layer on Besov scales have been established in Theorem 4.3.4 and Theorem 4.3.5 for the type of open sets Ω considered here. Granted the current hypotheses, from [68, (5.2.4), (5.10.24), (5.11.26)] we also see that Ω is a UR domain. As such, Theorem 4.1.1 and Theorem 4.1.5 (dealing with mapping properties for the boundary-to-boundary double layer potential operator and its formal transpose on the scale of Besov spaces) are also valid in this context. The aforementioned theorems, together with the operator identities from item *(xiii)* of Theorem 1.5.1 and the density results from [69, Lemma 7.1.10] then yield (4.4.51)-(4.4.54), at least if $q < \infty$. The case when $q = \infty$ may be covered a posteriori, based on what we have just proved and loose embeddings (cf. items *(iii)-(iv)* in [69, Proposition 7.7.1]). Alternatively, in the case when Ω is also assumed to be an (ε, δ)-domain, the operator identities (4.4.51)-(4.4.54) may be justified starting from Green's representation formula from Theorem 4.4.1 on account of the boundary behavior of intervening layer potentials (cf. Theorem 4.3.2, (4.3.72), (4.3.44), (4.3.83)). Finally, that the operator identities in (4.4.51)-(4.4.54) continue to be valid when $p = q = \infty$ and $s \in (0, 1)$ if Ω is also assumed to be a uniform domain follows from what we have proved so far, loose embeddings, and the mapping properties for S, $\partial_\nu^A \mathcal{D}$, K, $K^\#_{A^\top}$ on Besov scales with $p = q = \infty$ established in Theorem 4.3.4, Theorem 4.3.5, Theorem 4.1.1, and Theorem 4.1.5. □

The relationship between Besov and Triebel-Lizorkin spaces exhibiting the same first integrability exponent and the same amount of smoothness has been described in [69, (9.2.11)]. For any homogeneous constant (complex) coefficient second-order weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, our next corollary shows that a new phenomenon occurs when considering the intersections of said spaces with $\operatorname{Ker} L := \Big\{u \in \big[\mathscr{C}^\infty(\Omega)\big]^M : Lu = 0 \text{ in } \Omega\Big\}$. Also, the first equality in (4.4.59) below may be interpreted as saying that

$$\begin{array}{l}\text{informally speaking, pointwise multiplication by } \delta^a_{\partial\Omega} \text{ "adds" } a \text{ units of} \\ \text{smoothness (on Besov and Triebel-Lizorkin scales), i.e., multiplication} \\ \text{by } \delta^a_{\partial\Omega} \text{ acts like "fractional integration" of order } a \text{ if } a \geq 0, \text{ and like} \\ \text{"fractional differentiation" of order } -a \text{ if } a \leq 0.\end{array} \tag{4.4.56}$$

Corollary 4.4.3 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded (ε, δ)-domain with an Ahlfors regular boundary and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix an exponent $p \in (1, \infty)$ along with a number $s \in (0, 1)$. Finally, consider a homogeneous constant (complex) coefficient second-order weakly elliptic $M \times M$ system L in $\mathbb{R}^n$, and recall the notation* $\operatorname{Ker} L := \Big\{u \in \big[\mathscr{C}^\infty(\Omega)\big]^M : Lu = 0 \textit{ in } \Omega\Big\}$. *Then*

$$\big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} L \ \textit{ is independent of } q \in \big(\tfrac{n}{n+s+1/p}, \infty\big). \tag{4.4.57}$$

In particular, corresponding to $q := p$,

$$\big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} L = \big[B^{p,p}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} L \textit{ for each } q \in \big(\tfrac{n}{n+s+1/p}, \infty\big). \tag{4.4.58}$$

Moreover, under the additional assumption that the set $\mathbb{R}^n \setminus \Omega$ is n-thick one has (compare with [69, Corollary 9.2.31] *and* [69, Corollary 9.2.38])

$$\begin{gathered}\big[W^{1,p}_a(\Omega)\big]^M \cap \operatorname{Ker} L = \big[B^{p,p}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} L = \big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} L \\ \textit{with } a := 1 - s - \tfrac{1}{p} \in \big(-\tfrac{1}{p}, 1 - \tfrac{1}{p}\big) \textit{ and } \tfrac{n}{n+s+1/p} < q < \infty.\end{gathered} \tag{4.4.59}$$

Proof Suppose $q \in \big(\frac{n}{n+s+1/p}, \infty\big)$ and consider a function $u \in \big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} L$. Pick an exponent $q_o \in (1, \infty)$ such that $q_o \geq q$ and recall from [69, Corollary 9.2.1] and [69, Theorem 9.1.1] that $F^{p,q}_{s+\frac{1}{p}}(\Omega) \hookrightarrow F^{p,q_o}_{s+\frac{1}{p}}(\Omega)$. In particular, $u \in \big[F^{p,q_o}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} L$. Then item *(2)* in Theorem 4.4.1 (used with $q := q_o$) ensures that

$$\begin{aligned} &u = \mathcal{D}\big(\operatorname{Tr}_{\Omega\to\partial\Omega} u\big) - \mathcal{S}\big(\partial^A_\nu(u,0)\big) \text{ in } \Omega, \text{ with} \\ &\operatorname{Tr}_{\Omega\to\partial\Omega} u \in \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \text{ and } \partial^A_\nu(u,0) \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M. \end{aligned} \tag{4.4.60}$$

Granted this, from item *(1)* of Theorem 4.3.1 and item *(1)* of Theorem 4.3.3 we see that u belongs to $\big[B^{p,p}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} L$. This proves the left-to-right inclusion in (4.4.58).

Conversely, given an arbitrary function u in the space $\big[B^{p,p}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} L$, we may invoke item *(1)* of Theorem 4.4.1 to conclude that the integral representation formula (4.4.60) holds. Based on this, item *(2)* of Theorem 4.3.1 and item *(2)* of Theorem 4.3.3 it follows that $u \in \big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} L$ for each $q \in \big(\frac{n}{n+s+1/p}, \infty\big)$, proving the right-to-left inclusion in (4.4.58). Hence (4.4.58) has been established, and the claim in (4.4.57) is a direct consequence of it.

Finally, that (4.4.59) holds under the additional assumption that the set $\mathbb{R}^n \setminus \Omega$ is n-thick may be seen by reasoning as above, plus the help of item *(3)* in Theorem 4.4.1, Theorem 4.2.3, and Theorem 4.2.10. □

As a consequence of (4.4.57), in the context of Corollary 4.4.3 the conormal derivative operator from [69, Proposition 9.5.2] induces a well-defined, linear, and continuous mapping

$$\begin{aligned} &\big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} L \ni u \longmapsto \partial^A_\nu(u,0) \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M \\ &\text{whenever } p \in (1,\infty), \ s \in (0,1), \text{ and } q \in \big(\tfrac{n}{n+s+1/p}, \infty\big). \end{aligned} \tag{4.4.61}$$

Moving on, recall the class of injectively elliptic first-order systems in $\mathbb{R}^n$ introduced in [70, Definition 1.3.4]. For further reference, let us also make here the following convention. Given a homogeneous constant (complex) first-order injectively elliptic $N \times M$ system in $\mathbb{R}^n$ along with some arbitrary open set $\Omega \subseteq \mathbb{R}^n$, we agree to abbreviate

$$\operatorname{Ker} D := \Big\{u \in \big[\mathscr{C}^\infty(\Omega)\big]^M : Du = 0 \text{ in } \Omega\Big\}. \tag{4.4.62}$$

We now discuss a basic integral representation formula for null-solutions of a first-order injectively elliptic system D, which belong to suitable Besov, Triebel-Lizorkin, or weighted Sobolev spaces in a bounded (ε,δ)-domain with an Ahlfors regular boundary.

Theorem 4.4.4 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded (ε,δ)-domain with an Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Next, let L be a homogeneous constant (complex)* *second-order weakly elliptic $M \times M$ system in $\mathbb{R}^n$. Consider a factorization of L of the form*

$$L = \widetilde{D}D \tag{4.4.63}$$

where

$$\widetilde{D} = \Big(\sum_{j=1}^{n} \widetilde{b}_j^{\alpha\gamma}\partial_j\Big)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq N}} \text{ and } D = \Big(\sum_{j=k}^{n} b_k^{\gamma\beta}\partial_k\Big)_{\substack{1\leq\gamma\leq N\\ 1\leq\beta\leq M}} \tag{4.4.64}$$

are homogeneous, constant (complex) coefficient, first-order systems in $\mathbb{R}^n$, and define the coefficient tensor (with the summation convention over repeated indices in effect)

$$A_{\widetilde{D},D} := \big(a_{jk}^{\alpha\beta}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq M}} \text{ where each } a_{jk}^{\alpha\beta} := \widetilde{b}_j^{\alpha\gamma} b_k^{\gamma\beta} \tag{4.4.65}$$

(so that, in particular, $L = L_{A_{\widetilde{D},D}}$). Let $\mathcal{D}$ be the boundary-to-domain double layer potential operator associated with the coefficient tensor $A_{\widetilde{D},D}$ and the set Ω as in (1.3.18). Also, let K be the boundary-to-boundary double layer potential operator associated with the coefficient tensor $A_{\widetilde{D},D}$ and the set Ω, acting on the scale of boundary Besov spaces as in (4.1.1). Finally, consider a vector-valued function $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ satisfying $Du = 0$ in Ω.

Then u admits the integral representation formula

$$u = \mathcal{D}\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u\big) \text{ in } \Omega \tag{4.4.66}$$

and one has

$$\big(-\tfrac{1}{2}I + K\big)\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u\big) = 0 \text{ on } \partial\Omega, \tag{4.4.67}$$

in any of the scenarios described below:

(1) assume the function u belongs to $\big[B^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M$ with $\frac{n-1}{n} < p < \infty$, $0 < q \leq \infty$, and $(n-1)\big(\frac{1}{p}-1\big)_+ < s < 1$;

(2) assume the function u belongs to the Triebel-Lizorkin space $\big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M$ with $\frac{n-1}{n} < p < \infty$, $0 < q \leq \infty$, and $(n-1)\big(\frac{1}{p}-1\big)_+ < s < 1$;

(3) make the additional assumption that $\mathbb{R}^n \setminus \Omega$ is n-thick and suppose the function u belongs to the weighted Sobolev space $\big[W^{1,p}_a(\Omega)\big]^M$ with $p \in (1,\infty)$ and $a \in \big(-\frac{1}{p}, 1-\frac{1}{p}\big)$.

Proof Consider first the scenario described in item *(1)*. We may then employ [69, Corollary 9.2.1] to conclude that there exist $p_o, q_o \in (1,\infty)$ and $s_o \in (0,1)$ such that $B^{p,q}_{s+\frac{1}{p}}(\Omega) \subseteq B^{p_o,q_o}_{s_o+\frac{1}{p_o}}(\Omega)$. Since $Lu = \widetilde{D}(Du) = 0$, it follows that $u \in \big[B^{p_o,q_o}_{s_o+\frac{1}{p_o}}(\Omega)\big]^M$ is a null-solution of the system L in Ω. Moreover, from [69, (9.5.36)] and the fact that $Du = 0$ we see that

$$\partial_\nu^{A_{\widetilde{D},D}}(u,0) = (-\mathrm{i})\mathrm{Sym}(\widetilde{D};\nu)(Du,0) = 0. \tag{4.4.68}$$

Granted these properties of u, from item *(1)* of Theorem 4.4.1 we see that Green's integral representation formula (4.4.2) is presently valid and, in light of (4.4.68), reduces precisely to (4.4.66). In turn, (4.4.67) readily follows from (4.4.66) and the jump-formula from item *(1)* of Theorem 4.3.2. The situations described in items *(2)*, *(3)* are handled similarly (making use of [69, (8.5.49)] in the latter scenario). □

Our next result is the version of Corollary 4.4.3 for first-order injectively elliptic systems (cf. also [69, (9.2.11)] and (4.4.56)).

Corollary 4.4.5 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded (ε, δ)-domain with an Ahlfors regular boundary and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, consider a homogeneous constant (complex) coefficient first-order $N \times M$ system D in $\mathbb{R}^n$ which is injectively elliptic, and recall (4.4.62). Finally, suppose $\frac{n-1}{n} < p < \infty$ and $(n-1)\big(\frac{1}{p} - 1\big)_+ < s < 1$. Then*

$$\big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} D \ \text{ is independent of } q \in \big(\tfrac{n}{n+s+1/p}, \infty\big) \tag{4.4.69}$$

hence, corresponding to $q := p$,

$$\begin{gathered}\big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} D = \big[B^{p,p}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} D \\ \text{for each } \ q \in \big(\tfrac{n}{n+s+1/p}, \infty\big).\end{gathered} \tag{4.4.70}$$

Finally, under the additional assumption that the set $\mathbb{R}^n \setminus \Omega$ is n-thick and $p \in (1, \infty)$, one has

$$\begin{gathered}\big[W^{1,p}_a(\Omega)\big]^M \cap \operatorname{Ker} D = \big[B^{p,p}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} D = \big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} D \\ \text{with } \ a := 1 - s - \tfrac{1}{p} \in \big(-\tfrac{1}{p}, 1 - \tfrac{1}{p}\big) \ \text{ and } \ \tfrac{n}{n+s+1/p} < q < \infty.\end{gathered} \tag{4.4.71}$$

Proof If D^* denotes the Hermitian adjoint of D, then $L := D^*D$ is a homogeneous, constant (complex) coefficient, weakly elliptic, second-order $M \times M$ system in $\mathbb{R}^n$ (cf. [70, (1.3.28)]). Intersecting (4.4.57)-(4.4.59) (written for the second-order weakly elliptic system L) with $\operatorname{Ker} D$ then yields all desired conclusions, since it is immediate that $\operatorname{Ker} L \cap \operatorname{Ker} D = \operatorname{Ker} D$.

Another, more direct, proof of the claims made in (4.4.69)-(4.4.71) goes as follows. Fix some exponent $q \in \big(\frac{n}{n+s+1/p}, \infty\big)$ and consider an arbitrary function $u \in \big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} D$. Then item *(2)* in Theorem 4.4.4 (applied for $L := D^*D$) guarantees that we have the integral representation formula

$$u = \mathcal{D}\big(\operatorname{Tr}_{\Omega\to\partial\Omega} u\big) \ \text{ in } \Omega. \tag{4.4.72}$$

Based on this, [69, (9.4.93) in item *(ii)* of Theorem 9.4.5], and item *(1)* in Theorem 4.3.1 (with $q := p$) we then conclude that u belongs to the Besov space $\big[B^{p,p}_{s+\frac{1}{p}}(\Omega)\big]^M$. This proves the left-to-right inclusion in (4.4.70). The opposite inclusion in (4.4.70) is justified in a similar manner, using item *(1)* in Theorem 4.4.4, [69, (9.4.91) in item *(ii)* of Theorem 9.4.5] (with $q := p$), and item *(2)* in Theorem 4.3.1. With (4.4.70) in hand, the claim in (4.4.69) also follows. Finally, under the additional assumption that $\mathbb{R}^n \setminus \Omega$ is n-thick and $p \in (1, \infty)$, the claim in (4.4.71) is justified in a similar manner, relying on item *(3)* in Theorem 4.4.4, [69, (8.3.38)], and Theorem 4.2.3. □

We also have the following versions of the Maximum Principle adapted to null-solutions of a first-order injectively elliptic system, belonging to Besov, Triebel-Lizorkin, or weighted Sobolev spaces in a bounded (ε, δ)-domain with an Ahlfors regular boundary.

Corollary 4.4.6 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded (ε, δ)-domain with an Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Finally, let D be a homogeneous constant (complex) first-order injectively elliptic $N \times M$ system in $\mathbb{R}^n$, and recall (4.4.62).*

(1) Assume $\frac{n-1}{n} < p < \infty$, $0 < q \leq \infty$, and $(n-1)\big(\frac{1}{p} - 1\big)_+ < s < 1$. Then one has the Besov-themed Maximum Principle

$$\|u\|_{[B^{p,q}_{s+\frac{1}{p}}(\Omega)]^M} \approx \big\|\mathrm{Tr}_{\Omega\to\partial\Omega}\, u\big\|_{[B^{p,q}_s(\partial\Omega,\sigma)]^M}, \quad \textit{uniformly for } u \in \big[B^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} D. \tag{4.4.73}$$

As a consequence,

$$\mathrm{Tr}_{\Omega\to\partial\Omega} : \big[B^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} D \longrightarrow \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M \quad \textit{is an injective operator with closed range}. \tag{4.4.74}$$

(2) Assume $\frac{n-1}{n} < p < \infty$, $(n-1)\big(\frac{1}{p} - 1\big)_+ < s < 1$, and $\frac{n}{n+s+1/p} < q < \infty$. Then one has the Triebel-Lizorkin-themed Maximum Principle

$$\|u\|_{[F^{p,q}_{s+\frac{1}{p}}(\Omega)]^M} \approx \big\|\mathrm{Tr}_{\Omega\to\partial\Omega}\, u\big\|_{[B^{p,p}_s(\partial\Omega,\sigma)]^M}, \quad \textit{uniformly for } u \in \big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} D. \tag{4.4.75}$$

In particular,

$$\mathrm{Tr}_{\Omega\to\partial\Omega} : \big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \cap \operatorname{Ker} D \longrightarrow \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M \quad \textit{is an injective operator with closed range}. \tag{4.4.76}$$

(3) Make the additional assumption that the set $\mathbb{R}^n \setminus \Omega$ is n-thick and suppose $p \in (1, \infty)$ and $a \in \big(-\frac{1}{p}, 1 - \frac{1}{p}\big)$. Then, with $s := 1 - a - \frac{1}{p}$ one has the weighted Sobolev-themed Maximum Principle

$$\|u\|_{[W^{1,p}_a(\Omega)]^M} \approx \big\|\mathrm{Tr}_{\Omega\to\partial\Omega}\, u\big\|_{[B^{p,p}_s(\partial\Omega,\sigma)]^M}, \quad \textit{uniformly for } u \in \big[W^{1,p}_a(\Omega)\big]^M \cap \operatorname{Ker} D, \tag{4.4.77}$$

and, as such,

$$\begin{gathered}\mathrm{Tr}_{\Omega\to\partial\Omega}:\big[W^{1,p}_a(\Omega)\big]^M\cap\operatorname{Ker}D\longrightarrow\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M\\ \text{is an injective operator with closed range.}\end{gathered}\tag{4.4.78}$$

Proof To set the stage, recall from [70, (1.3.28)] that if D^* is the Hermitian adjoint of D then $L := D^*D$ is a homogeneous, constant (complex) coefficient, second-order $M\times M$ system in $\mathbb{R}^n$, which is weakly elliptic (in the sense of [70, (1.3.3) in Definition 1.3.1]). Suppose first that $u\in\big[B^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M\cap\operatorname{Ker}D$ with p,q,s as described in item *(1)*. Then the right-pointing inequality (4.4.73) is a direct consequence of [69, (9.4.91) in item *(ii)* of Theorem 9.4.5], while the left-pointing inequality (4.4.73) is a consequence of the integral representation formula (4.4.66) (considered in the setting of item *(1)* in Theorem 4.4.4, applied with $\widetilde{D}:=D^*$), and the mapping properties of the double layer operator from (4.3.1). This establishes the Besov-themed Maximum Principle (4.4.73), and (4.4.74) is a direct consequence of it.

Next, the Triebel-Lizorkin-themed Maximum Principle (4.4.75) is justified in an analogous fashion, based on [69, (9.4.93) in item *(ii)* of Theorem 9.4.5], the integral representation formula (4.4.66) (in the setting of item *(2)* in Theorem 4.4.4, once again applied with $\widetilde{D}:=D^*$), and the mapping properties of the double layer operator from (4.3.2). Finally, the claim in item *(3)* is dealt with similarly, now relying on [69, (8.3.38)], the integral representation formula (4.4.66) (in the setting of item *(3)* in Theorem 4.4.4, with $\widetilde{D}:=D^*$), and the mapping properties of the double layer operator from Theorem 4.2.3. □

In the next theorem we shall establish a basic integral representation formula for null-solutions of injectively elliptic first-order systems allowed to have a low amount of regularity (measured on Besov, Triebel-Lizorkin, and weighted Lebesgue scales) to the point that the ordinary boundary trace may not exist. Instead, the integral representation formula (4.4.80) (which should be compared with (4.4.66) and the last line in (1.4.36)) employes the principal symbol map defined in [69, Proposition 9.5.5] and [69, Proposition 8.5.7].

Theorem 4.4.7 *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded (ε,δ)-domain with an Ahlfors regular boundary. Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Next, let L be a homogeneous constant (complex) second-order weakly elliptic $M\times M$ system in $\mathbb{R}^n$. Consider a factorization of L of the form*

$$L=\widetilde{D}D\tag{4.4.79}$$

where $\widetilde{D}$ is a homogeneous, constant (complex) coefficient, first-order $M\times N$ system in $\mathbb{R}^n$, and D is a homogeneous, constant (complex) coefficient, first-order $N\times M$ system in $\mathbb{R}^n$. Bring in the matrix-valued fundamental solution $E_{L^\top}$ associated with the transpose system $L^\top$ as in [70, Theorem 1.4.2]. *Finally, consider a vector-valued function $u\in\big[\mathscr{C}^\infty(\Omega)\big]^M$ satisfying $Du=0$ in Ω.*

Then u admits the integral representation formula

$$u(x) = \Big\langle \big(\widetilde{D}^{\top} E_{L^{\top}}\big)^{\top}(x-\cdot)\Big|_{\partial\Omega}, (-\mathrm{i})\mathrm{Sym}(D;\nu)(u,0)\Big\rangle$$

$$= \Big(\Big\langle \widetilde{D}_x^{\top}\big(E_{L^{\top}}(x-\cdot)\big)_{\bullet\,\gamma}\Big|_{\partial\Omega}, (-\mathrm{i})\mathrm{Sym}(D;\nu)(u,0)\Big\rangle\Big)_{1\leq\gamma\leq M} \tag{4.4.80}$$

for all $x \in \Omega$, in any of the scenarios described below.

(1) Assume u belongs to the Besov space $\big[B^{p,q}_{s+\frac{1}{p}-1}(\Omega)\big]^M$ with $p,q \in (1,\infty)$ and $s \in (0,1)$. Also, $(-\mathrm{i})\mathrm{Sym}(D;\nu)(u,0) \in \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^N$ is defined as in [69, Proposition 9.5.5] *(with $\mathbb{A} := B$) and the brackets in the right-hand side of* (4.4.80) *stand for the canonical duality pairing between the space $\Big(\big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^N\Big)^* = \big[B^{p',q'}_{1-s}(\partial\Omega,\sigma)\big]^N$ (with $p' := \big(1-\frac{1}{p}\big)^{-1}$ and $q' := \big(1-\frac{1}{q}\big)^{-1}$) and the space $\big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^N$.*

(2) Assume the function u belongs to the Triebel-Lizorkin space $\big[F^{p,q}_{s+\frac{1}{p}-1}(\Omega)\big]^M$ with $p,q \in (1,\infty)$ and $s \in (0,1)$. Also, $(-\mathrm{i})\mathrm{Sym}(D;\nu)(u,0) \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^N$ is defined as in [69, Proposition 9.5.5] *(with $\mathbb{A} := F$) and the brackets in the right-hand side of* (4.4.80) *stand for the canonical duality pairing between the space $\Big(\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^N\Big)^* = \big[B^{p',p'}_{1-s}(\partial\Omega,\sigma)\big]^N$ (with $p' := \big(1-\frac{1}{p}\big)^{-1}$) and the space $\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^N$.*

(3) Make the additional assumption that $\mathbb{R}^n \setminus \Omega$ is n-thick and suppose the function u belongs to the weighted Lebesgue space $\big[L^p\big(\Omega, \delta^{ap}_{\partial\Omega}\mathcal{L}^n\big)\big]^M$ with $p \in (1,\infty)$ and $a \in \big(-\frac{1}{p}, 1-\frac{1}{p}\big)$. Finally, $(-\mathrm{i})\mathrm{Sym}(D;\nu)(u,0) \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^N$ (with $s := 1-a-\frac{1}{p} \in (0,1)$) is defined as in [69, Proposition 8.5.7] *and the brackets in the right-hand side of* (4.4.80) *stand for the canonical duality pairing between $\Big(\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^N\Big)^* = \big[B^{p',p'}_{1-s}(\partial\Omega,\sigma)\big]^N$ (with $p' := \big(1-\frac{1}{p}\big)^{-1}$) and $\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^N$.*

Proof Work in the scenario described in item *(1)*. To fix notation, assume

$$\widetilde{D} = \Big(\sum_{j=1}^{n} \widetilde{b}^{\alpha\gamma}_j \partial_j\Big)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq N}} \text{ and } D = \Big(\sum_{k=1}^{n} b^{\gamma\beta}_k \partial_k\Big)_{\substack{1\leq\gamma\leq N\\ 1\leq\beta\leq M}}. \tag{4.4.81}$$

Also, define the coefficient tensor (with the summation convention over repeated indices in effect)

$$A_{\widetilde{D},D} := \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha,\beta\leq M}} \text{ where each } a^{\alpha\beta}_{jk} := \widetilde{b}^{\alpha\gamma}_j b^{\gamma\beta}_k, \tag{4.4.82}$$

so that

$$L = L_{A_{\widetilde{D},D}} = \big(a^{\alpha\beta}_{jk}\partial_j\partial_k\big)_{1\leq\alpha,\beta\leq M}. \tag{4.4.83}$$

To proceed, pick a point $x \in \Omega$ and select a scalar-valued function $\theta \in \mathscr{C}^\infty(\mathbb{R}^n)$ with the property that $\theta = 0$ on $B(0,1)$, and $\theta = 1$ on $\mathbb{R}^n \setminus B(0,2)$. Next, for each $\varepsilon \in \big(0, \frac{1}{2}\,\mathrm{dist}(x, \partial\Omega)\big)$ define $\theta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by setting

$$\theta_\varepsilon(y) := \theta\Big(\frac{y-x}{\varepsilon}\Big) \text{ for every } y \in \mathbb{R}^n. \tag{4.4.84}$$

Hence, $\theta_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n)$ is a bounded function satisfying

$$\lim_{\varepsilon\to 0^+} \theta_\varepsilon(y) = 1 \text{ for every } y \in \mathbb{R}^n \setminus \{x\}, \tag{4.4.85}$$

and there exists a constant $C \in (0,\infty)$ such that for each $\varepsilon \in \big(0, \frac{1}{2}\,\mathrm{dist}(x, \partial\Omega)\big)$ we have

$$\begin{aligned} 1-\theta_\varepsilon \in \mathscr{C}^\infty_c(\Omega), \quad \theta_\varepsilon \equiv 0 \text{ on } B(x,\varepsilon), \quad \mathrm{supp}\,(\nabla\theta_\varepsilon) \subseteq \overline{B(x,2\varepsilon)} \setminus B(x,\varepsilon), \\ \text{and } |(\nabla^j\theta_\varepsilon)(y)| \le C\varepsilon^{-j} \text{ for every } j \in \mathbb{N}_0 \text{ and every } y \in \mathbb{R}^n. \end{aligned} \tag{4.4.86}$$

Going further, fix $\mu \in \{1,\dots,M\}$ and define

$$w := \Big(\theta_\varepsilon\big(\widetilde{D}^\top E_{L^\top}\big)_{\gamma\mu}(x-\cdot)\Big)_{1\le\gamma\le N} \in \big[\mathscr{C}^\infty(\overline{\Omega})\big]^N. \tag{4.4.87}$$

Then

$$\begin{aligned} D^\top w &= -\Big(\sum_{k=1}^n b_k^{\gamma\beta}\partial_k w_\gamma\Big)_{1\le\beta\le M} \\ &= -\Big(\sum_{k=1}^n b_k^{\gamma\beta}(\partial_k\theta_\varepsilon)\big(\widetilde{D}^\top E_{L^\top}\big)_{\gamma\mu}(x-\cdot)\Big)_{1\le\beta\le M}, \end{aligned} \tag{4.4.88}$$

since

$$\begin{aligned} &\Big(\sum_{k=1}^n \theta_\varepsilon b_k^{\gamma\beta}\partial_k\big(\widetilde{D}^\top E_{L^\top}\big)_{\gamma\mu}(x-\cdot)\Big)_{1\le\beta\le M} \\ &\qquad = \theta_\varepsilon\Big(\big(D^\top\widetilde{D}^\top E_{L^\top}\big)_{\beta\mu}(x-\cdot)\Big)_{1\le\beta\le M} \\ &\qquad = \theta_\varepsilon\Big(\big(L^\top E_{L^\top}\big)_{\beta\mu}(x-\cdot)\Big)_{1\le\beta\le M} = 0. \end{aligned} \tag{4.4.89}$$

In particular,

$$D^\top w \in \big[\mathscr{C}^\infty_c(\Omega)\big]^M. \tag{4.4.90}$$

Writing the generalized integration by parts formula [69, (9.5.32)] for the present u, with $f := 0$ (so that $Du = f\lfloor_\Omega$ in $\big[\mathcal{D}'(\Omega)\big]^N$), and with w as in (4.4.87) then gives

$$\begin{aligned}
&\text{the } \mu\text{-th component of } \Big\langle \big(\widetilde{D}^\top E_{L^\top}\big)^\top (x-\cdot)\Big|_{\partial\Omega}, (-\mathrm{i})\mathrm{Sym}(D;\nu)(u,0)\Big\rangle \\
&\quad = {}_{([B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^N)^*}\big\langle (-\mathrm{i})\mathrm{Sym}(D;\nu)(u,0), \mathrm{Tr}_{\Omega\to\partial\Omega}\, w\big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^N} \\
&\quad = -\big\langle u, D^\top w\big\rangle = \int_\Omega b^{\gamma\beta}_k (\partial_k\theta_\varepsilon)\big(\widetilde{D}^\top E_{L^\top}\big)_{\gamma\mu}(x-\cdot) u_\beta \, \mathrm{d}\mathcal{L}^n \\
&\quad =: \mathrm{I}_\varepsilon + \mathrm{II}_\varepsilon,
\end{aligned} \tag{4.4.91}$$

where

$$\mathrm{I}_\varepsilon := \int_\Omega b^{\gamma\beta}_k (\partial_k\theta_\varepsilon)(y)\big(\widetilde{D}^\top E_{L^\top}\big)_{\gamma\mu}(x-y)\big(u_\beta(y)-u_\beta(x)\big)\, \mathrm{d}y \tag{4.4.92}$$

and

$$\mathrm{II}_\varepsilon := u_\beta(x)\int_\Omega b^{\gamma\beta}_k (\partial_k\theta_\varepsilon)(y)\big(\widetilde{D}^\top E_{L^\top}\big)_{\gamma\mu}(x-y)\, \mathrm{d}y. \tag{4.4.93}$$

Next, based on [70, Theorem 1.4.2] and (4.4.86) we may estimate

$$\limsup_{\varepsilon\to 0^+} \big|\mathrm{I}_\varepsilon\big| \leq C \limsup_{\varepsilon\to 0^+} ⨍_{B(x,2\varepsilon)} \big|u(y)-u(x)\big|\, \mathrm{d}y = 0. \tag{4.4.94}$$

Also, since $1-\theta_\varepsilon \in \mathscr{C}^\infty_c(\Omega)$ (cf. (4.4.86)), we may write

$$\begin{aligned}
\int_\Omega b^{\gamma\beta}_k (\partial_k\theta_\varepsilon)(y)\big(\widetilde{D}^\top E_{L^\top}\big)_{\gamma\mu}(x-y)\, \mathrm{d}y &= \Big\langle \big(\widetilde{D}^\top E_{L^\top}\big)_{\gamma\mu}(x-\cdot), b^{\gamma\beta}_k \partial_k[\theta_\varepsilon - 1]\Big\rangle \\
&= -\Big\langle b^{\gamma\beta}_k \partial_k\big[\big(\widetilde{D}^\top E_{L^\top}\big)_{\gamma\mu}(x-\cdot)\big], \theta_\varepsilon - 1\Big\rangle \\
&= -\Big\langle \big(D^\top \widetilde{D}^\top E_{L^\top}\big)_{\beta\mu}(x-\cdot), \theta_\varepsilon - 1\Big\rangle \\
&= -\Big\langle \big(L^\top E_{L^\top}\big)_{\beta\mu}(x-\cdot), \theta_\varepsilon - 1\Big\rangle \\
&= -\big\langle \delta_x \delta_{\beta\mu}, \theta_\varepsilon - 1\big\rangle = -(\theta_\varepsilon(x)-1)\delta_{\beta\mu} \\
&= \delta_{\beta\mu}.
\end{aligned} \tag{4.4.95}$$

In concert with (4.4.93) this proves that

$$\mathrm{II}_\varepsilon = u_\mu(x) \ \text{ for each } \ \varepsilon \in \big(0, \tfrac{1}{2}\,\mathrm{dist}(x,\partial\Omega)\big). \tag{4.4.96}$$

At this stage, (4.4.80) follows from (4.4.91), (4.4.92), (4.4.93), (4.4.94), and (4.4.96). Finally, the validity of the integral representation formula (4.4.80) in the scenarios described in items *(2)-(3)* is established in a similar fashion (we only wish to note that for item *(3)* the generalized integration by parts formula [69, (8.5.45)] is used).□

The results in the corollary below should be compared with Corollary 4.4.3 and Corollary 4.4.5 (cf. also [69, (9.2.11)] and (4.4.56)).

Corollary 4.4.8 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded (ε,δ)-domain with an Ahlfors regular boundary and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, consider a homogeneous constant (complex) coefficient first-order $N\times M$ system D in $\mathbb{R}^n$, which is injectively elliptic. Finally, fix $p\in(1,\infty)$ along with $s\in(0,1)$ and recall (4.4.62). Then*

$$\big[F^{p,q}_{s+\frac{1}{p}-1}(\Omega)\big]^M \cap \operatorname{Ker} D \ \text{ is independent of } \ q\in\big(\tfrac{n}{n+s+1/p},\infty\big). \tag{4.4.97}$$

In particular, corresponding to $q := p$,

$$\begin{gathered}\big[F^{p,q}_{s+\frac{1}{p}-1}(\Omega)\big]^M \cap \operatorname{Ker} D = \big[B^{p,p}_{s+\frac{1}{p}-1}(\Omega)\big]^M \cap \operatorname{Ker} D\\ \text{for each } \ q\in\big(\tfrac{n}{n+s+1/p},\infty\big).\end{gathered} \tag{4.4.98}$$

Furthermore, under the additional assumption that the set $\mathbb{R}^n\setminus\Omega$ is n-thick one has

$$\begin{gathered}\big[L^p\big(\Omega,\delta^{ap}_{\partial\Omega}\mathcal{L}^n\big)\big]^M \cap \operatorname{Ker} D = \big[B^{p,p}_{s+\frac{1}{p}-1}(\Omega)\big]^M \cap \operatorname{Ker} D = \big[F^{p,q}_{s+\frac{1}{p}-1}(\Omega)\big]^M \cap \operatorname{Ker} D\\ \text{with } \ a := 1-s-\tfrac{1}{p}\in\big(-\tfrac{1}{p},1-\tfrac{1}{p}\big) \ \text{ and } \ \tfrac{n}{n+s+1/p}<q<\infty.\end{gathered} \tag{4.4.99}$$

Proof For starters, recall from [70, (1.3.28)] that if D^* is the Hermitian adjoint of D then $L := D^*D$ is a homogeneous, constant (complex) coefficient, second-order $M\times M$ system in $\mathbb{R}^n$, which is weakly elliptic (in the sense of [70, (1.3.3) in Definition 1.3.1]). To proceed, suppose $q\in\big(\frac{n}{n+s+1/p},\infty\big)$ and consider an arbitrary function $u\in\big[F^{p,q}_{s+\frac{1}{p}-1}(\Omega)\big]^M\cap\operatorname{Ker} D$. Select an exponent $q_o\in(1,\infty)$ such that $q_o\geq q$ and recall from [69, Corollary 9.2.1] that $F^{p,q}_{s+\frac{1}{p}-1}(\Omega)\hookrightarrow F^{p,q_o}_{s+\frac{1}{p}-1}(\Omega)$. Consequently, $u\in\big[F^{p,q_o}_{s+\frac{1}{p}-1}(\Omega)\big]^M\cap\operatorname{Ker} D$. Placing ourselves in the scenario described in item *(2)* in Theorem 4.4.7 (used with $q := q_o$) permits us to express

$$\begin{gathered}u(x) = \Big\langle \big(\overline{D}E_{L^\top}\big)^\top(x-\cdot)\big|_{\partial\Omega}, (-\mathrm{i})\mathrm{Sym}(D;\nu)(u,0)\Big\rangle, \qquad \forall\,x\in\Omega,\\ \text{with } \ (-\mathrm{i})\mathrm{Sym}(D;\nu)(u,0)\in\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^N.\end{gathered} \tag{4.4.100}$$

With this in hand, from item *(1)* of Theorem 4.3.3 and [69, (9.2.8)] we see that the function u belongs to the space $\big[B^{p,p}_{s+\frac{1}{p}-1}(\Omega)\big]^M\cap\operatorname{Ker} D$. This proves the left-to-right inclusion in (4.4.98).

In the converse direction, whenever $u \in \big[B^{p,p}_{s+\frac{1}{p}-1}(\Omega)\big]^M \cap \operatorname{Ker} D$ we are in the the context described in item *(1)* of Theorem 4.4.7 (with $q := p$) and this permits us to conclude that the integral representation formula (4.4.60) holds. Together with item *(2)* of Theorem 4.3.3 and [69, (9.2.8)], this implies that u belongs to $\big[F^{p,q}_{s+\frac{1}{p}-1}(\Omega)\big]^M \cap \operatorname{Ker} D$ for each $q \in \big(\frac{n}{n+s+1/p}, \infty\big)$, proving the right-to-left inclusion in (4.4.98). This finishes the proof of (4.4.98), and the claim in (4.4.97) is a direct consequence of (4.4.98). Finally, the fact that (4.4.99) holds under the additional assumption that $\mathbb{R}^n \setminus \Omega$ is n-thick may be seen by reasoning as above, now invoking item *(3)* in Theorem 4.4.7 and Theorem 4.2.10. □

From Theorem 4.4.7 and [70, (1.3.40), (1.3.41)] we conclude that the gradient of any null-solution of a homogeneous constant coefficient weakly elliptic second-order system in a given bounded (ε,δ)-domain with an Ahlfors regular boundary satisfies an integral representation formula. The aforementioned formula is identified in the theorem below (via a conceptually different proof).

Theorem 4.4.9 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded (ε,δ)-domain with an Ahlfors regular boundary with the property that*

$$\mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial_*\Omega\big) = 0. \tag{4.4.101}$$

Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Next, for some $M \in \mathbb{N}$, consider a coefficient tensor $A = \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ with complex entries, with the property that the $M \times M$ homogeneous second-order system $L = L_A$ associated with A in $\mathbb{R}^n$ as in (1.3.2) *is weakly elliptic* (*in the sense of* [70, (1.3.3) in Definition 1.3.1]), *and bring in the matrix-valued fundamental solution $E = \big(E_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$ associated with L as in* [70, Theorem 1.4.2]. *Finally, consider a null-solution $u = (u_\alpha)_{1\leq\alpha\leq M} \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ of the system L in Ω.*

Then, for each $\ell \in \{1,\dots,n\}$ and $\gamma \in \{1,\dots,M\}$, the integral representation formula

$$\begin{aligned}(\partial_\ell u_\gamma)(x) &= \Big\langle a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}, \partial_{\tau_{\ell s}}\big(\operatorname{Tr}_{\Omega\to\partial\Omega} u_\alpha\big)\Big\rangle \\ &\quad - \Big\langle(\partial_\ell E_{\gamma\bullet})(x-\cdot)\big|_{\partial\Omega}, \partial^A_\nu(u,0)\Big\rangle, \qquad \forall x \in \Omega,\end{aligned} \tag{4.4.102}$$

(*where the summation convention over repeated indices is in effect, and where $E_{\gamma\bullet}$ denotes the γ-th row of the matrix-valued function E*) *holds in any of the following scenarios:*

(1) One assumes that

$$\begin{gathered} u \text{ belongs to the Besov space } \big[B^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^M \text{ with} \\ 1 < p < \infty, \quad 0 < s < 1, \quad 1 < q < \infty, \end{gathered} \tag{4.4.103}$$

$$\begin{array}{l}\mathrm{Tr}_{\Omega\to\partial\Omega} : B^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,q}_{s}(\partial\Omega,\sigma) \textit{ is the boundary trace} \\ \textit{operator from } [69, (9.4.91) \text{ in item } (ii) \text{ of Theorem 9.4.5}] \\ (\textit{thus } \mathrm{Tr}_{\Omega\to\partial\Omega}\, u_\alpha \in B^{p,q}_{s}(\partial\Omega,\sigma) \textit{ for each index } \alpha),\end{array} \tag{4.4.104}$$

$$\begin{array}{l}\partial_{\tau_{\ell s}} \textit{ is the tangential derivative operator considered as in } [69, \\ (11.12.76)]\ (\textit{thus } \partial_{\tau_{\ell s}}\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u_\alpha\big) \in B^{p,q}_{s-1}(\partial\Omega,\sigma) \textit{ for all } \alpha, \ell, s),\end{array} \tag{4.4.105}$$

$$\begin{array}{l}\partial^{A}_{\nu} \textit{ is the conormal derivative associated with } A \textit{ and } \Omega \textit{ as in } [69, \\ \text{Proposition 9.5.2] } \textit{with } A := B\ (\textit{so } \partial^{A}_{\nu}(u,0) \in \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^{M}),\end{array} \tag{4.4.106}$$

and

$$\begin{array}{l}\textit{the brackets in the right-hand side of (4.4.102) are the duality pair-} \\ \textit{ings between } B^{p',q'}_{1-s}(\partial\Omega,\sigma) = \big(B^{p,q}_{s-1}(\partial\Omega,\sigma)\big)^{*}\ (\textit{with } p', q' \textit{ denoting} \\ \textit{the Hölder conjugate exponents of } p, q) \textit{ and } B^{p,q}_{s-1}(\partial\Omega,\sigma).\end{array} \tag{4.4.107}$$

(2) One assumes that

$$\begin{array}{c} u \textit{ belongs to the Triebel-Lizorkin space } \big[F^{p,q}_{s+\frac{1}{p}}(\Omega)\big]^{M} \textit{ with} \\ 1 < p < \infty, \quad 0 < s < 1, \quad 1 < q < \infty,\end{array} \tag{4.4.108}$$

$$\begin{array}{l}\mathrm{Tr}_{\Omega\to\partial\Omega} : F^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,p}_{s}(\partial\Omega,\sigma) \textit{ is the boundary trace} \\ \textit{operator from } [69, (9.4.93) \text{ in item } (ii) \text{ of Theorem 9.4.5}] \\ (\textit{hence } \mathrm{Tr}_{\Omega\to\partial\Omega}\, u_\alpha \in \big[B^{p,p}_{s}(\partial\Omega,\sigma)\big]^{M} \textit{ for each } \alpha),\end{array} \tag{4.4.109}$$

$$\begin{array}{l}\partial_{\tau_{\ell s}} \textit{ is the tangential derivative operator as in } [69, (11.12.76)] \textit{ with} \\ q := p\ (\textit{so that } \partial_{\tau_{\ell s}}\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u_\alpha\big) \in B^{p,p}_{s-1}(\partial\Omega,\sigma) \textit{ for all } \alpha, \ell, s),\end{array} \tag{4.4.110}$$

$$\begin{array}{l}\partial^{A}_{\nu} \textit{ is the conormal derivative associated with } A \textit{ and } \Omega \textit{ as in } [69, \\ \text{Proposition 9.5.2] } \textit{with } A := F\ (\textit{so } \partial^{A}_{\nu}(u,0) \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^{M}),\end{array} \tag{4.4.111}$$

and

$$\begin{array}{l}\textit{the brackets appearing in the right-hand side of (4.4.102) are the} \\ \textit{duality pairings between } B^{p',p'}_{1-s}(\partial\Omega,\sigma) = \big(B^{p,p}_{s-1}(\partial\Omega,\sigma)\big)^{*}\ (\textit{with} \\ p' \textit{ denoting the conjugate exponent of } p) \textit{ and } B^{p,p}_{s-1}(\partial\Omega,\sigma).\end{array} \tag{4.4.112}$$

(3) Strengthen (4.4.101) by assuming that $\mathbb{R}^n \setminus \Omega$ is n-thick, and suppose that

$$\begin{aligned}&\textit{u belongs to the weighted Sobolev space } \big[W^{1,p}_a(\Omega)\big]^M, \textit{ with}\\ &\qquad 1<p<\infty, \quad 0<s<1, \quad a:=1-s-\tfrac{1}{p},\end{aligned} \tag{4.4.113}$$

$\mathrm{Tr}_{\Omega\to\partial\Omega} : W^{1,p}_a(\Omega) \longrightarrow B^{p,p}_s(\partial\Omega,\sigma)$ *is the boundary trace from* [69, Theorem 8.3.6] (*thus* $\mathrm{Tr}_{\Omega\to\partial\Omega}\, u_\alpha \in B^{p,p}_s(\partial\Omega,\sigma)$ *for each index* α), (4.4.114)

$\partial_{\tau_{\ell s}}$ *is the tangential derivative operator as in* [69, (11.12.76)] *with* $q := p$ (*so that* $\partial_{\tau_{\ell s}}\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u_\alpha\big) \in B^{p,p}_{s-1}(\partial\Omega,\sigma)$ *for all* α, ℓ, s), (4.4.115)

∂^A_ν *is the conormal derivative operator associated with* A *and* Ω *as in* [69, Proposition 8.5.3] (*so,* $\partial^A_\nu(u,0)$ *ends up in* $\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^M$), (4.4.116)

and

the brackets appearing in the right-hand side of (4.4.102) *are the duality pairings between* $B^{p',p'}_{1-s}(\partial\Omega,\sigma) = \big(B^{p,p}_{s-1}(\partial\Omega,\sigma)\big)^*$ (*with* p' *denoting the conjugate exponent of* p) *and* $B^{p,p}_{s-1}(\partial\Omega,\sigma)$. (4.4.117)

Proof Let us establish the integral representation formula (4.4.102) in the scenario described in the current item *(1)*. Starting with the Green-type formula (4.4.2) in the context specified in item *(1)* of Theorem 4.4.1 and applying ∂_ℓ, for some fixed $\ell \in \{1,\dots,n\}$, to both sides yields

$$(\partial_\ell u)(x) = \partial_\ell \mathcal{D}\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u\big)(x) - \partial_\ell \mathcal{S}\big(\partial^A_\nu(u,0)\big)(x) \text{ for all } x\in\Omega. \tag{4.4.118}$$

To proceed, denote by $\nu = (\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω. Having also fixed $\gamma \in \{1,\dots,M\}$ and $x \in \Omega$ we may then compute

$$\begin{aligned}&\partial_\ell\big(\mathcal{D}(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u)\big)_\gamma(x)\\ &\quad = -\int_{\partial\Omega} \nu_s(y) a^{\beta\alpha}_{rs}(\partial_\ell\partial_r E_{\gamma\beta})(x-y)\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u_\alpha\big)(y)\,d\sigma(y)\\ &\quad = -\int_{\partial\Omega} a^{\beta\alpha}_{rs}\,\partial_{\tau_{\ell s}(y)}[(\partial_r E_{\gamma\beta})(x-y)]\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u_\alpha\big)(y)\,d\sigma(y)\\ &\quad = {}_{B^{p',q'}_{1-s}(\partial\Omega,\sigma)}\Big\langle a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-\cdot)\big|_{\partial\Omega}, \partial_{\tau_{\ell s}}\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u_\alpha\big)\Big\rangle_{B^{p,q}_{1-s}(\partial\Omega,\sigma)}.\end{aligned} \tag{4.4.119}$$

Above, the first equality comes from (1.3.18), (4.4.101), and differentiation under the integral sign. The second equality in (4.4.119) uses the definition of $\partial_{\tau_{\ell s}(y)}$ and [70, (1.4.33)]. The final equality in (4.4.119) is implied by [69, (7.6.9)] and [69, Proposition 11.12.5)] (bearing in mind (4.4.104)-(4.4.105)). Moreover, on account of (4.4.106), Proposition 4.2.8 presently gives

$$\partial_\ell\Big(\mathscr{S}\big(\partial_\nu^A(u,0)\big)\Big)_\gamma(x) = \Big\langle (\partial_\ell E_{\gamma\,\bullet})(x-\cdot)\big|_{\partial\Omega}, \partial_\nu^A(u,0)\Big\rangle. \tag{4.4.120}$$

At this stage, (4.4.102) follows by combining (4.4.118) with (4.4.119)-(4.4.120).

Finally, that the integral representation formula (4.4.102) also holds in the scenarios described in items *(2)-(3)* in the statement of the theorem is justified in a similar fashion. □

Our next result explores the nature of spaces of monogenic functions (i.e., null-solutions of the Dirac operator in $\mathbb{R}^n$) which exhibit a certain prescribed amount of smoothness measured on the scales of Besov, Triebel-Lizorkin, and weighted Sobolev spaces in a bounded NTA domain with an Ahlfors regular boundary.

Theorem 4.4.10 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded* NTA *domain with an Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Recall the formalism associated with the Clifford algebra $\mathcal{C}\ell_n$ from* [68, §6.4]*, and the Dirac operator $D := \sum_{j=1}^n \mathbf{e}_j \odot \partial_j$ from* (A.0.55)*. In this context, define the Besov-Hardy spaces*

$$\mathscr{B}^{p,q}_\alpha(\Omega; D) := \Big\{u \in B^{p,q}_\alpha(\Omega)\otimes\mathcal{C}\ell_n : Du = 0 \text{ in } \Omega\Big\} \qquad 0<p\leq\infty,\quad 0<q\leq\infty,\quad \alpha\in\mathbb{R}, \tag{4.4.121}$$

equipped in each instance with the quasi-norm inherited from $B^{p,q}_\alpha(\Omega)\otimes\mathcal{C}\ell_n$. Then, in relation to this scale of spaces, the following statements are true.

(1) Whenever

$$\tfrac{n-1}{n} < p < \infty, \quad 0<q\leq\infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \tag{4.4.122}$$

the boundary-to-domain Cauchy-Clifford integral operator (*cf.* (4.3.19))

$$\mathcal{C} : B^{p,q}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \longrightarrow \mathscr{B}^{p,q}_{s+\frac{1}{p}}(\Omega; D) \tag{4.4.123}$$

is well defined, linear, bounded, and surjective. In addition, one has the following Plemelj-type jump-formula

$$\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathcal{C} = \tfrac{1}{2}I + \mathfrak{C} \text{ on } B^{p,q}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n \tag{4.4.124}$$

where $\mathfrak{C}$ is the boundary-to-boundary Cauchy-Clifford integral operator from (4.1.13).

(2) For each p, q, s as in (4.4.122) *define the boundary Besov-Hardy space*

$$\mathscr{B}^{p,q}_s(\partial\Omega; D) := \Big\{\mathrm{Tr}_{\Omega\to\partial\Omega}\, u : u \in \mathscr{B}^{p,q}_{s+\frac{1}{p}}(\Omega; D)\Big\} \tag{4.4.125}$$

and equip it with the quasi-norm inherited from $B^{p,q}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n$. Then $\mathscr{B}^{p,q}_s(\partial\Omega; D)$ is a closed linear subspace of $B^{p,q}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n$ and the operator

$\frac{1}{2}I + \mathfrak{C}$ *is a projection*[1] *of* $B^{p,q}_s(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ *onto* $\mathscr{B}^{p,q}_s(\partial\Omega; D)$. *Furthermore,*

$$\mathscr{B}^{p,q}_s(\partial\Omega; D) = \mathrm{Ker}\Big[\tfrac{1}{2}I - \mathfrak{C} : B^{p,q}_s(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \to B^{p,q}_s(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n\Big]$$
$$= \mathrm{Im}\Big[\tfrac{1}{2}I + \mathfrak{C} : B^{p,q}_s(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n \to B^{p,q}_s(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n\Big] \tag{4.4.126}$$

and

$$\begin{aligned} &C : \mathscr{B}^{p,q}_s(\partial\Omega; D) \longrightarrow \mathscr{B}^{p,q}_{s+\frac{1}{p}}(\Omega; D) \ \textit{isomorphically,} \\ &\textit{with inverse } \mathrm{Tr}_{\Omega\to\partial\Omega} : \mathscr{B}^{p,q}_{s+\frac{1}{p}}(\Omega; D) \longrightarrow \mathscr{B}^{p,q}_s(\partial\Omega; D). \end{aligned} \tag{4.4.127}$$

(3) *If* p, q, s *are as in* (4.4.122), *then for each function* $u \in \mathscr{B}^{p,q}_{s+\frac{1}{p}}(\Omega; D)$ *one has the Cauchy Reproducing Formula*

$$u = C\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u\big) \ \textit{in} \ \Omega, \tag{4.4.128}$$

and the Besov-themed Maximum Principle

$$\|u\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)\otimes\mathcal{C}\ell_n} \approx \big\|\mathrm{Tr}_{\Omega\to\partial\Omega}\, u\big\|_{B^{p,q}_s(\partial\Omega,\sigma)\otimes\mathcal{C}\ell_n} \tag{4.4.129}$$

(*where the intervening proportionality constants are independent of* u).

(4) *Similar properties*[2] *to those described in items (1)-(3) are valid for the Triebel-Lizorkin-Hardy spaces*

$$\begin{aligned} &\mathscr{F}^{p,q}_\alpha(\Omega; D) := \Big\{u \in F^{p,q}_\alpha(\Omega) \otimes \mathcal{C}\ell_n : \ Du = 0 \ \textit{in} \ \Omega\Big\} \\ &0 < p < \infty, \quad 0 < q \le \infty, \quad \alpha \in \mathbb{R}, \end{aligned} \tag{4.4.130}$$

and for the weighted Sobolev-Hardy spaces

$$\begin{aligned} &\mathscr{W}^{1,p}_a(\Omega; D) := \Big\{u \in W^{1,p}_a(\Omega) \otimes \mathcal{C}\ell_n : \ Du = 0 \ \textit{in} \ \Omega\Big\} \\ &1 < p < \infty, \quad a \in \big(-\tfrac{1}{p}, 1 - \tfrac{1}{p}\big). \end{aligned} \tag{4.4.131}$$

(5) *Whenever* $\frac{n-1}{n} < p < \infty$ *and* $(n-1)\big(\frac{1}{p} - 1\big)_+ < s < 1$ *it follows that*

$$\mathscr{F}^{p,q}_{s+\frac{1}{p}}(\Omega; D) \ \textit{is independent of} \ q \in \big(\tfrac{n}{n+s+1/p}, \infty\big), \tag{4.4.132}$$

in particular, corresponding to $q := p$,

---

[1] i.e., a linear, bounded, and idempotent operator

[2] now employing diagonal Besov spaces on the boundary

$$\mathscr{F}^{p,q}_{s+\frac{1}{p}}(\Omega; D) = \mathscr{B}^{p,p}_{s+\frac{1}{p}}(\Omega; D) \textit{ for each } q \in \big(\tfrac{n}{n+s+1/p}, \infty\big). \tag{4.4.133}$$

Moreover,

$$\begin{aligned} &\mathscr{W}^{1,p}_a(\Omega; D) = \mathscr{F}^{p,q}_{s+\frac{1}{p}}(\Omega; D) = \mathscr{B}^{p,p}_{s+\frac{1}{p}}(\Omega; D) \\ &\quad \textit{whenever } 1 < p < \infty, \quad 0 < s < 1, \\ &a := 1 - s - \tfrac{1}{p}, \textit{ and } \tfrac{n}{n+s+1/p} < q < \infty. \end{aligned} \tag{4.4.134}$$

Proof For starters, observe that the present geometric assumptions imply that Ω is both a bounded (ε, δ)-domain whose complement is n-thick, as well as a UR domain (cf. [68, (5.11.66)], [68, (5.10.24)], [68, (5.2.4)], and [68, (5.1.6)]). That the operator (4.4.123) is well defined, linear, bounded is clear from (4.3.19), (4.4.121), and the fact that $D(\mathcal{C}f) = 0$ in Ω for each $f \in B^{p,q}_s(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ (cf. (2.1.78)). The jump-formula (4.4.124) is implied by item *(1)* of Theorem 4.3.2. Next, the Cauchy Reproducing Formula (4.4.128) is a particular case of the integral representation formula (4.4.66) in the context described in item *(1)* of Theorem 4.4.4 (taking D to be the Dirac operator (A.0.55)). In concert with [69, (9.4.91) in item *(ii)* of Theorem 9.4.5], the Cauchy Reproducing Formula (4.4.128) then proves that the Cauchy-Clifford integral operator (4.4.123) is surjective. Going further, (4.4.74) (used with D the Dirac operator (A.0.55)) presently implies that $\mathscr{B}^{p,q}_s(\partial\Omega; D)$ is a closed linear subspace of $B^{p,q}_s(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$. Also, knowing that the Cauchy-Clifford integral operator (4.4.123) is surjective and having established the Plemelj-type jump-formula (4.4.124) permits us to recast (4.4.125) as

$$\mathscr{B}^{p,q}_s(\partial\Omega; D) = \Big\{\big(\tfrac{1}{2}I + \mathfrak{C}\big)f : f \in B^{p,q}_s(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n\Big\}. \tag{4.4.135}$$

Since from (4.1.13), (4.1.15) we see that $\big(\frac{1}{2}I + \mathfrak{C}\big)^2 = \frac{1}{2}I + \mathfrak{C}$ on $B^{p,q}_s(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$, we ultimately conclude (bearing (4.4.135) in mind) that $\frac{1}{2}I + \mathfrak{C}$ is a projection of $B^{p,q}_s(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ onto $\mathscr{B}^{p,q}_s(\partial\Omega; D)$. Furthermore, (4.4.126) follows from (4.4.67) and (4.4.135), while (4.4.129) comes from (4.4.73). Next, that the Cauchy-Clifford operator $\mathcal{C}$ is an isomorphism in the context of (4.4.127), with the trace operator serving as inverse, is clear from (4.4.128), (4.4.125), (4.4.124), and (4.4.126). The argument so far yields all claims made in items *(1)-(3)*, and the claim made in item *(4)* is dealt with in a very similar fashion. Finally, the claims in item *(5)* are consequences of Corollary 4.4.5 (applied with D the Dirac operator (A.0.55)). □

To state our final result in this section, recall the Clifford-Riesz transform $\mathcal{R}_{\mathcal{C}\ell}$, originally defined as in (2.1.77) for a given Ahlfors regular domain $\Omega \subseteq \mathbb{R}^n$ with compact boundary. Its action may then be extended to distributions by setting

$$\begin{aligned} &\big(\mathcal{R}_{\mathcal{C}\ell} f\big)(x) := 2\sum_J \sum_{j=1}^{n} {}_{\mathrm{Lip}\,(\partial\Omega)}\big\langle \Phi_j(x - \cdot)\big|_{\partial\Omega}, f_J\big\rangle_{(\mathrm{Lip}\,(\partial\Omega))'}\, \mathbf{e}_j \odot e_J \\ &\quad \text{for each } f = \textstyle\sum_J f_J e_J \in \big(\mathrm{Lip}\,(\partial\Omega)\big)' \otimes \mathcal{C}\ell_n \text{ and } x \in \Omega, \end{aligned} \tag{4.4.136}$$

where $\Phi_j(x) := \frac{1}{\omega_{n-1}} \frac{x_j}{|x|^n}$ for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\}$, with $1 \leq j \leq n$.

Theorem 4.4.11 *Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded* NTA *domain with an Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Recall the Dirac operator D from* (A.0.55)*, the scale of Besov-Hardy spaces from* (4.4.121)*, and the Clifford-Riesz transform $\mathcal{R}_{C\ell}$ associated with Ω as in* (4.4.136)*. Then the following claims are true.*

(1) The operator

$$\begin{gathered} \mathcal{R}_{C\ell} : B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes C\ell_n \longrightarrow \mathscr{B}^{p,q}_{s+\frac{1}{p}-1}(\Omega; D) \\ \tfrac{n-1}{n} < p < \infty, \quad 0 < q \leq \infty, \quad (n-1)\big(\tfrac{1}{p} - 1\big)_+ < s < 1, \end{gathered} \tag{4.4.137}$$

is well defined, linear, and bounded.

(2) If $1 < p, q < \infty$ and $0 < s < 1$, then for each $f \in B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes C\ell_n$ one has the jump-formula

$$(-\mathrm{i})\mathrm{Sym}(D; \nu)\big(\mathcal{R}_{C\ell} f, 0\big) = \big(I - 2\mathfrak{C}^{\#}\big) f. \tag{4.4.138}$$

where the principal symbol map (*associated with the Dirac operator D*) *is defined as in* [69, Proposition 9.5.5]*, and where $\mathfrak{C}^{\#}$ is the boundary-to-boundary "transpose" Cauchy-Clifford operator* (*cf.* (4.1.49)).

(3) Assume $1 < p, q < \infty$ and $0 < s < 1$. Then the principal symbol map

$$\mathscr{B}^{p,q}_{s+\frac{1}{p}-1}(\Omega; D) \ni u \longmapsto (-\mathrm{i})\mathrm{Sym}(D; \nu)(u, 0) \in B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes C\ell_n \tag{4.4.139}$$

(*associated with the Dirac operator D as in* [69, Proposition 9.5.5]) *is well defined, linear, and bounded. Moreover, if one defines*

$$\mathscr{B}^{p,q}_{s-1}(\partial\Omega; D) := \Big\{(-\mathrm{i})\mathrm{Sym}(D; \nu)(u, 0) : \ u \in \mathscr{B}^{p,q}_{s+\frac{1}{p}-1}(\Omega; D)\Big\} \tag{4.4.140}$$

and equip it with the quasi-norm inherited from $B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes C\ell_n$, then $\mathscr{B}^{p,q}_{s-1}(\partial\Omega; D)$ is a closed linear subspace of $B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes C\ell_n$ and the operator $\frac{1}{2}I - \mathfrak{C}^{\#}$ is a projection[3] *of $B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes C\ell_n$ onto $\mathscr{B}^{p,q}_{s-1}(\partial\Omega; D)$. In addition,*

$$\begin{aligned} \mathscr{B}^{p,q}_{s-1}(\partial\Omega; D) &= \mathrm{Ker}\Big[\tfrac{1}{2}I + \mathfrak{C}^{\#} : B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes C\ell_n \to B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes C\ell_n\Big] \\ &= \mathrm{Im}\Big[\tfrac{1}{2}I - \mathfrak{C}^{\#} : B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes C\ell_n \to B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes C\ell_n\Big] \end{aligned} \tag{4.4.141}$$

and

[3] i.e., a linear, bounded, and idempotent operator

$$\begin{aligned}&\textit{the operator } \mathcal{R}_{C\ell} : \mathscr{B}^{p,q}_{s-1}(\partial\Omega; D) \longrightarrow \mathscr{B}^{p,q}_{s+\frac{1}{p}-1}(\Omega; D)\\ &\textit{is an isomorphism whose inverse is the principal symbol map}\\ &\mathscr{B}^{p,q}_{s+\frac{1}{p}-1}(\Omega; D) \ni u \mapsto (-\mathrm{i}/2)\mathrm{Sym}(D;\nu)(u,0) \in \mathscr{B}^{p,q}_{s-1}(\partial\Omega; D).\end{aligned} \tag{4.4.142}$$

(4) If $1 < p, q < \infty$ *and* $0 < s < 1$, *then each* $u \in \mathscr{B}^{p,q}_{s+\frac{1}{p}-1}(\Omega; D)$ *has the integral representation formula*

$$u = \tfrac{1}{2}\mathcal{R}_{C\ell}\big((-\mathrm{i})\mathrm{Sym}(D;\nu)(u,0)\big) \ \textit{in} \ \Omega, \tag{4.4.143}$$

and the following Besov-themed Maximum Principle holds

$$\|u\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)\otimes C\ell_n} \approx \big\|(-\mathrm{i})\mathrm{Sym}(D;\nu)(u,0)\big\|_{B^{p,q}_{s-1}(\partial\Omega,\sigma)\otimes C\ell_n} \tag{4.4.144}$$

uniformly for $u \in \mathscr{B}^{p,q}_{s+\frac{1}{p}-1}(\Omega; D)$.

(5) Similar properties[4] *to those described in items (1)-(4) are valid for the scale of Triebel-Lizorkin-Hardy spaces* (*cf.* (4.4.130)). *In fact, whenever* $1 < p < \infty$ *and* $0 < s < 1$ *it follows that*

$$\mathscr{F}^{p,q}_{s+\frac{1}{p}-1}(\Omega; D) \ \textit{is independent of} \ q \in \big(\tfrac{n}{n+s+1/p}, \infty\big), \tag{4.4.145}$$

in particular, corresponding to $q := p$,

$$\mathscr{F}^{p,q}_{s+\frac{1}{p}-1}(\Omega; D) = \mathscr{B}^{p,p}_{s+\frac{1}{p}-1}(\Omega; D) \ \textit{for each} \ q \in \big(\tfrac{n}{n+s+1/p}, \infty\big). \tag{4.4.146}$$

Moreover,

$$\begin{aligned}&\big[L^p\big(\Omega, \delta^{ap}_{\partial\Omega}\mathcal{L}^n\big) \otimes C\ell_n\big] \cap \mathrm{Ker}\, D = \mathscr{F}^{p,q}_{s+\frac{1}{p}-1}(\Omega; D) = \mathscr{B}^{p,p}_{s+\frac{1}{p}-1}(\Omega; D)\\ &\textit{if} \ 1 < p < \infty, \quad 0 < s < 1, \quad a := 1 - s - \tfrac{1}{p}, \ \textit{and} \ \tfrac{n}{n+s+1/p} < q < \infty.\end{aligned} \tag{4.4.147}$$

Proof In concert with [68, (5.11.66)], [68, (5.10.24)], [68, (5.2.4)], and [68, (5.1.6)], the present geometric assumptions imply that Ω is both a bounded (ε,δ)-domain whose complement is n-thick, as well as a UR domain. To proceed, bring in the boundary-to-domain single layer potential operator $\mathscr{S}_\Delta$ associated with the Laplacian in Ω, and observe that

$$\mathcal{R}_{C\ell} = 2D\mathscr{S}_\Delta \ \text{ on } \ \big(\mathrm{Lip}\,(\partial\Omega)\big)' \otimes C\ell_n. \tag{4.4.148}$$

In particular,

$$D\mathcal{R}_{C\ell} = 2D^2\mathscr{S}_\Delta = -2\Delta\mathscr{S}_\Delta = 0 \ \text{ on } \ \big(\mathrm{Lip}\,(\partial\Omega)\big)' \otimes C\ell_n. \tag{4.4.149}$$

[4] now employing diagonal Besov spaces on the boundary

From (4.4.148), (4.4.149), item *(1)* of Theorem 4.3.3, and [69, (9.2.8)] we may then conclude that the Clifford-Riesz transform induces a well-defined, linear, and bounded operator in the context of (4.4.137). Next, if $f \in B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ with $1 < p, q < \infty$ and $0 < s < 1$, then we may compute

$$\begin{aligned}(-\mathrm{i})\mathrm{Sym}(D;\nu)\big(\mathcal{R}_{\mathcal{C}\ell} f, 0\big) &= (-2\mathrm{i})\mathrm{Sym}(D;\nu)\big(D\mathscr{S}_{\Delta} f, 0\big) \\ &= -2\partial_\nu^{A_{-D,D}}\big(\mathscr{S}_{\Delta} f, 0\big) = \big(I - 2\mathfrak{C}^{\#}\big) f, \qquad (4.4.150)\end{aligned}$$

thanks to (4.4.148), [69, (9.5.36)], and the jump-formula (4.3.83) in the context of item *(b)* in Theorem 4.3.6 (also bearing in mind the discussion in Example 1.4.12; cf. (1.4.109) and the comment following it). Also, the fact that the principal symbol map (4.4.139) is well defined, linear, and bounded is a direct consequence of (4.4.121) and [69, Proposition 9.5.5]. Since from (4.1.53) we also see that $\big(\frac{1}{2}I + \mathfrak{C}^{\#}\big)^2 = \frac{1}{2}I + \mathfrak{C}^{\#}$ on $B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$, we deduce that $\frac{1}{2}I + \mathfrak{C}^{\#}$ is a projection of $B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$. In particular, the second equality in (4.4.141) follows.

Going further, for each $u \in \mathscr{B}^{p,q}_{s+\frac{1}{p}-1}(\Omega; D)$ with $p, q \in (1, \infty)$ and $s \in (0, 1)$ the integral representation formula (4.4.80) presently becomes

$$u = D\mathscr{S}_{\Delta}\Big((-\mathrm{i})\mathrm{Sym}(D;\nu)(u, 0)\Big) \text{ in } \Omega. \qquad (4.4.151)$$

On account of this and (4.4.148) wee then conclude that the integral representation formula (4.4.143) holds. In addition, from (4.4.151) and (4.4.138) we see that

$$(-\mathrm{i})\mathrm{Sym}(D;\nu)(u, 0) = \big(\tfrac{1}{2}I - \mathfrak{C}^{\#}\big)\big((-\mathrm{i})\mathrm{Sym}(D;\nu)(u, 0)\big), \qquad (4.4.152)$$

hence

$$\big(\tfrac{1}{2}I + \mathfrak{C}^{\#}\big)\big((-\mathrm{i})\mathrm{Sym}(D;\nu)(u, 0)\big) = 0. \qquad (4.4.153)$$

In concert with (4.4.140), this proves the left-to-right inclusion in the first equality in (4.4.141). Since (4.4.140), (4.4.137), and (4.4.138) ensure that the space in the last line of (4.4.141) is contained in $\mathscr{B}^{p,q}_{s-1}(\partial\Omega; D)$, the circular chain of inclusions shown so far simultaneously proves all equalities in (4.4.141). This has several consequences. First, as a byproduct of (4.4.141) and the continuity of (4.1.49) we conclude that $\mathscr{B}^{p,q}_{s-1}(\partial\Omega; D)$ is a closed linear subspace of $B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$. Second, we conclude from (4.4.141) that $\frac{1}{2}I - \mathfrak{C}^{\#}$ projects $B^{p,q}_{s-1}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ onto $\mathscr{B}^{p,q}_{s-1}(\partial\Omega; D)$.

Pressing on, the integral representation formula recorded in (4.4.143) readily implies (in view of (4.4.137), (4.4.139), and (4.4.140)) that the Clifford-Riesz transform is an isomorphism in the context of (4.4.142). Next, the right-pointing inequality in (4.4.144) is a direct consequence of the boundedness of the principal symbol map in (4.4.139), while the left-pointing inequality in (4.4.144) is implied by (4.4.143), (4.4.139), and the boundedness of the Clifford-Riesz transform in (4.4.137). The proof so far addresses all claims made in items *(1)*-*(4)*, and the very first claim made in item *(5)* is dealt with in a similar fashion. Finally, the claims in (4.4.145)-(4.4.147) are consequences of Corollary 4.4.8. □

Chapter 5
Generalized Double Layers in Uniformly Rectifiable Domains

Matching classes of singular integrals with function spaces on which they behave in a natural fashion is a topic at the core of the classical Calderón-Zygmund theory in the entire Euclidean space. See, for instance, [65] where the focus is on Calderón-Zygmund operators considered on Lebesgue (plain and Muckenhoupt weighted), Hölder, and Sobolev spaces in $\mathbb{R}^n$. The main tools employed are wavelets and the $T(1)$ Theorem of David and Journé. In this chapter we are interested in the situation when the Euclidean space is replaced by a more general "surface" and the philosophy that emerges is that *not all singular integral operators are created equal*. For example, not every singular integral operator bounded on Lebesgue spaces L^p (with $1 < p < \infty$) is bounded on Sobolev spaces L^p_1, as a certain algebraic structure is needed, linking said operator to the underlying surface. This is the case for double layer potential operators we have studied earlier. In fact, these enjoy a host of rather specialized properties, which are not generally shared by "ordinary", garden variety SIO's. Here the goal is to further nuance such distinctions.

The trade-mark characteristic of what we shall call a *generalized double layer operator* is the fact that its integral kernel is the inner product of the outward unit normal (to the "surface" on which this integral operator is defined) with a divergence-free vector-valued kernel; see (5.1.1)-(5.1.2) and (5.1.4)-(5.1.6) below. The algebraic structure just described confers excellent cancelation properties (brought to fruition by the Divergence Theorem) which, in turn, permit us to establish boundedness results for these generalized double layers for a multitude of basic scales of function spaces which, in addition to standard L^p spaces with $p \in (1, \infty)$, now also includes boundary Sobolev spaces, Hardy spaces, Hölder spaces, the John-Nirenberg space BMO, the Sarason space VMO, Besov spaces, and Triebel-Lizorkin spaces, among others.

In fact, our generalized double layer operators make up the largest class of singular integral operators enjoying the aforementioned properties.

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5.1 Theory of Generalized Double Layers

A theory of generalized double layers, of the sort advertised earlier, is presented in Theorem 5.1.1 below (see also Theorem 5.1.8 and Theorem 5.1.15 further in this section). Through its proof, this establishes a firm link between our versions of the Divergence Theorem from [68, Chapter 1] and singular integral operators on UR sets acting on scales of function spaces which, directly or implicitly, involve cancelations.

Theorem 5.1.1 (Theory of Generalized Double Layers) *Fix $n \in \mathbb{N}$ with $n \geq 2$, and suppose $\Omega \subseteq \mathbb{R}^n$ is an arbitrary* UR *domain*[1]. *Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Having fixed a sufficiently large integer $N = N(n) \in \mathbb{N}$, consider a vector-valued function*

$$\begin{gathered} \vec{k} = (k_j)_{1 \leq j \leq n} \in \big[\mathscr{C}^N(\mathbb{R}^n \setminus \{0\})\big]^n \\ \textit{odd, positive homogeneous of degree } 1-n, \end{gathered} \tag{5.1.1}$$

and satisfying (*with the summation convention over repeated indices in effect*)

$$\mathrm{div}\vec{k} = \partial_j k_j = 0 \ \textit{ in } \ \mathbb{R}^n \setminus \{0\}. \tag{5.1.2}$$

Finally, set

$$\vartheta := \int_{S^{n-1}} \langle \omega, \vec{k}(\omega) \rangle \, \mathrm{d}\mathcal{H}^{n-1}(\omega) \in \mathbb{C}. \tag{5.1.3}$$

In this setting, introduce integral operators[2] *acting on $f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ according to*

$$\mathcal{T} f(x) := \int_{\partial\Omega} \langle \nu(y), \vec{k}(x-y) \rangle f(y) \, \mathrm{d}\sigma(y) \ \textit{ for all } \ x \in \Omega, \tag{5.1.4}$$

and for each $f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ consider

$$T f(x) := \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y| > \varepsilon}} \langle \nu(y), \vec{k}(x-y) \rangle f(y) \, \mathrm{d}\sigma(y) \ \textit{ for } \sigma\textit{-a.e. } \ x \in \partial\Omega, \tag{5.1.5}$$

as well as

$$T^{\#} f(x) := \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y| > \varepsilon}} \langle \nu(x), \vec{k}(y-x) \rangle f(y) \, \mathrm{d}\sigma(y) \ \textit{ for } \sigma\textit{-a.e. } \ x \in \partial\Omega. \tag{5.1.6}$$

---

[1] If $\Omega \subseteq \mathbb{R}^n$ is merely an open set with a UR boundary, a large number of conclusions continue to hold, with at most minor alterations (namely, eventually replacing $\partial\Omega$ with $\partial_*\Omega$).

[2] which we think of as "generalized double layer potential operators"

Then the following statements are true.

(1) *For each aperture parameter* $\kappa > 0$, *each function* $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$, *and* σ*-a.e. point* $x \in \partial\Omega$ *one has*

$$\Big(\mathcal{T} f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = -\frac{\vartheta}{2} f(x) + (Tf)(x). \tag{5.1.7}$$

Also, if for each $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ *one defines*

$$\mathcal{W} f(x) := \int_{\partial\Omega} f(y)\vec{k}(x-y)\,\mathrm{d}\sigma(y) \ \textit{for all}\ \ x \in \Omega, \tag{5.1.8}$$

then for each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ *one has*

$$\mathcal{W} f \in \big[\mathscr{C}^N(\Omega)\big]^n \ \textit{and}\ \operatorname{div}\big(\mathcal{W} f\big) = 0 \ \textit{in}\ \Omega. \tag{5.1.9}$$

Moreover, for each $\kappa > 0$ *and* $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$, *the nontangential boundary trace*

$$(\mathcal{W} f)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists at } \sigma\textit{-a.e. point on}\ \partial\Omega, \tag{5.1.10}$$

and at σ*-a.e. point* $x \in \partial\Omega$ *one has*

$$\nu(x)\cdot\Big(\mathcal{W} f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = -\frac{\vartheta}{2} f(x) - (T^{\#} f)(x). \tag{5.1.11}$$

Finally, if $\partial\Omega$ *is compact then both* $\mathcal{T}$ *and* T *map constant functions to constant functions and, in fact, at each point in* Ω *we have*

$$\mathcal{T}1 = \begin{cases} -\vartheta & \textit{if } \Omega \textit{ is bounded,} \\ 0 & \textit{if } \Omega \textit{ is an exterior domain,} \end{cases} \tag{5.1.12}$$

(*recall that an exterior domain is the complement of a compact set in* $\mathbb{R}^n$), *while at* σ*-a.e. point on* $\partial\Omega$ *we have*

$$T1 = \begin{cases} -\frac{\vartheta}{2} & \textit{if } \Omega \textit{ is bounded,} \\ +\frac{\vartheta}{2} & \textit{if } \Omega \textit{ is an exterior domain.} \end{cases} \tag{5.1.13}$$

(2) *For each* $p \in [1,\infty)$ *and* $\kappa > 0$ *there exists some finite constant* $C > 0$, *depending only on* $\partial\Omega$, $\vec{k}$, n, p, *and* κ, *such that for each function* $f \in L^p(\partial\Omega,\sigma)$ *one has*

$$\max\Big\{\big\|\mathcal{N}_\kappa(\mathcal{T} f)\big\|_{L^p(\partial\Omega,\sigma)}, \big\|\mathcal{N}_\kappa(\mathcal{W} f)\big\|_{L^p(\partial\Omega,\sigma)}\Big\} \leq C\|f\|_{L^p(\partial\Omega,\sigma)} \ \textit{if}\ p > 1, \tag{5.1.14}$$

plus similar estimates in the case when $p = 1$ in which scenario the corresponding L^1-norms in the left-hand side are now replaced by the quasi-norm in $L^{1,\infty}(\partial\Omega, \sigma)$.

Moreover, the action of the operator $\mathcal{W}$, originally considered as in (5.1.8)*, may be further extended in a unique and coherent fashion* (*cf.* [70, (2.4.15), (2.4.16), (2.4.24)]) *to the scale of Lorentz-based Hardy spaces $H^{p,q}(\partial\Omega, \sigma)$ with $p \in \big(\frac{n-1}{n}, \infty\big)$ and $q \in (0, \infty]$ and said extension satisfies* (*for some constant $C = C(\partial\Omega, \vec{k}, n, p, q) \in (0, \infty)$*)

$$\big\|\mathcal{N}_\kappa(\mathcal{W} f)\big\|_{L^{p,q}(\partial\Omega,\sigma)} \leq C\|f\|_{H^{p,q}(\partial\Omega,\sigma)} \quad \text{for all } f \in H^{p,q}(\partial\Omega, \sigma). \tag{5.1.15}$$

Also, whenever $f \in H^{p,q}(\partial\Omega, \sigma)$ has compact support (*as a distribution, which is automatically the case if $\partial\Omega$ is compact*) *and $\psi \in \mathrm{Lip}_c(\partial\Omega)$ is identically one near* supp f *one has*

$$\big(\mathcal{W} f\big)(x) = {}_{\mathrm{Lip}_c(\partial\Omega)}\Big\langle \psi\,\vec{k}(x - \cdot)\big|_{\partial\Omega}, f\Big\rangle_{(\mathrm{Lip}_c(\partial\Omega))'} \quad \text{for each } x \in \Omega. \tag{5.1.16}$$

(3) For each $p \in (1, \infty)$, the operators

$$T : L^p(\partial\Omega, \sigma) \to L^p(\partial\Omega, \sigma) \text{ and } T^\# : L^p(\partial\Omega, \sigma) \to L^p(\partial\Omega, \sigma) \tag{5.1.17}$$

are well defined, linear, and bounded. Moreover, given any $p, p' \in (1, \infty)$ with $1/p + 1/p' = 1$ it follows that

$$\text{the transpose of } T \text{ acting on } L^p(\partial\Omega, \sigma) \text{ is the operator } T^\# \text{ acting on } L^{p'}(\partial\Omega, \sigma). \tag{5.1.18}$$

Corresponding to the end-point $p = 1$, both operators T and $T^\#$ map $L^1(\partial\Omega, \sigma)$ linearly and boundedly into $L^{1,\infty}(\partial\Omega, \sigma)$. Next, if for each vector-valued function $\vec{g} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\Big]^n$ one considers

$$V\vec{g}(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle \vec{k}(x - y), \vec{g}(y)\rangle \,\mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.1.19}$$

then

$$V(f\,\nu) = Tf \text{ for each function } f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big) \tag{5.1.20}$$

and for each $p \in (1, \infty)$ the following operator is well defined, linear, and bounded:

$$V : \left[L^p(\partial\Omega, \sigma)\right]^n \longrightarrow L^p(\partial\Omega, \sigma). \tag{5.1.21}$$

Finally, similar results are valid for Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, and Morrey spaces (as well as their duals and their preduals) on $\partial\Omega$*.*

(4) *Fix* $p \in \left(\frac{n-1}{n}, 1\right]$*. Then the operator* $T^\#$*, originally acting on Lebesgue spaces as in (the first part of) item (3), extends to a linear and bounded mapping from the Hardy space* $H^p(\partial\Omega, \sigma)$ *into itself,*

$$T^\# : H^p(\partial\Omega, \sigma) \longrightarrow H^p(\partial\Omega, \sigma) \tag{5.1.22}$$

and the operators corresponding to various choices of the index $p \in \left(\frac{n-1}{n}, 1\right]$ *are compatible with one another. As far as the dependence of the operator norm for* $T^\#$ *in (5.1.22) on the kernel* $\vec{k}$ *is concerned, homogeneity considerations dictate that*

$$\|T^\#\|_{H^p(\partial\Omega,\sigma)\to H^p(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha \vec{k}|\Big), \tag{5.1.23}$$

where the constant $C \in (0, \infty)$ *depends only on* n*,* p*, and the* UR *character of* $\partial\Omega$*. In addition,*

$$\begin{array}{l}\textit{if } p \in \left(\frac{n-1}{n}, \infty\right)\textit{, the composition to the left of } T^\# \textit{ from (5.1.22) with}\\ \textit{the } L^p\textit{-filtering operator } \mathfrak{H} : H^p(\partial\Omega, \sigma) \to L^p(\partial\Omega, \sigma) \textit{ (cf. [69,}\\ \text{(4.9.2)])} \textit{ is the operator } T^\# : H^p(\partial\Omega, \sigma) \to L^p(\partial\Omega, \sigma) \textit{ considered}\\ \textit{as in } \text{[70, (2.3.27)]}.\end{array} \tag{5.1.24}$$

In fact, $T^\#$ *maps the scale of Lorentz-based Hardy spaces on* $\partial\Omega$ *continuously into itself, specifically*

$$\begin{array}{c} T^\# : H^{p,q}(\partial\Omega, \sigma) \longrightarrow H^{p,q}(\partial\Omega, \sigma) \textit{ is well defined,}\\ \textit{linear, and bounded for } p \in \left(\frac{n-1}{n}, \infty\right) \textit{ and } q \in (0, \infty]. \end{array} \tag{5.1.25}$$

For example, for σ*-a.e.* $x_o \in \partial\Omega$ *one has (with* $\delta_{x_o} \in H^{1,\infty}(\partial\Omega, \sigma)$ *and* $T^\#$ *acting as in (5.1.25) with* $p = 1$ *and* $q = \infty$*)*

$$\begin{array}{c} T^\#\delta_{x_o} = -\text{P.V.}\big\langle \nu, \vec{k}(x_o - \cdot)\big|_{\partial\Omega}\big\rangle \\ \textit{as distributions in } \big(\text{Lip}_c(\partial\Omega)\big)', \end{array} \tag{5.1.26}$$

and for σ*-a.e.* $x_0, x_1 \in \partial\Omega$ *one has (with* $\delta_{x_0} - \delta_{x_1} \in \bigcap_{\frac{n-1}{n}<p<1} H^p(\partial\Omega, \sigma)$ *and* $T^\#$ *acting as in (5.1.25) with* $p \in \left(\frac{n-1}{n}, 1\right)$ *and* $q = p$*)*

$$\begin{array}{c} T^\#(\delta_{x_0} - \delta_{x_1}) = -\text{P.V.}\big\langle \nu, \vec{k}(x_0 - \cdot)\big|_{\partial\Omega}\big\rangle + \text{P.V.}\big\langle \nu, \vec{k}(x_1 - \cdot)\big|_{\partial\Omega}\big\rangle \\ \textit{as distributions in } \big(\text{Lip}_c(\partial\Omega)\big)'. \end{array} \tag{5.1.27}$$

In addition, for each $f \in H^{p,q}(\partial\Omega,\sigma)$ *with* $p \in \big(\frac{n-1}{n},\infty\big)$ *and* $q \in (0,\infty]$ *one has*

$$\nu \bullet \mathcal{W} f = -\frac{\vartheta}{2} f - T^{\#} f, \tag{5.1.28}$$

where the "bullet product" is defined as in [68, Proposition 4.2.3].
Finally, for each function $f \in H^1(\partial\Omega,\sigma)$ *one has*

$$f \in L^1(\partial\Omega,\sigma), \qquad T^{\#} f \in L^1(\partial\Omega,\sigma), \tag{5.1.29}$$

and, with all integrals involved absolutely convergent,

$$\int_{\partial\Omega} T^{\#} f \, d\sigma = \begin{cases} -\dfrac{\vartheta}{2}\displaystyle\int_{\partial\Omega} f \, d\sigma & \text{if } \Omega \text{ is bounded,} \\ +\dfrac{\vartheta}{2}\displaystyle\int_{\partial\Omega} f \, d\sigma & \text{if } \Omega \text{ is unbounded and } \partial\Omega \text{ is bounded,} \\ 0 & \text{if } \partial\Omega \text{ is unbounded.} \end{cases} \tag{5.1.30}$$

(5) Select $q \in (1,\infty)$ *along with* $\lambda \in (0, n-1)$, *and recall* $\mathcal{H}^{q,\lambda}(\partial\Omega,\sigma)$, *the pre-dual to the Morrey-Campanato space, defined as in* (A.0.84) (*with* $\Sigma := \partial\Omega$). *Then the operator*

$$T^{\#} : L^r(\partial\Omega,\sigma) \longrightarrow L^r(\partial\Omega,\sigma) \ \text{ with } \ r := \frac{q(n-1)}{n-1+\lambda(q-1)} \tag{5.1.31}$$

has $\mathcal{H}^{q,\lambda}(\partial\Omega,\sigma)$ *as an invariant subspace* (*cf.* (5.1.17) *and* [69, (6.1.22)]), *and*

$$T^{\#} : \mathcal{H}^{q,\lambda}(\partial\Omega,\sigma) \longrightarrow \mathcal{H}^{q,\lambda}(\partial\Omega,\sigma) \tag{5.1.32}$$

is a well defined, linear, and bounded operator.

(6) For each given function f *belonging to the weighted boundary Sobolev space* $L^1_1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ (*cf.* (A.0.131)), *each index* $\ell \in \{1,\dots,n\}$, *and each aperture parameter* $\kappa \in (0,\infty)$, *the pointwise nontangential boundary trace*

$$\big(\partial_\ell \mathcal{T} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega. \tag{5.1.33}$$

As a consequence of [69, (11.7.22)], *this is true whenever* $f \in L^{p,q}_1(\partial\Omega,\sigma)$ *with* $p,q \in [1,\infty)$.
Furthermore, for each $p,q \in [1,\infty)$ *and* $\kappa > 0$ *there exists some finite constant* $C > 0$, *depending only on* $\partial\Omega$, $\vec{k}$, n, p, q, *and* κ, *such that for each function* $f \in L^{p,q}_1(\partial\Omega,\sigma)$ *one has*

$$\big\|\mathcal{N}_\kappa(\mathcal{T} f)\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\mathcal{T} f)\big\|_{L^q(\partial\Omega,\sigma)} \leq C\|f\|_{L^{p,q}_1(\partial\Omega,\sigma)} \ \text{ if } p,q > 1, \tag{5.1.34}$$

plus similar estimates in the case when $p = 1$ *or* $q = 1$*, in which scenario the corresponding* L^1*-norms in the left-hand side are now replaced by the quasi-norm* $L^{1,\infty}(\partial\Omega, \sigma)$*.*

(7) The operator

$$T : L_1^{p,q}(\partial\Omega, \sigma) \longrightarrow L_1^{p,q}(\partial\Omega, \sigma) \tag{5.1.35}$$

is well defined, linear, and bounded for each $p, q \in (1, \infty)$*. In particular, the operator*

$$T : L_1^{p}(\partial\Omega, \sigma) \longrightarrow L_1^{p}(\partial\Omega, \sigma) \tag{5.1.36}$$

is well defined, linear, and bounded for each $p \in (1, \infty)$*. In addition,*

$$T : L_1^{p}(\partial\Omega, w) \longrightarrow L_1^{p}(\partial\Omega, w) \text{ is well defined, linear, and bounded for each exponent } p \in (1, \infty) \text{ and each weight } w \in A_p(\partial\Omega, \sigma). \tag{5.1.37}$$

Moreover, for each function

$$f \in L_1^1\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^{n-1}}\Big) \cap L^p_{1,\mathrm{loc}}(\partial\Omega, \sigma) \text{ for some } p \in (1, \infty) \tag{5.1.38}$$

it follows that

$$Tf \in L^p_{1,\mathrm{loc}}(\partial\Omega, \sigma), \tag{5.1.39}$$

and for each pair of indices $r, s \in \{1, \dots, n\}$ *one has (with* V *as in (5.1.19))*

$$\begin{aligned}\partial_{\tau_{rs}}(Tf) = T\big(\partial_{\tau_{rs}} f\big) &+ \big[M_{\nu_r}, T\big](\nabla_{\tan} f)_s - \big[M_{\nu_s}, T\big](\nabla_{\tan} f)_r \\ &- \big[M_{\nu_r}, V\big](\nu_s \nabla_{\tan} f) + \big[M_{\nu_s}, V\big](\nu_r \nabla_{\tan} f)\end{aligned} \tag{5.1.40}$$

at σ*-a.e. point on* $\partial\Omega$*. In particular, formula (5.1.40) holds for every function* $f \in L_1^{p,q}(\partial\Omega, \sigma)$ *with* $p, q \in (1, \infty)$*, as well as for every* $f \in L_1^p(\partial\Omega, w)$ *with* $p \in (1, \infty)$ *and* $w \in A_p(\partial\Omega, \sigma)$*. Formula (5.1.40) also holds for each function in* $M_1^{p,\lambda}(\partial\Omega, \sigma)$ *or in* $\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)$ *with* $p, q \in (1, \infty)$ *and* $\lambda \in (0, n - 1)$ *(cf. (A.0.150), (A.0.33)). In fact,*

the operator T *maps each of the Sobolev spaces* $M_1^{p,\lambda}(\partial\Omega, \sigma)$*,* $\mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma)$*,* $\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)$ *with* $p, q \in (1, \infty)$ *and* $\lambda \in (0, n-1)$ *(cf. (A.0.150), (A.0.155), (A.0.33)) boundedly into themselves.* (5.1.41)

(8) For each $p \in (1, \infty)$ *it follows that* $T^{\#}$*, originally acting on functions from* $L^p(\partial\Omega, \sigma)$*, further extends uniquely to a linear, bounded operator, from the negative boundary Sobolev space* $L^p_{-1}(\partial\Omega, \sigma)$ *into itself. Furthermore, if one retains the same notation* $T^{\#}$ *for said extension, then the transpose of (5.1.36) is*

$$T^{\#} : L^{p'}_{-1}(\partial\Omega, \sigma) \longrightarrow L^{p'}_{-1}(\partial\Omega, \sigma) \tag{5.1.42}$$

where $p' \in (1, \infty)$ *is such that* $1/p + 1/p' = 1$.
More generally, for each $p, q \in (1, \infty)$ *it follows that* $T^{\#}$ *acting on functions from* $L^p(\partial\Omega, \sigma)$ *further extends uniquely to a linear and bounded operator from the negative off-diagonal boundary Sobolev space* $L^{p,q}_{-1}(\partial\Omega, \sigma)$ *into itself and, adopting the same notation* $T^{\#}$ *for this extension, the transpose of* (5.1.35) *is*

$$T^{\#} : L^{p',q'}_{-1}(\partial\Omega, \sigma) \longrightarrow L^{p',q'}_{-1}(\partial\Omega, \sigma) \tag{5.1.43}$$

where $p', q' \in (1, \infty)$ *are such that* $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$.
In addition, for each exponent $p \in (1, \infty)$ *and each Muckenhoupt weight* w *in* $A_p(\partial\Omega, \sigma)$*, it follows that* $T^{\#}$*, originally acting on* $L^p(\partial\Omega, w)$*, further extends uniquely to a linear, bounded operator, from the negative boundary Sobolev space* $L^p_{-1}(\partial\Omega, w)$ *into itself which, in fact, is the transpose of* T *acting on* $L^{p'}_1(\partial\Omega, w')$ *where* $p' := (1 - 1/p)^{-1} \in (1, \infty)$ *is the conjugate exponent of* p *and* $w' := w^{1-p'} \in A_{p'}(\partial\Omega, \sigma)$ *is the conjugate weight of* w (*cf.* [68, item *(2)* in Lemma 7.7.1]).

(9) Consider the following modified version of the generalized double layer operator in (5.1.4) *acting on each function*[3] $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ *according to*

$$\big(\mathcal{T}_{\rm mod} f\big)(x) := \int_{\partial\Omega} \Big\langle \nu(y), \vec{k}(x-y) - \vec{k}(-y) \cdot \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(0,1)}}(y) \Big\rangle f(y)\, {\rm d}\sigma(y) \tag{5.1.44}$$

for all $x \in \Omega$*. Then the operator* $\mathcal{T}_{\rm mod}$ *is meaningfully defined, and is compatible with* $\mathcal{T}$ *from* (5.1.4) *in the sense that for each function* f *belonging to the smaller space* $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ (*hence, in particular, for each function* $f \in L^p(\partial\Omega, \sigma)$ *with* $p \in [1, \infty)$) *the difference*

$$C_f := \mathcal{T}_{\rm mod} f - \mathcal{T} f \;\text{ is a constant in }\; \Omega. \tag{5.1.45}$$

As a consequence,

$$\nabla \mathcal{T}_{\rm mod} f = \nabla \mathcal{T} f \text{ in } \Omega \text{ for each } f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big). \tag{5.1.46}$$

In addition, at each point $x \in \Omega$ *one may express*

$$\partial_\ell \big(\mathcal{T}_{\rm mod} f\big)(x) = \int_{\partial\Omega} \big\langle \nu(y), (\partial_\ell \vec{k})(x-y) \big\rangle f(y)\, {\rm d}\sigma(y) \quad \text{for each } \ell \in \{1, \dots, n\} \text{ and } f \in L^1\big(\partial\Omega, \tfrac{\sigma(y)}{1+|y|^n}\big). \tag{5.1.47}$$

[3] The reader is alerted to the change in power (from $n-1$ to n) for the weight intervening in $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$, compared with $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$. In particular, the former space is more inclusive than the latter one.

Moreover,

$$\mathcal{T}_{\rm mod} \text{ maps constant functions on } \partial\Omega \text{ into constant functions in the set } \Omega. \tag{5.1.48}$$

In fact, if in place of (5.1.44) *one now considers the modified operator acting on each given* $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ *according to*

$$\big(\widetilde{\mathcal{T}}_{\rm mod} f\big)(x) := \int_{\partial\Omega} \Big\langle \nu(y), \vec{k}(x-y) - \vec{k}(-y)(1-\psi(y))\Big\rangle f(y)\,{\rm d}\sigma(y) \tag{5.1.49}$$

for all $x \in \Omega$*, where* $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ *is a fixed function satisfying* $\psi \equiv 1$ *near* $0 \in \mathbb{R}^n$*, then*

$$\big(\widetilde{\mathcal{T}}_{\rm mod} 1\big)(x) = -\vartheta - \int_{\Omega} \nabla\psi \cdot \vec{k}\,{\rm d}\mathcal{L}^n \text{ for each } x \in \Omega. \tag{5.1.50}$$

Next, given any function

$$\begin{aligned} f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) & \text{ with the property that} \\ \partial_{\tau_{j\ell}} f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) & \text{ for each } j,\ell \in \{1,\dots,n\}, \end{aligned} \tag{5.1.51}$$

(*hence, in particular, for each* $f \in \dot{L}^p_1(\partial\Omega,\sigma)$ *with* $1 < p < \infty$) *it follows that for each index* $\ell \in \{1,\dots,n\}$ *and each point* $x \in \Omega$ *one has* (*using the summation convention*)

$$\partial_\ell\big(\mathcal{T}_{\rm mod} f\big)(x) = \int_{\partial\Omega} k_j(x-y)(\partial_{\tau_{\ell j}} f)(y)\,{\rm d}\sigma(y). \tag{5.1.52}$$

As a consequence of (5.1.52) *and* [70, Theorem 2.5.1]*, given any aperture parameter* $\kappa \in (0,\infty)$ *it follows that the nontangential boundary trace*

$$\begin{aligned} &\big(\partial_\ell \mathcal{T}_{\rm mod} f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \\ &\text{for each function } f \text{ as in (5.1.51) and each index } \ell \in \{1,\dots,n\}. \end{aligned} \tag{5.1.53}$$

Another corollary of (5.1.52) *and* [70, (2.4.8)] *is the fact that for each aperture parameter* $\kappa > 0$*, each truncation parameter* $\varepsilon \in (0,\infty)$*, and each exponent* $p \in (1,\infty)$*,*

$$\begin{aligned} &\mathcal{N}^\varepsilon_\kappa\big(\nabla(\mathcal{T}_{\rm mod} f)\big) \in L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for each function} \\ &f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \text{ with the property that} \\ &\partial_{\tau_{jk}} f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for all } j,k \in \{1,\dots,n\}. \end{aligned} \tag{5.1.54}$$

In addition, as seen from (5.1.44) *and* [70, (2.5.32)], *for each integrability exponent* $p \in (1,\infty)$*, each aperture parameter* $\kappa \in (0,\infty)$*, and each truncation parameter* $\varepsilon \in (0,\infty)$ *one has*

$$\begin{gathered}\mathcal{N}_\kappa^\varepsilon(\mathcal{T}_{\rm mod} f) \in L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for each function} \\ f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma).\end{gathered} \tag{5.1.55}$$

Finally, for each $f \in \dot{H}^p_1(\partial\Omega,\sigma)$ *with* $p \in \big(\frac{n-1}{n},1\big]$*, each* $\ell \in \{1,\dots,n\}$*, and each* $x \in \Omega$ *one has* (*using the summation convention*)

$$\partial_\ell\big(\mathcal{T}_{\rm mod} f\big)(x) = \begin{cases} \Big\langle k_j(x-\cdot)\big|_{\partial\Omega}, \partial_{\tau_{\ell j}} f\Big\rangle & \text{if } \partial\Omega \text{ bounded,} \\ \Big\langle \big[k_j(x-\cdot)\big|_{\partial\Omega}\big], \partial_{\tau_{\ell j}} f\Big\rangle & \text{if } \partial\Omega \text{ unbounded,}\end{cases} \tag{5.1.56}$$

where the pairings in (5.1.56) *are understood in the sense of* [69, Theorem 4.6.1] (*keeping in mind that each tangential derivative* $\partial_{\tau_{\ell j}} f$ *belongs to the Hardy space* $H^p(\partial\Omega,\sigma)$). *As a consequence of this and* [70, (2.4.14)] *in* [70, Theorem 2.4.1], *for each* $p \in \big(\frac{n-1}{n},1\big]$ *and each aperture parameter* $\kappa \in (0,\infty)$ *there exists some finite constant* $C > 0$*, depending only on* $\partial\Omega$*, k, n,* κ*, and p, such that for each distribution* $f \in \dot{H}^p_1(\partial\Omega,\sigma)$ *one has*

$$\big\|\mathcal{N}_\kappa(\nabla\mathcal{T}_{\rm mod} f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{\dot{H}^p_1(\partial\Omega,\sigma)}. \tag{5.1.57}$$

Also, as a consequence of (5.1.56) *and* [70, Corollary 2.5.4],

$$\begin{gathered}\text{for each } f \in \dot{H}^p_1(\partial\Omega,\sigma) \text{ with } p \in \big(\tfrac{n-1}{n},1\big], \text{ the nontangential} \\ \text{trace } (\nabla\mathcal{T}_{\rm mod} f)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ exists (in } \mathbb{C}^n) \text{ at } \sigma\text{-a.e. point on } \partial\Omega.\end{gathered} \tag{5.1.58}$$

(10) For each $\alpha \in (0,1)$ *there exists a constant* $C \in (0,\infty)$ *with the property that*

$$\sup_{x\in\Omega}\Big\{{\rm dist}(x,\partial\Omega)^{1-\alpha}\big|\nabla\big(\mathcal{T}_{\rm mod} f\big)(x)\big|\Big\} \leq C\|f\|_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \tag{5.1.59}$$

for every function $f \in \dot{\mathscr{C}}^\alpha(\partial\Omega)$*. Moreover,*

$$\begin{gathered}\text{if } \Omega \subseteq \mathbb{R}^n \text{ is a uniform domain with the property that } \partial\Omega \text{ is an Ahlfors} \\ \text{regular set then } \mathcal{T}_{\rm mod} : \dot{\mathscr{C}}^\alpha(\partial\Omega) \to \dot{\mathscr{C}}^\alpha\big(\overline{\Omega}\big) \text{ is a well-defined, linear,} \\ \text{and bounded operator for each given exponent } \alpha \in (0,1),\end{gathered} \tag{5.1.60}$$

whereas

if $\Omega \subseteq \mathbb{R}^n$ *is an* NTA *domain with an upper Ahlfors regular boundary then* $\mathscr{T}_{\text{mod}} : \dot{\mathscr{C}}^{\alpha}_{\text{van}}(\partial\Omega) \to \dot{\mathscr{C}}^{\alpha}_{\text{van}}\big(\overline{\Omega}\big)$ *is a well-defined, linear, and bounded operator for each given exponent* $\alpha \in (0,1)$, (5.1.61)

where the homogeneous vanishing Hölder spaces are defined as in (A.0.48) (*with* $\Sigma := \partial\Omega$ *and* $\Sigma := \overline{\Omega}$, *respectively*). *Also, for each* $\alpha \in (0,1)$ *and each* $p \in (1,\infty)$ *there exists some* $C \in (0,\infty)$ *with the property that for each function* $f \in \dot{\mathscr{C}}^{\alpha}(\partial\Omega)$ *one has*

$$\sup_{\substack{x\in\partial\Omega\\ r\in(0,\infty)}} \left(\frac{1}{r^{n-1+\alpha p}} \int_{B(x,r)\cap\Omega} \big|\nabla \mathscr{T}_{\text{mod}} f\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n\right)^{1/p} \leq C\|f\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)} \tag{5.1.62}$$

and

$$\lim_{R\to 0^+} \left\{ \sup_{\substack{x\in\partial\Omega\\ r\in(0,R)}} \left(\frac{1}{r^{n-1+\alpha p}} \int_{B(x,r)\cap\Omega} \big|\nabla \mathscr{T}_{\text{mod}} f\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n\right)^{1/p} \right\} \leq C \operatorname{dist}\big(f, \dot{\mathscr{C}}^{\alpha}_{\text{van}}(\partial\Omega)\big), \tag{5.1.63}$$

where the distance is measured in the space $\big(\dot{\mathscr{C}}^{\alpha}(\partial\Omega), \|\cdot\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)}\big)$. *As a corollary, if the function* f *actually belongs to the homogeneous vanishing Hölder space* $\dot{\mathscr{C}}^{\alpha}_{\text{van}}(\partial\Omega)$ *for some* $\alpha \in (0,1)$, *then for each* $p \in (1,\infty)$ *one has*

$$\lim_{R\to 0^+} \left\{ \sup_{\substack{x\in\partial\Omega\\ r\in(0,R)}} \left(\frac{1}{r^{n-1+\alpha p}} \int_{B(x,r)\cap\Omega} \big|\nabla \mathscr{T}_{\text{mod}} f\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n\right)^{1/p} \right\} = 0. \tag{5.1.64}$$

(11) Let T_{mod} *be the modified version of the singular integral operator* (5.1.5), *acting on each function* $f \in L^1\big(\partial\Omega, \frac{\sigma(y)}{1+|y|^n}\big)$ *according to*[4]

$$T_{\text{mod}} f(x) := \lim_{\varepsilon\to 0^+} \int_{\partial\Omega} \big\langle \nu(y), \vec{k}_\varepsilon(x-y) - \vec{k}_1(-y)\big\rangle f(y)\, d\sigma(y) \tag{5.1.65}$$

at σ*-a.e.* $x \in \partial\Omega$, *where*

$$\vec{k}_\varepsilon := \vec{k} \cdot \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(0,\varepsilon)}} \text{ for each } \varepsilon > 0. \tag{5.1.66}$$

Then T_{mod} *is compatible with* T *from* (5.1.5), *in the sense that*

[4] alternative descriptions of T_{mod} are seen from [70, Proposition 2.3.3]

for each function f belonging to the space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ (hence, in particular, for each $f \in L^p(\partial\Omega, \sigma)$ with $p \in [1, \infty)$) the difference $C_f := T_{\rm mod} f - Tf$ is a constant function on $\partial\Omega$ with $|C_f| \leq C_{\partial\Omega, \vec{k}} \cdot \|f\|_{L^1(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}})}$ for some finite $C_{\partial\Omega, \vec{k}} > 0$. (5.1.67)

Also, for each aperture parameter $\kappa > 0$ and each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ the following jump-formula holds at σ-a.e. point on $\partial\Omega$:

$$\mathcal{T}_{\rm mod} f\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} = -\frac{\vartheta}{2} f + T_{\rm mod} f. \tag{5.1.68}$$

In particular, (5.1.68) holds for each $f \in \dot{L}^p_1(\partial\Omega, \sigma)$ with $1 < p < \infty$. As a consequence of (5.1.68) and (5.1.48),

$$\begin{aligned} &T_{\rm mod} \text{ maps constant functions on } \partial\Omega \\ &\text{into constant functions on } \partial\Omega. \end{aligned} \tag{5.1.69}$$

Moreover,

$$\begin{gathered} \text{if } p, p' \in (1, \infty) \text{ are such that } 1/p + 1/p' = 1 \text{ then given any functions} \\ f \in L^p_{\rm loc}(\partial\Omega, \sigma) \cap L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \text{ and } g \in L^{p'}_{\rm comp}(\partial\Omega, \sigma) \text{ with } \int_{\partial\Omega} g \, {\rm d}\sigma = 0, \\ \text{it follows that } \int_{\partial\Omega} |T_{\rm mod} f||g| \, {\rm d}\sigma < +\infty, \quad \int_{\partial\Omega} |f||T^{\#} g| \, {\rm d}\sigma < +\infty, \\ \text{and } \int_{\partial\Omega} (T_{\rm mod} f) g \, {\rm d}\sigma = \int_{\partial\Omega} f(T^{\#} g) \, {\rm d}\sigma. \end{gathered} \tag{5.1.70}$$

Finally, if $p \in (1, \infty)$ then for each function

$$\begin{gathered} f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\rm loc}(\partial\Omega, \sigma) \text{ such that} \\ \partial_{\tau_{jk}} f \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\rm loc}(\partial\Omega, \sigma) \text{ for all } j, k \in \{1, \dots, n\} \end{gathered} \tag{5.1.71}$$

(hence for each $f \in \dot{L}^p_1(\partial\Omega, \sigma)$ if Ω satisfies a two-sided local John condition) it follows that

$$T_{\rm mod} f \in L^p_{1,{\rm loc}}(\partial\Omega, \sigma), \tag{5.1.72}$$

and for each pair of indices $r, s \in \{1, \dots, n\}$ one has (with V as in (5.1.19))

$$\begin{aligned} \partial_{\tau_{rs}} \big(T_{\rm mod} f\big) = T\big(\partial_{\tau_{rs}} f\big) &+ \big[M_{\nu_r}, T\big](\nabla_{\tan} f)_s - \big[M_{\nu_s}, T\big](\nabla_{\tan} f)_r \\ &- \big[M_{\nu_r}, V\big](\nu_s \nabla_{\tan} f) + \big[M_{\nu_s}, V\big](\nu_r \nabla_{\tan} f) \end{aligned} \tag{5.1.73}$$

at σ-a.e. point on $\partial\Omega$.

(12) *For each* $p \in (1,\infty)$ *there exists a constant* $C \in (0,\infty)$ *with the property that for each function* $f \in \mathrm{BMO}(\partial\Omega,\sigma)$ *the measure*[5] $\big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^p \,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n$ *is Carleson in* Ω *in the quantitative sense that*

$$\sup_{x\in\partial\Omega,\,r>0}\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n \leq C\|f\|^p_{\dot{\mathrm{BMO}}(\partial\Omega,\sigma)}. \tag{5.1.74}$$

In particular, corresponding to $p=2$, *it follows that*[6]

$$\begin{gathered}\big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n \text{ is a Carleson measure in } \Omega,\\ \text{for each function } f\in\mathrm{BMO}(\partial\Omega,\sigma).\end{gathered} \tag{5.1.75}$$

Moreover, for each $p \in (1,\infty)$ *there exists a constant* $C \in (0,\infty)$ *with the property that for each function* $f \in \mathrm{BMO}(\partial\Omega,\sigma)$ *one has*

$$\lim_{R\to 0^+}\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}} \leq C\,\mathrm{dist}\big(f,\mathrm{VMO}(\partial\Omega,\sigma)\big) \tag{5.1.76}$$

where the distance in the right-hand side is considered in the space $\mathrm{BMO}(\partial\Omega,\sigma)$. *As a corollary,*

$$\begin{gathered}\big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega,\\ \text{for each function } f\in\mathrm{VMO}(\partial\Omega,\sigma) \text{ and each } p\in(1,\infty)\end{gathered} \tag{5.1.77}$$

and, corresponding to $p=2$,

$$\begin{gathered}\big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega,\\ \text{for each function } f\in\mathrm{VMO}(\partial\Omega,\sigma).\end{gathered} \tag{5.1.78}$$

(13) *Make the additional assumption that* $\partial\Omega$ *is bounded. Then all properties listed in items (9)-(12) above are valid for the operator* $\mathcal{T}$, *as originally defined in* (5.1.4), *in place of its modified version* $\mathcal{T}_{\rm mod}$. *In particular, for each* $p \in (1,\infty)$ *there exists a constant* $C \in (0,\infty)$ *with the property that for each function* $f \in \mathrm{BMO}(\partial\Omega,\sigma)$ *one has*

[5] i.e., the Littlewood-Paley measure associated with f via the modified generalized double layer potential operator $\mathcal{T}_{\rm mod}$

[6] it is natural to refer to $\big|\nabla\mathcal{T}_{\rm mod} f\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n$ as the Littlewood-Paley measure associated with f via the operator $\mathcal{T}_{\rm mod}$

$$\sup_{x\in\partial\Omega,\, r>0} \frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n \leq C\|f\|^p_{dt\mathrm{BMO}(\partial\Omega,\sigma)}, \tag{5.1.79}$$

and

$$\lim_{R\to 0^+} \sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}} \Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{\frac{1}{p}} \leq C\operatorname{dist}\big(f,\mathrm{VMO}(\partial\Omega,\sigma)\big) \tag{5.1.80}$$

where the distance in the right-hand side is considered in the space $\mathrm{BMO}(\partial\Omega,\sigma)$. *In particular,*

$$\begin{gathered}\big|\nabla\big(\mathcal{T}f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega,\\ \text{for each function } f\in\mathrm{VMO}(\partial\Omega,\sigma) \text{ and each } p\in(1,\infty).\end{gathered} \tag{5.1.81}$$

(14) The operators

$$T_{\rm mod} : \mathrm{BMO}(\partial\Omega,\sigma) \longrightarrow \mathrm{BMO}(\partial\Omega,\sigma), \tag{5.1.82}$$

$$T_{\rm mod} : \mathrm{VMO}(\partial\Omega,\sigma) \longrightarrow \mathrm{VMO}(\partial\Omega,\sigma), \tag{5.1.83}$$

are well defined, linear, and bounded. In particular, (5.1.82)-(5.1.83) *together with* (5.1.69) *imply that both*

$$\begin{gathered}\big[T_{\rm mod}\big] : \mathrm{BMO}(\partial\Omega,\sigma)/\sim\, \longrightarrow \mathrm{BMO}(\partial\Omega,\sigma)/\sim \text{ defined as}\\ \big[T_{\rm mod}\big][f] := \big[T_{\rm mod}f\big] \text{ for each function } f\in\mathrm{BMO}(\partial\Omega,\sigma),\end{gathered} \tag{5.1.84}$$

and

$$\begin{gathered}\big[T_{\rm mod}\big] : \mathrm{VMO}(\partial\Omega,\sigma)/\sim\, \longrightarrow \mathrm{VMO}(\partial\Omega,\sigma)/\sim \text{ defined as}\\ \big[T_{\rm mod}\big][f] := \big[T_{\rm mod}f\big] \text{ for each function } f\in\mathrm{VMO}(\partial\Omega,\sigma),\end{gathered} \tag{5.1.85}$$

are well-defined, linear, and bounded operators. Also, if $\partial\Omega$ *is unbounded one has*

$$\big\langle\big[T_{\rm mod}f\big],g\big\rangle = \big\langle[f],T^{\#}g\big\rangle,\ \forall f\in\mathrm{BMO}(\partial\Omega,\sigma),\ \forall g\in H^1(\partial\Omega,\sigma), \tag{5.1.86}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality bracket between the John-Nirenberg space of functions of bounded mean oscillations on* $\partial\Omega$*, modulo constants, and the Hardy space* H^1 *on* $\partial\Omega$ *(cf.* [69, Theorem 4.6.1]*), and where* $T^{\#}$ *is presently considered as in* (5.1.22) *with* $p := 1$. *In addition, if* $\partial\Omega$ *is unbounded, the operators*

$$T_{\rm mod} : \mathrm{CMO}(\partial\Omega,\sigma) \longrightarrow \mathrm{CMO}(\partial\Omega,\sigma), \tag{5.1.87}$$

and

$$\begin{aligned}&\big[T_{\rm mod}\big] : \mathrm{CMO}(\partial\Omega,\sigma)/\sim\longrightarrow \mathrm{CMO}(\partial\Omega,\sigma)/\sim \ \textit{ defined as}\\ &\big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \textit{ for each function } f\in\mathrm{CMO}(\partial\Omega,\sigma),\end{aligned}\tag{5.1.88}$$

are also well defined, linear, and bounded.
In the case when $\partial\Omega$ *is bounded, a scenario in which one has*

$$\mathrm{BMO}(\partial\Omega,\sigma)\subseteq\bigcap_{0<p<\infty}L^p(\partial\Omega,\sigma),\tag{5.1.89}$$

the operator T *acting on the Lebesgue scale* $L^p(\partial\Omega,\sigma)$ *with* $p\in(1,\infty)$ (*cf.* (5.1.17)) *has* $\mathrm{BMO}(\partial\Omega,\sigma)$ *as an invariant subspace, and its restriction*

$$T:\mathrm{BMO}(\partial\Omega,\sigma)\longrightarrow\mathrm{BMO}(\partial\Omega,\sigma)\tag{5.1.90}$$

is a well-defined, linear, and bounded operator, which satisfies

$$\big\langle Tf,g\big\rangle=\big\langle f,T^{\#}g\big\rangle,\ \ \forall f\in\mathrm{BMO}(\partial\Omega,\sigma),\ \ \forall g\in H^1(\partial\Omega,\sigma)\tag{5.1.91}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality bracket between the John-Nirenberg space of functions of bounded mean oscillations on* $\partial\Omega$ *and the Hardy space* H^1 *on* $\partial\Omega$ (*cf.* [69, Theorem 4.6.1]) *and where* $T^{\#}$ *is presently considered as in* (5.1.22) *with* $p:=1$.
Moreover, when $\partial\Omega$ *is bounded the operator* T *acting on* $\mathrm{BMO}(\partial\Omega,\sigma)$ (*cf.* (5.1.90)) *has* $\mathrm{VMO}(\partial\Omega,\sigma)$ *as an invariant subspace, hence its restriction*

$$T:\mathrm{VMO}(\partial\Omega,\sigma)\longrightarrow\mathrm{VMO}(\partial\Omega,\sigma)\tag{5.1.92}$$

is a well-defined, linear, and bounded operator.

(15) Fix $p\in\big(\frac{n-1}{n},1\big)$ *and set* $\alpha:=(n-1)\big(\frac{1}{p}-1\big)\in(0,1)$. *Then the operators*

$$T_{\rm mod}:\dot{\mathscr{C}}^{\alpha}(\partial\Omega)\longrightarrow\dot{\mathscr{C}}^{\alpha}(\partial\Omega),\tag{5.1.93}$$

$$T_{\rm mod}:\dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)\longrightarrow\dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega),\tag{5.1.94}$$

$$\dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim\ni[f]\mapsto\big[T_{\rm mod}f\big]\in\dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim,\tag{5.1.95}$$

$$\dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)/\sim\ni[f]\mapsto\big[T_{\rm mod}f\big]\in\dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)/\sim,\tag{5.1.96}$$

are well defined, linear, bounded. As a consequence of (5.1.93)-(5.1.96), *the operators*

$$\begin{aligned}&\big[T_{\rm mod}\big]:\dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim\longrightarrow\dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim\ \textit{ defined as}\\ &\big[T_{\rm mod}\big][f]:=\big[T_{\rm mod}f\big] \textit{ for each function } f\in\dot{\mathscr{C}}^{\alpha}(\partial\Omega),\end{aligned}\tag{5.1.97}$$

$$
\begin{aligned}
&\big[T_{\rm mod}\big] : \dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)/\sim \longrightarrow \dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)/\sim \ \ \text{defined as} \\
&\big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \ \text{for each function} \ f \in \dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega),
\end{aligned}
\tag{5.1.98}
$$

are well defined, linear, and bounded. In addition, if $\partial\Omega$ is unbounded one has

$$
\big\langle \big[T_{\rm mod} f\big], g\big\rangle = \big\langle [f], T^{\#} g\big\rangle, \quad \forall f \in \dot{\mathscr{C}}^{\alpha}(\partial\Omega), \quad \forall g \in H^p(\partial\Omega,\sigma) \tag{5.1.99}
$$

with $\langle\cdot,\cdot\rangle$ denoting the duality bracket between a homogeneous Hölder space on $\partial\Omega$, modulo constants, and the corresponding Hardy space on $\partial\Omega$ (cf. [69, Theorem 4.6.1]*), and where $T^{\#}$ is presently considered as in* (5.1.22)*. Finally, in the case when $\partial\Omega$ is bounded, a scenario in which one has*

$$
\dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega) = \mathscr{C}^{\alpha}_{\rm van}(\partial\Omega) \subseteq \dot{\mathscr{C}}^{\alpha}(\partial\Omega) = \mathscr{C}^{\alpha}(\partial\Omega) \subseteq \bigcap_{0<q\leq\infty} L^q(\partial\Omega,\sigma), \tag{5.1.100}
$$

the operator T acting on the Lebesgue scale as in (5.1.17) *has both $\mathscr{C}^{\alpha}_{\rm van}(\partial\Omega)$ and $\mathscr{C}^{\alpha}(\partial\Omega)$ as invariant subspaces, its restrictions*

$$
T : \mathscr{C}^{\alpha}_{\rm van}(\partial\Omega) \longrightarrow \mathscr{C}^{\alpha}_{\rm van}(\partial\Omega) \tag{5.1.101}
$$

and

$$
T : \mathscr{C}^{\alpha}(\partial\Omega) \longrightarrow \mathscr{C}^{\alpha}(\partial\Omega) \tag{5.1.102}
$$

are well-defined, linear, bounded operators, and the latter operator satisfies

$$
\big\langle Tf, g\big\rangle = \big\langle f, T^{\#} g\big\rangle, \quad \forall f \in \mathscr{C}^{\alpha}(\partial\Omega), \quad \forall g \in H^p(\partial\Omega,\sigma) \tag{5.1.103}
$$

with $\langle\cdot,\cdot\rangle$ denoting the duality bracket between Hölder and Hardy spaces on $\partial\Omega$ (cf. [69, Theorem 4.6.1]*) and with $T^{\#}$ considered as in* (5.1.22)*.*

(16) Select $p,q \in (1,\infty)$ with $1/p + 1/q = 1$ along with $\lambda \in (0, n-1)$. Then the operator T from (5.1.17) *has the inhomogeneous Morrey-Campanato space $\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)$ (defined as in* (A.0.119) *with $\Sigma := \partial\Omega$) as an invariant subspace, and*

$$
T : \Big(\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma), \|\cdot\|_{\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)}\Big) \longrightarrow \Big(\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma), \|\cdot\|_{\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)}\Big) \tag{5.1.104}
$$

is a linear and bounded mapping. Moreover, if $T_{\rm mod}$ is the modified version of the singular integral operator T defined in (5.1.65)*, the assignment*

$$
\begin{aligned}
&[T_{\rm mod}] : \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim \longrightarrow \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim \\
&[T_{\rm mod}][f] := [T_{\rm mod} f] \ \text{for each} \ f \in \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)
\end{aligned}
\tag{5.1.105}
$$

is well defined, linear and bounded. In particular,

$$T_{\text{mod}} : \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma) \longrightarrow \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma) \text{ is well defined, linear, and there exists some constant } C \in (0,\infty) \text{ with the property that } \|T_{\text{mod}} f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)} \text{ for each } f \in \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma). \tag{5.1.106}$$

Finally,

the (real) transpose of the operator $T^{\#}$ from (5.1.32) is, respectively, $\big[T_{\text{mod}}\big]$ from (5.1.105) if $\partial\Omega$ is unbounded, and T from (5.1.104) if $\partial\Omega$ is bounded, (5.1.107)

that is, for each $f \in \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)$ and $g \in \mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)$ one has

$$\big\langle [T_{\text{mod}} f], g \big\rangle = \big\langle [f], T^{\#} g \big\rangle \text{ if } \partial\Omega \text{ is unbounded,} \tag{5.1.108}$$

$$\big\langle Tf, g \big\rangle = \big\langle f, T^{\#} g \big\rangle \text{ if } \partial\Omega \text{ is bounded,} \tag{5.1.109}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between the Morrey-Campanato space and its pre-dual (cf. [69, (6.1.25)]).

(17) Fix $q \in (1,\infty)$ and $\eta \in (0,1)$ and recall the Calderón spaces defined in (3.1.10)-(3.1.15). Then there exists a constant $C \in (0,\infty)$ with the property that, with the maximal operator defined as in (3.1.1) (with $\Sigma := \partial\Omega$), one has

$$\big(T_{\text{mod}} f\big)^{\#}_{q,\eta}(x) \leq C\, f^{\#}_{q,\eta}(x) \text{ at } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.1.110}$$

for every function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)$. As a consequence, given any $p \in [1,\infty)$, it follows that

$$[T_{\text{mod}}] : \dot{C}^{p}_{q,\eta}(\partial\Omega,\sigma)/\sim \longrightarrow \dot{C}^{p}_{q,\eta}(\partial\Omega,\sigma)/\sim \text{ defined by } [T_{\text{mod}}][f] := \big[T_{\text{mod}} f\big] \text{ for each } f \in \dot{C}^{p}_{q,\eta}(\partial\Omega,\sigma) \tag{5.1.111}$$

is a well-defined, linear, and bounded operator. Moreover,

$$\big(Tf\big)^{\#}_{q,\eta}(x) \leq C\, f^{\#}_{q,\eta}(x) \text{ at } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.1.112}$$

for every function $f \in L^p(\partial\Omega,\sigma)$ with $p \in [1,\infty)$. Finally, whenever $p \in (1,\infty)$ it follows that

$$T : C^{p}_{q,\eta}(\partial\Omega,\sigma) \longrightarrow C^{p}_{q,\eta}(\partial\Omega,\sigma) \tag{5.1.113}$$

is a well-defined, linear, and bounded operator.

(18) If

$$\ell \in \{1,\dots,N\}, \quad \tfrac{n-1}{n} < p \leq \infty \text{ and } (n-1)\big(\tfrac{1}{p}-1\big)_{+} < s < 1, \tag{5.1.114}$$

then for each $\theta \in (0,1)$ *there exists a finite constant* $C = C(\Omega, \vec{k}, \ell, p, s, \theta) > 0$ *such that*

$$\Big\| \delta_{\partial\Omega}^{\ell - \frac{1}{p} - s} \big|\nabla^\ell \mathcal{T}_{\mathrm{mod}} f\big|_{\star,\theta} \Big\|_{L^p(\Omega, \mathcal{L}^n)} \leq C \|f\|_{B^{p,p}_s(\partial\Omega,\sigma)} \tag{5.1.115}$$

for all $f \in B^{p,p}_s(\partial\Omega,\sigma)$, *with the understanding that when* $p > 1$ *the solid maximal function* $\big|\nabla^\ell \mathcal{T}_{\mathrm{mod}} f\big|_{\star,\theta}$ *is replaced by* $\big|\nabla^\ell \mathcal{T}_{\mathrm{mod}} f\big|$.

(19) *Make the additional assumption that* $\partial\Omega$ *is compact. Then the operator* T, *originally acting on Lebesgue spaces on* $\partial\Omega$ (*cf.* (5.1.17)), *extends uniquely to linear and bounded mappings*

$$\begin{gathered} T : B^{p,q}_s(\partial\Omega,\sigma) \longrightarrow B^{p,q}_s(\partial\Omega,\sigma), \\ p \in \big(\tfrac{n-1}{n}, \infty\big], \quad q \in (0,\infty], \quad (n-1)\big(\tfrac{1}{p} - 1\big)_+ < s < 1, \end{gathered} \tag{5.1.116}$$

and

$$\begin{gathered} T : F^{p,q}_s(\partial\Omega,\sigma) \longrightarrow F^{p,q}_s(\partial\Omega,\sigma), \\ p \in \big(\tfrac{n-1}{n}, \infty\big), \quad q \in \big(\tfrac{n-1}{n}, \infty\big], \quad (n-1)\big(\tfrac{1}{\min\{p,q\}} - 1\big)_+ < s < 1. \end{gathered} \tag{5.1.117}$$

Moreover, various choices of the exponents yield operators which are compatible with one another. In addition, the operator $T^\#$, *originally considered acting on Lebesgue spaces on* $\partial\Omega$ (*cf.* (5.1.17)) *further extends, in a unique fashion, to linear and bounded mappings*

$$\begin{gathered} T^\# : B^{p,q}_{-s}(\partial\Omega,\sigma) \longrightarrow B^{p,q}_{-s}(\partial\Omega,\sigma) \\ \textit{with } s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s}, \infty\big], \quad q \in (0,\infty], \end{gathered} \tag{5.1.118}$$

and

$$\begin{gathered} T^\# : F^{p,q}_{-s}(\partial\Omega,\sigma) \longrightarrow F^{p,q}_{-s}(\partial\Omega,\sigma), \\ \textit{with } s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s}, \infty\big), \quad q \in \big(\tfrac{n-1}{n-s}, \infty\big]. \end{gathered} \tag{5.1.119}$$

Again, various choices of the parameters p, q, s *yield operators which are compatible with one another. In all cases,*

> *the operator norms of* T *from* (5.1.116)-(5.1.117) *as well as* $T^\#$ *from* (5.1.118)-(5.1.119) *are dominated by* $C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha \vec{k}|\Big)$ *with* $C \in (0,\infty)$ *depending only on* n, p, q, s, *and the* UR *character of* $\partial\Omega$. (5.1.120)

Finally, if $p, q, p', q' \in (1,\infty)$ *satisfy* $1/p + 1/p' = 1 = 1/q + 1/q'$ *and* $s \in (0,1)$, *then*

$$\begin{aligned} &{}_{B^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle T^{\#} f, g\big\rangle_{B^{p',q'}_{s}(\partial\Omega,\sigma)} = {}_{B^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle f, Tg\big\rangle_{B^{p',q'}_{s}(\partial\Omega,\sigma)} \\ &\quad \textit{for each } f \in B^{p,q}_{-s}(\partial\Omega,\sigma) \textit{ and } g \in B^{p',q'}_{s}(\partial\Omega,\sigma), \end{aligned} \tag{5.1.121}$$

and

$$\begin{aligned} &{}_{F^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle T^{\#} f, g\big\rangle_{F^{p',q'}_{s}(\partial\Omega,\sigma)} = {}_{F^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle f, Tg\big\rangle_{F^{p',q'}_{s}(\partial\Omega,\sigma)} \\ &\quad \textit{for each } f \in F^{p,q}_{-s}(\partial\Omega,\sigma) \textit{ and } g \in F^{p',q'}_{s}(\partial\Omega,\sigma). \end{aligned} \tag{5.1.122}$$

(20) *Strengthen the original hypotheses on the underlying domain by assuming that* Ω *is a* UR *domain satisfying a local John condition. Also, fix some* $p \in (1,\infty)$*. Then the operator*

$$T_{\text{mod}} : \dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\text{loc}}(\partial\Omega,\sigma) \longrightarrow \dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\text{loc}}(\partial\Omega,\sigma) \tag{5.1.123}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.128)*. Moreover, formula* (5.1.73) *holds for each function* $f \in \dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\text{loc}}(\partial\Omega,\sigma)$*.*
Finally, if it is assumed that $\Omega \subseteq \mathbb{R}^n$ *is an* NTA *domain with an upper Ahlfors regular boundary*[7] *then for each integrability exponent* $p \in (1,\infty)$ *the operator*

$$T_{\text{mod}} : \dot{L}^p_1(\partial\Omega,\sigma) \longrightarrow \dot{L}^p_1(\partial\Omega,\sigma) \tag{5.1.124}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.128)*. In this case, it follows from* (5.1.124) *and* (5.1.69) *that for each* $p \in (1,\infty)$ *the operator*

$$\begin{aligned} &\big[T_{\text{mod}}\big] : \dot{L}^p_1(\partial\Omega,\sigma)/\sim \longrightarrow \dot{L}^p_1(\partial\Omega,\sigma)/\sim \ \textit{ defined as} \\ &\big[T_{\text{mod}}\big][f] := \big[T_{\text{mod}} f\big] \in \dot{L}^p_1(\partial\Omega,\sigma)/\sim \ \textit{ for all } f \in \dot{L}^p_1(\partial\Omega,\sigma), \end{aligned} \tag{5.1.125}$$

is well defined, linear, and bounded, when all quotient spaces are endowed with the natural semi-norm[8] *introduced in* [69, (11.5.138)].

(21) *Assume* Ω *is a* UR *domain satisfying a local John condition*[9]*, and fix an integrability exponent* $p \in (1,\infty)$ *along with a parameter* $\lambda \in (0, n-1)$*. Then the operator*

$$T_{\text{mod}} : \dot{M}^{p,\lambda}_1(\partial\Omega,\sigma) \longrightarrow \dot{M}^{p,\lambda}_1(\partial\Omega,\sigma) \tag{5.1.126}$$

---

[7] in particular, this is the case if Ω is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular; cf. (1.8.157)

[8] [69, Proposition 11.5.14] tells us that this semi-norm is fact a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

[9] for example, this is the case if $\Omega \subseteq \mathbb{R}^n$ is an NTA domain with an upper Ahlfors regular boundary

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.158)*. Also,* (5.1.73) *holds for each function* $f \in \dot{M}_1^{p,\lambda}(\partial\Omega, \sigma)$*. As a consequence of* (5.1.126) *and* (5.1.69)*, the operator*

$$\big[T_{\rm mod}\big] : \dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\ \longrightarrow \dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim \ \text{ defined as}$$
$$\big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \in \dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)/\sim\ \text{ for all } f \in \dot{M}_1^{p,\lambda}(\partial\Omega,\sigma), \tag{5.1.127}$$

is well defined, linear, and bounded, when all quotient spaces are endowed with the semi-norm[10] *introduced in* [69, (11.13.51)].

In fact, analogous properties are valid for vanishing Morrey-based homogeneous Sobolev spaces $\mathring{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)$ (*cf.* [69, Definition 11.13.15], *or* (A.0.159)-(A.0.160)) *in place of Morrey-based homogeneous Sobolev spaces. Finally, similar properties are valid for block-based homogeneous Sobolev spaces* $\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)$ *with* $q \in (1,\infty)$ *in place of Morrey-based homogeneous Sobolev spaces.*

(22) *Strengthen the hypotheses on the underlying domain by assuming that* $\Omega \subseteq \mathbb{R}^n$ *is an* NTA *domain with an upper Ahlfors regular boundary*[11]*. Then the modified boundary-to-boundary operator* $T_{\rm mod}$*, originally defined as in* (5.1.65)*, induces a linear and bounded mapping*

$$T_{\rm mod} : \dot{H}_1^p(\partial\Omega,\sigma) \longrightarrow \dot{H}_1^p(\partial\Omega,\sigma) \ \text{for each } p \in \big(\tfrac{n-1}{n}, 1\big]. \tag{5.1.128}$$

Before presenting the proof of Theorem 5.1.1 we shall discuss some relevant examples.

Example 5.1.2 (The Cauchy Operator in the Plane) *Work in the two-dimensional setting, and identify* $\mathbb{R}^2 \equiv \mathbb{C}$*. Let* $\Omega \subseteq \mathbb{C}$ *be a* UR *domain, set* $\sigma := \mathcal{H}^1\lfloor\partial\Omega$*, and denote by* $\nu = (\nu_1,\nu_2) \equiv \nu_1 + i\nu_2$ *the geometric measure theoretic outward unit normal to* Ω*. Also, consider*

$$\vec{k}(z) := -\frac{1}{2\pi}\Big(\frac{1}{z}, \frac{i}{z}\Big) \ \text{for each } z \in \mathbb{C}\setminus\{0\}. \tag{5.1.129}$$

This is a smooth vector-valued function which is odd, positive homogeneous of degree -1*, and satisfies*

$$\begin{aligned}\big(\mathrm{div}\vec{k}\big)(z) &= -\frac{1}{2\pi}\partial_x\Big(\frac{1}{z}\Big) - \frac{1}{2\pi}\partial_y\Big(\frac{i}{z}\Big) = -\frac{1}{2\pi}(\partial_x + i\partial_y)\Big(\frac{1}{z}\Big) \\ &= -\frac{1}{\pi}\overline{\partial}\Big(\frac{1}{z}\Big) = 0 \ \text{for each } z \in \mathbb{C}\setminus\{0\},\end{aligned} \tag{5.1.130}$$

---

[10] From [69, Proposition 11.13.10] it is known that this semi-norm is actually a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

[11] in particular, this is the case if Ω is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular; cf. (1.8.157)

where $\overline{\partial} := \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$ *is the Cauchy-Riemann operator in the plane. Hence all hypotheses of Theorem 5.1.1 are satisfied in this case, and* (5.1.3) *presently becomes*

$$\begin{aligned}\vartheta &= \int_{S^1} \langle \omega, \vec{k}(\omega)\rangle \, d\mathcal{H}^1(\omega) = -\frac{1}{2\pi}\int_{S^1}\Big\{\omega_1\Big(\frac{1}{\omega}\Big) + \omega_2\Big(\frac{\mathrm{i}}{\omega}\Big)\Big\}\, d\mathcal{H}^1(\omega) \\ &= -\frac{1}{2\pi}\int_{S^1}(\omega_1 + \mathrm{i}\omega_2)\Big(\frac{1}{\omega}\Big)\, d\mathcal{H}^1(\omega) = -\frac{1}{2\pi}\int_{S^1}\omega\frac{1}{\omega}\, d\mathcal{H}^1(\omega) \\ &= -\frac{1}{2\pi}\int_{S^1} d\mathcal{H}^1(\omega) = -1. \end{aligned} \tag{5.1.131}$$

Note since for each $z \in \Omega$ *we have*

$$\begin{aligned}\big\langle \nu(\zeta), \vec{k}(z-\zeta)\big\rangle &= -\frac{1}{2\pi}\nu_1(\zeta)\Big(\frac{1}{z-\zeta}\Big) - \frac{1}{2\pi}\nu_2(\zeta)\Big(\frac{\mathrm{i}}{z-\zeta}\Big) \\ &= -\frac{1}{2\pi}\big(\nu_1(\zeta) + \mathrm{i}\nu_2(\zeta)\big)\Big(\frac{1}{z-\zeta}\Big) \\ &= \frac{1}{2\pi}\frac{\nu(\zeta)}{\zeta - z} \ \textit{for } \sigma\textit{-a.e. } \zeta \in \partial\Omega, \end{aligned} \tag{5.1.132}$$

and since $i\nu(\zeta)\, d\sigma(\zeta) = d\zeta$ *on* $\partial\Omega$ *(cf.* (A.0.62)*), the operator* $\mathcal{T}$ *constructed as in* (5.1.4) *for* $\vec{k}$ *as in* (5.1.129) *becomes precisely the boundary-to-domain Cauchy integral operator, acting on each function* $f \in L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\big)$ *according to (cf.* (1.6.35)*)*

$$\mathscr{C} f(z) = \frac{1}{2\pi\mathrm{i}}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}\, d\zeta \ \textit{for all } z \in \Omega. \tag{5.1.133}$$

Also, the principal-value singular integral operator T from (5.1.5) *presently becomes*

$$Cf(z) := \lim_{\varepsilon\to 0^+}\frac{1}{2\pi\mathrm{i}}\int_{\substack{\zeta\in\partial\Omega \\ |z-\zeta|>\varepsilon}}\frac{f(\zeta)}{\zeta - z}\, d\zeta \ \textit{for } \sigma\textit{-a.e. } z \in \partial\Omega, \tag{5.1.134}$$

i.e., the principal-value Cauchy integral operator (1.6.36)*, while* $T^{\#}$ *from* (5.1.6) *currently acquires the format*

$$C^{\#} f(z) := -\nu(z)\Bigg(\lim_{\varepsilon\to 0^+}\frac{1}{2\pi}\int_{\substack{\zeta\in\partial\Omega \\ |z-\zeta|>\varepsilon}}\frac{f(\zeta)}{\zeta - z}\, d\sigma(\zeta)\Bigg) \ \textit{for } \sigma\textit{-a.e. } z \in \partial\Omega, \tag{5.1.135}$$

i.e., $T^{\#}$ *is the transpose Cauchy singular integral operator from* (1.6.37)*. In particular, Theorem 5.1.1 specialized to this setting becomes compatible with the results established earlier in Proposition 1.6.7.*

Example 5.1.3 (The Cauchy-Clifford Operator) *Let* $\Omega \subseteq \mathbb{R}^n$ *be an arbitrary* UR *domain* (*where* $n \in \mathbb{N}$ *with* $n \geq 2$)*. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Recall the Clifford algebra with n imaginary units* $\mathcal{C}\ell_n$ *from* [68, §6.4] *and consider the vector field with Clifford algebra-valued components*

$$\vec{k} = (k_j)_{1 \leq j \leq n} \in \left[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\}) \otimes \mathcal{C}\ell_n\right]^n \tag{5.1.136}$$

given by

$$k_j(x) := \frac{1}{\omega_{n-1}} \frac{x}{|x|^n} \odot \mathbf{e}_j \text{ for } 1 \leq j \leq n \text{ and } x \in \mathbb{R}^n \setminus \{0\}. \tag{5.1.137}$$

In particular, $\vec{k}$ *is odd and positive homogeneous of degree* $1-n$ *in* $\mathbb{R}^n \setminus \{0\}$*, and satisfies* (*with the summation convention over repeated indices in effect*)

$$\begin{aligned}(\mathrm{div}\vec{k})(x) = (\partial_j k_j)(x) &= \frac{1}{\omega_{n-1}}\partial_j\Big(\frac{x}{|x|^n}\Big) \odot \mathbf{e}_j \\ &= \frac{1}{\omega_{n-1}} D_R\Big(\frac{x}{|x|^n}\Big) = 0 \text{ in } \mathbb{R}^n \setminus \{0\},\end{aligned} \tag{5.1.138}$$

where D_R *is the version of the Dirac operator acting from the right (cf.* (A.0.59)*). Also,* (5.1.3) *presently becomes*

$$\begin{aligned}\vartheta &= \int_{S^{n-1}} \langle \omega, \vec{k}(\omega)\rangle \, d\mathcal{H}^{n-1}(\omega) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \omega_j \Big(\frac{\omega}{|\omega|^n} \odot \mathbf{e}_j\Big) \, d\mathcal{H}^{n-1}(\omega) \\ &= \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \frac{\omega}{|\omega|^n} \odot \omega \, d\mathcal{H}^{n-1}(\omega) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} (-1) \, d\mathcal{H}^{n-1} = -1,\end{aligned} \tag{5.1.139}$$

on account of [68, (6.4.1)]*. Given that for each* $x \in \Omega$ *we have*

$$\begin{aligned}\big\langle \nu(y), \vec{k}(x-y)\big\rangle &= \nu_j(y)\Big(\frac{1}{\omega_{n-1}} \frac{x-y}{|x-y|^n} \odot \mathbf{e}_j\Big) \\ &= \frac{1}{\omega_{n-1}} \frac{x-y}{|x-y|^n} \odot \nu(y) \text{ at } \sigma\text{-a.e. } y \in \partial\Omega,\end{aligned} \tag{5.1.140}$$

the operator $\mathcal{T}$ *from* (5.1.4) *for* $\vec{k}$ *as in* (5.1.129) *becomes precisely the boundary-to-domain Cauchy-Clifford integral operator, acting on each* $f \in L^1\big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\big) \otimes \mathcal{C}\ell_n$ *according to (cf.* (A.0.53)*)*

$$\mathcal{C}f(x) = \frac{1}{\omega_{n-1}} \int_{\partial\Omega} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y) \, d\sigma(y) \text{ for all } x \in \Omega. \tag{5.1.141}$$

Moreover, the principal-value singular integral operator T *from* (5.1.5) *presently becomes*

$$\mathfrak{C} f(x) = \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, d\sigma(y) \tag{5.1.142}$$

for σ-a.e. $x \in \partial\Omega$, i.e., the principal-value Cauchy-Clifford integral operator (A.0.54)*, while $T^\#$ from* (5.1.6) *currently becomes*

$$\mathfrak{C}^\# f(x) = -\lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu(x) \odot \frac{x-y}{|x-y|^n} \odot f(y)\, d\sigma(y) \tag{5.1.143}$$

for σ-a.e. $x \in \partial\Omega$, i.e., $T^\#$ is the transpose Cauchy-Clifford singular integral operator from (1.6.1)*. In particular, Theorem 5.1.1 specialized to this setting becomes compatible with the results established earlier in Proposition 1.6.1, Theorem 2.1.5,* (2.1.179)*-*(2.1.192)*,* (4.1.13)*-*(4.1.14)*, and* (4.1.49)*-*(4.1.50)*.*

Example 5.1.4 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider a complex-valued function $b \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ such that ∇b is odd and positive homogeneous of degree $1-n$ (this is the case if, for example, b is even and positive homogeneous of degree $2-n$). For each given pair of indices $i, j \in \{1, \dots, n\}$ define the vector field*

$$\vec{k}_{ij} := (\partial_i b)\mathbf{e}_j - (\partial_j b)\mathbf{e}_i \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^n \tag{5.1.144}$$

which is odd, positive homogeneous of degree $1-n$, and divergence-free in $\mathbb{R}^n \setminus \{0\}$. Note that (5.1.3) *presently becomes*

$$\begin{aligned}\vartheta &= \int_{S^{n-1}} \langle \omega, \vec{k}_{ij}(\omega)\rangle \, d\mathcal{H}^{n-1}(\omega) \\ &= \int_{S^{n-1}} \big\{\omega_j(\partial_i b)(\omega) - \omega_i(\partial_j b)(\omega)\big\}\, d\mathcal{H}^{n-1}(\omega) \\ &= \int_{S^{n-1}} \big(\partial_{\tau_{ji}} b\big)(\omega)\, d\mathcal{H}^{n-1}(\omega) = 0, \end{aligned} \tag{5.1.145}$$

thanks to (A.0.183) *and* [69, (11.1.7)]*. The operators $\mathcal{T}$, T, $T^\#$, associated as in* (5.1.4)*,* (5.1.5)*,* (5.1.6) *with the vector field from* (5.1.144) *have been considered earlier in* (1.2.1)*,* (1.2.2)*,* (1.2.3)*. The results in Theorem 5.1.1 are then compatible with those in Proposition 1.2.1,* (2.1.159)*-*(2.1.168)*,* [70, Corollary 2.4.2]*, Theorem 4.1.3, and Theorem 4.1.7. In closing, we wish to note that*

> *when $\Omega := \mathbb{R}^n_+$ the (non-zero) principal-value generalized double layer operators corresponding to the kernels* (5.1.144) *for the choice $b := E_\Delta$, the standard fundamental solution for the Laplacian in $\mathbb{R}^n$, coincide (up to normalization) with the family of ordinary Riesz transforms on $\partial\Omega \equiv \mathbb{R}^{n-1}$.* (5.1.146)

Example 5.1.5 (The Harmonic Double Layer) *Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary* UR *domain* (*where $n \in \mathbb{N}$ with $n \geq 2$*). *Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. If we consider the vector field*

$$\vec{k} = (k_j)_{1\leq j\leq n} \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^n \tag{5.1.147}$$

given by

$$\vec{k}(x) := -\frac{1}{\omega_{n-1}}\frac{x}{|x|^n} \ \textit{for all}\ \ x \in \mathbb{R}^n \setminus \{0\}, \tag{5.1.148}$$

then $\vec{k}$ is odd, positive homogeneous of degree $1-n$, and divergence-free in $\mathbb{R}^n \setminus \{0\}$. Also, (5.1.3) *presently becomes*

$$\vartheta = \int_{S^{n-1}} \langle\omega, \vec{k}(\omega)\rangle\, \mathrm{d}\mathcal{H}^{n-1}(\omega) = -\frac{1}{\omega_{n-1}}\int_{S^{n-1}}\frac{\langle\omega,\omega\rangle}{|\omega|^n}\,\mathrm{d}\mathcal{H}^{n-1}(\omega) = -1. \tag{5.1.149}$$

Given that for each $x \in \Omega$ we have

$$\big\langle\nu(y), \vec{k}(x-y)\big\rangle = \frac{1}{\omega_{n-1}}\frac{\langle\nu(y), y-x\rangle}{|x-y|^n} \ \textit{at}\ \sigma\textit{-a.e.}\ \ y \in \partial\Omega, \tag{5.1.150}$$

the operator $\mathcal{T}$ from (5.1.4) *for $\vec{k}$ as in* (5.1.147)-(5.1.148) *becomes precisely the classical boundary-to-domain harmonic double layer operator, acting on each function $f \in L^1\big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\big)$ according to* (*cf.* [70, (2.5.202)])

$$\mathcal{D}f(x) = \frac{1}{\omega_{n-1}}\int_{\partial\Omega}\frac{\langle\nu(y), y-x\rangle}{|x-y|^n}f(y)\,\mathrm{d}\sigma(y) \ \textit{for all}\ \ x \in \Omega. \tag{5.1.151}$$

Moreover, the principal-value singular integral operator T from (5.1.5) *presently becomes*

$$Kf(x) = \lim_{\varepsilon\to 0^+}\frac{1}{\omega_{n-1}}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\frac{\langle\nu(y), y-x\rangle}{|x-y|^n}f(y)\,\mathrm{d}\sigma(y) \tag{5.1.152}$$

for σ-a.e. $x \in \partial\Omega$, i.e., the harmonic double layer operator [70, (2.5.203)], *while $T^\#$ from* (5.1.6) *currently becomes*

$$K^\# f(x) = \lim_{\varepsilon\to 0^+}\frac{1}{\omega_{n-1}}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\frac{\langle\nu(x), x-y\rangle}{|x-y|^n}f(y)\,\mathrm{d}\sigma(y) \tag{5.1.153}$$

for σ-a.e. $x \in \partial\Omega$, i.e., $T^\#$ is the transpose harmonic double layer operator from (A.0.102). *In particular, Theorem 5.1.1 is applicable to these integral operators.*

Example 5.1.6 (Genuine Double Layers Associated with Second-Order Systems) *Let* $\Omega \subseteq \mathbb{R}^n$ *(where* $n \in \mathbb{N}$*,* $n \geq 2$*) be a* UR *domain and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$*. For some* $M \in \mathbb{N}$*, consider a coefficient tensor* $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ *with complex entries, with the property that the* $M\times M$ *homogeneous second-order system* $L = L_A$ *associated with* A *in* $\mathbb{R}^n$ *as in* (1.3.2) *is weakly elliptic* (*in the sense of* [70, (1.3.3) in Definition 1.3.1]). *Let* $E = (E_{\gamma\beta})_{1\leq\gamma,\beta\leq M}$ *be the matrix-valued fundamental solution associated with* L *as in* [70, Theorem 1.4.2].

For each fixed $\alpha, \gamma \in \{1, \dots, M\}$ *define the vector field* (*recall that the summation convention over repeated indices is presently in effect*)

$$\vec{k}_{\alpha\gamma} := -\big(a_{rs}^{\beta\alpha}\partial_r E_{\gamma\beta}\big)_{1\leq s\leq n} \in \big[\mathscr{C}^\infty(\mathbb{R}^n\setminus\{0\})\big]^n \tag{5.1.154}$$

which is odd and positive homogeneous of degree $1-n$ (*cf.* [70, Theorem 1.4.2]), *and which satisfies*

$$\operatorname{div}\vec{k}_{\alpha\gamma} = -a_{rs}^{\beta\alpha}\partial_r\partial_s E_{\gamma\beta} = 0 \text{ in } \mathbb{R}^n\setminus\{0\}, \tag{5.1.155}$$

thanks to [70, (1.4.33)]. *Also,* (5.1.3) *presently becomes*

$$\begin{aligned}\vartheta &= \int_{S^{n-1}} \langle\omega, \vec{k}_{\alpha\gamma}(\omega)\rangle\, d\mathcal{H}^{n-1}(\omega)\\ &= -\int_{S^{n-1}} \omega_s a_{rs}^{\beta\alpha}(\partial_r E_{\gamma\beta})(\omega)\, d\mathcal{H}^{n-1}(\omega) = -\delta_{\alpha\gamma}\end{aligned} \tag{5.1.156}$$

by virtue of the second equality in [70, (1.4.25)].

If $\mathcal{T}_{\alpha\gamma}$ *denotes the integral operator associated as in* (5.1.4) *with* $\vec{k}_{\alpha\gamma}$ *defined in* (5.1.154), *then it becomes apparent from* (1.3.18) *that for each given function* $f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M$ *we have*

$$\mathcal{D}f = \big(\mathcal{T}_{\alpha\gamma}f_\alpha\big)_{1\leq\gamma\leq M} \text{ in } \Omega. \tag{5.1.157}$$

Also, if $T_{\alpha\gamma}$ *denotes the principal-value singular integral operator associated as in* (5.1.5) *with* $\vec{k}_{\alpha\gamma}$ *defined in* (5.1.154), *then from* (1.3.68) *it follows that for each given function* $f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M$ *we have*

$$Kf = \big(T_{\alpha\gamma}f_\alpha\big)_{1\leq\gamma\leq M} \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{5.1.158}$$

hence

$$K = \big(T_{\alpha\gamma}\big)_{1\leq\gamma,\alpha\leq M}. \tag{5.1.159}$$

Finally, if $T_{\alpha\gamma}^{\#}$ *is defined as in* (5.1.6) *in relation to* $\vec{k}_{\alpha\gamma}$ *from* (5.1.154) *then, as seen from* (1.3.72), *for each* $f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M$ *we have*

$$K^{\#} f = \big(T^{\#}_{\alpha\gamma} f_\gamma\big)_{1\leq\alpha\leq M} \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{5.1.160}$$

thus,

$$K^{\#} = \big(T^{\#}_{\alpha\gamma}\big)_{1\leq\alpha,\gamma\leq M}. \tag{5.1.161}$$

As such, Theorem 5.1.1 is compatible with results established earlier for $\mathcal{D}$, K, $K^{\#}$ (and their modified versions) in Theorem 1.5.1, Theorem 1.8.2, Theorem 2.1.1, Corollary 2.1.2, Theorem 2.1.7, Corollary 2.1.9, Theorem 2.1.10, Corollary 2.1.13, Theorem 3.1.1, Theorem 3.2.2, Theorem 3.3.1, Theorem 4.1.1, Corollary 4.1.4, and Theorem 4.1.5.

Example 5.1.7 (Chord-Dot-Normal Singular Integral Operators) *Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary* UR *domain (where $n \in \mathbb{N}$ with $n \geq 2$). Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Consider a function $\theta \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ which is even and positive homogeneous of degree $-n$, and define the vector field*

$$\vec{k}(z) := z\,\theta(z) \text{ for each } z \in \mathbb{R}^n \setminus \{0\}. \tag{5.1.162}$$

Then $\vec{k} \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\big]^n$ is odd and positive homogeneous of degree $1-n$. The homogeneity property of θ entails Euler's identity

$$\langle z, (\nabla\theta)(z)\rangle = -n\,\theta(z) \text{ for each } z \in \mathbb{R}^n \setminus \{0\}. \tag{5.1.163}$$

As such,

$$\begin{aligned} \operatorname{div}\vec{k}(z) &= \partial_j\big(z_j\,\theta(z)\big) = n\,\theta(z) + z_j(\partial_j\theta)(z) \\ &= n\,\theta(z) + \langle z, (\nabla\theta)(z)\rangle = 0 \text{ for each } z \in \mathbb{R}^n \setminus \{0\}, \end{aligned} \tag{5.1.164}$$

which shows that $\vec{k}$ from (5.1.162) satisfies (5.1.2). In particular, Theorem 5.1.1 applies to the integral operators associated with kernels of the form (5.1.162). We shall consider this class of integral operators in detail a little later, in Theorem 5.2.2. For now, we wish to offer a couple of examples of this nature.

First, consider the case when, for some fixed pair of indices $j, k \in \{1, \dots, n\}$ we take $\theta := \theta_{jk}$, where $\theta(z) = z_j z_k/|z|^{n+2}$ for each $z \in \mathbb{R}^n \setminus \{0\}$. For this choice, the operator $\mathcal{T}$ from (5.1.4) becomes[12]

$$\begin{aligned} \Theta_{jk} f(x) &= \int_{\partial\Omega} \langle \nu(y), x-y\rangle \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^{n+2}} f(y)\,\mathrm{d}\sigma(y) \\ &\text{for each } f \in L^1\Big(\partial\Omega, \tfrac{\sigma(y)}{1+|y|^{n-1}}\Big) \text{ and each } x \in \Omega. \end{aligned} \tag{5.1.165}$$

[12] this is relevant in relation to double layer potential operators for the Lamé and Stokes systems

Second, work when $n = 2$ under the identification $\mathbb{R}^2 \equiv \mathbb{C}$. For each integers $j, k \in \{1, \dots, n\}$ consider the principal-value singular integral operator[13] *acting on each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\Big)$ according to*

$$U_{jk} f(z) := \lim_{\varepsilon \to 0^+} \int\limits_{\substack{y \in \partial\Omega \\ |x-y|>\varepsilon}} \frac{\langle \nu(z), z-\zeta \rangle \langle \nu(\zeta), z-\zeta \rangle}{(z-\zeta)^3} f(\zeta)\, \mathrm{d}\sigma(\zeta) \tag{5.1.166}$$

for σ-a.e. $z \in \partial\Omega$. Then U_{jk} may be expressed as a finite linear combination (*with coefficients scalar components of ν*) *of operators associated as in* (5.1.5) *with vector-valued kernels of the form*

$$\mathbb{C} \setminus \{0\} \ni \xi \longmapsto \xi_\ell(\xi_1, \xi_2)/\xi^3 \in \mathbb{C}^2, \qquad \ell \in \{1, 2\}. \tag{5.1.167}$$

We are now ready to present the proof of Theorem 5.1.1.

Proof of Theorem 5.1.1 Before commencing the proof in earnest, we make several observations. First, in view of the decay of $\vec{k}$ at infinity, $L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ is the largest space guaranteeing that the integral in (5.1.4) is absolutely convergent. Second, that for each $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ the limits in (5.1.5), (5.1.6) exist for σ-a.e. point on $\partial\Omega$ is guaranteed by [70, (2.3.15)], (the last condition in) the definition of a UR domain from [68, Definition 5.10.6], and [68, (5.6.32)]. Third, as noted in [68, (4.5.46)] we have

$$\mathrm{div}\vec{k} = \vartheta\delta \ \text{ in } \ \mathcal{D}'(\mathbb{R}^n), \tag{5.1.168}$$

where δ is Dirac's distribution with mass at the origin in $\mathbb{R}^n$. Fourth, each k_j induces (via integration against Schwartz functions) a tempered distribution, and recall from [70, (2.5.7)] that each $\widehat{k_j}$ is given by a continuous function in $\mathbb{R}^n \setminus \{0\}$. As a consequence of (5.1.168) we further obtain

$$\widehat{k_j}(\xi)\xi_j = \frac{1}{\mathrm{i}}\widehat{\partial_j k_j}(\xi) = \frac{1}{\mathrm{i}}\widehat{\mathrm{div}\vec{k}}(\xi) = \frac{\vartheta}{\mathrm{i}} \ \text{ for each } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{5.1.169}$$

After this preamble, we now turn to the actual proof in earnest.

Proof of claims in item (1): Denote by $\nu_1, \dots, \nu_n$ the scalar components of the geometric measure theoretic outward unit normal to Ω, and fix an aperture parameter $\kappa > 0$ along with a function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$. At σ-a.e. $x \in \partial\Omega$ we may then compute

[13] which is related to the radiosity singular integral operator

$$\Big(\mathcal{T} f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = \frac{1}{2\mathrm{i}}\widehat{k_j}\big(\nu(x)\big)\nu_j(x)f(x) + (Tf)(x)$$
$$= -\frac{\vartheta}{2}f(x) + (Tf)(x), \tag{5.1.170}$$

by [70, (2.5.4)], (5.1.4), (5.1.5), and (5.1.169). This establishes (5.1.7). Likewise, at σ-a.e. point $x \in \partial\Omega$ the nontangential boundary trace $(\mathcal{W} f)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists and we have (also keeping in mind that $\vec{k}$ is odd)

$$\nu(x)\cdot\Big(\mathcal{W} f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = \frac{1}{2\mathrm{i}}\widehat{k_j}\big(\nu(x)\big)\nu_j(x)f(x)$$
$$+ \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \langle \nu(x), \vec{k}(x-y)\rangle f(y)\,\mathrm{d}\sigma(y)$$
$$= -\frac{\vartheta}{2}f(x) - (T^{\#}f)(x). \tag{5.1.171}$$

This prove (5.1.10)-(5.1.11). Also, the claims in (5.1.9) are direct consequences of (5.1.8), (5.1.1), (5.1.2).

Assume next that Ω is bounded. Fix an aperture parameter $\kappa > 0$ and pick an arbitrary point $x \in \Omega$. Then [68, Theorem 1.4.1] applies to the vector field defined at $\mathcal{L}^n$-a.e. $y \in \Omega$ as $\vec{F}(y) := \vec{k}(x-y)$ and, on account of (5.1.168), the Divergence Formula [68, (1.4.6)] gives

$$(\mathcal{T}1)(x) = \int_{\partial\Omega} \nu\cdot\big(\vec{F}\,\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\,\mathrm{d}\sigma = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)}$$
$$= -{}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\vartheta\delta_x, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = -\vartheta. \tag{5.1.172}$$

Hence, $\mathcal{T}1 \equiv -\vartheta$ in Ω. With this in hand, (5.1.170) then gives that $T1 = -\frac{\vartheta}{2}$ at σ-a.e. point on $\partial\Omega$. In the case when Ω is an exterior domain, the same type of reasoning applies except that now we also need to take into account the contribution of $\vec{F}$ at infinity. According to [68, Proposition 4.7.1], the latter quantity may be expressed as the limit

$$[\vec{F}]_\infty = \lim_{R\to\infty}\int_{|x-y|=R}\Big(\frac{y-x}{R}\Big)\cdot\vec{k}(x-y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$$
$$= \int_{S^{n-1}}\langle\omega, \vec{k}(\omega)\rangle\,\mathrm{d}\mathcal{H}^{n-1}(\omega) = -\vartheta, \tag{5.1.173}$$

where we have changed variables $\omega := (y-x)/R$, used the homogeneity and parity of $\vec{k}$, and also recalled (5.1.3). Granted this, the Divergence Formula [68, (1.4.5)] permits us to compute

$$(\mathcal{T}1)(x) = \int_{\partial\Omega} \nu \cdot \big(\vec{F}\,\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)\,d\sigma = {}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\text{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\Omega)} - [\vec{F}]_\infty$$
$$= -{}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\vartheta\delta_x, 1\big)_{\mathscr{C}^\infty_b(\Omega)} + \vartheta = 0. \tag{5.1.174}$$

We therefore have $\mathcal{T}1 \equiv 0$ in Ω this time which, in concert with (5.1.170), shows that now $T1 = +\frac{\vartheta}{2}$ at σ-a.e. point on $\partial\Omega$. This completes the proof of (5.1.12)-(5.1.13).

Proof of claims in item (2): The claim in (5.1.16) is dealt with much as in the proof of (2.2.54), while all other results are consequences of items *(3)* and *(5)* in [70, Theorem 2.4.1].

Proof of claims in item (3): All desired results are consequences of items *(3)* and *(5)* in [70, Theorem 2.3.2].

Proof of claims in item (4): Fix two exponents, $p \in \big(\frac{n-1}{n}, 1\big]$ and $q \in (1, \infty)$, then pick an arbitrary function $f \in H^p(\partial\Omega, \sigma) \cap L^q(\partial\Omega, \sigma)$. Also, select an aperture parameter $\kappa > 0$, and denote by I the identity operator. Then, on the one hand, in the sense of distributions on $\partial\Omega$ we may write

$$\nu \bullet \mathcal{W}f = \nu \cdot \Big(\mathcal{W}f\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) = -\big((\vartheta/2)I + T^\#\big)f, \tag{5.1.175}$$

thanks to [69, Proposition 10.2.9] (whose applicability is ensured by (5.1.10) and the fact that we currently have $\mathcal{N}_\kappa(\mathcal{W}f) \in L^q(\partial\Omega, \sigma) \subset L^1_{\text{loc}}(\partial\Omega, \sigma)$; cf. (5.1.14)), and the jump-formula proved in (5.1.11) (bearing in mind (5.1.3)). On the other hand, [69, Theorem 10.2.1] (whose applicability in the present with $\vec{F} := \mathcal{W}f$ is guaranteed by (5.1.9) and (5.1.15)) gives that

$$\nu \bullet \mathcal{W}f \in H^p(\partial\Omega, \sigma) \text{ and } \|\nu \bullet \mathcal{W}f\|_{H^p(\partial\Omega,\sigma)} \leq C\|f\|_{H^p(\partial\Omega,\sigma)} \tag{5.1.176}$$

for some constant $C = C(\partial\Omega, \vec{k}, p) \in (0, \infty)$ independent of f. From (5.1.175) and (5.1.176) we conclude that

$$\begin{gathered}\|T^\# f\|_{H^p(\partial\Omega,\sigma)} \leq C\|f\|_{H^p(\partial\Omega,\sigma)} \\ \text{for each } f \in H^p(\partial\Omega, \sigma) \cap L^q(\partial\Omega, \sigma).\end{gathered} \tag{5.1.177}$$

In view of the density of $H^p(\partial\Omega, \sigma) \cap L^q(\partial\Omega, \sigma)$ in $H^p(\partial\Omega, \sigma)$ (cf. the last claim in [69, (4.4.114)]), we conclude that $T^\#$, originally acting on $L^q(\partial\Omega, \sigma)$, extends to a linear and bounded operator from $H^p(\partial\Omega, \sigma)$ into itself. The fact that $T^\#$ further extends to the scale of Lorentz-based Hardy spaces $H^{p,q}(\partial\Omega, \sigma)$ as indicated in (5.1.25) is then a consequence of what we have proved so far and the interpolation results from [69, Theorem 4.3.1].

To justify the identity claimed in (5.1.28) fix $p \in \big(\frac{n-1}{n}, \infty\big)$ along with $q \in (0, \infty]$ and observe that the assignment

$$H^{p,q}(\partial\Omega,\sigma) \ni f \longmapsto \nu \bullet \mathcal{W} f \in H^{p,q}(\partial\Omega,\sigma) \text{ is well defined, linear, and bounded,} \tag{5.1.178}$$

as may be seen from [69, Theorem 10.2.1], (5.1.9), and (5.1.15). From [69, Proposition 10.2.9] and (5.1.11) we also know that (5.1.175) continues to hold for each $f \in H^{p,q}(\partial\Omega,\sigma) \cap L^2(\partial\Omega,\sigma)$. Based on this, (5.1.178), (5.1.25), and [69, Lemma 4.3.3] we may then conclude that (5.1.28) holds whenever $f \in H^{p,q}(\partial\Omega,\sigma)$ with $q < 1$. Finally, that the end-point $q = \infty$ may also be allowed follows from what we have just proved and the fact that $H^{p,\infty}(\partial\Omega,\sigma)$ embeds into $H^{p_0}(\partial\Omega,\sigma) + H^{p_1}(\partial\Omega,\sigma)$ whenever $\frac{n-1}{n} < p_0 < p < p_1 < \infty$ (cf. [69, (4.3.146)]).

To prove the formula claimed in (5.1.26), at σ-a.e. point $x_o \in \partial\Omega$ we write, in the sense of distributions (i.e., in $\big(\mathrm{Lip}_c(\partial\Omega)\big)'$),

$$\begin{aligned} -\frac{\vartheta}{2}\delta_{x_o} + \text{P.V.}\big\langle \nu, \vec{k}(x_o - \cdot)\big|_{\partial\Omega}\big\rangle &= \nu \bullet \vec{k}(\cdot - x_o) = \nu \bullet \big(\mathcal{W}\delta_{x_o}\big) \\ &= -\frac{\vartheta}{2}\delta_{x_o} - T^{\#}\delta_{x_o}, \end{aligned} \tag{5.1.179}$$

thanks to [69, (11.9.30)] and the fact that $\vec{k}$ is odd, (5.1.16) with $f := \delta_{x_o}$, as well as (5.1.28) written for $f := \delta_{x_o} \in H^{1,\infty}(\partial\Omega,\sigma)$ (cf. [69, Example 4.2.4]). Now, (5.1.26) readily follows from (5.1.179). Also, (5.1.27) is a direct consequence of (5.1.26) and [69, (4.2.17)].

It is instructive to give a direct proof of the claims made in relation to the operator in (5.1.22). To this end, fix $p \in \big(\frac{n-1}{n}, 1\big]$. As in the proof of Theorem 2.1.1, the strategy is to show that $T^{\#}$ maps atoms into a fixed multiple of molecules for the Hardy space in question. Concretely, pick $q \in (1,\infty)$ and consider a (p,q)-atom a on $\partial\Omega$. Recall from [69, (4.4.167)-(4.4.168)] that this means that $a : \partial\Omega \to \mathbb{C}$ is some σ-measurable function with the property that there exist a point $x_o \in \partial\Omega$ and some number $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ such that

$$\begin{gathered} \operatorname{supp} a \subseteq B(x_o,r) \cap \partial\Omega, \\ \|a\|_{L^q(\partial\Omega,\sigma)} \leq \sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1/q-1/p}, \\ \int_{\partial\Omega} a \, d\sigma = 0. \end{gathered} \tag{5.1.180}$$

In addition, in the case when $\partial\Omega$ is compact, we agree that a constant of absolute value $\leq \sigma(\partial\Omega)^{-1/p}$ is also considered, by definition, to be a (p,q)-atom on $\partial\Omega$. The claim that we make is that

$$\begin{gathered} m := T^{\#}a \text{ is a fixed multiple of a } (p,q,\varepsilon)\text{-molecule on } \partial\Omega \\ \text{(i.e., a function as in [69, Definition 4.5.1]), with } \varepsilon := \tfrac{1}{n-1}. \end{gathered} \tag{5.1.181}$$

To prove this, first note that according to the current item *(3)* the function m is meaningfully defined and belongs to the space $L^q(\partial\Omega,\sigma)$. In fact, thanks to the current item *(3)* and (5.1.180), we have

$$\|m\|_{L^q(\partial\Omega,\sigma)} = \|T^\# a\|_{L^q(\partial\Omega,\sigma)} \leq C\|a\|_{L^q(\partial\Omega,\sigma)}$$
$$\leq C\sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1/q-1/p}, \tag{5.1.182}$$

for some finite constant $C = C(\partial\Omega, \vec{k}, q) > 0$ independent of the atom.

To study the decay of m, observe that based on (5.1.180) and the Mean Value Theorem, for each $x \in \partial\Omega \setminus B(x_o, 2r)$, we may estimate

$$|m(x)| \leq C\int_{B(x_o,r)\cap\partial\Omega} |\vec{k}(x-y) - \vec{k}(x-x_o)||a(y)|\,\mathrm{d}\sigma(y)$$
$$\leq C\frac{r}{|x-x_o|^n}\sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1-1/p}. \tag{5.1.183}$$

To proceed, for each $\ell \in \mathbb{N}$ define the boundary annulus

$$A_\ell(x_o,r) := \big[B(x_o,2^{\ell+1}r)\setminus B(x_o,2^\ell r)\big]\cap\partial\Omega. \tag{5.1.184}$$

We may then rely on (5.1.183) and the Ahlfors regularity of $\partial\Omega$ to obtain to that, for each $\ell \in \mathbb{N}$,

$$\Big(\int_{A_\ell(x_o,r)} |m|^q\,\mathrm{d}\sigma\Big)^{1/q}$$
$$\leq C\frac{r}{(2^\ell r)^n}\sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1-1/p}\sigma\big(B(x_o,2^{\ell+1}r)\cap\partial\Omega\big)^{1/q}$$
$$\leq C2^{\ell(n-1)[1/q-1-1/(n-1)]}\sigma\big(B(x_o,r)\cap\partial\Omega\big)^{1/q-1/p} \tag{5.1.185}$$

for some finite constant $C = C(\partial\Omega, \vec{k}) > 0$ independent of the atom. In particular, from (5.1.182) and (5.1.185) we conclude that

$$\int_{\partial\Omega} |m|\,\mathrm{d}\sigma < +\infty. \tag{5.1.186}$$

In view of (5.1.182), (5.1.185), and [69, Definition 4.5.1] the claim in (5.1.181) follows as soon as we check that

$$\int_{\partial\Omega} m\,\mathrm{d}\sigma = 0. \tag{5.1.187}$$

To this end, fix some background parameter $\kappa > 0$ and bring in the vector field

$$\vec{F} := \mathcal{W}a \ \text{ in } \Omega. \tag{5.1.188}$$

Thanks to (5.1.9) we have

$$\vec{F} \in \big[\mathscr{C}^N(\Omega)\big]^n \ \text{ and } \operatorname{div}\vec{F} = 0 \ \text{ in } \Omega. \tag{5.1.189}$$

Also, (5.1.11) gives

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \tag{5.1.190}$$

and

$$\nu\cdot\big(\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) = -\frac{\vartheta}{2}a - T^{\#}a = -\frac{\vartheta}{2}a - m \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{5.1.191}$$

Since, as is apparent from (5.1.180), the function $a : \partial\Omega \to \mathbb{C}$ is a multiple of an $(1, q)$-atom on $\partial\Omega$, we may invoke (5.1.15) (with $p = q = 1$) to conclude that

$$\mathcal{N}_\kappa\vec{F} \in L^1(\partial\Omega, \sigma). \tag{5.1.192}$$

Finally, the vanishing moment property of the atom (cf. the last line in (5.1.180)) together with (5.1.188) imply that

in the case when Ω is an exterior domain we have $\vec{F}(x) = O(|x|^{-n})$ as $x \in \Omega$ satisfies $|x| \to \infty$; hence, the pointwise decay property [68, (1.2.9)] is satisfied. (5.1.193)

Collectively, (5.1.189), (5.1.190), (5.1.192), (5.1.193) guarantee the validity of the Divergence Formula [68, (1.2.2)] which, in light of (5.1.189) and (5.1.191), presently gives

$$\begin{aligned} 0 &= \int_\Omega \operatorname{div}\vec{F}\,d\mathcal{L}^n = \int_{\partial\Omega} \nu\cdot\big(\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\,d\sigma \\ &= -\frac{\vartheta}{2}\int_{\partial\Omega} a\,d\sigma - \int_{\partial\Omega} m\,d\sigma = -\int_{\partial\Omega} m\,d\sigma, \end{aligned} \tag{5.1.194}$$

bearing in mind the vanishing moment property in (5.1.180). Having established this, we then conclude that $\int_{\partial\Omega} m\,d\sigma = 0$, finishing the proof of (5.1.187).

Let us record our progress: from (5.1.182), (5.1.185), and (5.1.187) we deduce that, whenever a is as in (5.1.180), the function $m := T^{\#}a$ is a fixed multiple of a molecule for the Hardy space $H^p(\partial\Omega, \sigma)$, in the sense of [69, Definition 4.5.1]. Granted this, it follows from [69, (4.5.6)] that

$$m \in H^p(\partial\Omega, \sigma) \ \text{ and } \ \|m\|_{H^p(\partial\Omega,\sigma)} \le C(\partial\Omega, \vec{k}, p) \in (0, \infty). \tag{5.1.195}$$

Moreover, in the case when $\partial\Omega$ is compact and the (p, q)-atom a is some constant function on $\partial\Omega$, of absolute value $\le \sigma(\partial\Omega)^{-1/p}$, it follows from the current item *(3)* that the function $m := T^{\#}a$ belongs to $L^2(\partial\Omega, \sigma)$ and satisfies $\|m\|_{L^2(\partial\Omega,\sigma)} \le C(\partial\Omega, \vec{k}, p) \in (0, \infty)$. Keeping in mind that, in the present setting, $L^2(\partial\Omega, \sigma)$ embeds continuously into $H^p(\partial\Omega, \sigma)$ (cf. [69, (4.2.13)]), we deduce that the conclusions in (5.1.195) are valid in this case as well. Having established (5.1.195) in all circumstances, we may now invoke [69, Theorem 4.4.7] (whose applicability in the present setting makes use of the current item *(3)*) to conclude that, indeed, the mapping $T^{\#}$, originally considered as in the present item *(3)*, extends uniquely to a linear and bounded operator from the Hardy space $H^p(\partial\Omega, \sigma)$ into itself. Finally, that various choices of $p \in \big(\frac{n-1}{n}, 1\big]$ in (5.1.22) yield operators which are compatible with one another may now be seen with the help of [69, Theorem 4.4.3].

Moving on, for each $f \in H^1(\partial\Omega, \sigma)$ the memberships in (5.1.29) are immediate consequences of [69, (4.2.10)] and (5.1.22) (used with $p := 1$). To justify (5.1.30), by continuity (cf. [69, (4.2.10)], (5.1.22)) and [69, Theorem 4.4.1] it suffices to consider the case when f is an atom for the Hardy space $H^1(\partial\Omega, \sigma)$, say $f = a$ where a is as in [69, (4.4.2)-(4.4.3)] (with $\Sigma := \partial\Omega$ and $p := 1$). When $\partial\Omega$ is bounded we may rely on (5.1.18) and (5.1.13) to write

$$\int_{\partial\Omega} T^{\#}a\,d\sigma = \int_{\partial\Omega} aT1\,d\sigma = \begin{cases} -\dfrac{\vartheta}{2}\displaystyle\int_{\partial\Omega} a\,d\sigma & \text{if } \Omega \text{ is bounded,} \\ +\dfrac{\vartheta}{2}\displaystyle\int_{\partial\Omega} a\,d\sigma & \text{if } \Omega \text{ is an exterior domain,} \end{cases} \tag{5.1.196}$$

which is in agreement with (5.1.30). When $\partial\Omega$ is unbounded, we may invoke the vanishing moment condition in [69, (4.4.2)] and (5.1.11) to write (for some arbitrary aperture parameter $\kappa > 0$)

$$\int_{\partial\Omega} T^{\#}a\,d\sigma = \int_{\partial\Omega} \big(\tfrac{\vartheta}{2}a + T^{\#}a\big)\,d\sigma = -\int_{\partial\Omega} \nu \cdot \Big(\mathcal{W}a\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)\,d\sigma = 0, \tag{5.1.197}$$

thanks to (5.1.9) and the Divergence Formula [68, (1.2.2)] (whose applicability with $\vec{F} := \mathcal{W}a$ is guaranteed by (5.1.9), (5.1.10), and (5.1.15) with $p = q = 1$). This finishes the proof of (5.1.30).

Proof of claims in item (5): The claim made in item *(5)* may be justified by reasoning much as in the proof of Theorem 3.2.1, now relying on what we have proved already in the current item *(4)*.

Proof of claims in item (6): Suppose f is an arbitrary function belonging to the weighted boundary Sobolev space $L^1_1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ and fix $\ell \in \{1, \dots, n\}$ arbitrary. Then for each $x \in \Omega$ we may write

$$
\begin{aligned}
\partial_\ell(\mathcal{T} f)(x) \\
&= \int_{\partial\Omega} \nu_j(y)\partial_{x_\ell}[k_j(x-y)]f(y)\,\mathrm{d}\sigma(y) \\
&= -\int_{\partial\Omega} \nu_j(y)\partial_{y_\ell}[k_j(x-y)]f(y)\,\mathrm{d}\sigma(y) \\
&= \int_{\partial\Omega} \partial_{\tau_{\ell j}(y)}[k_j(x-y)]f(y)\,\mathrm{d}\sigma(y) \\
&= \int_{\partial\Omega} k_j(x-y)\big(\partial_{\tau_{j\ell}} f\big)(y)\,\mathrm{d}\sigma(y). \qquad (5.1.198)
\end{aligned}
$$

Above, the first equality is obtained by differentiating under the integral sign in (5.1.4), the second inequality is a simple consequence of chain rule, the third equality takes into account (A.0.183) as well as the fact that $\partial_{y_j}[k_j(x-y)] = 0$ by (5.1.2), and the last equality is justified by [69, Lemma 11.1.7]. Then (5.1.33) follows from (5.1.198) and [70, Theorem 2.5.1]. Collectively, (5.1.198), (5.1.4), and item *(3)* in [70, Theorem 2.4.1] also prove the estimates claimed in (5.1.34) and the subsequent comment.

Proof of claims in item (7): Start by choosing an arbitrary function

$$
f \in L^1_1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{1,\mathrm{loc}}(\partial\Omega,\sigma) \ \text{ for some } \ p \in (1,\infty). \qquad (5.1.199)
$$

First, from (5.1.4), (5.1.198), and item *(2)* in [70, Theorem 2.4.1] (cf. [70, (2.4.8)]) we see that

$$
\text{if } f \text{ is as in (5.1.199) then } \ \mathcal{N}_\kappa(\mathcal{T} f), \mathcal{N}_\kappa(\nabla\mathcal{T} f) \in L^p_{\mathrm{loc}}(\partial\Omega,\sigma). \qquad (5.1.200)
$$

Second, based on (5.1.198), [69, (11.4.8)], and (5.1.4), for each $\ell \in \{1,\dots,n\}$ we may compute

$$
\begin{aligned}
\partial_\ell(\mathcal{T} f)(x) &= \mathcal{T}\big((\nabla_{\tan} f)_\ell\big)(x) \\
&\quad - \int_{\partial\Omega} \nu_\ell(y)\langle \vec{k}(x-y), (\nabla_{\tan} f)(y)\rangle\,\mathrm{d}\sigma(y) \ \text{ for all } \ x \in \Omega. \qquad (5.1.201)
\end{aligned}
$$

As a consequence of (5.1.201) and (5.1.170), at σ-a.e. $x \in \partial\Omega$ we then have

$$\Big(\partial_\ell\big(\mathcal{T}f\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = -\frac{\vartheta}{2}(\nabla_{\tan}f)_\ell + T\big((\nabla_{\tan}f)_\ell\big)(x)$$
$$-\frac{1}{2\mathrm{i}}\widehat{k_j}\big(\nu(x)\big)\nu_\ell(x)(\nabla_{\tan}f)_j(x)$$
$$-\lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} k_j(x-y)\nu_\ell(y)(\nabla_{\tan}f)_j(y)\,\mathrm{d}\sigma(y). \quad (5.1.202)$$

From (5.1.7), (5.1.202), (5.1.200), and [69, Proposition 11.3.2] (used with $u := \mathcal{T}f$) we conclude that $Tf \in L^p_{1,\mathrm{loc}}(\partial\Omega, \sigma)$, hence (5.1.39) holds. Moreover, for each $r, s \in \{1, \ldots, n\}$ we may then compute, at σ-a.e. $x \in \partial\Omega$,

$$
\begin{aligned}
&\partial_{\tau_{rs}}(Tf)(x) \qquad\qquad (5.1.203)\\
&= \partial_{\tau_{rs}}\Big(-\frac{\vartheta}{2}f + Tf\Big)(x) + \frac{\vartheta}{2}\big(\partial_{\tau_{rs}}f\big)(x)\\
&= \partial_{\tau_{rs}}\Big(\mathcal{T}f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) + \frac{\vartheta}{2}\partial_{\tau_{rs}}f(x)\\
&= \nu_r(x)\Big(\partial_s\big(\mathcal{T}f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) - \nu_s(x)\Big(\partial_r\big(\mathcal{T}f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) + \frac{\vartheta}{2}\big(\partial_{\tau_{rs}}f\big)(x)\\
&= -\frac{\vartheta}{2}\nu_r(x)(\nabla_{\tan}f)_s(x) + \nu_r(x)T\big((\nabla_{\tan}f)_s\big)(x)\\
&\quad - \frac{1}{2\mathrm{i}}\widehat{k_j}\big(\nu(x)\big)\nu_r(x)\nu_s(x)(\nabla_{\tan}f)_j(x)\\
&\quad - \lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} k_j(x-y)\nu_r(x)\nu_s(y)(\nabla_{\tan}f)_j(y)\,d\sigma(y)\\
&\quad + \frac{\vartheta}{2}\nu_s(x)(\nabla_{\tan}f)_r(x) - \nu_s(x)T\big((\nabla_{\tan}f)_r\big)(x)\\
&\quad + \frac{1}{2\mathrm{i}}\widehat{k_j}\big(\nu(x)\big)\nu_s(x)\nu_r(x)(\nabla_{\tan}f)_j(x)\\
&\quad + \lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} k_j(x-y)\nu_s(x)\nu_r(y)(\nabla_{\tan}f)_j(y)\,d\sigma(y)\\
&\quad + \frac{\vartheta}{2}\big(\partial_{\tau_{rs}}f\big)(x)\\
&= T\big(\partial_{\tau_{rs}}f\big)(x) + \big(\big[M_{\nu_r},T\big](\nabla_{\tan}f)_s\big)(x) - \big(\big[M_{\nu_s},T\big](\nabla_{\tan}f)_r\big)(x)\\
&\quad - \lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} k_j(x-y)\big(\nu_r(x)\nu_s(y) - \nu_s(x)\nu_r(y)\big)(\nabla_{\tan}f)_j(y)\,d\sigma(y).
\end{aligned}
$$

The first equality in (5.1.203) is just algebra, the second equality uses the jump-formula (5.1.7), the third equality is provided by [69, Proposition 11.3.2] used with $u := \mathcal{T}f$ (whose present applicability is ensured by (5.1.7), (5.1.202), and (5.1.200)), the fourth equality is based on (5.1.202), while the final equality is a consequence of [69, (11.4.8)]. Ultimately, (5.1.203) shows that at σ-a.e. $x \in \partial\Omega$ we have

$$\begin{aligned}
&\partial_{\tau_{rs}}(Tf)(x) \qquad (5.1.204)\\
&\quad = T\big(\partial_{\tau_{rs}}f\big)(x) + \big(\big[M_{\nu_r},T\big](\nabla_{\tan}f)_s\big)(x) - \big(\big[M_{\nu_s},T\big](\nabla_{\tan}f)_r\big)(x)\\
&\quad\quad - \lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big\langle \vec{k}(x-y),(\nabla_{\tan}f)(y)\big\rangle\big(\nu_r(x)\nu_s(y)-\nu_s(x)\nu_r(y)\big)\,d\sigma(y),
\end{aligned}$$

and (5.1.40) readily follows from this and (5.1.19). The claims pertaining to (5.1.35)-(5.1.37) are then seen from this and the current item *(3)*.

That formula (5.1.40) also holds for each function f belonging to $M_1^{p,\lambda}(\partial\Omega,\sigma)$ or $\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)$ with $p,q\in(1,\infty)$ and $\lambda\in(0,n-1)$ is clear from (5.1.38), (A.0.150), (A.0.33), [69, (6.2.7)], and [69, (6.2.71)]. In turn, this also ensures that the claims in (5.1.41) are valid.

Proof of claims in item (8): All results in item *(8)* are implied by the current item *(7)*, item *(3)*, (A.0.136), (A.0.137), and duality.

Proof of claims in item (9): All claims in item *(9)* with the exception of (5.1.48), (5.1.49)-(5.1.50), and (5.1.51)-(5.1.52) are direct consequences of definitions and [70, Corollary 2.5.3]. As far as the claim made in (5.1.48) is concerned, having fixed two arbitrary points $x_0,x_1\in\Omega$, we need to show that $\big(\mathcal{T}_{\rm mod}1\big)(x_0)=\big(\mathcal{T}_{\rm mod}1\big)(x_1)$, which is further equivalent to proving that

$$\int_{\partial\Omega}\big\langle \nu(y),\vec{k}(x_0-y)-\vec{k}(x_1-y)\big\rangle\,d\sigma(y)=0. \tag{5.1.205}$$

To this end, consider the vector field defined by

$$\vec{F}(y):=\vec{k}(x_0-y)-\vec{k}(x_1-y)\ \text{ at } \mathcal{L}^n\text{-a.e. point } y\in\Omega. \tag{5.1.206}$$

Hence

$$\vec{F}\in\big[L^1_{\rm loc}(\Omega,\mathcal{L}^n)\big]^n \tag{5.1.207}$$

and (5.1.168) gives that, with the divergence taken in the sense of distributions in Ω,

$$\operatorname{div}\vec{F}=-\vartheta\delta_{x_0}+\vartheta\delta_{x_1}\in\mathscr{E}'(\Omega) \tag{5.1.208}$$

where, generally speaking, δ_x is the Dirac distribution in Ω with mass at $x\in\Omega$.

Fix a compact set $K\subset\Omega$ whose interior contains both x_0 and x_1. Then the same argument which has established [70, (2.3.117)] presently gives that there exists a constant $C_K\in(0,\infty)$ such that

$$|\vec{F}(y)|\leq\frac{C_K}{1+|y|^n},\qquad \forall\,y\in\Omega\setminus K. \tag{5.1.209}$$

Granted this, [68, Lemma 8.3.7] shows that there exists $C=C(K,\vec{k},\kappa,n)\in(0,\infty)$ such that

$$\Big(\mathcal{N}_\kappa^{\Omega\setminus K}\vec{F}\Big)(y) \leq \frac{C}{1+|y|^n} \text{ for each } y \in \partial\Omega. \tag{5.1.210}$$

In turn, from (5.1.210), [68, (8.2.26)], and [68, Lemma 7.2.1], we conclude that

$$\mathcal{N}_\kappa^{\Omega\setminus K}\vec{F} \in L^1(\partial\Omega, \sigma). \tag{5.1.211}$$

In addition, since $\vec{F}$ is continuous in a neighborhood of $\partial\Omega$ it follows that the pointwise nontangential boundary trace

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at every point in } \partial_{\text{nta}}\Omega, \tag{5.1.212}$$

and at σ-a.e. $y \in \partial\Omega$ we have

$$\nu(y)\cdot\Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(y) = \big\langle \nu(y), \vec{k}(x_0-y) - \vec{k}(x_1-y)\big\rangle. \tag{5.1.213}$$

Finally, we note that in the case when Ω is unbounded we have $|\vec{F}(y)| = O(|y|^{-n})$ for $y \in \Omega$ with $|y| \to \infty$. Hence, in such a scenario,

$$\int_{[B(0,2R)\setminus B(0,R)]\cap\Omega} |y\cdot\vec{F}(y)|\,\mathrm{d}\mathcal{L}^n(y) = O(R) \text{ as } R \to \infty. \tag{5.1.214}$$

At this stage, we may write

$$\begin{aligned}\int_{\partial\Omega} \big\langle \nu(y), \vec{k}(x_0-y) - \vec{k}(x_1-y)\big\rangle\,\mathrm{d}\sigma(y)&\\ &= \int_{\partial\Omega}\nu(y)\cdot\Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(y)\,\mathrm{d}\sigma(y) = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)}\\ &= -\vartheta + \vartheta = 0.\end{aligned} \tag{5.1.215}$$

Above, the first equality comes from (5.1.213), the second equality is formula [68, (1.4.6)] (keeping in mind that the hypotheses of [68, Theorem 1.4.1] are satisfied, thanks to (5.1.207), (5.1.208), (5.1.212), (5.1.214)), the third equality is seen from (5.1.208), and the final equality is obvious. In turn, (5.1.215) establishes (5.1.205). This finishes the proof of (5.1.48).

Next, the identity claimed in (5.1.52) for any function f as in (5.1.51) is established staring from (5.1.47), then reasoning as in (5.1.198) based on the integration by parts formula on the boundary from [69, Lemma 11.1.7].

Consider now the task of justifying (5.1.50) for the operator (5.1.49). To this end, fix a function $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ with $\psi \equiv 1$ near the origin, and pick an arbitrary point $x \in \Omega$. Then the vector field

$$\vec{F}_x(y) := \vec{k}(x-y) - \vec{k}(-y)(1-\psi(y)) \text{ for } \mathcal{L}^n\text{-a.e. } y \in \Omega \tag{5.1.216}$$

belongs to $\big[L^1_{\rm loc}(\Omega,\mathcal{L}^n)\cap\mathscr{C}^\infty\big(\overline{\Omega}\setminus\{x\}\big)\big]^n$. In particular, $\vec{F}_x\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ exists at every point in $\partial_{\rm nta}\Omega$. Also, since $\nabla\psi\cdot\vec{k}\in\mathscr{C}^\infty_c(\mathbb{R}^n)$, from (5.1.168) and (5.1.2) we see that the divergence of $\vec{F}_x$, taken in the sense of distributions in Ω, is given by

$$\begin{aligned}{\rm div}\vec{F}_x &= -\vartheta\delta_x + {\rm div}[(1-\psi)\vec{k}]\\ &= -\vartheta\delta_x - \nabla\psi\cdot\vec{k}\in\mathscr{E}'(\Omega)+L^1(\Omega,\mathcal{L}^n).\end{aligned}\tag{5.1.217}$$

Finally, much as in (5.1.211), (5.1.214), we presently have

$$\mathcal{N}^{\Omega\setminus K}_\kappa\vec{F}_x\in L^1(\partial\Omega,\sigma)\ \text{ for any compact neighborhood } K\subseteq\Omega \text{ of } x,\tag{5.1.218}$$

and

$$\int_{[B(0,2R)\setminus B(0,R)]\cap\Omega}|y\cdot\vec{F}_x(y)|\,{\rm d}\mathcal{L}^n(y)=O(R)\ \text{ as }\ R\to\infty.\tag{5.1.219}$$

Granted these properties, we may then write

$$\begin{aligned}\big(\widetilde{\mathcal{T}}_{\rm mod}1\big)(x) &= \int_{\partial\Omega}\nu(y)\cdot\Big(\vec{F}_x\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)(y)\,{\rm d}\sigma(y) = {}_{(\mathscr{C}^\infty_b(\Omega))^*}\big({\rm div}\vec{F}_x,1\big)_{\mathscr{C}^\infty_b(\Omega)}\\ &= -\vartheta-\int_\Omega\nabla\psi\cdot\vec{k}\,{\rm d}\mathcal{L}^n,\end{aligned}\tag{5.1.220}$$

as seen from (5.1.49), (5.1.216), [68, (1.4.6)], and (5.1.217). This establishes (5.1.50), so the justification of the claims in item *(9)* is now complete.

Proof of claims in item (10): The claim pertaining to (5.1.59) follows from [70, Lemma 2.1.2], used with $\mathcal{Q}:=\nabla\mathcal{T}_{\rm mod}$, bearing in mind that the constant C_2 defined in [70, (2.1.18)] presently vanishes, thanks to (5.1.48). The claim concerning (5.1.60) then follows by combining (5.1.59) with [68, (5.11.78)]. Finally, the claims in (5.1.61) and (5.1.62)-(5.1.64) are justified in a manner very similar to the proof of (1.8.16) and (1.8.17)-(1.8.19), respectively.

Proof of claims in item (11): The claim in (5.1.67) is implied by [70, (2.3.34)]. The jump-formula (5.1.68) follows from definitions and [70, Corollary 2.5.3]. The claim made in (5.1.70) is a consequence of [70, (2.3.36)], bearing in mind (5.1.65) and (5.1.6). To prove (5.1.73), pick a function f as in (5.1.71). Then for each pair of indices $r,s\in\{1,\dots,n\}$ we may write

$$\begin{aligned}\partial_{\tau_{rs}}\big(T_{\rm mod}f\big) &= \partial_{\tau_{rs}}\big(-\tfrac{\vartheta}{2}f+T_{\rm mod}f\big)+\tfrac{\vartheta}{2}\partial_{\tau_{rs}}f\\ &= \nu_r\big(\partial_s(\mathcal{T}_{\rm mod}f)\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}-\nu_s\big(\partial_r(\mathcal{T}_{\rm mod}f)\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}+\tfrac{\vartheta}{2}\partial_{\tau_{rs}}f\end{aligned}\tag{5.1.221}$$

where the second equality is justified by invoking [69, Proposition 11.3.2] with $u:=\mathcal{T}_{\rm mod}f$ and $p:=1$ (its present applicability is ensured by (5.1.68), (5.1.53), (5.1.54), and (5.1.55)). Next, observe from (5.1.52) and [69, (11.4.8)] that for each index $\ell \in \{1, \dots, n\}$ we have

$$\begin{aligned}\partial_\ell\big(\mathcal{T}_{\rm mod} f\big)(x) &= \int_{\partial\Omega} k_j(x-y)(\partial_{\tau_{\ell j}} f)(y)\,\mathrm{d}\sigma(y) \\ &= \mathcal{T}\big((\nabla_{\tan} f)_\ell\big)(x) \\ &\quad - \int_{\partial\Omega} \nu_\ell(y)\langle \vec{k}(x-y), (\nabla_{\tan} f)(y)\rangle\,\mathrm{d}\sigma(y) \ \text{ for all } \ x \in \Omega.\end{aligned} \tag{5.1.222}$$

With (5.1.221) and (5.1.222) in hand, the same type of argument as in (5.1.203)-(5.1.204) then establishes (5.1.73). Finally, (5.1.72) follows from [70, (2.3.35)], (5.1.73), and [70, (2.3.17)].

Proof of claims in item (12): The main step is to prove the estimate stated in (5.1.74), from which all other claims in this item then follow. However, having established (5.1.48), the argument proceeds very much as in the case of the proof of [70, Corollary 2.4.2]. Specifically, having fixed a point $x_o \in \partial\Omega$ and a scale $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$, the same argument which have produced [70, (2.4.136)] (based on the estimate in [70, (2.4.34)] and the decay of the integral kernel of $\nabla\mathcal{T}_{\rm mod}$) now gives

$$\begin{aligned}&\int_{B(x_o,r)\cap\Omega}\Big|\nabla\big(\mathcal{T}_{\rm mod} f\big)(x)\Big|^p \mathrm{dist}(x,\partial\Omega)^{p-1}\,\mathrm{d}x \\ &\qquad \le C\sigma\big(B(x_o,r)\cap\partial\Omega\big) f_p^{\#}(x_o)^p + C\sigma\big(B(x_o,r)\cap\partial\Omega\big) f_1^{\#}(x_o)^p,\end{aligned} \tag{5.1.223}$$

where the L^q-based Fefferman-Stein maximal function $f_q^{\#}$ has been defined in (A.0.195). Granted this, the version of (5.1.74) with the supremum taken in the regime $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ follows on account of [68, (7.4.111)]. Finally, the case when Ω is an exterior domain and $r \ge 2\,\mathrm{diam}(\partial\Omega)$ is handled much as in [70, (2.4.142), (2.4.143)].

Next, consider the claim made in (5.1.76). To this end, pick $p \in (1, \infty)$ and select an arbitrary function $f \in \mathrm{BMO}(\partial\Omega, \sigma)$. Also, fix some $\alpha \in (0, 1)$ choose some arbitrary function

$$g \in \dot{\mathscr{C}}^\alpha(\partial\Omega) \cap \mathrm{BMO}(\partial\Omega, \sigma). \tag{5.1.224}$$

Then for each $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ and $x \in \partial\Omega$ we may estimate

$$\begin{aligned}&\bigg(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}(f-g)\big)\big|^p \mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\bigg)^{\frac{1}{p}} \\ &\qquad \le C\|f-g\|_{\mathrm{BMO}(\partial\Omega,\sigma)},\end{aligned} \tag{5.1.225}$$

thanks to (5.1.74) written with $f - g$ in place of f. In addition, having fixed some arbitrary $\alpha \in (0, 1)$, we may write

$$\begin{aligned}
&\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}\,g\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{\frac{1}{p}}\\
&\qquad\le C\|g\|_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\operatorname{dist}(\cdot,\partial\Omega)^{p\alpha-1}\,d\mathcal{L}^n\Big)^{\frac{1}{p}}\\
&\qquad\le C\|g\|_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}\,r^\alpha.
\end{aligned}\tag{5.1.226}$$

Indeed, the first inequality above uses (5.1.59) (written for g in place of f), while the second inequality is based on [68, (8.6.101)] used with $\lambda := 1 - p\alpha$, $\alpha := 1$, $\beta := n - 1$, and $E := B(x,r)\cap\Omega$. Collectively, (5.1.225) and (5.1.226) imply that

$$\begin{aligned}
\lim_{R\to 0^+}\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,R)}}&\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}\,f\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{\frac{1}{p}}\\
&\le C\|f-g\|_{\mathrm{BMO}(\partial\Omega,\sigma)},
\end{aligned}\tag{5.1.227}$$

for some constant $C\in(0,\infty)$ independent of f and g. With this in hand, (5.1.76) follows on account of [69, Theorem 3.1.3] and the ability to choose g arbitrary as in (5.1.224). In turn, (5.1.76) readily implies (5.1.77)-(5.1.78).

Proof of claims in item (13): All claims clear from what we have proved in the current items *(9)-(12)* and (5.1.45).

Proof of claims in item (14): Recall from [68, (7.4.118)] that

$$\mathrm{BMO}(\partial\Omega,\sigma)\subseteq L^1\Big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\Big). \tag{5.1.228}$$

If we now pick an arbitrary function $f\in\mathrm{BMO}(\partial\Omega,\sigma)$, from (5.1.228), [70, (2.3.35)], (5.1.65), and [68, (7.4.105)] we see that

$$T_{\rm mod}\,f\in\bigcap_{1\le p<\infty}L^p_{\rm loc}(\partial\Omega,\sigma). \tag{5.1.229}$$

Let us also consider an arbitrary $(1,\infty)$-atom $a:\partial\Omega\to\mathbb{C}$ with vanishing moment[14]. This is some σ-measurable function with the property that there exist $x_o\in\partial\Omega$ and $r\in\big(0,2\operatorname{diam}(\partial\Omega)\big)$ such that

$$\begin{aligned}
&\operatorname{supp}a\subseteq B(x_o,r)\cap\partial\Omega,\qquad \|a\|_{L^\infty(\partial\Omega,\sigma)}\le\sigma\big(B(x_o,r)\cap\partial\Omega\big)^{-1},\\
&\qquad\text{and}\int_{\partial\Omega}a\,d\sigma=0.
\end{aligned}\tag{5.1.230}$$

From (5.1.181) and [69, (4.5.33)] we know that

[14] this is always the case if $\partial\Omega$ is unbounded

$$\int_{\partial\Omega} |f||T^{\#}a| \, d\sigma < +\infty, \tag{5.1.231}$$

and we make the additional claim that $\int_{\partial\Omega} \big|T_{\text{mod}} f\big||a| \, d\sigma < +\infty$ and

$$\int_{\partial\Omega} f(T^{\#}a) \, d\sigma = \int_{\partial\Omega} \big(T_{\text{mod}} f\big) a \, d\sigma. \tag{5.1.232}$$

Indeed, these are all consequences of (5.1.70). A direct proof of (5.1.232) goes as follows. First, bearing (5.1.231) in mind, we write

$$\begin{aligned}\int_{\partial\Omega} f\, T^{\#} a \, d\sigma &= \int_{\partial\Omega\cap B(x_o,2r)} f\, T^{\#} a \, d\sigma + \int_{\partial\Omega\setminus B(x_o,2r)} f\, T^{\#} a \, d\sigma \\ &=: \mathrm{I} + \mathrm{II},\end{aligned} \tag{5.1.233}$$

with the last equality defining I and II. In relation to these, note that on account of the current item *(3)* and [68, (7.4.105)] we may re-write I as

$$\begin{aligned}\mathrm{I} &= \int_{\partial\Omega} \big(f \cdot \mathbf{1}_{\partial\Omega\cap B(x_o,2r)}\big)\, T^{\#} a \, d\sigma = \int_{\partial\Omega} \Big(T\big(f \cdot \mathbf{1}_{\partial\Omega\cap B(x_o,2r)}\big)\Big)\, a \, d\sigma \\ &= \int_{\partial\Omega} \Big(T_{\text{mod}}\big(f \cdot \mathbf{1}_{\partial\Omega\cap B(x_o,2r)}\big)\Big)\, a \, d\sigma\end{aligned} \tag{5.1.234}$$

since $T\big(f \cdot \mathbf{1}_{\partial\Omega\cap B(x_o,2r)}\big)$ differs from $T_{\text{mod}}\big(f \cdot \mathbf{1}_{\partial\Omega\cap B(x_o,2r)}\big)$ by a constant on $\partial\Omega$, thanks to [70, (2.3.34)] and the fact that the atom has integral zero (cf. the last property in (5.1.230)). Also,

$$
\begin{aligned}
\mathrm{II} &= \int_{\partial\Omega\setminus B(x_o,2r)} f(x)\Bigg(\int_{\partial\Omega}\langle\nu(x),\vec{k}(y-x)\rangle a(y)\,d\sigma(y)\Bigg)\,d\sigma(x)\\
&= \int_{\partial\Omega\setminus B(x_o,2r)} f(x)\Bigg(\int_{\partial\Omega}\langle\nu(x),\vec{k}_\varepsilon(y-x)\rangle a(y)\,d\sigma(y)\Bigg)\,d\sigma(x)\\
&= \int_{\partial\Omega\setminus B(x_o,2r)} f(x)\Bigg(\int_{\partial\Omega}\big\langle\nu(x),\vec{k}_\varepsilon(y-x)-\vec{k}_1(-x)\big\rangle a(y)\,d\sigma(y)\Bigg)\,d\sigma(x)\\
&= \int_{\partial\Omega\setminus B(x_o,2r)} f(x)\Bigg(\int_{\partial\Omega\cap B(x_o,r)}\big\langle\nu(x),\vec{k}_\varepsilon(y-x)-\vec{k}_1(-x)\big\rangle a(y)\,d\sigma(y)\Bigg)\,d\sigma(x)\\
&= \int_{\partial\Omega\cap B(x_o,r)}\Bigg(\int_{\partial\Omega\setminus B(x_o,2r)}\big\langle\nu(x),\vec{k}_\varepsilon(y-x)-\vec{k}_1(-x)\big\rangle f(x)\,d\sigma(x)\Bigg)a(y)\,d\sigma(y)\\
&= \int_{\partial\Omega}\lim_{\varepsilon\to 0^+}\Bigg(\int_{\partial\Omega\setminus B(x_o,2r)}\big\langle\nu(x),\vec{k}_\varepsilon(y-x)-\vec{k}_1(-x)\big\rangle f(x)\,d\sigma(x)\Bigg)a(y)\,d\sigma(y)\\
&= \int_{\partial\Omega}\Big(T_{\text{mod}}\big(f\cdot\mathbf{1}_{\partial\Omega\setminus B(x_o,2r)}\big)\Big)(y)\,a(y)\,d\sigma(y).
\end{aligned}
\tag{5.1.235}
$$

The first equality in (5.1.235) is implied by (5.1.6) bearing in mind that, thanks to the first property in (5.1.230), the variables x, y are uniformly separated. The second equality in (5.1.235) uses (5.1.66) and is valid for each choice $\varepsilon \in (0, r)$. The third equality in (5.1.235) is a consequence of the cancelation property of the atom (cf. the last property in (5.1.230)), while the fourth equality in (5.1.235) is seen from the first property in (5.1.230). The fifth equality in (5.1.235) follows from Fubini's Theorem whose applicability is presently ensured by the fact that the double integral is absolutely convergent, thanks to the properties listed in the first line of (5.1.230), the estimate in [70, (2.3.117)] (with a constant which stays bounded for x in a compact subset of $\partial\Omega$), and (5.1.228). The sixth equality in (5.1.235) uses the fact that the inner integral is actually independent of $\varepsilon \in (0, r)$, and also the support condition for the atom. Finally, the last equality in (5.1.235) is seen from (5.1.65). At this stage, from (5.1.233)-(5.1.235) we conclude that (5.1.232) holds for each a as in (5.1.230).

Let us now suppose that $\partial\Omega$ is unbounded. Then, on the one hand, based on (5.1.232) and [69, Proposition 4.8.6] (whose applicability with $g := T^{\#}a$ is ensured by (5.1.22) and (5.1.231)), for each $(1, \infty)$-atom a on $\partial\Omega$ we may write

$$
\int_{\partial\Omega}\big(T_{\text{mod}} f\big)\,a\,d\sigma = \big\langle [f], T^{\#}a\big\rangle \tag{5.1.236}
$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between the John-Nirenberg space of functions of bounded mean oscillations on $\partial\Omega$, modulo constants, and the Hardy space

H^1 on Σ (cf. [69, Theorem 4.6.1]). In concert with [69, (4.6.9)], this permits us to estimate

$$\begin{aligned}\Big|\int_{\partial\Omega}(T_{\text{mod}}f)\,a\,d\sigma\Big| &= \big|\langle [f], T^{\#}a\rangle\big| \\ &\leq C\|f\|_{\mathrm{BMO}(\partial\Omega,\sigma)}\cdot\|T^{\#}a\|_{H^1(\partial\Omega,\sigma)} \\ &\leq C\|f\|_{\mathrm{BMO}(\partial\Omega,\sigma)}, \end{aligned}\tag{5.1.237}$$

where the last inequality is based on (5.1.22) and [69, (4.5.5)-(4.5.6)]. On the other hand, from (5.1.229), (A.0.20), [68, Proposition 7.4.12], and (5.1.230) we see that

$$\|T_{\text{mod}}f\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \leq C\cdot\sup\left\{\Big|\int_{\partial\Omega}(T_{\text{mod}}f)\,a\,d\sigma\Big| : a\ (1,\infty)\text{-atom on }\partial\Omega\right\}. \tag{5.1.238}$$

Together, (5.1.237) and (5.1.238) give

$$\|T_{\text{mod}}f\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \leq C\|f\|_{\mathrm{BMO}(\partial\Omega,\sigma)}, \tag{5.1.239}$$

for some $C\in(0,\infty)$ independent of f. Hence, the operator T_{mod} is a well-defined, linear, bounded mapping from $\mathrm{BMO}(\partial\Omega,\sigma)$ into itself when $\partial\Omega$ is unbounded. In view of the format of the norm on $\widetilde{\mathrm{BMO}}(\partial\Omega,\sigma)$ (cf. [68, (7.4.95)]) and (5.1.69), we then also conclude that the mapping in (5.1.84) is well defined, linear, and bounded, when $\partial\Omega$ is unbounded. With these properties in hand, in the case when $\partial\Omega$ is unbounded, proving (5.1.86) comes down (thanks to the last property in [69, (4.4.114)] and the continuity of $T^{\#}$ on the Hardy scale) to showing that

$$\big\langle [T_{\text{mod}}f], a\big\rangle = \big\langle [f], T^{\#}a\big\rangle \tag{5.1.240}$$

for each $f\in\mathrm{BMO}(\partial\Omega,\sigma)$ and each $(1,\infty)$-atom a on $\partial\Omega$. This, however, is clear from (5.1.236) and the duality result in [69, Theorem 4.6.1, (4.6.8)] (bearing in mind that we already know that $T_{\text{mod}}f\in\mathrm{BMO}(\partial\Omega,\sigma)$).

At this stage, all claims pertaining to (5.1.82), (5.1.84), and (5.1.86) have been justified when $\partial\Omega$ is unbounded. From (5.1.82), (5.1.93) (whose proof is independent of the present considerations), and [69, Theorem 3.1.3] we then conclude that the operator (5.1.83) is also well defined, linear, and bounded. In concert with (5.1.69), this also takes care of (5.1.85) in the case when $\partial\Omega$ is unbounded.

Let us now treat the case when $\partial\Omega$ is bounded. In such a scenario, (5.1.89) holds thanks to [68, (7.4.105)]. As a consequence of this and (5.1.67), for each a as in (5.1.230) we have

$$\int_{\partial\Omega} f\,T^{\#}a\,d\sigma = \int_{\partial\Omega}(Tf)\,a\,d\sigma. \tag{5.1.241}$$

Then the same argument which, starting with (5.1.232), has produced (5.1.239) presently gives

$$\|Tf\|_{\dot{\mathrm{BMO}}(\partial\Omega,\sigma)} \leq C\|f\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \text{ for each } f \in \mathrm{BMO}(\partial\Omega,\sigma). \tag{5.1.242}$$

From this, (5.1.17), and [69, (4.6.18)] we then conclude that T is a well-defined, linear, and bounded operator in the context of (5.1.90).

Next, (5.1.89) and (5.1.67) currently give that $\big[T_{\text{mod}}f\big] = [Tf]$ for each function $f \in \mathrm{BMO}(\partial\Omega,\sigma)$. With this in hand, it follows from (5.1.90) that the mapping in (5.1.84) is bounded. Granted this, for each $f \in \mathrm{BMO}(\partial\Omega,\sigma)$ we may estimate, bearing in mind (A.0.20), the fact that $C_f := T_{\text{mod}}f - Tf$ is a constant as in (5.1.67), and [69, (4.6.18)]:

$$\begin{aligned}
\big\|T_{\text{mod}}f\big\|_{\mathrm{BMO}(\partial\Omega,\sigma)} &\leq \big\|T_{\text{mod}}f - Tf\big\|_{\mathrm{BMO}(\partial\Omega,\sigma)} + \|Tf\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \\
&\leq \sigma(\partial\Omega)|C_f| + C\|f\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \\
&\leq C\|f\|_{L^1(\partial\Omega,\sigma)} + C\|f\|_{\mathrm{BMO}(\partial\Omega,\sigma)} \\
&\leq C\|f\|_{\mathrm{BMO}(\partial\Omega,\sigma)}.
\end{aligned} \tag{5.1.243}$$

This proves that the mapping (5.1.82) is well defined and bounded. In turn, from (5.1.82), [69, (3.1.50)], and (5.1.102) (whose proof is independent of the present considerations), we also see that (5.1.83) is well-defined and bounded operator.

Pressing on, for each $f \in \mathrm{BMO}(\partial\Omega,\sigma)$ and each $(1,\infty)$-atom a on $\partial\Omega$ we may write

$$\big\langle Tf, a\big\rangle = \int_{\partial\Omega}(Tf)a\,\mathrm{d}\sigma = \int_{\partial\Omega} fT^{\#}a\,\mathrm{d}\sigma = \big\langle f, T^{\#}a\big\rangle \tag{5.1.244}$$

by [69, Proposition 4.8.6] (keeping in mind (5.1.89) and (5.1.17)), (5.1.18), and (5.1.22). In concert with the last property in [69, (4.4.114)] and the continuity of $T^{\#}$ on the Hardy scale, this proves (5.1.91). The claims pertaining to (5.1.92) are consequences of (5.1.90), (5.1.102), and [69, (3.1.50)].

Finally, the claims made in relation to (5.1.87) and (5.1.88) may be justified based on (5.1.82), (5.1.84), and (5.1.93), as in the proof of Corollary 2.1.14.

Proof of claims in item (15): All results in item *(15)* dealing with the ordinary Hölder scale are established much as their counterparts have been dealt with in the current item *(14)*. More specifically, from [68, (7.4.119)] we know that

$$\dot{\mathscr{C}}^{\alpha}(\partial\Omega) \subseteq L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big). \tag{5.1.245}$$

Then for any function $f \in \dot{\mathscr{C}}^{\alpha}(\partial\Omega)$, from (5.1.245), [70, (2.3.35)], and (5.1.65) we see that

$$T_{\text{mod}}f \in \bigcap_{1\leq q<\infty} L^q_{\text{loc}}(\partial\Omega,\sigma). \tag{5.1.246}$$

Also, from (5.1.181) and [69, Lemma 4.5.5] we know that

$$\int_{\partial\Omega} |f||T^{\#}a|\,\mathrm{d}\sigma < +\infty. \tag{5.1.247}$$

With this in hand, the same argument that has produced (5.1.232) currently gives

$$\int_{\partial\Omega} f\,T^{\#}a\,\mathrm{d}\sigma = \int_{\partial\Omega} \big(T_{\rm mod} f\big)\,a\,\mathrm{d}\sigma \tag{5.1.248}$$

for each (p,∞)-atom a on $\partial\Omega$ with vanishing moment. Granted this, we may reason as in (5.1.236)-(5.1.239), now relying on [69, Proposition 4.8.7] in place of [69, Proposition 4.8.6], and [68, Proposition 7.4.8] in place of [68, Proposition 7.4.12], to conclude that there exists $C \in (0,\infty)$ such that

$$\big\|T_{\rm mod} f\big\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)} \leq C\|f\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)} \text{ for each } f \in \dot{\mathscr{C}}^{\alpha}(\partial\Omega). \tag{5.1.249}$$

From this and (5.1.69), the claims concerning the operators (5.1.93)-(5.1.95) follow. Once these have been established, in the case when $\partial\Omega$ is unbounded, proving (5.1.99) reduces (in view of the last property in [69, (4.4.114)]) to checking that

$$\begin{gathered}\big\langle \big[T_{\rm mod} f\big], a\big\rangle = \big\langle [f], T^{\#}a\big\rangle \text{ for each} \\ f \in \dot{\mathscr{C}}^{\alpha}(\partial\Omega) \text{ and each } (p,\infty)\text{-atom } a \text{ on } \partial\Omega.\end{gathered} \tag{5.1.250}$$

Since we already know that $T_{\rm mod} f \in \dot{\mathscr{C}}^{\alpha}(\partial\Omega)$, from the duality result in [69, Theorem 4.6.1, (4.6.8)] we conclude that

$$\big\langle \big[T_{\rm mod} f\big], a\big\rangle = \int_{\partial\Omega} (T_{\rm mod} f)a\,\mathrm{d}\sigma. \tag{5.1.251}$$

Observe that since a is a (p,∞)-atom, hence also a multiple of a $(1,\infty)$-atom, from (5.1.22) we have

$$T^{\#}a \in H^1(\partial\Omega,\sigma) \cap H^p(\partial\Omega,\sigma). \tag{5.1.252}$$

Thanks to (5.1.247) and (5.1.252), [69, Proposition 4.8.7] applies and gives

$$\big\langle [f], T^{\#}a\big\rangle = \int_{\partial\Omega} f T^{\#}a\,\mathrm{d}\sigma. \tag{5.1.253}$$

Now (5.1.250) follows from (5.1.251), (5.1.253), and (5.1.248).

In the case when $\partial\Omega$ is bounded, starting with the fact that (5.1.241) holds for each $f \in \mathscr{C}^{\alpha}(\partial\Omega)$ and each (p,∞)-atom a on $\partial\Omega$, the same argument which has proved (5.1.249), outlined above, now gives

$$\|Tf\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)} \leq C\|f\|_{\mathscr{C}^{\alpha}(\partial\Omega)} \text{ for each } f \in \mathscr{C}^{\alpha}(\partial\Omega). \tag{5.1.254}$$

Moreover, for each function $f \in \mathscr{C}^{\alpha}(\partial\Omega)$ and σ-a.e. point $x \in \partial\Omega$ we may use (5.1.13) and [68, (7.2.5)] (with $X := \partial\Omega$, $r := 2\,\mathrm{diam}(\partial\Omega)$, $d := n-1$, and $\delta := \alpha$) to estimate

$$
\begin{aligned}
|(Tf)(x)| &\leq \big|\big(T(f - f(x))\big)(x)\big| + \tfrac{|\vartheta|}{2}|f(x)| \\
&\leq C\|f\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)} \int_{\partial\Omega} \frac{d\sigma(y)}{|x-y|^{n-1-\alpha}} + \tfrac{|\vartheta|}{2}|f(x)| \\
&\leq C\big[\mathrm{diam}(\partial\Omega)\big]^{\alpha}\|f\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)} + \tfrac{|\vartheta|}{2}\sup_{\partial\Omega}|f|,
\end{aligned}
\tag{5.1.255}
$$

for some $C \in (0, \infty)$ which depends only on $\vec{k}$, n, and the lower ADR constant of Ω. Keeping in mind that, as seen from (5.1.254), the operator T maps $\mathscr{C}^{\alpha}(\partial\Omega)$ into continuous functions on $\partial\Omega$, we then conclude from (5.1.255) that there exists some constant $C \in (0, \infty)$ with the property that

$$
\sup_{x\in\partial\Omega} |(Tf)(x)| \leq C\|f\|_{\mathscr{C}^{\alpha}(\partial\Omega)} \text{ for each } f \in \mathscr{C}^{\alpha}(\partial\Omega). \tag{5.1.256}
$$

Ultimately, (5.1.255) and (5.1.256) imply that there exists $C \in (0, \infty)$ such that

$$
\|Tf\|_{\mathscr{C}^{\alpha}(\partial\Omega)} \leq C\|f\|_{\mathscr{C}^{\alpha}(\partial\Omega)} \text{ for each } f \in \mathscr{C}^{\alpha}(\partial\Omega). \tag{5.1.257}
$$

This proves that T is well defined and bounded in the context of (5.1.102). Lastly, (5.1.103) is justified as before, based on density (cf. [69, (4.4.114)]) and the fact that (5.1.244) continues to hold for each $f \in \mathscr{C}^{\alpha}(\partial\Omega)$ and each (p, ∞)-atom a on $\partial\Omega$ (cf. [69, Corollary 4.8.11], keeping in mind (5.1.181), and the bounded set version of the duality result from [69, Theorem 4.6.1, (4.6.8)]).

Finally, all results in item *(15)* dealing with the scale of vanishing Hölder spaces follow from what we have proved so far and the density result from [69, Theorem 3.2.2].

Proof of claims in item (16): All desired conclusions may be justified by reasoning as in the proof of Theorem 3.2.2, making use of the results established so far.

Proof of claims in item (17): The claims in item *(17)* are established by closely following the argument presented in the proof of Theorem 3.1.1, and relying the results obtained in earlier items.

Proof of claims in item (18): Fix $\ell \in \{1, \dots, N\}$ and pick a multi-index $\alpha \in \mathbb{N}_0$ with $|\alpha| = \ell$. From (5.1.47) we know that

$$
\begin{gathered}
\partial^{\alpha}\big(\mathcal{T}_{\rm mod} f\big)(x) = \int_{\partial\Omega} \big\langle \nu(y), (\partial^{\alpha}\vec{k})(x-y)\big\rangle f(y)\, d\sigma(y) \\
\text{for each } f \in L^1\big(\partial\Omega, \tfrac{\sigma(y)}{1+|y|^n}\big) \text{ and } x \in \Omega.
\end{gathered}
\tag{5.1.258}
$$

In addition, (5.1.48) implies

$$
\partial^{\alpha}(\mathcal{T}_{\rm mod} 1) = 0 \text{ in } \Omega. \tag{5.1.259}
$$

Granted (5.1.258)-(5.1.259), the claims made in item *(18)* then become consequences of Theorem 4.2.1 (used with $k := \ell - 1$ and $\varepsilon := 0$).

Proof of claims in item (19): All results may be justified by reasoning as in the proofs of Theorem 4.1.1, Corollary 4.1.4, and Theorem 4.1.5. The key cancelation property, formulated as in (4.1.5) with $T^{\#}$ in place of $K^{\#}$, follows from (4.1.9) and (5.1.30).

Proof of claims in item (20): The justification of the claims in item *(20)* proceed along the lines of the proof of Theorem 1.8.14, making use of what we have established earlier.

Proof of claims in item (21): All claims in item *(21)* may be justified by reasoning much as in the proof of Theorem 3.3.8, employing results that are available to us from earlier work.

Proof of claims in item (22): Reason as in the proof of Theorem 2.3.9, this time making use of (5.1.14), (5.1.45), (5.1.57), (5.1.68), and (5.1.69). □

We next discuss a basic estimate which, in view of its specific format, can be thought of as a "space-less" Carleson-type estimate for modified generalized double layer operators in uniformly rectifiable domains.

Theorem 5.1.8 *Pick $n \in \mathbb{N}$ with $n \geq 2$, and suppose $\Omega \subseteq \mathbb{R}^n$ is an arbitrary* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. For a sufficiently large integer $N = N(n) \in \mathbb{N}$, consider a vector-valued function*

$$\vec{k} \in \big[\mathscr{C}^N(\mathbb{R}^n \setminus \{0\})\big]^n \text{ odd, positive homogeneous of degree } 1-n, \quad \text{and satisfying } \operatorname{div}\vec{k} = 0 \text{ in } \mathbb{R}^n \setminus \{0\}. \tag{5.1.260}$$

With the set Ω and the kernel $\vec{k}$, associate as in (5.1.44) *the modified generalized double layer operator $\mathcal{T}_{\rm mod}$, i.e., the mapping sending each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ into the function defined at each point $x \in \Omega$ according to*

$$\big(\mathcal{T}_{\rm mod} f\big)(x) := \int_{\partial\Omega} \Big\langle \nu(y), \vec{k}(x-y) - \vec{k}(-y)\cdot \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,1)}}(y)\Big\rangle f(y)\,\mathrm{d}\sigma(y). \tag{5.1.261}$$

Then for each $p \in (1,\infty)$ there exists a constant $C \in (0,\infty)$, which depends only on n, p, $\vec{k}$, and the UR *constants of $\partial\Omega$, with the property that for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$, each point $x_o \in \partial\Omega$, and each radius $r \in (0,\infty)$ one has*

$$\left(\frac{1}{\sigma\big(\Delta(x_o,r)\big)}\int_{B(x_o,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod} f\big)(x)\big|^p \operatorname{dist}(x,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n(x)\right)^{1/p} \tag{5.1.262}$$
$$\leq C\int_1^\infty \Big(⨍_{\Delta(x_o,\lambda r)}\Big|f - ⨍_{\Delta(x_o,\lambda r)} f\,\mathrm{d}\sigma\Big|^p\,\mathrm{d}\sigma\Big)^{1/p}\frac{\mathrm{d}\lambda}{\lambda^2}$$

where $\Delta(x_o, R) := B(x_o, R) \cap \partial\Omega$ *for each* $R \in (0, \infty)$.

As a corollary of (5.1.262) *and the definition of the Fefferman-Stein sharp maximal operator* (*cf.* (A.0.195)), *it follows that for each* $p \in (1, \infty)$ *there exists some constant* $C \in (0, \infty)$ *such that the pointwise inequality*

$$\sup_{r>0}\left(\frac{1}{\sigma\big(\Delta(x,r)\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\right)^{1/p}\leq C\cdot f_p^{\#}(x) \tag{5.1.263}$$

holds for each point $x \in \partial\Omega$ *and each function* $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$.

The membership of f to $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ is required simply to ensure that the action of $\mathcal{T}_{\rm mod}$ on f (as defined in (5.1.261)) is meaningful (so the formulation of (5.1.262) makes sense). The absence of a recognizable norm in the formulation of the estimate recorded in (5.1.262), together with the specific format of its left-hand side, validate the point of view (espoused earlier), to the effect that (5.1.262) can be thought of as a "space-less" Carleson-type estimate for modified generalized double layer operators in uniformly rectifiable domains.

In terms of the maximal operator $\mathfrak{P}$ mapping any given $\mathcal{L}^n$-measurable function $u : \Omega \to \mathbb{C}$ into the function defined at every point $x \in \partial\Omega$ as

$$(\mathfrak{P}u)(x) := \sup_{r>0}\left\{\frac{1}{\sigma\big(\partial\Omega\cap B(x,r)\big)}\int_{\Omega\cap B(x,r)}|u|\,d\mathcal{L}^n\right\}\in[0,\infty], \tag{5.1.264}$$

the estimate recorded in (5.1.263) simply reads

$$\mathfrak{P}\Big(\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\Big)\leq C\cdot\big(f_p^{\#}\big)^p\ \text{ on }\ \partial\Omega, \tag{5.1.265}$$

for any given exponent $p \in (1, \infty)$ and each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$.

Let us now present the proof of Theorem 5.1.8.

Proof of Theorem 5.1.8 Pick an arbitrary function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$, and fix some integrability exponent $p \in (1, \infty)$, some point $x_o \in \partial\Omega$, and some radius $r > 0$. Since $\mathcal{T}_{\rm mod}$ maps constant functions on $\partial\Omega$ into constant functions in Ω (cf. (5.1.48)), we see that

$$\nabla\mathcal{T}_{\rm mod}1\equiv 0\ \text{ in }\ \Omega. \tag{5.1.266}$$

Going forward, for each $R > 0$ abbreviate

$$\Delta_R := B(x_o,R)\cap\partial\Omega,\ \ T(\Delta_R) := B(x_o,R)\cap\Omega,\ \text{ and }\ f_{\Delta_R} := \fint_{\Delta_R} f\,d\sigma. \tag{5.1.267}$$

In addition, pick a scale $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ and consider a cutoff function η in $\mathbb{R}^n$ satisfying

$$
\begin{aligned}
&\eta \in \mathscr{C}_c^\infty(\mathbb{R}^n), \quad 0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ on } B(x_o, 2r), \\
&\eta \equiv 0 \text{ outside } B(x_o, 4r), \quad |\partial^\alpha \eta(x)| \leq C_\alpha r^{-|\alpha|}, \\
&\qquad \text{for all } x \in \mathbb{R}^n \text{ and for all } \alpha \in \mathbb{N}_0^n.
\end{aligned} \tag{5.1.268}
$$

We then proceed to split

$$
f = \eta(f - f_{\Delta_{4r}}) + (1 - \eta)(f - f_{\Delta_{4r}}) + f_{\Delta_{4r}}. \tag{5.1.269}
$$

From (5.1.269) and the cancelation property (5.1.266) we see that

$$
\nabla \mathcal{T}_{\rm mod} f = \nabla \mathcal{T}_{\rm mod}\big(\eta(f - f_{\Delta_{4r}})\big) + \nabla \mathcal{T}_{\rm mod}\big((1 - \eta)(f - f_{S_{4r}})\big). \tag{5.1.270}
$$

As such,

$$
\begin{aligned}
\int_{T(\Delta_r)} \Big|\nabla \mathcal{T}_{\rm mod} f(x)\Big|^p \operatorname{dist}(x, \partial\Omega)^{p-1}\, dx \\
&\leq C \int_{T(\Delta_r)} \Big|\nabla \mathcal{T}_{\rm mod}\big(\eta(f - f_{\Delta_{4r}})\big)(x)\Big|^p \operatorname{dist}(x, \partial\Omega)^{p-1}\, dx \\
&\quad + C \int_{T(\Delta_r)} \Big|\nabla \mathcal{T}_{\rm mod}\big((1 - \eta)(f - f_{\Delta_{4r}})\big)(x)\Big|^p \operatorname{dist}(x, \partial\Omega)^{p-1}\, dx \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{5.1.271}
$$

Next, write

$$
\begin{aligned}
\mathrm{I} &\leq C \int_{\Omega} \Big|\nabla \mathcal{T}_{\rm mod}\big(\eta(f - f_{\Delta_{4r}})\big)(x)\Big|^p \operatorname{dist}(x, \partial\Omega)^{p-1}\, dx \\
&\leq C \int_{\partial\Omega} \big|\eta(f - f_{\Delta_{4r}})\big|^p\, d\sigma \leq C \int_{\Delta_{4r}} \big|f - f_{\Delta_{4r}}\big|^p\, d\sigma \\
&\leq C\sigma(\Delta_r) \fint_{\Delta_{4r}} \big|f - f_{\Delta_{4r}}\big|^p\, d\sigma.
\end{aligned} \tag{5.1.272}
$$

Above, the first inequality follows from the definition of I in (5.1.271), the second inequality follows from [70, (2.4.34)], the third inequality is clear from the support properties of the function η introduced in (5.1.268), and the last inequality is due to the fact that σ is doubling (itself, a consequence of the Ahlfors regularity of $\partial\Omega$).

As regards II, we first observe that for each $x \in T(\Delta_r)$ we have

$$
\Big|\nabla \mathcal{T}_{\rm mod}\big((1 - \eta)(f - f_{\Delta_{4r}})\big)(x)\Big| \leq C \int_{\partial\Omega \setminus \Delta_{2r}} \frac{|f(y) - f_{\Delta_{4r}}|}{r + |x_o - y|^n}\, d\sigma(y) \tag{5.1.273}
$$

thanks to the definition of $\nabla \mathcal{T}_{\rm mod}\big((1 - \eta)(f - f_{\Delta_{4r}})\big)$, the properties of the function η from (5.1.268), and the fact that

$$|x-y| \approx |x_o - y| + r \text{ uniformly for } x \in T(\Delta_r) \text{ and } y \in \partial\Omega \setminus \Delta_{2r}. \tag{5.1.274}$$

In turn, from (5.1.271) and (5.1.273) we obtain

$$\begin{aligned}
\mathrm{II} &\le C\Big(\int_{\partial\Omega\setminus\Delta_{2r}} \frac{|f(y)-f_{\Delta_{4r}}|}{r+|x_o-y|^n}\,\mathrm{d}\sigma(y)\Big)^p \int_{T(\Delta_r)} \mathrm{dist}(x,\partial\Omega)^{p-1}\,\mathrm{d}x \\
&\le Cr^{p-1+n}\Big(\int_{\partial\Omega} \frac{|f(y)-f_{\Delta_{4r}}|}{r+|x_o-y|^n}\,\mathrm{d}\sigma(y)\Big)^p \\
&\le C\sigma(\Delta_r)\Big(r\int_{\partial\Omega} \frac{|f(y)-f_{\Delta_{4r}}|}{r+|x_o-y|^n}\,\mathrm{d}\sigma(y)\Big)^p,
\end{aligned} \tag{5.1.275}$$

where the last equality is a consequence of the lower Ahlfors regularity of $\partial\Omega$ and the fact that we are presently assuming $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$.

We continue by noting that the first inequality in [68, (7.4.115)], used with $X := \partial\Omega$, $\mu := \sigma$, $\rho := |\cdot - \cdot|$, $d := n-1$, $p := 1$, $q := 1$, and $\varepsilon := 1$, yields

$$\begin{aligned}
r\int_{\partial\Omega} \frac{|f(y)-f_{\Delta_{4r}}|}{r+|x_o-y|^n}\,\mathrm{d}\sigma(y) &\le C\int_1^\infty \Big(⨍_{\Delta_{\lambda r}} \big|f-f_{\Delta_{\lambda r}}\big|\,\mathrm{d}\sigma\Big)\frac{\mathrm{d}\lambda}{\lambda^2} \\
&\le C\int_1^\infty \Big(⨍_{\Delta_{\lambda r}} \big|f-f_{\Delta_{\lambda r}}\big|^p\,\mathrm{d}\sigma\Big)^{1/p}\frac{\mathrm{d}\lambda}{\lambda^2},
\end{aligned} \tag{5.1.276}$$

where the last step comes from Hölder's inequality. Combining (5.1.275) with (5.1.276) then gives

$$\mathrm{II} \le C\sigma(\Delta_r)\bigg(\int_1^\infty \Big(⨍_{\Delta_{\lambda r}} \big|f-f_{\Delta_{\lambda r}}\big|^p\,\mathrm{d}\sigma\Big)^{1/p}\frac{\mathrm{d}\lambda}{\lambda^2}\bigg)^p. \tag{5.1.277}$$

Also, from [68, (7.4.125)] used with $X := \partial\Omega$, $\mu := \sigma$, $\rho := |\cdot - \cdot|$, $j := 2$, $p := 1$, $q := p$, and $\varepsilon := 1$ we see that

$$\begin{aligned}
\Big(⨍_{\Delta_{4r}} \big|f-f_{\Delta_{4r}}\big|^p\,\mathrm{d}\sigma\Big)^{1/p} &\le C\int_4^8 \Big(⨍_{\Delta_{\lambda r}} \big|f-f_{\Delta_{\lambda r}}\big|^p\,\mathrm{d}\sigma\Big)^{1/p}\frac{\mathrm{d}\lambda}{\lambda^2} \\
&\le C\int_1^\infty \Big(⨍_{\Delta_{\lambda r}} \big|f-f_{\Delta_{\lambda r}}\big|^p\,\mathrm{d}\sigma\Big)^{1/p}\frac{\mathrm{d}\lambda}{\lambda^2}.
\end{aligned} \tag{5.1.278}$$

From (5.1.272) and (5.1.278) we then conclude that

$$\mathrm{I} \le C\sigma(\Delta_r)\bigg(\int_1^\infty \Big(⨍_{\Delta_{\lambda r}} \big|f-f_{\Delta_{\lambda r}}\big|^p\,\mathrm{d}\sigma\Big)^{1/p}\frac{\mathrm{d}\lambda}{\lambda^2}\bigg)^p. \tag{5.1.279}$$

Collectively, (5.1.271), (5.1.277) and (5.1.279) prove that there exists $C \in (0,\infty)$ such that

$$\left(\frac{1}{\sigma(\Delta_r)}\int_{B(x_o,r)\cap\Omega}\left|\nabla\mathcal{T}_{\text{mod}}f\right|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{1/p}$$

$$\leq C\int_1^\infty\left(\fint_{\Delta_{\lambda r}}\left|f-f_{\Delta_{\lambda r}}\right|^p\,\mathrm{d}\sigma\right)^{1/p}\frac{\mathrm{d}\lambda}{\lambda^2} \qquad (5.1.280)$$

whenever $0<r<2\operatorname{diam}(\partial\Omega)$. From this, the desired conclusion immediately follows in the case when $\partial\Omega$ is unbounded, or Ω is bounded. To complete the proof of the theorem, there remains to establish a similar estimate in the case when Ω is an exterior domain and when the supremum is taken in the regime $r\in\big[2\operatorname{diam}(\partial\Omega),\infty\big)$. In such a scenario, the inequality we seek becomes

$$\left(\frac{1}{\sigma(\partial\Omega)}\int_{B(x_o,r)\cap\Omega}\left|\nabla\mathcal{T}_{\text{mod}}f\right|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{1/p}$$

$$\leq C\Big(\fint_{\partial\Omega}\left|f-f_{\partial\Omega}\right|^p\,\mathrm{d}\sigma\Big)^{1/p}, \qquad (5.1.281)$$

where $f_{\partial\Omega}:=\fint_{\partial\Omega}f\,\mathrm{d}\sigma$. Equivalently, we may recast this as

$$\left(\int_{B(x_o,r)\cap\Omega}\left|\nabla\mathcal{T}_{\text{mod}}f\right|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{1/p}\leq C\left\|f-f_{\partial\Omega}\right\|_{L^p(\partial\Omega,\sigma)}. \qquad (5.1.282)$$

Observe that if f does not belong to $L^p(\partial\Omega,\sigma)$, then there is nothing to prove (since the right-hand side of (5.1.282) is infinite). In the case when $f\in L^p(\partial\Omega,\sigma)$ we rely on [70, (2.4.34)] (with $\Sigma:=\partial\Omega$ and $f-f_{\partial\Omega}$ in place of f) together with (5.1.46) and (5.1.266) to write

$$\Big(\int_{\mathbb{R}^n\setminus\partial\Omega}|(\nabla\mathcal{T}_{\text{mod}}f)(x)|^p\operatorname{dist}(x,\partial\Omega)^{p-1}\,\mathrm{d}x\Big)^{1/p}\leq C\left\|f-f_{\partial\Omega}\right\|_{L^p(\partial\Omega,\sigma)}. \qquad (5.1.283)$$

In turn, (5.1.283) readily implies (5.1.282), so the proof of Theorem 5.1.8 is complete. □

Our next corollary amounts to a user-friendly version of Theorem 5.1.8, to the effect that control of the mean oscillations of a given function at a certain scale implies control of the Carleson-like semi-norm of the "Littlewood-Paley measure" associated with said function (in relation to some generalized double layer potential operator).

Corollary 5.1.9 *Suppose Ω is an arbitrary* UR *domain in $\mathbb{R}^n$ (where $n\in\mathbb{N}$ with $n\geq 2$). Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. For a sufficiently large integer $N=N(n)\in\mathbb{N}$, consider a vector-valued function $\vec{k}\in\big[\mathscr{C}^N(\mathbb{R}^n\setminus\{0\})\big]^n$ which is odd, positive homogeneous of degree $1-n$, and divergence-free in $\mathbb{R}^n\setminus\{0\}$. Associate with the set Ω and the kernel*

$\vec{k}$ the modified generalized double layer operator $\mathcal{T}_{\text{mod}}$ acting on functions from the space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ as in (5.1.261). Next, assume

$$\begin{gathered}\phi : (0,\infty) \to [0,\infty) \text{ is a function satisfying} \\ \phi \in L^1_{\text{loc}}\big((0,\infty), \mathcal{L}^1\big) \text{ and } \int_1^\infty \phi(t)\,\frac{dt}{t^2} < +\infty,\end{gathered} \tag{5.1.284}$$

then define $\widetilde{\phi} : (0,\infty) \to [0,\infty)$ by setting

$$\widetilde{\phi}(r) := r\int_r^\infty \phi(t)\,\frac{dt}{t^2} \text{ for each } r \in (0,\infty). \tag{5.1.285}$$

Finally, fix some $p \in (1,\infty)$ and assume $f \in L^1_{\text{loc}}(\partial\Omega, \sigma)$ is such that

$$\sup_{x\in\partial\Omega}\Big(⨍_{\Delta(x,r)}\Big|f - ⨍_{\Delta(x,r)} f\,d\sigma\Big|^p\,d\sigma\Big)^{1/p} \le \phi(r) \text{ for each } r \in (0,\infty), \tag{5.1.286}$$

where $\Delta(x,r) := B(x,r) \cap \partial\Omega$.

Then f belongs to the space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ and there exists a constant $C \in (0,\infty)$, which depends only on n, p, $\vec{k}$, and the UR *constants of $\partial\Omega$, with the property that for each $r \in (0,\infty)$ one has*

$$\sup_{x\in\partial\Omega}\left(\frac{1}{\sigma\big(\Delta(x,r)\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{T}_{\text{mod}}f\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\right)^{1/p} \le C\widetilde{\phi}(r). \tag{5.1.287}$$

Proof From (5.1.284) we see that

$$\int_r^\infty \phi(t)\,\frac{dt}{t^2} < +\infty \text{ for each } r \in (0,\infty). \tag{5.1.288}$$

In concert with [68, (7.4.115)], used with $X := \partial\Omega$, $\mu := \sigma$, $\rho := |\cdot - \cdot|$, $d := n-1$, $p := 1$, $q := 1$, and $\varepsilon := 1$, for each $x_o \in \partial\Omega$ and $r \in (0,\infty)$ this allows us to estimate

$$\begin{aligned}\int_{\partial\Omega}\frac{|f(x) - f_{\Delta(x_o,r)}|}{r + |x-x_o|^n}\,d\sigma(x) &\le \frac{C}{r}\int_1^\infty \phi(\lambda r)\,\frac{d\lambda}{\lambda^2} \\ &= C\int_r^\infty \phi(t)\,\frac{dt}{t^2} < +\infty.\end{aligned} \tag{5.1.289}$$

In turn, this shows that f belongs to the space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$. Having established this, Theorem 5.1.8 applies and (5.1.262) readily implies (5.1.287), on account of (5.1.285). □

Corollary 5.1.9 brings into focus the quantity $\phi(r)$, which bounds the L^p-based mean oscillations of a given function at scale r. There are many function spaces in which such a bound occurs naturally, something we elaborate on in a series of examples, below. In all these cases we adopt the background context employed in Corollary 5.1.9.

The first such paradigm, has to do with controlling the mean oscillations of a given function in terms of its BMO semi-norm.

Example 5.1.10 *Given an arbitrary function* $f \in \mathrm{BMO}(\partial\Omega, \sigma)$, *define*

$$\phi : (0, \infty) \to [0, \infty), \quad \phi(t) := \|f\|_{\dot{\mathrm{BMO}}(\partial\Omega,\sigma)} \text{ for each } t > 0. \tag{5.1.290}$$

Hence, ϕ *satisfies the conditions in* (5.1.284), (5.1.286), *and since* $r\int_r^\infty \frac{dt}{t^2} = 1$ *it follows that* $\widetilde{\phi}$ *defined as in* (5.1.285) *is presently given by*

$$\widetilde{\phi}(r) := \|f\|_{\dot{\mathrm{BMO}}(\partial\Omega,\sigma)} \text{ for each } r > 0. \tag{5.1.291}$$

Then (5.1.287) *implies, after also taking the supremum over all* $r \in (0, \infty)$, *it follows that for each* $p \in (1, \infty)$ *we have*

$$\sup_{\substack{x\in\partial\Omega \\ r\in(0,\infty)}} \left(\frac{1}{\sigma\big(\Delta(x,r)\big)} \int_{B(x,r)\cap\Omega} \big|\nabla \mathcal{T}_{\mathrm{mod}} f\big|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1}\, d\mathcal{L}^n \right)^{1/p} \leq C\|f\|_{\dot{\mathrm{BMO}}(\partial\Omega,\sigma)}. \tag{5.1.292}$$

This corresponds to the Carlson estimate recorded in (5.1.74).

Next we consider the situation in which the mean oscillations of a given function are controlled in terms of its Lebesgue norm.

Example 5.1.11 *Pick an integrability exponent* $p \in (1, \infty)$. *Having fixed an arbitrary function* $f \in L^p(\partial\Omega, \sigma)$, *introduce*

$$\phi : (0, \infty) \to [0, \infty), \quad \phi(t) := C\|f\|_{L^p(\partial\Omega,\sigma)} \cdot t^{(1-n)/p} \text{ for each } t > 0, \tag{5.1.293}$$

where $C \in (0, \infty)$ *is a fixed constant which depends only on* p *and the upper Ahlfors regularity constant of* $\partial\Omega$. *If* C *is sufficiently large, it follows that the conditions stipulated in* (5.1.284), (5.1.286) *are satisfied by our present choice of* ϕ. *In addition, since*

$$r\int_r^\infty t^{(1-n)/p} \frac{dt}{t^2} = \left(\frac{p}{n-1+p}\right) r^{(1-n)/p} \tag{5.1.294}$$

we see that $\widetilde{\phi}$ *defined as in* (5.1.285) *currently takes the form*

$$\widetilde{\phi}(r) := C\left(\frac{p}{n-1+p}\right)\|f\|_{L^p(\partial\Omega,\sigma)} \cdot r^{(1-n)/p} \text{ for each } r > 0. \tag{5.1.295}$$

Then (5.1.287) *implies, in view of* (5.1.46), [70, (2.3.32)], *the lower Ahlfors regularity of* $\partial\Omega$, *and after also taking the supremum over* $r > 0$, *that the boundary-to-domain generalized double layer potential operator* $\mathcal{T}$ *from* (5.1.4) *satisfies*

$$\left(\int_{\Omega} \left| \nabla \mathcal{T} f \right|^{p} \operatorname{dist}(\cdot, \partial\Omega)^{p-1} \, d\mathcal{L}^{n} \right)^{1/p} \leq C \| f \|_{L^{p}(\partial\Omega, \sigma)}. \tag{5.1.296}$$

This corresponds to the L^p-based area-function estimate from [70, (2.4.34)] (*written for* $\Sigma := \partial\Omega$).

We continue by considering the scenario in which one controls the mean oscillations of a given function in terms of its Morrey norm.

Example 5.1.12 *Having fixed an integrability exponent $p \in (1, \infty)$ and some number $\lambda \in (0, n-1)$, pick an arbitrary function $f \in M^{p,\lambda}(\partial\Omega, \sigma)$ and define*

$$\phi : (0, \infty) \longrightarrow [0, \infty), \quad \phi(t) := 2\| f \|_{M^{p,\lambda}(\partial\Omega,\sigma)} \cdot t^{(\lambda-n+1)/p} \text{ for each } t > 0. \tag{5.1.297}$$

Thus, ϕ satisfies the conditions in (5.1.284), (5.1.286), *and since*

$$r \int_{r}^{\infty} t^{(\lambda-n+1)/p} \frac{dt}{t^2} = \left(\frac{p}{n-1-\lambda+p} \right) r^{(\lambda-n+1)/p} \tag{5.1.298}$$

it follows that $\widetilde{\phi}$ defined as in (5.1.285) *is presently given by*

$$\widetilde{\phi}(r) := \left(\frac{2p}{n-1-\lambda+p} \right) \| f \|_{M^{p,\lambda}(\partial\Omega,\sigma)} \cdot r^{(\lambda-n+1)/p} \text{ for each } r > 0. \tag{5.1.299}$$

Then (5.1.287) *implies, in view of* (5.1.46), [69, (6.2.25)], *and after also taking the supremum over $r > 0$, that the boundary-to-domain generalized double layer potential operator $\mathcal{T}$ from* (5.1.4) *satisfies*

$$\sup_{\substack{x \in \partial\Omega \\ r \in (0,\infty)}} \left(\frac{1}{r^{\lambda}} \int_{B(x,r)\cap\Omega} \left| \nabla \mathcal{T} f \right|^{p} \operatorname{dist}(\cdot, \partial\Omega)^{p-1} \, d\mathcal{L}^{n} \right)^{1/p} \leq C \| f \|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{5.1.300}$$

This is a generalization of the fractional Carleson measure estimates on Morrey spaces from Theorem 3.3.3.

It is clear from the above discussion that this machinery extends to other (Morrey-like) spaces, such as generalized Morrey spaces, and generalized Orlicz-Morrey spaces. One can also control the mean oscillations of a function in terms of its Morrey-Campanato semi-norm. This is analyzed below:

Example 5.1.13 *Again, start by fixing $p \in (1, \infty)$ along with some $\lambda \in (0, n-1)$. This time, pick an arbitrary function $f \in \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega, \sigma)$ and define*

$$\phi : (0, \infty) \longrightarrow [0, \infty), \quad \phi(t) := 2\| f \|_{\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)} \cdot t^{(\lambda-n+1)/p} \text{ for each } t > 0. \tag{5.1.301}$$

Consequently, ϕ satisfies the conditions in (5.1.284), (5.1.286), *and in view of* (5.1.298) *it follows that $\widetilde{\phi}$ defined as in* (5.1.285) *is presently given by*

$$\widetilde{\phi}(r) := \Big(\frac{2p}{n-1-\lambda+p}\Big)\|f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)} \cdot r^{(\lambda-n+1)/p} \text{ for each } r > 0. \qquad (5.1.302)$$

Then (5.1.287) *implies, after first dividing by* $r^{(\lambda-n+1)/p}$ *and then taking the supremum over* $r > 0$*, that*

$$\sup_{\substack{x\in\partial\Omega\\ r\in(0,\infty)}} \left(\frac{1}{r^{\lambda}}\int_{B(x,r)\cap\Omega} \big|\nabla\mathcal{T}_{\text{mod}} f\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n\right)^{1/p} \leq C\|f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)}. \qquad (5.1.303)$$

This is a generalization of the fractional Carleson measure estimate on Morrey-Campanato spaces from Theorem 3.2.4.

Furthermore, a similar result is valid for generalized Morrey-Campanato spaces. We conclude with an example in which the mean oscillations are controlled in terms of the Hölder semi-norm.

Example 5.1.14 *To set the stage, fix some exponent* $\alpha \in (0,1)$ *along with some arbitrary function* $f \in \dot{\mathscr{C}}^{\alpha}(\partial\Omega)$*, and introduce*

$$\phi : (0,\infty) \to [0,\infty), \quad \phi(t) := \|f\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)} \cdot (2t)^{\alpha} \text{ for each } t > 0. \qquad (5.1.304)$$

Note that ϕ *satisfies the conditions in* (5.1.284), (5.1.286)*, and since*

$$r\int_r^{\infty} t^{\alpha}\frac{dt}{t^2} = \frac{r^{\alpha}}{1-\alpha} \text{ for each } r > 0, \qquad (5.1.305)$$

we conclude that $\widetilde{\phi}$ *defined as in* (5.1.285) *is presently given by*

$$\widetilde{\phi}(r) := \Big(\frac{2^{\alpha}}{1-\alpha}\Big)\|f\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)} \cdot r^{\alpha} \text{ for each } r > 0. \qquad (5.1.306)$$

Then (5.1.287) *implies, after first dividing by* r^{α} *and then taking the supremum over* $r > 0$*, that*

$$\sup_{\substack{x\in\partial\Omega\\ r\in(0,\infty)}} \left(\frac{1}{r^{n-1+\alpha p}}\int_{B(x,r)\cap\Omega} \big|\nabla\mathcal{T}_{\text{mod}} f\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n\right)^{1/p} \leq C\|f\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)}. \qquad (5.1.307)$$

This corresponds precisely to the fractional Carleson measure estimate on Hölder spaces from item (10) in Theorem 5.1.1 (see (5.1.62)).

We wish to augment the "space-less" Carleson-type estimate for boundary-to-domain modified generalized double layer operators established in Theorem 5.1.8 with a companion "space-less" estimate[15] for *boundary-to-boundary* modified generalized double layer operators, of the sort described in the theorem below.

[15] in the sense that said estimate does not involve any recognizable norm, employed in standard function spaces

Theorem 5.1.15 *Fix $n \in \mathbb{N}$ with $n \geq 2$, and suppose $\Omega \subseteq \mathbb{R}^n$ is an arbitrary* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. For a sufficiently large integer $N = N(n) \in \mathbb{N}$, consider a vector-valued function*

$$\vec{k} \in \big[\mathscr{C}^N(\mathbb{R}^n \setminus \{0\})\big]^n \text{ odd, positive homogeneous of degree } 1-n, \\ \text{and satisfying } \operatorname{div}\vec{k} = 0 \text{ in } \mathbb{R}^n \setminus \{0\}. \tag{5.1.308}$$

With the set Ω and the kernel $\vec{k}$, associate as in (5.1.65) the boundary-to-boundary modified generalized double layer operator $T_{\rm mod}$, i.e., the mapping sending each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ into the function defined at σ-a.e. point $x \in \partial\Omega$ according to

$$T_{\rm mod} f(x) := \lim_{\varepsilon\to 0^+} \int_{\partial\Omega} \big\langle \nu(y), \vec{k}_\varepsilon(x-y) - \vec{k}_1(-y)\big\rangle f(y)\,d\sigma(y) \tag{5.1.309}$$

where

$$\vec{k}_\varepsilon := \vec{k} \cdot \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(0,\varepsilon)}} \text{ for each } \varepsilon > 0. \tag{5.1.310}$$

Then for each $p \in (1,\infty)$ there exists a constant $C \in (0,\infty)$, which depends only on n, p, $\vec{k}$, and the UR *constants of $\partial\Omega$, with the property that for each function*

$$f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \cap L^p_{\rm loc}(\partial\Omega, \sigma), \tag{5.1.311}$$

each point $x_o \in \partial\Omega$, and each radius $r \in (0,\infty)$ one has

$$\Big(⨍_{\Delta(x_o,r)} \Big|T_{\rm mod} f - ⨍_{\Delta(x_o,r)} T_{\rm mod} f\,d\sigma\Big|^p d\sigma\Big)^{1/p} \\ \leq C \int_1^\infty \Big(⨍_{\Delta(x_o,\lambda r)} \Big|f - ⨍_{\Delta(x_o,\lambda r)} f\,d\sigma\Big|^p d\sigma\Big)^{1/p} \frac{d\lambda}{\lambda^2}, \tag{5.1.312}$$

where $\Delta(x_o, R) := B(x_o, R) \cap \partial\Omega$ for each $R \in (0,\infty)$.

As a consequence of (5.1.312) and the definition of the Fefferman-Stein sharp maximal operator (cf. (A.0.195)), it follows that for each $p \in (1,\infty)$ there exists some constant $C \in (0,\infty)$ such that the following pointwise inequality holds:

$$\big(T_{\rm mod} f\big)^\#_p \leq C \cdot f^\#_p \text{ on } \partial\Omega, \text{ for every} \\ \text{function } f \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \cap L^p_{\rm loc}(\partial\Omega, \sigma). \tag{5.1.313}$$

From [70, (2.3.35)] we know that for each f as in (5.1.311) the function $T_{\rm mod} f$ is well-defined and belongs to the space $L^p_{\rm loc}(\partial\Omega, \sigma)$. As such, the estimate in (5.1.312) is meaningfully formulated.

Proof of Theorem 5.1.15 Fix a function f as in (5.1.311), along with some point $x_o \in \partial\Omega$ and a radius $r \in (0, \infty)$. Decompose

$$\begin{aligned} T_{\rm mod} f &= T_{\rm mod}\big[(f - f_{\Delta(x_o,2r)}) \cdot \mathbf{1}_{\partial\Omega\setminus\Delta(x_o,2r)}\big] \\ &\quad + T_{\rm mod}\big[(f - f_{\Delta(x_o,2r)}) \cdot \mathbf{1}_{\Delta(x_o,2r)}\big] + T_{\rm mod} f_{\Delta(x_o,2r)}, \end{aligned} \tag{5.1.314}$$

where $f_{\Delta(x_o,2r)} := ⨍_{\Delta(x_o,2r)} f\, d\sigma$. For each fixed $x \in \Delta(x_o, r)$ we may then further decompose

$$T_{\rm mod}\big[(f - f_{\Delta(x_o,2r)}) \cdot \mathbf{1}_{\partial\Omega\setminus\Delta(x_o,2r)}\big](x) = g(x) + C_f^{(1)}, \tag{5.1.315}$$

where

$$g(x) := \int_{\partial\Omega\setminus\Delta(x_o,2r)} \big\langle \nu(y), \vec{k}(x-y) - \vec{k}(x_o - y)\big\rangle \big(f(y) - f_{\Delta(x_o,2r)}\big)\, d\sigma(y) \tag{5.1.316}$$

and $C_f^{(1)}$ is a constant (depending on f, x_o, r), defined as

$$C_f^{(1)} := \int_{\partial\Omega\setminus\Delta(x_o,2r)} \big\langle \nu(y), \vec{k}(x_o - y) - \vec{k}_1(-y)\big\rangle \big(f(y) - f_{\Delta(x_o,2r)}\big)\, d\sigma(y). \tag{5.1.317}$$

Then

$$\begin{aligned} &\Big(⨍_{\Delta(x_o,r)} |g(x)|^p\, d\sigma(x)\Big)^{\frac{1}{p}} \\ &= \frac{1}{\sigma\big(\Delta(x_o,r)\big)^{\frac{1}{p}}} \bigg\| \int_{\partial\Omega\setminus\Delta(x_o,2r)} \big\langle \nu(y), \vec{k}(x-y) - \vec{k}(x_o - y)\big\rangle \times \\ &\qquad\qquad \times \big(f(y) - f_{\Delta(x_o,2r)}\big)\, d\sigma(y) \bigg\|_{L^p(\Delta(x_o,r),\sigma)} \\ &\leq \frac{1}{\sigma\big(\Delta(x_o,r)\big)^{\frac{1}{p}}} \int_{\partial\Omega\setminus\Delta(x_o,2r)} \Big(\int_{\Delta(x_o,r)} \big|\vec{k}(x-y) - \vec{k}(x_o - y)\big|^p\, d\sigma(x)\Big)^{\frac{1}{p}} \times \\ &\qquad\qquad \times \big|f(y) - f_{\Delta(x_o,2r)}\big|\, d\sigma(y) \\ &\leq Cr \int_{\partial\Omega\setminus\Delta(x_o,2r)} \frac{\big|f(y) - f_{\Delta(x_o,2r)}\big|}{|y - x_o|^n}\, d\sigma(y) \\ &\leq C \int_1^\infty \Big(⨍_{\Delta(x_o,\lambda r)} \Big|f - ⨍_{\Delta(x_o,\lambda r)} f\, d\sigma\Big|^p\, d\sigma\Big)^{1/p} \frac{d\lambda}{\lambda^2}. \end{aligned} \tag{5.1.318}$$

Above, the first step is obvious, the second one is Minkowski's inequality, the third one uses the Mean Value Theorem and the properties of $\vec{k}$, and the fourth is comes from the first inequality in [68, (7.4.115)] (used with $X := \partial\Omega$, $\rho := |\cdot - \cdot|$, $\mu := \sigma$, $p := 1$, $q := p$, $\varepsilon := 1$, $d := n-1$, and r replaced by $2r$).

Next, since the function $(f - f_{\Delta(x_o,2r)}) \cdot \mathbf{1}_{\Delta(x_o,2r)}$ belongs to $L^p(\partial\Omega, \sigma)$, from (5.1.67) we see that there exists a constant $C_f^{(2)}$, depending on f, x_o, r, such that

$$T_{\text{mod}}\big[(f - f_{\Delta(x_o,2r)}) \cdot \mathbf{1}_{\Delta(x_o,2r)}\big] = T\big[(f - f_{\Delta(x_o,2r)}) \cdot \mathbf{1}_{\Delta(x_o,2r)}\big] + C_f^{(2)}. \tag{5.1.319}$$

In addition,

$$\begin{aligned}
&\Big(\fint_{\Delta(x_o,r)} \Big|T\big[(f - f_{\Delta(x_o,2r)}) \cdot \mathbf{1}_{\Delta(x_o,2r)}\big]\Big|^p \, \mathrm{d}\sigma\Big)^{\frac{1}{p}} \\
&\qquad \le \frac{1}{\sigma\big(\Delta(x_o,r)\big)^{\frac{1}{p}}} \Big\|T\big[(f - f_{\Delta(x_o,2r)}) \cdot \mathbf{1}_{\Delta(x_o,2r)}\big]\Big\|_{L^p(\partial\Omega,\sigma)} \\
&\qquad \le \frac{C}{\sigma\big(\Delta(x_o,r)\big)^{\frac{1}{p}}} \Big\|\big(f - f_{\Delta(x_o,2r)}\big) \cdot \mathbf{1}_{\Delta(x_o,2r)}\Big\|_{L^p(\partial\Omega,\sigma)} \\
&\qquad \le C\Big(\fint_{\Delta(x_o,2r)} \big|f - f_{\Delta(x_o,2r)}\big|^p \, \mathrm{d}\sigma\Big)^{\frac{1}{p}} \\
&\qquad \le C\int_1^\infty \Big(\fint_{\Delta(x_o,\lambda r)} \Big|f - \fint_{\Delta(x_o,\lambda r)} f \, \mathrm{d}\sigma\Big|^p \, \mathrm{d}\sigma\Big)^{1/p} \frac{\mathrm{d}\lambda}{\lambda^2},
\end{aligned} \tag{5.1.320}$$

where the first inequality is obvious, the second one is a consequence of the boundedness of T on $L^p(\partial\Omega, \sigma)$ (see (5.1.17)), the third one uses the fact that σ is a doubling measure on $\partial\Omega$, and the last one is implied by [68, (7.4.125)]. Finally, we recall from (5.1.69) that

$$C_f^{(3)} := T_{\text{mod}} f_{\Delta(x_o,2r)} \tag{5.1.321}$$

is a constant, depending on f, x_o, r. If we now introduce

$$C_f := C_f^{(1)} + C_f^{(2)} + C_f^{(3)}, \tag{5.1.322}$$

it follows that C_f is a constant, depending on f, x_o, r, and from (5.1.314)-(5.1.321) we see that

$$\Big(\fint_{\Delta(x_o,r)}\Big|T_{\rm mod}f-\fint_{\Delta(x_o,r)}T_{\rm mod}f\,d\sigma\Big|^p\,d\sigma\Big)^{1/p} \tag{5.1.323}$$
$$\leq 2\Big(\fint_{\Delta(x_o,r)}\big|T_{\rm mod}f-C_f\big|^p\,d\sigma\Big)^{1/p}$$
$$\leq C\int_1^\infty\Big(\fint_{\Delta(x_o,\lambda r)}\Big|f-\fint_{\Delta(x_o,\lambda r)}f\,d\sigma\Big|^p\,d\sigma\Big)^{1/p}\frac{d\lambda}{\lambda^2},$$

as wanted. □

It is of interest to record a user-friendly version of Theorem 5.1.15, to the effect that control of the mean oscillations of a given function at a certain scale implies (a suitably correlated) control of the mean oscillations for any modified generalized double layer acting on said function. This is done in our next corollary.

Corollary 5.1.16 *Suppose Ω is an arbitrary* UR *domain in $\mathbb{R}^n$ (where $n\in\mathbb{N}$ with $n\geq 2$). Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. For a sufficiently large integer $N=N(n)\in\mathbb{N}$, consider a vector-valued function $\vec{k}\in\big[\mathscr{C}^N(\mathbb{R}^n\setminus\{0\})\big]^n$ which is odd, positive homogeneous of degree $1-n$, and divergence-free in $\mathbb{R}^n\setminus\{0\}$. With the set Ω and the kernel $\vec{k}$, associate the boundary-to-boundary modified generalized double layer operator $T_{\rm mod}$ acting on functions from the space $L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\big)$ as in (5.1.65)-(5.1.66). Next, assume*

$$\phi:(0,\infty)\to[0,\infty)\ \textit{is a function satisfying}$$
$$\phi\in L^1_{\rm loc}\big((0,\infty),\mathcal{L}^1\big)\ \textit{and}\ \int_1^\infty\phi(t)\,\frac{dt}{t^2}<+\infty, \tag{5.1.324}$$

then define $\widetilde{\phi}:(0,\infty)\to[0,\infty)$ by setting

$$\widetilde{\phi}(r):=r\int_r^\infty\phi(t)\,\frac{dt}{t^2}\ \textit{for each}\ r\in(0,\infty). \tag{5.1.325}$$

Finally, fix some $p\in(1,\infty)$ and assume $f\in L^1_{\rm loc}(\partial\Omega,\sigma)$ is such that

$$\sup_{x\in\partial\Omega}\Big(\fint_{\Delta(x,r)}\Big|f-\fint_{\Delta(x,r)}f\,d\sigma\Big|^p\,d\sigma\Big)^{1/p}\leq\phi(r)\ \textit{for each}\ r\in(0,\infty), \tag{5.1.326}$$

where $\Delta(x,r):=B(x,r)\cap\partial\Omega$.

Then f belongs to the space $L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\big)\cap L^p_{\rm loc}(\partial\Omega,\sigma)$ and there exists a constant $C\in(0,\infty)$, which depends only on n, p, $\vec{k}$, and the UR *constants of $\partial\Omega$, with the property that for each $r\in(0,\infty)$ one has*

$$\sup_{x\in\partial\Omega}\Big(\fint_{\Delta(x,r)}\Big|T_{\rm mod}f-\fint_{\Delta(x,r)}T_{\rm mod}f\,d\sigma\Big|^p\,d\sigma\Big)^{1/p}\leq C\widetilde{\phi}(r). \tag{5.1.327}$$

Proof That f belongs to the space $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ is seen from (5.1.289), while the membership of f to $L^p_{\rm loc}(\partial\Omega, \sigma)$ is clear from (5.1.324) and (5.1.326). Finally, (5.1.327) follows from (5.1.312), (5.1.326), a natural change of variables, and (5.1.325). □

Retaining the background context employed in Corollary 5.1.16, we with to elaborate on the estimate (5.1.327) in a series of examples, included below. In the first such example, we start by controlling the mean oscillations of a given function in terms of its BMO semi-norm.

Example 5.1.17 *Fix an arbitrary function* $f \in \mathrm{BMO}(\partial\Omega, \sigma)$ *and define*

$$\phi : (0, \infty) \to [0, \infty), \quad \phi(t) := \|f\|_{\dot{\mathrm{BMO}}(\partial\Omega,\sigma)} \text{ for each } t > 0. \tag{5.1.328}$$

Hence, ϕ *satisfies the conditions in* (5.1.324), (5.1.326), *and since* $r\int_r^\infty \frac{dt}{t^2} = 1$ *it follows that* $\widetilde{\phi}$ *defined as in* (5.1.325) *is currently given by*

$$\widetilde{\phi}(r) := \|f\|_{\dot{\mathrm{BMO}}(\partial\Omega,\sigma)} \text{ for each } r > 0. \tag{5.1.329}$$

Then (5.1.327) *implies, after also taking the supremum over all* $r \in (0, \infty)$*, that for each* $p \in (1, \infty)$ *we have*

$$\sup_{x\in\partial\Omega,\, r>0} \Big(⨍_{\Delta(x,r)} \Big| T_{\rm mod} f - ⨍_{\Delta(x,r)} T_{\rm mod} f \, d\sigma \Big|^p \, d\sigma \Big)^{1/p} \leq C\|f\|_{\dot{\mathrm{BMO}}(\partial\Omega,\sigma)}. \tag{5.1.330}$$

Ultimately, this provides a new proof of the fact that $T_{\rm mod}$ *is a well-defined, linear, and bounded operator on* $\mathrm{BMO}(\partial\Omega, \sigma)$ (*cf.* (5.1.82)).

We next consider the scenario in which one controls the mean oscillations of a given function in terms of its Morrey-Campanato semi-norm (a similar result is valid for generalized Morrey-Campanato spaces).

Example 5.1.18 *Start by fixing some* $p \in (1, \infty)$ *along with some* $\lambda \in (0, n-1)$*. Pick an arbitrary function* $f \in \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega, \sigma)$ *and define*

$$\phi : (0, \infty) \to [0, \infty), \quad \phi(t) := 2\|f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)} \cdot t^{(\lambda-n+1)/p} \text{ for each } t > 0. \tag{5.1.331}$$

Note that ϕ *satisfies the conditions in* (5.1.324), (5.1.326), *and in view of* (5.1.298) *it follows that* $\widetilde{\phi}$ *defined as in* (5.1.325) *is presently given by*

$$\widetilde{\phi}(r) := \Big(\frac{2p}{n-1-\lambda+p}\Big)\|f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)} \cdot r^{(\lambda-n+1)/p} \text{ for each } r > 0. \tag{5.1.332}$$

Then the estimate recorded in (5.1.327) *implies, after first dividing by* $r^{(\lambda-n+1)/p}$ *and then taking the supremum over* $r \in \big(0, 2\,\mathrm{diam}\,\partial\Omega\big)$*, that*

$$\sup_{\substack{x\in\partial\Omega \text{ and}\\ 0<r<2\,\mathrm{diam}\,\partial\Omega}} \left\{ r^{\frac{n-1-\lambda}{p}} \left(⨍_{\Delta(x,r)} \left| T_{\mathrm{mod}} f - ⨍_{\Delta(x,r)} T_{\mathrm{mod}} f \, d\sigma \right|^p d\sigma \right)^{\frac{1}{p}} \right\} \leq C\|f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)}. \tag{5.1.333}$$

In view of (A.0.118)*, this gives a new proof of the fact that* T_{mod} *is a well defined, linear, and bounded operator on* $\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)$ (*cf.* (5.1.106)).

In our final example, the mean oscillations are controlled in terms of the Hölder semi-norm.

Example 5.1.19 *To set the stage, fix some exponent* $\alpha \in (0,1)$ *along with some arbitrary function* $f \in \dot{\mathscr{C}}^{\alpha}(\partial\Omega)$*, and introduce*

$$\phi : (0,\infty) \to [0,\infty), \quad \phi(t) := \|f\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)} \cdot (2t)^{\alpha} \text{ for each } t > 0. \tag{5.1.334}$$

Observe that ϕ *satisfies the conditions in* (5.1.324), (5.1.326)*, and in view of* (5.1.305) *we conclude that* $\widetilde{\phi}$ *defined as in* (5.1.325) *is currently given by*

$$\widetilde{\phi}(r) := \left(\frac{2^{\alpha}}{1-\alpha}\right)\|f\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)} \cdot r^{\alpha} \text{ for each } r > 0. \tag{5.1.335}$$

Then (5.1.327) *implies, after first dividing by* r^{α} *and then taking the supremum over* $r \in (0,\infty)$*, that*

$$\sup_{\substack{x\in\partial\Omega \text{ and}\\ 0<r<\infty}} \left\{ \frac{1}{r^{\alpha}} \left(⨍_{\Delta(x,r)} \left| T_{\mathrm{mod}} f - ⨍_{\Delta(x,r)} T_{\mathrm{mod}} f \, d\sigma \right|^p d\sigma \right)^{1/p} \right\} \leq C\|f\|_{\dot{\mathscr{C}}^{\alpha}(\partial\Omega)}. \tag{5.1.336}$$

Bearing in mind the characterization of the class of Hölder functions from [68, Proposition 7.4.9] (*whose present applicability is guaranteed by* [68, Lemma 3.6.4] *used with* $s := n-1$)*, this ultimately provides a new proof of the fact that* T_{mod} *is a well defined, linear, and bounded operator on* $\dot{\mathscr{C}}^{\alpha}(\partial\Omega)$ (*see* (5.1.93)).

All the above examples are subsumed by the following corollary, generalizing work in the entire Euclidean setting from [83].

Corollary 5.1.20 *Suppose* Ω *is an arbitrary* UR *domain in* $\mathbb{R}^n$ (*where* $n \in \mathbb{N}$ *with* $n \geq 2$)*. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. For a sufficiently large integer* $N = N(n) \in \mathbb{N}$*, consider a vector-valued function* $\vec{k} \in \big[\mathscr{C}^N(\mathbb{R}^n \setminus \{0\})\big]^n$ *which is odd, positive homogeneous of degree* $1-n$*, and divergence-free in* $\mathbb{R}^n \setminus \{0\}$*. With the set* Ω *and the kernel* $\vec{k}$*, associate the boundary-to-boundary modified generalized double layer operator* T_{mod} *acting on functions from the space* $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ *as in* (5.1.65)-(5.1.66)*. Next, assume*

$$\begin{gathered}\phi : (0,\infty) \to (0,\infty) \text{ is } \mathcal{L}^1\text{-measurable and there exists } C \in (0,\infty)\\ \text{such that } r\int_r^{\infty} \phi(t)\,\frac{dt}{t^2} \leq C\phi(r) \text{ for each } r \in (0,\infty).\end{gathered} \tag{5.1.337}$$

Finally, fix some $p \in (1,\infty)$ and introduce the generalized BMO *space*[16]

$$\mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma) := \Big\{ f \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma) : \|f\|_{\mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma)} < +\infty \Big\}, \tag{5.1.338}$$

where for each $f \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$ one defines

$$\|f\|_{\mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma)} := \sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,\infty)}} \left\{ \frac{1}{\phi(r)} \Big(⨍_{\Delta(x,r)} \Big| f - ⨍_{\Delta(x,r)} f \, d\sigma \Big|^p \, d\sigma \Big)^{1/p} \right\}. \tag{5.1.339}$$

Then one has

$$\mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma) \subseteq L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma), \tag{5.1.340}$$

and the operator

$$T_{\mathrm{mod}} : \mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma) \longrightarrow \mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma) \tag{5.1.341}$$

is well defined, linear, and bounded.

Parenthetically, we wish to note that functional inequalities in the spirit of the inequality in (5.1.337) are discussed in [102, Lemma 2.3, p. 515] and [81, Lemma 2, p. 98].

Let us also remark that, whenever ϕ is as in (5.1.337), we may employ notation introduced in (5.1.338)-(5.1.339) to recast (5.1.287) as the following generalized Carleson measure estimate[17] valid for each function $f \in \mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma)$:

$$\sup_{\substack{x\in\partial\Omega \text{ and}\\ r\in(0,\infty)}} \left\{ \frac{1}{\phi(r)} \bigg(\frac{1}{\sigma\big(\Delta(x,r)\big)} \int_{B(x,r)\cap\Omega} \big|\nabla \mathcal{T}_{\mathrm{mod}} f\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1} \, d\mathcal{L}^n \bigg)^{1/p} \right\}$$
$$\leq C\|f\|_{\mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma)}. \tag{5.1.342}$$

It is relevant to remark that the brand of BMO space introduced in (5.1.338)-(5.1.339) may be considered on an arbitrary closed Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$ (rather than the topological boundary of a UR domain), and for an arbitrary function $\phi : (0,\infty) \to (0,\infty)$. Specifically, with $\sigma := \mathcal{H}^{n-1} \lfloor \Sigma$, for each given $p \in [1,\infty)$ set

$$\mathrm{BMO}_{\phi,p}(\Sigma,\sigma) := \Big\{ f \in L^1_{\mathrm{loc}}(\Sigma,\sigma) : \|f\|_{\mathrm{BMO}_{\phi,p}(\Sigma,\sigma)} < +\infty \Big\} \tag{5.1.343}$$

where, for each $f \in L^1_{\mathrm{loc}}(\Sigma,\sigma)$, we define

[16] alternatively, (ϕ, p)-BMO space, or simply ϕ-BMO space

[17] alternatively, (ϕ, p)-Carleson measure estimate, or simply ϕ-Carleson measure estimate

$$\|f\|_{\mathrm{BMO}_{\phi,p}(\Sigma,\sigma)} := \sup_{\substack{x\in\Sigma \text{ and} \\ r\in(0,\infty)}} \left\{ \frac{1}{\phi(r)} \left(⨍_{\Delta(x,r)} \left| f - ⨍_{\Delta(x,r)} f \, d\sigma \right|^p d\sigma \right)^{1/p} \right\}. \tag{5.1.344}$$

In relation to this we wish to note that if the function ϕ is quasi-increasing and doubling (see (5.1.349)) then the space $\mathrm{BMO}_{\phi,p}(\Sigma,\sigma)$ is actually independent of $p \in [1,\infty)$. More specifically, with $\phi : (0,\infty) \to (0,\infty)$ arbitrary define

$$\mathrm{BMO}_{\phi}(\Sigma,\sigma) := \left\{ f \in L^1_{\mathrm{loc}}(\Sigma,\sigma) : \|f\|_{\mathrm{BMO}_{\phi}(\Sigma,\sigma)} < +\infty \right\}, \tag{5.1.345}$$

where, for each $f \in L^1_{\mathrm{loc}}(\Sigma,\sigma)$, we have set

$$\|f\|_{\mathrm{BMO}_{\phi}(\Sigma,\sigma)} := \sup_{\substack{x\in\Sigma \text{ and} \\ r\in(0,\infty)}} \left\{ \frac{1}{\phi(r)} \left(⨍_{\Delta(x,r)} \left| f - ⨍_{\Delta(x,r)} f \, d\sigma \right| d\sigma \right) \right\}. \tag{5.1.346}$$

Let us now assume that

$$\begin{gathered} \text{there exist two constants } C_1, C_2 \in (1,\infty) \text{ such that} \\ \phi(t) \le C_1 \phi(r) \text{ for each } r \in (0,\infty) \text{ and } t \in (0, C_2 r). \end{gathered} \tag{5.1.347}$$

Via iterations, this is equivalent to asking that

$$\begin{gathered} \text{for each given } C \in (0,\infty) \text{ there exists } \widetilde{C} \in (0,\infty) \text{ such that} \\ \phi(t) \le \widetilde{C} \phi(r) \text{ whenever } r \in (0,\infty) \text{ and } t \in (0, Cr). \end{gathered} \tag{5.1.348}$$

It is also straightforward to check that (5.1.347) (or (5.1.348)) is equivalent to demanding that

ϕ is quasi-increasing (i.e., there exists $C \in (0,\infty)$ such that $\phi(t_1) \le C\phi(t_2)$ whenever $0 < t_1 \le t_2 < \infty$), and ϕ is doubling (i.e., there exist $C_1, C_2 \in (1,\infty)$ such that $\phi(C_1 t) \le C_2 \phi(t)$ for each $t \in (0,\infty)$). (5.1.349)

The claim we make is that if we assume (5.1.347) (or, equivalently, (5.1.348), or (5.1.349)) then

$$\begin{gathered} \mathrm{BMO}_{\phi,p}(\Sigma,\sigma) = \mathrm{BMO}_{\phi}(\Sigma,\sigma) \text{ for each } p \in (1,\infty) \\ \text{(equal vector spaces with equivalent semi-norms).} \end{gathered} \tag{5.1.350}$$

To justify (5.1.350), fix some $p \in (1,\infty)$ along with some function $f \in L^1_{\mathrm{loc}}(\Sigma,\sigma)$. Also, pick an arbitrary $x \in \Sigma$ together with an arbitrary $r \in (0,\infty)$. Consider a surface ball $\Delta' \subseteq \Delta(x,5r)$ and having a radius $t \in \big(0, 2\,\mathrm{diam}\,\Sigma\big)$. Then the Ahlfors regularity of Σ implies that $t < C_3 r$ for some $C_3 \in (0,\infty)$ which depends only on n and the Ahlfors regularity constants of Σ. Bearing this in mind, we may then estimate

$$\begin{aligned}
\frac{1}{\phi(r)}\Big(&⨍_{\Delta(x,r)}\Big|f-⨍_{\Delta(x,r)} f\,\mathrm{d}\sigma\Big|^p\,\mathrm{d}\sigma\Big)^{1/p}\\
&\leq \frac{C}{\phi(r)}\sup_{\Delta'\subseteq\Delta(x,5r)}\Big(⨍_{\Delta'}\Big|f-⨍_{\Delta'} f\,\mathrm{d}\sigma\Big|\mathrm{d}\sigma\Big)\\
&\leq \frac{C}{\phi(r)}\Big(\sup_{t\in(0,C_3 r)}\phi(t)\Big)\cdot\|f\|_{\mathrm{BMO}_\phi(\Sigma,\sigma)}\\
&\leq C\|f\|_{\mathrm{BMO}_\phi(\Sigma,\sigma)}, \qquad (5.1.351)
\end{aligned}$$

thanks to [68, Lemma 7.4.10], the observation made in [68, (7.4.61)] (according to which the supremum in the first line of (5.1.351) may be taken over all surface balls Δ' on Σ contained in $\Delta(x, 5r)$ and having radii $< 2\,\mathrm{diam}\,\Sigma$), and the additional assumption made in (5.1.348). After taking the supremum over all $x \in \Sigma$ and $r \in (0,\infty)$ we arrive at

$$\|f\|_{\mathrm{BMO}_{\phi,p}(\Sigma,\sigma)} \leq C\|f\|_{\mathrm{BMO}_\phi(\Sigma,\sigma)}. \qquad (5.1.352)$$

The opposite inequality in (5.1.352) is also true (with $C = 1$), as a consequence of Hölder's inequality. Altogether, this establishes the claim made in (5.1.350).

For example,

$$\begin{gathered}
\phi : (0,\infty) \to (0,\infty) \text{ given by } \phi(t) := 1 \text{ for each } t \in (0,\infty)\\
\text{has the properties described in (5.1.337), (5.1.347), and}\\
\mathrm{BMO}_\phi(\partial\Omega,\sigma) = \Big(\mathrm{BMO}(\partial\Omega,\sigma), \|\cdot\|_{\dot{\mathrm{B}}\mathrm{MO}(\partial\Omega,\sigma)}\Big),
\end{gathered} \qquad (5.1.353)$$

while for each $\alpha \in (0,1)$,

$$\begin{gathered}
\phi : (0,\infty) \to (0,\infty) \text{ with } \phi(t) := t^\alpha \text{ for all } t > 0\\
\text{is as in (5.1.337), (5.1.347), and we have}\\
\mathrm{BMO}_\phi(\partial\Omega,\sigma) = \dot{\mathscr{C}}^\alpha(\partial\Omega).
\end{gathered} \qquad (5.1.354)$$

On the other hand, for each $p \in (1,\infty)$ and $\lambda \in (0, n-1)$,

$$\begin{gathered}
\phi : (0,\infty) \to (0,\infty) \text{ with } \phi(t) := t^{(\lambda-n+1)/p} \text{ for each } t > 0\\
\text{is as in (5.1.337) and } \mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma) = \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma).
\end{gathered} \qquad (5.1.355)$$

This being said, the function ϕ from (5.1.355) fails to satisfy (5.1.347) (specifically, ϕ fails to be quasi-increasing), and the space $\mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma) = \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)$ is strongly dependent on the choice of the parameter $p \in (1,\infty)$.

We now turn to the task of presenting the proof of Corollary 5.1.20.

Proof of Corollary 5.1.20 The inclusion (5.1.340) is a consequence of the first conclusion in Corollary 5.1.16. As regards (5.1.341), pick an arbitrary function

$f \in \mathrm{BMO}_{\phi,p}(\partial\Omega, \sigma)$ and note that this entails

$$\sup_{x\in\partial\Omega}\left\{\left(\fint_{\Delta(x,r)}\left|f - \fint_{\Delta(x,r)} f\,\mathrm{d}\sigma\right|^p \mathrm{d}\sigma\right)^{1/p}\right\} \leq \|f\|_{\mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma)} \cdot \phi(r) \quad (5.1.356)$$

for each $r \in (0,\infty)$. Granted this, we may then invoke (5.1.327), (5.1.325), and the inequality in (5.1.337) to write

$$\begin{aligned} &\big\|T_{\mathrm{mod}} f\big\|_{\mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma)} \\ &\qquad = \sup_{\substack{x\in\partial\Omega \text{ and} \\ r\in(0,\infty)}} \left\{\frac{1}{\phi(r)}\left(\fint_{\Delta(x,r)}\left|T_{\mathrm{mod}} f - \fint_{\Delta(x,r)} T_{\mathrm{mod}} f\,\mathrm{d}\sigma\right|^p \mathrm{d}\sigma\right)^{1/p}\right\} \\ &\qquad \leq C\|f\|_{\mathrm{BMO}_{\phi,p}(\partial\Omega,\sigma)}. \end{aligned} \quad (5.1.357)$$

This shows that the operator (5.1.341) is indeed well defined and bounded. □

We close this section with a result essentially stating that modified boundary-to-domain generalized double layers map CMO functions into (certain densities of) super vanishing Carleson measures (a notion introduced in Definition 1.8.1). We shall actually show more in Theorem 5.1.22, stated a little further below. To set the stage, we first prove the following lemma.

Lemma 5.1.21 *Pick $n \in \mathbb{N}$ with $n \geq 2$, and suppose $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain for which $\partial\Omega$ is unbounded. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. For a sufficiently large $N = N(n) \in \mathbb{N}$, consider a vector-valued function*

$$\begin{gathered} \vec{k} \in \left[\mathscr{C}^N(\mathbb{R}^n \setminus \{0\})\right]^n \text{ odd, positive homogeneous of degree } 1-n, \\ \text{and satisfying } \operatorname{div}\vec{k} = 0 \text{ in } \mathbb{R}^n \setminus \{0\}. \end{gathered} \quad (5.1.358)$$

With the set Ω and the kernel $\vec{k}$, associate as in (5.1.44) the modified generalized double layer operator $\mathcal{T}_{\mathrm{mod}}$, i.e., the mapping sending each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ into the function defined at each point $x \in \Omega$ according to

$$(\mathcal{T}_{\mathrm{mod}} f)(x) := \int_{\partial\Omega}\Big\langle \nu(y),\, \vec{k}(x-y) - \vec{k}(-y)\cdot \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,1)}}(y)\Big\rangle f(y)\,\mathrm{d}\sigma(y). \quad (5.1.359)$$

Then for each $p \in (1,\infty)$ and each function $f \in L^\infty_{\mathrm{comp}}(\partial\Omega,\sigma)$ one has

$$\lim_{R\to\infty}\sup_{x\in\partial\Omega,\, r>R}\left(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla(\mathcal{T}_{\mathrm{mod}} f)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{1/p} = 0 \quad (5.1.360)$$

and

$$\lim_{R\to\infty} \sup_{\substack{x\in\partial\Omega,\ r>0\\ B(x,r)\subseteq\mathbb{R}^n\setminus B(0,R)}} \left(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{1/p}$$
$$= 0. \tag{5.1.361}$$

Proof Start by fixing an integrability exponent $p\in(1,\infty)$ and pick an arbitrary function $f\in L^\infty_{\rm comp}(\partial\Omega,\sigma)$. To simplify notation, let Δ denote an arbitrary surface ball on $\partial\Omega$, i.e., a set of the form $\Delta=B(x,r)\cap\partial\Omega$ for some $x\in\partial\Omega$ and $r>0$. For any such surface ball abbreviate $f_\Delta:=\fint_\Delta f\,\mathrm{d}\sigma$. We claim that

$$\begin{gathered}\text{there exists some constant } C_f\in(0,\infty) \text{ such that}\\ \Big(\fint_\Delta \big|f-f_\Delta\big|^p\,\mathrm{d}\sigma\Big)^{1/p}\leq\frac{C_f}{\sigma(\Delta)^{1/p}}\ \text{ for all }\ \Delta.\end{gathered} \tag{5.1.362}$$

Indeed, since f is compactly supported and bounded there exists some surface ball $\Delta_*\subseteq\partial\Omega$ (depending on f) such that

$$|f|\leq\|f\|_{L^\infty(\partial\Omega,\sigma)}\cdot\mathbf{1}_{\Delta_*}\ \text{ pointwise on }\ \partial\Omega. \tag{5.1.363}$$

Hence, for every Δ we may estimate

$$\Big(\fint_\Delta |f|^p\,\mathrm{d}\sigma\Big)^{1/p}\leq\|f\|_{L^\infty(\partial\Omega,\sigma)}\Big(\frac{\sigma(\Delta_*)}{\sigma(\Delta)}\Big)^{1/p}. \tag{5.1.364}$$

Consequently, invoking the triangle inequality and Hölder's inequality it follows that, for every Δ,

$$\begin{aligned}\Big(\fint_\Delta \big|f-f_\Delta\big|^p\,\mathrm{d}\sigma\Big)^{1/p}&\leq 2\Big(\fint_\Delta |f|^p\,\mathrm{d}\sigma\Big)^{1/p}\\ &\leq 2\|f\|_{L^\infty(\partial\Omega,\sigma)}\Big(\frac{\sigma(\Delta_*)}{\sigma(\Delta)}\Big)^{1/p}.\end{aligned} \tag{5.1.365}$$

From this the claim in (5.1.362) follows by choosing $C_f:=2\sigma(\Delta_*)^{1/p}\cdot\|f\|_{L^\infty(\partial\Omega,\sigma)}$.

Next, given any ball B centered on $\partial\Omega$, say $B:=B(x,r)$ for some $x\in\partial\Omega$ and $r>0$, we agree to abbreviate

$$\begin{gathered}\lambda B:=B(x,\lambda r)\ \text{ for each }\ \lambda>0,\\ \Delta_B:=\Delta(x,r):=B(x,r)\cap\partial\Omega,\\ \text{and }\ C_B:=B(x,r)\cap\Omega.\end{gathered} \tag{5.1.366}$$

Fix now a ball B centered on $\partial\Omega$ and define the function

$$F_B(\lambda):=\Big(\fint_{\Delta_{\lambda B}}\big|f-f_{\Delta_{\lambda B}}\big|^p\,\mathrm{d}\sigma\Big)^{1/p},\quad \forall\,\lambda\in(0,\infty). \tag{5.1.367}$$

Then, by (5.1.362) and the Ahlfors regularity of $\partial\Omega$, there exists a finite constant $C_f > 0$ such that

$$F_B(\lambda) \le \frac{C_f}{\sigma(\Delta_{\lambda B})^{1/p}} \le \frac{C_f}{\lambda^{\frac{n-1}{p}}\sigma(\Delta_B)^{1/p}} \quad \text{for each } \lambda \in (0,\infty). \tag{5.1.368}$$

By also invoking (5.1.262), it follows that

$$\begin{aligned}
&\left(\frac{1}{\sigma(\Delta_B)}\int_{C_B}\big|\nabla\big(\mathcal{T}_{\mathrm{mod}}f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{1/p} \\
&\qquad \le C\int_1^\infty F_B(\lambda)\frac{\mathrm{d}\lambda}{\lambda^2} \le \frac{C_f}{\sigma(\Delta_B)^{1/p}}\int_1^\infty \frac{1}{\lambda^{\frac{n-1}{p}}}\cdot\frac{\mathrm{d}\lambda}{\lambda^2} \\
&\qquad = \frac{C_f}{\sigma(\Delta_B)^{1/p}}.
\end{aligned} \tag{5.1.369}$$

In particular, (5.1.369) implies (5.1.360), bearing in mind that $\partial\Omega$ is an Ahlfors regular set.

Our next goal is to show that

$$\text{for each fixed } \lambda > 0, \text{ one has } \lim_{R\to\infty}\Bigg(\sup_{\substack{B\subseteq\mathbb{R}^n\setminus B(0,R)\\ B \text{ ball centered on } \partial\Omega}} F_B(\lambda)\Bigg) = 0. \tag{5.1.370}$$

To see why this is true, fix $\lambda > 0$ and pick $\varepsilon \in (0,\infty)$ arbitrary. Let $R > 0$ and consider a ball B in $\mathbb{R}^n \setminus B(0,R)$ centered on $\partial\Omega$. If $\sigma(\Delta_B) \ge \frac{(C_f)^p}{\varepsilon^p\cdot\lambda^{n-1}}$, then by (5.1.368) we have $F_B(\lambda) \le \varepsilon$. On the other hand, if $\sigma(\Delta_B) \le \frac{(C_f)^p}{\varepsilon^p\cdot\lambda^{n-1}}$, then the radius of the ball B is bounded by a constant and since $B \subseteq \mathbb{R}^n \setminus B(0,R)$ we must have that λB is disjoint from the support of f provided $R > R_0$ for some sufficiently large R_0 (relative to λ, f, ε, n, p, and $\partial\Omega$). This forces $F_B(\lambda) = 0$ when $B \subseteq \mathbb{R}^n \setminus B(0,R)$ with $R \ge R_0$. Consequently, $\sup_{B\subseteq\mathbb{R}^n\setminus B(0,R)} F_B(\lambda) \le \varepsilon$ for $R \ge R_0$. Since $\varepsilon \in (0,\infty)$ is arbitrary, (5.1.370) follows.

Having proved (5.1.370), let us fix $R > 0$ and use (5.1.262) to write

$$\begin{aligned}
&\sup_{\substack{B\subseteq\mathbb{R}^n\setminus B(0,R)\\ B \text{ ball centered on } \partial\Omega}}\left(\frac{1}{\sigma(\Delta_B)}\int_{C_B}\big|\nabla\big(\mathcal{T}_{\mathrm{mod}}f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{1/p} \\
&\qquad \le C\cdot\sup_{\substack{B\subseteq\mathbb{R}^n\setminus B(0,R)\\ B \text{ ball centered on } \partial\Omega}}\int_1^\infty F_B(\lambda)\,\frac{\mathrm{d}\lambda}{\lambda^2} \\
&\qquad \le C\int_1^\infty\Bigg(\sup_{\substack{B\subseteq\mathbb{R}^n\setminus B(0,R)\\ B \text{ ball centered on } \partial\Omega}} F_B(\lambda)\Bigg)\frac{\mathrm{d}\lambda}{\lambda^2}.
\end{aligned} \tag{5.1.371}$$

It is clear from (5.1.367) and (5.1.365) that for each $\lambda > 0$,

$$\sup_{R>0}\left(\sup_{\substack{B\subseteq\mathbb{R}^n\setminus B(0,R)\\ B\text{ ball centered on }\partial\Omega}} F_B(\lambda)\right)\leq 2\|f\|_{L^\infty(\partial\Omega,\sigma)}, \tag{5.1.372}$$

and we note that the constant $2\|f\|_{L^\infty(\partial\Omega,\sigma)}$ is absolutely integrable on $(1,\infty)$ with respect to the measure $\frac{d\lambda}{\lambda^2}$. The latter, (5.1.372), (5.1.370), and Lebesgue's Dominated Convergence Theorem imply

$$\lim_{R\to\infty}\int_1^\infty\left(\sup_{\substack{B\subseteq\mathbb{R}^n\setminus B(0,R)\\ B\text{ ball centered on }\partial\Omega}} F_B(\lambda)\right)\frac{d\lambda}{\lambda^2}=0. \tag{5.1.373}$$

In turn, (5.1.373) may be used in connection with (5.1.371) to conclude that (5.1.361) holds. □

Before stating our final result in this section, the reader is advised to recall the notion of super vanishing Carleson measure we have introduced in Definition 1.8.1.

Theorem 5.1.22 *Pick $n\in\mathbb{N}$ with $n\geq 2$, and suppose $\Omega\subseteq\mathbb{R}^n$ is a* UR *domain with $\partial\Omega$ an unbounded set. Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. For a sufficiently large $N=N(n)\in\mathbb{N}$, consider a vector-valued function*

$$\vec{k}\in\big[\mathscr{C}^N(\mathbb{R}^n\setminus\{0\})\big]^n \text{ odd, positive homogeneous of degree } 1-n, \\ \text{and satisfying } \operatorname{div}\vec{k}=0 \text{ in } \mathbb{R}^n\setminus\{0\}. \tag{5.1.374}$$

With the set Ω and the kernel $\vec{k}$, associate as in (5.1.44) the modified generalized double layer operator $\mathcal{T}_{\rm mod}$, i.e., the mapping sending each $f\in L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\big)$ into the function defined at each point $x\in\Omega$ according to

$$\big(\mathcal{T}_{\rm mod}f\big)(x):=\int_{\partial\Omega}\Big\langle\nu(y),\,\vec{k}(x-y)-\vec{k}(-y)\cdot\mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,1)}}(y)\Big\rangle f(y)\,d\sigma(y). \tag{5.1.375}$$

Then for each $p\in(1,\infty)$ there exists a constant $C\in(0,\infty)$, which depends only on n, p, $\vec{k}$, and the UR *constants of $\partial\Omega$, with the property that for each function $f\in\mathrm{BMO}(\partial\Omega,\sigma)$ one has*

$$\max\Bigg\{\lim_{R\to\infty}\sup_{\substack{x\in\partial\Omega,\, r>0\\ B(x,r)\subseteq\mathbb{R}^n\setminus B(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times$$

$$\times\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{1/p},$$

$$\lim_{R\to 0^+}\sup_{x\in\partial\Omega,\, r\in(0,R)}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times$$

$$\times\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{1/p},$$

$$\lim_{R\to\infty}\sup_{x\in\partial\Omega,\, r>R}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times$$

$$\times\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{1/p}\Bigg\}$$

$$\leq C\operatorname{dist}\big(f,\,\mathrm{CMO}(\partial\Omega,\sigma)\big), \tag{5.1.376}$$

where the distance in the right-hand side is considered in BMO$(\partial\Omega,\sigma)$*. As a consequence of* (5.1.376) *and Definition 1.8.1,*

$$\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\ \textit{ is a super vanishing Carleson measure in }\Omega,$$
$$\textit{for each function } f\in\mathrm{CMO}(\partial\Omega,\sigma)\textit{ and each } p\in(1,\infty). \tag{5.1.377}$$

Proof Fix an exponent $p\in(1,\infty)$ and a function $f\in\mathrm{BMO}(\partial\Omega,\sigma)$. Then, for each $g\in\mathrm{Lip}_c(\partial\Omega)$ we have $f-g\in\mathrm{BMO}(\partial\Omega,\sigma)$ and, as already proved in (5.1.227),

$$\lim_{R\to 0^+}\sup_{x\in\partial\Omega,\ r\in(0,R)}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{\frac{1}{p}}$$

$$\leq C\|f-g\|_{\mathrm{BMO}(\partial\Omega,\sigma)}, \tag{5.1.378}$$

for some constant $C\in(0,\infty)$ independent of f and g. On the other hand, the triangle inequality and estimate (5.1.74) show that there exists a constant $C\in(0,\infty)$, independent of f and g, with the property that for each $x\in\partial\Omega$ and $r>0$ we have

$$\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{1/p}$$

$$\leq\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}g\big)\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\Big)^{1/p}$$

$$+C\|f-g\|_{\mathrm{BMO}(\partial\Omega,\sigma)}. \tag{5.1.379}$$

Taking now the supremum over $x \in \partial\Omega$ and $r > 0$ such that $B(x, r) \subseteq \mathbb{R}^n \setminus B(0, R)$ of both sides of (5.1.379) and then passing to limit as $R \to \infty$ in the resulting inequality yields, on account of (5.1.361),

$$\lim_{R\to\infty} \sup_{\substack{x\in\partial\Omega,\, r>0 \\ B(x,r)\subseteq\mathbb{R}^n\setminus B(0,R)}} \left(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1}\, \mathrm{d}\mathcal{L}^n\right)^{1/p}$$
$$\leq C\|f - g\|_{\mathrm{BMO}(\partial\Omega,\sigma)}. \tag{5.1.380}$$

Likewise, taking the supremum over $x \in \partial\Omega$ and $r > R$ of both sides of (5.1.379), and passing to limit as $R \to \infty$ in the resulting inequality we obtain, this time relying on (5.1.360),

$$\lim_{R\to\infty} \sup_{x\in\partial\Omega,\, r>R} \left(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1}\, \mathrm{d}\mathcal{L}^n\right)^{1/p}$$
$$\leq C\|f - g\|_{\mathrm{BMO}(\partial\Omega,\sigma)}. \tag{5.1.381}$$

Now (5.1.376) follows by combining (5.1.378), (5.1.380), (5.1.381) and the fact that the space $\mathrm{CMO}(\partial\Omega, \sigma)$ is the closure of $\mathrm{Lip}_c(\partial\Omega)$ in $\mathrm{BMO}(\partial\Omega, \sigma)$ (cf. [69, (4.6.13)]). Finally, (5.1.377) is clear from (5.1.376) and Definition 1.8.1, so the proof of Theorem 5.1.22 is complete. □

5.2 Generalized Double Layers with Matrix-Valued Kernels, and Chord-Dot-Normal SIO's

Our first result in this section may be regarded as a version of Theorem 5.1.1 for *matrix-valued* kernels.

Theorem 5.2.1 *Fix a dimension $n \in \mathbb{N}$ with $n \geq 2$, and suppose $\Omega \subseteq \mathbb{R}^n$ is an arbitrary* UR *domain*[18]*. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Let D be a homogeneous first-order $M \times M'$ system* (*where $M, M' \in \mathbb{N}$ are arbitrary*) *with constant complex coefficients*

$$D = \Big(\sum_{j=1}^{n} a_j^{\alpha\beta}\partial_j\Big)_{\substack{1\leq\alpha\leq M \\ 1\leq\beta\leq M'}} \tag{5.2.1}$$

(*with the summation convention over repeated indices in effect*) *and recall that its* (*principal*) *symbol is the $M \times M'$ matrix-valued function*

$$\mathrm{Sym}(D; \xi) := \mathrm{i}\Big(a_j^{\alpha\beta}\xi_j\Big)_{\substack{1\leq\alpha\leq M \\ 1\leq\beta\leq M'}} \quad \textit{for each } \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n. \tag{5.2.2}$$

[18] When $\Omega \subseteq \mathbb{R}^n$ is just an open set with a UR boundary, a large number of conclusions are still valid, with at most minor alterations (namely, eventually replacing $\partial\Omega$ with $\partial_*\Omega$).

Having fixed some $M'' \in \mathbb{N}$ together with a sufficiently large integer $N = N(n) \in \mathbb{N}$, consider a matrix-valued function

$$\Theta = \big(\Theta_{\beta\gamma}\big)_{\substack{1\leq\beta\leq M'\\ 1\leq\gamma\leq M''}} \in \big[\mathscr{C}^N(\mathbb{R}^n \setminus \{0\})\big]^{M'\times M''} \tag{5.2.3}$$

which is odd, positive homogeneous of degree $1-n$ *in* $\mathbb{R}^n \setminus \{0\}$, *and satisfies* (*with* D *acting on the columns of* Θ)

$$D\,\Theta = 0 \cdot I_{M\times M''} \text{ in } \mathbb{R}^n \setminus \{0\}. \tag{5.2.4}$$

Finally, set

$$\vartheta := \int_{S^{n-1}} (-\mathrm{i})\mathrm{Sym}(D;\omega)\Theta(\omega)\,\mathrm{d}\mathcal{H}^{n-1}(\omega) \in \mathbb{C}^{M\times M''}. \tag{5.2.5}$$

In this setting, for each vector-valued function $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\Big]^M$ *define*

$$\mathcal{T} f(x) := \int_{\partial\Omega} \Big[(-\mathrm{i})\mathrm{Sym}(D;\nu(y))\Theta(x-y)\Big]^{\top} f(y)\,\mathrm{d}\sigma(y) \text{ for all } x \in \Omega, \tag{5.2.6}$$

and

$$Tf(x) := \lim_{\varepsilon\to0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \Big[(-\mathrm{i})\mathrm{Sym}(D;\nu(y))\Theta(x-y)\Big]^{\top} f(y)\,\mathrm{d}\sigma(y) \tag{5.2.7}$$

for σ*-a.e.* $x \in \partial\Omega$, *where the superscript* $\top$ *indicates* (*real*) *transposition. In addition, for each given vector-valued function* $g \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\Big]^{M''}$ *consider*

$$T^{\#} f(x) := \lim_{\varepsilon\to0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} (-\mathrm{i})\mathrm{Sym}(D;\nu(x))\Theta(y-x)g(y)\,\mathrm{d}\sigma(y) \tag{5.2.8}$$

for σ*-a.e.* $x \in \partial\Omega$.

Then, for each fixed aperture parameter $\kappa > 0$, *the following statements are true.*

(1) For each function $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M$ *one has*

$$\Big(\mathcal{T} f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = -\frac{1}{2}\vartheta^{\top} f(x) + (Tf)(x) \text{ at } \sigma\text{-a.e. } x \in \partial\Omega. \tag{5.2.9}$$

Also, if for each $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^{M''}$ *one defines*

$$\mathcal{W}f(x) := \int_{\partial\Omega} \Theta(x-y)f(y)\,d\sigma(y) \text{ for all } x \in \Omega, \tag{5.2.10}$$

then for each function $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^{M''}$ *one has*

$$\mathcal{W}f \in \big[\mathscr{C}^N(\Omega)\big]^{M'} \text{ and } D\big(\mathcal{W}f\big) = 0 \in \mathbb{C}^M \text{ in } \Omega. \tag{5.2.11}$$

Moreover, for each $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^{M''}$ *the nontangential boundary trace*

$$(\mathcal{W}f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists (in } \mathbb{C}^{M'}) \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{5.2.12}$$

and at σ*-a.e. point* $x \in \partial\Omega$ *one has*

$$(-\mathrm{i})\mathrm{Sym}\big(D;\nu(x)\big)\Big(\mathcal{W}f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = -\frac{1}{2}\vartheta f(x) - (T^{\#}f)(x). \tag{5.2.13}$$

Finally, if $\partial\Omega$ *is compact then for each* $\lambda \in \mathbb{C}^M$ *both* $\mathcal{T}\lambda$ *and* $T\lambda$ *are constant functions and, in fact, at each point in* Ω *we have*

$$\mathcal{T}\lambda = \begin{cases} -\vartheta^{\top}\lambda & \text{if } \Omega \text{ is bounded,} \\ 0 \in \mathbb{C}^{M''} & \text{if } \Omega \text{ is an exterior domain}^{19}, \end{cases} \tag{5.2.14}$$

while at σ*-a.e. point on* $\partial\Omega$ *we have*

$$T\lambda = \begin{cases} -\frac{1}{2}\vartheta^{\top}\lambda & \text{if } \Omega \text{ is bounded,} \\ +\frac{1}{2}\vartheta^{\top}\lambda & \text{if } \Omega \text{ is an exterior domain.} \end{cases} \tag{5.2.15}$$

(2) For each $p \in [1,\infty)$ *there exists some finite constant* $C > 0$*, depending only on* $\partial\Omega$*,* Θ*,* D*,* n*,* p*, and* κ*, such that for each function* $f \in \big[L^p(\partial\Omega,\sigma)\big]^{M''}$ *one has*

$$\max\Big\{\big\|\mathcal{N}_\kappa(\mathcal{T}f)\big\|_{L^p(\partial\Omega,\sigma)}, \big\|\mathcal{N}_\kappa(\mathcal{W}f)\big\|_{L^p(\partial\Omega,\sigma)}\Big\} \leq C\|f\|_{[L^p(\partial\Omega,\sigma)]^{M''}} \tag{5.2.16}$$

if $p > 1$*, plus a similar estimate when* $p = 1$ *in which case the corresponding* L^1*-norms in the left-hand side are now replaced by the quasi-norm in the space* $\big[L^{1,\infty}(\partial\Omega,\sigma)\big]^{M''}$.

Moreover, the action of the operator $\mathcal{W}$*, originally considered as in* (5.2.10)*, may be further extended in a unique and coherent fashion* (*cf.* [70, (2.4.15), (2.4.16), (2.4.24)]) *to the scale of Lorentz-based Hardy spaces* $\big[H^{p,q}(\partial\Omega,\sigma)\big]^{M''}$ *with* $p \in \big(\frac{n-1}{n},\infty\big)$ *and* $q \in (0,\infty]$ *and for each* $\kappa > 0$ *said extension satisfies* (*for some constant* $C = C(\partial\Omega,\Theta,D,n,p,q,\kappa) \in (0,\infty)$)

$$\begin{gathered} \big\|\mathcal{N}_\kappa(\mathcal{W}f)\big\|_{L^{p,q}(\partial\Omega,\sigma)} \leq C\|f\|_{[H^{p,q}(\partial\Omega,\sigma)]^{M''}} \\ \text{for all } f \in \big[H^{p,q}(\partial\Omega,\sigma)\big]^{M''}. \end{gathered} \tag{5.2.17}$$

(3) For each $p \in (1, \infty)$*, the operators*

$$T : \big[L^p(\partial\Omega, \sigma)\big]^M \to \big[L^p(\partial\Omega, \sigma)\big]^{M''}, \tag{5.2.18}$$

$$T^\# : \big[L^p(\partial\Omega, \sigma)\big]^{M''} \to \big[L^p(\partial\Omega, \sigma)\big]^M, \tag{5.2.19}$$

are well defined, linear, and bounded. Also, given any $p, p' \in (1, \infty)$ *with* $1/p + 1/p' = 1$ *it follows that*

$$\begin{aligned} &\text{the transpose of } T : \big[L^p(\partial\Omega, \sigma)\big]^M \to \big[L^p(\partial\Omega, \sigma)\big]^{M''} \\ &\text{is the operator } T^\# : \big[L^{p'}(\partial\Omega, \sigma)\big]^{M''} \to \big[L^{p'}(\partial\Omega, \sigma)\big]^M. \end{aligned} \tag{5.2.20}$$

Corresponding to the end-point $p = 1$*, both operators* T *and* $T^\#$ *map vector-valued functions with components in* $L^1(\partial\Omega, \sigma)$ *linearly and continuously into vector-valued functions with components in* $L^{1,\infty}(\partial\Omega, \sigma)$*. Next, if for each vector-valued function* $\vec{g} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\Big]^n$ *one considers*

$$V\vec{g}(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} (-\mathrm{i})\mathrm{Sym}(D; \vec{g}(y))\Theta(x-y)\,\mathrm{d}\sigma(y) \tag{5.2.21}$$

for σ*-a.e.* $x \in \partial\Omega$*, then for each* $p \in (1, \infty)$ *the following operator is well defined, linear, and bounded:*

$$V : \big[L^p(\partial\Omega, \sigma)\big]^n \longrightarrow \big[L^p(\partial\Omega, \sigma)\big]^{M\times M''}. \tag{5.2.22}$$

Finally, similar results are valid for Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, and Morrey spaces (as well as their duals and their preduals) on $\partial\Omega$.

(4) Fix $p \in \big(\frac{n-1}{n}, 1\big]$*. Then the operator* $T^\#$*, originally considered as in (5.2.19), extends to a linear and bounded mapping on Hardy spaces, i.e.,*

$$T^\# : \big[H^p(\partial\Omega, \sigma)\big]^{M''} \longrightarrow \big[H^p(\partial\Omega, \sigma)\big]^M \tag{5.2.23}$$

and the operators corresponding to various choices of the index $p \in \big(\frac{n-1}{n}, 1\big]$ *are compatible with one another. In fact,* $T^\#$ *maps the scale of Lorentz-based Hardy spaces on* $\partial\Omega$ *continuously into itself, specifically*

$$\begin{aligned} &T^\# : \big[H^{p,q}(\partial\Omega, \sigma)\big]^{M''} \longrightarrow \big[H^{p,q}(\partial\Omega, \sigma)\big]^M \text{ is well defined,} \\ &\qquad \text{linear, and bounded for } p \in \big(\tfrac{n-1}{n}, \infty\big) \text{ and } q \in (0, \infty]. \end{aligned} \tag{5.2.24}$$

In addition, for each $f \in \big[H^{p,q}(\partial\Omega, \sigma)\big]^{M''}$ *with* $p \in \big(\frac{n-1}{n}, \infty\big)$ *and* $q \in (0, \infty]$ *one has*

$$(-\mathrm{i})\mathrm{Sym}(D;\nu) \bullet \mathcal{W} f = -\frac{1}{2}\vartheta f - T^{\#} f, \tag{5.2.25}$$

where the "bullet product" is defined as in [69, (10.2.94) in Proposition 10.2.11]. *Finally, for each function* $f \in H^1(\partial\Omega, \sigma)$ *one has*

$$f \in \big[L^1(\partial\Omega, \sigma)\big]^{M''}, \qquad T^{\#} f \in \big[L^1(\partial\Omega, \sigma)\big]^{M}, \tag{5.2.26}$$

and, with all integrals involved absolutely convergent,

$$\int_{\partial\Omega} T^{\#} f \,\mathrm{d}\sigma = \begin{cases} -\dfrac{1}{2}\displaystyle\int_{\partial\Omega} \vartheta f \,\mathrm{d}\sigma & \text{if } \Omega \text{ is bounded,} \\ +\dfrac{1}{2}\displaystyle\int_{\partial\Omega} \vartheta f \,\mathrm{d}\sigma & \text{if } \Omega \text{ is unbounded and } \partial\Omega \text{ is bounded,} \\ \quad 0 \in \mathbb{C}^M & \text{if } \partial\Omega \text{ is unbounded.} \end{cases} \tag{5.2.27}$$

(5) *Pick* $q \in (1, \infty)$ *and* $\lambda \in (0, n-1)$ *and recall* $\mathscr{H}^{q,\lambda}(\partial\Omega, \sigma)$, *the pre-dual to the Morrey-Campanato space, defined as in* (A.0.84) (*with* $\Sigma := \partial\Omega$). *Then the operator*

$$T^{\#} : \big[L^r(\partial\Omega, \sigma)\big]^{M''} \longrightarrow \big[L^r(\partial\Omega, \sigma)\big]^{M} \ \text{ with } r := \frac{q(n-1)}{n-1+\lambda(q-1)} \tag{5.2.28}$$

induces a well-defined, linear, and bounded mapping in the context

$$T^{\#} : \big[\mathscr{H}^{q,\lambda}(\partial\Omega, \sigma)\big]^{M''} \longrightarrow \big[\mathscr{H}^{q,\lambda}(\partial\Omega, \sigma)\big]^{M}. \tag{5.2.29}$$

(6) *For each given function* f *in the space* $\Big[L^1_1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^M$, *the vector version of the weighted boundary Sobolev space defined in* (A.0.131), *each index* $\ell \in \{1, \dots, n\}$, *and each aperture parameter* $\kappa > 0$, *the pointwise nontangential boundary trace*

$$\big(\partial_\ell \mathcal{T} f\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \text{ exists (in } \mathbb{C}^{M''}\text{) at } \sigma\text{-a.e. point on } \partial\Omega. \tag{5.2.30}$$

As a consequence of [69, (11.7.22)], *this is true whenever* $f \in \big[L^{p,q}_1(\partial\Omega, \sigma)\big]^{M''}$ *with* $p, q \in [1, \infty)$.
Furthermore, for each $p, q \in [1, \infty)$ *and* $\kappa > 0$ *there exists some finite constant* $C > 0$, *depending only on* $\partial\Omega$, Θ, D, n, p, q, *and* κ, *such that for each function* $f \in \big[L^{p,q}_1(\partial\Omega, \sigma)\big]^{M''}$ *one has*

$$\big\|\mathcal{N}_\kappa(\mathcal{T} f)\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla \mathcal{T} f)\big\|_{L^q(\partial\Omega,\sigma)} \leq C\|f\|_{[L^{p,q}_1(\partial\Omega,\sigma)]^{M''}} \tag{5.2.31}$$

if $p, q > 1$, plus similar estimates in the case when $p = 1$ or $q = 1$, in which scenario the corresponding L^1-norms in the left-hand side are now replaced by the quasi-norm $L^{1,\infty}(\partial\Omega, \sigma)$.

(7) *The operator*

$$T : \big[L_1^{p,q}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L_1^{p,q}(\partial\Omega, \sigma)\big]^{M''} \tag{5.2.32}$$

is well defined, linear, and bounded for each $p, q \in (1, \infty)$. In particular, the operator

$$T : \big[L_1^{p}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[L_1^{p}(\partial\Omega, \sigma)\big]^{M''} \tag{5.2.33}$$

is well defined, linear, and bounded for each $p \in (1, \infty)$. Actually,

$$T : \big[L_1^{p}(\partial\Omega, w)\big]^M \longrightarrow \big[L_1^{p}(\partial\Omega, w)\big]^{M''} \text{ is well defined, linear, bounded}$$
$$\text{for each exponent } p \in (1, \infty) \text{ and each weight } w \in A_p(\partial\Omega, \sigma). \tag{5.2.34}$$

Moreover, for each function

$$f \in \Big[L_1^1\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^{n-1}}\Big) \cap L^p_{1,\mathrm{loc}}(\partial\Omega, \sigma)\Big]^M \text{ for some } p \in (1, \infty) \tag{5.2.35}$$

it follows that

$$Tf \in \big[L^p_{1,\mathrm{loc}}(\partial\Omega, \sigma)\big]^{M''} \tag{5.2.36}$$

and for each pair of indices $r, s \in \{1, \dots, n\}$ at σ-a.e. point on $\partial\Omega$ one has (*with V as in* (5.2.21))

$$\begin{aligned} \partial_{\tau_{rs}}(Tf) = T\big(\partial_{\tau_{rs}} f\big) &+ \big[M_{\nu_r}, T\big](\nabla_{\tan} f)_s - \big[M_{\nu_s}, T\big](\nabla_{\tan} f)_r \\ &+ \big[M_{\nu_r}, V\big]^\top(\nu_s \nabla_{\tan} f) - \big[M_{\nu_s}, V\big]^\top(\nu_r \nabla_{\tan} f). \end{aligned} \tag{5.2.37}$$

In particular, formula (5.2.37) *holds for every function $f \in \big[L_1^{p,q}(\partial\Omega, \sigma)\big]^M$ with $p, q \in (1, \infty)$, as well as for every function $f \in \big[L_1^p(\partial\Omega, w)\big]^M$ with p in $(1, \infty)$ and $w \in A_p(\partial\Omega, \sigma)$. Formula* (5.2.37) *also holds for each function in $\big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M$ or $\big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M$ with $p, q \in (1, \infty)$ and $\lambda \in (0, n-1)$* (*cf.* (A.0.150), (A.0.33)). *In fact,*

$$T : \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[M_1^{p,\lambda}(\partial\Omega, \sigma)\big]^{M''}, \tag{5.2.38}$$

$$T : \big[\mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma)\big]^{M''}, \tag{5.2.39}$$

$$T : \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^M \longrightarrow \big[\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)\big]^{M''}, \tag{5.2.40}$$

are all well-defined, linear, and continuous operators whenever $p, q \in (1, \infty)$ *and* $\lambda \in (0, n-1)$ *(cf.* (A.0.150)*,* (A.0.155)*,* (A.0.33)*).*

(8) For each $p \in (1, \infty)$ *it follows that* $T^{\#}$*, originally acting as in* (5.2.19)*, further extends uniquely to a linear, bounded operator on negative boundary Sobolev spaces:*

$$T^{\#} : \big[L^p_{-1}(\partial\Omega, \sigma)\big]^{M''} \to \big[L^p_{-1}(\partial\Omega, \sigma)\big]^{M}. \tag{5.2.41}$$

Furthermore, if one retains the same notation $T^{\#}$ *for said extension, then the transpose of* (5.2.33) *is*

$$T^{\#} : \big[L^{p'}_{-1}(\partial\Omega, \sigma)\big]^{M''} \longrightarrow \big[L^{p'}_{-1}(\partial\Omega, \sigma)\big]^{M} \tag{5.2.42}$$

where $p' \in (1, \infty)$ *is such that* $1/p + 1/p' = 1$*.*
More generally, for each $p, q \in (1, \infty)$ *it follows that* $T^{\#}$ *acting as in* (5.2.19) *further extends uniquely to a linear and bounded operator from the negative off-diagonal boundary Sobolev space* $\big[L^{p,q}_{-1}(\partial\Omega, \sigma)\big]^{M''}$ *into* $\big[L^{p,q}_{-1}(\partial\Omega, \sigma)\big]^{M}$ *and, adopting the same notation* $T^{\#}$ *for this extension, the transpose of* (5.2.32) *is*

$$T^{\#} : \big[L^{p',q'}_{-1}(\partial\Omega, \sigma)\big]^{M''} \longrightarrow \big[L^{p',q'}_{-1}(\partial\Omega, \sigma)\big]^{M} \tag{5.2.43}$$

where $p', q' \in (1, \infty)$ *are such that* $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$*.*
Also, for each integrability exponent $p \in (1, \infty)$ *and each Muckenhoupt weight* $w \in A_p(\partial\Omega, \sigma)$*, it follows that* $T^{\#}$*, originally acting from* $\big[L^p(\partial\Omega, w)\big]^{M''}$ *into* $\big[L^p(\partial\Omega, w)\big]^{M}$*, further extends uniquely to a linear, bounded operator, from the negative boundary Sobolev space* $\big[L^p_{-1}(\partial\Omega, w)\big]^{M''}$ *into* $\big[L^p_{-1}(\partial\Omega, w)\big]^{M}$ *which, in fact, is the transpose of* T *acting from* $\big[L^{p'}_1(\partial\Omega, w')\big]^{M''}$ *into* $\big[L^{p'}_1(\partial\Omega, w')\big]^{M}$ *where* $p' := (1 - 1/p)^{-1} \in (1, \infty)$ *is the Hölder conjugate exponent of* p *and* $w' := w^{1-p'} \in A_{p'}(\partial\Omega, \sigma)$ *is the conjugate weight of* w *(cf.* [68, item *(2)* in Lemma 7.7.1]*).*

(9) Consider the following modified version of the generalized double layer potential operator in (5.2.6)*, acting on each function* $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ *according to*

$$\big(\mathcal{T}_{\rm mod} f\big)(x) \tag{5.2.44}$$
$$:= \int_{\partial\Omega} \Big[(-\mathrm{i})\mathrm{Sym}(D; \nu(y))\big\{\Theta(x-y) - \Theta(-y)\cdot \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,1)}}(y)\big\}\Big]^{\top} f(y)\,\mathrm{d}\sigma(y)$$

for all $x \in \Omega$*. Then the operator* $\mathcal{T}_{\rm mod}$ *is meaningfully defined, and is compatible with* $\mathcal{T}$ *from* (5.2.6) *in the sense that for each* f *belonging to the smaller space* $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ *(hence, in particular, for each function* $f \in \big[L^p(\partial\Omega, \sigma)\big]^M$

with $p \in [1,\infty)$) *the difference*

$$C_f := \mathcal{T}_{\rm mod} f - \mathcal{T} f \ \text{ is a constant (belonging to } \mathbb{C}^{M''}\text{) in } \Omega. \tag{5.2.45}$$

As a consequence,

$$\nabla \mathcal{T}_{\rm mod} f = \nabla \mathcal{T} f \text{ in } \Omega \text{ for each } f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M. \tag{5.2.46}$$

Moreover,

$$\begin{gathered} \mathcal{T}_{\rm mod} \text{ maps } (\mathbb{C}^M\text{-valued) constant functions on } \partial\Omega \\ \text{into } (\mathbb{C}^{M''}\text{-valued) constant functions in } \Omega. \end{gathered} \tag{5.2.47}$$

In addition, at each point $x \in \Omega$ *one may express*

$$\begin{gathered} \partial_\ell\big(\mathcal{T}_{\rm mod} f\big)(x) = \int_{\partial\Omega} \Big[(-\mathrm{i})\mathrm{Sym}(D;\nu(y))(\partial_\ell\Theta)(x-y)\Big]^\top f(y)\,\mathrm{d}\sigma(y) \\ \text{for each } \ell \in \{1,\dots,n\} \text{ and } f \in \big[L^1\big(\partial\Omega, \tfrac{\sigma(y)}{1+|y|^n}\big)\big]^M. \end{gathered} \tag{5.2.48}$$

Also,

$$\begin{gathered} \big(\partial_\ell \mathcal{T}_{\rm mod} f\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists (in } \mathbb{C}^{M''}\text{) at } \sigma\text{-a.e. point on } \partial\Omega, \\ \text{for each } \ell \in \{1,\dots,n\} \text{ and each } f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M \text{ satisfying} \\ \partial_{\tau_{jk}} f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M \text{ for all } j,k \in \{1,\dots,n\}. \end{gathered} \tag{5.2.49}$$

Furthermore, for each truncation parameter $\varepsilon \in (0,\infty)$ *and each integrability exponent* $p \in (1,\infty)$ *it follows that*

$$\begin{gathered} \mathcal{N}_\kappa^\varepsilon\big(\nabla(\mathcal{T}_{\rm mod} f)\big) \in L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for each function} \\ f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M \text{ with the property that} \\ \partial_{\tau_{jk}} f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma)\Big]^M \text{ for all } j,k \in \{1,\dots,n\}, \end{gathered} \tag{5.2.50}$$

and

$$\begin{gathered} \mathcal{N}_\kappa^\varepsilon(\mathcal{T}_{\rm mod} f) \in L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for each function} \\ f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma)\Big]^M. \end{gathered} \tag{5.2.51}$$

Finally, for each $p \in \left(\frac{n-1}{n}, 1\right]$ *and each aperture parameter* $\kappa \in (0, \infty)$ *there exists some finite constant* $C > 0$*, depending only on* $\partial\Omega$*,* n*,* Θ*,* κ*, and* p*, with the property that for each distribution* $f \in \left[\dot{H}^p_1(\partial\Omega, \sigma)\right]^M$ *one has*

$$\left\| \mathcal{N}_\kappa(\nabla \mathcal{T}_{\text{mod}} f)\right\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^M}. \tag{5.2.52}$$

(10) For each $\alpha \in (0, 1)$ *there exists a constant* $C \in (0, \infty)$ *with the property that*

$$\sup_{x\in\Omega} \left\{ \operatorname{dist}(x, \partial\Omega)^{1-\alpha} \left|\nabla\big(\mathcal{T}_{\text{mod}} f\big)(x)\right|\right\} \leq C\|f\|_{[\dot{\mathscr{C}}^\alpha(\partial\Omega)]^M} \tag{5.2.53}$$

for every function $f \in \left[\dot{\mathscr{C}}^\alpha(\partial\Omega)\right]^M$*. Moreover,*

> *if* $\Omega \subseteq \mathbb{R}^n$ *is a uniform domain with the property that* $\partial\Omega$ *is an Ahlfors regular set then* $\mathcal{T}_{\text{mod}} : \left[\dot{\mathscr{C}}^\alpha(\partial\Omega)\right]^M \to \left[\dot{\mathscr{C}}^\alpha\big(\overline{\Omega}\big)\right]^{M''}$ *is a well-defined, linear, and bounded operator for each given exponent* $\alpha \in (0, 1)$*,* (5.2.54)

whereas

> *if* $\Omega \subseteq \mathbb{R}^n$ *is an* NTA *domain with an upper Ahlfors regular boundary then* $\mathcal{T}_{\text{mod}} : \left[\dot{\mathscr{C}}^\alpha_{\text{van}}(\partial\Omega)\right]^M \to \left[\dot{\mathscr{C}}^\alpha_{\text{van}}\big(\overline{\Omega}\big)\right]^{M''}$ *is a well-defined, linear, and bounded operator for each given exponent* $\alpha \in (0, 1)$*,* (5.2.55)

with the vanishing Hölder spaces defined as in (A.0.48) (*with* $\Sigma := \partial\Omega$ *and* $\Sigma := \overline{\Omega}$*, respectively*)*. Also, for each* $\alpha \in (0, 1)$ *and each* $p \in (1, \infty)$ *there exists some* $C \in (0, \infty)$ *with the property that for each function* $f \in \left[\dot{\mathscr{C}}^\alpha(\partial\Omega)\right]^M$ *one has*

$$\sup_{\substack{x\in\partial\Omega\\ r\in(0,\infty)}} \left(\frac{1}{r^{n-1+\alpha p}} \int_{B(x,r)\cap\Omega} \left|\nabla\mathcal{T}_{\text{mod}} f\right|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1}\, d\mathcal{L}^n\right)^{1/p} \leq C\|f\|_{[\dot{\mathscr{C}}^\alpha(\partial\Omega)]^M} \tag{5.2.56}$$

and

$$\lim_{R\to 0^+} \left\{ \sup_{\substack{x\in\partial\Omega\\ r\in(0,R)}} \left(\frac{1}{r^{n-1+\alpha p}} \int_{B(x,r)\cap\Omega} \left|\nabla\mathcal{T}_{\text{mod}} f\right|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1}\, d\mathcal{L}^n\right)^{1/p}\right\} \leq C \operatorname{dist}\big(f, \left[\dot{\mathscr{C}}^\alpha_{\text{van}}(\partial\Omega)\right]^M\big), \tag{5.2.57}$$

where the distance is measured in the space $\Big(\left[\dot{\mathscr{C}}^\alpha(\partial\Omega)\right]^M, \|\cdot\|_{[\dot{\mathscr{C}}^\alpha(\partial\Omega)]^M}\Big)$*. As a corollary, if the function* f *actually belongs to the homogeneous vanishing Hölder space* $\left[\dot{\mathscr{C}}^\alpha_{\text{van}}(\partial\Omega)\right]^M$ *for some* $\alpha\in(0, 1)$*, then for each* $p\in(1, \infty)$ *one has*

$$\lim_{R\to 0^+}\left\{\sup_{\substack{x\in\partial\Omega\\ r\in(0,R)}}\left(\frac{1}{r^{n-1+\alpha p}}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{T}_{\rm mod}f\big|^p\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\right)^{1/p}\right\}=0. \tag{5.2.58}$$

(11) Let $T_{\rm mod}$ be the modified version of the singular integral operator (5.2.7), acting on each function $f\in\big[L^1\big(\partial\Omega,\frac{\sigma(y)}{1+|y|^n}\big)\big]^M$ according to[20]

$$T_{\rm mod}f(x):=\lim_{\varepsilon\to 0^+}\int_{\partial\Omega}\Big[(-i)\mathrm{Sym}(D;\nu(y))\big\{\Theta_\varepsilon(x-y)-\Theta_1(-y)\big\}\Big]^\top f(y)\,d\sigma(y) \tag{5.2.59}$$

at σ-a.e. $x\in\partial\Omega$, where

$$\Theta_\varepsilon:=\Theta\cdot\mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,\varepsilon)}}\ \text{ for each }\ \varepsilon>0. \tag{5.2.60}$$

Then for each function $f\in\big[L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ one has the jump-formula

$$\mathcal{T}_{\rm mod}f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}=-\frac{1}{2}\vartheta^\top f+T_{\rm mod}f\ \text{ at }\sigma\text{-a.e. point on }\ \partial\Omega. \tag{5.2.61}$$

In particular, (5.2.61) holds for each $f\in\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$ with $1<p<\infty$. As a consequence of (5.2.61) and (5.2.47),

$$\begin{gathered}T_{\rm mod}\text{ maps }(\mathbb{C}^M\text{-valued) constant functions on }\partial\Omega\\ \text{into }(\mathbb{C}^{M''}\text{-valued) constant functions on }\partial\Omega.\end{gathered} \tag{5.2.62}$$

Moreover,

$$\begin{gathered}\text{if } p,p'\in(1,\infty)\text{ are such that } 1/p+1/p'=1 \text{ then given}\\ \text{any functions } f\in\Big[L^p_{\rm loc}(\partial\Omega,\sigma)\cap L^1\Big(\partial\Omega,\tfrac{\sigma(x)}{1+|x|^n}\Big)\Big]^M \text{ and}\\ g\in\big[L^{p'}_{\rm comp}(\partial\Omega,\sigma)\big]^{M''}\text{ satisfying }\int_{\partial\Omega}g\,d\sigma=0\in\mathbb{C}^{M''},\\ \text{one has }\int_{\partial\Omega}|T_{\rm mod}f||g|\,d\sigma<+\infty,\quad\int_{\partial\Omega}|f||T^\# g|\,d\sigma<+\infty,\\ \text{as well as }\int_{\partial\Omega}\langle T_{\rm mod}f,g\rangle\,d\sigma=\int_{\partial\Omega}\langle f,T^\# g\rangle\,d\sigma.\end{gathered} \tag{5.2.63}$$

Lastly, given any $p\in(1,\infty)$, for each function

[20] alternative descriptions of $T_{\rm mod}$ are seen from [70, Proposition 2.3.3]

$$f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\rm loc}(\partial\Omega,\sigma)\Big]^M \ \textit{such that}$$
$$\partial_{\tau_{jk}} f \in \Big[L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big) \cap L^p_{\rm loc}(\partial\Omega,\sigma)\Big]^M \ \textit{for all}\ j,k\in\{1,\dots,n\} \tag{5.2.64}$$

(hence for each $f \in \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$ *if* Ω *also satisfies a two-sided local John condition) it follows that*

$$T_{\rm mod} f \in \big[L^p_{1,{\rm loc}}(\partial\Omega,\sigma)\big]^{M''} \tag{5.2.65}$$

and for each pair of indices $r,s\in\{1,\dots,n\}$ *one has (with* V *as in (5.2.21))*

$$\begin{aligned}\partial_{\tau_{rs}}\big(T_{\rm mod} f\big) &= T\big(\partial_{\tau_{rs}} f\big) + \big[M_{\nu_r}, T\big](\nabla_{\tan} f)_s - \big[M_{\nu_s}, T\big](\nabla_{\tan} f)_r \\ &\quad + \big[M_{\nu_r}, V\big]^\top(\nu_s\nabla_{\tan} f) - \big[M_{\nu_s}, V\big]^\top(\nu_r\nabla_{\tan} f)\end{aligned} \tag{5.2.66}$$

at σ*-a.e. point on* $\partial\Omega$*. In particular, formula (5.2.66) holds for every function* $f\in\big[L^{p,q}_1(\partial\Omega,\sigma)\big]^M$ *with exponents* $p,q\in(1,\infty)$*, as well as for every function* $f\in\big[L^p_1(\partial\Omega,w)\big]^M$ *with* $p\in(1,\infty)$ *and* $w\in A_p(\partial\Omega,\sigma)$*.*

(12) For each $p\in(1,\infty)$ *there exists a constant* $C\in(0,\infty)$ *with the property that for each* $f\in\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$ *the measure* $\big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n$ *is Carleson in* Ω *in the quantitative sense that*

$$\begin{aligned}\sup_{x\in\partial\Omega,\, r>0} \frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^p \mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n \\ \leq C\|f\|^p_{[\dot{\mathrm{BMO}}(\partial\Omega,\sigma)]^M}.\end{aligned} \tag{5.2.67}$$

In particular, corresponding to $p=2$*, it follows that*[21]

$$\begin{aligned}\big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n \ \textit{is a Carleson measure in}\ \Omega, \\ \textit{for each function}\ f\in\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M.\end{aligned} \tag{5.2.68}$$

Moreover, for each $p\in(1,\infty)$ *there exists a constant* $C\in(0,\infty)$ *with the property that for each function* $f\in\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$ *one has*

$$\begin{aligned}\lim_{R\to 0^+} \sup_{\substack{x\in\partial\Omega\ \textit{and}\\ r\in(0,R)}} \Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^p \mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}} \\ \leq C\,\mathrm{dist}\big(f, [\mathrm{VMO}(\partial\Omega,\sigma)]^M\big)\end{aligned} \tag{5.2.69}$$

[21] it is natural to regard $\big|\nabla\mathcal{T}_{\rm mod} f\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n$ as the Littlewood-Paley measure associated with f via the operator $\mathcal{T}_{\rm mod}$

where the distance in the right-hand side is considered in $\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$. *As a corollary,*

$$\begin{gathered}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega,\\ \text{for each function } f\in\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M \text{ and each } p\in(1,\infty)\end{gathered}\tag{5.2.70}$$

and, corresponding to $p=2$,

$$\begin{gathered}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega,\\ \text{for each function } f\in\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M.\end{gathered}\tag{5.2.71}$$

Finally, for each $p\in(1,\infty)$ *there exists a constant* $C\in(0,\infty)$ *with the property that for each function* $f\in\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$ *one has*

$$\begin{aligned}\max\Bigg\{&\lim_{R\to\infty}\sup_{\substack{x\in\partial\Omega,\,r>0\\ B(x,r)\subseteq\mathbb{R}^n\setminus B(0,R)}}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times\\ &\qquad\times\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\,\mathrm{dist}\,(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Big)^{1/p},\\ &\lim_{R\to0^+}\sup_{x\in\partial\Omega,\,r\in(0,R)}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times\\ &\qquad\times\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\,\mathrm{dist}\,(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Big)^{1/p},\\ &\lim_{R\to\infty}\sup_{x\in\partial\Omega,\,r>R}\Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times\\ &\qquad\times\int_{B(x,r)\cap\Omega}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\,\mathrm{dist}\,(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Big)^{1/p}\Bigg\}\\ &\leq C\,\mathrm{dist}\big(f,\ [\mathrm{CMO}(\partial\Omega,\sigma)]^M\big),\end{aligned}\tag{5.2.72}$$

where the distance in the right-hand side is considered in $\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$. *As a consequence of* (5.1.376) *and Definition 1.8.1,*

$$\begin{gathered}\big|\nabla\big(\mathcal{T}_{\rm mod}f\big)\big|^p\,\mathrm{dist}\,(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n \text{ is a super vanishing Carleson measure in } \Omega,\\ \text{for each function } f\in\big[\mathrm{CMO}(\partial\Omega,\sigma)\big]^M \text{ and each } p\in(1,\infty).\end{gathered}\tag{5.2.73}$$

(13) *Make the additional assumption that* $\partial\Omega$ *is bounded. Then all properties listed in items (9)-(12) above are valid for the operator* $\mathcal{T}$, *as originally defined in* (5.2.6), *in place of its modified version* $\mathcal{T}_{\rm mod}$. *In particular, for each* $p\in(1,\infty)$

there exists a constant $C \in (0,\infty)$ *with the property that for each function* $f \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$ *one has*

$$\sup_{x\in\partial\Omega,\, r>0} \frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}_{\rm mod} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n \leq C\|f\|^p_{[\dot{\mathrm{BMO}}(\partial\Omega,\sigma)]^M}. \tag{5.2.74}$$

and

$$\lim_{R\to 0^+} \sup_{\substack{x\in\partial\Omega \text{ and} \\ r\in(0,R)}} \Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n\Big)^{\frac{1}{p}} \leq C \operatorname{dist}\big(f, [\mathrm{VMO}(\partial\Omega,\sigma)]^M\big) \tag{5.2.75}$$

where the distance in the right-hand side is considered in $\big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M$*. In particular,*

$$\big|\nabla\big(\mathcal{T} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega, \text{ for each function } f \in \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M \text{ and each } p\in(1,\infty). \tag{5.2.76}$$

(14) The operators

$$T_{\rm mod} : \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^{M''} \text{ and} \tag{5.2.77}$$

$$T_{\rm mod} : \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^{M''} \tag{5.2.78}$$

are well defined, linear, and bounded. As a corollary, (5.2.77)-(5.2.78) *and* (5.2.62) *imply that both*

$$\big[T_{\rm mod}\big] : \big[\mathrm{BMO}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\mathrm{BMO}(\partial\Omega,\sigma)/\sim\big]^{M''} \text{ defined as } \big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \text{ for each function } f \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^M \tag{5.2.79}$$

and

$$\big[T_{\rm mod}\big] : \big[\mathrm{VMO}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\mathrm{VMO}(\partial\Omega,\sigma)/\sim\big]^{M''} \text{ defined as } \big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \text{ for each function } f \in \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^M \tag{5.2.80}$$

are well-defined, linear, and bounded operators. Moreover, if $\partial\Omega$ *is unbounded then*

$$\begin{gathered}\big\langle \big[T_{\text{mod}} f\big], g\big\rangle = \big\langle [f], T^{\#} g\big\rangle,\\ \forall f \in \big[\text{BMO}(\partial\Omega, \sigma)\big]^{M}, \quad \forall g \in \big[H^1(\partial\Omega, \sigma)\big]^{M''}\end{gathered} \tag{5.2.81}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality bracket between the John-Nirenberg space of functions of bounded mean oscillations on* $\partial\Omega$*, modulo constants, and the Hardy space* H^1 *on* $\partial\Omega$ (*cf.* [69, Theorem 4.6.1])*, the operator* T_{mod} *is as in* (5.2.77)*, and* $T^{\#}$ *is presently considered as in* (5.2.23) *with* $p = 1$*. In addition, if* $\partial\Omega$ *is unbounded, the operators*

$$T_{\text{mod}} : \big[\text{CMO}(\partial\Omega, \sigma)\big]^{M} \longrightarrow \big[\text{CMO}(\partial\Omega, \sigma)\big]^{M''}, \tag{5.2.82}$$

and

$$\begin{gathered}\big[T_{\text{mod}}\big] : \big[\text{CMO}(\partial\Omega, \sigma)/\sim\big]^{M} \longrightarrow \big[\text{CMO}(\partial\Omega, \sigma)/\sim\big]^{M''} \text{ defined as}\\ \big[T_{\text{mod}}\big][f] := \big[T_{\text{mod}} f\big] \text{ for each function } f \in \big[\text{CMO}(\partial\Omega, \sigma)\big]^{M},\end{gathered} \tag{5.2.83}$$

are also well defined, linear, and bounded.

Furthermore, in the case when the set $\partial\Omega$ *is bounded* (*a scenario in which the inclusion* $\text{BMO}(\partial\Omega, \sigma) \subseteq L^p(\partial\Omega, \sigma)$ *holds for each* $p \in (1, \infty)$)*, the operator* T *acting as in* (5.2.18) *induces a well-defined, linear, and bounded mapping*

$$T : \big[\text{BMO}(\partial\Omega, \sigma)\big]^{M} \longrightarrow \big[\text{BMO}(\partial\Omega, \sigma)\big]^{M''} \tag{5.2.84}$$

which satisfies

$$\begin{gathered}\big\langle T f, g\big\rangle = \big\langle f, T^{\#} g\big\rangle,\\ \forall f \in \big[\text{BMO}(\partial\Omega, \sigma)\big]^{M}, \quad \forall g \in \big[H^1(\partial\Omega, \sigma)\big]^{M''}\end{gathered} \tag{5.2.85}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality bracket between the John-Nirenberg space of functions of bounded mean oscillations on* $\partial\Omega$ *and the Hardy space* H^1 *on* $\partial\Omega$ (*cf.* [69, Theorem 4.6.1])*, and* $T^{\#}$ *is presently considered as in* (5.2.23) *with* $p = 1$.

Finally, when $\partial\Omega$ *is bounded the operator* T *acting as in* (5.2.84) *induces a linear and bounded mapping*

$$T : \big[\text{VMO}(\partial\Omega, \sigma)\big]^{M} \longrightarrow \big[\text{VMO}(\partial\Omega, \sigma)\big]^{M''}. \tag{5.2.86}$$

(15) Fix $p \in \big(\frac{n-1}{n}, 1\big)$ *and set* $\alpha := (n-1)\big(\frac{1}{p} - 1\big) \in (0, 1)$*. Then the operators*

$$T_{\rm mod} : \big[\dot{\mathscr{C}}^{\alpha}(\partial\Omega)\big]^{M} \longrightarrow \big[\dot{\mathscr{C}}^{\alpha}(\partial\Omega)\big]^{M''}, \tag{5.2.87}$$

$$T_{\rm mod} : \big[\dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)\big]^{M} \longrightarrow \big[\dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)\big]^{M''}, \tag{5.2.88}$$

$$\big[\dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim\big]^{M} \ni [f] \mapsto \big[T_{\rm mod} f\big] \in \big[\dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim\big]^{M''} \tag{5.2.89}$$

are well defined, linear, bounded. In particular, (5.2.87)-(5.2.89) *imply that*

$$\begin{aligned} &\big[T_{\rm mod}\big] : \big[\dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim\big]^{M} \longrightarrow \big[\dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim\big]^{M''} \text{ defined as} \\ &\big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \text{ for each function } f \in \big[\dot{\mathscr{C}}^{\alpha}(\partial\Omega)\big]^{M} \end{aligned} \tag{5.2.90}$$

and

$$\begin{aligned} &\big[T_{\rm mod}\big] : \big[\dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)/\sim\big]^{M} \longrightarrow \big[\dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)/\sim\big]^{M''} \text{ defined as} \\ &\big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \text{ for each function } f \in \big[\dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)\big]^{M} \end{aligned} \tag{5.2.91}$$

are well-defined, linear, and bounded operators. Also, if $\partial\Omega$ is unbounded one has

$$\begin{aligned} &\big\langle [T_{\rm mod} f], g\big\rangle := \big\langle [f], T^{\#} g\big\rangle, \\ &\forall f \in \big[\dot{\mathscr{C}}^{\alpha}(\partial\Omega)\big]^{M}, \quad \forall g \in \big[H^{p}(\partial\Omega,\sigma)\big]^{M''}, \end{aligned} \tag{5.2.92}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between a homogeneous Hölder space on $\partial\Omega$, modulo constants, and the corresponding Hardy space on $\partial\Omega$ (cf. [69, Theorem 4.6.1]*), the operator $T_{\rm mod}$ is as in* (5.2.87)*, and where the operator $T^{\#}$ is presently considered as in* (5.2.23) *with exponent $p \in \big(\frac{n-1}{n}, 1\big)$. Finally, in the situation when the set $\partial\Omega$ is bounded (a scenario in which one actually has $\dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega) = \mathscr{C}^{\alpha}_{\rm van}(\partial\Omega) \subseteq \dot{\mathscr{C}}^{\alpha}(\partial\Omega) = \mathscr{C}^{\alpha}(\partial\Omega) \subseteq L^{p}(\partial\Omega,\sigma)$ for each $p \in (1,\infty)$), the operator T acting as in* (5.2.18) *induces well-defined, linear and bounded mappings*

$$T : \big[\mathscr{C}^{\alpha}_{\rm van}(\partial\Omega)\big]^{M} \longrightarrow \big[\mathscr{C}^{\alpha}_{\rm van}(\partial\Omega)\big]^{M''} \tag{5.2.93}$$

and

$$T : \big[\mathscr{C}^{\alpha}(\partial\Omega)\big]^{M} \longrightarrow \big[\mathscr{C}^{\alpha}(\partial\Omega)\big]^{M''} \tag{5.2.94}$$

the latter of which satisfies

$$\big\langle Tf, g\big\rangle = \big\langle f, T^{\#} g\big\rangle, \quad \forall f \in \big[\mathscr{C}^{\alpha}(\partial\Omega)\big]^{M}, \quad \forall g \in \big[H^{p}(\partial\Omega,\sigma)\big]^{M''} \tag{5.2.95}$$

with $\langle\cdot,\cdot\rangle$ denoting the duality bracket between Hölder and Hardy spaces on $\partial\Omega$ (cf. [69, Theorem 4.6.1]*), the operator T is as in* (5.2.94)*, and the operator $T^{\#}$ is as in* (5.2.23) *with $p \in \big(\frac{n-1}{n}, 1\big)$.*

(16) Select $p, q \in (1,\infty)$ with $1/p + 1/q = 1$ along with $\lambda \in (0, n-1)$, and recall the Morrey-Campanato space $\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)$ (defined as in (A.0.119) *with $\Sigma := \partial\Omega$). Then the operator T from* (5.2.18) *induces a linear and bounded mapping in the context*

$$T : \Big(\big[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)\big]^M, \|\cdot\|_{[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)]^M}\Big) \longrightarrow \Big(\big[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)\big]^{M''}, \|\cdot\|_{[\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)]^{M''}}\Big) \tag{5.2.96}$$

Moreover, if $T_{\rm mod}$ *is the modified version of the singular integral operator* T *defined in* (5.2.59)*, the assignment*

$$\begin{gathered}[T_{\rm mod}] : \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)/\sim\big]^{M''}\\ [T_{\rm mod}][f] := [T_{\rm mod} f] \text{ for each } f \in \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^M\end{gathered} \tag{5.2.97}$$

is well defined, linear and bounded. In particular,

$T_{\rm mod} : \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^M \to \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^{M''}$ *is a well-defined, linear, operator and there exists some constant* $C \in (0,\infty)$ *with the property that* $\|T_{\rm mod} f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^{M''}} \leq C\|f\|_{[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)]^M}$ *for all* $f \in \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$. (5.2.98)

Finally,

the (*real*) *transpose of the operator* $T^{\#}$ *from* (5.2.29) *is, respectively,* $\big[T_{\rm mod}\big]$ *from* (5.2.97) *if* $\partial\Omega$ *is unbounded, and* T *from* (5.2.96) *if* $\partial\Omega$ *is bounded,* (5.2.99)

that is, for each $f \in \big[\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)\big]^M$ *and* $g \in \big[\mathscr{H}^{q,\lambda}(\partial\Omega,\sigma)\big]^{M''}$ *one has*

$$\big\langle \big[T_{\rm mod} f\big], g\big\rangle = \big\langle [f], T^{\#} g\big\rangle \text{ if } \partial\Omega \text{ is unbounded,} \tag{5.2.100}$$

$$\big\langle Tf, g\big\rangle = \big\langle f, T^{\#} g\big\rangle \text{ if } \partial\Omega \text{ is bounded,} \tag{5.2.101}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality bracket between vector Morrey-Campanato spaces and their vector pre-duals* (*cf.* [69, (6.1.25)]).

(17) Fix $q \in (1,\infty)$ *and* $\eta \in (0,1)$ *and recall the Calderón spaces defined in* (3.1.10)-(3.1.15)*. Then there exists a constant* $C \in (0,\infty)$ *with the property that, with the maximal operator defined as in* (3.1.1) (*with* $\Sigma := \partial\Omega$)*, one has*

$$\big(T_{\rm mod} f\big)^{\#}_{q,\eta}(x) \leq C\, f^{\#}_{q,\eta}(x) \text{ at } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.2.102}$$

for every function $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^M$*. As a consequence, given any exponent* $p \in [1,\infty)$*, it follows that*

$$\begin{aligned} &[T_{\rm mod}] : \big[\dot{C}^p_{q,\eta}(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{C}^p_{q,\eta}(\partial\Omega,\sigma)/\sim\big]^{M''} \\ &\textit{defined by } [T_{\rm mod}][f] := \big[T_{\rm mod} f\big] \textit{ for each } f \in \big[\dot{C}^p_{q,\eta}(\partial\Omega,\sigma)\big]^M \end{aligned} \tag{5.2.103}$$

is a well-defined, linear, and bounded operator. Moreover,

$$\big(Tf\big)^{\#}_{q,\eta}(x) \leq C\, f^{\#}_{q,\eta}(x) \textit{ at } \sigma\textit{-a.e. } x \in \partial\Omega, \tag{5.2.104}$$

for every function $f \in \big[L^p(\partial\Omega,\sigma)\big]^M$ *with* $p \in [1,\infty)$. *Finally, whenever the exponent* $p \in (1,\infty)$ *it follows that*

$$T : \big[C^p_{q,\eta}(\partial\Omega,\sigma)\big]^M \longrightarrow \big[C^p_{q,\eta}(\partial\Omega,\sigma)\big]^{M''} \tag{5.2.105}$$

is a well-defined, linear, and bounded operator.

(18) Assume

$$\ell \in \{1,\dots,N\}, \quad \tfrac{n-1}{n} < p \leq \infty \textit{ and } (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1. \tag{5.2.106}$$

Then for each $\theta \in (0,1)$ *there exists a finite* $C = C(\Omega,\Theta,D,\ell,p,s,\theta) > 0$ *such that*

$$\Big\| \delta_{\partial\Omega}^{\ell-\frac{1}{p}-s} \big|\nabla^\ell \mathscr{T}_{\rm mod} f\big|_{\star,\theta} \Big\|_{L^p(\Omega,\mathcal{L}^n)} \leq C\|f\|_{[B^{p,p}_s(\partial\Omega,\sigma)]^M} \tag{5.2.107}$$

for all $f \in \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^M$, *with the understanding that when* $p > 1$ *the solid maximal function* $\big|\nabla^\ell \mathscr{T}_{\rm mod} f\big|_{\star,\theta}$ *is replaced by* $\big|\nabla^\ell \mathscr{T}_{\rm mod} f\big|$.

(19) Make the additional assumption that $\partial\Omega$ *is compact. Then the operator* T, *originally acting on Lebesgue spaces on* $\partial\Omega$ (*cf.* (5.2.18)), *extends uniquely to linear and bounded mappings*

$$\begin{aligned} &T : \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^M \longrightarrow \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^{M''}, \\ &p \in \big(\tfrac{n-1}{n},\infty\big], \quad q \in (0,\infty], \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \end{aligned} \tag{5.2.108}$$

and

$$\begin{aligned} &T : \big[F^{p,q}_s(\partial\Omega,\sigma)\big]^M \longrightarrow \big[F^{p,q}_s(\partial\Omega,\sigma)\big]^{M''}, \\ &p \in \big(\tfrac{n-1}{n},\infty\big), \quad q \in \big(\tfrac{n-1}{n},\infty\big], \quad (n-1)\big(\tfrac{1}{\min\{p,q\}}-1\big)_+ < s < 1. \end{aligned} \tag{5.2.109}$$

Moreover, various choices of the exponents yield operators which are compatible with one another. In addition, the operator $T^{\#}$, *originally considered acting on Lebesgue spaces on* $\partial\Omega$ (*cf.* (5.2.19)) *further extends, in a unique fashion, to linear and bounded mappings*

$$T^{\#}:\big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^{M''}\longrightarrow\big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^{M}$$
$$\textit{with}\ \ s\in(0,1),\quad p\in\big(\tfrac{n-1}{n-s},\infty\big],\quad q\in(0,\infty],\tag{5.2.110}$$

and

$$T^{\#}:\big[F^{p,q}_{-s}(\partial\Omega,\sigma)\big]^{M''}\longrightarrow\big[F^{p,q}_{-s}(\partial\Omega,\sigma)\big]^{M},$$
$$\textit{with}\ \ s\in(0,1),\quad p\in\big(\tfrac{n-1}{n-s},\infty\big),\quad q\in\big(\tfrac{n-1}{n-s},\infty\big].\tag{5.2.111}$$

Again, various choices of the parameters p, q, s yield operators which are compatible with one another. Finally, if the exponents $p, q, p', q' \in (1,\infty)$ satisfy $1/p + 1/p' = 1 = 1/q + 1/q'$ and $s \in (0,1)$, then

$$\begin{aligned}&{}_{[B^{p,q}_{-s}(\partial\Omega,\sigma)]^M}\big\langle T^{\#}f,g\big\rangle_{[B^{p',q'}_{s}(\partial\Omega,\sigma)]^M}\\&\qquad={}_{[B^{p,q}_{-s}(\partial\Omega,\sigma)]^{M''}}\big\langle f,Tg\big\rangle_{[B^{p',q'}_{s}(\partial\Omega,\sigma)]^{M''}}\\&\textit{for each}\ \ f\in\big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^{M''}\ \textit{and}\ g\in\big[B^{p',q'}_{s}(\partial\Omega,\sigma)\big]^{M},\end{aligned}\tag{5.2.112}$$

and

$$\begin{aligned}&{}_{[F^{p,q}_{-s}(\partial\Omega,\sigma)]^M}\big\langle T^{\#}f,g\big\rangle_{[F^{p',q'}_{s}(\partial\Omega,\sigma)]^M}\\&\qquad={}_{[F^{p,q}_{-s}(\partial\Omega,\sigma)]^{M''}}\big\langle f,Tg\big\rangle_{[F^{p',q'}_{s}(\partial\Omega,\sigma)]^{M''}}\\&\textit{for each}\ \ f\in\big[F^{p,q}_{-s}(\partial\Omega,\sigma)\big]^{M''}\ \textit{and}\ g\in\big[F^{p',q'}_{s}(\partial\Omega,\sigma)\big]^{M}.\end{aligned}\tag{5.2.113}$$

(20) Strengthen the original hypotheses on the underlying domain by assuming that Ω is a UR *domain satisfying a local John condition. Then the operator*

$$T_{\rm mod}:\big[\dot{L}^p_1(\partial\Omega,\sigma)\cap L^p_{\rm loc}(\partial\Omega,\sigma)\big]^M\to\big[\dot{L}^p_1(\partial\Omega,\sigma)\cap L^p_{\rm loc}(\partial\Omega,\sigma)\big]^{M''}\tag{5.2.114}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.128). In addition, (5.2.66) holds for each function $f\in\big[\dot{L}^p_1(\partial\Omega,\sigma)\cap L^p_{\rm loc}(\partial\Omega,\sigma)\big]^M$.
Finally, under the assumption that $\Omega\subseteq\mathbb{R}^n$ is an NTA *domain with an upper Ahlfors regular boundary*[22]*, for each integrability exponent $p\in(1,\infty)$ the operator*

$$T_{\rm mod}:\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M\longrightarrow\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^{M''}\tag{5.2.115}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.128). In this case, it follows from (5.2.115) and (5.2.62) that for each $p\in(1,\infty)$ the operator

[22] in particular, this is the case if Ω is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular; cf. (1.8.157)

$$
\begin{aligned}
&\big[T_{\rm mod}\big] : \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^{M''} \text{ defined as}\\
&\big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \in \dot{L}^p_1(\partial\Omega,\sigma)/\sim \text{ for all } f \in \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M,
\end{aligned}
\tag{5.2.116}
$$

is well defined, linear, and bounded, when all quotient spaces are endowed with the natural semi-norm[23] *introduced in* [69, (11.5.138)].

(21) *Assume* Ω *is a* UR *domain satisfying a local John condition*[24]*, and fix an integrability exponent* $p \in (1,\infty)$ *along with a parameter* $\lambda \in (0, n-1)$*. Then the operator*

$$
T_{\rm mod} : \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^{M''} \tag{5.2.117}
$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.158)*. As a consequence of* (5.2.117) *and* (5.2.62)*, the operator*

$$
\begin{aligned}
&\big[T_{\rm mod}\big] : \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^M \longrightarrow \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^{M''} \text{ defined as}\\
&\big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)/\sim\big]^{M''} \text{ for all } f \in \big[\dot{M}^{p,\lambda}_1(\partial\Omega,\sigma)\big]^M
\end{aligned}
\tag{5.2.118}
$$

is well defined, linear, and bounded when all quotient spaces are equipped with the semi-norm[25] *introduced in* [69, (11.13.51)].
Finally, analogous properties are valid for vanishing Morrey-based homogeneous Sobolev spaces $\mathring{\mathbb{M}}^{p,\lambda}_1(\partial\Omega,\sigma)$ *(cf.* [69, Definition 11.13.15]*, or* (A.0.159)-(A.0.160)*) with* $p \in (1,\infty)$ *and for block-based homogeneous Sobolev spaces* $\dot{\mathcal{B}}^{q,\lambda}_1(\partial\Omega,\sigma)$ *with* $q \in (1,\infty)$ *in place of Morrey-based homogeneous Sobolev spaces.*

(22) *Strengthen the hypotheses on the underlying domain by assuming that* $\Omega \subseteq \mathbb{R}^n$ *is an* NTA *domain with an upper Ahlfors regular boundary*[26]*. Then the modified boundary-to-boundary operator* $T_{\rm mod}$ *(originally defined as in* (5.2.59)-(5.2.60)*) induces a linear and bounded mapping*

$$
T_{\rm mod} : \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^M \longrightarrow \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^{M''} \text{ for all } p \in \big(\tfrac{n-1}{n}, 1\big]. \tag{5.2.119}
$$

(23) *The vector-valued versions of Theorem 5.1.8 and Corollary 5.1.9 are valid for the boundary-to-domain modified generalized double layer potential operator*

---

[23] [69, Proposition 11.5.14] tells us that this semi-norm is fact a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

[24] for example, this is the case if $\Omega \subseteq \mathbb{R}^n$ is an NTA domain with an upper Ahlfors regular boundary

[25] Recall from [69, Proposition 11.13.10] that this semi-norm is actually a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

[26] again, this is the case if Ω is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular; cf. (1.8.157)

$\mathcal{T}_{\rm mod}$ *from* (5.2.44)*, while the vector-valued versions of Theorem 5.1.15, Corollary 5.1.16, and Corollary 5.1.20 are valid for the boundary-to-boundary modified generalized double layer potential operator* $T_{\rm mod}$ *defined in* (5.2.59)-(5.2.60).

To offer an example, work under the assumption $n = 2$ and identify $\mathbb{R}^2 \equiv \mathbf{C}$. Consider $D := \overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$, the Cauchy-Riemann operator in the plane, and take $\Theta(z) := -1/(2\pi z)$ for each $z \in \mathbb{C} \setminus \{0\}$ (hence, $M' = M'' = 1$, and $N = \infty$). Then Θ is odd, positive homogeneous of degree -1 in $\mathbb{C} \setminus \{0\}$, and satisfies $D\,\Theta = 0$ in $\mathbb{C} \setminus \{0\}$ (hence (5.2.4) holds). In this scenario, the action of the integral operator (5.2.6) on each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|}\Big)$ then becomes

$$\mathcal{T} f(z) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\nu(\zeta)\,d\sigma(\zeta) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\,d\zeta \ \text{ for all } z \in \Omega. \tag{5.2.120}$$

Thus, $\mathcal{T}$ becomes precisely the boundary-to-domain Cauchy integral operator $\mathscr{C}$ associated with Ω (as in (1.6.35)).

Here is the proof of Theorem 5.2.1.

Proof of Theorem 5.2.1 For each $\alpha \in \{1, \dots, M\}$ and $\gamma \in \{1, \dots, M''\}$ define the vector field

$$\vec{k}_{\alpha\gamma} := \big(a_j^{\alpha\beta}\Theta_{\beta\gamma}\big)_{1\leq j\leq n} \in \big[\mathscr{C}^N(\mathbb{R}^n \setminus \{0\})\big]^n \tag{5.2.121}$$

which is odd, positive homogeneous of degree $1 - n$, and satisfies

$$\mathrm{div}\vec{k}_{\alpha\gamma} = a_j^{\alpha\beta}\partial_j\Theta_{\beta\gamma} = \big(D\,\Theta_{\bullet\gamma}\big)_\alpha = 0 \ \text{ in } \mathbb{R}^n \setminus \{0\}, \tag{5.2.122}$$

thanks to (5.2.4). Associate the integral operators $\mathcal{T}_{\alpha\gamma}, T_{\alpha\gamma}, T^{\#}_{\alpha\gamma}$ with the kernel $\vec{k}_{\alpha\gamma}$ as in (5.1.4), (5.1.5), and (5.1.6), respectively. Henceforth we also agree to abbreviate

$$\vartheta_{\alpha\gamma} := \int_{S^{n-1}} \big\langle \omega, \vec{k}_{\alpha\gamma}(\omega)\big\rangle\, d\mathcal{H}^{n-1}(\omega) \in \mathbb{C}. \tag{5.2.123}$$

In particular, from (5.2.5), (5.2.2), and (5.2.121) we see that

$$\begin{aligned}\vartheta &= \left(\int_{S^{n-1}} a_j^{\alpha\beta}\omega_j\Theta_{\beta\gamma}(\omega)\,d\mathcal{H}^{n-1}(\omega)\right)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq M''}}\\ &= \left(\int_{S^{n-1}} \big\langle \omega, \vec{k}_{\alpha\gamma}(\omega)\big\rangle\,d\mathcal{H}^{n-1}(\omega)\right)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq M''}}\\ &= \big(\vartheta_{\alpha\gamma}\big)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq M''}}.\end{aligned} \tag{5.2.124}$$

To proceed, let $(\nu_j)_{1\leq j\leq n}$ be the scalar components of the geometric measure theoretic outward unit normal ν to Ω. Then, keeping (5.2.2) in mind, we see that for each given vector-valued function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\Big]^M$ we may recast (5.2.6) in terms of the vector fields introduced in (5.2.121) as

$$\begin{aligned}\mathcal{T} f(x) &= \left(\int_{\partial\Omega} a_j^{\alpha\beta}\nu_j(y)\Theta_{\beta\gamma} f_\alpha(y)\,d\sigma(y)\right)_{1\leq\gamma\leq M''}\\ &= \left(\int_{\partial\Omega} \big\langle \nu(y), \vec{k}_{\alpha\gamma}(x-y)\big\rangle f_\alpha(y)\,d\sigma(y)\right)_{1\leq\gamma\leq M''}\\ &= \big(\mathcal{T}_{\alpha\gamma} f_\alpha(x)\big)_{1\leq\gamma\leq M''} \ \text{ for all } \ x\in\Omega,\end{aligned}\tag{5.2.125}$$

and we may reformulate (5.2.7) as

$$\begin{aligned}T f(x) &= \left(\lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big\langle \nu(y), \vec{k}_{\alpha\gamma}(x-y)\big\rangle f_\alpha(y)\,d\sigma(y)\right)_{1\leq\gamma\leq M''}\\ &= \big(T_{\alpha\gamma} f_\alpha(x)\big)_{1\leq\gamma\leq M''} \ \text{ for } \sigma\text{-a.e. } \ x\in\partial\Omega.\end{aligned}\tag{5.2.126}$$

Similarly, for each vector-valued function $g = (g_\gamma)_{1\leq\gamma\leq M''} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\Big]^{M''}$ we may refashion (5.2.8) as

$$\begin{aligned}T^\# g(x) &= \left(\lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big\langle \nu(x), \vec{k}_{\alpha\gamma}(y-x)\big\rangle g_\gamma(y)\,d\sigma(y)\right)_{1\leq\alpha\leq M}\\ &= \big(T^\#_{\alpha\gamma} g_\gamma(x)\big)_{1\leq\alpha\leq M} \ \text{ for } \sigma\text{-a.e. } \ x\in\partial\Omega.\end{aligned}\tag{5.2.127}$$

The above identifications make it possible to rely on Theorem 5.1.1 to deal with the current claims. First, a combination of (5.2.125), (5.2.126), (5.1.7) and (5.2.124) gives that for each vector-valued function $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^M$ we have

$$\begin{aligned}\Big(\mathcal{T} f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) &= \Big(\mathcal{T}_{\alpha\gamma} f_\alpha\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x)\Big)_{1\leq\gamma\leq M''}\\ &= \Big(-\frac{1}{2}\vartheta_{\alpha\gamma} f_\alpha(x) + \big(T_{\alpha\gamma} f_\alpha\big)(x)\Big)_{1\leq\gamma\leq M''}\\ &= -\frac{1}{2}\vartheta^\top f(x) + (T f)(x) \ \text{ at } \sigma\text{-a.e. } \ x\in\partial\Omega,\end{aligned}\tag{5.2.128}$$

which establishes (5.2.9). Next, the claims in (5.2.11) are implied by (5.2.10), (5.2.4), and (5.2.3). Also, (5.2.12) is a direct consequence of assumptions and [70, Theorem 2.5.1]. To justify (5.2.13), first note that at σ-a.e. point $x \in \partial\Omega$ we have

$$
\begin{aligned}
(-\mathrm{i})\mathrm{Sym}\big(D;\nu(x)\big)\Big(\frac{1}{2\mathrm{i}}\widehat{\Theta}\big(\nu(x)\big)\Big) &= (-\mathrm{i})\mathrm{i}\Big(a_j^{\alpha\beta}\nu_j(x)\frac{1}{2\mathrm{i}}\widehat{\Theta}_{\beta\gamma}\big(\nu(x)\big)\Big)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq M''}} \\
&= \frac{1}{2\mathrm{i}}\Big(\nu(x)\cdot\widehat{\vec{k}_{\alpha\gamma}}\big(\nu(x)\big)\Big)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq M''}} \\
&= -\frac{1}{2}\Big(\widehat{\mathrm{div}\vec{k}_{\alpha\gamma}}\big(\nu(x)\big)\Big)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq M''}} \\
&= -\Big(\frac{1}{2}\vartheta_{\alpha\gamma}\Big)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq M''}} = -\frac{1}{2}\vartheta, \qquad (5.2.129)
\end{aligned}
$$

thanks to (5.2.2), (5.2.121), (5.2.123), (5.1.168) (keeping in mind that $\widehat{\delta} = 1$), and (5.2.124). Pick now an arbitrary function $f \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^{M''}$ and use [70, Theorem 2.5.1] to write, at σ-a.e. point $x \in \partial\Omega$,

$$
\begin{aligned}
&(-\mathrm{i})\mathrm{Sym}\big(D;\nu(x)\big)\Big(\mathcal{W}f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) \\
&\quad = (-\mathrm{i})\mathrm{Sym}\big(D;\nu(x)\big)\Big(\frac{1}{2\mathrm{i}}\widehat{\Theta}\big(\nu(x)\big)f(x)\Big) \\
&\qquad + (-\mathrm{i})\mathrm{Sym}(D;\nu(x))\Bigg(\lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\Theta(x-y)f(y)\,\mathrm{d}\sigma(y)\Bigg) \\
&\quad = -\frac{1}{2}\vartheta f(x) - (T^{\#}f)(x), \qquad (5.2.130)
\end{aligned}
$$

where the last equality uses (5.2.129) and (5.2.8) (plus the fact that Θ is odd). This finishes the justification of (5.2.13).

For each $\alpha \in \{1,\dots,M\}$ and $\gamma \in \{1,\dots,M''\}$ consider the operator acting on each vector-valued function $\vec{g} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\Big]^n$ according to

$$
V_{\alpha\gamma}\,\vec{g}(x) := \lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}}\big\langle\vec{k}_{\alpha\gamma}(x-y),\vec{g}(y)\big\rangle\,\mathrm{d}\sigma(y) \ \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \qquad (5.2.131)
$$

where $\vec{k}_{\alpha\gamma}$ is as in (5.2.121). Then, based on (5.2.21), (5.2.131), and (5.2.121) for each given vector-valued function $\vec{g} = (g_j)_{1\leq j\leq n} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\Big]^n$ we have

$$
\begin{aligned}
V\vec{g}(x) &= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} (-\mathrm{i})\mathrm{Sym}(D;\vec{g}(y))\Theta(x-y)\,\mathrm{d}\sigma(y)\\
&= \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} a_j^{\alpha\beta}\Theta_{\beta\gamma}(x-y)g_j(y)\,\mathrm{d}\sigma(y)\Bigg)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq M''}}\\
&= \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \big\langle\vec{k}_{\alpha\gamma}(x-y),\vec{g}(y)\big\rangle\,\mathrm{d}\sigma(y)\Bigg)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq M''}}\\
&= \big(V_{\alpha\gamma}\,\vec{g}(x)\big)_{\substack{1\leq\alpha\leq M\\ 1\leq\gamma\leq M''}} \ \text{ for } \sigma\text{-a.e. } x\in\partial\Omega.
\end{aligned}
\tag{5.2.132}
$$

In turn, given any $r,s\in\{1,\dots,n\}$ and any function $f=(f_\alpha)_{1\leq\alpha\leq M}$ as in (5.2.35), based on (5.2.126), (5.1.40), and (5.2.132) we may compute

$$
\begin{aligned}
\partial_{\tau_{rs}}(Tf) &= \partial_{\tau_{rs}}(T_{\alpha\gamma}f_\alpha)_\gamma = \big(\partial_{\tau_{rs}}T_{\alpha\gamma}f_\alpha\big)_\gamma\\
&= \Big(T_{\alpha\gamma}\big(\partial_{\tau_{rs}}f_\alpha\big)\Big)_\gamma + \Big(\big[M_{\nu_r},T_{\alpha\gamma}\big](\nabla_{\tan}f_\alpha)_s\Big)_\gamma - \Big(\big[M_{\nu_s},T_{\alpha\gamma}\big](\nabla_{\tan}f_\alpha)_r\Big)_\gamma\\
&\quad - \Big(\big[M_{\nu_r},V_{\alpha\gamma}\big](\nu_s\nabla_{\tan}f_\alpha)\Big)_\gamma + \Big(\big[M_{\nu_s},V_{\alpha\gamma}\big](\nu_r\nabla_{\tan}f_\alpha)\Big)_\gamma\\
&= T\big(\partial_{\tau_{rs}}f\big) + \big[M_{\nu_r},T\big](\nabla_{\tan}f)_s - \big[M_{\nu_s},T\big](\nabla_{\tan}f)_r\\
&\quad + \big[M_{\nu_r},V\big]^\top(\nu_s\nabla_{\tan}f) - \big[M_{\nu_s},V\big]^\top(\nu_r\nabla_{\tan}f)
\end{aligned}
\tag{5.2.133}
$$

at σ-a.e. point on $\partial\Omega$. This establishes (5.2.37).

All other claims made in the statement of Theorem 5.2.1 are justified in a similar fashion, based on (5.2.121)-(5.2.127) and the corresponding properties in Theorem 5.1.1. □

Our next theorem amounts to a comprehensive theory (including nontangential maximal function estimates, jump-formulas, boundedness results on a large variety of basic function spaces, duality results, Carleson measure estimates, among many other things), for "chord-dot-normal" singular integral operators in arbitrary UR domains, of the sort discussed in (5.2.135)-(5.2.137) below.

Theorem 5.2.2 (Chord-Dot-Normal Singular Integral Operators) *Fix an integer $n\in\mathbb{N}$ with $n\geq 2$, and suppose $\Omega\subseteq\mathbb{R}^n$ is an arbitrary* UR *domain*[27]*. Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$, denote by ν the geometric measure theoretic outward unit normal to Ω, and pick an aperture parameter $\kappa>0$. Also, having fixed a sufficiently large*

[27] If $\Omega\subseteq\mathbb{R}^n$ is merely an open set with a UR boundary, a large number of conclusions continue to hold, with at most minor alterations (namely, eventually replacing $\partial\Omega$ with $\partial_*\Omega$).

integer $N = N(n) \in \mathbb{N}$, consider a function $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is even and positive homogeneous of degree $-n$, and define

$$\vartheta := \int_{S^{n-1}} k(\omega)\,\mathrm{d}\mathcal{H}^{n-1}(\omega) \in \mathbb{C}. \tag{5.2.134}$$

In this setting, introduce chord-dot-normal singular integral operators of the following sort. For each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ define the boundary-to-domain integral operator

$$\mathcal{T} f(x) := \int_{\partial\Omega} \langle \nu(y), x-y\rangle k(x-y) f(y)\,\mathrm{d}\sigma(y) \text{ for all } x \in \Omega, \tag{5.2.135}$$

and for each $f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ consider the boundary-to-boundary singular integral operators

$$T f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \langle \nu(y), x-y\rangle k(x-y) f(y)\,\mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.2.136}$$

as well as

$$T^{\#} f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \langle \nu(x), y-x\rangle k(y-x) f(y)\,\mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{5.2.137}$$

Then, in relation to these chord-dot-normal singular integral operators, the following statements are true.

(1) For each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ one has

$$\Big(\mathcal{T} f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) = -\frac{\vartheta}{2} f(x) + (T f)(x) \text{ at } \sigma\text{-a.e. } x \in \partial\Omega. \tag{5.2.138}$$

In addition, if for each $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ one defines

$$\mathcal{W} f(x) := \int_{\partial\Omega} f(y) k(x-y)(x-y)\,\mathrm{d}\sigma(y) \text{ for all } x \in \Omega, \tag{5.2.139}$$

then for each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ one has

$$\mathcal{W} f \in \big[\mathscr{C}^N(\Omega)\big]^n \text{ and } \operatorname{div}\big(\mathcal{W} f\big) = 0 \text{ in } \Omega. \tag{5.2.140}$$

Furthermore, for each $\kappa > 0$ *and* $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$, *the nontangential boundary trace*

$$(\mathcal{W}f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \tag{5.2.141}$$

and satisfies

$$\nu(x)\cdot\Big(\mathcal{W}f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = -\frac{\vartheta}{2}f(x) - (T^{\#}f)(x) \text{ at } \sigma\text{-a.e. } x \in \partial\Omega. \tag{5.2.142}$$

Finally, if $\partial\Omega$ *is bounded then both* $\mathcal{T}1$ *and* $T1$ *are constant functions and, in fact, at each point in* Ω *we have*

$$\mathcal{T}1 = \begin{cases} -\vartheta & \text{if } \Omega \text{ is bounded,} \\ 0 & \text{if } \Omega \text{ is an exterior domain}^{28}, \end{cases} \tag{5.2.143}$$

while at σ*-a.e. point on* $\partial\Omega$ *we have*

$$T1 = \begin{cases} -\frac{\vartheta}{2} & \text{if } \Omega \text{ is bounded,} \\ +\frac{\vartheta}{2} & \text{if } \Omega \text{ is an exterior domain.} \end{cases} \tag{5.2.144}$$

(2) *For each* $p \in [1,\infty)$ *there exists some finite constant* $C > 0$, *depending only on* $\partial\Omega$, k, n, p, *and* κ, *such that for each function* $f \in L^p(\partial\Omega,\sigma)$ *one has*

$$\max\Big\{\big\|\mathcal{N}_\kappa(\mathcal{T}f)\big\|_{L^p(\partial\Omega,\sigma)}, \big\|\mathcal{N}_\kappa(\mathcal{W}f)\big\|_{L^p(\partial\Omega,\sigma)}\Big\} \leq C\|f\|_{L^p(\partial\Omega,\sigma)} \tag{5.2.145}$$

if $p > 1$, *plus similar estimates in the case when* $p = 1$ *in which scenario the corresponding* L^1*-norms in the left-hand side are now replaced by the quasi-norm* $L^{1,\infty}(\partial\Omega,\sigma)$.

Moreover, the action of the operator $\mathcal{W}$, *originally considered as in* (5.2.139), *may be further extended in a unique and coherent fashion* (*cf.* [70, (2.4.15), (2.4.16), (2.4.24)]) *to the scale of Lorentz-based Hardy spaces* $H^{p,q}(\partial\Omega,\sigma)$ *with* $p \in \big(\frac{n-1}{n},\infty\big)$ *and* $q \in (0,\infty]$ *and for each* $\kappa > 0$ *said extension satisfies* (*for some constant* $C = C(\partial\Omega,k,n,p,q,\kappa) \in (0,\infty)$)

$$\big\|\mathcal{N}_\kappa(\mathcal{W}f)\big\|_{L^{p,q}(\partial\Omega,\sigma)} \leq C\|f\|_{H^{p,q}(\partial\Omega,\sigma)} \tag{5.2.146}$$

for all $f \in H^{p,q}(\partial\Omega,\sigma)$.

(3) *For each* $p \in (1,\infty)$, *the operators*

$$T : L^p(\partial\Omega,\sigma) \to L^p(\partial\Omega,\sigma) \text{ and } T^{\#} : L^p(\partial\Omega,\sigma) \to L^p(\partial\Omega,\sigma) \tag{5.2.147}$$

are both well defined, linear, and bounded. Also, given any $p, p' \in (1,\infty)$ *satisfying* $1/p + 1/p' = 1$ *it follows that*

$$\begin{aligned}&\textit{the transpose of } T \textit{ acting on } L^p(\partial\Omega,\sigma)\\ &\textit{is the operator } T^{\#} \textit{ acting on } L^{p'}(\partial\Omega,\sigma).\end{aligned} \tag{5.2.148}$$

Corresponding to the end-point $p = 1$, *both operators* T *and* $T^{\#}$ *map* $L^1(\partial\Omega,\sigma)$ *linearly and boundedly into* $L^{1,\infty}(\partial\Omega,\sigma)$. *Next, if for each vector-valued function* $\vec{g} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)\Big]^n$ *one considers*

$$V\vec{g}(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} k(x-y)\langle x-y, \vec{g}(y)\rangle \, d\sigma(y) \ \textit{for } \sigma\textit{-a.e. } x \in \partial\Omega, \tag{5.2.149}$$

then

$$V(f\,\nu) = Tf \ \textit{ for each function } f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big), \tag{5.2.150}$$

and for each $p \in (1,\infty)$ *the following operator is well defined, linear, and bounded:*

$$V : \big[L^p(\partial\Omega,\sigma)\big]^n \longrightarrow L^p(\partial\Omega,\sigma) \tag{5.2.151}$$

Finally, similar results are valid for Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, and Morrey spaces (as well as their duals and their preduals) on $\partial\Omega$.

(4) *Fix* $p \in \big(\frac{n-1}{n}, 1\big]$. *Then the operator* $T^{\#}$, *originally acting on Lebesgue spaces as in (the first part of) item (3), extends to a linear and bounded mapping from the Hardy space* $H^p(\partial\Omega,\sigma)$ *into itself,*

$$T^{\#} : H^p(\partial\Omega,\sigma) \longrightarrow H^p(\partial\Omega,\sigma) \tag{5.2.152}$$

and the operators corresponding to various choices of the index $p \in \big(\frac{n-1}{n}, 1\big]$ *are compatible with one another. In fact,* $T^{\#}$ *maps the scale of Lorentz-based Hardy spaces on* $\partial\Omega$ *continuously into itself, specifically*

$$\begin{aligned}&T^{\#} : H^{p,q}(\partial\Omega,\sigma) \longrightarrow H^{p,q}(\partial\Omega,\sigma) \textit{ is well defined,}\\ &\textit{linear, and bounded for } p \in \big(\tfrac{n-1}{n}, \infty\big) \textit{ and } q \in (0,\infty].\end{aligned} \tag{5.2.153}$$

As far as the dependence of the operator norm for $T^{\#}$ *in (5.2.153) on the kernel* k *is concerned, homogeneity considerations dictate that*

$$\big\|T^{\#}\big\|_{H^{p,q}(\partial\Omega,\sigma)\to H^{p,q}(\partial\Omega,\sigma)} \leq C\Big(\sum_{|\alpha|\leq N} \sup_{S^{n-1}} |\partial^\alpha k|\Big), \tag{5.2.154}$$

where the constant $C \in (0,\infty)$ *depends only on* n, p, q, *and the* UR *character of* $\partial\Omega$. *In addition, for each* $f \in H^{p,q}(\partial\Omega,\sigma)$ *with* $p \in \big(\frac{n-1}{n}, \infty\big)$ *and* $q \in (0,\infty]$

one has

$$\nu \bullet \mathcal{W} f = -\frac{\vartheta}{2} f - T^{\#} f. \tag{5.2.155}$$

Finally, for each function $f \in H^1(\partial\Omega, \sigma)$ *one has*

$$f \in L^1(\partial\Omega, \sigma), \qquad T^{\#} f \in L^1(\partial\Omega, \sigma), \tag{5.2.156}$$

and, with all integrals involved absolutely convergent,

$$\int_{\partial\Omega} T^{\#} f \, \mathrm{d}\sigma = \begin{cases} -\dfrac{\vartheta}{2} \displaystyle\int_{\partial\Omega} f \, \mathrm{d}\sigma & \text{if } \Omega \text{ is bounded,} \\ +\dfrac{\vartheta}{2} \displaystyle\int_{\partial\Omega} f \, \mathrm{d}\sigma & \text{if } \Omega \text{ is unbounded and } \partial\Omega \text{ is bounded,} \\ \quad 0 & \text{if } \partial\Omega \text{ is unbounded.} \end{cases} \tag{5.2.157}$$

(5) *Pick* $q \in (1, \infty)$ *and* $\lambda \in (0, n-1)$ *and recall* $\mathscr{H}^{q,\lambda}(\partial\Omega, \sigma)$, *the pre-dual to the Morrey-Campanato space, defined as in* (A.0.84) (*with* $\Sigma := \partial\Omega$). *Then the operator*

$$T^{\#} : L^r(\partial\Omega, \sigma) \longrightarrow L^r(\partial\Omega, \sigma) \text{ with } r := \frac{q(n-1)}{n-1+\lambda(q-1)} \tag{5.2.158}$$

has $\mathscr{H}^{q,\lambda}(\partial\Omega, \sigma)$ *as an invariant subspace* (*cf.* (5.2.147) *and* [69, (6.1.22)]), *and*

$$T^{\#} : \mathscr{H}^{q,\lambda}(\partial\Omega, \sigma) \longrightarrow \mathscr{H}^{q,\lambda}(\partial\Omega, \sigma) \tag{5.2.159}$$

is a well defined, linear, and bounded operator.

(6) *For each given function* f *belonging to the weighted boundary Sobolev space* $L^1_1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ (*cf.* (A.0.131)), *and each index* $\ell \in \{1, \dots, n\}$, *the pointwise nontangential boundary trace*

$$\big(\partial_\ell \mathcal{T} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega. \tag{5.2.160}$$

As a consequence of [69, (11.7.22)], *this is true whenever* $f \in L^{p,q}_1(\partial\Omega, \sigma)$ *with* $p, q \in [1, \infty)$.
Furthermore, for each $p, q \in [1, \infty)$ *there exists* $C \in (0, \infty)$, *depending only on* $\partial\Omega$, k, n, p, q, *and* κ, *such that for each function* $f \in L^{p,q}_1(\partial\Omega, \sigma)$ *one has*

$$\big\|\mathcal{N}_\kappa(\mathcal{T} f)\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla \mathcal{T} f)\big\|_{L^q(\partial\Omega,\sigma)} \leq C \|f\|_{L^{p,q}_1(\partial\Omega,\sigma)} \tag{5.2.161}$$

if $p, q > 1$, *plus similar estimates in the case when* $p = 1$ *or* $q = 1$, *in which scenario the corresponding* L^1-*norms in the left-hand side are now replaced by the quasi-norm in* $L^{1,\infty}(\partial\Omega, \sigma)$.

(7) The operator

$$T : L_1^{p,q}(\partial\Omega, \sigma) \longrightarrow L_1^{p,q}(\partial\Omega, \sigma) \tag{5.2.162}$$

is well defined, linear, and bounded for each $p, q \in (1, \infty)$. In particular, the operator

$$T : L_1^p(\partial\Omega, \sigma) \longrightarrow L_1^p(\partial\Omega, \sigma) \tag{5.2.163}$$

is well defined, linear, and bounded for each $p \in (1, \infty)$. In fact,

$$\begin{gathered} T : L_1^p(\partial\Omega, w) \longrightarrow L_1^p(\partial\Omega, w) \text{ is well defined, linear, and bounded} \\ \text{for each exponent } p \in (1, \infty) \text{ and each weight } w \in A_p(\partial\Omega, \sigma). \end{gathered} \tag{5.2.164}$$

Moreover, for each function

$$f \in L_1^1\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^{n-1}}\Big) \cap L_{1,\mathrm{loc}}^p(\partial\Omega, \sigma) \text{ for some } p \in (1, \infty) \tag{5.2.165}$$

it follows that

$$Tf \in L_{1,\mathrm{loc}}^p(\partial\Omega, \sigma), \tag{5.2.166}$$

and for each pair of indices $r, s \in \{1, \dots, n\}$ one has (with V as in (5.2.149))

$$\begin{aligned} \partial_{\tau_{rs}}(Tf) &= T\big(\partial_{\tau_{rs}} f\big) + \big[M_{\nu_r}, T\big](\nabla_{\tan} f)_s - \big[M_{\nu_s}, T\big](\nabla_{\tan} f)_r \\ &\quad - \big[M_{\nu_r}, V\big](\nu_s \nabla_{\tan} f) + \big[M_{\nu_s}, V\big](\nu_r \nabla_{\tan} f) \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \end{aligned} \tag{5.2.167}$$

In particular, formula (5.2.167) holds for every function $f \in L_1^{p,q}(\partial\Omega, \sigma)$ with exponents $p, q \in (1, \infty)$, as well as for every function $f \in L_1^p(\partial\Omega, w)$ with exponent $p \in (1, \infty)$ and weight $w \in A_p(\partial\Omega, \sigma)$. Formula (5.2.167) also holds for each function in $M_1^{p,\lambda}(\partial\Omega, \sigma)$ or in $\mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma)$ with $p, q \in (1, \infty)$ and $\lambda \in (0, n-1)$ (cf. (A.0.150), (A.0.33)). In fact,

$$\begin{gathered} \text{the operator } T \text{ maps each of the Sobolev spaces } M_1^{p,\lambda}(\partial\Omega, \sigma), \\ \mathring{M}_1^{p,\lambda}(\partial\Omega, \sigma),\ \mathcal{B}_1^{q,\lambda}(\partial\Omega, \sigma) \text{ with } p, q \in (1, \infty) \text{ and } \lambda \in (0, n-1) \\ \text{(cf. (A.0.150), (A.0.155), (A.0.33)) boundedly into themselves.} \end{gathered} \tag{5.2.168}$$

(8) For each $p \in (1, \infty)$ it follows that $T^\#$, originally acting on functions from $L^p(\partial\Omega, \sigma)$, further extends uniquely to a linear, bounded operator, from the negative boundary Sobolev space $L_{-1}^p(\partial\Omega, \sigma)$ into itself. Furthermore, if one retains the same notation $T^\#$ for said extension, then the transpose of (5.2.163) is

$$T^\# : L_{-1}^{p'}(\partial\Omega, \sigma) \longrightarrow L_{-1}^{p'}(\partial\Omega, \sigma) \tag{5.2.169}$$

where $p' \in (1,\infty)$ *is such that* $1/p + 1/p' = 1$.
More generally, for each $p, q \in (1,\infty)$ *it follows that* $T^{\#}$ *acting on functions from* $L^p(\partial\Omega,\sigma)$ *further extends uniquely to a linear and bounded operator from the negative off-diagonal boundary Sobolev space* $L^{p,q}_{-1}(\partial\Omega,\sigma)$ *into itself and, adopting the same notation* $T^{\#}$ *for this extension, the transpose of* (5.2.162) *is*

$$T^{\#} : L^{p',q'}_{-1}(\partial\Omega,\sigma) \longrightarrow L^{p',q'}_{-1}(\partial\Omega,\sigma) \tag{5.2.170}$$

where $p', q' \in (1,\infty)$ *are such that* $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$.
Also, for each exponent $p \in (1,\infty)$ *and each weight* $w \in A_p(\partial\Omega,\sigma)$*, it follows that the operator* $T^{\#}$*, originally acting on* $L^p(\partial\Omega, w)$*, further extends uniquely to a linear, bounded operator, from the negative boundary Sobolev space* $L^p_{-1}(\partial\Omega, w)$ *into itself which, in fact, is the transpose of* T *acting on the boundary Sobolev space* $L^{p'}_1(\partial\Omega, w')$ *where* $p' := (1-1/p)^{-1} \in (1,\infty)$ *is the conjugate exponent of* p *and* $w' := w^{1-p'} \in A_{p'}(\partial\Omega,\sigma)$ *is the conjugate weight of* w (*cf.* [68, item *(2)* in Lemma 7.7.1]).

(9) Consider the following modified version of the double layer operator in (5.2.135) *acting on each function* $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ *according to*

$$\big(\mathcal{T}_{\rm mod} f\big)(x) := \int_{\partial\Omega} \Big\langle \nu(y), (x-y)k(x-y) - (-y)k(-y)\cdot \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,1)}}(y)\Big\rangle f(y)\,{\rm d}\sigma(y) \tag{5.2.171}$$

for all $x \in \Omega$*. Then the operator* $\mathcal{T}_{\rm mod}$ *is meaningfully defined, and is compatible with* $\mathcal{T}$ *from* (5.1.4) *in the sense that for each function* f *belonging to the smaller space* $L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ (*hence, in particular, for each function* $f \in L^p(\partial\Omega,\sigma)$ *with* $p \in [1,\infty)$) *the difference*

$$C_f := \mathcal{T}_{\rm mod} f - \mathcal{T} f \ \text{ is a constant in } \Omega. \tag{5.2.172}$$

As a consequence,

$$\nabla\mathcal{T}_{\rm mod} f = \nabla\mathcal{T} f \text{ in } \Omega \text{ for each } f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big). \tag{5.2.173}$$

Moreover,

$$\begin{gathered}\mathcal{T}_{\rm mod} \text{ maps constant functions on } \partial\Omega \\ \text{into constant functions in } \Omega.\end{gathered} \tag{5.2.174}$$

In addition, at each point $x \in \Omega$ *one may express*

$$\partial_\ell\big(\mathcal{T}_{\rm mod} f\big)(x) = \int_{\partial\Omega} \big\langle \nu(y), \partial_{x_\ell}[(x-y)k(x-y)]\big\rangle f(y)\,d\sigma(y)$$

$$= \int_{\partial\Omega} \Big\{\nu_\ell(y)k(x-y) + \langle\nu(x), x-y\rangle(\partial_\ell k)(x-y)\Big\} f(y)\,d\sigma(y) \tag{5.2.175}$$

for each $\ell \in \{1,\dots,n\}$ *and each function* $f \in L^1\big(\partial\Omega, \frac{\sigma(y)}{1+|y|^n}\big)$.

Also, given any function

$$f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \text{ with the property that}$$
$$\partial_{\tau_{j\ell}} f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \text{ for each } j,\ell \in \{1,\dots,n\}, \tag{5.2.176}$$

(*in particular, for each function* $f \in \dot{L}^p_1(\partial\Omega,\sigma)$ *with* $1 < p < \infty$) *it follows that for each index* $\ell \in \{1,\dots,n\}$ *and each point* $x \in \Omega$ *one has*

$$\partial_\ell\big(\mathcal{T}_{\rm mod} f\big)(x) = \int_{\partial\Omega} (x_j - y_j)k(x-y)(\partial_{\tau_{\ell j}} f)(y)\,d\sigma(y). \tag{5.2.177}$$

As a consequence of (5.2.177) *and* [70, Theorem 2.5.1],

$$\text{the boundary trace } \big(\partial_\ell \mathcal{T}_{\rm mod} f\big)\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega,$$
$$\text{for each function } f \text{ as in (5.2.176) and each index } \ell \in \{1,\dots,n\}. \tag{5.2.178}$$

Another corollary of (5.2.177) *and* [70, (2.4.8)] *is the fact that for each truncation parameter* $\varepsilon \in (0,\infty)$, *and each exponent* $p \in (1,\infty)$,

$$\mathcal{N}^{\varepsilon}_\kappa\big(\nabla(\mathcal{T}_{\rm mod} f)\big) \text{ belongs to the space } L^p_{\rm loc}(\partial\Omega,\sigma)$$

$$\text{for each function } f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \text{ with the property that}$$

$$\partial_{\tau_{jk}} f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for all } j,k \in \{1,\dots,n\}. \tag{5.2.179}$$

Also, as seen from (5.2.171) *and* [70, (2.5.32)], *for each integrability exponent* $p \in (1,\infty)$ *and each truncation parameter* $\varepsilon \in (0,\infty)$ *one has*

$$\mathcal{N}^{\varepsilon}_\kappa(\mathcal{T}_{\rm mod} f) \in L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for each function}$$
$$f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma). \tag{5.2.180}$$

Finally, for each exponent $p \in \big(\frac{n-1}{n}, 1\big]$ *and each aperture parameter* $\kappa \in (0,\infty)$, *there exists some finite constant* $C > 0$, *depending only on* $\partial\Omega$, k, n, κ, *and* p, *with the property that for each distribution* $f \in \dot{H}^p_1(\partial\Omega, \sigma)$ *one has*

$$\big\|\mathcal{N}_\kappa(\nabla \mathcal{T}_{\text{mod}} f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{\dot{H}^p_1(\partial\Omega,\sigma)}. \tag{5.2.181}$$

(10) Given any exponent $\alpha \in (0,1)$ *there exists a constant* $C \in (0,\infty)$ *with the property that*

$$\sup_{x\in\Omega}\Big\{\operatorname{dist}(x,\partial\Omega)^{1-\alpha}\big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)(x)\big|\Big\} \leq C\|f\|_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \tag{5.2.182}$$

for each function $f \in \dot{\mathscr{C}}^\alpha(\partial\Omega)$*. Moreover,*

> *if* $\Omega \subseteq \mathbb{R}^n$ *is a uniform domain with the property that* $\partial\Omega$ *is an Ahlfors regular set then* $\mathcal{T}_{\text{mod}} : \dot{\mathscr{C}}^\alpha(\partial\Omega) \to \dot{\mathscr{C}}^\alpha\big(\overline{\Omega}\big)$ *is a well-defined, linear, and bounded operator for each* $\alpha \in (0,1)$*,* (5.2.183)

whereas

> *if* $\Omega \subseteq \mathbb{R}^n$ *is an* NTA *domain with an upper Ahlfors regular boundary then* $\mathcal{T}_{\text{mod}} : \dot{\mathscr{C}}^\alpha_{\text{van}}(\partial\Omega) \to \dot{\mathscr{C}}^\alpha_{\text{van}}\big(\overline{\Omega}\big)$ *is a well-defined, linear, and bounded operator for each exponent* $\alpha \in (0,1)$*,* (5.2.184)

with the homogeneous vanishing Hölder spaces defined as in (A.0.48) (*with* $\Sigma := \partial\Omega$ *and* $\Sigma := \overline{\Omega}$*, respectively). Also, for each* $\alpha \in (0,1)$ *and each exponent* $p \in (1,\infty)$ *there exists some* $C \in (0,\infty)$ *with the property that for each function* $f \in \dot{\mathscr{C}}^\alpha(\partial\Omega)$ *one has*

$$\sup_{\substack{x\in\partial\Omega\\ r\in(0,\infty)}}\left(\frac{1}{r^{n-1+\alpha p}}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{T}_{\text{mod}} f\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\right)^{1/p} \leq C\|f\|_{\dot{\mathscr{C}}^\alpha(\partial\Omega)} \tag{5.2.185}$$

and

$$\lim_{R\to 0^+}\left\{\sup_{\substack{x\in\partial\Omega\\ r\in(0,R)}}\left(\frac{1}{r^{n-1+\alpha p}}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{T}_{\text{mod}} f\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\right)^{1/p}\right\}$$
$$\leq C\operatorname{dist}\big(f,\dot{\mathscr{C}}^\alpha_{\text{van}}(\partial\Omega)\big), \tag{5.2.186}$$

where the distance is measured in the space $\big(\dot{\mathscr{C}}^\alpha(\partial\Omega), \|\cdot\|_{\dot{\mathscr{C}}^\alpha(\partial\Omega)}\big)$*. As a corollary, if the function* f *actually belongs to the homogeneous vanishing Hölder space* $\dot{\mathscr{C}}^\alpha_{\text{van}}(\partial\Omega)$ *for some* $\alpha \in (0,1)$*, then for each* $p \in (1,\infty)$ *one has*

$$\lim_{R\to 0^+}\left\{\sup_{\substack{x\in\partial\Omega\\ r\in(0,R)}}\left(\frac{1}{r^{n-1+\alpha p}}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{T}_{\text{mod}} f\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\right)^{1/p}\right\} = 0. \tag{5.2.187}$$

(11) Let $T_{\rm mod}$ be the modified version of the singular integral operator (5.2.136), *acting on each function $f \in L^1\big(\partial\Omega, \frac{\sigma(y)}{1+|y|^n}\big)$ according to*[29]

$$T_{\rm mod} f(x) := \lim_{\varepsilon\to 0^+} \int_{\partial\Omega} \big\langle \nu(y), \vec{k}_\varepsilon(x-y) - \vec{k}_1(-y)\big\rangle f(y)\,{\rm d}\sigma(y) \tag{5.2.188}$$

at σ-a.e. $x \in \partial\Omega$, where

$$\vec{k}_\varepsilon(z) := z\,k(z)\cdot \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,\varepsilon)}}(z) \text{ for each } \varepsilon > 0 \text{ and } z \in \mathbb{R}^n. \tag{5.2.189}$$

Then for each function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ one has the jump-formula

$$\mathcal{T}_{\rm mod} f\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} = -\frac{\vartheta}{2} f + T_{\rm mod} f \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{5.2.190}$$

In particular, (5.2.190) *holds for each $f \in \dot{L}^p_1(\partial\Omega,\sigma)$ with $1 < p < \infty$. As a consequence of* (5.2.190) *and* (5.2.174)*,*

$$\begin{gathered} T_{\rm mod} \text{ maps constant functions on } \partial\Omega \\ \text{into constant functions on } \partial\Omega. \end{gathered} \tag{5.2.191}$$

Furthermore,

$$\begin{gathered} \text{if } p, p' \in (1,\infty) \text{ are such that } 1/p + 1/p' = 1, \\ \text{then given any } f \in L^p_{\rm loc}(\partial\Omega,\sigma) \cap L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \\ \text{and } g \in L^{p'}_{\rm comp}(\partial\Omega,\sigma) \text{ with } \textstyle\int_{\partial\Omega} g\,{\rm d}\sigma = 0, \\ \text{it follows that } \displaystyle\int_{\partial\Omega} |T_{\rm mod} f||g|\,{\rm d}\sigma < +\infty, \quad \int_{\partial\Omega} |f||T^\# g|\,{\rm d}\sigma < +\infty, \\ \text{and } \displaystyle\int_{\partial\Omega} (T_{\rm mod} f) g\,{\rm d}\sigma = \int_{\partial\Omega} f(T^\# g)\,{\rm d}\sigma. \end{gathered} \tag{5.2.192}$$

Finally, given any integrability exponent $p \in (1,\infty)$, for each function

$$\begin{gathered} f \in L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma) \text{ with the property that} \\ \partial_{\tau_{jk}} f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\rm loc}(\partial\Omega,\sigma) \\ \text{for all } j,k \in \{1,\dots,n\} \end{gathered} \tag{5.2.193}$$

[29] alternative descriptions of $T_{\rm mod}$ may be obtained from [70, Proposition 2.3.3]

(*hence for each* $f \in \dot{L}^p_1(\partial\Omega, \sigma)$ *if* Ω *also satisfies a two-sided local John condition*) *it follows that*

$$T_{\text{mod}} f \in L^p_{1,\text{loc}}(\partial\Omega, \sigma), \tag{5.2.194}$$

and for each pair of indices $r, s \in \{1, \dots, n\}$ *one has* (*with* V *as in* (5.2.149))

$$\begin{aligned}\partial_{\tau_{rs}}\big(T_{\text{mod}} f\big) = T\big(\partial_{\tau_{rs}} f\big) + \big[M_{\nu_r}, T\big](\nabla_{\tan} f)_s - \big[M_{\nu_s}, T\big](\nabla_{\tan} f)_r \\ - \big[M_{\nu_r}, V\big](\nu_s \nabla_{\tan} f) + \big[M_{\nu_s}, V\big](\nu_r \nabla_{\tan} f)\end{aligned} \tag{5.2.195}$$

at σ*-a.e. point on* $\partial\Omega$.

(12) For each $p \in (1, \infty)$ *there exists a constant* $C \in (0, \infty)$ *with the property that for each function* $f \in \text{BMO}(\partial\Omega, \sigma)$ *the measure* $\big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1}\, d\mathcal{L}^n$ *is Carleson in* Ω *in the quantitative sense that*

$$\begin{aligned}\sup_{x\in\partial\Omega,\, r>0} \frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1}\, d\mathcal{L}^n \\ \leq C\|f\|^p_{\dot{\text{BMO}}(\partial\Omega,\sigma)}.\end{aligned} \tag{5.2.196}$$

In particular, corresponding to $p = 2$, *it follows that*[30]

$$\begin{aligned}\big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^2 \operatorname{dist}(\cdot, \partial\Omega)\, d\mathcal{L}^n \ \text{ is a Carleson measure in } \Omega, \\ \text{for each function } f \in \text{BMO}(\partial\Omega, \sigma).\end{aligned} \tag{5.2.197}$$

In addition, for each $p \in (1, \infty)$ *there exists a constant* $C \in (0, \infty)$ *with the property that for each function* $f \in \text{BMO}(\partial\Omega, \sigma)$ *one has*

$$\begin{aligned}\lim_{R\to 0^+} \sup_{\substack{x\in\partial\Omega \text{ and} \\ r\in(0,R)}} \Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1}\, d\mathcal{L}^n\Big)^{\frac{1}{p}} \\ \leq C \operatorname{dist}\big(f, \text{VMO}(\partial\Omega, \sigma)\big)\end{aligned} \tag{5.2.198}$$

where the distance in the right-hand side is considered in the space $\text{BMO}(\partial\Omega, \sigma)$. *As a corollary,*

$$\begin{aligned}\big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot, \partial\Omega)^{p-1}\, d\mathcal{L}^n \\ \text{is a vanishing Carleson measure in } \Omega, \\ \text{for each } f \in \text{VMO}(\partial\Omega, \sigma) \text{ and each } p \in (1, \infty)\end{aligned} \tag{5.2.199}$$

and, corresponding to $p = 2$,

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[30] it is natural to refer to $\big|\nabla\mathcal{T}_{\text{mod}} f\big|^2 \operatorname{dist}(\cdot, \partial\Omega)\, d\mathcal{L}^n$ as the Littlewood-Paley measure associated with f via the operator $\mathcal{T}_{\text{mod}}$

$$\begin{gathered}\big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^2 \operatorname{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^n \\ \textit{is a vanishing Carleson measure in } \Omega, \\ \textit{for each } f \in \mathrm{VMO}(\partial\Omega,\sigma).\end{gathered} \tag{5.2.200}$$

Finally, for each $p \in (1,\infty)$ *there exists a constant* $C \in (0,\infty)$ *with the property that for each function* $f \in \mathrm{BMO}(\partial\Omega,\sigma)$ *one has*

$$\begin{aligned}\max\Bigg\{ &\lim_{R\to\infty} \sup_{\substack{x\in\partial\Omega,\, r>0\\ B(x,r)\subseteq\mathbb{R}^n\setminus B(0,R)}} \Bigg(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times \\ &\qquad\qquad \times\int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Bigg)^{1/p}, \\ &\lim_{R\to 0^+} \sup_{x\in\partial\Omega,\, r\in(0,R)} \Bigg(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times \\ &\qquad\qquad \times\int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Bigg)^{1/p}, \\ &\lim_{R\to\infty} \sup_{x\in\partial\Omega,\, r>R} \Bigg(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times \\ &\qquad\qquad \times\int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\Bigg)^{1/p}\Bigg\} \\ &\qquad\qquad\qquad\qquad \le C \operatorname{dist}\big(f\,,\ \mathrm{CMO}(\partial\Omega,\sigma)\big),\end{aligned} \tag{5.2.201}$$

where the distance in the right-hand side is considered in the space $\mathrm{BMO}(\partial\Omega,\sigma)$*. As a consequence of* (5.2.201) *and Definition 1.8.1,*

$$\begin{gathered}\big|\nabla\big(\mathcal{T}_{\text{mod}} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n \\ \textit{is a super vanishing Carleson measure in } \Omega, \\ \textit{for each } f \in \mathrm{CMO}(\partial\Omega,\sigma) \textit{ and each } p \in (1,\infty).\end{gathered} \tag{5.2.202}$$

(13) Make the additional assumption that $\partial\Omega$ *is bounded. Then all properties listed in items (9)-(12) above are valid for the operator* $\mathcal{T}$*, as originally defined in* (5.2.135)*, in place of its modified version* $\mathcal{T}_{\text{mod}}$*. In particular, for each* $p \in (1,\infty)$ *there exists a constant* $C \in (0,\infty)$ *with the property that for each function* $f \in \mathrm{BMO}(\partial\Omega,\sigma)$ *one has*

$$\begin{aligned}\sup_{x\in\partial\Omega,\, r>0} \frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n \\ \le C\|f\|^p_{\dot{\mathrm{BMO}}(\partial\Omega,\sigma)}.\end{aligned} \tag{5.2.203}$$

and

$$\lim_{R\to 0^+} \sup_{\substack{x\in\partial\Omega \text{ and} \\ r\in(0,R)}} \left(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{T} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{p}}$$
$$\leq C \operatorname{dist}\big(f, \mathrm{VMO}(\partial\Omega,\sigma)\big) \tag{5.2.204}$$

where the distance in the right-hand side is considered in $\mathrm{BMO}(\partial\Omega,\sigma)$. *As a consequence,*

$$\begin{gathered} \big|\nabla\big(\mathcal{T} f\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^n \\ \textit{is a vanishing Carleson measure in } \Omega, \\ \textit{for each } f\in \mathrm{VMO}(\partial\Omega,\sigma) \textit{ and each } p\in(1,\infty). \end{gathered} \tag{5.2.205}$$

(14) The operators

$$T_{\rm mod} : \mathrm{BMO}(\partial\Omega,\sigma) \longrightarrow \mathrm{BMO}(\partial\Omega,\sigma) \ \textit{ and} \tag{5.2.206}$$

$$T_{\rm mod} : \mathrm{VMO}(\partial\Omega,\sigma) \longrightarrow \mathrm{VMO}(\partial\Omega,\sigma) \tag{5.2.207}$$

are well defined, linear, and bounded. As a consequence of (5.2.206)-(5.2.207), *the operators*

$$\begin{gathered} \big[T_{\rm mod}\big] : \mathrm{BMO}(\partial\Omega,\sigma)/\sim \longrightarrow \mathrm{BMO}(\partial\Omega,\sigma)/\sim \ \ \textit{defined as} \\ \big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \ \textit{for each function } f\in \mathrm{BMO}(\partial\Omega,\sigma) \end{gathered} \tag{5.2.208}$$

and

$$\begin{gathered} \big[T_{\rm mod}\big] : \mathrm{VMO}(\partial\Omega,\sigma)/\sim \longrightarrow \mathrm{VMO}(\partial\Omega,\sigma)/\sim \ \ \textit{defined as} \\ \big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \ \textit{for each function } f\in \mathrm{VMO}(\partial\Omega,\sigma) \end{gathered} \tag{5.2.209}$$

are well defined, linear, and bounded. Also, if $\partial\Omega$ *is unbounded one has*

$$\begin{gathered} \big\langle \big[T_{\rm mod} f\big], g\big\rangle = \big\langle [f], T^{\#} g\big\rangle, \\ \forall f\in \mathrm{BMO}(\partial\Omega,\sigma), \quad \forall g\in H^1(\partial\Omega,\sigma), \end{gathered} \tag{5.2.210}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality bracket between the John-Nirenberg space of functions of bounded mean oscillations on* $\partial\Omega$*, modulo constants, and the Hardy space* H^1 *on* $\partial\Omega$ *(cf.* [69, Theorem 4.6.1]), *and where* $T^{\#}$ *is considered here as in* (5.2.152) *with* $p := 1$. *In addition, if* $\partial\Omega$ *is unbounded, the operators*

$$T_{\rm mod} : \mathrm{CMO}(\partial\Omega,\sigma) \longrightarrow \mathrm{CMO}(\partial\Omega,\sigma), \tag{5.2.211}$$

and

$$\begin{aligned}&\big[T_{\rm mod}\big] : {\rm CMO}(\partial\Omega,\sigma)/\sim \longrightarrow {\rm CMO}(\partial\Omega,\sigma)/\sim \ \ \textit{defined as}\\&\big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \ \textit{for each function} \ \ f\in {\rm CMO}(\partial\Omega,\sigma),\end{aligned}\tag{5.2.212}$$

are also well defined, linear, and bounded.
Corresponding to the case when $\partial\Omega$ *is bounded (a scenario in which one has* ${\rm BMO}(\partial\Omega,\sigma)\subseteq L^p(\partial\Omega,\sigma)$ *for each* $p\in(1,\infty)$*), the operator* T *acting on the Lebesgue scale* $L^p(\partial\Omega,\sigma)$ *with* $p\in(1,\infty)$ *has* ${\rm BMO}(\partial\Omega,\sigma)$ *as an invariant subspace, and its restriction*

$$T : {\rm BMO}(\partial\Omega,\sigma)\longrightarrow {\rm BMO}(\partial\Omega,\sigma)\tag{5.2.213}$$

is a well-defined, linear, and bounded operator, which satisfies

$$\big\langle Tf,g\big\rangle = \big\langle f,T^{\#}g\big\rangle,\quad \forall f\in {\rm BMO}(\partial\Omega,\sigma),\quad \forall g\in H^1(\partial\Omega,\sigma)\tag{5.2.214}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality bracket between the John-Nirenberg space of functions of bounded mean oscillations on* $\partial\Omega$ *and the Hardy space* H^1 *on* $\partial\Omega$ *(cf.* [69, Theorem 4.6.1]*) and where* $T^{\#}$ *is presently considered as in* (5.2.152) *with* $p:=1$.
Moreover, when $\partial\Omega$ *is bounded the operator* T *acting on* ${\rm BMO}(\partial\Omega,\sigma)$ *(cf.* (5.2.213)*) has* ${\rm VMO}(\partial\Omega,\sigma)$ *as an invariant subspace, hence its restriction*

$$T : {\rm VMO}(\partial\Omega,\sigma)\longrightarrow {\rm VMO}(\partial\Omega,\sigma)\tag{5.2.215}$$

is a well-defined, linear, and bounded operator.

(15) Fix $p\in\big(\frac{n-1}{n},1\big)$ *and set* $\alpha:=(n-1)\big(\frac{1}{p}-1\big)\in(0,1)$*. Then the operators*

$$T_{\rm mod} : \dot{\mathscr{C}}^{\alpha}(\partial\Omega)\longrightarrow \dot{\mathscr{C}}^{\alpha}(\partial\Omega),\tag{5.2.216}$$

$$T_{\rm mod} : \dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)\longrightarrow \dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega),\tag{5.2.217}$$

$$\dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim\ \ni [f]\mapsto \big[T_{\rm mod} f\big]\in \dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim\tag{5.2.218}$$

are well defined, linear, and bounded. As a consequence of (5.2.216)*-*(5.2.218)*, the operators*

$$\begin{aligned}&\big[T_{\rm mod}\big] : \dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim \longrightarrow \dot{\mathscr{C}}^{\alpha}(\partial\Omega)/\sim \ \ \textit{defined as}\\&\big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \ \textit{for each function} \ \ f\in \dot{\mathscr{C}}^{\alpha}(\partial\Omega),\end{aligned}\tag{5.2.219}$$

and

$$\begin{aligned}&\big[T_{\rm mod}\big] : \dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)/\sim \longrightarrow \dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega)/\sim \ \ \textit{defined as}\\&\big[T_{\rm mod}\big][f] := \big[T_{\rm mod} f\big] \ \textit{for each function} \ \ f\in \dot{\mathscr{C}}^{\alpha}_{\rm van}(\partial\Omega),\end{aligned}\tag{5.2.220}$$

are well defined, linear, and bounded. Also, if $\partial\Omega$ *is unbounded one has*

$$\big\langle [T_{\text{mod}} f], g\big\rangle := \big\langle [f], T^{\#} g\big\rangle, \quad \forall f \in \dot{\mathscr{C}}^{\alpha}(\partial\Omega), \quad \forall g \in H^{p}(\partial\Omega, \sigma), \tag{5.2.221}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between a homogeneous Hölder space on $\partial\Omega$, modulo constants, and the corresponding Hardy space on $\partial\Omega$ (cf. [69, Theorem 4.6.1]*), and where $T^{\#}$ is presently considered as in (5.2.152) (with $p \in \big(\frac{n-1}{n}, 1\big)$). Finally, in the case when the set $\partial\Omega$ is bounded (a scenario in which one has $\dot{\mathscr{C}}^{\alpha}_{\text{van}}(\partial\Omega) = \mathscr{C}^{\alpha}_{\text{van}}(\partial\Omega) \subseteq \dot{\mathscr{C}}^{\alpha}(\partial\Omega) = \mathscr{C}^{\alpha}(\partial\Omega) \subseteq L^{p}(\partial\Omega, \sigma)$ for each $p \in (1, \infty)$), the operator T acting on the Lebesgue scale as in item (3) has both $\mathscr{C}^{\alpha}_{\text{van}}(\partial\Omega)$ and $\mathscr{C}^{\alpha}(\partial\Omega)$ as invariant subspaces, and its restrictions*

$$T : \mathscr{C}^{\alpha}_{\text{van}}(\partial\Omega) \longrightarrow \mathscr{C}^{\alpha}_{\text{van}}(\partial\Omega) \tag{5.2.222}$$

and

$$T : \mathscr{C}^{\alpha}(\partial\Omega) \longrightarrow \mathscr{C}^{\alpha}(\partial\Omega) \tag{5.2.223}$$

are well-defined, linear and bounded operators, the latter of which satisfies

$$\big\langle Tf, g\big\rangle = \big\langle f, T^{\#} g\big\rangle, \quad \forall f \in \mathscr{C}^{\alpha}(\partial\Omega), \quad \forall g \in H^{p}(\partial\Omega, \sigma) \tag{5.2.224}$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket between Hölder and Hardy spaces on $\partial\Omega$ (cf. [69, Theorem 4.6.1]*), and where $T^{\#}$ is currently considered as in (5.2.152) (with $p \in \big(\frac{n-1}{n}, 1\big)$).*

(16) Select $p, q \in (1, \infty)$ with $1/p + 1/q = 1$ along with $\lambda \in (0, n-1)$. Then the operator T from (5.2.147) has the inhomogeneous Morrey-Campanato space $\mathfrak{L}^{p,\lambda}(\partial\Omega, \sigma)$ (defined as in (A.0.119) with $\Sigma := \partial\Omega$) as an invariant subspace, and

$$T : \Big(\mathfrak{L}^{p,\lambda}(\partial\Omega, \sigma), \|\cdot\|_{\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)}\Big) \longrightarrow \Big(\mathfrak{L}^{p,\lambda}(\partial\Omega, \sigma), \|\cdot\|_{\mathfrak{L}^{p,\lambda}(\partial\Omega,\sigma)}\Big) \tag{5.2.225}$$

is a linear and bounded mapping. Moreover, if T_{mod} is the modified version of the singular integral operator T defined in (5.2.188), the assignment

$$\begin{aligned} &[T_{\text{mod}}] : \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega, \sigma)/\sim \longrightarrow \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega, \sigma)/\sim \\ &[T_{\text{mod}}][f] := [T_{\text{mod}} f] \text{ for each } f \in \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega, \sigma) \end{aligned} \tag{5.2.226}$$

is well defined, linear and bounded. In particular,

the operator $T_{\text{mod}} : \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega, \sigma) \longrightarrow \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega, \sigma)$ is well defined, linear, and there exists some constant $C \in (0, \infty)$ with the property that $\|T_{\text{mod}} f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)} \leq C\|f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega,\sigma)}$ for each $f \in \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega, \sigma)$. (5.2.227)

Finally,

$$\text{the (real) transpose of the operator } T^{\#} \text{ from (5.2.159) is, respectively, } \big[T_{\text{mod}}\big] \text{ from (5.2.226) if } \partial\Omega \text{ is unbounded, and } T \text{ from (5.2.225) if } \partial\Omega \text{ is bounded,} \tag{5.2.228}$$

that is, for each $f \in \dot{\mathfrak{L}}^{p,\lambda}(\partial\Omega, \sigma)$ *and* $g \in \mathscr{H}^{q,\lambda}(\partial\Omega, \sigma)$ *one has*

$$\big\langle \big[T_{\text{mod}} f\big], g\big\rangle = \big\langle [f], T^{\#} g\big\rangle \ \text{ if } \partial\Omega \text{ is unbounded,} \tag{5.2.229}$$

$$\big\langle T f, g\big\rangle = \big\langle f, T^{\#} g\big\rangle \ \text{ if } \partial\Omega \text{ is bounded,} \tag{5.2.230}$$

where $\langle\cdot,\cdot\rangle$ *denotes the duality bracket between the Morrey-Campanato space and its pre-dual (cf.* [69, (6.1.25)]).

(17) Select $q \in (1, \infty)$ *and* $\eta \in (0, 1)$ *and recall the Calderón spaces defined in* (3.1.10)-(3.1.15)*. Then there exists a constant* $C \in (0, \infty)$ *with the property that, with the maximal operator defined as in* (3.1.1) *(with* $\Sigma := \partial\Omega$*), one has*

$$\big(T_{\text{mod}} f\big)^{\#}_{q,\eta}(x) \leq C\, f^{\#}_{q,\eta}(x) \ \text{ at } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.2.231}$$

for each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)$*. Consequently, given any* $p \in [1, \infty)$*, it follows that*

$$\begin{gathered} [T_{\text{mod}}] : \dot{C}^{p}_{q,\eta}(\partial\Omega, \sigma)/\sim \longrightarrow \dot{C}^{p}_{q,\eta}(\partial\Omega, \sigma)/\sim \\ \text{defined by } [T_{\text{mod}}][f] := \big[T_{\text{mod}} f\big] \text{ for each } f \in \dot{C}^{p}_{q,\eta}(\partial\Omega, \sigma) \end{gathered} \tag{5.2.232}$$

is a well-defined, linear, and bounded operator. Furthermore, for each exponent $p \in [1, \infty)$ *there exists some constant* $C \in (0, \infty)$ *such that*

$$\big(T f\big)^{\#}_{q,\eta}(x) \leq C\, f^{\#}_{q,\eta}(x) \ \text{ at } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.2.233}$$

for each function $f \in L^p(\partial\Omega, \sigma)$*. Finally, whenever* $p \in (1, \infty)$ *it follows that*

$$T : C^{p}_{q,\eta}(\partial\Omega, \sigma) \longrightarrow C^{p}_{q,\eta}(\partial\Omega, \sigma) \tag{5.2.234}$$

is a well-defined, linear, and bounded operator.

(18) If

$$\ell \in \{1, \dots, N\}, \quad \tfrac{n-1}{n} < p \leq \infty \text{ and } (n-1)\big(\tfrac{1}{p} - 1\big)_{+} < s < 1, \tag{5.2.235}$$

it follows that for each $\theta \in (0, 1)$ *there exists* $C = C(\Omega, \vec{k}, \ell, p, s, \theta) \in (0, \infty)$ *such that*

$$\Big\|\delta_{\partial\Omega}^{\ell - \frac{1}{p} - s}\big|\nabla^{\ell}\mathcal{T}_{\text{mod}} f\big|_{\star,\theta}\Big\|_{L^p(\Omega, \mathcal{L}^n)} \leq C\|f\|_{B^{p,p}_{s}(\partial\Omega,\sigma)} \tag{5.2.236}$$

for all $f \in B^{p,p}_{s}(\partial\Omega, \sigma)$*, with the understanding that when* $p > 1$ *the solid maximal function* $\big|\nabla^{\ell}\mathcal{T}_{\text{mod}} f\big|_{\star,\theta}$ *is replaced by* $\big|\nabla^{\ell}\mathcal{T}_{\text{mod}} f\big|$.

(19) *Make the additional assumption that $\partial\Omega$ is compact. Then the operator T, originally acting on Lebesgue spaces on $\partial\Omega$ (cf. (5.2.147)), extends uniquely to linear and bounded mappings*

$$\begin{gathered} T : B^{p,q}_s(\partial\Omega,\sigma) \longrightarrow B^{p,q}_s(\partial\Omega,\sigma), \\ p \in \big(\tfrac{n-1}{n},\infty\big], \quad q \in (0,\infty], \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \end{gathered} \tag{5.2.237}$$

and

$$\begin{gathered} T : F^{p,q}_s(\partial\Omega,\sigma) \longrightarrow F^{p,q}_s(\partial\Omega,\sigma), \\ p \in \big(\tfrac{n-1}{n},\infty\big), \quad q \in \big(\tfrac{n-1}{n},\infty\big], \quad (n-1)\big(\tfrac{1}{\min\{p,q\}}-1\big)_+ < s < 1. \end{gathered} \tag{5.2.238}$$

Moreover, various choices of the exponents yield operators which are compatible with one another. In addition, the operator $T^{\#}$, originally considered acting on Lebesgue spaces on $\partial\Omega$ (cf. (5.2.147)) further extends, in a unique fashion, to linear and bounded mappings

$$\begin{gathered} T^{\#} : B^{p,q}_{-s}(\partial\Omega,\sigma) \longrightarrow B^{p,q}_{-s}(\partial\Omega,\sigma) \\ \textit{with } s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s},\infty\big], \quad q \in (0,\infty], \end{gathered} \tag{5.2.239}$$

and

$$\begin{gathered} T^{\#} : F^{p,q}_{-s}(\partial\Omega,\sigma) \longrightarrow F^{p,q}_{-s}(\partial\Omega,\sigma), \\ \textit{with } s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s},\infty\big), \quad q \in \big(\tfrac{n-1}{n-s},\infty\big]. \end{gathered} \tag{5.2.240}$$

Again, various choices of the parameters p, q, s yield operators which are compatible with one another. In all cases,

> *the operator norms of T in (5.2.237)-(5.2.238) as well as of $T^{\#}$ in (5.2.239)-(5.2.240) are all dominated by $C\Big(\sum_{|\alpha|\le N}\sup_{S^{n-1}}|\partial^\alpha k|\Big)$ where the constant $C \in (0,\infty)$ depends only on n, p, q, s, and the* UR *character of $\partial\Omega$.* (5.2.241)

Finally, if $p, q, p', q' \in (1,\infty)$ satisfy $1/p + 1/p' = 1 = 1/q + 1/q'$ and $s \in (0,1)$, then

$$\begin{gathered} {}_{B^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle T^{\#} f, g\big\rangle_{B^{p',q'}_{s}(\partial\Omega,\sigma)} = {}_{B^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle f, Tg\big\rangle_{B^{p',q'}_{s}(\partial\Omega,\sigma)} \\ \textit{for each } f \in B^{p,q}_{-s}(\partial\Omega,\sigma) \textit{ and } g \in B^{p',q'}_{s}(\partial\Omega,\sigma), \end{gathered} \tag{5.2.242}$$

and

$$\begin{gathered} {}_{F^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle T^{\#} f, g\big\rangle_{F^{p',q'}_{s}(\partial\Omega,\sigma)} = {}_{F^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle f, Tg\big\rangle_{F^{p',q'}_{s}(\partial\Omega,\sigma)} \\ \textit{for each } f \in F^{p,q}_{-s}(\partial\Omega,\sigma) \textit{ and } g \in F^{p',q'}_{s}(\partial\Omega,\sigma). \end{gathered} \tag{5.2.243}$$

(20) Impose stronger hypotheses on the underlying domain by now assuming that Ω *is a* UR *domain satisfying a local John condition. Also, fix some* $p \in (1, \infty)$. *Then the operator*

$$T_{\text{mod}} : \dot{L}^p_1(\partial\Omega, \sigma) \cap L^p_{\text{loc}}(\partial\Omega, \sigma) \longrightarrow \dot{L}^p_1(\partial\Omega, \sigma) \cap L^p_{\text{loc}}(\partial\Omega, \sigma) \tag{5.2.244}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.128). *Moreover, identity* (5.2.195) *holds for each function* $f \in \dot{L}^p_1(\partial\Omega, \sigma) \cap L^p_{\text{loc}}(\partial\Omega, \sigma)$.
Finally, if one now assumes that $\Omega \subseteq \mathbb{R}^n$ *is an* NTA *domain with an upper Ahlfors regular boundary*[31], *then for each integrability exponent* $p \in (1, \infty)$ *the operator*

$$T_{\text{mod}} : \dot{L}^p_1(\partial\Omega, \sigma) \longrightarrow \dot{L}^p_1(\partial\Omega, \sigma) \tag{5.2.245}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.128). *In this case, it follows from* (5.2.245) *and* (5.2.191) *that for each* $p \in (1, \infty)$ *the operator*

$$\begin{aligned} &\big[T_{\text{mod}}\big] : \dot{L}^p_1(\partial\Omega, \sigma)/\sim \longrightarrow \dot{L}^p_1(\partial\Omega, \sigma)/\sim \;\; \textit{defined as} \\ &\big[T_{\text{mod}}\big][f] := \big[T_{\text{mod}} f\big] \in \dot{L}^p_1(\partial\Omega, \sigma)/\sim \;\; \textit{for all } f \in \dot{L}^p_1(\partial\Omega, \sigma), \end{aligned} \tag{5.2.246}$$

is well defined, linear, and bounded, when all quotient spaces are endowed with the natural semi-norm[32] *introduced in* [69, (11.5.138)].

(21) Suppose Ω *is a* UR *domain satisfying a local John condition*[33], *and fix an integrability exponent* $p \in (1, \infty)$ *along with a parameter* $\lambda \in (0, n-1)$. *Then the operator*

$$T_{\text{mod}} : \dot{M}^{p,\lambda}_1(\partial\Omega, \sigma) \longrightarrow \dot{M}^{p,\lambda}_1(\partial\Omega, \sigma) \tag{5.2.247}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.158). *In turn,* (5.2.247) *and* (5.2.191) *imply that the operator*

$$\begin{aligned} &\big[T_{\text{mod}}\big] : \dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)/\sim \longrightarrow \dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)/\sim \;\; \textit{defined as} \\ &\big[T_{\text{mod}}\big][f] := \big[T_{\text{mod}} f\big] \in \dot{M}^{p,\lambda}_1(\partial\Omega, \sigma)/\sim \quad \forall f \in \dot{M}^{p,\lambda}_1(\partial\Omega, \sigma) \end{aligned} \tag{5.2.248}$$

[31] recall that this is the case if Ω is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular; cf. (1.8.157)

[32] [69, Proposition 11.5.14] tells us that this semi-norm is fact a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

[33] for example, this is the case if $\Omega \subseteq \mathbb{R}^n$ is an NTA domain with an upper Ahlfors regular boundary

is well defined, linear, and bounded, when the quotient spaces are endowed with the semi-norm[34] *introduced* [69, (11.13.51)].

Furthermore, similar properties are valid for vanishing Morrey-based homogeneous Sobolev spaces $\mathring{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)$ (*cf.* [69, Definition 11.13.15], *or* (A.0.159)-(A.0.160)) *with* $p \in (1,\infty)$, $\lambda \in (0, n-1)$, *and for block-based homogeneous Sobolev spaces* $\mathring{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)$ *with* $q \in (1,\infty)$, $\lambda \in (0, n-1)$, *in place of Morrey-based homogeneous Sobolev spaces.*

(22) Strengthen the hypotheses on the underlying domain by assuming that $\Omega \subseteq \mathbb{R}^n$ *is an* NTA *domain with an upper Ahlfors regular boundary*[35]. *Then the modified boundary-to-boundary operator* T_{mod} (*originally defined as in* (5.2.188)-(5.2.189)) *induces a linear and bounded mapping*

$$T_{\text{mod}} : \dot{H}_1^p(\partial\Omega,\sigma) \longrightarrow \dot{H}_1^p(\partial\Omega,\sigma) \text{ for each } p \in \big(\tfrac{n-1}{n}, 1\big]. \tag{5.2.249}$$

(23) Theorem 5.1.8 and Corollary 5.1.9 are valid for the boundary-to-domain modified chord-dot-normal integral operator $\mathcal{T}_{\text{mod}}$ *from* (5.2.171). *In addition, Theorem 5.1.15, Corollary 5.1.16, and Corollary 5.1.20 are valid for the boundary-to-boundary modified chord-dot-normal singular integral operator* T_{mod} *from* (5.2.188)-(5.2.189).

Proof If we define the vector-valued function

$$\vec{k}(z) := z\,k(z) \text{ for each } z \in \mathbb{R}^n \setminus \{0\}, \tag{5.2.250}$$

then $\vec{k} \in \big[\mathscr{C}^N(\mathbb{R}^n \setminus \{0\})\big]^n$ is odd and positive homogeneous of degree $1-n$, since the current scalar-valued function k is even and positive homogeneous of degree $-n$ in $\mathbb{R}^n \setminus \{0\}$. The latter condition implies that for each $z \in \mathbb{R}^n \setminus \{0\}$ and $t \in (0,\infty)$ we have $k(tz) = t^{-n}k(z)$. Differentiating in t then yields $\langle z, (\nabla k)(tz)\rangle = -n\,t^{-n-1}k(z)$ and, after specializing this by taking $t = 1$, we arrive at Euler's identity

$$\langle z, (\nabla k)(z)\rangle = -n\,k(z) \text{ for each } z \in \mathbb{R}^n \setminus \{0\}. \tag{5.2.251}$$

Consequently,

$$\begin{aligned} \operatorname{div}\vec{k}(z) &= \partial_j\big(z_j\,k(z)\big) = n\,k(z) + z_j(\partial_j k)(z) \\ &= n\,k(z) + \langle z, (\nabla k)(z)\rangle = 0 \text{ for each } z \in \mathbb{R}^n \setminus \{0\}, \end{aligned} \tag{5.2.252}$$

which ultimately shows that $\vec{k}$ from (5.2.250) satisfies (5.1.2). Note that

[34] We know from [69, Proposition 11.13.10] that this semi-norm is actually a genuine norm if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

[35] in particular, this is the case if Ω is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular; cf. (1.8.157)

$$\int_{S^{n-1}} \langle \omega, \vec{k}(\omega) \rangle \, \mathrm{d}\mathcal{H}^{n-1}(\omega) = \int_{S^{n-1}} k(\omega) \, \mathrm{d}\mathcal{H}^{n-1}(\omega), \tag{5.2.253}$$

so ϑ from (5.1.3) agrees with ϑ from (5.2.134). Also, the integral operators (5.1.4)-(5.1.6) for $\vec{k}$ as in (5.2.250) agree with (5.2.135)-(5.2.137). Granted these, all results become direct consequences of Theorem 5.1.1. □

Of course, there is a natural version of Theorem 5.2.2 for chord-dot-normal singular integral operators with matrix-valued kernels, i.e., as in (5.2.135)-(5.2.137) where now k is matrix-valued (and the respective SIO's now acting on vector-valued functions). Double layer potential operators associated with distinguished coefficient tensors fall under this category (we shall elaborate on this later, in the next volume).

It is also of interest to work out a version of Theorem 5.2.2 corresponding to variable coefficient kernels, of the sort presented below.

Theorem 5.2.3 *For each $n \in \mathbb{N}$ with $n \geq 2$ there exists a positive integer $M = M(n)$ with the following significance. Let $b(x, z)$ be a function which is even and positive homogeneous of degree $-n$ in the variable $z \in \mathbb{R}^n \setminus \{0\}$, and such that $\partial_z^\alpha b(x, z)$ is continuous and bounded on $\mathbb{R}^n \times S^{n-1}$ for each multi-index $\alpha \in \mathbb{N}_0^n$ satisfying $|\alpha| \leq M$. Also, let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set with the property that $\partial\Omega$ is a* UR *set; in particular, Ω is a set of locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Define*

$$\vartheta(x) := \int_{S^{n-1}} b(x, \omega) \, \mathrm{d}\mathcal{H}^{n-1}(\omega) \in \mathbb{C} \text{ for each } x \in \mathbb{R}^n. \tag{5.2.254}$$

In this setting, introduce variable coefficient chord-dot-normal singular integral operators of the following sort. For each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ define the boundary-to-domain integral operators

$$\mathcal{T} f(x) := \int_{\partial\Omega} \langle \nu(y), x - y \rangle b(x, x - y) f(y) \, \mathrm{d}\sigma(y) \text{ for all } x \in \Omega, \tag{5.2.255}$$

and

$$\widetilde{\mathcal{T}} f(x) := \int_{\partial\Omega} \langle \nu(y), x - y \rangle b(y, x - y) f(y) \, \mathrm{d}\sigma(y) \text{ for all } x \in \Omega. \tag{5.2.256}$$

Also, for each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ consider the boundary-to-boundary singular integral operators

$$T f(x) := \lim_{\varepsilon \to 0^+} \int_{\substack{y \in \partial\Omega \\ |x-y| > \varepsilon}} \langle \nu(y), x - y \rangle b(x, x - y) f(y) \, \mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.2.257}$$

$$T^{\#}f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle\nu(x), y-x\rangle b(y, y-x) f(y)\,\mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x\in\partial\Omega, \tag{5.2.258}$$

as well as

$$\widetilde{T}f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle\nu(y), x-y\rangle b(y, x-y) f(y)\,\mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x\in\partial\Omega, \tag{5.2.259}$$

and

$$\widetilde{T}^{\#}f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle\nu(x), y-x\rangle b(x, y-x) f(y)\,\mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x\in\partial\Omega. \tag{5.2.260}$$

Then, in relation to these chord-dot-normal singular integral operators, the following statements are true.

(1) For each function $f\in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ *and* $\kappa>0$ *one has*

$$\Big(\mathcal{T}f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = -\frac{\vartheta(x)}{2}f(x) + (Tf)(x) \text{ at } \sigma\text{-a.e. } x\in\partial\Omega, \tag{5.2.261}$$

and

$$\Big(\widetilde{\mathcal{T}}f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = -\frac{\vartheta(x)}{2}f(x) + (\widetilde{T}f)(x) \text{ at } \sigma\text{-a.e. } x\in\partial\Omega. \tag{5.2.262}$$

In addition, if for each function $f\in L^1\Big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\Big)$ *one defines the vector-valued functions*

$$\mathcal{W}f(x) := \int_{\partial\Omega} f(y) b(y, x-y)(x-y)\,\mathrm{d}\sigma(y) \text{ for all } x\in\Omega, \tag{5.2.263}$$

and

$$\widetilde{\mathcal{W}}f(x) := \int_{\partial\Omega} f(y) b(x, x-y)(x-y)\,\mathrm{d}\sigma(y) \text{ for all } x\in\Omega, \tag{5.2.264}$$

then for each $\kappa>0$ *the nontangential boundary traces*

$$(\mathcal{W}f)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ and } (\widetilde{\mathcal{W}}f)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega, \tag{5.2.265}$$

and satisfy

$$\nu(x)\cdot\Big(\mathcal{W}f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = -\frac{\vartheta(x)}{2}f(x) - (T^{\#}f)(x) \text{ at } \sigma\text{-a.e. } x\in\partial\Omega, \tag{5.2.266}$$

$$\nu(x)\cdot\Big(\widetilde{\mathcal{W}}f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x)=-\frac{\vartheta(x)}{2}f(x)-(\widetilde{T}^{\#}f)(x)\ \text{at } \sigma\text{-a.e. } x\in\partial\Omega. \tag{5.2.267}$$

(2) For each $p\in[1,\infty)$ and $\kappa>0$ there exists some finite constant $C>0$, depending only on $\partial\Omega$, b, n, p, and κ, such that for each function $f\in L^p(\partial\Omega,\sigma)$ one has

$$\max\Big\{\big\|\mathcal{N}_\kappa(\mathcal{T}f)\big\|_{L^p(\partial\Omega,\sigma)},\big\|\mathcal{N}_\kappa(\mathcal{W}f)\big\|_{L^p(\partial\Omega,\sigma)}\Big\}\leq C\|f\|_{L^p(\partial\Omega,\sigma)} \tag{5.2.268}$$

and

$$\max\Big\{\big\|\mathcal{N}_\kappa(\widetilde{\mathcal{T}}f)\big\|_{L^p(\partial\Omega,\sigma)},\big\|\mathcal{N}_\kappa(\widetilde{\mathcal{W}}f)\big\|_{L^p(\partial\Omega,\sigma)}\Big\}\leq C\|f\|_{L^p(\partial\Omega,\sigma)} \tag{5.2.269}$$

if $p>1$, plus similar estimates in the case when $p=1$ in which scenario the corresponding L^1-norms in the left-hand side are now replaced by the quasi-norm $L^{1,\infty}(\partial\Omega,\sigma)$.
Moreover, the action of the operator $\widetilde{\mathcal{W}}$, originally considered as in (5.2.264), may be further extended in a unique and coherent fashion (cf. [70, (2.4.15), (2.4.16), (2.4.24)]) to the scale of Lorentz-based Hardy spaces $H^{p,q}(\partial\Omega,\sigma)$ with $p\in\big(\frac{n-1}{n},\infty\big)$ and $q\in(0,\infty]$ and for each $\kappa>0$ said extension satisfies (for some constant $C=C(\partial\Omega,b,n,p,q,\kappa)\in(0,\infty)$)

$$\big\|\mathcal{N}_\kappa(\widetilde{\mathcal{W}}f)\big\|_{L^{p,q}(\partial\Omega,\sigma)}\leq C\|f\|_{H^{p,q}(\partial\Omega,\sigma)} \tag{5.2.270}$$

for all $f\in H^{p,q}(\partial\Omega,\sigma)$.
Finally, a similar extension result and estimate holds for the operator $\mathcal{W}$ in a slightly more restrictive setting than originally assumed, namely when

$$\begin{array}{l}\partial\Omega \text{ is compact, } p\in\big(\frac{n-1}{n},1\big],\ r>(n-1)\big(\frac{1}{p}-1\big), \text{ and whenever}\\ \alpha\in\mathbb{N}_0^n \text{ has } |\alpha|\leq M \text{ the function } \partial_z^\alpha b(\cdot,z) \text{ belongs to } \mathscr{C}^r(\mathbb{R}^n),\\ \text{uniformly for } z\in S^{n-1}.\end{array} \tag{5.2.271}$$

(3) For each $p\in(1,\infty)$, the operators

$$T,T^{\#},\widetilde{T},\widetilde{T}^{\#}:L^p(\partial\Omega,\sigma)\longrightarrow L^p(\partial\Omega,\sigma) \tag{5.2.272}$$

are all well defined, linear, and bounded. Also, given any $p,p'\in(1,\infty)$ satisfying $1/p+1/p'=1$ it follows that

$$\begin{array}{c}\text{the transpose of } T \text{ acting on } L^p(\partial\Omega,\sigma)\\ \text{is the operator } T^{\#} \text{ acting on } L^{p'}(\partial\Omega,\sigma),\end{array} \tag{5.2.273}$$

while

$$\begin{aligned}&\textit{the transpose of } \widetilde{T} \textit{ acting on } L^p(\partial\Omega,\sigma)\\&\textit{is the operator } \widetilde{T}^{\#} \textit{ acting on } L^{p'}(\partial\Omega,\sigma).\end{aligned} \tag{5.2.274}$$

Corresponding to the end-point $p=1$, the operators $T, T^{\#}, \widetilde{T}, \widetilde{T}^{\#}$ map $L^1(\partial\Omega,\sigma)$ linearly and boundedly into $L^{1,\infty}(\partial\Omega,\sigma)$. Next, if for each vector-valued function $\vec{g} \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^n$ one considers

$$V\vec{g}(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} b(x, x-y)\langle x-y, \vec{g}(y)\rangle \, d\sigma(y) \tag{5.2.275}$$

for σ-a.e. $x \in \partial\Omega$, and

$$\widetilde{V}\vec{g}(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} b(y, x-y)\langle x-y, \vec{g}(y)\rangle \, d\sigma(y) \tag{5.2.276}$$

for σ-a.e. $x \in \partial\Omega$, then the operators

$$\begin{aligned}V, \widetilde{V} &: \big[L^p(\partial\Omega,\sigma)\big]^n \longrightarrow L^p(\partial\Omega,\sigma), \quad p\in(1,\infty),\\ V, \widetilde{V} &: \big[L^1(\partial\Omega,\sigma)\big]^n \longrightarrow L^{1,\infty}(\partial\Omega,\sigma),\end{aligned} \tag{5.2.277}$$

are well defined, linear, bounded, and for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ one has

$$V(f\nu) = Tf \ \text{ and } \ \widetilde{V}(f\nu) = \widetilde{T}f. \tag{5.2.278}$$

Finally, similar results are valid for Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, and Morrey spaces (as well as their duals and their preduals) on $\partial\Omega$.

(4) *Fix $p \in \big(\frac{n-1}{n}, 1\big]$. Then the operators $T^{\#}$ and $\widetilde{T}^{\#}$, originally acting on Lebesgue spaces as in (the first part of) item (3), extend to linear and bounded mappings*

$$T^{\#}, \widetilde{T}^{\#} : H^p(\partial\Omega,\sigma) \longrightarrow L^p(\partial\Omega,\sigma) \tag{5.2.279}$$

and the operators corresponding to various choices of the index $p \in \big(\frac{n-1}{n}, 1\big]$ are compatible with one another.

(5) *Work under the stronger assumption that whenever $\alpha, \beta \in \mathbb{N}_0^n$ satisfy $|\alpha| \le M$ and $|\beta| \le 1$ the function $\partial_x^\beta \partial_z^\alpha b(x,z)$ is continuous and bounded on $\mathbb{R}^n \times S^{n-1}$. Then for each function f belonging to the boundary Sobolev space $L_1^p(\partial\Omega,\sigma)$ with $1<p<\infty$, each index $j \in \{1,\dots,n\}$, and each aperture parameter $\kappa > 0$, the pointwise nontangential boundary traces*

$$\big(\partial_j \mathcal{T} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ and } \ \big(\partial_j \widetilde{\mathcal{T}} f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \textit{ exist at } \sigma\textit{-a.e. point on } \partial\Omega. \tag{5.2.280}$$

Furthermore, for each $p \in (1,\infty)$ and $\kappa > 0$ there exists some finite constant $C > 0$, depending only on $\partial\Omega$, b, n, p, and κ, such that for each function $f \in L^p_1(\partial\Omega,\sigma)$ one has

$$\big\|\mathcal{N}_\kappa(\mathcal{T}f)\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\mathcal{T}f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p_1(\partial\Omega,\sigma)} \tag{5.2.281}$$

and

$$\big\|\mathcal{N}_\kappa(\widetilde{\mathcal{T}}f)\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\widetilde{\mathcal{T}}f)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p_1(\partial\Omega,\sigma)}. \tag{5.2.282}$$

(6) Continue to assume that whenever $\alpha,\beta \in \mathbb{N}_0^n$ satisfy $|\alpha| \leq M$ and $|\beta| \leq 1$ the function $\partial_x^\beta\partial_z^\alpha b(x,z)$ is continuous and bounded on $\mathbb{R}^n \times S^{n-1}$. Then the operators

$$T,\widetilde{T} : L^p_1(\partial\Omega,\sigma) \longrightarrow L^p_1(\partial\Omega,\sigma) \tag{5.2.283}$$

are well defined, linear, and bounded for each $p \in (1,\infty)$. In fact,

$$\begin{aligned} &T,\widetilde{T} : L^p_1(\partial\Omega,w) \to L^p_1(\partial\Omega,w) \text{ are well defined, linear, bounded} \\ &\quad\text{for each exponent } p \in (1,\infty) \text{ and each weight } w \in A_p(\partial\Omega,\sigma). \end{aligned} \tag{5.2.284}$$

Moreover, for each function $f \in L^p_1(\partial\Omega,\sigma)$ with $1 \leq p < \infty$ and for any given pair of indices $r,s \in \{1,\dots,n\}$ one has (with V as in (5.2.149))

$$\begin{aligned} \partial_{\tau_{rs}}(Tf) = T\big(\partial_{\tau_{rs}}f\big) &+ \big[M_{\nu_r},T\big](\nabla_{\tan}f)_s - \big[M_{\nu_s},T\big](\nabla_{\tan}f)_r \\ &- \big[M_{\nu_r},V\big](\nu_s\nabla_{\tan}f) + \big[M_{\nu_s},V\big](\nu_r\nabla_{\tan}f) \\ &+ B_{rs}f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \end{aligned} \tag{5.2.285}$$

where, at σ-a.e. point $x \in \partial\Omega$,

$$B_{rs}f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle\nu(y),x-y\rangle(\partial_{\tau_{rs}(x)}b)(x,x-y)f(y)\,d\sigma(y), \tag{5.2.286}$$

(here $(\partial_{\tau_{rs}(x)}b)(x,x-y)$ indicates that the tangential derivative $\partial_{\tau_{rs}}$ is applied to the function $b(x,z)$ in the variable x and subsequently z taken to be $x-y$) induces linear and bounded mappings

$$\begin{aligned} &B_{rs} : L^p(\partial\Omega,\sigma) \longrightarrow L^p(\partial\Omega,\sigma) \quad \text{if } p \in (1,\infty), \\ &B_{rs} : L^1(\partial\Omega,\sigma) \longrightarrow L^{1,\infty}(\partial\Omega,\sigma) \text{ if } p = 1. \end{aligned} \tag{5.2.287}$$

A similar result is valid for $\widetilde{T}$. Finally, *the operators T and $\widetilde{T}$ map the Sobolev spaces $M_1^{p,\lambda}(\partial\Omega,\sigma)$, $\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)$, and $\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)$ with exponents $p,q \in (1,\infty)$ and $\lambda \in (0,n-1)$ (cf. (A.0.150), (A.0.155), (A.0.33)) boundedly into themselves.* (5.2.288)

(7) Once again make the assumption that whenever $\alpha,\beta \in \mathbb{N}_0^n$ satisfy $|\alpha| \leq M$ and $|\beta| \leq 1$ the function $\partial_x^\beta \partial_z^\alpha b(x,z)$ is continuous and bounded on $\mathbb{R}^n \times S^{n-1}$. Then for each $p \in (1,\infty)$ it follows that $T^\#$ and $\widetilde{T}^\#$, originally acting on functions from $L^p(\partial\Omega,\sigma)$, further extend uniquely to linear, bounded operators, from the negative boundary Sobolev space $L^p_{-1}(\partial\Omega,\sigma)$ into itself. Furthermore, if one retains the same notation for said extensions, then the transpose operators of (5.2.283) *are*

$$T^\#, \widetilde{T}^\# : L^{p'}_{-1}(\partial\Omega,\sigma) \longrightarrow L^{p'}_{-1}(\partial\Omega,\sigma) \tag{5.2.289}$$

where $p' \in (1,\infty)$ is such that $1/p + 1/p' = 1$.
Also, for each exponent $p \in (1,\infty)$ and each weight $w \in A_p(\partial\Omega,\sigma)$, it follows that $T^\#$ and $\widetilde{T}^\#$, originally acting on $L^p(\partial\Omega,w)$, further extend uniquely to linear, bounded operators, from the negative boundary Sobolev space $L^p_{-1}(\partial\Omega,w)$ into itself which, in fact, are the transpose of T and $\widetilde{T}$ acting on the Sobolev space $L_1^{p'}(\partial\Omega,w')$ where $p' := (1-1/p)^{-1} \in (1,\infty)$ is the conjugate exponent of p and $w' := w^{1-p'} \in A_{p'}(\partial\Omega,\sigma)$ is the conjugate weight of w (cf. [68, item *(2)* in Lemma 7.7.1]).

(8) Adopt the stronger assumption that whenever $\alpha,\beta \in \mathbb{N}_0^n$ satisfy $|\alpha| \leq M$ and $|\beta| \leq 1$ the function $\partial_x^\beta \partial_z^\alpha b(x,z)$ is continuous and bounded on $\mathbb{R}^n \times S^{n-1}$. Then the operators T and $\widetilde{T}$ originally acting on Lebesgue spaces on $\partial\Omega$ (cf. (5.2.272)*), extend uniquely to linear and bounded mappings*

$$\begin{gathered} T, \widetilde{T} : B_s^{p,q}(\partial\Omega,\sigma) \longrightarrow B_s^{p,q}(\partial\Omega,\sigma), \\ p \in \left(\tfrac{n-1}{n},\infty\right], \quad q \in (0,\infty], \quad (n-1)\left(\tfrac{1}{p}-1\right)_+ < s < 1, \end{gathered} \tag{5.2.290}$$

and

$$\begin{gathered} T, \widetilde{T} : F_s^{p,q}(\partial\Omega,\sigma) \longrightarrow F_s^{p,q}(\partial\Omega,\sigma), \\ p \in \left(\tfrac{n-1}{n},\infty\right), \quad q \in \left(\tfrac{n-1}{n},\infty\right], \quad (n-1)\left(\tfrac{1}{\min\{p,q\}}-1\right)_+ < s < 1. \end{gathered} \tag{5.2.291}$$

Moreover, various choices of the exponents yield operators which are compatible with one another. In addition, the operators $T^\#$ and $\widetilde{T}^\#$, originally considered acting on Lebesgue spaces on $\partial\Omega$ (cf. (5.2.272)*) further extend, in a unique fashion, to linear and bounded mappings*

$$\begin{gathered} T^\# : B_{-s}^{p,q}(\partial\Omega,\sigma) \longrightarrow B_{-s}^{p,q}(\partial\Omega,\sigma) \\ \text{with } s \in (0,1), \quad p \in \left(\tfrac{n-1}{n-s},\infty\right], \quad q \in (0,\infty], \end{gathered} \tag{5.2.292}$$

and

$$T^{\#}: F^{p,q}_{-s}(\partial\Omega,\sigma) \longrightarrow F^{p,q}_{-s}(\partial\Omega,\sigma),$$
$$\text{with } s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s},\infty\big), \quad q \in \big(\tfrac{n-1}{n-s},\infty\big]. \tag{5.2.293}$$

Again, various choices of the parameters p, q, s yield operators which are compatible with one another. Finally, if the exponents $p, q, p', q' \in (1,\infty)$ satisfy $1/p + 1/p' = 1 = 1/q + 1/q'$ and $s \in (0,1)$, then

$$ {}_{B^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle T^{\#} f, g\big\rangle_{B^{p',q'}_{s}(\partial\Omega,\sigma)} = {}_{B^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle f, Tg\big\rangle_{B^{p',q'}_{s}(\partial\Omega,\sigma)}$$
$$\text{for each } f \in B^{p,q}_{-s}(\partial\Omega,\sigma) \text{ and } g \in B^{p',q'}_{s}(\partial\Omega,\sigma), \tag{5.2.294}$$

and

$$ {}_{F^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle T^{\#} f, g\big\rangle_{F^{p',q'}_{s}(\partial\Omega,\sigma)} = {}_{F^{p,q}_{-s}(\partial\Omega,\sigma)}\big\langle f, Tg\big\rangle_{F^{p',q'}_{s}(\partial\Omega,\sigma)}$$
$$\text{for each } f \in F^{p,q}_{-s}(\partial\Omega,\sigma) \text{ and } g \in F^{p',q'}_{s}(\partial\Omega,\sigma), \tag{5.2.295}$$

plus two other similar duality formulas now involving the operators $\widetilde{T}$ and $\widetilde{T}^{\#}$.

Proof The idea is to reduce matters to the case of "constant coefficient" kernel already treated in Theorem 5.2.2 via a spherical harmonics expansion, much as in the proof of [70, Theorem 2.5.38]. We shall freely use notation and results from the proof on [70, Theorem 2.5.38]. To get started, suppose

$$\{\Psi_{i\ell}\}_{\ell\in\mathbb{N}_0,\,1\le i\le H_\ell} \text{ is an orthonormal basis for } L^2(S^{n-1},\mathcal{H}^{n-1}),$$
$$\text{consisting of spherical harmonics as in [70, (2.5.483)].} \tag{5.2.296}$$

In particular,

$$\|\Psi_{i\ell}\|_{L^2(S^{n-1},\mathcal{H}^{n-1})} = 1 \text{ for each } \ell \in \mathbb{N}_0 \text{ and } 1 \le i \le H_\ell, \tag{5.2.297}$$

and

$$\Psi_{i\ell} : S^{n-1} \to \mathbb{R} \text{ is an odd function whenever } \ell \text{ is odd.} \tag{5.2.298}$$

Also, recall from [70, (2.5.481), (2.5.482)] that

$$H_0 = 1 \text{ and } H_\ell \le C_n\,\ell^{n-1} \text{ for } \ell \ge 1. \tag{5.2.299}$$

If for each $\ell \in \mathbb{N}_0$ and $1 \le i \le H_\ell$ we define

$$a_{i\ell}(x) := \int_{S^{n-1}} b(x,\omega)\Psi_{i\ell}(\omega)\,d\mathcal{H}^{n-1}(\omega), \text{ for each } x \in \mathbb{R}^n, \tag{5.2.300}$$

it follows from (5.2.298) and the current assumptions on $b(x,z)$ that

$$\text{the function } a_{i\ell} \text{ is continuous in } \mathbb{R}^n \text{ and}$$
$$\text{is identically zero whenever } \ell \text{ is odd.} \tag{5.2.301}$$

As in [70, (2.5.497)], we may then expand (bearing in mind (5.2.301), and the fact that now $b(x,z)$ is positive homogeneous of degree $-n$ in the variable z)

$$b(x,z) = \sum_{\ell\in 2\mathbb{N}_0}\sum_{i=1}^{H_\ell} a_{i\ell}(x)k_{i\ell}(z), \quad \forall\,(x,z)\in\mathbb{R}^n\times\big(\mathbb{R}^n\setminus\{0\}\big), \tag{5.2.302}$$

where for each $\ell\in 2\mathbb{N}_0$ and $1\leq i\leq H_\ell$ we have set

$$k_{i\ell}(z) := \Psi_{i\ell}\Big(\frac{z}{|z|}\Big)|z|^{-n} \text{ for all } z\in\mathbb{R}^n\setminus\{0\}, \tag{5.2.303}$$

and where the coefficients $a_{i\ell}$ are rapidly decreasing, in a uniform manner, in the sense that for each number $m\in\mathbb{N}$ with $2m\leq M$ there exists a constant $C_{n,m}\in(0,\infty)$ such that

$$\sup_{x\in\mathbb{R}^n}\big|a_{i\ell}(x)\big| \leq C_{n,m}\cdot C_b\cdot\max\{1,\ell\}^{-2m} \text{ for } \ell\in 2\mathbb{N}_0 \text{ and } 1\leq i\leq H_\ell, \tag{5.2.304}$$

where

$$C_b := C_n\cdot\sup_{\substack{(x,z)\in\mathbb{R}^n\times S^{n-1}\\ |\alpha|\leq M}}\big|\big(\partial_z^\alpha b\big)(x,z)\big|\in(0,\infty). \tag{5.2.305}$$

Also, much as in [70, (2.5.498)], if the number d is as in [70, (2.5.487)] then there exists some constant $C_{n,N}\in(0,\infty)$ with the property that

$$\big\|k_{i\ell}\big|_{S^{n-1}}\big\|_{\mathscr{C}^N(S^{n-1})}\leq C_{n,N}\cdot\max\{1,\ell\}^d \text{ if } \ell\in\mathbb{N}_0 \text{ and } 1\leq i\leq H_\ell. \tag{5.2.306}$$

Consequently, if for each $\ell\in 2\mathbb{N}_0$ and $1\leq i\leq H_\ell$ we introduce

$$\vartheta_{i\ell} := \int_{S^{n-1}} k_{i\ell}(\omega)\,d\mathcal{H}^{n-1}(\omega) = \int_{S^{n-1}}\Psi_{i\ell}(\omega)\,d\mathcal{H}^{n-1}(\omega)\in\mathbb{C}, \tag{5.2.307}$$

where the second equality comes from (5.2.303), then the normalization condition recorded in (5.2.297) implies that there exists a constant $C_n\in(0,\infty)$ such that

$$|\vartheta_{i\ell}|\leq C_n \text{ for all } \ell\in\mathbb{N}_0 \text{ and } 1\leq i\leq H_\ell. \tag{5.2.308}$$

Based on (5.2.302) and (5.2.254), for each $x\in\mathbb{R}^n$ we may compute

$$\begin{aligned}\sum_{\ell\in 2\mathbb{N}_0}\sum_{1\leq i\leq H_\ell} a_{i\ell}(x)\vartheta_{i\ell} &= \sum_{\ell\in 2\mathbb{N}_0}\sum_{1\leq i\leq H_\ell}\int_{S^{n-1}} a_{i\ell}(x)k_{i\ell}(\omega)\,d\mathcal{H}^{n-1}(\omega)\\ &= \int_{S^{n-1}} b(x,\omega)\,d\mathcal{H}^{n-1}(\omega) = \vartheta(x),\end{aligned} \tag{5.2.309}$$

where all series are absolutely convergent and may be interchanged with integration thanks to (5.2.299), (5.2.304), and (5.2.308).

Moving on, for each $\ell \in 2\mathbb{N}_0$ and $1 \le i \le H_\ell$, define the boundary-to-domain integral operator $\mathcal{T}^{i\ell}$ acting on each function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ according to

$$\mathcal{T}^{i\ell} f(x) := \int_{\partial\Omega} \langle \nu(y), x-y\rangle k_{i\ell}(x-y) f(y)\,\mathrm{d}\sigma(y) \ \text{ for all } \ x \in \Omega, \tag{5.2.310}$$

and consider the boundary-to-boundary singular integral operator $T^{i\ell}$ on $\partial\Omega$ acting on any given function $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ according to

$$T^{i\ell} f(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle \nu(y), x-y\rangle k_{i\ell}(x-y) f(y)\,\mathrm{d}\sigma(y) \tag{5.2.311}$$

for σ-a.e. $x \in \partial\Omega$.

Then, with justifications for the convergence of the series involved as in the proof of [70, Theorem 2.5.38], we have

$$T = \sum_{\ell\in 2\mathbb{N}_0} \sum_{1\le i\le H_\ell} a_{i\ell} T^{i\ell} \tag{5.2.312}$$

with convergence in $\mathcal{B}\Big(L^p(\partial\Omega,\sigma) \to L^p(\partial\Omega,\sigma)\Big)$ if $1 < p < \infty$ and, corresponding to $p = 1$, in $\mathcal{B}\Big(L^1(\partial\Omega,\sigma) \to L^{1,\infty}(\partial\Omega,\sigma)\Big)$. Also, for each $f \in L^p(\partial\Omega,\sigma)$ with $1 \le p < \infty$ and each $\kappa > 0$, at σ-a.e. $x \in \partial\Omega$ we may compute

$$\begin{aligned}
\Big(\mathcal{T} f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) &= \sum_{\ell\in 2\mathbb{N}_0} \sum_{i=1}^{H_\ell} a_{i\ell}(x)\Big(\mathcal{T}^{i\ell} f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) \\
&= -\frac{1}{2} \sum_{\ell\in 2\mathbb{N}_0} \sum_{i=1}^{H_\ell} a_{i\ell}(x)\vartheta_{i\ell}(x) f(x) + \sum_{\ell\in 2\mathbb{N}_0} \sum_{i=1}^{H_\ell} a_{i\ell}(x)(T^{i\ell} f)(x) \\
&= -\frac{\vartheta(x)}{2} f(x) + (Tf)(x),
\end{aligned} \tag{5.2.313}$$

where the first equality is justified much as in [70, (2.5.520)-(2.5.525)], and the subsequent steps follow from (5.2.138), (5.2.312), and (5.2.309).

We shall now use the above result to prove the jump-formula stated in (5.2.261). To get started, select a function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$, fix a point $x_0 \in \partial\Omega$ and pick an arbitrary number $r \in (0,\infty)$. Decompose

$$\begin{aligned}
& f = f_1 + f_2 \ \text{ on } \ \partial\Omega, \ \text{ where} \\
& f_1 := \mathbf{1}_{\partial\Omega\cap B(x_0,2r)} \cdot f \ \text{ and } \ f_2 := \mathbf{1}_{\partial\Omega\setminus B(x_0,2r)} \cdot f,
\end{aligned} \tag{5.2.314}$$

then split

$$\mathcal{T}f = \mathcal{T}f_1 + \mathcal{T}f_2 \ \text{ in } \ \Omega. \tag{5.2.315}$$

Choose an aperture parameter $\kappa > 0$. In view of the fact that $\mathcal{T}f_2$ has a continuous extension to $B(x_0, r)$, it trivially follows that the nontangential trace $\mathcal{T}f_2\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at every point in $\partial_{\text{nta}}\Omega \cap B(x_0, r)$. Specifically,

$$\begin{aligned}\big(\mathcal{T}f_2\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) &= \int_{\partial\Omega} \langle \nu(y), x-y\rangle b(x, x-y) f_2(y)\, d\sigma(y) \\ &= \big(Tf_2\big)(x) \ \text{ for each } \ x \in \partial_{\text{nta}}\Omega \cap B(x_0, r).\end{aligned} \tag{5.2.316}$$

From (5.2.316) and item *(iii)* in [68, Proposition 8.8.6] we then see that

$$\big(\mathcal{T}f_2\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) = \big(Tf_2\big)(x) \ \text{ at } \sigma\text{-a.e. } \ x \in \partial_*\Omega \cap B(x_0, r). \tag{5.2.317}$$

Next, since $f_1 \in L^1(\partial\Omega, \sigma)$, we may rely on (5.2.313) to conclude that, at σ-a.e. point $x \in \partial_*\Omega \cap B(x_0, r)$, we have

$$\big(\mathcal{T}f_1\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) = -\frac{\vartheta(x)}{2} f_1(x) + (Tf_1)(x). \tag{5.2.318}$$

Finally, from (5.2.317), (5.2.318), and (5.2.314) we conclude (bearing in mind the arbitrariness of $r > 0$) that (5.2.261) holds.

Next, the goal is to prove (5.2.281). As a preamble, for each $\ell \in 2\mathbb{N}_0$ and $1 \leq i \leq H_\ell$ we recall from (5.1.201) (written for $\vec{k}(z) := k_{i\ell}(z)z$) that for any $j \in \{1, \dots, n\}$ and at each $x \in \Omega$, we have

$$\begin{aligned}\partial_j(\mathcal{T}^{i\ell} f)(x) &= \mathcal{T}^{i\ell}\big((\nabla_{\tan} f)_j\big)(x) \\ &\quad - \int_{\partial\Omega} \nu_j(y)\langle x-y, (\nabla_{\tan} f)(y)\rangle k_{i\ell}(x-y)\, d\sigma(y)\end{aligned} \tag{5.2.319}$$

for any function $f \in L^p_1(\partial\Omega, \sigma)$ with $p \in (1, \infty)$. Based on this, (5.2.310), and [70, (2.4.9)] we then conclude that for each $p \in (1, \infty)$ and $\kappa > 0$ there exists a constant $C_{\partial\Omega,p,n,\kappa} \in (0, \infty)$ with the property that for any $f \in L^p_1(\partial\Omega, \sigma)$ we have

$$\begin{aligned}&\big\|\mathcal{N}_\kappa(\mathcal{T}^{i\ell} f)\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\mathcal{T}^{i\ell} f)\big\|_{L^p(\partial\Omega,\sigma)} \\ &\qquad\qquad \leq C_{\partial\Omega,p,n,\kappa} \cdot \big\|k_{i\ell}\big|_{S^{n-1}}\big\|_{\mathscr{C}^N(S^{n-1})} \|f\|_{L^p_1(\partial\Omega,\sigma)}.\end{aligned} \tag{5.2.320}$$

From (5.2.310), (5.2.311), (5.2.138), (5.2.161) (wit $q := p$), and [69, Proposition 11.3.4] we conclude that for any function $f \in L^p_1(\partial\Omega, \sigma)$ with $p \in (1, \infty)$ and any $\kappa > 0$ we have

$$T^{i\ell} f = \big(\mathcal{T}^{i\ell} f\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} + \frac{\vartheta_{i\ell}}{2} f \in L^p_1(\partial\Omega,\sigma) \tag{5.2.321}$$

and there exists some $C = C(\partial\Omega, p, n, \kappa) \in (0,\infty)$ such that

$$\begin{aligned} \|T^{i\ell} f\|_{L^p_1(\partial\Omega,\sigma)} \leq C\big\|\mathcal{N}_\kappa(\mathcal{T}^{i\ell} f)\big\|_{L^p(\partial\Omega,\sigma)} + C\big\|\mathcal{N}_\kappa(\nabla\mathcal{T}^{i\ell} f)\big\|_{L^p(\partial\Omega,\sigma)} \\ + C|\vartheta_{i\ell}|\|f\|_{L^p_1(\partial\Omega,\sigma)}. \end{aligned} \tag{5.2.322}$$

Combining (5.2.322) with (5.2.320), (5.2.308), and (5.2.306) we arrive at the conclusion that for each $p \in (1,\infty)$ there exists a constant $C_{\partial\Omega,p,n} \in (0,\infty)$ with the property that

$$\begin{aligned} &\|T^{i\ell} f\|_{L^p_1(\partial\Omega,\sigma)} \leq C_{\partial\Omega,p,n} \cdot \max\{1,\ell\}^{d} \cdot \|f\|_{L^p_1(\partial\Omega,\sigma)} \\ &\text{for each } f \in L^p_1(\partial\Omega,\sigma), \text{ if } \ell \in 2\mathbb{N}_0 \text{ and } 1 \leq i \leq H_\ell. \end{aligned} \tag{5.2.323}$$

Let us now work under the stronger assumptions adopted in item *(5)*. Much as in [70, (2.5.542)], these imply that for each $m \in \mathbb{N}$ with $2m \leq M$ there exists a constant $C_{n,m} \in (0,\infty)$ such that

$$\begin{aligned} \sup_{x\in\mathbb{R}^n} \big|(\nabla a_{i\ell})(x)\big| \leq C_{n,m} \cdot \Big(\sup_{\substack{|\alpha|\leq M \\ |\beta|\leq 1}} \sup_{\substack{x\in\mathbb{R}^n \\ z\in S^{n-1}}} \big|(\partial_x^\beta \partial_z^\alpha b)(x,z)\big|\Big) \cdot \max\{1,\ell\}^{-2m} \\ \text{whenever } \ell \in \mathbb{N}_0 \text{ and } 1 \leq i \leq H_\ell. \end{aligned} \tag{5.2.324}$$

From [69, Corollary 11.1.19] we see that

$$\begin{aligned} &\text{given any scalar function } a \in \mathscr{C}^1(\mathbb{R}^n) \text{ with the property that } \\ &\sup_{x\in\mathbb{R}^n}|a(x)| < \infty \text{ as well as } \sup_{x\in\mathbb{R}^n}|(\nabla a)(x)| < \infty, \text{ for each } \\ &f \in L^p_1(\partial\Omega,\sigma) \text{ with } p \in [1,\infty] \text{ it follows that } a\cdot f \text{ belongs to } \\ &L^p_1(\partial\Omega,\sigma) \text{ and for each pair of indices } j,k \in \{1,\dots,n\} \text{ one has } \\ &\partial_{\tau_{jk}}(a\cdot f) = \big[\nu_j(\partial_k a)\big|_{\partial\Omega} - \nu_k(\partial_j a)\big|_{\partial\Omega}\big]\cdot f + \big(a\big|_{\partial\Omega}\big)\cdot\partial_{\tau_{jk}} f. \end{aligned} \tag{5.2.325}$$

Then a combination of (5.2.312), (5.2.324), (5.2.304), and (5.2.325) proves that, for each integrability exponent $p \in (1,\infty)$, the series

$$\sum_{\ell\in 2\mathbb{N}_0}\sum_{1\leq i\leq H_\ell} a_{i\ell} T^{i\ell} \text{ converges to } T \text{ in } \mathcal{B}\Big(L^p_1(\partial\Omega,\sigma) \to L^p_1(\partial\Omega,\sigma)\Big). \tag{5.2.326}$$

As a consequence, T defines a linear and bounded operator from $L^p_1(\partial\Omega,\sigma)$ into itself, for each $p \in (1,\infty)$. A similar argument is valid for $\widetilde{T}$, and this establishes (5.2.283). The same circle of ideas may be employed to justify all claims made in item *(5)*.

Let us now turn our attention to (5.2.285). We shall work under the hypotheses adopted in item *(6)* so, in particular, (5.2.324) continues to hold. To set the stage, for each $\ell \in \mathbb{N}_0$ and $1 \leq i \leq H_\ell$ consider the singular integral operator acting on each vector-valued function $\vec{g} \in \big[L^p(\partial\Omega,\sigma)\big]^n$ with $1 \leq p < \infty$ according to

$$V^{i\ell}\vec{g}(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} k_{i\ell}(x-y)\langle x-y, \vec{g}(y)\rangle \,\mathrm{d}\sigma(y) \tag{5.2.327}$$

for σ-a.e. $x \in \partial\Omega$. Thanks to (5.2.302) and (5.2.275), much as with (5.2.312) we have

$$V = \sum_{\ell\in 2\mathbb{N}_0} \sum_{1\leq i\leq H_\ell} a_{i\ell} V^{i\ell} \tag{5.2.328}$$

with convergence in $\mathscr{B}\Big(L^p(\partial\Omega,\sigma) \to L^p(\partial\Omega,\sigma)\Big)$ if $1 < p < \infty$ and, corresponding to $p = 1$, in $\mathscr{B}\Big(L^1(\partial\Omega,\sigma) \to L^{1,\infty}(\partial\Omega,\sigma)\Big)$.

To proceed, observe that (5.2.324) permits us to differentiate the series in (5.2.302) term by term and obtain that, for each $j \in \{1,\dots,n\}$,

$$(\partial_{x_j} b)(x,z) = \sum_{\ell\in 2\mathbb{N}_0} \sum_{i=1}^{H_\ell} (\partial_{x_j} a_{i\ell})(x) k_{i\ell}(z), \ \ \forall\,(x,z) \in \mathbb{R}^n \times \big(\mathbb{R}^n \setminus \{0\}\big). \tag{5.2.329}$$

Next, given any $r, s \in \{1,\dots,n\}$, for each $f \in L^p(\partial\Omega,\sigma)$ with $1 \leq p < \infty$ we may use (5.2.286) to write

$$\begin{aligned}
B_{rs} f(x) &= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle \nu(y), x-y\rangle \Big(\nu_r(x)(\partial_{x_s} b)(x,x-y) \\
&\qquad\qquad - \nu_s(x)(\partial_{x_r} b)(x,x-y)\Big) f(y)\,\mathrm{d}\sigma(y) \\
&= \nu_r(x)\Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle \nu(y), x-y\rangle (\partial_{x_s} b)(x,x-y) f(y)\,\mathrm{d}\sigma(y)\Bigg) \\
&\quad - \nu_s(x)\Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \langle \nu(y), x-y\rangle (\partial_{x_r} b)(x,x-y) f(y)\,\mathrm{d}\sigma(y)\Bigg) \\
&= \sum_{\ell\in 2\mathbb{N}_0} \sum_{1\leq i\leq H_\ell} \nu_r(x)(\partial_{x_s} a_{i\ell})(x)(T^{i\ell} f)(x) \\
&\quad - \sum_{\ell\in 2\mathbb{N}_0} \sum_{1\leq i\leq H_\ell} \nu_s(x)(\partial_{x_r} a_{i\ell})(x)(T^{i\ell} f)(x) \\
&= \sum_{\ell\in 2\mathbb{N}_0} \sum_{1\leq i\leq H_\ell} \Big[\nu_r(x)(\partial_{x_s} a_{i\ell})(x) - \nu_s(x)(\partial_{x_r} a_{i\ell})(x)\Big] \cdot (T^{i\ell} f)(x)
\end{aligned} \tag{5.2.330}$$

at σ-a.e. point $x \in \partial\Omega$. Finally, for each function $f \in L^p_1(\partial\Omega, \sigma)$ with $1 < p < \infty$ we may now compute

$$
\begin{aligned}
\partial_{\tau_{rs}}(Tf) &= \sum_{\ell\in 2\mathbb{N}_0}\sum_{1\le i\le H_\ell}\Big[\nu_r(\partial_s a_{i\ell})\big|_{\partial\Omega} - \nu_s(\partial_r a_{i\ell})\big|_{\partial\Omega}\Big]\cdot T^{i\ell} f \\
&\quad + \sum_{\ell\in 2\mathbb{N}_0}\sum_{1\le i\le H_\ell} a_{i\ell}\partial_{\tau_{rs}}(T^{i\ell} f) \\
&= B_{rs} f + \sum_{\ell\in 2\mathbb{N}_0}\sum_{1\le i\le H_\ell} a_{i\ell} T^{i\ell}\big(\partial_{\tau_{rs}} f\big) \\
&\quad + \sum_{\ell\in 2\mathbb{N}_0}\sum_{1\le i\le H_\ell} a_{i\ell}\big[M_{\nu_r}, T^{i\ell}\big](\nabla_{\tan} f)_s \\
&\quad - \sum_{\ell\in 2\mathbb{N}_0}\sum_{1\le i\le H_\ell} a_{i\ell}\big[M_{\nu_s}, T^{i\ell}\big](\nabla_{\tan} f)_r \\
&\quad - \sum_{\ell\in 2\mathbb{N}_0}\sum_{1\le i\le H_\ell} a_{i\ell}\big[M_{\nu_r}, V^{i\ell}\big](\nu_s\nabla_{\tan} f) \\
&\quad + \sum_{\ell\in 2\mathbb{N}_0}\sum_{1\le i\le H_\ell} a_{i\ell}\big[M_{\nu_s}, V^{i\ell}\big](\nu_r\nabla_{\tan} f) \\
&= B_{rs} f + T\big(\partial_{\tau_{rs}} f\big) + \big[M_{\nu_r}, T\big](\nabla_{\tan} f)_s - \big[M_{\nu_s}, T\big](\nabla_{\tan} f)_r \\
&\quad - \big[M_{\nu_r}, V\big](\nu_s\nabla_{\tan} f) + \big[M_{\nu_s}, V\big](\nu_r\nabla_{\tan} f).
\end{aligned}
\tag{5.2.331}
$$

Above, the first equality uses (5.2.326), item *(iv)* in [69, Proposition 11.1.9] (used with $q := p$), (5.2.163) (used with $T := T^{i\ell}$), and (5.2.325). The second equality in (5.2.331) is based on (5.2.330) and (5.2.167) (used with $T := T^{i\ell}$ and $V := V^{i\ell}$). The last equality in (5.2.331) is a consequence of (5.2.312) and (5.2.328). The proof of (5.2.285) is therefore complete.

All other claims in the statement of Theorem 5.2.3 are proved in a similar fashion, using the spherical harmonic decomposition technique employed in the proof of [70, Theorem 2.5.38] together with the constant coefficient kernel case already dealt with in Theorem 5.2.2. □

To offer examples of chord-dot-normal singular integral operators of a different nature, work in the two-dimensional setting, and employ the notation

$$x^* := (x_2, -x_1) \text{ for each } x = (x_1, x_2) \in \mathbb{R}^2. \tag{5.2.332}$$

Note that, with 'dot' denoting the ordinary scalar product in $\mathbb{R}^2$, we have

$$x \cdot x^* = 0 \text{ and } x \cdot y = x^* \cdot y^*, \qquad \forall x, y \in \mathbb{R}^2, \tag{5.2.333}$$

which further entails the following useful orthogonal decomposition:

$$y = \frac{y \cdot x^*}{|x|^2} x^* + \frac{y \cdot x}{|x|^2} x, \qquad \forall x \in \mathbb{R}^2 \setminus \{0\}, \quad \forall y \in \mathbb{R}^2. \tag{5.2.334}$$

Lemma 5.2.4 *Let Ω be an Ahlfors regular domain in $\mathbb{R}^2$. Denote by ν geometric measure theoretic outward unit normal to Ω, and let $\tau = \mathrm{i}\nu$ be the geometric measure theoretic unit tangent along $\partial\Omega$* (*cf.* [68, (5.6.29)-(5.6.31)]). *Also, assume that $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ is a function of class $\mathscr{C}^N$, for some $N \in \mathbb{N}$, which is positive homogeneous of degree zero.*

Then the function

$$\begin{gathered} k : \mathbb{R}^2 \setminus \{0\} \longrightarrow \mathbb{R} \text{ given by} \\ k(x) := -\frac{(\nabla F)(x) \cdot x^*}{|x|^2} \text{ for each } x \in \mathbb{R}^2 \setminus \{0\} \end{gathered} \tag{5.2.335}$$

is of class $\mathscr{C}^{N-1}$, positive homogeneous of degree -2, and one has[36]

$$\begin{gathered} \partial_{\tau(y)}[F(x-y)] = \langle x-y, \nu(y)\rangle\, k(x-y) \\ \text{for } \mathcal{H}^1\text{-a.e. point } y \in \partial\Omega \text{ and each } x \in \mathbb{R}^2 \setminus \{y\}, \end{gathered} \tag{5.2.336}$$

as well as

$$\begin{gathered} \partial_{\tau(x)}[F(x-y)] = -\langle x-y, \nu(x)\rangle\, k(x-y) \\ \text{for } \mathcal{H}^1\text{-a.e. point } x \in \partial\Omega \text{ and each } y \in \mathbb{R}^2 \setminus \{x\}. \end{gathered} \tag{5.2.337}$$

Finally, if F is also even, then so is k.

Proof Since F is positive homogeneous of degree zero we have

$$F(tx) = F(x) \qquad \forall t > 0, \quad \forall x \in \mathbb{R}^2 \setminus \{0\}. \tag{5.2.338}$$

Differentiating with respect to t in the above equality implies $(\nabla F)(tx) \cdot x = 0$ for all $t \in (0, \infty)$ and $x \in \mathbb{R}^2 \setminus \{0\}$. In particular, corresponding to $t = 1$, we have

$$(\nabla F)(x) \cdot x = 0, \qquad \forall x \in \mathbb{R}^2 \setminus \{0\}. \tag{5.2.339}$$

Recall from [68, (5.6.30)] that if (ν_1, ν_2) are the scalar components of the vector ν then $\tau = (-\nu_2, \nu_1)$ at $\mathcal{H}^1$-a.e. point on $\partial\Omega$. In the notation introduced in (5.2.332), this translates into

$$\tau = \nu^* \text{ at } \mathcal{H}^1\text{-a.e. point on } \partial\Omega. \tag{5.2.340}$$

For $\mathcal{H}^1$-a.e. point $y \in \partial\Omega$ and each $x \in \mathbb{R}^2 \setminus \{y\}$ we may then write

[36] where ∂_τ denotes the directional derivative operator along the unit vector τ

$$\begin{aligned}\partial_{\tau(y)}[F(x-y)] &= \tau(y)\cdot\nabla_y[F(x-y)] = -\tau(y)\cdot(\nabla F)(x-y)\\ &= -\nu^*(y)\cdot\Big[\frac{(\nabla F)(x-y)\cdot(x-y)^*}{|x-y|^2}(x-y)^*\\ &\qquad\qquad + \frac{(\nabla F)(x-y)\cdot(x-y)}{|x-y|^2}(x-y)\Big]\\ &= -\left[\nu^*(y)\cdot(x-y)^*\right]\frac{(\nabla F)(x-y)\cdot(x-y)^*}{|x-y|^2}\\ &= -\langle x-y,\nu(y)\rangle\frac{(\nabla F)(x-y)\cdot(x-y)^*}{|x-y|^2}\\ &= \langle x-y,\nu(y)\rangle\, k(x-y). \end{aligned}\tag{5.2.341}$$

Above, the first equality is simply the definition of ∂_τ, the directional derivative operator along the unit vector τ. The second equality in (5.2.341) is just Chain Rule, and the third equality comes from (5.2.334). The fourth equality in (5.2.341) is then implied by (5.2.339), while the fifth equality follows from the second identity in (5.2.333) (used with $\nu(y)$ in place of x and with $x-y$ in place of y). The last equality in (5.2.341) is seen from (5.2.335). This establishes (5.2.336), and (5.2.337) is justified in a very similar fashion.

Next, differentiating (5.2.338) with respect to x yields $t(\nabla F)(tx) = (\nabla F)(x)$ for each $t\in(0,\infty)$ and each $x\in\mathbb{R}^2\setminus\{0\}$, ergo ∇F is a positive homogeneous function of degree -1 in $\mathbb{R}^2\setminus\{0\}$. Thus, the fact that the assignment

$$\mathbb{R}^2\setminus\{0\}\ni x\mapsto\frac{(\nabla F)(x)\cdot x^*}{|x|^2}\ \text{ is positive homogeneous of degree } -2 \tag{5.2.342}$$

follows easily from the observation above and the fact that $\mathbb{R}^2\setminus\{0\}\ni x\mapsto x^*$ is positive homogeneous of degree 1, while $\mathbb{R}^2\setminus\{0\}\ni x\mapsto|x|^2$ is positive homogeneous of degree 2. In view of (5.2.335), this analysis proves that k is positive homogeneous of degree -2 in $\mathbb{R}^2\setminus\{0\}$. It is also clear from definitions that k is a function of class $\mathscr{C}^{N-1}$.

Finally, it is clear from (5.2.335) that k is an even function whenever F is so. □

From Lemma 5.2.4 we see that:

singular integral operators whose kernels are the directional derivative, along the unit tangent vector to a two-dimensional UR domain, of smooth even functions which are positive homogeneous of degree zero in $\mathbb{R}^2\setminus\{0\}$ are singular integral operators of "chord-dot-normal" type. (5.2.343)

To all such singular integral operators the results from Theorem 5.2.2 then apply. Examples of functions F which are smooth, even, and positive homogeneous of degree zero in $\mathbb{R}^2\setminus\{0\}$ include

$$F(x) := \frac{P_m(x)}{|x|^m} \text{ for each } x \in \mathbb{R}^2 \setminus \{0\}, \tag{5.2.344}$$

where $m \in \mathbb{N}$ is even, and P_m is a homogeneous polynomial of degree m in $\mathbb{R}^2$. In particular, we may take

$$F(x) := \frac{x_1 x_2}{|x|^2} \text{ for each } x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}, \tag{5.2.345}$$

or, using complex notation,

$$F(z) := \frac{\overline{z}^k}{z^k} \text{ for each } z \in \mathbb{C} \setminus \{0\} \text{ and } k \in \mathbb{N}. \tag{5.2.346}$$

5.3 Another Look at Standard and Modified Riesz Transforms

We first consider standard and modified Riesz transforms in the entire Euclidean space. Throughout, fix $n \in \mathbb{N}$ satisfying $n \geq 2$. For each $j \in \{1, \dots, n-1\}$ define the (ordinary) j-th Riesz transform in $\mathbb{R}^{n-1}$ as the singular integral operator R_j acting on any given function $f \in L^1\big(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1}}\big)$ according to

$$R_j f(x') := \lim_{\varepsilon \to 0^+} \frac{2}{\omega_{n-1}} \int\limits_{\substack{y' \in \mathbb{R}^{n-1} \\ |x'-y'|>\varepsilon}} \frac{x_j - y_j}{|x' - y'|^n} f(y')\, dy' \tag{5.3.1}$$

for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$. These operators fall withing the scope of [70, Theorem 2.3.2], [70, Theorem 2.6.1], and Theorem 2.1.4 stated for the closed UR set

$$\Sigma := \mathbb{R}^{n-1} \times \{0\} \subseteq \mathbb{R}^n, \tag{5.3.2}$$

canonically identified with the Euclidean space $\mathbb{R}^{n-1}$, and the kernels

$$k_j \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\}), \quad k_j(x) := x_j/|x|^n \text{ for each } x \in \mathbb{R}^n \setminus \{0\}, \tag{5.3.3}$$

for $j \in \{1, \dots, n-1\}$, which are smooth, odd, and positive homogeneous of degree $1-n$ in $\mathbb{R}^n \setminus \{0\}$. The aforementioned Riesz transforms are also special instances of the singular integral operators discussed in Proposition 1.2.1, corresponding to the particular case when $\Omega := \mathbb{R}^n_+$, a scenario in which we have the identifications $\partial\Omega = \mathbb{R}^{n-1} \times \{0\} \equiv \mathbb{R}^{n-1}$, $\sigma \equiv \mathcal{L}^{n-1}$, and $\nu = -\mathbf{e}_n$. If we also take the function b from the statement of Proposition 1.2.1 to be

$$b \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\}), \quad b(x) := |x|^{n-2} \text{ for each } x \in \mathbb{R}^n \setminus \{0\} \tag{5.3.4}$$

then, up to normalization, the "tangential" singular integral operators T_{nj} corresponding to $1 \leq j \leq n-1$, defined as in (1.2.2) but for the present setting, agree with the Riesz transforms R_j with $1 \leq j \leq n-1$, defined as in (5.3.1). In addition, the singular integral operators $T^{\#}_{nj}$ with $1 \leq j \leq n-1$, defined as in (1.2.3) for the choice of Ω and b as above, also agree, up to normalization, with the Riesz transforms R_j with $1 \leq j \leq n-1$, defined in (5.3.1).

Significantly, for each $j \in \{1, \dots, n-1\}$ the j-th Riesz transform R_j also falls under the umbrella of Theorem 5.1.1 specialized to the case when $\Omega := \mathbb{R}^n_+$, with $\partial\Omega = \mathbb{R}^{n-1} \times \{0\}$ canonically identified with $\mathbb{R}^{n-1}$. Specifically, with b as in (5.3.4), when the vector-valued function (5.1.1) is presently taken to be

$$\vec{k} := (\partial_j b)\mathbf{e}_n - (\partial_n b)\mathbf{e}_j \in \left[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})\right]^n \tag{5.3.5}$$

(which is odd, positive homogeneous of degree $1-n$, and divergence-free in $\mathbb{R}^n \setminus \{0\}$), then the generalized double layers T and $T^{\#}$, defined as in (5.1.5)-(5.1.6) for these choices, agree, up to normalization, with R_j. See also Example 5.1.4 in this regard.

From [70, Theorem 2.3.2], Proposition 1.2.1, Theorem 2.1.4, and [70, Theorem 2.6.1] we then see that each Riesz transform acts naturally on a multitude of function spaces in $\mathbb{R}^{n-1}$. For example, for each index $j \in \{1, \dots, n-1\}$, the following are well-defined, linear, and bounded operators:

$$R_j : L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < p < \infty, \tag{5.3.6}$$

$$R_j : L^p(\mathbb{R}^{n-1}, w) \to L^p(\mathbb{R}^{n-1}, w),\ 1 < p < \infty,\ w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{5.3.7}$$

$$R_j : L^1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to L^{1,\infty}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{5.3.8}$$

$$R_j : L^{p,q}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to L^{p,q}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < p < \infty,\ 0 < q \le \infty, \tag{5.3.9}$$

$$R_j : H^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to H^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ \tfrac{n-1}{n} < p < \infty, \tag{5.3.10}$$

$$R_j : H^{p,q}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to H^{p,q}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ \tfrac{n-1}{n} < p < \infty,\ 0 < q \le \infty, \tag{5.3.11}$$

$$R_j : L^p_1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to L^p_1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < p < \infty, \tag{5.3.12}$$

$$R_j : L^p_{-1}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to L^p_{-1}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < p < \infty, \tag{5.3.13}$$

$$R_j : L^p_1(\mathbb{R}^{n-1}, w) \to L^p_1(\mathbb{R}^{n-1}, w),\ 1 < p < \infty,\ w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{5.3.14}$$

$$R_j : L^p_{-1}(\mathbb{R}^{n-1}, w) \to L^p_{-1}(\mathbb{R}^{n-1}, w),\ 1 < p < \infty,\ w \in A_p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{5.3.15}$$

$$R_j : M^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to M^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < p < \infty,\ 0 < \lambda < n-1, \tag{5.3.16}$$

$$R_j : M^{p,\lambda}_1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to M^{p,\lambda}_1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < p < \infty,\ 0 < \lambda < n-1, \tag{5.3.17}$$

$$R_j : \mathring{M}^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \mathring{M}^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < p < \infty,\ 0 < \lambda < n-1, \tag{5.3.18}$$

$$R_j : \mathring{M}^{p,\lambda}_1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \mathring{M}^{p,\lambda}_1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < p < \infty,\ 0 < \lambda < n-1, \tag{5.3.19}$$

$$R_j : \mathcal{B}^{q,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \mathcal{B}^{q,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < q < \infty,\ 0 < \lambda < n-1, \tag{5.3.20}$$

$$R_j : \mathcal{B}^{q,\lambda}_1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \mathcal{B}^{q,\lambda}_1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < q < \infty,\ 0 < \lambda < n-1, \tag{5.3.21}$$

$$R_j : \mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \mathfrak{L}^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < p < \infty,\ 0 < \lambda < n-1, \tag{5.3.22}$$

$$R_j : C^p_{q,\eta}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to C^p_{q,\eta}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 < p, q < \infty,\ 0 < \eta < 1. \tag{5.3.23}$$

Higher order versions of the smoothness spaces above are also allowed.

Slightly digressing, here is an invertibility result involving linear combinations of Riesz transforms in the entire Euclidean setting which is going to be relevant later on.

Proposition 5.3.1 *Fix $n \in \mathbb{N}$ with $n \geq 2$. Also, pick a family of n numbers $\lambda_0, \lambda_1 \dots, \lambda_{n-1} \in \mathbb{C}$, not all zero, and define*

$$T := \lambda_0 I + \sum_{j=1}^{n-1} \lambda_j R_j, \tag{5.3.24}$$

where I is the identity operator and R_j is the j-th Riesz transform in $\mathbb{R}^{n-1}$ (cf. (5.3.1)). Then

$$\text{if } n \geq 3, \text{ then } T \text{ is a linear, bounded, injective operator with dense range from } L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ into itself.} \tag{5.3.25}$$

In addition, the following properties are equivalent:

(a) The operator T is invertible on $L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ for each $p \in (1, \infty)$.

(b) The operator T is invertible on $L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$.

(c) One has

$$\sum_{j=1}^{n-1} \lambda_j \xi_j \neq (-\mathrm{i})\lambda_0 |\xi'| \text{ for each } \xi' = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1} \setminus \{0\}. \tag{5.3.26}$$

Proof For starters, from (5.3.24) and (5.3.6) we see that T is a well-defined linear and bounded operator on each $L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ with $p \in (1, \infty)$. Bring in the Fourier transform $\mathcal{F}'$ in $\mathbb{R}^{n-1}$. Since, as is well known (see, e.g., [66, (4.9.15), p. 183]), for each $f \in L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ and each $j \in \{1, \dots, n-1\}$ we have

$$\mathcal{F}'\big(R_j f\big)(\xi') = (-\mathrm{i})\frac{\xi_j}{|\xi'|}\big(\mathcal{F}' f\big)(\xi') \text{ for all } \xi' \in \mathbb{R}^{n-1} \setminus \{0\}, \tag{5.3.27}$$

it follows that

$$\mathcal{F}'\big(Tf\big) = m\mathcal{F}' f \text{ for each } f \in L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{5.3.28}$$

where

$$m(\xi') := \lambda_0 + (-\mathrm{i})\sum_{j=1}^{n-1} \frac{\lambda_j \xi_j}{|\xi'|} \text{ for all } \xi' = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1} \setminus \{0\}. \tag{5.3.29}$$

In particular,

$$m \text{ belongs to } \mathscr{C}^\infty\big(\mathbb{R}^{n-1} \setminus \{0\}\big) \cap L^\infty(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ and is positive homogeneous of degree zero.} \tag{5.3.30}$$

We also claim that

$$\text{if } n \geq 3 \text{ then } m(\xi') \neq 0 \text{ for } \mathcal{L}^{n-1}\text{-a.e. } \xi' \in \mathbb{R}^{n-1}. \tag{5.3.31}$$

When $\lambda_0 = 0$, this simply follows by observing that the zeros $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ of m are contained in the intersection of the following two hyperplanes in $\mathbb{R}^{n-1}$,

$$\sum_{j=1}^{n-1} (\mathrm{Re}\,\lambda_j)\xi_j = 0 \ \text{ and } \ \sum_{j=1}^{n-1} (\mathrm{Im}\,\lambda_j)\xi_j = 0, \tag{5.3.32}$$

at least one of which is non-degenerate. There remains to justify (5.3.31) in the case when $\lambda_0 \neq 0$. Note that if $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$ is such that $m(\xi') = 0$ then formula (5.3.29) implies that we necessarily have $(-\mathrm{i})\lambda_0|\xi'| = \sum_{j=1}^{n-1} \lambda_j\xi_j$. Hence, the zeros of m are among the solutions of the equation

$$-\lambda_0^2|\xi'|^2 = \Big(\sum_{j=1}^{n-1} \lambda_j\xi_j\Big)^2 \tag{5.3.33}$$

or, equivalently, the polynomial equation

$$\lambda_0^2\big(\xi_1^2 + \cdots + \xi_{n-1}^2\big) + \sum_{j,k=1}^{n-1} \lambda_j\lambda_k\xi_j\xi_k = 0. \tag{5.3.34}$$

If $n \geq 3$, the polynomial in the left-side of (5.3.34) is not identically zero, since the latter eventuality amounts to having

$$\lambda_0^2 + \lambda_\ell^2 = 0 \ \text{ and } \ \lambda_j\lambda_k = 0 \ \text{ for all } \ \ell, j, k \in \{1, \ldots, n-1\} \ \text{ with } \ j \neq k, \tag{5.3.35}$$

itself an impossibility for $n \geq 3$ (given that we are presently assuming $\lambda_0 \neq 0$). As such, the claim made in (5.3.31) follows as soon as we show that, for each $n \in \mathbb{N}$ with $n \geq 2$,

$$\begin{gathered}\mathcal{L}^{n-1}\big(\phi^{-1}(\{0\})\big) = 0 \ \text{ for any polynomial function} \\ \phi : \mathbb{R}^{n-1} \to \mathbb{C} \ \text{ not identically zero.}\end{gathered} \tag{5.3.36}$$

We shall prove (5.3.36) by induction on n. If $n = 2$, this is readily implied by the Fundamental Theorem of Algebra, which ensures that $\phi^{-1}(\{0\})$ has finite cardinality in this case. Assume (5.3.36) holds for a given $n \in \mathbb{N}$ with $n \geq 2$, and consider a polynomial function $\phi : \mathbb{R}^{n-1} \to \mathbb{C}$ which is not identically zero. Write ϕ as

$$\phi(x', x_n) = \sum_{j=0}^{N} \phi_j(x')x_n^j \ \text{ for each } \ (x', x_n) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}, \tag{5.3.37}$$

for some integer $N \in \mathbb{N}_0$ and some polynomial functions $\phi_j : \mathbb{R}^{n-1} \to \mathbb{C}$ with index $j \in \{0, 1, \ldots, N\}$, at least one of which is not identically zero. Denote by j_* the index for which the latter property holds. Also, for each $x' \in \mathbb{R}^{n-1}$ define the polynomial of one variable

$$\psi_{x'}(t) := \sum_{j=0}^{N} \phi_j(x')t^j \text{ for each } t \in \mathbb{R}. \tag{5.3.38}$$

Next, decompose the closed set $\phi^{-1}(\{0\})$ as

$$\phi^{-1}(\{0\}) = Z_1 \cup Z_2 \tag{5.3.39}$$

where

$$Z_1 := \Big\{(x', x_n) \in \mathbb{R}^n : \phi_j(x') = 0 \text{ for each } j \in \{0, 1, \dots, N\}\Big\}, \tag{5.3.40}$$

and

$$Z_2 := \Big\{(x', x_n) \in \mathbb{R}^n : \text{there exists some } j_o \in \{0, 1, \dots, N\} \text{ with } \phi_{j_o}(x') \neq 0 \text{ and } x_n \in \psi_{x'}^{-1}(\{0\})\Big\}. \tag{5.3.41}$$

The inductive hypothesis implies $\mathcal{L}^{n-1}\big(\phi_{j_*}^{-1}(\{0\})\big) = 0$ and since $Z_1 \subseteq \phi_{j_*}^{-1}(\{0\}) \times \mathbb{R}$ we conclude (based on Fubini's Theorem) that

$$\mathcal{L}^n(Z_1) = 0. \tag{5.3.42}$$

Also, for each $(x', x_n) \in Z_2$ the polynomial $\psi_{x'}$ is not identically zero, so the Fundamental Theorem of Algebra guarantees that $\psi_{x'}^{-1}(\{0\})$ has finite cardinality. In particular, if we set

$$Z_2' := \Big\{x' \in \mathbb{R}^{n-1} : \text{ there exists } x_n \in \mathbb{R} \text{ such that } (x', x_n) \in Z_2\Big\}, \tag{5.3.43}$$

then $\mathcal{L}^1\big(\psi_{x'}^{-1}(\{0\})\big) = 0$ for each $x' \in Z_2'$. Based on this and Fubini's Theorem we may now compute

$$\mathcal{L}^n(Z_2) = \int_{Z_2} 1 \, \mathrm{d}\mathcal{L}^n = \int_{Z_2'} \Big(\int_{\psi_{z'}^{-1}(\{0\})} 1 \, \mathrm{d}\mathcal{L}^1\Big) \, \mathrm{d}\mathcal{L}^{n-1}(z') = 0. \tag{5.3.44}$$

From (5.3.39), (5.3.42), and (5.3.44) we ultimately conclude that $\mathcal{L}^n\big(\phi(\{0\})\big) = 0$. This finishes the proof of (5.3.36), so (5.3.31) is now fully justified.

In turn, from (5.3.28), (5.3.31), and the fact that $\mathcal{F}'$ is an isomorphism of $L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ we deduce that

$$T : L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ is an injective operator if } n \geq 3. \tag{5.3.45}$$

With $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ replaced by $\overline{\lambda_0}, -\overline{\lambda_1}, \dots, -\overline{\lambda_{n-1}}$, this also proves that

$$T^* = \overline{\lambda_0} I - \sum_{j=1}^{n-1} \overline{\lambda_j} R_j : L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ is injective if } n \geq 3. \tag{5.3.46}$$

Having established (5.3.46) we then conclude from [69, (2.1.45)] that the operator

$$T : L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ has dense range if } n \geq 3. \tag{5.3.47}$$

Together, (5.3.45) and (5.3.47) finish the proof of (5.3.25).

Let us now turn to the equivalences claimed in the last part of the statement. First, it is clear that *(a)* $\Rightarrow$ *(b)*. To prove the implication *(b)* $\Rightarrow$ *(c)*, assume T is invertible on $L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$. Since $\mathcal{F}'$ is an isomorphism of $L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$, from (5.3.28) we conclude that the mapping

$$L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \ni f \longmapsto mf \in L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \tag{5.3.48}$$

is an isomorphism. This is equivalent to having

$$m(\xi') \neq 0 \text{ for } \mathcal{L}^{n-1}\text{-a.e. } \xi' \in \mathbb{R}^{n-1} \text{ and } m^{-1} \in L^\infty(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}). \tag{5.3.49}$$

Keeping (5.3.29) in mind, (5.3.26) is readily implied by (5.3.49).

There remains to justify *(c)* $\Rightarrow$ *(a)*. To this end, fix an arbitrary $p \in (1, \infty)$ and assume (5.3.26) holds. The latter property ensures that

$$m^{-1} \text{ belongs to } \mathscr{C}^\infty\big(\mathbb{R}^{n-1} \setminus \{0\}\big) \cap L^\infty(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ and is positive homogeneous of degree zero.} \tag{5.3.50}$$

Consequently, for each multi-index $\alpha \in \mathbb{N}_0^{n-1}$ the function $\partial^\alpha m^{-1}$ is continuous in $\mathbb{R}^{n-1} \setminus \{0\}$ and positive homogeneous of degree $-|\alpha|$ hence, in particular, there exists $C_\alpha \in (0, \infty)$ such that

$$\big|\partial^\alpha m^{-1}(\xi')\big| \leq C_\alpha |\xi'|^{-|\alpha|} \text{ for each } \xi' \in \mathbb{R}^{n-1} \setminus \{0\}. \tag{5.3.51}$$

Granted this, Mikhlin's Multiplier Theorem (cf., e.g., [27, Theorem 6.3, p. 210]) applies and gives that there exists a linear operator

$$Q \text{ bounded both on } L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ and on } L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \text{ with } \mathcal{F}'(Qf) = m^{-1}\mathcal{F}'f \text{ if } f \in L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \cap L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}). \tag{5.3.52}$$

From this, (5.3.28), the fact that $\mathcal{F}'$ is an isomorphism of $L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$, and that T is bounded both on $L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ and on $L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$, we conclude that $Q(Tf) = f$ and $T(Qf) = f$ for each $f \in L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \cap L^2(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$. By density, it follows that $QT = I = TQ$ on the entire space $L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$, hence T is invertible on $L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$. □

Moving on, for each $j \in \{1, \dots, n-1\}$ we shall denote by R_j^{mod} the j-th modified Riesz transform in $\mathbb{R}^{n-1}$, i.e., the singular integral operator acting on any given function $f \in L^1\big(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\big)$ at $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ according to

$$\begin{aligned}\big(R_j^{\text{mod}} f\big)(x') \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad & (5.3.53)\\ := \lim_{\varepsilon\to 0^+} \frac{2}{\omega_{n-1}} \int_{\mathbb{R}^{n-1}} \Bigg\{ \frac{x_j - y_j}{|x'-y'|^n} \mathbf{1}_{\mathbb{R}^{n-1}\setminus\overline{B_{n-1}(x',\varepsilon)}}(y') & \\ - \frac{-y_j}{|-y'|^n} \mathbf{1}_{\mathbb{R}^{n-1}\setminus\overline{B_{n-1}(0',1)}}(y') \Bigg\} f(y')\, dy' &\end{aligned}$$

where, generally speaking, $B_{n-1}(z', r)$ is the $(n-1)$-dimensional ball centered at $z' \in \mathbb{R}^{n-1}$ and of radius r.

For each $j \in \{1, \dots, n-1\}$, this agrees, up to normalization, with the operator T_{nj}^{mod} defined as in (1.8.168)-(1.8.170) in the case when $\Omega := \mathbb{R}^n_+$ and for b as in (5.3.4).

The format of the j-th modified Riesz transform R_j^{mod} also agrees with the general recipe from (A.0.202), presently implemented for Σ as in (5.3.2), which is a closed UR subset of $\mathbb{R}^n$, and the integral kernel k_j as in (5.3.3). It also agrees, up to normalization, with the modified generalized double layer defined as in (5.1.65) in the case when $\Omega := \mathbb{R}^n_+$ and for the vector-valued kernel $\vec{k}$ as in (5.3.5). In particular, (5.1.69) implies that

$$\begin{gathered} R_j^{\text{mod}} \text{ maps constant functions on } \mathbb{R}^{n-1} \\ \text{into constant functions on } \mathbb{R}^{n-1}, \text{ for each } j \in \{1, \dots, n-1\}. \end{gathered} \tag{5.3.54}$$

This makes it possible to consider the action of the modified Riesz transforms on quotient spaces (of equivalence classes modulo constants), a scenario in which we define

$$\big[R_j^{\text{mod}}\big][f] := \big[R_j^{\text{mod}} f\big]. \tag{5.3.55}$$

With this convention in mind, from items *(14)*, *(15)*, *(16)*, *(17)*, *(20)*, *(21)*, *(22)* in Theorem 5.1.1 we see that for each $j \in \{1, \dots, n-1\}$ the following are well-defined, linear, and bounded operators:

$$R_j^{\text{mod}} : \dot{L}_1^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \dot{L}_1^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ with } 1 < p < \infty, \tag{5.3.56}$$

$$\big[R_j^{\text{mod}}\big] : \dot{L}_1^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \,\to \dot{L}_1^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \;\text{ with } 1 < p < \infty, \tag{5.3.57}$$

$$R_j^{\text{mod}} : \dot{M}_1^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \dot{M}_1^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ with } 1 < p < \infty,\ 0 < \lambda < n-1, \tag{5.3.58}$$

$$\big[R_j^{\text{mod}}\big] : \dot{M}_1^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \,\to \dot{M}_1^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \text{ with } 1 < p < \infty,\ 0 < \lambda < n-1, \tag{5.3.59}$$

$$R_j^{\text{mod}} : \dot{\mathbb{M}}_1^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \dot{\mathbb{M}}_1^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ with } 1 < p < \infty,\ 0 < \lambda < n-1, \tag{5.3.60}$$

$$\big[R_j^{\text{mod}}\big] : \dot{\mathbb{M}}_1^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \,\to \dot{\mathbb{M}}_1^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \text{ with } 1 < p < \infty,\ 0 < \lambda < n-1, \tag{5.3.61}$$

$$R_j^{\text{mod}} : \dot{\mathcal{B}}_1^{q,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \dot{\mathcal{B}}_1^{q,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ with } 1 < p < \infty,\ 0 < \lambda < n-1, \tag{5.3.62}$$

$$\big[R_j^{\text{mod}}\big] : \dot{\mathcal{B}}_1^{q,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \,\to \dot{\mathcal{B}}_1^{q,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \text{ with } 1 < p < \infty,\ 0 < \lambda < n-1, \tag{5.3.63}$$

$$R_j^{\text{mod}} : \dot{H}_1^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \dot{H}_1^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ with } \tfrac{n-1}{n} < p < \infty, \tag{5.3.64}$$

$$\big[R_j^{\text{mod}}\big] : \dot{H}_1^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \,\to \dot{H}_1^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \text{ with } \tfrac{n-1}{n} < p < \infty, \tag{5.3.65}$$

$$R_j^{\text{mod}} : \dot{\mathscr{C}}^\alpha(\mathbb{R}^{n-1}) \longrightarrow \dot{\mathscr{C}}^\alpha(\mathbb{R}^{n-1}) \text{ with } 0 < \alpha < 1, \tag{5.3.66}$$

$$\big[R_j^{\text{mod}}\big] : \dot{\mathscr{C}}^\alpha(\mathbb{R}^{n-1})/\sim \,\to \dot{\mathscr{C}}^\alpha(\mathbb{R}^{n-1})/\sim \;\text{ with } 0 < \alpha < 1, \tag{5.3.67}$$

$$R_j^{\text{mod}} : \dot{\mathscr{C}}^\alpha_{\text{van}}(\mathbb{R}^{n-1}) \longrightarrow \dot{\mathscr{C}}^\alpha_{\text{van}}(\mathbb{R}^{n-1}) \text{ with } 0 < \alpha < 1, \tag{5.3.68}$$

$$\big[R_j^{\text{mod}}\big] : \dot{\mathscr{C}}^\alpha_{\text{van}}(\mathbb{R}^{n-1})/\sim \,\to \dot{\mathscr{C}}^\alpha_{\text{van}}(\mathbb{R}^{n-1})/\sim \;\text{ with}\,,\, 0 < \alpha < 1, \tag{5.3.69}$$

as well as

$$R_j^{\rm mod} : \mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \longrightarrow \mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{5.3.70}$$

$$\big[R_j^{\rm mod}\big] : \widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \widetilde{\mathrm{BMO}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{5.3.71}$$

$$R_j^{\rm mod} : \mathrm{VMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \longrightarrow \mathrm{VMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{5.3.72}$$

$$\big[R_j^{\rm mod}\big] : \widetilde{\mathrm{VMO}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \widetilde{\mathrm{VMO}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{5.3.73}$$

$$R_j^{\rm mod} : \mathrm{CMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \longrightarrow \mathrm{CMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{5.3.74}$$

$$\big[R_j^{\rm mod}\big] : \widetilde{\mathrm{CMO}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \widetilde{\mathrm{CMO}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \tag{5.3.75}$$

$$\begin{aligned} R_j^{\rm mod} &: \dot{\mathfrak{L}}^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \dot{\mathfrak{L}}^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \\ &\qquad \text{with } 1 < p < \infty,\ 0 < \lambda < n-1, \end{aligned} \tag{5.3.76}$$

$$\begin{aligned} \big[R_j^{\rm mod}\big] &: \dot{\mathfrak{L}}^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \,\to \dot{\mathfrak{L}}^{p,\lambda}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \\ &\qquad \text{with } 1 < p < \infty,\ 0 < \lambda < n-1, \end{aligned} \tag{5.3.77}$$

$$\begin{aligned} R_j^{\rm mod} &: \dot{\mathcal{C}}^{p}_{q,\eta}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \to \dot{\mathcal{C}}^{p}_{q,\eta}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\ 1 \le p < \infty \\ &\qquad \text{with } 1 < q < \infty,\ 0 < \eta < 1, \end{aligned} \tag{5.3.78}$$

$$\begin{aligned} \big[R_j^{\rm mod}\big] &: \dot{\mathcal{C}}^{p}_{q,\eta}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \,\to \dot{\mathcal{C}}^{p}_{q,\eta}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})/\sim \\ &\qquad \text{with } 1 < p, q < \infty,\ 0 < \eta < 1. \end{aligned} \tag{5.3.79}$$

In addition, there are natural mapping properties of the Riesz transforms acting on Besov and Triebel-Lizorkin spaces in the entire Euclidean ambient, like the ones in item *(19)* of Theorem 5.1.1.

Recall that for each $j \in \{1, \dots, n-1\}$ we have identified (up to a common normalization constant) the "ordinary" j-th Riesz transform R_j with the operator $T^{\#}_{nj}$, and the j-th modified Riesz transform $R_j^{\rm mod}$ with the operator $T_{nj}^{\rm mod}$. In view of these identifications and (2.1.161)-(2.1.162) we then conclude that for each index $j \in \{1, \dots, n-1\}$ we have (with the duality brackets as in [69, Theorem 4.6.1])

$$\begin{gathered} \big\langle \big[R_j^{\rm mod} f\big], g\big\rangle = -\big\langle [f], R_j g\big\rangle \ \text{ for any two functions} \\ f \in \mathrm{BMO}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \subset L^1\Big(\mathbb{R}^{n-1}, \tfrac{dx'}{1+|x'|^n}\Big) \ \text{ and } \ g \in H^1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \end{gathered} \tag{5.3.80}$$

as well as

$$\begin{gathered}\left\langle \left[R_j^{\text{mod}} f \right], g \right\rangle = -\left\langle [f], R_j g \right\rangle \text{ for each} \\ f \in \dot{\mathscr{C}}^\alpha(\mathbb{R}^{n-1}) \subset L^1\Big(\mathbb{R}^{n-1}, \tfrac{dx'}{1+|x'|^n}\Big) \text{ and } g \in H^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \\ \text{with } p \in \big(\tfrac{n-1}{n}, 1\big) \text{ and } \alpha := (n-1)\big(\tfrac{1}{p} - 1\big) \in (0,1). \end{gathered} \tag{5.3.81}$$

From (5.1.146) and (5.1.71)-(5.1.73) we also see that if $p \in (1, \infty)$ then for each function

$$\begin{gathered} f \in L^1\big(\mathbb{R}^{n-1}, \tfrac{dx'}{1+|x'|^n}\big) \cap L^p_{\text{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \text{ such that} \\ \partial_k f \in L^1\big(\mathbb{R}^{n-1}, \tfrac{dx'}{1+|x'|^{n-1}}\big) \cap L^p_{\text{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \\ \text{for all } \ k \in \{1, \dots, n-1\} \end{gathered} \tag{5.3.82}$$

it follows that for each $j \in \{1, \dots, n-1\}$ we have

$$R_j^{\text{mod}} f \in L^p_{1,\text{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \tag{5.3.83}$$

and

$$\partial_k \big(R_j^{\text{mod}} f\big) = R_j\big(\partial_k f\big) \text{ for each } k \in \{1, \dots, n-1\}. \tag{5.3.84}$$

In particular, for each $j, k \in \{1, \dots, n-1\}$ and each $p \in (1, \infty)$ it follows that

$$\begin{gathered} \partial_k \big(R_j^{\text{mod}} f\big) = R_j\big(\partial_k f\big) \text{ for each } f \in \dot{L}^p_1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \\ \text{and } \partial_k(R_j f) = R_j(\partial_k f) \text{ for each } f \in L^p_1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}). \end{gathered} \tag{5.3.85}$$

Shifting perspectives, bring in the modified boundary-to-boundary Cauchy-Clifford integral operator $\mathfrak{C}_{\text{mod}}$, originally introduced in (1.8.112) and re-branded as such in Remark 1.8.27. When specialized to the case when $\Omega := \mathbb{R}^n_+$, with $\partial\Omega = \mathbb{R}^{n-1} \times \{0\}$ canonically identified with $\mathbb{R}^{n-1}$ and with $\nu = -\mathbf{e}_n$, this becomes the principal-value singular integral operator acting on any given function $f \in L^1\big(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\big) \otimes C\ell_n$ at $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ according to

$$\begin{aligned} &\big(\mathfrak{C}_{\text{mod}} f\big)(x') \\ &= \lim_{\varepsilon \to 0^+} \frac{-1}{\omega_{n-1}} \int_{\mathbb{R}^{n-1}} \Bigg\{ \frac{x'-y'}{|x'-y'|^n} \mathbf{1}_{\mathbb{R}^n \setminus \overline{B_{n-1}(x',\varepsilon)}}(y') \\ &\qquad - \frac{-y'}{|-y'|^n} \mathbf{1}_{\mathbb{R}^n \setminus \overline{B_{n-1}(0',1)}}(y') \Bigg\} \odot \mathbf{e}_n \odot f(y')\, dy'. \end{aligned} \tag{5.3.86}$$

Hence, in terms of the modified Riesz transforms introduced earlier, we may express this as

$$\mathfrak{C}_{\text{mod}} = -\frac{1}{2} \sum_{j=1}^{n-1} \mathbf{e}_j \odot \mathbf{e}_n \odot R_j^{\text{mod}} \text{ on } L^1\Big(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\Big) \otimes C\ell_n. \tag{5.3.87}$$

Then the fact that the formula noted in (1.8.281) presently becomes (with I denoting the identity operator)

$$\big[\mathfrak{C}_{\rm mod}\big]^2 = \tfrac{1}{4}I \ \text{ on } \ \big(\dot{L}^p_1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})\otimes \mathcal{C}\!\ell_n\big)\big/\sim \ \text{ with } \ p\in(1,\infty), \tag{5.3.88}$$

implies that, when acting on equivalence classes (modulo constants) of scalar-valued functions in the space $\dot{L}^p_1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ with $p\in(1,\infty)$, we have

$$\begin{aligned} I &= 4\big[\mathfrak{C}_{\rm mod}\big]^2 = \Big(\sum_{j=1}^{n-1}\mathbf{e}_j\odot\mathbf{e}_n\big[R_j^{\rm mod}\big]\Big)\Big(\sum_{k=1}^{n-1}\mathbf{e}_k\odot\mathbf{e}_n\big[R_k^{\rm mod}\big]\Big) \\ &= \sum_{j,k=1}^{n-1}\mathbf{e}_j\odot\mathbf{e}_n\odot\mathbf{e}_k\odot\mathbf{e}_n\big[R_j^{\rm mod}\big]\big[R_k^{\rm mod}\big] = \sum_{j,k=1}^{n-1}\mathbf{e}_j\odot\mathbf{e}_k\big[R_j^{\rm mod}\big]\big[R_k^{\rm mod}\big] \\ &= -\sum_{j=1}^{n-1}\big[R_j^{\rm mod}\big]^2 + \sum_{1\le j<k\le n-1}\mathbf{e}_j\odot\mathbf{e}_k\Big[\big[R_j^{\rm mod}\big],\big[R_k^{\rm mod}\big]\Big], \end{aligned}\tag{5.3.89}$$

where the last set of brackets stand for the commutator $[A,B]:=AB-BA$. In turn, (5.3.89) amounts to saying that, on the quotient space $\dot{L}^p_1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})/\sim$ with $p\in(1,\infty)$, we have

$$\sum_{j=1}^{n-1}\big[R_j^{\rm mod}\big]^2 = -I, \tag{5.3.90}$$

and

$$\big[R_j^{\rm mod}\big]\big[R_k^{\rm mod}\big] = \big[R_k^{\rm mod}\big]\big[R_j^{\rm mod}\big] \ \text{ for each } \ j,k\in\{1,\dots,n-1\}. \tag{5.3.91}$$

In view of (5.3.55) and (5.3.54), we may equivalently recast these operator identities as the statement that, on the space $\dot{L}^p_1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ with $p\in(1,\infty)$,

$$\sum_{j=1}^{n-1}\big(R_j^{\rm mod}\big)^2 = -I \ \text{ modulo constants,} \tag{5.3.92}$$

and

$$R_j^{\rm mod}R_k^{\rm mod} = R_k^{\rm mod}R_j^{\rm mod} \ \text{ modulo constants, for each } \ j,k\in\{1,\dots,n-1\}. \tag{5.3.93}$$

Moreover, we observe that

the very same identities (5.3.92)-(5.3.93) are also valid on a variety of other function spaces of interest, in place of the homogeneous Sobolev space employed above, namely on the homogeneous Morrey-based Sobolev space $\dot{M}^{p,\lambda}_1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ with $1<p<\infty$ and $0<\lambda<n-1$, on the homogeneous block-based Sobolev space $\dot{\mathcal{B}}^{q,\lambda}_1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ with $1<q<\infty$ and $0<\lambda<n-1$, on the homogeneous Hardy-based Sobolev space $\dot{H}^p_1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ with $\frac{n-1}{n}<p<\infty$, on the homogeneous Hölder space $\dot{\mathscr{C}}^{\alpha}(\mathbb{R}^{n-1})$ with $0<\alpha<1$, and on the John-Nirenberg space $\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$. (5.3.94)

Indeed, the same proof which, starting with (5.3.88), has produced (5.3.92)-(5.3.93) continues to work in any of those settings, thanks to Remark 3.3.13, Remark 2.3.12, (2.1.189), and (2.1.186).

In particular, from (5.3.92) written on the space $\mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ and (5.3.72) we conclude that

$$\text{for any function } f\in \mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1}) \text{ one has}$$
$$f\in \mathrm{VMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1}) \Leftrightarrow \begin{cases} R^{\mathrm{mod}}_j f\in \mathrm{VMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1}) \\ \text{for all } j\in\{1,\dots,n-1\}. \end{cases} \tag{5.3.95}$$

This should be compared with the similar result, in the unit, sphere established in Proposition 5.3.2 (cf. (5.3.107)). Also, from (5.3.92) written on the homogeneous Morrey-based Sobolev space $\dot{M}^{p,\lambda}_1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ and (5.3.60) we conclude that

$$\text{for any } f\in \dot{M}^{p,\lambda}_1(\mathbb{R}^{n-1},\mathcal{L}^{n-1}) \text{ with } 1<p<\infty \text{ and } 0<\lambda<n-1 \text{ one has}$$
$$f\in \dot{\mathbb{M}}^{p,\lambda}_1(\mathbb{R}^{n-1},\mathcal{L}^{n-1}) \Leftrightarrow R^{\mathrm{mod}}_j f\in \dot{\mathbb{M}}^{p,\lambda}_1(\mathbb{R}^{n-1},\mathcal{L}^{n-1}) \ \text{ for all } j\in\{1,\dots,n-1\}. \tag{5.3.96}$$

From (5.3.92) and (5.3.56)-(5.3.71) we may also obtain “mixed” regularity results of the following sort:

$$\text{for any given } f\in \mathrm{BMO}(\mathbb{R}^{n-1},\mathcal{L}^{n-1}) \text{ and } \alpha\in(0,1) \text{ one has}$$
$$f\in \dot{\mathscr{C}}^{\alpha}(\mathbb{R}^{n-1}) \Leftrightarrow R^{\mathrm{mod}}_j f\in \dot{\mathscr{C}}^{\alpha}(\mathbb{R}^{n-1}) \ \text{ for all } j\in\{1,\dots,n-1\}. \tag{5.3.97}$$

All the considerations so far in this section apply to the Hilbert transform on the real line, i.e., the singular integral operator H acting on any given function $f\in L^1\big(\mathbb{R},\frac{dx}{1+|x|}\big)$ according to

$$Hf(x):=\lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int\limits_{\substack{y\in\mathbb{R}\\ |x-y|>\varepsilon}}\frac{f(y)}{x-y}\,dy \quad \text{for } \mathcal{L}^1\text{-a.e. } x\in\mathbb{R}, \tag{5.3.98}$$

and its modified version, H_{mod}, which is the singular integral operator acting on any given function $f\in L^1\big(\mathbb{R},\frac{dx}{1+|x|^2}\big)$ at $\mathcal{L}^1$-a.e. point $x\in\mathbb{R}$ according to

$$\big(H_{\rm mod}f\big)(x) := \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{\mathbb{R}}\left\{\frac{1}{x-y}\mathbf{1}_{\mathbb{R}\setminus[x-\varepsilon,x+\varepsilon]}(y)+\frac{1}{y}\mathbf{1}_{\mathbb{R}\setminus[-1,1]}(y)\right\}f(y)\,{\rm d}y. \tag{5.3.99}$$

In particular,

$$\big(H_{\rm mod}\big)^2 = -I \ \text{ modulo constants,} \tag{5.3.100}$$

on any of the spaces mentioned in relation to (5.3.92) above (cf. (5.3.94)).

In particular, from (5.3.100) written on the John-Nirenberg space $\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)$ and the fact that $H_{\rm mod}$ maps $\mathrm{VMO}(\mathbb{R},\mathcal{L}^1)$ into itself we conclude that

$$\begin{gathered}\text{for any given function } f\in \mathrm{BMO}(\mathbb{R},\mathcal{L}^1) \text{ one has}\\ f\in \mathrm{VMO}(\mathbb{R},\mathcal{L}^1) \Longleftrightarrow H_{\rm mod}f\in \mathrm{VMO}(\mathbb{R},\mathcal{L}^1).\end{gathered} \tag{5.3.101}$$

Our last result in this section elaborates on the mapping properties of the Riesz transforms on smooth surfaces, and contains a remarkable characterization of the Sarason space VMO in this setting (cf. (5.3.107)).

Proposition 5.3.2 *Fix $n\in\mathbb{N}$ with $n\geq 2$, and consider the Riesz transforms $(R_j)_{1\leq j\leq n}$ associated as in (A.0.187) with $\Sigma := S^{n-1}$. Then for each $j\in\{1,\dots,n\}$, the operator R_j induces mappings*

$$R_j : \mathrm{BMO}(S^{n-1},\mathcal{H}^{n-1}) \longrightarrow \mathrm{BMO}(S^{n-1},\mathcal{H}^{n-1}), \tag{5.3.102}$$

$$R_j : \mathrm{VMO}(S^{n-1},\mathcal{H}^{n-1}) \longrightarrow \mathrm{VMO}(S^{n-1},\mathcal{H}^{n-1}), \tag{5.3.103}$$

$$R_j : L^p_1(S^{n-1},\mathcal{H}^{n-1}) \longrightarrow L^p_1(S^{n-1},\mathcal{H}^{n-1}) \ \textit{with } \ p\in(1,\infty), \tag{5.3.104}$$

$$R_j : \mathscr{C}^\alpha(S^{n-1}) \longrightarrow \mathscr{C}^\alpha(S^{n-1}) \ \textit{with } \ \alpha\in(0,1), \tag{5.3.105}$$

$$R_j : \mathscr{C}^\alpha_{\rm van}(S^{n-1}) \longrightarrow \mathscr{C}^\alpha_{\rm van}(S^{n-1}) \ \textit{with } \ \alpha\in(0,1), \tag{5.3.106}$$

that are well defined, linear, and bounded. Moreover,

$$\begin{gathered}\textit{for any given function } f\in \mathrm{BMO}(S^{n-1},\mathcal{H}^{n-1}) \textit{ one has}\\ f\in \mathrm{VMO}(S^{n-1},\mathcal{H}^{n-1}) \Leftrightarrow \begin{cases} R_jf\in \mathrm{VMO}(S^{n-1},\mathcal{H}^{n-1})\\ \textit{for all } \ j\in\{1,\dots,n\},\end{cases}\end{gathered} \tag{5.3.107}$$

$$\begin{gathered}\textit{for each given function } f\in \mathscr{C}^\alpha(S^{n-1}) \textit{ with } \alpha\in(0,1) \textit{ one has}\\ f\in \mathscr{C}^\alpha_{\rm van}(S^{n-1}) \Longleftrightarrow R_jf\in \mathscr{C}^\alpha_{\rm van}(S^{n-1}) \textit{ for all } \ j\in\{1,\dots,n\},\end{gathered} \tag{5.3.108}$$

$$\begin{gathered}\textit{for each given function } f\in L^p(S^{n-1},\mathcal{H}^{n-1}) \textit{ with } p\in(1,\infty) \textit{ one has}\\ f\in L^p_1(S^{n-1},\mathcal{H}^{n-1}) \Leftrightarrow R_jf\in L^p_1(S^{n-1},\mathcal{H}^{n-1}) \textit{ for all } \ j\in\{1,\dots,n\}.\end{gathered} \tag{5.3.109}$$

Finally, all results are valid with the unit sphere $S^{n-1} = \partial\Omega$ *replaced by the boundary* $\partial\Omega$ *of any bounded domain* $\Omega \subseteq \mathbb{R}^n$ *of class* $\mathscr{C}^{1+\varepsilon}$, $\varepsilon \in (0,1)$, *with the understanding that one now takes* $\alpha \in (0,\varepsilon)$ *in* (5.3.105)-(5.3.106) *and* (5.3.108).

Proof Work in the Clifford algebra context. Let M_ν be the operator of pointwise left-multiplication in $\mathcal{C}\ell_n$ by ν, where $\nu(x) \equiv x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$ for each $x \in S^{n-1}$. Then (1.6.22) tells us that

$$\mathfrak{C}M_\nu = -\frac{1}{2}\sum_{j=1}^{n}\mathbf{e}_j R_j \ \text{ on } L^1\big(S^{n-1},\mathcal{H}^{n-1}\big)\otimes\mathcal{C}\ell_n. \tag{5.3.110}$$

From [69, Proposition 4.4.8] we know that $H^1(S^{n-1},\mathcal{H}^{n-1})$ is a module over the ring of smooth functions on S^{n-1}. Via duality (cf. [69, Theorem 4.6.1] and [69, Lemma 4.6.9]) we then see that

$$\begin{gathered}\text{BMO}(S^{n-1},\mathcal{H}^{n-1}) \text{ is a module over the ring}\\ \text{of smooth functions on } S^{n-1} \text{ (in a quantitative fashion).}\end{gathered} \tag{5.3.111}$$

Obviously,

$$\begin{gathered}\mathscr{C}^\alpha(S^{n-1}) \text{ and } \mathscr{C}^\alpha_{\text{van}}(S^{n-1}) \text{ with } \alpha\in(0,1) \text{ are modules over the ring}\\ \text{of smooth functions on } S^{n-1} \text{ (in a quantitative fashion).}\end{gathered} \tag{5.3.112}$$

From (5.3.111)-(5.3.112) and [69, (3.1.50)] we then conclude that

$$\begin{gathered}\text{VMO}(S^{n-1},\mathcal{H}^{n-1}) \text{ is a module over the ring}\\ \text{of smooth functions on } S^{n-1} \text{ (in a quantitative fashion).}\end{gathered} \tag{5.3.113}$$

Finally, item *(v)* in [69, Proposition 11.1.9] implies that

$$\begin{gathered}L^p_1(S^{n-1},\mathcal{H}^{n-1}) \text{ with } p\in(1,\infty) \text{ is a module over the ring}\\ \text{of smooth functions on } S^{n-1} \text{ (in a quantitative fashion).}\end{gathered} \tag{5.3.114}$$

Collectively, (2.1.191), (5.3.110), and (5.3.111) then prove that each Riesz transform induces a well-defined, linear, and bounded mapping in the context of (5.3.102). The claim pertaining to (5.3.103)-(5.3.106) are dealt with similarly, making use of (5.3.110), (5.3.112)-(5.3.114), (2.1.192), (2.1.193), (2.1.194), and (1.6.10).

Consider next the claim made in (5.3.107). The left-to-right implication is a consequence of (5.3.103). To prove the opposite implication, pick an arbitrary function f in the space $\text{BMO}(S^{n-1},\mathcal{H}^{n-1})$ with the property that $R_j f \in \text{VMO}(S^{n-1},\mathcal{H}^{n-1})$ for all $j \in \{1,\dots,n\}$. Then $f \in L^2(S^{n-1},\mathcal{H}^{n-1})$ (cf. [68, (7.4.106)]) and, as such, we may write

$$f = -\sum_{j=1}^{n} R_j(R_j f) \in \text{VMO}(S^{n-1},\mathcal{H}^{n-1}) \tag{5.3.115}$$

where the equality comes from (1.6.27) and the membership is a consequence of assumptions and (5.3.103). Properties (5.3.108) and (5.3.109) are dealt with similarly.

Finally, the very last claim in the statement is proved in a completely similar fashion, bearing in mind that now $\nu \in \left[\mathscr{C}^{\varepsilon}(\partial\Omega)\right]^n$. □

Chapter 6
Green Formulas and Layer Potential Operators for the Stokes System

To say that a pair, consisting of vector-valued function $\vec{u}$, playing the role of velocity, together with a scalar-valued function π, playing the role of pressure, is a null-solution of the Stokes system of linear hydrostatics in an open set $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, amounts to having

$$\begin{gathered}\vec{u} \in \left[\mathscr{C}^\infty(\Omega)\right], \quad \pi \in \mathscr{C}^\infty(\Omega), \text{ with} \\ \Delta\vec{u} - \nabla\pi = 0 \text{ in } \Omega \text{ and } \operatorname{div}\vec{u} = 0 \text{ in } \Omega,\end{gathered} \tag{6.0.1}$$

where the Laplacian $\Delta = \sum_{j=1}^n \partial_j^2$ acts on $\vec{u}$ componentwise. Due to its special algebraic format (specifically, the divergence-free condition imposed on $\vec{u}$ and the fact that the pressure function π plays a different role than the scalar components of $\vec{u}$), the Stokes system does not fit directly into the general framework of homogeneous constant coefficient second-order systems, treated earlier in §1.5, §1.7, and Chapter 3 in Volume III ([70]), as well as Chapter 1, Chapter 3, and Chapter 4 in the current volume. As such, the partial differential equations (6.0.1) warrant separate consideration. In this chapter we shall examine aspects of the theory built around the Stokes system in which our brand of Divergence Theorem developed in Volume I ([68]) plays a prominent role, such as Green-type formulas, boundary layer potential operators, and Fatou-type theorems, with the goal of producing results which are sharp from a geometric/analytic point of view.

More specifically, in §6.1 we derive a number of basic Green-type formulas for the Stokes system in open subsets of $\mathbb{R}^n$ with a lower Ahlfors regular boundary and a doubling "surface" measure. In §6.2 we treat boundary layer potential operators for the Stokes system in open sets with uniformly rectifiable boundaries, acting from Lebesgue, Sobolev, and Hardy spaces. In addition to other integral representation formulas of interest, in §6.3 we establish quantitative Fatou-type theorems for the Stokes system in UR domains. Lastly, in §6.4 we deal with boundary layer potentials for the Stokes system on Besov, Triebel-Lizorkin, and weighted Sobolev spaces.

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6.1 Green-Type Formulas for the Stokes System

We begin by elaborating on the algebraic formalism associated with the Stokes system. For a parameter $\lambda \in \mathbb{C}$ fixed, let

$$a_{jk}^{\alpha\beta}(\lambda) := \delta_{jk}\delta_{\alpha\beta} + \lambda\delta_{j\beta}\delta_{k\alpha}, \qquad 1 \leq j, k, \alpha, \beta \leq n, \tag{6.1.1}$$

and, adopting the summation convention over repeated indices, consider the second-order, homogeneous, constant (complex) coefficient, $n \times n$ system L_λ given by

$$L_\lambda := \Big(a_{jk}^{\alpha\beta}(\lambda)\partial_j\partial_k\Big)_{1\leq\alpha,\beta\leq n}. \tag{6.1.2}$$

Then for any vector $\vec{u} = (u_1, \dots, u_n)$ with components distributions in an open subset of $\mathbb{R}^n$ we have

$$\begin{aligned} L_\lambda\vec{u} &= \Big(a_{jk}^{\alpha\beta}(\lambda)\partial_j\partial_k u_\beta\Big)_{1\leq\alpha\leq n} = \Big(\Delta u_\alpha + \lambda\,\partial_\alpha(\partial_\beta u_\beta)\Big)_{1\leq\alpha\leq n} \\ &= \Delta\vec{u} + \lambda\,\nabla\mathrm{div}\vec{u}, \end{aligned} \tag{6.1.3}$$

where the Laplacian is applied to $\vec{u}$ componentwise. Note that when acting on divergence-free vectors, L_λ simply becomes the (vector) Laplacian.

One aspect directly affected by special algebraic format of the Stokes system is the manner in which one associates a notion of conormal derivative with this system. Specifically, $\Omega \subseteq \mathbb{R}^n$ is an open set with the property that $\partial\Omega$ is lower Ahlfors regular and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal $\nu = (\nu_1, \dots, \nu_n)$ is defined σ-a.e. on $\partial_*\Omega$. Fix $\lambda \in \mathbb{C}$ and $\kappa \in (0, \infty)$. Assume two functions,

$$\begin{gathered} \vec{u} = (u_\beta)_{1\leq\beta\leq n} \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^n \text{ with} \\ \big(a_{jk}^{\alpha\beta}(\lambda)\partial_k u_\beta\big)_{\alpha,j} \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^{n\times n} \text{ and such that} \\ \big(a_{jk}^{\alpha\beta}(\lambda)\partial_k u_\beta\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial_*\Omega \text{ for } 1 \leq \alpha, j \leq n, \end{gathered} \tag{6.1.4}$$

together with

$$\pi \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \text{ such that } \pi\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial_*\Omega, \tag{6.1.5}$$

have been given. Then various considerations dictate that the conormal derivative for the Stokes system, in relation to the manner in which the system in (6.1.3) has been written, be defined as[1]

[1] the choice $\lambda := 1$ is ubiquitous in the literature; for example, this is used in Ladyzhenskaya's book [54]

$$\partial_\nu^\lambda(\vec{u},\pi) := \Big(\nu_j\big(a_{jk}^{\alpha\beta}(\lambda)\partial_k u_\beta\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} - \nu_\alpha\pi\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)_{1\leq\alpha\leq n}$$
$$= \Big[\nabla\vec{u} + \lambda(\nabla\vec{u})^\top\Big]\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\nu - \big(\pi\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\nu, \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{6.1.6}$$

Above, the coefficients $a_{jk}^{\alpha\beta}(\lambda)$ are as in (6.1.1), $\nabla\vec{u} := (\partial_k u_j)_{1\leq j,k\leq n}$ denotes the Jacobian matrix of the vector-valued function $\vec{u}$, and the superscript $\top$ indicates transposition of matrices.

In the next two theorems we present sharp versions of Green-type formulas for the Stokes system. To state our first result on this topic, for each given $\lambda \in \mathbb{C}$ we introduce the bilinear form

$$A_\lambda(\xi,\zeta) := a_{jk}^{\alpha\beta}(\lambda)\xi_j^\alpha\zeta_k^\beta \text{ for all}$$
$$\xi = (\xi_j^\alpha)_{j,\alpha} \in \mathbb{C}^{n\times n},\ \zeta = (\zeta_k^\beta)_{k,\beta} \in \mathbb{C}^{n\times n}, \tag{6.1.7}$$

where the coefficients $a_{jk}^{\alpha\beta}(\lambda)$ are as in (6.1.1).

Theorem 6.1.1 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \geq 2$) is an open set with a lower Ahlfors regular boundary and with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure. Hence, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Next, fix $\lambda \in \mathbb{C}$ and recall the homogeneous constant (complex) coefficient second-order $n \times n$ system L_λ in $\mathbb{R}^n$ defined as in (6.1.3). Also, fix $p \in [1,\infty]$, denote by p' its Hölder conjugate exponent, pick some aperture parameters $\kappa, \kappa' > 0$, and consider three functions*

$$\vec{u}, \vec{w} \in \big[W^{1,1}_{\text{loc}}(\Omega)\big]^n \text{ and } \pi \in W^{1,1}_{\text{loc}}(\Omega) \tag{6.1.8}$$

satisfying (with all derivatives taken in the sense of distributions in Ω and the bilinear form $A_\lambda(\cdot,\cdot)$ as in (6.1.7)):

$$\begin{gathered}\mathcal{N}_\kappa\pi,\ \mathcal{N}_\kappa(\nabla\vec{u}),\ \mathcal{N}_{\kappa'}\vec{w} < \infty \text{ at } \sigma\text{-a.e. point on } \partial\Omega,\\ \big[\mathcal{N}_\kappa\pi + \mathcal{N}_\kappa(\nabla\vec{u})\big]\cdot\mathcal{N}_{\kappa'}\vec{w} \text{ belongs to } L^1(\partial\Omega,\sigma),\\ (\nabla\vec{u})\Big|_{\partial\Omega}^{\kappa-\text{n.t.}},\ \pi\Big|_{\partial\Omega}^{\kappa-\text{n.t.}},\text{ and } \vec{w}\Big|_{\partial\Omega}^{\kappa'-\text{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega,\\ A_\lambda(\nabla\vec{u},\nabla\vec{w}) - \pi(\operatorname{div}\vec{w}) \text{ belongs to } L^1(\Omega,\mathcal{L}^n), \text{ and}\\ L_\lambda\vec{u} \in \big[L^1_{\text{loc}}(\Omega,\mathcal{L}^n)\big]^n,\quad \big\langle L_\lambda\vec{u} - \nabla\pi, \vec{w}\big\rangle \in L^1(\Omega,\mathcal{L}^n).\end{gathered} \tag{6.1.9}$$

Then the following Green-type formula (involving absolutely convergent integrals, and the conormal derivative defined as in (6.1.6)) holds

$$\int_{\Omega} \langle L_\lambda \vec{u} - \nabla\pi, \vec{w}\rangle \, d\mathcal{L}^n = \int_{\partial_*\Omega} \Big\langle \partial_\nu^\lambda(\vec{u},\pi), \vec{w}\big|_{\partial\Omega}^{\kappa'-\text{n.t.}} \Big\rangle \, d\sigma$$
$$- \int_{\Omega} \big\{A_\lambda(\nabla\vec{u}, \nabla\vec{w}) - \pi(\operatorname{div}\vec{w})\big\} \, d\mathcal{L}^n, \tag{6.1.10}$$

in the case when either Ω *is bounded, or* $\partial\Omega$ *is unbounded. Moreover, formula* (6.1.10) *also holds if* Ω *is unbounded and* $\partial\Omega$ *is bounded* (*i.e., when* Ω *is an exterior domain*) *provided there exists* $\mu \in (1,\infty)$ *such that*

$$\int_{B(0,\mu R)\setminus B(0,R)} \big\{|\nabla\vec{u}| + |\pi|\big\}|\vec{w}| \, d\mathcal{L}^n = o(R) \ \text{ as } \ R \to \infty. \tag{6.1.11}$$

For examples, all conditions in the first two lines of (6.1.9) are satisfied if

$$\begin{gathered} \mathcal{N}_\kappa\pi, \ \mathcal{N}_\kappa(\nabla\vec{u}) \in L^p(\partial\Omega,\sigma), \ \ \mathcal{N}_{\kappa'}\vec{w} \in L^{p'}(\partial\Omega,\sigma), \\ \text{for some } \ p, p' \in [1,\infty] \ \text{ with } \ 1/p + 1/p' = 1. \end{gathered} \tag{6.1.12}$$

In such a scenario, [68, Proposition 8.9.8] implies that for any other given aperture parameter $\kappa'' > 0$ we have

$$\mathcal{N}_{\kappa''}\pi, \ \mathcal{N}_{\kappa''}(\nabla\vec{u}) \in L^p(\partial\Omega,\sigma), \quad \mathcal{N}_{\kappa''}\vec{w} \in L^{p'}(\partial\Omega,\sigma), \tag{6.1.13}$$

and the nontangential boundary traces

$$\begin{gathered} \pi\big|_{\partial\Omega}^{\kappa''-\text{n.t.}}, \ (\nabla\vec{u})\big|_{\partial\Omega}^{\kappa''-\text{n.t.}}, \ \vec{w}\big|_{\partial\Omega}^{\kappa''-\text{n.t.}} \ \text{exist at } \sigma\text{-a.e. point on } \partial_{\text{nta}}\Omega, \\ \text{and agree with the nontangential boundary traces in (6.1.9).} \end{gathered} \tag{6.1.14}$$

We now present the proof of Theorem 6.1.1.

Proof of Theorem 6.1.1 The first line in (6.1.9) together with [68, Lemma 8.3.1] guarantee that

$$\pi \in L^\infty_{\text{loc}}(\Omega,\mathcal{L}^n), \quad \nabla\vec{u} \in \big[L^\infty_{\text{loc}}(\Omega,\mathcal{L}^n)\big]^{n\times n}, \quad \vec{w} \in \big[L^\infty_{\text{loc}}(\Omega,\mathcal{L}^n)\big]^n. \tag{6.1.15}$$

To proceed, consider the vector field $\vec{F} = (F_j)_{1\leq j\leq n}$ with scalar components given by

$$F_j := a^{\alpha\beta}_{jk}(\lambda)(\partial_k u_\beta)w_\alpha - \pi w_j, \qquad 1 \leq j \leq n. \tag{6.1.16}$$

In particular, from (6.1.16) and (6.1.15) we see that

$$\vec{F} \in \big[L^\infty_{\text{loc}}(\Omega,\mathcal{L}^n)\big]^n \subset \big[L^1_{\text{loc}}(\Omega,\mathcal{L}^n)\big]^n. \tag{6.1.17}$$

To compute the divergence of $\vec{F}$, in the sense of distributions in Ω, fix an arbitrary scalar-valued function $\varphi \in \mathscr{C}^\infty_c(\Omega)$ and write

$$
\begin{aligned}
{}_{\mathcal{D}'(\Omega)}\big\langle \mathrm{div}\vec{F}, \varphi\big\rangle_{\mathcal{D}(\Omega)} &= -\int_\Omega F_j \partial_j \varphi \, \mathrm{d}\mathcal{L}^n \\
&= -\int_\Omega a^{\alpha\beta}_{jk}(\lambda)(\partial_k u_\beta) w_\alpha \partial_j \varphi \, \mathrm{d}\mathcal{L}^n + \int_\Omega \pi w_j \partial_j \varphi \, \mathrm{d}\mathcal{L}^n.
\end{aligned}
\tag{6.1.18}
$$

As in the past, for each sufficiently small parameter $\varepsilon > 0$ consider the region defined by $\Omega_\varepsilon := \big\{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \varepsilon\big\}$ and use a Friedrichs mollifier to construct a sequence $\vec{w}^\varepsilon = (w^\varepsilon_\alpha)_{1\leq\alpha\leq n} \in \big[\mathscr{C}^\infty(\Omega_\varepsilon)\big]^n$ such that

$$
\begin{aligned}
&\text{for each compact set } K \subset \Omega \text{ we have} \\
&\vec{w}^\varepsilon \xrightarrow[\varepsilon\to 0^+]{} \vec{w} \ \text{ in } \big[L^1(K, \mathcal{L}^n)\big]^n \text{ and} \\
&\vec{w}^\varepsilon \xrightarrow[\varepsilon\to 0^+]{} \vec{w} \ \text{ at } \mathcal{L}^n\text{-a.e. point in } \Omega, \\
&\nabla\vec{w}^\varepsilon \xrightarrow[\varepsilon\to 0^+]{} \nabla\vec{w} \ \text{ in } \big[L^1(K, \mathcal{L}^n)\big]^{n\times n}, \\
&\text{and there exists some number } \varepsilon_K > 0 \\
&\text{so that } \sup_{0<\varepsilon<\varepsilon_K} \|\vec{w}^\varepsilon\|_{[L^\infty(K,\mathcal{L}^n)]^n} < \infty.
\end{aligned}
\tag{6.1.19}
$$

Based on (6.1.15), (6.1.19), the fact that $L_\lambda \vec{u} \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^n$ (cf. the last line in (6.1.9)), and Lebesgue's Dominated Convergence Theorem, we may then compute

$$
\begin{aligned}
&-\int_\Omega a^{\alpha\beta}_{jk}(\lambda)(\partial_k u_\beta) w_\alpha \partial_j\varphi \,\mathrm{d}\mathcal{L}^n \\
&\quad = -\lim_{\varepsilon\to 0^+}\int_\Omega a^{\alpha\beta}_{jk}(\lambda)(\partial_k u_\beta) w^\varepsilon_\alpha \partial_j\varphi \,\mathrm{d}\mathcal{L}^n \\
&\quad = -\lim_{\varepsilon\to 0^+}\int_\Omega a^{\alpha\beta}_{jk}(\lambda)(\partial_k u_\beta) \partial_j(w^\varepsilon_\alpha\varphi) \,\mathrm{d}\mathcal{L}^n \\
&\qquad + \lim_{\varepsilon\to 0^+}\int_\Omega a^{\alpha\beta}_{jk}(\lambda)(\partial_k u_\beta) (\partial_j w^\varepsilon_\alpha)\varphi \,\mathrm{d}\mathcal{L}^n \\
&\quad = -\lim_{\varepsilon\to 0^+} {}_{\mathcal{D}'(\Omega)}\Big\langle a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta, \partial_j(w^\varepsilon_\alpha\varphi)\Big\rangle_{\mathcal{D}(\Omega)} \\
&\qquad + \int_\Omega a^{\alpha\beta}_{jk}(\lambda)(\partial_k u_\beta)(\partial_j w_\alpha)\varphi \,\mathrm{d}\mathcal{L}^n \\
&\quad = \lim_{\varepsilon\to 0^+} {}_{\mathcal{D}'(\Omega)}\Big\langle a^{\alpha\beta}_{jk}(\lambda)\partial_j\partial_k u_\beta, w^\varepsilon_\alpha\varphi\Big\rangle_{\mathcal{D}(\Omega)} + \int_\Omega A_\lambda(\nabla\vec{u},\nabla\vec{w})\varphi \,\mathrm{d}\mathcal{L}^n \\
&\quad = \lim_{\varepsilon\to 0^+} {}_{\mathcal{D}'(\Omega)}\Big\langle (L_\lambda\vec{u})_\alpha, w^\varepsilon_\alpha\varphi\Big\rangle_{\mathcal{D}(\Omega)} + \int_\Omega A_\lambda(\nabla\vec{u},\nabla\vec{w})\varphi \,\mathrm{d}\mathcal{L}^n \\
&\quad = \lim_{\varepsilon\to 0^+} \int_\Omega (L_\lambda\vec{u})_\alpha w^\varepsilon_\alpha\varphi \,\mathrm{d}\mathcal{L}^n + \int_\Omega A_\lambda(\nabla\vec{u},\nabla\vec{w})\varphi \,\mathrm{d}\mathcal{L}^n \\
&\quad = \int_\Omega (L_\lambda\vec{u})_\alpha w_\alpha\varphi \,\mathrm{d}\mathcal{L}^n + \int_\Omega A_\lambda(\nabla\vec{u},\nabla\vec{w})\varphi \,\mathrm{d}\mathcal{L}^n \\
&\quad = \int_\Omega \big\langle L_\lambda\vec{u},\vec{w}\big\rangle\varphi \,\mathrm{d}\mathcal{L}^n + \int_\Omega A_\lambda(\nabla\vec{u},\nabla\vec{w})\varphi \,\mathrm{d}\mathcal{L}^n. \qquad (6.1.20)
\end{aligned}
$$

Also, (6.1.15), (6.1.19), and Lebesgue's Dominated Convergence Theorem permit us to compute

$$
\begin{aligned}
\int_\Omega \pi w_j\partial_j\varphi \,\mathrm{d}\mathcal{L}^n &= \lim_{\varepsilon\to 0^+}\int_\Omega \pi w^\varepsilon_j\partial_j\varphi \,\mathrm{d}\mathcal{L}^n \\
&= -\lim_{\varepsilon\to 0^+}\int_\Omega (\partial_j\pi) w^\varepsilon_j\varphi \,\mathrm{d}\mathcal{L}^n - \lim_{\varepsilon\to 0^+}\int_\Omega \pi(\partial_j w^\varepsilon_j)\varphi \,\mathrm{d}\mathcal{L}^n \\
&= -\int_\Omega (\partial_j\pi) w_j\varphi \,\mathrm{d}\mathcal{L}^n - \int_\Omega \pi(\partial_j w_j)\varphi \,\mathrm{d}\mathcal{L}^n \\
&= -\int_\Omega \langle\nabla\pi,\vec{w}\rangle\varphi \,\mathrm{d}\mathcal{L}^n - \int_\Omega \pi(\mathrm{div}\vec{w})\varphi \,\mathrm{d}\mathcal{L}^n. \qquad (6.1.21)
\end{aligned}
$$

Collectively, (6.1.18), (6.1.20), and (6.1.21) prove that

$$\mathrm{div}\vec{F} = \big\langle L_\lambda \vec{u} - \nabla\pi, \vec{w}\big\rangle + A_\lambda(\nabla\vec{u}, \nabla\vec{w}) - \pi(\mathrm{div}\vec{w}) \ \text{ in } \ \mathcal{D}'(\Omega). \tag{6.1.22}$$

In concert with the last two lines in (6.1.9), this implies

$$\mathrm{div}\vec{F} \in L^1(\Omega, \mathcal{L}^n). \tag{6.1.23}$$

Moving on, define $\kappa'' := \min\{\kappa, \kappa'\} > 0$ and observe from (6.1.16) that there exists a constant $C_\lambda \in (0, \infty)$ such that

$$\begin{aligned} 0 \le \mathcal{N}_{\kappa''}\vec{F} &\le C_\lambda\Big(\mathcal{N}_{\kappa''}(\nabla\vec{u}) + \mathcal{N}_{\kappa''}\pi\Big)\mathcal{N}_{\kappa''}\vec{w} \\ &\le C_\lambda\Big(\mathcal{N}_{\kappa}(\nabla\vec{u}) + \mathcal{N}_{\kappa}\pi\Big)\mathcal{N}_{\kappa'}\vec{w} \ \text{ on } \ \partial\Omega. \end{aligned} \tag{6.1.24}$$

From this, (6.1.9), and [68, (8.2.26)] we then conclude that

$$\mathcal{N}_{\kappa''}\vec{F} \in L^1(\partial\Omega, \sigma). \tag{6.1.25}$$

Furthermore, (6.1.9) and (6.1.16) ensure that the nontangential boundary trace

$$\vec{F}\Big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \ \partial_{\mathrm{nta}}\Omega \tag{6.1.26}$$

and, in fact, at σ-a.e. point on $\partial_{\mathrm{nta}}\Omega$ we have

$$\vec{F}\Big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}} = \Big(a^{\alpha\beta}_{jk}(\lambda)\big(\partial_k u_\beta\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)\Big(w_\alpha\Big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\Big) - \Big(\pi\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)\Big(w_j\Big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\Big)\Big)_{1\le j\le n}. \tag{6.1.27}$$

From this, (A.0.184), (6.1.6), and [68, Proposition 8.8.6] we conclude that

$$\nu\cdot\Big(\vec{F}\Big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}}\Big) = \Big\langle \partial^\lambda_\nu(\vec{u}, \pi), \vec{w}\Big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\Big\rangle \ \text{ at } \sigma\text{-a.e. point on } \ \partial_*\Omega. \tag{6.1.28}$$

Finally, we remark that when Ω is an exterior domain, (6.1.16) and (6.1.11) guarantee the validity of the integral growth condition [68, (1.2.3)] for the current vector field $\vec{F}$.

Granted the aforementioned properties of $\vec{F}$, [68, Theorem 1.2.1] applies and the Divergence Formula [68, (1.2.2)] presently yields [70, (1.7.121)] on account of (6.1.22) and (6.1.28). □

Here is our second Green-type formula for the Stokes system.

Theorem 6.1.2 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \ge 2$) is an open set with a lower Ahlfors regular boundary and with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure. Thus, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Fix a parameter $\lambda \in \mathbb{C}$ and recall the homogeneous constant (complex) coefficient second-order $n \times n$ system L_λ in $\mathbb{R}^n$ from (6.1.3). In addition, pick some aperture parameters $\kappa, \kappa' > 0$, and consider four functions*

$$\vec{u}, \vec{w} \in \big[W^{1,1}_{\mathrm{loc}}(\Omega)\big]^n \ \ \textit{and} \ \ \pi, \rho \in W^{1,1}_{\mathrm{loc}}(\Omega) \tag{6.1.29}$$

satisfying (*with all derivatives taken in the sense of distributions in* Ω *and the bilinear form* $A_\lambda(\cdot,\cdot)$ *as in* (6.1.7)):

$$\begin{gathered}
\mathcal{N}_\kappa \pi,\ \mathcal{N}_\kappa(\nabla\vec{u}),\ \mathcal{N}_\kappa \vec{u},\ \mathcal{N}_{\kappa'}\rho,\ \mathcal{N}_{\kappa'}(\nabla\vec{w}),\ \mathcal{N}_{\kappa'}\vec{w} < \infty \ \textit{at } \sigma\textit{-a.e. point on } \partial\Omega,\\
\big[\mathcal{N}_\kappa\pi + \mathcal{N}_\kappa(\nabla\vec{u})\big]\cdot \mathcal{N}_{\kappa'}\vec{w} \in L^1(\partial\Omega,\sigma) \textit{ and } \mathcal{N}_\kappa\vec{u}\cdot\big[\mathcal{N}_{\kappa'}\rho + \mathcal{N}_{\kappa'}(\nabla\vec{w})\big] \in L^1(\partial\Omega,\sigma),\\
\vec{u}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ (\nabla\vec{u})\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ \pi\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ \vec{w}\Big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega},\ (\nabla\vec{w})\Big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega},\ \rho\Big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega} \ \textit{exist } \sigma\textit{-a.e. on } \partial_{\mathrm{nta}}\Omega,\\
\textit{both } L_\lambda\vec{u} \textit{ and } L_\lambda\vec{w} \textit{ belong to the space } \big[L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n)\big]^n, \textit{ and, finally,}\\
\big\langle L_\lambda\vec{u} - \nabla\pi, \vec{w}\big\rangle - \pi(\mathrm{div}\vec{w}) - \big\langle \vec{u}, L_\lambda\vec{w} - \nabla\rho\big\rangle + \rho(\mathrm{div}\vec{u}) \in L^1(\Omega,\mathcal{L}^n).
\end{gathered} \tag{6.1.30}$$

Then the following Green-type formula (*involving absolutely convergent integrals, and with the conormal derivatives defined as in* (6.1.6)) *holds*

$$\begin{aligned}
&\int_\Omega \Big\{\big\langle L_\lambda\vec{u} - \nabla\pi, \vec{w}\big\rangle - \pi(\mathrm{div}\vec{w}) - \big\langle \vec{u}, L_\lambda\vec{w} - \nabla\rho\big\rangle + \rho(\mathrm{div}\vec{u})\Big\}\,\mathrm{d}\mathcal{L}^n\\
&\qquad = \int_{\partial_*\Omega} \Big\{\Big\langle \partial^\lambda_\nu(\vec{u},\pi), \vec{w}\Big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle - \Big\langle \vec{u}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, \partial^\lambda_\nu(\vec{w},\rho)\Big\rangle\Big\}\,\mathrm{d}\sigma,
\end{aligned} \tag{6.1.31}$$

in the case when either Ω *is bounded, or* $\partial\Omega$ *is unbounded. Furthermore, formula* (6.1.31) *continues to be valid in the case when* Ω *is an exterior domain provided there exists some* $\mu \in (1,\infty)$ *such that*

$$\int\limits_{B(0,\mu R)\setminus B(0,R)} \Big\{\big[|\nabla\vec{u}| + |\pi|\big]|\vec{w}| + \big[|\nabla\vec{w}| + |\rho|\big]|\vec{u}|\Big\}\,\mathrm{d}\mathcal{L}^n = o(R) \ \textit{ as } \ R \to \infty. \tag{6.1.32}$$

For example, the assumptions in the first two lines of (6.1.30) are satisfied if

$$\begin{gathered}
\mathcal{N}_\kappa\pi,\ \mathcal{N}_\kappa(\nabla\vec{u}) \in L^p(\partial\Omega,\sigma), \quad \mathcal{N}_\kappa\vec{u} \in L^q(\partial\Omega,\sigma),\\
\mathcal{N}_{\kappa'}\rho,\ \mathcal{N}_{\kappa'}(\nabla\vec{w}) \in L^{q'}(\partial\Omega,\sigma), \quad \mathcal{N}_{\kappa'}\vec{w} \in L^{p'}(\partial\Omega,\sigma),\\
\text{with } p, q, p', q' \in [1,\infty] \text{ such that } 1/p + 1/p' = 1/q + 1/q' = 1.
\end{gathered} \tag{6.1.33}$$

In such a scenario, [68, Proposition 8.9.8] ensure that for any given aperture parameter $\kappa'' > 0$ we have

$$\begin{gathered}
\mathcal{N}_{\kappa''}\pi,\ \mathcal{N}_{\kappa''}(\nabla\vec{u}) \in L^p(\partial\Omega,\sigma), \quad \mathcal{N}_{\kappa''}\vec{u} \in L^q(\partial\Omega,\sigma),\\
\mathcal{N}_{\kappa''}\rho,\ \mathcal{N}_{\kappa''}(\nabla\vec{w}) \in L^{q'}(\partial\Omega,\sigma), \quad \mathcal{N}_{\kappa''}w \in L^{p'}(\partial\Omega,\sigma),
\end{gathered} \tag{6.1.34}$$

and the nontangential boundary traces

$$
\begin{aligned}
&\pi\Big|_{\partial\Omega}^{\kappa''-\text{n.t.}},\ \vec{u}\Big|_{\partial\Omega}^{\kappa''-\text{n.t.}},\ (\nabla\vec{u})\Big|_{\partial\Omega}^{\kappa''-\text{n.t.}},\ \rho\Big|_{\partial\Omega}^{\kappa''-\text{n.t.}},\ \vec{w}\Big|_{\partial\Omega}^{\kappa''-\text{n.t.}},\ (\nabla\vec{w})\Big|_{\partial\Omega}^{\kappa''-\text{n.t.}} \\
&\qquad\text{exist } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega \text{ and agree with the respective} \\
&\qquad\qquad\text{pointwise boundary traces from (6.1.30).}
\end{aligned}
\tag{6.1.35}
$$

Here is the proof of Theorem 6.1.2.

Proof of Theorem 6.1.2 For starters, observe that (6.1.29), the first line in (6.1.30), and [68, Lemma 8.3.1] imply

$$
\vec{u}, \vec{w} \in \big[W^{1,\infty}_{\text{loc}}(\Omega)\big]^n = \big[\text{Lip}_{\text{loc}}(\Omega)\big]^n \ \text{ and } \ \pi, \rho \in L^\infty_{\text{loc}}(\Omega, \mathcal{L}^n). \tag{6.1.36}
$$

To proceed, consider the vector field $\vec{F} = (F_j)_{1\le j\le n} : \Omega \to \mathbb{C}^n$ with components given by (with the summation convention over repeated indices enforced throughout the proof)

$$
F_j := a^{\alpha\beta}_{jk}(\lambda)(\partial_k u_\beta)w_\alpha - \pi w_j - a^{\beta\alpha}_{jk}(\lambda)(\partial_k w_\alpha)u_\beta + \rho u_j. \tag{6.1.37}
$$

In particular, from (6.1.37) and (6.1.36) we see that

$$
\vec{F} \in \big[L^\infty_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^n \subset \big[L^1_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^n. \tag{6.1.38}
$$

The next order of business is to compute $\text{div}\vec{F}$ in the sense of distributions in Ω. To this end, fix an arbitrary scalar-valued function $\varphi \in \mathscr{C}^\infty_c(\Omega)$ and write

$$
\begin{aligned}
{}_{\mathcal{D}'(\Omega)}\big\langle \text{div}\vec{F}, \varphi\big\rangle_{\mathcal{D}(\Omega)} &= -\int_\Omega F_j \partial_j\varphi \, \mathrm{d}\mathcal{L}^n \\
&= -\int_\Omega a^{\alpha\beta}_{jk}(\lambda)(\partial_k u_\beta)w_\alpha \partial_j\varphi \, \mathrm{d}\mathcal{L}^n + \int_\Omega \pi w_j \partial_j\varphi \, \mathrm{d}\mathcal{L}^n \\
&\quad + \int_\Omega a^{\beta\alpha}_{jk}(\lambda)(\partial_k w_\alpha)u_\beta \partial_j\varphi \, \mathrm{d}\mathcal{L}^n - \int_\Omega \rho u_j \partial_j\varphi \, \mathrm{d}\mathcal{L}^n \\
&=: I + II + III + IV.
\end{aligned}
\tag{6.1.39}
$$

For each sufficiently small $\varepsilon > 0$ consider $\Omega_\varepsilon := \big\{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\big\}$. Using a Friedrichs mollifier, we may construct a sequence $\vec{u}^\varepsilon = (u^\varepsilon_\beta)_{1\le\beta\le n} \in \big[\mathscr{C}^\infty(\Omega_\varepsilon)\big]^n$ such that

$$\begin{aligned}
&\vec{u}^{\varepsilon} \xrightarrow[\varepsilon\to 0^+]{} \vec{u} \text{ uniformly on compact subsets of } \Omega,\\
&\text{and } \nabla\vec{u}^{\varepsilon} \xrightarrow[\varepsilon\to 0^+]{} \nabla\vec{u} \text{ at } \mathcal{L}^n\text{-a.e. point in } \Omega;\\
&\text{moreover, for each fixed compact set } K\subset\Omega\\
&\text{we have } L_\lambda\vec{u}^{\varepsilon} \xrightarrow[\varepsilon\to 0^+]{} L_\lambda\vec{u} \text{ in } \big[L^1(K,\mathcal{L}^n)\big]^n\\
&\text{and there exists some small } \varepsilon_K>0 \text{ such that}\\
&\sup_{0<\varepsilon<\varepsilon_K}\Big\{\|\vec{u}^{\varepsilon}\|_{[L^\infty(K,\mathcal{L}^n)]^n}+\big\|\nabla\vec{u}^{\varepsilon}\big\|_{[L^\infty(K,\mathcal{L}^n)]^{n\times n}}\Big\}<\infty,
\end{aligned}\tag{6.1.40}$$

along with a sequence $\vec{w}^{\varepsilon}=(w_\alpha^{\varepsilon})_{1\le\alpha\le n}\in\big[\mathscr{C}^\infty(\Omega_\varepsilon)\big]^n$ such that

$$\begin{aligned}
&\vec{w}^{\varepsilon} \xrightarrow[\varepsilon\to 0^+]{} \vec{w} \text{ uniformly on compact subsets of } \Omega,\\
&\text{and } \nabla\vec{w}^{\varepsilon} \xrightarrow[\varepsilon\to 0^+]{} \nabla\vec{w} \text{ at } \mathcal{L}^n\text{-a.e. point in } \Omega;\\
&\text{moreover, for each fixed compact set } K\subset\Omega\\
&\text{we have } L_\lambda\vec{w}^{\varepsilon} \xrightarrow[\varepsilon\to 0^+]{} L_\lambda\vec{w} \text{ in } \big[L^1(K,\mathcal{L}^n)\big]^n\\
&\text{and there exists some small } \varepsilon_K>0 \text{ such that}\\
&\sup_{0<\varepsilon<\varepsilon_K}\Big\{\|\vec{w}^{\varepsilon}\|_{[L^\infty(K,\mathcal{L}^n)]^n}+\big\|\nabla\vec{w}^{\varepsilon}\big\|_{[L^\infty(K,\mathcal{L}^n)]^{n\times n}}\Big\}<\infty.
\end{aligned}\tag{6.1.41}$$

By relying on [70, (1.7.97)], (6.1.40), (6.1.41), (6.1.3), the penultimate line in (6.1.30), and (6.1.7) we may then compute

$$\begin{aligned}
I&=-\lim_{\varepsilon\to 0^+}\int_\Omega a_{jk}^{\alpha\beta}(\lambda)(\partial_k u_\beta^{\varepsilon})w_\alpha^{\varepsilon}\partial_j\varphi\,\mathrm{d}\mathcal{L}^n\\
&=-\lim_{\varepsilon\to 0^+}\int_\Omega a_{jk}^{\alpha\beta}(\lambda)(\partial_k u_\beta^{\varepsilon})\partial_j(w_\alpha^{\varepsilon}\varphi)\,\mathrm{d}\mathcal{L}^n+\lim_{\varepsilon\to 0^+}\int_\Omega a_{jk}^{\alpha\beta}(\lambda)(\partial_k u_\beta^{\varepsilon})(\partial_j w_\alpha^{\varepsilon})\varphi\,\mathrm{d}\mathcal{L}^n\\
&=\lim_{\varepsilon\to 0^+}\int_\Omega (L_\lambda\vec{u}^{\varepsilon})_\alpha w_\alpha^{\varepsilon}\varphi\,\mathrm{d}\mathcal{L}^n+\lim_{\varepsilon\to 0^+}\int_\Omega A_\lambda(\nabla\vec{u}^{\varepsilon},\nabla\vec{w}^{\varepsilon})\varphi\,\mathrm{d}\mathcal{L}^n\\
&=\int_\Omega\big\langle L_\lambda\vec{u},\vec{w}\big\rangle\varphi\,\mathrm{d}\mathcal{L}^n+\lim_{\varepsilon\to 0^+}\int_\Omega A_\lambda(\nabla\vec{u}^{\varepsilon},\nabla\vec{w}^{\varepsilon})\varphi\,\mathrm{d}\mathcal{L}^n.
\end{aligned}\tag{6.1.42}$$

In a similar fashion,

$$\begin{aligned}
III&=-\int_\Omega u_\beta(L_\lambda\vec{w})_\beta\varphi\,\mathrm{d}\mathcal{L}^n-\lim_{\varepsilon\to 0^+}\int_\Omega a_{jk}^{\beta\alpha}(\lambda)(\partial_j u_\beta^{\varepsilon})(\partial_k w_\alpha^{\varepsilon})\varphi\,\mathrm{d}\mathcal{L}^n\\
&=-\int_\Omega\big\langle\vec{u},L_\lambda\vec{w}\big\rangle\varphi\,\mathrm{d}\mathcal{L}^n-\lim_{\varepsilon\to 0^+}\int_\Omega A_\lambda(\nabla\vec{u}^{\varepsilon},\nabla\vec{w}^{\varepsilon})\varphi\,\mathrm{d}\mathcal{L}^n,
\end{aligned}\tag{6.1.43}$$

so that

$$I + III = \int_\Omega \big\langle L_\lambda \vec{u}, \vec{w} \big\rangle \varphi \,\mathrm{d}\mathcal{L}^n - \int_\Omega \big\langle \vec{u}, L_\lambda \vec{w} \big\rangle \varphi \,\mathrm{d}\mathcal{L}^n. \tag{6.1.44}$$

Also, (6.1.36), (6.1.40)-(6.1.41), and Lebesgue's Dominated Convergence Theorem permit us to compute

$$\begin{aligned} II &= \lim_{\varepsilon\to 0^+} \int_\Omega \pi w_j^\varepsilon \partial_j \varphi \,\mathrm{d}\mathcal{L}^n \\ &= -\lim_{\varepsilon\to 0^+} \int_\Omega (\partial_j \pi) w_j^\varepsilon \varphi \,\mathrm{d}\mathcal{L}^n - \lim_{\varepsilon\to 0^+} \int_\Omega \pi (\partial_j w_j^\varepsilon) \varphi \,\mathrm{d}\mathcal{L}^n \\ &= -\int_\Omega \langle \nabla \pi, \vec{w} \rangle \varphi \,\mathrm{d}\mathcal{L}^n - \int_\Omega \pi (\mathrm{div} \vec{w}) \varphi \,\mathrm{d}\mathcal{L}^n, \end{aligned} \tag{6.1.45}$$

and, likewise,

$$IV = \int_\Omega \langle \nabla \rho, \vec{u} \rangle \varphi \,\mathrm{d}\mathcal{L}^n + \int_\Omega \rho (\mathrm{div} \vec{u}) \varphi \,\mathrm{d}\mathcal{L}^n. \tag{6.1.46}$$

By combining (6.1.39), and (6.1.44)-(6.1.46) we conclude that

$$\mathrm{div} \vec{F} = \big\langle L_\lambda \vec{u} - \nabla \pi, \vec{w} \big\rangle - \pi (\mathrm{div} \vec{w}) - \big\langle \vec{u}, L_\lambda \vec{w} - \nabla \rho \big\rangle + \rho (\mathrm{div} \vec{u}) \text{ in } \mathcal{D}'(\Omega). \tag{6.1.47}$$

Granted this, the last line in (6.1.30) ensures that

$$\mathrm{div} \vec{F} \in L^1(\Omega, \mathcal{L}^n). \tag{6.1.48}$$

Moreover, if $\kappa'' := \min\{\kappa, \kappa'\} > 0$ then from (6.1.37) we see that at each point on $\partial\Omega$ we have

$$\begin{aligned} \mathcal{N}_{\kappa''} \vec{F} &\leq C \Big\{ \big(\mathcal{N}_{\kappa''}(\nabla \vec{u}) + \mathcal{N}_{\kappa''} \pi \big) \mathcal{N}_{\kappa''} \vec{w} + \big(\mathcal{N}_{\kappa''}(\nabla \vec{w}) + \mathcal{N}_{\kappa''} \rho \big) \mathcal{N}_{\kappa''} \vec{u} \Big\} \\ &\leq C \Big\{ \big(\mathcal{N}_{\kappa}(\nabla \vec{u}) + \mathcal{N}_{\kappa} \pi \big) \mathcal{N}_{\kappa'} \vec{w} + \big(\mathcal{N}_{\kappa'}(\nabla \vec{w}) + \mathcal{N}_{\kappa'} \rho \big) \mathcal{N}_{\kappa} \vec{u} \Big\}, \end{aligned} \tag{6.1.49}$$

for some constant $C = C(\lambda) \in (0, \infty)$ which depends only on λ. In turn, from (6.1.49), (6.1.34), and [68, (8.2.26)] we conclude that

$$\mathcal{N}_{\kappa''} \vec{F} \in L^1(\partial\Omega, \sigma). \tag{6.1.50}$$

Let us also observe that (6.1.37) and (6.1.30) imply that the nontangential boundary trace $\vec{F}\big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial_{\mathrm{nta}}\Omega$. Concretely, at σ-a.e. point on $\partial_{\mathrm{nta}}\Omega$ we have

$$\vec{F}\Big|_{\partial\Omega}^{\kappa''-\text{n.t.}} = \Bigg(a_{jk}^{\alpha\beta}(\lambda)\big((\partial_k u_\beta)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\big(w_\alpha\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big) - \big(\pi\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\big(w_j\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big)$$

$$- \big(u_\beta\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)a_{jk}^{\beta\alpha}(\lambda)\big((\partial_k w_\alpha)\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big) + \big(\rho\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big)\big(u_j\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\Bigg)_{1\leq j\leq n}. \tag{6.1.51}$$

In turn, from (6.1.51), (6.1.6), and [68, Proposition 8.8.6], we conclude that

$$\nu \cdot \big(\vec{F}\big|_{\partial\Omega}^{\kappa''-\text{n.t.}}\big) = \big\langle \partial_\nu^\lambda(\vec{u},\pi), \vec{w}\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big\rangle - \big\langle \vec{u}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}, \partial_\nu^\lambda(\vec{w},\rho)\big\rangle \tag{6.1.52}$$

at σ-a.e. point on $\partial_*\Omega$. Finally, in the case when Ω is an exterior domain, (6.1.32) guarantees that the vector field $\vec{F}$ satisfies the growth condition [68, (1.2.3)]. Granted these properties of $\vec{F}$, [68, Theorem 1.2.1] applies and the Divergence Formula [68, (1.2.2)] currently yields (6.1.31) on account of (6.1.47) and (6.1.52). □

6.2 Boundary Layer Potential Operators for the Stokes System: Lebesgue, Sobolev, and Hardy Spaces

The goal here is to introduce and study boundary layer potential operators for the Stokes system considered in a general class of subsets of $\mathbb{R}^n$, with $n \geq 2$. To set the stage, recall the Kelvin matrix-valued fundamental solution $E = \big(E_{jk}\big)_{1\leq j,k\leq n}$ of the Stokes system in $\mathbb{R}^n$, whose (j,k)-entry is defined at each $x = (x_j)_{1\leq j\leq n} \in \mathbb{R}^n \setminus \{0\}$ by

$$E_{jk}(x) := \begin{cases} -\dfrac{1}{2\omega_{n-1}}\left(\dfrac{1}{n-2}\dfrac{\delta_{jk}}{|x|^{n-2}} + \dfrac{x_j x_k}{|x|^n}\right) & \text{if } \ n \geq 3, \\[2ex] -\dfrac{1}{4\pi}\left(-\delta_{jk}\ln|x| + \dfrac{x_j x_k}{|x|^2}\right) & \text{if } \ n = 2, \end{cases} \tag{6.2.1}$$

and the accompanying pressure vector $\vec{q}$ given by

$$\vec{q}(x) = \big(q_j(x)\big)_{1\leq j\leq n} := -\frac{1}{\omega_{n-1}}\frac{x}{|x|^n}, \qquad \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{6.2.2}$$

See, e.g., [66, Theorem 10.29, p. 382] when $n \geq 3$ and [41, (2.3.3), p. 63] when $n = 2$. Regarding them as functions defined $\mathcal{L}^n$-a.e. in $\mathbb{R}^n$, these are locally integrable functions of (at most) slow growth at infinity. As such, the distributions they induce in $\mathbb{R}^n$ are tempered, i.e.,

$$E = \big(E_{jk}\big)_{1\leq j,k\leq n} \in \big[\mathcal{S}'(\mathbb{R}^n)\big]^{n\times n} \text{ and } \vec{q} = \big(q_j\big)_{1\leq j\leq n} \in \big[\mathcal{S}'(\mathbb{R}^n)\big]^n. \tag{6.2.3}$$

Interpreted as such, they satisfy in the sense of distributions in $\mathbb{R}^n$,

$$\Delta E_{jk} - \partial_j q_k = \delta_{jk}\delta \text{ for each } j, k \in \{1, \dots, n\}, \tag{6.2.4}$$

$$\partial_j E_{jk} = 0 \text{ for each } k \in \{1, \dots, n\} \text{ and } \partial_k E_{jk} = 0 \text{ for each } j \in \{1, \dots, n\}, \tag{6.2.5}$$

$$E_{jk} = E_{kj} \text{ and } \partial_j q_k = \partial_k q_j \text{ for each } j, k \in \{1, \dots, n\}, \tag{6.2.6}$$

$$\vec{q} = -\nabla E_{\Delta} \text{ and } \operatorname{div}\vec{q} = \partial_j q_j = -\delta \text{ and } \Delta\vec{q} = -\nabla\delta, \tag{6.2.7}$$

where δ is the Dirac delta-function and E_{Δ} is the standard fundamental solution for the Laplacian in $\mathbb{R}^n$ (cf. (A.0.65)). Moreover, for each $j, k, \ell \in \{1, \dots, n\}$ we have

$$\widehat{q_j}(\xi) = \mathrm{i}\frac{\xi_j}{|\xi|^2} \text{ and } \widehat{\partial_\ell E_{jk}}(\xi) = \mathrm{i}\xi_\ell\Big(\frac{\xi_j\xi_k}{|\xi|^4} - \frac{\delta_{jk}}{|\xi|^2}\Big) \\ \text{for each } \xi = (\xi_r)_{1\leq r\leq n} \in \mathbb{R}^n \setminus \{0\}. \tag{6.2.8}$$

Indeed, the first formula above is a particular case of [66, Corollary 4.65, p. 147]. As regards the second formula in (6.2.8), the case $n \geq 3$ is clear from [66, (10.6.15), p. 382], while the case $n = 2$ may be treated using [66, Proposition 4.73, pp. 153-154].

In particular, (6.2.4)-(6.2.7) imply that at each $x \in \mathbb{R}^n \setminus \{0\}$ we have

$$\begin{gathered} \partial_k E_{jk}(x) = 0 \text{ for } 1 \leq j \leq n, \ \partial_j E_{jk}(x) = 0 \text{ for } 1 \leq k \leq n, \\ \Delta E_{jk}(x) = \Delta E_{kj}(x) = \partial_k q_j(x) = \partial_j q_k(x) \text{ for } 1 \leq j, k \leq n, \\ \Delta\vec{q}(x) = 0, \text{ and } \partial_j q_j(x) = 0. \end{gathered} \tag{6.2.9}$$

Also,

$$\Delta^2 E_{jk}(x) = 0 \text{ for } 1 \leq j, k \leq n, \text{ and } x \in \mathbb{R}^n \setminus \{0\}. \tag{6.2.10}$$

Consider next an open set $\Omega \subseteq \mathbb{R}^n$ and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In this setting, define the action of the boundary-to-domain `single layer potential operator` for the Stokes system $\mathcal{S}$ on each function[2]

$$\vec{f} = (f_j)_{1\leq j\leq n} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\Big)\Big]^n \tag{6.2.11}$$

according to

$$\begin{aligned} \mathcal{S}\vec{f}(x) &:= \int_{\partial\Omega} E(x-y)\vec{f}(y)\,\mathrm{d}\sigma(y) \\ &= \Big(\int_{\partial\Omega} E_{jk}(x-y) f_k(y)\,\mathrm{d}\sigma(y)\Big)_{1\leq j\leq n} \text{ for each } x \in \Omega. \end{aligned} \tag{6.2.12}$$

[2] when $n = 2$, the weight $(1 + |x|^{n-2})^{-1}$ should be replaced by $\ln(2 + |x|)$

The corresponding (single layer) `pressure potential` acts on functions

$$\vec{f} = (f_j)_{1\leq j\leq n} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n \tag{6.2.13}$$

according to

$$\begin{aligned} \mathcal{Q}\vec{f}(x) &:= \int_{\partial\Omega} \langle \vec{q}(x-y), \vec{f}(y)\rangle \, \mathrm{d}\sigma(y) \\ &= \int_{\partial\Omega} q_j(x-y) f_j(y)\, \mathrm{d}\sigma(y) \ \text{ for each } \ x\in\Omega. \end{aligned} \tag{6.2.14}$$

Then, on account of (6.2.9), it follows that for each function $\vec{f}$ as in (6.2.11) we have

$$\Delta\mathcal{S}\vec{f} - \nabla\mathcal{Q}\vec{f} = 0 \ \text{ and } \ \mathrm{div}\mathcal{S}\vec{f} = 0 \ \text{ in } \ \Omega. \tag{6.2.15}$$

This may be interpreted as saying that, for each function $\vec{f}$ as in (6.2.11), the pair $\big(\mathcal{S}\vec{f}, \mathcal{Q}\vec{f}\big)$ is a null-solution for the Stokes system in Ω.

Let us now consider an open set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω, which is well defined at σ-a.e. point on $\partial_*\Omega$. In such a setting, the action of the boundary-to-domain `double layer potential operator` for the Stokes system $\mathcal{D}_\lambda$ on an arbitrary function

$$\vec{f} = (f_j)_{1\leq j\leq n} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n \tag{6.2.16}$$

is given at each $x \in \Omega$ by[3]

[3] with $E_{\bullet\gamma}$ denoting the γ-th column of the matrix-valued function E

$$\mathcal{D}_\lambda \vec{f}(x) := \left(\int_{\partial_*\Omega} \left\langle \partial^\lambda_{\nu(y)} \big(E_{\bullet\gamma}(y-x), q_\gamma(y-x)\big), \vec{f}(y) \right\rangle \mathrm{d}\sigma(y) \right)_{1\leq\gamma\leq n}$$

$$= \left(\int_{\partial_*\Omega} \left\langle \partial^\lambda_{\nu(y)} \big(E_{\bullet\gamma}(x-y), -q_\gamma(x-y)\big), \vec{f}(y) \right\rangle \mathrm{d}\sigma(y) \right)_{1\leq\gamma\leq n}$$

$$= \left(\int_{\partial_*\Omega} \Big\{ -\nu_j(y) a^{\alpha\beta}_{jk}(\lambda)(\partial_k E_{\beta\gamma})(x-y) + \nu_\alpha(y) q_\gamma(x-y) \Big\} f_\alpha(y)(y)\, \mathrm{d}\sigma(y) \right)_{1\leq\gamma\leq n}$$

$$= \left(\int_{\partial_*\Omega} \Big\{ -\nu_k(y)(\partial_k E_{j\gamma})(x-y) - \lambda \nu_k(y)(\partial_j E_{k\gamma})(x-y) + \nu_j(y) q_\gamma(x-y) \Big\} f_j(y)\, \mathrm{d}\sigma(y) \right)_{1\leq\gamma\leq n}. \tag{6.2.17}$$

In a more explicit fashion, for each vector-valued function $\vec{f}$ as in (6.2.16) we have

$$\mathcal{D}_\lambda \vec{f}(x) \tag{6.2.18}$$

$$= \left(\int_{\partial_*\Omega} \left\{ (\lambda-1)\frac{\delta_{j\gamma}}{2\omega_{n-1}} \frac{\langle x-y, \nu(y)\rangle}{|x-y|^n} - (\lambda+1)\frac{n}{2\omega_{n-1}} \frac{\langle x-y, \nu(y)\rangle (x_j-y_j)(x_\gamma-y_\gamma)}{|x-y|^{n+2}} + (\lambda-1)\frac{1}{2\omega_{n-1}} \frac{\nu_j(y)(x_\gamma-y_\gamma) - \nu_\gamma(y)(x_j-y_j)}{|x-y|^n} \right\} f_j(y)\, \mathrm{d}\sigma(y) \right)_{1\leq\gamma\leq n}.$$

We wish to note here that, as a consequence of (6.2.17), (6.2.4), (6.2.5), and the Divergence Formula in [68, Corollary 1.5.2], much as in the case of (1.3.45), for each constant $\vec{c} \in \mathbb{C}^n$ we have

$$\text{if } \partial\Omega \text{ is compact then } \ \mathcal{D}_\lambda \vec{c} = \begin{cases} \vec{c} \ \text{ in } \ \Omega, \text{ if } \Omega \text{ is bounded,} \\ 0 \ \text{ in } \ \Omega, \text{ if } \Omega \text{ is unbounded.} \end{cases} \tag{6.2.19}$$

Let us also define the action of the corresponding (double layer) `pressure potential` on functions

$$\vec{f} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^n \tag{6.2.20}$$

by setting, for each $x \in \Omega$,

$$\begin{aligned}\mathcal{P}_\lambda \vec{f}(x) &:= -(1+\lambda)\int_{\partial_*\Omega} \nu_j(y)\langle(\partial_j\vec{q})(x-y), \vec{f}(y)\rangle \, d\sigma(y)\\ &= -(1+\lambda)\int_{\partial_*\Omega} \nu_j(y)(\partial_j q_k)(x-y) f_k(y)\, d\sigma(y).\end{aligned} \tag{6.2.21}$$

Then (6.2.9) implies that for each function $\vec{f}$ as in (6.2.16) we have

$$\Delta\mathcal{D}_\lambda\vec{f} - \nabla\mathcal{P}_\lambda\vec{f} = 0 \text{ and } \mathrm{div}\mathcal{D}_\lambda\vec{f} = 0 \text{ in } \Omega. \tag{6.2.22}$$

Hence, for each function $\vec{f}$ as in (6.2.16), the pair $\big(\mathcal{D}_\lambda\vec{f}, \mathcal{P}_\lambda\vec{f}\big)$ is a null-solution for the Stokes system in Ω.

In relation to these operators we have the following boundary layer integral representation formulas for null-solutions of the Stokes system.

Theorem 6.2.1 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, be an open set with a lower Ahlfors regular boundary, and with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Also, fix an aperture parameter $\kappa > 0$. Suppose the functions $\vec{u} \in \big[\mathscr{C}^\infty(\Omega)\big]^n$ and $\pi \in \mathscr{C}^\infty(\Omega)$ satisfy*

$$\begin{aligned}&\Delta\vec{u} - \nabla\pi = 0 \text{ in } \Omega, \quad \mathrm{div}\vec{u} = 0 \text{ in } \Omega, \text{ and}\\ &\vec{u}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, (\nabla\vec{u})\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \pi\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial_{\mathrm{nta}}\Omega.\end{aligned} \tag{6.2.23}$$

Then the following conclusions are true.

(a) If $n \geq 3$ and one has

$$\int_{\partial\Omega} \frac{(\mathcal{N}_\kappa\vec{u})(y)}{1+|y|^{n-1}}\, d\sigma(y) < \infty \text{ and } \int_{\partial\Omega} \frac{\big(\mathcal{N}_\kappa(\nabla\vec{u})\big)(y) + (\mathcal{N}_\kappa\pi)(y)}{1+|y|^{n-2}}\, d\sigma(y) < \infty, \tag{6.2.24}$$

then for any $\kappa' > 0$ the nontangential traces $u\Big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ $(\nabla\vec{u})\Big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$, $\pi\Big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ also exist at σ-a.e. point on $\partial_{\mathrm{nta}}\Omega$ and are actually independent of the aperture parameter κ'. Moreover, with the dependence on the parameter κ' dropped, for each $\lambda \in \mathbb{C}$ one has[4]

$$\vec{u}(x) = \mathcal{D}_\lambda\Big(\vec{u}\Big|_{\partial\Omega}^{\mathrm{n.t.}}\Big)(x) - \mathscr{S}\Big(\partial_\nu^\lambda(\vec{u},\pi)\Big)(x), \text{ for all } x \in \Omega, \tag{6.2.25}$$

[4] here we allow the operator $\mathscr{S}$ to act on functions originally defined only on $\partial_*\Omega$ by extending them by zero to the entire topological boundary $\partial\Omega$

if either Ω *is bounded, or* $\partial\Omega$ *is unbounded. In the case when* Ω *is an exterior domain, formula* (6.2.25) *continues to be true under the additional assumption that there exists* $\mu \in (1,\infty)$ *such that*

$$\fint_{B(0,\mu R)\setminus B(0,R)} \big\{|\vec{u}| + R|\pi|\big\}\, \mathrm{d}\mathcal{L}^n = o(1) \ \ \textit{as}\ \ R \to \infty. \tag{6.2.26}$$

(b) If $n \geq 3$ *and one has*

$$\int\limits_{\partial\Omega} \frac{(\mathcal{N}_\kappa\vec{u})(y)}{1+|y|^n}\, \mathrm{d}\sigma(y) < \infty \ \ \textit{and} \ \ \int\limits_{\partial\Omega} \frac{\big(\mathcal{N}_\kappa(\nabla\vec{u})\big)(y) + (\mathcal{N}_\kappa\pi)(y)}{1+|y|^{n-1}}\, \mathrm{d}\sigma(y) < \infty, \tag{6.2.27}$$

then for any $\kappa' > 0$ *the nontangential traces* $u\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$ $(\nabla\vec{u})\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$, $\pi\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$ *also exist at* σ*-a.e. point on* $\partial_{\mathrm{nta}}\Omega$ *and are actually independent of* κ'*. Furthermore, with the dependence on the parameter* κ' *dropped, for each* $\lambda \in \mathbb{C}$ *one has*[5]

$$\pi(x) = \mathcal{P}_\lambda\Big(\vec{u}\big|^{\mathrm{n.t.}}_{\partial\Omega}\Big)(x) - \mathcal{Q}\Big(\partial^\lambda_\nu(\vec{u},\pi)\Big)(x), \ \ \textit{for all}\ \ x \in \Omega, \tag{6.2.28}$$

if either Ω *is bounded, or* $\partial\Omega$ *is unbounded. In the case when* Ω *is an exterior domain, the same conclusion holds true under the additional assumption that there exists* $\mu \in (1,\infty)$ *such that*

$$\fint_{B(0,\mu R)\setminus B(0,R)} \big\{|\vec{u}| + R|\pi|\big\}\, \mathrm{d}\mathcal{L}^n = o(R) \ \ \textit{as}\ \ R \to \infty. \tag{6.2.29}$$

(c) Assume $n \geq 2$ *and* Ω *is an exterior domain. Also, suppose there exists a truncation parameter* $\varepsilon > 0$ *for which*

$$\mathcal{N}^\varepsilon_\kappa\vec{u},\ \mathcal{N}^\varepsilon_\kappa(\nabla\vec{u}),\ \mathcal{N}^\varepsilon_\kappa\pi \in L^1(\partial\Omega,\sigma). \tag{6.2.30}$$

Finally, make the assumption that there exists some $\mu \in (1,\infty)$ *such that*

$$\fint_{B(0,\mu R)\setminus B(0,R)} \big\{|\vec{u}| + R|\pi|\big\}\, \mathrm{d}\mathcal{L}^n = o(R) \ \ \textit{as}\ \ R \to \infty. \tag{6.2.31}$$

Then for any $\kappa' > 0$ *the nontangential traces* $u\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$ $(\nabla\vec{u})\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$, $\pi\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$ *also exist at* σ*-a.e. point on* $\partial_{\mathrm{nta}}\Omega$ *and are actually independent of* κ' (*so the dependence on the aperture parameter may be dropped*).
In addition, there exists a constant $c \in \mathbb{C}^n$ *with the property that for each* $\lambda \in \mathbb{C}$ *one has*[6]

[5] again, allowing the operator $\mathcal{Q}$ to act on functions originally defined only on $\partial_*\Omega$ by extending them by zero to the entire topological boundary $\partial\Omega$

[6] once more allowing the operators $\mathcal{S}$, $\mathcal{Q}$ to act on functions originally defined only on $\partial_*\Omega$ by extending them by zero to the entire topological boundary $\partial\Omega$

$$\vec{u}(x) = \mathcal{D}_\lambda\Big(\vec{u}\big|^{\rm n.t.}_{\partial\Omega}\Big)(x) - \mathcal{S}\Big(\partial^\lambda_\nu(\vec{u},\pi)\Big)(x) + c, \quad \textit{for all } x \in \Omega, \tag{6.2.32}$$

and

$$\pi(x) = \mathcal{P}_\lambda\Big(\vec{u}\big|^{\rm n.t.}_{\partial\Omega}\Big)(x) - \mathcal{Q}\Big(\partial^\lambda_\nu(\vec{u},\pi)\Big)(x), \quad \textit{for all } x \in \Omega, \tag{6.2.33}$$

(d) *Make the assumption that* $n = 2$*, the set* $\partial\Omega$ *is compact, and there exists a truncation parameter* $\varepsilon > 0$ *for which* (6.2.30) *holds.*
Then for any $\kappa' > 0$ *the nontangential traces* $u\big|^{\kappa'-\rm n.t.}_{\partial\Omega}$ $(\nabla\vec{u})\big|^{\kappa'-\rm n.t.}_{\partial\Omega}$, $\pi\big|^{\kappa'-\rm n.t.}_{\partial\Omega}$ *also exist at* σ*-a.e. point on* $\partial_{\rm nta}\Omega$ *and are actually independent of* κ'*.*
Moreover, if Ω *is bounded then the integral representation formulas in* (6.2.25) *and* (6.2.28) *are valid. Finally, if* Ω *is unbounded then* (6.2.25) *holds provided there exists* $\mu \in (1,\infty)$ *such that*

$$⨍_{B(0,\mu R)\setminus B(0,R)} \big\{|\vec{u}| + R|\pi|\big\}\, d\mathcal{L}^2 = o\big(\tfrac{1}{\ln R}\big) \ \textit{as} \ R \to \infty, \tag{6.2.34}$$

while (6.2.28) *holds provided there exists some* $\mu \in (1,\infty)$ *such that*

$$⨍_{B(0,\mu R)\setminus B(0,R)} \big\{|\vec{u}| + R|\pi|\big\}\, d\mathcal{L}^2 = o(R) \ \textit{as} \ R \to \infty. \tag{6.2.35}$$

Proof Let us first deal with item *(a)*, under the additional assumption made in (6.2.24). First, from (6.2.24) and [68, Corollary 8.9.9] we deduce that for any $\kappa' > 0$ the nontangential traces

$$\begin{gathered} \vec{u}\big|^{\kappa'-\rm n.t.}_{\partial\Omega},\ (\nabla\vec{u})\big|^{\kappa'-\rm n.t.}_{\partial\Omega},\ \pi\big|^{\kappa'-\rm n.t.}_{\partial\Omega} \ \text{exist at } \sigma\text{-a.e. point on } \partial_{\rm nta}\Omega, \\ \text{and are actually independent of the aperture parameter } \kappa'. \end{gathered} \tag{6.2.36}$$

In addition, with the dependence on κ' dropped, from (6.2.24), [68, (8.9.8)], [68, (8.9.44)], and [68, (8.8.52)] we conclude that

$$\int_{\partial_*\Omega} \frac{\big|\big(\vec{u}\big|^{\rm n.t.}_{\partial\Omega}\big)(y)\big|}{1+|y|^{n-1}}\, d\sigma(y) < \infty \ \text{ and } \int_{\partial_*\Omega} \frac{\big|\big((\nabla\vec{u})\big|^{\rm n.t.}_{\partial\Omega}\big)(y)\big| + \big|\big(\pi\big|^{\rm n.t.}_{\partial\Omega}\big)(y)\big|}{1+|y|^{n-2}}\, d\sigma(y) < \infty. \tag{6.2.37}$$

In particular, having selected some $\lambda \in \mathbb{C}$, from (6.2.37) and (6.2.1)-(6.2.2) it follows that, for each fixed point $x \in \Omega$, the integrals defining $\mathcal{D}_\lambda\big(\vec{u}\big|^{\rm n.t.}_{\partial\Omega}\big)(x)$ and $\mathcal{S}\big(\partial^\lambda_\nu(\vec{u},\pi)\big)(x)$ are absolutely convergent.

To proceed, fix an arbitrary index $\gamma \in \{1,\dots,n\}$, pick an arbitrary point $x \in \Omega$, and consider the vector field $\vec{F}_x = \big(F_j\big)_{1\le j\le n}$ with components given at $\mathcal{L}^n$-a.e. point in Ω by (recall that, throughout, the summation convention over repeated indices is in effect)

$$F_j := -a^{\alpha\beta}_{jk}(\lambda)(\partial_k E_{\beta\gamma})(x-\cdot)u_\alpha + q_\gamma(x-\cdot)u_j$$
$$- E_{\gamma\alpha}(x-\cdot)a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta + E_{\gamma j}(x-\cdot)\pi. \tag{6.2.38}$$

Then from (6.2.38) and (6.2.1)-(6.2.2) we see that

$$\vec{F}_x \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n. \tag{6.2.39}$$

To compute $\operatorname{div}\vec{F}_x$ in the sense of distributions in the open set Ω, first observe from definitions that we have $E(x-\cdot) = E(\cdot - x)$ and $\vec{q}(x-\cdot) = -\vec{q}(\cdot - x)$. Also, we find it convenient to express $(\partial_k E_{\beta\gamma})(x-\cdot)$ as $-\partial_k[E_{\beta\gamma}(\cdot - x)]$ then, using (6.2.4)-(6.2.7), the first line in (6.2.23), and (6.1.3), write

$$\begin{aligned}
\operatorname{div}\vec{F}_x &= \partial_j F_j = \Big(a^{\alpha\beta}_{jk}(\lambda)\partial_j\partial_k[E_{\beta\gamma}(\cdot - x)]\Big)u_\alpha + \partial_k[E_{\beta\gamma}(\cdot - x)]a^{\alpha\beta}_{jk}(\lambda)(\partial_j u_\alpha) \\
&\quad - \partial_j[q_\gamma(\cdot - x)]u_j - q_\gamma(\cdot - x)(\partial_j u_j) \\
&\quad - \partial_j[E_{\gamma\alpha}(\cdot - x)]a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta - E_{\gamma\alpha}(\cdot - x)a^{\alpha\beta}_{jk}(\lambda)\partial_j\partial_k u_\beta \\
&\quad + \partial_j[E_{\gamma j}(\cdot - x)]\pi + E_{\gamma j}(\cdot - x)\partial_j\pi \\
&= \big(L_\lambda[E_{\bullet\gamma}(\cdot - x)]\big)_\alpha u_\alpha - \partial_j[q_\gamma(\cdot - x)]u_j - E_{\gamma\alpha}(\cdot - x)(L_\lambda\vec{u})_\alpha \\
&\quad + E_{\gamma j}(\cdot - x)\partial_j\pi \\
&= \Big(L_\lambda[E_{\bullet\gamma}(\cdot - x)] - \nabla[q_\gamma(\cdot - x)]\Big)_\alpha u_\alpha - E_{\gamma\alpha}(\cdot - x)(L_\lambda\vec{u} - \nabla\pi)_\alpha \\
&= \delta_{\alpha\gamma}u_\alpha\delta_x = u_\gamma(x)\delta_x
\end{aligned} \tag{6.2.40}$$

where δ_x is Dirac's delta distribution with mass at x. Hence,

$$\operatorname{div}\vec{F}_x = u_\gamma(x)\delta_x \in \mathscr{E}'(\Omega). \tag{6.2.41}$$

Next, consider $K := \overline{B\big(x, \tfrac{1}{2}\operatorname{dist}(x, \partial\Omega)\big)}$ which is a compact subset of Ω, and based on (6.2.38), (6.2.1)-(6.2.2), and [68, Lemma 8.3.7] estimate, at each $y \in \partial\Omega$,

$$\begin{aligned}
\big(\mathcal{N}^{\Omega\setminus K}_\kappa \vec{F}_x\big)(y) &\leq C\big(\mathcal{N}_\kappa\vec{u}\big)(y)\cdot \sup_{z\in\Gamma_\kappa(y)\setminus K}\big[|x-z|^{1-n}\big] \\
&\quad + C\Big\{\big(\mathcal{N}_\kappa(\nabla\vec{u})\big)(y) + \big(\mathcal{N}_\kappa\pi\big)(y)\Big\}\cdot \sup_{z\in\Gamma_\kappa(y)\setminus K}\big[|x-z|^{2-n}\big] \\
&\leq C\left\{\frac{\big(\mathcal{N}_\kappa\vec{u}\big)(y)}{|x-y|^{n-1}} + \frac{\big(\mathcal{N}_\kappa(\nabla\vec{u})\big)(y) + \big(\mathcal{N}_\kappa\pi\big)(y)}{|x-y|^{n-2}}\right\},
\end{aligned} \tag{6.2.42}$$

for some constant $C = C(\Omega, n, \lambda, \kappa) \in (0, \infty)$. From (6.2.42), (6.2.24), and [68, (8.2.26)] it follows that

$$\mathcal{N}_\kappa^{\Omega\setminus K} \vec{F}_x \in L^1(\partial\Omega, \sigma). \tag{6.2.43}$$

Moreover, from (6.2.38), [68, (8.9.10)-(8.9.11)], and (6.2.36) we conclude that

$$\vec{F}_x\Big|_{\partial\Omega}^{\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial_{\text{nta}}\Omega \tag{6.2.44}$$

and, in fact, for each $j \in \{1, \dots, n\}$ at σ-a.e. point $y \in \partial_{\text{nta}}\Omega$ we have

$$\begin{aligned}\Big(F_j\Big|_{\partial\Omega}^{\text{n.t.}}\Big)(y) &= -a_{jk}^{\alpha\beta}(\lambda)(\partial_k E_{\beta\gamma})(x-y)\big(u_\alpha\big|_{\partial\Omega}^{\text{n.t.}}\big)(y) + q_\gamma(x-y)\big(u_j\big|_{\partial\Omega}^{\text{n.t.}}\big)(y)\\ &\quad - E_{\gamma\alpha}(x-y)a_{jk}^{\alpha\beta}(\lambda)\big((\partial_k u_\beta)\big|_{\partial\Omega}^{\text{n.t.}}\big)(y)\\ &\quad + E_{\gamma j}(x-y)\big(\pi\big|_{\partial\Omega}^{\text{n.t.}}\big)(y).\end{aligned} \tag{6.2.45}$$

From (6.2.45), (6.1.6), and [68, (8.8.52)] we then conclude that at σ-a.e. point $y \in \partial_*\Omega$ we have

$$\begin{aligned}\nu(y)\cdot\Big(\vec{F}_x\Big|_{\partial\Omega}^{\text{n.t.}}\Big)(y) &= \nu_j(y)\Big(F_j\Big|_{\partial\Omega}^{\text{n.t.}}\Big)(y)\\ &= \Big\{-\nu_j(y)a_{jk}^{\alpha\beta}(\lambda)(\partial_k E_{\beta\gamma})(x-y) + \nu_\alpha(y)q_\gamma(x-y)\Big\}\big(u_\alpha\big|_{\partial\Omega}^{\text{n.t.}}\big)(y)\\ &\quad - E_{\gamma\alpha}(x-y)\Big\{\nu_j(y)a_{jk}^{\alpha\beta}(\lambda)\big((\partial_k u_\beta)\big|_{\partial\Omega}^{\text{n.t.}}\big)(y) - \nu_\alpha(y)\big(\pi\big|_{\partial\Omega}^{\text{n.t.}}\big)(y)\Big\}\\ &= \Big\langle \partial_{\nu(y)}^{\lambda}\big(E_{\bullet\gamma}(x-y), -q_\gamma(x-y)\big), \big(\vec{u}\big|_{\partial\Omega}^{\text{n.t.}}\big)(y)\Big\rangle\\ &\quad - \Big\langle E_{\gamma\bullet}(x-y), \partial_\nu^\lambda(\vec{u},\pi)(y)\Big\rangle.\end{aligned} \tag{6.2.46}$$

In the case when Ω is an exterior domain, condition (6.2.26) self-improves thanks to [70, Lemma 1.5.6] (bearing in mind that $\Delta^2\vec{u} = 0$) to

$$\fint_{B(0,\mu R)\setminus B(0,R)} \Big\{|\vec{u}| + R\big(|\nabla\vec{u}| + |\pi|\big)\Big\}\, d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{6.2.47}$$

In turn, (6.2.47) together with (6.2.1)-(6.2.2) guarantee that

$$\text{the vector field } \vec{F}_x \text{ satisfies the decay condition [68, (1.4.8)].} \tag{6.2.48}$$

Collectively, (6.2.39), (6.2.41), (6.2.44), and (6.2.48) ensure that, for each point $x \in \Omega$ such that [70, (1.5.12)] holds, the vector field $\vec{F}_x$ satisfies the hypotheses of [68, Theorem 1.4.1]. On account of [68, (4.6.21)], (6.2.41), and (6.2.46), the

Divergence Formula [68, (1.4.6)] presently yields

$$\begin{aligned}
u_\gamma(x) &= {}_{\mathscr{E}'(\Omega)}\big\langle u_\gamma(x)\delta_x, 1\big\rangle_{\mathscr{E}(\Omega)} = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\Big(u_\gamma(x)\delta_x, 1\Big)_{\mathscr{C}_b^\infty(\Omega)} \\
&= {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}_x, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}_x\big|^{\mathrm{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma \\
&= \int_{\partial_*\Omega} \Big\langle \partial^\lambda_{\nu(y)}\big(E_{\bullet\gamma}(x-y), -q_\gamma(x-y)\big), \big(\vec{u}\big|^{\mathrm{n.t.}}_{\partial\Omega}\big)(y)\Big\rangle\,\mathrm{d}\sigma(y) \\
&\quad - \int_{\partial_*\Omega} \Big\langle E_{\gamma\bullet}(x-y), \partial^\lambda_\nu(\vec{u},\pi)(y)\Big\rangle\,\mathrm{d}\sigma(y), \\
&= \Big\{\mathcal{D}_\lambda\Big(\vec{u}\big|^{\mathrm{n.t.}}_{\partial\Omega}\Big)(x) - \mathscr{S}\Big(\partial^\lambda_\nu(\vec{u},\pi)\Big)(x)\Big\}_\gamma
\end{aligned} \tag{6.2.49}$$

for each point $x \in \Omega$ and each $\gamma \in \{1, \dots, n\}$. This establishes (6.2.25).

Moving on, the first claims in item *(b)* are dealt with in a similar manner, so we focus on establishing the integral representation formula (6.2.28), under the assumption made in (6.2.27). To this end, fix an arbitrary point $x \in \Omega$ and consider the vector field $\vec{G}_x = \big(G_j\big)_{1\le j\le n}$ with components given at $\mathcal{L}^n$-a.e. point in Ω by

$$G_j := -(1+\lambda)(\partial_j q_k)(x-\cdot)u_k - q_\alpha(x-\cdot)a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta + q_j(x-\cdot)\pi. \tag{6.2.50}$$

Note that (6.2.50) and (6.2.1)-(6.2.2) guarantee that

$$\vec{G}_x \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^n. \tag{6.2.51}$$

The next order of business is to compute $\mathrm{div}\vec{G}_x$ in the sense of distributions in Ω. In this vein, first recall that $\vec{q}(x-\cdot) = -\vec{q}(\cdot - x)$. Also, express $(\partial_j q_k)(x-\cdot)$ as $\partial_j[q_k(\cdot - x)]$ then rely on (6.1.1), (6.1.3), the first line in (6.2.23), and (6.2.6)-(6.2.7) to write

$$
\begin{aligned}
\operatorname{div}\vec{G}_x = \partial_j G_j &= -(1+\lambda)\big(\partial_j\partial_j[q_k(\cdot - x)]\big)u_k - (1+\lambda)\big(\partial_j[q_k(\cdot - x)]\big)(\partial_j u_k)\\
&\quad + \partial_j[q_\alpha(\cdot - x)]a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta + q_\alpha(\cdot - x)a^{\alpha\beta}_{jk}(\lambda)\partial_j\partial_k u_\beta\\
&\quad - \partial_j[q_j(\cdot - x)]\pi - q_j(\cdot - x)\partial_j\pi\\
&= -(1+\lambda)\big(\Delta[q_k(\cdot - x)]\big)u_k - (1+\lambda)\big(\partial_j[q_k(\cdot - x)]\big)(\partial_j u_k)\\
&\quad + \partial_j[q_\alpha(\cdot - x)]\big(\delta_{\alpha\beta}\delta_{jk} + \lambda\delta_{j\beta}\delta_{k\alpha}\big)\partial_k u_\beta + q_\alpha(\cdot - x)\big(L_\lambda\vec{u}\big)_\alpha\\
&\quad + \pi\delta_x - q_j(\cdot - x)\partial_j\pi\\
&= (1+\lambda)(\partial_k\delta_x)u_k - (1+\lambda)\big(\partial_j[q_k(\cdot - x)]\big)(\partial_j u_k)\\
&\quad + \partial_j[q_\alpha(\cdot - x)]\partial_j u_\alpha + \lambda\partial_\beta[q_\alpha(\cdot - x)]\partial_\alpha u_\beta + q_\alpha(\cdot - x)\partial_\alpha\pi\\
&\quad + \pi\delta_x - q_j(\cdot - x)\partial_j\pi\\
&= (1+\lambda)u_k(\partial_k\delta_x) + \pi(x)\delta_x.
\end{aligned}
\tag{6.2.52}
$$

This shows that

$$
\operatorname{div}\vec{G}_x = (1+\lambda)u_k(\partial_k\delta_x) + \pi(x)\delta_x \in \mathscr{E}'(\Omega). \tag{6.2.53}
$$

For further reference, let us observe that

$$
\begin{aligned}
{}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\operatorname{div}\vec{G}_x, 1\big)_{\mathscr{C}^\infty_b(\Omega)} &= {}_{\mathscr{E}'(\Omega)}\big\langle(1+\lambda)u_k(\partial_k\delta_x) + \pi(x)\delta_x, 1\big\rangle_{\mathscr{E}(\Omega)}\\
&= (1+\lambda)\,{}_{\mathscr{E}'(\Omega)}\big\langle u_k(\partial_k\delta_x), 1\big\rangle_{\mathscr{E}(\Omega)} + \pi(x)\,{}_{\mathscr{E}'(\Omega)}\big\langle\delta_x, 1\big\rangle_{\mathscr{E}(\Omega)}\\
&= (1+\lambda)\,{}_{\mathscr{E}'(\Omega)}\big\langle\partial_k\delta_x, u_k\big\rangle_{\mathscr{E}(\Omega)} + \pi(x)\\
&= -(1+\lambda)\,{}_{\mathscr{E}'(\Omega)}\big\langle\delta_x, \operatorname{div}\vec{u}\big\rangle_{\mathscr{E}(\Omega)} + \pi(x)\\
&= \pi(x),
\end{aligned}
\tag{6.2.54}
$$

thanks to [68, (4.6.21)] and the fact that $\vec{u}$ is divergence-free (cf. (6.2.23)).

Next, with the compact $K := \overline{B\big(x, \tfrac{1}{2}\operatorname{dist}(x, \partial\Omega)\big)}$ as before, based on (6.2.50), (6.2.2), and [68, Lemma 8.3.7], at each $y \in \partial\Omega$ we may estimate

$$
\begin{aligned}
\big(\mathcal{N}_\kappa^{\Omega\setminus K}\vec{G}_x\big)(y) &\leq \big(\mathcal{N}_\kappa\vec{u}\big)(y)\cdot\sup_{z\in\Gamma_\kappa(y)\setminus K}\big[|x-z|^{-n}\big]\\
&\quad + \Big\{\big(\mathcal{N}_\kappa(\nabla\vec{u})\big)(y) + \big(\mathcal{N}_\kappa\pi\big)(y)\Big\}\cdot\sup_{z\in\Gamma_\kappa(y)\setminus K}\big[|x-z|^{1-n}\big]\\
&\leq C\left\{\frac{\big(\mathcal{N}_\kappa\vec{u}\big)(y)}{|x-y|^n} + \frac{\big(\mathcal{N}_\kappa(\nabla\vec{u})\big)(y) + \big(\mathcal{N}_\kappa\pi\big)(y)}{|x-y|^{n-1}}\right\},
\end{aligned}
\tag{6.2.55}
$$

for some constant $C = C(\Omega, \lambda, \kappa) \in (0, \infty)$. From (6.2.55), (6.2.27), and [68, (8.2.26)] it follows that

$$\mathcal{N}_\kappa^{\Omega\setminus K} \vec{G}_x \in L^1(\partial\Omega, \sigma). \tag{6.2.56}$$

In addition, from (6.2.50), the second line in (6.2.23), [68, (8.9.10)-(8.9.11)], and [70, (1.5.18)] we conclude that

$$\vec{G}_x\Big|_{\partial\Omega}^{\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial_{\text{nta}}\Omega \tag{6.2.57}$$

and, in fact, for each $j \in \{1, \dots, n\}$ at σ-a.e. point $y \in \partial_{\text{nta}}\Omega$ we have

$$\begin{aligned}\Big(G_j\Big|_{\partial\Omega}^{\text{n.t.}}\Big)(y) &= -(1+\lambda)(\partial_j q_k)(x-y)\big(u_k\big|_{\partial\Omega}^{\text{n.t.}}\big)(y)\\ &\quad - q_\alpha(x-y)a_{jk}^{\alpha\beta}(\lambda)\big((\partial_k u_\beta)\big|_{\partial\Omega}^{\text{n.t.}}\big)(y) + q_j(x-\cdot)\big(\pi\big|_{\partial\Omega}^{\text{n.t.}}\big)(y).\end{aligned} \tag{6.2.58}$$

Using (6.2.58), (6.1.6), and [68, (8.8.52)] we then see that at σ-a.e. point $y \in \partial_*\Omega$ we have

$$\begin{aligned}\nu(y)\cdot\Big(\vec{G}_x\Big|_{\partial\Omega}^{\text{n.t.}}\Big)(y) &= -(1+\lambda)\nu_j(y)(\partial_j q_k)(x-y)\big(u_k\big|_{\partial\Omega}^{\text{n.t.}}\big)(y)\\ &\quad - q_\alpha(x-y)\nu_j(y)a_{jk}^{\alpha\beta}(\lambda)\big((\partial_k u_\beta)\big|_{\partial\Omega}^{\text{n.t.}}\big)(y)\\ &\quad + q_j(x-y)\nu_j(y)\big(\pi\big|_{\partial\Omega}^{\text{n.t.}}\big)(y)\\ &= -(1+\lambda)\nu_j(y)\big\langle(\partial_j\vec{q})(x-y), \big(\vec{u}\big|_{\partial\Omega}^{\text{n.t.}}\big)(y)\big\rangle\\ &\quad - q_\alpha(x-y)\big(\partial_\nu^\lambda(\vec{u},\pi)\big)_\alpha(y).\end{aligned} \tag{6.2.59}$$

Finally, when Ω is an exterior domain condition (6.2.29) self-improves (much as (6.2.26) implied (6.2.47) via interior estimates) to

$$⨍_{B(0,\mu R)\setminus B(0,R)} \Big\{|\vec{u}| + R\big(|\nabla\vec{u}| + |\pi|\big)\Big\}\, d\mathcal{L}^n = o(R) \text{ as } R \to \infty. \tag{6.2.60}$$

Together with (6.2.2), this guarantees that

the vector field $\vec{G}_x$ satisfies the decay condition [68, (1.4.8)]. (6.2.61)

Granted (6.2.51), (6.2.53), (6.2.56), (6.2.57), (6.2.61), it follows that the vector field $\vec{G}_x$ satisfies the hypotheses of [68, Theorem 1.4.1]. As such we may rely on (6.2.54), and the Divergence Formula [68, (1.4.6)] to write, on account of (6.2.59),

$$
\begin{aligned}
\pi(x) &= {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{G}_x, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_{\partial_*\Omega} \nu \cdot \big(\vec{G}_x\big|^{\text{n.t.}}_{\partial\Omega}\big)\, d\sigma \\
&= -(1+\lambda)\int_{\partial_*\Omega} \nu_j(y)\big\langle(\partial_j\vec{q})(x-y), \big(\vec{u}\big|^{\text{n.t.}}_{\partial\Omega}\big)(y)\big\rangle\, d\sigma(y) \\
&\quad - \int_{\partial_*\Omega} q_j(x-y)\big(\partial^\lambda_\nu(\vec{u},\pi)\big)_j(y)\, d\sigma(y) \\
&= \mathcal{P}_\lambda\Big(\vec{u}\Big|^{\text{n.t.}}_{\partial\Omega}\Big)(x) - \mathcal{Q}\Big(\partial^\lambda_\nu(\vec{u},\pi)\Big)(x)
\end{aligned}
\tag{6.2.62}
$$

for each point $x \in \Omega$. This establishes (6.2.28) and completes the treatment of item *(b)*.

In fact, a very similar treatment applies to item *(d)*, since the assumptions made on that occasion guarantee that all hypotheses of [68, Corollary 1.5.2] are satisfied, and that [68, (1.5.22)] holds (with $n = 2$).

There remains to deal with item *(c)*. Hence, we shall work under the assumption that Ω is an exterior domain (in particular, $\partial\Omega$ is compact). The opening claims in item *(c)* are justified as before, so we focus on the integral representation formulas claimed in (6.2.32) and (6.2.33). In the case of (6.2.33), the very same argument as in the proof of item *(b)* works in the current setting, if we now employ the Divergence Theorem recorded in [68, Corollary 1.5.2], bearing in mind that (6.2.31) is identical to (6.2.29).

To justify the integral representation formula claimed in (6.2.32), select an arbitrary index $\gamma \in \{1,\dots,n\}$ and fix two arbitrary points $x_0, x_1 \in \Omega$. In relation to these, define the vector field

$$
\vec{F}_{x_0,x_1} := \vec{F}_{x_0} - \vec{F}_{x_1} \tag{6.2.63}
$$

where the vector fields $\vec{F}_{x_0}$ and $\vec{F}_{x_1}$ are associated with the points x_0 and x_1, respectively, like the vector field $\vec{F}_x = \big(F_j\big)_{1\leq j\leq n}$ has been associated with the point $x \in \Omega$ in (6.2.38). By (6.2.63), (6.2.39), and (6.2.40) we have

$$
\vec{F}_{x_0,x_1} \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^n \text{ and } \mathrm{div}\,\vec{F}_{x_0,x_1} = u_\gamma(x_0)\delta_{x_0} - u_\gamma(x_1)\delta_{x_1} \in \mathscr{E}'(\Omega). \tag{6.2.64}
$$

From definitions, (6.2.1)-(6.2.2), and (6.2.30) we also see that

$$
\mathcal{N}^\varepsilon_\kappa \vec{F}_{x_0,x_1} \in L^1(\partial\Omega, \sigma). \tag{6.2.65}
$$

Together, (6.2.63), (6.2.44), (6.2.46) imply

$$
\vec{F}_{x_0,x_1}\Big|^{\text{n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial_{\text{nta}}\Omega, \tag{6.2.66}
$$

and at σ-a.e. point $y \in \partial_{\text{nta}}\Omega$ we have

$$\nu(y) \cdot \Big(\vec{F}_{x_0,x_1}\Big|^{\text{n.t.}}_{\partial\Omega}\Big)(y) = \Big\langle \partial^{\lambda}_{\nu(y)}\big(E_{\bullet\gamma}(x_0 - y), -q_\gamma(x_0 - y)\big), \big(\vec{u}\big|^{\text{n.t.}}_{\partial\Omega}\big)(y)\Big\rangle$$
$$- \Big\langle E_{\gamma\bullet}(x_0 - y), \partial^{\lambda}_{\nu}(\vec{u}, \pi)(y)\Big\rangle$$
$$- \Big\langle \partial^{\lambda}_{\nu(y)}\big(E_{\bullet\gamma}(x_1 - y), -q_\gamma(x_1 - y)\big), \big(\vec{u}\big|^{\text{n.t.}}_{\partial\Omega}\big)(y)\Big\rangle$$
$$+ \Big\langle E_{\gamma\bullet}(x_1 - y), \partial^{\lambda}_{\nu}(\vec{u}, \pi)(y)\Big\rangle. \tag{6.2.67}$$

Finally, from (6.2.63), (6.2.38), (6.2.1)-(6.2.2), the Mean Value Theorem, and (6.2.31) we deduce that

$$⨍_{B(0,\mu R)\setminus B(0,R)} \big|\vec{F}_{x_0,x_1}\big| \, d\mathcal{L}^n = o(R^{1-n}) \text{ as } R \to \infty. \tag{6.2.68}$$

Collectively, (6.2.64), (6.2.65), (6.2.66), and (6.2.68) ensure that all hypotheses of [68, Corollary 1.5.2] are satisfied, and that [68, (1.5.22)] holds for the vector field $\vec{F}_{x_0,x_1}$. Consequently, for this vector field we may use the Divergence Formula in the version recorded in [68, (1.5.20)] which, thanks to (6.2.64), (6.2.67), (6.2.17), and (6.2.12), presently gives

$$u_\gamma(x_0) - u_\gamma(x_1) = \Big(\mathcal{D}_\lambda\big(\vec{u}\big|^{\text{n.t.}}_{\partial\Omega}\big)\Big)_\gamma(x_0) - \Big(\mathcal{S}\big(\partial^{\lambda}_{\nu}(\vec{u}, \pi)\big)\Big)_\gamma(x_0)$$
$$- \Big(\mathcal{D}_\lambda\big(\vec{u}\big|^{\text{n.t.}}_{\partial\Omega}\big)_\gamma\Big)(x_1) + \Big(\mathcal{S}\big(\partial^{\lambda}_{\nu}(\vec{u}, \pi)\big)\Big)_\gamma(x_1). \tag{6.2.69}$$

In view of the arbitrariness of $x_0, x_1 \in \Omega$, this may be interpreted as saying that the function

$$\Omega \ni x \mapsto u_\gamma(x) - \Big(\mathcal{D}_\lambda\big(\vec{u}\big|^{\text{n.t.}}_{\partial\Omega}\big)\Big)_\gamma(x) + \Big(\mathcal{S}\big(\partial^{\lambda}_{\nu}(\vec{u}, \pi)\big)\Big)_\gamma(x) \in \mathbb{C} \tag{6.2.70}$$

is constant in Ω. Denoting its value by $c_\gamma \in \mathbb{C}$ and then defining $c := (c_\gamma)_{1 \le \gamma \le n} \in \mathbb{C}^n$ ultimately establishes (6.2.32). This finishes the proof of Theorem 6.2.1. □

There is yet another set of integral representation formulas for null-solutions of the Stokes system, of the sort described in the theorem below.

Theorem 6.2.2 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \ge 2$, be an open set with a lower Ahlfors regular boundary, and with the property that $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ is a doubling measure on $\partial\Omega$. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω, and fix an aperture parameter $\kappa \in (0, \infty)$. Suppose the functions $\vec{u} = (u_1, \dots, u_n) \in \big[\mathscr{C}^\infty(\Omega)\big]^n$ and $\pi \in \mathscr{C}^\infty(\Omega)$ satisfy*

$$\begin{aligned}
&\Delta\vec{u} - \nabla\pi = 0 \ \ in \ \ \Omega, \quad \operatorname{div}\vec{u} = 0 \ \ in \ \ \Omega, \\
&(\nabla\vec{u})\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ and \ \pi\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ exist \ \sigma\text{-a.e. on } \partial_{\mathrm{nta}}\Omega, \\
&\int_{\partial\Omega} \frac{\big(\mathcal{N}_\kappa(\nabla\vec{u})\big)(y) + (\mathcal{N}_\kappa\pi)(y)}{1+|y|^{n-1}}\,\mathrm{d}\sigma(y) < +\infty.
\end{aligned} \tag{6.2.71}$$

In the case when Ω is an exterior domain, make the additional assumption that there exists some $\mu \in (1,\infty)$ such that

$$⨍_{B(0,\mu R)\setminus B(0,R)} \big\{|\nabla u| + |\pi|\big\}\,\mathrm{d}\mathcal{L}^n = o(1) \ \ as \ \ R \to \infty. \tag{6.2.72}$$

Then, given any $r,\gamma \in \{1,\dots,n\}$ and $\lambda \in \mathbb{C}$, one has

$$\begin{aligned}
(\partial_r u_\gamma)(x) = &\int_{\partial_*\Omega} (\partial_k E_{j\gamma})(x-y)\Big\{\nu_r(y)\big((\partial_k u_j)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y) \\
&\qquad\qquad - \nu_k(y)\big((\partial_r u_j)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)\Big\}\,\mathrm{d}\sigma(y) \\
&+ \lambda\int_{\partial_*\Omega} (\partial_j E_{k\gamma})(x-y)\Big\{\nu_r(y)\big((\partial_k u_j)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y) \\
&\qquad\qquad - \nu_k(y)\big((\partial_r u_j)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)\Big\}\,\mathrm{d}\sigma(y) \\
&+ \int_{\partial_*\Omega} q_\gamma(x-y)\nu_j(y)\big((\partial_r u_j)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)\,\mathrm{d}\sigma(y) \\
&- \int_{\partial_*\Omega} (\partial_r E_{\gamma\alpha})(x-y)\big(\partial_\nu^\lambda(\vec{u},\pi)\big)_\alpha(y)\,\mathrm{d}\sigma(y), \qquad \forall\, x \in \Omega,
\end{aligned} \tag{6.2.73}$$

and

$$\begin{aligned}
\pi(x) = (1+\lambda)&\int_{\partial_*\Omega} q_j(x-y)\nu_k(y)\big(\partial_j u_k\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)\,\mathrm{d}\sigma(y) \\
&- \int_{\partial_*\Omega} q_\alpha(x-y)\big(\partial_\nu^\lambda(\vec{u},\pi)\big)_\alpha(y)\,\mathrm{d}\sigma(y), \qquad \forall\, x \in \Omega.
\end{aligned} \tag{6.2.74}$$

Proof Fix an arbitrary point $x \in \Omega$ along with two indexes $r,\gamma \in \{1,\dots,n\}$. Consider the vector field defined at $\mathcal{L}^n$-a.e. point in Ω as

$$\begin{aligned}
\vec{F}_x := &\big\{(\partial_k E_{j\gamma})(x-\cdot) + \lambda(\partial_j E_{k\gamma})(x-\cdot)\big\}\big\{(\partial_k u_j)\mathbf{e}_r - (\partial_r u_j)\mathbf{e}_k\big\} \\
&+ q_\gamma(x-\cdot)(\partial_r u_j)\mathbf{e}_j - (\partial_r E_{\gamma\alpha})(x-\cdot)\big\{a_{jk}^{\alpha\beta}(\lambda)(\partial_k u_\beta)\mathbf{e}_j - \pi\mathbf{e}_\alpha\big\}.
\end{aligned} \tag{6.2.75}$$

From (6.2.75) and (6.2.1)-(6.2.2) we see that

$$\vec{F}_x \in \big[L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n) \cap \mathscr{C}^\infty(\Omega\setminus\{x\})\big]^n. \tag{6.2.76}$$

while from (6.2.75), the second line in (6.2.71) we conclude that

$$\vec{F}_x\Big|_{\partial\Omega}^{\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial_{\text{nta}}\Omega. \tag{6.2.77}$$

Moreover, a direct computation based on (6.2.75), (6.1.1), (6.1.3), (6.1.6), (6.2.4)-(6.2.7), (6.2.9), and the first line in (6.2.71) gives that

$$\operatorname{div}\vec{F}_x = (\partial_r u_\gamma)(x)\delta_x \text{ in } \mathcal{D}'(\Omega), \tag{6.2.78}$$

hence, in particular,

$$\operatorname{div}\vec{F}_x \in \mathscr{E}'(\Omega). \tag{6.2.79}$$

Abbreviate $K := \overline{B\big(x, \frac{1}{2}\operatorname{dist}(x, \partial\Omega)\big)}$ and note that K is a compact subset of Ω. Based on (6.2.75), (6.2.1)-(6.2.2), and [68, Lemma 8.3.7] we may estimate

$$\begin{aligned}\big(\mathcal{N}_\kappa^{\Omega\setminus K}\vec{F}_x\big)(y) &\leq C\Big\{\big(\mathcal{N}_\kappa(\nabla\vec{u})\big)(y) + (\mathcal{N}_\kappa\pi)(y)\Big\} \cdot \sup_{z\in\Gamma_\kappa(y)\setminus K}\big[|x-z|^{1-n}\big] \\ &\leq C\frac{\big(\mathcal{N}_\kappa(\nabla\vec{u})\big)(y) + (\mathcal{N}_\kappa\pi)(y)}{|x-y|^{n-1}}, \qquad \forall\, y \in \partial\Omega,\end{aligned} \tag{6.2.80}$$

for some constant $C = C(\Omega, n, \lambda, \kappa) \in (0, \infty)$. From (6.2.80), the last line in (6.2.71), and [68, (8.2.26)] it follows that

$$\mathcal{N}_\kappa^{\Omega\setminus K}\vec{F}_x \in L^1(\partial\Omega, \sigma). \tag{6.2.81}$$

Finally, in the case when Ω is an exterior domain, it follows from (6.2.75), (6.2.72), and (6.2.1)-(6.2.2) that

$$\begin{gathered}\int_{A_{\mu,R}\cap\Omega} |x\cdot\vec{F}(x)|\,d\mathcal{L}^n(x) = o(R^2) \text{ as } R \to \infty, \\ \text{where } A_{\mu,R} := B(0, \mu R) \setminus B(0, R).\end{gathered} \tag{6.2.82}$$

Together, the above properties ensure that the vector field $\vec{F}_x$ satisfies the hypotheses of [68, Theorem 1.4.1]. As such, on account of (6.2.78), the Divergence Formula [68, (1.4.6)] currently gives (6.2.73).

The justification of (6.2.74) uses the same circle of ideas, this time starting with the vector field defined at $\mathcal{L}^n$-a.e. point in Ω by

$$\vec{G}_x := (1+\lambda)q_j(x-\cdot)(\partial_j u_k)\mathbf{e}_k - q_\alpha(x-\cdot)\big\{a_{jk}^{\alpha\beta}(\lambda)(\partial_k u_\beta)\mathbf{e}_j - \pi\mathbf{e}_\alpha\big\}, \tag{6.2.83}$$

where $x \in \Omega$ is an arbitrary fixed point. □

Our next major result is Theorem 6.2.4, stated a little later, which amounts to a brand of Calderón-Zygmund theory customized to fit the specific format of the boundary layer operators naturally associated with Stokes system. As a preamble, in the lemma below we study how derivatives applied to the Stokes double layer (both for the velocity and the pressure), acting on boundary Sobolev functions, may be absorbed under the integral sign as weak tangential derivatives.

Lemma 6.2.3 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set with an upper Ahlfors regular boundary and abbreviate $\sigma_* := \mathcal{H}^{n-1}\lfloor\partial_*\Omega$. Also, fix $\lambda \in \mathbb{C}$. Then for each $\vec{f} = (f_j)_{1\leq j\leq n}$ belonging to the weighted boundary Sobolev space $\big[L^1_1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big)\big]^n$ (cf. (A.0.131)) and each index $r \in \{1,\dots,n\}$ one has*

$$\partial_r \mathcal{D}_\lambda \vec{f}(x) = \left(\int_{\partial_*\Omega} \big\{(\partial_k E_{j\gamma})(x-y)\big(\partial_{\tau_{rk}} f_j\big)(y) + \lambda(\partial_j E_{k\gamma})(x-y)\big(\partial_{\tau_{rk}} f_j\big)(y) + q_\gamma(x-y)\big(\partial_{\tau_{jr}} f_j\big)(y)\big\}\,\mathrm{d}\sigma_*(y)\right)_{1\leq\gamma\leq n} \tag{6.2.84}$$

at every point $x \in \Omega$. In addition, for each function

$$\begin{aligned} &\vec{f} = (f_\ell)_{1\leq\ell\leq n} \in \Big[L^1\Big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big)\Big]^n \text{ such that} \\ &\partial_{\tau_{jk}} f_\ell \in L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \text{ for all } j,k,\ell \in \{1,\dots,n\} \end{aligned} \tag{6.2.85}$$

one has

$$\mathcal{P}_\lambda \vec{f}(x) = (1+\lambda)\int_{\partial_*\Omega} q_j(x-y)\big(\partial_{\tau_{kj}} f_k\big)(y)\,\mathrm{d}\sigma_*(y) \tag{6.2.86}$$

at every point $x \in \Omega$.

Proof Fix $r, \gamma \in \{1,\dots,n\}$ along with $x \in \Omega$ arbitrary. Given $\vec{f}$ as in the statement, from (6.2.17) we see that

$$\begin{aligned} \partial_r\big(\mathcal{D}_\lambda \vec{f}\big)_\gamma(x) = \int_{\partial_*\Omega} \big\{ &-\nu_k(y)(\partial_r\partial_k E_{j\gamma})(x-y) - \lambda\nu_k(y)(\partial_r\partial_j E_{k\gamma})(x-y) \\ &+ \nu_j(y)(\partial_r q_\gamma)(x-y)\big\} f_j(y)\,\mathrm{d}\sigma_*(y). \end{aligned} \tag{6.2.87}$$

Write

$$\begin{aligned} -\nu_k(y)(\partial_r\partial_k E_{j\gamma})(x-y) &= \nu_k(y)\partial_{y_r}\big[(\partial_k E_{j\gamma})(x-y)\big] \\ &= \partial_{\tau_{kr}(y)}\big[(\partial_k E_{j\gamma})(x-y)\big] + \nu_r(y)\partial_{y_k}\big[(\partial_k E_{j\gamma})(x-y)\big] \\ &= \partial_{\tau_{kr}(y)}\big[(\partial_k E_{j\gamma})(x-y)\big] - \nu_r(y)(\Delta E_{j\gamma})(x-y) \\ &= \partial_{\tau_{kr}(y)}\big[(\partial_k E_{j\gamma})(x-y)\big] - \nu_r(y)(\partial_j q_\gamma)(x-y), \end{aligned} \tag{6.2.88}$$

and

$$
\begin{aligned}
-\lambda\nu_k(y)(\partial_r\partial_j E_{k\gamma})(x-y)& \\
&= \lambda\nu_k(y)\partial_{y_r}\big[(\partial_j E_{k\gamma})(x-y)\big]\\
&= \lambda\partial_{\tau_{kr}(y)}\big[(\partial_j E_{k\gamma})(x-y)\big] + \lambda\nu_r(y)\partial_{y_k}\big[(\partial_j E_{k\gamma})(x-y)\big]\\
&= \lambda\partial_{\tau_{kr}(y)}\big[(\partial_j E_{k\gamma})(x-y)\big] + \lambda\nu_r(y)\partial_{y_j}\big[(\partial_k E_{k\gamma})(x-y)\big]\\
&= \lambda\partial_{\tau_{kr}(y)}\big[(\partial_j E_{k\gamma})(x-y)\big]. \qquad (6.2.89)
\end{aligned}
$$

Also, write

$$
\begin{aligned}
\nu_j(y)(\partial_r q_\gamma)(x-y) &= -\nu_j(y)\partial_{y_r}\big[q_\gamma(x-y)\big]\\
&= \partial_{\tau_{rj}(y)}[q_\gamma(x-y)] - \nu_r(y)\partial_{y_j}[q_\gamma(x-y)]\\
&= \partial_{\tau_{rj}(y)}[q_\gamma(x-y)] + \nu_r(y)(\partial_j q_\gamma)(x-y) \qquad (6.2.90)
\end{aligned}
$$

and observe that the last term above cancels the last term in (6.2.88). With this in mind, we conclude from (6.2.87)-(6.2.90) that

$$
\begin{aligned}
\partial_r\big(\mathcal{D}_\lambda\vec{f}\,\big)_\gamma(x) = \int_{\partial_*\Omega}\Big\{&\partial_{\tau_{kr}(y)}\big[(\partial_k E_{j\gamma})(x-y)\big]f_j(y)\\
&+\lambda\partial_{\tau_{kr}(y)}\big[(\partial_j E_{k\gamma})(x-y)\big]f_j(y)\\
&+\partial_{\tau_{rj}(y)}\big[q_\gamma(x-y)\big]f_j(y)\Big\}\,d\sigma_*(y). \qquad (6.2.91)
\end{aligned}
$$

On account of this and [69, Lemma 11.1.7] (whose applicability, with φ taken to be one of the components in $(\nabla E)(x-\cdot)$ or $\vec{q}(x-\cdot)$, is ensured by (6.2.1)-(6.2.2)) we therefore obtain

$$
\begin{aligned}
\partial_r\big(\mathcal{D}_\lambda\vec{f}\big)_\gamma(x) = \int_{\partial_*\Omega}\big\{&(\partial_k E_{j\gamma})(x-y)\big(\partial_{\tau_{rk}}f_j\big)(y) + \lambda(\partial_j E_{k\gamma})(x-y)\big(\partial_{\tau_{rk}}f_j\big)(y)\\
&+q_\gamma(x-y)\big(\partial_{\tau_{jr}}f_j\big)(y)\big\}\,d\sigma_*(y). \qquad (6.2.92)
\end{aligned}
$$

This proves (6.2.84). As regards (6.2.86), fix an arbitrary vector-valued function $\vec{f}$ as in (6.2.85). Based on (6.2.21) we may write

$$\begin{aligned}\mathcal{P}_\lambda\vec{f}(x) &= -(1+\lambda)\int_{\partial_*\Omega}\nu_j(y)(\partial_j q_k)(x-y)f_k(y)\,\mathrm{d}\sigma_*(y)\\ &= -(1+\lambda)\int_{\partial_*\Omega}\nu_j(y)(\partial_k q_j)(x-y)f_k(y)\,\mathrm{d}\sigma_*(y)\\ &= (1+\lambda)\int_{\partial_*\Omega}\nu_j(y)\partial_{y_k}\big[q_j(x-y)\big]f_k(y)\,\mathrm{d}\sigma_*(y)\\ &= (1+\lambda)\int_{\partial_*\Omega}\partial_{\tau_{jk}(y)}\big[q_j(x-y)\big]f_k(y)\,\mathrm{d}\sigma_*(y),\end{aligned} \tag{6.2.93}$$

from which (6.2.86) follows by once again appealing to [69, Lemma 11.1.7] (with φ now taken to be one of the components of $\vec{q}(x-\cdot)$). □

We shall also need to consider the principal-value version of the Stokes double layer operator. To be specific, suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) is a set of locally finite perimeter. Much as before, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu = (\nu_j)_{1\leq j\leq n}$ the geometric measure theoretic outward unit normal to Ω. Also, fix some $\lambda \in \mathbb{C}$. In this setting, define the action of the `boundary-to-boundary` (or `principal-value`) `double layer potential operator for the Stokes system` on each function

$$\vec{f} = (f_j)_{1\leq j\leq n} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n \tag{6.2.94}$$

at σ-a.e. $x \in \partial_*\Omega$ according to

$$\begin{aligned}K_\lambda\vec{f}(x) := \Bigg(\lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}}\big\{-\nu_k(y)(\partial_k E_{j\gamma})(x-y) - \lambda\nu_k(y)(\partial_j E_{k\gamma})(x-y)\\ + \nu_j(y)q_\gamma(x-y)\big\}f_j(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\leq\gamma\leq n}.\end{aligned} \tag{6.2.95}$$

Then [68, Proposition 5.6.7] ensures that this limit exists and $K_\lambda\vec{f}$ is a σ-measurable function on $\partial_*\Omega$. Furthermore, the last result in [68, Proposition 5.6.7] ensures that if Ω is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (hence, in particular, if $\partial\Omega$ is a UR set), then for each function $\vec{f}$ as in (6.2.94) the limit in (6.2.95) actually exists for σ-a.e. $x \in \partial\Omega$ and gives rise to a σ-measurable $\mathbb{C}^n$-valued function on $\partial\Omega$. Let us also note here that, for each vector-valued function $\vec{f}$ as in (6.2.94), at σ-a.e. $x \in \partial_*\Omega$ we may write

$$K_\lambda \vec{f}(x) \tag{6.2.96}$$
$$= \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \Big\{ (\lambda-1)\frac{\delta_{j\gamma}}{2\omega_{n-1}} \frac{\langle x-y, \nu(y)\rangle}{|x-y|^n}$$
$$- (\lambda+1)\frac{n}{2\omega_{n-1}} \frac{\langle x-y, \nu(y)\rangle(x_j-y_j)(x_\gamma-y_\gamma)}{|x-y|^{n+2}}$$
$$+ (\lambda-1)\frac{1}{2\omega_{n-1}} \frac{\nu_j(y)(x_\gamma-y_\gamma)-\nu_\gamma(y)(x_j-y_j)}{|x-y|^n} \Big\} f_j(y)\, d\sigma(y) \Bigg)_{1\le\gamma\le n}.$$

Assume next that $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure (this is the case if, e.g., $\partial\Omega$ is a UR set). Once again abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_j)_{1\le j\le n}$ the geometric measure theoretic outward unit normal to Ω. In this context, let us define the action of the `transpose double layer potential operator` for the Stokes system on each vector-valued function

$$\vec{f} = (f_\gamma)_{1\le\gamma\le n} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n \tag{6.2.97}$$

according to

$$K_\lambda^\# \vec{f}(x) := \Bigg(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \Big\{ \nu_k(x)(\partial_k E_{j\gamma})(x-y) + \lambda\nu_k(x)(\partial_j E_{k\gamma})(x-y)$$
$$- \nu_j(x) q_\gamma(x-y) \Big\} f_\gamma(y)\, d\sigma(y) \Bigg)_{1\le j\le n} \tag{6.2.98}$$

at σ-a.e. point $x \in \partial_*\Omega$. From [68, (5.6.23)] and [68, Corollary 5.3.6] we know that this definition is indeed meaningful in the present geometric context. In a more explicit fashion, for each vector-valued function $\vec{f}$ as in (6.2.97), at σ-a.e. $x \in \partial_*\Omega$ we may write

$$K_\lambda^{\#}\vec{f}(x) \tag{6.2.99}$$
$$= \Bigg(\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \Bigg\{ -(\lambda-1)\frac{\delta_{j\gamma}}{2\omega_{n-1}}\frac{\langle x-y,\nu(x)\rangle}{|x-y|^n}$$
$$+(\lambda+1)\frac{n}{2\omega_{n-1}}\frac{\langle x-y,\nu(x)\rangle(x_j-y_j)(x_\gamma-y_\gamma)}{|x-y|^{n+2}}$$
$$-(\lambda-1)\frac{1}{2\omega_{n-1}}\frac{\nu_j(x)(x_\gamma-y_\gamma)-\nu_\gamma(x)(x_j-y_j)}{|x-y|^n}\Bigg\} f_\gamma(y)\,d\sigma(y)\Bigg)_{1\le j\le n}.$$

Here is the version of Calderón-Zygmund theory designed to accommodate the specific format of the boundary layer operators naturally associated with Stokes system, advertised earlier.

Theorem 6.2.4 *Assume $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \ge 2$) is an open set with the property that $\partial\Omega$ is a* UR *set. Abbreviate $\sigma_* := \mathcal{H}^{n-1}\lfloor\partial_*\Omega$ and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by $\nu = (\nu_1,\dots,\nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, fix some $\lambda \in \mathbb{C}$ and consider the boundary layer potential operators $\mathscr{S}$, $\mathcal{Q}$, $\mathcal{D}_\lambda$, $\mathcal{P}_\lambda$, K_λ, $K_\lambda^{\#}$ associated with Ω as in* (6.2.12)*,* (6.2.14)*,* (6.2.17)*,* (6.2.21)*,* (6.2.95)*,* (6.2.98)*. Then the following properties hold.*

(i) For each $p \in [1,\infty)$ and $\kappa > 0$ there exists a finite constant $C > 0$ with the property that for every $\vec{f} \in \big[L^p(\partial_*\Omega,\sigma_*)\big]^n$ one has

$$\big\|\mathcal{N}_\kappa(\mathcal{D}_\lambda\vec{f})\big\|_{L^p(\partial\Omega,\sigma)} \le C\|\vec{f}\|_{[L^p(\partial_*\Omega,\sigma_*)]^n} \text{ if } 1<p<\infty, \tag{6.2.100}$$

$$\big\|\mathcal{N}_\kappa(\mathcal{D}_\lambda\vec{f})\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \le C\|\vec{f}\|_{[L^1(\partial_*\Omega,\sigma_*)]^n} \text{ if } p=1, \tag{6.2.101}$$

and for every $\vec{f} \in \big[L^p(\partial\Omega,\sigma)\big]^n$ one has

$$\big\|\mathcal{N}_\kappa(\mathcal{Q}\vec{f})\big\|_{L^p(\partial\Omega,\sigma)} \le C\|\vec{f}\|_{[L^p(\partial\Omega,\sigma)]^n} \text{ if } 1<p<\infty, \tag{6.2.102}$$

$$\big\|\mathcal{N}_\kappa(\mathcal{Q}\vec{f})\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \le C\|\vec{f}\|_{[L^1(\partial\Omega,\sigma)]^n} \text{ if } p=1. \tag{6.2.103}$$

(ii) For each function $\vec{f}$ belonging to the weighted boundary Sobolev space $\Big[L_1^1\big(\partial_*\Omega,\frac{\sigma_*(x)}{1+|x|^{n-1}}\big)\Big]^n$ (cf. (A.0.131)*), aperture parameter $\kappa \in (0,\infty)$, and index $r \in \{1,\dots,n\}$, the pointwise nontangential boundary traces*

$$\mathcal{P}_\lambda\vec{f}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}},\ \big(\partial_r\mathcal{D}_\lambda\vec{f}\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exist (in } \mathbb{C}^n\text{) at } \sigma_*\text{-a.e. point on } \partial_*\Omega. \tag{6.2.104}$$

Also, for each aperture parameter $\kappa \in (0,\infty)$ and exponents $p,q \in [1,\infty)$ there exists some constant $C \in (0,\infty)$, depending only on $\partial\Omega$, λ, n, κ, p, q, such that every function $\vec{f} \in \big[L_1^{p,q}(\partial_*\Omega,\sigma_*)\big]^n$ one has

$$\big\|\mathcal{N}_\kappa(\mathcal{D}_\lambda\vec{f})\big\|_{L^p(\partial\Omega,\sigma)}+\big\|\mathcal{N}_\kappa(\nabla\mathcal{D}_\lambda\vec{f})\big\|_{L^q(\partial\Omega,\sigma)}+\big\|\mathcal{N}_\kappa(\mathcal{P}_\lambda\vec{f})\big\|_{L^q(\partial\Omega,\sigma)}$$
$$\leq C\|f\|_{[L_1^{p,q}(\partial_*\Omega,\sigma_*)]^M} \quad \textit{if } p,q>1, \tag{6.2.105}$$

plus similar estimates in the case when either $p=1$ or $q=1$, this time with the corresponding L^1-norm in the left side replaced by the weak-L^1 (quasi-)norm.

(iii) Fix $p,p'\in(1,\infty)$ with $1/p+1/p'=1$. Then the operators

$$\begin{aligned} K_\lambda &: \big[L^p(\partial_*\Omega,\sigma_*)\big]^n \longrightarrow \big[L^p(\partial\Omega,\sigma)\big]^n, \\ K_\lambda^\# &: \big[L^p(\partial\Omega,\sigma)\big]^n \longrightarrow \big[L^p(\partial_*\Omega,\sigma_*)\big]^n, \end{aligned} \tag{6.2.106}$$

are well-defined, linear, and bounded. Moreover, corresponding to $p=1$, the operators in (6.2.95)-(6.2.98) induce well-defined, linear, and bounded mappings

$$\begin{aligned} K_\lambda &: \big[L^1(\partial_*\Omega,\sigma_*)\big]^n \longrightarrow \big[L^{1,\infty}(\partial\Omega,\sigma)\big]^n, \\ K_\lambda^\# &: \big[L^1(\partial\Omega,\sigma)\big]^n \longrightarrow \big[L^{1,\infty}(\partial_*\Omega,\sigma_*)\big]^n. \end{aligned} \tag{6.2.107}$$

In addition, the transpose of $K^\#$ in (6.2.106) is

$$K_\lambda : \big[L^{p'}(\partial_*\Omega,\sigma_*)\big]^n \longrightarrow \big[L^{p'}(\partial_*\Omega,\sigma_*)\big]^n. \tag{6.2.108}$$

(iv) Having fixed some $\kappa\in(0,\infty)$, for each $\vec{f}\in\big[L^1\big(\partial_*\Omega,\frac{\sigma_*(x)}{1+|x|^{n-1}}\big)\big]^n$ the following nontangential boundary trace formula (aka jump-formula) holds:

$$\Big(\mathcal{D}_\lambda\vec{f}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x)=\Big(\big(\tfrac{1}{2}I+K_\lambda\big)\vec{f}\Big)(x) \ \textit{ at } \sigma_*\textit{-a.e. point } x\in\partial_*\Omega. \tag{6.2.109}$$

In particular, (6.2.109) holds for each $\vec{f}\in\big[L^p(\partial_*\Omega,\sigma_*)\big]^n$ with $p\in[1,\infty)$. Moreover, as a consequence of (6.2.19) and (6.2.109), it follows that

if Ω has a compact boundary, then for each $\vec{c}\in\mathbb{C}^n$ one has

$$K_\lambda\vec{c}=\begin{cases} +\frac{1}{2}\vec{c} \ \textit{ at } \sigma\textit{-a.e. point on } \partial_*\Omega, \textit{ if } \Omega \textit{ is bounded}, \\ -\frac{1}{2}\vec{c} \ \textit{ at } \sigma\textit{-a.e. point on } \partial_*\Omega, \textit{ if } \Omega \textit{ is unbounded}. \end{cases} \tag{6.2.110}$$

(v) For each aperture parameter $\kappa\in(0,\infty)$ and each $\vec{f}\in\big[L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$, at σ-a.e. point $x\in\partial_*\Omega$ one has the jump-formula

$$\Big(\mathcal{Q}\vec{f}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x)=\frac{1}{2}\langle\nu(x),\vec{f}(x)\rangle+\lim_{\varepsilon\to0^+}\int_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}}\langle\vec{q}(x-y),\vec{f}(y)\rangle\,d\sigma(y). \tag{6.2.111}$$

In particular, (6.2.111) holds for each $\vec{f}\in\big[L^p(\partial\Omega,\sigma)\big]^n$ with $p\in[1,\infty)$.

(vi) The single layer potential operator $\mathscr{S}$ for the Stokes system, defined in (6.2.12), along with its boundary-to-boundary version

$$S\vec{f}(x) := \int_{\partial\Omega} E(x-y)\vec{f}(y)\,\mathrm{d}\sigma(y) \text{ for } x \in \partial\Omega, \tag{6.2.112}$$

(where $E = \big(E_{jk}\big)_{1\leq j,k\leq n}$ is the Kelvin matrix-valued fundamental solution for the Stokes system in $\mathbb{R}^n$ recalled in (6.2.1)) satisfy similar properties as those of the single layers for generic weakly elliptic, homogeneous, constant (complex) coefficient, second-order $M \times M$ systems described in items (ix)-(xii) of Theorem 1.5.1, as well as Theorem 2.2.3 and Theorem 2.2.6. This time, the jump-formula (1.5.59) should be interpreted as

$$\partial_\nu^\lambda\big(\mathscr{S}\vec{f}, \mathcal{Q}\vec{f}\big) = \big(-\tfrac{1}{2}I + K_\lambda^{\#}\big)\vec{f} \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega, \tag{6.2.113}$$

if $n \geq 3$ and $\vec{f} \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\big)\Big]^n$ (hence, in particular, whenever we have $\vec{f} \in \big[L^p(\partial\Omega,\sigma)\big]^n$ with $p \in [1, n-1)$).

(vii) Strengthen the original hypotheses by assuming that Ω is actually a UR *domain. Then the operator*

$$K_\lambda : \big[L_1^{p,q}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[L_1^{p,q}(\partial\Omega,\sigma)\big]^n \tag{6.2.114}$$

is well defined, linear, and bounded for each $p, q \in (1,\infty)$. In particular, the operator

$$K_\lambda : \big[L_1^{p}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[L_1^{p}(\partial\Omega,\sigma)\big]^n \tag{6.2.115}$$

is well defined, linear, and bounded for each $p \in (1,\infty)$.

(viii) Retain the assumption that Ω is a UR *domain. Then for each $p, q \in (1,\infty)$ it follows that $K_\lambda^{\#}$ in (6.2.106) extends uniquely to a linear and bounded operator from the negative boundary Sobolev space $\big[L_{-1}^p(\partial\Omega,\sigma)\big]^n$ into itself and, more generally, from the off-diagonal negative Sobolev space $\big[L_{-1}^{p,q}(\partial\Omega,\sigma)\big]^n$ into itself. Furthermore, if one adopts the same notation $K_\lambda^{\#}$ for said extensions, and if $p', q' \in (1,\infty)$ are such that $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$, then the transpose of (6.2.114) is*

$$K_\lambda^{\#} : \big[L_{-1}^{p',q'}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[L_{-1}^{p'q'}(\partial\Omega,\sigma)\big]^n, \tag{6.2.116}$$

while the transpose of (6.2.115) is

$$K_\lambda^{\#} : \big[L_{-1}^{p'}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[L_{-1}^{p'}(\partial\Omega,\sigma)\big]^n. \tag{6.2.117}$$

(ix) Continue to assume that Ω is actually a UR *domain. Then given any vector-valued function $\vec{f} = (f_j)_{1\leq j\leq n}$ in the Sobolev space $\big[L_1^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$ (cf.*

(A.0.131)), *for each* $\alpha \in \{1, \dots, n\}$ *one has*

$$
\begin{aligned}
&\Big(\partial_\nu^\lambda\big(\mathcal{D}_\lambda \vec{f}, \mathcal{P}_\lambda \vec{f}\big)\Big)_\alpha(x) \qquad (6.2.118)\\
&= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\ |x-y|>\varepsilon}} \Big\{\nu_\ell(x)(\partial_k E_{j\alpha})(x-y)\big(\partial_{\tau_{\ell k}} f_j\big)(y) \\
&\qquad + \lambda\nu_\ell(x)(\partial_j E_{k\alpha})(x-y)\big(\partial_{\tau_{\ell k}} f_j\big)(y) \\
&\qquad + \nu_\ell(x) q_\alpha(x-y)\big(\partial_{\tau_{j\ell}} f_j\big)(y) \\
&\qquad + \lambda\nu_\ell(x)(\partial_k E_{j\ell})(x-y)\big(\partial_{\tau_{\alpha k}} f_j\big)(y) \\
&\qquad + \lambda^2\nu_\ell(x)(\partial_j E_{k\ell})(x-y)\big(\partial_{\tau_{\alpha k}} f_j\big)(y) \\
&\qquad + \lambda\nu_\ell(x) q_\ell(x-y)\big(\partial_{\tau_{j\alpha}} f_j\big)(y) \\
&\qquad - (1+\lambda)\nu_\alpha(x) q_j(x-y)\big(\partial_{\tau_{kj}} f_k\big)(y)\Big\}\,\mathrm{d}\sigma(y)
\end{aligned}
$$

at σ*-a.e. point* $x \in \partial\Omega$, *where the conormal derivative* $\partial_\nu^\lambda\big(\mathcal{D}_\lambda \vec{f}, \mathcal{P}_\lambda \vec{f}\big)$ *is defined as in* (6.1.6). *As a corollary of* [69, (11.7.22)], *this is true whenever we have* $\vec{f} \in \big[L_1^{p,q}(\partial\Omega,\sigma)\big]^n$ *with* $p, q \in [1,\infty)$.
Hence, in such a setting, for any given exponents $p, q \in (1,\infty)$ *the conormal derivative* (6.1.6) *of the pair* $\big(\mathcal{D}_\lambda, \mathcal{P}_\lambda\big)$ *induces a well-defined, linear, and bounded operator*

$$
\partial_\nu^\lambda\big(\mathcal{D}_\lambda, \mathcal{P}_\lambda\big) : \big[L_1^{p,q}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[L^q(\partial\Omega,\sigma)\big]^n. \qquad (6.2.119)
$$

Finally, (6.2.118) *implies that* $\partial_\nu^\lambda\big(\mathcal{D}_\lambda \vec{f}, \mathcal{P}_\lambda \vec{f}\big)$ *does not jump across the boundary* (*in the sense that it has the same nontangential boundary trace when considered from* $\Omega_+ := \Omega$ *and* $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$).

(x) *Continue to assume that* Ω *is actually a* UR *domain. Also, pick exponents* $p, p', q, q' \in (1,\infty)$ *satisfying* $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$. *Then for all functions* $\vec{f} \in \big[L_1^{p,q}(\partial\Omega,\sigma)\big]^n$ *and* $\vec{g} \in \big[L_1^{q',p'}(\partial\Omega,\sigma)\big]^n$ *one has*

$$
\int_{\partial\Omega} \big\langle \partial_\nu^\lambda\big(\mathcal{D}_\lambda \vec{f}, \mathcal{P}_\lambda \vec{f}\big), \vec{g}\big\rangle\,\mathrm{d}\sigma = \int_{\partial\Omega} \big\langle \vec{f}, \big(\mathcal{D}_\lambda \vec{g}, \mathcal{P}_\lambda \vec{g}\big)\big\rangle\,\mathrm{d}\sigma. \qquad (6.2.120)
$$

As a consequence, whenever Ω *is a* UR *domain, the operator* (6.2.119) *has a unique extension to a well-defined, linear, and bounded mapping*

$$
\partial_\nu^\lambda\big(\mathcal{D}_\lambda, \mathcal{P}_\lambda\big) : \big[L^p(\partial\Omega,\sigma)\big]^n \longrightarrow \big[L_{-1}^{q,p}(\partial\Omega,\sigma)\big]^n, \qquad (6.2.121)
$$

namely the (*real*) *transpose of the operator*

$$\partial_\nu^\lambda(\mathcal{D}_\lambda, \mathcal{P}_\lambda) : \big[L_1^{q',p'}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[L^{p'}(\partial\Omega,\sigma)\big]^n. \tag{6.2.122}$$

(xi) Make the assumption that Ω *is a* UR *domain. Then the following operator identities hold:*

$$\begin{gathered}\big(\tfrac12 I + K_\lambda\big) \circ \big(-\tfrac12 I + K_\lambda\big) = S \circ \partial_\nu^\lambda(\mathcal{D}_\lambda, \mathcal{P}_\lambda) \\ \textit{on } \big[L_1^{p,q}(\partial\Omega,\sigma)\big]^n \textit{ with } p \in (1,\infty) \textit{ and } q \in (1, n-1), \\ \textit{as well as on } \big[L^p(\partial\Omega,\sigma)\big]^n \textit{ with } p \in \big(\tfrac{n-1}{n-2}, \infty\big),\end{gathered} \tag{6.2.123}$$

$$\begin{gathered}\big(\tfrac12 I + K_\lambda^\#\big) \circ \big(-\tfrac12 I + K_\lambda^\#\big) = \partial_\nu^\lambda(\mathcal{D}_\lambda, \mathcal{P}_\lambda) \circ S \\ \textit{on } \big[L^p(\partial\Omega,\sigma)\big]^n \textit{ with } p \in (1, n-1), \textit{ as well as} \\ \textit{on } \big[L_{-1}^{p,p^*}(\partial\Omega,\sigma)\big]^n \textit{ with } p \in (1, n-1) \textit{ and } \tfrac{1}{p^*} = \tfrac1p - \tfrac{1}{n-1},\end{gathered} \tag{6.2.124}$$

$$\begin{gathered}S \circ K_\lambda^\# = K_\lambda \circ S \\ \textit{on } \big[L^p(\partial\Omega,\sigma)\big]^n \textit{ with } p \in (1, n-1), \textit{ as well as} \\ \textit{on } \big[L_{-1}^{p,p^*}(\partial\Omega,\sigma)\big]^n \textit{ with } p \in (1, n-1) \textit{ and } \tfrac{1}{p^*} = \tfrac1p - \tfrac{1}{n-1},\end{gathered} \tag{6.2.125}$$

$$\begin{gathered}K_\lambda^\# \circ \partial_\nu^\lambda(\mathcal{D}_\lambda, \mathcal{P}_\lambda) = \partial_\nu^\lambda(\mathcal{D}_\lambda, \mathcal{P}_\lambda) \circ K_\lambda \\ \textit{on } \big[L_1^{p,q}(\partial\Omega,\sigma)\big]^n \textit{ with } p \in (1,\infty) \textit{ and } q \in (1, n-1), \\ \textit{as well as on } \big[L^p(\partial\Omega,\sigma)\big]^n \textit{ with } p \in (1, \infty).\end{gathered} \tag{6.2.126}$$

Moreover, if $\partial\Omega$ *is bounded then one may allow any* $p, q \in (1,\infty)$*, this time taking* $p^* \in (1,\infty)$ *arbitrary (and unrelated to* p*).*

(xii) Under the assumption that Ω *is a* UR *domain, similar results to those presented in Theorem 3.3.1, Theorem 3.3.2, and Theorem 3.3.3 are valid for the boundary layer operators associated with Stokes system considered earlier in this section acting on Morrey spaces and their pre-duals.*

Proof The nontangential maximal function estimates in item *(i)* are consequences of (6.2.17), (6.2.14), (6.2.1)-(6.2.2), and [70, Theorem 2.4.1]. Likewise, all claims in item *(ii)* may be justified based on Lemma 6.2.3, (6.2.1)-(6.2.2), and [70, Theorem 2.4.1]. Next, the claims in item *(iii)* are readily implied by (6.2.95)-(6.2.98), (6.2.1)-(6.2.2), [70, Theorem 2.3.2], and [70, (2.3.25)].

To prove the jump-formula from item *(iv)*, we first introduce some notation. Specifically, for each given $\ell, j, k \in \{1, \dots, n\}$ and each complex-valued function

$$g \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1 + |x|^{n-1}}\Big) \tag{6.2.127}$$

define

$$(\mathcal{T}_{\ell,j,k}g)(x) := \int_{\partial\Omega} (\partial_\ell E_{jk})(x-y)g(y)\,\mathrm{d}\sigma(y) \ \text{ for each } \ x \in \Omega, \tag{6.2.128}$$

$$(\mathcal{Q}_j g)(x) := \int_{\partial\Omega} q_j(x-y)g(y)\,\mathrm{d}\sigma(y) \ \text{ for each } \ x \in \Omega, \tag{6.2.129}$$

along with their principal-value versions

$$(T_{\ell,j,k}g)(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}} (\partial_\ell E_{jk})(x-y)g(y)\,\mathrm{d}\sigma(y) \ \text{ for } \sigma\text{-a.e. } \ x \in \partial\Omega, \tag{6.2.130}$$

$$(Q_j g)(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}} q_j(x-y)g(y)\,\mathrm{d}\sigma(y) \ \text{ for } \sigma\text{-a.e. } \ x \in \partial\Omega. \tag{6.2.131}$$

Also, fix an aperture parameter $\kappa > 0$. Then from the jump-formula recorded in [70, (2.5.4)] and (6.2.8) we conclude that for σ-a.e. point $x \in \partial_*\Omega$ we have

$$\big(\mathcal{T}_{\ell,j,k}g\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x) = \frac{1}{2}\nu_\ell(x)\big(\nu_j(x)\nu_k(x) - \delta_{jk}\big)g(x) + (T_{\ell,j,k}g)(x), \tag{6.2.132}$$

$$\big(\mathcal{Q}_j g\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x) = \frac{1}{2}\nu_j(x)g(x) + (Q_j g)(x). \tag{6.2.133}$$

In turn, from (6.2.17) and (6.2.130)-(6.2.133) we deduce that for each given vector-valued function $\vec{f} = (f_j)_{1\leq j\leq n}$ belonging to the space $\big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$ and for each index $\gamma \in \{1,\dots,n\}$ we have

$$\begin{aligned}
\big(\mathcal{D}_\lambda \vec{f}\big)_\gamma\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x) &= -\frac{1}{2}\nu_k(x)\nu_k(x)\big(\nu_j(x)\nu_\gamma(x) - \delta_{j\gamma}\big)f_j(x) - \big(T_{k,j,\gamma}(\nu_k f_j)\big)(x)\\
&\quad - \frac{\lambda}{2}\nu_k(x)\nu_j(x)\big(\nu_k(x)\nu_\gamma(x) - \delta_{k\gamma}\big)f_j(x) - \lambda\big(T_{k,j,\gamma}(\nu_k f_j)\big)(x)\\
&\quad + \frac{1}{2}\nu_j(x)\nu_\gamma(x)f_j(x) + \big(Q_\gamma(\nu_j f_j)\big)(x)\\
&= \frac{1}{2}f_\gamma(x) + \big(K_\lambda \vec{f}\big)_\gamma(x) \ \text{ at } \sigma\text{-a.e. point } \ x \in \partial_*\Omega.
\end{aligned} \tag{6.2.134}$$

On account of the arbitrariness of γ, this proves the jump-formula (6.2.109). The jump-formula claimed in item *(v)* is seen directly from (6.2.133) and (6.2.14).

Going further, the first claim in item *(vi)* is a consequence of the fact that the integral kernel of the single layer potential operators for the Stokes system have the same analytical properties as in the case of the single layers for generic weakly elliptic, homogeneous, constant (complex) coefficient, second-order $M \times M$ systems treated earlier in Theorem 1.5.1.

As regards the jump-formula (6.2.113), if $n \geq 3$ and $\vec{f} \in \Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\big)\Big]^n$ we may write

$$
\begin{aligned}
&\Big(\partial^{\lambda}_{\nu}\big(\mathscr{S}\vec{f}, \mathcal{Q}\vec{f}\big)\Big)_{\alpha}(x)\\
&\quad = \nu_j(x)a^{\alpha\beta}_{jk}(\lambda)\Big(\big(\partial_k\mathscr{S}\vec{f}\big)_{\beta}\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) - \nu_{\alpha}(x)\big(\mathcal{Q}\vec{f}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x)\\
&\quad = \nu_j(x)\Big(\big(\partial_j\mathscr{S}\vec{f}\big)_{\alpha}\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) + \lambda\nu_j(x)\Big(\big(\partial_{\alpha}\mathscr{S}\vec{f}\big)_{j}\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x)\\
&\qquad - \nu_{\alpha}(x)\big(\mathcal{Q}\vec{f}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x)\\
&\quad = \frac{1}{2}\nu_j(x)\nu_j(x)\big(\nu_{\alpha}(x)\nu_k(x) - \delta_{\alpha k}\big)f_k(x) + \nu_j(x)(T_{j,\alpha,k}f_k)(x)\\
&\qquad + \frac{\lambda}{2}\nu_j(x)\nu_{\alpha}(x)\big(\nu_j(x)\nu_k(x) - \delta_{jk}\big)f_k(x) + \lambda\nu_j(x)(T_{\alpha,j,k}f_k)(x)\\
&\qquad - \frac{1}{2}\nu_{\alpha}(x)\nu_j(x)f_j(x) - \nu_{\alpha}(x)(Q_jf_j)(x)\\
&\quad = -\frac{1}{2}f_{\alpha}(x) + \big(K^{\#}_{\lambda}\vec{f}\big)_{\alpha}(x) \text{ at } \sigma\text{-a.e. point } x \in \partial_*\Omega, \qquad (6.2.135)
\end{aligned}
$$

thanks to (6.1.6), (6.1.1), (6.2.132)-(6.2.133), and (6.2.98).

Henceforth strengthen the hypotheses on the underlying set by assuming that Ω is actually a UR domain. The claim that for each $p, q \in (1, \infty)$ the operator K_{λ} is well defined and bounded in the context of (6.2.114) is a consequence of (6.2.109), (6.2.105), and [69, Proposition 11.3.2]. This takes care of item *(vii)*. Next, the claims in the current item *(viii)* may be dealt with based on what we have proved so far, reasoning as in the treatment of item *(vi)* in Theorem 1.5.1.

Consider next the claims in item *(ix)*. To get started, fix a vector-valued function $\vec{f} = (f_j)_{1\leq j\leq n} \in \big[L^1_1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$, along with some $\alpha \in \{1, \dots, n\}$. For starters, from (6.1.6), (6.1.1), we see that at σ-a.e. point on $\partial\Omega$ we have

$$
\begin{aligned}
\Big(\partial^{\lambda}_{\nu}\big(\mathscr{D}_{\lambda}\vec{f}, \mathcal{P}_{\lambda}\vec{f}\big)\Big)_{\alpha} &= \nu_j a^{\alpha\beta}_{jk}(\lambda)\Big(\big(\partial_k\mathscr{D}_{\lambda}\vec{f}\big)_{\beta}\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) - \nu_{\alpha}\big(\mathcal{P}_{\lambda}\vec{f}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)\\
&= \nu_{\ell}\Big(\big(\partial_{\ell}\mathscr{D}_{\lambda}\vec{f}\big)_{\alpha}\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) + \lambda\nu_{\ell}\Big(\big(\partial_{\alpha}\mathscr{D}_{\lambda}\vec{f}\big)_{\ell}\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)\\
&\quad - \nu_{\alpha}\big(\mathcal{P}_{\lambda}\vec{f}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big). \qquad (6.2.136)
\end{aligned}
$$

Let us consider the terms above containing nontangential traces separately. First, based on (6.2.84) and (6.2.132)-(6.2.133), at σ-a.e. point on $\partial\Omega$ we may write

$$
\begin{aligned}
\nu_\ell\Big(\big(\partial_\ell \mathcal{D}_\lambda \vec{f}\big)_\alpha\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) &= \frac{1}{2}\nu_\ell\nu_k\big(\nu_j\nu_\alpha - \delta_{j\alpha}\big)\big(\partial_{\tau_{\ell k}} f_j\big) + \nu_\ell T_{k,j,\alpha}\big(\partial_{\tau_{\ell k}} f_j\big) \\
&\quad + \frac{\lambda}{2}\nu_\ell\nu_j\big(\nu_k\nu_\alpha - \delta_{k\alpha}\big)\big(\partial_{\tau_{\ell k}} f_j\big) + \lambda\nu_\ell T_{j,k,\alpha}\big(\partial_{\tau_{\ell k}} f_j\big) \\
&\quad + \frac{1}{2}\nu_\ell\nu_\alpha\big(\partial_{\tau_{j\ell}} f_j\big) + \nu_\ell Q_\alpha\big(\partial_{\tau_{j\ell}} f_j\big),
\end{aligned}
\tag{6.2.137}
$$

and

$$
\begin{aligned}
\lambda\nu_\ell\Big(\big(\partial_\alpha \mathcal{D}_\lambda \vec{f}\big)_\ell\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) &= \frac{\lambda}{2}\nu_\ell\nu_k\big(\nu_j\nu_\ell - \delta_{j\ell}\big)\big(\partial_{\tau_{\alpha k}} f_j\big) + \lambda\nu_\ell T_{k,j,\ell}\big(\partial_{\tau_{\alpha k}} f_j\big) \\
&\quad + \frac{\lambda^2}{2}\nu_\ell\nu_j\big(\nu_k\nu_\ell - \delta_{k\ell}\big)\big(\partial_{\tau_{\alpha k}} f_j\big) + \lambda^2\nu_\ell T_{j,k,\ell}\big(\partial_{\tau_{\alpha k}} f_j\big) \\
&\quad + \frac{\lambda}{2}\nu_\ell\nu_\ell\big(\partial_{\tau_{j\alpha}} f_j\big) + \lambda\nu_\ell Q_\ell\big(\partial_{\tau_{j\alpha}} f_j\big).
\end{aligned}
\tag{6.2.138}
$$

In addition, (6.2.86) and (6.2.133) imply that at σ-a.e. point on $\partial\Omega$ we have

$$
\big(\mathcal{P}_\lambda \vec{f}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big) = \frac{1+\lambda}{2}\nu_j\big(\partial_{\tau_{kj}} f_k\big) + (1+\lambda)Q_j\big(\partial_{\tau_{kj}} f_k\big),
\tag{6.2.139}
$$

hence

$$
-\nu_\alpha\big(\mathcal{P}_\lambda \vec{f}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big) = -\frac{1+\lambda}{2}\nu_\alpha\nu_j\big(\partial_{\tau_{kj}} f_k\big) - (1+\lambda)\nu_\alpha Q_j\big(\partial_{\tau_{kj}} f_k\big).
\tag{6.2.140}
$$

By making repeated use of [69, (11.4.3), (11.4.8)] we may express the jump-terms appearing in (6.2.137) and (6.2.138) as

$$
\begin{aligned}
&\frac{1}{2}\nu_\ell\nu_k\big(\nu_j\nu_\alpha - \delta_{j\alpha}\big)\big\{\nu_\ell(\nabla_{\tan} f_j)_k - \nu_k(\nabla_{\tan} f_j)_\ell\big\} \\
&\qquad + \frac{\lambda}{2}\nu_\ell\nu_j\big(\nu_k\nu_\alpha - \delta_{k\alpha}\big)\big\{\nu_\ell(\nabla_{\tan} f_j)_k - \nu_k(\nabla_{\tan} f_j)_\ell\big\} \\
&\qquad + \frac{1}{2}\nu_\ell\nu_\alpha\big\{\nu_j(\nabla_{\tan} f_j)_\ell - \nu_\ell(\nabla_{\tan} f_j)_j\big\} \\
&\qquad = -\frac{\lambda}{2}\nu_j(\nabla_{\tan} f_j)_\alpha - \frac{1}{2}\nu_\alpha(\nabla_{\tan} f_j)_j
\end{aligned}
\tag{6.2.141}
$$

and, respectively,

$$\frac{\lambda}{2}\nu_\ell\nu_k\big(\nu_j\nu_\ell-\delta_{j\ell}\big)\big\{\nu_\alpha(\nabla_{\tan}f_j)_k-\nu_k(\nabla_{\tan}f_j)_\alpha\big\}$$
$$+\frac{\lambda^2}{2}\nu_\ell\nu_j\big(\nu_k\nu_\ell-\delta_{k\ell}\big)\big\{\nu_\alpha(\nabla_{\tan}f_j)_k-\nu_k(\nabla_{\tan}f_j)_\alpha\big\}$$
$$+\frac{\lambda}{2}\nu_\ell\nu_\ell\big\{\nu_j(\nabla_{\tan}f_j)_\alpha-\nu_\alpha(\nabla_{\tan}f_j)_j\big\}$$
$$=\frac{\lambda}{2}\big\{\nu_j(\nabla_{\tan}f_j)_\alpha-\nu_\alpha(\nabla_{\tan}f_j)_j\big\}. \tag{6.2.142}$$

Combining (6.2.141) with (6.2.142) yields

$$-\frac{\lambda}{2}\nu_j(\nabla_{\tan}f_j)_\alpha-\frac{1}{2}\nu_\alpha(\nabla_{\tan}f_j)_j+\frac{\lambda}{2}\big\{\nu_j(\nabla_{\tan}f_j)_\alpha-\nu_\alpha(\nabla_{\tan}f_j)_j\big\}$$
$$=-\frac{1+\lambda}{2}\nu_\alpha(\nabla_{\tan}f_j)_j=-\frac{1+\lambda}{2}\nu_\alpha\nu_k\partial_{\tau_{kj}}f_j$$
$$=\frac{1+\lambda}{2}\nu_\alpha\nu_j\partial_{\tau_{kj}}f_k, \tag{6.2.143}$$

where we have also made use of (A.0.78) and [69, (11.1.24)]. Observe that the last expression in (6.2.143) cancels the jump-term in (6.2.140). Keeping this in mind, we may now conclude from (6.2.136)-(6.2.140) that at σ-a.e. point $x\in\partial\Omega$ we have

$$\Big(\partial_\nu^\lambda\big(\mathcal{D}_\lambda\vec{f},\mathcal{P}_\lambda\vec{f}\big)\Big)_\alpha(x)$$
$$=\nu_\ell(x)\Big\{T_{k,j,\alpha}\big(\partial_{\tau_{\ell k}}f_j\big)(x)+\lambda T_{j,k,\alpha}\big(\partial_{\tau_{\ell k}}f_j\big)(x)+Q_\alpha\big(\partial_{\tau_{j\ell}}f_j\big)(x)\Big\}$$
$$+\lambda\nu_\ell(x)\Big\{T_{k,j,\ell}\big(\partial_{\tau_{\alpha k}}f_j\big)(x)+\lambda T_{j,k,\ell}\big(\partial_{\tau_{\alpha k}}f_j\big)(x)+Q_\ell\big(\partial_{\tau_{j\alpha}}f_j\big)(x)\Big\}$$
$$-(1+\lambda)\nu_\alpha(x)Q_j\big(\partial_{\tau_{kj}}f_k\big)(x). \tag{6.2.144}$$

On account of (6.2.130)-(6.2.131), this proves (6.2.118). Granted this, the fact that (6.2.119) is a well-defined, linear, and bounded operator becomes a consequence of (6.2.1)-(6.2.2) and [70, Theorem 2.3.2].

Let us now turn our attention to the claims made in item *(x)*. Assume $\Omega\neq\mathbb{R}^n$, else there is nothing to prove. Define $\Omega_+:=\Omega$ and $\Omega_-:=\mathbb{R}^n\setminus\overline{\Omega}$. Item *(7)* in [68, Lemma 5.10.9] then guarantees that $\Omega_\pm$ are two UR domains, whose topological and geometric measure theoretic boundaries agree with those of Ω, and whose geometric measure theoretic outward unit normals are $\pm\nu$ at σ-a.e. point on $\partial\Omega$. Next, pick $\vec{f}\in\big[L_1^{p,q}(\partial\Omega,\sigma)\big]^n$ and $\vec{g}\in\big[L_1^{q',p'}(\partial\Omega,\sigma)\big]^n$ with $p,p',q,q'\in(1,\infty)$ satisfying $1/p+1/p'=1$ and $1/q+1/q'=1$, then define

$$\begin{aligned} \vec{u}_{\pm} &:= \mathcal{D}_\lambda \vec{f} \text{ and } \pi_{\pm} := \mathcal{P}_\lambda \vec{f} \text{ in } \Omega_{\pm}, \\ \vec{w}_{\pm} &:= \mathcal{D}_\lambda \vec{g} \text{ and } \rho_{\pm} := \mathcal{P}_\lambda \vec{g} \text{ in } \Omega_{\pm}. \end{aligned} \tag{6.2.145}$$

Then our earlier results in the current items *(ii)* and *(iv)*, together with (6.2.22), imply that the pairs $\big((\vec{u}_+, \pi_+), (\vec{w}_+, \rho_+)\big)$ and $\big((\vec{u}_-, \pi_-), (\vec{w}_-, \rho_-)\big)$ satisfy (6.1.29)-(6.1.30) (with the roles of p, q interchanged), relative to Ω_+ and to Ω_-. In addition, in the case when $\Omega_{\pm}$ is an exterior domain, it is clear from (6.2.17), (6.2.21), and (6.2.1)-(6.2.2) that the functions $\vec{u}_{\pm}, \pi_{\pm}, \vec{w}_{\pm}, \rho_{\pm}$ satisfy, for each multi-index $\alpha \in \mathbb{N}_0^n$,

$$\begin{gathered} \big(\partial^\alpha \vec{u}_{\pm}\big)(x) = O(|x|^{1-n-|\alpha|}) \text{ and } \big(\partial^\alpha \vec{w}_{\pm}\big)(x) = O(|x|^{1-n-|\alpha|}) \text{ as } |x| \to \infty, \\ \pi_{\pm}(x) = O(|x|^{-n}) \text{ and } \rho_{\pm}(x) = O(|x|^{-n}) \text{ as } |x| \to \infty. \end{gathered} \tag{6.2.146}$$

In turn, these properties imply that condition (6.1.32) formulated for the exterior domain $\Omega_{\pm}$ holds in such a scenario. As such, we may invoke Green's formula (6.1.31) which, in light of (6.2.109), (6.2.118), and (6.2.22), permits us to write

$$\begin{aligned} &\int_{\partial\Omega} \Big\langle \partial_\nu^\lambda (\mathcal{D}_\lambda \vec{f}, \mathcal{P}_\lambda \vec{f}), \big(\pm \tfrac{1}{2} I + K_\lambda\big)\vec{g} \Big\rangle \, d\sigma \\ &\qquad = \int_{\partial\Omega} \Big\langle \big(\pm \tfrac{1}{2} I + K_\lambda\big)\vec{f}, \partial_\nu^\lambda (\mathcal{D}_\lambda \vec{g}, \mathcal{P}_\lambda \vec{g}) \Big\rangle \, d\sigma. \end{aligned} \tag{6.2.147}$$

Subtracting the two versions of this equality yields (6.2.120).

On to the claims in item *(xi)*. As a preamble, it is useful to observe that, in concert with (6.2.113), formula (6.2.28) written for $\vec{u} := \mathcal{S}\vec{f}$ and $\pi := \mathcal{Q}\vec{f}$, where $\vec{f} \in \big[L^p(\partial\Omega, \sigma)\big]^n$ with $p \in [1, n-1)$, implies

$$\mathcal{Q}\Big(\big(\tfrac{1}{2} I + K_\lambda^\#\big)\vec{f}\Big) = \mathcal{P}_\lambda(S\vec{f}) \text{ in } \Omega. \tag{6.2.148}$$

Likewise, writing formula (6.2.28) for the pair $\vec{u} := \mathcal{D}_\lambda \vec{f}$ and $\pi := \mathcal{P}_\lambda \vec{f}$, where $\vec{f} \in \big[L^p(\partial\Omega, \sigma)\big]^n$ with $p \in [1, \infty)$, and then making use of (6.2.109) yields

$$\mathcal{Q}\Big(\partial_\nu^\lambda (\mathcal{D}_\lambda \vec{f}, \mathcal{P}_\lambda \vec{f})\Big) = \mathcal{P}_\lambda\Big(\big(-\tfrac{1}{2} I + K_\lambda\big)\vec{f}\Big) \text{ in } \Omega. \tag{6.2.149}$$

The operator identities claimed in current item *(xi)* may now be justified in an analogous fashion to those in item *(xiii)* of Theorem 1.5.1, making use of the integral representation formulas from Theorem 6.2.1, as well as the jump-formulas for the Stokes layer potential operators established earlier in this proof, and (6.2.148)-(6.2.149).

Finally, the claim in item *(xii)* may be justified by reasoning as in the proofs of Theorems 3.3.1-3.3.3, making use of [70, Theorem 2.6.1], [70, Proposition 2.6.2], and our earlier results for the boundary layer operators associated with Stokes system in this section. □

The machinery developed so far permits us to show that tangential derivatives commute with any of the principal-value double layer potential operators for the S-tokes system introduced before at the expense of commutators of Calderón-Zygmund singular integrals with operators of pointwise multiplication with scalar components of the geometric measure theoretic outward unit normal. This claim is made precise in the theorem below.

Theorem 6.2.5 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set with the property that $\partial\Omega$ is a* UR *set and*

$$\mathcal{H}^{n-1}\big(\partial_{\text{nta}}\Omega \setminus \partial_*\Omega\big) = 0 \tag{6.2.150}$$

(this condition is automatically satisfied if the set Ω is a UR *domain to begin with). Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Next, pick some $\lambda \in \mathbb{C}$ and consider the principal-value double layer potential operator K_λ associated with Ω as in (6.2.95). Finally, fix an integrability exponent $p \in (1, \infty)$. Then for each vector-valued function*

$$\vec{f} = (f_\mu)_{1 \leq \mu \leq n} \in \Big[L^1\Big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\text{loc}}(\partial_*\Omega, \sigma)\Big]^n \text{ such that}$$

$$\partial_{\tau_{jk}} f_\mu \in L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\text{loc}}(\partial_*\Omega, \sigma) \text{ for all } j, k, \mu \in \{1, \dots, n\} \tag{6.2.151}$$

it follows that

$$K_\lambda \vec{f} \in \big[L^p_{1,\text{loc}}(\partial_*\Omega, \sigma)\big]^n \tag{6.2.152}$$

and for each pair of indices $j, k \in \{1, \dots, n\}$ one has

$$\begin{aligned}
\partial_{\tau_{jk}}(K_\lambda \vec{f})_\gamma = {} & \Big(K_\lambda\big(\partial_{\tau_{jk}}\vec{f}\big)\Big)_\gamma + \big[M_{\nu_j}, T_{\ell,\mu,\gamma}\big](\partial_{\tau_{k\ell}} f_\mu) - \big[M_{\nu_k}, T_{\ell,\mu,\gamma}\big](\partial_{\tau_{j\ell}} f_\mu) \\
& + \lambda\big[M_{\nu_j}, T_{\mu,\ell,\gamma}\big](\partial_{\tau_{k\ell}} f_\mu) - \lambda\big[M_{\nu_k}, T_{\mu,\ell,\gamma}\big](\partial_{\tau_{j\ell}} f_\mu) \\
& + \big[M_{\nu_j}, Q_\gamma\big](\partial_{\tau_{\mu,k}} f_\mu) - \big[M_{\nu_k}, Q_\gamma\big](\partial_{\tau_{\mu,j}} f_\mu),
\end{aligned} \tag{6.2.153}$$

where M_{ν_ℓ} is the operator of pointwise multiplication with ν_ℓ for each $\ell \in \{1, \dots, n\}$, and the family of operators $T_{\ell,j,k}$, Q_ℓ has been defined in (6.2.130)-(6.2.131).

Proof Pick a vector-valued function $\vec{f}$ as in (6.2.151) and select an aperture parameter $\kappa > 0$. Recall the boundary-to-domain double layer potential operator $\mathcal{D}_\lambda$ associated with Ω as in (6.2.17). From Lemma 6.2.3, [70, (2.4.8)] in [70, Theorem 2.4.1], and [70, Theorem 2.5.1] we then see that

$$\mathcal{N}_\kappa\big(\mathcal{D}_\lambda \vec{f}\big) \in L^p_{\text{loc}}(\partial\Omega, \sigma), \quad \mathcal{N}_\kappa\big(\nabla\mathcal{D}_\lambda \vec{f}\big) \in L^p_{\text{loc}}(\partial\Omega, \sigma), \tag{6.2.154}$$

and (bearing in mind (6.2.150)) the nontangential traces

$$\begin{gathered}
\mathcal{D}_\lambda \vec{f}\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ and } \big(\partial_j \mathcal{D}_\lambda \vec{f}\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ for } j \in \{1, \dots, n\} \\
\text{exist at } \sigma\text{-a.e. point on } \partial_{\text{nta}}\Omega.
\end{gathered} \tag{6.2.155}$$

Granted these properties, [69, Proposition 11.3.2] applies and, in light of item *(iv)* in Theorem 6.2.4, its first conclusion guarantees that (6.2.152) holds.

To proceed, fix $j, k, \gamma \in \{1, \dots, n\}$. Then, thanks to (6.2.109) and [69, Proposition 11.3.2, (11.3.26)] (whose applicability in the current setting has been justified above), we may compute

$$\begin{aligned} \partial_{\tau_{jk}}(K_\lambda \vec{f}\,)_\gamma &= \partial_{\tau_{jk}}\big(\tfrac{1}{2}\vec{f} + K_\lambda \vec{f}\,\big)_\gamma - \tfrac{1}{2}\partial_{\tau_{jk}} f_\gamma \\ &= \nu_j\big(\partial_k \mathcal{D}_\lambda \vec{f}\,\big)_\gamma\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - \nu_k\big(\partial_j \mathcal{D}_\lambda \vec{f}\,\big)_\gamma\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - \tfrac{1}{2}\partial_{\tau_{jk}} f_\gamma. \end{aligned} \tag{6.2.156}$$

Using notation and results from Lemma 6.2.3 and (6.2.128)-(6.2.133) we may then compute

$$\begin{aligned} &\nu_j\big(\partial_k \mathcal{D}_\lambda \vec{f}\,\big)_\gamma\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \\ &\qquad = \nu_j\bigg\{\Big[\mathcal{T}_{\ell,\mu,\gamma}(\partial_{\tau_{k\ell}} f_\mu) + \lambda \mathcal{T}_{\mu,\ell,\gamma}(\partial_{\tau_{k\ell}} f_\mu)\Big]\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} + \mathcal{Q}_\gamma(\partial_{\tau_{\mu k}} f_\mu)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\bigg\} \\ &\qquad = \nu_j\Big\{\tfrac{1}{2}\nu_\ell\big(\nu_\mu\nu_\gamma - \delta_{\mu\gamma}\big)\partial_{\tau_{k\ell}} f_\mu + T_{\ell,\mu,\gamma}(\partial_{\tau_{k\ell}} f_\mu)\Big\} \\ &\qquad\quad + \lambda\nu_j\Big\{\tfrac{1}{2}\nu_\mu\big(\nu_\ell\nu_\gamma - \delta_{\ell\gamma}\big)\partial_{\tau_{k\ell}} f_\mu + T_{\mu,\ell,\gamma}(\partial_{\tau_{k\ell}} f_\mu)\Big\} \\ &\qquad\quad + \nu_j\Big\{\tfrac{1}{2}\nu_\gamma\partial_{\tau_{\mu k}} f_\mu + Q_\gamma(\partial_{\tau_{\mu,k}} f_\mu)\Big\} \end{aligned} \tag{6.2.157}$$

and, similarly,

$$\begin{aligned} \nu_k\big(\partial_j \mathcal{D}_\lambda \vec{f}\,\big)_\gamma\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} &= \nu_k\Big\{\tfrac{1}{2}\nu_\ell\big(\nu_\mu\nu_\gamma - \delta_{\mu\gamma}\big)\partial_{\tau_{j\ell}} f_\mu + T_{\ell,\mu,\gamma}(\partial_{\tau_{j\ell}} f_\mu)\Big\} \\ &\quad + \lambda\nu_k\Big\{\tfrac{1}{2}\nu_\mu\big(\nu_\ell\nu_\gamma - \delta_{\ell\gamma}\big)\partial_{\tau_{j\ell}} f_\mu + T_{\mu,\ell,\gamma}(\partial_{\tau_{j\ell}} f_\mu)\Big\} \\ &\quad + \nu_k\Big\{\tfrac{1}{2}\nu_\gamma\partial_{\tau_{\mu j}} f_\mu + Q_\gamma(\partial_{\tau_{\mu,j}} f_\mu)\Big\}. \end{aligned} \tag{6.2.158}$$

By also making use of [69, Proposition 11.4.2], we see that the jump-terms in $\nu_j\big(\partial_k \mathcal{D}_\lambda \vec{f}\,\big)_\gamma\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ are

$$\tfrac{1}{2}\nu_j\nu_\ell\nu_\mu\nu_\gamma\nu_k(\nabla_{\tan}f_\mu)_\ell - \tfrac{1}{2}\nu_j\nu_\mu\nu_\ell\nu_\gamma\nu_\ell(\nabla_{\tan}f_\gamma)_k - \tfrac{1}{2}\nu_j\nu_\ell\nu_k(\nabla_{\tan}f_\gamma)_\ell$$
$$+ \tfrac{1}{2}\nu_j\nu_\ell\nu_\ell(\nabla_{\tan}f_\gamma)_k + \tfrac{\lambda}{2}\nu_j\nu_\mu\nu_\ell\nu_\gamma\nu_k(\nabla_{\tan}f_\mu)_\ell - \tfrac{\lambda}{2}\nu_j\nu_\mu\nu_\ell\nu_\gamma\nu_\ell(\nabla_{\tan}f_\mu)_k$$
$$- \tfrac{\lambda}{2}\nu_j\nu_\mu\nu_k(\nabla_{\tan}f_\mu)_\gamma + \tfrac{\lambda}{2}\nu_j\nu_\mu\nu_\gamma(\nabla_{\tan}f_\mu)_k + \tfrac{1}{2}\nu_j\nu_\gamma\nu_\mu(\nabla_{\tan}f_\mu)_k$$
$$- \tfrac{1}{2}\nu_j\nu_\gamma\nu_k(\nabla_{\tan}f_\mu)_\mu. \tag{6.2.159}$$

Re-write (6.2.159) with j and k interchanged, then subtract it from (6.2.159). After canceling like terms (in which j, k play symmetric roles) and making repeated use of [69, Proposition 11.4.2], we arrive at the conclusion that the jump-terms in the expression $\nu_j\big(\partial_k\mathcal{D}_\lambda\vec{f}\,\big)_\gamma\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - \nu_k\big(\partial_j\mathcal{D}_\lambda\vec{f}\,\big)_\gamma\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ simply amount to $\frac{1}{2}\partial_{\tau_{jk}}f_\gamma$. In the ultimate analysis, this is going to cancel the very last term in (6.2.156). Returning to (6.2.156) and putting it altogether we therefore obtain

$$\partial_{\tau_{jk}}(K_\lambda\vec{f}\,)_\gamma = \nu_j T_{\ell,\mu,\gamma}(\partial_{\tau_{k\ell}}f_\mu) + \lambda\nu_j T_{\mu,\ell,\gamma}(\partial_{\tau_{k\ell}}f_\mu) + \nu_j Q_\gamma(\partial_{\tau_{\mu,k}}f_\mu)$$
$$- \nu_k T_{\ell,\mu,\gamma}(\partial_{\tau_{j\ell}}f_\mu) - \lambda\nu_k T_{\mu,\ell,\gamma}(\partial_{\tau_{j\ell}}f_\mu) - \nu_k Q_\gamma(\partial_{\tau_{\mu,j}}f_\mu). \tag{6.2.160}$$

Note that, by once again employing [69, Proposition 11.4.2], we have

$$\begin{aligned}
\nu_j T_{\ell,\mu,\gamma}(\partial_{\tau_{k\ell}}f_\mu) &= \big[M_{\nu_j}, T_{\ell,\mu,\gamma}\big](\partial_{\tau_{k\ell}}f_\mu)\\
&\quad + T_{\ell,\mu,\gamma}\Big(\nu_j\nu_k(\nabla_{\tan}f_\mu)_\ell - \nu_j\nu_\ell(\nabla_{\tan}f_\mu)_k\Big),\\
\nu_j T_{\mu,\ell,\gamma}(\partial_{\tau_{k\ell}}f_\mu) &= \big[M_{\nu_j}, T_{\mu,\ell,\gamma}\big](\partial_{\tau_{k\ell}}f_\mu)\\
&\quad + T_{\mu,\ell,\gamma}\Big(\nu_j\nu_k(\nabla_{\tan}f_\mu)_\ell - \nu_j\nu_\ell(\nabla_{\tan}f_\mu)_k\Big),\\
\nu_k T_{\ell,\mu,\gamma}(\partial_{\tau_{j\ell}}f_\mu) &= \big[M_{\nu_k}, T_{\ell,\mu,\gamma}\big](\partial_{\tau_{j\ell}}f_\mu)\\
&\quad + T_{\ell,\mu,\gamma}\Big(\nu_k\nu_j(\nabla_{\tan}f_\mu)_\ell - \nu_k\nu_\ell(\nabla_{\tan}f_\mu)_j\Big),\\
\nu_k T_{\mu,\ell,\gamma}(\partial_{\tau_{j\ell}}f_\mu) &= \big[M_{\nu_k}, T_{\mu,\ell,\gamma}\big](\partial_{\tau_{j\ell}}f_\mu)\\
&\quad + T_{\mu,\ell,\gamma}\Big(\nu_k\nu_j(\nabla_{\tan}f_\mu)_\ell - \nu_k\nu_\ell(\nabla_{\tan}f_\mu)_j\Big),
\end{aligned} \tag{6.2.161}$$

and

$$\begin{aligned}
\nu_j Q_\gamma(\partial_{\tau_{\mu,k}}f_\mu) &= \big[M_{\nu_j}, Q_\gamma\big](\partial_{\tau_{\mu,k}}f_\mu) + Q_\gamma\Big(\nu_j\nu_\mu(\nabla_{\tan}f_\mu)_k - \nu_j\nu_k(\nabla_{\tan}f_\mu)_\mu\Big),\\
\nu_k Q_\gamma(\partial_{\tau_{\mu,j}}f_\mu) &= \big[M_{\nu_k}, Q_\gamma\big](\partial_{\tau_{\mu,j}}f_\mu) + Q_\gamma\Big(\nu_k\nu_\mu(\nabla_{\tan}f_\mu)_j - \nu_k\nu_j(\nabla_{\tan}f_\mu)_\mu\Big).
\end{aligned} \tag{6.2.162}$$

Then (6.2.160)-(6.2.162) and [69, Proposition 11.4.2] yield

$$\begin{aligned}
&\partial_{\tau_{jk}}(K_\lambda \vec{f})_\gamma \qquad (6.2.163)\\
&\quad = \big[M_{\nu_j}, T_{\ell,\mu,\gamma}\big](\partial_{\tau_{k\ell}} f_\mu) - \big[M_{\nu_k}, T_{\ell,\mu,\gamma}\big](\partial_{\tau_{j\ell}} f_\mu) - T_{\ell,\mu,\gamma}\big(\nu_\ell \partial_{\tau_{jk}} f_\mu\big)\\
&\qquad + \lambda\big[M_{\nu_j}, T_{\mu,\ell,\gamma}\big](\partial_{\tau_{k\ell}} f_\mu) - \lambda\big[M_{\nu_k}, T_{\mu,\ell,\gamma}\big](\partial_{\tau_{j\ell}} f_\mu) - \lambda T_{\mu,\ell,\gamma}\big(\nu_\ell \partial_{\tau_{jk}} f_\mu\big)\\
&\qquad + \big[M_{\nu_j}, Q_\gamma\big](\partial_{\tau_{\mu,k}} f_\mu) - \big[M_{\nu_k}, Q_\gamma\big](\partial_{\tau_{\mu,j}} f_\mu) + Q_\gamma\big(\nu_\mu \partial_{\tau_{jk}} f_\mu\big).
\end{aligned}$$

Combining (6.2.163) with (6.2.95) and (6.2.130)-(6.2.131) leads to the conclusion that (6.2.153) holds. □

Recall the Kelvin matrix-valued fundamental solution $E = \big(E_{jk}\big)_{1\le j,k\le n}$ for the Stokes system in $\mathbb{R}^n$ from (6.2.1), and the pressure vector $\vec{q} = (q_j)_{1\le j\le n}$ from (6.2.2). Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In this setting, define the `modified boundary-to-domain single layer potential operator` for the Stokes system

$$\begin{aligned}
&\mathscr{S}_{\rm mod}\vec{f}(x) := \int_{\partial\Omega} \big\{E(x-y) - E_*(-y)\big\}\vec{f}(y)\,d\sigma(y) \ \text{ for each } \ x\in\Omega,\\
&\text{for each } \ \vec{f} \in \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n, \ \text{ where } \ E_* := E\cdot \mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}.
\end{aligned} \tag{6.2.164}$$

This definition implies that for each $\vec{f} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n$ the function $\mathscr{S}_{\rm mod}\vec{f}$ is well defined, belongs to the space $\big[\mathscr{C}^\infty(\Omega)\big]^n$, and for each multi-index $\alpha\in\mathbb{N}_0^n$ with $|\alpha|\ge 1$ one has

$$\partial^\alpha(\mathscr{S}_{\rm mod} f)(x) = \int_{\partial\Omega} (\partial^\alpha E)(x-y)\vec{f}(y)\,d\sigma(y) \ \text{ for each } \ x\in\Omega. \tag{6.2.165}$$

Moreover, for each function $\vec{f} \in \Big[L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n$ we have

$$\begin{aligned}
&\mathscr{S}_{\rm mod}\vec{f} \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \quad Q\vec{f} \in \mathscr{C}^\infty(\Omega),\\
&\Delta\big(\mathscr{S}_{\rm mod}\vec{f}\big) - \nabla Q\vec{f} = 0 \ \text{ and } \ \operatorname{div}\mathscr{S}_{\rm mod}\vec{f} = 0 \ \text{ in } \ \Omega.
\end{aligned} \tag{6.2.166}$$

In analogy with (6.2.164), we also consider the following modified version of the boundary-to-boundary single layer operator for the Stokes system

$$\begin{aligned}
&S_{\rm mod}\vec{f}(x) := \int_{\partial\Omega} \big\{E(x-y) - E_*(-y)\big\} f(y)\,d\sigma(y) \ \text{ at } \sigma\text{-a.e. } \ x\in\partial\Omega,\\
&\text{for each } \ \vec{f} \in \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n, \ \text{ where } \ E_* := E\cdot \mathbf{1}_{\mathbb{R}^n\setminus B(0,1)},
\end{aligned} \tag{6.2.167}$$

which is meaningfully defined, via an absolutely convergent integral, at σ-a.e. point in $\partial\Omega$.

To proceed, we agree to abbreviate, for each $\varepsilon > 0$,

$$\Theta_\varepsilon^{(\ell,j,k)} := (\partial_\ell E_{jk}) \cdot \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(0,\varepsilon)}} \quad \text{for every } \ \ell, j, k \in \{1, \dots, n\}, \tag{6.2.168}$$

as well as

$$\theta_\varepsilon^{(j)} := q_j \cdot \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(0,\varepsilon)}} \quad \text{for every } \ j \in \{1, \dots, n\}. \tag{6.2.169}$$

Let us now consider an open set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter and define $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω, which is well defined at σ-a.e. point on $\partial_*\Omega$. Also, pick an arbitrary $\lambda \in \mathbb{C}$. Use this parameter and the pieces of notation from (6.2.168)-(6.2.169) to define the action of the `modified boundary-to-domain double layer potential operator` for the Stokes system (compare with (6.2.16)-(6.2.17)) on each function

$$\vec{f} = (f_j)_{1 \le j \le n} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1 + |x|^n}\Big)\Big]^n \tag{6.2.170}$$

by setting at each $x \in \Omega$

$$\begin{aligned} &\mathcal{D}_\lambda^{\text{mod}} \vec{f}(x) \qquad\qquad (6.2.171)\\ &:= \Bigg(\int_{\partial_*\Omega} \Big\{ - \nu_k(y)\big((\partial_k E_{j\gamma})(x-y) - \Theta_1^{(k,j,\gamma)}(-y)\big) \\ &\qquad - \lambda \nu_k(y)\big((\partial_j E_{k\gamma})(x-y) - \Theta_1^{(j,k,\gamma)}(-y)\big) \\ &\qquad + \nu_j(y)\big(q_\gamma(x-y) - \theta_1^{(\gamma)}(-y)\big)\Big\} f_j(y)\, \mathrm{d}\sigma(y) \Bigg)_{1 \le \gamma \le n}. \end{aligned}$$

We also define the `modified boundary-to-boundary double layer potential operator` for the Stokes system (compare with (6.2.94)-(6.2.95)), acting on each function $\vec{f} = (f_j)_{1 \le j \le n}$ as in (6.2.170) according to

$$\begin{aligned} &K_\lambda^{\text{mod}} \vec{f}(x) \qquad\qquad (6.2.172)\\ &:= \Bigg(\lim_{\varepsilon \to 0^+} \int_{\partial_*\Omega} \Big\{ - \nu_k(y)\big(\Theta_\varepsilon^{(k,j,\gamma)}(x-y) - \Theta_1^{(k,j,\gamma)}(-y)\big) \\ &\qquad - \lambda \nu_k(y)\big(\Theta_\varepsilon^{(j,k,\gamma)}(x-y) - \Theta_1^{(j,k,\gamma)}(-y)\big) \\ &\qquad + \nu_j(y)\big(\theta_\varepsilon^{(\gamma)}(x-y) - \theta_1^{(\gamma)}(-y)\big)\Big\} f_j(y)\, \mathrm{d}\sigma(y) \Bigg)_{1 \le \gamma \le n} \end{aligned}$$

at σ-a.e. $x \in \partial_*\Omega$. It is then apparent from definitions and [68, Proposition 5.6.7] that K_λ^{mod} is compatible with K_λ (acting on functions from $\Big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^n$ as in (6.2.95)) in the sense that

$$\text{for each function } \vec{f} \in \big[L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n \text{ the difference } c_{\vec{f}} := K_\lambda^{\mathrm{mod}} \vec{f} - K_\lambda \vec{f} \text{ is a constant (belonging to } \mathbb{C}^n\text{) on } \partial_*\Omega. \tag{6.2.173}$$

The immediate goal is to develop a versatile Calderón-Zygmund theory for the modified boundary layer operators associated with Stokes system as above, in the spirit of Theorem 6.2.4.

Theorem 6.2.6 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set with the property that $\partial\Omega$ is an Ahlfors regular set; in particular, Ω is a set of locally finite perimeter. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix an aperture parameter κ and a number $\lambda \in \mathbb{C}$. Then the following properties hold.*

(1) The operator $\mathcal{D}_\lambda^{\mathrm{mod}}$ is meaningfully defined, and

$$\text{for each } \vec{f} \in \Big[L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^n \text{ one has } \mathcal{D}_\lambda^{\mathrm{mod}} \vec{f} \in \big[\mathscr{C}^\infty(\Omega)\big]^n \text{ as well as } \Delta(\mathcal{D}_\lambda^{\mathrm{mod}} \vec{f}) - \nabla \mathcal{P}_\lambda \vec{f} = 0 \text{ and } \operatorname{div} \mathcal{D}_\lambda^{\mathrm{mod}} \vec{f} = 0 \text{ in } \Omega. \tag{6.2.174}$$

In addition, the operator $\mathcal{D}_\lambda^{\mathrm{mod}}$ is compatible with $\mathcal{D}_\lambda$ from (6.2.17), in the sense that for each function $\vec{f}$ belonging to the smaller space $\big[L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$ the difference

$$C_f := \mathcal{D}_\lambda^{\mathrm{mod}} \vec{f} - \mathcal{D}_\lambda \vec{f} \text{ is a constant (belonging to } \mathbb{C}^n\text{) in } \Omega. \tag{6.2.175}$$

Consequently,

$$\nabla \mathcal{D}_\lambda^{\mathrm{mod}} \vec{f} = \nabla \mathcal{D}_\lambda \vec{f} \text{ in } \Omega \text{ for each } \vec{f} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n. \tag{6.2.176}$$

Moreover,

$$\mathcal{D}_\lambda^{\mathrm{mod}} \text{ maps constant } (\mathbb{C}^n\text{-valued) functions on } \partial_*\Omega \text{ into constant } (\mathbb{C}^n\text{-valued) functions in the set } \Omega. \tag{6.2.177}$$

In addition,

$$\text{for each multi-index } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| > 0 \text{ and each function } \vec{f} = (f_j)_{1 \leq j \leq n} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^n, \tag{6.2.178}$$

at each point $x \in \Omega$ one may express

$$\partial^\alpha\big(\mathcal{D}_\lambda^{\text{mod}}\vec{f}\big)(x) \tag{6.2.179}$$
$$= \Bigg(\int_{\partial_*\Omega}\Big\{-\nu_k(y)(\partial^\alpha\partial_k E_{j\gamma})(x-y)-\lambda\nu_k(y)(\partial^\alpha\partial_j E_{k\gamma})(x-y)$$
$$+\nu_j(y)(\partial^\alpha q_\gamma)(x-y)\Big\}f_j(y)\,\mathrm{d}\sigma(y)\Bigg)_{1\leq\gamma\leq n}.$$

Finally, given any function

$$\vec{f}=(f_j)_{1\leq j\leq n}\in\Big[L^1\Big(\partial_*\Omega,\ \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^n \text{ with the property that}$$
$$\partial_{\tau_{rs}}f_j\in L^1\Big(\partial_*\Omega,\ \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \text{ for all } r,s,j\in\{1,\dots,n\}, \tag{6.2.180}$$

it follows that for each index $r\in\{1,\dots,n\}$ *and each point* $x\in\Omega$ *one has*

$$\partial_r\big(\mathcal{D}_\lambda^{\text{mod}}\vec{f}\big)(x)=\Bigg(\int_{\partial_*\Omega}\big\{(\partial_k E_{j\gamma})(x-y)\big(\partial_{\tau_{rk}}f_j\big)(y)$$
$$+\lambda(\partial_j E_{k\gamma})(x-y)\big(\partial_{\tau_{rk}}f_j\big)(y)$$
$$+q_\gamma(x-y)\big(\partial_{\tau_{jr}}f_j\big)(y)\big\}\,\mathrm{d}\sigma(y)\Bigg)_{1\leq\gamma\leq n} \tag{6.2.181}$$

as well as

$$\mathcal{P}_\lambda\vec{f}(x)=(1+\lambda)\int_{\partial_*\Omega}q_j(x-y)\big(\partial_{\tau_{kj}}f_k\big)(y)\,\mathrm{d}\sigma(y). \tag{6.2.182}$$

(2) For each $\eta\in(0,1)$ *there exists a constant* $C\in(0,\infty)$ *with the property that*

$$\sup_{x\in\Omega}\Big\{\operatorname{dist}(x,\partial\Omega)^{1-\eta}\big|\nabla\big(\mathcal{D}_\lambda^{\text{mod}}\vec{f}\big)(x)\big|\Big\}\leq C\|\vec{f}\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^n} \tag{6.2.183}$$

for every function $\vec{f}\in\big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^n$*. Moreover,*

if $\Omega\subseteq\mathbb{R}^n$ *is a uniform domain with the property that* $\partial\Omega$ *is an Ahlfors regular set then* $\mathcal{D}_\lambda^{\text{mod}}:\big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^n\to\big[\dot{\mathscr{C}}^\eta\big(\overline{\Omega}\big)\big]^n$ *is a well-defined, linear, bounded operator for any* $\eta\in(0,1)$*,* (6.2.184)

whereas

if $\Omega\subseteq\mathbb{R}^n$ *is an* NTA *domain with an upper Ahlfors regular boundary then* $\mathcal{D}_\lambda^{\text{mod}}:\big[\dot{\mathscr{C}}^\eta_{\text{van}}(\partial\Omega)\big]^n\to\big[\dot{\mathscr{C}}^\eta_{\text{van}}\big(\overline{\Omega}\big)\big]^n$ *is a well-defined, linear, and bounded operator for each* $\eta\in(0,1)$*,* (6.2.185) *where the homogeneous vanishing Hölder spaces intervening above are defined as in (A.0.48) (with $\Sigma := \partial\Omega$ and $\Sigma := \overline{\Omega}$, respectively). Also, for each $\eta \in (0,1)$ and each $p \in (1,\infty)$ there exists some $C \in (0,\infty)$ with the property that for each function $\vec{f} \in \big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^n$ one has*

$$\sup_{\substack{x\in\partial\Omega\\ r\in(0,\infty)}}\left(\frac{1}{r^{n-1+\eta p}}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{D}_\lambda^{\rm mod}\vec{f}\big|^p\,{\rm dist}\,(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\right)^{1/p} \leq C\|\vec{f}\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^n} \tag{6.2.186}$$

and

$$\lim_{R\to 0^+}\left\{\sup_{\substack{x\in\partial\Omega\\ r\in(0,R)}}\left(\frac{1}{r^{n-1+\eta p}}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{D}_\lambda^{\rm mod}\vec{f}\big|^p{\rm dist}(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\right)^{1/p}\right\} \leq C\,{\rm dist}\,\Big(\vec{f},\ \big[\dot{\mathscr{C}}^\eta_{\rm van}(\partial\Omega)\big]^n\Big), \tag{6.2.187}$$

where the distance is measured in the space $\Big(\big[\dot{\mathscr{C}}^\eta(\partial\Omega)\big]^n,\ \|\cdot\|_{[\dot{\mathscr{C}}^\eta(\partial\Omega)]^n}\Big)$. As a corollary, if the function $\vec{f}$ actually belongs to the homogeneous vanishing Hölder space $\big[\dot{\mathscr{C}}^\eta_{\rm van}(\partial\Omega)\big]^n$ for some $\eta \in (0,1)$, then for each $p \in (1,\infty)$ one has

$$\lim_{R\to 0^+}\left\{\sup_{\substack{x\in\partial\Omega\\ r\in(0,R)}}\left(\frac{1}{r^{n-1+\eta p}}\int_{B(x,r)\cap\Omega}\big|\nabla\mathcal{D}_\lambda^{\rm mod}\vec{f}\big|^p\,{\rm dist}\,(\cdot,\partial\Omega)^{p-1}\,d\mathcal{L}^n\right)^{1/p}\right\} = 0. \tag{6.2.188}$$

(3) *Strengthen the original geometric hypotheses by assuming that $\partial\Omega$ is actually a* UR *set. Also, fix an aperture parameter $\kappa \in (0,\infty)$. Then, as a consequence of* (6.2.181) *and* [70, Theorem 2.5.1]*, the nontangential boundary trace*

$$\begin{gathered}\big(\partial_\ell\mathcal{D}_\lambda^{\rm mod}\vec{f}\big)\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \text{ exists (in } \mathbb{C}^n\text{) at } \sigma\text{-a.e. point on } \partial_*\Omega,\\ \text{for all functions } \vec{f} \text{ as in (6.2.180) and all } \ell\in\{1,\dots,n\}.\end{gathered} \tag{6.2.189}$$

Another corollary of (6.2.181) *and* [70, (2.4.8)] *is the fact that for each $\varepsilon > 0$ and each $p \in (1,\infty)$*

$$\begin{gathered}\mathcal{N}_\kappa^\varepsilon\big(\nabla(\mathcal{D}_\lambda^{\rm mod}\vec{f})\big) \in L^p_{\rm loc}(\partial\Omega,\sigma) \text{ for each function}\\ \vec{f} = (f_\alpha)_{1\leq\alpha\leq n} \in \left[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\right]^n \text{ such that}\\ \partial_{\tau_{jk}} f_\alpha \in L^1\Big(\partial_*\Omega,\ \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\rm loc}(\partial_*\Omega,\sigma)\\ \text{for all } j,k\in\{1,\dots,n\} \text{ and all } \alpha\in\{1,\dots,n\}.\end{gathered} \tag{6.2.190}$$

In addition, as seen from (6.2.171) *and* [70, (2.5.32)], *for each truncation parameter* $\varepsilon \in (0,\infty)$ *one has*

$$\begin{aligned}&\mathcal{N}_\kappa^\varepsilon(\overset{\text{mod}}{\mathcal{D}_\lambda}\vec{f}) \in L^p_{\text{loc}}(\partial\Omega,\sigma) \text{ for each function}\\ &\vec{f} \in \Big[L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big) \cap L^p_{\text{loc}}(\partial_*\Omega,\sigma)\Big]^n \text{ with } p\in(1,\infty).\end{aligned} \tag{6.2.191}$$

Next, given any function $\vec{f} = (f_j)_{1\le j\le n}$ *as in* (6.2.180) *and given any index* $\alpha \in \{1,\dots,n\}$, *at* σ*-a.e. point* $x \in \partial_*\Omega$ *one has* (*where the conormal derivative is considered as in* (6.1.6))

$$\begin{aligned}\Big(\partial_\nu^\lambda\big(\overset{\text{mod}}{\mathcal{D}_\lambda}\vec{f},\mathcal{P}_\lambda\vec{f}\big)\Big)_\alpha(x) \quad &\qquad\qquad (6.2.192)\\ = \lim_{\varepsilon\to 0^+}\int_{\substack{y\in\partial_*\Omega\\ |x-y|>\varepsilon}} \Big\{&\nu_\ell(x)(\partial_k E_{j\alpha})(x-y)\big(\partial_{\tau_{\ell k}} f_j\big)(y)\\ &+\lambda\nu_\ell(x)(\partial_j E_{k\alpha})(x-y)\big(\partial_{\tau_{\ell k}} f_j\big)(y)\\ &+\nu_\ell(x)q_\alpha(x-y)\big(\partial_{\tau_{j\ell}} f_j\big)(y)\\ &+\lambda\nu_\ell(x)(\partial_k E_{j\ell})(x-y)\big(\partial_{\tau_{\alpha k}} f_j\big)(y)\\ &+\lambda^2\nu_\ell(x)(\partial_j E_{k\ell})(x-y)\big(\partial_{\tau_{\alpha k}} f_j\big)(y)\\ &+\lambda\nu_\ell(x)q_\ell(x-y)\big(\partial_{\tau_{j\alpha}} f_j\big)(y)\\ &-(1+\lambda)\nu_\alpha(x)q_j(x-y)\big(\partial_{\tau_{kj}} f_k\big)(y)\Big\}\,d\sigma(y).\end{aligned}$$

(4) Once more strengthen the original geometric hypotheses by assuming that $\partial\Omega$ *is actually a* UR *set. Then the following jump-formula holds:*

$$\begin{aligned}&\big(\overset{\text{mod}}{\mathcal{D}_\lambda}\vec{f}\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\tfrac12 I + \overset{\text{mod}}{K_\lambda}\big)\vec{f} \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega,\\ &\text{for each given function } \vec{f} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\Big)\Big]^n,\end{aligned} \tag{6.2.193}$$

where, as usual, I is the identity operator. As a consequence of (6.2.193) *and* (6.2.177),

$$\begin{aligned}&\text{if } \partial\Omega \text{ is a UR set, the operator } \overset{\text{mod}}{K_\lambda} \text{ maps constant } (\mathbb{C}^n\text{-valued})\\ &\text{functions on } \partial_*\Omega \text{ into constant } (\mathbb{C}^n\text{-valued}) \text{ functions on } \partial_*\Omega.\end{aligned} \tag{6.2.194}$$

Moreover, if Ω *is actually a* UR *domain and if* $p, p' \in (1,\infty)$ *are such that* $1/p + 1/p' = 1$, *then given any functions*

$$\vec{f} \in \Big[L^p_{\rm loc}(\partial\Omega,\sigma) \cap L^1\Big(\partial\Omega,\,\tfrac{\sigma(x)}{1+|x|^n}\Big)\Big]^n \text{ together with}$$
$$\vec{g} \in \big[L^{p'}_{\rm comp}(\partial\Omega,\sigma)\big]^n \text{ satisfying } \int_{\partial\Omega} \vec{g}\,{\rm d}\sigma = 0 \in \mathbb{C}^n, \tag{6.2.195}$$

it follows that

$$\int_{\partial\Omega} \big|K^{\rm mod}_\lambda \vec{f}\big||\vec{g}|\,{\rm d}\sigma < +\infty, \quad \int_{\partial\Omega} |\vec{f}||K^{\#}_\lambda \vec{g}|\,{\rm d}\sigma < +\infty,$$
$$\text{and } \int_{\partial\Omega} \big\langle K^{\rm mod}_\lambda \vec{f}, \vec{g}\big\rangle\,{\rm d}\sigma = \int_{\partial\Omega} \langle \vec{f}, K^{\#}_\lambda \vec{g}\rangle\,{\rm d}\sigma. \tag{6.2.196}$$

(5) Work under the stronger assumption that $\partial\Omega$ *is a* UR *set. For each vector-valued function* $\vec{f}$ *belonging to the space* $\Big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\Big]^n$ *(hence, in particular, for each vector-valued function* $\vec{f} \in \big[L^p(\partial\Omega,\sigma)\big]^n$ *with* $p \in [1,\infty)$*) the following jump-formula holds:*

$$\partial^\lambda_\nu\big(\mathscr{S}_{\rm mod}\vec{f}, \mathcal{Q}\vec{f}\big) = \big(-\tfrac{1}{2}I + K^{\#}_\lambda\big)\vec{f} \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega, \tag{6.2.197}$$

where the conormal derivative is defined as in (6.1.6).

(6) Continue to work under the stronger assumption that $\partial\Omega$ *is a* UR *set. Then for each* $p \in (1,\infty)$ *there exists a constant* $C \in (0,\infty)$ *with the property that for each* $\vec{f} \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^n$ *the measure* $\big|\nabla\big(\mathcal{D}^{\rm mod}_\lambda \vec{f}\big)\big|^p\,{\rm dist}\,(\cdot,\partial\Omega)^{p-1}\,{\rm d}\mathcal{L}^n$ *is Carleson in* Ω*, in the quantitative sense that*

$$\sup_{x\in\partial\Omega,\,r>0} \frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{D}^{\rm mod}_\lambda \vec{f}\big)\big|^p {\rm dist}\,(\cdot,\partial\Omega)^{p-1}\,{\rm d}\mathcal{L}^n$$
$$\leq C\|\vec{f}\|^p_{[\dot{\mathrm{BMO}}(\partial\Omega,\sigma)]^n} \tag{6.2.198}$$

(*with the piece of notation introduced in* (A.0.19)). *Moreover, for each* $p \in (1,\infty)$ *there exists a constant* $C \in (0,\infty)$ *with the property that for each function* $\vec{f} \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^n$ *one has*

$$\lim_{R\to 0^+} \sup_{\substack{x\in\partial\Omega \text{ and} \\ r\in(0,R)}} \Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{D}^{\rm mod}_\lambda \vec{f}\big)\big|^p {\rm dist}\,(\cdot,\partial\Omega)^{p-1}\,{\rm d}\mathcal{L}^n\Big)^{\frac{1}{p}}$$
$$\leq C\,{\rm dist}\,\big(\vec{f},\,[\mathrm{VMO}(\partial\Omega,\sigma)]^n\big) \tag{6.2.199}$$

where the distance in the right-hand side is considered in $[\mathrm{BMO}(\partial\Omega,\sigma)]^n$. *In particular,*

$$\left|\nabla\big(\mathcal{D}_{\lambda}^{\mathrm{mod}}\vec{f}\big)\right|^{p}\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^{n}\ \text{ is a vanishing Carleson measure in } \Omega,\\ \text{for each function } \vec{f}\in\big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^{n} \text{ and each } p\in(1,\infty). \tag{6.2.200}$$

Furthermore, for each $p \in (1,\infty)$ there exists a constant $C \in (0,\infty)$, which depends only on n, p, and the UR *constants of $\partial\Omega$, with the property that for each function $\vec{f} \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^n$ one has*

$$\max\Bigg\{\lim_{R\to\infty}\sup_{\substack{x\in\partial\Omega,\ r>0\\ B(x,r)\subseteq\mathbb{R}^n\setminus B(0,R)}}\Bigg(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times\\ \times\int_{B(x,r)\cap\Omega}\left|\nabla\big(\mathcal{D}_{\lambda}^{\mathrm{mod}}\vec{f}\big)\right|^{p}\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^{n}\Bigg)^{1/p},\\ \lim_{R\to0^{+}}\sup_{x\in\partial\Omega,\ r\in(0,R)}\Bigg(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times\\ \times\int_{B(x,r)\cap\Omega}\left|\nabla\big(\mathcal{D}_{\lambda}^{\mathrm{mod}}\vec{f}\big)\right|^{p}\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^{n}\Bigg)^{1/p},\\ \lim_{R\to\infty}\sup_{x\in\partial\Omega,\ r>R}\Bigg(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)}\times\\ \times\int_{B(x,r)\cap\Omega}\left|\nabla\big(\mathcal{D}_{\lambda}^{\mathrm{mod}}\vec{f}\big)\right|^{p}\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^{n}\Bigg)^{1/p}\Bigg\}\\ \leq C\operatorname{dist}\Big(\vec{f},\ \big[\mathrm{CMO}(\partial\Omega,\sigma)\big]^{n}\Big), \tag{6.2.201}$$

where the distance in the right-hand side is considered in $[\mathrm{BMO}(\partial\Omega,\sigma)]^n$ (*recall that* $\mathrm{CMO}(\partial\Omega,\sigma)$ *has been introduced in* (A.0.52)). *As a consequence of* (6.2.201) *and Definition 1.8.1,*

$$\left|\nabla\big(\mathcal{D}_{\lambda}^{\mathrm{mod}}\vec{f}\big)\right|^{p}\operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^{n}\ \text{ is a super vanishing Carleson measure in } \Omega,\\ \text{for each function } \vec{f}\in\big[\mathrm{CMO}(\partial\Omega,\sigma)\big]^{n} \text{ and each } p\in(1,\infty). \tag{6.2.202}$$

(7) If $\partial\Omega$ is bounded, then all properties listed in items (1)-(4) are valid for the ordinary double layer operator $\mathcal{D}_\lambda$, as originally defined in (6.2.17)*, in place of its modified version $\mathcal{D}_{\lambda}^{\mathrm{mod}}$. In particular, if $\partial\Omega$ is a compact* UR *set then for each $p \in (1,\infty)$ there exists a constant $C \in (0,\infty)$ with the property that for each function $\vec{f} \in \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^n$ one has*

$$\lim_{R\to 0^+} \sup_{\substack{x\in\partial\Omega \text{ and} \\ r\in(0,R)}} \Big(\frac{1}{\sigma\big(B(x,r)\cap\partial\Omega\big)} \int_{B(x,r)\cap\Omega} \big|\nabla\big(\mathcal{D}_\lambda \vec{f}\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n \Big)^{\frac{1}{p}}$$
$$\leq C \operatorname{dist}\big(\vec{f},\ [\mathrm{VMO}(\partial\Omega,\sigma)]^n\big) \tag{6.2.203}$$

where the distance in the right-hand side is considered in $[\mathrm{BMO}(\partial\Omega,\sigma)]^n$. *In particular, if* $\partial\Omega$ *is a compact* UR *set then*

$$\big|\nabla\big(\mathcal{D}_\lambda \vec{f}\big)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^n \text{ is a vanishing Carleson measure in } \Omega,$$
$$\text{for each function } \vec{f} \in \big[\mathrm{VMO}(\partial\Omega,\sigma)\big]^n \text{ and each } p\in(1,\infty). \tag{6.2.204}$$

(8) *Strengthen the original assumptions by demanding now that* Ω *is a* UR *domain. Then, having fixed an integrability exponent* $p\in(1,\infty)$, *for each vector-valued function*

$$\vec{f} = (f_\mu)_{1\leq\mu\leq n} \in \Big[L^1\Big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^n}\Big) \cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma)\Big]^n \text{ such that}$$
$$\partial_{\tau_{jk}} f_\mu \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma) \text{ for all } j,k,\mu\in\{1,\dots,n\} \tag{6.2.205}$$

it follows that

$$K_\lambda^{\mathrm{mod}} \vec{f} \in \big[L^p_{1,\mathrm{loc}}(\partial\Omega,\sigma)\big]^n \tag{6.2.206}$$

and for each pair of indices $j,k\in\{1,\dots,n\}$ *one has*

$$\begin{aligned} &\partial_{\tau_{jk}}\big(K_\lambda^{\mathrm{mod}}\vec{f}\,\big)_\gamma \\ &\quad = \Big(K_\lambda\big(\partial_{\tau_{jk}}\vec{f}\,\big)\Big)_\gamma + \big[M_{\nu_j}, T_{\ell,\mu,\gamma}\big](\partial_{\tau_{k\ell}} f_\mu) - \big[M_{\nu_k}, T_{\ell,\mu,\gamma}\big](\partial_{\tau_{j\ell}} f_\mu) \\ &\qquad + \lambda\big[M_{\nu_j}, T_{\mu,\ell,\gamma}\big](\partial_{\tau_{k\ell}} f_\mu) - \lambda\big[M_{\nu_k}, T_{\mu,\ell,\gamma}\big](\partial_{\tau_{j\ell}} f_\mu) \\ &\qquad + \big[M_{\nu_j}, Q_\gamma\big](\partial_{\tau_{\mu,k}} f_\mu) - \big[M_{\nu_k}, Q_\gamma\big](\partial_{\tau_{\mu,j}} f_\mu), \end{aligned} \tag{6.2.207}$$

where, for each $\ell\in\{1,\dots,n\}$, M_{ν_ℓ} *is the operator of pointwise multiplication with* ν_ℓ *and the family of operators* $T_{\ell,j,k}$, Q_ℓ *has been defined in* (6.2.130)-(6.2.131).

(9) *Make the stronger assumption that* $\partial\Omega$ *is a* UR *set. Fix an integrability exponent* $p\in(1,\infty)$. *Then the operator*

$$S_{\mathrm{mod}} : \big[L^p(\partial\Omega,\sigma)\big]^n \longrightarrow \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^n \tag{6.2.208}$$

is well defined, linear, and bounded, when the target space is endowed with the semi-norm induced by (A.0.128). *In addition,*

$$\begin{gathered}\big[S_{\rm mod}\big] : \big[L^p(\partial\Omega,\sigma)\big]^n \longrightarrow \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^n \ \textit{defined as}\\ \big[S_{\rm mod}\big]f := \big[S_{\rm mod}f\big] \in \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^n, \quad \forall\,\vec{f}\in\big[L^p(\partial\Omega,\sigma)\big]^n\end{gathered} \tag{6.2.209}$$

is also a well-defined, linear, and bounded operator, when the quotient space is endowed with the natural semi-norm[7] *introduced in* [69, (11.5.138)]. *Furthermore, with* $\mathscr{S}_{\rm mod}$ *denoting the modified version of the single layer operator for the Stokes system acting on functions from* $\big[L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$ *as in* (6.2.164)*, there exists some constant* $C = C(\Omega,n,p,\kappa)\in(0,\infty)$ *with the property that for each function* $\vec{f}\in\big[L^p(\partial\Omega,\sigma)\big]^n$ *and each truncation parameter* $\varepsilon\in(0,\infty)$ *one has:*

$$\begin{gathered}\mathscr{S}_{\rm mod}\vec{f}\in\big[\mathscr{C}^\infty(\Omega)\big]^n,\quad \Delta\big(\mathscr{S}_{\rm mod}\vec{f}\big)-\nabla Q\vec{f}=0 \ \textit{and}\ \operatorname{div}\mathscr{S}_{\rm mod}\vec{f}=0\ \textit{in}\ \Omega,\\ \mathcal{N}_\kappa\big(\nabla\mathscr{S}_{\rm mod}\vec{f}\big)\in L^p(\partial\Omega,\sigma),\ \big\|\mathcal{N}_\kappa\big(\nabla\mathscr{S}_{\rm mod}\vec{f}\big)\big\|_{L^p(\partial\Omega,\sigma)}\le C\|\vec{f}\|_{[L^p(\partial\Omega,\sigma)]^n},\\ \mathcal{N}_\kappa\big(Q\vec{f}\big)\in L^p(\partial\Omega,\sigma),\ \big\|\mathcal{N}_\kappa\big(Q\vec{f}\big)\big\|_{L^p(\partial\Omega,\sigma)}\le C\|\vec{f}\|_{[L^p(\partial\Omega,\sigma)]^n},\\ \mathcal{N}^\varepsilon_\kappa(\mathscr{S}_{\rm mod}\vec{f})\in L^q_{\rm loc}(\partial\Omega,\sigma)\ \textit{for each}\ q\in\big(0,\tfrac{n-1}{n-2}\big),\\ \nabla(\mathscr{S}_{\rm mod}\vec{f})\big|^{\kappa-\rm n.t.}_{\partial\Omega}\ \textit{exists at}\ \sigma\textit{-a.e. point on}\ \partial_*\Omega,\\ \partial^\lambda_\nu\big(\mathscr{S}_{\rm mod}\vec{f},Q\vec{f}\big)=\big(-\tfrac12 I+K^\#_\lambda\big)\vec{f}\ \textit{at}\ \sigma\textit{-a.e. point on}\ \partial_*\Omega,\\ \textit{and}\ \Big(\big(\mathscr{S}_{\rm mod}\vec{f}\big)\Big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big)(x)=(S_{\rm mod}f)(x)\ \textit{at}\ \sigma\textit{-a.e. point}\ x\in A_\kappa(\partial\Omega),\\ \textit{in particular (cf.}\ [68,\ \text{Proposition 8.8.4}]\textit{), at}\ \sigma\textit{-a.e. point}\ x\in\partial_*\Omega.\end{gathered} \tag{6.2.210}$$

(10) Assume the set Ω *is actually a* UR *domain. Fix some integrability exponent* $p\in(1,\infty)$*. Then there exists some constant* $C=C(\Omega,n,p,\kappa)\in(0,\infty)$ *with the property that for each function* $\vec{f}\in\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^n$ *one has*

$$\begin{gathered}\mathcal{D}^{\rm mod}_\lambda\vec{f}\in\big[\mathscr{C}^\infty(\Omega)\big]^n,\quad \Delta(\mathcal{D}^{\rm mod}_\lambda\vec{f})-\nabla\mathcal{P}_\lambda\vec{f}=0\ \textit{and}\ \operatorname{div}\mathcal{D}^{\rm mod}_\lambda\vec{f}=0\ \textit{in}\ \Omega,\\ \big(\mathcal{D}^{\rm mod}_\lambda\vec{f}\big)\big|^{\kappa-\rm n.t.}_{\partial\Omega}\ \textit{and}\ \big(\nabla\mathcal{D}^{\rm mod}_\lambda\vec{f}\big)\big|^{\kappa-\rm n.t.}_{\partial\Omega}\ \textit{exist at}\ \sigma\textit{-a.e. point on}\ \partial\Omega,\\ \mathcal{N}_\kappa\big(\nabla\mathcal{D}^{\rm mod}_\lambda\vec{f}\big)\in L^p(\partial\Omega,\sigma),\ \big\|\mathcal{N}_\kappa\big(\nabla\mathcal{D}^{\rm mod}_\lambda\vec{f}\big)\big\|_{L^p(\partial\Omega,\sigma)}\le C\|\vec{f}\|_{[\dot{L}^p_1(\partial\Omega,\sigma)]^n},\\ \mathcal{N}_\kappa\big(\mathcal{P}_\lambda\vec{f}\big)\in L^p(\partial\Omega,\sigma),\ \big\|\mathcal{N}_\kappa\big(\mathcal{P}_\lambda\vec{f}\big)\big\|_{L^p(\partial\Omega,\sigma)}\le C\|\vec{f}\|_{[\dot{L}^p_1(\partial\Omega,\sigma)]^n}.\end{gathered} \tag{6.2.211}$$

In fact, for each function $\vec{f}\in\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^n$ *one has*

$$(\mathcal{D}^{\rm mod}_\lambda\vec{f})\big|^{\kappa-\rm n.t.}_{\partial\Omega}=\big(\tfrac12 I+K^{\rm mod}_\lambda\big)\vec{f}\ \textit{at}\ \sigma\textit{-a.e. point on}\ \partial\Omega, \tag{6.2.212}$$

[7] recall from [69, Proposition 11.5.14] that said semi-norm is actually a genuine norm if $\Omega\subseteq\mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

where I is the identity operator on $\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^M$, and $K^{\rm mod}_\lambda$ is the modified boundary-to-boundary double layer potential operator for the Stokes system from (6.2.172)*. Also,*

$$\begin{array}{l}\text{if } p > n-1 \text{ then } \mathcal{D}^{\rm mod}_\lambda : \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^n \to \big[\dot{\mathscr{C}}^\eta(\overline{\Omega})\big]^n \text{ is a well-}\\ \text{defined, linear, and bounded operator, with } \eta := 1-\tfrac{n-1}{p}\in(0,1),\\ \text{provided either } \Omega\subseteq\mathbb{R}^n \text{ is an open set satisfying a two-sided local}\\ \text{John condition and whose boundary is Ahlfors regular, or } \Omega \text{ is}\\ \text{simultaneously a uniform domain and a UR domain in } \mathbb{R}^n.\end{array}\tag{6.2.213}$$

Moreover, formula (6.2.192) *holds for any function $\vec{f}=(f_j)_{1\le j\le n}$ belonging to the homogeneous boundary Sobolev space $\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^n$. Furthermore, the operator*

$$\begin{array}{l}\partial^\lambda_\nu\big(\mathcal{D}^{\rm mod}_\lambda,\mathcal{P}_\lambda\big) : \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^n \longrightarrow \big[L^p(\partial\Omega,\sigma)\big]^n \text{ defined as}\\ \partial^\lambda_\nu\big(\mathcal{D}^{\rm mod}_\lambda,\mathcal{P}_\lambda\big)\vec{f} := \partial^\lambda_\nu\big(\mathcal{D}^{\rm mod}_\lambda\vec{f},\mathcal{P}_\lambda\vec{f}\big) \text{ for each } \vec{f}\in\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^n\end{array}\tag{6.2.214}$$

is well defined, linear, and bounded, when the domain space is equipped with the semi-norm induced by (A.0.128)*. In addition,*

$$\begin{array}{l}\big[\partial^\lambda_\nu\big(\mathcal{D}^{\rm mod}_\lambda,\mathcal{P}_\lambda\big)\big] : \big[\dot{L}^p_1(\partial\Omega,\sigma)/\sim\big]^n \longrightarrow \big[L^p(\partial\Omega,\sigma)\big]^n \text{ defined as}\\ \big[\partial^\lambda_\nu\big(\mathcal{D}^{\rm mod}_\lambda,\mathcal{P}_\lambda\big)\big][\vec{f}] := \partial^\lambda_\nu\big(\mathcal{D}^{\rm mod}_\lambda\vec{f},\mathcal{P}_\lambda\vec{f}\big) \text{ for each } \vec{f}\in\big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^n\end{array}\tag{6.2.215}$$

is a well-defined, linear, and bounded operator, when the quotient space is equipped with the natural semi-norm[8] *introduced in* [69, (11.5.138)].
Finally, similar properties to those described in Theorem 2.3.1 and Theorem 2.3.7 hold for the layer potentials associated with the Stokes system in a UR *domain $\Omega\subseteq\mathbb{R}^n$ acting on $\big[\mathrm{BMO}_{-1}(\partial\Omega,\sigma)\big]^n$ and $\big[\mathrm{VMO}_{-1}(\partial\Omega,\sigma)\big]^n$.*

(11) If Ω is a UR *domain satisfying a local John condition, then the operator*

$$K^{\rm mod}_\lambda : \big[\dot{L}^p_1(\partial\Omega,\sigma)\cap L^p_{\rm loc}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[\dot{L}^p_1(\partial\Omega,\sigma)\cap L^p_{\rm loc}(\partial\Omega,\sigma)\big]^n \tag{6.2.216}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.128)*. As a consequence of this and* (6.2.194)*,*

$$\begin{array}{l}\big[K^{\rm mod}_\lambda\big] : \Big[\Big(\dot{L}^p_1(\partial\Omega,\sigma)\cap L^p_{\rm loc}(\partial\Omega,\sigma)\Big)\Big/\sim\Big]^n\\ \qquad\qquad\longrightarrow \Big[\Big(\dot{L}^p_1(\partial\Omega,\sigma)\cap L^p_{\rm loc}(\partial\Omega,\sigma)\Big)\Big/\sim\Big]^n\end{array}\tag{6.2.217}$$

---

[8] [69, Proposition 11.5.14] tells us that this semi-norm is fact a genuine norm if $\Omega\subseteq\mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set

defined as

$$\begin{gathered}\big[K_\lambda^{\mathrm{mod}}\big][\vec{f}] := \big[K_\lambda^{\mathrm{mod}}\vec{f}\big] \in \Big[\Big(\dot{L}_1^p(\partial\Omega,\sigma)\cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma)\Big)\Big/\sim\Big]^n \\ \textit{for each function } \vec{f}\in\big[\dot{L}_1^p(\partial\Omega,\sigma)\cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma)\big]^n\end{gathered} \tag{6.2.218}$$

is also a well-defined linear and bounded operator, when the quotient spaces are equipped with the natural semi-norm introduced in [69, (11.5.138)]. *In addition,*

$$\begin{gathered}\textit{the identity recorded in } (6.2.207) \textit{ is true for each} \\ \textit{function } \vec{f}\in\big[\dot{L}_1^p(\partial\Omega,\sigma)\cap L^p_{\mathrm{loc}}(\partial\Omega,\sigma)\big]^n.\end{gathered} \tag{6.2.219}$$

(12) Impose the stronger assumption that Ω *is an open set in* $\mathbb{R}^n$ *satisfying a two-sided local John condition and whose boundary is Ahlfors regular*[9]*. Then the operator*

$$K_\lambda^{\mathrm{mod}} : \big[\dot{L}_1^p(\partial\Omega,\sigma)\big]^n \longrightarrow \big[\dot{L}_1^p(\partial\Omega,\sigma)\big]^n \tag{6.2.220}$$

is well defined, linear, and bounded, when the spaces involved are endowed with the semi-norm (A.0.128). *As a corollary of* (6.2.220) *and* (6.2.194)*, the following is a well-defined linear and bounded*[10] *operator:*

$$\begin{gathered}\big[K_\lambda^{\mathrm{mod}}\big] : \big[\dot{L}_1^p(\partial\Omega,\sigma)/\sim\big]^n \longrightarrow \big[\dot{L}_1^p(\partial\Omega,\sigma)/\sim\big]^n \ \textit{ defined as} \\ \big[K_\lambda^{\mathrm{mod}}\big][\vec{f}\,] := \big[K_\lambda^{\mathrm{mod}}\vec{f}\big] \in \big[\dot{L}_1^p(\partial\Omega,\sigma)/\sim\big]^n, \quad \forall\,\vec{f}\in\big[\dot{L}_1^p(\partial\Omega,\sigma)\big]^n.\end{gathered} \tag{6.2.221}$$

Furthermore,

$$\begin{gathered}\textit{the identity recorded in } (6.2.207) \textit{ holds for each} \\ \textit{vector-valued function } \vec{f}\in\big[\dot{L}_1^p(\partial\Omega,\sigma)\big]^n.\end{gathered} \tag{6.2.222}$$

(13) Strengthen the original hypotheses by now assuming that Ω *is an* NTA *domain with an Ahlfors regular boundary*[11]*. Then the operator* K_λ^{mod} *is still well defined, linear, bounded in the context of* (6.2.220)*, and* (6.2.221) *continues to be a well-defined linear and bounded operator.*

(14) Similar properties to those discussed in items (9)-(13) above hold for the modified layer potential operators for the Stokes system acting on Morrey-based and block-based Sobolev spaces.

Proof All claims may be justified employing the same ideas as in the proofs of Theorem 1.8.2, Proposition 1.8.8, Theorem 1.8.9, Theorem 1.8.12, and Theorem 1.8.14, now making use of the definitions given in (6.2.164), (6.2.167), (6.2.171), (6.2.172)

---

[9] in which scenario, Ω is known to be a UR domain; see [68, (5.11.27)]

[10] if $\partial\Omega$ is unbounded then the natural semi-norm introduced in [69, (11.5.138)] endowing the quotient spaces in (6.2.221) is actually a genuine norm and the homogeneous Sobolev spaces of order one, modulo constants, becomes Banach spaces; see [69, Proposition 11.5.14]

[11] a scenario in which Ω is known to be a UR domain; cf. [68, (5.11.5)]

and relying on the formalism associated with the Stokes system developed so far in this chapter. □

We shall now use the modified layer potential operators for the Stokes system to prove a basic integral representation result.

Theorem 6.2.7 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set with the property that $\partial\Omega$ is an Ahlfors regular set; in particular, Ω is a set of locally finite perimeter. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix an arbitrary number $\lambda \in \mathbb{C}$.*

In this setting, recall the modified version of the double layer operator $\mathcal{D}_\lambda^{\text{mod}}$ for the Stokes system acting on vector-valued functions from $\big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^n$ as in (6.2.171), and the modified version of the single layer operator $\mathcal{S}_{\text{mod}}$ for the Stokes system acting on vector-valued functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$ as in (6.2.164). Finally, fix an aperture parameter $\kappa \in (0,\infty)$, a truncation parameter $\varepsilon \in (0,\infty)$, and consider a pair of functions, $\vec{u} : \Omega \to \mathbb{C}^n$ and $\pi : \Omega \to \mathbb{C}$, satisfying

$$
\begin{gathered}
\vec{u} \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \quad \pi \in \mathscr{C}^\infty(\Omega), \\
\Delta\vec{u} - \nabla\pi = 0 \ \textit{and} \ \operatorname{div}\vec{u} = 0 \ \textit{in} \ \Omega, \\
\mathcal{N}_\kappa^\varepsilon \vec{u} \in L^1_{\text{loc}}(\partial\Omega, \sigma), \quad \mathcal{N}_\kappa(\nabla\vec{u}),\ \mathcal{N}_\kappa \pi \in L^1\big(\partial\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big), \\
\textit{the traces } \vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}, \quad (\nabla\vec{u})\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}, \quad \pi\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \textit{exist } \sigma\textit{-a.e. on } \partial_{\text{nta}}\Omega, \\
\textit{and the function } \vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \textit{belongs to the space} \ \Big[L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\Big]^n.
\end{gathered}
\tag{6.2.223}
$$

In the case when Ω is an exterior domain make the additional assumption that there exists $\mu \in (1,\infty)$ such that

$$
⨍_{B(0,\mu R)\setminus B(0,R)} \big\{|\nabla\vec{u}| + |\pi|\big\}\, d\mathcal{L}^n = o(1) \ \textit{as} \ R \to \infty. \tag{6.2.224}
$$

Then the conormal derivative $\partial_\nu^\lambda(\vec{u},\pi)$, extended to the entire topological boundary by setting it to be zero outside $\partial_*\Omega$, belongs to $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$, and the nontangential trace $\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ considered as a function on $\partial_*\Omega$ (cf. [68, (8.8.52)]) belongs to the space

$$
\Big\{\vec{f} \in \big[L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^n}\big)\big]^n : \partial_{\tau_{jk}}\vec{f} \in \big[L^1\big(\partial_*\Omega, \tfrac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n,\ 1 \leq j,k \leq n\Big\}. \tag{6.2.225}
$$

Furthermore, there exists some $\mathbb{C}^n$-valued locally constant function $c_{\vec{u},\pi}$ in Ω with the property that

$$
\vec{u} = \mathcal{D}_\lambda^{\text{mod}}\Big(\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \mathcal{S}_{\text{mod}}\big(\partial_\nu^\lambda(\vec{u},\pi)\big) + c_{\vec{u},\pi} \ \textit{in} \ \Omega, \tag{6.2.226}
$$

and

$$\pi(x) = (1+\lambda)\int_{\partial_*\Omega} q_j(x-y)\partial_{\tau_{kj}}\big(u_k\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)\,d\sigma(y)$$
$$-\mathcal{Q}\big(\partial_\nu^\lambda(\vec{u},\pi)\big)(x) \ \textit{at each point}\ x\in\Omega. \tag{6.2.227}$$

Proof Work under the assumption that the functions $\vec{u} = (u_j)_{1\leq j\leq n}$ and π are as in (6.2.223). The current hypotheses imply that $\mathcal{N}_\kappa^\varepsilon\vec{u}$, $\mathcal{N}_\kappa^\varepsilon(\nabla\vec{u}) \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$, so we may rely on (6.2.223) and [69, Proposition 11.3.2] to conclude that the function $\vec{f} := \vec{u}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ considered on $\partial_*\Omega$ (cf. [68, (8.8.52)]) belongs to the space $\big[L^1_{1,\mathrm{loc}}(\partial_*\Omega,\sigma)\big]^n$ and satisfies

$$\partial_{\tau_{jk}}\vec{f} = \nu_j\Big((\partial_k\vec{u})\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big) - \nu_k\Big((\partial_j\vec{u})\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)$$
$$\text{at } \sigma\text{-a.e. point on } \partial_*\Omega, \ \text{ for each } j,k\in\{1,\dots,n\}, \tag{6.2.228}$$

where $(\nu_1,\dots,\nu_n)$ are the scalar components of the geometric measure theoretic outward unit normal ν to Ω. In concert with (6.2.223) this also entails

$$\vec{f}\in\Big[L^1\Big(\partial_*\Omega,\,\frac{\sigma(x)}{1+|x|^n}\Big)\Big]^n \ \text{and}\ \partial_{\tau_{jk}}\vec{f}\in\Big[L^1\Big(\partial_*\Omega,\,\frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n$$
$$\text{for each pair of indices } j,k\in\{1,\dots,n\}. \tag{6.2.229}$$

To proceed, abbreviate (cf. (6.1.6))

$$\vec{g} := \partial_\nu^\lambda(\vec{u},\pi) = \big[\nabla\vec{u}+\lambda(\nabla\vec{u})^\top\big]\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\nu - \big(\pi\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)\nu \tag{6.2.230}$$

at σ-a.e. point on $\partial_*\Omega$, and note that (6.2.223) together with [68, (8.8.52), (8.9.8), (8.9.44)] ensure that

$$\vec{g}\in\Big[L^1\Big(\partial_*\Omega,\,\frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n. \tag{6.2.231}$$

Going further, extend $\vec{g}$ to the entire topological boundary by setting it to be zero outside $\partial_*\Omega$, and define

$$\vec{w} := \mathcal{D}_\lambda^{\mathrm{mod}}\vec{f} - \mathcal{S}_{\mathrm{mod}}\vec{g}\in\big[\mathscr{C}^\infty(\Omega)\big]^n. \tag{6.2.232}$$

Also, denote by $(w_\gamma)_{1\leq\gamma\leq n}$ the scalar components of $\vec{w}$. Finally, recall from (6.2.1) the Kelvin matrix-valued fundamental solution $E = \big(E_{jk}\big)_{1\leq j,k\leq n}$ for the Stokes system in $\mathbb{R}^n$, and from (6.2.2) the pressure vector $\vec{q} = (q_j)_{1\leq j\leq n}$. Then for each index $r\in\{1,\dots,n\}$, each index $\gamma\in\{1,\dots,n\}$, and each point $x\in\Omega$ we may compute

$$
\begin{aligned}
&(\partial_r w_\gamma)(x)\\
&= \partial_r \big(\mathcal{D}_\lambda^{\text{mod}} \vec{f}\big)_\gamma(x) - \partial_r \big(\mathcal{S}_{\text{mod}} \vec{g}\big)_\gamma(x)\\
&= \int_{\partial_*\Omega} (\partial_k E_{j\gamma})(x-y)\Big\{\nu_r(y)\big((\partial_k u_j)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(y) - \nu_k(y)\big((\partial_r u_j)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(y)\Big\}\, d\sigma(y)\\
&+ \lambda \int_{\partial_*\Omega} (\partial_j E_{k\gamma})(x-y)\Big\{\nu_r(y)\big((\partial_k u_j)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(y) - \nu_k(y)\big((\partial_r u_j)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(y)\Big\}\, d\sigma(y)\\
&+ \int_{\partial_*\Omega} q_\gamma(x-y)\Big\{\nu_j(y)\big((\partial_r u_j)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(y) - \nu_r(y)\big((\partial_j u_j)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(y)\Big\}\, d\sigma(y)\\
&- \int_{\partial_*\Omega} (\partial_r E_{\gamma\alpha})(x-y)\big(\partial_\nu^\lambda(\vec{u},\pi)\big)_\alpha(y)\, d\sigma(y)\\
&= (\partial_r u_\gamma)(x).
\end{aligned}
\tag{6.2.233}
$$

The first equality above comes from (6.2.232), the second equality uses (6.2.180)-(6.2.181), (6.2.229), (6.2.165), (6.2.231), the third equality utilizes (6.2.228), and the last equality is provided by (6.2.73) (bearing in mind that $(\partial_j u_j)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = 0$ since $\vec{u}$ is divergence-free). From (6.2.233) we then conclude that $\nabla\vec{w} = \nabla\vec{u}$ in Ω, which shows that the difference $c_{\vec{u},\pi} := \vec{u} - \vec{w}$ is a $\mathbb{C}^n$-valued locally constant function in Ω. This concludes the proof of (6.2.226). Finally, (6.2.227) is a consequence of (6.2.74), the fact that $\vec{u}$ is divergence-free plus (6.2.228), and the definition made in (6.2.14). □

We are now prepared to augment the results in item *(xi)* of Theorem 6.2.4 by proving some remarkable composition identities involving the modified layer potentials for the Stokes system.

Theorem 6.2.8 *Pick $n \in \mathbb{N}$ with $n \geq 2$ and assume $\Omega \subseteq \mathbb{R}^n$ is a* UR *domain. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Select a number $\lambda \in \mathbb{C}$ along with an integrability exponent $p \in (1,\infty)$, and recall the modified boundary-to-boundary layer potential operators for the Stokes system: S_{mod} from* (6.2.208)*, $\partial_\nu^\lambda\big(\mathcal{D}_\lambda^{\text{mod}}, \mathcal{P}_\lambda\big)$ from* (6.2.214)*, and K_λ^{mod} from item (11) of Theorem* 6.2.6 *(cf. also* (6.2.172)*). Finally, let $K_\lambda^{\#}$ be the operator associated with the parameter λ and the set Ω as in* (6.2.98)*. Then the following statements are valid.*

(1) Given any vector-valued function $\vec{f} \in \big[L^p(\partial\Omega,\sigma)\big]^n$, at σ-a.e. point on $\partial\Omega$ one has

$$
\big(\tfrac{1}{2}I + K_\lambda^{\#}\big)\Big(\big(-\tfrac{1}{2}I + K_\lambda^{\#}\big)\vec{f}\Big) = \Big(\partial_\nu^\lambda\big(\mathcal{D}_\lambda^{\text{mod}}, \mathcal{P}_\lambda\big)\Big)\big(S_{\text{mod}}\vec{f}\big) \tag{6.2.234}
$$

and there exists $c_{\vec{f}}$, which is the nontangential trace on $\partial\Omega$ of some $\mathbb{C}^n$-valued locally constant function in Ω, such that

$$S_{\rm mod}\big(K^{\#}_{\lambda}\vec{f}\big) = K^{\rm mod}_{\lambda}\big(S_{\rm mod}\vec{f}\big) + c_{\vec{f}} \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{6.2.235}$$

(2) *Make the additional hypothesis that Ω satisfies a local John condition, and recall the operator $K^{\rm mod}_{\lambda}$ from (6.2.216). Then, for each vector-valued function $\vec{f} \in \big[\dot{L}^p_1(\partial\Omega,\sigma) \cap L^p_{\rm loc}(\partial\Omega,\sigma)\big]^n$, at σ-a.e. point on $\partial\Omega$ one has*

$$K^{\#}_{\lambda}\Big(\partial^{\lambda}_{\nu}\big(\mathcal{D}^{\rm mod}_{\lambda},\mathcal{P}_{\lambda}\big)\vec{f}\Big) = \Big(\partial^{\lambda}_{\nu}\big(\mathcal{D}^{\rm mod}_{\lambda},\mathcal{P}_{\lambda}\big)\Big)\big(K^{\rm mod}_{\lambda}\vec{f}\big) \tag{6.2.236}$$

and there exists $c_{\vec{f}}$, which is the nontangential trace on $\partial\Omega$ of some $\mathbb{C}^n$-valued locally constant function in Ω, with the property that at σ-a.e. point on $\partial\Omega$ one has

$$\big(\tfrac{1}{2}I + K^{\rm mod}_{\lambda}\big)\Big(\big(-\tfrac{1}{2}I + K^{\rm mod}_{\lambda}\big)\vec{f}\Big) = S_{\rm mod}\Big(\partial^{\lambda}_{\nu}\big(\mathcal{D}^{\rm mod}_{\lambda},\mathcal{P}_{\lambda}\big)\vec{f}\Big) + c_{\vec{f}}. \tag{6.2.237}$$

(3) *Strengthen the original geometric assumptions by now asking that $\Omega \subseteq \mathbb{R}^n$ is an* NTA *domain with an upper Ahlfors regular boundary*[12]. *Then, with the operator $K^{\rm mod}_{\lambda}$ now interpreted as in (6.2.220), both formula (6.2.236) as well as formula (6.2.237) actually hold for each vector-valued function $\vec{f} \in \big[\dot{L}^p_1(\partial\Omega,\sigma)\big]^n$.*

Proof All claims are established by reasoning as in the proof of Theorem 1.8.26, now making use of the integral representation formulas from Theorem 6.2.7 and the functional analytic properties of the modified boundary layer potential operators for the Stokes system from Theorem 6.2.6. □

Shifting perspectives, we shall now define the Stokes weak conormal derivative a distribution given by a suitable bullet product. For each $\lambda \in \mathbb{C}$, recall the system L_λ from (6.1.1)-(6.1.2).

Definition 6.2.9 *Fix $\lambda \in \mathbb{C}$ and let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary open set. Consider a vector-valued function $\vec{u} = (u_\alpha)_{1\leq\alpha\leq n}$ along with a complex-valued function π satisfying (with all derivatives taken in the sense of distributions in Ω)*

$$\begin{gathered}\vec{u} \in \big[L^1_{\rm loc}(\Omega,\mathcal{L}^n)\big]^n, \quad \nabla\vec{u} \in \big[L^1_{\rm bdd}(\Omega,\mathcal{L}^n)\big]^{n\times n},\\ \pi \in L^1_{\rm bdd}(\Omega,\mathcal{L}^n) \text{ and } L_\lambda\vec{u} - \nabla\pi \in \big[L^1_{\rm bdd}(\Omega,\mathcal{L}^n)\big]^n.\end{gathered} \tag{6.2.238}$$

Introduce the family of vector fields

$$\vec{F}^{\alpha} := \nabla u_\alpha + \lambda\,\partial_\alpha\vec{u} - \pi\,\mathbf{e}_\alpha \text{ for each } \alpha \in \{1,\dots,n\}. \tag{6.2.239}$$

In this setting, define the `Stokes weak conormal derivative` *of the pair $\vec{u},\pi$ as the distribution*

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[12] in particular, this is the case if Ω is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular; cf. (1.8.157)

$$\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u},\pi) := \big(\nu \bullet \vec{F}^{\alpha}\big)_{1\leq\alpha\leq n} \in \Big[\big(\mathrm{Lip}_c(\partial\Omega)\big)'\Big]^n. \tag{6.2.240}$$

Since (6.2.238) implies that for each $\alpha \in \{1, \dots, n\}$ we have

$$\vec{F}^{\alpha} \in \big[L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)\big]^n \text{ and } \mathrm{div}\vec{F}^{\alpha} = \big(L_\lambda\vec{u} - \nabla\pi\big)_\alpha \in L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n), \tag{6.2.241}$$

from [68, Proposition 4.2.3] we see that $\big(\nu \bullet \vec{F}^{\alpha}\big)_{1\leq\alpha\leq n} \in \Big[\big(\mathrm{Lip}_c(\partial\Omega)\big)'\Big]^n$. Thus, Definition 6.2.9 is meaningful.

Our next theorem concerns finer integrability and compatibility properties of the Stokes weak conormal derivative introduced in Definition 6.2.9.

Theorem 6.2.10 *Fix $n \in \mathbb{N}$ with $n \geq 2$, and suppose $\Omega \subseteq \mathbb{R}^n$ is an open set with an Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\nu = (\nu_j)_{1\leq j\leq n}$ the geometric measure theoretic outward unit normal to Ω. Also, fix some $\lambda \in \mathbb{C}$ and define the system L_λ as in (6.1.1)-(6.1.2). Finally, consider a $\mathbb{C}^n$-valued function $\vec{u} = (u_\alpha)_{1\leq\alpha\leq n}$ along with a complex-valued function π which is $\mathcal{L}^n$-measurable in Ω satisfying, for some integrability exponents $p \in \big(\tfrac{n-1}{n}, \infty\big)$, $q \in (0, \infty]$, and some aperture parameter $\kappa > 0$,*

$$\begin{gathered} \vec{u} \text{ belongs to the local Sobolev space } \big[W^{1,1}_{\mathrm{loc}}(\Omega)\big]^n, \\ L_\lambda\vec{u} - \nabla\pi \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^n, \quad \mathfrak{P}\big(L_\lambda\vec{u} - \nabla\pi\big) \in L^{p,q}(\partial\Omega, \sigma), \\ \mathcal{N}_\kappa(\nabla\vec{u}) \in L^{p,q}(\partial\Omega, \sigma) \text{ and } \mathcal{N}_\kappa\pi \in L^{p,q}(\partial\Omega, \sigma). \end{gathered} \tag{6.2.242}$$

Then the following statements are true.

(1) There exists a constant $C(\Omega, \lambda, \kappa, p, q) \in (0, \infty)$ with the property that the Stokes weak conormal derivative of the pair $(\vec{u}, \pi)$, taken in the sense of Definition 6.2.9, satisfies

$$\begin{aligned} &\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u},\pi) \text{ belongs to the Lorentz-based Hardy space } \big[H^{p,q}(\partial\Omega, \sigma)\big]^n \text{ and} \\ &\big\|\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u},\pi)\big\|_{[H^{p,q}(\partial\Omega,\sigma)]^n} \leq C\big\|\mathcal{N}_\kappa(\nabla\vec{u})\big\|_{L^{p,q}(\partial\Omega,\sigma)} + C\big\|\mathcal{N}_\kappa\pi\big\|_{L^{p,q}(\partial\Omega,\sigma)} \\ &\qquad\qquad + C\big\|\mathfrak{P}\big(L_\lambda\vec{u} - \nabla\pi\big)\big\|_{L^{p,q}(\partial\Omega,\sigma)}. \end{aligned} \tag{6.2.243}$$

(2) Whenever

$$\begin{gathered} \Omega \text{ is actually an Ahlfors regular domain and, in addition to (6.2.242), one assumes that } \mathcal{N}_\kappa\pi,\ \mathcal{N}_\kappa(\nabla\vec{u}) \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma) \text{ and} \\ \text{the nontangential pointwise traces } \pi\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ and } (\nabla\vec{u})\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exist at } \sigma\text{-a.e. point on } \partial\Omega, \end{gathered} \tag{6.2.244}$$

it follows that the Stokes weak conormal derivative of the pair $(\vec{u}, \pi)$ agrees with the distribution associated (as in [68, Proposition 4.1.4]*) with the conormal derivative of $(\vec{u}, \pi)$ taken in a pointwise sense as in (6.1.6), i.e.,*

$$\begin{aligned}\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u},\pi) &= \Big(\nu_j a^{\alpha\beta}_{jk}(\lambda)(\partial_k u_\beta)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - \nu_\alpha \pi\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)_{1\leq\alpha\leq n}\\ &= \Big(\nu_j(\partial_j u_\alpha)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} + \lambda\,\nu_j(\partial_\alpha u_j)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - \nu_\alpha\pi\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)_{1\leq\alpha\leq n}\\ &= \Big[\nabla\vec{u} + \lambda(\nabla\vec{u})^{\top}\Big]\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\nu - \big(\pi\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\nu \ \ \textit{in}\ \Big[\big(\mathrm{Lip}_c(\partial\Omega)\big)'\Big]^n. \end{aligned}\tag{6.2.245}$$

(3) If, in fact,

$$\begin{array}{l}\Omega \textit{ is actually an Ahlfors regular domain, the conditions in } (6.2.242) \textit{ are assumed to hold with } p\in[1,\infty) \textit{ and } q\in(0,p],\\ \textit{and one also asks that the nontangential traces } \pi\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \textit{ and } (\nabla\vec{u})\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \textit{ exist } \sigma\textit{-a.e. on } \partial\Omega,\end{array}\tag{6.2.246}$$

then the Stokes weak conormal derivative $\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u},\pi)\in\big[H^{p,q}(\partial\Omega,\sigma)\big]^n$ *from* (6.2.243) *actually belongs to the Lebesgue space* $\big[L^p(\partial\Omega,\sigma)\big]^n$ *and one has the pointwise formula*

$$\begin{aligned}\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u},\pi) &= \Big(\nu_j(\partial_j u_\alpha)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} + \lambda\,\nu_j(\partial_\alpha u_j)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - \nu_\alpha\pi\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)_{1\leq\alpha\leq n}\\ &= \Big[\nabla\vec{u} + \lambda(\nabla\vec{u})^{\top}\Big]\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\nu - \big(\pi\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\nu\end{aligned}\tag{6.2.247}$$

at σ-a.e. point on $\partial\Omega$.

Proof Given $\vec{u},\pi$ as in (6.2.242) for some $p\in\big(\frac{n-1}{n},\infty\big)$, $q\in(0,\infty]$, we have

$$\begin{aligned}\nabla\vec{u}\in\big[L^1_{\mathrm{bdd}}(\Omega,\mathcal{L}^n)\big]^{n\times n},\quad \pi\in L^1_{\mathrm{bdd}}(\Omega,\mathcal{L}^n),\\ \text{and } L_\lambda\vec{u}-\nabla\pi\in\big[L^1_{\mathrm{bdd}}(\Omega,\mathcal{L}^n)\big]^n,\end{aligned}\tag{6.2.248}$$

thanks to [68, (8.6.51) in Proposition 8.6.3] and [69, (10.1.4)]. Also, each of the vector fields $\vec{F}^\alpha$ defined as in (6.2.239) belongs to $\big[L^1_{\mathrm{bdd}}(\Omega,\mathcal{L}^n)\big]^n$ and satisfies

$$\begin{aligned}\mathcal{N}_\kappa\vec{F}^\alpha\in L^{p,q}(\partial\Omega,\sigma),\quad \mathrm{div}\vec{F}^\alpha=\big(L_\lambda\vec{u}-\nabla\pi\big)_\alpha\in L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n),\\ \text{and } \mathfrak{P}\big(\mathrm{div}\vec{F}^\alpha\big)=\mathfrak{P}\Big(\big(L_\lambda\vec{u}-\nabla\pi\big)_\alpha\Big)\in L^{p,q}(\partial\Omega,\sigma).\end{aligned}\tag{6.2.249}$$

As such, [69, Theorem 10.2.1] applies and from [69, (10.2.4)-(10.2.5)] we conclude that (6.2.243) holds. Next, the claim in item *(2)* is implied by (6.2.240) and the compatibility property established in [69, Proposition 10.2.9]. In turn, the claim in item *(3)* is a consequence of *(2)* in view of the fact that the present conditions on p,q entail $L^{p,q}(\partial\Omega,\sigma)\subseteq L^p(\partial\Omega,\sigma)\subseteq L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$; cf. [68, (6.2.25)-(6.2.26)]. □

We augment Theorem 6.2.4 with results pertaining to the action of boundary layer potential operators associated with the Stokes system on Hardy spaces.

Theorem 6.2.11 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be a* UR *domain and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix some $\lambda \in \mathbb{C}$ and consider the boundary layer potential operators $\mathcal{S}$, $\mathcal{Q}$, $\mathcal{D}_\lambda$, $\mathcal{P}_\lambda$, K_λ, $K_\lambda^\#$ associated with Ω as in* (6.2.12), (6.2.14), (6.2.17), (6.2.21), (6.2.95), (6.2.98). *Also, recall the modified versions $\mathcal{D}_\lambda^{\text{mod}}$ from* (6.2.171), *and K_λ^{mod} from* (6.2.172). *Then the following properties hold.*

(1) The operator $K_\lambda^\#$, originally acting on Lebesgue spaces on $\partial\Omega$ (as in (6.2.106)), *extends uniquely to a linear and bounded operator*

$$K_\lambda^\# : \big[H^p(\partial\Omega,\sigma)\big]^n \longrightarrow \big[H^p(\partial\Omega,\sigma)\big]^n, \qquad p \in \big(\tfrac{n-1}{n}, 1\big]. \tag{6.2.250}$$

Moreover, various choices of p yield operators which are compatible with one another.

(2) More generally, for each $p \in \big(\frac{n-1}{n}, \infty\big)$ and $q \in (0,\infty]$, the operator $K_\lambda^\#$, originally acting on Lebesgue spaces on $\partial\Omega$ (as in (6.2.106)), *induces a linear and bounded mapping*

$$K_\lambda^\# : \big[H^{p,q}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[H^{p,q}(\partial\Omega,\sigma)\big]^n \tag{6.2.251}$$

whose action continues to be compatible with that of $K_\lambda^\#$ in (6.2.106).

(3) Given $p \in \big(\frac{n-1}{n}, 1\big]$, there exists $C \in (0,\infty)$ with the property that for every $\vec{f} \in [H^p(\partial\Omega,\sigma)]^n$ one has

$$\begin{gathered} \mathcal{S}\vec{f} \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \quad \mathcal{Q}\vec{f} \in \mathscr{C}^\infty(\Omega), \\ \Delta\big(\mathcal{S}\vec{f}\big) - \nabla\mathcal{Q}\vec{f} = 0 \ \ \textit{and} \ \ \operatorname{div}\mathcal{S}\vec{f} = 0 \ \ \textit{in} \ \ \Omega, \end{gathered} \tag{6.2.252}$$

as well as

$$\big\|\mathcal{N}_\kappa(\nabla\mathcal{S}\vec{f})\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\mathcal{Q}\vec{f})\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|\vec{f}\|_{[H^p(\partial\Omega,\sigma)]^n}. \tag{6.2.253}$$

Also, for every $\vec{f} \in [H^p(\partial\Omega,\sigma)]^n$ with $p \in \big(\frac{n-1}{n}, 1\big]$ the following jump-formula holds:

$$\overset{\bullet}{\partial}{}_\nu^\lambda\big(\mathcal{S}\vec{f},\mathcal{Q}\vec{f}\big) = \big(-\tfrac{1}{2}I + K_\lambda^\#\big)\vec{f} \ \ \textit{in} \ \ \big[H^p(\partial\Omega,\sigma)\big]^n. \tag{6.2.254}$$

(4) For each $p \in \big(\frac{n-1}{n}, 1\big]$ and $q \in [1,\infty)$ there exists some finite constant $C > 0$, depending only on $\partial\Omega$, λ, n, κ, and p, such that for each $\vec{f} \in \big[H_1^{q,p}(\partial\Omega,\sigma)\big]^n$ one has

$$\big\|\mathcal{N}_\kappa(\mathcal{P}_\lambda\vec{f})\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\nabla\mathcal{D}_\lambda\vec{f})\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|\vec{f}\|_{[\dot{H}_1^p(\partial\Omega,\sigma)]^n}. \tag{6.2.255}$$

(5) For each $p \in \big(\frac{n-1}{n}, 1\big]$ and $q \in [1,\infty)$, the operator $\partial_\nu^\lambda\big(\mathcal{D}_\lambda,\mathcal{P}_\lambda\big)$ from (6.2.119) *extends to a bounded linear mapping*

$$\dot{\partial}_\nu^\lambda(\mathcal{D}_\lambda, \mathcal{P}_\lambda) : \big[H_1^{q,p}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[H^p(\partial\Omega,\sigma)\big]^n. \tag{6.2.256}$$

Furthermore, for each $p \in \left(\frac{n-1}{n}, 1\right]$ *the following operator is well defined, linear, and bounded:*

$$\begin{aligned} &\dot{\partial}_\nu^\lambda(\mathcal{D}_\lambda^{\text{mod}}, \mathcal{P}_\lambda) : \big[\dot{H}_1^p(\partial\Omega,\sigma)\big]^n \longrightarrow \big[H^p(\partial\Omega,\sigma)\big]^n \text{ defined as} \\ &\dot{\partial}_\nu^\lambda(\mathcal{D}_\lambda^{\text{mod}}, \mathcal{P}_\lambda)\vec{f} := \dot{\partial}_\nu^\lambda\big(\mathcal{D}_\lambda^{\text{mod}}\vec{f}, \mathcal{P}_\lambda\vec{f}\big) \text{ for each } \vec{f} \in \big[\dot{H}_1^p(\partial\Omega,\sigma)\big]^n, \end{aligned} \tag{6.2.257}$$

where the weak conormal derivative is considered in the sense of Definition 6.2.9. As a consequence, the operator

$$\begin{aligned} &\Big[\dot{\partial}_\nu^\lambda(\mathcal{D}_\lambda^{\text{mod}}, \mathcal{P}_\lambda)\Big] : \big[\dot{H}_1^p(\partial\Omega,\sigma)/\sim\big]^n \to \big[H^p(\partial\Omega,\sigma)\big]^n \text{ given by} \\ &\Big[\dot{\partial}_\nu^\lambda(\mathcal{D}_\lambda^{\text{mod}}, \mathcal{P}_\lambda)\Big][\vec{f}\,] := \dot{\partial}_\nu^\lambda\big(\mathcal{D}_\lambda^{\text{mod}}\vec{f}, \mathcal{P}_\lambda\vec{f}\big) \text{ for each } \vec{f} \in \big[\dot{H}_1^p(\partial\Omega,\sigma)\big]^n, \end{aligned} \tag{6.2.258}$$

is well defined, linear, and bounded, when the quotient space is equipped with the semi-quasinorm[13] *introduced in* (A.0.92)*.*

(6) *If* $p \in \left(\frac{n-1}{n}, 1\right]$ *and* $q \in (1,\infty)$ *then the operator* K_λ*, originally acting on boundary Sobolev spaces as in* (6.2.115)*, extends to a linear and bounded mapping*

$$K_\lambda : \big[H_1^{q,p}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[H_1^{q,p}(\partial\Omega,\sigma)\big]^n. \tag{6.2.259}$$

In addition, if $\Omega \subseteq \mathbb{R}^n$ *is an* NTA *domain with an upper Ahlfors regular boundary*[14]*, it follows that for each* $p \in \left(\frac{n-1}{n}, 1\right]$ *the modified boundary-to-boundary double layer potential operator* K_λ^{mod} *induces a linear and bounded mapping*

$$K_\lambda^{\text{mod}} : \big[\dot{H}_1^p(\partial\Omega,\sigma)\big]^n \longrightarrow \big[\dot{H}_1^p(\partial\Omega,\sigma)\big]^n. \tag{6.2.260}$$

As a consequence of this and (6.2.194)*, the operator*

$$\begin{aligned} &\big[K_\lambda^{\text{mod}}\big] : \big[\dot{H}_1^p(\partial\Omega,\sigma)/\sim\big]^n \longrightarrow \big[\dot{H}_1^p(\partial\Omega,\sigma)/\sim\big]^n \text{ defined as} \\ &\big[K_\lambda^{\text{mod}}\big][\vec{f}\,] := \big[K_\lambda^{\text{mod}}\vec{f}\,\big] \in \big[\dot{H}_1^p(\partial\Omega,\sigma)/\sim\big]^n, \quad \forall\,\vec{f} \in \big[\dot{H}_1^p(\partial\Omega,\sigma)\big]^n, \end{aligned} \tag{6.2.261}$$

[13] if in fact $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then Proposition 2.3.8 guarantees that said semi-quasinorm becomes a genuine quasinorm, making $\dot{H}_1^p(\partial\Omega,\sigma)/\sim$ a quasi-Banach space

[14] in particular, this is the case if Ω is an open set satisfying a two-sided local John condition and whose boundary is Ahlfors regular; cf. (1.8.157)

is well defined, linear, and bounded, when all quotient spaces are equipped with the semi-quasinorm[15] *introduced in* (A.0.92).

(7) *The boundary-to-boundary version of the single layer potential operator S for the Stokes system, defined in* (6.2.112)*, satisfies properties analogous to those for single layers associated with generic weakly elliptic second-order systems, described in Theorem* 2.2.6*. In addition, if $p \in \big(\frac{n-1}{n}, n-1\big)$ and $p^* \in (1,\infty)$ is such that $1/p^* = 1/p - 1/(n-1)$, then the operator S from* (6.2.112) *induces a linear and bounded mapping in the context*

$$S : \big[H^p(\partial\Omega,\sigma)\big]^n \longrightarrow \big[H_1^{p^*,p}(\partial\Omega,\sigma)\big]^n, \tag{6.2.262}$$

and, corresponding to the case when $p = 1$ in the two-dimensional setting, the following operator is well defined, linear, and bounded:

$$S : \big[H^1(\partial\Omega,\sigma)\big]^2 \longrightarrow \big[L^\infty(\partial\Omega,\sigma)\cap \mathscr{C}^0(\partial\Omega)\big]^2 \ \text{ if } n = 2. \tag{6.2.263}$$

Also, if $p \in \big(\frac{n-1}{n}, 1\big]$ then the boundary-to-boundary single layer potential operator S for the Stokes system induces a linear and bounded mapping

$$S : \big[H^p(\partial\Omega,\sigma)\big]^n \longrightarrow \big[\dot{H}_1^p(\partial\Omega,\sigma)\big]^n. \tag{6.2.264}$$

As a consequence, for each $p \in \big(\frac{n-1}{n}, 1\big]$ the following operator is well-defined, linear, and bounded:

$$\begin{gathered}[S] : \big[H^p(\partial\Omega,\sigma)\big]^n \longrightarrow \big[\dot{H}_1^p(\partial\Omega,\sigma)/\sim\big]^n \ \text{ defined as} \\ [S]\vec{f} := [S\vec{f}\,] \in \big[\dot{H}_1^p(\partial\Omega,\sigma)/\sim\big]^n, \quad \forall\,\vec{f} \in \big[H^p(\partial\Omega,\sigma)\big]^n,\end{gathered} \tag{6.2.265}$$

when the quotient space is equipped with the semi-quasinorm introduced in (A.0.92).

(8) *Recall the Kelvin matrix-valued fundamental solution $E = \big(E_{jk}\big)_{1\le j,k\le n}$ of the Stokes system in $\mathbb{R}^n$, and the accompanying pressure vector $\vec{q} = (q_j)_{1\le j\le n}$, from* (6.2.1)-(6.2.2)*. Given $p \in \big(\frac{n-1}{n}, 1\big]$, for each $\vec{f} = (f_j)_{1\le j\le n} \in \big[\dot{H}_1^p(\partial\Omega,\sigma)\big]^n$, each $\gamma, r \in \{1,\dots,n\}$, and each $x \in \Omega$ one has* (*using the summation convention*)

$$\begin{aligned}\partial_r\big(\mathcal{D}_\lambda^{\text{mod}}\vec{f}\big)_\gamma(x) = &\Big\langle \big[(\partial_k E_{j\gamma})(x-\cdot)\big|_{\partial\Omega}\big], \partial_{\tau_{rk}} f_j\Big\rangle \\ &+ \lambda\Big\langle \big[(\partial_j E_{k\gamma})(x-\cdot)\big|_{\partial\Omega}\big], \partial_{\tau_{rk}} f_j\Big\rangle \\ &+ \Big\langle \big[q_\gamma(x-\cdot)\big|_{\partial\Omega}\big], \partial_{\tau_{jr}} f_j\Big\rangle\end{aligned} \tag{6.2.266}$$

[15] Proposition 2.3.8 tells us that if $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying a two-sided local John condition and whose boundary is an unbounded Ahlfors regular set, then said semi-quasinorm is actually a genuine quasinorm, and $\dot{H}_1^p(\partial\Omega,\sigma)/\sim$ becomes a quasi-Banach space

and

$$\mathcal{P}_\lambda \vec{f}(x) = (1+\lambda)\Big\langle \big[q_j(x-\cdot)\big|_{\partial\Omega}\big], \partial_{\tau_{kj}} f_k \Big\rangle \tag{6.2.267}$$

if $\partial\Omega$ is unbounded, plus similar formulas without equivalence classes of functions modulo constants if $\partial\Omega$ is bounded, with the pairings in (6.2.266)-(6.2.267) *understood in the sense of* [69, Theorem 4.6.1] (*keeping in mind that all tangential derivatives belong to the Hardy space $H^p(\partial\Omega,\sigma)$*). *If some aperture parameter $\kappa > 0$ has also been fixed then, as a consequence of* (6.2.266)-(6.2.267), (6.2.1)-(6.2.2), *and* [70, Corollary 2.5.4],

$$\begin{array}{l}\textit{for each given } \vec{f} \in \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^n, \textit{ the nontangential} \\ \textit{boundary traces } (\nabla \mathcal{D}^{\rm mod}_\lambda \vec{f})\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \textit{ and } (\mathcal{P}_\lambda \vec{f})\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \textit{ exist (in} \\ \mathbb{C}^{n\cdot n} \textit{ and } \mathbb{C}^n\textit{, respectively) } \sigma\textit{-a.e. on the set } \partial\Omega. \end{array} \tag{6.2.268}$$

From (6.2.266)-(6.2.267), (6.2.1)-(6.2.2), *and item (3) in* [70, Theorem 2.4.1], *one also sees that there exists a finite constant $C > 0$, depending only on $\partial\Omega$, n, κ, and p, such that for each $\vec{f} \in \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^n$ one has*

$$\big\|\mathcal{N}_\kappa(\nabla \mathcal{D}^{\rm mod}_\lambda \vec{f})\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa(\mathcal{P}_\lambda \vec{f})\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|\vec{f}\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^n}. \tag{6.2.269}$$

Proof To deal with item *(1)*, fix $p \in \big(\frac{n-1}{n}, 1\big]$ and $q \in (1,\infty)$. As in the case of Theorem 2.1.1, the gist of the proof is to show that if $\vec{a} : \partial\Omega \to \mathbb{C}^n$ is an arbitrary (p,q)-atom with vanishing moment,

$$\int_{\partial\Omega} \vec{a}\, d\sigma = 0, \tag{6.2.270}$$

then $\vec{m} := K^\#_\lambda \vec{a}$ is a fixed multiple of a molecule for the Hardy space $\big[H^p(\partial\Omega,\sigma)\big]^n$ (cf. [69, Definition 4.5.1]). The estimates in [69, (4.5.1)-(4.5.2)] may be established by closely mimicking the arguments used in the proof (2.1.9) and (2.1.12). To justify the vanishing moment condition

$$\int_{\partial\Omega} \vec{m}\, d\sigma = 0, \tag{6.2.271}$$

observe that

$$\vec{m} = K^\#_\lambda \vec{a} = \partial^\lambda_\nu(\mathscr{S}\vec{a}, \mathcal{Q}\vec{a}) + \tfrac{1}{2}\vec{a}. \tag{6.2.272}$$

Thanks to this and (6.2.270), we see that (6.2.271) follows as soon as we establish that

$$\int_{\partial\Omega} \partial^\lambda_\nu(\mathscr{S}\vec{a}, \mathcal{Q}\vec{a})\, d\sigma = 0. \tag{6.2.273}$$

To this end, recall that

$$\partial_\nu^\lambda(\mathcal{S}\vec{a},\mathcal{Q}\vec{a}) \tag{6.2.274}$$
$$= \Big(\nu_j(\partial_j(\mathcal{S}\vec{a})_\alpha)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} + \lambda\,\nu_j(\partial_\alpha(\mathcal{S}\vec{a})_j)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - \nu_\alpha(\mathcal{Q}\vec{a})\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)_{1\le\alpha\le n}.$$

To proceed, fix $\alpha \in \{1,\dots,n\}$ and consider the vector field

$$\vec{F} := \partial_j(\mathcal{S}\vec{a})_\alpha\mathbf{e}_j + \lambda\,\partial_\alpha(\mathcal{S}\vec{a})_j\mathbf{e}_j - (\mathcal{Q}\vec{a})\mathbf{e}_\alpha \in \big[\mathscr{C}^\infty(\Omega)\big]^n. \tag{6.2.275}$$

Then (6.2.275) and (6.2.15) permit us to compute

$$\begin{aligned}\mathrm{div}\vec{F} &= \partial_j\partial_j(\mathcal{S}\vec{a})_\alpha + \lambda\,\partial_\alpha\partial_j(\mathcal{S}\vec{a})_j - \partial_\alpha(\mathcal{Q}\vec{a})\\ &= \Big(\Delta\mathcal{S}\vec{a} + \lambda\,\mathrm{div}\mathcal{S}\vec{a} - \nabla\mathcal{Q}\vec{a}\Big)_\alpha = 0 \ \text{ in } \ \Omega.\end{aligned} \tag{6.2.276}$$

Also, having fixed a background aperture parameter $\kappa > 0$, [70, Theorem 2.5.1] implies that

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\Omega \tag{6.2.277}$$

and, in fact, the design of $\vec{F}$ is such that

$$\nu\cdot\Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big) = \Big(\partial_\nu^\lambda(\mathcal{S}\vec{a},\mathcal{Q}\vec{a})\Big)_\alpha \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{6.2.278}$$

Since the components of the function $a : \partial\Omega \to \mathbb{C}^n$ are multiples of $(1,q)$-atoms on $\partial\Omega$, we may invoke [70, (2.4.14)] with $p = 1$ to also conclude that

$$\mathcal{N}_\kappa\vec{F} \in L^1(\partial\Omega,\sigma). \tag{6.2.279}$$

Finally, the vanishing moment property of the atom together with (6.2.275) and (6.2.1)-(6.2.2) imply that

$$\begin{aligned}&\text{in the case when } \Omega \text{ is an exterior domain we have}\\ &\vec{F}(x) = O(|x|^{-n}) \text{ as } x\in\Omega \text{ satisfies } |x|\to\infty; \text{ as such,}\\ &\text{condition [68, (1.2.9)] is currently satisfied.}\end{aligned} \tag{6.2.280}$$

Collectively, (6.2.275), (6.2.276), (6.2.277), (6.2.279), (6.2.280), guarantee the validity of the Divergence Formula [68, (1.2.2)] which, in light of (6.2.278), (2.1.20), and the arbitrariness of $\alpha \in \{1,\dots,n\}$, presently gives (6.2.273). This establishes (6.2.271) which, in concert with [69, Theorem 4.4.7], ultimately implies that $K_\lambda^\#$ extends uniquely a linear and bounded operator in the context of (6.2.250). Finally, [69, Theorem 4.4.3] ensures that various choices of $p \in \big(\frac{n-1}{n}, 1\big]$ in (6.2.250) yield operators which are compatible with one another. This concludes the treatment of item *(1)*. Next, the claims in item *(2)* are consequences of what we have just proved and real interpolation (cf. [69, Theorem 4.3.1]).

Turning attention to item *(3)*, the properties claimed in (6.2.252) follows from Lemma 2.2.1 and definitions. Next, the estimate in (6.2.253) is seen directly from

(6.2.1)-(6.2.2) and [70, (2.4.14)]. As regards the jump-formula (6.2.254), we first observe that, thanks to (6.2.243), (6.2.15), and (6.2.253),

$$\big[H^p(\partial\Omega,\sigma)\big]^n \ni \vec{f} \longmapsto \dot{\partial}_\nu^{\lambda}(\mathscr{S}\vec{f}, \mathcal{Q}\vec{f}\ ) \in \big[H^p(\partial\Omega,\sigma)\big]^n \tag{6.2.281}$$

is a well-defined, linear, and bounded assignment. Moreover, from item *(3)* in Theorem 6.2.10, (6.2.113), and (6.1.6) we know that this assignment agrees with $-\frac{1}{2}I+K^{\#}_{\lambda}$ when acting on arbitrary $\mathbb{C}^n$-valued (p,q)-atoms on $\partial\Omega$ (with $q \in (1,\infty)$). Since from the current item *(1)* we also know that $-\frac{1}{2}I+K^{\#}_{\lambda}$ is a well-defined linear and bounded operator on $\big[H^p(\partial\Omega,\sigma)\big]^n$, the jump-formula (6.2.254) now follows via a standard density argument (based on [69, (4.4.114)]). The claims in items *(4)-(7)* are justified by reasoning much as in the proof of Theorem 2.3.1, making use of Theorem 6.2.4 and Theorem 6.2.10 (also reasoning as in the proof of Theorem 2.2.6 to justify (6.2.262) and (6.2.263)).

As regards item *(8)*, formula (6.2.266) follows from identity (6.2.91) and [69, Lemma 11.10.4] (currently applied with φ one of the entries in $(\nabla E)(x-\cdot)$), while formula (6.2.267) is implied by (6.2.93) and [69, Lemma 11.10.4] (now used with φ one of the entries in $(\nabla\vec{q})(x-\cdot)$). □

6.3 Other Integral Representations and Fatou-Type Results for the Stokes System

We debut by presenting integral representation formulas for derivatives of the components of the velocity function as well as for the pressure function, working solely with integrability properties of the nontangential maximal operator, *without* making any assumptions on the existence of nontangential pointwise boundary traces for said functions.

Theorem 6.3.1 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, be an Ahlfors regular domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix some aperture parameter $\kappa > 0$. Assume the pair consisting of $\vec{u} = (u_j)_{1\leq j\leq n} \in \big[\mathscr{C}^\infty(\Omega)\big]^n$ and $\pi \in \mathscr{C}^\infty(\Omega)$ is a null-solution of the Stokes system in Ω, i.e.,*

$$\Delta\vec{u} - \nabla\pi = 0 \ \ in\ \ \Omega, \quad \operatorname{div}\vec{u} = 0 \ \ in\ \ \Omega, \tag{6.3.1}$$

which also satisfies

$$\mathcal{N}_\kappa(\nabla\vec{u}) \in L^p(\partial\Omega,\sigma) \ \ and\ \ \mathcal{N}_\kappa\pi \in L^p(\partial\Omega,\sigma) \ \ for\ some\ \ p \in \big(\tfrac{n-1}{n},\infty\big). \tag{6.3.2}$$

Finally, recall the Kelvin matrix-valued fundamental solution $E = \big(E_{jk}\big)_{1\leq j,k\leq n}$ of the Stokes system in $\mathbb{R}^n$, and the accompanying pressure vector $\vec{q} = (q_j)_{1\leq j\leq n}$, from (6.2.1)-(6.2.2).

Then for each $\lambda \in \mathbb{C}$ and each $\ell, s, j \in \{1,\dots,n\}$ one has

$$\dot{\partial}_{\tau_{\ell s}} u_j \in H^p(\partial\Omega, \sigma) \text{ and } \dot{\partial}_\nu^\lambda(\vec{u}, \pi) \in \big[H^p(\partial\Omega, \sigma)\big]^n. \tag{6.3.3}$$

Moreover, given any $r, \gamma \in \{1, \dots, n\}$ *and* $\lambda \in \mathbb{C}$, *one has* (*with the duality pairings understood in the sense of* [69, Theorem 4.6.1] *with* $\Sigma := \partial\Omega$, *and the summation convention over repeated indices in effect*)

$$\begin{aligned}(\partial_r u_\gamma)(x) &= \Big\langle \Big[(\partial_k E_{j\gamma})(x - \cdot)\big|_{\partial\Omega}\Big], \dot{\partial}_{\tau_{rk}} u_j \Big\rangle \\ &\quad + \lambda \Big\langle \Big[(\partial_j E_{k\gamma})(x - \cdot)\big|_{\partial\Omega}\Big], \dot{\partial}_{\tau_{rk}} u_j \Big\rangle \\ &\quad + \Big\langle \big[q_\gamma(x - \cdot)\big|_{\partial\Omega}\big], \dot{\partial}_{\tau_{jr}} u_j \Big\rangle \\ &\quad - \Big\langle \Big[(\partial_r E_{\gamma\alpha})(x - \cdot)\big|_{\partial\Omega}\Big], \big(\dot{\partial}_\nu^\lambda(\vec{u}, \pi)\big)_\alpha \Big\rangle, \quad \text{for all } x \in \Omega,\end{aligned} \tag{6.3.4}$$

and

$$\begin{aligned}\pi(x) &= (1 + \lambda)\Big\langle \big[q_j(x - \cdot)\big|_{\partial\Omega}\big], \dot{\partial}_{\tau_{kj}} u_k \Big\rangle \\ &\quad - \Big\langle \big[q_\alpha(x - \cdot)\big|_{\partial\Omega}\big], \big(\dot{\partial}_\nu^\lambda(\vec{u}, \pi)\big)_\alpha \Big\rangle, \quad \text{for all } x \in \Omega,\end{aligned} \tag{6.3.5}$$

assuming that $p \in \big(\frac{n-1}{n}, 1\big]$ *and* $\partial\Omega$ *is unbounded. Similar formulas hold when* $p \in \big(\frac{n-1}{n}, 1\big]$ *and* Ω *is bounded, this time omitting taking equivalence classes of functions modulo constants in the duality pairings in* (6.3.4)-(6.3.5). *In the case when* $p \in \big(\frac{n-1}{n}, 1\big]$ *and* Ω *is an exterior domain, these formula continue to be valid under the additional assumption that there exists some* $\mu \in (1, \infty)$ *such that*

$$⨍_{B(0,\mu R)\setminus B(0,R)} \big\{|\nabla u| + |\pi|\big\}\, \mathrm{d}\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{6.3.6}$$

Finally, similar integral representation formulas hold when $p \in (1, \infty)$, *this time interpreting all duality pairings as integration on* $\partial\Omega$ *with respect to* σ, *i.e.*,

$$\begin{aligned}(\partial_r u_\gamma)(x) &= \int_{\partial\Omega} (\partial_k E_{j\gamma})(x - y)\big(\dot{\partial}_{\tau_{rk}} u_j\big)(y)\, \mathrm{d}\sigma(y) \\ &\quad + \lambda \int_{\partial\Omega} (\partial_j E_{k\gamma})(x - y)\big(\dot{\partial}_{\tau_{rk}} u_j\big)(y)\, \mathrm{d}\sigma(y) \\ &\quad + \int_{\partial\Omega} q_\gamma(x - y)\big(\dot{\partial}_{\tau_{jr}} u_j\big)(y)\, \mathrm{d}\sigma(y) \\ &\quad - \int_{\partial\Omega} (\partial_r E_{\gamma\alpha})(x - y)\big(\dot{\partial}_\nu^\lambda(\vec{u}, \pi)\big)_\alpha(y)\, \mathrm{d}\sigma(y), \quad \text{for all } x \in \Omega,\end{aligned} \tag{6.3.7}$$

and

$$\pi(x) = (1+\lambda)\int_{\partial\Omega} q_j(x-y)\big(\overset{\bullet}{\partial}_{\tau_{kj}} u_k\big)(y)\,\mathrm{d}\sigma(y) - \int_{\partial\Omega} q_\alpha(x-y)\big(\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u},\pi)\big)_\alpha(y)\,\mathrm{d}\sigma(y), \quad \textit{for all } x\in\Omega, \tag{6.3.8}$$

with the same caveat as before when Ω *is an exterior domain.*

Proof The claims in (6.3.3) are consequences of assumptions, [69, (10.2.14)], and Theorem 6.2.10. To prove the integral representation formula (6.3.4), fix some arbitrary $r,\gamma\in\{1,\dots,n\}$, $\lambda\in\mathbb{C}$, and $x\in\Omega$. The strategy is to use the version of the Divergence Formula established in [68, Theorem 1.9.4] to the vector field define at $\mathcal{L}^n$-a.e. point in Ω as

$$\begin{aligned}\vec{F} := &\Big\{(\partial_k E_{j\gamma})(x-\cdot) + \lambda(\partial_j E_{k\gamma})(x-\cdot)\Big\}\big\{(\partial_k u_j)\mathbf{e}_r - (\partial_r u_j)\mathbf{e}_k\big\}\\ &+ q_\gamma(x-\cdot)(\partial_r u_j)\mathbf{e}_j\\ &- (\partial_r E_{\gamma\alpha})(x-\cdot)\Big\{(\partial_j u_\alpha)\mathbf{e}_j + \lambda\,(\partial_\alpha u_j)\mathbf{e}_j - \pi\mathbf{e}_\alpha\Big\}.\end{aligned} \tag{6.3.9}$$

Upon recalling (6.2.1)-(6.2.2), we clearly have $\vec{F}\in\big[L^1_{\rm bdd}(\Omega,\mathcal{L}^n)\big]^n$. With the divergence considered in the sense of distributions in Ω, we also have

$$\begin{aligned}\mathrm{div}\vec{F} = &-\Big\{\partial_r\partial_k[E_{j\gamma}(x-\cdot)] + \lambda\partial_r\partial_j[E_{k\gamma}(x-\cdot)]\Big\}(\partial_k u_j)\\ &+\Big\{(\partial_k E_{j\gamma})(x-\cdot) + \lambda(\partial_j E_{k\gamma})(x-\cdot)\Big\}(\partial_r\partial_k u_j)\\ &+\Big\{\partial_k\partial_k[E_{j\gamma}(x-\cdot)] + \lambda\partial_k\partial_j[E_{k\gamma}(x-\cdot)]\Big\}(\partial_r u_j)\\ &-\Big\{(\partial_k E_{j\gamma})(x-\cdot) + \lambda(\partial_j E_{k\gamma})(x-\cdot)\Big\}(\partial_k\partial_r u_j)\\ &-(\partial_j q_\gamma)(x-\cdot)(\partial_r u_j) + q_\gamma(x-\cdot)(\partial_j\partial_r u_j)\\ &+\partial_j\partial_r[E_{\gamma\alpha}(x-\cdot)]\Big\{(\partial_j u_\alpha) + \lambda\,(\partial_\alpha u_j)\Big\}\\ &-(\partial_r E_{\gamma\alpha})(x-\cdot)\Big\{(\partial_j\partial_j u_\alpha) + \lambda\,(\partial_j\partial_\alpha u_j)\Big\}\\ &-\partial_\alpha\partial_r[E_{\gamma\alpha}(x-\cdot)]\pi + (\partial_r E_{\gamma\alpha})(x-\cdot)\partial_\alpha\pi.\end{aligned} \tag{6.3.10}$$

Using (6.2.6), the first line and the sixth line cancel. The second line and fourth line also cancel. In the third line, write $\partial_k\partial_k[E_{j\gamma}(x-\cdot)] = \Delta[E_{j\gamma}(x-\cdot)]$ and use the fact that $\lambda\partial_k\partial_j[E_{k\gamma}(x-\cdot)] = 0$ (cf. (6.2.5)). In the fifth line, we have $\partial_j\partial_r u_j = 0$. In the seventh line, write $\partial_j\partial_j u_\alpha = \Delta u_\alpha$ and note that $\lambda\,(\partial_j\partial_\alpha u_j) = \lambda\,\partial_\alpha(\mathrm{div}\vec{u}) = 0$. Finally, in the eighth line $\partial_\alpha\partial_r[E_{\gamma\alpha}(x-\cdot)] = 0$ (cf. (6.2.5)). Implementing these observations leads to

$$\begin{aligned}\operatorname{div}\vec{F} &= \Big\{(\Delta E_{j\gamma})(x-\cdot)] - (\partial_j q_\gamma)(x-\cdot)\Big\}(\partial_r u_j)\\ &\quad - (\partial_r E_{\gamma\alpha})(x-\cdot)\Big\{\Delta u_\alpha - \partial_\alpha \pi\Big\}\\ &= \delta_{j\gamma}\delta_x(\partial_r u_j) = (\partial_r u_\gamma)(x)\delta_x, \end{aligned} \tag{6.3.11}$$

thanks to (6.2.4) and (6.3.1). In particular, $\operatorname{div}\vec{F} \in \mathscr{E}'(\Omega)$, so condition [68, (1.9.29)] is presently satisfied.

Next, with μ as in (6.3.6) if Ω is an exterior domain and $\mu := 2$ otherwise, pick a function $\phi \in \mathscr{C}_c^\infty\big(B(0,\mu)\big)$ with the property that $\phi \equiv 1$ on $B(0,1)$. Then the family

$$\mathscr{F} := \{\phi_R\}_{R>0}, \text{ where } \phi_R := \phi(\cdot/R) \text{ for each } R \in (0,\infty), \tag{6.3.12}$$

becomes a system of auxiliary functions (in the sense of [68, (1.3.3)]). From [68, (1.9.30)] we know that

$$[\vec{F}]_{\mathscr{F}} = -\lim_{R\to\infty} R^{-1}\int_\Omega (\nabla\phi)(y/R)\cdot\vec{F}(y)\,\mathrm{d}\mathcal{L}^n(y) \tag{6.3.13}$$

Let us assume for the moment that either Ω is bounded, or $\partial\Omega$ is unbounded. We may then rely on (6.2.1)-(6.2.2), [68, Proposition 8.6.3], (6.3.2), and Hölder's inequality to estimate, for R large,

$$\begin{aligned} R^{-1}\int_\Omega &|(\nabla\phi)(y/R)|\big\{|(\nabla E)(x-y)| + |\vec{q}(x-y)|\big\}\big\{|(\nabla\vec{u})(y)| + |\pi(y)|\big\}\,\mathrm{d}\mathcal{L}^n(y)\\ &\le CR^{-n}\int_{\Omega\cap[B(0,\mu R)\setminus B(0,R)]}\big\{|\nabla\vec{u}| + |\pi|\big\}\,\mathrm{d}\mathcal{L}^n\\ &\le CR^{-n}\Big(\int_\Omega\big\{|\nabla\vec{u}| + |\pi|\big\}^{\frac{np}{n-1}}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{np}}\big(R^n\big)^{1-\frac{n-1}{np}}\\ &\le CR^{-\frac{n-1}{p}}\Big\{\|\mathcal{N}_\kappa(\nabla\vec{u})\|_{L^p(\partial\Omega,\sigma)} + \|\mathcal{N}_\kappa\pi\|_{L^p(\partial\Omega,\sigma)}\Big\}\\ &= o(1) \text{ as } R\to\infty. \end{aligned} \tag{6.3.14}$$

From (6.3.9), (6.3.13), and (6.3.14) we then conclude that

$$\text{if either } \Omega \text{ is bounded, or } \partial\Omega \text{ is unbounded, then } [\vec{F}]_{\mathscr{F}} = 0. \tag{6.3.15}$$

In the case when Ω is an exterior domain, thanks to (6.3.6) we may directly estimate

$$\begin{aligned} R^{-1}\int_{\Omega} |(\nabla\phi)(y/R)|\big\{|(\nabla E)(x-y)|+|q(x-y)|\big\}\big\{|(\nabla\vec{u})(y)|+|\pi(y)|\big\}\,d\mathcal{L}^n(y) \\ \leq C\fint_{B(0,\mu R)\setminus B(0,R)}\big\{|\nabla\vec{u}|+|\pi|\big\}\,d\mathcal{L}^n \\ = o(1) \ \text{ as } \ R\to\infty. \end{aligned} \tag{6.3.16}$$

Thus,

$$\text{if } \Omega \text{ is an exterior domain then } [\vec{F}]_{\mathscr{F}} = 0. \tag{6.3.17}$$

Collectively, (6.3.15) and (6.3.17) show that in all cases the limit in [68, (1.9.30)] written for $\vec{F}$ exists, and is actually zero.

To proceed, assume first that $p \in \big(\frac{n-1}{n}, 1\big]$ and $\partial\Omega$ is unbounded. Choose a function $\eta \in \mathscr{C}^\infty_c(\Omega)$ satisfying $\eta \equiv 1$ near x. Granted [70, (3.3.118), (3.3.119)], [68, Definition 4.2.6] permits us to define $\nu \bullet \vec{F}$ as

$$\nu\bullet\vec{F} = \nu\bullet\big((1-\eta)\vec{F}\big) \ \text{ in } \ \big(\mathrm{Lip}_c(\partial\Omega)\big)' \tag{6.3.18}$$

where $\nu\bullet\big((1-\eta)\vec{F}\big)$ is now interpreted in the sense of [68, Proposition 4.2.3] (bearing in mind that both the components of the vector field $(1-\eta)\vec{F}$ and its divergence are absolutely integrable on arbitrary bounded measurable subsets of Ω). Hence, with the piece of notation introduced in [68, (1.9.31)], we have

$$\begin{aligned} \big(\nu\bullet\vec{F},1\big)_{\mathscr{F}} &= \Big(\nu\bullet\big((1-\eta)\vec{F}\big),1\Big)_{\mathscr{F}} \\ &= \lim_{R\to\infty} {}_{(\mathrm{Lip}_c(\partial\Omega))'}\Big\langle \nu\bullet\big((1-\eta)\vec{F}\big), \phi_R\big|_{\partial\Omega}\Big\rangle_{\mathrm{Lip}_c(\partial\Omega)} \\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}, \end{aligned} \tag{6.3.19}$$

where

$$\mathrm{I} := \lim_{R\to\infty} {}_{(\mathrm{Lip}_c(\partial\Omega))'}\Big\langle \nu \bullet \Big((1-\eta)(\partial_k E_{j\gamma})(x-\cdot)\Big)\big((\partial_k u_j)\mathbf{e}_r - (\partial_r u_j)\mathbf{e}_k\big), \phi_R\big|_{\partial\Omega}\Big\rangle_{\mathrm{Lip}_c(\partial\Omega)}$$

$$\mathrm{II} := \lim_{R\to\infty} {}_{(\mathrm{Lip}_c(\partial\Omega))'}\Big\langle \nu \bullet \Big(\lambda(1-\eta)(\partial_j E_{k\gamma})(x-\cdot)\Big)\big((\partial_k u_j)\mathbf{e}_r - (\partial_r u_j)\mathbf{e}_k\big), \phi_R\big|_{\partial\Omega}\Big\rangle_{\mathrm{Lip}_c(\partial\Omega)}$$

$$\mathrm{III} := \lim_{R\to\infty} {}_{(\mathrm{Lip}_c(\partial\Omega))'}\Big\langle \nu \bullet \Big((1-\eta)(\partial_r E_{\gamma\alpha})(x-\cdot)\big((\partial_j u_\alpha + \lambda\,\partial_\alpha u_j)\mathbf{e}_j - \pi\mathbf{e}_\alpha\big)\Big), \phi_R\big|_{\partial\Omega}\Big\rangle_{\mathrm{Lip}_c(\partial\Omega)}$$

$$\mathrm{IV} := \lim_{R\to\infty} {}_{(\mathrm{Lip}_c(\partial\Omega))'}\Big\langle \nu \bullet \Big((1-\eta)q_\gamma(x-\cdot)(\partial_r u_j)\mathbf{e}_j\Big), \phi_R\big|_{\partial\Omega}\Big\rangle_{\mathrm{Lip}_c(\partial\Omega)}. \tag{6.3.20}$$

As regards the first term above, we may write

$$\begin{aligned}
\mathrm{I} &= \lim_{R\to\infty} {}_{(\mathrm{Lip}_c(\partial\Omega))'}\Big\langle \Big((1-\eta)(\partial_k E_{j\gamma})(x-\cdot)\Big)\Big|_{\partial\Omega} \nu \bullet \big((\partial_k u_j)\mathbf{e}_r - (\partial_r u_j)\mathbf{e}_k\big), \phi_R\big|_{\partial\Omega}\Big\rangle_{\mathrm{Lip}_c(\partial\Omega)} \\
&= \lim_{R\to\infty} {}_{(\mathrm{Lip}_c(\partial\Omega))'}\Big\langle \big((\partial_k E_{j\gamma})(x-\cdot)\big)\big|_{\partial\Omega} \dot{\partial}_{\tau_{rk}} u_j, \phi_R\big|_{\partial\Omega}\Big\rangle_{\mathrm{Lip}_c(\partial\Omega)} \\
&= \lim_{R\to\infty} {}_{(\mathrm{Lip}_c(\partial\Omega))'}\Big\langle \dot{\partial}_{\tau_{rk}} u_j, \big(\phi_R(\partial_k E_{j\gamma})(x-\cdot)\big)\big|_{\partial\Omega}\Big\rangle_{\mathrm{Lip}_c(\partial\Omega)} \\
&= \lim_{R\to\infty} {}_{H^p(\partial\Omega,\sigma)}\Big\langle \dot{\partial}_{\tau_{rk}} u_j, \Big[\big(\phi_R(\partial_k E_{j\gamma})(x-\cdot)\big)\big|_{\partial\Omega}\Big]\Big\rangle_{(H^p(\partial\Omega,\sigma))^*} \\
&= {}_{H^p(\partial\Omega,\sigma)}\Big\langle \dot{\partial}_{\tau_{rk}} u_j, \Big[(\partial_k E_{j\gamma})(x-\cdot)\big|_{\partial\Omega}\Big]\Big\rangle_{(H^p(\partial\Omega,\sigma))^*}.
\end{aligned} \tag{6.3.21}$$

In (6.3.21), the first equality uses [68, (4.2.14)]. The second equality relies on two things, namely the observation that $1-\eta = 1$ on $\partial\Omega$, and the definition of the weak tangential derivative from [68, Example 4.2.4] (cf. (A.0.175)-(A.0.176) in the Glossary). The third equality is implied by [68, (4.1.43)]. The fourth equality follows on account of (6.3.3) and [69, Lemma 4.6.4] ((2.2.2) is also helpful in this regard). Finally, the last equality in (6.3.21) is a consequence of the fact that for each $j,k \in \{1,\dots,n\}$ we have

$$\lim_{R\to\infty}\Big[\big((\partial_k E_{j\gamma})(x-\cdot)\phi_R\big)\big|_{\partial\Omega}\Big] = \Big[(\partial_k E_{j\gamma})(x-\cdot)\big|_{\partial\Omega}\Big] \text{ weak-}* \text{ in}$$

$$\big(H^p(\partial\Omega,\sigma)\big)^* = \begin{cases} \dot{\mathscr{C}}^{(n-1)(1/p-1)}(\partial\Omega)\big/\sim & \text{if } p<1, \\ \widetilde{\mathrm{BMO}}(\partial\Omega,\sigma) & \text{if } p=1. \end{cases} \tag{6.3.22}$$

In turn, this is implied by the general weak-$*$ convergence results established in [69, Lemma 4.8.4] (here (2.2.2) helps) and, respectively, [69, Lemma 4.8.1] (also bearing in mind the trivial bounded embedding $L^\infty(\partial\Omega,\sigma)\hookrightarrow \mathrm{BMO}(\partial\Omega,\sigma)$ in the latter case). This finishes the justification of (6.3.21).

Going further, in a similar fashion we also obtain

$$\mathrm{II} = {}_{H^p(\partial\Omega,\sigma)}\Big\langle \dot{\partial}_{\tau_{rk}} u_j, \Big[\lambda(\partial_j E_{k\gamma})(x-\cdot)\big|_{\partial\Omega}\Big]\Big\rangle_{(H^p(\partial\Omega,\sigma))^*} \tag{6.3.23}$$

and

$$\mathrm{III} = {}_{H^p(\partial\Omega,\sigma)}\Big\langle \big(\dot{\partial}^{\lambda}_{\nu}(\vec{u},\pi)\big)_\alpha, \Big[(\partial_r E_{\gamma\alpha})(x-\cdot)\big|_{\partial\Omega}\Big]\Big\rangle_{(H^p(\partial\Omega,\sigma))^*}. \tag{6.3.24}$$

In addition, upon noting that $(\partial_r u_j)\mathbf{e}_j = (\partial_r u_j)\mathbf{e}_j - (\partial_j u_j)\mathbf{e}_r$, the same type of argument gives

$$\mathrm{IV} = {}_{H^p(\partial\Omega,\sigma)}\Big\langle \dot{\partial}_{\tau_{jr}} u_j, \big[q_\gamma(x-\cdot)\big|_{\partial\Omega}\big]\Big\rangle_{(H^p(\partial\Omega,\sigma))^*}. \tag{6.3.25}$$

In fact, similar arguments may be employed to prove (6.3.21), (6.3.23), (6.3.24), (6.3.25) in the case when $p\in\big(\frac{n-1}{n},1\big]$ and $\partial\Omega$ is bounded, a scenario in which the dual of H^p and the corresponding duality bracket no longer involve classes of functions modulo constants. Thus, in all cases, if $p\in\big(\frac{n-1}{n},1\big]$ then from (6.3.19), (6.3.21), and (6.3.23)-(6.3.25) we obtain

$$\begin{aligned}\big(\nu\bullet\vec{F},1\big)_{\mathscr{F}} &= \Big\langle\Big[(\partial_k E_{j\gamma})(x-\cdot)\big|_{\partial\Omega}\Big], \dot{\partial}_{\tau_{rk}} u_j\Big\rangle \\ &\quad + \Big\langle\Big[\lambda(\partial_j E_{k\gamma})(x-\cdot)\big|_{\partial\Omega}\Big], \dot{\partial}_{\tau_{rk}} u_j\Big\rangle \\ &\quad + \Big\langle\big[q_\gamma(x-\cdot)\big|_{\partial\Omega}\big], \dot{\partial}_{\tau_{jr}} u_j\Big\rangle \\ &\quad + \Big\langle\Big[(\partial_r E_{\gamma\alpha})(x-\cdot)\big|_{\partial\Omega}\Big], \big(\dot{\partial}^{\lambda}_{\nu}(\vec{u},\pi)\big)_\alpha\Big\rangle \end{aligned} \tag{6.3.26}$$

where all brackets in the right-hand side refer to the duality pairing from [69, Theorem 4.6.1].

At this stage, if $p\in\big(\frac{n-1}{n},1\big]$ then [68, Theorem 1.9.4] applies and, on account of (6.3.11), (6.3.15), (6.3.17), and (6.3.26), the Divergence Formula [68, (1.9.32)] presently yields (6.3.4). Finally, the case when $p\in(1,\infty)$ is handled analogously, keeping in mind the compatibility of the duality pairing with the ordinary integral pairing on $\partial\Omega$, and using Lebesgue's Dominated convergence Theorem in place of

(6.3.22). In such a scenario we arrive at the integral representation formula claimed in (6.3.7).

To finish the proof of Theorem 6.3.1, there remains to deal with the integral representations for the pressure, claimed in (6.3.5) and (6.3.8). This time, the idea is to apply the version of the Divergence Formula from [68, Theorem 1.9.4] to the vector field

$$\vec{G} := (1+\lambda)q_j(x-\cdot)(\partial_j u_k)\mathbf{e}_k - q_\alpha(x-\cdot)\Big\{(\partial_j u_\alpha)\mathbf{e}_j + \lambda\,(\partial_\alpha u_j)\mathbf{e}_j - \pi\mathbf{e}_\alpha\Big\}, \tag{6.3.27}$$

where $x \in \Omega$ is an arbitrary, fixed point. In this regard, retaining the system of auxiliary functions $\mathscr{F}$ from (6.3.12), by reasoning much as in (6.3.14), (6.3.16) we arrive at the conclusion that

$$[\vec{G}]_{\mathscr{F}} = -\lim_{R\to\infty} R^{-1}\int_\Omega (\nabla\phi)(y/R)\cdot\vec{G}(y)\,\mathrm{d}\mathcal{L}^n(y) = 0. \tag{6.3.28}$$

Also,

$$\begin{aligned}
\mathrm{div}\vec{G} &= (1+\lambda)\partial_k[q_j(x-\cdot)](\partial_j u_k) + (1+\lambda)q_j(x-\cdot)(\partial_k\partial_j u_k)\\
&\quad - \partial_j[q_\alpha(x-\cdot)]\Big\{(\partial_j u_\alpha) + \lambda\,(\partial_\alpha u_j)\Big\} - q_\alpha(x-\cdot)\Big\{(\partial_j\partial_j u_\alpha) + \lambda\,(\partial_j\partial_\alpha u_j)\Big\}\\
&\quad + \partial_\alpha[q_\alpha(x-\cdot)]\pi + q_\alpha(x-\cdot)\partial_\alpha\pi\\
&= \pi(x)\delta_x,
\end{aligned} \tag{6.3.29}$$

where the last equality makes use of (6.2.6)-(6.2.7). Moreover, if $p \in \big(\tfrac{n-1}{n}, 1\big]$ and $\partial\Omega$ is unbounded then the same type of argument as in (6.3.19)-(6.3.26) now gives

$$\begin{aligned}
\big(\nu \bullet \vec{G}, 1\big)_{\mathscr{F}} &= (1+\lambda)\Big\langle \big[q_j(x-\cdot)\big|_{\partial\Omega}\big], \dot{\partial}_{\tau_{kj}} u_k\Big\rangle\\
&\quad - \Big\langle \big[q_\alpha(x-\cdot)\big|_{\partial\Omega}\big], \big(\dot{\partial}^{\lambda}_{\nu}(\vec{u},\pi)\big)_\alpha\Big\rangle
\end{aligned} \tag{6.3.30}$$

where all brackets in the right-hand side refer to the duality pairing from [69, Theorem 4.6.1]. In fact, similar formulas valid are valid when $\partial\Omega$ is bounded, or $p \in (1,\infty)$. Thanks to them, (6.3.28), and (6.3.29), the Divergence Formula [68, (1.9.32)] now yields (6.3.5) if $p \in \big(\tfrac{n-1}{n}, 1\big]$ and $\partial\Omega$ is unbounded (plus a similar version when $\partial\Omega$ is bounded), as well as (6.3.8) when $p \in (1,\infty)$. □

In turn, Theorem 6.3.1 is one of the key ingredients in the proof of our first Fatou-type result for null-solutions of the Stokes system in arbitrary UR domains, presented below.

Theorem 6.3.2 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Assume the functions $\vec{u} \in \big[\mathscr{C}^\infty(\Omega)\big]^n$ and $\pi \in \mathscr{C}^\infty(\Omega)$ satisfy, for some aperture parameter $\kappa > 0$,*

$$\Delta\vec{u} - \nabla\pi = 0 \text{ in } \Omega, \quad \operatorname{div}\vec{u} = 0 \text{ in } \Omega, \text{ and}$$
$$\mathcal{N}_\kappa(\nabla\vec{u}),\ \mathcal{N}_\kappa\pi \in L^p(\partial\Omega,\sigma) \text{ for some } p \in \big(\tfrac{n-1}{n},\infty\big). \tag{6.3.31}$$

Then

$$\text{the nontangential pointwise traces } \nabla\vec{u}\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ and } \pi\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ exist, in } \mathbb{C}^{n\times n} \text{ and } \mathbb{C}, \text{ respectively, at } \sigma\text{-a.e. point on } \partial\Omega. \tag{6.3.32}$$

In addition, these traces are independent of the aperture parameter $\kappa \in (0,\infty)$, *the function* $(\nabla\vec{u})\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ *belongs to* $\big[L^p(\partial\Omega,\sigma)\big]^{n\times n}$, *the function* $\pi\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ *belongs to the space* $L^p(\partial\Omega,\sigma)$, *and*

$$\Big\|(\nabla\vec{u})\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big\|_{[L^p(\partial\Omega,\sigma)]^{n\times n}} \leq \big\|\mathcal{N}_\kappa(\nabla\vec{u})\big\|_{L^p(\partial\Omega,\sigma)}, \tag{6.3.33}$$

$$\Big\|\pi\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big\|_{L^p(\partial\Omega,\sigma)} \leq \big\|\mathcal{N}_\kappa\pi\big\|_{L^p(\partial\Omega,\sigma)}. \tag{6.3.34}$$

Furthermore, having fixed some arbitrary $\lambda \in \mathbb{C}$,

$$\text{if } p \in \big(\tfrac{n-1}{n},1\big] \text{ then } \dot{\partial}^{\lambda}_{\nu}(\vec{u},\pi) \in \big[H^p(\partial\Omega,\sigma)\big]^n \text{ and}$$
$$\dot{\partial}_{\tau_{jk}}\vec{u} \in \big[H^p(\partial\Omega,\sigma)\big]^n \text{ for each } j,k \in \{1,\dots,n\}, \tag{6.3.35}$$

and the following H^p/L^p*-boundary maximum principle holds:*

$$\big\|\mathcal{N}_\kappa(\nabla\vec{u})\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa\pi\big\|_{L^p(\partial\Omega,\sigma)} \tag{6.3.36}$$
$$\approx \big\|\dot{\partial}^{\lambda}_{\nu}(\vec{u},\pi)\big\|_{[H^p(\partial\Omega,\sigma)]^n} + \sum_{j,k=1}^{n} \big\|\dot{\partial}_{\tau_{jk}}\vec{u}\big\|_{[H^p(\partial\Omega,\sigma)]^n} \text{ if } p \in \big(\tfrac{n-1}{n},1\big]$$

and

$$\big\|\mathcal{N}_\kappa(\nabla\vec{u})\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa\pi\big\|_{L^p(\partial\Omega,\sigma)} \tag{6.3.37}$$
$$\approx \Big\|(\nabla\vec{u})\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big\|_{[L^p(\partial\Omega,\sigma)]^{n\times n}} + \Big\|\pi\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big\|_{L^p(\partial\Omega,\sigma)} \text{ if } p \in (1,\infty)$$

provided, in the case when Ω *is an exterior domain, it is also assumed that there exists some number* $\mu \in (1,\infty)$ *such that*

$$\fint_{B(0,\mu R)\setminus B(0,R)} \big\{|\nabla u| + |\pi|\big\}\,d\mathcal{L}^n = o(1) \text{ as } R \to \infty. \tag{6.3.38}$$

Proof From (6.3.4)-(6.3.5) (plus their counterparts when Ω is bounded), as well as (6.3.7)-(6.3.8), on account of (6.3.3), [70, Theorem 2.5.1], and item *(6)* in Theorem 2.2.6 we conclude that

$$\begin{array}{l}\text{the nontangential pointwise traces } (\nabla\vec{u})\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ and } \pi\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exist at } \sigma\text{-a.e.}\\ \text{point on } \partial\Omega \text{ if either } \Omega \text{ is bounded, or } \partial\Omega \text{ is unbounded, or } \Omega \text{ is an}\\ \text{exterior domain and the decay condition (6.3.6) holds.}\end{array} \tag{6.3.39}$$

This Fatou-type property further self-improves. Specifically, in the case when Ω is an exterior domain, working with $\Omega_R := B(0, R) \cap \Omega$ in place of Ω (where $R \in (0, \infty)$ is a sufficiently large number), much as in the proof of [70, Theorem 3.1.6], eliminates the need of asking that the decay condition (6.3.6) holds. Simply put, (6.3.32) holds as stated. The integrability properties claimed next in the statement, along with the independence of the nontangential traces on the aperture parameter and the estimates in (6.3.33)-(6.3.34), follow from assumptions, [68, Proposition 8.9.8], [68, Corollary 8.9.6], [68, (8.9.8)], [68, (8.9.8)], and [68, Proposition 8.8.6].

Going further, (6.3.35) is implied by (6.3.3). The left-pointing inequality in (6.3.36) is a consequence of (6.3.4), (6.3.5), (6.3.3), [70, (2.4.14)], and (6.2.1)-(6.2.2). The right-pointing inequality in (6.3.36) comes from (6.2.243) and [69, (10.2.14)]. Next, the left-pointing inequality in (6.3.37) follows from (6.3.7), (6.3.8), and [70, (2.4.9)]. Finally, granted what we have proved already, the right-pointing inequality in (6.3.37) is seen from [68, (8.9.8)]. □

It is possible to further expand upon the approach taken to prove Theorem 6.3.1 as to allow membership of the nontangential maximal functions of null-solutions of the Stokes system to Generalized Banach Function Spaces in place of Lebesgue spaces. Here is a formal statement:

Theorem 6.3.3 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, be an Ahlfors regular domain. Abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Suppose $\mathbb{X}$ is a Generalized Banach Function Space on $(\partial\Omega, \sigma)$ with the property that*

$$\begin{array}{c}\mathcal{M}_{\partial\Omega} : \mathbb{X} \to \mathbb{X} \textit{ and } \mathcal{M}_{\partial\Omega} : \mathbb{X}' \to \mathbb{X}'\\ \textit{are well-defined bounded mappings,}\end{array} \tag{6.3.40}$$

where $\mathcal{M}_{\partial\Omega}$ is the Hardy-Littlewood maximal operator on $\partial\Omega$, and $\mathbb{X}'$ is the associated space of $\mathbb{X}$ (cf. [69, Definitions 5.1.4, 5.1.11]).

Assume the pair consisting of $\vec{u} = (u_j)_{1\leq j\leq n} \in \big[\mathscr{C}^\infty(\Omega)\big]^n$ and $\pi \in \mathscr{C}^\infty(\Omega)$ is a null-solution of the Stokes system in Ω, i.e.,

$$\Delta\vec{u} - \nabla\pi = 0 \textit{ in } \Omega, \quad \operatorname{div}\vec{u} = 0 \textit{ in } \Omega, \tag{6.3.41}$$

which also satisfies, for some aperture parameter $\kappa > 0$,

$$\mathcal{N}_\kappa(\nabla\vec{u}) \in \mathbb{X} \textit{ and } \mathcal{N}_\kappa\pi \in \mathbb{X}. \tag{6.3.42}$$

In the case when Ω is an exterior domain make the additional assumption that there exists some $\mu \in (1, \infty)$ such that

$$\fint_{B(0,\mu R)\setminus B(0,R)} \big\{|\nabla u| + |\pi|\big\}\, d\mathcal{L}^n = o(1) \ \ as \ \ R \to \infty. \tag{6.3.43}$$

Finally, recall the Kelvin matrix-valued fundamental solution $E = \big(E_{jk}\big)_{1\leq j,k\leq n}$ *of the Stokes system in* $\mathbb{R}^n$*, and the accompanying pressure vector* $\vec{q} = (q_j)_{1\leq j\leq n}$*, from* (6.2.1)-(6.2.2).

Then for each $\lambda \in \mathbb{C}$ *and each* $\ell, s, j \in \{1, \dots, n\}$ *one has*

$$\overset{\bullet}{\partial}_{\tau_{\ell s}} u_j \in \mathbb{X} \ \ and \ \ \overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u}, \pi) \in \big[\mathbb{X}\big]^n. \tag{6.3.44}$$

Moreover, given any $r, \gamma \in \{1, \dots, n\}$ *and* $\lambda \in \mathbb{C}$*, one has* (*with absolutely convergent integrals*)

$$\begin{aligned}(\partial_r u_\gamma)(x) &= \int_{\partial\Omega} (\partial_k E_{j\gamma})(x-y)\big(\overset{\bullet}{\partial}_{\tau_{rk}} u_j\big)(y)\, d\sigma(y)\\ &\quad + \lambda \int_{\partial\Omega} (\partial_j E_{k\gamma})(x-y)\big(\overset{\bullet}{\partial}_{\tau_{rk}} u_j\big)(y)\, d\sigma(y)\\ &\quad + \int_{\partial\Omega} q_\gamma(x-y)\big(\overset{\bullet}{\partial}_{\tau_{jr}} u_j\big)(y)\, d\sigma(y)\\ &\quad - \int_{\partial\Omega} (\partial_r E_{\gamma\alpha})(x-y)\big(\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u},\pi)\big)_\alpha(y)\, d\sigma(y)\end{aligned} \tag{6.3.45}$$

for all $x \in \Omega$*, and*

$$\begin{aligned}\pi(x) &= (1+\lambda)\int_{\partial\Omega} q_j(x-y)\big(\overset{\bullet}{\partial}_{\tau_{kj}} u_k\big)(y)\, d\sigma(y)\\ &\quad - \int_{\partial\Omega} q_\alpha(x-y)\big(\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u},\pi)\big)_\alpha(y)\, d\sigma(y), \quad for\ all \ \ x \in \Omega.\end{aligned} \tag{6.3.46}$$

Proof Thanks to (6.3.40), we may invoke [69, Proposition 5.2.7] which guarantees that

$$\begin{gathered}\text{there exist } q \in (1,\infty) \text{ and } \varepsilon \in (0,1) \text{ such that}\\ \mathbb{X} \hookrightarrow L^q\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1-\varepsilon}}\Big) \text{ continuously.}\end{gathered} \tag{6.3.47}$$

From [69, Proposition 10.2.6], Definition 6.2.9, [69, Example 10.2.2], (6.3.40), and (6.3.47) we then conclude that the memberships claimed in (6.3.44) are indeed true.

Having established this, we now run the same argument as in the proof of Theorem 6.3.1, with two key differences singled out below. First, in the case when either Ω is bounded, or $\partial\Omega$ is unbounded, in place of (6.3.14) we now make use of (6.2.1)-(6.2.2), Hölder's inequality, [68, Proposition 8.6.3], [68, (8.1.17)], (6.3.42), and (6.3.47) to estimate (with q and ε as in (6.3.47)):

$$
\begin{aligned}
R^{-1}\int_{\Omega} &|(\nabla\phi)(y/R)|\big\{|(\nabla E)(x-y)|+|\vec{q}(x-y)|\big\}\big\{|(\nabla\vec{u})(y)|+|\pi(y)|\big\}\,\mathrm{d}\mathcal{L}^n(y)\\
&\leq CR^{-n}\int_{\Omega\cap[B(0,\mu R)\setminus B(0,R)]}\big\{|\nabla\vec{u}|+|\pi|\big\}\,\mathrm{d}\mathcal{L}^n\\
&\leq CR^{-n}\Big(\int_{\Omega\cap[B(0,\mu R)\setminus B(0,R)]}\big\{|\nabla\vec{u}|+|\pi|\big\}^{\frac{nq}{n-1}}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{nq}}\big(R^n\big)^{1-\frac{n-1}{nq}}\\
&\leq CR^{-\frac{n-1}{q}}\bigg\{\Big(\int_{\partial\Omega\cap B(0,\mu(2+\kappa)R)}|\mathcal{N}_\kappa(\nabla\vec{u})|^q\,\mathrm{d}\sigma\Big)^{1/q}\\
&\qquad\qquad+\Big(\int_{\partial\Omega\cap B(0,\mu(2+\kappa)R)}|\mathcal{N}_\kappa\pi|^q\,\mathrm{d}\sigma\Big)^{1/q}\bigg\}\\
&\leq CR^{-\frac{n-1}{q}}\Big(\int_{\partial\Omega\cap B(0,\mu(2+\kappa)R)}R^{n-1-\varepsilon}\frac{|\mathcal{N}_\kappa(\nabla\vec{u})(z)|^q}{1+|z|^{n-1-\varepsilon}}\,\mathrm{d}\sigma(z)\Big)^{1/q}\\
&\quad+CR^{-\frac{n-1}{q}}\Big(\int_{\partial\Omega\cap B(0,\mu(2+\kappa)R)}R^{n-1-\varepsilon}\frac{|(\mathcal{N}_\kappa\pi)(z)|^q}{1+|z|^{n-1-\varepsilon}}\,\mathrm{d}\sigma(z)\Big)^{1/q}\\
&\leq CR^{-\frac{\varepsilon}{q}}\bigg\{\big\|\mathcal{N}_\kappa(\nabla\vec{u})\big\|_{L^q\left(\partial\Omega,\,\frac{\sigma(z)}{1+|z|^{n-1-\varepsilon}}\right)}+\big\|\mathcal{N}_\kappa\pi\big\|_{L^q\left(\partial\Omega,\,\frac{\sigma(z)}{1+|z|^{n-1-\varepsilon}}\right)}\bigg\}\\
&\leq CR^{-\frac{\varepsilon}{q}}\Big\{\|\mathcal{N}_\kappa(\nabla\vec{u})\|_{\mathbb{X}}+\|\mathcal{N}_\kappa\pi\|_{\mathbb{X}}\Big\}\\
&=o(1)\ \text{ as }\ R\to\infty.
\end{aligned}
\tag{6.3.48}
$$

As in the case of (6.3.14), this suits our purposes. The second significant adjustment we have to make in the proof Theorem 6.3.1 regards (6.3.22). In lieu of this we shall presently use the fact that for each $j,k\in\{1,\dots,n\}$, each fixed $x\in\Omega$, and any $f\in\mathbb{X}$ (ultimately playing the role of functions appearing in (6.3.44)) we have

$$
\lim_{R\to\infty}\int_{\partial\Omega}(\partial_k E_{j\gamma})(x-\cdot)\phi_R\,f\,\mathrm{d}\sigma=\int_{\partial\Omega}(\partial_k E_{j\gamma})(x-\cdot)\,f\,\mathrm{d}\sigma.
\tag{6.3.49}
$$

In turn, (6.3.49) is a consequence of the fact that $f\in L^1\big(\partial\Omega,\,\frac{\sigma(y)}{1+|y|^{n-1}}\big)$ as seen from (6.3.47), that $|(\nabla E)(x-y)|\leq C_x(1+|y|^{n-1})^{-1}$ for all $y\in\partial\Omega$ as seen from (6.2.1), and Lebesgue's Dominated Convergence Theorem.

With these alterations implemented, the rest of the argument in the proof of Theorem 6.3.1 goes through and gives (6.3.45). The justification of (6.3.46) is similarly based on the proof of Theorem 6.3.1. □

In turn, the integral representation formulas established in Theorem 6.3.3 allow us to prove the following versatile Fatou-type theorem for the Stokes system in arbitrary

UR domains, in which control of the nontangential maximal function is in terms of a Generalized Banach Function Space.

Theorem 6.3.4 *Let* $\Omega \subseteq \mathbb{R}^n$*, where* $n \in \mathbb{N}$ *with* $n \geq 2$*, be a* UR *domain. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Assume* $\mathbb{X}$ *is a Generalized Banach Function Space on* $(\partial\Omega, \sigma)$ *with the property that*

$$\begin{gathered}\mathcal{M}_{\partial\Omega} : \mathbb{X} \to \mathbb{X} \text{ and } \mathcal{M}_{\partial\Omega} : \mathbb{X}' \to \mathbb{X}' \\ \text{are well-defined bounded mappings,}\end{gathered} \tag{6.3.50}$$

where $\mathcal{M}_{\partial\Omega}$ *is the Hardy-Littlewood maximal operator on* $\partial\Omega$*, and* $\mathbb{X}'$ *is the associated space of* $\mathbb{X}$ (*cf.* [69, Definitions 5.1.4, 5.1.11]). *Finally, having fixed some aperture parameter* $\kappa > 0$*, assume*

$$\begin{gathered}\vec{u} \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \quad \pi \in \mathscr{C}^\infty(\Omega), \\ \Delta\vec{u} - \nabla\pi = 0 \text{ in } \Omega, \quad \operatorname{div}\vec{u} = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa(\nabla\vec{u}) \in \mathbb{X}, \quad \mathcal{N}_\kappa\pi \in \mathbb{X}.\end{gathered} \tag{6.3.51}$$

Then

$$\begin{gathered}\text{the nontangential pointwise traces } \nabla\vec{u}\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ and } \pi\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exist,} \\ \text{in } \mathbb{C}^{n\times n} \text{ and } \mathbb{C}, \text{ respectively, at } \sigma\text{-a.e. point on } \partial\Omega.\end{gathered} \tag{6.3.52}$$

In addition, these traces are actually independent of the parameter $\kappa \in (0, \infty)$*, the function* $(\nabla\vec{u})\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ *belongs to* $\big[\mathbb{X}\big]^{n\times n}$*, the function* $\pi\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ *belongs to* $\mathbb{X}$*, and*

$$\Big\|(\nabla\vec{u})\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big\|_{[\mathbb{X}]^{n\times n}} \leq \big\|\mathcal{N}_\kappa(\nabla\vec{u})\big\|_{\mathbb{X}}, \tag{6.3.53}$$

$$\Big\|\pi\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big\|_{\mathbb{X}} \leq \big\|\mathcal{N}_\kappa\pi\big\|_{\mathbb{X}}. \tag{6.3.54}$$

Proof All claims follow from Theorem 6.3.4, [69, Proposition 5.2.7], [70, Theorem 2.5.1], (6.2.1)-(6.2.2), [68, Corollary 8.9.9], [69, (5.1.12)], [68, (8.9.8)], and [68, Corollary 8.9.6]. □

We continue by presenting another brand of Fatou-type result for null-solutions of the Stokes system in arbitrary UR domains, which augments that in Theorem 6.3.2.

Theorem 6.3.5 *Let* $\Omega \subseteq \mathbb{R}^n$*, where* $n \in \mathbb{N}$ *with* $n \geq 2$*, be a* UR *domain. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and fix some aperture parameter* $\kappa > 0$*. Assume the functions* $\vec{u} \in \big[\mathscr{C}^\infty(\Omega)\big]^n$ *and* $\pi \in \mathscr{C}^\infty(\Omega)$ *satisfy*

$$\begin{gathered}\Delta\vec{u} - \nabla\pi = 0 \text{ in } \Omega, \quad \operatorname{div}\vec{u} = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa(\nabla\vec{u}),\ \mathcal{N}_\kappa\pi \in L^p(\partial\Omega, \sigma) \text{ for some } p \in \big(\tfrac{n-1}{n}, n-1\big), \\ \text{and } \mathcal{N}_\kappa\vec{u} \in L^{p^*}(\partial\Omega, \sigma) \text{ where } p^* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1}.\end{gathered} \tag{6.3.55}$$

Then

$$\text{the nontangential pointwise boundary traces } \vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}},\ \nabla\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}},\ \text{and } \pi\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial\Omega. \tag{6.3.56}$$

Moreover, these traces are actually independent of $\kappa \in (0,\infty)$*, and*

$$\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in \big[H_1^{p^*,p}(\partial\Omega,\sigma)\big]^n,\ (\nabla\vec{u})\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in \big[L^p(\partial\Omega,\sigma)\big]^{n\times n},\ \pi\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \in L^p(\partial\Omega,\sigma), \tag{6.3.57}$$

in a quantitative sense, i.e., there exists some $C = C(\Omega,\kappa,p) \in (0,\infty)$ *such that*

$$\Big\|\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big\|_{[H_1^{p^*,p}(\partial\Omega,\sigma)]^n} \leq C\big\|\mathcal{N}_\kappa(\nabla\vec{u})\big\|_{L^p(\partial\Omega,\sigma)} + C\big\|\mathcal{N}_\kappa\vec{u}\big\|_{L^{p^*}(\partial\Omega,\sigma)}, \tag{6.3.58}$$

$$\Big\|(\nabla\vec{u})\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big\|_{[L^p(\partial\Omega,\sigma)]^{n\times n}} \leq \big\|\mathcal{N}_\kappa(\nabla\vec{u})\big\|_{L^p(\partial\Omega,\sigma)}, \tag{6.3.59}$$

$$\Big\|\pi\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big\|_{L^p(\partial\Omega,\sigma)} \leq C\big\|\mathcal{N}_\kappa\pi\big\|_{L^p(\partial\Omega,\sigma)}. \tag{6.3.60}$$

Finally, for each given $\lambda \in \mathbb{C}$*, the Stokes weak conormal derivative of the pair* $\vec{u},\pi$ *satisfies*

$$\begin{gathered}\dot{\partial}_\nu^\lambda(\vec{u},\pi) \text{ belongs to the Hardy space } \big[H^p(\partial\Omega,\sigma)\big]^n \text{ and} \\ \big\|\dot{\partial}_\nu^\lambda(\vec{u},\pi)\big\|_{[H^p(\partial\Omega,\sigma)]^n} \leq C\big\|\mathcal{N}_\kappa(\nabla\vec{u})\big\|_{L^p(\partial\Omega,\sigma)} + C\big\|\mathcal{N}_\kappa\pi\big\|_{L^p(\partial\Omega,\sigma)}\end{gathered} \tag{6.3.61}$$

for some constant $C \in (0,\infty)$ *which depends only on* $\Omega, n, \lambda, \kappa, p$ *and, with the layer potential operators* $\mathcal{D}_\lambda$, $\mathcal{P}_\lambda$, $\mathcal{S}$, $\mathcal{Q}$ *associated with* Ω *and* λ *as in* (6.2.17)*,* (6.2.21)*,* (6.2.12)*,* (6.2.14)*, respectively, one has the integral representation formulas*

$$\vec{u} = \mathcal{D}_\lambda\Big(\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \mathcal{S}\big(\dot{\partial}_\nu^\lambda(\vec{u},\pi)\big) \text{ in } \Omega, \tag{6.3.62}$$

$$\pi = \mathcal{P}_\lambda\Big(\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \mathcal{Q}\big(\dot{\partial}_\nu^\lambda(\vec{u},\pi)\big) \text{ in } \Omega, \tag{6.3.63}$$

with the understanding that if Ω *is an exterior domain one also assumes that there exists* $\mu \in (1,\infty)$ *such that*

$$⨍_{B(0,\mu R)\setminus B(0,R)} \big\{|\vec{u}| + R|\pi|\big\}\,\mathrm{d}\mathcal{L}^n = o(1) \text{ as } R \to \infty, \tag{6.3.64}$$

in the case of (6.3.62)*, and*

$$⨍_{B(0,\mu R)\setminus B(0,R)} \big\{|\vec{u}| + R|\pi|\big\}\,\mathrm{d}\mathcal{L}^n = o(R) \text{ as } R \to \infty, \tag{6.3.65}$$

in the case of (6.3.63)*.*

Proof For starters, the claims in (6.3.61) are consequences of (6.2.243). Also, reasoning as in the proof of [70, (3.3.111)], the current assumptions on $\vec{u} = (u_j)_{1\leq j\leq n}$ imply that

$$\text{for all } j,k\in\{1,\dots,n\} \text{ the distribution } \nu\bullet(u_j\mathbf{e}_k) \text{ belongs to } L^{p^*}(\partial\Omega,\sigma)$$
$$\text{and } \big\|\nu\bullet(u_j\mathbf{e}_k)\big\|_{L^{p^*}(\partial\Omega,\sigma)}\leq C\|\mathcal{N}_\kappa\vec{u}\|_{L^{p^*}(\partial\Omega,\sigma)}+C\|\mathcal{N}_\kappa(\nabla\vec{u})\|_{L^p(\partial\Omega,\sigma)} \tag{6.3.66}$$

for some constant $C\in(0,\infty)$ depending only on Ω,n,κ,p.

To proceed, fix $\gamma\in\{1,\dots,n\}$ along with $x\in\Omega$ and consider the vector field $\vec{F}_x=(F_j)_{1\leq j\leq n}$ whose scalar components are defined as in (6.2.38). Specifically, with the summation convention over repeated indices assumed throughout, at $\mathcal{L}^n$-a.e. point in Ω define the j-th component of $\vec{F}_x$ as

$$\begin{aligned} F_j := & -a^{\alpha\beta}_{jk}(\lambda)(\partial_k E_{\beta\gamma})(x-\cdot)u_\alpha+q_\gamma(x-\cdot)u_j \\ & - E_{\gamma\alpha}(x-\cdot)a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta+E_{\gamma j}(x-\cdot)\pi. \end{aligned} \tag{6.3.67}$$

The strategy is to apply [68, Theorem 1.9.4] to the vector field $\vec{F}_x$ just introduced. As a prelude, from (6.3.67) and (6.2.1)-(6.2.2) we note that

$$\vec{F}_x\in\big[L^1_{\rm bdd}(\Omega,\mathcal{L}^n)\big]^n\subseteq\big[L^1_{\rm bdd}(\Omega,\mathcal{L}^n)+\mathscr{E}'(\Omega)\big]^n, \tag{6.3.68}$$

and recall from (6.2.40) that the divergence of $\vec{F}_x$, taken in the sense of distributions in Ω, is

$$\mathrm{div}\vec{F}_x=u_\gamma(x)\delta_x\in\mathscr{E}'(\Omega)\subseteq L^1(\Omega,\mathcal{L}^n)+\mathscr{E}'(\Omega). \tag{6.3.69}$$

Together, (6.3.68) and (6.3.69) prove that the vector field $\vec{F}_x$ satisfies [68, (1.9.29)]. Next, with μ as in (6.3.64)-(6.3.65) if Ω is an exterior domain and $\mu:=2$ otherwise, pick a function $\phi\in\mathscr{C}^\infty_c\big(B(0,\mu)\big)$ with the property that $\phi\equiv 1$ on $B(0,1)$. Then the family

$$\mathscr{F}:=\{\phi_R\}_{R>0},\ \text{ where }\ \phi_R:=\phi(\cdot/R)\ \text{ for each }\ R\in(0,\infty), \tag{6.3.70}$$

becomes a system of auxiliary functions (in the sense of [68, (1.3.3)]). Estimates similar in nature to [70, (3.3.122), (3.3.123)] prove that

$$\text{if either } \Omega \text{ is bounded, or } \partial\Omega \text{ is unbounded, then } [\vec{F}_x]_{\mathscr{F}}=0. \tag{6.3.71}$$

In the case when Ω is an exterior domain, condition (6.3.64) self-improves (thanks to interior estimates, bearing in mind that $\Delta^2\vec{u}=0$) to

$$⨍_{B(0,\mu R)\setminus B(0,R)}\Big\{|\vec{u}|+R\big(|\nabla\vec{u}|+|\pi|\big)\Big\}\,d\mathcal{L}^n=o(1)\ \text{ as }\ R\to\infty. \tag{6.3.72}$$

In turn, (6.3.72) together with (6.2.1)-(6.2.2) guarantee that

$$\text{if } \Omega \text{ is an exterior domain then } [\vec{F}_x]_{\mathscr{F}}=0. \tag{6.3.73}$$

From (6.3.71) and (6.3.73) we then conclude that, in all cases, the limit in [68, (1.9.30)] written for $\vec{F}_x$ exists, and is actually zero.

Next, reasoning as in [70, (3.3.129)-(3.3.133)] (while availing ourselves of (6.3.61) and (6.3.66)), in place of [70, (3.3.134)] we presently obtain

$$\begin{aligned}&\big(\nu \bullet \vec{F}_x, 1\big)_{\mathscr{F}}\\&\quad = \int_{\partial\Omega} \Big\{ - \big(\nu \bullet (u_j \mathbf{e}_k)\big)(y)(\partial_k E_{j\gamma})(x-y) - \lambda\big(\nu \bullet (u_j \mathbf{e}_k)\big)(y)(\partial_j E_{k\gamma})(x-y)\\&\qquad\qquad + \big(\nu \bullet (u_j \mathbf{e}_j)\big)(y) q_\gamma(x-y)\Big\}\, \mathrm{d}\sigma(y)\\&\quad - \Big(\mathscr{S}(\dot{\partial}_\nu^\lambda(\vec{u}, \pi))(x)\Big)_\gamma.\end{aligned} \tag{6.3.74}$$

Granted (6.3.68)-(6.3.69), (6.3.71), (6.3.73), it follows that [68, Theorem 1.9.4] may be applied to $\vec{F}_x$. On account of (6.3.74), (6.3.69), (6.3.71), and (6.3.73), the Divergence Formula [68, (1.9.32)] written for $\vec{F}_x$ then gives

$$\begin{aligned}u_\gamma(x) &= \int_{\partial\Omega} \Big\{ - \big(\nu \bullet (u_j \mathbf{e}_k)\big)(y)(\partial_k E_{j\gamma})(x-y) - \lambda\big(\nu \bullet (u_j \mathbf{e}_k)\big)(y)(\partial_j E_{k\gamma})(x-y)\\&\qquad\qquad + \big(\nu \bullet (u_j \mathbf{e}_j)\big)(y) q_\gamma(x-y)\Big\}\, \mathrm{d}\sigma(y)\\&\quad - \Big(\mathscr{S}(\dot{\partial}_\nu^\lambda(\vec{u}, \pi))(x)\Big)_\gamma.\end{aligned} \tag{6.3.75}$$

In view of the arbitrariness of $\gamma \in \{1, \dots, n\}$ and $x \in \Omega$, from (6.3.75), (6.3.66), [70, Theorem 2.5.1], item *(6)* in Theorem 2.2.6, and (6.3.61), we conclude that

the nontangential trace $\vec{u}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$ if either Ω is bounded, or $\partial\Omega$ is unbounded, or Ω is an exterior domain and the decay condition (6.3.64) holds. (6.3.76)

In fact, this Fatou-type property further self-improves. Specifically, in the case when Ω is an exterior domain, working with $\Omega_R := B(0, R) \cap \Omega$ in place of Ω (where the radius $R \in (0, \infty)$ is sufficiently large), much as in the proof of [70, Theorem 3.1.6], eliminates the need of asking that the decay condition (6.3.64) holds. Simply put, at this stage we have shown that, in all cases,

$$\text{the nontangential trace } \vec{u}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega. \tag{6.3.77}$$

With this in hand, we may now apply [69, Proposition 10.2.9] to each vector field $u_j \mathbf{e}_k$ with $j, k \in \{1, \dots, n\}$ to conclude that if $\nu = (\nu_1, \dots, \nu_n)$ is the geometric measure theoretic outward unit normal to Ω then

$$\nu \bullet \big(u_j \mathbf{e}_k\big) = \nu_k \big(u_j\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) \text{ for each } j, k \in \{1, \dots, n\}. \tag{6.3.78}$$

Returning with (6.3.78) in (6.3.75) ultimately produces (6.3.62), in view of (6.2.17). Having proved (6.3.77), we may now rely on [69, Definition 11.10.6] and [69, Corollary 11.10.11] to conclude that the first claim in (6.3.57) along with the estimate in (6.3.58) hold as well.

From Theorem 6.3.2 we also know that the nontangential pointwise traces $\nabla\vec{u}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ and $\pi\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exist at σ-a.e. point on $\partial\Omega$, they are independent of the aperture parameter as well as p-th power integrable on $\partial\Omega$ with respect to the measure σ, and the estimates (6.3.33)-(6.3.34) hold. In particular, this takes care of the last two memberships in (6.3.57) and (6.3.59)-(6.3.60).

We are therefore left with proving the integral representation formula claimed in (6.3.63). With this goal in mind, fix an arbitrary point $x \in \Omega$ and bring back the vector field $\vec{G}_x = \big(G_j\big)_{1\leq j\leq n}$ with components defined as in (6.2.50), i.e.,

$$G_j := -(1+\lambda)(\partial_j q_k)(x-\cdot)u_k - q_\alpha(x-\cdot)a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta + q_j(x-\cdot)\pi \tag{6.3.79}$$

at $\mathcal{L}^n$-a.e. point in Ω. Note that (6.3.79) and (6.2.1)-(6.2.2) guarantee that

$$\vec{G}_x \in \big[L^1_{\mathrm{bdd}}(\Omega,\mathcal{L}^n)\big]^n. \tag{6.3.80}$$

Also, from (6.2.52) we know that the divergence of $\vec{G}_x$ computed in the sense of distributions in Ω is given by

$$\mathrm{div}\vec{G}_x = (1+\lambda)u_k(\partial_k\delta_x) + \pi(x)\delta_x \in \mathscr{E}'(\Omega) \subseteq L^1(\Omega,\mathcal{L}^n) + \mathscr{E}'(\Omega). \tag{6.3.81}$$

In particular, much as in (6.2.54), we have

$$_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\mathrm{div}\vec{G}_x, 1\big)_{\mathscr{C}^\infty_b(\Omega)} = \pi(x). \tag{6.3.82}$$

Based on estimates similar to [70, (3.3.122), (3.3.123)], and also relying on (6.3.65) in the case when Ω is an exterior domain, we obtain

$$[\vec{G}_x]_{\mathscr{F}} = 0. \tag{6.3.83}$$

In an analogous fashion to (6.3.74) we have

$$\begin{aligned}\big(\nu\bullet\vec{G}_x, 1\big)_{\mathscr{F}} = -(1+\lambda)\int_{\partial\Omega}\big(\nu\bullet(u_k\mathbf{e}_j)\big)(y)(\partial_j q_k)(x-y)\,d\sigma(y)\\ - \mathcal{Q}(\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u},\pi))(x).\end{aligned} \tag{6.3.84}$$

Hence, thanks to (6.3.84), (6.3.78), and (6.2.21),

$$
\begin{aligned}
\big(\nu \bullet \vec{G}_x, 1\big)_{\mathscr{F}} &= -(1+\lambda)\int_{\partial\Omega} \nu_j(y)\big(u_k\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(y)(\partial_j q_k)(x-y)\,\mathrm{d}\sigma(y) \\
&\quad - \mathcal{Q}(\dot{\partial}_\nu^\lambda(\vec{u},\pi))(x) \\
&= \mathcal{P}_\lambda\big(\vec{u}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) - \mathcal{Q}(\dot{\partial}_\nu^\lambda(\vec{u},\pi))(x). \qquad (6.3.85)
\end{aligned}
$$

At this stage, the Divergence Formula [68, (1.9.32)] written for $\vec{G}_x$ ultimately yields (6.3.63), on account of (6.3.83), (6.3.82), and (6.3.85). □

Lastly, here is a powerful result which uses our earlier Fatou-type results for null-solutions of the Stokes system to produce integral representation formulas in an inclusive setting.

Theorem 6.3.6 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an* NTA *domain with the property that $\partial\Omega$ is an Ahlfors regular set. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix an arbitrary number $\lambda \in \mathbb{C}$ along with an aperture parameter $\kappa \in (0,\infty)$.*

In this setting, recall the modified version of the double layer operator $\mathcal{D}_\lambda^{\mathrm{mod}}$ for the Stokes system acting on vector-valued functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^n$ as in (6.2.171), and the modified version of the single layer operator $\mathscr{S}_{\mathrm{mod}}$ for the Stokes system acting on vector-valued functions from $\big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$ as in (6.2.164). The reader is also reminded that actions of the boundary–to-domain single layer operator $\mathscr{S}$ for the Stokes system and the corresponding pressure potential $\mathcal{Q}$ have been extended to Hardy spaces in item (3) of Theorem 6.2.11. Finally, consider a pair of functions, $\vec{u} = (u_j)_{1\leq j\leq n} : \Omega \to \mathbb{C}^n$ and $\pi : \Omega \to \mathbb{C}$, satisfying

$$
\begin{gathered}
\vec{u} \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \quad \pi \in \mathscr{C}^\infty(\Omega), \\
\Delta\vec{u} - \nabla\pi = 0 \ \text{ and } \ \operatorname{div}\vec{u} = 0 \ \text{ in } \Omega, \qquad (6.3.86) \\
\mathcal{N}_\kappa(\nabla\vec{u}),\ \mathcal{N}_\kappa\pi \in L^p(\partial\Omega,\sigma) \ \text{ for some } \ p \in \big(\tfrac{n-1}{n}, \infty\big).
\end{gathered}
$$

In the case when Ω is an exterior domain make the additional assumption that there exists $\mu \in (1,\infty)$ such that

$$
\fint_{B(0,\mu R)\setminus B(0,R)} \big\{|\nabla\vec{u}| + |\pi|\big\}\,\mathrm{d}\mathcal{L}^n = o(1) \ \text{ as } \ R \to \infty. \qquad (6.3.87)
$$

Then the following properties hold: First,

$$
\text{the nontangential pointwise traces } \nabla\vec{u}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ and } \pi\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exist, in } \mathbb{C}^{n\times n} \text{ and } \mathbb{C}, \text{ respectively, at } \sigma\text{-a.e. point on } \partial\Omega, \qquad (6.3.88)
$$

and these traces are actually independent of the aperture parameter $\kappa \in (0,\infty)$. Second, the function $(\nabla\vec{u})\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ belongs to $\big[L^p(\partial\Omega,\sigma)\big]^{n\times n}$, the function $\pi\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ belongs to $L^p(\partial\Omega,\sigma)$, and

$$\left\|(\nabla\vec{u})\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\right\|_{[L^p(\partial\Omega,\sigma)]^{n\times n}} \leq \left\|\mathcal{N}_\kappa(\nabla\vec{u})\right\|_{L^p(\partial\Omega,\sigma)}, \tag{6.3.89}$$

$$\left\|\pi\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\right\|_{L^p(\partial\Omega,\sigma)} \leq \left\|\mathcal{N}_\kappa\pi\right\|_{L^p(\partial\Omega,\sigma)}. \tag{6.3.90}$$

In particular, the pointwise conormal derivative $\partial_\nu^\lambda(\vec{u},\pi)$ *belongs to* $\big[L^p(\partial\Omega,\sigma)\big]^n$. *Third, the nontangential boundary trace* $\vec{u}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ *exists (in* $\mathbb{C}^n$*) at* σ*-a.e. point on* $\partial\Omega$ *and is independent of the aperture parameter* κ*. As a function,*

$$\begin{gathered}\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ belongs to } \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^n \text{ and satisfies}\\ \partial_{\tau_{jk}}\Big(\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) = \dot{\partial}_{\tau_{jk}}\vec{u} \in \big[H^p(\partial\Omega,\sigma)\big]^n \text{ for all } j,k\in\{1,\dots,n\},\end{gathered} \tag{6.3.91}$$

as well as

$$\left\|\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\right\|_{[\dot{H}^p_1(\partial\Omega,\sigma)]^n} \leq C\left\|\mathcal{N}_\kappa(\nabla\vec{u})\right\|_{L^p(\partial\Omega,\sigma)} \tag{6.3.92}$$

for some constant $C = C(\Omega,n,\kappa,p)\in(0,\infty)$*. Fourth, the weak conormal derivative* $\dot{\partial}_\nu^\lambda(\vec{u},\pi)$ *belongs to the Hardy space* $\big[H^p(\partial\Omega,\sigma)\big]^n$*, and there exists a constant* $C = C(\Omega,n,\kappa,p,\lambda)\in(0,\infty)$ *such that*

$$\left\|\dot{\partial}_\nu^\lambda(\vec{u},\pi)\right\|_{[H^p(\partial\Omega,\sigma)]^n} \leq C\left\|\mathcal{N}_\kappa(\nabla\vec{u})\right\|_{L^p(\partial\Omega,\sigma)}. \tag{6.3.93}$$

Fifth, there exists some $\mathbb{C}^n$*-valued locally constant function* $c_{\vec{u},\pi}$ *in* Ω *with the property that whenever* $p\in\big(\frac{n-1}{n},1\big]$ *one has*

$$\vec{u} = \mathcal{D}_\lambda^{\text{mod}}\Big(\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \mathcal{S}\big(\dot{\partial}_\nu^\lambda(\vec{u},\pi)\big) + c_{\vec{u},\pi} \text{ in } \Omega, \tag{6.3.94}$$

and

$$\begin{aligned}\pi &= (1+\lambda)\mathcal{Q}\Big(\Big(\partial_{\tau_{kj}}\Big(u_k\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)\Big)_{1\leq j\leq n}\Big) - \mathcal{Q}\big(\dot{\partial}_\nu^\lambda(\vec{u},\pi)\big)\\ &= \mathcal{P}_\lambda\Big(\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \mathcal{Q}\big(\dot{\partial}_\nu^\lambda(\vec{u},\pi)\big) \text{ in } \Omega,\end{aligned} \tag{6.3.95}$$

while for $p\in(1,\infty)$ *one has*

$$\vec{u} = \mathcal{D}_\lambda^{\text{mod}}\Big(\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) - \mathcal{S}_{\text{mod}}\big(\partial_\nu^\lambda(\vec{u},\pi)\big) + c_{\vec{u},\pi} \text{ in } \Omega, \tag{6.3.96}$$

and

$$\begin{aligned}\pi(x) &= (1+\lambda)\int_{\partial\Omega} q_j(x-y)\partial_{\tau_{kj}}\Big(u_k\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(y)\,\mathrm{d}\sigma(y) - \mathcal{Q}\big(\partial_\nu^\lambda(\vec{u},\pi)\big)(x)\\ &= \mathcal{P}_\lambda\Big(\vec{u}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) - \mathcal{Q}\big(\partial_\nu^\lambda(\vec{u},\pi)\big)(x) \text{ at each point } x\in\Omega.\end{aligned} \tag{6.3.97}$$

Sixth, for each truncation parameter $\varepsilon > 0$ *one has*

$$\begin{aligned}&\mathcal{N}_\kappa^\varepsilon \vec{u} \textit{ belongs to the space } L^{p^*}_{\mathrm{loc}}(\partial\Omega,\sigma)\\ &\textit{if } p \in \big(\tfrac{n-1}{n}, n-1\big) \textit{ and } p^* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1} \in (1,\infty),\end{aligned} \tag{6.3.98}$$

and

$$\begin{aligned}&\mathcal{N}_\kappa^\varepsilon \vec{u} \textit{ belongs to the space } L^{q}_{\mathrm{loc}}(\partial\Omega,\sigma)\\ &\textit{if } n = 2,\ p = 1, \textit{ and } q \in (1,\infty) \textit{ is arbitrary.}\end{aligned} \tag{6.3.99}$$

Proof The properties claimed in (6.3.88)-(6.3.90), and (6.3.93) are consequences of the Fatou-type result for the Stokes system from Theorem 6.3.2 (bearing in mind that any NTA domain with an Ahlfors regular boundary is a UR domain), while the properties claimed in (6.3.91)-(6.3.92) are seen from Theorem 2.2.8. Next, assume that $p \in \big(\frac{n-1}{n}, 1\big]$, with the goal of proving (6.3.94). To this end, define

$$\vec{w} = (w_\gamma)_{1\le\gamma\le n} := \mathcal{D}_\lambda^{\mathrm{mod}}\big(\vec{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) - \mathcal{S}\big(\dot{\partial}_\nu^\lambda(\vec{u},\pi)\big) \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \tag{6.3.100}$$

and recall the Kelvin matrix-valued fundamental solution $E = \big(E_{jk}\big)_{1\le j,k\le n}$ for the Stokes system in $\mathbb{R}^n$ from (6.2.1), and the pressure vector $\vec{q} = (q_j)_{1\le j\le n}$ from (6.2.2). For the clarity of the exposition, assume first that $\partial\Omega$ is unbounded. Then for each index $r \in \{1,\dots,n\}$ and each index $\gamma \in \{1,\dots,n\}$ we may use (6.3.100), (6.2.266), and Lemma 2.2.1 to compute (with all pointy brackets indicating duality in the sense of [69, Theorem 4.6.1])

$$\begin{aligned}(\partial_r w_\gamma)(x) &= \partial_r\Big(\mathcal{D}_\lambda^{\mathrm{mod}}\big(\vec{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\Big)_\gamma(x) - \partial_r\Big(\mathcal{S}\big(\dot{\partial}_\nu^\lambda(\vec{u},\pi)\big)\Big)_\gamma(x)\\ &= \Big\langle\big[(\partial_k E_{j\gamma})(x-\cdot)\big|_{\partial\Omega}\big],\ \partial_{\tau_{rk}}\big(\vec{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)_j\Big\rangle\\ &\quad + \lambda\Big\langle\big[(\partial_j E_{k\gamma})(x-\cdot)\big|_{\partial\Omega}\big],\ \partial_{\tau_{rk}}\big(\vec{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)_j\Big\rangle\\ &\quad + \Big\langle\big[q_\gamma(x-\cdot)\big|_{\partial\Omega}\big],\ \partial_{\tau_{jr}}\big(\vec{u}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)_j\Big\rangle\\ &\quad - \Big\langle\big[(\partial_r E_{\gamma j})(x-\cdot)\big|_{\partial\Omega}\big],\ \big(\dot{\partial}_\nu^\lambda(\vec{u},\pi)\big)_j\Big\rangle\end{aligned} \tag{6.3.101}$$

at each point $x \in \Omega$. On account of the last line in (6.3.91) this further shows that, at each point $x \in \Omega$,

$$
\begin{aligned}
(\partial_r w_\gamma)(x) = &\Big\langle \big[(\partial_k E_{j\gamma})(x-\cdot)\big|_{\partial\Omega}\big],\, \overset{\bullet}{\partial}_{\tau_{rk}} u_j \Big\rangle \\
&+ \lambda\Big\langle \big[(\partial_j E_{k\gamma})(x-\cdot)\big|_{\partial\Omega}\big],\, \overset{\bullet}{\partial}_{\tau_{rk}} u_j \Big\rangle \\
&+ \Big\langle \big[q_\gamma(x-\cdot)\big|_{\partial\Omega}\big],\, \overset{\bullet}{\partial}_{\tau_{jr}} u_j \Big\rangle \\
&- \Big\langle \big[(\partial_r E_{\gamma j})(x-\cdot)\big|_{\partial\Omega}\big],\, \big(\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\vec{u},\pi)\big)_j \Big\rangle \\
= &\,(\partial_r u_\gamma)(x),
\end{aligned}
\tag{6.3.102}
$$

with the last equality provided by (6.3.4). From (6.3.102) we then conclude that $\nabla\vec{w} = \nabla\vec{u}$ in Ω, which shows that the difference $c_{\vec{u},\pi} := \vec{u} - \vec{w}$ is a $\mathbb{C}^n$-valued locally constant function in Ω. This concludes the proof of (6.3.94) in the case when $\partial\Omega$ is unbounded. Note that, in the case when $\partial\Omega$ is unbounded, the first equality in formula (6.3.95) is directly implied by (6.3.5) and the last line in (6.3.91). The second equality in (6.3.95) is then a consequence of the first, (6.3.91), and [69, Lemma 11.10.4]. The proofs of (6.3.94)-(6.3.95) in the case when $\partial\Omega$ is bounded is carried out in a similar fashion.

In addition, from Theorem 6.2.7 we see that (6.3.96)-(6.3.97) are true when $p \in (1,\infty)$. Finally, the validity of (6.3.98)-(6.3.99) in the range $p \in \big(\frac{n-1}{n}, 1\big]$ may be justified based on (6.3.94) reasoning as in the proof of item *(d)* in Theorem 2.2.7, while the validity of (6.3.98)-(6.3.99) in the range $p \in (1,\infty)$ may be justified based on (6.3.96) by reasoning as in the proof of the last claim in Theorem 1.8.19. □

We conclude this section by complementing the operator identities from item *(xi)* in Theorem 6.2.4 and Theorem 6.2.8 by now considering similar formulas on Hardy spaces and Hardy-based Sobolev spaces. We do this in two installments, in Theorems 6.3.7-6.3.8 below.

Theorem 6.3.7 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Finally, fix $\lambda \in \mathbb{C}$ and consider the Stokes boundary layer potential operators $\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\mathcal{D}_\lambda, \mathcal{P}_\lambda)$, K_λ, $K^{\#}_\lambda$, S, associated with λ and Ω as in* (6.2.256), (6.2.259), (6.2.250)*, and* (6.2.262)*. Then for any two exponents*

$$
p \in \big(\tfrac{n-1}{n}, 1\big] \ \text{ and } \ q \in (1,\infty) \tag{6.3.103}
$$

the following operator identities hold:

$$
\big(\tfrac12 I + K_\lambda\big) \circ \big(-\tfrac12 I + K_\lambda\big) = S \circ \big(\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\mathcal{D}_\lambda, \mathcal{P}_\lambda)\big) \ \text{ on } \ \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^n, \tag{6.3.104}
$$

$$
\big(\tfrac12 I + K^{\#}_\lambda\big) \circ \big(-\tfrac12 I + K^{\#}_\lambda\big) = \big(\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\mathcal{D}_\lambda, \mathcal{P}_\lambda)\big) \circ S \ \text{ on } \ \big[H^{p}(\partial\Omega,\sigma)\big]^n, \tag{6.3.105}
$$

$$
S \circ K^{\#}_\lambda = K_\lambda \circ S \ \text{ on } \ \big[H^{p}(\partial\Omega,\sigma)\big]^n, \tag{6.3.106}
$$

$$
K^{\#}_\lambda \circ \big(\overset{\bullet}{\partial}{}^{\lambda}_{\nu}\mathcal{D}_\lambda\big) = \big(\overset{\bullet}{\partial}{}^{\lambda}_{\nu}(\mathcal{D}_\lambda, \mathcal{P}_\lambda)\big) \circ K_\lambda \ \text{ on } \ \big[H^{q,p}_1(\partial\Omega,\sigma)\big]^n. \tag{6.3.107}
$$

Proof All claims may be justified in a manner analogous to the proof of Theorem 2.3.18 (cf. also the proof of item *(xi)* in Theorem 6.2.4), making use of Theorem 6.2.11, Theorem 6.2.10, and the integral representation formulas from Theorem 6.3.5. □

Here is the final installment of composition identities, now also involving modified layer potentials for the Stokes system on Hardy spaces.

Theorem 6.3.8 *Pick $n \in \mathbb{N}$ with $n \geq 2$ and assume $\Omega \subseteq \mathbb{R}^n$ is an* NTA *domain with an Ahlfors regular boundary. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Select a number $\lambda \in \mathbb{C}$ and fix an exponent*

$$p \in \big(\tfrac{n-1}{n}, 1\big]. \tag{6.3.108}$$

Bring back the following operators associated with the Stokes system in the set Ω, for the given parameter λ: S from (6.2.264), $\dot{\partial}^{\lambda}_{\nu}\big(\mathcal{D}^{\text{mod}}_{\lambda}, \mathcal{P}_\lambda\big)$ from (6.2.257), K^{mod}_{λ} from (6.2.260), and $K^{\#}_{\lambda}$ from (6.2.250). Then, in relation to these, the following results are valid.

(1) Given any $\vec{f} \in \big[H^p(\partial\Omega,\sigma)\big]^n$, at σ-a.e. point on $\partial\Omega$ one has

$$\big(\tfrac12 I + K^{\#}_{\lambda}\big)\Big(\big(-\tfrac12 I + K^{\#}_{\lambda}\big)\vec{f}\Big) = \Big(\dot{\partial}^{\lambda}_{\nu}\big(\mathcal{D}^{\text{mod}}_{\lambda}, \mathcal{P}_\lambda\big)\Big)\big(S\vec{f}\big) \tag{6.3.109}$$

and there exists $c_{\vec{f}}$, which is the nontangential trace on $\partial\Omega$ of some $\mathbb{C}^n$-valued locally constant function in Ω, such that

$$S\big(K^{\#}_{\lambda}\vec{f}\big) = K^{\text{mod}}_{\lambda}\big(S\vec{f}\big) + c_{\vec{f}} \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{6.3.110}$$

(2) Recall the operator K^{mod}_{λ} from (6.2.216) and (6.2.220). Then, given any distribution $\vec{f} \in \big[\dot{H}^p_1(\partial\Omega,\sigma)\big]^n$, at σ-a.e. point on $\partial\Omega$ one has

$$K^{\#}_{\lambda}\Big(\dot{\partial}^{\lambda}_{\nu}\big(\mathcal{D}^{\text{mod}}_{\lambda}, \mathcal{P}_\lambda\big)\vec{f}\Big) = \Big(\dot{\partial}^{\lambda}_{\nu}\big(\mathcal{D}^{\text{mod}}_{\lambda}, \mathcal{P}_\lambda\big)\Big)\big(K^{\text{mod}}_{\lambda}\vec{f}\big) \tag{6.3.111}$$

and there exists $c_{\vec{f}}$, which is the nontangential trace on $\partial\Omega$ of some $\mathbb{C}^n$-valued locally constant function in Ω, with the property that at σ-a.e. point on $\partial\Omega$ one has

$$\big(\tfrac12 I + K^{\text{mod}}_{\lambda}\big)\Big(\big(-\tfrac12 I + K^{\text{mod}}_{\lambda}\big)\vec{f}\Big) = S\Big(\dot{\partial}^{\lambda}_{\nu}\big(\mathcal{D}^{\text{mod}}_{\lambda}, \mathcal{P}_\lambda\big)\vec{f}\Big) + c_{\vec{f}}. \tag{6.3.112}$$

Proof The desired conclusions may be justified by reasoning as in the proof of Theorem 2.3.11, now making use of the integral representation formulas from Theorem 6.3.6 and availing ourselves of the functional analytic properties enjoyed by the boundary layer potential operators associated with the Stokes system on Hardy spaces from Theorem 6.2.11. □

6.4 Layer Potentials for the Stokes System on Besov, Triebel-Lizorkin, and Weighted Sobolev Spaces

The vast majority of the results from §4 pertaining to the action of boundary layer potentials associated with generic constant coefficient homogeneous second-order systems on Besov, Triebel-Lizorkin, and weighted Sobolev spaces have natural counterparts for the Stokes system[16] as well. Here, our goal is to briefly elaborate on this topic. Our first result of this nature reads as follows.

Theorem 6.4.1 *Suppose* $\Omega \subseteq \mathbb{R}^n$ (*where* $n \in \mathbb{N}$ *satisfies* $n \geq 2$) *is a* UR *domain with compact boundary and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Also, fix* $\lambda \in \mathbb{C}$ *arbitrary and recall the boundary layer potential operators* K_λ *and* $K^\#_\lambda$ *from* (6.2.95), (6.2.98).

Then the operator K_λ, *originally acting on Lebesgue spaces on* $\partial\Omega$ (*cf. item (iii) in Theorem 6.2.4*), *extends uniquely to linear and bounded mappings*

$$\begin{gathered} K_\lambda : \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^n \longrightarrow \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^n, \\ p \in \big(\tfrac{n-1}{n},\infty\big], \quad q \in (0,\infty], \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \end{gathered} \tag{6.4.1}$$

and

$$\begin{gathered} K_\lambda : \big[F^{p,q}_s(\partial\Omega,\sigma)\big]^n \longrightarrow \big[F^{p,q}_s(\partial\Omega,\sigma)\big]^n, \\ p \in \big(\tfrac{n-1}{n},\infty\big), \quad q \in \big(\tfrac{n-1}{n},\infty\big], \quad (n-1)\big(\tfrac{1}{\min\{p,q\}}-1\big)_+ < s < 1. \end{gathered} \tag{6.4.2}$$

Also, the operator $K^\#_\lambda$, *originally acting on Lebesgue spaces on* $\partial\Omega$ (*cf. item (iii) in Theorem 6.2.4*), *extends uniquely to linear and bounded mappings*

$$\begin{gathered} K^\#_\lambda : \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^n \\ \textit{provided } \ s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s},\infty\big], \quad q \in (0,\infty], \end{gathered} \tag{6.4.3}$$

and

$$\begin{gathered} K^\#_\lambda : \big[F^{p,q}_{-s}(\partial\Omega,\sigma)\big]^n \longrightarrow \big[F^{p,q}_{-s}(\partial\Omega,\sigma)\big]^n, \\ \textit{provided } \ s \in (0,1), \quad p \in \big(\tfrac{n-1}{n-s},\infty\big), \quad q \in \big(\tfrac{n-1}{n-s},\infty\big]. \end{gathered} \tag{6.4.4}$$

In all cases, various choices of the exponents yield operators which are compatible with one another. Finally, if $p, q, p', q' \in (1,\infty)$ *satisfy* $1/p + 1/p' = 1 = 1/q + 1/q'$ *and* $s \in (0,1)$, *then*

$$\begin{gathered} {}_{[B^{p,q}_{-s}(\partial\Omega,\sigma)]^n}\big\langle K^\#_\lambda \vec{f}, \vec{g}\big\rangle_{[B^{p',q'}_{s}(\partial\Omega,\sigma)]^n} = {}_{[B^{p,q}_{-s}(\partial\Omega,\sigma)]^n}\big\langle \vec{f}, K_\lambda \vec{g}\big\rangle_{[B^{p',q'}_{s}(\partial\Omega,\sigma)]^n} \\ \textit{for each } \ \vec{f} \in \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^n \ \textit{ and } \ \vec{g} \in \big[B^{p',q'}_{s}(\partial\Omega,\sigma)\big]^n, \end{gathered} \tag{6.4.5}$$

[16] due to its special algebraic format, the Stokes system does not fit directly into the category of generic constant coefficient homogeneous second-order systems

and

$$
{}_{[F^{p,q}_{-s}(\partial\Omega,\sigma)]^n}\big\langle K^{\#}_{\lambda}\vec{f},\vec{g}\big\rangle_{[F^{p',q'}_{s}(\partial\Omega,\sigma)]^n} = {}_{[F^{p,q}_{-s}(\partial\Omega,\sigma)]^n}\big\langle \vec{f}, K_{\lambda}\vec{g}\big\rangle_{[F^{p',q'}_{s}(\partial\Omega,\sigma)]^n}
\\ \text{for each } \vec{f}\in\big[F^{p,q}_{-s}(\partial\Omega,\sigma)\big]^n \text{ and } \vec{g}\in\big[F^{p',q'}_{s}(\partial\Omega,\sigma)\big]^n. \tag{6.4.6}
$$

Proof All claims are justified by reasoning as in the proofs of Theorem 4.1.1 and Theorem 4.1.5, bearing in mind (6.2.271), item *(1)* in Theorem 6.2.11, and item *(iii)* in Theorem 6.2.4. □

We next describe the smoothing effect of the boundary-to-domain double layer potential operator associated with the Stokes system, measured on weighted Sobolev, Besov, and Triebel-Lizorkin scales.

Theorem 6.4.2 *Let $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$, $n\geq 2$) be an open set whose boundary is compact and Ahlfors regular. Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ and pick $\lambda\in\mathbb{C}$. In this setting, recall the boundary-to-domain double layer $\mathcal{D}_\lambda$ for the Stokes system from (6.2.17), and the corresponding potential for the pressure $\mathcal{P}_\lambda$ from (6.2.21). Finally, fix*

$$
p\in(1,\infty),\ \ s\in(0,1),\ \ \text{and}\ \ a:=1-s-\tfrac{1}{p}\in\big(-\tfrac{1}{p},1-\tfrac{1}{p}\big). \tag{6.4.7}
$$

Then the following operators are well-defined, linear, and bounded:

$$
\mathcal{D}_\lambda:\big[B^{p,p}_{s}(\partial\Omega,\sigma)\big]^n\longrightarrow\big[W^{1,p}_{a}(\Omega)_{\text{bdd}}\big]^n\cap\operatorname{Ker}\Delta^2, \tag{6.4.8}
$$

$$
\mathcal{P}_\lambda:\big[B^{p,p}_{s}(\partial\Omega,\sigma)\big]^n\longrightarrow\big[L^{p}\big(\Omega,\delta^{ap}_{\partial\Omega}\mathcal{L}^n\big)\big]^n\cap\operatorname{Ker}\Delta, \tag{6.4.9}
$$

$$
\mathcal{S}:\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^n\longrightarrow\big[W^{1,p}_{a}(\Omega)_{\text{bdd}}\big]^n\cap\operatorname{Ker}\Delta^2, \tag{6.4.10}
$$

$$
\mathcal{Q}:\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^n\longrightarrow\big[L^{p}_{\text{bdd}}\big(\Omega,\delta^{ap}_{\partial\Omega}\mathcal{L}^n\big)\big]^n\cap\operatorname{Ker}\Delta. \tag{6.4.11}
$$

In addition, assuming

$$
\tfrac{n-1}{n}<p<\infty,\quad (n-1)\big(\tfrac{1}{p}-1\big)_+<s<1,\ \ \text{and with}\ \ a:=1-s-\tfrac{1}{p}, \tag{6.4.12}
$$

the operators

$$
\mathcal{D}_\lambda:\big[B^{p,p}_{s}(\partial\Omega,\sigma)\big]^n\longrightarrow\big[W^{1,p}_{a,\star}(\Omega)\big]^n, \tag{6.4.13}
$$

$$
\mathcal{P}_\lambda:\big[B^{p,p}_{s}(\partial\Omega,\sigma)\big]^n\longrightarrow\big[L^{p}_{\star}\big(\Omega,\delta^{ap}_{\partial\Omega}\mathcal{L}^n\big)\big]^n, \tag{6.4.14}
$$

$$
\mathcal{S}:\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^n\longrightarrow\big[W^{1,p}_{a,\star}(\Omega)\big]^n, \tag{6.4.15}
$$

$$
\mathcal{Q}:\big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^n\longrightarrow\big[L^{p}_{\star}\big(\Omega,\delta^{ap}_{\partial\Omega}\mathcal{L}^n\big)\big]^n, \tag{6.4.16}
$$

are well-defined, linear, and bounded, provided Ω is bounded.

Finally, strengthen the original hypotheses by assuming this time that Ω is an (ε,δ)-domain whose boundary is a compact UR *set with the additional property* *that* $\mathcal{H}^{n-1}\big(\partial\Omega\setminus\partial_*\Omega\big)=0$*. Then, whenever* s,p,s *are as in* (6.4.7)*, the double layer operator* $\mathcal{D}_\lambda$*, acting in the context of* (6.4.8)*, satisfies the jump-formula*

$$\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathcal{D}_\lambda=\tfrac{1}{2}I+K_\lambda \ \ on \ \ \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^n, \tag{6.4.17}$$

where $\mathrm{Tr}_{\Omega\to\partial\Omega}:\big[W^{1,p}_a(\Omega)_{\rm bdd}\big]^n\to\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^n$ *is the boundary trace operator from* [69, Theorem 8.3.6]*,* I *denotes the identity operator on* $\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^n$*, and* K_λ *is the boundary-to-boundary double layer operator acting on* $\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^n$ *as in* (6.4.1) *of Theorem 6.4.1.*

Proof The claim pertaining to (6.4.8) is seen from Theorem 4.2.1, much as in the proof of Theorem 4.2.3 (recalling from (6.2.84) that $\nabla\mathcal{D}_\lambda$ annihilates all constant vectors). As regards the operator $\mathcal{P}_\lambda$ in (6.4.9), upon noting that thanks to (6.2.6)-(6.2.7) for each $\vec{f}\in\big[L^1(\partial_*\Omega,\sigma)\big]^n$ and each $x\in\Omega$ we may express

$$\begin{aligned}\mathcal{P}_\lambda\vec{f}(x)&=-(1+\lambda)\int_{\partial_*\Omega}\nu_j(y)(\partial_j q_k)(x-y)f_k(y)\,d\sigma(y)\\&=-(1+\lambda)\int_{\partial_*\Omega}\nu_j(y)(\partial_k q_j)(x-y)f_k(y)\,d\sigma(y)\\&=(1+\lambda)\int_{\partial_*\Omega}\Big\{\nu_j(y)\partial_{y_k}\big[q_j(x-y)\big]-\nu_k(y)\partial_{y_j}\big[q_j(x-y)\big]\Big\}f_k(y)\,d\sigma(y),\end{aligned} \tag{6.4.18}$$

the desired conclusion follows from Corollary 4.2.4. The claim about the operator (6.4.10) is justified by reasoning much as in the proof of Theorem 4.2.10 (bearing in mind (6.2.10)). Next, if $\mathscr{S}_\Delta$ denotes the boundary-to-domain single layer potential operator associated with $L:=\Delta$, the Laplacian in $\mathbb{R}^n$, and the set Ω, then we extend the action of Q, originally considered as in (6.2.14), to any vector distribution on $\partial\Omega$ according to

$$\begin{aligned}&Q\vec{f}=-\mathrm{div}\mathscr{S}_\Delta\vec{f}=-\partial_j\mathscr{S}_\Delta f_j \ \text{ in } \Omega,\\&\text{for each } \vec{f}=(f_j)_{1\le j\le n}\in\big[\big(\mathrm{Lip}\,(\partial\Omega)\big)'\big]^n.\end{aligned} \tag{6.4.19}$$

From this and Theorem 4.2.10 it follows that the operator Q is well defined, linear, and bounded in the context of (6.4.11).

When p,s,a are as in (6.4.12), the fact that the operators (6.4.13)-(6.4.16) are well-defined, linear and bounded may be justified much as in the proof of Theorem 4.2.3 and Theorem 4.2.10.

Finally, the claim regarding the validity of the jump-formula (6.4.17) may be established by reasoning as in the proof of Theorem 4.2.5, based on what we have proved so far and item *(iv)* in Theorem 6.2.4. □

We now turn to the issue of boundedness of hydrostatic layer potentials on Besov and Triebel-Lizorkin scales.

Theorem 6.4.3 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) is an (ε, δ)-domain whose boundary is a compact Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix $\lambda \in \mathbb{C}$ and recall the boundary-to-domain double layer $\mathcal{D}_\lambda$ for the Stokes system from (6.2.17), and the corresponding potential for the pressure $\mathcal{P}_\lambda$ from (6.2.21). Then the following assertions are true.*

(1) The following operators are well-defined, linear, and bounded:

$$
\begin{aligned}
\mathcal{D}_\lambda &: \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^n \longrightarrow \Big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{P}_\lambda &: \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^n \longrightarrow \Big[B^{p,q}_{s+\frac{1}{p}-1}(\Omega)_{\rm bdd}\Big]^n,\\
\mathscr{S} &: \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^n \longrightarrow \Big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{Q} &: \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^n \longrightarrow \Big[B^{p,q}_{s+\frac{1}{p}-1}(\Omega)_{\rm bdd}\Big]^n,\\
&\textit{if } \tfrac{n-1}{n} < p < \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \quad 0 < q \leq \infty.
\end{aligned}
\tag{6.4.20}
$$

(2) The following operators are well-defined, linear, and bounded:

$$
\begin{aligned}
\mathcal{D}_\lambda &: \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^n \longrightarrow \Big[F^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{P}_\lambda &: \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^n \longrightarrow \Big[F^{p,q}_{s+\frac{1}{p}-1}(\Omega)_{\rm bdd}\Big]^n,\\
\mathscr{S} &: \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^n \longrightarrow \Big[F^{p,q}_{s+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{Q} &: \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^n \longrightarrow \Big[F^{p,q}_{s+\frac{1}{p}-1}(\Omega)_{\rm bdd}\Big]^n,\\
&\textit{if } \tfrac{n-1}{n} < p < \infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \quad \tfrac{n}{n+s+1/p} < q \leq \infty.
\end{aligned}
\tag{6.4.21}
$$

(3) Corresponding to taking $s := 2 - \frac{1}{p}$ and $q := 2$ in (4.3.2), one has the following well-defined, linear, and bounded mappings

$$
\begin{aligned}
\mathcal{D}_\lambda &: \Big[B^{p,p}_{2-\frac{1}{p}}(\partial\Omega,\sigma)\Big]^n \longrightarrow \big[h^p_2(\Omega)_{\rm bdd}\big]^n,\\
\mathcal{P}_\lambda &: \Big[B^{p,p}_{2-\frac{1}{p}}(\partial\Omega,\sigma)\Big]^n \longrightarrow \big[h^p_1(\Omega)_{\rm bdd}\big]^n,\\
\mathscr{S} &: \Big[B^{p,p}_{1-\frac{1}{p}}(\partial\Omega,\sigma)\Big]^n \longrightarrow \big[h^p_2(\Omega)_{\rm bdd}\big]^n,\\
\mathcal{Q} &: \Big[B^{p,p}_{1-\frac{1}{p}}(\partial\Omega,\sigma)\Big]^n \longrightarrow \big[h^p_1(\Omega)_{\rm bdd}\big]^n,\\
&\textit{provided } \tfrac{n}{n+1} < p < 1,
\end{aligned}
\tag{6.4.22}
$$

where $h^p_k(\Omega)$, $k \in \mathbb{N}$, is the scale of local Hardy-based Sobolev spaces in Ω (cf. [69, (9.2.43)]).

(4) Strengthen the original hypotheses on Ω *by assuming that* $\partial\Omega$ *is actually a* UR *set. Then, as a limiting case of* (6.4.20)-(6.4.21)*, formally corresponding to making* $s := 0$*, the operators*

$$
\begin{aligned}
\mathcal{D}_\lambda &: \big[L^p(\partial_*\Omega,\sigma)\big]^n \longrightarrow \Big[B^{p,q}_{\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{P}_\lambda &: \big[L^p(\partial_*\Omega,\sigma)\big]^n \longrightarrow \Big[B^{p,q}_{\frac{1}{p}-1}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{S} &: \big[L^p_{-1}(\partial_*\Omega,\sigma)\big]^n \longrightarrow \Big[B^{p,q}_{\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{Q} &: \big[L^p_{-1}(\partial_*\Omega,\sigma)\big]^n \longrightarrow \Big[B^{p,q}_{\frac{1}{p}-1}(\Omega)_{\rm bdd}\Big]^n,\\
&\textit{provided } 1 < p < \infty \textit{ and } p \leq q \leq \infty,
\end{aligned}
\tag{6.4.23}
$$

as well as the operators

$$
\begin{aligned}
\mathcal{D}_\lambda &: \big[L^p(\partial_*\Omega,\sigma)\big]^n \longrightarrow \Big[F^{p,q}_{\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{P}_\lambda &: \big[L^p(\partial_*\Omega,\sigma)\big]^n \longrightarrow \Big[F^{p,q}_{\frac{1}{p}-1}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{S} &: \big[L^p_{-1}(\partial_*\Omega,\sigma)\big]^n \longrightarrow \Big[F^{p,q}_{\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{Q} &: \big[L^p_{-1}(\partial_*\Omega,\sigma)\big]^n \longrightarrow \Big[F^{p,q}_{\frac{1}{p}-1}(\Omega)_{\rm bdd}\Big]^n,\\
&\textit{provided } 1 < p < \infty \textit{ and } \tfrac{n}{n+1/p} < q \leq \infty,
\end{aligned}
\tag{6.4.24}
$$

are all well-defined, linear, and bounded. In particular, corresponding to the case $p = q = 2$*, it follows that* (*recall* (A.0.86) *and* [69, (9.2.22)])

$$
\begin{aligned}
\mathcal{D}_\lambda &: \big[L^2(\partial_*\Omega,\sigma)\big]^n \longrightarrow \big[H^{1/2}_{\rm bdd}(\Omega)\big]^n,\\
\mathcal{P}_\lambda &: \big[L^2(\partial_*\Omega,\sigma)\big]^n \longrightarrow \big[H^{-1/2}_{\rm bdd}(\Omega)\big]^n,\\
\mathcal{S} &: \big[L^2_{-1}(\partial_*\Omega,\sigma)\big]^n \longrightarrow \big[H^{1/2}_{\rm bdd}(\Omega)\big]^n,\\
\mathcal{Q} &: \big[L^2_{-1}(\partial_*\Omega,\sigma)\big]^n \longrightarrow \big[H^{-1/2}_{\rm bdd}(\Omega)\big]^n,
\end{aligned}
\tag{6.4.25}
$$

are well-defined, linear, and bounded operators. Furthermore, as a limiting case of (6.4.20)-(6.4.21)*, formally corresponding to making* $s := 1$*, the operators*

$$
\begin{aligned}
\mathcal{D}_\lambda &: \big[L^p_1(\partial_*\Omega,\sigma)\big]^n \longrightarrow \Big[B^{p,q}_{1+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{P}_\lambda &: \big[L^p_1(\partial_*\Omega,\sigma)\big]^n \longrightarrow \Big[B^{p,q}_{\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^n,\\
\mathcal{S} &: \big[L^p(\partial\Omega,\sigma)\big]^n \longrightarrow \Big[B^{p,q}_{1+\frac{1}{p}}(\Omega)_{\rm bdd}\Big]^n,\\
&\textit{provided } 1 < p < \infty \textit{ and } p \leq q \leq \infty,
\end{aligned}
\tag{6.4.26}
$$

as well as

$$\begin{aligned}
\mathcal{D}_\lambda &: \left[L^p_1(\partial_*\Omega,\sigma)\right]^n \longrightarrow \left[F^{p,q}_{1+\frac{1}{p}}(\Omega)_{\rm bdd}\right]^n,\\
\mathcal{P}_\lambda &: \left[L^p_1(\partial_*\Omega,\sigma)\right]^n \longrightarrow \left[F^{p,q}_{\frac{1}{p}}(\Omega)_{\rm bdd}\right]^n,\\
\mathcal{S} &: \left[L^p(\partial\Omega,\sigma)\right]^n \longrightarrow \left[F^{p,q}_{1+\frac{1}{p}}(\Omega)_{\rm bdd}\right]^n,\\
\mathcal{Q} &: \left[L^p(\partial\Omega,\sigma)\right]^n \longrightarrow \left[F^{p,q}_{\frac{1}{p}}(\Omega)_{\rm bdd}\right]^n,\\
&\text{provided } 1<p<\infty \text{ and } \tfrac{n}{n+1+1/p}<q\leq\infty,
\end{aligned} \tag{6.4.27}$$

are well-defined, linear, and bounded. In particular, corresponding to $p = q = 2$, *it follows that* (*recall* (A.0.86) *and* [69, (9.2.22)])

$$\begin{aligned}
\mathcal{D}_\lambda &: \left[L^2_1(\partial_*\Omega,\sigma)\right]^n \longrightarrow \left[H^{3/2}_{\rm bdd}(\Omega)\right]^n,\\
\mathcal{P}_\lambda &: \left[L^2_1(\partial_*\Omega,\sigma)\right]^n \longrightarrow \left[H^{1/2}_{\rm bdd}(\Omega)\right]^n,\\
\mathcal{S} &: \left[L^2(\partial\Omega,\sigma)\right]^n \longrightarrow \left[H^{3/2}_{\rm bdd}(\Omega)\right]^n,\\
\mathcal{Q} &: \left[L^2(\partial\Omega,\sigma)\right]^n \longrightarrow \left[H^{1/2}_{\rm bdd}(\Omega)\right]^n,
\end{aligned} \tag{6.4.28}$$

are well-defined, linear, and bounded operators.

(5) Continue to enforce the additional assumption that $\partial\Omega$ *is a* UR *set and also suppose that* $\mathcal{H}^{n-1}\big(\partial\Omega\setminus\partial_*\Omega\big)=0$ (*e.g., any* NTA *domain with a compact Ahlfors regular boundary will do*)*. Then the jump-formula*

$$\begin{aligned}
&\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathcal{D}_\lambda = \tfrac{1}{2}I + K_\lambda \ \text{ on } \left[B^{p,q}_s(\partial\Omega,\sigma)\right]^n, \ \text{ whenever}\\
&\tfrac{n-1}{n}<p<\infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \quad 0<q\leq\infty,
\end{aligned} \tag{6.4.29}$$

holds if $\mathcal{D}_\lambda : \left[B^{p,q}_s(\partial\Omega,\sigma)\right]^n \to \left[B^{p,q}_{s+1/p}(\Omega)_{\rm bdd}\right]^n$ *is the double layer potential operator considered in* (6.4.20)*,* $\mathrm{Tr}_{\Omega\to\partial\Omega} : \left[B^{p,q}_{s+1/p}(\Omega)_{\rm bdd}\right]^n \to \left[B^{p,q}_s(\partial\Omega,\sigma)\right]^n$ *is the boundary trace operator from* [69, (9.4.91) in item *(ii)* of Theorem 9.4.5]*,* I *denotes the identity operator on* $\left[B^{p,q}_s(\partial\Omega,\sigma)\right]^n$*, and* K_λ *is the boundary-to-boundary double layer operator acting on* $\left[B^{p,q}_s(\partial\Omega,\sigma)\right]^n$ *as in* (6.4.1) *of Theorem 6.4.1. Furthermore, the jump-formula*

$$\begin{aligned}
&\mathrm{Tr}_{\Omega\to\partial\Omega}\circ\mathcal{D}_\lambda = \tfrac{1}{2}I + K_\lambda \ \text{ on } \left[B^{p,p}_s(\partial\Omega,\sigma)\right]^n, \ \text{ whenever}\\
&\tfrac{n-1}{n}<p<\infty, \quad (n-1)\big(\tfrac{1}{p}-1\big)_+ < s < 1, \quad \tfrac{n}{n+s+1/p}<q\leq\infty,
\end{aligned} \tag{6.4.30}$$

holds if, this time, $\mathrm{Tr}_{\Omega\to\partial\Omega} : \left[F^{p,q}_{s+1/p}(\Omega)_{\rm bdd}\right]^n \to \left[B^{p,p}_s(\partial\Omega,\sigma)\right]^n$ *is the boundary trace operator from* [69, (9.4.93) in item *(ii)* of Theorem 9.4.5]*,* $\mathcal{D}_\lambda : \left[B^{p,p}_s(\partial\Omega,\sigma)\right]^n \to \left[F^{p,q}_{s+1/p}(\Omega)_{\rm bdd}\right]^n$ *is the double layer potential operator considered in* (6.4.21)*,* I *denotes the identity operator on* $\left[B^{p,p}_s(\partial\Omega,\sigma)\right]^n$

and, finally, K_λ is the boundary-to-boundary double layer operator acting on $\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^n$ as in (6.4.1) of Theorem 6.4.1.

Proof The mapping properties for the double layer potential operator $\mathcal{D}_\lambda$ may be dealt with as in the treatment of items *(1)-(4)* in the proof of Theorem 4.3.1. As regards the operator $\mathcal{P}_\lambda$, from (6.4.18) and (6.2.2) we see that for each function $\vec{f}=(f_k)_{1\leq k\leq n}\in\big[L^1(\partial_*\Omega,\sigma)\big]^n$ we have

$$\begin{aligned}\mathcal{P}_\lambda\vec{f}(x)&=(1+\lambda)\int_{\partial_*\Omega}\Big\{\nu_j(y)\partial_{y_k}\big[q_j(x-y)\big]-\nu_k(y)\partial_{y_j}\big[q_j(x-y)\big]\Big\}f_k(y)\,\mathrm{d}\sigma(y)\\&=-(1+\lambda)\partial_j\big(\mathcal{U}_{jk}f_k\big)(x),\qquad\text{for all }\ x\in\Omega,\end{aligned}\tag{6.4.31}$$

where, for each $j,k\in\{1,\dots,n\}$, the integral operator $\mathcal{U}_{jk}$ is associated as in (4.2.76) to the Laplace operator $L:=\Delta$ and the set Ω. From this, item *(5)* in Theorem 4.3.1, and [69, (9.2.8)] then all desired mapping properties for $\mathcal{P}_\lambda$ follow. Next, the mapping properties of the operator $\mathcal{S}$ are justified by the same argument used in the proof of Theorem 4.3.3, while the mapping properties of the operator $\mathcal{Q}$ are seen from (6.4.19) and Theorem 4.3.3. Finally, the jump-formulas claimed in (6.4.29)-(6.4.30) may be established based on the mapping properties for $\mathcal{D}_\lambda$ in the context of (6.4.20)-(6.4.21), [69, Theorem 9.4.5], Theorem 6.4.1, and (6.2.109). □

Remark 6.4.4 *The boundary-to-boundary version of the single layer potential operator S associated with the Stokes system* (*cf.* (6.2.112)) *satisfies the mapping properties described in Theorem 4.3.4* (*with $M:=n$*). *Indeed, this may be justified in the manner as in the proof of Theorem 4.3.4, now relying on item (7) of Theorem 6.2.11, item (vi) of Theorem 6.2.4, and the mapping properties of the boundary-to-domain version of the single layer potential operator $\mathcal{S}$ from Theorem 6.4.3.*

We next introduce and study the conormal derivative operators ∂^λ_ν, with $\lambda\in\mathbb{C}$, associated with the Stokes system in the context of Besov and Triebel-Lizorkin spaces.

Proposition 6.4.5 *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded (ε,δ)-domain with an Ahlfors regular boundary, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, pick $p,q\in(1,\infty)$ along with $s\in(0,1)$ then set $p':=\big(1-\frac{1}{p}\big)^{-1}$ and $q':=\big(1-\frac{1}{q}\big)^{-1}$. Finally, fix $\lambda\in\mathbb{C}$ and recall the system L_λ from* (6.1.3). *In this context, define the conormal derivative operator ∂^λ_ν associated with the Stokes system as the mapping from the space*

$$\Big\{(\vec{u},\pi;\vec{f}\,)\in\big[B^{p,q}_{s+1/p}(\Omega)\big]^n\oplus B^{p,q}_{s+1/p-1}(\Omega)\oplus\Big[\big(B^{p',q'}_{1-s+1/p'}(\Omega)\big)^*\Big]^n:\\ L_\lambda\vec{u}-\nabla\pi=\vec{f}\lfloor_\Omega\ \text{in}\ \Omega\Big\}\tag{6.4.32}$$

(*where the convention introduced in* [69, (9.5.1)] *has been used*), *with values in $\big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^n$, associating to each triplet $(\vec{u},\pi;\vec{f}\,)$ from* (6.4.32) *the functional*

$$\partial_\nu^\lambda(\vec{u},\pi;\vec{f}\,)\in\Big(\big[B^{p',q'}_{1-s}(\partial\Omega,\sigma)\big]^n\Big)^*=\big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^n \tag{6.4.33}$$

acting according to (recall that the coefficients $a^{\alpha\beta}_{jk}(\lambda)$ are as in (6.1.1), and that the summation convention over repeated indices is presently in effect)

$$\begin{aligned}&{}_{([B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n)^*}\big\langle\partial_\nu^\lambda(\vec{u},\pi;\vec{f}\,),\vec{\varphi}\big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n}\\&\qquad:=\big\langle a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta,\partial_j\Phi_\alpha\big\rangle-\big\langle\pi,\mathrm{div}\vec{\Phi}\big\rangle+\langle\vec{f},\vec{\Phi}\rangle\\&\textit{for all }\ \vec{\varphi}\in\big[B^{p',q'}_{1-s}(\partial\Omega,\sigma)\big]^n\ \textit{ and }\ \vec{\Phi}=(\Phi_\alpha)_\alpha\in\big[B^{p',q'}_{1-s+1/p'}(\Omega)\big]^n\\&\qquad\qquad\textit{satisfying }\ \mathrm{Tr}_{\Omega\to\partial\Omega}\vec{\Phi}=\vec{\varphi},\end{aligned} \tag{6.4.34}$$

where the first two sets of brackets in the right side of the first line above are understood as the duality pairing between $B^{p,q}_{s+1/p-1}(\Omega)$ and $B^{p',q'}_{-s+1/p'}(\Omega)$ (cf. [69, (9.2.141)]) and the final set of brackets in the right side of the first line above is the canonical duality pairing between $\Big[\big(B^{p',q'}_{1-s+1/p'}(\Omega)\big)^*\Big]^n$ and $\big[B^{p',q'}_{1-s+1/p'}(\Omega)\big]^n$.

Then the conormal derivative operator ∂_ν^A considered in (6.4.32)-(6.4.34) is well defined, linear, and bounded in the sense that there exists a constant $C\in(0,\infty)$ with the property that

$$\begin{aligned}&\big\|\partial_\nu^\lambda(\vec{u},\pi;\vec{f}\,)\big\|_{[B^{p,q}_{s-1}(\partial\Omega,\sigma)]^n}\\&\qquad\le C\Big\{\|\vec{u}\|_{[B^{p,q}_{s+1/p}(\Omega)]^n}+\|\pi\|_{B^{p,q}_{s+1/p-1}(\Omega)}+\|\vec{f}\|_{[(B^{p',q'}_{1-s+1/p'}(\Omega))^*]^n}\Big\}\end{aligned} \tag{6.4.35}$$

for each triplet $(\vec{u},\pi;\vec{f}\,)$ belonging to the domain of ∂_ν^λ (cf. (6.4.32)).

Moreover, for each triplet $(\vec{u},\pi;\vec{f}\,)$ as in (6.4.32) and each $\vec{w}\in\big[B^{p',q'}_{1-s+1/p'}(\Omega)\big]^n$ one has the following generalized "half" Green's formula:

$$\begin{aligned}&{}_{([B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n)^*}\big\langle\partial_\nu^\lambda(\vec{u},\pi;\vec{f}\,),\mathrm{Tr}_{\Omega\to\partial\Omega}\vec{w}\big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n}\\&\qquad={}_{B^{p,q}_{s+1/p-1}(\Omega)}\big\langle a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta,\partial_j w_\alpha\big\rangle_{B^{p',q'}_{-s+1/p'}(\Omega)}\\&\qquad\quad-{}_{B^{p,q}_{s+1/p-1}(\Omega)}\big\langle\pi,\mathrm{div}\vec{w}\big\rangle_{B^{p',q'}_{-s+1/p'}(\Omega)}\\&\qquad\quad+{}_{[(B^{p',q'}_{1-s+1/p'}(\Omega))^*]^n}\big\langle\vec{f},\vec{w}\big\rangle_{[B^{p',q'}_{1-s+1/p'}(\Omega)]^n}\end{aligned} \tag{6.4.36}$$

where $(u_\beta)_{1\le\beta\le n}$ are the scalar components of $\vec{u}$ and $(w_\alpha)_{1\le\alpha\le n}$ are the scalar components of $\vec{w}$.

Furthermore, for any two triplets, $(\vec{u},\pi;\vec{f}\,)$ belonging to (6.4.32), and $(\vec{w},\rho;\vec{g}\,)$ belonging to the analogue of (6.4.32) with p,q,s replaced by $p',q',1-s$, one has the following generalized "full" (or "symmetric") Green's formula:

$$
\begin{aligned}
&{}_{([B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n)^*}\Big\langle \partial^\lambda_\nu(\vec{u},\pi;\vec{f}\,), \mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{w}\Big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n}\\
&\qquad - {}_{([B^{p,q}_{s}(\partial\Omega,\sigma)]^n)^*}\Big\langle \partial^\lambda_\nu(\vec{w},\rho;\vec{g}\,), \mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\Big\rangle_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^n}\\
&\quad = {}_{B^{p,q}_{s+1/p-1}(\Omega)}\Big\langle \mathrm{div}\vec{u},\rho\Big\rangle_{B^{p',q'}_{-s+1/p'}(\Omega)} - {}_{B^{p,q}_{s+1/p-1}(\Omega)}\Big\langle \pi,\mathrm{div}\vec{w}\Big\rangle_{B^{p',q'}_{-s+1/p'}(\Omega)}\\
&\qquad + {}_{[(B^{p',q'}_{1-s+1/p'}(\Omega))^*]^n}\Big\langle \vec{f},\vec{w}\Big\rangle_{[B^{p',q'}_{1-s+1/p'}(\Omega)]^n}\\
&\qquad - {}_{[(B^{p,q}_{s+1/p}(\Omega))^*]^n}\Big\langle \vec{g},\vec{u}\Big\rangle_{[B^{p,q}_{s+1/p}(\Omega)]^n}. \qquad (6.4.37)
\end{aligned}
$$

Finally, similar results are valid on the scale of Triebel-Lizorkin spaces (*in the spirit of* [69, Proposition 9.5.2] *with* $\mathbb{A} := F$) *and the scale of weighted Sobolev spaces* (*in the spirit of* [69, Proposition 8.5.3]).

Proof All claims are justified by reasoning as in the proofs of [69, Proposition 8.5.3] and [69, Proposition 9.5.2]. □

Much as in [69, Remark 9.5.3] (see also the subsequent comments), it is possible to define the conormal derivative in the more general setting described below.

Remark 6.4.6 *Retain the assumptions on* Ω, λ, p, q, s *from Proposition 6.4.5, along with the conventions made there. For* $\mathbb{A} \in \{B, F\}$, *set* $q_* := q$ *if* $\mathbb{A} = B$ *and* $q_* := p$ *if* $\mathbb{A} = F$, *and denote by* q'_* *the Hölder conjugate exponent of* q_*. *In addition, fix an arbitrary cutoff function* $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ *with* $\psi \equiv 1$ *near* $\partial\Omega$. *Recall* [69, Convention 8.3.7] (*cf. also* (A.0.217)). *Assume*

$$
\begin{aligned}
\vec{u} = (u_\beta)_{1\le\beta\le n} \in \big[\mathbb{A}^{p,q}_{s+1/p}(\Omega)_{\mathrm{bdd}}\big]^n, \quad \pi \in \mathbb{A}^{p,q}_{s+1/p-1}(\Omega)_{\mathrm{bdd}},\\
\vec{f} \in \big[\big(\mathbb{A}^{p',q'}_{1-s+1/p'}(\Omega)\big)^*\big]^n
\end{aligned}
\qquad (6.4.38)
$$

with the property that

$$
L_\lambda\vec{u} - \nabla\pi = \vec{f}\lfloor_\Omega \ \textit{in} \ \big[\mathcal{D}'(\Omega)\big]^n \qquad (6.4.39)
$$

and define

$$
\partial^\lambda_\nu(\vec{u},\pi;\vec{f}\,) := \partial^\lambda_\nu(\vec{u}_\psi,\pi_\psi;\vec{f}_\psi) \in \big[B^{p,q_*}_{s-1}(\partial\Omega,\sigma)\big]^{\widetilde{M}} \qquad (6.4.40)
$$

with

$$
\begin{gathered}
\vec{u}_\psi := \psi\vec{u} \in \big[\mathbb{A}^{p,q}_{s+1/p}(\Omega)\big]^n, \quad \pi_\psi := \psi\pi \in \mathbb{A}^{p,q}_{s+1/p-1}(\Omega)\\
\textit{and } \ \vec{f}_\psi \in \big[\big(\mathbb{A}^{p',q'}_{1-s+1/p'}(\Omega)\big)^*\big]^n \ \textit{ given by}\\
\vec{f}_\psi := \Big(a^{\alpha\beta}_{jk}(\lambda)(\partial_j\partial_k\psi)u_\beta + a^{\alpha\beta}_{jk}(\lambda)(\partial_j\psi)(\partial_k u_\beta) + a^{\alpha\beta}_{jk}(\lambda)(\partial_k\psi)(\partial_j u_\beta)\Big)_{1\le\alpha\le n}\\
+\psi\vec{f} + \pi\nabla\psi.
\end{gathered}
\qquad (6.4.41)
$$

Then the above definition is meaningful, and does not depend on the particular cutoff function ψ. *Furthermore, for each* $\vec{w} = (w_\alpha)_{1\leq\alpha\leq n} \in \big[\mathbb{A}^{p',q'}_{1-s+1/p'}(\Omega)\big]^n$ *which vanishes outside a bounded subset of* Ω *the generalized "half" Green's formula*

$$
\begin{aligned}
{}_{([B^{p',q'_*}_{1-s}(\partial\Omega,\sigma)]^n)^*}&\Big\langle \partial^\lambda_\nu(\vec{u},\pi;\vec{f}\,), \mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{w}\Big\rangle_{[B^{p',q'_*}_{1-s}(\partial\Omega,\sigma)]^n} \\
&= {}_{\mathbb{A}^{p,q}_{s+1/p-1}(\Omega)}\Big\langle a^{\alpha\beta}_{jk}(\lambda)\psi\,\partial_k u_\beta, \partial_j w_\alpha\Big\rangle_{\mathbb{A}^{p',q'}_{-s+1/p'}(\Omega)} \\
&\quad - {}_{\mathbb{A}^{p,q}_{s+1/p-1}(\Omega)}\Big\langle \psi\pi, \mathrm{div}\vec{w}\Big\rangle_{\mathbb{A}^{p',q'}_{-s+1/p'}(\Omega)} \\
&\quad + {}_{[(\mathbb{A}^{p',q'}_{1-s+1/p'}(\Omega))^*]^n}\Big\langle \vec{f}, \vec{w}\Big\rangle_{[\mathbb{A}^{p',q'}_{1-s+1/p'}(\Omega)]^n}
\end{aligned}
\tag{6.4.42}
$$

holds for each cutoff function $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ *with* $\psi \equiv 1$ *near both* $\partial\Omega$ *and the support of* $\vec{w}$.

In addition, for any triplet $(\vec{w},\rho;\vec{g}\,)$ *satisfying analogous properties to* (6.4.38)-(6.4.39) *with* p,q,s *replaced by* $p',q',1-s$, *and such that both* $\vec{w}$ *and* ρ *vanish outside a bounded subset of* Ω, *the generalized "full" Green's formula*

$$
\begin{aligned}
&{}_{([B^{p',q'_*}_{1-s}(\partial\Omega,\sigma)]^n)^*}\Big\langle \partial^\lambda_\nu(\vec{u},\pi;\vec{f}\,), \mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{w}\Big\rangle_{[B^{p',q'_*}_{1-s}(\partial\Omega,\sigma)]^n} \\
&\qquad - {}_{([B^{p,q_*}_{s}(\partial\Omega,\sigma)]^n)^*}\Big\langle \partial^\lambda_\nu(\vec{w},\rho;\vec{g}\,), \mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\Big\rangle_{[B^{p,q_*}_{s}(\partial\Omega,\sigma)]^n} \\
&= {}_{\mathbb{A}^{p,q}_{s+1/p-1}(\Omega)}\Big\langle \mathrm{div}\vec{u}, \rho\Big\rangle_{\mathbb{A}^{p',q'}_{-s+1/p'}(\Omega)} - {}_{\mathbb{A}^{p,q}_{s+1/p-1}(\Omega)}\Big\langle \pi, \mathrm{div}\vec{w}\Big\rangle_{\mathbb{A}^{p',q'}_{-s+1/p'}(\Omega)} \\
&\qquad + {}_{[(\mathbb{A}^{p',q'}_{1-s+1/p'}(\Omega))^*]^n}\Big\langle \vec{f}, \vec{w}\Big\rangle_{[\mathbb{A}^{p',q'}_{1-s+1/p'}(\Omega)]^n} \\
&\qquad - {}_{[(\mathbb{A}^{p,q}_{s+1/p}(\Omega))^*]^n}\Big\langle \vec{g}, \vec{u}\Big\rangle_{[\mathbf{A}^{p,q}_{s+1/p}(\Omega)]^n}
\end{aligned}
\tag{6.4.43}
$$

holds for each cutoff function $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ *with* $\psi \equiv 1$ *near both* $\partial\Omega$ *as well as the supports of* $\vec{w}$ *and* ρ.

Finally, similar results are valid on the scale of weighted Sobolev spaces.

Our next result deals with the conormal derivative of the pair consisting of the hydrostatic double layer and the associated pressure potential.

Theorem 6.4.7 *Let* $\Omega \subseteq \mathbb{R}^n$ (*where* $n \in \mathbb{N}$, $n \geq 2$) *be an open set satisfying a two-sided local John condition and whose boundary is compact and Ahlfors regular. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω. *Also, fix* $\lambda \in \mathbb{C}$ *along with*

$$
1 < p < \infty, \quad 0 < q \leq \infty, \quad 0 < s < 1. \tag{6.4.44}
$$

Then the operator $\partial^\lambda_\nu\big(\mathcal{D}_\lambda, \mathcal{P}_\lambda\big)$ *from item (ix) of Theorem* 6.2.4 *extends in a unique fashion to a bounded linear mapping*

$$\big[B^{p,q}_s(\partial\Omega,\sigma)\big]^n \ni \vec{f} \longmapsto \partial^\lambda_\nu\big(\mathcal{D}_\lambda\vec{f},\mathcal{P}_\lambda\vec{f}\big) \in \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^n \tag{6.4.45}$$

and various choices of p,q,s *as in* (6.4.44) *yield operators which are compatible with one another. Moreover, if* $s\in(0,1)$ *and the exponents* $p,q,p',q'\in(1,\infty)$ *satisfy* $1/p+1/p'=1=1/q+1/q'$ *then*

$$\begin{aligned}
&{}_{[B^{p,q}_{s-1}(\partial\Omega,\sigma)]^n}\Big\langle\partial^\lambda_\nu\big(\mathcal{D}_\lambda\vec{f},\mathcal{P}_\lambda\vec{f}\big),\vec{g}\Big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n}\\
&\qquad={}_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^n}\Big\langle\vec{f},\partial^\lambda_\nu\big(\mathcal{D}_\lambda\vec{g},\mathcal{P}_\lambda\vec{g}\big)\Big\rangle_{[B^{p',q'}_{-s}(\partial\Omega,\sigma)]^n}\\
&\text{for each } \vec{f}\in\big[B^{p,q}_s(\partial\Omega,\sigma)\big]^n \text{ and } \vec{g}\in\big[B^{p',q'}_{1-s}(\partial\Omega,\sigma)\big]^n.
\end{aligned}\tag{6.4.46}$$

Finally, under the additional assumption that Ω *is an* (ε,δ)*-domain, one has the following compatibility result*

$$\partial^\lambda_\nu\big(\mathcal{D}_\lambda\vec{f},\mathcal{P}_\lambda\vec{f}\big)=\partial^\lambda_\nu\big(\mathcal{D}_\lambda\vec{f},\mathcal{P}_\lambda\vec{f};0\big) \text{ for each } \vec{f}\in\big[B^{p,q}_s(\partial\Omega,\sigma)\big]^n, \tag{6.4.47}$$

where the expression in the left-hand side is considered as in (6.4.45)*, and the conormal derivative in the right-hand side is taken in either of the scenarios described in Proposition 6.4.5.*

Proof The same type of argument as in the proof of Theorem 4.3.5, now making use of item *(ix)* of Theorem 6.2.4 and Proposition 6.4.5, yields all desired conclusions. □

The jump-formula for the conormal derivative of the pair $(\mathcal{S},\mathcal{Q})$ is discussed in our next theorem.

Theorem 6.4.8 *Let* $\Omega\subseteq\mathbb{R}^n$ *be an* NTA *domain with a compact Ahlfors regular boundary. Abbreviate* $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Also, select* $\lambda\in\mathbb{C}$ *and pick*

$$1<p<\infty,\quad 1<q<\infty,\quad 0<s<1. \tag{6.4.48}$$

Then, with I *denoting the identity operator, one has the jump-formula*

$$\partial^\lambda_\nu\big(\mathcal{S}\vec{f},\mathcal{Q}\vec{f};0\big)=\big(-\tfrac{1}{2}I+K^\#_\lambda\big)\vec{f} \text{ for each } \vec{f}\in\big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^n, \tag{6.4.49}$$

where the conormal derivative in the left-hand side of (6.4.49) *is considered in either of the scenarios described in Proposition 6.4.5.*

Proof This parallels the proof of Theorem 4.3.6, now making use of item *(vi)* in Theorem 6.2.4, Proposition 6.4.5, and Theorem 6.4.1. □

The basic Green-type integral representation formulas involving hydrostatic layer potentials for functions belonging to Besov, Triebel-Lizorkin, and weighted Sobolev spaces make the object of the theorem below.

Theorem 6.4.9 *Let* $\Omega\subseteq\mathbb{R}^n$ *be an* (ε,δ)*-domain with a compact Ahlfors regular boundary. Abbreviate* $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure* *theoretic outward unit normal to* Ω*. Also, fix some* $\lambda \in \mathbb{C}$ *and let* $\vec{u} \in \big[\mathscr{C}^\infty(\Omega)\big]^n$ *and* $\pi \in \mathscr{C}^\infty(\Omega)$ *solve the Stokes system in* Ω *(cf.* (6.0.1)*). Then the integral representation formulas*

$$\vec{u} = \mathcal{D}_\lambda\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\big) - \mathscr{S}\big(\partial_\nu^\lambda(\vec{u},\pi;0)\big) \ \ in \ \ \Omega, \tag{6.4.50}$$

and

$$\pi = \mathcal{P}_\lambda\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\big) - \mathcal{Q}\big(\partial_\nu^\lambda(\vec{u},\pi;0)\big) \ \ in \ \ \Omega, \tag{6.4.51}$$

hold with the understanding that if Ω *is an exterior domain one also assumes that there exists* $\mu \in (1,\infty)$ *such that*

$$⨍_{B(0,\mu R)\setminus B(0,R)} \big\{|\vec{u}| + R|\pi|\big\}\,\mathrm{d}\mathcal{L}^n = o(1) \ \ as \ \ R \to \infty, \tag{6.4.52}$$

in the case of (6.4.50)*, and*

$$⨍_{B(0,\mu R)\setminus B(0,R)} \big\{|\vec{u}| + R|\pi|\big\}\,\mathrm{d}\mathcal{L}^n = o(R) \ \ as \ \ R \to \infty, \tag{6.4.53}$$

in the case of (6.4.51)*, in any for the following scenarios:*

(1) One assumes that

$$\begin{aligned} &\vec{u} \in \Big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\mathrm{bdd}}\Big]^n \ \ and \ \ \pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega)_{\mathrm{bdd}} \ \ with \\ &\quad 1 < p < \infty, \quad 0 < s < 1, \quad 1 < q < \infty, \end{aligned} \tag{6.4.54}$$

and

$\mathrm{Tr}_{\Omega\to\partial\Omega} : \Big[B^{p,q}_{s+\frac{1}{p}}(\Omega)_{\mathrm{bdd}}\Big]^n \longrightarrow \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^n$ *is the boundary trace operator from* [69, (9.4.91) in item *(ii)* of Theorem 9.4.5] (*hence* $\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u} \in \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^n$*), the layer potential operators* $\mathcal{D}_\lambda$*,* $\mathcal{P}_\lambda$*,* $\mathscr{S}$*,* $\mathcal{Q}$ *are as in* (6.4.20)*, and* $\partial_\nu^\lambda(\vec{u},\pi;0) \in \big[B^{p,q}_{s-1}(\partial\Omega,\sigma)\big]^n$ *is the conormal derivative defined as in Proposition* 6.4.5 *and further extended in Remark* 6.4.6*.* (6.4.55)

(2) One assumes that

$$\begin{aligned} &\vec{u} \in \Big[F^{p,q}_{s+\frac{1}{p}}(\Omega)_{\mathrm{bdd}}\Big]^n \ \ and \ \ \pi \in F^{p,q}_{s+\frac{1}{p}-1}(\Omega)_{\mathrm{bdd}} \ \ with \\ &\quad 1 < p < \infty, \quad 0 < s < 1, \quad 1 < q < \infty, \end{aligned} \tag{6.4.56}$$

and

$\mathrm{Tr}_{\Omega\to\partial\Omega} : \big[F^{p,q}_{s+\frac{1}{p}}(\Omega)_{\mathrm{bdd}}\big]^n \to \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^n$ *is the boundary trace operator from* [69, (9.4.93) in item *(ii)* of Theorem 9.4.5] (*hence* $\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u} \in \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^n$*), the layer potential operators* $\mathcal{D}_\lambda$*,* $\mathcal{P}_\lambda$*,* $\mathcal{S}$*,* $\mathcal{Q}$ *are as in* (6.4.21)*, and* $\partial^\lambda_\nu(\vec{u},\pi;0) \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^n$ *is the conormal derivative defined as in Proposition 6.4.5 and further extended in Remark 6.4.6.* (6.4.57)

(3) One also assumes that $\mathbb{R}^n \setminus \Omega$ *is n-thick, that*

$$\vec{u} \in \big[W^{1,p}_a(\Omega)_{\mathrm{bdd}}\big]^n \ \textit{and} \ \pi \in L^p_{\mathrm{bdd}}\big(\Omega, \delta^{ap}_{\partial\Omega}\mathcal{L}^n\big) \ \textit{with} \quad 1 < p < \infty, \quad 0 < s < 1, \quad a := 1 - s - \tfrac{1}{p}, \tag{6.4.58}$$

and

$\mathrm{Tr}_{\Omega\to\partial\Omega} : \big[W^{1,p}_a(\Omega)_{\mathrm{bdd}}\big]^n \to \big[B^{p,p}_s(\partial\Omega,\sigma)\big]^n$ *is the boundary trace operator from* [69, Theorem 8.3.6] (*hence* $\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}$ *belongs to* $\big[B^{p,p}_s(\partial\Omega,\sigma)\big]^n$*), the layer potential operators* $\mathcal{D}_\lambda$*,* $\mathcal{P}_\lambda$*,* $\mathcal{S}$*,* $\mathcal{Q}$ *are as in* (6.4.8)-(6.4.11) *and, finally,* $\partial^\lambda_\nu(\vec{u},\pi;0) \in \big[B^{p,p}_{s-1}(\partial\Omega,\sigma)\big]^n$ *is the conormal derivative defined as in the very last part of Proposition 6.4.5 and further extended in Remark 6.4.6.* (6.4.59)

Finally, if Ω *is an exterior domain and in place of* (6.4.52) *one imposes the weaker assumption*

$$\fint_{B(0,\mu R)\setminus B(0,R)} \big\{|\vec{u}| + R|\pi|\big\}\,\mathrm{d}\mathcal{L}^n = o(R) \ \textit{as} \ R \to \infty, \tag{6.4.60}$$

then in place of (6.4.50) *one now concludes that there exists a constant* $c \in \mathbb{C}^n$ *with the property that*

$$\vec{u} = \mathcal{D}_\lambda\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\big) - \mathcal{S}\big(\partial^\lambda_\nu(\vec{u},\pi;0)\big) + c \ \textit{in} \ \Omega, \tag{6.4.61}$$

in any of the scenarios described in items (1)-(3) above.

Proof We shall carry out the proof in the case when Ω is assumed to be bounded, since the case when Ω is an exterior domain largely proceeds along similar lines.

Consider the scenario specified in item *(1)*. Fix a point $x \in \Omega$ and choose a scalar-valued function $\theta \in \mathscr{C}^\infty(\mathbb{R}^n)$ with the property that $\theta = 0$ identically on $B(0,1)$ and $\theta = 1$ identically on $\mathbb{R}^n \setminus B(0,2)$. For each $\varepsilon \in \big(0, \frac{1}{2}\,\mathrm{dist}(x,\partial\Omega)\big)$ define $\theta_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by setting

$$\theta_\varepsilon(y) := \theta\Big(\frac{y-x}{\varepsilon}\Big) \ \text{for every} \ y \in \mathbb{R}^n, \tag{6.4.62}$$

so that

$$\begin{aligned} &\theta_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n), \quad 1-\theta_\varepsilon \in \mathscr{C}^\infty_c(\Omega), \\ &\theta_\varepsilon \equiv 0 \text{ on } B(x,\varepsilon) \text{ and } \theta_\varepsilon \equiv 1 \text{ on } \mathbb{R}^n \setminus B(x,2\varepsilon). \end{aligned} \tag{6.4.63}$$

Next, bring in the fundamental solution for the Stokes system $E = (E_{jk})_{1\leq j,k\leq n}$ with entries as in (6.2.1), and recall the accompanying pressure vector $\vec{q} = (q_j)_{1\leq j\leq n}$ from (6.2.2). Fix $j \in \{1,\dots,n\}$ and introduce

$$\vec{w} := \big(E_{jk}(x-\cdot)\theta_\varepsilon\big)_{1\leq k\leq n} \in \big[\mathscr{C}^\infty(\overline{\Omega})\big]^n \text{ and } \rho := -q_j(x-\cdot)\theta_\varepsilon \in \mathscr{C}^\infty(\overline{\Omega}). \tag{6.4.64}$$

Then a direct computation (based on (6.2.4)-(6.2.5) and (6.4.63)) gives that

$$\mathrm{div}\vec{w} = E_{jk}(x-\cdot)(\partial_k\theta_\varepsilon) \in \mathscr{C}^\infty_c(\Omega), \tag{6.4.65}$$

and

$$\begin{aligned} L_\lambda\vec{w} - \nabla\rho = \Big\{ &-2(\partial_\ell E_{jk})(x-\cdot)(\partial_\ell\theta_\varepsilon) + E_{jk}(x-\cdot)(\Delta\theta_\varepsilon) - q_j(x-\cdot)(\partial_k\theta_\varepsilon) \\ &-\lambda(\partial_k E_{j\ell})(x-\cdot)(\partial_\ell\theta_\varepsilon) + \lambda E_{j\ell}(x-\cdot)(\partial_k\partial_\ell\theta_\varepsilon)\Big\}_{1\leq k\leq n} \end{aligned} \tag{6.4.66}$$

is a vector with components in $\mathscr{C}^\infty_c(\Omega)$. Furthermore, $\vec{w} \in \big[B^{p',q'}_{1-s+\frac{1}{p'}}(\Omega)\big]^n$ and $\rho \in B^{p',q'}_{-s+\frac{1}{p'}}(\Omega)$ where p', q' are the Hölder conjugate exponents of p, q. Define next

$$\vec{g} := L_\lambda\vec{w} - \nabla\rho \in \big[\mathscr{C}^\infty_c(\Omega)\big]^n. \tag{6.4.67}$$

When naturally regarded as a functional $\vec{g} \in \Big[\big(B^{p,q}_{s+1/p}(\Omega)\big)^*\Big]^n$ via integral pairing over Ω, this trivially satisfies

$$L_\lambda\vec{w} - \nabla\rho = \vec{g}\lfloor_\Omega \text{ in } \big[\mathcal{D}'(\Omega)\big]^n. \tag{6.4.68}$$

In addition, if $(\nu_1,\dots,\nu_n)$ are the components of ν, then from the memberships in (6.4.64), Proposition 6.4.5, and (6.1.10) we see that

$$\partial^\lambda_\nu(\vec{w},\rho;\vec{g}) = \Big(\nu_r a^{\alpha\beta}_{rs}(\lambda)\big(\partial_s w_\beta\big)\big|_{\partial\Omega} - \nu_\alpha\rho\big|_{\partial\Omega}\Big)_{1\leq\alpha\leq n}. \tag{6.4.69}$$

Granted these properties, the generalized "full" Green's formula (6.4.37) for Besov spaces, applied with $\vec{u}, \pi$ as the statement of the theorem (which are currently assumed to be as in (6.4.54)), with $\vec{f} := 0 \in \Big[\big(B^{p',q'}_{1-s+1/p'}(\Omega)\big)^*\Big]^n$, and with $\vec{w}$, ρ, and $\vec{g}$ as above, yields

$$
\begin{aligned}
&{}_{([B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n)^*}\Big\langle \partial^\lambda_\nu(\vec{u},\pi;0),\,\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{w}\Big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n}\\
&\qquad -{}_{([B^{p,q}_{s}(\partial\Omega,\sigma)]^n)^*}\Big\langle \partial^\lambda_\nu(\vec{w},\rho;\vec{g}),\,\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\Big\rangle_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^n}\\
&\qquad = -{}_{B^{p,q}_{s+1/p-1}(\Omega)}\Big\langle \pi,\mathrm{div}\vec{w}\Big\rangle_{B^{p',q'}_{-s+1/p'}(\Omega)} - {}_{[(B^{p,q}_{s+1/p}(\Omega))^*]^n}\Big\langle \vec{g},\vec{u}\Big\rangle_{[B^{p,q}_{s+1/p}(\Omega)]^n}\\
&\qquad = -{}_{\mathcal{D}'(\Omega)}\Big\langle \pi,\mathrm{div}\vec{w}\Big\rangle_{\mathcal{D}(\Omega)} - \int_\Omega \Big\langle L_\lambda\vec{w}-\nabla\rho,\vec{u}\Big\rangle\,d\mathcal{L}^n. \qquad (6.4.70)
\end{aligned}
$$

Note that, thanks to (6.4.64) and Proposition 4.2.8,

$$
\begin{aligned}
&{}_{([B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n)^*}\Big\langle \partial^\lambda_\nu(\vec{u},\pi;0),\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{w}\Big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n}\\
&\qquad\qquad = \Big(\mathscr{S}\big(\partial^\lambda_\nu(\vec{u},\pi;0)\big)\Big)_j(x). \qquad (6.4.71)
\end{aligned}
$$

By virtue of (6.4.69), (6.4.64), and (6.2.17) we also have

$$
\begin{aligned}
&{}_{([B^{p,q}_{s}(\partial\Omega,\sigma)]^n)^*}\Big\langle \partial^\lambda_\nu(\vec{w},\rho;\vec{g}),\,\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\Big\rangle_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^n}\\
&\qquad = \int_{\partial_*\Omega} \Big(\nu_r a^{\alpha\beta}_{rs}(\lambda)\big(\partial_s w_\beta\big)\big|_{\partial\Omega} - \nu_\alpha\rho\big|_{\partial\Omega}\Big)\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\big)_\alpha\,d\sigma\\
&\qquad = \int_{\partial_*\Omega} \Big\{-\nu_r(y)a^{\alpha\beta}_{rs}(\lambda)(\partial_s E_{j\beta})(x-y) + \nu_\alpha(y)q_j(x-y)\Big\}\times\\
&\qquad\qquad\qquad \times\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\big)_\alpha(y)\,d\sigma(y)\\
&\qquad = \Big(\mathcal{D}_\lambda\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\big)\Big)_j(x). \qquad (6.4.72)
\end{aligned}
$$

There remains to consider the last line of (6.4.70). In this regard, based on (6.4.65)-(6.4.66) we may express

$$
\begin{aligned}
&-{}_{\mathcal{D}'(\Omega)}\big\langle \pi\,, \mathrm{div}\vec{w}\big\rangle_{\mathcal{D}(\Omega)} - \int_\Omega \big\langle L_\lambda \vec{w} - \nabla\rho, \vec{u}\big\rangle\, \mathrm{d}\mathcal{L}^n \\
&\quad = -{}_{\mathcal{D}'(\Omega)}\big\langle E_{jk}(x-\cdot), (\partial_k\theta_\varepsilon)\pi\big\rangle_{\mathcal{D}(\Omega)} \\
&\qquad + {}_{\mathcal{D}'(\Omega)}\big\langle 2(\partial_\ell E_{jk})(x-\cdot), (\partial_\ell\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)} \\
&\qquad - {}_{\mathcal{D}'(\Omega)}\big\langle E_{jk}(x-\cdot), (\Delta\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)} \\
&\qquad + {}_{\mathcal{D}'(\Omega)}\big\langle q_j(x-\cdot), (\partial_k\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)} \\
&\qquad - {}_{\mathcal{D}'(\Omega)}\big\langle \lambda(\partial_k E_{j\ell})(x-\cdot), (\partial_\ell\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)} \\
&\qquad + {}_{\mathcal{D}'(\Omega)}\big\langle \lambda E_{j\ell}(x-\cdot), (\partial_k\partial_\ell\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)} \\
&\quad =: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI}.
\end{aligned}
\tag{6.4.73}
$$

Observe that since $\nabla\theta_\varepsilon \in \big[\mathscr{C}_c^\infty(\Omega)\big]^n$ and $\theta_\varepsilon - 1 \in \mathscr{C}_c^\infty(\Omega)$, we may use (6.2.5) to write

$$
\begin{aligned}
\mathrm{I} &= -{}_{\mathcal{D}'(\Omega)}\big\langle E_{jk}(x-\cdot), (\partial_k\theta_\varepsilon)\pi\big\rangle_{\mathcal{D}(\Omega)} \\
&= -{}_{\mathcal{D}'(\Omega)}\big\langle E_{jk}(x-\cdot), \partial_k\big[(\theta_\varepsilon-1)\pi\big]\big\rangle_{\mathcal{D}(\Omega)} \\
&\quad + {}_{\mathcal{D}'(\Omega)}\big\langle E_{jk}(x-\cdot), (\theta_\varepsilon-1)(\partial_k\pi)\big\rangle_{\mathcal{D}(\Omega)} \\
&= {}_{\mathcal{D}'(\Omega)}\big\langle E_{jk}(x-\cdot), (\theta_\varepsilon-1)(\partial_k\pi)\big\rangle_{\mathcal{D}(\Omega)}.
\end{aligned}
\tag{6.4.74}
$$

In a similar fashion,

$$
\begin{aligned}
\mathrm{II} + \mathrm{III} &= {}_{\mathcal{D}'(\Omega)}\big\langle E_{jk}(x-\cdot), (\Delta\theta_\varepsilon)u_k + 2(\partial_\ell\theta_\varepsilon)(\partial_\ell u_k)\big\rangle_{\mathcal{D}(\Omega)} \\
&= {}_{\mathcal{D}'(\Omega)}\big\langle E_{jk}(x-\cdot), \Delta\big[(\theta_\varepsilon-1)u_k\big]\big\rangle_{\mathcal{D}(\Omega)} \\
&\quad - {}_{\mathcal{D}'(\Omega)}\big\langle E_{jk}(x-\cdot), (\theta_\varepsilon-1)(\Delta u_k)\big\rangle_{\mathcal{D}(\Omega)} \\
&= {}_{\mathcal{D}'(\Omega)}\big\langle \Delta[E_{jk}(x-\cdot)], (\theta_\varepsilon-1)u_k\big\rangle_{\mathcal{D}(\Omega)} \\
&\quad - {}_{\mathcal{D}'(\Omega)}\big\langle E_{jk}(x-\cdot), (\theta_\varepsilon-1)(\Delta u_k)\big\rangle_{\mathcal{D}(\Omega)}.
\end{aligned}
\tag{6.4.75}
$$

Next, given that $\vec{u}$ is divergence-free, we have

$$
\begin{aligned}
\mathrm{IV} &= {}_{\mathcal{D}'(\Omega)}\big\langle q_j(x-\cdot), (\partial_k\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)} \\
&= {}_{\mathcal{D}'(\Omega)}\big\langle q_j(x-\cdot), \partial_k\big((\theta_\varepsilon-1)u_k\big)\big\rangle_{\mathcal{D}(\Omega)} \\
&= -{}_{\mathcal{D}'(\Omega)}\big\langle \partial_k[q_j(x-\cdot)], (\theta_\varepsilon-1)u_k\big\rangle_{\mathcal{D}(\Omega)}.
\end{aligned}
\tag{6.4.76}
$$

Finally, by once again using the fact that $\vec{u}$ is divergence-free we obtain

$$\mathrm{V}+\mathrm{VI} = -{}_{\mathcal{D}'(\Omega)}\Big\langle \lambda E_{j\ell}(x-\cdot), \partial_k\big[(\partial_\ell\theta_\varepsilon)u_k\big]\Big\rangle_{\mathcal{D}(\Omega)} + {}_{\mathcal{D}'(\Omega)}\Big\langle \lambda E_{j\ell}(x-\cdot), (\partial_k\partial_\ell\theta_\varepsilon)u_k\Big\rangle_{\mathcal{D}(\Omega)} = 0. \tag{6.4.77}$$

Bearing in mind (6.2.4) and the fact that $\Delta\vec{u} = \nabla\pi$ in Ω, from (6.4.74)-(6.4.77) we conclude that

$$\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI} = {}_{\mathcal{D}'(\Omega)}\Big\langle \delta_{jk}\delta_x, (\theta_\varepsilon - 1)u_k\Big\rangle_{\mathcal{D}(\Omega)} = -u_j(x). \tag{6.4.78}$$

At this stage, the integral representation formula (6.4.50) follows from (6.4.70), (6.4.71), (6.4.72), (6.4.87), and (6.4.78), on account of the arbitrariness of the index $j \in \{1,\dots,n\}$ and point $x \in \Omega$.

Let us now turn our attention to the integral representation formula claimed in (6.4.51). To get started, introduce

$$\vec{\omega} := \vec{q}(x-\cdot)\theta_\varepsilon = \big\{q_k(x-\cdot)\theta_\varepsilon\big\}_{1\le k\le n} \in \big[\mathscr{C}^\infty(\overline{\Omega})\big]^n \subseteq \big[B^{p',q'}_{1-s+\frac{1}{p'}}(\Omega)\big]^n, \tag{6.4.79}$$

and observe that, thanks to (6.2.7) and (6.4.63),

$$\mathrm{div}\vec{\omega} = q_k(x-\cdot)(\partial_k\theta_\varepsilon) \in \mathscr{C}^\infty_c(\Omega), \tag{6.4.80}$$

and

$$\vec{h} := L_\lambda\vec{\omega} = \Big\{2\partial_\ell[q_k(x-\cdot)](\partial_\ell\theta_\varepsilon) + q_k(x-\cdot)(\Delta\theta_\varepsilon) + \lambda\partial_k[q_j(x-\cdot)](\partial_j\theta_\varepsilon) + \lambda q_j(x-\cdot)(\partial_k\partial_j\theta_\varepsilon)\Big\}_{1\le k\le n} \in \big[\mathscr{C}^\infty_c(\Omega)\big]^n. \tag{6.4.81}$$

When naturally regarded as a functional $\vec{h} \in \Big[\big(B^{p,q}_{s+1/p}(\Omega)\big)^*\Big]^n$ via integral pairing over Ω, this trivially satisfies

$$L_\lambda\vec{\omega} = \vec{h}\lfloor_\Omega \ \text{ in } \ \big[\mathcal{D}'(\Omega)\big]^n. \tag{6.4.82}$$

In addition, from the memberships in (6.4.64), Proposition 6.4.5, (6.1.10), and (6.1.1) we see that

$$\begin{aligned}
\partial^\lambda_\nu(\vec{\omega},0;\vec{h}) &= \Big(\nu_r a^{\alpha\beta}_{rs}(\lambda)\big(\partial_s\omega_\beta\big)\big|_{\partial\Omega}\Big)_{1\le\alpha\le n} \\
&= \Big(\nu_s\partial_s[q_\alpha(x-\cdot)]\big|_{\partial\Omega}\Big)_{1\le\alpha\le n} + \Big(\lambda\nu_\beta\partial_\alpha[q_\beta(x-\cdot)]\big|_{\partial\Omega}\Big)_{1\le\alpha\le n} \\
&= (1+\lambda)\Big(\nu_s\partial_s[q_\alpha(x-\cdot)]\big|_{\partial\Omega}\Big)_{1\le\alpha\le n},
\end{aligned} \tag{6.4.83}$$

where the last equality uses (6.2.6). The idea is now to invoke the version of the generalized "full" Green's formula (6.4.37) for Besov spaces, applied with $\vec{u}, \pi$ as the statement of the theorem (which are currently assumed to be as in (6.4.54)), with $\vec{f} := 0 \in \Big[\big(B^{p',q'}_{1-s+1/p'}(\Omega)\big)^*\Big]^n$, and with $\vec{w} := \vec{\omega}$, $\rho := 0$, and $\vec{g} := \vec{h}$ as above. Specifically, we obtain

$$
\begin{aligned}
&{}_{([B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n)^*}\big\langle \partial^\lambda_\nu(\vec{u},\pi;0), \mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{\omega}\big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n}\\
&\qquad - {}_{([B^{p,q}_{s}(\partial\Omega,\sigma)]^n)^*}\big\langle \partial^\lambda_\nu(\vec{\omega},0;\vec{h}), \mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\big\rangle_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^n}\\
&\quad = -{}_{B^{p,q}_{s+1/p-1}(\Omega)}\big\langle \pi, \mathrm{div}\vec{\omega}\big\rangle_{B^{p',q'}_{-s+1/p'}(\Omega)} - {}_{[(B^{p,q}_{s+1/p}(\Omega))^*]^n}\big\langle \vec{h},\vec{u}\big\rangle_{[B^{p,q}_{s+1/p}(\Omega)]^n}\\
&\quad = -{}_{\mathcal{D}'(\Omega)}\big\langle \pi, \mathrm{div}\vec{\omega}\big\rangle_{\mathcal{D}(\Omega)} - \int_\Omega \big\langle \vec{h},\vec{u}\big\rangle\,\mathrm{d}\mathcal{L}^n. \qquad (6.4.84)
\end{aligned}
$$

Note that thanks to (6.4.79) and the manner the action of the integral operator $\mathcal{Q}$ from (6.2.14) is extended to distributions on $\partial\Omega$ (cf. (6.4.19)) we have

$$
{}_{([B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n)^*}\big\langle \partial^\lambda_\nu(\vec{u},\pi;0), \mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{\omega}\big\rangle_{[B^{p',q'}_{1-s}(\partial\Omega,\sigma)]^n} = \mathcal{Q}\big(\partial^\lambda_\nu(\vec{u},\pi;0)\big)(x). \qquad (6.4.85)
$$

On account of (6.4.83) and (6.2.21) we also have

$$
\begin{aligned}
&{}_{([B^{p,q}_{s}(\partial\Omega,\sigma)]^n)^*}\big\langle \partial^\lambda_\nu(\vec{\omega},0;\vec{h}), \mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\big\rangle_{[B^{p,q}_{s}(\partial\Omega,\sigma)]^n}\\
&\qquad = (1+\lambda)\int_{\partial_*\Omega} \nu_s(y)\partial_{y_s}[q_\alpha(x-y)]\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\big)_\alpha(y)\,\mathrm{d}\sigma(y)\\
&\qquad = \mathcal{P}_\lambda\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\,\vec{u}\big)(x). \qquad (6.4.86)
\end{aligned}
$$

Let us take a closer look at the last line of (6.4.70). For starters, use (6.4.65)-(6.4.66) to expand

$$
\begin{aligned}
&-{}_{\mathcal{D}'(\Omega)}\big\langle \pi\,, \mathrm{div}\vec{\omega}\big\rangle_{\mathcal{D}(\Omega)} - \int_\Omega \big\langle \vec{h},\vec{u}\big\rangle\,\mathrm{d}\mathcal{L}^n\\
&\qquad = -{}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\partial_k\theta_\varepsilon)\pi\big\rangle_{\mathcal{D}(\Omega)}\\
&\qquad\quad - {}_{\mathcal{D}'(\Omega)}\big\langle 2\partial_\ell[q_k(x-\cdot)], (\partial_\ell\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)}\\
&\qquad\quad - {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\Delta\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)}\\
&\qquad\quad - {}_{\mathcal{D}'(\Omega)}\big\langle \lambda\partial_k[q_j(x-\cdot)], (\partial_j\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)}\\
&\qquad\quad - {}_{\mathcal{D}'(\Omega)}\big\langle \lambda q_j(x-\cdot), (\partial_k\partial_j\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)}\\
&\qquad =: \mathrm{I}' + \mathrm{II}' + \mathrm{III}' + \mathrm{IV}' + \mathrm{V}'. \qquad (6.4.87)
\end{aligned}
$$

Since from (6.4.63) we know that $\nabla\theta_\varepsilon \in \big[\mathscr{C}_c^\infty(\Omega)\big]^n$ and $\theta_\varepsilon - 1 \in \mathscr{C}_c^\infty(\Omega)$, while from (6.2.7) we see that $\partial_k[q_k(x-\cdot)] = -\delta_x$, we may compute

$$\begin{aligned}
\mathrm{I}' &= -{}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\partial_k\theta_\varepsilon)\pi\big\rangle_{\mathcal{D}(\Omega)} \\
&= -{}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), \partial_k\big((\theta_\varepsilon-1)\pi\big)\big\rangle_{\mathcal{D}(\Omega)} + {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\theta_\varepsilon-1)(\partial_k\pi)\big\rangle_{\mathcal{D}(\Omega)} \\
&= {}_{\mathcal{D}'(\Omega)}\big\langle \partial_k[q_k(x-\cdot)], (\theta_\varepsilon-1)\pi\big\rangle_{\mathcal{D}(\Omega)} + {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\theta_\varepsilon-1)(\partial_k\pi)\big\rangle_{\mathcal{D}(\Omega)} \\
&= {}_{\mathcal{D}'(\Omega)}\big\langle \delta_x, (\theta_\varepsilon-1)\pi\big\rangle_{\mathcal{D}(\Omega)} + {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\theta_\varepsilon-1)(\partial_k\pi)\big\rangle_{\mathcal{D}(\Omega)} \\
&= -\pi(x) + {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\theta_\varepsilon-1)(\partial_k\pi)\big\rangle_{\mathcal{D}(\Omega)}.
\end{aligned} \tag{6.4.88}$$

Making use of (6.2.7), the fact that $\vec{u}$ is divergence-free, that $\Delta\vec{u} = \nabla\pi$, and that $\nabla\theta_\varepsilon \equiv 0$ on $B(x,\varepsilon)$ (cf. (6.4.63)), we also obtain

$$\begin{aligned}
\mathrm{II}' + \mathrm{III}' &= {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), 2\partial_\ell\big((\partial_\ell\theta_\varepsilon)u_k\big)\big\rangle_{\mathcal{D}(\Omega)} \\
&\quad - {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\Delta\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)} \\
&= {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\Delta\theta_\varepsilon)u_k + 2(\partial_\ell\theta_\varepsilon)(\partial_\ell u_k)\big\rangle_{\mathcal{D}(\Omega)} \\
&= {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), \Delta\big((\theta_\varepsilon-1)u_k\big) - (\theta_\varepsilon-1)(\Delta u_k)\big\rangle_{\mathcal{D}(\Omega)} \\
&= {}_{\mathcal{D}'(\Omega)}\big\langle \Delta[q_k(x-\cdot)], (\theta_\varepsilon-1)u_k\big\rangle_{\mathcal{D}(\Omega)} \\
&\quad - {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\theta_\varepsilon-1)(\Delta u_k)\big\rangle_{\mathcal{D}(\Omega)} \\
&= -{}_{\mathcal{D}'(\Omega)}\big\langle \partial_k\delta_x, (\theta_\varepsilon-1)u_k\big\rangle_{\mathcal{D}(\Omega)} - {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\theta_\varepsilon-1)(\partial_k\pi)\big\rangle_{\mathcal{D}(\Omega)} \\
&= {}_{\mathcal{D}'(\Omega)}\big\langle \delta_x, (\partial_k\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)} - {}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\theta_\varepsilon-1)(\partial_k\pi)\big\rangle_{\mathcal{D}(\Omega)} \\
&= -{}_{\mathcal{D}'(\Omega)}\big\langle q_k(x-\cdot), (\theta_\varepsilon-1)(\partial_k\pi)\big\rangle_{\mathcal{D}(\Omega)}.
\end{aligned} \tag{6.4.89}$$

Lastly, given that $\vec{u}$ is divergence-free, we have

$$\begin{aligned}
\mathrm{IV}' + \mathrm{V}' &= -{}_{\mathcal{D}'(\Omega)}\big\langle \lambda\partial_k[q_j(x-\cdot)], (\partial_j\theta_\varepsilon)u_k\big\rangle_{\mathcal{D}(\Omega)} \\
&\quad - {}_{\mathcal{D}'(\Omega)}\big\langle \lambda q_j(x-\cdot), \partial_k\big((\partial_j\theta_\varepsilon)u_k\big)\big\rangle_{\mathcal{D}(\Omega)} = 0.
\end{aligned} \tag{6.4.90}$$

Collectively, (6.4.88)-(6.4.90) prove that

$$\mathrm{I}' + \mathrm{II}' + \mathrm{III}' + \mathrm{IV}' + \mathrm{V}' = -\pi(x). \tag{6.4.91}$$

In concert with (6.4.84), (6.4.85), (6.4.86), and (6.4.87), this ultimately establishes the integral representation formula in (6.4.51). This takes care of the claims made in the context of item *(1)* in the statement of the theorem. Finally, the validity of (6.4.50)-(6.4.51) in items *(2)-(3)* in the statement is proved in a similar fashion.

At this stage, there remains to justify formula (6.4.61), working now under the assumption that Ω is an exterior domains and, in place of (6.4.52), imposing the weaker assumption formulated in (6.4.60). With this goal in mind, fix $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and, in place of (6.4.64), now consider

$$\begin{gathered}\vec{w} := \Big(\big(E_{jk}(x-\cdot) - E_{jk}(x_0-\cdot)\big)\theta_\varepsilon\Big)_{1\leq k\leq n} \in \big[\mathscr{C}^\infty(\overline{\Omega})\big]^n,\\ \text{and } \rho := -\big(q_j(x-\cdot) - q_j(x_0-\cdot)\big)\theta_\varepsilon \in \mathscr{C}^\infty(\overline{\Omega}).\end{gathered} \tag{6.4.92}$$

Thanks to (6.2.1)-(6.2.2) and the Mean Value Theorem, these altered functions have one extra unit of decay at infinity, compared to their original counterparts in (6.4.64). In turn, this permits us to run the same argument as above for this choice of $(\vec{w}, \rho)$, even though we are now only assuming (6.4.60). Collecting the contributions made by the introduction of the terms containing x_0 into a constant $c \in \mathbb{C}^n$ then proves (6.4.61). The proof of Theorem 6.4.9 is now complete. □

Lastly, we deduce some basic functional analytic identities involving hydrostatic boundary layer potential operators acting on Besov spaces, akin to those discussed in Theorem 4.4.2.

Theorem 6.4.10 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) be an open set satisfying a two-sided local John condition and whose boundary is a compact Ahlfors regular set. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix $\lambda \in \mathbb{C}$ and consider $p, q \in (1,\infty)$ and $s \in (0,1)$.*

Then the following operator identities hold:

$$\big(\tfrac{1}{2}I + K_\lambda\big) \circ \big(-\tfrac{1}{2}I + K_\lambda\big) = S \circ \partial_\nu^\lambda\big(\mathcal{D}_\lambda, \mathcal{P}_\lambda\big) \text{ on } \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^n, \tag{6.4.93}$$

$$\big(\tfrac{1}{2}I + K^\#_\lambda\big) \circ \big(-\tfrac{1}{2}I + K^\#_\lambda\big) = \partial_\nu^\lambda\big(\mathcal{D}_\lambda, \mathcal{P}_\lambda\big) \circ S \text{ on } \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^n, \tag{6.4.94}$$

$$K^\#_\lambda \circ \partial_\nu^\lambda\big(\mathcal{D}_\lambda, \mathcal{P}_\lambda\big) = \partial_\nu^\lambda\big(\mathcal{D}_\lambda, \mathcal{P}_\lambda\big) \circ K_\lambda \text{ on } \big[B^{p,q}_s(\partial\Omega,\sigma)\big]^n, \tag{6.4.95}$$

$$S \circ K^\#_\lambda = K_\lambda \circ S \text{ on } \big[B^{p,q}_{-s}(\partial\Omega,\sigma)\big]^n. \tag{6.4.96}$$

Proof One way to see this is to rely on the operator identities from item *(xi)* of Theorem 6.2.4, the density result from [69, Lemma 7.1.10], as well as the mapping properties of the boundary layer potential operators from Theorem 6.4.3 and Theorem 6.4.7. Another way of justifying the operator identities from (6.4.93)-(6.4.96) is to carry out the same type of argument as in the proof of the operator identities from item *(xi)* of Theorem 6.2.4, now starting with the Green-type representation formula from Theorem 6.4.9 and then using the jump-relations from item *(5)* of Theorem 6.4.3, Theorem 6.4.7, and Theorem 6.4.8. □

Chapter 7
Applications to Analysis in Several Complex Variables

In this chapter we shall present a multitude of applications of the body of results developed so far in the area of Geometric Harmonic Analysis to the field of Complex Analysis of Several Variables. As is well known, Complex Analysis, Geometric Measure Theory, and Harmonic Analysis interface tightly in the complex plane (see, e.g., J. Garnett's book [28] and the references therein). The complexity of these connections is quite fascinating, and there are many aspects yet to be fully understood. At the same time, progress continues to be registered, as this area of mathematics continues to undergo seismic transformations thanks to spectacular advances made in the last decades. These includes G. David's characterization of the L^2 boundedness of the Cauchy operator in terms of Ahlfors regularity, and X. Tolsa's results on analytic capacity, to name just a couple. Nonetheless, this fruitful interplay between these branches of mathematics appears to have been much less explored in the higher-dimensional setting, involving several complex variables.

Here we wish to venture into such territory, employing our brand of Divergence Theorem from Volume I ([68]), the type of function space theory developed in Volume II ([69]), and the sort of Calderón-Zygmund theory for singular integral operators of boundary layer type devised in the current volume. Concrete topics addressed in this chapter include: CR-functions and differential forms on boundaries of sets of locally finite perimeter (in §7.1), integration by parts formulas involving the $\bar{\partial}$-operator on sets of locally finite perimeter (in §7.2), the Bochner-Martinelli integral operator (in §7.3), a sharp version of the Bochner-Martinelli-Koppelman formula (in §7.4), the Extension Problem for Hölder CR-functions on boundaries of Ahlfors regular domains (in §7.5), the Extension Problem for Lebesgue and bounded (or vanishing) mean oscillation functions on boundaries of uniformly rectifiable domains (in §7.6), as well as the $\bar{\partial}$-operator and the Dolbeault complex on uniformly rectifiable sets (in §7.7).

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7.1 CR-Functions and Differential Forms on Boundaries of Locally Finite Perimeter Sets

Having fixed $n \in \mathbb{N}$, we have a natural identification

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C} \equiv (\mathbb{R} \times \mathbb{R}) \times \cdots \times (\mathbb{R} \times \mathbb{R}) \equiv \mathbb{R}^{2n}. \tag{7.1.1}$$

Specifically, if for each $j \in \{1, \ldots, n\}$ we let z_j denote the complex variable in the j-th factor of the Cartesian product $\mathbb{C}^n$, then set $x_j := \operatorname{Re} z_j$ and $y_j := \operatorname{Im} z_j$, we shall identify

$$\mathbb{C}^n \ni (z_1, \ldots, z_n) \equiv (x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n}. \tag{7.1.2}$$

In the converse direction, given a real vector $\xi = (\xi_1, \xi_2, \ldots, \xi_{2n-1}, \xi_{2n}) \in \mathbb{R}^{2n}$, we agree to identify it with its complex version

$$\xi_{\mathbb{C}} := (\xi_1 + \mathrm{i}\xi_2, \ldots, \xi_{2n-1} + \mathrm{i}\xi_{2n}) \in \mathbb{C}^n. \tag{7.1.3}$$

For each $j \in \{1, \ldots, n\}$, the relationship between the individual complex variables $z_j \in \mathbb{C}$ and their real components $(x_j, y_j) \in \mathbb{R} \times \mathbb{R}$ implies

$$\begin{aligned}
&z_j = x_j + \mathrm{i}y_j, \quad \mathrm{d}z_j = \mathrm{d}x_j + \mathrm{i}\mathrm{d}y_j, \quad \mathrm{d}\bar{z}_j = \mathrm{d}x_j - \mathrm{i}\mathrm{d}y_j, \\
&\mathrm{d}x_j = 2^{-1}(\mathrm{d}z_j + \mathrm{d}\bar{z}_j), \quad \mathrm{d}y_j = (-\mathrm{i})2^{-1}(\mathrm{d}z_j - \mathrm{d}\bar{z}_j), \\
&\partial_{z_j} = 2^{-1}(\partial_{x_j} - \mathrm{i}\partial_{y_j}) = 2^{-1}(\partial_{2j-1} - \mathrm{i}\partial_{2j}), \\
&\partial_{\bar{z}_j} = 2^{-1}(\partial_{x_j} + \mathrm{i}\partial_{y_j}) = 2^{-1}(\partial_{2j-1} + \mathrm{i}\partial_{2j}), \\
&\partial_{x_j} = \partial_{z_j} + \partial_{\bar{z}_j}, \quad \partial_{y_j} = \mathrm{i}(\partial_{z_j} - \partial_{\bar{z}_j}).
\end{aligned} \tag{7.1.4}$$

Going further, assume that Ω is a set of locally finite perimeter in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. Abbreviate $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$, denoted by ν its geometric measure theoretic outward unit normal, and recall that $\nu_{\mathbb{C}}$ stands for its complexified version (defined as in (7.1.3)). In such a setting, for each $j, k \in \{1, \ldots, n\}$ we introduce the `complex tangential derivative operator` $\partial^{\mathbb{C}}_{\tau_{jk}}$ by setting

$$\begin{gathered}
\partial^{\mathbb{C}}_{\tau_{jk}} \psi := (\nu_{\mathbb{C}})_j \big[\partial_{\bar{z}_k} \psi\big]\Big|_{\partial_*\Omega} - (\nu_{\mathbb{C}})_k \big[\partial_{\bar{z}_j} \psi\big]\Big|_{\partial_*\Omega} \\
\text{for every complex-valued function } \psi \text{ of class} \\
\mathscr{C}^1 \text{ in some open neighborhood of } \partial_*\Omega \text{ in } \mathbb{C}^n.
\end{gathered} \tag{7.1.5}$$

We aim to extend the action of the complex tangential derivative operators to other classes of functions than those just described.

Definition 7.1.1 *Let Ω be a set of locally finite perimeter in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. Denote by ν its geometric measure theoretic outward unit normal, and abbreviate $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$. Also, fix two arbitrary indices $j, k \in \{1, \ldots, n\}$. Say that the complex tangential derivative operator $\partial^{\mathbb{C}}_{\tau_{jk}}$ maps a given complex-valued function $f \in L^1_{\mathrm{loc}}(\partial_*\Omega, \sigma)$*

into $L^1_{\rm loc}(\partial_*\Omega,\sigma)$ *provided that there exists some complex-valued function* h *in* $L^1_{\rm loc}(\partial_*\Omega,\sigma)$ *such that*

$$\int_{\partial_*\Omega} f\big(\partial^{\mathbb{C}}_{\tau_{jk}}\psi\big)\,{\rm d}\sigma=-\int_{\partial_*\Omega} h\psi\,{\rm d}\sigma \text{ for each } \psi\in\mathscr{C}^1_c(\mathbb{C}^n), \tag{7.1.6}$$

with $\partial^{\mathbb{C}}_{\tau_{jk}}\psi$ *in the left-hand side understood in the sense of* (7.1.5).

Finally, given any $p\in[1,\infty]$ *define*

$$L^{p,1}_{\mathbb{C},\rm loc}(\partial_*\Omega,\sigma):=\Big\{f\in L^p_{\rm loc}(\partial_*\Omega,\sigma):\partial^{\mathbb{C}}_{\tau_{jk}}f \text{ belongs to } L^p_{\rm loc}(\partial_*\Omega,\sigma) \\ \text{for each } j,k\in\{1,\dots,n\}\Big\}. \tag{7.1.7}$$

From [68, Proposition 3.7.2] (whose applicability in the present setting is ensured by [68, Lemma 3.6.4] and [68, (5.6.33)]) it follows that there could be at most one function $h\in L^1_{\rm loc}(\partial_*\Omega,\sigma)$ doing the job in (7.1.6). As such, there is no ambiguity if for a function $f\in L^1_{\rm loc}(\partial_*\Omega,\sigma)$ which is mapped by the complex tangential derivative operator $\partial^{\mathbb{C}}_{\tau_{jk}}$ into $L^1_{\rm loc}(\partial_*\Omega,\sigma)$ we define

$$\partial^{\mathbb{C}}_{\tau_{jk}}f:=h \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{7.1.8}$$

In particular, in the context of the above definition, formula (7.1.6) becomes

$$\int_{\partial_*\Omega} f\big(\partial^{\mathbb{C}}_{\tau_{jk}}\psi\big)\,{\rm d}\sigma=-\int_{\partial_*\Omega}\big(\partial^{\mathbb{C}}_{\tau_{jk}}f\big)\psi\,{\rm d}\sigma \text{ for each } \psi\in\mathscr{C}^1_c(\mathbb{C}^n). \tag{7.1.9}$$

We wish to stress that using the symbol $\partial^{\mathbb{C}}_{\tau_{jk}}f$ for the function h doing the job in (7.1.6) creates no ambiguity with the definition given in (7.1.5) in the case when $f\in\mathscr{C}^1_c(\mathbb{C}^n)$. Indeed, in such a scenario, (7.1.9) holds by virtue of the De Giorgi-Federer version of the Divergence Theorem recorded in [68, Theorem 1.1.1] (see Lemma 7.1.6 for details in similar circumstances).

We are going to be particularly interested in the case when, for a given integrability exponent $p\in(1,\infty)$, the complex tangential derivative operator $\partial^{\mathbb{C}}_{\tau_{jk}}$ maps a certain function $f\in L^p(\partial_*\Omega,\sigma)$ into the space $L^p(\partial_*\Omega,\sigma)$. Thanks to (7.1.9), Riesz's Duality Theorem, and [68, Proposition 3.7.1] (whose applicability is presently ensured by [68, Lemma 3.6.4] and [68, (5.6.33)]), if $p'\in(1,\infty)$ is such that $1/p+1/p'=1$, this amounts to having

$$\sup_{\substack{\psi\in\mathscr{C}^1_c(\mathbb{C}^n)\\ \|\psi|_{\partial_*\Omega}\|_{L^{p'}(\partial_*\Omega,\sigma)}\le 1}}\left|\int_{\partial_*\Omega} f\big(\partial^{\mathbb{C}}_{\tau_{jk}}\psi\big)\,{\rm d}\sigma\right|<+\infty. \tag{7.1.10}$$

Moreover, in (7.1.10) we may restrict ourselves to functions $\psi\in\mathscr{C}^N_c(U)$, where $U\subseteq\mathbb{C}^n$ is an open neighborhood of $\partial\Omega$, and $N\in\mathbb{N}\cup\{\infty\}$.

Here is a formal definition of the partial Sobolev spaces involving the action of the complex tangential differential operators considered in the sense of Definition 7.1.1 (cf. also (7.1.8)).

Definition 7.1.2 *Assume Ω is a set of locally finite perimeter in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. Abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ and denote by ν its geometric measure theoretic outward unit normal. In this context, for each $p \in [1, \infty]$ define the* `complex boundary Sobolev space`

$$L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma) := \Big\{ f \in L^p(\partial_*\Omega, \sigma) : \partial^{\mathbb{C}}_{\tau_{jk}} f \text{ belongs to } L^p(\partial_*\Omega, \sigma) \\ \text{for each } j, k \in \{1, \dots, n\} \Big\} \tag{7.1.11}$$

and equip it with the norm

$$\|f\|_{L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)} := \|f\|_{L^p(\partial_*\Omega,\sigma)} + \sum_{j,k=1}^{n} \big\|\partial^{\mathbb{C}}_{\tau_{jk}} f\big\|_{L^p(\partial_*\Omega,\sigma)} \\ \text{for each } f \in L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma). \tag{7.1.12}$$

In particular, in the above setting, for each $j, k \in \{1, \dots, n\}$ the operator

$$\partial^{\mathbb{C}}_{\tau_{jk}} : L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma) \longrightarrow L^p(\partial_*\Omega, \sigma) \tag{7.1.13}$$

is well defined, linear, and bounded. Moreover, we have

$$\partial^{\mathbb{C}}_{\tau_{jk}} = -\partial^{\mathbb{C}}_{\tau_{kj}}, \quad 1 \le j, k \le n, \text{ hence } \partial^{\mathbb{C}}_{\tau_{jj}} = 0, \quad \forall j \in \{1, \dots, n\}. \tag{7.1.14}$$

The operators introduced in (7.1.13) should be contrasted with the family $\{\partial_{\tau_{\ell m}}\}_{1 \le \ell, m \le 2n}$ of `real tangential derivative operators` on $\partial_*\Omega$. On smooth functions, the latter operators act according to

$$\partial_{\tau_{\ell m}}\psi := \nu_\ell\big[\partial_m\psi\big]\Big|_{\partial_*\Omega} - \nu_m\big[\partial_\ell\psi\big]\Big|_{\partial_*\Omega} \text{ for every } \psi \in \mathscr{C}^1(\mathbb{C}^n), \tag{7.1.15}$$

(for each $\ell, m \in \{1, \dots, 2n\}$) and their action is further extended to functions in the boundary Sobolev space $L^p_1(\partial_*\Omega, \sigma)$. See [69, Chapter 11] for details. It is then clear from definitions that

$$L^p_1(\partial_*\Omega, \sigma) \subseteq L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma) \tag{7.1.16}$$

and that

$$\partial^{\mathbb{C}}_{\tau_{jk}} = \frac{1}{2}\Big\{\big(\partial_{\tau_{2k-1,\,2j-1}} - \partial_{\tau_{2k,\,2j}}\big) + \mathrm{i}\big(\partial_{\tau_{2k,\,2j-1}} + \partial_{\tau_{2k-1,\,2j}}\big)\Big\} \tag{7.1.17}$$

for all $j, k \in \{1, \dots, n\}$.

Proposition 7.1.3 *Suppose Ω is a set of locally finite perimeter in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ and, as usual, abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. Also, select an open neighborhood $U \subseteq \mathbb{C}^n$ of*

$\partial\Omega$ and consider two function, $f \in L^{1,1}_{\mathbb{C},\mathrm{loc}}(\partial_*\Omega,\sigma)$ along with $\psi \in \mathscr{C}^1(U)$, satisfying

$$\int_{\partial_*\Omega} |f| \sum_{j=1}^{n} \left|\partial_{\bar{z}_j}\psi\right| \,\mathrm{d}\sigma < +\infty, \qquad \int_{\partial_*\Omega} \sum_{j,k=1}^{n} \left|\partial^{\mathbb{C}}_{\tau_{jk}} f\right| |\varphi| \,\mathrm{d}\sigma < +\infty,$$
$$\textit{as well as} \int_{\partial_*\Omega} |f(z)||\psi(z)|(1+|z|)^{-1} \,\mathrm{d}\sigma(z) < +\infty. \tag{7.1.18}$$

Then for each $j,k \in \{1,\dots,n\}$ one has

$$\int_{\partial_*\Omega} f\left(\partial^{\mathbb{C}}_{\tau_{jk}}\psi\right) \mathrm{d}\sigma = -\int_{\partial_*\Omega} \left(\partial^{\mathbb{C}}_{\tau_{jk}} f\right)\psi \,\mathrm{d}\sigma. \tag{7.1.19}$$

Consequently, the boundary integration by parts formula (7.1.19) is true for all $j,k \in \{1,\dots,n\}$ if $f \in L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)$ with $p \in [1,\infty]$ and, with $p' \in [1,\infty]$ such that $1/p + 1/p' = 1$,

$$\psi \in \mathscr{C}^1(U) \ \textit{satisfies} \ \psi\big|_{\partial_*\Omega} \in L^{p'}(\partial_*\Omega,\sigma) \ \textit{and}$$
$$(\partial_{\bar{z}_j}\psi)\big|_{\partial_*\Omega} \in L^{p'}(\partial_*\Omega,\sigma) \ \textit{for each} \ j \in \{1,\dots,n\}. \tag{7.1.20}$$

In particular, (7.1.19) holds for every $f \in L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)$ with $p \in [1,\infty]$ and $\psi \in \mathscr{C}^1(U)$ in the case when $\partial_*\Omega$ is bounded.

Proof The same argument as in the proof of [69, Lemma 11.1.7] works virtually verbatim in the present setting. □

In contrast with the real tangential derivative operators $\{\partial_{\tau_{jk}}\}_{1\le j,k\le n}$ (from [69, Definition 11.1.2]) which, at least when considered on Sobolev spaces defined on the boundary of an open set $\Omega \subseteq \mathbb{R}^n$ with an Ahlfors regular boundary and which satisfies a two-sided local John condition, annihilate precisely the locally constant functions on $\partial\Omega$ (cf. [69, Proposition 11.4.3]), the common null-space for the family of complex tangential derivative operators $\{\partial^{\mathbb{C}}_{\tau_{jk}}\}_{1\le j,k\le n}$ is much richer, consisting of CR-functions on $\partial\Omega$ as defined next.

Definition 7.1.4 *Let Ω be a set of locally finite perimeter in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ and, as usual, abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. In this setting, call $f \in L^1_{\mathrm{loc}}(\partial_*\Omega,\sigma)$ a* `CR-function` *provided*

$$\partial^{\mathbb{C}}_{\tau_{jk}} f = 0 \ \textit{at } \sigma\textit{-a.e. point on } \partial_*\Omega \textit{ for all } \ j,k \in \{1,\dots,n\}. \tag{7.1.21}$$

In the context of the above definition, it follows from Definition 7.1.2 that

$$\text{any CR-function } f \in L^p(\partial_*\Omega,\sigma) \text{ with } p \in [1,\infty] \text{ actually belongs to the Sobolev space } L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma). \tag{7.1.22}$$

Trivially, the last formula in (7.1.14) implies that

$$\text{when } n = 1 \text{ it follows that any } f \in L^p(\partial_*\Omega, \sigma) \text{ with } p \in [1, \infty] \text{ is actually a CR-function.} \tag{7.1.23}$$

Also, from (7.1.21) and (7.1.5) we deduce that, in the context of Definition 7.1.4,

$$\text{if } U \text{ is an open neighborhood of } \partial_*\Omega \text{ in } \mathbb{C}^n \text{ and } \psi \in \mathscr{C}^1(U) \text{ is a holomorphic function in } U, \text{ then } \psi\big|_{\partial_*\Omega} \text{ is a CR-function.} \tag{7.1.24}$$

An incisive generalization of this observation is contained in the proposition below. Before stating it, the reader is reminded that, given an open set $\Omega \subseteq \mathbb{C}^n$, a function $F : \Omega \to \mathbb{C}$ is said to be `holomorphic` provided F is separately holomorphic (or separately analytic) in the sense of one complex variable analysis. That is, for each $j \in \{1, \dots, n\}$ and each $z = (z_1, \dots, z_n) \in \Omega$, the function

$$\begin{gathered} \mathcal{O}_{z,j} \ni \zeta \longmapsto F\big(z_1, \dots, z_{j-1}, \zeta, z_{j+1}, \dots, z_n\big) \in \mathbb{C} \\ \text{where } \mathcal{O}_{z,j} := \big\{\zeta \in \mathbb{C} : \big(z_1, \dots, z_{j-1}, \zeta, z_{j+1}, \dots, z_n\big) \in \Omega\big\} \end{gathered} \tag{7.1.25}$$

is holomorphic, in the classical one-variable sense. In this vein, it is worth recalling that Hartogs' classical theorem on separate analyticity asserts that any such "separately analytic" function F turns out to be continuous and, ultimately, $\mathscr{C}^\infty$ smooth in Ω (see, e.g., [37], [51, §2.4, pp. 107–110]).

Proposition 7.1.5 *Let Ω be an open nonempty proper subset of $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ with a lower Ahlfors regular boundary and with the property that $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Fix an aperture parameter $\kappa \in (0, \infty)$ and a truncation parameter $\varepsilon > 0$. Suppose $F \in L^1_{\rm loc}(\Omega, \mathcal{L}^{2n})$ is a complex-valued function satisfying*

$$\mathcal{N}^\varepsilon_\kappa F \in L^1_{\rm loc}(\partial\Omega, \sigma), \ \textit{the trace } F\big|^{\kappa-\rm n.t.}_{\partial\Omega} \textit{ exists } \sigma\textit{-a.e. on } \partial_{\rm nta}\Omega \tag{7.1.26}$$

and such that, for some fixed $j, k \in \{1, \dots, n\}$, (with derivatives taken in the sense of distributions)

$$\begin{gathered} \partial_{\bar z_j}F,\ \partial_{\bar z_k}F \in L^1_{\rm loc}(\Omega, \mathcal{L}^{2n}), \quad \mathcal{N}^\varepsilon_\kappa(\partial_{\bar z_j}F),\ \mathcal{N}^\varepsilon_\kappa(\partial_{\bar z_k}F) \in L^1_{\rm loc}(\partial\Omega, \sigma), \ \textit{and} \\ \textit{the traces } (\partial_{\bar z_j}F)\big|^{\kappa-\rm n.t.}_{\partial\Omega},\ (\partial_{\bar z_k}F)\big|^{\kappa-\rm n.t.}_{\partial\Omega} \textit{ exist at } \sigma\textit{-a.e. point on } \partial_{\rm nta}\Omega. \end{gathered} \tag{7.1.27}$$

Then for any other $\kappa' > 0$ the nontangential traces $F\big|^{\kappa'-\rm n.t.}_{\partial\Omega}$, $(\partial_{\bar z_j}F)\big|^{\kappa'-\rm n.t.}_{\partial\Omega}$, $(\partial_{\bar z_k}F)\big|^{\kappa'-\rm n.t.}_{\partial\Omega}$ exist at σ-a.e. point on $\partial_{\rm nta}\Omega$, and are actually independent of κ'. Also, when the dependence on κ' is dropped, and when considered on $\partial_*\Omega$, the function $F\big|^{\rm n.t.}_{\partial\Omega}$ belongs to $L^1_{\rm loc}(\partial_*\Omega, \sigma)$ and is mapped by $\partial^{\mathbb{C}}_{\tau_{jk}}$ into $L^1_{\rm loc}(\partial_*\Omega, \sigma)$, the functions $(\partial_{\bar z_j}F)\big|^{\rm n.t.}_{\partial\Omega}$, $(\partial_{\bar z_k}F)\big|^{\rm n.t.}_{\partial\Omega}$ belong to $L^1_{\rm loc}(\partial_*\Omega, \sigma)$, and

$$\partial^{\mathbb{C}}_{\tau_{jk}}\Big(F\big|^{\rm n.t.}_{\partial\Omega}\Big) = (\nu_{\mathbb{C}})_j\Big((\partial_{\bar z_k}F)\big|^{\rm n.t.}_{\partial\Omega}\Big) - (\nu_{\mathbb{C}})_k\Big((\partial_{\bar z_j}F)\big|^{\rm n.t.}_{\partial\Omega}\Big) \tag{7.1.28}$$

at σ-a.e. point on $\partial_*\Omega$.

As a corollary, if $F : \Omega \to \mathbb{C}$ *is a holomorphic function satisfying* (7.1.26) *then* $F\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ *considered on* $\partial_*\Omega$ *is a* CR*-function.*

Proof The present assumptions imply that Ω is a set of locally finite perimeter. In particular, it is meaningful to consider its geometric measure theoretic outward unit normal ν. Consider $U := \{z \in \mathbb{C}^n : \operatorname{dist}(z, \partial\Omega) < \varepsilon\}$, an open neighborhood of $\partial\Omega$ in $\mathbb{C}^n$, and pick an arbitrary function $\psi \in \mathscr{C}_c^1(U)$. Also, select $j, k \in \{1, \dots, n\}$ and, with $\{\mathbf{e}_\ell\}_{1 \le \ell \le 2n}$ denoting the standard orthonormal basis in $\mathbb{R}^{2n}$, define the vector field

$$\begin{aligned}\vec{H} &:= (\psi\, \partial_{\bar{z}_k} F + F\, \partial_{\bar{z}_k}\psi)\,(\mathbf{e}_{2j-1} + \mathbf{i}\mathbf{e}_{2j}) \\ &\quad - (\psi\, \partial_{\bar{z}_j} F + F\, \partial_{\bar{z}_j}\psi)\,(\mathbf{e}_{2k-1} + \mathbf{i}\mathbf{e}_{2k}) \ \text{ in } \Omega.\end{aligned} \tag{7.1.29}$$

Then $\vec{H} \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n})\big]^{2n}$ and, with the divergence considered as an operator in $\mathbb{R}^{2n}$,

$$\operatorname{div}\vec{H} = 2\,\partial_{\bar{z}_j}\partial_{\bar{z}_k}\big(F\,\psi\big) - 2\,\partial_{\bar{z}_k}\partial_{\bar{z}_j}\big(F\,\psi\big) = 0 \ \text{ in } \mathcal{D}'(\Omega). \tag{7.1.30}$$

Going further, if we abbreviate

$$f := F\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \quad f_j := (\partial_{\bar{z}_j} F)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \quad f_k := (\partial_{\bar{z}_k} F)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \tag{7.1.31}$$

then $f, f_j, f_k \in L^1_{\mathrm{loc}}(\partial_*\Omega, \sigma)$, thanks to (7.1.26), (7.1.27), [68, (8.9.8)], [68, (8.9.44)], and [68, (8.8.52)]. Also

$$\vec{H}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega \tag{7.1.32}$$

and, by design,

$$\nu \cdot \Big(\vec{H}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big) = \big((\nu_{\mathbb{C}})_j f_k - (\nu_{\mathbb{C}})_k f_j\big)\psi + f\,\partial^{\mathbb{C}}_{\tau_{jk}}\psi \ \text{ at } \sigma\text{-a.e. on } \partial_*\Omega. \tag{7.1.33}$$

Note that since

$$\mathcal{N}_\kappa \vec{H} \le \sqrt{2}\big(\mathcal{N}_\kappa^\varepsilon F\big)\mathcal{N}_\kappa(\nabla\psi) + \sqrt{2}\big(\mathcal{N}_\kappa^\varepsilon(\partial_{\bar{z}_j} F) + \mathcal{N}_\kappa^\varepsilon(\partial_{\bar{z}_k} F)\big)\mathcal{N}_\kappa\psi \tag{7.1.34}$$

pointwise on $\partial\Omega$, and since $\mathcal{N}_\kappa\psi, \mathcal{N}_\kappa(\nabla\psi) \in L^\infty_{\mathrm{comp}}(\partial\Omega, \sigma)$, the first condition in (7.1.26) together with the second condition in (7.1.27) and [68, (8.2.26)] imply that $\mathcal{N}_\kappa\vec{H} \in L^1(\partial\Omega, \sigma)$. Lastly, if Ω is an exterior domain then the compact support condition on ψ ensures that $\vec{H}$ vanishes in a neighborhood of infinity. Granted these properties, [68, Theorem 1.2.1] applies and the Divergence Formula [68, (1.2.2)] written for $\vec{H}$ yields (on account of (7.1.30) and (7.1.33))

$$\int_{\partial_*\Omega} f\,\partial^{\mathbb{C}}_{\tau_{jk}}\psi\,\mathrm{d}\sigma = \int_{\partial_*\Omega} \big((\nu_{\mathbb{C}})_j f_k - (\nu_{\mathbb{C}})_k f_j\big)\psi\,\mathrm{d}\sigma. \tag{7.1.35}$$

In light of Definition 7.1.1 (cf. also (7.1.8)) from (7.1.35) and the arbitrariness of ψ in $\mathscr{C}_c^1(U)$ we ultimately conclude that $\partial^{\mathbb{C}}_{\tau_{jk}} f = (\nu_{\mathbb{C}})_j f_k - (\nu_{\mathbb{C}})_k f_j$ on $\partial_*\Omega$. From this, (7.1.28) follows on account of (7.1.31).

Finally, when F is actually a holomorphic function in Ω, the right-hand side of (7.1.28) vanishes for each indices $j,k \in \{1,\dots,n\}$. As noted in (7.1.21), this amounts to saying that $F\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ is indeed a CR-function. □

We next review notation pertaining to differential forms used frequently in the remainder of this volume. For more background on this and related issues, the reader is referred to the monographs [13], [59], and [88]. Throughout, the wedge symbol $\wedge$ will denote the exterior product of differential forms. Given $\alpha,\beta \in \{0,1,\dots,n\}$ denote by $\Lambda^{\alpha,\beta}\,\mathbb{C}^n$ the space of differential forms of (type) degree (α,β) with complex coefficients, i.e., objects represented as

$$u = \sum_{|I|=\alpha,\,|J|=\beta} u_{I,J}\, dz^I \wedge d\bar{z}^J. \tag{7.1.36}$$

The sum in (7.1.36) is performed over strictly increasing arrays. That is, we have $I = (i_1,\dots,i_\alpha) \in \{1,\dots,n\}^\alpha$ satisfying $i_1 < \cdots < i_\alpha$ of length $|I| = \alpha$, and $J = (j_1,\dots,j_\beta) \in \{1,\dots,n\}^\beta$ satisfying $j_1 < \cdots < j_\beta$ of length $|J| = \beta$, for which we have set $dz^I := dz_{i_1} \wedge \cdots \wedge dz_{i_\alpha}$ and $d\bar{z}^J := d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_\beta}$, and $u_{I,J} \in \mathbb{C}$. The complex conjugate of a form $u \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n$ written as in (7.1.36) is defined to be

$$\bar{u} := \sum_{|I|=\alpha}\sum_{|J|=\beta} \overline{u_{I,J}}\, d\bar{z}^I \wedge dz^J = (-1)^{\alpha\beta} \sum_{|J|=\beta}\sum_{|I|=\alpha} \overline{u_{I,J}}\, dz^J \wedge d\bar{z}^I. \tag{7.1.37}$$

Hence,

$$\bar{u} \in \Lambda^{\beta,\alpha}\,\mathbb{C}^n \ \text{ and } \ \bar{\bar{u}} = u \ \text{ for each } \ u \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n. \tag{7.1.38}$$

Also, for each $u \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n$ and $w \in \Lambda^{\alpha',\beta'}\,\mathbb{C}^n$ we have

$$\overline{u \wedge w} = \bar{u} \wedge \bar{w}. \tag{7.1.39}$$

Pressing on, recall from (A.0.67) that for any two arrays J, K (not necessarily ordered), the generalized Kronecker symbol ε^J_K is given by

$$\varepsilon^J_K := \begin{cases} \det\big((\delta_{jk})_{j\in J,\,k\in K}\big) & \text{if } J,K \text{ agree as sets,} \\ 0 & \text{otherwise,} \end{cases} \tag{7.1.40}$$

where $\delta_{jk} := 1$ if $j = k$, and zero if $j \neq k$. Put differently,

$$\varepsilon^J_K = \begin{cases} 0 & \text{if } J,K \text{ do not agree as sets,} \\ +1 & \text{if } J \text{ differs from } K \text{ by an even permutation,} \\ -1 & \text{if } J \text{ differs from } K \text{ by an odd permutation.} \end{cases} \tag{7.1.41}$$

We shall employ the Hermitian inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ on differential forms uniquely defined by the requirement that

$$\left\langle dz^I \wedge d\bar{z}^J, dz^A \wedge d\bar{z}^B \right\rangle_{\mathbb{C}} = 2^{|I|+|J|}\varepsilon^I_A \varepsilon^J_B \text{ for all arrays } I, J, A, B. \tag{7.1.42}$$

The power of 2 is an artifact of $dz_j = dx_j + i dy_j$ having length $2^{1/2}$ (rather than being of unit length). Thus, in particular, if $\alpha, \beta \in \{0, 1, \dots, n\}$ then

$$\begin{gathered}\langle f, g\rangle_{\mathbb{C}} = 2^{\alpha+\beta} \sum_{|I|=\alpha,\,|J|=\beta} f_{I,J}\overline{g_{I,J}} \text{ if } f, g \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n \text{ are given by} \\ f = \sum_{|I|=\alpha,\,|J|=\beta} f_{I,J}\, dz^I \wedge d\bar{z}^J \text{ and } g = \sum_{|I|=\alpha,\,|J|=\beta} g_{I,J}\, dz^I \wedge d\bar{z}^J.\end{gathered} \tag{7.1.43}$$

Note that if $f, g \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n$ then

$$\langle f, g\rangle_{\mathbb{C}} = \overline{\langle g, f\rangle_{\mathbb{C}}} = \langle \bar{g}, \bar{f}\,\rangle_{\mathbb{C}} \tag{7.1.44}$$

and that

$$\begin{gathered}\langle f, g\rangle_{\mathbb{C}} = 0 \text{ if } f \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n \text{ and } g \in \Lambda^{\alpha',\beta'}\,\mathbb{C}^n \\ \text{with either } \alpha \neq \alpha' \text{ or } \beta \neq \beta'.\end{gathered} \tag{7.1.45}$$

As customary, write

$$\begin{gathered}|f|_{\mathbb{C}} := \sqrt{\langle f, f\,\rangle_{\mathbb{C}}} = \Big(2^{\alpha+\beta}\sum_{|I|=\alpha}\sum_{|J|=\beta}|f_{I,J}|^2\Big)^{1/2} \\ \text{for each } f = \sum_{|I|=\alpha,\,|J|=\beta} f_{I,J}\, dz^I \wedge d\bar{z}^J \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n.\end{gathered} \tag{7.1.46}$$

The volume element in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ is given by

$$\begin{aligned} dV &= dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \\ &= (-2i)^{-n} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \\ &= (2i)^{-n}(-1)^{n(n-1)/2} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n \wedge dz_1 \wedge \cdots \wedge dz_n. \end{aligned} \tag{7.1.47}$$

In the context of integration over open subsets of $\mathbb{C}^n \equiv \mathbb{R}^{2n}$, we shall tacitly identify dV with the Lebesgue measure $\mathcal{L}^{2n}$. Next, consider the Hodge star operator $*$ in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$. For each given $\alpha, \beta \in \{0, 1, \dots, n\}$ this may be characterized as the unique linear isomorphism

$$* : \Lambda^{\alpha,\beta}\,\mathbb{C}^n \longrightarrow \Lambda^{n-\beta,n-\alpha}\,\mathbb{C}^n \tag{7.1.48}$$

with the property that

$$u \wedge (*\bar{u}) = |u|^2_{\mathbb{C}} dV \text{ for every } u \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n. \tag{7.1.49}$$

In particular, $*1 = \mathrm{d}V$. In fact, it can be checked that

if I, J, K are increasing, mutually disjoint subsets of $\{1, \dots, n\}$ then

$$*\left(\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J \wedge \big(\mathrm{d}z \wedge \mathrm{d}\bar{z}\big)^K\right) = \mathrm{i}^n 2^{M-n}(-1)^{M(M-1)/2+|I|}\times \\ \times \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J \wedge \big(\mathrm{d}z \wedge \mathrm{d}\bar{z}\big)^{\{1,\dots,n\}\setminus(I\cup J\cup K)} \tag{7.1.50}$$

where $M := |I| + |J| + 2|K|$ and, if $K = (k_1, \dots, k_\ell)$, we have abbreviated

$$\big(\mathrm{d}z \wedge \mathrm{d}\bar{z}\big)^K := (\mathrm{d}z_{k_1} \wedge \mathrm{d}\bar{z}_{k_1}) \wedge \cdots \wedge (\mathrm{d}z_{k_\ell} \wedge \mathrm{d}\bar{z}_{k_\ell}). \tag{7.1.51}$$

Using the Hodge-star operator, we define the interior product between a 1-form θ and an ℓ-form u by setting

$$\theta \vee u := *(\theta \wedge *u). \tag{7.1.52}$$

For further reference as well as for the convenience of the reader, some basic, elementary properties of these objects are summarized in the following lemma.

Lemma 7.1.6 *For arbitrary one-forms θ, η, and any ℓ-form u, ℓ-form ω, $(\ell+1)$-form w, and $(2n-\ell)$-form λ in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$, the following are true:*

(1) $**u = (-1)^\ell u$, $\langle u, *\lambda\rangle_{\mathbb{C}} = (-1)^\ell \langle *u, \lambda\rangle_{\mathbb{C}}$, and $\langle *u, *\lambda\rangle_{\mathbb{C}} = \langle u, \lambda\rangle_{\mathbb{C}}$;
(2) $\theta \wedge (\theta \wedge u) = 0$ and $\theta \vee (\theta \vee u) = 0$;
(3) $\theta \wedge (\eta \vee u) + \eta \vee (\theta \wedge u) = \langle \theta, \bar{\eta}\rangle_{\mathbb{C}} u$ and $\langle \theta \wedge u, w\rangle_{\mathbb{C}} = \langle u, \bar{\theta} \vee w\rangle_{\mathbb{C}}$;
(4) $*(\theta \wedge u) = (-1)^\ell \theta \vee (*u)$ and $*(\theta \vee u) = (-1)^{\ell-1}\theta \wedge (*u)$;
(5) $*\bar{u} = \overline{*u}$ and $u \wedge *\bar{\omega} = \langle u, \omega\rangle_{\mathbb{C}} \mathrm{d}V$.

Moreover, if θ is normalized such that $\langle \theta, \theta\rangle_{\mathbb{C}} = 1$, then also:

(6) $u = \theta \wedge (\bar{\theta} \vee u) + \bar{\theta} \vee (\theta \wedge u)$ and $|u|_{\mathbb{C}}^2 = |\theta \wedge u|_{\mathbb{C}}^2 + |\bar{\theta} \vee u|_{\mathbb{C}}^2$;
(7) $|\bar{\theta} \wedge (\theta \vee u)|_{\mathbb{C}} = |\theta \vee u|_{\mathbb{C}}$ and $|\theta \vee (\bar{\theta} \wedge u)|_{\mathbb{C}} = |\bar{\theta} \wedge u|_{\mathbb{C}}$.

Lastly, if $\theta \in \Lambda^{1,0}\mathbb{C}^n$ and $\eta \in \Lambda^{0,1}\mathbb{C}^n$ then

$$\begin{aligned} &\theta\wedge : \Lambda^{\alpha,\beta}\,\mathbb{C}^n \longrightarrow \Lambda^{\alpha+1,\beta}\,\mathbb{C}^n, \qquad \theta\vee : \Lambda^{\alpha,\beta}\,\mathbb{C}^n \longrightarrow \Lambda^{\alpha,\beta-1}\mathbb{C}^n, \\ &\eta\wedge : \Lambda^{\alpha,\beta}\,\mathbb{C}^n \longrightarrow \Lambda^{\alpha,\beta+1}\mathbb{C}^n, \qquad \eta\vee : \Lambda^{\alpha,\beta}\,\mathbb{C}^n \longrightarrow \Lambda^{\alpha-1,\beta}\,\mathbb{C}^n. \end{aligned} \tag{7.1.53}$$

For practical purposes it is relevant to point out that if $\alpha, \beta \in \{0, 1, \dots, n\}$ then

$$\eta \wedge u = (-1)^\alpha \sum_{|M|=\alpha} \sum_{|J|=\beta+1} \sum_{|I|=\beta} \sum_{j=1}^{n} \varepsilon^{jI}_J \eta_j\, u_{M,I}\, \mathrm{d}z^M \wedge \mathrm{d}\bar{z}^J \in \Lambda^{\alpha,\beta+1}\mathbb{C}^n \text{ if} \\ u = \sum_{|M|=\alpha} \sum_{|I|=\beta} u_{M,I}\, \mathrm{d}z^M \wedge \mathrm{d}\bar{z}^I \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n \text{ and } \eta = \sum_{j=1}^{n} \eta_j \mathrm{d}\bar{z}_j \in \Lambda^{0,1}\mathbb{C}^n, \tag{7.1.54}$$

and

$$\theta \wedge w = \sum_{|M|=\alpha} \sum_{|J|=\alpha+1} \sum_{|I|=\beta} \sum_{j=1}^{n} \varepsilon_J^{jM} \theta_j\, w_{M,I}\, \mathrm{d}z^J \wedge \mathrm{d}\bar{z}^I \in \Lambda^{\alpha+1,\beta}\mathbb{C}^n \text{ if}$$
$$w = \sum_{|M|=\alpha} \sum_{|I|=\beta} w_{M,I}\, \mathrm{d}z^M \wedge \mathrm{d}\bar{z}^I \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n \text{ and } \theta = \sum_{j=1}^{n} \theta_j\, \mathrm{d}z_j \in \Lambda^{1,0}\mathbb{C}^n. \tag{7.1.55}$$

These identities can be seen directly from definitions. In addition, given any two indices $\alpha, \beta \in \{0, 1, \dots, n\}$, we may conclude from (7.1.50) and (7.1.52) that

$$\eta \vee w = 2 \sum_{|K|=\alpha} \sum_{|J|=\beta} \sum_{|I|=\alpha-1} \sum_{j=1}^{n} \varepsilon_K^{jI} \eta_j w_{K,J}\, \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J \in \Lambda^{\alpha-1,\beta}\,\mathbb{C}^n \text{ if}$$
$$w = \sum_{|K|=\alpha} \sum_{|J|=\beta} w_{K,J}\, \mathrm{d}z^K \wedge \mathrm{d}\bar{z}^J \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n \text{ and } \eta = \sum_{j=1}^{n} \eta_j\, \mathrm{d}\bar{z}_j \in \Lambda^{0,1}\mathbb{C}^n, \tag{7.1.56}$$

and

$$\theta \vee u = 2(-1)^{\alpha} \sum_{|K|=\alpha} \sum_{|J|=\beta} \sum_{|I|=\beta-1} \sum_{j=1}^{n} \varepsilon_J^{jI} \theta_j u_{K,J}\, \mathrm{d}z^K \wedge \mathrm{d}\bar{z}^I \in \Lambda^{\alpha,\beta-1}\mathbb{C}^n \text{ if}$$
$$u = \sum_{|K|=\alpha} \sum_{|J|=\beta} u_{K,J}\, \mathrm{d}z^K \wedge \mathrm{d}\bar{z}^J \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n \text{ and } \theta = \sum_{j=1}^{n} \theta_j\, \mathrm{d}z_j \in \Lambda^{1,0}\mathbb{C}^n. \tag{7.1.57}$$

Before moving on, we wish to remark that the operators $\wedge$, $\vee$, $*$ extend to differential forms with variable coefficients by considering their action in a natural pointwise fashion.

Given any real vector $\xi = (\xi_1, \xi_2, \dots, \xi_{2n-1}, \xi_{2n}) \in \mathbb{R}^{2n}$, recall its complex version $\xi_{\mathbb{C}} \in \mathbb{C}^n$ from (7.1.3), and define

$$\xi^{1,0} := \sum_{j=1}^{n} \overline{(\xi_{\mathbb{C}})_j}\, \mathrm{d}z_j \in \Lambda^{1,0}\mathbb{C}^n, \quad \xi^{0,1} := \sum_{j=1}^{n} (\xi_{\mathbb{C}})_j\, \mathrm{d}\bar{z}_j \in \Lambda^{0,1}\mathbb{C}^n. \tag{7.1.58}$$

From definitions it follows that for each $\xi \in \mathbb{R}^{2n}$ we have

$$2\xi = \xi^{1,0} + \xi^{0,1}, \quad \overline{\xi^{0,1}} = \xi^{1,0}, \quad \overline{\xi^{1,0}} = \xi^{0,1},$$
$$\langle \xi^{1,0}, \xi^{0,1} \rangle_{\mathbb{C}} = 0, \quad |\xi^{1,0}|_{\mathbb{C}} = |\xi^{0,1}|_{\mathbb{C}} = 2^{1/2}|\xi|. \tag{7.1.59}$$

In particular, all the above considerations apply to the geometric measure theoretic unit normal $\nu = (\nu_1, \nu_2, \dots, \nu_{2n-1}, \nu_{2n}) \in \mathbb{R}^{2n}$ of a set of locally finite perimeter $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$. In such a setting, recall from (7.1.3) that the complex outward unit normal to Ω is given by

$$\nu_{\mathbb{C}} = \big(\nu_1 + \mathrm{i}\nu_2, \dots, \nu_{2n-1} + \mathrm{i}\nu_{2n}\big) = \big(\nu_{2j-1} + \mathrm{i}\nu_{2j}\big)_{1 \le j \le n} \in \mathbb{C}^n. \tag{7.1.60}$$

Also, in keeping with (7.1.58), let us define

$$\nu^{1,0} := \sum_{j=1}^{n} \overline{(\nu_{\mathbb{C}})_j}\, \mathrm{d}z_j \in \Lambda^{1,0}\mathbb{C}^n, \qquad \nu^{0,1} := \sum_{j=1}^{n} (\nu_{\mathbb{C}})_j\, \mathrm{d}\overline{z}_j \in \Lambda^{0,1}\mathbb{C}^n. \tag{7.1.61}$$

If the vector $\nu = (\nu_1, \nu_2, \dots, \nu_{2n-1}, \nu_{2n})$ is further identified with the 1-form

$$\nu = \nu_1 \mathrm{d}x_1 + \nu_2\, \mathrm{d}y_1 + \cdots + \nu_{2n-1}\, \mathrm{d}x_n + \nu_{2n}\, \mathrm{d}y_n, \tag{7.1.62}$$

then from (7.1.59) we see that

$$\begin{aligned} &2\nu = \nu^{1,0} + \nu^{0,1}, \quad \overline{\nu^{0,1}} = \nu^{1,0}, \quad \overline{\nu^{1,0}} = \nu^{0,1}, \\ &\langle \nu^{1,0}, \nu^{0,1} \rangle_{\mathbb{C}} = 0, \qquad |\nu^{1,0}|_{\mathbb{C}} = |\nu^{0,1}|_{\mathbb{C}} = 2^{1/2}. \end{aligned} \tag{7.1.63}$$

Definition 7.1.7 *Assume Ω is a set of locally finite perimeter in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. Abbreviate $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$ and denote by ν its geometric measure theoretic outward unit normal. Also, fix some $\alpha, \beta \in \{0, 1, \dots, n\}$ and consider a $\Lambda^{\alpha,\beta}\,\mathbb{C}^n$-valued function f defined σ-a.e. on $\partial_*\Omega$. Then call*

$$f \text{ complex tangential} \overset{def}{\Longleftrightarrow} \nu^{1,0} \vee f = 0 \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega, \tag{7.1.64}$$

and

$$f \text{ complex normal} \overset{def}{\Longleftrightarrow} \nu^{0,1} \wedge f = 0 \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{7.1.65}$$

In the context of the above definition, based on item *(6)* of Lemma 7.1.6 and (7.1.63) we may write

$$2f = \left\langle \nu^{0,1}, \overline{\nu^{1,0}} \right\rangle_{\mathbb{C}} f = \nu^{0,1} \wedge (\nu^{1,0} \vee f) + \nu^{1,0} \vee (\nu^{0,1} \wedge f), \tag{7.1.66}$$

hence

$$f = f_{\tan,\mathbb{C}} + f_{\mathrm{nor},\mathbb{C}} \tag{7.1.67}$$

where

$$f_{\tan,\mathbb{C}} := \tfrac{1}{2}\, \nu^{1,0} \vee (\nu^{0,1} \wedge f) \ \text{ and } \quad f_{\mathrm{nor},\mathbb{C}} := \tfrac{1}{2}\, \nu^{0,1} \wedge (\nu^{1,0} \vee f). \tag{7.1.68}$$

Also, a direct consequence of the definitions in (7.1.68) and (7.1.66) is that

$$\nu^{0,1} \wedge f = \nu^{0,1} \wedge f_{\tan,\mathbb{C}} \ \text{ and } \ \nu^{1,0} \vee f = \nu^{1,0} \vee f_{\mathrm{nor},\mathbb{C}}. \tag{7.1.69}$$

Moving on, recall that the d-bar operator acts on a continuously differentiable complex-valued function f defined in an open subset of $\mathbb{C}^n$ according to

$$\bar{\partial} f := \sum_{j=1}^{n} (\partial_{\bar{z}_j} f)\, \mathrm{d}\bar{z}_j. \tag{7.1.70}$$

Continue to assume that Ω is a set of locally finite perimeter in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. As in the past, denote by ν its geometric measure theoretic outward unit normal, and abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. In this context, given any complex-valued function f of class $\mathscr{C}^1$ in a neighborhood of $\partial_*\Omega$, we have

$$\begin{aligned}\nu^{0,1} \wedge \bar{\partial} f &= \sum_{1\leq j<k\leq n} \big\{(\nu_{\mathbb{C}})_j \partial_{\bar{z}_k} f - (\nu_{\mathbb{C}})_k \partial_{\bar{z}_j} f\big\}\, \mathrm{d}\bar{z}_j \wedge \mathrm{d}\bar{z}_k \\ &= \sum_{1\leq j<k\leq n} \partial^{\mathbb{C}}_{\tau_{jk}} f\, \mathrm{d}\bar{z}_j \wedge \mathrm{d}\bar{z}_k \ \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \end{aligned} \tag{7.1.71}$$

From this, Definition 7.1.7, Definition 7.1.4, and (7.1.70) it follows that

$$\begin{aligned}&\text{if } f \text{ is a scalar function of class } \mathscr{C}^1 \text{ in a neighborhood of } \partial_*\Omega\\ &\text{in } \mathbb{C}^n \text{ then its restriction to } \partial_*\Omega \text{ is a CR-function if and only if}\\ &\text{the restriction of } \bar{\partial} f \text{ to } \partial_*\Omega \text{ is a complex normal form.}\end{aligned} \tag{7.1.72}$$

A significantly more refined version of this observation is described in our next proposition.

Proposition 7.1.8 *Let Ω be an open nonempty proper subset of $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ with a lower Ahlfors regular boundary and with the property that $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Fix an integrability exponent $p \in [1,\infty]$, an aperture parameter $\kappa \in (0,\infty)$, and a truncation parameter $\varepsilon > 0$. In this setting, assume $F \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n})$ is a complex-valued function satisfying*

$$\mathcal{N}^{\varepsilon}_{\kappa} F \in L^p_{\mathrm{loc}}(\partial\Omega,\sigma), \ \textit{the trace } F\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \textit{ exists } \sigma\textit{-a.e. on } \partial_{\mathrm{nta}}\Omega \tag{7.1.73}$$

and such that (with the operator $\bar{\partial}$ applied in the sense of distributions)

$$\begin{aligned}&\bar{\partial} F \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{0,1}, \quad \mathcal{N}^{\varepsilon}_{\kappa}(\bar{\partial} F) \in L^p_{\mathrm{loc}}(\partial\Omega,\sigma), \ \textit{and}\\ &\textit{the trace } (\bar{\partial} F)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \textit{ exists at } \sigma\textit{-a.e. point on } \partial_{\mathrm{nta}}\Omega.\end{aligned} \tag{7.1.74}$$

Then for any other $\kappa' > 0$ the nontangential traces $F\Big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$, $(\bar{\partial} F)\Big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$ exist σ-a.e. on $\partial_{\mathrm{nta}}\Omega$, and are actually independent of κ'. Also, when the dependence on κ' is dropped, and when considered on $\partial_*\Omega$, the function $F\Big|^{\mathrm{n.t.}}_{\partial\Omega}$ belongs to $L^{p,1}_{\mathbb{C},\mathrm{loc}}(\partial_*\Omega,\sigma)$, the form $(\bar{\partial} F)\Big|^{\mathrm{n.t.}}_{\partial\Omega}$ belong to $L^p_{\mathrm{loc}}(\partial_*\Omega,\sigma) \otimes \Lambda^{0,1}$ and

$$\nu^{0,1} \wedge \Big((\bar{\partial} F)\Big|^{\mathrm{n.t.}}_{\partial\Omega}\Big) = \sum_{1\leq j<k\leq n} \partial^{\mathbb{C}}_{\tau_{jk}} \Big(F\Big|^{\mathrm{n.t.}}_{\partial\Omega}\Big)\, \mathrm{d}\bar{z}_j \wedge \mathrm{d}\bar{z}_k \ \textit{ at } \sigma\textit{-a.e. point on } \partial_*\Omega. \tag{7.1.75}$$

As a corollary, $(\bar{\partial} F)\Big|^{\mathrm{n.t.}}_{\partial\Omega}$ is complex normal if and only if $F\Big|^{\mathrm{n.t.}}_{\partial\Omega}$ is a CR*-function.*

Proof The fact that the nontangential traces do not depend on the aperture and belong to the specified spaces is seen as in Proposition 7.1.5. Together with (7.1.61), the latter result (cf. (7.1.28)) also permits us to compute

$$\nu^{0,1}\wedge\Big((\bar\partial F)\Big|^{\text{n.t.}}_{\partial\Omega}\Big)=\sum_{1\le j<k\le n}\Big\{(\nu_{\mathbb C})_j\Big((\partial_{\bar z_k}F)\Big|^{\text{n.t.}}_{\partial\Omega}\Big)-(\nu_{\mathbb C})_k\Big((\partial_{\bar z_j}F)\Big|^{\text{n.t.}}_{\partial\Omega}\Big)\Big\}\,d\bar z_j\wedge d\bar z_k$$
$$=\sum_{1\le j<k\le n}\partial^{\mathbb C}_{\tau_{jk}}\Big(F\Big|^{\text{n.t.}}_{\partial\Omega}\Big)\,d\bar z_j\wedge d\bar z_k \tag{7.1.76}$$

at σ-a.e. point on $\partial_*\Omega$. This establishes (7.1.75), and the very last claim in the statement is a consequence of it. □

Going forward, in keeping with an earlier notational convention, if (X,μ) is a given measure space then for each $\alpha,\beta\in\{0,1,\dots,n\}$ and each $p\in(0,\infty]$ we shall abbreviate

$$L^p(X,\mu)\otimes\Lambda^{\alpha,\beta}:=L^p(X,\mu)\otimes\Lambda^{\alpha,\beta}\,\mathbb C^n. \tag{7.1.77}$$

In other words, the space $L^p(X,\mu)\otimes\Lambda^{\alpha,\beta}$ is the is the space of differential forms of type (α,β) with coefficients from $L^p(X,\mu)$. This becomes a quasi-Banach space when equipped with the quasi-norm

$$\begin{aligned}\|f\|_{L^p(X,\mu)\otimes\Lambda^{\alpha,\beta}}&:=\sum_{|I|=\alpha,\ |J|=\beta}\|f_{I,J}\|_{L^p(X,\mu)}\\ \text{if } f&=\sum_{|I|=\alpha,|J|=\beta}f_{I,J}\,dz^I\wedge d\bar z^J\in L^p(X,\mu)\otimes\Lambda^{\alpha,\beta}.\end{aligned} \tag{7.1.78}$$

Definition 7.1.9 *Suppose Ω is a set of locally finite perimeter in $\mathbb R^{2n}\equiv\mathbb C^n$. Abbreviate $\sigma:=\mathcal H^{2n-1}\lfloor\partial\Omega$ and denote by ν its geometric measure theoretic outward unit normal. For each pair of degrees $\alpha,\beta\in\{0,1,\dots,n\}$ and each exponent $p\in(0,\infty]$ consider*

$$L^p_{\tan,\mathbb C}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}:=\{f\in L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}:\ f\text{ is complex tangential}\}, \tag{7.1.79}$$

and

$$L^p_{\text{nor},\mathbb C}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}:=\{f\in L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}:\ f\text{ is complex normal}\}, \tag{7.1.80}$$

where complex tangentiality and complex normality are as introduced in Definition 7.1.7.

It follows from definitions that $L^p_{\tan,\mathbb C}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ and $L^p_{\text{nor},\mathbb C}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ are closed subspaces of $L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$, satisfying

$$\big[L^p_{\tan,\mathbb C}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}\big]\cap\big[L^p_{\text{nor},\mathbb C}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}\big]=\{0\}. \tag{7.1.81}$$

Consequently, the following decomposition holds both algebraically and topologically:

$$L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta} = \left[L^p_{\tan,\mathbb{C}}(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}\right] \oplus \left[L^p_{\mathrm{nor},\mathbb{C}}(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}\right]. \tag{7.1.82}$$

When $p = 2$ the above direct sum is in fact orthogonal. For further use, let us also note here that, as is apparent from (7.1.53), (7.1.67)-(7.1.68), and simple degree considerations,

$$\begin{aligned} L^p_{\tan,\mathbb{C}}(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,0} &= L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,0}, \\ L^p_{\mathrm{nor},\mathbb{C}}(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,n} &= L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,n}. \end{aligned} \tag{7.1.83}$$

Moving on, in the same context as in Definition 7.1.2, for each $\alpha, \beta \in \{0, 1, \dots, n\}$ let us now consider the space

$$\begin{aligned} &L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta} \\ &\quad := \Big\{ f = \sum_{|K|=\alpha} \sum_{|I|=\beta} f_{K,I}\, \mathrm{d}z^K \wedge \mathrm{d}\bar{z}^I \,:\, f_{K,I} \in L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma) \\ &\qquad\qquad \text{for each } K \in \{1, \dots, n\}^\alpha \text{ and } I \in \{1, \dots, n\}^\beta \Big\} \end{aligned} \tag{7.1.84}$$

equipped with the norm

$$\begin{aligned} \|f\|_{L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} &:= \|f\|_{L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} + \sum_{|K|=\alpha} \sum_{|I|=\beta} \sum_{j,k=1}^{n} \left\|\partial^{\mathbb{C}}_{\tau_{jk}} f_{K,I}\right\|_{L^p(\partial_*\Omega,\sigma)} \\ \text{whenever } f &= \sum_{|K|=\alpha} \sum_{|I|=\beta} f_{K,I}\, \mathrm{d}z^K \wedge \mathrm{d}\bar{z}^I \in L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}. \end{aligned} \tag{7.1.85}$$

The space $L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}$ can be thought of as a partial Sobolev space of (α, β)-differential forms on the geometric measure theoretic boundary of Ω which is well-adapted to the family of operators $\big\{\partial^{\mathbb{C}}_{\tau_{jk}}\big\}_{1\leq j,k\leq n}$.

7.2 Integration by Parts Formulas Involving the $\overline{\partial}$ Operator on Sets of Locally Finite Perimeter

We begin by briefly discussing the algebraic formalism associated with the $\bar{\partial}$-operator (for more on this topic see, e.g., [13], [59], [88]). To facilitate the subsequent discussion we find it useful to introduce the following piece of notation. Given an arbitrary open Ω subset of $\mathbb{C}^n$ and an arbitrary subspace $\mathscr{X}$ of $\mathcal{D}'(\Omega)$, we agree to denote by $\mathscr{X} \otimes \Lambda^{\alpha,\beta}$ the space of differential forms $u = \sum_{|I|=\alpha, |J|=\beta} u_{I,J}\, \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J$

with each $u_{I,J}$ belonging to the space $\mathscr{X}$. Whenever dealing with differential forms whose coefficients are actually functions, earlier operations with forms with complex coefficients are now understood in a pointwise sense.

To start in earnest, recall that the exterior derivative operator d in $\mathbb{R}^{2n}$ may be decomposed as

$$\begin{aligned} \mathrm{d} &= \sum_{j=1}^{n} \partial_{x_j} \mathrm{d}x_j \wedge \cdot \; + \sum_{j=1}^{n} \partial_{y_j} \mathrm{d}y_j \wedge \cdot \\ &= \sum_{j=1}^{n} \partial_{z_j} \mathrm{d}z_j \wedge \cdot + \sum_{j=1}^{n} \partial_{\bar{z}_j} \mathrm{d}\bar{z}_j \wedge \cdot = \partial + \bar{\partial}, \end{aligned} \tag{7.2.1}$$

where, as customary, we have set

$$\bar{\partial} := \sum_{j=1}^{n} \partial_{\bar{z}_j} \mathrm{d}\bar{z}_j \wedge \cdot \;\text{ and }\; \partial := \sum_{j=1}^{n} \partial_{z_j} \mathrm{d}z_j \wedge \cdot \tag{7.2.2}$$

for the standard d-bar operator and its complex conjugate, respectively. Their principal symbols are given by

$$\begin{gathered} \mathrm{Sym}(\bar{\partial};\xi) := \frac{\mathrm{i}}{2}\,\xi^{0,1} \wedge \cdot \;\text{ and }\; \mathrm{Sym}(\partial;\xi) := \frac{\mathrm{i}}{2}\,\xi^{1,0} \wedge \cdot \\ \text{for each } \xi \in \mathbb{R}^{2n}. \end{gathered} \tag{7.2.3}$$

For each $\alpha, \beta \in \{0, 1 \dots, n\}$ and each open subset Ω of $\mathbb{C}^n$, the operators

$$\begin{aligned} \bar{\partial} &: \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha,\beta} \longrightarrow \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha,\beta+1}, \\ \partial &: \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha,\beta} \longrightarrow \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha+1,\beta}, \end{aligned} \tag{7.2.4}$$

are well-defined, linear, and continuous. Explicitly, if

$$u = \sum_{|M|=\alpha} \sum_{|I|=\beta} u_{M,I}\, \mathrm{d}z^M \wedge \mathrm{d}\bar{z}^I \in \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha,\beta}, \tag{7.2.5}$$

it follows that

$$\bar{\partial} u = (-1)^{\alpha} \sum_{|J|=\beta+1} \sum_{|M|=\alpha} \sum_{|I|=\beta} \sum_{j=1}^{n} \varepsilon^{jI}_{J}\, \partial_{\bar{z}_j} u_{M,I}\, \mathrm{d}z^M \wedge \mathrm{d}\bar{z}^J, \tag{7.2.6}$$

and

$$\partial u = \sum_{|K|=\alpha+1} \sum_{|M|=\alpha} \sum_{|I|=\beta} \sum_{j=1}^{n} \varepsilon^{jM}_{K}\, \partial_{z_j} u_{M,I}\, \mathrm{d}z^K \wedge \mathrm{d}\bar{z}^I. \tag{7.2.7}$$

Note that

$$\bar{\partial} u = \overline{\partial \bar{u}} \;\text{ for each }\; u \in \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha,\beta}. \tag{7.2.8}$$

Also, the fact that $d \circ d = 0$ forces (via simple degree considerations)

$$\partial \circ \partial = 0, \quad \bar{\partial} \circ \bar{\partial} = 0, \quad \partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0. \tag{7.2.9}$$

Next, if we set

$$\vartheta := - * \partial * \;\text{ and }\; \bar{\vartheta} := - * \bar{\partial} * \tag{7.2.10}$$

then, for each $\alpha, \beta \in \{0, 1 \dots, n\}$ and each open subset Ω of $\mathbb{C}^n$, the operators

$$\begin{aligned} \vartheta &: \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha,\beta} \longrightarrow \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha,\beta-1}, \\ \bar{\vartheta} &: \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha,\beta} \longrightarrow \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha-1,\beta}, \end{aligned} \tag{7.2.11}$$

are well-defined, linear, and continuous. More transparently, if $\alpha, \beta \in \{0, 1, \dots, n\}$ then for each open set $\Omega \subseteq \mathbb{C}^n$ and for each form

$$\psi = \sum_{|M|=\alpha,\, |J|=\beta} \psi_{M,J} \mathrm{d}z^M \wedge \mathrm{d}\bar{z}^J \in \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha,\beta} \tag{7.2.12}$$

we have

$$\vartheta\psi = 2(-1)^{\alpha+1} \sum_{|M|=\alpha} \sum_{|I|=\beta-1} \sum_{|J|=\beta} \sum_{j=1}^{n} \varepsilon_J^{jI} \partial_{z_j} \psi_{M,J} \mathrm{d}z^M \wedge \mathrm{d}\bar{z}^I. \tag{7.2.13}$$

Also,

$$\partial = *\vartheta*, \quad \bar{\partial} = *\bar{\vartheta}*, \quad \vartheta \circ \vartheta = 0, \quad \bar{\vartheta} \circ \bar{\vartheta} = 0, \quad \vartheta \circ \bar{\vartheta} + \bar{\vartheta} \circ \vartheta = 0. \tag{7.2.14}$$

Consider next an arbitrary open set $\Omega \subseteq \mathbb{C}^n$ and fix some $\alpha, \beta \in \{0, 1, \dots, n\}$. Given two arbitrary forms $u \in \mathscr{C}^1(\Omega) \otimes \Lambda^{\alpha,\beta-1}$ and $w \in \mathscr{C}^1(\Omega) \otimes \Lambda^{\alpha,\beta}$, we may then compute

$$\begin{aligned} \mathrm{d}(u \wedge *\bar{w}) &= (\mathrm{d}u) \wedge *\bar{w} + (-1)^{\alpha+\beta+1} u \wedge \mathrm{d}(*\bar{w}) \\ &= (\partial u + \bar{\partial} u) \wedge *\bar{w} + u \wedge * * (\partial + \bar{\partial})(*\bar{w}) \\ &= (\partial u) \wedge *\bar{w} + (\bar{\partial} u) \wedge *\bar{w} - u \wedge *\vartheta\bar{w} - u \wedge *\bar{\vartheta}\bar{w} \\ &= \big\langle \partial u, w \big\rangle_{\mathbb{C}} \mathrm{d}V + \big\langle \bar{\partial} u, w \big\rangle_{\mathbb{C}} \mathrm{d}V - \big\langle u, \bar{\vartheta} w \big\rangle_{\mathbb{C}} \mathrm{d}V - \big\langle u, \vartheta w \big\rangle_{\mathbb{C}} \mathrm{d}V \\ &= \big\langle \bar{\partial} u, w \big\rangle_{\mathbb{C}} \mathrm{d}V - \big\langle u, \vartheta w \big\rangle_{\mathbb{C}} \mathrm{d}V. \end{aligned} \tag{7.2.15}$$

Above, the first equality is the Leibniz product formula for the exterior derivative operator, the second equality is implied by the decomposition in (7.2.1) and item *(1)* in Lemma 7.1.6, the third equality uses (7.2.10), the third equality is a consequence of item *(5)* in Lemma 7.1.6, and the final equality is seen from simple degree considerations (taking into account (7.2.4), (7.2.11), and (7.1.45)). In turn, from (7.2.15) and a most elementary version of Stokes' theorem we conclude that

$$\int_\Omega \big\langle \bar\partial u, w\big\rangle_{\mathbb{C}}\,\mathrm{d}V - \int_\Omega \big\langle u, \vartheta w\big\rangle_{\mathbb{C}}\,\mathrm{d}V = \int_\Omega \mathrm{d}(u\wedge *\bar w) = 0$$
$$\text{whenever } u\in\mathscr{C}^1_c(\Omega)\otimes\Lambda^{\alpha,\beta-1} \text{ and } w\in\mathscr{C}^1(\Omega)\otimes\Lambda^{\alpha,\beta}.$$

This shows that

$$(\bar\partial)^\top = \bar\vartheta \text{ and } \partial^\top = \vartheta \tag{7.2.16}$$

in the sense that the real transpose of $\bar\partial$ (respectively, ∂) acting on $(\alpha,\beta-1)$-forms is the operator $\bar\vartheta$ (respectively, ϑ) acting on (α,β)-forms. Let us also note that, as is apparent from (7.2.16), [68, (1.7.17)-(1.7.18)], (7.2.3), and (7.1.58),

$$\begin{gathered}\mathrm{Sym}(\vartheta;\xi) = -\frac{\mathrm{i}}{2}\,\xi^{1,0}\vee\cdot \text{ and } \mathrm{Sym}(\bar\vartheta;\xi) = -\frac{\mathrm{i}}{2}\,\xi^{0,1}\vee\cdot\\ \text{for each given real vector } \xi\in\mathbb{R}^{2n}.\end{gathered}\tag{7.2.17}$$

Below, we discuss a basic integration by parts formula for the $\bar\partial$ operator in a very general setting.

Theorem 7.2.1 *Let Ω be an open nonempty proper subset of $\mathbb{C}^n\equiv\mathbb{R}^{2n}$ with a lower Ahlfors regular boundary, and such that $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Fix an aperture parameter $\kappa\in(0,\infty)$ and assume that*

$$u\in L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta} \text{ and } w\in L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta+1}, \tag{7.2.18}$$

where $\alpha,\beta\in\{0,1,\dots,n\}$, satisfy

$$\begin{gathered}\mathcal{N}_\kappa u<\infty \text{ and } \mathcal{N}_\kappa w<\infty \text{ at } \sigma\text{-a.e. point on } \partial\Omega,\\ \mathcal{N}_\kappa u\cdot\mathcal{N}_\kappa w \text{ belongs to the space } L^1(\partial\Omega,\sigma),\\ \text{the nontangential traces } u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ w\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exist } \sigma\text{-a.e. on } \partial_{\mathrm{nta}}\Omega,\\ \bar\partial u\in L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta+1},\quad \vartheta w\in L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta},\\ \text{and } \big\langle\bar\partial u, w\big\rangle_{\mathbb{C}} - \big\langle u,\vartheta w\big\rangle_{\mathbb{C}} \text{ belongs to } L^1(\Omega,\mathcal{L}^{2n}),\end{gathered}\tag{7.2.19}$$

with all partial differential operators considered in the sense of distributions in Ω. In the case when Ω is an exterior domain make the additional assumption that there exists $\lambda\in(1,\infty)$ such that

$$\int_{B(0,\lambda R)\setminus B(0,R)} |u|_{\mathbb{C}}|w|_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n} = o(R) \text{ as } R\to\infty. \tag{7.2.20}$$

Then

$$\int_\Omega \Big\{\langle\bar\partial u, w\rangle_{\mathbb{C}} - \langle u, \vartheta w\rangle_{\mathbb{C}}\Big\}\,\mathrm{d}V = \frac{1}{2}\int_{\partial_*\Omega}\Big\langle \nu^{0,1}\wedge u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma$$

$$= \frac{1}{2}\int_{\partial_*\Omega}\Big\langle u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \nu^{1,0}\vee w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma. \qquad (7.2.21)$$

As a corollary, if the integrability condition in the last line of (7.2.19) *is strengthened to*

$$\big\langle\bar\partial u, w\big\rangle_{\mathbb{C}} \in L^1(\Omega, \mathcal{L}^{2n}) \text{ and } \big\langle u, \vartheta w\big\rangle_{\mathbb{C}} \in L^1(\Omega, \mathcal{L}^{2n}) \qquad (7.2.22)$$

then (7.2.21) *may be recast as the integration by parts formula*

$$\int_\Omega \langle\bar\partial u, w\rangle_{\mathbb{C}}\,\mathrm{d}V = \int_\Omega \langle u, \vartheta w\rangle_{\mathbb{C}}\,\mathrm{d}V + \frac{1}{2}\int_{\partial_*\Omega}\Big\langle \nu^{0,1}\wedge u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma$$

$$= \int_\Omega \langle u, \vartheta w\rangle_{\mathbb{C}}\,\mathrm{d}V + \frac{1}{2}\int_{\partial_*\Omega}\Big\langle u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \nu^{1,0}\vee w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma. \qquad (7.2.23)$$

For example, all hypotheses demanded in the first two lines of (7.2.19) are satisfied if we assume $\mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma)$ and $\mathcal{N}_\kappa w \in L^{p'}(\partial\Omega,\sigma)$ for some integrability exponents $p, p' \in [1,\infty]$ with $1/p + 1/p' = 1$. In such a scenario, it follows that for any other $\kappa' > 0$ the nontangential traces $u\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$, $w\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ exist at σ-a.e. point on $\partial_{\mathrm{nta}}\Omega$ and are actually independent of κ'.

By taking conjugates, a number of related versions of (7.2.21) may be established from it. For example, (7.2.21) and (7.1.44) it follows that if the differential forms u, w are as in (7.2.18)-(7.2.20) then

$$\int_\Omega \Big\{\langle\vartheta w, u\rangle_{\mathbb{C}} - \langle w, \bar\partial u\rangle_{\mathbb{C}}\Big\}\,\mathrm{d}V = -\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \nu^{1,0}\vee w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma$$

$$= -\frac{1}{2}\int_{\partial_*\Omega}\Big\langle w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \nu^{0,1}\wedge u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma. \qquad (7.2.24)$$

Proof of Theorem 7.2.1 The idea is to invoke [68, Theorem 1.7.2] for the choice $D := \bar\partial$ (and with w replaced by $\bar w$). As may seen from (7.2.16), such a choice implies that $D^\top = \bar\vartheta$. Since in the present setting $(-i)\mathrm{Sym}(D;\nu)$ may be identified with $(-i)\mathrm{Sym}(\bar\partial;\nu) = \frac{1}{2}\nu^{0,1}\wedge$ (cf. (7.2.3)), formula (7.2.21) then follows directly from [68, (1.7.24)]. □

Our next goal is to introduce a partial Sobolev space of differential forms on the geometric measure theoretic boundary of a set of locally finite perimeter in $\mathbb{C}^n$ which is well-adapted to the $\bar\partial$-formalism. In order to be able to do so, we first make the definition below.

Definition 7.2.2 *Let Ω be a set of locally finite perimeter in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. Denote by ν its geometric measure theoretic outward unit normal, and abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. Also, fix a pair of arbitrary degrees $\alpha, \beta \in \{0, 1, \dots, n\}$. In this context, say that*

$f \in L^1_{\rm loc}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ has the property that $\bar\partial_\tau f \in L^1_{\rm loc}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+2}$ provided there exists some $g \in L^1_{\rm loc}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+2}$ such that

$$\int_{\partial_*\Omega}\left\langle \nu^{0,1}\wedge f,\vartheta\psi\right\rangle_{\mathbb{C}}\,{\rm d}\sigma = \int_{\partial_*\Omega}\langle g,\psi\rangle_{\mathbb{C}}\,{\rm d}\sigma \ \textit{for all}\ \psi\in\mathscr{C}^\infty_c(\mathbb{C}^n)\otimes\Lambda^{\alpha,\beta+2}. \tag{7.2.25}$$

In the setting of Definition 7.2.2, from [68, (5.2.6)] and [68, Lemma 3.6.4] we see that $\sigma\lfloor\partial_*\Omega$ is a complete, locally finite, Borel-regular measure on $\partial_*\Omega$ (where the latter set is endowed with the topology inherited from the Euclidean ambient). Granted this, [68, Proposition 3.7.2] applies and gives that the form g is unambiguously defined by the condition imposed in (7.2.25). In order to stress the dependence of such g on the given form f we shall henceforth employ the notation

$$\bar\partial_\tau f := g. \tag{7.2.26}$$

In the same setting as in Definition 7.2.2, given any $p\in[1,\infty]$ we then define

$$L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega,\sigma) := \left\{f\in L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}\,:\,\bar\partial_\tau f\in L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+2}\right\}. \tag{7.2.27}$$

In this notation, we may now recast (7.2.25) as the integration by parts formula on the boundary

$$\begin{gathered}\int_{\partial_*\Omega}\left\langle \nu^{0,1}\wedge f,\vartheta\psi\right\rangle_{\mathbb{C}}\,{\rm d}\sigma = \int_{\partial_*\Omega}\left\langle \bar\partial_\tau f,\psi\right\rangle_{\mathbb{C}}\,{\rm d}\sigma \\ \text{for all } f\in L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega,\sigma) \text{ and } \psi\in\mathscr{C}^\infty_c(\mathbb{C}^n)\otimes\Lambda^{\alpha,\beta+2}.\end{gathered} \tag{7.2.28}$$

We also agree to equip the space $L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega,\sigma)$ with the natural norm

$$\|f\|_{L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega,\sigma)} := \|f\|_{L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} + \|\bar\partial_\tau f\|_{L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+2}} \tag{7.2.29}$$

for each $f\in L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega,\sigma)$.

Proposition 7.2.3 *Consider a set of locally finite perimeter $\Omega\subseteq\mathbb{R}^{2n}\equiv\mathbb{C}^n$. Abbreviate $\sigma:=\mathcal{H}^{2n-1}\lfloor\partial\Omega$ and denote by ν its geometric measure theoretic outward unit normal. Also, pick some function $f\in L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ with exponent $p\in[1,\infty]$ and $\alpha,\beta\in\{0,1,\dots,n\}$. Then*

$$f\in L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega,\sigma)\ \textit{if and only if}\ f_{{\rm tan},\mathbb{C}}\in L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega,\sigma), \tag{7.2.30}$$

and if either of the above memberships holds then

$$\bar\partial_\tau f = \bar\partial_\tau(f_{{\rm tan},\mathbb{C}})\ \textit{at}\ \sigma\textit{-a.e. point on}\ \partial_*\Omega. \tag{7.2.31}$$

Proof This is a direct consequence of (7.2.25) and (7.1.69). □

As is apparent from our next result, the partial Sobolev space $L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega,\sigma)$ introduced in (7.2.27) turns out to be rather rather rich.

Proposition 7.2.4 *Suppose Ω is a set of locally finite perimeter in $\mathbb{R}^{2n}\equiv\mathbb{C}^n$. As usual, abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$, and pick $p\in[1,\infty]$ along with $\alpha,\beta\in\{0,1,\dots,n\}$. Then one has the inclusion*

$$L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}\subseteq L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega,\sigma) \tag{7.2.32}$$

and for each differential form

$$f=\sum_{|K|=\alpha}\sum_{|I|=\beta} f_{K,I}\,\mathrm{d}z^K\wedge\mathrm{d}\bar z^I\in L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta} \tag{7.2.33}$$

there holds

$$\bar\partial_\tau f=2^{-1}(-1)^{\alpha+1}\sum_{|K|=\alpha}\sum_{|I|=\beta}\sum_{1\le j\neq k\le n}(\partial^{\mathbb{C}}_{\tau_{jk}}f_{K,I})\mathrm{d}z^K\wedge\mathrm{d}\bar z_j\wedge\mathrm{d}\bar z_k\wedge\mathrm{d}\bar z^I. \tag{7.2.34}$$

In particular, corresponding to $\alpha=\beta=0$, one has

$$\bar\partial_\tau f=-\sum_{1\le j<k\le n}\partial^{\mathbb{C}}_{\tau_{jk}}f\,\mathrm{d}\bar z_j\wedge\mathrm{d}\bar z_k\ \text{ for each }\ f\in L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma). \tag{7.2.35}$$

Proof Pick $\psi=\sum_{|M|=\alpha,|J|=\beta+2}\psi_{M,J}\,\mathrm{d}z^M\wedge\mathrm{d}\bar z^J\in\mathscr{C}^\infty_c(\mathbb{C}^n)\otimes\Lambda^{\alpha,\beta+2}$ arbitrary along with some increasing multi-indices $K\in\{1,\dots,n\}^\alpha$ and $I\in\{1,\dots,n\}^\beta$. Then, if ν denotes the geometric measure theoretic outward unit normal to Ω, the (K,I)-th component of $\overline{\nu^{0,1}}\vee\vartheta\psi$ on $\partial_*\Omega$ is given at σ-a.e. point by

$$\begin{aligned}
\big(\overline{\nu^{0,1}}\vee\vartheta\psi\big)_{K,I}&=2\sum_{|M|=\beta+1}\sum_{j=1}^{n}\varepsilon^{jI}_{M}(\overline{\nu^{0,1}})_j(\vartheta\psi)_{K,M}\\
&=4(-1)^{\alpha+1}\sum_{|M|=\beta+1}\sum_{j=1}^{n}\sum_{|J|=\beta+2}\sum_{k=1}^{n}\varepsilon^{jI}_{M}\varepsilon^{kM}_{J}(\overline{\nu^{0,1}})_j\,\partial_{z_k}\psi_{K,J}\\
&=4(-1)^{\alpha+1}\sum_{|J|=\beta+2}\sum_{1\le j\neq k\le n}\varepsilon^{kjI}_{J}(\overline{\nu^{0,1}})_j\,\partial_{z_k}\psi_{K,J}\\
&=2(-1)^{\alpha+1}\sum_{|J|=\beta+2}\sum_{1\le j\neq k\le n}\varepsilon^{kjI}_{J}\Big[(\overline{\nu^{0,1}})_j\,\partial_{z_k}-(\overline{\nu^{0,1}})_k\,\partial_{z_j}\Big]\psi_{K,J}\\
&=2(-1)^{\alpha+1}\sum_{|J|=\beta+2}\sum_{1\le j\neq k\le n}\varepsilon^{kjI}_{J}\overline{\Big[(\nu_{\mathbb{C}})_j\,\partial_{\bar z_k}-(\nu_{\mathbb{C}})_k\,\partial_{\bar z_j}\Big]\overline{\psi}_{K,J}}\\
&=2(-1)^{\alpha+1}\sum_{|J|=\beta+2}\sum_{1\le j\neq k\le n}\varepsilon^{kjI}_{J}\overline{(\partial^{\mathbb{C}}_{\tau_{jk}}\overline{\psi}_{K,J})}.
\end{aligned}\tag{7.2.36}$$

Above, the first equality is a consequence of (7.1.57) and the second equality follows from (7.2.13). Furthermore, the third identity in (7.2.36) is a consequence of the fact that

$$\sum_{|M|=\beta+1} \varepsilon_J^{kM} \varepsilon_M^{jI} = \sum_{|M|=\beta+1} \varepsilon_J^{kM} \varepsilon_{kM}^{kjI} = \varepsilon_J^{kjI}. \tag{7.2.37}$$

The fourth identity in (7.2.36) follows from the fact that $\varepsilon_J^{kjI} = -\varepsilon_J^{jkI}$ while the fifth one follows from (7.1.61). Finally, the sixth equality follows easily from the properties of complex conjugation while the last equality uses the definition (7.1.5) of the complex tangential derivative operator $\partial_{\tau_{jk}}^{\mathbb{C}}$.

Hence, for any given differential form f as in (7.2.33) we may write

$$\begin{aligned}
&\int_{\partial_*\Omega} \left\langle \nu^{0,1} \wedge f, \vartheta\psi \right\rangle_{\mathbb{C}} \,\mathrm{d}\sigma \\
&\quad = \int_{\partial_*\Omega} \left\langle f, \overline{\nu^{0,1}} \vee \vartheta\psi \right\rangle_{\mathbb{C}} \,\mathrm{d}\sigma \\
&\quad = 2^{\alpha+\beta} \sum_{|K|=\alpha} \sum_{|I|=\beta} \int_{\partial_*\Omega} f_{K,I} \overline{(\overline{\nu^{0,1}} \vee \vartheta\psi)_{K,I}} \,\mathrm{d}\sigma \\
&\quad = 2^{\alpha+\beta+1}(-1)^{\alpha+1} \sum_{|K|=\alpha} \sum_{|I|=\beta} \sum_{|J|=\beta+2} \sum_{1\leq j\neq k\leq n} \varepsilon_J^{kjI} \int_{\partial_*\Omega} f_{K,I} (\partial_{\tau_{jk}}^{\mathbb{C}} \overline{\psi}_{K,J}) \,\mathrm{d}\sigma,
\end{aligned} \tag{7.2.38}$$

where the first identity above is a consequence of item *(3)* in Lemma 7.1.6, the second equality follows from (7.1.43), and the last equality uses (7.2.36).

To proceed, let us define

$$g := \sum_{|K|=\alpha} \sum_{|I|=\beta} \sum_{1\leq j\neq k\leq n} (\partial_{\tau_{jk}}^{\mathbb{C}} f_{K,I}) \mathrm{d}z^K \wedge \mathrm{d}\bar{z}_j \wedge \mathrm{d}\bar{z}_k \wedge \mathrm{d}\bar{z}^I \in L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta+2}. \tag{7.2.39}$$

Making use of (7.2.38), (7.1.9), (7.1.14), and (7.2.39) we may then compute

$$\begin{aligned}
&\int_{\partial_*\Omega} \left\langle \nu^{0,1} \wedge f, \vartheta\psi \right\rangle_{\mathbb{C}} \,\mathrm{d}\sigma \\
&\qquad = 2^{\alpha+\beta+1}(-1)^{\alpha} \sum_{|K|=\alpha} \sum_{|I|=\beta} \sum_{|J|=\beta+2} \sum_{1\leq j\neq k\leq n} \varepsilon_J^{kjI} \int_{\partial_*\Omega} (\partial_{\tau_{jk}}^{\mathbb{C}} f_{K,I}) \overline{\psi}_{K,J} \,\mathrm{d}\sigma \\
&\qquad = 2^{\alpha+\beta+1}(-1)^{\alpha+1} \sum_{|K|=\alpha} \sum_{|J|=\beta+2} \int_{\partial_*\Omega} g_{K,J} \overline{\psi}_{K,J} \,\mathrm{d}\sigma \\
&\qquad = 2^{-1}(-1)^{\alpha+1} \int_{\partial_*\Omega} \langle g, \psi \rangle_{\mathbb{C}} \,\mathrm{d}\sigma.
\end{aligned} \tag{7.2.40}$$

In light of (7.2.27), this proves that $f \in L^{p,\bar{\partial}_\tau}_{\alpha,\beta}(\partial_*\Omega, \sigma)$ (ultimately establishing the inclusion in (7.2.32)) and that $\bar{\partial}_\tau f = 2^{-1}(-1)^{\alpha+1} g \in L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta+2}$, (hence (7.2.34) holds). □

We augment the result established in Proposition 7.2.4 by showing that the inclusion in (7.2.32) actually becomes an equality when $\beta = 0$.

Proposition 7.2.5 *Assume Ω is a set of locally finite perimeter in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ and, as usual, abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. Then for each degree $\alpha \in \{0, 1, \dots, n\}$ and integrability exponent $p \in [1, \infty]$ one has*

$$L^{p,\bar{\partial}_\tau}_{\alpha,0}(\partial_*\Omega, \sigma) = L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,0}. \tag{7.2.41}$$

Proof The right-to-left inclusion in (7.2.41) is a special case of Proposition 7.2.4. To prove the opposite inclusion in (7.2.41), pick $f = \sum_{|K|=\alpha} f_K \, dz^K \in L^{p,\bar{\partial}_\tau}_{\alpha,0}(\partial_*\Omega, \sigma)$. Also, select a test function $\varphi \in \mathscr{C}^\infty_c(\mathbb{C}^n)$ along with two indices $j_o, k_o \in \{1, \dots, n\}$ satisfying $j_o < k_o$ and some ordered array $K_o \in \{1, \dots, n\}^\alpha$. Consider next the differential form

$$\psi := \bar{\varphi} \, dz^{K_o} \wedge d\bar{z}_{j_o} \wedge d\bar{z}_{k_o} \in \mathscr{C}^\infty_c(\mathbb{C}^n) \otimes \Lambda^{\alpha,2}. \tag{7.2.42}$$

Then, if ν denotes the geometric measure theoretic outward unit normal to Ω, by specializing formula (7.2.38) to the present choice of f and ψ we obtain

$$\int_{\partial_*\Omega} \big\langle \nu^{0,1} \wedge f, \vartheta\psi \big\rangle_{\mathbb{C}} \, d\sigma = 2^{\alpha+2}(-1)^{\alpha+1} \int_{\partial_*\Omega} f_{K_o} \, \partial^{\mathbb{C}}_{\tau_{j_o k_o}} \varphi \, d\sigma. \tag{7.2.43}$$

Next, in light of (7.2.27), the membership of f to the space $L^{p,\bar{\partial}_\tau}_{\alpha,0}(\partial_*\Omega, \sigma)$ ensures the existence of some $g = \sum_{|K|=\alpha} \sum_{1 \le j < k \le n} g_{K,jk} \, dz^K \wedge d\bar{z}_j \wedge d\bar{z}_k \in L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,2}$ with the property that

$$\int_{\partial_*\Omega} \big\langle \nu^{0,1} \wedge f, \vartheta\psi \big\rangle_{\mathbb{C}} \, d\sigma = \int_{\partial_*\Omega} \langle g, \psi \rangle_{\mathbb{C}} \, d\sigma = 2^{\alpha+2} \int_{\partial_*\Omega} g_{K_o, j_o k_o} \varphi \, d\sigma, \tag{7.2.44}$$

where the last equality uses (7.1.42) and the specific formats of g and ψ. In concert, (7.2.43) and (7.2.44) ultimately prove that

$$(-1)^{\alpha+1} \int_{\partial_*\Omega} f_{K_o} \, \partial^{\mathbb{C}}_{\tau_{j_o k_o}} \varphi \, d\sigma = \int_{\partial_*\Omega} g_{j_o k_o} \varphi \, d\sigma, \qquad \forall \varphi \in \mathscr{C}^\infty_c(\mathbb{C}^n). \tag{7.2.45}$$

Having established (7.2.45), from Definition 7.1.1 (cf. also (7.1.8)) we may then conclude that $\partial^{\mathbb{C}}_{\tau_{j_o k_o}} f_{K_o} = (-1)^\alpha g_{K_o, j_o k_o} \in L^p(\partial_*\Omega, \sigma)$. Given that $f_{K_o} \in L^p(\partial_*\Omega, \sigma)$ and the indeces j_o, k_o are arbitrary, Definition 7.1.2 shows that we necessarily have $f_{K_o} \in L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma)$. Ultimately, since K_o has been arbitrarily chosen this finishes the proof of (7.2.41). □

In applications, it is useful to know that a version of the integration by parts formula on the boundary recorded in (7.2.28) also holds if the compact support condition for the intervening test functions is replaced by suitable integrability conditions. The specific format of this result is recorded below.

Proposition 7.2.6 *Let Ω be a set of locally finite perimeter in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. Abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ and denote by ν its geometric measure theoretic outward unit normal. Also, fix $p \in [1,\infty]$ and $\alpha, \beta \in \{0, 1, \dots, n\}$, along with an open neighborhood $U \subseteq \mathbb{C}^n$ of $\partial\Omega$. Consider a differential form $f \in L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega, \sigma)$ and suppose*

$$\psi \in \mathscr{C}^1(U) \otimes \Lambda^{\alpha,\beta+2} \text{ is such that } \psi\big|_{\partial_*\Omega} \in L^{p'}(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta+2} \\ \text{and } (\vartheta\psi)\big|_{\partial_*\Omega} \in L^{p'}(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta+1}, \tag{7.2.46}$$

where $p' \in [1,\infty]$ is the Hölder conjugate exponent of p. Then

$$\int_{\partial_*\Omega} \big\langle \nu^{0,1} \wedge f, \vartheta\psi \big\rangle_{\mathbb{C}} \, d\sigma = \int_{\partial_*\Omega} \big\langle \bar\partial_\tau f, \psi \big\rangle_{\mathbb{C}} \, d\sigma. \tag{7.2.47}$$

In particular, (7.2.47) holds for every $\psi \in \mathscr{C}^1(U) \otimes \Lambda^{\alpha,\beta+2}$ in the case when $\partial_*\Omega$ is bounded.

Proof Pick a scalar-valued function $\theta \in \mathscr{C}^\infty_c(\mathbb{C}^n)$ with the property that $\theta \equiv 1$ on $B(0,1)$ and, for each $R > 0$, define $\theta_R(z) := \theta(z/R)$ for every $z \in \mathbb{C}^n$. The idea is to write

$$\begin{aligned} \int_{\partial_*\Omega} \big\langle \nu^{0,1} \wedge f, \vartheta\psi \big\rangle_{\mathbb{C}} \, d\sigma &= \lim_{R\to\infty} \int_{\partial_*\Omega} \big\langle \nu^{0,1} \wedge f, \theta_R \vartheta\psi \big\rangle_{\mathbb{C}} \, d\sigma \\ &= \lim_{R\to\infty} \int_{\partial_*\Omega} \big\langle \nu^{0,1} \wedge f, \vartheta(\theta_R\psi) \big\rangle_{\mathbb{C}} \, d\sigma \\ &\quad - \lim_{R\to\infty} \int_{\partial_*\Omega} \big\langle \nu^{0,1} \wedge f, (\partial\theta_R) \vee \psi \big\rangle_{\mathbb{C}} \, d\sigma \\ &= \lim_{R\to\infty} \int_{\partial_*\Omega} \big\langle \bar\partial_\tau f, \theta_R\psi \big\rangle_{\mathbb{C}} \, d\sigma \\ &= \lim_{R\to\infty} \int_{\partial_*\Omega} \big\langle \bar\partial_\tau f, \psi \big\rangle_{\mathbb{C}} \, d\sigma. \end{aligned} \tag{7.2.48}$$

Indeed, the first equality above is a consequence of Lebesgue's Dominated Convergence Theorem (bearing in mind that the last condition in (7.2.46) and the fact that $f \in L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}$). The second equality uses a Leibniz product rule to the effect that

$$\vartheta(\theta_R\psi) = \theta_R\vartheta\psi - (\partial\theta_R) \vee \psi \tag{7.2.49}$$

which, in turn, is seen from (7.2.10) and items *(1)* and *(4)* in in Lemma 7.1.6. The third equality is based on two facts. First, since $\theta_R\psi \in \mathscr{C}^1_c(U) \otimes \Lambda^{\alpha,\beta+2}$ for each $R > 0$, we may use a standard mollifier argument which ultimately allows us to invoke (7.2.28) in order to conclude that

$$\int_{\partial_*\Omega} \big\langle \nu^{0,1} \wedge f, \vartheta(\theta_R\psi)\big\rangle_{\mathbb{C}}\, d\sigma = \int_{\partial_*\Omega} \big\langle \bar{\partial}_\tau f, \theta_R\psi\big\rangle_{\mathbb{C}}\, d\sigma \text{ for each } R > 0. \tag{7.2.50}$$

Second, since $\big|\partial\theta_R\big|_{\mathbb{C}} \le CR^{-1}$ at σ-a.e. point in $\partial_*\Omega$, we may estimate

$$\begin{aligned} &\Big|\int_{\partial_*\Omega} \big\langle \nu^{0,1} \wedge f, (\partial\theta_R) \vee \psi\big\rangle_{\mathbb{C}}\, d\sigma\Big| \\ &\qquad \le CR^{-1}\|f\|_{L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{0,\beta}} \big\|\psi\big|_{\partial_*\Omega}\big\|_{L^{p'}(\partial_*\Omega,\sigma)\otimes\Lambda^{0,\beta+2}} \longrightarrow 0 \text{ as } R \to \infty. \end{aligned} \tag{7.2.51}$$

As such, the second limit in the second line of (7.2.48) vanishes. There remains to observe that the last equality in (7.2.48) is implied by Lebesgue's Dominated Convergence Theorem (bearing in mind that $\bar{\partial}_\tau f \in L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta+2}$ and $\psi\big|_{\partial_*\Omega} \in L^{p'}(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta+2}$. □

In our next proposition we study how the operator $\bar{\partial}_\tau$ interacts with the pointwise nontangential boundary trace operator.

Proposition 7.2.7 *Consider an open nonempty proper subset Ω of $\mathbb{R}^n$ with a lower Ahlfors regular boundary and such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$; in particular, Ω is a set of locally finite perimeter. Denote by ν the geometric measure theoretic outward unit normal to Ω. Also, fix $\alpha, \beta \in \{0, 1, \dots, n\}$, an aperture parameter $\kappa \in (0, \infty)$, an integrability exponent $p \in [1, \infty]$, and a truncation parameter $\varepsilon > 0$. In this context, pick a differential form u satisfying*

$$u \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta}, \quad \bar{\partial}u \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{0,\beta+1}, \tag{7.2.52}$$

$$\mathcal{N}^{\varepsilon}_{\kappa} u \in L^p(\partial\Omega, \sigma), \quad \mathcal{N}^{\varepsilon}_{\kappa}(\bar{\partial}u) \in L^p(\partial\Omega, \sigma), \text{ and} \tag{7.2.53}$$

$$\text{the traces } u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ and } (\bar{\partial}u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exist } \sigma\text{-a.e. on } \partial_{\mathrm{nta}}\Omega. \tag{7.2.54}$$

Then

$$\begin{aligned} &u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ belongs to the space } L^{p,\bar{\partial}_\tau}_{\alpha,\beta}(\partial_*\Omega, \sigma) \text{ and} \\ &\bar{\partial}_\tau\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) = -\nu^{0,1} \wedge \Big[(\bar{\partial}u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big] \text{ on } \partial_*\Omega. \end{aligned} \tag{7.2.55}$$

In particular, at σ-a.e. point on $\partial_*\Omega$ one has

$$\nu^{1,0} \vee \bar{\partial}_\tau\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) = -\nu^{1,0} \vee \Big(\nu^{0,1} \wedge \big[(\bar{\partial}u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big]\Big) = -2\Big[(\bar{\partial}u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big]_{\tan,\mathbb{C}}. \tag{7.2.56}$$

Moreover, there exists a finite constant $C > 0$, independent of u and $\varepsilon > 0$, such that

$$\Big\| u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \Big\|_{L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial\Omega,\sigma)} \le C\Big(\big\|\mathcal{N}_\kappa^\varepsilon u\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathcal{N}_\kappa^\varepsilon(\bar\partial u)\big\|_{L^p(\partial\Omega,\sigma)}\Big). \tag{7.2.57}$$

Proof With the function Φ_ε as in [68, Lemma 6.1.2], define $\Psi_\varepsilon := 1 - \Phi_\varepsilon$. Then [68, (6.1.5)-(6.1.6)] imply

$$\Psi_\varepsilon \in \mathscr{C}^\infty(\Omega), \quad 0 \le \Psi_\varepsilon(x) \le 1 \text{ for all } x \in \Omega, \tag{7.2.58}$$

$$\sup_{x\in\Omega} \big|(\partial^\alpha \Psi_\varepsilon)(x)\big| \le C_\alpha \varepsilon^{-|\alpha|} \text{ for each } \alpha \in \mathbb{N}_0^n, \tag{7.2.59}$$

$$\Psi_\varepsilon \equiv 0 \text{ on } \Omega \setminus \mathcal{O}_\varepsilon \text{ and } \Psi_\varepsilon \equiv 1 \text{ on } \mathcal{O}_{\varepsilon/N}. \tag{7.2.60}$$

Consequently, $\Psi_\varepsilon \in W^{1,\infty}(\Omega)$ and $\operatorname{supp}(\nabla\Psi_\varepsilon) \subseteq \mathcal{O}_\varepsilon \setminus \mathcal{O}_{\varepsilon/N}$. If we now introduce

$$u_\varepsilon := \Psi_\varepsilon u \text{ in } \Omega, \tag{7.2.61}$$

then

$$u_\varepsilon \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta} \text{ and } \bar\partial u_\varepsilon = \bar\partial \Psi_\varepsilon \wedge u + \Psi_\varepsilon \bar\partial u. \tag{7.2.62}$$

Note that we have $\mathcal{N}_\kappa u_\varepsilon \le \mathcal{N}_\kappa^\varepsilon u$ pointwise on $\partial\Omega$ which, thanks to (7.2.53) and [68, (8.2.26)], gives that $\mathcal{N}_\kappa u_\varepsilon \in L^p(\partial\Omega,\sigma)$. Similarly, since the indentity in (7.2.62) implies $\mathcal{N}_\kappa(\bar\partial u_\varepsilon) \le C\big(\mathcal{N}_\kappa^\varepsilon u + \mathcal{N}_\kappa^\varepsilon(\bar\partial u)\big)$ pointwise on $\partial\Omega$, we may also infer that $\mathcal{N}_\kappa(\bar\partial u_\varepsilon) \in L^p(\partial\Omega,\sigma)$. Given that $u_\varepsilon = u$ and $\bar\partial u_\varepsilon = \bar\partial u$ in $\mathcal{O}_{\varepsilon/N}$, we conclude that at σ-a.e. point on $\partial_{\mathrm{nta}}\Omega$ we also have

$$u_\varepsilon\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ and } (\bar\partial u_\varepsilon)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = (\bar\partial u)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}. \tag{7.2.63}$$

To proceed, consider $U := \big\{z \in \mathbb{C}^n : \operatorname{dist}(z,\partial\Omega) < \varepsilon/N\big\}$, which is an open neighborhood of $\partial\Omega$ in $\mathbb{C}^n$, and pick an arbitrary form $\psi \in \mathscr{C}^1_c(U) \otimes \Lambda^{\alpha,\beta+2}$. Then $\vartheta\psi$ has compact support, $\vartheta(\vartheta\psi) = 0$ in U, and $\mathcal{N}_\kappa(\vartheta\psi) \in L^\infty_{\mathrm{comp}}(\partial\Omega,\sigma)$. Moreover, estimate [68, (8.6.51) in Proposition 8.6.3] ensures that

$$\big\langle \bar\partial u_\varepsilon, \vartheta\psi\big\rangle_{\mathbb{C}} \in L^1(\Omega, \mathcal{L}^{2n}). \tag{7.2.64}$$

From these we conclude that conditions (7.2.19) for the forms u_ε (playing the role of u) and $w := \vartheta\psi$ are satisfied, hence Theorem 7.2.1 permits us to write (on account of (7.2.63))

$$\begin{aligned}\int_{\partial_*\Omega} \Big\langle \nu^{0,1} \wedge \Big(u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big), \vartheta\psi\Big\rangle_{\mathbb{C}}\, d\sigma &= \int_{\partial_*\Omega} \Big\langle \nu^{0,1} \wedge \Big(u_\varepsilon\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big), \vartheta\psi\Big\rangle_{\mathbb{C}}\, d\sigma \\ &= 2\int_\Omega \big\langle \bar\partial u_\varepsilon, \vartheta\psi\big\rangle_{\mathbb{C}}\, dV.\end{aligned} \tag{7.2.65}$$

Integrating by parts once more, now moving the partial derivatives from ψ to u_ε (i.e., employing (7.2.21) with $\bar\partial u_\varepsilon$ playing the role of u and with ψ playing the role of w, since once again the conditions (7.2.19) can be easily verified), we obtain

$$2\int_{\Omega}\left\langle\bar{\partial}u_{\varepsilon},\vartheta\psi\right\rangle_{\mathbb{C}}\,\mathrm{d}V=-\int_{\partial_*\Omega}\left\langle\nu^{0,1}\wedge\left(\bar{\partial}u_{\varepsilon}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\right),\psi\right\rangle_{\mathbb{C}}\,\mathrm{d}\sigma$$
$$=-\int_{\partial_*\Omega}\left\langle\nu^{0,1}\wedge\left(\bar{\partial}u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\right),\psi\right\rangle_{\mathbb{C}}\,\mathrm{d}\sigma,\tag{7.2.66}$$

where the last equality uses (7.2.63). All in all, introducing $f:=u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$, the above reasoning shows that $f\in L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ and

$$\int_{\partial_*\Omega}\langle\nu^{0,1}\wedge f,\vartheta\psi\rangle_{\mathbb{C}}\,\mathrm{d}\sigma=-\int_{\partial_*\Omega}\left\langle\nu^{0,1}\wedge\left(\bar{\partial}u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\right),\psi\right\rangle_{\mathbb{C}}\,\mathrm{d}\sigma\tag{7.2.67}$$

for all $\psi\in\mathscr{C}^1_c(U)\otimes\Lambda^{\alpha,\beta+2}$. In addition, thanks to the hypotheses made in (7.2.53)-(7.2.54), the membership $\nu^{0,1}\wedge\left(\bar{\partial}u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\right)\in L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+2}$ holds. On account of 7.2.27, this establishes (7.2.55). With (7.2.55) in hand, all remaining conclusions in the statement of the proposition now easily follow. □

To proceed, in the definition below we introduce a very general notion of CR-form (aka, Cauchy-Riemann differential form).

Definition 7.2.8 *Suppose Ω is an open set of locally finite perimeter in $\mathbb{R}^{2n}\equiv\mathbb{C}^n$. Abbreviate $\sigma:=\mathcal{H}^{2n-1}\lfloor\partial\Omega$ and denote by ν its geometric measure theoretic outward unit normal. Also, pick an arbitrary pair of degrees $\alpha,\beta\in\{0,1,\dots,n\}$. In this context, call $f\in L^1_{\mathrm{loc}}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ a* CR-`form` *provided*

$$\int_{\partial_*\Omega}\left\langle\nu^{0,1}\wedge f,\vartheta\psi\right\rangle_{\mathbb{C}}\,\mathrm{d}\sigma=0\ \textit{for each}\ \psi\in\mathscr{C}^\infty_c(\mathbb{C}^n)\otimes\Lambda^{\alpha,\beta+2}.\tag{7.2.68}$$

A standard mollifier argument shows that an equivalent definition is obtained by considering test forms $\psi\in\mathscr{C}^1_c(U)\otimes\Lambda^{\alpha,\beta+2}$, for some $U\subseteq\mathbb{C}^n$ open neighborhood of $\partial\Omega$. Moreover, by (7.2.28) and [68, Proposition 3.7.2] (whose applicability in the present setting is ensures by [68, (5.2.6)] and [68, Lemma 3.6.4]) we see that for each $\alpha,\beta\in\{0,1,\dots,n\}$ the following holds:

$$\begin{aligned}&\text{if }\Omega\subseteq\mathbb{R}^{2n}\equiv\mathbb{C}^n\text{ is an open set of locally finite perimeter, and}\\&f\in L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}\text{ with }p\in[1,\infty]\text{ and }\sigma:=\mathcal{H}^{2n-1}\lfloor\partial\Omega,\text{ then}\\&f\text{ is a CR-form}\iff f\in L^{p,\bar{\partial}_\tau}_{\alpha,\beta}(\partial\Omega,\sigma)\text{ and }\bar{\partial}_\tau f=0\text{ on }\partial_*\Omega.\end{aligned}\tag{7.2.69}$$

In particular, corresponding to the case when $\alpha=\beta=0$, from (7.2.69), Proposition 7.2.4, and Proposition 7.2.5 we see that a complex-valued scalar function $f\in L^p(\partial_*\Omega,\sigma)$ with $p\in[1,\infty]$ is a CR-form provided $f\in L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)$ and $\partial^{\mathbb{C}}_{\tau_{jk}}f=0$ for each $j,k\in\{1,\dots,n\}$. In view of Definition 7.1.4 and (7.1.22) this ultimately shows that

in the scalar case, the concept of CR-form naturally reduces to the notion of CR-function. (7.2.70)

Let us momentarily digress for the purpose of considering in greater detail the particular case when Ω is a bounded $\mathscr{C}^1$ domain in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. In such a scenario, if ν and σ retain their earlier significance, then the latter may be naturally identified with the "volume" element $d\sigma$ on the $\mathscr{C}^1$ manifold $\partial\Omega$. Lastly, denote by $\iota : \partial\Omega \hookrightarrow \mathbb{C}^n$ the canonical inclusion mapping. In particular, ι^* denotes the pull-back from $\mathbb{C}^n$ to $\partial\Omega$. Next, fix $\alpha, \beta \in \{0, 1, \dots, n\}$ and assume a differential form $f \in \mathscr{C}^0(\partial\Omega) \otimes \Lambda^{\alpha,\beta}$ has been given. Below, we shall tacitly identify f with an arbitrary extension of itself to a form with continuous coefficients in an open neighborhood of $\partial\Omega$. Pick an arbitrary test form $\psi \in \mathscr{C}_c^\infty(\mathbb{C}^n) \otimes \Lambda^{\alpha,\beta+2}$ and introduce

$$\varphi := 2(-1)^\beta * \bar{\psi} \in \mathscr{C}_c^\infty(\mathbb{C}^n) \otimes \Lambda^{n-\alpha,n-2-\beta}. \tag{7.2.71}$$

The stage is set for us to compute

$$\begin{aligned} \big\langle \nu^{0,1} \wedge f, \vartheta\psi \big\rangle_{\mathbb{C}}\, d\sigma &= 2\big\langle \nu \wedge f, \vartheta\psi \big\rangle_{\mathbb{C}}\, d\sigma = 2\iota^*\big(f \wedge \overline{*\vartheta\psi}\big) \\ &= 2(-1)^{\beta+2}\iota^*\big(f \wedge \overline{*\vartheta * *\psi}\big) = 2(-1)^\beta \iota^*\big(f \wedge \bar{\partial}(*\bar{\psi})\big) \\ &= \iota^*\big(f \wedge \bar{\partial}\varphi\big) \ \text{ on } \partial\Omega, \end{aligned} \tag{7.2.72}$$

where the first equality is based on (7.1.63) and simple degree considerations, the second equality uses the fact that, generally speaking, for any continuous ℓ-form u and $(\ell + 1)$-form ω in a neighborhood of $\partial\Omega$ we have (cf., e.g., [79, Lemma 3.1])

$$\iota^*(u \wedge *\bar{\omega}) = \Big\langle \nu \wedge \big(u\big|_{\partial\Omega}\big), \omega\big|_{\partial\Omega} \Big\rangle_{\mathbb{C}}\, d\sigma, \tag{7.2.73}$$

the third equality in (7.2.72) is implied by item *(1)* in Lemma 7.1.6, the fourth equality relies on (7.2.14), and the final equality simply takes (7.2.71) into account. In concert, (7.2.68) and (7.2.71)-(7.2.72) prove the following:

if Ω is a bounded $\mathscr{C}^1$ domain in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ and $f \in \mathscr{C}^0(\partial\Omega) \otimes \Lambda^{\alpha,\beta}$, having f be a CR-*form in the sense of Definition 7.2.8 becomes equivalent to the demand that $\int_{\partial\Omega} \iota^*\big(f \wedge \bar{\partial}\varphi\big) = 0$ for each differential form $\varphi \in \mathscr{C}_c^\infty(\mathbb{C}^n) \otimes \Lambda^{n-\alpha,n-2-\beta}$.* (7.2.74)

In particular, corresponding to the scalar case $\alpha = \beta = 0$, if Ω is a bounded $\mathscr{C}^1$ domain in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$, then having some given $f \in \mathscr{C}^0(\partial\Omega)$ be a CR-function in the sense of Definition 7.1.4 becomes equivalent to asking that

$$\int_{\partial\Omega} \iota^*\big(f\,\bar{\partial}\varphi\big) = 0 \ \text{ for each } \ \varphi \in \mathscr{C}_c^\infty(\mathbb{C}^n) \otimes \Lambda^{n,n-2}. \tag{7.2.75}$$

Retaining the assumption that Ω is a bounded $\mathscr{C}^1$ domain in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$, we may further rephrase the condition that a given $f \in \mathscr{C}^1(\partial\Omega)$ is a CR-function as follows. Identifying f with an extension of itself to a function of class $\mathscr{C}^1$ in an open neighborhood of $\partial\Omega$, for each $\varphi \in \mathscr{C}_c^\infty(\mathbb{C}^n) \otimes \Lambda^{n,n-2}$ we may write

$$\int_{\partial\Omega} \iota^*\big(f\,\bar{\partial}\varphi\big) = \int_{\partial\Omega} \iota^*\big(f\,\mathrm{d}\varphi\big) = \int_{\partial\Omega} \iota^*\big(d(f\,\varphi)\big) - \int_{\partial\Omega} \iota^*\big(df \wedge \varphi\big)$$

$$= \int_{\partial\Omega} d_{\partial\Omega}\big(\iota^*(f\,\varphi)\big) - \int_{\partial\Omega} \iota^*\big(df \wedge \varphi\big) = -\int_{\partial\Omega} \iota^*\big(df \wedge \varphi\big), \quad (7.2.76)$$

using (7.2.1) (and keeping in mind that φ is of degree $(n, n-2)$), the product rule for d, the fact that pull-back commutes with the exterior derivative operator ($\mathrm{d}_{\partial\Omega}$ denoting its version on the manifold $\partial\Omega$), and Stokes' Theorem. Expressing $\varphi \in \mathscr{C}^\infty_c(\mathbb{C}^n) \otimes \Lambda^{n,n-2}$ as $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \psi$ with $\psi \in \mathscr{C}^\infty_c(\mathbb{C}^n) \otimes \Lambda^{0,n-2}$ arbitrary, from (7.2.76) and (7.2.70) we ultimately conclude that

if Ω is a bounded $\mathscr{C}^1$ domain in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ and $f \in \mathscr{C}^1(\partial\Omega)$, then having f be a CR*-function in the sense of Definition 7.1.4 becomes equivalent to the demand that the pull-back to the manifold $\partial\Omega$ of the differential form $\mathrm{d}f \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ vanishes identically on $\partial\Omega$.* (7.2.77)

The latter vanishing property is sometimes referred to as Wirtinger's condition (cf. [90]).

The point of view adopted in (7.2.75) is very popular. For example, this is how the concept of CR-function is introduced within the class of continuous functions defined on boundaries of $\mathscr{C}^1$ domains in $\mathbb{C}^n$ in [59, Definition 8.17, p. 44], [52, (6.9), p. 62], and [53, Definition 3.1.2, p. 76]. We wish to emphasize that this commonly held point of view (based on Stokes formula and pull-back to $\partial\Omega$, assumed to have a manifold structure of at least class $\mathscr{C}^1$) is no longer practical if $\partial\Omega$ is lacking regularity. In this regard, the progress that we register here is dispensing with any manifold structure assumption on the boundary of the given set Ω.

We close this section by proving that, in a suitable context, nontangential boundary traces of null-solutions of the $\bar{\partial}$-operator are CR-forms.

Proposition 7.2.9 *Let $\Omega \subseteq \mathbb{C}^n \equiv \mathbb{R}^{2n}$ be a* UR *domain. Abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. Also, fix a pair of degrees $\alpha, \beta \in \{0, 1, \dots, n\}$, an exponent $p \in [1, \infty)$, an aperture $\kappa > 0$, and a truncation parameter $\varepsilon > 0$. In this context, if $u \in \mathscr{C}^1(\Omega) \otimes \Lambda^{\alpha,\beta}$ is a differential form satisfying*

$$\mathcal{N}^{\varepsilon}_{\kappa} u \in L^p(\partial\Omega, \sigma), \quad \bar{\partial}u = 0 \ \text{ in } \ \Omega, \qquad \text{and } \ u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \quad (7.2.78)$$

then

$$u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \in L^{p,\bar{\partial}_\tau}_{\alpha,\beta}(\partial\Omega, \sigma) \ \text{ and } \ \bar{\partial}_\tau\Big(u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big) = 0. \quad (7.2.79)$$

In particular,

$$u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \text{ is a CR-form.} \quad (7.2.80)$$

Proof The claims in (7.2.79) are immediate from (7.2.55). The claim in (7.2.80) then follows from (7.2.79) and (7.2.69). □

7.3 The Bochner-Martinelli Integral Operator

Having fixed some $n \in \mathbb{N}$, we begin by considering double forms in the variables ζ, z in $\mathbb{C}^n$. Given integers $\alpha, \beta, \gamma, \delta \in \{0, 1, \dots, n\}$, a double form of type $\big((\alpha,\beta),(\gamma,\delta)\big)$ is an object which may be expressed as

$$\Theta = \sum_{|I|=\alpha} \sum_{|J|=\beta} \sum_{|K|=\gamma} \sum_{|L|=\delta} \theta^{IJ}_{KL} \big(\mathrm{d}\zeta^I \wedge \mathrm{d}\bar{\zeta}^J\big) \otimes \big(\mathrm{d}z^K \wedge \mathrm{d}\bar{z}^L\big), \tag{7.3.1}$$

where each θ^{IJ}_{KL} is a function in the variables ζ, z. These objects can then be made subject to natural operations with ordinary forms, simply by viewing them as differential forms in the variable ζ whose coefficients are themselves differential forms in the variable z, or the other way around. For example, the complex conjugate of Θ in (7.3.1) is

$$\begin{aligned}\overline{\Theta} &:= \sum_{|I|=\alpha} \sum_{|J|=\beta} \sum_{|K|=\gamma} \sum_{|L|=\delta} \overline{\theta^{IJ}_{KL}} \big(\mathrm{d}\bar{\zeta}^I \wedge \mathrm{d}\zeta^J\big) \otimes \big(\mathrm{d}\bar{z}^K \wedge \mathrm{d}z^L\big) \\ &= (-1)^{\alpha\beta+\gamma\delta} \sum_{|J|=\beta} \sum_{|I|=\alpha} \sum_{|L|=\delta} \sum_{|K|=\gamma} \overline{\theta^{IJ}_{KL}} \big(\mathrm{d}\zeta^J \wedge \mathrm{d}\bar{\zeta}^I\big) \otimes \big(\mathrm{d}z^L \wedge \mathrm{d}\bar{z}^K\big), \end{aligned} \tag{7.3.2}$$

which is a double form of type $\big((\beta,\alpha),(\delta,\gamma)\big)$. We may also apply differential operators, such as $\bar\partial$, ∂, ϑ, and $\bar\vartheta$, in either the variable ζ, or the variable z (each time we agree to indicate this by appending the variable to the operator as a subscript). For example, if Θ is as in (7.3.1) then

$$\begin{aligned}\bar\partial_\zeta \Theta &:= \sum_{j=1}^{n} \sum_{|I|=\alpha} \sum_{|J|=\beta} \sum_{|K|=\gamma} \sum_{|L|=\delta} \big(\partial_{\bar\zeta_j} \theta^{IJ}_{KL}\big) \big(\mathrm{d}\bar\zeta_j \wedge \mathrm{d}\zeta^I \wedge \mathrm{d}\bar{\zeta}^J\big) \otimes \big(\mathrm{d}z^K \wedge \mathrm{d}\bar{z}^L\big) \\ &= (-1)^{\alpha} \sum_{\substack{|I|=\alpha \\ |M|=\beta+1}} \sum_{\substack{|K|=\gamma \\ |L|=\delta}} \Big(\sum_{j=1}^{n} \sum_{|J|=\beta} \varepsilon^{jJ}_{M} \big(\partial_{\bar\zeta_j} \theta^{IJ}_{KL}\big)\Big) \big(\mathrm{d}\zeta^I \wedge \mathrm{d}\bar{\zeta}^M\big) \otimes \big(\mathrm{d}z^K \wedge \mathrm{d}\bar{z}^L\big), \end{aligned} \tag{7.3.3}$$

which is a double form of type $\big((\alpha,\beta+1),(\gamma,\delta)\big)$. Similar conventions are in effect for the action of the Hodge star operator $*$, the exterior product, and the interior product with one forms (considered either in the variable ζ, or the variable z) on double forms. One natural venue through which double forms arise is taking the tensor product of two ordinary forms. Concretely, suppose $u = \sum_{|I|=\alpha,|J|=\beta} u_{I,J}\, \mathrm{d}\zeta^I \wedge \mathrm{d}\bar\zeta^J$ is an ordinary (α,β)-form and $w = \sum_{|K|=\gamma,|L|=\delta} w_{K,L}\, \mathrm{d}z^K \wedge \mathrm{d}\bar z^L$ is an ordinary (γ,δ)-form, we define $u \otimes w$ as the $\big((\alpha,\beta),(\gamma,\delta)\big)$ double form given by

$$u \otimes w := \sum_{|I|=\alpha} \sum_{|J|=\beta} \sum_{|K|=\gamma} \sum_{|L|=\delta} u_{I,J} w_{K,L} \big(\mathrm{d}\zeta^I \wedge \mathrm{d}\bar{\zeta}^J\big) \otimes \big(\mathrm{d}z^K \wedge \mathrm{d}\bar{z}^L\big). \tag{7.3.4}$$

Going further, we wish to extend the Hermitian inner product (7.1.43) to the case when one of the forms participating in the inner product is now a double form. Specifically, if Θ is as in (7.3.1) and $u = \sum_{|I|=\alpha', |J|=\beta'} u_{I,J}\, dz^I \wedge d\bar{z}^J$ is an ordinary (α', β')-form we define

$$\langle u, \Theta \rangle_{\mathbb{C}} := 0 \text{ if } (\alpha', \beta') \neq (\alpha, \beta) \tag{7.3.5}$$

and, corresponding to the case when $(\alpha', \beta') = (\alpha, \beta)$, we set

$$\begin{aligned}\langle u, \Theta \rangle_{\mathbb{C}} &:= 2^{\alpha+\beta} \sum_{|K|=\gamma} \sum_{|L|=\delta} \Big(\sum_{|I|=\alpha} \sum_{|J|=\beta} u_{I,J}\, \overline{\theta^{IJ}_{KL}} \Big) d\bar{z}^K \wedge dz^L \\ &= 2^{\alpha+\beta}(-1)^{\gamma\delta} \sum_{|L|=\delta} \sum_{|K|=\gamma} \Big(\sum_{|I|=\alpha} \sum_{|J|=\beta} u_{I,J}\, \overline{\theta^{IJ}_{KL}} \Big) dz^L \wedge d\bar{z}^K. \end{aligned} \tag{7.3.6}$$

In particular,

the inner product of an ordinary (α, β)-form with a double form of type $\big((\alpha, \beta), (\gamma, \delta)\big)$ is an ordinary (δ, γ)-form. (7.3.7)

In fact, there is a natural Hermitian inner product for double forms of the following sort. If the double form Θ is as in (7.3.1) and

$$\widetilde{\Theta} = \sum_{|\widetilde{I}|=\alpha} \sum_{|\widetilde{J}|=\beta} \sum_{|\widetilde{K}|=\gamma} \sum_{|\widetilde{L}|=\delta} \widetilde{\theta}^{\widetilde{I}\widetilde{J}}_{\widetilde{K}\widetilde{L}} \big(d\zeta^{\widetilde{I}} \wedge d\bar{\zeta}^{\widetilde{J}}\big) \otimes \big(dz^{\widetilde{K}} \wedge d\bar{z}^{\widetilde{L}}\big), \tag{7.3.8}$$

we set

$$\big\langle\!\big\langle \widetilde{\Theta}, \Theta \big\rangle\!\big\rangle_{\mathbb{C}} := 2^{\alpha+\beta+\gamma+\delta} \sum_{|I|=\alpha} \sum_{|J|=\beta} \sum_{|K|=\gamma} \sum_{|L|=\delta} \widetilde{\theta}^{IJ}_{KL} \overline{\theta^{IJ}_{KL}}. \tag{7.3.9}$$

In this vein, it is useful to note that for any double form Θ of type $\big((\alpha, \beta), (\gamma, \delta)\big)$, and any two ordinary forms, u of type (α, β) and w of type (γ, δ), we have

$$\big\langle\!\big\langle u \otimes w, \Theta \big\rangle\!\big\rangle_{\mathbb{C}} = \Big\langle w, \langle \bar{u}, \overline{\Theta} \rangle_{\mathbb{C}} \Big\rangle_{\mathbb{C}}. \tag{7.3.10}$$

Finally, the Hermitian inner product of double forms defined in (7.3.9) may be further extended to the case when, for some given open subset $\Omega \subseteq \mathbb{C}^n$, the coefficients of $\widetilde{\Theta}$ in (7.3.8) belong to $\mathcal{D}'(\Omega)$ while the coefficients of Θ in (7.3.1) belong to $\mathcal{D}(\Omega)$ by setting

$$_{\mathcal{D}'(\Omega)}\big\langle\!\big\langle \widetilde{\Theta}, \Theta \big\rangle\!\big\rangle_{\mathcal{D}(\Omega)} := 2^{\alpha+\beta+\gamma+\delta} \sum_{|I|=\alpha} \sum_{|J|=\beta} \sum_{|K|=\gamma} \sum_{|L|=\delta} {}_{\mathcal{D}'(\Omega)}\big\langle \widetilde{\theta}^{IJ}_{KL}, \overline{\theta^{IJ}_{KL}} \big\rangle_{\mathcal{D}(\Omega)}. \tag{7.3.11}$$

Similarly, when the coefficients of $\widetilde{\Theta}$ in (7.3.8) belong to $\mathscr{E}'(\Omega)$ while the coefficients of Θ in (7.3.1) belong to $\mathscr{E}(\Omega)$, we define

]quad

$$
{}_{\mathscr{E}'(\Omega)}\Big\langle\!\!\Big\langle \widetilde{\Theta}, \Theta \Big\rangle\!\!\Big\rangle_{\mathscr{E}(\Omega)} := 2^{\alpha+\beta+\gamma+\delta} \sum_{|I|=\alpha} \sum_{|J|=\beta} \sum_{|K|=\gamma} \sum_{|L|=\delta} {}_{\mathscr{E}'(\Omega)}\big\langle \widetilde{\theta}^{IJ}_{KL}, \overline{\theta^{IJ}_{KL}} \big\rangle_{\mathscr{E}(\Omega)}. \tag{7.3.12}
$$

Based on these definitions, it may be checked that if $\widetilde{\Theta}$ is a double form of type $\big((\alpha,\beta),(\gamma,\delta)\big)$ with coefficients in the space $\mathcal{D}'(\Omega)$ and Θ is a double form of type $\big((\alpha,\beta+1),(\gamma,\delta)\big)$ with coefficients in $\mathcal{D}(\Omega)$ then

$$
{}_{\mathcal{D}'(\Omega)}\Big\langle\!\!\Big\langle \bar{\partial}_\zeta \widetilde{\Theta}, \Theta \Big\rangle\!\!\Big\rangle_{\mathcal{D}(\Omega)} = {}_{\mathcal{D}'(\Omega)}\Big\langle\!\!\Big\langle \widetilde{\Theta}, \vartheta_\zeta \Theta \Big\rangle\!\!\Big\rangle_{\mathcal{D}(\Omega)}, \tag{7.3.13}
$$

plus a similar formula when $\mathcal{D}'(\Omega)$, $\mathcal{D}(\Omega)$ are replaced by $\mathscr{E}'(\Omega)$ and $\mathscr{E}(\Omega)$, respectively.

Changing topics, consider the `complex Laplacian` in $\mathbb{C}^n$ defined as

$$
\square := \bar{\partial}\vartheta + \vartheta\bar{\partial}. \tag{7.3.14}
$$

In particular, from (7.3.14), (7.2.9), and (7.2.14) we see that

$$
\square = \big(\bar{\partial} + \vartheta\big)^2. \tag{7.3.15}
$$

A direct computation also shows that

$$
\square = -2 \sum_{k=1}^{n} \partial_{z_k} \partial_{\bar{z}_k} = -\tfrac{1}{2}\Delta \tag{7.3.16}
$$

where Δ denotes the real Laplacian in $\mathbb{R}^{2n}$, i.e.,

$$
\Delta := \sum_{j=1}^{n} (\partial^2_{x_j} + \partial^2_{y_j}) \text{ in } \mathbb{R}^{2n}. \tag{7.3.17}
$$

We are interested in manufacturing a fundamental solution for $\square$. To this end, for each $z, \xi \in \mathbb{C}^n$ with $z \neq \xi$ consider

$$
E_n(\zeta, z) := \begin{cases} -\dfrac{1}{2\pi} \ln|\zeta - z|^2 & \text{for } n = 1, \\[2ex] \dfrac{(n-2)!}{2\pi^n} |\zeta - z|^{2-2n} & \text{for } n \geq 2. \end{cases} \tag{7.3.18}
$$

Since the surface area of the unit ball in $\mathbb{R}^{2n}$ is given by $\omega_{2n-1} = 2\pi^n/(n-1)!$ it follows that E_n is -2 times the standard fundamental solution for the real Laplacian Δ in $\mathbb{R}^{2n}$. Given the goal we have in mind, this shows that the choice of $E_n(\zeta, z)$ in (7.3.18) is indeed natural. Next, for each $\alpha, \beta \in \{0, 1, \ldots, n\}$ consider the double form of type $\big((\alpha,\beta),(\beta,\alpha)\big)$ given by

$$\Gamma_{\alpha,\beta}(\zeta,z) := 2^{-\alpha-\beta} E_n(\zeta,z) \sum_{|J|=\alpha} \sum_{|I|=\beta} (\mathrm{d}\zeta^J \wedge \mathrm{d}\bar{\zeta}^I) \otimes (\mathrm{d}\bar{z}^J \wedge \mathrm{d}z^I)$$

$$= 2^{-\alpha-\beta}(-1)^{\alpha\beta} E_n(\zeta,z) \sum_{|J|=\alpha} \sum_{|I|=\beta} (\mathrm{d}\zeta^J \wedge \mathrm{d}\bar{\zeta}^I) \otimes (\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J). \quad (7.3.19)$$

Then, by design,

$$\square_\zeta \Gamma_{\alpha,\beta}(\zeta,z) = 2^{-\alpha-\beta}(-1)^{\alpha\beta} \delta_z(\zeta) \sum_{|J|=\alpha} \sum_{|I|=\beta} (\mathrm{d}\zeta^J \wedge \mathrm{d}\bar{\zeta}^I) \otimes (\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J), \quad (7.3.20)$$

$$\square_z \Gamma_{\alpha,\beta}(\zeta,z) = 2^{-\alpha-\beta}(-1)^{\alpha\beta} \delta_\zeta(z) \sum_{|J|=\alpha} \sum_{|I|=\beta} (\mathrm{d}\zeta^J \wedge \mathrm{d}\bar{\zeta}^I) \otimes (\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J), \quad (7.3.21)$$

where $\delta_z(\zeta)$ the Dirac distribution in the variable $\zeta \in \mathbb{C}^n$ with mass at z, and where $\delta_\zeta(z)$ the Dirac distribution in the variable $z \in \mathbb{C}^n$ with mass at ζ. Moreover, it is clear from (7.3.19) that

$$\Gamma_{\alpha,\beta}(z,\zeta) = \overline{\Gamma_{\beta,\alpha}(\zeta,z)}. \quad (7.3.22)$$

We claim that we also have

$$\vartheta_\zeta \Gamma_{\alpha,\beta}(\zeta,z) = \partial_z \Gamma_{\alpha,\beta-1}(\zeta,z), \quad (7.3.23)$$

$$\partial_\zeta \overline{\Gamma_{\alpha,\beta}(\zeta,z)} = \vartheta_z \overline{\Gamma_{\alpha,\beta+1}(\zeta,z)}. \quad (7.3.24)$$

Indeed, by (7.2.12)-(7.2.13) and (7.3.19) we have

$$\vartheta_\zeta \Gamma_{\alpha,\beta}(\zeta,z) = 2^{1-\alpha-\beta}(-1)^{\alpha\beta+\alpha+1} \times \quad (7.3.25)$$

$$\times \sum_{|M|=\beta-1} \sum_{j=1}^{n} \sum_{|J|=\alpha} \sum_{|I|=\beta} \varepsilon_I^{jM} \partial_{\zeta_j}\big[E_n(\zeta,z)\big] (\mathrm{d}\zeta^J \wedge \mathrm{d}\bar{\zeta}^M) \otimes (\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J),$$

whereas (7.2.7)-(7.2.5) imply that

$$\partial_z \Gamma_{\alpha,\beta-1}(\zeta,z) = 2^{1-\alpha-\beta}(-1)^{\alpha(\beta-1)} \times \quad (7.3.26)$$

$$\times \sum_{|K|=\beta} \sum_{j=1}^{n} \sum_{|J|=\alpha} \sum_{|I|=\beta-1} \varepsilon_K^{jI} \partial_{z_j}\big[E_n(\zeta,z)\big] (\mathrm{d}\zeta^J \wedge \mathrm{d}\bar{\zeta}^I) \otimes (\mathrm{d}z^K \wedge \mathrm{d}\bar{z}^J).$$

Upon interchanging M with I, and noting that $\partial_{\zeta_j}\big[E_n(\zeta,z)\big] = -\partial_{z_j}\big[E_n(\zeta,z)\big]$ for each j, it follows that the two right-hand sides of (7.3.25) and (7.3.26) match. This establishes (7.3.23), and (7.3.24) follows from it by interchanging z, ζ (bearing in mind (7.3.22)), then readjusting notation.

The `Bochner-Martinelli kernel` for $(0,\beta)$-forms in $\mathbb{C}^n$ with $\beta \in \{0,1,\dots,n\}$ is then defined as the double differential form

$$K_{n\beta}(\zeta,z) := -*\,\partial_\zeta \overline{\Gamma_{0,\beta}(\zeta,z)}, \quad (7.3.27)$$

where the Hodge star isomorphism is applied in the variable ζ. If Ω is a bounded $\mathscr{C}^1$ domain in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$, then $\partial\Omega$ is a $\mathscr{C}^1$-smooth submanifold of $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. In such a context, the Bochner-Martinelli integral operator is defined on a continuous $(0,\beta)$-form f on $\partial\Omega$ as

$$\mathcal{B}_{0,\beta} f(z) := \int_{\partial\Omega} \iota_\zeta^*\Big(f(\zeta) \wedge K_{n\beta}(\zeta, z)\Big), \qquad \forall z \in \mathbb{C}^n \setminus \partial\Omega, \tag{7.3.28}$$

where $\iota : \partial\Omega \hookrightarrow \mathbb{C}^n$ is the canonical inclusion and ι_ζ^* indicates pull-back in the variable ζ. In view of (7.2.73), an equivalent way of defining the Bochner-Martinelli integral operator on a continuous $(0,\beta)$-form f on the boundary of a bounded $\mathscr{C}^1$ domain $\Omega \subseteq \mathbb{C}^n$ is

$$\mathcal{B}_{0,\beta} f(z) := -\int_{\partial\Omega} \big\langle \nu(\zeta) \wedge f(\zeta), \bar{\partial}_\zeta \Gamma_{0,\beta}(\zeta, z)\big\rangle_{\mathbb{C}} \, d\sigma(\zeta), \tag{7.3.29}$$

at each $z \in \mathbb{C}^n \setminus \partial\Omega$. As explained before, it is this expression which we find most suitable for extending the Bochner-Martinelli integral operator to situations when Ω is lacking smoothness in a traditional sense.

While this is done in Definition 7.3.1 below, for now we make one other observation regarding the structure of (7.3.29). Specifically, in the case when f is a $\Lambda^{0,\beta}$-valued function, using the second identity in (7.1.63) we may write

$$\nu(\zeta) \wedge f(\zeta) = \tfrac{1}{2}\nu^{1,0}(\zeta) \wedge f(\zeta) + \tfrac{1}{2}\nu^{0,1}(\zeta) \wedge f(\zeta) \in \Lambda_\zeta^{1,\beta} + \Lambda_\zeta^{0,\beta+1}. \tag{7.3.30}$$

From (7.3.19) we see that the double form $\Gamma_{0,\beta}(\zeta, z)$ is of type $\big((0,\beta),(\beta,0)\big)$ and, hence, (7.3.3) gives that $\bar{\partial}_\zeta \Gamma_{0,\beta}(\zeta, z)$ is of type $\big((0,\beta+1),(\beta,0)\big)$. Based on degree considerations (see (7.3.7)) we may therefore conclude that, whenever Ω is a bounded $\mathscr{C}^1$ domain in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ with outward unit normal ν and surface measure σ, we may further express the action of the Bochner-Martinelli integral operator (7.3.29) on a continuous $\Lambda^{0,\beta}$-valued function f defined on $\partial\Omega$ as

$$\mathcal{B}_{0,\beta} f(z) = -\frac{1}{2}\int_{\partial\Omega} \big\langle \nu^{0,1}(\zeta) \wedge f(\zeta), \bar{\partial}_\zeta \Gamma_{0,\beta}(\zeta, z)\big\rangle_{\mathbb{C}} \, d\sigma(\zeta), \tag{7.3.31}$$

at each $z \in \mathbb{C}^n \setminus \partial\Omega$, where the Hermitian inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ is taken in the sense of (7.3.6).

In the most general geometric and algebraic setting, we shall introduce the (`higher-degree`) `Bochner-Martinelli integral operator` as follows.

Definition 7.3.1 *Given a set $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ (where $n \in \mathbb{N}$) of locally finite perimeter, denote by ν its geometric measure theoretic outward unit normal, and abbreviate $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$. Also, fix $\alpha, \beta \in \{0, 1, \dots, n\}$. In this setting, define the action of the (higher-degree) Bochner-Martinelli integral operator $\mathcal{B}_{\alpha,\beta}$ associated with Ω on an arbitrary σ-measurable function $f : \partial_*\Omega \to \Lambda^{\alpha,\beta}\,\mathbb{C}^n$ satisfying*

$$\int_{\partial_*\Omega} \frac{|f(\zeta)|_{\mathbb{C}}}{1+|\zeta|^{2n-1}}\, d\sigma(\zeta) < +\infty \tag{7.3.32}$$

according to

$$\mathcal{B}_{\alpha,\beta}f(z) := -\frac{1}{2}\int_{\partial_*\Omega} \big\langle \nu^{0,1}(\zeta) \wedge f(\zeta), \bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta, z)\big\rangle_{\mathbb{C}}\, d\sigma(\zeta), \tag{7.3.33}$$

at each $z \in \mathbb{C}^n \setminus \partial\Omega$*, where the Hermitian inner product* $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ *is taken in the sense of* (7.3.6).

As regards Definition 7.3.1, note that the conditions imposed on f ensure that the integral in the right-hand side of (7.3.33) is absolutely convergent for each $z \in \mathbb{C}^n \setminus \partial\Omega$. In fact,

$$\mathcal{B}_{\alpha,\beta}f \in \mathscr{C}^\infty(\Omega) \otimes \Lambda^{\alpha,\beta} \text{ and } \Delta(\mathcal{B}_{\alpha,\beta}f) = 0 \text{ in } \Omega, \tag{7.3.34}$$

where $\Delta := \partial_1^2 + \cdots + \partial_{2n}^2$ is the Laplacian in $\mathbb{R}^{2n}$. Moreover, in the case when $\partial\Omega$ is an upper Ahlfors regular set, [68, Lemma 7.2.1] implies that any differential form $f \in L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}$ with $p \in [1,\infty)$ satisfies (7.3.32).

The reader is also reminded that (1.4.199) elaborates on the relationship between the higher-degree Bochner-Martinelli integral operator $\mathcal{B}_{\alpha,\beta}$ from Definition 7.3.1 and the class of double layer potential operators, constructed according to the general recipe described in (1.4.36).

The main point of our next proposition is that, when acting on differential forms belonging to an appropriate space on the boundary, the Bochner-Martinelli integral operator interacts well with the $\bar\partial$-operator.

Proposition 7.3.2 *Let* $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ *be an open set with the property that* $\partial\Omega$ *is upper Ahlfors regular; in particular,* Ω *is a set of locally finite perimeter. Abbreviate* $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ *and denote by* ν *its geometric measure theoretic outward unit normal. Also, fix* $\alpha, \beta \in \{0, 1, \dots, n\}$. *Then for each differential form* $f \in L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega, \sigma)$ *with* $p \in [1,\infty)$ *one has*

$$\bar\partial(\mathcal{B}_{\alpha,\beta}f)(z) = \frac{1}{2}\int_{\partial_*\Omega} \big\langle (\bar\partial_\tau f)(\zeta), \bar\partial_\zeta \Gamma_{\alpha,\beta+1}(\zeta, z)\big\rangle_{\mathbb{C}}\, d\sigma(\zeta) \tag{7.3.35}$$

at every $z \in \mathbb{C}^n \setminus \partial\Omega$.

Proof By taking $\bar\partial$ in both sides in the formula (7.3.33) it follows that for every $z \in \mathbb{C}^n \setminus \partial\Omega$ we may write

$$
\begin{aligned}
\bar{\partial}\big(\mathcal{B}_{\alpha,\beta} f\big)(z) &= -\frac{1}{2}\int_{\partial_*\Omega} \big\langle \nu^{0,1}(\zeta)\wedge f(\zeta), \bar{\partial}_\zeta \partial_z \Gamma_{\alpha,\beta}(\zeta,z)\big\rangle_{\mathbb{C}}\, \mathrm{d}\sigma(\zeta) \\
&= -\frac{1}{2}\int_{\partial_*\Omega} \big\langle \nu^{0,1}(\zeta)\wedge f(\zeta), \bar{\partial}_\zeta \vartheta_\zeta \Gamma_{\alpha,\beta+1}(\zeta,z)\big\rangle_{\mathbb{C}}\, \mathrm{d}\sigma(\zeta) \\
&= -\frac{1}{2}\int_{\partial_*\Omega} \big\langle \nu^{0,1}(\zeta)\wedge f(\zeta), (\square_\zeta - \vartheta_\zeta\bar{\partial}_\zeta) \Gamma_{\alpha,\beta+1}(\zeta,z)\big\rangle_{\mathbb{C}}\, \mathrm{d}\sigma(\zeta) \\
&= \frac{1}{2}\int_{\partial_*\Omega} \big\langle \nu^{0,1}(\zeta)\wedge f(\zeta), \vartheta_\zeta\bar{\partial}_\zeta \Gamma_{\alpha,\beta+1}(\zeta,z)\big\rangle_{\mathbb{C}}\, \mathrm{d}\sigma(\zeta) \\
&= \frac{1}{2}\int_{\partial_*\Omega} \big\langle (\bar{\partial}_\tau f)(\zeta), \bar{\partial}_\zeta \Gamma_{\alpha,\beta+1}(\zeta,z)\big\rangle_{\mathbb{C}}\, \mathrm{d}\sigma(\zeta),
\end{aligned}
\tag{7.3.36}
$$

where the second equality follows from (7.3.23), the third equality is a consequence of (7.3.14), the fourth equality uses (7.3.20) (bearing in mind that $z \in \mathbb{C}^n \setminus \partial\Omega$ and $\zeta \in \partial_*\Omega \subseteq \partial\Omega$ ensure that we have $z \neq \zeta$), and the last equality may be justified making use of Proposition 7.2.6 (whose present applicability is ensured by (7.3.19) and [68, Lemma 7.2.1]). This completes the proof of the proposition. □

A significant consequence of Proposition 7.3.2 is that, when applied to CR-forms, the Bochner-Martinelli integral operator yields null-solution of the $\bar{\partial}$-operator.

Proposition 7.3.3 *Consider an open set $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ with the property that $\partial\Omega$ is upper Ahlfors regular; in particular, Ω is a set of locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ and fix $\alpha, \beta \in \{0, 1, \dots, n\}$ along with $p \in [1, \infty)$. In this context, if $f \in L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}$ is a* CR*-form then*

$$
\mathcal{B}_{\alpha,\beta} f \text{ is } \bar{\partial}\text{-closed in } \Omega, \tag{7.3.37}
$$

i.e., $\mathcal{B}_{\alpha,\beta} f$ is a null-solution of the $\bar{\partial}$-operator in Ω.

Proof This is a direct consequence of (7.2.69) and Proposition 7.3.2. □

Pressing on, in the proposition below we introduce an integral operator which is closely connected to the Bochner-Martinelli integral operator, through the action of $\bar{\partial}$ and $\bar{\partial}_\tau$ (cf. (7.3.41)), and study its mapping properties and boundary behavior.

Proposition 7.3.4 *Let $n \in \mathbb{N}$ and suppose $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ is a nonempty open set such that $\partial\Omega$ is a* UR *set; in particular, Ω has locally finite perimeter. Denote by ν its geometric measure theoretic outward unit normal and abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. Also, fix a pair of arbitrary degrees $\alpha, \beta \in \{0, 1, \dots, n\}$, along with an aperture parameter $\kappa > 0$. Lastly, for each differential form $f \in L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta+1}$, define*

$$\mathscr{A}_{\alpha,\beta} f(z) := -\int_{\partial\Omega} \big\langle f(\zeta), \bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z)\big\rangle_{\mathbb{C}}\, d\sigma(\zeta), \qquad \forall z \in \Omega, \tag{7.3.38}$$

$$A_{\alpha,\beta} f(z) := -\lim_{\varepsilon\to 0^+} \int_{\substack{\zeta\in\partial\Omega\\ |z-\zeta|>\varepsilon}} \big\langle f(\zeta), \bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z)\big\rangle_{\mathbb{C}}\, d\sigma \text{ for } \sigma\text{-a.e. } z \in \partial\Omega. \tag{7.3.39}$$

Then the following properties hold.

(i) For each differential form $f \in L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta+1}$ *(hence, in particular, for each differential form* $f \in L^p(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta+1}$ *with* $p \in [1,\infty)$*) the limit in* (7.3.39) *exists for* σ*-a.e.* $z \in \partial\Omega$*. Moreover,*

$$\mathscr{A}_{\alpha,\beta} f \in \mathscr{C}^\infty(\Omega) \otimes \Lambda^{\alpha,\beta} \text{ and } \Delta(\mathscr{A}_{\alpha,\beta} f) = 0 \text{ in } \Omega, \tag{7.3.40}$$

where $\Delta := \partial_1^2 + \cdots + \partial_{2n}^2$ *is the Laplacian in* $\mathbb{R}^{2n}$*.*

(ii) Pick an arbitrary differential form $f \in L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega,\sigma)$ *with integrability exponent* $p \in [1,\infty)$*, and canonically regard* $\bar\partial_\tau f \in L^p(\partial_*\Omega,\sigma) \otimes \Lambda^{\alpha,\beta+2}$ *as being a form in* $L^p(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta+2}$ *by extending it by zero from* $\partial_*\Omega$ *to the entire topological boundary* $\partial\Omega$*. Then, with this convention, one has*

$$\bar\partial(\mathscr{B}_{\alpha,\beta} f) = -\tfrac{1}{2}\mathscr{A}_{\alpha,\beta+1}(\bar\partial_\tau f) \text{ in } \Omega. \tag{7.3.41}$$

Also, given any $f \in L^p(\partial_*\Omega,\sigma) \otimes \Lambda^{\alpha,\beta}$ *with* $p \in [1,\infty)$*, upon agreeing to regard* $\nu^{0,1} \wedge f$ *as a form in* $L^p(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta+1}$ *by extending it by zero from* $\partial_*\Omega$ *to* $\partial\Omega$*, one has*

$$\mathscr{B}_{\alpha,\beta} f = \tfrac{1}{2}\mathscr{A}_{\alpha,\beta}(\nu^{0,1} \wedge f) \text{ in } \Omega. \tag{7.3.42}$$

(iii) The operators

$$A_{\alpha,\beta} : L^p(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta+1} \longrightarrow L^p(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta}, \quad p \in (1,\infty), \tag{7.3.43}$$

$$A_{\alpha,\beta} : L^1(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta+1} \longrightarrow L^{1,\infty}(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta}, \tag{7.3.44}$$

are well-defined, linear, and bounded. In addition, corresponding to the end-point $p = \infty$*, the operator* $A_{\alpha,\beta}$ *induces a linear and bounded mapping*

$$A_{\alpha,\beta} : L^\infty(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta+1} \longrightarrow \mathrm{BMO}(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta}. \tag{7.3.45}$$

The operator (7.3.39) *further extends to a linear and bounded mapping*

$$A_{\alpha,\beta} : H^p(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta+1} \longrightarrow L^p(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta}, \quad \tfrac{n-1}{n} < p \le 1. \tag{7.3.46}$$

(iv) For each $p \in \big(\frac{n-1}{n}, \infty\big)$ *there exists a constant* $C = C(\Omega, p, \kappa, n) \in (0,\infty)$ *such that*

$$\big\|\mathcal{N}_\kappa(\mathcal{A}_{\alpha,\beta}f)\big\|_{L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} \le C\|f\|_{L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}} \text{ if } 1<p<\infty, \tag{7.3.47}$$

$$\big\|\mathcal{N}_\kappa(\mathcal{A}_{\alpha,\beta}f)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} \le C\|f\|_{L^1(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}}, \tag{7.3.48}$$

$$\big\|\mathcal{N}_\kappa(\mathcal{A}_{\alpha,\beta}f)\big\|_{L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} \le C\|f\|_{H^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}} \text{ if } \tfrac{n-1}{n}<p\le 1. \tag{7.3.49}$$

(v) For each $p\in(1,\infty)$ there exists a constant $C=C(\Omega,p,n)\in(0,\infty)$ with the property that for each differential form $f\in L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}$ one has

$$\Big(\int_\Omega |(\nabla\mathcal{A}_{\alpha,\beta}f)|_{\mathbb{C}}^p\,\mathrm{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^{2n}\Big)^{1/p} \le C\|f\|_{L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}}, \tag{7.3.50}$$

where $\nabla\mathcal{A}_{\alpha,\beta}f$ stands for the collection of all first-order partial derivatives of all components of the differential form $\mathcal{A}_{\alpha,\beta}f$. In particular, corresponding to $p=2$, one has the following L^2-square function estimate

$$\int_\Omega |(\nabla\mathcal{A}_{\alpha,\beta}f)|_{\mathbb{C}}^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^{2n} \le C\int_{\partial\Omega}|f|_{\mathbb{C}}^2\,\mathrm{d}\sigma. \tag{7.3.51}$$

(vi) There exists a constant $C=C(\Omega,n)\in(0,\infty)$ with the property that for each differential form $f\in L^\infty(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}$ one has the following Carleson measure estimate:

$$\begin{aligned}\sup_{x\in\partial\Omega,\,r>0}\Big(\frac{1}{\sigma\big(\partial\Omega\cap B(x,r)\big)}\int_{B(x,r)\cap\Omega}\big|(\nabla\mathcal{A}_{\alpha,\beta})f\big|_{\mathbb{C}}^2\,\mathrm{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^{2n}\Big)^{1/2}&\\ \le C\|f\|_{L^\infty(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}}.&\end{aligned} \tag{7.3.52}$$

(vii) For each form $f\in L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}$ with $1\le p<\infty$ the following jump-formula holds:

$$\mathcal{A}_{\alpha,\beta}f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \frac{1}{2}\,\nu^{1,0}\vee f + A_{\alpha,\beta}f \text{ for } \sigma\text{-a.e. point in } \partial_*\Omega. \tag{7.3.53}$$

(viii) If Ω is actually a bounded NTA *domain in $\mathbb{R}^{2n}\equiv\mathbb{C}^n$ with an Ahlfors regular boundary and $p\in(1,\infty)$, then for each $q\in\big(\frac{n}{n+1/p},\infty\big]$ the operators*

$$\begin{aligned}\mathcal{A}_{\alpha,\beta}&: L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}\longrightarrow F^{p,q}_{1/p}(\Omega)\otimes\Lambda^{\alpha,\beta},\\ \mathcal{A}_{\alpha,\beta}&: L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}\longrightarrow B^{p,p}_{1/p}(\Omega)\otimes\Lambda^{\alpha,\beta},\\ \mathcal{A}_{\alpha,\beta}&: L^2(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}\longrightarrow H^{1/2}(\Omega)\otimes\Lambda^{\alpha,\beta},\end{aligned} \tag{7.3.54}$$

are well-defined, linear, and bounded.

In preparation for the proof of the above proposition, we introduce some notation. Given an open set of locally finite perimeter $\Omega\subseteq\mathbb{R}^{2n}\equiv\mathbb{C}^n$, let $\sigma:=\mathcal{H}^{2n-1}\lfloor\partial\Omega$. In this setting, for each $j \in \{1, \dots, n\}$ we define the `complex Riesz transform` $\mathcal{R}_{\mathbb{C},j}$ as the integral operator acting on σ-measurable functions $f : \partial\Omega \to \mathbb{C}$ satisfying

$$\int_{\partial\Omega} \frac{|f(\zeta)|}{1 + |\zeta|^{2n-1}} \, \mathrm{d}\sigma(\zeta) < +\infty \tag{7.3.55}$$

according to

$$\mathcal{R}_{\mathbb{C},j} f(z) := \frac{2}{\omega_{2n-1}} \int_{\partial\Omega} \frac{z_j - \zeta_j}{|z - \zeta|^{2n}} f(\zeta) \, \mathrm{d}\sigma(\zeta), \quad \text{for all } z \in \Omega. \tag{7.3.56}$$

Its complex conjugate is the operator $\mathcal{R}_j^c$ given by $\mathcal{R}_{\mathbb{C},j}^c f := \overline{\mathcal{R}_{\mathbb{C},j} \bar{f}}$. Explicitly,

$$\mathcal{R}_{\mathbb{C},j}^c f(z) := \frac{2}{\omega_{2n-1}} \int_{\partial\Omega} \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}} f(\zeta) \, \mathrm{d}\sigma(\zeta), \quad \text{for all } z \in \Omega. \tag{7.3.57}$$

When $\partial\Omega$ is a UR set it also makes sense to define their boundary-to-boundary versions, i.e.,

$$R_{\mathbb{C},j} f(x) := \lim_{\varepsilon \to 0^+} \frac{2}{\omega_{2n-1}} \int_{\substack{|z-\zeta|>\varepsilon \\ \zeta \in \partial\Omega}} \frac{z_j - \zeta_j}{|z - \zeta|^{2n}} f(\zeta) \, \mathrm{d}\sigma(\zeta) \tag{7.3.58}$$

for σ-a.e. $z \in \partial\Omega$, where $f \in L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big)$. Finally, consider the complex conjugate of the operators $R_{\mathbb{C},j}$, that is, $R_{\mathbb{C},j}^c f := \overline{R_{\mathbb{C},j} \bar{f}}$ for $1 \le j \le n$. Specifically, for each $j \in \{1, \dots, n\}$ we have

$$R_{\mathbb{C},j}^c f(x) = \lim_{\varepsilon \to 0^+} \frac{2}{\omega_{2n-1}} \int_{\substack{|z-\zeta|>\varepsilon \\ \zeta \in \partial\Omega}} \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}} f(\zeta) \, \mathrm{d}\sigma(\zeta) \tag{7.3.59}$$

for σ-a.e. $z \in \partial\Omega$, where $f \in L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big)$.

Proof of Proposition 7.3.4 To set the stage, we note that

$$
\begin{aligned}
&\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z) &(7.3.60)\\
&= \Big(\sum_{j=1}^{n} \partial_{\bar\zeta_j}\, \mathrm{d}\bar\zeta_j \wedge \cdot\Big)\Big\{2^{-\alpha-\beta}(-1)^{\alpha\beta} E_n(\zeta,z) \sum_{|J|=\alpha}\sum_{|I|=\beta} (\mathrm{d}\zeta^J \wedge \mathrm{d}\bar\zeta^I)\otimes(\mathrm{d}z^I \wedge \mathrm{d}\bar z^J)\Big\}\\
&= 2^{-\alpha-\beta}(-1)^{\alpha\beta} \sum_{j=1}^{n}\sum_{|J|=\alpha}\sum_{|I|=\beta} \partial_{\bar\zeta_j}[E_n(\zeta,z)](\mathrm{d}\bar\zeta_j \wedge \mathrm{d}\zeta^J \wedge \mathrm{d}\bar\zeta^I)\otimes(\mathrm{d}z^I \wedge \mathrm{d}\bar z^J)\\
&= 2^{-\alpha-\beta}(-1)^{\alpha\beta+\alpha}\times\\
&\qquad \times \sum_{j=1}^{n}\sum_{|J|=\alpha}\sum_{|I|=\beta}\sum_{|K|=\beta+1} \varepsilon^{jI}_K \partial_{\bar\zeta_j}[E_n(\zeta,z)](\mathrm{d}\zeta^J \wedge \mathrm{d}\bar\zeta^K)\otimes(\mathrm{d}z^I \wedge \mathrm{d}\bar z^J).
\end{aligned}
$$

In particular, $\bar\partial_\zeta \Gamma_\beta(\zeta,z)$ is a double form of type $\big((\alpha,\beta+1),(\beta,\alpha)\big)$. As such, if the differential form $f \in L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}$, with $p\in[1,\infty)$, is explicitly written as

$$
f(\zeta) = \sum_{|J|=\alpha}\sum_{|K|=\beta+1} f_{J,K}(\zeta)\,\mathrm{d}\zeta^J \wedge \mathrm{d}\bar\zeta^K \ \text{ for } \sigma\text{-a.e. } \zeta\in\partial\Omega, \tag{7.3.61}
$$

then

$$
\begin{aligned}
&-\big\langle f(\zeta), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\big\rangle_{\mathbb{C}} &(7.3.62)\\
&\qquad = 2(-1)^{\alpha\beta+\alpha+1}\sum_{j=1}^{n}\sum_{|J|=\alpha}\sum_{|I|=\beta}\sum_{|K|=\beta+1} \varepsilon^{jI}_K\, \overline{\partial_{\bar\zeta_j}[E_n(\zeta,z)]} f_{J,K}(\zeta)\,\mathrm{d}\bar z^I \wedge \mathrm{d}z^J\\
&\qquad = 2(-1)^{\alpha+1}\sum_{j=1}^{n}\sum_{|J|=\alpha}\sum_{|I|=\beta}\sum_{|K|=\beta+1} \varepsilon^{jI}_K\, \partial_{\zeta_j}[E_n(\zeta,z)] f_{J,K}(\zeta)\,\mathrm{d}z^J \wedge \mathrm{d}\bar z^I\\
&\qquad = \frac{2(-1)^\alpha}{\omega_{2n-1}}\sum_{j=1}^{n}\sum_{|J|=\alpha}\sum_{|I|=\beta}\sum_{|K|=\beta+1} \varepsilon^{jI}_K\, \frac{\bar\zeta_j-\bar z_j}{|\zeta-z|^{2n}} f_{J,K}(\zeta)\,\mathrm{d}z^J \wedge \mathrm{d}\bar z^I.
\end{aligned}
$$

Consequently, in terms of the complex Riesz transforms discussed in (7.3.56)-(7.3.59), for f as in (7.3.61) we have

$$
\mathcal{A}_{\alpha,\beta} f = (-1)^{\alpha+1}\sum_{j=1}^{n}\sum_{|J|=\alpha}\sum_{|I|=\beta}\sum_{|K|=\beta+1} \varepsilon^{jI}_K\, \mathcal{R}^c_{\mathbb{C},j} f_{J,K}\,\mathrm{d}z^J \wedge \mathrm{d}\bar z^I \ \text{ in } \Omega, \tag{7.3.63}
$$

and

$$
A_{\alpha,\beta} f = (-1)^{\alpha+1}\sum_{j=1}^{n}\sum_{|J|=\alpha}\sum_{|I|=\beta}\sum_{|K|=\beta+1} \varepsilon^{jI}_K\, R^c_{\mathbb{C},j} f_{J,K}\,\mathrm{d}z^J \wedge \mathrm{d}\bar z^I \ \text{ on } \partial\Omega. \tag{7.3.64}
$$

The first claim in item *(i)* in the statement of the proposition is then clear from (7.3.64) and [70, Theorem 2.3.2]. The second claim in item *(i)* is seen straight from definitions (with [68, Lemma 7.2.1] ensuring the absolute convergence of the integral defining $\mathcal{A}_\beta f$ in (7.3.38)). Next, the first claim in item *(ii)* follows from Proposition 7.3.2, in view of (7.3.38), while the second claim in item *(ii)* is implied by Definition 7.3.1 and (7.3.38).

Going further, all claims in item *(iii)* are direct consequences of (7.3.64) and [70, Theorem 2.3.2] (bearing in mind that the complex Riesz transforms do fall under the scope of the latter theorem). Likewise, all claims in items *(iv)*, *(v)*, and *(vi)* are immediate consequences of (7.3.63) and the corresponding results for the complex Riesz transforms implied by [70, Theorem 2.4.1].

As regards the jump-formula in item *(vii)*, we first observe that in the present context [70, Theorem 2.5.1] implies that for each $j \in \{1,\dots,n\}$ and each scalar function $g \in L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big)$, the j-th complex Riesz transform satisfies

$$\big[\mathcal{R}_{\mathbb{C},j}\, g\big]\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = -(\nu_{\mathbb{C}})_j\, g + R_{\mathbb{C},j}\, g \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{7.3.65}$$

Taking complex conjugates this yields (after re-denoting $\bar{g}$ by g)

$$\big[\mathcal{R}^c_{\mathbb{C},j}\, g\big]\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = -\overline{(\nu_{\mathbb{C}})}_j\, g + R^c_{\mathbb{C},j}\, g \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{7.3.66}$$

To proceed, fix an arbitrary differential form $f \in L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{0,\beta+1}$. Based on (7.3.63)-(7.3.64), (7.1.57), (7.3.66), and (7.1.61), we may then compute

$$\begin{aligned}
\big[\mathcal{A}_{\alpha,\beta} f\big]\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} &= (-1)^{\alpha+1} \sum_{j=1}^{n} \sum_{|J|=\alpha} \sum_{|I|=\beta} \sum_{|K|=\beta+1} \varepsilon^{jI}_{K} \big[\mathcal{R}^c_{\mathbb{C},j} f_{J,K}\big]\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} dz^J \wedge d\bar{z}^I \\
&= (-1)^{\alpha} \sum_{j=1}^{n} \sum_{|J|=\alpha} \sum_{|I|=\beta} \sum_{|K|=\beta+1} \varepsilon^{jI}_{K}\, \overline{(\nu_{\mathbb{C}})}_j f_{J,K}\, dz^J \wedge d\bar{z}^I \\
&\quad + (-1)^{\alpha+1} \sum_{j=1}^{n} \sum_{|J|=\alpha} \sum_{|I|=\beta} \sum_{|K|=\beta+1} \varepsilon^{jI}_{K}\, R^c_{\mathbb{C},j} f_{J,K}\, dz^J \wedge d\bar{z}^I \\
&= \tfrac{1}{2}\, \nu^{1,0} \vee f + A_{\alpha,\beta} f \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega.
\end{aligned} \tag{7.3.67}$$

This proves (7.3.53). Lastly, the claim in item *(viii)* is a consequence of item *(4)* of [70, Theorem 2.4.1] (bearing in mind (7.3.40)). □

Our main result pertaining to the nature of the Bochner-Martinelli operator $\mathcal{B}_{\alpha,\beta}$ defined in (7.3.33) is contained in the theorem below.

Theorem 7.3.5 *Let $n \in \mathbb{N}$ and suppose $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ is a nonempty open set with the property that $\partial\Omega$ is a* UR *set; in particular, Ω has locally finite perimeter.*

Abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω. *Also, fix a pair of arbitrary degrees* $\alpha, \beta \in \{0, 1, \dots, n\}$, *along with an aperture parameter* $\kappa > 0$. *Finally, for each differential form* $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta}$, *define the principal-value (boundary-to-boundary) Bochner-Martinelli integral operator*

$$B_{\alpha,\beta}f(z) := -\frac{1}{2}\lim_{\varepsilon\to 0^+}\int_{\substack{\zeta\in\partial_*\Omega\\|z-\zeta|>\varepsilon}} \big\langle \nu^{0,1}(\zeta)\wedge f(\zeta), \bar{\partial}_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\big\rangle_{\mathbb{C}}\,d\sigma(\zeta) \qquad (7.3.68)$$

for σ*-a.e. point* $z \in \partial\Omega$. *Then the following properties hold.*

(i) For each differential form $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta}$ *(hence, in particular, for each differential form* $f \in L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}$ *with* $p \in [1, \infty)$*) the limit in* (7.3.39) *exists for* σ*-a.e.* $z \in \partial\Omega$. *Moreover, the operators*

$$B_{\alpha,\beta} : L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta} \longrightarrow L^p(\partial\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}, \quad p \in (1, \infty), \qquad (7.3.69)$$

$$B_{\alpha,\beta} : L^1(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta} \longrightarrow L^{1,\infty}(\partial\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}, \qquad (7.3.70)$$

are well-defined, linear, and bounded. In addition, if $\partial\Omega$ *is compact, then corresponding to the end-point* $p = \infty$ *the operator* $B_{\alpha,\beta}$ *induces a linear and bounded mapping*

$$B_{\alpha,\beta} : L^\infty(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta} \longrightarrow \mathrm{BMO}(\partial\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}. \qquad (7.3.71)$$

(ii) For each $p \in [1, \infty)$ *there exists a constant* $C = C(\Omega, p, \kappa, n) \in (0, \infty)$ *such that for each differential form* $f \in L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}$ *one has*

$$\big\|\mathcal{N}_\kappa(\mathcal{B}_{\alpha,\beta}f)\big\|_{L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} \leq C\|f\|_{L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} \text{ if } 1 < p < \infty, \qquad (7.3.72)$$

and, corresponding to $p = 1$,

$$\big\|\mathcal{N}_\kappa(\mathcal{B}_{\alpha,\beta}f)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} \leq C\|f\|_{L^1(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}}. \qquad (7.3.73)$$

(iii) For each $p \in [1, \infty)$ *there exists some constant* $C = C(\Omega, p, \kappa, n) \in (0, \infty)$ *with the property that if the differential form* f *belongs to the partial Sobolev space* $L^{p,\bar{\partial}_\tau}_{\alpha,\beta}(\partial_*\Omega, \sigma)$ *then*

$$\big\|\mathcal{N}_\kappa(\bar{\partial}\mathcal{B}_{\alpha,\beta}f)\big\|_{L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} \leq C\|\bar{\partial}_\tau f\|_{L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+2}} \text{ if } 1 < p < \infty, \qquad (7.3.74)$$

and, corresponding to $p = 1$,

$$\big\|\mathcal{N}_\kappa(\bar{\partial}\mathcal{B}_{\alpha,\beta}f)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} \leq C\|\bar{\partial}_\tau f\|_{L^1(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+2}}. \qquad (7.3.75)$$

Also, the nontangential boundary limit

$$\bar{\partial}\mathcal{B}_{\alpha,\beta}f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \quad \text{exists at } \sigma\text{-a.e. point on } \partial_*\Omega. \tag{7.3.76}$$

(iv) For each $p \in (1,\infty)$ there exists a constant $C = C(\Omega, p, n) \in (0,\infty)$ with the property that for each form $f \in L^p(\partial_*\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}$ one has

$$\Big(\int_\Omega |(\nabla\mathcal{B}_{\alpha,\beta}f)|_{\mathbb{C}}^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\,\mathrm{d}\mathcal{L}^{2n}\Big)^{1/p} \leq C\|f\|_{L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}}, \tag{7.3.77}$$

where $\nabla\mathcal{B}_{\alpha,\beta}f$ stands for the collection of all first-order partial derivatives of all components of the differential form $\mathcal{B}_{\alpha,\beta}f$. In particular, corresponding to $p = 2$, one has the following L^2-square function estimate:

$$\int_\Omega |(\nabla\mathcal{B}_{\alpha,\beta}f)|_{\mathbb{C}}^2 \operatorname{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^{2n} \leq C\int_{\partial_*\Omega} |f|_{\mathbb{C}}^2\,\mathrm{d}\sigma. \tag{7.3.78}$$

(v) There exists a constant $C = C(\Omega, n) \in (0,\infty)$ with the property that for each differential form $f \in L^\infty(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ one has the following Carleson measure estimate:

$$\begin{aligned}\sup_{x\in\partial\Omega,\,r>0}\Big(\frac{1}{\sigma\big(\partial\Omega\cap B(x,r)\big)}\int_{B(x,r)\cap\Omega}\big|(\nabla\mathcal{B}_{\alpha,\beta})f\big|_{\mathbb{C}}^2 \operatorname{dist}(\cdot,\partial\Omega)\,\mathrm{d}\mathcal{L}^{2n}\Big)^{1/2}&\\ \leq C\|f\|_{L^\infty(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}}.&\end{aligned} \tag{7.3.79}$$

(vi) If Ω is actually a bounded NTA *domain in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ with an Ahlfors regular boundary, then whenever $p \in (1,\infty)$ and $q \in \big(\frac{n}{n+1/p},\infty\big]$ the operators*

$$\begin{aligned}\mathcal{B}_{\alpha,\beta} &: L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta} \longrightarrow F^{p,q}_{1/p}(\Omega)\otimes\Lambda^{\alpha,\beta},\\ \mathcal{B}_{\alpha,\beta} &: L^p(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta} \longrightarrow B^{p,p}_{1/p}(\Omega)\otimes\Lambda^{\alpha,\beta},\\ \mathcal{B}_{\alpha,\beta} &: L^2(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta} \longrightarrow H^{1/2}(\Omega)\otimes\Lambda^{\alpha,\beta},\end{aligned} \tag{7.3.80}$$

are well-defined, linear, and bounded.

(vii) For each form $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big)\otimes\Lambda^{\alpha,\beta}$ the following jump-formula holds:

$$\begin{aligned}\mathcal{B}_{\alpha,\beta}f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} &= \frac{1}{4}\,\nu^{1,0}\vee(\nu^{0,1}\wedge f) + B_{\alpha,\beta}f\\ &= \frac{1}{2}f_{\tan,\mathbb{C}} + B_{\alpha,\beta}f \quad \text{at } \sigma\text{-a.e. point on } \partial_*\Omega.\end{aligned} \tag{7.3.81}$$

In particular, (with I denoting the identity operator),

$$\begin{aligned}&\text{if } f \in L^p_{\tan,\mathbb{C}}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta} \text{ with } 1 \leq p < \infty \text{ then}\\ &\mathcal{B}_{\alpha,\beta}f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + B_{\alpha,\beta}\big)f \quad \text{at } \sigma\text{-a.e. point on } \partial_*\Omega.\end{aligned} \tag{7.3.82}$$

(viii) Suppose the set $\Omega \subsetneq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ *is actually a* UR *domain, and fix an arbitrary differential form* $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta}$. *Then, if one defines*

$$\Omega_+ := \Omega, \quad \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}, \textit{ and } \mathcal{B}^{\pm}_{\alpha,\beta} f := \mathcal{B}_{\alpha,\beta} f\Big|_{\Omega_\pm}, \tag{7.3.83}$$

it follows that Ω_- *is also a* UR *domain whose topological and geometric measure theoretic boundaries agree with those of* Ω, *and whose geometric measure theoretic outward unit normal is* $-\nu$ *at* σ*-a.e. point on* $\partial\Omega$. *In addition, one has the jump-formulas:*

$$\mathcal{B}^{\pm}_{\alpha,\beta} f\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \pm\tfrac{1}{2} f_{\tan,\mathbb{C}} + B_{\alpha,\beta} f \ \textit{ at } \sigma\textit{-a.e. point on } \partial\Omega. \tag{7.3.84}$$

As a consequence,

$$\begin{gathered} \textit{if } \Omega \textit{ is a } \mathrm{UR} \textit{ domain and } f \in L^p_{\tan,\mathbb{C}}(\partial_*\Omega,\sigma) \otimes \Lambda^{\alpha,\beta} \textit{ with } 1 \le p < \infty \\ \textit{then } \mathcal{B}^{\pm}_{\alpha,\beta} f\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = \big(\pm\tfrac{1}{2} I + B_{\alpha,\beta}\big) f \ \textit{ at } \sigma\textit{-a.e. point on } \partial\Omega. \end{gathered} \tag{7.3.85}$$

Thanks to [70, Theorem 2.6.1], under the assumption that Ω is a UR domain we also have mapping properties on Morrey spaces which are similar to those for the double layer operator presented in Theorem 3.3.1. Also, thanks to (1.4.199) and Theorem 3.3.3, we see that if Ω is a UR domain, $p \in (1,\infty)$, and $\lambda \in (0, 2n-1)$, then there exists a constant $C \in (0,\infty)$ with the property that the fractional Carleson measure estimate

$$\begin{aligned} \sup_{\substack{z\in\partial\Omega \text{ and} \\ 0<r<2\,\mathrm{diam}(\partial\Omega)}} & \Big\{ r^{-\lambda} \int_{B(z,r)\cap\Omega} \big|\nabla\big(\mathcal{B}_{\alpha,\beta} f\big)\big|^p \mathrm{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^{2n} \Big\}^{\frac{1}{p}} \\ & \le C\|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}} \end{aligned} \tag{7.3.86}$$

holds for each differential form $f \in M^{p,\lambda}(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ which is complex tangential.

Proof of Theorem 7.3.5 The first claim in item *(i)* is a consequence of [70, Theorem 2.3.2]. Moreover, given any differential form $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta}$, if we agree to regard $\nu^{0,1} \wedge f$ as a form in the space $L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta+1}$ by extending it by zero from $\partial_*\Omega$ to $\partial\Omega$ then (7.3.68) and (7.3.39) imply

$$B_{\alpha,\beta} f = \tfrac{1}{2} A_{\alpha,\beta}(\nu^{0,1} \wedge f) \text{ on } \partial\Omega. \tag{7.3.87}$$

Bearing this in mind, all other claims in the current item *(i)* are implied by part *(iii)* of Proposition 7.3.4. In fact, granted (7.3.87) and given the relationship between the family of operators $\mathcal{B}_{\alpha,\beta}$ and $\mathcal{A}_{\alpha,\beta}$ identified in (7.3.41)-(7.3.42) (and also keeping in mind that the exterior product with $\nu^{0,1}$ is a bounded operator on Lebesgue spaces on $\partial_*\Omega$ with respect to the measure σ), all claims in the current items *(ii)*-*(vi)* are direct consequences of Proposition 7.3.4.

As regards the first claim in item *(vii)*, having fixed $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta}$ we may invoke (7.3.42), (7.3.53), (7.3.87), and (7.1.68) in order to write

$$\begin{aligned}\mathcal{B}_{\alpha,\beta} f\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} &= \frac{1}{2}\mathcal{A}_{\alpha,\beta}(\nu^{0,1}\wedge f)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \\ &= \frac{1}{2}\Big\{\frac{1}{2}\nu^{1,0}\vee(\nu^{0,1}\wedge f) + A_{\alpha,\beta}(\nu^{0,1}\wedge f)\Big\} \\ &= \frac{1}{2} f_{\tan,\mathbb{C}} + B_{\alpha,\beta} f \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \end{aligned} \tag{7.3.88}$$

This proves the jump-formula (7.3.81), and (7.3.82) follows from it. Lastly, all the claims in item *(viii)* are consequences of (7.3.81)-(7.3.82) and part *(7)* of [68, Lemma 5.10.9]. □

7.4 A Sharp Version of the Bochner-Martinelli-Koppelman Formula and Related Topics

In its classical form, the Bochner-Martinelli-Koppelman formula involves smooth domains and differential forms, and here the main goal is to establish a geometrically and analytically sharp version of this basic result. See Theorem 7.4.3 and the comments preceding its statement. For now, we discuss the following generalization of the Cauchy-Pompeiu integral representation formula from [70, Theorem 1.1.1] to functions of several complex variables.

Theorem 7.4.1 *Let $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ be an open set with a lower Ahlfors regular boundary, and with the property that $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Also, fix a pair of degrees $\alpha, \beta \in \{0, 1, \dots, n\}$ along with some aperture parameter $\kappa > 0$.*

In this context, suppose $u \in L^1_{\rm loc}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta}$ is a differential form satisfying (with all partial differential operators considered in the sense of distributions in Ω)

$$\bar\partial u \in L^1\Big(\Omega, \frac{\mathcal{L}^{2n}(\zeta)}{1+|\zeta|^{2n-1}}\Big) \otimes \Lambda^{\alpha,\beta+1}, \quad \vartheta u \in L^1\Big(\Omega, \frac{\mathcal{L}^{2n}(\zeta)}{1+|\zeta|^{2n-1}}\Big) \otimes \Lambda^{\alpha,\beta-1}, \tag{7.4.1}$$
$$\text{the nontangential trace } u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial_{\rm nta}\Omega,$$

and

$$\int_{\partial\Omega} \frac{(\mathcal{N}_\kappa u)(\zeta)}{1+|\zeta|^{2n-1}}\, d\sigma(\zeta) < +\infty. \tag{7.4.2}$$

Then for any $\kappa' > 0$ *the nontangential trace* $u\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ *also exists* σ*-a.e. on* $\partial_{\mathrm{nta}}\Omega$ *and is actually independent of* κ'. *Moreover, with the dependence on the parameter* κ' *dropped, for* $\mathcal{L}^{2n}$*-a.e. point* $z \in \Omega$ *one has (with absolutely convergent integrals)*

$$\begin{aligned} u(z) = &-\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \nu^{0,1}(\zeta)\wedge \big(u\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\ &+\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \nu^{1,0}(\zeta)\vee \big(u\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\ &+\int_\Omega \Big\langle (\bar\partial u)(\zeta), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\\ &+\int_\Omega \Big\langle (\vartheta u)(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta) \end{aligned} \tag{7.4.3}$$

if either Ω *is bounded, or* $\partial\Omega$ *is unbounded. In the remaining case, i.e., when* Ω *is an exterior domain, the same conclusion holds true under the additional assumption that there exists* $\lambda \in (1,\infty)$ *such that*

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |u|_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n} = o(1) \ \text{as}\ R\to\infty. \tag{7.4.4}$$

Furthermore, for each $z \in \mathbb{C}^n \setminus \overline{\Omega}$ *one has*

$$\begin{aligned} 0 = &-\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \nu^{0,1}(\zeta)\wedge \big(u\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\ &+\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \nu^{1,0}(\zeta)\vee \big(u\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\ &+\int_\Omega \Big\langle (\bar\partial u)(\zeta), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\\ &+\int_\Omega \Big\langle (\vartheta u)(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta), \end{aligned} \tag{7.4.5}$$

with the same caveat as in (7.4.4) when Ω *is an exterior domain.*

Prior to presenting the proof of this theorem we make several comments. First, if $\partial\Omega$ is upper Ahlfors regular then the integrability condition in (7.4.2) is satisfied whenever $\mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma)$ for some $p \in [1,\infty)$. Moreover, if $\partial\Omega$ is actually compact, then the integrability conditions in (7.4.2) simply reduce to the membership

$$\mathcal{N}_\kappa u \in L^1(\partial\Omega,\sigma). \tag{7.4.6}$$

Second, recall from [68, Lemma 3.5.7] that the integrability conditions in the first line of (7.4.1) are equivalent with having for $\mathcal{L}^{2n}$-a.e. point $z \in \mathbb{C}^n$ the finiteness condition

$$\int_{\Omega} \frac{|(\bar{\partial} u)(\zeta)|_{\mathbb{C}} + |(\vartheta u)(\zeta)|_{\mathbb{C}}}{|\zeta - z|^{2n-1}}\, d\mathcal{L}^{2n}(\zeta) < +\infty. \tag{7.4.7}$$

Third, if [68, Theorem 1.5.1] is employed in lieu of [68, Theorem 1.4.1] in the proof of Theorem 7.4.1, it is possible to relax the doubling assumption on $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$ to merely demanding that this is a locally finite measure. In such a scenario, we need to impose the condition that the aperture parameter κ is sufficiently large (depending on Ω), and the flexibility of changing κ when considering nontangential boundary traces may be lost. Nonetheless, modulo these nuances, the format of the main results (i.e., formulas (7.4.3) and (7.4.5)) remains the same.

Fourth, in the case when $n = 1$ and $\alpha = \beta = 0$, Theorem 7.4.1 becomes [70, Theorem 1.1.1]. In particular, (7.4.3) reduces in this case precisely to the Cauchy-Pompeiu formula [70, (1.1.8)].

After this preamble, we now turn to the proof of Theorem 7.4.1.

Proof of Theorem 7.4.1 We debut by observing that, in concert, (7.4.2) and [68, Lemma 8.3.1] imply that

$$u \in L^{\infty}_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta}. \tag{7.4.8}$$

Moreover, (7.4.1), (7.4.2), and [68, Corollary 8.9.9] ensure that for any $\kappa' > 0$ the nontangential trace $u\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$ exists at σ-a.e. point on $\partial_{\mathrm{nta}}\Omega$ and is actually independent of the parameter $\kappa' > 0$. In addition, with the dependence on κ' dropped, from [68, (8.9.8)], [68, (8.9.44)], and (7.4.2) we conclude that

$$\int_{\partial_*\Omega} \frac{\Big|\big(u\big|^{\mathrm{n.t.}}_{\partial\Omega}\big)(\zeta)\Big|_{\mathbb{C}}}{1 + |\zeta|^{2n-1}}\, d\sigma(\zeta) < +\infty. \tag{7.4.9}$$

In particular, from (7.4.9) and (7.3.18)-(7.3.19) it follows that, for each point $z \in \Omega$, the boundary integrals in the first two lines of (7.4.3) are absolutely convergent.

The strategy for actually proving formula (7.4.3) is to apply [68, Theorem 1.2.1] to a suitably constructed vector field. Specifically, fix a Lebesgue point $z \in \Omega$ for (all the coefficients of) u with the property that (7.4.7) holds, and define $\vec{F} : \Omega \to \big[\Lambda^{\alpha,\beta}\mathbb{C}^n\big]^n$ by requiring that

$$\xi \cdot \vec{F}(\zeta) = -\frac{1}{2}\Big\langle \xi^{0,1} \wedge u(\zeta), \bar{\partial}_{\zeta}\Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}} + \frac{1}{2}\Big\langle \xi^{1,0} \vee u(\zeta), \vartheta_{\zeta}\Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}}$$
$$\text{for all real vectors } \xi \in \mathbb{R}^{2n}, \text{ at } \mathcal{L}^{2n}\text{-a.e. point } \zeta \in \Omega. \tag{7.4.10}$$

Above, as in (7.1.58), for each real vector $\xi = (\xi_1, \xi_2, \dots, \xi_{2n-1}, \xi_{2n}) \in \mathbb{R}^{2n}$ we have denoted

$$\xi^{1,0} := \sum_{j=1}^{n} \overline{(\xi_{\mathbb{C}})_j}\, \mathrm{d}z_j \in \Lambda^{1,0}\mathbb{C}^n \ \text{ and } \ \xi^{0,1} := \sum_{j=1}^{n} (\xi_{\mathbb{C}})_j\, \mathrm{d}\bar{z}_j \in \Lambda^{0,1}\mathbb{C}^n, \tag{7.4.11}$$

where $\xi_{\mathbb{C}} := (\xi_1 + \mathrm{i}\xi_2, \dots, \xi_{2n-1} + \mathrm{i}\xi_{2n}) \in \mathbb{C}^n$ is the complex version of $\xi \in \mathbb{R}^{2n}$. With this piece of notation, it is clear that the right-hand side of (7.4.10) depends linearly in the variable $\xi \in \mathbb{R}^{2n}$, hence the demand in (7.4.10) determines $\vec{F}$ uniquely (as a function defined $\mathcal{L}^{2n}$-a.e. in Ω) and unambiguously. Moreover, from (7.4.8), (7.4.10), and (7.3.18)-(7.3.19) we have

$$\vec{F} \in \left[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta}\right]^n. \tag{7.4.12}$$

Granted this, it is meaningful to compute the distributional divergence of $\vec{F}$ in Ω. To identify this explicitly, fix an arbitrary real-valued test function $\varphi \in \mathscr{C}^\infty_c(\Omega)$ and observe that

$$\begin{aligned}(\nabla\varphi)_{\mathbb{C}} &= \left(\partial_{x_1}\varphi + \mathrm{i}\partial_{y_1}\varphi, \dots, \partial_{x_n}\varphi + \mathrm{i}\partial_{y_n}\varphi\right)\\ &= 2\left(\partial_{\bar{z}_1}\varphi, \dots, \partial_{\bar{z}_n}\varphi\right).\end{aligned} \tag{7.4.13}$$

As a consequence of (7.4.13), (7.4.11), and (7.2.2) we therefore have

$$(\nabla\varphi)^{1,0} = 2\,\partial\varphi \ \text{ and } \ (\nabla\varphi)^{0,1} = 2\,\bar{\partial}\varphi. \tag{7.4.14}$$

We may then write

$$\begin{aligned}&{}_{\mathcal{D}'(\Omega)\otimes\Lambda^{\alpha,\beta}}\left\langle \mathrm{div}\vec{F}, \varphi\right\rangle_{\mathcal{D}(\Omega)}\\ &\quad = -{}_{\mathcal{D}'(\Omega)\otimes\Lambda^{\alpha,\beta}}\left\langle \vec{F}, \nabla\varphi\right\rangle_{\mathcal{D}(\Omega)} = -\int_\Omega \nabla\varphi\cdot\vec{F}\,\mathrm{d}\mathcal{L}^{2n}\\ &\quad = \frac{1}{2}\int_\Omega \Big\{\Big\langle \left(\nabla\varphi(\zeta)\right)^{0,1} \wedge u(\zeta), \bar{\partial}_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\\ &\qquad\quad - \Big\langle \left(\nabla\varphi(\zeta)\right)^{1,0} \vee u(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\Big\}\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\\ &\quad = \int_\Omega \Big\{\Big\langle (\bar{\partial}\varphi)(\zeta) \wedge u(\zeta), \bar{\partial}_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\\ &\qquad\quad - \Big\langle (\partial\varphi)(\zeta) \vee u(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\Big\}\,\mathrm{d}\mathcal{L}^{2n}(\zeta),\end{aligned} \tag{7.4.15}$$

thanks to (7.4.10) (presently used with $\xi := \nabla\varphi$ viewed as a real vector in $\mathbb{R}^{2n}$), (7.4.11), (7.1.4), and (7.4.14). Since the Leibniz product rule for $\bar{\partial}, \vartheta$ currently gives that, in the sense of distributions in Ω,

$$(\bar{\partial}\varphi)\wedge u = \bar{\partial}(\varphi u) - \varphi\bar{\partial}u, \qquad (\partial\varphi)\vee u = -\vartheta(\varphi u) + \varphi\vartheta u, \tag{7.4.16}$$

it follows from (7.4.8), (7.4.16), and the first line in (7.4.1) that $\varphi u \in L^\infty_{\rm comp}(\Omega)\otimes\Lambda^{\alpha,\beta}$ satisfies $\bar\partial(\varphi u) \in L^1_{\rm comp}(\Omega,\mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta+1}$ and $\vartheta(\varphi u) \in L^1_{\rm comp}(\Omega,\mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta-1}$. Furthermore, in light of (7.4.7) we also have

$$\int_\Omega \frac{|\bar\partial(\varphi u)(\zeta)|_{\mathbb{C}} + |\vartheta(\varphi u)(\zeta)|_{\mathbb{C}}}{|\zeta - z|^{2n-1}}\,d\mathcal{L}^{2n}(\zeta) < +\infty. \tag{7.4.17}$$

As such, we may further express

$${}_{\mathcal{D}'(\Omega)\otimes\Lambda^{\alpha,\beta}}\big\langle \operatorname{div}\vec{F}, \varphi\big\rangle_{\mathcal{D}(\Omega)} = I + II, \tag{7.4.18}$$

where

$$I := -\int_\Omega \varphi(\zeta)\Big\{\Big\langle(\bar\partial u)(\zeta), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}} + \Big\langle(\vartheta u)(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\Big\}\,d\mathcal{L}^{2n}(\zeta), \tag{7.4.19}$$

and

$$II := \int_\Omega \Big\{\Big\langle\bar\partial(\varphi u)(\zeta), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}} + \Big\langle\vartheta(\varphi u)(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\Big\}\,d\mathcal{L}^{2n}(\zeta). \tag{7.4.20}$$

To handle term II, bring in a scalar-valued function $\theta \in \mathscr{C}^\infty(\mathbb{R}^{2n})$ with the property that $\theta = 0$ on $B(0,1)$ and $\theta = 1$ on $\mathbb{R}^{2n}\setminus B(0,2)$. For each $\varepsilon \in \big(0, \frac{1}{2}\operatorname{dist}(z,\partial\Omega)\big)$ define $\theta_\varepsilon : \mathbb{R}^{2n} \equiv \mathbb{C}^n \to \mathbb{R}$ by setting

$$\theta_\varepsilon(\zeta) := \theta\Big(\frac{\zeta - z}{\varepsilon}\Big) \text{ for every } \zeta \in \mathbb{C}^n \equiv \mathbb{R}^{2n}. \tag{7.4.21}$$

Then

$$\begin{gathered}\theta_\varepsilon \in \mathscr{C}^\infty(\mathbb{C}^n) \text{ is a bounded function uniformly in } \varepsilon,\\ \text{satisfying } \lim_{\varepsilon\to 0^+}\theta_\varepsilon(\zeta) = 1 \text{ for every fixed } \zeta \in \mathbb{C}^n\setminus\{z\},\end{gathered} \tag{7.4.22}$$

and there exists a constant $C \in (0,\infty)$ such that for each $\varepsilon \in \big(0, \frac{1}{2}\operatorname{dist}(z,\partial\Omega)\big)$ we have

$$\begin{gathered}\theta_\varepsilon - 1 \in \mathscr{C}^\infty_c(\Omega),\ \ \theta_\varepsilon \equiv 0 \text{ on } B(z,\varepsilon),\\ \operatorname{supp}(\nabla\theta_\varepsilon) \subseteq \overline{B(z,2\varepsilon)}\setminus B(z,\varepsilon), \text{ and}\\ |(\nabla\theta_\varepsilon)(\zeta)| \le C\varepsilon^{-1} \text{ for every } \zeta \in \mathbb{C}^n.\end{gathered} \tag{7.4.23}$$

In particular, both $\theta_\varepsilon(\zeta)\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)$ and $\theta_\varepsilon(\zeta)\vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)$ are $\mathscr{C}^\infty$ in the variable $\zeta \in \Omega$. Based on Lebesgue's Dominated Convergence Theorem (whose applicability is presently ensures by (7.4.22) and (7.4.17)) may therefore write

$$II = \lim_{\varepsilon\to 0^+} \int_\Omega \Big\{ \Big\langle \bar\partial(\varphi u)(\zeta), \theta_\varepsilon(\zeta)\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z) \Big\rangle_{\mathbb{C}} + \Big\langle \vartheta(\varphi u)(\zeta), \theta_\varepsilon(\zeta)\vartheta_\zeta \Gamma_{\alpha,\beta}(\zeta,z) \Big\rangle_{\mathbb{C}} \Big\} \, d\mathcal{L}^{2n}(\zeta)$$

$$= \lim_{\varepsilon\to 0^+} \int_\Omega \Big\{ \Big\langle (\varphi u)(\zeta), \vartheta_\zeta \big[\theta_\varepsilon(\zeta)\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z)\big] \Big\rangle_{\mathbb{C}} + \Big\langle (\varphi u)(\zeta), \bar\partial_\zeta \big[\theta_\varepsilon(\zeta)\vartheta_\zeta \Gamma_{\alpha,\beta}(\zeta,z)\big] \Big\rangle_{\mathbb{C}} \Big\} \, d\mathcal{L}^{2n}(\zeta). \tag{7.4.24}$$

Note that, on account of Leibniz' product rule for $\bar\partial, \vartheta$, (7.3.14), and (7.3.20), we may compute in the sense of distributions

$$\begin{aligned} &\vartheta_\zeta \big[\theta_\varepsilon(\zeta)\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z)\big] + \bar\partial_\zeta \big[\theta_\varepsilon(\zeta)\vartheta_\zeta \Gamma_{\alpha,\beta}(\zeta,z)\big] \\ &\qquad = \theta_\varepsilon(\zeta)\big(\vartheta_\zeta\bar\partial_\zeta + \bar\partial_\zeta\vartheta_\zeta\big)\Gamma_{\alpha,\beta}(\zeta,z) \\ &\qquad\quad + (\bar\partial\theta_\varepsilon)(\zeta) \wedge \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z) - (\partial\theta_\varepsilon)(\zeta) \vee \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z) \\ &\qquad = (\bar\partial\theta_\varepsilon)(\zeta) \wedge \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z) - (\partial\theta_\varepsilon)(\zeta) \vee \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z). \end{aligned} \tag{7.4.25}$$

In turn, this suggests that we split

$$II = III + IV \tag{7.4.26}$$

where

$$III := \lim_{\varepsilon\to 0^+} \int_\Omega \Big\langle (\varphi u)(\zeta) - (\varphi u)(z), (\bar\partial\theta_\varepsilon)(\zeta) \wedge \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z) - (\partial\theta_\varepsilon)(\zeta) \vee \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z) \Big\rangle_{\mathbb{C}} \Big\} \, d\mathcal{L}^{2n}(\zeta), \tag{7.4.27}$$

and

$$IV := \lim_{\varepsilon\to 0^+} \int_\Omega \Big\langle (\varphi u)(z), (\bar\partial\theta_\varepsilon)(\zeta) \wedge \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z) - (\partial\theta_\varepsilon)(\zeta) \vee \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z) \Big\rangle_{\mathbb{C}} \Big\} \, d\mathcal{L}^{2n}(\zeta). \tag{7.4.28}$$

Since z is a Lebesgue point for u, we may estimate (based on (7.4.23) and (7.3.18)-(7.3.19))

$$|III| \leq C \limsup_{\varepsilon\to 0^+} ⨍_{B(z,2\varepsilon)} \big|(\varphi u)(\zeta) - (\varphi u)(z)\big|_{\mathbb{C}} \, d\mathcal{L}^{2n}(\zeta) = 0, \tag{7.4.29}$$

thus

$$III = 0. \tag{7.4.30}$$

Also, since $\eta_\varepsilon := \theta_\varepsilon - 1$ belongs to $\mathscr{C}_c^\infty(\Omega)$ and since $(\varphi u)(z)$ does not depend on ζ, we may reverse-engineer the first equality in (7.4.25) in order to write

$$\begin{aligned} IV &= \lim_{\varepsilon\to 0^+} \int_\Omega \Big\langle (\varphi u)(z), (\bar\partial \eta_\varepsilon)(\zeta) \wedge \vartheta_\zeta \Gamma_{\alpha,\beta}(\zeta, z) \\ &\qquad\qquad - (\partial \eta_\varepsilon)(\zeta) \vee \vartheta_\zeta \Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}} \Big\} \, d\mathcal{L}^{2n}(\zeta) \\ &= \lim_{\varepsilon\to 0^+} {}_{\mathscr{E}(\Omega)\otimes\Lambda^{\alpha,\beta}}\Big\langle (\varphi u)(z), \overline{(\bar\partial \eta_\varepsilon) \wedge \vartheta_\cdot \Gamma_{\alpha,\beta}(\cdot, z) - (\partial\eta_\varepsilon) \vee \vartheta_\cdot \Gamma_{\alpha,\beta}(\cdot, z)} \Big\rangle_{\mathscr{E}'(\Omega)\otimes\Lambda^{\alpha,\beta}} \\ &= -\lim_{\varepsilon\to 0^+} {}_{\mathscr{E}(\Omega)\otimes\Lambda^{\alpha,\beta}}\Big\langle (\varphi u)(z), \overline{\eta_\varepsilon \big(\vartheta_\cdot \bar\partial_\cdot + \bar\partial_\cdot \vartheta_\cdot\big) \Gamma_{\alpha,\beta}(\cdot, z)} \Big\rangle_{\mathscr{E}'(\Omega)\otimes\Lambda^{\alpha,\beta}} \\ &= -\lim_{\varepsilon\to 0^+} \eta_\varepsilon(z)(\varphi u)(z) = (\varphi u)(z), \end{aligned} \tag{7.4.31}$$

(with all distributional pairings in the “dot" variable), where in the penultimate equality we have also made use of (7.3.20), and the last equality relies on the fact that $\eta_\varepsilon(z) = -1$ for each $\varepsilon \in (0, \varepsilon_o)$ (cf. (7.4.23)).

Collectively, (7.4.18)-(7.4.20), (7.4.26)-(7.4.28), (7.4.30)-(7.4.31) establish that

$$\begin{aligned} \operatorname{div}\vec F = &-\big\langle \bar\partial u, \bar\partial_\cdot \Gamma_{\alpha,\beta}(\cdot, z)\big\rangle_{\mathbb{C}} - \big\langle \vartheta u, \vartheta_\cdot \Gamma_{\alpha,\beta}(\cdot, z)\big\rangle_{\mathbb{C}} \\ &+ u(z)\delta_z \ \text{ in } \mathcal{D}'(\Omega) \otimes \Lambda^{\alpha,\beta}, \end{aligned} \tag{7.4.32}$$

where δ_z is the Dirac distribution in Ω with mass at z. In particular, from (7.4.32) and (7.4.7) we conclude that

$$\operatorname{div}\vec F \in \mathscr{E}'(\Omega) \otimes \Lambda^{\alpha,\beta} + L^1(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta}. \tag{7.4.33}$$

Next, $K := \overline{B\big(z, \frac12 \operatorname{dist}(z, \partial\Omega)\big)}$ is a compact subset of Ω, and based on (7.4.10), (7.3.18)-(7.3.19), and [68, Lemma 8.3.7] we may estimate

$$\begin{aligned} \big(\mathcal{N}_\kappa^{\Omega\setminus K} \vec F\big)(\zeta) &\le C\big(\mathcal{N}_\kappa u\big)(\zeta) \cdot \sup_{\xi\in\Gamma_\kappa(\zeta)\setminus K} \big[|z-\xi|^{1-2n}\big] \\ &\le C \frac{\big(\mathcal{N}_\kappa u\big)(\zeta)}{|z-\zeta|^{2n-1}}, \quad \text{for all } \zeta \in \partial\Omega, \end{aligned} \tag{7.4.34}$$

for some constant $C = C(\Omega, n, \kappa) \in (0, \infty)$. In turn, from (7.4.34), (7.4.2), and [68, (8.2.26)] it follows that

$$\mathcal{N}_\kappa^{\Omega\setminus K} \vec F \in L^1(\partial\Omega, \sigma). \tag{7.4.35}$$

Moreover, from (7.4.10), the second line in (7.4.1), and [68, (8.9.10)-(8.9.11)] we conclude that

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega \tag{7.4.36}$$

and at σ-a.e. point $\zeta \in \partial_*\Omega$ we have

$$\nu(\zeta)\cdot\Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(\zeta) = -\frac{1}{2}\Big\langle \nu^{0,1}(\zeta)\wedge\big(u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(\zeta), \bar{\partial}_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}$$
$$+\frac{1}{2}\Big\langle \nu^{1,0}(\zeta)\vee\big(u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}. \tag{7.4.37}$$

Let us also remark that condition (7.4.4) together with (7.3.18)-(7.3.19) guarantee that

if the open set $\Omega \subseteq \mathbb{R}^{2n}$ is an exterior domain, then the vector field $\vec{F}$ satisfies the growth condition [68, (1.4.8)]. (7.4.38)

Together, (7.4.12), (7.4.33), and (7.4.36) guarantee that the vector field $\vec{F}$ satisfies the hypotheses of [68, Theorem 1.4.1]. On account of [68, (4.6.19)], (7.4.32), and (7.4.37), the Divergence Formula recorded in [68, (1.4.6)] presently yields (7.4.3).

Finally, formula (7.4.5) is established in a similar (and simpler) fashion, keeping in mind that if $z \in \mathbb{C}^n \setminus \overline{\Omega}$ then $\Gamma_{\alpha,\beta}(\cdot,z)$ has coefficients in $\mathscr{C}^\infty(\Omega)$. As such, there is no need to bring in the cutoff function θ_ε and, this time, the Dirac distribution in (7.4.32) is no longer present. □

Theorem 7.4.1 contains as a particular case the following geometrically sharp version of the Bochner-Martinelli integral representation formula.

Corollary 7.4.2 *Suppose $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ is an open set with a lower Ahlfors regular boundary, and with the property that $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Fix some aperture parameter $\kappa > 0$.*

In this context, suppose $F \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n})$ is a complex–valued function satisfying (with all partial derivatives considered in the sense of distributions in Ω)

$$\partial_{\bar{z}_j}F \in L^1\Big(\Omega, \frac{\mathcal{L}^{2n}(\zeta)}{1+|\zeta|^{2n-1}}\Big) \text{ for each } j \in \{1,\dots,n\},$$
$$\text{the trace } F\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega, \tag{7.4.39}$$

and

$$\int_{\partial\Omega}\frac{(\mathcal{N}_\kappa F)(\zeta)}{1+|\zeta|^{2n-1}}\,d\sigma(\zeta) < +\infty. \tag{7.4.40}$$

Then for any $\kappa' > 0$ the nontangential trace $F\Big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ also exists σ-a.e. on $\partial_{\mathrm{nta}}\Omega$ and is in fact independent of κ'. Furthermore, with the dependence on the parameter κ' dropped, for $\mathcal{L}^{2n}$-a.e. point $z \in \Omega$ one has (with absolutely convergent integrals)

$$F(z) = \mathscr{B}_{0,0}\big(F\big|^{\text{n.t.}}_{\partial\Omega}\big)(z) - \frac{2}{\omega_{2n-1}} \int_{\Omega} \sum_{j=1}^{n} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} (\partial_{\bar{z}_j} F)(\zeta)\, \mathrm{d}\mathcal{L}^{2n}(\zeta) \tag{7.4.41}$$

if either Ω is bounded, or $\partial\Omega$ is unbounded. In the remaining case, i.e., when Ω is an exterior domain, the same conclusion holds true under the additional assumption that there exists $\lambda \in (1, \infty)$ such that

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |F|\, \mathrm{d}\mathcal{L}^{2n} = o(1) \ \text{ as } \ R \to \infty. \tag{7.4.42}$$

Furthermore, for each $z \in \mathbb{C}^n \setminus \overline{\Omega}$ holds one has

$$0 = \mathscr{B}_{0,0}\big(F\big|^{\text{n.t.}}_{\partial\Omega}\big)(z) - \frac{2}{\omega_{2n-1}} \int_{\Omega} \sum_{j=1}^{n} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} (\partial_{\bar{z}_j} F)(\zeta)\, \mathrm{d}\mathcal{L}^{2n}(\zeta) \tag{7.4.43}$$

with the same provision as in (7.4.42) when Ω is an exterior domain.

As a consequence, if F is actually a holomorphic function in Ω with the property that

$$\begin{aligned} &F\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \text{ exists at } \sigma\text{-a.e. point on } \partial_{\text{nta}}\Omega, \\ &\text{and } \int_{\partial\Omega} \frac{(\mathcal{N}_\kappa F)(\zeta)}{1 + |\zeta|^{2n-1}}\, \mathrm{d}\sigma(\zeta) < +\infty, \end{aligned} \tag{7.4.44}$$

then for any other $\kappa' > 0$ the nontangential trace $F\big|^{\kappa'-\text{n.t.}}_{\partial\Omega}$ also exists σ-a.e. on $\partial_{\text{nta}}\Omega$, is in fact independent of κ' and, with the dependence on the parameter κ' dropped, one has

$$\mathscr{B}_{0,0}\big(F\big|^{\text{n.t.}}_{\partial\Omega}\big)(z) = \begin{cases} F(z) & \text{if } z \in \Omega, \\ 0 & \text{if } z \in \mathbb{C}^n \setminus \overline{\Omega}, \end{cases} \quad \text{for each } z \in \mathbb{C}^n \setminus \partial\Omega, \tag{7.4.45}$$

provided one also assumes the decay condition (7.4.42) in the case when Ω is an exterior domain.

Proof All claims follow by specializing Theorem 7.4.1 to the case when $\alpha = \beta = 0$. □

Theorem 7.4.3 stated below is a sharp rendition of the Bochner-Martinelli-Koppelman formula. Our version generalizes [88, Theorem 1.10, p. 154] and [59, Theorem 4.11, p. 23] which assume that underlying set Ω is a bounded $\mathscr{C}^1$ domain and the differential form u is of class $\mathscr{C}^1(\overline{\Omega})$, and [36, Theorem 1.11.1, p. 57] where it is assumed that Ω is a bounded domain with piecewise $\mathscr{C}^2$ boundary, and that both u and $\bar{\partial}u$ are continuous on $\overline{\Omega}$.

Theorem 7.4.3 *Suppose $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ is an open set with a lower Ahlfors regular boundary, and with the property that $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Also, fix a pair of degrees $\alpha, \beta \in \{0, 1, \dots, n\}$ along with some aperture parameter $\kappa > 0$.*

In this context, assume

$$u \in L^1\Big(\Omega, \frac{\mathcal{L}^{2n}(\zeta)}{1+|\zeta|^{2n-1}}\Big) \otimes \Lambda^{\alpha,\beta} \tag{7.4.46}$$

is a differential form satisfying (with all partial differential operators considered in the sense of distributions in Ω*)*

$$\bar\partial u \in L^1\Big(\Omega, \frac{\mathcal{L}^{2n}(\zeta)}{1+|\zeta|^{2n-1}}\Big) \otimes \Lambda^{\alpha,\beta+1}, \tag{7.4.47}$$

$$u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega, \tag{7.4.48}$$

$$\int_{\partial\Omega} \frac{(\mathcal{N}_\kappa u)(\zeta)}{1+|\zeta|^{2n-1}}\, \mathrm{d}\sigma(\zeta) < +\infty. \tag{7.4.49}$$

Then the differential form

$$\begin{aligned} &U(z) := \int_\Omega \Big\langle u(\zeta), \bar\partial_\zeta \Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle_{\mathbb{C}} \, \mathrm{d}\mathcal{L}^{2n}(\zeta) \text{ for } \mathcal{L}^{2n}\text{-a.e. } z\in\Omega, \\ &\text{is locally integrable in } \Omega \text{ and satisfies } \bar\partial U \in L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta}. \end{aligned} \tag{7.4.50}$$

Also, for any $\kappa' > 0$ *the nontangential trace* $u\Big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ *exists* σ*-a.e. on* $\partial_{\mathrm{nta}}\Omega$ *and is actually independent of* κ'*. Moreover, with the dependence on the parameter* κ' *dropped, for* $\mathcal{L}^{2n}$*-a.e. point* $z\in\Omega$ *one has*

$$\begin{aligned} u(z) = &-\frac{1}{2}\int_{\partial_*\Omega} \Big\langle \nu^{0,1}(\zeta)\wedge \big(u\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}} \, \mathrm{d}\sigma(\zeta) \\ &+ \int_\Omega \Big\langle (\bar\partial u)(\zeta), \bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}} \, \mathrm{d}\mathcal{L}^{2n}(\zeta) \\ &+ \bar\partial_z \Big\{ \int_\Omega \Big\langle u(\zeta), \bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle_{\mathbb{C}} \, \mathrm{d}\mathcal{L}^{2n}(\zeta)\Big\}, \end{aligned} \tag{7.4.51}$$

where the action of $\bar\partial_z$ *is considered in the sense of distributions in* Ω *(cf. (7.4.50)).*

Proof We debut by observing that, in concert, (7.4.49) and [68, Lemma 8.3.1] imply that

$$u \in L^\infty_{\mathrm{loc}}(\Omega,\mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta}. \tag{7.4.52}$$

Moreover, (7.4.48), (7.4.49), and [68, Corollary 8.9.9] ensure that for any $\kappa' > 0$ the nontangential trace $u\Big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial_{\mathrm{nta}}\Omega$ and is actually independent of the parameter $\kappa' > 0$. In addition, with the dependence on κ' dropped, from [68, (8.9.8), (8.9.44)] and (7.4.49) we conclude that

$$\int_{\partial_*\Omega} \frac{\big|\big(u\big|^{\text{n.t.}}_{\partial\Omega}\big)(\zeta)\big|_{\mathbb{C}}}{1+|\zeta|^{2n-1}}\,\mathrm{d}\sigma(\zeta) < +\infty. \tag{7.4.53}$$

In particular, from (7.4.53) and (7.3.18)-(7.3.19) it follows that, for each fixed point $z \in \Omega$, the boundary integral in the first line of (7.4.51) is absolutely convergent. Finally, we observe from the proof of [68, Lemma 3.5.7] (cf. [68, (3.5.32)]) that, collectively, the integrability conditions in (7.4.46) and the first line of (7.4.47) imply that

$$\text{the expression } \int_{\Omega} \frac{|u(\zeta)|_{\mathbb{C}} + |(\bar{\partial} u)(\zeta)|_{\mathbb{C}}}{|\zeta - z|^{2n-1}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta) \\ \text{belongs to } L^1_{\text{loc}}(\mathbb{C}^n, \mathcal{L}^{2n}) \text{ as a function of } z. \tag{7.4.54}$$

The strategy for actually proving formula (7.4.51) is to apply [68, Theorem 1.2.1] to a suitable domain and vector field. Specifically, define

$$D := \Omega \times \mathbb{C}^n \tag{7.4.55}$$

which is an open set in $\mathbb{R}^{4n} \equiv \mathbb{C}^{2n}$. Next,

$$\text{fix } w \in \mathscr{C}^\infty_c(\Omega) \otimes \Lambda^{\beta,\alpha} \text{ and define } \vec{F}_w : D \longrightarrow \mathbb{C}^{2n} \tag{7.4.56}$$

by requiring (using notation introduced in (7.3.4) and (7.3.9)) that

$$\begin{aligned}(\xi,\eta)\cdot \vec{F}_w(\zeta,z) &= -\frac{1}{2}\Big\langle\Big\langle \big(\xi^{0,1} \wedge u(\zeta)\big) \otimes w(z), \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\Big\rangle_{\mathbb{C}} \\ &\quad + \frac{1}{2}\Big\langle\Big\langle u(\zeta) \otimes \big(\eta^{0,1} \vee w(z)\big), \bar{\partial}_\zeta \Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle\Big\rangle_{\mathbb{C}} \end{aligned} \tag{7.4.57}$$

$$\text{for all vectors } (\xi,\eta) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \text{ at } \mathcal{L}^{2n} \otimes \mathcal{L}^{2n}\text{-a.e. point } (\zeta,z) \in D.$$

Indeed, since the rems in the right-hand side of (7.4.57) depend linearly in the variables $\xi, \eta \in \mathbb{R}^{2n}$, the demand in (7.4.57) determines $\vec{F}_w$ unambiguously as an $\mathcal{L}^{2n} \otimes \mathcal{L}^{2n}$-a.e. defined function in D which is $\mathbb{C}^{2n}$-valued. This also shows that $\vec{F}$ is $\mathcal{L}^{2n} \otimes \mathcal{L}^{2n}$-measurable. Observe next that (7.4.57) and (7.3.18)-(7.3.19) imply the existence of a purely dimensional constant $C \in (0,\infty)$ such that

$$\big|\vec{F}_w(\zeta,z)\big| \leq C\frac{|u(\zeta)|_{\mathbb{C}}|w(z)|_{\mathbb{C}}}{|\zeta - z|^{2n-1}} \text{ for all } (\zeta,z) \in D \text{ with } \zeta \neq z. \tag{7.4.58}$$

In concert with (7.4.54) and (7.4.55)-(7.4.56) this allows us to conclude that

$$\vec{F}_w \in \big[L^1(D, \mathcal{L}^{2n} \otimes \mathcal{L}^{2n})\big]^{4n}. \tag{7.4.59}$$

Granted this, it is meaningful to compute the distributional divergence of $\vec{F}_w$ in D. To identify this explicitly, fix two arbitrary real-valued test functions $\varphi \in \mathscr{C}^\infty_c(\Omega)$ and $\psi \in \mathscr{C}^\infty_c(\mathbb{C}^n)$. Upon recalling from (7.4.14) that $(\nabla\varphi)^{0,1} = 2\,\bar{\partial}\varphi$ and $(\nabla\psi)^{0,1} = 2\,\bar{\partial}\psi$, we may then write

$$
\begin{aligned}
&{}_{\mathcal{D}'(D)}\big\langle \mathrm{div}\vec{F}_w, \varphi\otimes\psi\big\rangle_{\mathcal{D}(D)} \\
&= -{}_{\mathcal{D}'(D)}\Big\langle \vec{F}_w, \big((\nabla\varphi)\otimes\psi, \varphi\otimes(\nabla\psi)\big)\Big\rangle_{\mathcal{D}(D)} \\
&= -\int_D \big((\nabla\varphi)\otimes\psi, \varphi\otimes(\nabla\psi)\big)\cdot\vec{F}_w \,\mathrm{d}\mathcal{L}^{2n}\otimes\mathrm{d}\mathcal{L}^{2n} \\
&= \frac{1}{2}\int_D \Big\langle\!\Big\langle \big(\big(\nabla\varphi(\zeta)\big)^{0,1}\wedge u(\zeta)\big)\otimes(\psi w)(z), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} \mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z) \\
&\quad - \frac{1}{2}\int_D \Big\langle\!\Big\langle (\varphi u)(\zeta)\otimes\big(\big(\nabla\psi(\zeta)\big)^{0,1}\vee w(z)\big), \bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} \mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z) \\
&= \int_D \Big\langle\!\Big\langle \big((\bar\partial\varphi)(\zeta)\wedge u(\zeta)\big)\otimes(\psi w)(z), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} \mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z) \\
&\quad - \int_D \Big\langle\!\Big\langle (\varphi u)(\zeta)\otimes\big((\bar\partial\psi)(z)\vee w(z)\big), \bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} \mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z) \\
&=: \mathrm{I}+\mathrm{II},
\end{aligned}
\tag{7.4.60}
$$

thanks to (7.4.57) presently used with $\xi := (\nabla\varphi)\otimes\psi$ and $\eta := \varphi\otimes(\nabla\psi)$ viewed as reals vector in $\mathbb{R}^{2n}$. Making use of Leibniz' product rule for $\bar\partial$ (cf. (7.4.16)) we may further split

$$
\mathrm{I} = \mathrm{I}_a + \mathrm{I}_b, \tag{7.4.61}
$$

where

$$
\begin{aligned}
\mathrm{I}_a &:= -\int_D \Big\langle\!\Big\langle \big(\varphi(\zeta)(\bar\partial u)(\zeta)\big)\otimes(\psi w)(z), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} \mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z) \\
&= -\int_D \Big\langle\!\Big\langle (\bar\partial u)(\zeta)\otimes w(z), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} (\varphi\otimes\psi)(\zeta,z)\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z)
\end{aligned}
\tag{7.4.62}
$$

and

$$
\mathrm{I}_b := \int_D \Big\langle\!\Big\langle \big(\bar\partial(\varphi u)(\zeta)\big)\otimes(\psi w)(z), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} \mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z). \tag{7.4.63}
$$

To handle term I_b, bring in a scalar-valued function $\theta\in\mathscr{C}^\infty(\mathbb{R}^{2n})$ with the property that $\theta = 0$ on $B(0,1)$ and $\theta = 1$ on $\mathbb{R}^{2n}\setminus B(0,2)$. For each $\varepsilon > 0$ define

$$
\theta_\varepsilon : \mathbb{R}^{2n}\times\mathbb{R}^{2n} \equiv \mathbb{C}^n\times\mathbb{C}^n \longrightarrow \mathbb{R} \tag{7.4.64}
$$

by setting

$$\theta_\varepsilon(\zeta, z) := \theta\Big(\frac{\zeta - z}{\varepsilon}\Big) \text{ for every } \zeta, z \in \mathbb{C}^n \equiv \mathbb{R}^{2n}. \tag{7.4.65}$$

Then

$$\begin{aligned} &\theta_\varepsilon \in \mathscr{C}^\infty(\mathbb{C}^n \times \mathbb{C}^n) \text{ is a bounded function uniformly in } \varepsilon, \\ &\text{satisfying } \lim_{\varepsilon \to 0^+} \theta_\varepsilon(\zeta, z) = 1 \text{ for each fixed } \zeta, z \in \mathbb{C}^n \text{ with } \zeta \neq z, \end{aligned} \tag{7.4.66}$$

and there exists a constant $C \in (0, \infty)$ such that for each $\varepsilon > 0$ we have

$$\begin{aligned} &\theta_\varepsilon(\zeta, z) = 0 \text{ if } |\zeta - z| < \varepsilon, \;\; \theta_\varepsilon(\zeta, z) = 1 \text{ if } |\zeta - z| \geq 2\varepsilon, \\ &\operatorname{supp}(\nabla\theta_\varepsilon) \subseteq \big\{(\zeta, z) \in \mathbb{C}^n \times \mathbb{C}^n : \varepsilon \leq |\zeta - z| \leq 2\varepsilon\big\}, \\ &\text{and } |(\nabla\theta_\varepsilon)(\zeta, z)| \leq C\varepsilon^{-1} \text{ for all } (\zeta, z) \in \mathbb{C}^n \times \mathbb{C}^n. \end{aligned} \tag{7.4.67}$$

Based on Lebesgue's Dominated Convergence Theorem (whose applicability in the present setting is ensured by (7.4.66) and (7.4.54)) and (7.3.13) we may then write

$$\begin{aligned} \mathrm{I}_b &= \lim_{\varepsilon \to 0^+} \int_D \Big\langle\!\Big\langle \big(\bar\partial(\varphi u)(\zeta)\big) \otimes (\psi w)(z), \theta_\varepsilon(\zeta, z)\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta, z) \Big\rangle\!\Big\rangle_{\mathbb{C}} \, d\mathcal{L}^{2n}(\zeta)\, d\mathcal{L}^{2n}(z) \\ &= \lim_{\varepsilon \to 0^+} {}_{\mathscr{E}'(D)}\Big\langle\!\Big\langle \big(\bar\partial(\varphi u)(\zeta)\big) \otimes (\psi w)(z), \theta_\varepsilon(\zeta, z)\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta, z) \Big\rangle\!\Big\rangle_{\mathscr{E}(D)} \\ &= \lim_{\varepsilon \to 0^+} {}_{\mathscr{E}'(D)}\Big\langle\!\Big\langle (\varphi u)(\zeta) \otimes (\psi w)(z), \vartheta_\zeta\big[\theta_\varepsilon(\zeta, z)\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta, z)\big] \Big\rangle\!\Big\rangle_{\mathscr{E}(D)}. \\ &= \lim_{\varepsilon \to 0^+} \int_D \Big\langle\!\Big\langle (\varphi u)(\zeta) \otimes (\psi w)(z), \vartheta_\zeta\big[\theta_\varepsilon(\zeta, z)\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta, z)\big] \Big\rangle\!\Big\rangle_{\mathbb{C}} \, d\mathcal{L}^{2n}(\zeta)\, d\mathcal{L}^{2n}(z). \end{aligned} \tag{7.4.68}$$

On the other hand, on account of Leibniz' product rule for $\bar\partial$, ϑ, (7.3.14), and (7.3.20), we may compute

$$
\begin{aligned}
&\vartheta_\zeta\big[\theta_\varepsilon(\zeta,z)\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\big] \qquad (7.4.69)\\
&\qquad = -\partial_\zeta\theta_\varepsilon(\zeta,z)\vee\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)+\theta_\varepsilon(\zeta,z)\vartheta_\zeta\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\\
&\qquad = -\partial_\zeta\theta_\varepsilon(\zeta,z)\vee\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)+\theta_\varepsilon(\zeta,z)\big(\vartheta_\zeta\bar\partial_\zeta+\bar\partial_\zeta\vartheta_\zeta\big)\Gamma_{\alpha,\beta}(\zeta,z)\\
&\qquad\quad -\theta_\varepsilon(\zeta,z)\bar\partial_\zeta\vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\\
&\qquad = -\partial_\zeta\theta_\varepsilon(\zeta,z)\vee\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)+\theta_\varepsilon(\zeta,z)\Box_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\\
&\qquad\quad -\theta_\varepsilon(\zeta,z)\bar\partial_\zeta\partial_z\Gamma_{\alpha,\beta-1}(\zeta,z)\\
&\qquad = -\partial_\zeta\theta_\varepsilon(\zeta,z)\vee\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\\
&\qquad\quad -\partial_z\big[\theta_\varepsilon(\zeta,z)\bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z)\big]+\partial_z\theta_\varepsilon(\zeta,z)\wedge\bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z).
\end{aligned}
$$

In turn, this suggests that we split

$$
\mathrm{I}_b=\mathrm{III}-\mathrm{IV}+\mathrm{V}. \tag{7.4.70}
$$

Above, $\mathrm{III} := -\lim_{\varepsilon\to0^+}\mathrm{III}_\varepsilon$ where, for each $\varepsilon>0$,

$$
\mathrm{III}_\varepsilon := \int_D \Big\langle\!\Big\langle(\varphi u)(\zeta)\otimes(\psi w)(z),\partial_\zeta\theta_\varepsilon(\zeta,z)\vee\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z). \tag{7.4.71}
$$

Also, IV is given by

$$
\begin{aligned}
&\lim_{\varepsilon\to0^+}\int_D \Big\langle\!\Big\langle(\varphi u)(\zeta)\otimes(\psi w)(z),\partial_z\big[\theta_\varepsilon(\zeta,z)\bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z)\big]\Big\rangle\!\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\mathrm{d}\mathcal{L}^{2n}(z)\\
&\quad=\lim_{\varepsilon\to0^+}\int_D \Big\langle\!\Big\langle(\varphi u)(\zeta)\otimes\bar\vartheta(\psi w)(z),\theta_\varepsilon(\zeta,z)\bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\mathrm{d}\mathcal{L}^{2n}(z)\\
&\quad=\int_D \Big\langle\!\Big\langle(\varphi u)(\zeta)\otimes\bar\vartheta(\psi w)(z),\bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\mathrm{d}\mathcal{L}^{2n}(z), \qquad (7.4.72)
\end{aligned}
$$

with the last step justified by Lebesgue's Dominated Convergence Theorem (which, in turn, relies on (7.4.66) and (7.4.54)). Finally, $\mathrm{V} := \lim_{\varepsilon\to0^+}\mathrm{V}_\varepsilon$ where, for each $\varepsilon>0$,

$$
\mathrm{V}_\varepsilon := \int_D \Big\langle\!\Big\langle(\varphi u)(\zeta)\otimes(\psi w)(z),\partial_z\theta_\varepsilon(\zeta,z)\wedge\bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\mathrm{d}\mathcal{L}^{2n}(z). \tag{7.4.73}
$$

Observe that since $(\bar\partial\psi)\vee w+\bar\vartheta(\psi w)=\psi\bar\vartheta w$ we may combine

$$\mathrm{II} - \mathrm{IV} = -\int\limits_D \Big\langle\!\Big\langle (\varphi u)(\zeta) \otimes (\psi \bar{\vartheta} w)(z), \bar{\partial}_\zeta \Gamma_{\alpha,\beta-1}(\zeta, z) \Big\rangle\!\Big\rangle_{\mathbb{C}} \, \mathrm{d}\mathcal{L}^{2n}(\zeta) \, \mathrm{d}\mathcal{L}^{2n}(z)$$
$$= -\int\limits_D \Big\langle\!\Big\langle u(\zeta) \otimes (\bar{\vartheta} w)(z), \bar{\partial}_\zeta \Gamma_{\alpha,\beta-1}(\zeta, z) \Big\rangle\!\Big\rangle_{\mathbb{C}} (\varphi \otimes \psi)(\zeta, z) \mathrm{d}\mathcal{L}^{2n}(\zeta) \, \mathrm{d}\mathcal{L}^{2n}(z). \tag{7.4.74}$$

Pressing on, break up

$$\mathrm{V} = \mathrm{V}_a + \mathrm{V}_b \tag{7.4.75}$$

where V_a is the version of V in which we freeze the coefficients of the differential form φu at the point z (while retaining the $\mathrm{d}\zeta$'s and $\mathrm{d}\bar{\zeta}$'s originally present in the writing of u unaffected), i.e.,

$$\mathrm{V}_a := \lim_{\varepsilon \to 0^+} \int_D \Big\langle\!\Big\langle (\varphi u)(z) \otimes (\psi w)(z), \partial_z \theta_\varepsilon(\zeta, z) \wedge \bar{\partial}_\zeta \Gamma_{\alpha,\beta-1}(\zeta, z) \Big\rangle\!\Big\rangle_{\mathbb{C}} \, \mathrm{d}\mathcal{L}^{2n}(\zeta) \, \mathrm{d}\mathcal{L}^{2n}(z), \tag{7.4.76}$$

and $\mathrm{V}_b = \mathrm{V} - \mathrm{V}_a$ is the remainder given by

$$\mathrm{V}_b := \lim_{\varepsilon \to 0^+} \int_D \Big\langle\!\Big\langle \big((\varphi u)(\zeta) - (\varphi u)(z)\big) \otimes (\psi w)(z), \partial_z \theta_\varepsilon(\zeta, z) \wedge \bar{\partial}_\zeta \Gamma_{\alpha,\beta-1}(\zeta, z) \Big\rangle\!\Big\rangle_{\mathbb{C}} \, \mathrm{d}\mathcal{L}^{2n}(\zeta) \, \mathrm{d}\mathcal{L}^{2n}(z). \tag{7.4.77}$$

Note that if we set $\varepsilon_w := \frac{1}{2} \operatorname{dist}\big(\operatorname{supp} w, \partial\Omega\big) > 0$ and for each point $z \in \operatorname{supp} w$ and each $\varepsilon \in (0, \varepsilon_w)$ we define

$$f_\varepsilon(z) := \fint_{B(z,2\varepsilon)} \big|(\varphi u)(\zeta) - (\varphi u)(z)\big|_{\mathbb{C}} \, \mathrm{d}\mathcal{L}^{2n}(\zeta) \tag{7.4.78}$$

then by invoking Lebesgue's Dominated Convergence Theorem (whose applicability is ensured by Lebesgue's Differentiation Theorem and (7.4.52)) we conclude from (7.4.77), (7.4.67), and (7.3.18)-(7.3.19) that

$$|\mathrm{V}_b| \leq C \lim_{\varepsilon \to 0^+} \int_{\operatorname{supp} w} f_\varepsilon(z) \, \mathrm{d}\mathcal{L}^{2n}(z) = 0, \ \text{ hence } \ \mathrm{V}_b = 0. \tag{7.4.79}$$

Similarly, break up

$$\mathrm{III} = \mathrm{III}_a + \mathrm{III}_b \tag{7.4.80}$$

where III_a is the version of III in which we freeze the coefficients of the differential form φu at the point z, i.e.,

$$\mathrm{III}_a := -\lim_{\varepsilon\to 0^+}\int_D \Big\langle\!\!\Big\langle (\varphi u)(z)\otimes(\psi w)(z), \\ \partial_\zeta\theta_\varepsilon(\zeta,z)\vee\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\!\!\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z), \tag{7.4.81}$$

then estimate the disagreement term $\mathrm{III}_b := \mathrm{III} - \mathrm{III}_a$ as in (7.4.79) to conclude that

$$|\mathrm{III}_b| \le C\lim_{\varepsilon\to 0^+}\int_{\operatorname{supp} w} f_\varepsilon(z)\,\mathrm{d}\mathcal{L}^{2n}(z) = 0, \text{ hence } \mathrm{III}_b = 0. \tag{7.4.82}$$

At this stage, abbreviating

$$\eta_\varepsilon := \theta_\varepsilon - 1 \text{ and } \Theta := (\varphi u)(z)\otimes(\psi w)(z), \tag{7.4.83}$$

from (7.4.75)-(7.4.77), (7.4.79), and (7.4.80)-(7.4.82) we see that we may express III + V as

$$\begin{aligned} \mathrm{III}+\mathrm{V} &= \mathrm{III}_a + \mathrm{V}_a \\ &= \lim_{\varepsilon\to 0^+}\int_D \Big\langle\!\!\Big\langle \Theta, \partial_z\eta_\varepsilon(\zeta,z)\wedge\bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z) \\ &\qquad - \partial_\zeta\eta_\varepsilon(\zeta,z)\vee\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\!\!\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z). \end{aligned} \tag{7.4.84}$$

To proceed, the idea is to reverse-engineer (7.4.69) (with the role of θ_ε now played by η_ε) and compute, in the sense of distributions:

$$\begin{aligned} &\partial_z\eta_\varepsilon(\zeta,z)\wedge\bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z) - \partial_\zeta\eta_\varepsilon(\zeta,z)\vee\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z) \\ &= \partial_z\big[\eta_\varepsilon(\zeta,z)\bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z)\big] - \eta_\varepsilon(\zeta,z)\bar\partial_\zeta\partial_z\Gamma_{\alpha,\beta-1}(\zeta,z) \\ &\quad + \vartheta_\zeta\big[\eta_\varepsilon(\zeta,z)\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\big] - \eta_\varepsilon(\zeta,z)\vartheta_\zeta\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z) \\ &= \partial_z\big[\eta_\varepsilon(\zeta,z)\bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z)\big] + \vartheta_\zeta\big[\eta_\varepsilon(\zeta,z)\bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\big] \\ &\quad - \eta_\varepsilon(\zeta,z)(\bar\partial_\zeta\vartheta_\zeta + \vartheta_\zeta\bar\partial_\zeta)\Gamma_{\alpha,\beta}(\zeta,z). \end{aligned} \tag{7.4.85}$$

Let us also observe that, in the sense of distributions in D,

$$\begin{aligned} (\bar\partial_\zeta\vartheta_\zeta + \vartheta_\zeta\bar\partial_\zeta)\Gamma_{\alpha,\beta}(\zeta,z) &= \Box_\zeta\Gamma_{\alpha,\beta}(\zeta,z) \\ &= 2^{-\alpha-\beta}(-1)^{\alpha\beta}\delta(\zeta,z)\sum_{|J|=\alpha}\sum_{|I|=\beta}(\mathrm{d}\zeta^J\wedge\mathrm{d}\bar\zeta^I)\otimes(\mathrm{d}z^I\wedge\mathrm{d}\bar z^J) \end{aligned} \tag{7.4.86}$$

where $\delta(\zeta,z)\in\mathcal{D}'(D)$ is the diagonal Dirac distribution defined as

$${}_{\mathcal{D}'(D)}\big\langle\delta(\zeta,z),\Phi(\zeta,z)\big\rangle_{\mathcal{D}(D)} := \int_\Omega \Phi(z,z)\,\mathrm{d}\mathcal{L}^{2n}(z) \tag{7.4.87}$$

for each test function $\Phi \in \mathscr{C}_c^\infty(D)$. In particular, since $\eta_\varepsilon(z, z) = -1$ for each $z \in \Omega$, it follows that

$$-\eta_\varepsilon(\zeta, z)(\bar{\partial}_\zeta \vartheta_\zeta + \vartheta_\zeta \bar{\partial}_\zeta)\Gamma_{\alpha,\beta}(\zeta, z) \qquad (7.4.88)$$
$$= 2^{-\alpha-\beta}(-1)^{\alpha\beta}\delta(\zeta, z) \sum_{|J|=\alpha} \sum_{|I|=\beta} (d\zeta^J \wedge d\bar{\zeta}^I) \otimes (dz^I \wedge d\bar{z}^J).$$

Since, by assumption,

$$u \in L^1_{\rm loc}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta} \text{ and } \bar{\partial} u \in L^1_{\rm loc}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta+1}, \qquad (7.4.89)$$

using a standard mollifier argument yields a sequence

$$\{u_j\}_{j\in\mathbb{N}} \subset \mathscr{C}^\infty(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta} \text{ with } u_j \to u \text{ in } L^1_{\rm loc}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta},$$
$$\bar{\partial} u_j \to \bar{\partial} u \text{ in } L^1_{\rm loc}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta+1} \text{ as } j \to \infty, \text{ and} \qquad (7.4.90)$$
$$u_j(z) \to u(z) \text{ as } j \to \infty \text{ at each Lebesgue point } z \text{ of } u \text{ in } \Omega.$$

Moreover, thanks to (7.4.52), we may also assume that

$$\text{for each compact } K \subset \Omega \text{ there exists } j_K \in \mathbb{N}$$
$$\text{with the property that } \sup_{j \geq j_K} \sup_K |u_j|_{\mathbb{C}} < +\infty. \qquad (7.4.91)$$

Associated with each u_j then define the double form

$$\Theta_j := (\varphi u_j)(z) \otimes (\psi w)(z), \quad \text{for all } z \in \Omega. \qquad (7.4.92)$$

Returning to (7.4.84) then write

$$\text{III} + \text{V} = \lim_{\varepsilon\to 0^+} \lim_{j\to\infty} \int_D \Big\langle\!\Big\langle \Theta_j, \partial_z \eta_\varepsilon(\zeta, z) \wedge \bar{\partial}_\zeta \Gamma_{\alpha,\beta-1}(\zeta, z)$$
$$- \partial_\zeta \eta_\varepsilon(\zeta, z) \vee \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta, z) \Big\rangle\!\Big\rangle_{\mathbb{C}} \, d\mathcal{L}^{2n}(\zeta) \, d\mathcal{L}^{2n}(z)$$
$$= \lim_{\varepsilon\to 0^+} \lim_{j\to\infty} {}_{\mathcal{D}(D)}\Big\langle\!\Big\langle \Theta_j, \partial_z \eta_\varepsilon(\zeta, z) \wedge \bar{\partial}_\zeta \Gamma_{\alpha,\beta-1}(\zeta, z)$$
$$- \partial_\zeta \eta_\varepsilon(\zeta, z) \vee \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta, z) \Big\rangle\!\Big\rangle_{\mathcal{D}'(D)}$$
$$= \lim_{\varepsilon\to 0^+} \lim_{j\to\infty} \big\{\text{VI}_{\varepsilon,j} + \text{VII}_{\varepsilon,j} + \text{VIII}_j\big\}, \qquad (7.4.93)$$

where, for each $\varepsilon > 0$ and $j \in \mathbb{N}$, we have set

$$\text{VI}_{\varepsilon,j} := {}_{\mathcal{D}(D)}\Big\langle\!\Big\langle \Theta_j, \partial_z \big[\eta_\varepsilon(\zeta, z)\bar{\partial}_\zeta \Gamma_{\alpha,\beta-1}(\zeta, z)\big] \Big\rangle\!\Big\rangle_{\mathcal{D}'(D)} \qquad (7.4.94)$$

$$\mathrm{VII}_{\varepsilon,j} := {}_{\mathcal{D}(D)}\Big\langle\!\Big\langle \Theta_j, \vartheta_\zeta \big[\eta_\varepsilon(\zeta,z)\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z)\big]\Big\rangle\!\Big\rangle_{\mathcal{D}'(D)} \tag{7.4.95}$$

and

$$\begin{aligned}\mathrm{VIII}_j &:= {}_{\mathcal{D}(D)}\Big\langle\!\Big\langle \Theta_j, 2^{-\alpha-\beta}(-1)^{\alpha\beta}\delta(\zeta,z)\sum_{\substack{|J|=\alpha\\|I|=\beta}}(\mathrm{d}\zeta^J\wedge\mathrm{d}\bar\zeta^I)\otimes(\mathrm{d}z^I\wedge\mathrm{d}\bar z^J)\Big\rangle\!\Big\rangle_{\mathcal{D}'(D)}\\ &= \int_\Omega \big\langle \varphi u_j, \psi\bar w\big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}. \end{aligned}\tag{7.4.96}$$

Let us now consider the linear functional

$$\Lambda : \mathscr{C}^0_c(D)\to\mathbb{C},\qquad \Lambda\Phi := \int_\Omega \Phi(z,z)\,\mathrm{d}\mathcal{L}^{2n}(z),\quad \forall\,\Phi\in\mathscr{C}^0_c(D). \tag{7.4.97}$$

Since this obviously has the property that for each compact set $K\subseteq D$

$$\sup\big\{|\Lambda\Phi| :\ \Phi\in\mathscr{C}^0_c(D),\ |\Phi|\le 1 \text{ on } D, \text{ and } \operatorname{supp}\Phi\subseteq K\big\} < +\infty, \tag{7.4.98}$$

from Riesz' Representation Theorem (cf. [68, Proposition 3.9.1]) we conclude that there exists Borel measure on D, call it μ_{diag}, which is locally finite and Borel-regular, with the property that

$$\Lambda\Phi = \int_D \Phi(\zeta,z)\,\mathrm{d}\mu_{\text{diag}}(\zeta,z) \text{ for every } \Phi\in\mathscr{C}^0_c(D). \tag{7.4.99}$$

In particular, if we change u on a $\mathcal{L}^{2n}$-nullset to make it Borel measurable (which, as is well known, is always possible – see, e.g., [92, Exercise 5, p. 56] – and which does not affect any of the original properties of u) we may then write

$$\begin{aligned}\lim_{j\to\infty}\mathrm{VIII}_j &= \lim_{j\to\infty}\int_\Omega \big\langle\varphi u_j,\psi w\big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}\\ &= \lim_{j\to\infty}\int_\Omega \big\langle u_j(z),\bar w(z)\big\rangle_{\mathbb{C}}(\varphi\otimes\psi)(z,z)\,\mathrm{d}\mathcal{L}^{2n}(z)\\ &= \lim_{j\to\infty}\int_D \big\langle u_j(\zeta),\bar w(z)\big\rangle_{\mathbb{C}}(\varphi\otimes\psi)(\zeta,z)\,\mathrm{d}\mu_{\text{diag}}(\zeta,z)\\ &= \int_D \big\langle u(\zeta),\bar w(z)\big\rangle_{\mathbb{C}}(\varphi\otimes\psi)(\zeta,z)\,\mathrm{d}\mu_{\text{diag}}(\zeta,z), \end{aligned}\tag{7.4.100}$$

by relying on (7.4.97), (7.4.99), (7.4.90)-(7.4.91), and Lebesgue's Dominated Convergence Theorem. Note that for each $\varepsilon>0$ and $j\in\mathbb{N}$ we have

$$
\begin{aligned}
\mathrm{VI}_{\varepsilon,j} &= {}_{\mathcal{D}(D)}\Big\langle\!\Big\langle \bar\vartheta_z \Theta_j, \eta_\varepsilon(\zeta,z)\bar\partial_\zeta \Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathcal{D}'(D)} \\
&= {}_{\mathcal{D}(D)}\Big\langle\!\Big\langle (\varphi u_j)(z)\otimes\big(\bar\vartheta(\psi w)\big)(z), \eta_\varepsilon(\zeta,z)\bar\partial_\zeta \Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathcal{D}'(D)} \\
&= \int_D \Big\langle\!\Big\langle (\varphi u_j)(z)\otimes\big(\bar\vartheta(\psi w)\big)(z), \eta_\varepsilon(\zeta,z)\bar\partial_\zeta \Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z)
\end{aligned}
\tag{7.4.101}
$$

hence, as a simple application of Lebesgue's Dominated Convergence Theorem shows,

$$
\lim_{j\to\infty}\mathrm{VI}_{\varepsilon,j} = \int_D \Big\langle\!\Big\langle (\varphi u)(z)\otimes\big(\bar\vartheta(\psi w)\big)(z), \eta_\varepsilon(\zeta,z)\bar\partial_\zeta \Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z) \tag{7.4.102}
$$

and, ultimately,

$$
\lim_{\varepsilon\to 0^+}\lim_{j\to\infty}\mathrm{VI}_{\varepsilon,j} = 0, \tag{7.4.103}
$$

once again by Lebesgue's Dominated Convergence Theorem. Also, since the coefficients of the double form Θ_j are independent of ζ, we have

$$
\mathrm{VII}_{\varepsilon,j} = {}_{\mathcal{D}(D)}\Big\langle\!\Big\langle \bar\partial_\zeta \Theta_j, \eta_\varepsilon(\zeta,z)\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathcal{D}'(D)} = 0 \tag{7.4.104}
$$

for all $\varepsilon > 0$ and all $j \in \mathbb{N}$. Together, (7.4.60), (7.4.61), (7.4.70), (7.4.74), (7.4.75), (7.4.79), (7.4.80), (7.4.82), (7.4.93), (7.4.100), (7.4.103), and (7.4.104) prove that

$$
\begin{aligned}
&{}_{\mathcal{D}'(D)}\big\langle \mathrm{div}\vec F_w, \varphi\otimes\psi\big\rangle_{\mathcal{D}(D)} \\
&= -\int_D \Big\langle\!\Big\langle (\bar\partial u)(\zeta)\otimes w(z), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} (\varphi\otimes\psi)(\zeta,z)\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z) \\
&\quad -\int_D \Big\langle\!\Big\langle u(\zeta)\otimes(\bar\vartheta w)(z), \bar\partial_\zeta\Gamma_{\alpha,\beta-1}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} (\varphi\otimes\psi)(\zeta,z)\,\mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z) \\
&\quad +\int_D \big\langle u(\zeta), \bar w(z)\big\rangle_{\mathbb{C}} (\varphi\otimes\psi)(\zeta,z)\,\mathrm{d}\mu_{\mathrm{diag}}(\zeta,z).
\end{aligned}
\tag{7.4.105}
$$

In view of the arbitrariness of $\varphi \in \mathscr{C}^\infty_c(\Omega)$ and $\psi \in \mathscr{C}^\infty_c(\mathbb{C}^n)$, if we now define

$$
\Psi_w(\zeta,z) := \big\langle u(\zeta), \bar w(z)\big\rangle_{\mathbb{C}} \quad \text{for } (\zeta,z)\in D, \tag{7.4.106}
$$

this further implies that

$$\begin{aligned}\operatorname{div}\vec{F}_w = &-\Big\langle\!\!\Big\langle (\bar\partial u)\otimes w, \bar\partial_\zeta \Gamma_{\alpha,\beta}\Big\rangle\!\!\Big\rangle_{\mathbb{C}} \\ &- \Big\langle\!\!\Big\langle u\otimes(\bar\vartheta w), \bar\partial_\zeta\Gamma_{\alpha,\beta-1}\Big\rangle\!\!\Big\rangle_{\mathbb{C}} + \Psi_w\,\mu_{\rm diag} \ \text{ in } \mathcal{D}'(D).\end{aligned} \tag{7.4.107}$$

In particular, since w is smooth and compactly supported, u is locally essentially bounded, and $\mu_{\rm diag}$ is a locally finite Borel measure, it follows that

$$\Psi_w\mu_{\rm diag} \in \mathrm{CBM}(D), \tag{7.4.108}$$

from (7.4.107), (7.4.54), and (7.4.108) we conclude that

$$\operatorname{div}\vec{F}_w \in L^1(D, \mathcal{L}^{2n}\otimes\mathcal{L}^{2n}) + \mathrm{CBM}(D). \tag{7.4.109}$$

Going forward, straight from definitions we see that

$$D \subseteq \mathbb{C}^{2n} \ \text{ is an open set, with } \ \partial D = \partial\Omega\times\mathbb{C}^n, \ \text{ and } \ \partial_* D = \partial_*\Omega\times\mathbb{C}^n. \tag{7.4.110}$$

In particular, D has locally finite perimeter, and if we consider $\sigma_D := \mathcal{H}^{4n-1}\lfloor\partial D$ and let ν_D denote the geometric measure theoretic outward unit normal to D, then

$$\begin{gathered}\sigma_D = \sigma\otimes\mathcal{L}^{2n} \ \text{ and } \ \nu_D(\zeta,z) = \big(\nu(\zeta),0\big) \\ \text{at } \sigma_D\text{-a.e. point } (\zeta,z)\in\partial_* D = \partial_*\Omega\times\mathbb{C}^n.\end{gathered} \tag{7.4.111}$$

In addition, for each $\kappa > 0$ we have

$$\begin{gathered}\Gamma_{\Omega,\kappa}(\zeta_o)\times\{z_o\} \subseteq \Gamma_{D,\kappa}(\zeta_o,z_o) \subseteq \Gamma_{\Omega,\kappa}(\zeta_o)\times\mathbb{C}^n \\ \text{for all } (\zeta_o,z_o)\in\partial D = \partial\Omega\times\mathbb{C}^n,\end{gathered} \tag{7.4.112}$$

which, in particular, proves that

$$\partial_{\rm nta} D = \partial_{\rm nta}\Omega\times\mathbb{C}^n. \tag{7.4.113}$$

Next, if we now fix some compact neighborhood K of $\operatorname{supp} w$ in the open set Ω, based on (7.4.10), (7.4.112), and [68, Lemma 8.3.7] we may find some constant $C = C(\Omega, K, w, \kappa, n) \in (0,\infty)$ such that for all $(\zeta_o, z_o)\in\partial D = \partial\Omega\times\mathbb{C}^n$ we have

$$\begin{aligned}\big(\mathcal{N}_\kappa^{D\setminus(K\times K)}\vec{F}_w\big)(\zeta_o,z_o) &\leq C\big(\mathcal{N}_\kappa u\big)(\zeta_o)\cdot \sup_{z\in\operatorname{supp} w}\left(\sup_{\zeta\in\Gamma_\kappa(\zeta_o)\setminus K}\big[|\zeta-z|^{1-2n}\big]\right) \\ &\leq C\frac{\big(\mathcal{N}_\kappa u\big)(\zeta_o)}{1+|\zeta_o|^{2n-1}}.\end{aligned} \tag{7.4.114}$$

In turn, from (7.4.114), (7.4.49), and [68, (8.1.18), (8.2.26)] it follows that

$$\mathcal{N}_\kappa^{D\setminus(K\times K)}\vec{F}_w \in L^1_{\rm loc}(\partial D, \sigma_D). \tag{7.4.115}$$

Moreover, from (7.4.57), (7.4.47), and [68, (8.9.10)-(8.9.11)] we conclude that

$$\vec{F}_w\Big|_{\partial D}^{\kappa-\text{n.t.}} \text{ exists at } \sigma_D\text{-a.e. point on } \partial_{\text{nta}} D, \tag{7.4.116}$$

and, by also taking (7.4.111), (7.4.110), and [68, (8.8.52)] into account, that

$$\begin{aligned} \nu_D(\zeta,z)\cdot\Big(\vec{F}_w\Big|_{\partial D}^{\kappa-\text{n.t.}}\Big)(\zeta,z) &= \big(\nu(\zeta),0\big)\cdot\Big(\vec{F}_w\Big|_{\partial D}^{\kappa-\text{n.t.}}\Big)(\zeta,z) \\ &= -\frac{1}{2}\Big\langle\!\Big\langle\big(\nu^{0,1}(\zeta)\wedge\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(\zeta)\big)\otimes w(z),\bar{\partial}_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle\!\Big\rangle_{\mathbb{C}} \\ &\qquad \text{at } \sigma_D\text{-a.e. point } (\zeta,z)\in\partial_* D=\partial_*\Omega\times\mathbb{C}^n. \end{aligned} \tag{7.4.117}$$

As a consequence of (7.4.117), there exists a constant $C\in(0,\infty)$ with the property that for σ_D-a.e. point $(\zeta,z)\in\partial_* D$ we have

$$\Big|\nu_D(\zeta,z)\cdot\Big(\vec{F}_w\Big|_{\partial D}^{\kappa-\text{n.t.}}\Big)(\zeta,z)\Big| \leq C\frac{\big|\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(\zeta)\big|_{\mathbb{C}}|w(z)|_{\mathbb{C}}}{|\zeta-z|^{2n-1}}. \tag{7.4.118}$$

Together with (7.4.53) and [68, (8.9.44)], this goes to show (also keeping in mind that w is a smooth compactly supported function in Ω) that in fact

$$\nu_D\cdot\Big(\vec{F}_w\Big|_{\partial D}^{\kappa-\text{n.t.}}\Big) \text{ belongs to the space } L^1(\partial_* D,\sigma_D). \tag{7.4.119}$$

Together, (7.4.59), (7.4.109), (7.4.115), (7.4.116), and (7.4.119) guarantee that the vector field $\vec{F}_w$ satisfies the hypotheses of [68, Theorem 1.4.1]. Moreover, for each $R>0$ the membership in (7.4.59) allows us to estimate

$$\int_{\substack{(\zeta,z)\in D\\ R\leq|(\zeta,z)|<2R}} |(\zeta,z)\cdot\vec{F}_w(\zeta,z)|\,d\mathcal{L}^{2n}(\zeta)\,d\mathcal{L}^{2n}(z) \leq 2R\|\vec{F}_w\|_{L^1(D,\mathcal{L}^{2n}\otimes\mathcal{L}^{2n})} \tag{7.4.120}$$

which, in turn, ensures that [68, (1.4.8)] holds. On account of (7.4.107) and (7.4.117), the Divergence Formula recorded in [68, (1.4.6)] then currently yields

$${}_{(\mathscr{C}_b^\infty(D))^*}\big(\text{div}\vec{F}_w,1\big)_{\mathscr{C}_b^\infty(D)} = \int_{\partial_* D}\nu_D\cdot\big(\vec{F}_w\big|_{\partial D}^{\text{n.t.}}\big)\,d\sigma_D. \tag{7.4.121}$$

On the other hand, (7.4.107) and [68, (4.6.19)] imply

$$
\begin{aligned}
{}_{(\mathscr{C}_b^\infty(D))^*}\big(\mathrm{div}\vec{F}_w, 1\big)_{\mathscr{C}_b^\infty(D)} = &-\int_D \Big\langle\!\Big\langle (\bar\partial u)\otimes w, \bar\partial_\zeta \Gamma_{\alpha,\beta}\Big\rangle\!\Big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}\,\mathrm{d}\mathcal{L}^{2n} \\
&-\int_D \Big\langle\!\Big\langle u\otimes(\bar\vartheta w), \bar\partial_\zeta \Gamma_{\alpha,\beta-1}\Big\rangle\!\Big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}\,\mathrm{d}\mathcal{L}^{2n} \\
&+\int_D \Psi_w \,\mathrm{d}\mu_{\text{diag}} =: A_1 + A_2 + A_3. \qquad (7.4.122)
\end{aligned}
$$

Use (7.3.10) and Fubini's Theorem to expand (keeping in mind that w is compactly supported in Ω)

$$
\begin{aligned}
A_1 &= -\int_{\Omega\times\mathbb{C}^n} \Big\langle w(z), \big\langle \overline{(\bar\partial u)(\zeta)}, \overline{\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z)}\big\rangle_{\mathbb{C}}\Big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z) \\
&= -\int_\Omega \Big\langle w(z), \int_\Omega \big\langle \overline{(\bar\partial u)(\zeta)}, \overline{\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z)}\big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}(\zeta)\Big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}(z), \qquad (7.4.123)
\end{aligned}
$$

as well as

$$
\begin{aligned}
A_2 &= -\int_{\Omega\times\mathbb{C}^n} \Big\langle (\bar\vartheta w)(z), \big\langle \overline{u(\zeta)}, \overline{\bar\partial_\zeta \Gamma_{\alpha,\beta-1}(\zeta,z)}\big\rangle_{\mathbb{C}}\Big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}(\zeta)\,\mathrm{d}\mathcal{L}^{2n}(z) \\
&= -\int_\Omega \Big\langle (\bar\vartheta w)(z), \int_\Omega \big\langle \overline{u(\zeta)}, \overline{\bar\partial_\zeta \Gamma_{\alpha,\beta-1}(\zeta,z)}\big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}(\zeta)\Big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}(z). \qquad (7.4.124)
\end{aligned}
$$

Also,

$$
A_3 = \int_D \Psi_w \,\mathrm{d}\mu_{\text{diag}} = \int_\Omega \big\langle u, \bar w\big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n} = \int_\Omega \big\langle w(z), \bar u(z)\big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}(z). \qquad (7.4.125)
$$

In a similar fashion, (7.4.117) implies

$$
\begin{aligned}
&\int_{\partial_* D} \nu_D \cdot \big(\vec{F}_w\big|^{\text{n.t.}}_{\partial D}\big)\,\mathrm{d}\sigma_D \qquad (7.4.126) \\
&= -\frac{1}{2}\int_{\partial_* D} \Big\langle w(z), \big\langle \overline{\nu^{0,1}(\zeta)\wedge\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(\zeta)}, \overline{\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z)}\big\rangle_{\mathbb{C}}\Big\rangle_{\mathbb{C}} \,\mathrm{d}\sigma(\zeta)\mathrm{d}\mathcal{L}^{2n}(z) \\
&= -\frac{1}{2}\int_{\Omega} \Big\langle w(z), \int_{\partial_*\Omega} \big\langle \overline{\nu^{0,1}(\zeta)\wedge\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(\zeta)}, \overline{\bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta,z)}\,\big\rangle_{\mathbb{C}} \mathrm{d}\sigma(\zeta)\Big\rangle_{\mathbb{C}} \,\mathrm{d}\mathcal{L}^{2n}(z).
\end{aligned}
$$

Observe that (7.4.54) ensures that the differential form U, originally defined in (7.4.50), actually belongs to the space $L^1_{\text{loc}}(\Omega, \mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta-1}$. Keeping this in mind, from (7.4.121)-(7.4.126), (7.2.16), and the fact that $w \in \mathscr{C}_c^\infty(\Omega)\otimes\Lambda^{\beta,\alpha}$ is arbitrary we may now conclude that

$$
\begin{aligned}
\bar{\partial} U &= \bar{\partial}_z \Big\{ \int_\Omega \big\langle u(\zeta), \bar{\partial}_\zeta \Gamma_{\alpha,\beta-1}(\zeta, z) \big\rangle_{\mathbb{C}} \, d\mathcal{L}^{2n}(\zeta) \Big\} \\
&= - \int_\Omega \big\langle (\bar{\partial} u)(\zeta), \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta, z) \big\rangle_{\mathbb{C}} \, d\mathcal{L}^{2n}(\zeta) + u(z) \\
&\quad + \frac{1}{2} \int_{\partial_* \Omega} \Big\langle \nu^{0,1}(\zeta) \wedge \big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \big)(\zeta), \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta, z) \Big\rangle_{\mathbb{C}} \, d\sigma(\zeta) \qquad (7.4.127)
\end{aligned}
$$

in the sense of distributions in Ω. Since, once again thanks to (7.4.54), the right-hand side of (7.4.127) is locally integrable in Ω, we deduce that $\bar{\partial} U \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta}$. This proves (7.4.50) and ultimately formula (7.4.51) is a reinterpretation of (7.4.127) with this observation in mind. This finishes the proof of Theorem 7.4.3. □

Here is a companion to Theorem 7.4.1. The integral formulas in this result are going to be useful in the proof of Theorem 7.4.6, stated a little later.

Theorem 7.4.4 *Fix $n \in \mathbb{N}$ with $n \geq 2$ and let $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ be an open set with a lower Ahlfors regular boundary, with the property that $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Also, fix a pair of degrees $\alpha, \beta \in \{0, 1, \dots, n\}$ along with some aperture parameter $\kappa > 0$. Lastly, suppose $u \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta}$ is a differential form satisfying (with all partial differential operators considered in the sense of distributions in Ω):*

$$
\begin{gathered}
\bar{\partial} u \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta+1}, \;\; \vartheta u \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta-1}, \\
\Delta u \text{ belongs to the space } L^1\Big(\Omega, \frac{\mathcal{L}^{2n}(\zeta)}{1 + |\zeta|^{2n-2}}\Big) \otimes \Lambda^{\alpha,\beta}, \qquad (7.4.128) \\
u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, \; (\bar{\partial} u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, \; (\vartheta u)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exist at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega,
\end{gathered}
$$

and

$$
\int_{\partial\Omega} \frac{(\mathcal{N}_\kappa u)(\zeta)}{1 + |\zeta|^{2n-1}} \, d\sigma(\zeta) < \infty, \qquad \int_{\partial\Omega} \frac{\mathcal{N}_\kappa(\bar{\partial} u)(\zeta) + \mathcal{N}_\kappa(\vartheta u)(\zeta)}{1 + |\zeta|^{2n-2}} \, d\sigma(\zeta) < \infty. \quad (7.4.129)
$$

Then for any $\kappa' > 0$ the nontangential traces $u\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$, $(\bar{\partial} u)\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$, $(\vartheta u)\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$ also exist σ-a.e. on $\partial_{\mathrm{nta}}\Omega$ and are actually independent of κ'. Moreover, with the dependence on the parameter κ' dropped, for $\mathcal{L}^{2n}$-a.e. point $z \in \Omega$ one has (with absolutely convergent integrals)

$$
\begin{aligned}
u(z) = &-\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \nu^{0,1}(\zeta)\wedge\big(u\big|^{\text{n.t.}}_{\partial\Omega}\big)(\zeta), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\
&+\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \nu^{1,0}(\zeta)\vee\big(u\big|^{\text{n.t.}}_{\partial\Omega}\big)(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\
&+\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \big((\bar\partial u)\big|^{\text{n.t.}}_{\partial\Omega}\big)(\zeta), \nu^{0,1}(\zeta)\wedge\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\
&-\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \big((\vartheta u)\big|^{\text{n.t.}}_{\partial\Omega}\big)(\zeta), \nu^{1,0}(\zeta)\vee\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\
&-\frac{1}{2}\int_{\Omega}\Big\langle (\Delta u)(\zeta), \Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta)
\end{aligned}
\tag{7.4.130}
$$

if either Ω *is bounded, or* $\partial\Omega$ *is unbounded. In the remaining case, i.e., when* Ω *is an exterior domain, the same conclusion holds true under the additional assumption that there exists* $\lambda\in(1,\infty)$ *such that*

$$
\begin{aligned}
&⨍_{B(0,\lambda R)\setminus B(0,R)}|u|_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}=o(1)\ \text{ as }\ R\to\infty,\ \text{ and}\\
&⨍_{B(0,\lambda R)\setminus B(0,R)}\big\{|\bar\partial u|_{\mathbb{C}}+|\vartheta u|_{\mathbb{C}}\big\}\,\mathrm{d}\mathcal{L}^{2n}=o(R)\ \text{ as }\ R\to\infty.
\end{aligned}
\tag{7.4.131}
$$

Furthermore, for each $z\in\mathbb{C}^n\setminus\overline{\Omega}$ *one has*

$$
\begin{aligned}
0 = &-\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \nu^{0,1}(\zeta)\wedge\big(u\big|^{\text{n.t.}}_{\partial\Omega}\big)(\zeta), \bar\partial_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\
&+\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \nu^{1,0}(\zeta)\vee\big(u\big|^{\text{n.t.}}_{\partial\Omega}\big)(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\
&+\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \big((\bar\partial u)\big|^{\text{n.t.}}_{\partial\Omega}\big)(\zeta), \nu^{0,1}(\zeta)\wedge\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\
&-\frac{1}{2}\int_{\partial_*\Omega}\Big\langle \big((\vartheta u)\big|^{\text{n.t.}}_{\partial\Omega}\big)(\zeta), \nu^{1,0}(\zeta)\vee\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\sigma(\zeta)\\
&-\frac{1}{2}\int_{\Omega}\Big\langle (\Delta u)(\zeta), \Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}\,\mathrm{d}\mathcal{L}^{2n}(\zeta),
\end{aligned}
\tag{7.4.132}
$$

with the same caveat as in (7.4.131) *when* Ω *is an exterior domain.*

Finally, suitable versions of these results hold when $n=1$*, taking into account the fact that now the fundamental solution for the Laplacian in* $\mathbb{C}\equiv\mathbb{R}^2$ *has a logarithmic behavior.*

Proof For starters, (7.4.129) and [68, Lemma 8.3.1] imply

$$u \in L^{\infty}_{\rm loc}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta}, \quad \bar{\partial} u \in L^{\infty}_{\rm loc}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta+1}, \\ \text{and } \vartheta u \in L^{\infty}_{\rm loc}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta-1}. \tag{7.4.133}$$

In addition, from (7.4.128), (7.4.129), and [68, Corollary 8.9.9] it follows that for any $\kappa' > 0$ the nontangential traces $u\big|^{\kappa'-\rm n.t.}_{\partial\Omega}$, $(\bar{\partial}u)\big|^{\kappa'-\rm n.t.}_{\partial\Omega}$, $(\vartheta u)\big|^{\kappa'-\rm n.t.}_{\partial\Omega}$ also exist σ-a.e. on $\partial_{\rm nta}\Omega$ and are actually independent of the parameter κ'. Also, with the dependence on κ' dropped, from [68, (8.9.8)], [68, (8.9.44)], and the last condition in (7.4.128) we see that

$$\int_{\partial_*\Omega} \frac{\big|\big(u\big|^{\rm n.t.}_{\partial\Omega}\big)(\zeta)\big|_{\mathbb{C}}}{1+|\zeta|^{2n-1}}\, {\rm d}\sigma(\zeta) < +\infty \ \text{ and} \\ \int_{\partial_*\Omega} \frac{\big|\big((\bar{\partial}u)\big|^{\rm n.t.}_{\partial\Omega}\big)(\zeta)\big|_{\mathbb{C}} + \big|\big((\vartheta u)\big|^{\rm n.t.}_{\partial\Omega}\big)(\zeta)\big|_{\mathbb{C}}}{1+|\zeta|^{2n-2}}\, {\rm d}\sigma(\zeta) < +\infty. \tag{7.4.134}$$

In turn, this ensure that, for each fixed point $z \in \Omega$, the boundary integrals in the first four lines of (7.4.130) are absolutely convergent. Also, recall from [68, Lemma 3.5.7] that the integrability condition in the second line of (7.4.128) is equivalent with having for $\mathcal{L}^{2n}$-a.e. point $z \in \mathbb{C}^n$ the finiteness condition

$$\int_{\Omega} \frac{|(\Delta u)(\zeta)|_{\mathbb{C}}}{|\zeta - z|^{2n-2}}\, {\rm d}\mathcal{L}^{2n}(\zeta) < +\infty. \tag{7.4.135}$$

As in the proof of Theorem 7.4.1, the strategy for actually proving formula (7.4.130) is to apply [68, Theorem 1.2.1] to a suitable vector field. Concretely, fix a Lebesgue point $z \in \Omega$ for (all the coefficients of) u with the property that (7.4.135) holds, and define $\vec{F} : \Omega \to \big[\Lambda^{\alpha,\beta}\mathbb{C}^n\big]^n$ by requiring (with convention (7.4.11) in place) that

$$\xi \cdot \vec{F}(\zeta) = -\frac{1}{2}\Big\langle \xi^{0,1} \wedge u(\zeta), \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}} + \frac{1}{2}\Big\langle \xi^{1,0} \vee u(\zeta), \vartheta_\zeta \Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}} \\ + \frac{1}{2}\Big\langle (\bar{\partial}u)(\zeta), \xi^{0,1} \wedge \Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}} - \frac{1}{2}\Big\langle (\vartheta u)(\zeta), \xi^{1,0} \vee \Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}} \\ \text{for all real vectors } \xi \in \mathbb{R}^{2n}, \text{ at } \mathcal{L}^{2n}\text{-a.e. point } \zeta \in \Omega. \tag{7.4.136}$$

Note that the right-hand side of (7.4.136) depends linearly in $\xi \in \mathbb{R}^{2n}$. As such, the demand in (7.4.136) determines $\vec{F}$ uniquely and unambiguously. In addition, from (7.4.133), (7.4.136), and (7.3.18)-(7.3.19) we have

$$\vec{F} \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^{2n}) \otimes \Lambda^{\alpha,\beta}\big]^n. \tag{7.4.137}$$

A computation in the spirit of (7.4.32) (cf. also [70, (1.5.37)] for details in similar circumstances) then gives

$$\begin{aligned}\mathrm{div}\vec{F} &= -\big\langle \Box u, \Gamma_{\alpha,\beta}(\cdot,z)\big\rangle_{\mathbb{C}} + u(z)\delta_z \\ &= \tfrac{1}{2}\big\langle \Delta u, \Gamma_{\alpha,\beta}(\cdot,z)\big\rangle_{\mathbb{C}} + u(z)\delta_z \ \text{ in } \mathcal{D}'(\Omega)\otimes\Lambda^{\alpha,\beta},\end{aligned} \tag{7.4.138}$$

where δ_z is the Dirac distribution in Ω with mass at z. In particular, from (7.4.138) and (7.4.135) we conclude that

$$\mathrm{div}\vec{F} \in \mathscr{E}'(\Omega)\otimes\Lambda^{\alpha,\beta} + L^1(\Omega,\mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta}. \tag{7.4.139}$$

Introducing the compact subset of Ω given by $K := \overline{B\big(z, \frac{1}{2}\,\mathrm{dist}(z,\partial\Omega)\big)}$, we may conclude from [68, Lemma 8.3.7], (7.4.136), (7.3.18)-(7.3.19), (7.4.129), and [68, (8.2.26)] that

$$\mathcal{N}_\kappa^{\Omega\setminus K}\vec{F} \in L^1(\partial\Omega,\sigma). \tag{7.4.140}$$

Furthermore, from (7.4.136), the second line in (7.4.128), and [68, (8.9.10)-(8.9.11)] we conclude that

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega \tag{7.4.141}$$

and at σ-a.e. point $\zeta \in \partial_{\mathrm{nta}}\Omega$ we have

$$\begin{aligned}\nu(\zeta)\cdot\Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(\zeta) &= -\frac{1}{2}\Big\langle \nu^{0,1}(\zeta)\wedge\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(\zeta), \bar{\partial}_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}} \\ &\quad + \frac{1}{2}\Big\langle \nu^{1,0}(\zeta)\vee\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(\zeta), \vartheta_\zeta\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}} \\ &\quad + \frac{1}{2}\Big\langle \big((\bar{\partial}u)\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \nu^{0,1}(\zeta)\wedge\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}} \\ &\quad - \frac{1}{2}\Big\langle \big((\vartheta u)\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \nu^{1,0}(\zeta)\vee\Gamma_{\alpha,\beta}(\zeta,z)\Big\rangle_{\mathbb{C}}.\end{aligned} \tag{7.4.142}$$

Next, condition (7.4.131) together with (7.3.18)-(7.3.19) ensures that

if the open set $\Omega \subseteq \mathbb{R}^{2n}$ is an exterior domain, then the vector field $\vec{F}$ satisfies the growth condition [68, (1.4.8)]. (7.4.143)

Collectively, (7.4.137), (7.4.139), and (7.4.141) imply that the vector field $\vec{F}$ satisfies the hypotheses of [68, Theorem 1.4.1]. On account of [68, (4.6.19)], (7.4.138), and (7.4.142), the Divergence Formula recorded in [68, (1.4.6)] currently yields (7.4.130). Lastly, formula (7.4.132) is established similarly (the proof is actually simpler since for $z \in \mathbb{C}^n \setminus \overline{\Omega}$ the form $\Gamma_{\alpha,\beta}(\cdot,z)$ has coefficients in $\mathscr{C}^\infty(\Omega)$). □

Before going any further, we wish to comment on the nature of the integral representation formula (7.4.130). Specifically, this involves several integral operators which are worth singling out. First, we have what may be considered a `complex double layer operator`, whose action on a form $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta}$ is defined at each $z \in \Omega$ as

$$\mathcal{D}_{\alpha,\beta} f(z) := \frac{1}{2} \int\limits_{\partial_*\Omega} \Big\langle f(\zeta), \nu^{0,1}(\zeta) \wedge \vartheta_\zeta \Gamma_{\alpha,\beta}(\zeta, z) - \nu^{1,0}(\zeta) \vee \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta, z) \Big\rangle_{\mathbb{C}} \, d\sigma(\zeta). \tag{7.4.144}$$

In particular, with $\Delta := \partial_1^2 + \cdots + \partial_{2n}^2$ denoting the Laplacian in $\mathbb{R}^{2n}$, we have (compare with [70, (2.5.205)])

$$\mathcal{D}_{\alpha,\beta} f \in \mathscr{C}^\infty(\Omega) \otimes \Lambda^{\alpha,\beta} \text{ and } \Delta\big(\mathcal{D}_{\alpha,\beta} f\big) = 0 \text{ in } \Omega. \tag{7.4.145}$$

Also, corresponding to $\alpha = \beta = 0$,

$$\mathcal{D}_{0,0} = \mathcal{B}_{0,0}. \tag{7.4.146}$$

Second, for suitable differential forms $f : \partial_*\Omega \to \Lambda^{\alpha,\beta}\mathbb{C}^n$ let us define the boundary-to-domain single layer potential operator

$$\mathscr{S}_{\alpha,\beta} f(z) := \int_{\partial_*\Omega} \big\langle f(\zeta), \Gamma_{\alpha,\beta}(\zeta, z) \big\rangle_{\mathbb{C}} \, d\sigma(\zeta), \qquad \forall z \in \Omega, \tag{7.4.147}$$

Its boundary-to-boundary version, i.e.,

$$S_{\alpha,\beta} f(z) := \int_{\partial_*\Omega} \big\langle f(\zeta), \Gamma_{\alpha,\beta}(\zeta, z) \big\rangle_{\mathbb{C}} \, d\sigma(\zeta), \qquad z \in \partial\Omega. \tag{7.4.148}$$

is going to be important later on (in the proof of Theorem 7.4.6). Finally, in the same context as above, define the complex volume (Newtonian) potential operator acting on suitable differential forms $U : \Omega \to \Lambda^{\alpha,\beta}\mathbb{C}^n$ according to

$$\Pi_{\alpha,\beta} U(z) := -\frac{1}{2} \int_\Omega \big\langle U(\zeta), \Gamma_{\alpha,\beta}(\zeta, z) \big\rangle_{\mathbb{C}} \, d\mathcal{L}^{2n}(\zeta), \qquad z \in \Omega. \tag{7.4.149}$$

In this notation, the integral representation formula (7.4.130) may be succinctly recast as

$$\begin{aligned} u(z) &= \mathcal{D}_{\alpha,\beta}\big(u\big|^{\text{n.t.}}_{\partial\Omega}\big)(z) \\ &\quad - \mathscr{S}_{\alpha,\beta}\Big(\tfrac{1}{2}\nu^{0,1} \wedge \big((\vartheta u)\big|^{\text{n.t.}}_{\partial\Omega}\big) - \tfrac{1}{2}\nu^{1,0} \vee \big((\bar{\partial} u)\big|^{\text{n.t.}}_{\partial\Omega}\big)\Big)(z) \\ &\quad + \Pi_{\alpha,\beta}(\Delta u)(z). \end{aligned} \tag{7.4.150}$$

The density-form on which the single layer is acting in the second line above, i.e.,

$$\tfrac{1}{2}\nu^{0,1}\wedge\big((\vartheta u)\big|^{\text{n.t.}}_{\partial\Omega}\big)-\tfrac{1}{2}\nu^{1,0}\vee\big((\bar\partial u)\big|^{\text{n.t.}}_{\partial\Omega}\big)$$
$$=(-\mathrm{i})\,\mathrm{Sym}(\bar\partial;\nu)\big((\vartheta u)\big|^{\text{n.t.}}_{\partial\Omega}\big)+(-\mathrm{i})\,\mathrm{Sym}(\vartheta;\nu)\big((\bar\partial u)\big|^{\text{n.t.}}_{\partial\Omega}\big) \tag{7.4.151}$$

is precisely the co-normal derivative associated with the factorization of the complex Laplacian $\Box$ as $\bar\partial\vartheta+\vartheta\bar\partial$ (cf. (7.3.14)) acting on u. In this vein, it is worth noting that the integral kernel of the complex double layer (7.4.144) is precisely this conormal derivative acting on $\Gamma_{\alpha,\beta}(\zeta,z)$ in the variable ζ. Ultimately, formula (7.4.150) is the natural analogue of Green's third identity discussed in [70, Theorem 1.5.1] (cf. [70, (1.5.4)]) for the complex Laplacian.

In our next proposition we prove energy identities for the complex Laplacian $\Box$ in a very general geometric setting.

Proposition 7.4.5 *Suppose $\Omega\subseteq\mathbb{R}^{2n}\equiv\mathbb{C}^n$ is a bounded open set with a lower Ahlfors regular boundary, and with the property that $\sigma:=\mathcal{H}^{2n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Pick a pair of arbitrary degrees $\alpha,\beta\in\{0,1,\dots,n\}$ and an aperture parameter $\kappa>0$. In this context, suppose the differential form $u\in\mathscr{C}^\infty(\Omega)\otimes\Lambda^{\alpha,\beta}$ satisfies:*

$$\begin{gathered}\Delta u=0\ \text{ in }\ \Omega,\qquad \mathcal{N}_\kappa u\in L^{(2n-1)/(n-1)}(\partial\Omega,\sigma),\\ u\big|^{\kappa-\text{n.t.}}_{\partial\Omega},\ (\bar\partial u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega},\ (\vartheta u)\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\ \text{ exist } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega,\\ \text{and }\ \mathcal{N}_\kappa(\bar\partial u),\ \mathcal{N}_\kappa(\vartheta u)\in L^{(2n-1)/n}(\partial\Omega,\sigma).\end{gathered} \tag{7.4.152}$$

Then for any other specified aperture parameter $\kappa'\in(0,\infty)$ the nontangential boundary traces $u\big|^{\kappa'-\text{n.t.}}_{\partial\Omega}$, $(\bar\partial u)\big|^{\kappa'-\text{n.t.}}_{\partial\Omega}$, and $(\vartheta u)\big|^{\kappa'-\text{n.t.}}_{\partial\Omega}$ exist at σ-a.e. point on $\partial_{\text{nta}}\Omega$ and are actually independent of κ'. Moreover, with the dependence on the parameter κ' dropped, the following energy identity, involving absolutely convergent integrals, holds:

$$\begin{aligned}\int_\Omega\Big\{\big|\bar\partial u\big|^2_{\mathbb{C}}+\big|\vartheta u\big|^2_{\mathbb{C}}\Big\}\,d\mathcal{L}^{2n}&=-\frac{1}{2}\int_{\partial_*\Omega}\Big\langle\nu^{1,0}\vee(\bar\partial u)\big|^{\text{n.t.}}_{\partial\Omega},u\big|^{\text{n.t.}}_{\partial\Omega}\Big\rangle_{\mathbb{C}}\,d\sigma\\&\quad+\frac{1}{2}\int_{\partial_*\Omega}\Big\langle\nu^{0,1}\wedge(\vartheta u)\big|^{\text{n.t.}}_{\partial\Omega},u\big|^{\text{n.t.}}_{\partial\Omega}\Big\rangle_{\mathbb{C}}\,d\sigma.\end{aligned} \tag{7.4.153}$$

Proof To get started, observe that $(2n-1)/(n-1)$ and $(2n-1)/n$ are Hölder conjugate exponents and that, thanks to [68, (8.6.51)], the last line in (7.4.152) implies

$$\bar\partial u\in L^2(\Omega,\mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta+1},\qquad \vartheta u\in L^2(\Omega,\mathcal{L}^{2n})\otimes\Lambda^{\alpha,\beta-1}. \tag{7.4.154}$$

To proceed, introduce the first-order differential operators

$$D := \begin{pmatrix} \bar\partial \\ \vartheta \end{pmatrix} \quad \text{and} \quad \widetilde{D} := \big(\vartheta, \bar\partial\big). \tag{7.4.155}$$

Then (7.3.14)-(7.3.16) imply that

$$L := \widetilde{D}D = \vartheta\bar\partial + \bar\partial\vartheta = \square = -\tfrac{1}{2}\Delta. \tag{7.4.156}$$

Moreover, choosing $w := \bar u$ it follows from (7.2.3) and (7.2.17) that

$$\begin{aligned}
&\Big\langle (-\mathrm{i})\mathrm{Sym}(\widetilde{D};\nu)\big((Du)\big|^{\text{n.t.}}_{\partial\Omega}\big)\,,\, w\big|^{\text{n.t.}}_{\partial\Omega}\Big\rangle \\
&\qquad = \left\langle \Big(-\tfrac{1}{2}\nu^{1,0}\vee\cdot, \tfrac{1}{2}\nu^{0,1}\wedge\cdot\Big)\begin{pmatrix} (\bar\partial u)\big|^{\text{n.t.}}_{\partial\Omega} \\ (\vartheta u)\big|^{\text{n.t.}}_{\partial\Omega}\end{pmatrix}, u\big|^{\text{n.t.}}_{\partial\Omega}\right\rangle_{\mathbb{C}} \\
&\qquad = \Big\langle -\tfrac{1}{2}\nu^{1,0}\vee(\bar\partial u)\big|^{\text{n.t.}}_{\partial\Omega} + \tfrac{1}{2}\nu^{0,1}\wedge(\vartheta u)\big|^{\text{n.t.}}_{\partial\Omega}\,,\, u\big|^{\text{n.t.}}_{\partial\Omega}\Big\rangle_{\mathbb{C}}.
\end{aligned} \tag{7.4.157}$$

In addition, from (7.4.155) and (7.2.16) we see that

$$\widetilde{D}^\top = \begin{pmatrix} \vartheta^\top \\ \bar\partial^\top \end{pmatrix} = \begin{pmatrix} \partial \\ \bar\vartheta \end{pmatrix}, \tag{7.4.158}$$

hence with the same choice of w as above we have

$$\big\langle Du, \widetilde{D}^\top w\big\rangle = \left\langle \begin{pmatrix} \bar\partial u \\ \vartheta u \end{pmatrix}, \overline{\begin{pmatrix} \bar\partial u \\ \vartheta u \end{pmatrix}} \right\rangle_{\mathbb{C}} = \big|\bar\partial u\big|^2_{\mathbb{C}} + \big|\vartheta u\big|^2_{\mathbb{C}}. \tag{7.4.159}$$

With these identifications in hand, all desired conclusions now follow directly from [70, Theorem 1.7.18]. This finishes the proof of the proposition. □

Pressing on, recall the complex double layer and single layer operators, $\mathcal{D}_{\alpha,\beta}$ and $\mathscr{S}_{\alpha,\beta}$, $S_{\alpha,\beta}$, introduced in (7.4.144) and (7.4.147)-(7.4.148), respectively. The point of our next theorem is that even though the kernel of the complex double layer $\mathcal{D}_{\alpha,\beta}$ is merely harmonic in the variable z, having a form u reproduced by $\mathcal{D}_{\alpha,\beta}$ acting on its nontangential boundary trace turns out to be equivalent to having u simultaneously $\bar\partial$-closed and ϑ-closed.

Theorem 7.4.6 *Fix $n \in \mathbb{N}$ with $n \geq 2$ and suppose Ω is a bounded* UR *domain in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$. Abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$, select $\alpha, \beta \in \{0, 1, \dots, n\}$, and pick some $\kappa > 0$. In this context, assume $u \in \mathscr{C}^1(\Omega) \otimes \Lambda^{\alpha,\beta}$ is a differential form such that $\mathcal{N}_\kappa u \in L^{(2n-1)/(n-1)}(\partial\Omega, \sigma)$ and $u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists σ-a.e. on $\partial\Omega$. Then*

$$\left.\begin{array}{r} \mathcal{N}_\kappa(\bar\partial u),\ \mathcal{N}_\kappa(\vartheta u) \in L^{(2n-1)/n}(\partial\Omega, \sigma) \\ u = \mathcal{D}_{\alpha,\beta}\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big) \text{ at each point in } \Omega \end{array}\right\} \Longleftrightarrow \bar\partial u = 0 \text{ and } \vartheta u = 0 \text{ in } \Omega. \tag{7.4.160}$$

As a corollary (corresponding to the scalar case, when $\alpha = \beta = 0$), if $F \in \mathscr{C}^1(\Omega)$ is a complex-valued function with the property that $\mathcal{N}_\kappa F \in L^{(2n-1)/(n-1)}(\partial\Omega, \sigma)$ and $F\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists σ-a.e. on $\partial\Omega$, then

$$\left.\begin{array}{l}\mathcal{N}_\kappa(\bar\partial F) \in L^{(2n-1)/n}(\partial\Omega, \sigma)\\ \text{and } F = \mathscr{B}_{0,0}\big(F\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) \text{ in } \Omega\end{array}\right\} \Longleftrightarrow F \text{ is holomorphic in } \Omega. \tag{7.4.161}$$

Proof Consider the right-pointing implication in (7.4.160). The fact that, by assumption,

$$u = \mathcal{D}_{\alpha,\beta}\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) \text{ in } \Omega \tag{7.4.162}$$

implies (cf. (7.4.145)) that u is harmonic in Ω. Granted this and the hypotheses made in (7.4.160), we may invoke [70, (3.1.148)] to conclude that

$$(\bar\partial u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}},\ (\vartheta u)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial\Omega. \tag{7.4.163}$$

As such, all hypotheses of Theorem 7.4.4 are satisfied and, with the dependence on the aperture parameter κ dropped, at each point $z \in \Omega$ the integral representation formula (7.4.130) currently becomes

$$\begin{aligned} u(z) = &-\frac{1}{2}\int_{\partial\Omega}\Big\langle \nu^{0,1}(\zeta) \wedge \big(u\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \bar\partial_\zeta \Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}}\, d\sigma(\zeta)\\ &+\frac{1}{2}\int_{\partial\Omega}\Big\langle \nu^{1,0}(\zeta) \vee \big(u\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \vartheta_\zeta \Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}}\, d\sigma(\zeta)\\ &+\frac{1}{2}\int_{\partial\Omega}\Big\langle \big((\bar\partial u)\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \nu^{0,1}(\zeta) \wedge \Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}}\, d\sigma(\zeta)\\ &-\frac{1}{2}\int_{\partial\Omega}\Big\langle \big((\vartheta u)\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \nu^{1,0}(\zeta) \vee \Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}}\, d\sigma(\zeta). \end{aligned} \tag{7.4.164}$$

Contrasting this with (7.4.162) then leads to the conclusion that

$$\begin{aligned} &\frac{1}{2}\int_{\partial\Omega}\Big\langle \big((\bar\partial u)\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \nu^{0,1}(\zeta) \wedge \Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}}\, d\sigma(\zeta)\\ &\qquad -\frac{1}{2}\int_{\partial\Omega}\Big\langle \big((\vartheta u)\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)(\zeta), \nu^{1,0}(\zeta) \vee \Gamma_{\alpha,\beta}(\zeta, z)\Big\rangle_{\mathbb{C}}\, d\sigma(\zeta) = 0 \end{aligned} \tag{7.4.165}$$

at each point $z \in \Omega$. In terms of (7.4.147) this translates into saying that

$$\mathscr{S}_{\alpha,\beta} f = 0 \text{ in } \Omega, \tag{7.4.166}$$

where we have abbreviated

$$f := \frac{1}{2}\nu^{1,0} \vee \left((\bar{\partial} u)\Big|_{\partial\Omega}^{\text{n.t.}} \right) - \frac{1}{2}\nu^{0,1} \vee \left((\vartheta u)\Big|_{\partial\Omega}^{\text{n.t.}} \right) \in L^{(2n-1)/n}(\partial\Omega, \sigma) \otimes \Lambda^{\alpha,\beta}. \tag{7.4.167}$$

Going nontangentially to the boundary in (7.4.166) then proves (cf. [70, Proposition 2.5.39]) that, on the one hand,

$$S_{\alpha,\beta} f = 0 \text{ on } \partial\Omega. \tag{7.4.168}$$

On the other hand, by arguing as in the proof of [76, Proposition 3.5, p. 126] (while keeping part *(7)* of [68, Lemma 5.10.9] in mind) we conclude that the operator

$$\begin{gathered} S_{\alpha,\beta} : L^{(2n-1)/n}(\partial\Omega, \sigma) \otimes \Lambda^{\alpha,\beta} \longrightarrow L^{(2n-1)/(n-1)}(\partial\Omega, \sigma) \otimes \Lambda^{\alpha,\beta} \\ \text{is well defined, linear, bounded, injective, with dense range.} \end{gathered} \tag{7.4.169}$$

Collectively, (7.4.167)-(7.4.169) imply that $f = 0$, i.e.,

$$\frac{1}{2}\nu^{1,0} \vee \left((\bar{\partial} u)\Big|_{\partial\Omega}^{\text{n.t.}} \right) - \frac{1}{2}\nu^{0,1} \vee \left((\vartheta u)\Big|_{\partial\Omega}^{\text{n.t.}} \right) = 0 \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{7.4.170}$$

The working assumptions on u guarantee that the hypotheses of Proposition 7.4.5 are satisfied. As such, we may write the energy identity (7.4.153) which, given (7.4.170), simply reduces to

$$\int_\Omega \left\{ \left|\bar{\partial} u\right|_{\mathbb{C}}^2 + \left|\vartheta u\right|_{\mathbb{C}}^2 \right\} d\mathcal{L}^{2n} = 0. \tag{7.4.171}$$

In turn, this forces $\bar{\partial} u = 0$ and $\vartheta u = 0$ in Ω, finishing the proof of the right-pointing implication in (7.4.160).

As regards the left-pointing implication in (7.4.160), since the differential u now satisfies the hypotheses of Theorem 7.4.1, we may write the integral representation formula (7.4.3). Given that we are presently assuming $\bar{\partial} u = 0$ and $\vartheta u = 0$ in Ω, the latter reduces precisely to (7.4.162). Having established this, all other desired conclusions are immediate. □

7.5 The Extension Problem for Hölder CR-Functions on Boundaries of Ahlfors Regular Domains

As a prelude to discussing the Extension Problem for Hölder CR-functions on boundaries of Ahlfors regular domains, we shall first study the action of the Bochner-Martinelli integral operator in such a context. To set the stage, we make the following observation. Let $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ be a set of locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. In this setting, for each σ-measurable, complex-valued, function defined on $\partial_*\Omega$, satisfying

$$\int_{\partial_*\Omega} \frac{|f(\zeta)|}{1+|\zeta|^{2n-1}}\,\mathrm{d}\sigma(\zeta) < +\infty, \tag{7.5.1}$$

the scar case of formula (7.3.33) (i.e., when $\alpha = \beta = 0$) corresponds to

$$\begin{aligned}\mathcal{B}_{0,0}f(z) &= -\frac{1}{2}\int_{\partial_*\Omega} \big\langle \nu^{0,1}(\zeta), \bar{\partial}_\zeta \Gamma_{0,0}(\zeta,z)\big\rangle_{\mathbb{C}} f(\zeta)\,\mathrm{d}\sigma(\zeta) \\ &= -\int_{\partial_*\Omega} \sum_{j=1}^{n} (\nu_{\mathbb{C}})_j(\zeta)\partial_{\zeta_j}[E_n(\zeta,z)] f(\zeta)\,\mathrm{d}\sigma(\zeta) \\ &= \frac{1}{\omega_{2n-1}}\int_{\partial_*\Omega} \sum_{j=1}^{n} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} (\nu_{\mathbb{C}})_j(\zeta) f(\zeta)\,\mathrm{d}\sigma(\zeta) \\ &= \frac{1}{\omega_{2n-1}}\int_{\partial_*\Omega} \frac{\big\langle \nu_{\mathbb{C}}(\zeta), \zeta - z\big\rangle_{\mathbb{C}}}{|z-\zeta|^{2n}} f(\zeta)\,\mathrm{d}\sigma(\zeta), \qquad \forall z \in \mathbb{C}^n \setminus \partial\Omega. \end{aligned} \tag{7.5.2}$$

Theorem 7.3.5 specialized to the case $\alpha = \beta = 0$ yields a wealth of information about the nature of the operator (7.5.2). We wish to augment this body of results by establishing the properties in the proposition below.

Proposition 7.5.1 *Suppose $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ is an open set of locally finite perimeter, and abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. Then the following properties hold.*

(i) For every σ-measurable function $f : \partial_*\Omega \to \mathbb{C}$ satisfying the integrability condition stated in (7.5.1) one has (with Δ denoting the Laplacian in $\mathbb{R}^{2n}$)

$$\mathcal{B}_{0,0}f \in \mathscr{C}^\infty(\mathbb{C}^n \setminus \partial\Omega) \text{ and } \Delta(\mathcal{B}_{0,0}f) = 0 \text{ in } \mathbb{C}^n \setminus \partial\Omega. \tag{7.5.3}$$

(ii) Under the additional assumption that $\partial\Omega$ is bounded one has

$$\mathcal{B}_{0,0}1 \equiv \begin{cases} 1 \text{ in } \Omega, \text{ if } \Omega \text{ is bounded,} \\ 0 \text{ in } \Omega, \text{ if } \Omega \text{ is unbounded.} \end{cases} \tag{7.5.4}$$

(iii) Under the additional assumption that $\partial\Omega$ is bounded and $\partial_*\Omega$ is upper Ahlfors regular, it follows that for each $\alpha \in (0,1)$ there exists some $C \in (0,\infty)$, depending only on n, α, diam$(\partial_*\Omega)$*, and the upper Ahlfors regularity constant of $\partial_*\Omega$, such that for every function $f \in \mathscr{C}^\alpha(\partial_*\Omega)$ one has*

$$\sup_{z\in\Omega} \big|(\mathcal{B}_{0,0}f)(z)\big| + \sup_{z\in\Omega}\Big\{\mathrm{dist}(z,\partial_*\Omega)^{1-\alpha}\big|\nabla(\mathcal{B}_{0,0}f)(z)\big|\Big\} \le C\|f\|_{\mathscr{C}^\alpha(\partial_*\Omega)}. \tag{7.5.5}$$

In particular, if $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ is a uniform domain whose boundary is compact and upper Ahlfors regular then $\mathcal{B}_{0,0}$ induces a well-defined, linear, and bounded operator in the context

$$\mathcal{B}_{0,0} : \mathscr{C}^\alpha(\partial_*\Omega) \longrightarrow \mathscr{C}^\alpha\big(\overline{\Omega}\big), \qquad \forall \alpha \in (0,1). \tag{7.5.6}$$

As a corollary, if $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ *is simultaneously a uniform domain, and an Ahlfors regular domain with compact boundary, then for each* $\alpha \in (0, 1)$ *the operator* $\mathscr{B}_{0,0}$ *is well defined, linear, and bounded in the context*

$$\mathscr{B}_{0,0} : \mathscr{C}^{\alpha}(\partial\Omega) \longrightarrow \mathscr{C}^{\alpha}\big(\overline{\Omega}\big). \tag{7.5.7}$$

(iv) Strengthen the local finite perimeter assumption on Ω *by demanding that* $\partial\Omega$ *is a* UR *set. Then for each function* f *belonging to* $L^p_1(\partial_*\Omega, \sigma)$ *with* $p \in [1, \infty)$ *(the* L^p*-based Sobolev space of order one on* $\partial_*\Omega$*, relative to the ambient* $\mathbb{R}^{2n}$*; cf.* [69, Chapter 11]) *one has*

$$\big\|\mathcal{N}_\kappa\big(\nabla\mathscr{B}_{0,0}f\big)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p_1(\partial_*\Omega,\sigma)} \text{ if } 1 < p < \infty, \tag{7.5.8}$$

$$\big\|\mathcal{N}_\kappa\big(\nabla\mathscr{B}_{0,0}f\big)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \leq C\|f\|_{L^1_1(\partial_*\Omega,\sigma)} \text{ if } p = 1, \tag{7.5.9}$$

where ∇ *denotes the gradient operator in the ambient* $\mathbb{R}^{2n}$ *and* $C \in (0, \infty)$ *is a constant independent of* f*.*

(v) Under the additional assumption that $\partial_*\Omega$ *is an upper Ahlfors regular set, given any function* $f \in L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma)$ *with* $p \in [1, \infty)$ *and any index* $j \in \{1, \dots, n\}$*, for every* $z \in \mathbb{C}^n \setminus \partial\Omega$ *one has*

$$\partial_{\bar z_j}\big(\mathscr{B}_{0,0}f\big)(z) = \frac{1}{\omega_{2n-1}}\int_{\partial_*\Omega}\sum_{k=1}^{n}\frac{\bar\zeta_k - \bar z_k}{|\zeta - z|^{2n}}\big(\partial^{\mathbb{C}}_{\tau_{jk}}f\big)(\zeta)\,\mathrm{d}\sigma(\zeta). \tag{7.5.10}$$

As a corollary of (7.5.10)*, item (3) in* [70, Theorem 2.4.1]*, and* [70, Theorem 2.5.1]*, if* $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ *is open and* $\partial\Omega$ *is a* UR *set then for each* $f \in L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma)$ *with* $p \in [1, \infty)$ *and each* $\kappa > 0$ *the nontangential boundary trace*

$$\partial_{\bar z_j}\big(\mathscr{B}_{0,0}f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exists at } \sigma\text{-a.e. point on } \partial_*\Omega \text{ for each index } j \in \{1, \dots, n\}, \tag{7.5.11}$$

and there exists a constant $C \in (0, \infty)$*, independent of* f*, with the property that*

$$\sum_{j=1}^{n}\big\|\mathcal{N}_\kappa\big(\partial_{\bar z_j}\mathscr{B}_{0,0}f\big)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)} \text{ if } 1 < p < \infty, \tag{7.5.12}$$

$$\sum_{j=1}^{n}\big\|\mathcal{N}_\kappa\big(\partial_{\bar z_j}\mathscr{B}_{0,0}f\big)\big\|_{L^1(\partial\Omega,\sigma)} \leq C\|f\|_{L^{1,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)} \text{ if } p = 1. \tag{7.5.13}$$

(vi) Make the additional assumption that $\partial_*\Omega$ *is an upper Ahlfors regular set, and fix a function*

$$f \in L^p(\partial_*\Omega, \sigma) \text{ with } p \in [1,\infty) \text{ such that } \\ \partial^{\mathbb{C}}_{\tau_{jk}} f = 0 \text{ on } \partial_*\Omega \text{ for each } j,k \in \{1,\dots,n\}. \tag{7.5.14}$$

Then

$$\mathscr{B}_{0,0} f \text{ is holomorphic in } \mathbb{C}^n \setminus \partial\Omega \tag{7.5.15}$$

and

$$\text{if one also assumes that } \partial\Omega \text{ is bounded and } n > 1, \text{ then the function } \mathscr{B}_{0,0} f \text{ vanishes identically in the unbounded connected component of the set } \mathbb{C}^n \setminus \partial\Omega. \tag{7.5.16}$$

(vii) Under the additional assumption that $\partial\Omega$ is a compact UR *set, for each exponent $p \in (1,\infty)$ there exists some constant $C \in (0,\infty)$ with the property that for each function $f \in \mathrm{BMO}(\partial\Omega,\sigma)$ it follows that $\big|\nabla(\mathscr{B}_{0,0} f)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^{2n}$ is a Carleson measure in Ω in the quantitative sense that*

$$\sup_{z\in\partial\Omega,\, r>0} \frac{1}{\sigma\big(B(z,r)\cap\partial\Omega\big)} \int_{B(z,r)\cap\Omega} \big|\nabla(\mathscr{B}_{0,0} f)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^{2n} \\ \leq C\|f\|^p_{\mathrm{BMO}(\partial\Omega,\sigma)}. \tag{7.5.17}$$

Moreover, for each $p \in (1,\infty)$ and $f \in \mathrm{VMO}(\partial\Omega,\sigma)$ it follows that

$$\big|\nabla(\mathscr{B}_{0,0} f)\big|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^{2n} \\ \text{is a vanishing Carleson measure in } \Omega. \tag{7.5.18}$$

Proof Starting with (7.5.2) and unraveling definitions we see that

$$\mathscr{B}_{0,0} = \mathscr{D} + \mathrm{i}\sum_{j=1}^{n} \mathscr{R}_{(2j-1)(2j)} \tag{7.5.19}$$

where $\mathscr{D}$ is the (boundary-to-domain) harmonic double layer potential operator associated with the present domain Ω as in [70, Definition 2.5.17], and each $\mathscr{R}_{(2j-1)(2j)}$ is the operator associated with the given domain Ω as in (A.0.189). Granted this, all claims in items *(i)-(iii)* become direct consequences of [70, Lemmas 2.5.18, 2.5.19], and [70, Proposition 2.5.26]. Moreover, the claim in item *(vii)* also follows from (7.5.19), in light of [70, Propositions 2.5.24, 2.5.26].

Consider next the claim in item *(iv)*. Having fixed a function $f \in L^p_1(\partial_*\Omega,\sigma)$ with $p \in [1,\infty)$, it follows from (7.5.19) that for each $\ell \in \{1,\dots,2n\}$ the (real) spatial partial derivative $\partial_\ell \mathscr{B}_{0,0} f$ may be expressed as

$$\partial_\ell \mathscr{B}_{0,0} f = \partial_\ell \mathscr{D} f + \mathrm{i}\sum_{j=1}^{n} \partial_\ell \mathscr{R}_{(2j-1)(2j)} f \text{ in } \Omega. \tag{7.5.20}$$

On the other hand, from (A.0.189) and [69, Lemma 11.1.7] we see that for each $j, k, \ell \in \{1, \dots, 2n\}$ we have

$$\partial_\ell \mathcal{R}_{jk} f(z) = -\int_{\partial_*\Omega} (\partial_\ell E_\Delta)(z-\zeta)(\partial_{\tau_{jk}} f)(\zeta)\, d\sigma(\zeta), \qquad \forall z \in \Omega, \tag{7.5.21}$$

where E_Δ denotes the standard fundamental solution for the Laplacian in $\mathbb{R}^{2n}$ (defined as in (A.0.65) with n replaced by $2n$), and $\partial_{\tau_{jk}}$ is the real tangential partial differential operator (defined as in [69, Chapter 11] relative to the ambient $\mathbb{R}^{2n}$). Then the estimates in (7.5.8)-(7.5.9) are consequences of (7.5.20)-(7.5.21), estimate (1.5.8) and the subsequent comment in Theorem 1.5.1 (applied to the harmonic double layer introduced in [70, Definition 2.5.17]), and item *(3)* in [70, Theorem 2.4.1] (which provides nontangential maximal function estimates for the integral operators in the right-hand side of (7.5.21)).

Next we deal with the claim made in item *(v)*. To this end, having fixed an arbitrary function $f \in L^{p,1}_{\mathbb{C}}(\partial_*\Omega, \sigma)$ with $p \in [1, \infty)$, along with some $j \in \{1, \dots, n\}$, for each $z \in \mathbb{C}^n \setminus \partial\Omega$ we may compute

$$\begin{aligned}
\partial_{\bar{z}_j}\big(\mathcal{B}_{0,0} f\big)(z) &= -\int_{\partial_*\Omega} \sum_{k=1}^{n} (\nu_{\mathbb{C}})_k(\zeta)\partial_{\bar{z}_j}\partial_{\zeta_k}[E_n(\zeta,z)] f(\zeta)\, d\sigma(\zeta) \\
&= \int_{\partial_*\Omega} \sum_{k=1}^{n} (\nu_{\mathbb{C}})_k(\zeta)\partial_{\bar{\zeta}_j}\partial_{\zeta_k}[E_n(\zeta,z)] f(\zeta)\, d\sigma(\zeta) \\
&= \int_{\partial_*\Omega} \sum_{k=1}^{n} \partial^{\mathbb{C}}_{\tau_{kj}(\zeta)}\Big[\partial_{\zeta_k}[E_n(\zeta,z)]\Big] f(\zeta)\, d\sigma(\zeta) \\
&= \int_{\partial_*\Omega} \sum_{k=1}^{n} \partial_{\zeta_k}[E_n(\zeta,z)]\big(\partial^{\mathbb{C}}_{\tau_{jk}} f\big)(\zeta)\, d\sigma(\zeta).
\end{aligned} \tag{7.5.22}$$

Above, the first equality is obtained by differentiating under the integral sign in (7.5.2), the second equality is seen from (7.3.18), the third equality uses (7.1.5) and the fact that

$$\sum_{k=1}^{n} \partial_{\bar{\zeta}_k}\partial_{\zeta_k}[E_n(\zeta,z)] = \Delta_\zeta[E_n(\zeta,z)] = 0, \tag{7.5.23}$$

and the final equality is a consequence of Proposition 7.1.3 (whose applicability is presently ensured by the assumption that $\partial_*\Omega$ is upper Ahlfors regular). This justifies (7.5.22), and (7.5.10) readily follows from it on account of (7.3.18).

Let us now deal with the claims in item *(vi)*, under the additional assumption that $\partial_*\Omega$ is upper Ahlfors regular. First, that the function $\mathcal{B}_{0,0} f$ is holomorphic in $\mathbb{C}^n \setminus \partial\Omega$ if f is as in (7.5.14) is clear from item *(v)*. Second, to prove the claim in

(7.5.16), let us also assume that $\partial\Omega$ is bounded and that $n \geq 2$. With f as in (7.5.14), abbreviate $F := \mathscr{B}_{0,0} f$. Then F is holomorphic in $\mathbb{C}^n \setminus \partial\Omega$ and, as is apparent from (7.5.2), decays at infinity. Consider next a number $R \in (0, \infty)$ which is sufficiently large so that if

$$Q := B(0, R)^n := B(0, R) \times \cdots \times B(0, R) \subseteq \mathbb{C}^n \tag{7.5.24}$$

(where each factor in the Cartesian product is the origin-centered ball of radius R in $\mathbb{C}$), then $\partial\Omega \subseteq Q$. Pick an index $j \in \{1, \dots, n\}$ and fix a point $z_j^* \in \mathbb{C} \setminus \overline{B(0, R)}$, then define the function $G_j : \mathbb{C}^{n-1} \to \mathbb{C}$ by setting

$$\begin{aligned} G_j(z_1, \dots, z_{n-1}) &:= F(z_1, \dots, z_{j-1}, z_j^*, z_{j+1}, \dots, z_{n-1}) \\ &\text{for each } (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}. \end{aligned} \tag{7.5.25}$$

Note that $(z_1, \dots, z_{j-1}, z_j^*, z_{j+1}, \dots, z_{n-1}) \notin Q$ for each $(z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$ which, in turn, implies $(z_1, \dots, z_{j-1}, z_j^*, z_{j+1}, \dots, z_{n-1}) \in \mathbb{C}^n \setminus \partial\Omega$. This ensures that G_j is well defined. Also, by design, G_j is holomorphic in $\mathbb{C}^{n-1}$ and decays at infinity. Granted these (and bearing in mind that $n \geq 2$), Liouville's Theorem applies and yields that $G_j \equiv 0$ in $\mathbb{C}^{n-1}$. Given the arbitrariness of $j \in \{1, \dots, n\}$ and $z_j^* \in \mathbb{C} \setminus \overline{B(0, R)}$, this ultimately translates into saying that $F \equiv 0$ in $\mathbb{C}^n \setminus \overline{Q}$. Denote by U the unbounded connected component of the set $\mathbb{C}^n \setminus \partial\Omega$. Since U overlaps with $\mathbb{C}^n \setminus \overline{Q}$ on a nonempty open set, unique continuation implies that F necessarily vanishes in U. This concludes the proof of (7.5.16).

Lastly, the claims in the current item *(viii)* follows from items *(4)-(5)* in Theorem 1.8.2 bearing in mind that, as discussed in Example 1.4.16, the Bochner-Martinelli integral operator is a particular example of a double layer potential operator associated with a certain factorization of the Laplacian in $\mathbb{R}^{2n}$. □

Let $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ be a set of locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. In this (or a possibly more restrictive) geometric context, we would like to make sense of the principal-value Bochner-Martinelli integral operator acting on any function $f \in L^1\big(\partial_*\Omega, \frac{\sigma}{1+|\cdot|^{2n-1}}\big)$ according to:

$$B_{0,0}f(z) = -\lim_{\varepsilon\to0^+}\frac{1}{2}\int\limits_{\substack{\zeta\in\partial_*\Omega\\|z-\zeta|>\varepsilon}}\big\langle\nu^{0,1}(\zeta),\bar\partial_\zeta\Gamma_{0,0}(\zeta,z)\big\rangle_{\mathbb{C}}\,f(\zeta)\,\mathrm{d}\sigma(\zeta)$$

$$= \lim_{\varepsilon\to0^+}\frac{1}{\omega_{2n-1}}\int\limits_{\substack{\zeta\in\partial_*\Omega\\|z-\zeta|>\varepsilon}}\sum_{j=1}^{n}\frac{\bar\zeta_j-\bar z_j}{|\zeta-z|^{2n}}(\nu_{\mathbb{C}})_j(\zeta)f(\zeta)\,\mathrm{d}\sigma(\zeta) \tag{7.5.26}$$

$$= \lim_{\varepsilon\to0^+}\frac{1}{\omega_{2n-1}}\int\limits_{\substack{\zeta\in\partial_*\Omega\\|z-\zeta|>\varepsilon}}\frac{\big\langle\nu_{\mathbb{C}}(\zeta),\zeta-z\big\rangle_{\mathbb{C}}}{|z-\zeta|^{2n}}\,f(\zeta)\,\mathrm{d}\sigma(\zeta)\ \text{ for }\sigma\text{-a.e. }z\in\partial_*\Omega.$$

According to [68, Proposition 5.6.7], this limit exists and $B_{0,0}f$ is a σ-measurable function on $\partial_*\Omega$. Furthermore, the last claim in [68, Proposition 5.6.7] guarantees that if $\Omega\subseteq\mathbb{C}^n$ is a Lebesgue measurable set whose topological boundary $\partial\Omega$ is countably rectifiable (of dimension $2n-1$) and has locally finite $\mathcal{H}^{2n-1}$ measure (hence, in particular, if $\partial\Omega$ is a UR set), then for each function $f\in L^1\big(\partial_*\Omega,\frac{\sigma}{1+|\cdot|^{2n-1}}\big)$ the limit in (7.5.26) actually exists for σ-a.e. $x\in\partial\Omega$ and gives rise to a σ-measurable complex-valued function on $\partial\Omega$.

Theorem 7.3.5 (with $\alpha=\beta=0$) already provides a great deal of information about this singular integral operator, and our next result further elaborates on the properties enjoyed by the principal-value Bochner-Martinelli integral operator just considered. Before stating it we make one more definition. Specifically, in the same geometric context as above, for each $p,q\in[1,\infty]$ consider the off-diagonal (partial) Sobolev spaces

$$L^{1,p,q}_{\mathbb{C}}(\partial_*\Omega,\sigma) := \big\{f\in L^p(\partial_*\Omega,\sigma):\ \partial^{\mathbb{C}}_{\tau_{jk}}f\in L^q(\partial_*\Omega,\sigma)\ \text{ for }\ 1\le j,k\le n\big\} \tag{7.5.27}$$

equipped with the natural norm

$$\|f\|_{L^{1,p,q}_{\mathbb{C}}(\partial_*\Omega,\sigma)} := \|f\|_{L^p(\partial_*\Omega,\sigma)} + \sum_{1\le j,k\le n}\big\|\partial^{\mathbb{C}}_{\tau_{jk}}f\big\|_{L^q(\partial_*\Omega,\sigma)} \tag{7.5.28}$$

for each $f\in L^{1,p,q}_{\mathbb{C}}(\partial_*\Omega,\sigma)$. Note that, by design,

$$L^{1,p,p}_{\mathbb{C}}(\partial_*\Omega,\sigma_*) = L^{1,p}_{\mathbb{C}}(\partial_*\Omega,\sigma_*)\ \text{ for each }\ p\in[1,\infty]. \tag{7.5.29}$$

Proposition 7.5.2 *Let $\Omega\subsetneq\mathbb{R}^{2n}\equiv\mathbb{C}^n$ be a Lebesgue measurable set with a compact upper Ahlfors regular boundary and abbreviate $\sigma:=\mathcal{H}^{2n-1}\lfloor\partial\Omega$. Then for each $\alpha\in(0,1)$ the following statements are true.*

(i) For each given complex-valued function $f\in\mathscr{C}^\alpha(\partial_*\Omega)$ the limit defining the principal-value Bochner-Martinelli integral operator $B_{0,0}f(z)$ in (7.5.26) exists for σ-a.e. point $z\in\partial_*\Omega$.

(ii) After possibly redefining $B_{0,0}f$ on a σ-nullset contained in $\partial_*\Omega$, the assignment $f \mapsto B_{0,0}f$ thus defined induces a well-defined, linear, and bounded operator

$$B_{0,0} : \mathscr{C}^\alpha(\partial_*\Omega) \longrightarrow \mathscr{C}^\alpha(\partial_*\Omega) \tag{7.5.30}$$

with the property that

$$B_{0,0}1 \equiv \begin{cases} +\frac{1}{2} \text{ on } \partial_*\Omega, \text{ if } \Omega \text{ is bounded,} \\ -\frac{1}{2} \text{ on } \partial_*\Omega, \text{ if } \Omega \text{ is unbounded.} \end{cases} \tag{7.5.31}$$

(iii) Under the additional assumption that the set Ω is open and $\partial\Omega$ is also lower Ahlfors regular, the jump-formula (where I denotes the identity operator and $\kappa > 0$ is an arbitrary fixed number) is valid:

$$\begin{gathered} \text{for every complex-valued function } f \in \mathscr{C}^\alpha(\partial_*\Omega) \text{ one has} \\ \big(\mathcal{B}_{0,0}f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = (\tfrac{1}{2}I + B_{0,0})f \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega. \end{gathered} \tag{7.5.32}$$

(iv) In the case when it is also assumed that $\mathcal{H}^{2n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$, one may replace $\partial_*\Omega$ by $\partial\Omega$ in the formulation of (7.5.30), (7.5.31), and (7.5.32).

(v) Suppose the set Ω is actually an Ahlfors regular domain with compact boundary and define $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^{2n} \setminus \overline{\Omega}$. In this setting, let $\mathcal{B}^\pm_{0,0}$ be the Bochner-Martinelli integral operators associated with $\Omega_\pm$. That is, for each complex-valued function $f \in L^1(\partial_*\Omega, \sigma)$ define

$$\mathcal{B}^\pm_{0,0}f := \big[\mathcal{B}_{0,0}f\big]\Big|_{\Omega_\pm}. \tag{7.5.33}$$

Then the sets $\Omega_\pm$ are also Ahlfors regular domains with compact boundaries, $\partial(\Omega_\pm) = \partial\Omega$, and $\partial_*(\Omega_+) = \partial_*\Omega = \partial_*(\Omega_-)$. In addition, the geometric measure theoretic outward unit normals to $\Omega_\pm$ are $\pm\nu$ at σ-a.e. point on $\partial\Omega$, where ν is the geometric measure theoretic outward unit normal to Ω. Moreover, given any $\kappa > 0$,

$$\begin{gathered} \text{for every complex-valued function } f \in \mathscr{C}^\alpha(\partial\Omega) \text{ one has} \\ \big(\mathcal{B}^\pm_{0,0}f\big)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = (\pm\tfrac{1}{2}I + B_{0,0})f \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \end{gathered} \tag{7.5.34}$$

(vi) Suppose $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ is an open set with the property that

$$\partial\Omega \text{ is a UR set and } \mathcal{H}^{2n-1}\big(\partial_{\text{nta}}\Omega \setminus \partial_*\Omega\big) = 0. \tag{7.5.35}$$

Then for each $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big)$ the principal-value Bochner-Martinelli integral operator (7.5.26) is well defined. Moreover, for each $p \in (1,\infty)$ this induces a linear and bounded mapping in the following settings:

$$B_{0,0} : L_1^p(\partial_*\Omega, \sigma) \longrightarrow L_1^p(\partial_*\Omega, \sigma), \tag{7.5.36}$$

$$B_{0,0} : L_{\mathbb{C}}^{p,1}(\partial_*\Omega, \sigma) \to L_{\mathbb{C}}^{p,1}(\partial_*\Omega, \sigma). \tag{7.5.37}$$

More generally, if (7.5.35) *is satisfied then*

$$B_{0,0} : L_1^{p,q}(\partial_*\Omega, \sigma) \longrightarrow L_1^{p,q}(\partial_*\Omega, \sigma), \tag{7.5.38}$$

$$B_{0,0} : L_{\mathbb{C}}^{1,p,q}(\partial_*\Omega, \sigma) \longrightarrow L_{\mathbb{C}}^{1,p,q}(\partial_*\Omega, \sigma), \tag{7.5.39}$$

are well-defined, linear, and bounded operators whenever $p, q \in (1, \infty)$.

Proof From (7.5.26) and definitions it follows that

$$B_{0,0} = K + \mathrm{i}\sum_{j=1}^{n} R_{(2j-1)(2j)} \tag{7.5.40}$$

where K is the boundary-to-boundary (i.e., principal-value) harmonic double layer potential operator associated with the present domain Ω as in [70, (2.5.203)], and each $R_{(2j-1)(2j)}$ is the operator associated with the given domain Ω as in (A.0.190). Having noticed this, all claims in items *(i)-(iv)* become direct consequences of Proposition 7.5.2 and [70, Proposition 2.5.27]. Going further, the claims in item *(v)* readily follow from part *(6)* of [68, Lemma 5.10.9] and the current item *(iv)*.

To deal with the claims in item *(vi)*, assume Ω is an open set satisfying the properties listed in (7.5.35). Then [70, Theorem 2.3.2] implies that for each function $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big)$ the limit defining $B_{0,0}f(z)$ in (7.5.26) exists at σ-a.e. $z \in \partial\Omega$. Next, the fact that $B_{0,0}$ in (7.5.36) is well defined, linear, and bounded is a direct consequence of (7.5.40) and Theorem 1.5.1. As regards the corresponding claims for the operator $B_{0,0}$ in (7.5.37), observe first that (7.3.69) specialized to the case when $\alpha = \beta = 0$ gives that

$$B_{0,0} : L^p(\partial_*\Omega, \sigma) \longrightarrow L^p(\partial\Omega, \sigma) \text{ is well defined, linear, and bounded.} \tag{7.5.41}$$

Next, fix an arbitrary function $f \in L_{\mathbb{C}}^{p,1}(\partial_*\Omega, \sigma)$ along some $j \in \{1, \dots, n\}$ and some aperture parameter $\kappa > 0$. Then at σ-a.e. point on $\partial_*\Omega$ we may write

$$\begin{aligned} \partial^{\mathbb{C}}_{\tau_{jk}}\big(B_{0,0}f\big) &= \partial^{\mathbb{C}}_{\tau_{jk}}(\tfrac{1}{2}I + B_{0,0})f - \tfrac{1}{2}\partial^{\mathbb{C}}_{\tau_{jk}}f \\ &= (\nu_{\mathbb{C}})_j\Big[\partial_{\bar z_k}\big(\mathcal{B}_{0,0}f\big)\Big]\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - (\nu_{\mathbb{C}})_k\Big[\partial_{\bar z_j}\big(\mathcal{B}_{0,0}f\big)\Big]\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - \tfrac{1}{2}\partial^{\mathbb{C}}_{\tau_{jk}}f, \end{aligned} \tag{7.5.42}$$

where the second equality is a consequence of (7.3.82) (specialized to $\alpha = \beta = 0$ while bearing (7.1.83) in mind), and (7.1.28) in Proposition 7.1.5 used with $F := \mathcal{B}_{0,0}f$ (its present applicability being ensured by (7.5.35), item *(v)* in Proposition 7.5.1, together with items *(ii)* and *(vii)* in Theorem 7.3.5 used with $\alpha = \beta = 0$). Collectively, (7.5.41), (7.5.42), and (7.1.11)-(7.1.12) then prove that $B_{0,0}$ in (7.5.37) is indeed a well-defined, linear, and bounded operator. Finally, the claims about the operators (7.5.38)-(7.5.39) are implicit in what we have proved already. □

We momentarily digress to include the characterization of balls in terms of (Hermitian) self-adjointness of the associated Bochner-Martinelli singular integral operator in the class of bounded UR domains, given in the proposition below. This is a significant improvement over the main result in [5] where the more restrictive class of bounded $\mathscr{C}^1$ domains has been considered.

Proposition 7.5.3 *Let* $\Omega \subseteq \mathbb{C}^n$*, where* $n \geq 2$*, be a bounded* UR *domain and set* $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$*. Then the principal-value Bochner-Martinelli singular integral operator* $B_{0,0}$ *(cf.* (7.5.26)*) is Hermitian self-adjoint on* $L^2(\partial\Omega, \sigma)$ *if and only if* Ω *is a ball.*

Proof In one direction, if the operator $B_{0,0} : L^2(\partial\Omega, \sigma) \to L^2(\partial\Omega, \sigma)$ is Hermitian self-adjoint, then from (7.5.40) we conclude (upon noting that both K and the operators $R_{(2j-1)(2j)}$ with $1 \leq j \leq n$ have real-valued integral kernels) that the principal-value harmonic double layer potential operator K associated with the set $\Omega \subseteq \mathbb{C}^n \equiv \mathbb{R}^{2n}$ as in [70, (2.5.203)] is symmetric on $L^2(\partial\Omega, \sigma)$. Granted this, we may then invoke [38, Theorem 4.23] (bearing in mind [68, (5.10.52)]) to conclude that Ω is a ball. As far as the converse direction is concerned, if Ω is a ball we then trivially have $\big\langle \nu_{\mathbb{C}}(\zeta), \zeta - z\big\rangle_{\mathbb{C}} = \overline{\big\langle \nu_{\mathbb{C}}(z), z - \zeta\big\rangle_{\mathbb{C}}}$ for each $z, \zeta \in \partial\Omega$ (where ν denotes the outward unit normal to Ω). On account of (7.5.26), this goes to show that the operator $B_{0,0} : L^2(\partial\Omega, \sigma) \to L^2(\partial\Omega, \sigma)$ is Hermitian self-adjoint. □

Returning to the main topic of conversation, we shall prove that the square of the principal-value version of the Bochner-Martinelli integral operator is, up to normalization, the identity when acting on CR-functions in the context described in the next proposition.

Proposition 7.5.4 *Suppose* $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ *is an open set with the property that* $\partial\Omega$ *is a* UR *set and such that*

$$\mathcal{H}^{2n-1}\big(\partial_{{}_{\rm nta}}\Omega \setminus \partial_*\Omega\big) = 0. \tag{7.5.43}$$

Abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ *and fix an exponent* $p \in (1, \infty)$*. Then for each* CR*-function* $f \in L^p(\partial_*\Omega, \sigma)$ *one has*

$$B_{0,0}^2 f = \tfrac{1}{4} f \;\; \textit{at } \sigma\textit{-a.e. point on } \partial_*\Omega. \tag{7.5.44}$$

Proof Fix some $\kappa > 0$ and, having picked a CR-function $f \in L^p(\partial_*\Omega, \sigma)$, introduce $F := \mathscr{B}_{0,0} f$ in Ω. Then from items *(ii)*, *(vii)* in Theorem 7.3.5, and (7.5.15), we conclude that

$$\begin{aligned} &F\Big|_{\partial\Omega}^{\kappa-\rm n.t.} = \big(\tfrac{1}{2} I + B_{0,0}\big) f \;\text{ at } \sigma\text{-a.e. point on } \partial_*\Omega, \\ &\mathcal{N}_\kappa F \in L^p(\partial\Omega, \sigma), \quad \text{and } F \text{ is holomorphic in } \Omega. \end{aligned} \tag{7.5.45}$$

Moreover,

$$\text{if } \Omega \text{ is an exterior domain then } F(z) = O(|z|^{1-2n}) \text{ as } |z| \to \infty. \tag{7.5.46}$$

In turn, (7.5.45)-(7.5.46) and (7.5.43) ensure that the integral representation formula (7.4.45) holds for the current function F, i.e.,

$$F = \mathscr{B}_{0,0}\big(F\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) \text{ in } \Omega. \tag{7.5.47}$$

Going nontangentially to the boundary in (7.5.47) then yield

$$(\tfrac{1}{2}I + B_{0,0})f = (\tfrac{1}{2}I + B_{0,0})\big((\tfrac{1}{2}I + B_{0,0})f\big) \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega, \tag{7.5.48}$$

on account of (7.5.45) and (7.3.82). With this in hand, formula (7.5.44) follows after some simple algebra. □

We are now ready to discuss the main result in this section. To place matters in a broader perspective we first make a series of remarks pertaining to the history of extension phenomena in several complex variables. Virtually all results of this nature originate in the pioneering work of Friedrich Hartogs who, in 1906, by relying on Cauchy's integral formula for functions of several complex variables proved[1] the following:

[Hartogs' Kugelsatz; [34]] If $n \geq 2$ and Ω is an open bounded subset of $\mathbb{C}^n$ with connected boundary, then any holomorphic function f in a neighborhood U of $\partial\Omega$ extends to a holomorphic function in the set Ω. (7.5.49)

In the limiting case when the neighborhood U shrinks to $\partial\Omega$, the standard holomorphicity assumption for f should be replaced by the demand that f is a CR-function on $\partial\Omega$. The latter condition is meaningful under suitable assumptions on f and $\partial\Omega$, a scenario in which we shall refer to this question as the Extension Problem[2]. The first solution to the Extension Problem was given in 1931 by Francesco Severi in the case when both $\partial\Omega$ and f are real-analytic. Severi proved such a theorem in [93] for the case $n = 2$ for tangential analytic functions in the sense of Wirtinger (cf. the discussion in (7.2.77)) via a technique involving passing from real variables to complex variables at the level of power series which naturally extends to higher dimensions. He first established a local version which then was globalized using the classical Hartogs extension theorem (cf. (7.5.49)). Seemingly unaware of Severi's earlier work, in 1936 Helmuth Kneser solved the Extension Problem in [45] for bounded strictly pseudoconvex domains of class $\mathscr{C}^2$ in $\mathbb{C}^2$. In this paper Kneser also observed that Wirtinger's tangential analyticity condition $\mathrm{d}f \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n = 0$ on $\partial\Omega$ (which is meaningful only in the category of continuously differentiable functions) may be recast as a Morera type condition on $\partial\Omega$, which permitted him to

[1] It was actually not until 2007 that Hartogs' original analytic disk method was shown to work in full generality by J. Merker and E. Porten in [64]

[2] A positive resolution of the Extension Problem in any reasonable setting yields (7.5.49) as a corollary

holomorphically extend functions f which are merely continuous on $\partial\Omega$ and satisfy $\int_{bD} f \, dz_1 \wedge dz_2 = 0$ for every $\mathscr{C}^1$ subdomain D of the manifold $\partial\Omega$ (with bD denoting the topological boundary of D relative to this ambient). Apparently unaware of Kneser's work, in 1956 Hand Lewy reproved the extension theorem for continuously differentiable CR-functions in the setting of bounded strictly pseudoconvex domains of class $\mathscr{C}^2$ in $\mathbb{C}^2$ in [57][3].

Using potential theoretic methods (specifically, the solvability of the Dirichlet Problem), in 1957 Gaetano Fichera succeeded in extending Severi's 1931 global extension theorem without demanding real analyticity for the data and without assuming pseudoconvexity for the underlying domain. Concretely, in [25] Fichera solved the Extension Problem for bounded domains of class $\mathscr{C}^{1+\varepsilon}$, with $\varepsilon > 0$ arbitrary. Inspired by this work, in 1961 Enzo Martinelli re-visited his 1942 proof of the classical Hartogs theorem (based on the Bochner-Martinelli integral formula) and, in [62], adapted it to produce a conceptually simple proof of the global extension theorem of Severi-Fichera in the class of bounded $\mathscr{C}^1$ domains Ω and for CR-functions $f \in \mathscr{C}^1(\partial\Omega)$. In the late 1960's B. Weinstock dealt with the Extension Problem for continuous CR-functions defined on boundaries of bounded $\mathscr{C}^\infty$ domains in [104], [105].

For an informative account on the early history of this subject, see [89], [90]. As regards more recent work, in [59, Theorem 8.20, p. 45] a version of the Extension Problem is presented[4] which involves bounded $\mathscr{C}^1$ domains and continuous CR-functions. Also, the 1976 survey [35] of G. Henkin and E. Chirka contains a discussion of the Extension Problem for integrable functions on compact Lyapunov surfaces (which essentially are topological boundaries of bounded domains of class $\mathscr{C}^{1+\varepsilon}$ for some $\varepsilon > 0$). This continues to be an active area of research and the interested reader is referred to [55] for a review of related work up to the early 1990's.

In this volume we contribute to this line of research, aimed at identifying the most general geometric/analytic setting in which the Extension Problem may be solved, by considering domains so rough that their topological boundaries are lacking any type of manifold structure, and also consider CR-functions which are not differentiable in a traditional sense. Such a setting gives rise to a number of significant challenges. For example, one has to find a suitable notion of tangential Cauchy-Riemann equations on a rough "surface" and in Definition 7.1.4 we have done just that, assuming that the surface in question is the topological boundary of a set which is merely of locally finite perimeter. As regards our main results on this topic, in Theorem 7.5.5 we first discuss the solution to the Extension Problem for Hölder CR-functions in Ahlfors regular domains. Subsequently, in Theorem 7.6.1 we solve the Extension Problem in the class of uniformly rectifiable (UR) domains for CR-functions in Lebesgue,

[3] Motivated by this topic (specifically, in an effort to show that the Dolbeault complex for the boundary d-bar operator is almost never exact), one year later Lewy published his celebrated example of a smooth first-order linear partial differential equation with no solution in [58] (which shows that the Cauchy-Kovalevskaya theorem does not have a natural analog in the smooth category)

[4] this is referred to by I. Lieb and J. Michel as "a deep generalization of the Kugelsatz"

John-Nirenberg, and Sarason spaces (L^p/BMO/VMO). In both cases, our earlier results on the nature of the Bochner-Martinelli integral operator play a crucial role.

Theorem 7.5.5 *Having fixed $n \in \mathbb{N}$ with $n \geq 2$, assume $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ is a bounded Ahlfors regular domain with the property that $\mathbb{R}^{2n} \setminus \overline{\Omega}$ is a connected set, and abbreviate $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$.*

(1) Existence/Uniqueness: *Given any $\alpha \in (0,1)$, there exists a constant $C \in (0,\infty)$ with the property that for each* CR*-function $f \in \mathscr{C}^\alpha(\partial\Omega)$ one can find a unique holomorphic function F in Ω which extends f nontangentially, in the sense that for each $\kappa > 0$ one has*

$$F\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{7.5.50}$$

and such that

$$\sup_{z\in\Omega} |F(z)| + \sup_{z\in\Omega} \Big\{\text{dist}(z,\partial\Omega)^{1-\alpha}\big|(\nabla F)(z)\big|\Big\} \leq C\|f\|_{\mathscr{C}^\alpha(\partial\Omega)}. \tag{7.5.51}$$

(2) Strong Uniqueness: *Given any $f \in L^1(\partial\Omega,\sigma)$ and any $\kappa > 0$, there could be at most one holomorphic function F in Ω with $\mathcal{N}_\kappa F \in L^1(\partial\Omega,\sigma)$ and such that $F\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f$ at σ-a.e. point on $\partial\Omega$. In particular, any* CR*-function $f \in \mathscr{C}^\alpha(\partial\Omega)$ with $\alpha \in (0,1)$ has a unique bounded holomorphic nontangential extension to Ω* (*in the sense of* (7.5.50)).

(3) Further Regularity: *If Ω is also a uniform domain, then the holomorphic nontangential extension F of the* CR*-function $f \in \mathscr{C}^\alpha(\partial\Omega)$ with $\alpha \in (0,1)$ from part (1) actually belongs to $\mathscr{C}^\alpha\big(\overline{\Omega}\big)$ and satisfies*

$$\|F\|_{\mathscr{C}^\alpha(\overline{\Omega})} \leq C\|f\|_{\mathscr{C}^\alpha(\partial\Omega)}. \tag{7.5.52}$$

In such a scenario one has $f = F\big|_{\partial\Omega}$, the ordinary restriction of F to $\partial\Omega$.

In the context of the above theorem, recall from (7.1.21) that $f \in L^1(\partial\Omega,\sigma)$ is a CR-function provided $\partial^{\mathbb{C}}_{\tau_{jk}} f = 0$ on $\partial\Omega$ (in the sense of Definition 7.1.1) for all $j,k \in \{1,\dots,n\}$. From Proposition 7.1.5 we know that being a CR-function is a necessary condition for the existence of a holomorphic nontangential extension (with control of the nontangential maximal function).

Proof of Theorem 7.5.5 Define $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^{2n} \setminus \overline{\Omega}$. Then, having fixed some arbitrary CR-function $f \in \mathscr{C}^\alpha(\partial\Omega)$ with $\alpha \in (0,1)$, introduce

$$F^\pm := \big[\mathcal{B}_{0,0} f\big]\Big|_{\Omega_\pm}. \tag{7.5.53}$$

Note that $\Omega_\pm \subseteq \mathbb{C}^n \setminus \partial\Omega$, so (7.5.14)-(7.5.15) imply that $F^\pm$ are holomorphic in $\Omega_\pm$. Moreover, since $n > 1$ and Ω_- is assumed to be connected, from (7.5.16) we conclude that actually F^- vanishes identically in Ω_-. In concert with (7.5.34), for each $\kappa > 0$ this permits us to write

$$
\begin{aligned}
f &= (\tfrac{1}{2}I + B_{0,0})f - (-\tfrac{1}{2}I + B_{0,0})f \\
&= \big(\mathscr{B}^{+}_{0,0}f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} - \big(\mathscr{B}^{-}_{0,0}f\big)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \\
&= F^{+}\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega.
\end{aligned}
\tag{7.5.54}
$$

Hence, $F := F^{+} = \mathscr{B}^{+}_{0,0}f$ is a holomorphic nontangential extension of the given f which also satisfies (7.5.51), thanks to (7.5.5) (also bearing in mind the inequality $\text{dist}(\cdot, \partial\Omega) \leq \text{dist}(\cdot, \partial_*\Omega)$, which holds since $\partial_*\Omega \subseteq \partial\Omega$).

That for any given function in $L^1(\partial\Omega, \sigma)$, and any $\kappa > 0$, there could be at most one holomorphic function F in Ω with $\mathcal{N}_\kappa F \in L^1(\partial\Omega, \sigma)$ and such that $F\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ matches the given function at σ-a.e. point on $\partial\Omega$, is then clear from the reproducing formula the last part of Corollary 7.4.2 (cf. (7.4.45)). In particular, this establishes uniqueness for the extension F^{+} of f constructed above.

Lastly, if Ω is also a uniform domain, then from (7.5.6) we conclude that F^{+} belongs to $\mathscr{C}^{\alpha}\big(\overline{\Omega}\big)$ and $\|F^{+}\|_{\mathscr{C}^{\alpha}(\overline{\Omega})} \leq C\|f\|_{\mathscr{C}^{\alpha}(\partial\Omega)}$ for some constant $C \in (0, \infty)$ independent of f. □

As a byproduct of our solution to the Extension Problem from Theorem 7.5.5 we have the following result, to the effect that the class of Hölder CR-functions is stable under pointwise multiplication.

Corollary 7.5.6 *Assume $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is a bounded Ahlfors regular domain with the property that $\mathbb{R}^{2n} \setminus \overline{\Omega}$ is a connected set. Then*

$$
\begin{aligned}
&\mathscr{A}_\alpha := \big\{f \in \mathscr{C}^{\alpha}(\partial\Omega) : \ f \textit{ is a } \text{CR}\textit{-function}\big\} \\
&\textit{is a sub-algebra of } \mathscr{C}^{\alpha}(\partial\Omega) \textit{ for each } \alpha \in (0, 1).
\end{aligned}
\tag{7.5.55}
$$

Proof Fix $\alpha \in (0, 1)$. Clearly, $\mathscr{A}_\alpha$ is a linear subspace of $\mathscr{C}^{\alpha}(\partial\Omega)$. To show that this is actually an algebra, we need to check that it is stable under pointwise multiplication. With this goal in mind, pick two arbitrary CR-functions $f, g \in \mathscr{C}^{\alpha}(\partial\Omega)$ and use Theorem 7.5.5 to extend them holomorphically to Ω. Specifically, there exist two bounded holomorphic functions F, G in Ω with the property that for each $\kappa > 0$ we have

$$
F\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = f \ \text{ and } \ G\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = g \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{7.5.56}
$$

where $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. Then their product, $H := FG$, is a bounded holomorphic function in Ω which, for each $\kappa > 0$ satisfies

$$
H\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = fg \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{7.5.57}
$$

Granted this, the very last claim in the statement of Proposition 7.1.5 guarantees that fg is a CR-function. Since fg also belongs to $\mathscr{C}^{\alpha}(\partial\Omega)$, the desired conclusion follows. □

7.6 The Extension Problem for L^p/BMO/VMO/Morrey Functions on Boundaries of Uniformly Rectifiable Domains

In this section, the principal goal is to present the solution to the Extension Problem formulated for L^p/BMO/VMO/Morrey functions in UR domains.

Theorem 7.6.1 *Fix $n \in \mathbb{N}$ with $n \geq 2$ and assume that $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ is a bounded* UR *domain with the property that the set $\mathbb{R}^{2n} \setminus \overline{\Omega}$ is connected. Also, abbreviate $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$.*

(1) Existence: *Given any $p \in [1, \infty)$, there exists a constant $C \in (0, \infty)$ with the property that for each* CR*-function $f \in L^p(\partial\Omega, \sigma)$ one can find a holomorphic function F in Ω which extends f nontangentially, in the sense that for each $\kappa > 0$ one has*

$$F\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{7.6.1}$$

and such that

$$\big\|\mathcal{N}_\kappa F\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p(\partial\Omega,\sigma)} \text{ if } 1 < p < \infty, \tag{7.6.2}$$

$$\big\|\mathcal{N}_\kappa F\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \leq C\|f\|_{L^1(\partial\Omega,\sigma)} \text{ if } p = 1. \tag{7.6.3}$$

Also, corresponding to $p = \infty$, for each CR*-function $f \in \mathrm{BMO}(\partial\Omega, \sigma)$ there exists a holomorphic function F in Ω which extends f nontangentially (in the sense of (7.6.1)) and which, for each $q \in (1, \infty)$ and with some constant $C \in (0, \infty)$ independent of f, satisfies the following Carleson measure estimate:*

$$\sup_{z\in\partial\Omega,\, r>0} \frac{1}{\sigma\big(\partial\Omega\cap B(z,r)\big)} \int_{B(z,r)\cap\Omega} \big|\nabla F\big|^q \operatorname{dist}(\cdot, \partial\Omega)^{q-1} d\mathcal{L}^{2n} \leq C\|f\|^q_{\mathrm{BMO}(\partial\Omega,\sigma)}. \tag{7.6.4}$$

(2) Strong Uniqueness: *For each holomorphic function F in Ω with the property that $\mathcal{N}_\kappa F \in L^p(\partial\Omega, \sigma)$, for some $\kappa > 0$, the nontangential boundary trace $F\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$ and satisfies*

$$\Big\|F\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\|_{L^1(\partial\Omega,\sigma)} \approx \big\|\mathcal{N}_\kappa F\big\|_{L^1(\partial\Omega,\sigma)} \tag{7.6.5}$$

where the (implicit) proportionality constants are independent of F. As a consequence, given any $f \in L^1(\partial\Omega, \sigma)$ and any $\kappa > 0$, there could be at most one holomorphic function F in Ω with $\mathcal{N}_\kappa F \in L^1(\partial\Omega, \sigma)$ and such that $F\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f$ at σ-a.e. point on $\partial\Omega$. In particular, given any CR*-function $f \in L^p(\partial\Omega, \sigma)$ with $p \in (1, \infty)$ the holomorphic nontangential extension F of f to Ω constructed in part (1) is unique (in the class of holomorphic functions in Ω whose nontangential maximal functions are in $L^p(\partial\Omega, \sigma)$).*

(3) Further Regularity: *For each $p, q \in [1, \infty)$ there exists a constant $C \in (0, \infty)$ with the property that if f is a* CR*-function belonging to the off-diagonal Sobolev*

spaces $L^{p,q}_1(\partial\Omega,\sigma)$ *then, in addition to* (7.6.2)-(7.6.3)*, its holomorphic nontangential extension* F *to* Ω *constructed in part (1) also satisfies*

$$\big\|\mathcal{N}_\kappa(\nabla F)\big\|_{L^q(\partial\Omega,\sigma)} \leq C\|\nabla_{\tan} f\|_{L^q(\partial\Omega,\sigma)} \text{ if } 1 < q < \infty, \tag{7.6.6}$$

$$\big\|\mathcal{N}_\kappa(\nabla F)\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \leq C\|\nabla_{\tan} f\|_{L^1(\partial\Omega,\sigma)} \text{ if } q = 1. \tag{7.6.7}$$

Also, for each $p \in (1,\infty)$ *there exists a constant* $C = C(\Omega, p, n) \in (0,\infty)$ *such that for any* CR*-function* $f \in L^p(\partial\Omega,\sigma)$*, the holomorphic nontangential extension* F *of* f *to* Ω *constructed in part (1) has the additional property that*

$$\Big(\int_\Omega |\nabla F|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^{2n}\Big)^{1/p} \leq C\|f\|_{L^p(\partial\Omega,\sigma)}. \tag{7.6.8}$$

In addition, for any CR*-function* $f \in \mathrm{VMO}(\partial\Omega,\sigma)$ *the holomorphic nontangential extension* F *of* f *to* Ω *constructed in part (1) has the property that for each* $p \in (1,\infty)$ *the measure* $|\nabla F|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^{2n}$ *is a vanishing Carleson measure in* Ω*, in the sense that*

$$\lim_{R\to 0^+} \sup_{z\in\partial\Omega,\, r\in(0,R)} \frac{1}{\sigma\big(B(z,r)\cap\partial\Omega\big)} \int_{B(z,r)\cap\Omega} |\nabla F|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^{2n} = 0. \tag{7.6.9}$$

Moreover, given any integrability exponent $p \in (1,\infty)$ *along with a parameter* $\lambda \in (0, 2n-1)$*, there exists a constant* $C = C(\Omega, n, p, \lambda) \in (0,\infty)$ *with the property that for any* CR*-function* f *belonging to the Morrey space* $M^{p,\lambda}(\partial\Omega,\sigma)$ *the holomorphic nontangential extension* F *of* f *to* Ω *devised in part (1) satisfies the following fractional Carleson measure estimate:*

$$\sup_{\substack{z\in\partial\Omega \text{ and}\\ 0<r<2\operatorname{diam}(\partial\Omega)}} \Big\{ r^{-\lambda} \int_{B(z,r)\cap\Omega} |\nabla F|^p \operatorname{dist}(\cdot,\partial\Omega)^{p-1}\, d\mathcal{L}^{2n} \Big\}^{\frac{1}{p}} \leq C\|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{7.6.10}$$

Finally, when Ω *is actually a bounded* NTA *domain with Ahlfors regular boundary, then for each* $p \in (1,\infty)$ *and* $q \in \big(\frac{n}{n+1/p}, \infty\big]$ *there exists a constant* $C \in (0,\infty)$ *with the property that for any* CR*-function* $f \in L^p(\partial\Omega,\sigma)$ *the holomorphic nontangential extension* F *of* f *to* Ω *constructed in part (1) belongs to* $F^{p,q}_{1/p}(\Omega) \cap B^{p,p}_{1/p}(\Omega)$ *and satisfies*

$$\max\Big\{\|F\|_{F^{p,q}_{1/p}(\Omega)}, \|F\|_{B^{p,p}_{1/p}(\Omega)}\Big\} \leq C\|f\|_{L^p(\partial\Omega,\sigma)}. \tag{7.6.11}$$

In particular, corresponding to $p = q = 2$*,*

$$\|F\|_{H^{1/2}(\Omega)} \leq C\|f\|_{L^2(\partial\Omega,\sigma)}. \tag{7.6.12}$$

Recall from (7.1.21) that, in the context of the above theorem, $f \in L^1(\partial\Omega, \sigma)$ is a CR-function whenever $\partial^{\mathbb{C}}_{\tau_{jk}} f = 0$ on $\partial\Omega$ (in the sense of Definition 7.1.1) for all $j, k \in \{1, \dots, n\}$. In this vein, it is worth pointing out that, as is apparent from Proposition 7.1.5, being a CR-function is a necessary condition for the existence of a holomorphic nontangential extension with an integrable nontangential maximal function.

We also wish to note that, as far as extending arbitrary CR-functions in a nontangential fashion to holomorphic functions inside the given domain, assuming $n > 1$ is a necessary condition. Indeed, bearing (7.1.23) in mind, it is easy to see that such an extension may not exist even in the case of the unit disk in the plane. Also, simple counterexamples in an annulus (taking f to be two different constants on the two spheres making up the boundary) show that the connectivity hypothesis on the complement is a necessary condition as well.

Proof of Theorem 7.6.1 Fix some arbitrary CR-function $f \in L^p(\partial\Omega, \sigma)$ for some exponent $p \in [1, \infty)$. Having defined $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^{2n} \setminus \overline{\Omega}$, consider

$$F^{\pm} := \big[\mathcal{B}_{0,0} f\big]\Big|_{\Omega_\pm}. \tag{7.6.13}$$

Since $\Omega_\pm \subseteq \mathbb{C}^n \setminus \partial\Omega$, it follows from (7.5.14)-(7.5.15) imply that $F^\pm$ are holomorphic in $\Omega_\pm$. Given that $n > 1$ and Ω_- is connected, (7.5.16) actually forces the function F^- to vanish identically in the set Ω_-. Bearing this in mind, for each $\kappa > 0$ we may then invoke (7.3.85) (with $\alpha = \beta = 0$, in which case we have the natural identification $L^p_{\tan,\mathbb{C}}(\partial\Omega, \sigma) \otimes \Lambda^{0,0} \equiv L^p(\partial\Omega, \sigma)$; cf. (7.1.64) and (7.1.79) in this regard) to write

$$\begin{aligned} f &= \big(\tfrac{1}{2}I + B_{0,0}\big) f - \big(-\tfrac{1}{2}I + B_{0,0}\big) f \\ &= \big(\mathcal{B}^+_{0,0} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - \big(\mathcal{B}^-_{0,0} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \\ &= F^+\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \end{aligned} \tag{7.6.14}$$

Consequently, $F := F^+ = \mathcal{B}^+_{0,0} f$ is a holomorphic nontangential extension of the f which, by virtue of (7.3.72)-(7.3.73) (presently used with $\alpha = \beta = 0$), also satisfies (7.6.2)-(7.6.3). The argument in the case when $f \in \mathrm{BMO}(\partial\Omega, \sigma)$ is similar, relying on (7.5.17) in place of nontangential maximal function estimates.

Next, the claims in item *(2)* up to, and including, (7.6.5) are consequences of [70, Theorem 3.1.6] (cf. [70, (3.1.103)], [70, (3.1.112)]), and the discussion in Comment 6 (following the statement of said theorem). Parenthetically, we wish to note that we can also use the reproducing formula (7.4.45) to conclude that there could be at most one nontangential extension of a given $f \in L^1(\partial\Omega, \sigma)$ to a holomorphic function F in Ω satisfying $\mathcal{N}_\kappa F \in L^1(\partial\Omega, \sigma)$ for some $\kappa > 0$.

With the exception of (7.6.10), the additional regularity properties of the extension specified in the last part of the statement are direct consequences of the mapping properties of the operator $\mathcal{B}_{0,0}$ discussed in (7.5.8)-(7.5.9), (7.3.77), (7.3.80), and item *(vii)* in Proposition 7.5.1 (also keeping in mind the estimates established in [69, Proposition 11.4.2]).

Finally, that the fractional Carleson measure estimate recorded in (7.6.10) holds whenever the function f belonging to the Morrey space $M^{p,\lambda}(\partial\Omega,\sigma)$ is a consequence of (3.3.69), bearing in mind that $F = \mathscr{B}^{+}_{0,0}f$ and the fact that the operator $\mathscr{B}^{+}_{0,0}$ is a particular case of a double layer (as discussed in Example 1.4.16). □

In turn, the solution to the version of the Extension Problem from Theorem 7.6.1 allows us to establish the following result pertaining to the stability of the class of CR-functions in Lebesgue spaces under pointwise multiplication.

Corollary 7.6.2 *Assume $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is a bounded* UR *domain with the property that $\mathbb{R}^{2n} \setminus \overline{\Omega}$ is a connected set. Abbreviate $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$ and fix $p, q \in (1,\infty)$ with the property that $1/p + 1/q = 1/r$ for some $r \geq 1$. In this setting, consider two* CR*-functions, $f \in L^p(\partial\Omega,\sigma)$ and $g \in L^q(\partial\Omega,\sigma)$. Then fg is a* CR*-function in $L^r(\partial\Omega,\sigma)$.*

Proof Clearly, $fg \in L^r(\partial\Omega,\sigma)$ (in particular $fg \in L^1(\partial\Omega,\sigma)$ since $\partial\Omega$ has finite measure) so there remains to show that fg is a CR-function. Fix $\kappa > 0$. According to Theorem 7.6.1 there exist two holomorphic functions F, G in Ω satisfying

$$\begin{gathered} \mathcal{N}_\kappa F \in L^p(\partial\Omega,\sigma), \quad \mathcal{N}_\kappa G \in L^q(\partial\Omega,\sigma), \\ F\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = f \text{ and } G\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = g \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \end{gathered} \tag{7.6.15}$$

Then the product

$$\begin{gathered} H := FG \text{ is a holomorphic function in } \Omega \text{ with the property that} \\ \mathcal{N}_\kappa H \in L^r(\partial\Omega,\sigma) \text{ and } H\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} = fg \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \end{gathered} \tag{7.6.16}$$

Having established (7.6.16), the very last claim in Proposition 7.1.5 ensures that fg is indeed a CR-function. □

Together, the Fatou-type result from [70, Theorem 3.1.6] (cf. [70, (3.1.144)] in particular) and the solution to the version of the Extension Problem from Theorem 7.6.1 directly impact the theory of Hardy spaces for holomorphic functions of several complex variables. A concrete example is offered by the following theorem.

Theorem 7.6.3 *Let $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ be a* UR *domain and abbreviate $\sigma := \mathcal{H}^{2n-1} \lfloor \partial\Omega$. Having fixed a background aperture parameter $\kappa > 0$, for each $p \in \big(\tfrac{2n-1}{2n}, \infty\big)$ define the holomorphic Hardy space in Ω as*

$$\begin{aligned} \mathcal{H}^p(\Omega) := \Big\{ & F : \Omega \to \mathbb{C} : F \textit{ holomorphic}, \ \mathcal{N}_\kappa F \in L^p(\partial\Omega,\sigma), \\ & \textit{and } \lim_{|z|\to\infty} F(z) = 0 \textit{ if } \Omega \textit{ is an exterior domain} \Big\}, \end{aligned} \tag{7.6.17}$$

and equip it with the quasi-norm $\mathcal{H}^p(\Omega) \ni F \mapsto \|F\|_{\mathcal{H}^p(\Omega)} := \|\mathcal{N}_\kappa F\|_{L^p(\partial\Omega,\sigma)}$. *Then the following statements are true.*

(i) The space $\mathcal{H}^p(\Omega)$ *is quasi-Banach, and the nontangential trace mapping*

$$\mathcal{H}^p(\Omega) \ni F \longmapsto F\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \in L^p(\partial\Omega,\sigma) \tag{7.6.18}$$

is well defined, linear, and bounded. Moreover, if $p \in (1,\infty)$ *then the image of* (7.6.18) *is contained in* $\big\{f \in L^p(\partial\Omega,\sigma) : f$ *is a* CR-*function*$\big\}$.

(ii) Whenever $p \in (1,\infty)$*, the Bochner-Martinelli integral operator*

$$\mathcal{B}_{0,0} : \big\{f \in L^p(\partial\Omega,\sigma) : f \text{ is a CR-function}\big\} \longrightarrow \mathcal{H}^p(\Omega) \tag{7.6.19}$$

is well defined, linear, bounded, and surjective, with the nontangential trace map (7.6.18) *serving as a right-inverse.*

(iii) If $p \in (1,\infty)$*, the set* Ω *is bounded,* $n > 1$*, and* $\mathbb{C}^n \setminus \overline{\Omega}$ *is connected, then the operator* $\mathcal{B}_{0,0}$ *is actually an isomorphism in the context of* (7.6.19).

Proof By design, $\mathcal{H}^p(\Omega)$ is a subspace of $N^p_\kappa(\Omega;\sigma)$, defined as in (A.0.168). Keeping this in mind, we may then rely on [68, Proposition 8.3.5] to conclude that $\mathcal{H}^p(\Omega)$ is indeed a quasi-Banach space. Also, from [70, (3.1.144)] we know that for each $F \in \mathcal{H}^p(\Omega)$ the nontangential trace $F\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$. Finally, the very last claim in item *(i)* is a consequence of what we have proved so far and the last conclusion in the statement of Proposition 7.1.5.

Moving on, that the operator (7.6.19) is well defined, linear, bounded when $p \in (1,\infty)$ is seen from item *(vi)* of Proposition 7.5.1 together with items *(ii)* and *(vii)* of Theorem 7.3.5. In addition, the very last claim in the current item *(i)* together with the very last claim in Corollary 7.4.2 prove that the nontangential trace $\mathcal{H}^p(\Omega) \ni F \mapsto F\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ is indeed as a right-inverse for the operator (7.6.19).

Finally, assume $p \in (1,\infty)$, the set Ω is bounded, $n > 1$, and $\mathbb{C}^n \setminus \overline{\Omega}$ is connected. In this scenario, if $f \in L^p(\partial\Omega,\sigma)$ is a CR-function with the property that $\mathcal{B}_{0,0}f = 0$ in Ω, then (7.6.14) simply reduces to $f = 0$. This proves that the operator (7.6.19) is injective, hence ultimately an isomorphism, in this case. □

Our next three theorems are concerned with characterizing the quality of being a CR-functions in terms of the action of the principal-value version of the Bochner-Martinelli integral operator. The emerging philosophy is that, under favorable geometric and analytic assumptions, the space of CR-functions is precisely the null-space of the operator $-\frac{1}{2}I + B_{0,0}$.

Theorem 7.6.4 *Fix* $n \in \mathbb{N}$ *with* $n \geq 2$ *and suppose* $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ *is a bounded* UR *domain. Abbreviate* $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$*, select* $\alpha \in \{0,1,\dots,n\}$*, and pick a differential form*

$$\begin{gathered} f \in L^{(2n-1)/(n-1)}(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,0} \ \text{ with the property that} \\ \bar{\partial}_\tau f \in L^{(2n-1)/n}(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,2} \end{gathered} \tag{7.6.20}$$

(*in particular, the conditions in* (7.6.20) *are satisfied if* f *is an arbitrary differential form belonging to the space* $L^{p,\bar{\partial}_\tau}_{\alpha,0}(\partial\Omega,\sigma)$ *with* $(2n-1)/(n-1) \leq p \leq \infty$). *Then*

$$B_{\alpha,0}f = \tfrac{1}{2}f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega \implies f \text{ is a CR-form.} \tag{7.6.21}$$

Proof Given a form f as in (7.6.20) which also satisfies $B_{\alpha,0}f = \frac{1}{2}f$ at σ-a.e. point on $\partial\Omega$, introduce

$$u := \mathcal{B}_{\alpha,0}f \in \mathscr{C}^\infty(\Omega) \otimes \Lambda^{\alpha,0}. \tag{7.6.22}$$

Thanks to (7.6.20), items *(ii)*-*(iii)* in Theorem 7.3.5, and simple degree considerations (based on (7.2.11)) we then have

$$\begin{gathered} \mathcal{N}_\kappa u \in L^{(2n-1)/(n-1)}(\partial\Omega,\sigma), \quad \mathcal{N}_\kappa(\bar{\partial}u) \in L^{(2n-1)/n}(\partial\Omega,\sigma), \\ \text{and } \ \vartheta u = 0 \ \text{ in } \ \Omega. \end{gathered} \tag{7.6.23}$$

Moreover, from (7.3.82), (7.1.83), and the fact that we are assuming that $B_{\alpha,0}f = \frac{1}{2}f$ at σ-a.e. point on $\partial\Omega$ we obtain

$$u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\tfrac{1}{2}I + B_{\alpha,0}\big)f = f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{7.6.24}$$

In particular, (7.6.24) and (7.6.22) entail

$$u = \mathcal{B}_{\alpha,0}\Big(u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) \ \text{ in } \ \Omega. \tag{7.6.25}$$

Collectively, (7.6.23)-(7.6.25) and Theorem 7.4.6 then imply that u is $\bar{\partial}$-closed in Ω. Having established this, from Proposition 7.2.9 and (7.6.24) we may conclude that $f = u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ is indeed a CR-form. □

To state our next result, recall the off-diagonal (partial) Sobolev spaces defined in (7.5.27).

Theorem 7.6.5 *Fix* $n \in \mathbb{N}$ *with* $n \geq 2$ *and assume that* $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ *is a bounded* UR *domain such that* $\mathbb{R}^{2n} \setminus \overline{\Omega}$ *is connected. Abbreviate* $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ *and consider a complex-valued function*

$$f \in L^{1,p,q}_{\mathbb{C}}(\partial\Omega,\sigma) \ \text{ with } \ \tfrac{2n-1}{n-1} \leq p \leq \infty \ \text{ and } \ \tfrac{2n-1}{n} \leq q \leq \infty \tag{7.6.26}$$

(*in particular, this is the case if* f *is an arbitrary function belonging to the space* $L^{p,1}_{\mathbb{C}}(\partial\Omega,\sigma)$ *with* $(2n-1)/(n-1) \leq p \leq \infty$). *Then*

$$f \text{ is a CR-function} \iff B_{0,0}f = \tfrac{1}{2}f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{7.6.27}$$

It is relevant to note that, under the geometric assumptions on the set Ω made in first part of Theorem 7.6.5, we may rephrase (7.6.26)-(7.6.27) simply by saying that, for any given integrability exponents $p \in \big[\frac{2n-1}{n-1},\infty\big]$ and $q \in \big[\frac{2n-1}{n},\infty\big]$, the null-space of the operator $-\frac{1}{2}I + B_{0,0}$ acting from the off-diagonal (partial) Sobolev space $L^{1,p,q}_{\mathbb{C}}(\partial\Omega,\sigma)$ into itself consists precisely of all CR-functions belonging to $L^p(\partial\Omega,\sigma)$.

Proof of Theorem 7.6.5 As regards the right-pointing implication, if f is a CR-function then from Theorem 7.6.1 and its proof we know that f extends nontangentially to a holomorphic function in Ω which is actually given by $F := \mathscr{B}_{0,0}f$. In concert with (7.3.82) (used with $\alpha=\beta=0$), this further implies that for each $\kappa>0$ we have $f = F\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = \big(\frac{1}{2}I + B_{0,0}\big)f$ at σ-a.e. point on $\partial\Omega$, which ultimately goes to show that $B_{0,0}f = \frac{1}{2}f$ on $\partial\Omega$. Lastly, the left-pointing implication in (7.6.29) is a direct consequence of Theorem 7.6.4 (presently invoked with $\alpha=0$), Proposition 7.2.4, and Proposition 7.2.5. □

We conclude this section by discussing the following characterization of CR-functions in terms of the principal-value Bochner-Martinelli singular integral operator.

Theorem 7.6.6 *Fix $n\in\mathbb{N}$ with $n\geq 2$ and assume that $\Omega\subseteq\mathbb{R}^{2n}\equiv\mathbb{C}^n$ is a bounded* NTA *domain with an Ahlfors regular boundary and with the property that the set $\mathbb{R}^{2n}\setminus\overline{\Omega}$ is connected. Abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$ and recall the critical exponent p_Ω associated with Ω as in* [70, (5.7.46)]. *Finally, pick a complex-valued function*

$$f\in L^{1,p,q}_{\mathbb{C}}(\partial\Omega,\sigma)\ \text{ with }\ p\in\big[\tfrac{2n-1}{n-1},\infty\big]$$
$$\text{satisfying }\ p>p_\Omega,\ \text{ and }\ q\in\big[\tfrac{2n-1}{n},\infty\big] \tag{7.6.28}$$

$\big($*in particular, this is the case if f is an arbitrary function belonging to the space $L^{p,1}_{\mathbb{C}}(\partial\Omega,\sigma)$ with* $\max\{(2n-1)/(n-1),p_\Omega\}<p\leq\infty\big)$. *Then*

$$B_{0,0}f=\tfrac{1}{2}f\ \text{ on }\partial\Omega \iff f \text{ is a CR-function} \iff B^2_{0,0}f=\tfrac{1}{4}f\ \text{ on }\partial\Omega. \tag{7.6.29}$$

In the case when Ω is a bounded Lipschitz domain in $\mathbb{R}^{2n}\equiv\mathbb{C}^n$ with a connected complement, then all hypotheses of geometric nature in Theorem 7.6.6 are satisfied and, in addition, $p_\Omega\in[1,2)$ (cf. [18]). In particular, the demand in (7.6.28) that $p>p_\Omega$ becomes superfluous in this situation.

Proof of Theorem 7.6.6 For starters, we wish to note that the geometric assumptions imply that Ω is a UR domain (cf. [68, Proposition 5.10.4] and [68, (5.2.4)]). Granted this, the first equivalence in (7.6.29) is provided by Theorem 7.6.5. Also, the right-pointing implication in the second equivalence in (7.6.29) is seen directly from Proposition 7.5.4 (keeping in mind that condition (7.5.43) is presently satisfied thanks to [68, (5.2.4)] and [68, (8.8.52)]).

As regards the left-pointing implication in the second equivalence in (7.6.29), we start by defining the complex-valued function

$$F:=\mathscr{B}_{0,0}f\ \text{ in }\Omega. \tag{7.6.30}$$

Observe that, thanks to (7.3.34), (7.3.82), item *(v)* in Proposition 7.5.1, and (7.3.72), for each fixed $\kappa>0$ we have

$$
\begin{aligned}
&F \in \mathscr{C}^\infty(\Omega), \quad \mathcal{N}_\kappa F \in L^p(\partial\Omega, \sigma), \quad \mathcal{N}_\kappa(\bar\partial F) \in L^{(2n-1)/n}(\partial\Omega, \sigma), \\
&\text{both } F\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ and } \bar\partial F\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial\Omega, \text{ and } \Delta F = 0 \text{ in } \Omega,
\end{aligned}
\tag{7.6.31}
$$

where $\Delta := \partial_1^2 + \cdots + \partial_{2n}^2$ is the Laplacian in $\mathbb{R}^{2n}$. Moreover, (7.3.82), (7.1.83), and (7.3.69) imply that actually

$$
F\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + B_{0,0}\big)f \in L^p(\partial\Omega, \sigma). \tag{7.6.32}
$$

In particular, given that we are currently assuming that $B_{0,0}^2 f = \frac{1}{4}f$ on $\partial\Omega$, another application of (7.3.82) gives (once again bearing in mind (7.1.83))

$$
\begin{aligned}
\Big[\mathscr{B}_{0,0}\big(F\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\Big]\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} &= \big(\tfrac{1}{2}I + B_{0,0}\big)\big(\tfrac{1}{2}I + B_{0,0}\big)f \\
&= \big(\tfrac{1}{4}I + B_{0,0} + B_{0,0}^2\big)f = \big(\tfrac{1}{2}I + B_{0,0}\big)f.
\end{aligned}
\tag{7.6.33}
$$

At this stage, we make the claim that

$$
F = \mathscr{B}_{0,0}\Big(F\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) \text{ in } \Omega. \tag{7.6.34}
$$

To prove this, observe that by (7.6.31)-(7.6.32) on the one hand, and by (7.3.34), (7.3.72), (7.6.33) on the other hand, both sides of (7.6.34) solve the L^p Dirichlet Problem for the Laplacian in Ω with boundary datum $\big(\frac{1}{2}I + B_{0,0}\big)f \in L^p(\partial\Omega, \sigma)$, i.e.,

$$
\begin{cases}
u \in \mathscr{C}^\infty(\Omega), \\
\Delta u = 0 \text{ in } \Omega, \\
\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma), \\
u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\tfrac{1}{2}I + B_{0,0}\big)f \ \ \sigma\text{-a.e. on } \partial\Omega.
\end{cases}
\tag{7.6.35}
$$

Granted this, from the assumption on p and the uniqueness portion of [70, Theorem 5.7.7] we conclude that (7.6.34) holds, as claimed.

Together, (7.6.31), (7.6.34), and the very last claim in Theorem 7.4.6 then imply that F is holomorphic in Ω. Having established this, from Proposition 7.1.5 and (7.6.32) we may conclude that $f = F\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ is indeed a CR-function. □

7.7 The $\overline{\partial}$ Operator and the Dolbeault Complex on Uniformly Rectifiable Sets

In this section we introduce a suitable version of the $\bar\partial$ operator on the boundaries of UR domains and explain how this may be used to set up the Dolbeault cohomology complex in such a general geometric context. We begin establishing the following result, regarding the action of $\bar\partial_\tau$.

Proposition 7.7.1 *Consider a* UR *domain* $\Omega \subseteq \mathbb{C}^n \equiv \mathbb{R}^{2n}$ *and set* $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. *Also, fix* $\alpha, \beta \in \{0, 1, \dots, n\}$ *along with* $p \in [1, \infty)$. *Then the operator*

$$\bar{\partial}_\tau : L^{p,\bar{\partial}_\tau}_{\alpha,\beta}(\partial\Omega, \sigma) \longrightarrow L^p_{\mathrm{nor},\mathbb{C}}(\partial\Omega, \sigma) \otimes \Lambda^{\alpha,\beta+2} \tag{7.7.1}$$

is well defined, linear, and bounded.

Proof Pick an arbitrary form $f \in L^{p,\bar{\partial}_\tau}_{\alpha,\beta}(\partial\Omega, \sigma)$. Since $\bar{\partial}_\tau f \in L^p(\partial\Omega, \sigma) \otimes \Lambda^{\alpha,\beta+2}$ by design, we only need to show that $\bar{\partial}_\tau f$ is complex normal (cf. Definition 7.1.7 and the subsequent discussion).

With this goal in mind, denote by ν the geometric measure theoretic outward unit normal to Ω, and introduce $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^n \setminus \overline{\Omega}$. Next, for each $\alpha, \beta \in \{0, 1, \dots, n\}$ define the integral operators $\mathcal{A}^{\pm}_{\alpha,\beta}$ as in (7.3.38) with the domains $\Omega_\pm$ playing the role of Ω. Also, recall the operators $\mathcal{B}^{\pm}_{\alpha,\beta}$ introduced in (7.3.83). Having fixed some aperture parameter $\kappa > 0$, make use of Proposition 7.3.2 and Proposition 7.3.4 (both written for the UR domains $\Omega_\pm$, mindful of item *(7)* in [68, Lemma 5.10.9]) in order to write

$$\begin{aligned}
\big(\bar{\partial}\mathcal{B}^{\pm}_{\alpha,\beta} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} &= -\frac{1}{2}\Big[\mathcal{A}^{\pm}_{\alpha,\beta+1}(\bar{\partial}_\tau f)\Big]\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \\
&= -\frac{1}{2}\Big\{\pm\frac{1}{2}\,\nu^{1,0} \vee \bar{\partial}_\tau f + A_{\alpha,\beta}(\bar{\partial}_\tau f)\Big\} \\
&= \mp\tfrac{1}{4}\nu^{1,0} \vee \bar{\partial}_\tau f - \tfrac{1}{2}A_{\alpha,\beta+1}(\bar{\partial}_\tau f) \ \text{ at } \sigma\text{-a.e. point on } \partial\Omega.
\end{aligned} \tag{7.7.2}$$

If we now abbreviate

$$u^{\pm} := \mathcal{B}^{\pm}_{\alpha,\beta} f \in \mathscr{C}^\infty(\Omega_\pm) \otimes \Lambda^{\alpha,\beta}, \tag{7.7.3}$$

then (7.7.2) allows us to express

$$\begin{aligned}
-\tfrac{1}{2}\nu^{1,0} \vee \bar{\partial}_\tau f &= \big(\bar{\partial}\mathcal{B}^{+}_{\alpha,\beta} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - \big(\bar{\partial}\mathcal{B}^{-}_{\alpha,\beta} f\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \\
&= (\bar{\partial}u^+)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - (\bar{\partial}u^-)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}.
\end{aligned} \tag{7.7.4}$$

Observe that thanks to items *(ii)*, *(iii)*, and *(viii)* in Theorem 7.3.5, we may invoke the conclusions of Proposition 7.2.7 applied to the functions $u^\pm$ in $\Omega_\pm$. Concretely, based on (7.7.4), Proposition 7.2.7, the linearity of the operator $\bar{\partial}_\tau$, the jump-formulas recorded in (7.3.84), and (7.2.31), we may write

$$
\begin{aligned}
-\tfrac{1}{2}\nu^{0,1}\wedge(\nu^{1,0}\vee\bar\partial_\tau f) &= \nu^{0,1}\wedge\Big[(\bar\partial u^+)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big] - \nu^{0,1}\wedge\Big[(\bar\partial u^-)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big]\\
&= -\Big\{\bar\partial_\tau\big(u^+\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) - \bar\partial_\tau\big(u^-\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\Big\}\\
&= -\bar\partial_\tau\big(u^+\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - u^-\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) = -\bar\partial_\tau(f_{\mathrm{tan},\mathbb{C}})\\
&= -\bar\partial_\tau f \text{ at } \sigma\text{-a.e. point on } \partial\Omega.
\end{aligned}
\tag{7.7.5}
$$

In light of (7.1.69), the above identity entails

$$
\bar\partial_\tau f = \tfrac{1}{2}\nu^{0,1}\wedge(\nu^{1,0}\vee\bar\partial_\tau f) = (\bar\partial_\tau f)_{\mathrm{nor},\mathbb{C}} \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{7.7.6}
$$

This amounts to saying that $\bar\partial_\tau f$ is complex normal, as wanted. □

To proceed, in the definition below we introduce a very general version of the boundary $\bar\partial$-bar operator.

Definition 7.7.2 *Let $\Omega \subseteq \mathbb{C}^n \equiv \mathbb{R}^{2n}$ be a* UR *domain. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. Also, fix $\alpha, \beta \in \{0, 1, \dots, n\}$ along with $p \in [1, \infty)$. In this context, define*

$$
L^{p,\bar\partial_b}_{\alpha,\beta}(\partial\Omega,\sigma) := L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial\Omega,\sigma)\cap L^p_{\mathrm{tan},\mathbb{C}}(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}, \tag{7.7.7}
$$

and introduce the `boundary d-bar operator`

$$
\bar\partial_b : L^{p,\bar\partial_b}_{\alpha,\beta}(\partial\Omega,\sigma) \longrightarrow L^p_{\mathrm{tan},\mathbb{C}}(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1} \tag{7.7.8}
$$

by setting

$$
\bar\partial_b f := \nu^{1,0}\vee\bar\partial_\tau f \text{ for each } f \in L^{p,\bar\partial_b}_{\alpha,\beta}(\partial\Omega,\sigma). \tag{7.7.9}
$$

From (7.2.27), Definition 7.1.9, and item *(2)* in Lemma 7.1.6 it follows that the operator (7.7.8)-(7.7.9) is well defined, linear, and bounded. In addition, it satisfies the properties presented in the proposition below.

Proposition 7.7.3 *Let $\Omega \subseteq \mathbb{C}^n \equiv \mathbb{R}^{2n}$ be a* UR *domain and set $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. Also, fix $\alpha, \beta \in \{0, 1, \dots, n\}$ along with $p \in [1, \infty)$. Then the operator $\bar\partial_b$ in (7.7.8) is well defined, linear, bounded, and satisfies*

$$
\bar\partial_\tau f = \tfrac{1}{2}\nu^{0,1}\wedge\bar\partial_b f, \qquad \forall f \in L^{p,\bar\partial_b}_{\alpha,\beta}(\partial\Omega,\sigma). \tag{7.7.10}
$$

Furthermore,

$$
\bar\partial\mathcal{B}_{\alpha,\beta} f = -\tfrac{1}{2}\mathcal{B}_{\alpha,\beta+1}(\bar\partial_b f) \text{ in } \mathbb{C}^n\setminus\partial\Omega, \qquad \forall f \in L^{p,\bar\partial_b}_{\alpha,\beta}(\partial\Omega,\sigma). \tag{7.7.11}
$$

Proof The first claim in the statement, pertaining to the operator $\bar\partial_b$, follows from Definition 7.7.2, Proposition 7.7.1, and Lemma 7.1.6. Turning our attention to the second claim, fix an arbitrary form $f \in L^{p,\bar\partial_b}_{\alpha,\beta}(\partial\Omega,\sigma)$. Based on the definition of $\bar\partial_b$, item *(3)* in Lemma 7.1.6, (7.1.63), the fact that the differential form $\bar{\partial}_\tau f$ is complex normal (cf. Proposition 7.7.1), and (7.1.63), we may write

$$\begin{aligned}\nu^{0,1} \wedge \bar{\partial}_b f &= \nu^{0,1} \wedge (\nu^{1,0} \vee \bar{\partial}_\tau f) \\ &= \big\langle \nu^{0,1}, \overline{\nu^{1,0}} \big\rangle_{\mathbb{C}} \bar{\partial}_\tau f - \nu^{1,0} \vee (\nu^{0,1} \wedge \bar{\partial}_\tau f) \\ &= \big\langle \nu^{0,1}, \nu^{0,1} \big\rangle_{\mathbb{C}} \bar{\partial}_\tau f = |\nu^{0,1}|_{\mathbb{C}}^2 \, \bar{\partial}_\tau f = 2\, \bar{\partial}_\tau f. \end{aligned} \tag{7.7.12}$$

This proves (7.7.10). Lastly, for each $z \in \mathbb{C}^n \setminus \partial\Omega$, Proposition 7.3.2 and (7.7.10) allow us to compute

$$\begin{aligned}\bar{\partial}\big(\mathcal{B}_{\alpha,\beta} f\big)(z) &= \frac{1}{2}\int_{\partial_*\Omega} \big\langle (\bar{\partial}_\tau f)(\zeta), \bar{\partial}_\zeta \Gamma_{\alpha,\beta+1}(\zeta,z)\big\rangle_{\mathbb{C}}\, \mathrm{d}\sigma(\zeta) \\ &= \frac{1}{4}\int_{\partial\Omega} \big\langle \nu^{0,1}(\zeta) \wedge (\bar{\partial}_b f)(\zeta), \bar{\partial}_\zeta \Gamma_{\alpha,\beta+1}(\zeta,z)\big\rangle_{\mathbb{C}}\, \mathrm{d}\sigma(\zeta) \\ &= -\frac{1}{2}\mathcal{B}_{\alpha,\beta+1}(\bar{\partial}_b f)(z). \end{aligned} \tag{7.7.13}$$

This establishes (7.7.11), and finishes the proof of the proposition. □

In the class of uniformly rectifiable domains, (7.7.9) and (7.7.10) show that

$$\begin{aligned} &\text{whenever } \Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n \text{ is a given UR domain and} \\ &f \in L^{p,\bar{\partial}_b}_{\alpha,\beta}(\partial\Omega,\sigma) \text{ with } p \in [1,\infty) \text{ and } \sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega, \\ &f \text{ is a CR-form } \Longleftrightarrow\ \bar{\partial}_b f = 0 \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \end{aligned} \tag{7.7.14}$$

Our final result in this section opens the doors for considering the Dolbeault cohomology complex in the context of UR sets.

Proposition 7.7.4 *Consider a* UR *domain* $\Omega \subseteq \mathbb{C}^n \equiv \mathbb{R}^{2n}$, *and set* $\sigma := \mathcal{H}^{2n-1}\lfloor\partial\Omega$. *Also, fix* $\alpha, \beta \in \{0, 1, \dots, n\}$ *along with* $p \in [1,\infty)$. *Then the operator* $\bar{\partial}_b$ *from* (7.7.8) *actually induces a well-defined, linear, and bounded mapping*

$$\bar{\partial}_b : L^{p,\bar{\partial}_b}_{\alpha,\beta}(\partial\Omega,\sigma) \longrightarrow L^{p,\bar{\partial}_b}_{\alpha,\beta+1}(\partial\Omega,\sigma) \tag{7.7.15}$$

which satisfies

$$\bar{\partial}_b \circ \bar{\partial}_b = 0. \tag{7.7.16}$$

Proof Pick $f \in L^{p,\bar{\partial}_b}_{\alpha,\beta}(\partial\Omega,\sigma)$. If ν stands for the geometric measure theoretic outward unit normal to Ω, then (7.7.9) gives

$$g := \bar{\partial}_b f = \nu^{1,0} \vee \bar{\partial}_\tau f \in L^p_{\tan,\mathbb{C}}(\partial\Omega,\sigma) \otimes \Lambda^{\alpha,\beta+1}. \tag{7.7.17}$$

Having selected an arbitrary form $\psi \in \mathscr{C}^\infty_c(\mathbb{C}^n) \otimes \Lambda^{\alpha,\beta+3}$, we may compute

$$\begin{aligned}\int_{\partial_*\Omega} \big\langle \nu^{0,1} \wedge g, \vartheta\psi \big\rangle_{\mathbb{C}} \, \mathrm{d}\sigma &= \int_{\partial_*\Omega} \big\langle \nu^{0,1} \wedge (\nu^{1,0} \vee \bar{\partial}_\tau f), \vartheta\psi \big\rangle_{\mathbb{C}} \, \mathrm{d}\sigma \\ &= 2\int_{\partial_*\Omega} \big\langle (\bar{\partial}_\tau f)_{\mathrm{nor},\mathbb{C}}, \vartheta\psi \big\rangle_{\mathbb{C}} \, \mathrm{d}\sigma = 2\int_{\partial_*\Omega} \big\langle \bar{\partial}_\tau f, \vartheta\psi \big\rangle_{\mathbb{C}} \, \mathrm{d}\sigma \\ &= 2\int_{\partial_*\Omega} \big\langle f, \vartheta(\vartheta\psi) \big\rangle_{\mathbb{C}} \, \mathrm{d}\sigma = 0, \end{aligned} \tag{7.7.18}$$

thanks to (7.7.17), (7.1.68), (7.7.1), and (7.2.14). In concert with (7.7.17), (7.2.27) (and (7.2.26)), this proves that $\bar{\partial}_b f \in L^{p,\bar{\partial}_b}_{\alpha,\beta+1}(\partial\Omega, \sigma)$ and $\bar{\partial}_b(\bar{\partial}_b f) = 0$. All conclusions in the proposition then readily follow from this. □

Chapter 8
Hardy Spaces for Second-Order Weakly Elliptic Operators in the Complex Plane

In this chapter we shall work in the complex plane $\mathbb{C} \equiv \mathbb{R}^2$. Recall the Cauchy-Riemann operator $\partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$ and its complex conjugate $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$.

From the discussion in [70, §1.4] (cf. [70, (1.4.186)]) we know that there are three prototypes of scalar, constant (complex) coefficient, elliptic weakly elliptic operators in the plane, namely $\partial_z \partial_{\bar{z}}$ which, up to a multiplicative factor is the two-dimensional Laplacian, $\partial_{\bar{z}}^2$ also referred to as Bitsadze's operator (cf. [3], [4]), and its complex conjugate ∂_z^2.

We have already considered in [70, §5.5] Hardy spaces of harmonic functions (i.e., null-solutions of the Laplacian) in NTA domains. In this section the focus is the study of Hardy spaces of null-solutions for the operator $\partial_{\bar{z}}^2$ in the unit disk of the complex plane, a scale of spaces which interfaces tightly with the Dirichlet Problem for $\partial_{\bar{z}}^2$ in the unit disk. In this regard, it has been noted by A.V. Bitsadze that

$$\Big\{(1 - |z|^2)W : W \in \mathscr{C}^0(\overline{\mathbb{D}}) \text{ and } W \text{ holomorphic in } \mathbb{D}\Big\} \tag{8.0.1}$$

is an infinite dimensional subspace of the space of null-solutions for the classical Dirichlet Problem for $\partial_{\bar{z}}^2$ in the unit disk, i.e., the space of complex-valued functions u satisfying

$$\begin{gathered} u \in \mathscr{C}^\infty(\mathbb{D}) \cap \mathscr{C}^0(\overline{\mathbb{D}}), \quad \partial_{\bar{z}}^2 u = 0 \text{ in } \mathbb{D}, \\ \text{and } u\big|_{\partial\mathbb{D}} = 0 \text{ at each point on } \partial\mathbb{D}. \end{gathered} \tag{8.0.2}$$

It was this observation that really thrusted the operator $\partial_{\bar{z}}^2$ into the spotlight. Here our goal is to *precisely* describe the space of null-solutions, as well as the space of admissible boundary data, for the Dirichlet Problem for the Bitsadze operator in the unit disk. For example, our work in this chapter (see (8.1.32) and (8.1.38)) shows that the space

$$\Big\{u \in \mathscr{C}^\infty(\overline{\mathbb{D}}) : \partial_{\bar{z}}^2 u = 0 \text{ in } \mathbb{D} \text{ and } u\big|_{\partial\mathbb{D}} = 0\Big\} \tag{8.0.3}$$

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coincides with

$$\Big\{(1-|z|^2)W : W \in \mathscr{C}^\infty(\overline{\mathbb{D}}) \text{ and } W \text{ holomorphic in } \mathbb{D}\Big\}. \tag{8.0.4}$$

We will primarily concern ourselves with the L^p Dirichlet Problem for the Bitsadze operator in the unit disk, in which the boundary condition is taken in the nontangential sense, and for which the size of the solution is measured using the nontangential maximal operator. Ultimately, our analysis in §8.1 paints a very precise picture of the failure of Fredholm solvability of the Dirichlet and Regularity Problems for Bitsadze's operator in the unit disk of the complex plane. In §8.2 we carry out a program with similar aims for a more inclusive family of scalar second-order operators in the complex plane, namely $L_\lambda := \partial_{\bar{z}}^2 - \lambda^2\partial_z^2$, which contains Bitsadze's operator as a specal case (corresponding to $\lambda = 0$).

8.1 Null-Solutions and Boundary Traces for Bitsadze's Operator $\partial_{\bar{z}}^2$ in the Unit Disk

In this section we shall work with Bitsadze's operator $\partial_{\bar{z}}^2$ in the complex plane. The goal is to precisely describe its space of null-solutions as well as the corresponding spaces of boundary traces, assuming only nontangential maximal function control. Throughout, we abbreviate

$$\mathbb{D}_+ := \mathbb{D} := \big\{z \in \mathbb{C} : |z| < 1\big\}, \quad \mathbb{D}_- := \mathbb{C} \setminus \overline{\mathbb{D}}, \quad \sigma := \mathcal{H}^1 \lfloor \partial\mathbb{D}. \tag{8.1.1}$$

Fix some integrability exponent $p \in (1, \infty)$ along with some aperture parameter $\kappa \in (0, \infty)$, and recall the Hardy space of holomorphic functions in $\mathbb{D}_+$ possessing p-th power integrable nontangential maximal functions

$$\mathcal{H}^p(\mathbb{D}_+) := \mathcal{H}^p(\mathbb{D}) := \Big\{U \text{ holomorphic in } \mathbb{D} : \mathcal{N}_\kappa U \in L^p(\partial\mathbb{D}, \sigma)\Big\} \tag{8.1.2}$$

and, in the case of $\mathbb{D}_-$, also vanishing at infinity

$$\mathcal{H}^p(\mathbb{D}_-) = \Big\{U \text{ holomorphic in } \mathbb{D}_- : \mathcal{N}_\kappa U \in L^p(\partial\mathbb{D}, \sigma) \text{ and } U(\infty) = 0\Big\}. \tag{8.1.3}$$

We shall equip these spaces with the norms

$$\|U\|_{\mathcal{H}^p(\mathbb{D}_\pm)} := \|\mathcal{N}_\kappa U\|_{L^p(\partial\mathbb{D},\sigma)} \text{ for each } U \in \mathcal{H}^p(\mathbb{D}_\pm). \tag{8.1.4}$$

Next, denote by $\mathscr{C}$ the boundary-to-domain Cauchy operator associated with the unit disk, acting on each function $f \in L^1(\partial\mathbb{D}, \sigma)$ according to

$$\mathscr{C}f(z) := \frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta, \qquad \forall z \in \mathbb{C} \setminus \partial\mathbb{D}. \tag{8.1.5}$$

This induces well-defined, linear, bounded mappings

$$\mathscr{C} : L^p(\partial\mathbb{D}, \sigma) \longrightarrow \mathcal{H}^p(\mathbb{D}_\pm). \tag{8.1.6}$$

We shall also need the corresponding *boundary* Hardy spaces of nontangential traces of functions in $\mathcal{H}^p(\mathbb{D}_\pm)$, i.e.,

$$\mathcal{H}^p(\partial\mathbb{D}_\pm, \sigma) := \Big\{ U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{D}_\pm} : U \in \mathcal{H}^p(\mathbb{D}_\pm) \Big\}. \tag{8.1.7}$$

According to a classical theorem of Fatou, if u is a bounded harmonic function on the unit disc, $\mathbb{D}$ in the complex plane $\mathbb{C}$, then u has nontangential boundary traces at $\mathcal{H}^1$-a.e. point on the unit circle $\partial\mathbb{D}$. In fact, the same conclusion remains valid if u is merely assumed to be bounded from below. The latter result has a local analogue due to I. Privalov [85] (see also [109]). Specifically,

> if u is a harmonic function in the unit disc $\mathbb{D}$ with the property that for each point $e^{i\theta}$ of a measurable subset E of the unit circle $\partial\mathbb{D}$ there exists some $\kappa_\theta \in (0, \infty)$ such that u is bounded from below on the nontangential approach region $\Gamma_{\kappa_\theta}(e^{i\theta})$, then for any aperture parameter $\kappa \in (0, \infty)$ the nontangential boundary trace $\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{D}}\big)(z)$ exists for $\mathcal{H}^1$-a.e. point $z \in E$. (8.1.8)

See also A.P. Calderón [7] and E.M. Stein [96] for extensions to higher dimensions (with the unit disk replaced by the upper half-space), and D. Jerison and C.E. Kenig [42] for a local Fatou theorem in NTA domains.

The classical Fatou theorem mentioned above implies that

$$U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{D}_\pm} \text{ exists } \sigma\text{-a.e. on } \partial\mathbb{D} \text{ for each } U \in \mathcal{H}^p(\mathbb{D}_\pm), \tag{8.1.9}$$

so the boundary Hardy spaces $\mathcal{H}^p(\partial\mathbb{D}_\pm, \sigma)$ are indeed well defined. In fact, as is well known,

$$\begin{aligned} &\mathcal{H}^p(\partial\mathbb{D}_\pm, \sigma) \text{ are closed subspaces of } L^p(\partial\mathbb{D}, \sigma) \text{ and} \\ &L^p(\partial\mathbb{D}, \sigma) = \mathcal{H}^p(\partial\mathbb{D}_+, \sigma) \oplus \mathcal{H}^p(\partial\mathbb{D}_-, \sigma), \text{ direct sum.} \end{aligned} \tag{8.1.10}$$

Also, we have the Cauchy reproducing formulas

$$U = \mathscr{C}\big(U\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{D}_\pm}\big) \text{ in } \mathbb{D}_\pm \text{ for each } U \in \mathcal{H}^p(\mathbb{D}_\pm), \tag{8.1.11}$$

as well as the Cauchy vanishing formulas

$$\mathscr{C}f = 0 \text{ in } \mathbb{D}_\mp \text{ for each } f \in \mathcal{H}^p(\partial\mathbb{D}_\pm, \sigma). \tag{8.1.12}$$

In fact,

$$\Big\{ f \in L^p(\partial\mathbb{D}, \sigma) : \mathscr{C}f = 0 \text{ in } \mathbb{D}_\pm \Big\} = \mathcal{H}^p(\partial\mathbb{D}_\mp, \sigma). \tag{8.1.13}$$

We also claim that

$$\mathcal{H}^p(\partial\mathbb{D}_+,\sigma)=\Big\{f\in L^p(\partial\mathbb{D},\sigma):\ \int_{\partial\mathbb{D}} f(\zeta)\zeta^k\,\mathrm{d}\sigma(\zeta)=0\ \text{ for all }\ k\in\mathbb{N}\Big\},\tag{8.1.14}$$

and

$$\mathcal{H}^p(\partial\mathbb{D}_-,\sigma)=\Big\{f\in L^p(\partial\mathbb{D},\sigma):\ \int_{\partial\mathbb{D}} f(\zeta)\zeta^{-k}\,\mathrm{d}\sigma(\zeta)=0\ \text{ for all }\ k\in\mathbb{N}_0\Big\}.\tag{8.1.15}$$

To justify (8.1.14), note that

$$\left|\frac{\zeta}{z}\right|=\frac{1}{|z|}<1\ \text{ for all }\ z\in\mathbb{C}\setminus\overline{\mathbb{D}}\ \text{ and }\ \zeta\in\partial\mathbb{D},\tag{8.1.16}$$

hence for each $f\in L^p(\partial\mathbb{D},\sigma)$ and $z\in\mathbb{C}\setminus\overline{\mathbb{D}}$ we may write

$$\begin{aligned}(\mathscr{C}f)(z)&=\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathbb{D}}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta=\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathbb{D}}\frac{f(\zeta)}{(-z)(1-\zeta/z)}\,\mathrm{d}\zeta\\&=-\frac{1}{2\pi\mathrm{i}}\sum_{k=0}^{\infty}\int_{\partial\mathbb{D}}\frac{f(\zeta)}{z}\Big(\frac{\zeta}{z}\Big)^k\,\mathrm{d}\zeta=-\frac{1}{2\pi\mathrm{i}}\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\int_{\partial\mathbb{D}}f(\zeta)\zeta^k\,\mathrm{d}\zeta.\\&=-\frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\int_{\partial\mathbb{D}}f(\zeta)\zeta^{k+1}\,\mathrm{d}\sigma(\zeta),\end{aligned}\tag{8.1.17}$$

where the last equality above makes use of the identity $\mathrm{d}\zeta=\mathrm{i}\zeta\,\mathrm{d}\sigma(\zeta)$. In turn, the power series expansion just established in (8.1.17) proves that for each given function $f\in L^p(\partial\mathbb{D},\sigma)$ we have

$$\mathscr{C}f=0\ \text{ in }\ \mathbb{C}\setminus\overline{\mathbb{D}}\Longleftrightarrow\int_{\partial\mathbb{D}}f(\zeta)\zeta^k\,\mathrm{d}\sigma(\zeta)=0\ \text{ for all }\ k\in\mathbb{N},\tag{8.1.18}$$

so (8.1.14) follows from this and (8.1.13). The characterization in (8.1.15) may be justified in a similar fashion. Among other things, (8.1.14) readily yields

$$\bar{z}\cdot\mathcal{H}^p(\partial\mathbb{D}_+,\sigma)=\mathcal{H}^p(\partial\mathbb{D}_+,\sigma)\oplus\langle\bar{z}\rangle,\tag{8.1.19}$$

where we define $\langle\bar{z}\rangle$ to be the (one-dimensional) linear span over the field $\mathbb{C}$ of the function $\partial\mathbb{D}\ni z\mapsto\bar{z}\in\mathbb{C}$. Indeed, the left-to-right inclusion is seen by decomposing $\bar{z}\cdot f=\bar{z}\cdot\big(f-\fint_{\partial\mathbb{D}}f\,\mathrm{d}\sigma\big)+\bar{z}\cdot\fint_{\partial\mathbb{D}}f\,\mathrm{d}\sigma$ and using (8.1.14) for the first term, while the right-to-left inclusion is a consequence of the fact that we have $f=\bar{z}(zf)$ and that multiplication by z keeps you in $\mathcal{H}^p(\partial\mathbb{D}_+,\sigma)$.

Going further, we also consider Hardy spaces with regularity in $\mathbb{D}_\pm$, i.e.,

$$\begin{aligned}\mathcal{H}_1^p(\mathbb{D}) &:= \mathcal{H}_1^p(\mathbb{D}_+)\\ &:= \Big\{U \text{ holomorphic in } \mathbb{D} : \mathcal{N}_\kappa U,\ \mathcal{N}_\kappa(\nabla U) \in L^p(\partial\mathbb{D},\sigma)\Big\},\end{aligned} \tag{8.1.20}$$

and

$$\begin{aligned}\mathcal{H}_1^p(\mathbb{C}\setminus\overline{\mathbb{D}}) &:= \mathcal{H}_1^p(\mathbb{D}_-)\\ &:= \Big\{U \text{ holomorphic in } \mathbb{D} : \mathcal{N}_\kappa U,\ \mathcal{N}_\kappa(\nabla U) \in L^p(\partial\mathbb{D},\sigma),\ U(\infty)=0\Big\}.\end{aligned} \tag{8.1.21}$$

We can then define boundary Hardy spaces with regularity, namely

$$\mathcal{H}_1^p(\partial\mathbb{D}_\pm,\sigma) := \Big\{U\big|_{\partial\mathbb{D}_\pm}^{\kappa-\mathrm{n.t.}} : U \in \mathcal{H}_1^p(\mathbb{D}_\pm)\Big\}. \tag{8.1.22}$$

In relation to these, it is useful to note that

$$\mathcal{H}^p(\partial\mathbb{D}_\pm,\sigma)\cap L_1^p(\partial\mathbb{D},\sigma) = \mathcal{H}_1^p(\partial\mathbb{D}_\pm,\sigma). \tag{8.1.23}$$

Indeed, the left-to-right inclusion is seen from Cauchy's reproducing formula, (1.4.7), and (1.5.8). The right-to-left inclusion in (8.1.23) is implied by [69, Proposition 11.3.4] and definitions. In concert with (8.1.14)-(8.1.15), this implies

$$\mathcal{H}_1^p(\partial\mathbb{D}_+,\sigma) = \Big\{f \in L_1^p(\partial\mathbb{D},\sigma) : \int_{\partial\mathbb{D}} f(\zeta)\zeta^k\,d\sigma(\zeta) = 0 \text{ for all } k \in \mathbb{N}\Big\}, \tag{8.1.24}$$

and

$$\mathcal{H}_1^p(\partial\mathbb{D}_-,\sigma) = \Big\{f \in L_1^p(\partial\mathbb{D},\sigma) : \int_{\partial\mathbb{D}} f(\zeta)\zeta^{-k}\,d\sigma(\zeta) = 0 \text{ for all } k \in \mathbb{N}_0\Big\}. \tag{8.1.25}$$

Also, as seen from definitions, (1.4.7), and (6.2.105),

$$\begin{gathered}\text{the boundary-to-domain Cauchy operator } \mathscr{C} \text{ induces}\\ \text{well-defined mappings from } L_1^p(\partial\mathbb{D},\sigma) \text{ into } \mathcal{H}_1^p(\mathbb{D}_\pm).\end{gathered} \tag{8.1.26}$$

Finally, (8.1.10) implies that we have the direct sum decomposition

$$L_1^p(\partial\mathbb{D},\sigma) = \mathcal{H}_1^p(\partial\mathbb{D}_+,\sigma)\oplus\mathcal{H}_1^p(\partial\mathbb{D}_-,\sigma). \tag{8.1.27}$$

Our first result amounts to a "structure theorem" for null-solutions of the Bitsadze operator $\partial_{\bar z}^2$ in the unit disk of the complex plane. To facilitate its statement, we agree to abbreviate

$$\delta_{\partial\mathbb{D}}(z) := \mathrm{dist}\big(z,\partial\mathbb{D}\big) = 1-|z|, \text{ for each } z \in \mathbb{D}. \tag{8.1.28}$$

Theorem 8.1.1 *Let* $\mathbb{D} := \big\{z \in \mathbb{C} : |z| < 1\big\}$ *be the unit disk of the complex plane. Then, for a given complex-valued function u defined in* $\mathbb{D}$*, the conditions*

$$u \in \mathscr{C}^\infty(\mathbb{D}) \text{ and } \partial_{\bar z}^2 u = 0 \text{ in } \mathbb{D}, \tag{8.1.29}$$

are equivalent to having the decomposition

$$\begin{gathered} u = (1-|z|^2)W + U + \lambda \bar z \text{ in } \mathbb{D} \text{ for some} \\ \text{holomorphic functions } U, W \text{ in } \mathbb{D} \text{ and some } \lambda \in \mathbb{C}. \end{gathered} \tag{8.1.30}$$

Furthermore, U, W, λ are uniquely determined by the function u namely,

$$\lambda = (\partial_{\bar z} u)(0), \tag{8.1.31}$$

while, with 'prime' denoting the ordinary complex differentiation of holomorphic functions, for each $z \in \mathbb{D}$ one has

$$W(z) = \begin{cases} -\dfrac{(\partial_{\bar z} u)(z) - (\partial_{\bar z} u)(0)}{z} & \text{if } z \in \mathbb{D} \setminus \{0\}, \\ -(\partial_{\bar z} u)'(0) & \text{if } z = 0, \end{cases} \tag{8.1.32}$$

and

$$U(z) = \begin{cases} u(z) - \bar z (\partial_{\bar z} u)(z) + \dfrac{(\partial_{\bar z} u)(z) - (\partial_{\bar z} u)(0)}{z} & \text{if } z \in \mathbb{D} \setminus \{0\}, \\ u(0) + (\partial_{\bar z} u)'(0) & \text{if } z = 0. \end{cases} \tag{8.1.33}$$

In relation to the decomposition (8.1.30) of a function u as in (8.1.29), the following additional properties are valid for any integrability exponent $p \in (1, \infty)$ and any aperture parameter $\kappa \in (0, \infty)$:

(1) For any function u as in (8.1.29) one has (recall (8.1.1), (8.1.2), and (8.1.28))

$$\begin{gathered} \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma) \text{ if and only if} \\ U \in \mathcal{H}^p(\mathbb{D}) \text{ and } \mathcal{N}_\kappa\big(\delta_{\partial\mathbb{D}} \cdot W\big) \in L^p(\partial\mathbb{D}, \sigma). \end{gathered} \tag{8.1.34}$$

(2) For any function u as in (8.1.29) one has

$$\begin{gathered} u\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\mathbb{D} \text{ if and only if} \\ U\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \text{ exists and } \big[\delta_{\partial\mathbb{D}} \cdot W\big]\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} = 0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}. \end{gathered} \tag{8.1.35}$$

(3) For any function u as in (8.1.29) satisfying two additional properties, namely that $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma)$ and the nontangential boundary trace $u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\mathbb{D}$, it follows that

$$\begin{gathered} \text{the function } u\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} - \lambda \bar z \text{ belongs to} \\ \text{the boundary Hardy space } \mathcal{H}^p(\partial\mathbb{D}_+, \sigma), \end{gathered} \tag{8.1.36}$$

and, with $\mathscr{C}$ denoting the boundary-to-domain Cauchy operator associated with the unit disk (cf. (8.1.5)), one has

$$U = \mathscr{C}\big(u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\big) \ \text{in} \ \mathbb{D}. \tag{8.1.37}$$

(4) For any function u as in (8.1.29) and with the additional property that $\mathcal{N}_\kappa u$ belongs to $L^p(\partial\mathbb{D},\sigma)$ one has

$$\begin{gathered} u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0 \ \text{at } \sigma\text{-a.e. point on } \partial\mathbb{D} \ \text{if and only if} \\ U = 0, \ \lambda = 0, \ \text{and} \ \big[\delta_{\partial\mathbb{D}} \cdot W\big]\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0 \ \text{at } \sigma\text{-a.e. point on } \partial\mathbb{D}. \end{gathered} \tag{8.1.38}$$

(5) For any function u as in (8.1.29), the following equivalences are true: first,

$$\mathcal{N}_\kappa(\partial_{\bar z} u) \in L^p(\partial\mathbb{D},\sigma) \ \text{if and only if} \ W \in \mathcal{H}^p(\mathbb{D}), \tag{8.1.39}$$

second,

$$\begin{gathered} \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D},\sigma) \ \text{if and only if} \\ W \in \mathcal{H}^p(\mathbb{D}) \ \text{and} \ U \in \mathcal{H}^p_1(\mathbb{D}), \end{gathered} \tag{8.1.40}$$

third,

$$\begin{gathered} \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D},\sigma) \ \text{if and only if} \\ \partial_{\bar z} u \in \mathcal{H}^p(\mathbb{D}), \ \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D},\sigma), \ u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} \in L^p_1(\partial\mathbb{D},\sigma), \end{gathered} \tag{8.1.41}$$

and, finally,

$$\begin{gathered} \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D},\sigma) \ \text{if and only if} \\ \partial_{\bar z} u \in \mathcal{H}^p(\mathbb{D}), \ \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D},\sigma), \ u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} \in \mathcal{H}^p_1(\partial\mathbb{D},\sigma) \oplus \langle \bar z \rangle \end{gathered} \tag{8.1.42}$$

where, as before, $\langle \bar z \rangle$ denotes the (one-dimensional) linear span over $\mathbb{C}$ of the function $\partial\mathbb{D} \ni z \mapsto \bar z \in \mathbb{C}$.

We wish to remark that having $u \in \mathscr{C}^\infty(\mathbb{D})$ with $\partial_{\bar z}^2 u = 0$ in $\mathbb{D}$ implies that the function $\partial_{\bar z} u$ is holomorphic in $\mathbb{D}$. In particular, it makes sense to speak of the complex derivative $(\partial_{\bar z} u)'(0)$ in (8.1.32). Let also note that, under the assumptions in item *(3)*, $u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}$ is a function in $L^p(\partial\mathbb{D},\sigma)$, so the Cauchy operator acts in a meaningful manner in (8.1.37). Finally, we observe that

there exists a bounded function $u \in \mathscr{C}^\infty(\mathbb{D})$ with $\partial_{\bar z}^2 u = 0$ in $\mathbb{D}$ and such that for any aperture parameter $\kappa > 0$ the nontangential boundary trace $u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}$ fails to exist at every point on $\partial\mathbb{D}$. (8.1.43)

This is implicit in the discussion in [63, §2] where the function in question, defined (for some fixed even $\alpha \in \mathbb{N}$ sufficiently large) as

$$u(z) := (1-|z|^2)\sum_{k=1}^{\infty} \alpha^k z^{\alpha^k} \text{ for each } z \in \mathbb{D}, \tag{8.1.44}$$

is actually shown to fail to possess a radial limit at each point on $\partial\mathbb{D}$. Another counterexample is described in [1, §4.3, p. 233]. Thus, *in stark contrast with the case of harmonic functions, null-solutions u of the operator $\partial_{\bar z}^2$ in the unit disk may fail to have a nontangential boundary trace at any point on $\partial\mathbb{D}$ even when $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma)$ for each $\kappa > 0$ and each $p \in (0, \infty)$.*

We next give the proof of Theorem 8.1.1.

Proof of Theorem 8.1.1 Assume first that u is an in (8.1.30). A moment's reflection then shows that both conditions in (8.1.29) are satisfied. Furthermore, applying the Cauchy-Riemann operator to the decomposition formula in (8.1.30) yields

$$\partial_{\bar z} u = -zW + \lambda \text{ in } \mathbb{D} \tag{8.1.45}$$

which, in turn, forces $(\partial_{\bar z} u)(0) = \lambda$ (so we necessarily have (8.1.31)) and

$$W(z) = -\frac{(\partial_{\bar z} u)(z) - (\partial_{\bar z} u)(0)}{z} \text{ for each } z \in \mathbb{D} \setminus \{0\}. \tag{8.1.46}$$

In view of the fact that W is continuous (by virtue of being holomorphic) in $\mathbb{D}$ and that, as noted earlier, $\partial_{\bar z} u$ is holomorphic in $\mathbb{D}$, we conclude from (8.1.46) that

$$W(0) = -(\partial_{\bar z} u)'(0). \tag{8.1.47}$$

Collectively, (8.1.46) and (8.1.47) prove (8.1.32). Then a straightforward computation based on (8.1.31) and (8.1.32) shows that $U = u - (1-|z|^2)W - \lambda\bar z$ is necessarily as claimed in (8.1.33). In particular, U, W, λ are uniquely determined by the function u.

In the opposite direction, suppose u is as in (8.1.29). If we define U, W, λ as in (8.1.31)-(8.1.33) then basic algebra shows that the decomposition claimed in (8.1.30) holds. Moreover, the fact that $\partial_{\bar z} u$ is holomorphic in $\mathbb{D}$ renders W holomorphic in $\mathbb{D}$. In concert with the observation that, as seen from (8.1.29), the function

$$\mathbb{D} \ni x \longmapsto u(z) - \bar z(\partial_{\bar z} u)(z) \text{ is holomorphic in } \mathbb{D}, \tag{8.1.48}$$

this also proves that U defined as in (8.1.33) is holomorphic in $\mathbb{D}$. This establishes the equivalence between (8.1.29) and (8.1.30).

To deal with the claims in items *(1)-(5)*, fix some $p \in (1, \infty)$ along with some $\kappa \in (0, \infty)$.

Proof of item (1): In one direction, assume $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma)$. Based on this and the last part in [68, Corollary 8.9.13] we deduce that

$$\mathcal{N}_\kappa\big(\delta_{\partial\mathbb{D}} \cdot \nabla u\big) \in L^p(\partial\mathbb{D}, \sigma). \tag{8.1.49}$$

From (8.1.49) and (8.1.32) we then see that

$$\mathcal{N}_\kappa^\rho\big(\delta_{\partial\mathbb{D}}\cdot W\big)\in L^p(\partial\mathbb{D},\sigma) \text{ for each truncation parameter } \rho\in(0,1/2). \tag{8.1.50}$$

Since W is bounded on compact subsets of $\mathbb{D}$ (by virtue of being continuous in $\mathbb{D}$), we ultimately conclude that

$$\mathcal{N}_\kappa\big(\delta_{\partial\mathbb{D}}\cdot W\big)\in L^p(\partial\mathbb{D},\sigma). \tag{8.1.51}$$

Thus, the last property claimed in the second line of (8.1.34) holds. Next, observe that

$$1-|z|^2=(1+|z|)\delta_{\partial\mathbb{D}}(z) \text{ for each } z\in\mathbb{D}. \tag{8.1.52}$$

In particular, we may recast the decomposition formula from (8.1.30) as

$$U=u-(1+|z|)\big(\delta_{\partial\mathbb{D}}\cdot W\big)-\lambda\bar z \text{ in } \mathbb{D}. \tag{8.1.53}$$

From this, (8.1.51), and the current working assumption we then conclude that $\mathcal{N}_\kappa U\in L^p(\partial\mathbb{D},\sigma)$. In view of (8.1.2), this shows that the first property claimed in the second line of (8.1.34) holds as well. This proves the direct implication in the equivalence stated in (8.1.34).

In the opposite direction, from the second line of (8.1.34) and the decomposition formula from (8.1.30) (reformulated as in (8.1.53)) we readily conclude that $\mathcal{N}_\kappa u$ lies in $L^p(\partial\mathbb{D},\sigma)$.

Proof of item (2): In one direction, assume u is a function as in (8.1.29) with the additional property that the nontangential boundary trace $u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\mathbb{D}$. In concert with the very last part in [68, Proposition 8.9.11] this implies that

$$\begin{gathered}\text{for each aperture parameter } \kappa'\in(0,\kappa) \text{ we have}\\ \big[\delta_{\partial\mathbb{D}}\cdot\nabla u\big]\Big|_{\partial\mathbb{D}}^{\kappa'-\mathrm{n.t.}}=0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}.\end{gathered} \tag{8.1.54}$$

In particular,

$$\big[\delta_{\partial\mathbb{D}}\cdot(\partial_{\bar z}u)\big]\Big|_{\partial\mathbb{D}}^{\kappa'-\mathrm{n.t.}}=0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}, \text{ for each } \kappa'\in(0,\kappa). \tag{8.1.55}$$

From (8.1.55) and (8.1.32) we then deduce that

$$\big[\delta_{\partial\mathbb{D}}\cdot W\big]\Big|_{\partial\mathbb{D}}^{\kappa'-\mathrm{n.t.}}=0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}, \text{ for each } \kappa'\in(0,\kappa). \tag{8.1.56}$$

Based on (8.1.56), the fact that we are presently assuming that the nontangential boundary trace $u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\mathbb{D}$, and (8.1.53) we then conclude that

$$\begin{gathered}\text{for each } \kappa'\in(0,\kappa) \text{ the nontangential boundary trace}\\ U\Big|_{\partial\mathbb{D}}^{\kappa'-\mathrm{n.t.}} \text{ exists (in } \mathbb{C}) \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}.\end{gathered} \tag{8.1.57}$$

Having established this, we may invoke the local Fatou theorem recalled in (8.1.8) (bearing in mind that U is a holomorphic function in $\mathbb{D}$) and conclude that

$$U\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} \text{ exists at } \sigma\text{-a.e. point on } \partial\mathbb{D}. \tag{8.1.58}$$

From this, the fact that we are currently assuming that $u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial\mathbb{D}$, and (8.1.53) we see then that

$$\big[\delta_{\partial\mathbb{D}} \cdot W\big]\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\mathbb{D}. \tag{8.1.59}$$

Finally, combining (8.1.54) and (8.1.59) we arrive at the conclusion that

$$\big[\delta_{\partial\mathbb{D}} \cdot W\big]\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}. \tag{8.1.60}$$

Together, (8.1.58) and (8.1.60) prove the claims made in the second line of (8.1.35). This takes care of the direct implication in the equivalence (8.1.35). The opposite implication in the equivalence claimed in (8.1.35) is clear from (8.1.53).

Proof of item (3): Consider a function u as in (8.1.29), with $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D},\sigma)$, and such that $u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial\mathbb{D}$. From item *(2)* and (8.1.53) we then see that

$$\Big(U\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\Big)(z) = \Big(u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\Big)(z) - \lambda\bar{z} \text{ for } \sigma\text{-a.e. point } z \in \partial\mathbb{D}. \tag{8.1.61}$$

Let us also note that, given the current working assumptions, item *(1)* guarantees that $U \in \mathcal{H}^p(\mathbb{D})$. Based on this, (8.1.61), the Cauchy reproducing and vanishing formulas recorded in (8.1.11)-(8.1.12), and also keeping in mind that the function

$$\mathbb{D} \ni z \mapsto z^{-1} \in \mathbb{C} \text{ belongs to } \mathcal{H}^p(\partial\mathbb{D}_-,\sigma), \tag{8.1.62}$$

we may then write

$$\begin{aligned} U &= \mathscr{C}\Big(U\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\Big) = \mathscr{C}\Big(u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\Big) + \mathscr{C}\big(\lambda\bar{z}\big) \\ &= \mathscr{C}\Big(u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\Big) + \lambda\mathscr{C}\big(z^{-1}\big) = \mathscr{C}\Big(u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\Big) \text{ in } \mathbb{D}, \end{aligned} \tag{8.1.63}$$

establishing (8.1.37). Finally, since $U \in \mathcal{H}^p(\mathbb{D})$, from (8.1.61), (8.1.7), and (8.1.31) we conclude that the membership claimed in (8.1.36) holds as well.

Proof of item (4): Suppose u is as in (8.1.29) and has $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D},\sigma)$. In one direction, if $u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0$ at σ-a.e. point on $\partial\mathbb{D}$ then (8.1.37) gives at once that $U = 0$ in $\mathbb{D}$. Also, from the current working assumptions, (8.1.36), (8.1.62), the fact that the sum in (8.1.10) is direct, and (8.1.31) we deduce that $\lambda = 0$. Since item *(2)* presently implies that $\big[\delta_{\partial\mathbb{D}} \cdot W\big]\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0$ at σ-a.e. point on $\partial\mathbb{D}$, the direct implication in (8.1.38) is established. The converse implication in (8.1.38) is a simple consequence of (8.1.53).

Proof of item (5): Throughout, fix a function u as in (8.1.29). From the definition of W in (8.1.32) it is clear that

$$\text{for each truncation parameter } \rho \in (0, 1/2) \text{ one has} \\ \mathcal{N}_\kappa^\rho(\partial_{\bar{z}} u) \in L^p(\partial\mathbb{D}, \sigma) \text{ if and only if } \mathcal{N}_\kappa^\rho W \in L^p(\partial\mathbb{D}, \sigma). \tag{8.1.64}$$

In view of the fact that both $\partial_{\bar{z}} u$ and W are bounded near the origin, this readily implies that

$$\mathcal{N}_\kappa(\partial_{\bar{z}} u) \in L^p(\partial\mathbb{D}, \sigma) \text{ if and only if } \mathcal{N}_\kappa W \in L^p(\partial\mathbb{D}, \sigma). \tag{8.1.65}$$

Bearing in mind (8.1.2), this establishes the equivalence in (8.1.39).

Assume next that $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma)$. Then (8.1.65) guarantees that W belongs to the Hardy space $\mathcal{H}^p(\mathbb{D})$ which, in concert with the last part in [68, Corollary 8.9.13], further implies that

$$\mathcal{N}_\kappa\big(\delta_{\partial\mathbb{D}} \cdot \nabla W\big) \in L^p(\partial\mathbb{D}, \sigma). \tag{8.1.66}$$

Also, applying the gradient to both sides of the decomposition formula in (8.1.30) gives

$$\begin{aligned} \nabla u &= -2zW + (1 - |z|^2)\nabla W + \nabla U + \langle \lambda, -\mathrm{i}\lambda \rangle \\ &= -2zW + (1 + |z|)\big(\delta_{\partial\mathbb{D}} \cdot \nabla W\big) + \nabla U + \langle \lambda, -\mathrm{i}\lambda \rangle \text{ in } \mathbb{D}. \end{aligned} \tag{8.1.67}$$

From this, the fact that both $\mathcal{N}_\kappa(\nabla u)$ and $\mathcal{N}_\kappa W$ belong to $L^p(\partial\mathbb{D}, \sigma)$, and (8.1.66) we then conclude that $\mathcal{N}_\kappa(\nabla U)$ belongs to $L^p(\partial\mathbb{D}, \sigma)$. Since from item *(1)* we know that $U \in \mathcal{H}^p(\mathbb{D})$, we ultimately conclude that U actually belongs to $\mathcal{H}_1^p(\mathbb{D})$ (defined in (8.1.20)). Another, alternative way of showing that $U \in \mathcal{H}^p(\mathbb{D})$ is to rely on (8.1.37) and the fact that we presently have $u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} \in L_1^p(\partial\mathbb{D}, \sigma)$. This finishes the proof of the direct implication in (8.1.40). In the opposite direction, assume $W \in \mathcal{H}^p(\mathbb{D})$ and $U \in \mathcal{H}_1^p(\mathbb{D})$ Then the last part in [68, Corollary 8.9.13] implies that (8.1.66) holds, so we conclude from (8.1.67) that $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma)$. The equivalence in (8.1.40) is therefore justified.

Moving on, assume $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma)$. Then $\partial_{\bar{z}} u$ is a holomorphic function in $\mathbb{D}$ with $\mathcal{N}_\kappa(\partial_{\bar{z}} u) \in L^p(\partial\mathbb{D}, \sigma)$, hence $\partial_{\bar{z}} u \in \mathcal{H}^p(\mathbb{D})$ (cf. (8.1.2)). Also, [68, Proposition 8.4.9], [68, Proposition 8.9.22], and [69, Proposition 11.3.4] enure that $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma)$ and that the nontangential boundary trace $u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}$ exists and belongs to $L_1^p(\partial\mathbb{D}, \sigma)$. This proves the direct implication in (8.1.41).

In the opposite direction, assume u is a function as in (8.1.29) and has the following additional properties: $\partial_{\bar{z}} u \in \mathcal{H}^p(\mathbb{D})$, the nontangential boundary trace $u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}$ exists and belongs to $L_1^p(\partial\mathbb{D}, \sigma)$, and $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma)$. In concert with item *(3)* and (8.1.26), these properties imply that

$$U = \mathscr{C}\big(u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\big) \in \mathcal{H}_1^p(\mathbb{D}). \tag{8.1.68}$$

In addition, the current assumptions and (8.1.39) guarantee that $W \in \mathcal{H}^p(\mathbb{D})$. Based on this, (8.1.68), and (8.1.40) we conclude that $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma)$. This completes the proof of the equivalence claimed in (8.1.41).

Finally, consider the equivalence claimed in (8.1.42). In one direction, assume u is a function as in (8.1.29) with $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma)$. From (8.1.41) we then conclude that $\partial_{\bar z} u \in \mathcal{H}^p(\mathbb{D})$, $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma)$, and the nontangential boundary trace $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{D}}$ exists and belongs to $L^p_1(\partial\mathbb{D}, \sigma)$. Granted these properties, item *(3)* guarantees that the function $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{D}} - \lambda\bar z$ belongs to the boundary Hardy space $\mathcal{H}^p(\partial\mathbb{D}_+, \sigma)$. As such, said function belongs to

$$\mathcal{H}^p(\partial\mathbb{D}_+, \sigma) \cap L^p_1(\partial\mathbb{D}, \sigma) = \mathcal{H}^p_1(\partial\mathbb{D}_+, \sigma), \tag{8.1.69}$$

with the equality provided by (8.1.23). Ultimately, this shows that $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{D}}$ belongs to $\mathcal{H}^p_1(\partial\mathbb{D}, \sigma) \oplus \langle \bar z \rangle$, so the direct implication in (8.1.42) holds. The opposite implication is readily seen from (8.1.41) and (8.1.23), so the proof of the equivalence claimed in (8.1.42) is complete. □

In the theorem below we identify both the space of admissible boundary data and the space of null-solutions for the L^p Dirichlet Problem for the Bitsadze operator in the unit disk of the complex plane. The main ingredient in the proof is the structural result established earlier in Theorem 8.1.1, centered around the decomposition in (8.1.30). In particular, the nature of the piece U decisively determines the nature of the space of admissible boundary data, while the nature of the piece W is solely responsible for the nature of the space of null-solutions.

Theorem 8.1.2 *Denote by $\mathbb{D}$ be the unit disk in the complex plane, and abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\mathbb{D}$. Fix an integrability exponent $p \in (1, \infty)$ and some aperture parameter $\kappa \in (0, \infty)$. Then the space of admissible boundary data for the L^p Dirichlet Problem for the Bitsadze operator in the unit disk has the following description:*

$$\Big\{ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{D}} : \ u \in \mathscr{C}^\infty(\mathbb{D}),\ \partial^2_{\bar z} u = 0 \ \text{in } \mathbb{D},\ \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma), \ \text{and } u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{D}} \text{ exists } \sigma\text{-a.e. on } \partial\mathbb{D} \Big\} = \mathcal{H}^p(\partial\mathbb{D}_+, \sigma) \oplus \langle \bar z \rangle, \tag{8.1.70}$$

where $\langle \bar z \rangle$ is defined as the (one-dimensional) linear span over the field $\mathbb{C}$ of the function $\partial\mathbb{D} \ni z \mapsto \bar z \in \mathbb{C}$. Alternatively,

$$\Big\{ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{D}} : \ u \in \mathscr{C}^\infty(\mathbb{D}),\ \partial^2_{\bar z} u = 0 \ \text{in } \mathbb{D},\ \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma), \ \text{and } u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{D}} \text{ exists } \sigma\text{-a.e. on } \partial\mathbb{D} \Big\}$$
$$= \Big\{ f \in L^p(\partial\mathbb{D}, \sigma) : \int_{\partial\mathbb{D}} f(z) z^k \, \mathrm{d}\sigma(z) = 0 \ \text{for all } k \in \mathbb{N} \text{ with } k \geq 2 \Big\}. \tag{8.1.71}$$

In addition, the space of null-solutions for the L^p Dirichlet Problem for the Bitsadze operator in the unit disk, i.e.,

$$\Big\{u \in \mathscr{C}^\infty(\mathbb{D}) : \partial_{\bar z}^2 u = 0 \text{ in } \mathbb{D},\ \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D},\sigma),\\ \text{and } u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}\Big\}, \tag{8.1.72}$$

coincides with the infinite dimensional space

$$\Big\{(1-|z|^2)W : W \text{ holomorphic in } \mathbb{D},\ \mathcal{N}_\kappa\big(\delta_{\partial\mathbb{D}}\cdot W\big) \in L^p(\partial\mathbb{D},\sigma),\\ \text{and } \big[\delta_{\partial\mathbb{D}}\cdot W\big]\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}\Big\}. \tag{8.1.73}$$

In light of the classical decomposition (8.1.10), we conclude from (8.1.70) that the cokernel in the full space of boundary data $L^p(\partial\mathbb{D},\sigma)$ of the space of admissible boundary data for the L^p Dirichlet Problem for the Bitsadze operator in the unit disk (described in the left side of (8.1.70)) is isomorphic to

$$\mathcal{H}^p(\partial\mathbb{D}_-,\sigma)\big/\langle\bar z\rangle, \tag{8.1.74}$$

which is an infinite dimensional space. Also, since

$$\big\{(1-|z|^2)W : W \in \mathcal{H}^p(\mathbb{D})\big\} \tag{8.1.75}$$

is an infinite dimensional subspace of (8.1.73), it follows that

> the space of null-solutions for the L^p Dirichlet Problem for the Bitsadze operator in the unit disk (described in (8.1.72)) is infinite dimensional. (8.1.76)

The space (8.1.73) is far richer than the space identified by A.V. Bitsadze in (8.0.1). Indeed, as opposed to the latter, the former contains functions whose nontangential trace fails to be absolutely integrable. An example of this sort is offered by

$$W_* : \mathbb{D} \to \mathbb{C}, \quad W_*(z) := \frac{1}{1-z} \text{ for each } z \in \mathbb{D}. \tag{8.1.77}$$

Obviously, this is holomorphic in $\mathbb{D}$ and belongs to $\mathscr{C}^\infty\big(\overline{\mathbb{D}}\setminus\{1\}\big)$. In particular, $\big[\delta_{\partial\mathbb{D}}\cdot W_*\big]\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0$ at each point in $\partial\mathbb{D}\setminus\{1\}$, hence at σ-a.e. point on $\partial\mathbb{D}$. Since we also have $\delta_{\partial\mathbb{D}}(z) = 1-|z| \leq |1-z|$ for each $z \in \mathbb{D}$, it follows that $\big|\delta_{\partial\mathbb{D}}\cdot W_*\big| \leq 1$ in $\mathbb{D}$, hence $\mathcal{N}_\kappa\big(\delta_{\partial\mathbb{D}}\cdot W_*\big) \in L^\infty(\partial\mathbb{D},\sigma)$. Thus, W_* belongs to (8.1.73) but not to (8.0.1). As a byproduct, the function

$$u_* : \mathbb{D} \to \mathbb{C}, \quad u_*(z) := \frac{1-|z|^2}{1-z} \text{ for each } z \in \mathbb{D} \tag{8.1.78}$$

belongs to the space (8.1.72) but fails to be in $\mathscr{C}^0\big(\overline{\mathbb{D}}\big)$. More generally, for any two finite families $\theta_1,\dots,\theta_N \in [0,2\pi)$ and $c_1,\dots,c_N \in \mathbb{C}$, the function

$$W_o : \mathbb{D} \to \mathbb{C}, \quad W_o(z) := \sum_{j=1}^{N} \frac{c_j}{\mathrm{e}^{\mathrm{i}\theta_j} - z} \text{ for each } z \in \mathbb{D}, \tag{8.1.79}$$

belongs to (8.1.73) but not to (8.0.1).

Finally, we note that thanks to the result recorded in (8.1.43) it is actually necessary to *assume* the existence of the nontangential boundary trace $u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}}$ in (8.1.70) and (8.1.71).

Here is the proof of Theorem 8.1.2.

Proof of Theorem 8.1.2 The left-to-right inclusion in (8.1.70) follows from item *(3)* in Theorem 8.1.1 (cf. (8.1.36)). To justify the right-to-left inclusion in (8.1.70), observe (cf. (8.1.2)) that any function f belonging to $\mathcal{H}^p(\partial\mathbb{D}_+, \sigma) \oplus \langle \bar{z} \rangle$ is of the form

$$f = U\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} + \lambda\bar{z} \tag{8.1.80}$$

for some holomorphic function U in $\mathbb{D}$ with $\mathcal{N}_\kappa U \in L^p(\partial\mathbb{D}, \sigma)$ and some complex number λ. Then the function $u : \mathbb{D} \to \mathbb{C}$ given by $u(z) := U(z) + \lambda\bar{z}$ for each $z \in \mathbb{D}$ satisfies

$$\begin{aligned} & u \in \mathscr{C}^\infty(\mathbb{D}), \ \partial_{\bar{z}}^2 u = 0 \text{ in } \mathbb{D}, \ \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma), \\ & \text{and } u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} = U\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} + \lambda\bar{z} = f \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}. \end{aligned} \tag{8.1.81}$$

This places f in the space appearing in the left side of (8.1.70), so the proof of (8.1.70) is complete. The alternative characterization offered in (8.1.71) is a consequence of (8.1.70) and the fact that

$$\begin{aligned} & \mathcal{H}^p(\partial\mathbb{D}_+, \sigma) \oplus \langle \bar{z} \rangle \\ & = \Big\{ f \in L^p(\partial\mathbb{D}, \sigma) : \int_{\partial\mathbb{D}} f(z) z^k \, \mathrm{d}\sigma(z) = 0 \text{ for all } k \in \mathbb{N} \text{ with } k \geq 2 \Big\}. \end{aligned} \tag{8.1.82}$$

In turn, (8.1.82) is justified by observing that the left-to-right inclusion is directly implied by (8.1.14), while the right-to-left inclusion follows from the observation that if f belongs to the space in the right side of (8.1.82) then the function

$$\partial\mathbb{D} \ni z \longmapsto f(z) - \Big(\fint_{\partial\mathbb{D}} f(\zeta)\zeta \, \mathrm{d}\sigma(\zeta) \Big) \bar{z} \in \mathbb{C} \tag{8.1.83}$$

belongs to the space in the right side of (8.1.14), hence ultimately to $\mathcal{H}^p(\partial\mathbb{D}_+, \sigma)$. This places f in $\mathcal{H}^p(\partial\mathbb{D}_+, \sigma) \oplus \langle \bar{z} \rangle$, finishing the proof of (8.1.82). The claim made in (8.1.71) is therefore established.

Next, from the decomposition in (8.1.30) and items *(1)*, *(4)* of Theorem 8.1.1 we see that the space (8.1.72) is contained in (8.1.73). Conversely, given any holomorphic function W in $\mathbb{D}$ with $\mathcal{N}_\kappa\big(\delta_{\partial\mathbb{D}} \cdot W\big) \in L^p(\partial\mathbb{D}, \sigma)$ and such that $\big[\delta_{\partial\mathbb{D}} \cdot W\big]\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} = 0$ at σ-a.e. point on $\partial\mathbb{D}$, it follows that $u := (1 - |z|^2)W$ belongs to $\mathscr{C}^\infty(\mathbb{D})$, solves $\partial_{\bar{z}}^2 u = 0$ in $\mathbb{D}$, has $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma)$, and satisfies $u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} = 0$ at σ-a.e. point on

$\partial\mathbb{D}$. This places u in the space (8.1.72). Hence, the space (8.1.73) is contained in (8.1.72). □

We wish to augment Theorem 8.1.2 with a result of similar flavor, describing the space of admissible boundary data and the space of null-solutions for the L_1^p Regularity Problem for the Bitsadze operator in the unit disk of the complex plane.

Theorem 8.1.3 *Let $\mathbb{D}$ be the unit disk in the complex plane, and set $\sigma := \mathcal{H}^1 \lfloor \partial\mathbb{D}$. Also, pick an exponent $p \in (1,\infty)$ and some aperture parameter $\kappa \in (0,\infty)$. Then the space of admissible boundary data for the L_1^p Regularity Problem for the Bitsadze operator in the unit disk has the following description:*

$$\Big\{u\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} : u \in \mathscr{C}^\infty(\mathbb{D}),\ \partial_{\bar z}^2 u = 0 \ \textit{in}\ \mathbb{D},\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D},\sigma)\Big\}$$
$$= \mathcal{H}_1^p(\partial\mathbb{D}_+,\sigma) \oplus \langle \bar z\rangle \tag{8.1.84}$$

where, as in the past, $\langle \bar z\rangle$ is the (one-dimensional) linear span over $\mathbb{C}$ of the function $\partial\mathbb{D} \ni z \mapsto \bar z \in \mathbb{C}$. Alternatively,

$$\Big\{u\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} : u \in \mathscr{C}^\infty(\mathbb{D}),\ \partial_{\bar z}^2 u = 0 \ \textit{in}\ \mathbb{D},\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D},\sigma), \tag{8.1.85}$$
$$\textit{and}\ u\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \ \textit{exists}\ \sigma\textit{-a.e. on}\ \partial\mathbb{D}\Big\}$$
$$= \Big\{f \in L_1^p(\partial\mathbb{D},\sigma) : \int_{\partial\mathbb{D}} f(z) z^k \, d\sigma(z) = 0 \ \textit{for all}\ k \in \mathbb{N}\ \textit{with}\ k \geq 2\Big\}.$$

Moreover, the space of null-solutions for the L_1^p Regularity Problem for the Bitsadze operator in the unit disk, i.e.,

$$\Big\{u \in \mathscr{C}^\infty(\mathbb{D}) : \ \partial_{\bar z}^2 u = 0 \ \textit{in}\ \mathbb{D},\ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D},\sigma),$$
$$\textit{and}\ u\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} = 0 \ \textit{at}\ \sigma\textit{-a.e. point on}\ \partial\mathbb{D}\Big\}, \tag{8.1.86}$$

coincides with

$$\big\{(1-|z|^2)W : \ W \in \mathcal{H}^p(\mathbb{D})\big\}. \tag{8.1.87}$$

Before presenting the proof of this theorem we make a three comments. First, given any function $u \in \mathscr{C}^\infty(\mathbb{D})$ with $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D},\sigma)$, the result established in [68, (8.9.236)] implies that the nontangential trace $u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\mathbb{D}$. This is relevant in the context of (8.1.84).

Second, from (8.1.27) and (8.1.84) we see that

> the quotient space between the full space of boundary data $L_1^p(\partial\mathbb{D},\sigma)$ and the space of admissible boundary data for the L_1^p Regularity Problem for the Bitsadze operator in $\mathbb{D}$ (described in the left side of (8.1.84)) is isomorphic to the infinite dimensional space $\mathcal{H}_1^p(\partial\mathbb{D}_-,\sigma)\big/\langle \bar z\rangle$. (8.1.88)

Third, as seen from the last part in the statement of the above theorem,

$$\text{the space of null-solutions for the } L_1^p \text{ Regularity Problem for the Bitsadze operator in the unit disk (described in (8.1.86)) is infinite dimensional.} \tag{8.1.89}$$

The proof of Theorem 8.1.3 is given next:

Proof of Theorem 8.1.3 The left-to-right inclusion in (8.1.84) is seen from (8.1.42). As regards the right-to-left inclusion in (8.1.84), first note from (8.1.22) that any function f belonging to $\mathcal{H}_1^p(\partial\mathbb{D}_+, \sigma) \oplus \langle \bar{z} \rangle$ is of the form $f = U\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} + \lambda\bar{z}$ for some holomorphic function U in $\mathbb{D}$ with $\mathcal{N}_\kappa(\nabla U) \in L^p(\partial\mathbb{D}, \sigma)$ and some complex number λ. Then the function $u : \mathbb{D} \to \mathbb{C}$ given for each $z \in \mathbb{D}$ by $u(z) := U(z) + \lambda\bar{z}$ belongs to the space in the left side of (8.1.84) and satisfies

$$u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = U\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} + \lambda\bar{z} = f \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}. \tag{8.1.90}$$

As a consequence, f belongs to the space appearing in the left side of (8.1.84). This establishes (8.1.84). The alternative characterization given in (8.1.85) is then implied by (8.1.85) and the fact that

$$\begin{aligned} &\mathcal{H}_1^p(\partial\mathbb{D}_+, \sigma) \oplus \langle \bar{z} \rangle \\ &\quad = \Big\{ f \in L_1^p(\partial\mathbb{D}, \sigma) : \int_{\partial\mathbb{D}} f(z) z^k \, d\sigma(z) = 0 \text{ for all } k \in \mathbb{N} \text{ with } k \geq 2 \Big\}, \end{aligned} \tag{8.1.91}$$

itself a consequence of (8.1.91) and (8.1.23).

Next, let us justify the fact that the spaces in (8.1.86) and (8.1.87) coincide. First, from the decomposition in (8.1.30) item *(4)* of Theorem 8.1.1 and (8.1.40) it follows that the space (8.1.86) is contained in (8.1.87). For the opposite inclusion, given any function $W \in \mathcal{H}^p(\mathbb{D})$ it readily follows that $u := (1 - |z|^2)W$ belongs to $\mathscr{C}^\infty(\mathbb{D})$, solves $\partial_{\bar{z}}^2 u = 0$ in $\mathbb{D}$, and satisfies $u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0$ at σ-a.e. point on $\partial\mathbb{D}$. Also, (8.1.40) (and the uniqueness of the decomposition in (8.1.30)) gives that $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma)$. Hence, u belongs to the space (8.1.86), thus finishing the proof of the fact that the space (8.1.87) is contained in (8.1.86). □

Moving on, we present a "structure theorem" for null-solutions of the Bitsadze operator $\partial_{\bar{z}}^2$ in the *complement* of the closed unit disk in the complex plane (compare with Theorem 8.1.1). The reader is advised to recall (8.1.1), (8.1.3), (8.1.7), and (8.1.28).

Theorem 8.1.4 *Let* $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ *be the unit disk of the complex plane and abbreviate* $\mathbb{D}_- := \mathbb{C} \setminus \overline{\mathbb{D}}$. *Then, for a given complex-valued function u defined in* $\mathbb{D}_-$, *the conditions*

$$u \in \mathscr{C}^\infty(\mathbb{D}_-) \text{ and } \partial_{\bar{z}}^2 u = 0 \text{ in } \mathbb{D}_-, \tag{8.1.92}$$

are equivalent to having the decomposition

$$
\begin{aligned}
&u = (1 - |z|^2)W + U \ \textit{ in } \ \mathbb{D}_- \\
&\textit{for some holomorphic functions } \ U, W \ \textit{ in } \ \mathbb{D}_-.
\end{aligned}
\tag{8.1.93}
$$

Moreover, U, W are uniquely determined by the function u, namely

$$
W(z) = -\frac{(\partial_{\bar{z}} u)(z)}{z} \ \textit{ for each } \ z \in \mathbb{D}_-, \tag{8.1.94}
$$

and

$$
U(z) = u(z) - \bar{z}(\partial_{\bar{z}} u)(z) + \frac{(\partial_{\bar{z}} u)(z)}{z} \ \textit{ for each } \ z \in \mathbb{D}_-. \tag{8.1.95}
$$

Also,

$$
\begin{aligned}
&u(\infty) = 0 \ \textit{ if and only if} \\
&U(\infty) = 0 \ \textit{ and } \ W(z) = o(1/|z|^2) \ \textit{ as } \ z \to \infty,
\end{aligned}
\tag{8.1.96}
$$

and

$$
\begin{aligned}
&u(z) = O(1) \ \textit{ as } \ z \to \infty \ \textit{ if and only if there exists } \ c \in \mathbb{C} \ \textit{ such that} \\
&U(z) = c + O(1/|z|) \ \textit{ and } \ W(z) = O(1/|z|^2) \ \textit{ as } \ z \to \infty.
\end{aligned}
\tag{8.1.97}
$$

Regarding the decomposition (8.1.93) of a function u as in (8.1.92), the following additional properties are valid for any integrability exponent $p \in (1, \infty)$ and any aperture parameter $\kappa \in (0, \infty)$:

(1) Let u be a function as in (8.1.92). Then

$$
\begin{aligned}
&\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma) \ \textit{ if and only if} \\
&\mathcal{N}_\kappa U \in L^p(\partial\mathbb{D}, \sigma) \ \textit{ and } \ \mathcal{N}_\kappa\big((1 - |z|^2)W\big) \in L^p(\partial\mathbb{D}, \sigma),
\end{aligned}
\tag{8.1.98}
$$

Thus, as a consequence of (8.1.98), (8.1.96), and (8.1.3),

$$
\begin{aligned}
&\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma) \ \textit{ and } \ u(\infty) = 0 \ \textit{ if and only if} \\
&U \in \mathcal{H}^p(\mathbb{D}_-), \ \mathcal{N}_\kappa\big((1 - |z|^2)W\big) \in L^p(\partial\mathbb{D}, \sigma), \\
&\textit{and } \ W(z) = o(1/|z|^2) \ \textit{ as } \ z \to \infty,
\end{aligned}
\tag{8.1.99}
$$

while as a consequence of (8.1.98), (8.1.97), and (8.1.3),

$$
\begin{aligned}
&\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma) \ \textit{ and } \ u(z) = O(1) \ \textit{ as } \ z \to \infty \ \textit{ if and only if} \\
&U \in \mathcal{H}^p(\mathbb{D}_-) \oplus \mathbb{C}, \ \mathcal{N}_\kappa\big((1 - |z|^2)W\big) \in L^p(\partial\mathbb{D}, \sigma), \\
&\textit{and } \ W(z) = O(1/|z|^2) \ \textit{ as } \ z \to \infty.
\end{aligned}
\tag{8.1.100}
$$

(2) For any function u as in (8.1.92) one has

$$
\begin{aligned}
&u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \ \textit{ exists at } \sigma\textit{-a.e. point on } \ \partial\mathbb{D} \ \textit{ if and only if} \\
&U\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \ \textit{ exists and } \ \big[(1 - |z|^2)W\big]\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} = 0 \ \textit{ at } \sigma\textit{-a.e. point on } \ \partial\mathbb{D}.
\end{aligned}
\tag{8.1.101}
$$

(3) *For any given function u as in* (8.1.92) *satisfying the additional properties that* $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D},\sigma)$*, the nontangential boundary trace* $u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}}$ *exists at* σ*-a.e. point on* $\partial\mathbb{D}$*, and* $u(\infty)=0$ *it follows that*

$$u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \in \mathcal{H}^p(\partial\mathbb{D}_-,\sigma), \tag{8.1.102}$$

and, with $\mathscr{C}$ *denoting the boundary-to-domain Cauchy operator associated with* $\mathbb{D}_-$*, one has*

$$U = \mathscr{C}\big(u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}}\big) \ \text{ in } \mathbb{D}_-. \tag{8.1.103}$$

If in place of u vanishing at infinity one now assumes that $u(z)=O(1)$ *as* $z\to\infty$*, then in place of* (8.1.102)-(8.1.103) *one obtains*

$$u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \in \mathcal{H}^p(\partial\mathbb{D}_-,\sigma)\oplus\mathbb{C} \tag{8.1.104}$$

and, respectively,

$$U = c + \mathscr{C}\big(u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}}\big) \ \text{ in } \mathbb{D}_-, \ \text{ for some } c\in\mathbb{C}. \tag{8.1.105}$$

(4) *For any given function u as in* (8.1.92) *and with the additional properties that* $\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D},\sigma)$ *and* $u(z)=O(1)$ *as* $z\to\infty$ *(hence, in particular, if* $u(\infty)=0$*) one has*

$$\begin{gathered} u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} = 0 \ \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D} \ \text{ if and only if} \\ U=0 \ \text{ in } \mathbb{D}_- \ \text{ and } \ \big[(1-|z|^2)W\big]\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} = 0 \ \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}. \end{gathered} \tag{8.1.106}$$

(5) *For any function u as in* (8.1.92) *plus the additional property that* $u(\infty)=0$*, the following equivalences are true: first,*

$$\mathcal{N}_\kappa(\partial_{\bar z}u) \in L^p(\partial\mathbb{D},\sigma) \ \text{ if and only if } \ W\in\mathcal{H}^p(\mathbb{D}_-), \tag{8.1.107}$$

second,

$$\begin{gathered} \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D},\sigma) \ \text{ if and only if} \\ W\in\mathcal{H}^p(\mathbb{D}_-) \ \text{ and } \ U\in\mathcal{H}^p_1(\mathbb{D}_-), \end{gathered} \tag{8.1.108}$$

third,

$$\begin{gathered} \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D},\sigma) \ \text{ if and only if} \\ \partial_{\bar z}u\in\mathcal{H}^p(\mathbb{D}_-), \ \mathcal{N}_\kappa u\in L^p(\partial\mathbb{D},\sigma), \ u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \in L^p_1(\partial\mathbb{D},\sigma), \end{gathered} \tag{8.1.109}$$

and, finally,

$$\begin{gathered} \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D},\sigma) \ \text{ if and only if} \\ \partial_{\bar z}u\in\mathcal{H}^p(\mathbb{D}_-), \ \mathcal{N}_\kappa u\in L^p(\partial\mathbb{D},\sigma), \ u\big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \in \mathcal{H}^p_1(\partial\mathbb{D}_-,\sigma). \end{gathered} \tag{8.1.110}$$

Finally, if in place of u vanishing at infinity one now assumes that $u(z) = O(1)$ *as* $z \to \infty$, *then* (8.1.107) *and* (8.1.109) *remain valid, while in place of* (8.1.108) *and* (8.1.110) *one now has*

$$\begin{gathered}\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma) \text{ if and only if} \\ W \in \mathcal{H}^p(\mathbb{D}_-) \text{ and } U \in \mathcal{H}_1^p(\mathbb{D}_-) \oplus \mathbb{C},\end{gathered} \tag{8.1.111}$$

and, respectively,

$$\begin{gathered}\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma) \text{ if and only if} \\ \partial_{\bar{z}} u \in \mathcal{H}^p(\mathbb{D}_-),\ \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma),\ u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} \in \mathcal{H}_1^p(\partial\mathbb{D}_-, \sigma) \oplus \mathbb{C}.\end{gathered} \tag{8.1.112}$$

Proof Clearly, any function as in (8.1.93) satisfies both conditions in (8.1.92). If we now start with u as in (8.1.92) and define W, U as in (8.1.94)-(8.1.95), then since both $\partial_{\bar{z}} u$ and $u - \bar{z}\partial_{\bar{z}} u$ are holomorphic in $\mathbb{D}_-$, it follows that W, U are holomorphic functions in $\mathbb{D}_-$ and the decomposition in (8.1.93) holds. Moreover, applying $\partial_{\bar{z}}$ to said decomposition reveals that W, U are necessarily given by the formulas (8.1.94)-(8.1.95). In particular, W, U are uniquely determined by u.

Next, the equivalence claimed in (8.1.96) is a consequence of (8.1.93), the fact that W, U are holomorphic functions in $\mathbb{D}_-$, the function u is a null-solution of the weakly elliptic operator $\partial_{\bar{z}}^2$ in $\mathbb{D}_-$, and [70, Lemma 1.5.6]. The equivalence in (8.1.97) is justified similarly, now also making use of [70, Corollary 1.6.8] (with $D := \partial_{\bar{z}}$).

To deal with the equivalence claimed in (8.1.98), assume first that $\mathcal{N}_\kappa u$ belongs to $L^p(\partial\mathbb{D}, \sigma)$. In concert with the last part in [68, Corollary 8.9.13] this implies

$$\mathcal{N}_\kappa\big(\delta_{\partial\mathbb{D}} \cdot \partial_{\bar{z}} u\big) \in L^p(\partial\mathbb{D}, \sigma). \tag{8.1.113}$$

In view of (8.1.28) and the fact that the function $\mathbb{D}_- \ni z \mapsto (1 + |z|)/z \in \mathbb{C}$ is bounded, from (8.1.113) we conclude that

$$\mathcal{N}_\kappa\Big(\mathbb{D}_- \ni z \mapsto (1 - |z|^2)\tfrac{(\partial_{\bar{z}} u)(z)}{z}\Big) \in L^p(\partial\mathbb{D}, \sigma). \tag{8.1.114}$$

Glancing at (8.1.94) then shows that $\mathcal{N}_\kappa\big((1 - |z|^2)W\big) \in L^p(\partial\mathbb{D}, \sigma)$. From (8.1.93) and the working hypothesis we also see that $\mathcal{N}_\kappa U \in L^p(\partial\mathbb{D}, \sigma)$. This proves the direct implication in (8.1.98). The converse implication in (8.1.98) is clear from (8.1.93).

The claim in item *(2)* is justified by arguing much as in the proof of (8.1.35) (while keeping in mind (8.1.52)). The claims in the first part of item *(3)* are consequences of what we have proved in items *(1)*-*(2)* and Cauchy's reproducing formula (8.1.11). The claims in the second part of item *(3)*, corresponding to the case when we now assume that $u(z) = O(1)$ as $z \to \infty$, are dealt with similarly. The claims in item *(4)* are implied by items *(1)*-*(3)*.

There remains to justify the claims made in item *(5)*. Assume u is a function as in (8.1.92). If $u(z) = O(1)$ as $z \to \infty$, then (8.1.97) and [70, Lemma 1.5.6] imply

$$(\nabla u)(z) = O(1/|z|), \quad (\nabla U)(z) = O(1/|z|^2),$$
$$\text{and } W(z) = O(1/|z|^2) \text{ as } z \to \infty. \tag{8.1.115}$$

Consequently, for any fixed truncation parameter $\rho \in (0, \infty)$ we have:

$$\begin{aligned}
\mathcal{N}_\kappa(\partial_{\bar z} u) \in L^p(\partial\mathbb{D}, \sigma) &\text{ if and only if } \mathcal{N}_\kappa^\rho(\partial_{\bar z} u) \in L^p(\partial\mathbb{D}, \sigma), \\
\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma) &\text{ if and only if } \mathcal{N}_\kappa^\rho(\nabla u) \in L^p(\partial\mathbb{D}, \sigma), \\
\mathcal{N}_\kappa(\nabla U) \in L^p(\partial\mathbb{D}, \sigma) &\text{ if and only if } \mathcal{N}_\kappa^\rho(\nabla U) \in L^p(\partial\mathbb{D}, \sigma), \\
\mathcal{N}_\kappa W \in L^p(\partial\mathbb{D}, \sigma) &\text{ if and only if } \mathcal{N}_\kappa^\rho W \in L^p(\partial\mathbb{D}, \sigma).
\end{aligned} \tag{8.1.116}$$

Keeping these in mind, then the same type of argument as in the proof of (8.1.39)-(8.1.42), now making use of (8.1.94)-(8.1.95) and what we have proved already in the current items *(1)-(4)*, yields all desired conclusions. □

There are similar results to those established in Theorem 8.1.2, identifying both the space of admissible boundary data and the space of null-solutions for the L^p Dirichlet Problem for the Bitsadze operator in the complement of the closed unit disk of the complex plane.

Theorem 8.1.5 *Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, and set $\sigma := \mathcal{H}^1\lfloor\partial\mathbb{D}$. Also, pick an exponent $p \in (1, \infty)$ and some aperture parameter $\kappa \in (0, \infty)$. Then the space of admissible boundary data for the L^p Dirichlet Problem for the Bitsadze operator in the complement of the closed unit disk has the following description:*

$$\Big\{u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} : u \in \mathscr{C}^\infty(\mathbb{C} \setminus \overline{\mathbb{D}}),\ \partial_{\bar z}^2 u = 0 \text{ in } \mathbb{C} \setminus \overline{\mathbb{D}},\ \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma), \tag{8.1.117}$$
$$u(\infty) = 0, \text{ and } u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\mathbb{D}\Big\} = \mathcal{H}^p(\partial\mathbb{D}_-, \sigma).$$

Alternatively,

$$\Big\{u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} : u \in \mathscr{C}^\infty(\mathbb{C} \setminus \overline{\mathbb{D}}),\ \partial_{\bar z}^2 u = 0 \text{ in } \mathbb{C} \setminus \overline{\mathbb{D}},\ \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \sigma), \tag{8.1.118}$$
$$u(\infty) = 0, \text{ and } u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\mathbb{D}\Big\}$$
$$= \Big\{f \in L^p(\partial\mathbb{D}, \sigma) : \int_{\partial\mathbb{D}} f(z) z^{-k}\, d\sigma(z) = 0 \text{ for all } k \in \mathbb{N}_0\Big\}.$$

Also, if in place of the vanishing condition at infinity one now assumes mere boundedness for the function u, then in place of (8.1.117)-(8.1.118) *one obtains*

$$\Big\{u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} : u \in \mathscr{C}^\infty\big(\mathbb{C}\setminus\overline{\mathbb{D}}\big),\ \partial_{\bar z}^2 u = 0 \ \textit{in}\ \mathbb{C}\setminus\overline{\mathbb{D}},\ \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D},\sigma),$$
$$u(z) = O(1)\ \textit{as}\ z\to\infty,\ u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\ \textit{exists}\ \sigma\textit{-a.e. on}\ \partial\mathbb{D}\Big\}$$
$$= \mathcal{H}^p(\partial\mathbb{D}_-,\sigma)\oplus\mathbb{C}$$
$$= \Big\{f\in L^p(\partial\mathbb{D},\sigma) : \int_{\partial\mathbb{D}} f(z) z^{-k}\,d\sigma(z) = 0\ \textit{for all}\ k\in\mathbb{N}\Big\}. \tag{8.1.119}$$

Moreover, the space of null-solutions for the L^p Dirichlet Problem for the Bitsadze operator in the complement of the closed unit disk, i.e.,

$$\Big\{u\in\mathscr{C}^\infty\big(\mathbb{C}\setminus\overline{\mathbb{D}}\big) : \partial_{\bar z}^2 u = 0\ \textit{in}\ \mathbb{C}\setminus\overline{\mathbb{D}},\ \mathcal{N}_\kappa u\in L^p(\partial\mathbb{D},\sigma),$$
$$u(z) = o(1)\ \textit{as}\ z\to\infty,\ u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0\ \textit{at}\ \sigma\textit{-a.e. point on}\ \partial\mathbb{D}\Big\}, \tag{8.1.120}$$

coincides with

$$\Big\{(1-|z|^2)W : W\ \textit{holomorphic in}\ \mathbb{C}\setminus\overline{\mathbb{D}},\ \mathcal{N}_\kappa\big((1-|z|^2)W\big)\in L^p(\partial\mathbb{D},\sigma),$$
$$\big[(1-|z|^2)W\big]\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0\ \textit{at}\ \sigma\textit{-a.e. point on}\ \partial\mathbb{D},$$
$$\textit{and}\ W(z) = o(1/|z|^2)\ \textit{as}\ z\to\infty\Big\}. \tag{8.1.121}$$

Finally, the same characterization remains valid provided little "o" is replaced by big "O" both in (8.1.120) *and in* (8.1.121)*.*

From (8.1.10) and (8.1.117) we see that the quotient space between the full space of boundary data $L^p(\partial\mathbb{D},\sigma)$ and the space of admissible boundary data for the L^p Dirichlet Problem for the Bitsadze operator in the complement of the closed unit disk (described in the left side of (8.1.117)) is isomorphic to the boundary Hardy space $\mathcal{H}^p(\partial\mathbb{D}_+,\sigma)$, which is infinite dimensional. Also, since

$$\Big\{\frac{1-|z|^2}{z^{k+2}} : k\in\mathbb{N}\Big\} \tag{8.1.122}$$

is an infinite dimensional subspace of (8.1.121), the last part in the statement of the above theorem implies that

> the space of null-solutions for the L^p Dirichlet Problem for the Bitsadze operator in the complement of the closed unit disk is infinite dimensional. (8.1.123)

Here is the proof of Theorem 8.1.5:

Proof of Theorem 8.1.5 This is justified by reasoning much as in the proof of Theorem 8.1.2, now using Theorem 8.1.4 in place of Theorem 8.1.1, and also keeping in mind (8.1.15). □

Remark 8.1.6 *As a result of the jump-formula valid for any double layer across the boundary of a* UR *domain* $\Omega \subset \mathbb{R}^n$*, the sum of the spaces of admissible boundary data for the* L^p *Dirichlet Problems in* Ω *and* $\mathbb{R}^n \setminus \overline{\Omega}$*, for any given weakly elliptic, second-order, homogeneous, constant* (*complex*) *coefficient,* $M \times M$ *system in* $\mathbb{R}^n$ *is always the full space* $\big[L^p(\partial\Omega, \sigma)\big]^M$ (*where, as usual,* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$).

In the case $L = \partial_{\bar{z}}^2$*, the Bitsadze operator in the plane* (*hence* $n = 2$ *and* $M = 1$)*, and with* $\Omega := \mathbb{D}$*, the open unit disk of the complex plane, we see from* (8.1.70)*,* (8.1.117)*, and* (8.1.10) *that the sum of the two spaces of admissible boundary data referred to in the previous paragraph is*

$$\Big(\mathcal{H}^p(\partial\mathbb{D}_+, \sigma) \oplus \langle\bar{z}\rangle\Big) + \mathcal{H}^p(\partial\mathbb{D}_-, \sigma) = L^p(\partial\mathbb{D}, \sigma), \tag{8.1.124}$$

as anticipated.

Finally, we are also in a position to identify the space of null-solutions and the space of admissible boundary data for the L_1^p Regularity Problem for the Bitsadze operator in the complement of the closed unit disk in the complex plane.

Theorem 8.1.7 *Let* $\mathbb{D}$ *be the unit disk in* $\mathbb{C}$*, and set* $\sigma := \mathcal{H}^1\lfloor\partial\mathbb{D}$*. Also, pick an exponent* $p \in (1, \infty)$ *and some aperture parameter* $\kappa \in (0, \infty)$*. Then the space of admissible boundary data for the* L_1^p *Regularity Problem for the Bitsadze operator in the complement of the closed unit disk, i.e.,*

$$\Big\{u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} : \ u \in \mathscr{C}^\infty\big(\mathbb{C} \setminus \overline{\mathbb{D}}\big), \ \partial_{\bar{z}}^2 u = 0 \ \text{ in } \ \mathbb{C} \setminus \overline{\mathbb{D}}, \ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma), \ u(\infty) = 0\Big\} \tag{8.1.125}$$

is precisely the boundary Hardy space with regularity

$$\mathcal{H}_1^p(\partial\mathbb{D}_-, \sigma). \tag{8.1.126}$$

Alternatively,

$$\begin{aligned} &\Big\{u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} : u \in \mathscr{C}^\infty\big(\mathbb{C} \setminus \overline{\mathbb{D}}\big), \ \partial_{\bar{z}}^2 u = 0 \ \text{ in } \ \mathbb{C} \setminus \overline{\mathbb{D}}, \ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma), \ u(\infty) = 0\Big\} \\ &\qquad = \Big\{f \in L_1^p(\partial\mathbb{D}, \sigma) : \int_{\partial\mathbb{D}} f(z) z^{-k}\, d\sigma(z) = 0 \text{ for all } k \in \mathbb{N}_0\Big\}. \end{aligned} \tag{8.1.127}$$

In addition, if in place of the vanishing condition at infinity one now assumes mere boundedness for the function u, then in place of (8.1.125)-(8.1.127) *one has*

$$\begin{aligned} &\Big\{u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} : u \in \mathscr{C}^\infty\big(\mathbb{C} \setminus \overline{\mathbb{D}}\big), \ \partial_{\bar{z}}^2 u = 0 \ \text{ in } \ \mathbb{C} \setminus \overline{\mathbb{D}}, \\ &\qquad \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma), \ \text{ and } \ u(z) = O(1) \ \text{ as } \ z \to \infty\Big\} \\ &\quad = \mathcal{H}_1^p(\partial\mathbb{D}_-, \sigma) \oplus \mathbb{C} \\ &\quad = \Big\{f \in L_1^p(\partial\mathbb{D}, \sigma) : \int_{\partial\mathbb{D}} f(z) z^{-k}\, d\sigma(z) = 0 \text{ for all } k \in \mathbb{N}\Big\}. \end{aligned} \tag{8.1.128}$$

Furthermore, the space of null-solutions for the L^p_1 Regularity Problem for the Bitsadze operator in the complement of the closed unit disk, i.e.,

$$\Big\{u \in \mathscr{C}^\infty\big(\mathbb{C} \setminus \overline{\mathbb{D}}\big) : \partial_{\bar z}^2 u = 0 \ \text{in} \ \mathbb{C} \setminus \overline{\mathbb{D}}, \ \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma), \qquad (8.1.129)$$
$$u(z) = o(1) \ \text{as} \ z \to \infty, \ u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0 \ \text{at } \sigma\text{-a.e. point on } \partial\mathbb{D}\Big\},$$

coincides with

$$\Big\{(1-|z|^2)W : \ W \in \mathcal{H}^p(\mathbb{D}_-) \ \text{and} \ W(z) = o(1/|z|^2) \ \text{as} \ z \to \infty\Big\}. \qquad (8.1.130)$$

Finally, the same identification remains valid if little "o" is replaced by big "O" both in (8.1.129) and in (8.1.130).

Given any function $u \in \mathscr{C}^\infty\big(\mathbb{C} \setminus \overline{\mathbb{D}}\big)$ with $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\mathbb{D}, \sigma)$, the result established in [68, (8.9.236)] guarantees that the nontangential trace $u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial\mathbb{D}$. This observation is relevant in the context of (8.1.125) and (8.1.128).

Theorem 8.1.7 has two significant consequences. First, the quotient space between the full space of boundary data $L^p_1(\partial\mathbb{D}, \sigma)$ and the space of admissible boundary data for the L^p_1 Regularity Problem for the Bitsadze operator in the complement of the closed unit disk (described in (8.1.125)) is isomorphic to the boundary Hardy space $\mathcal{H}^p(\partial\mathbb{D}_+, \sigma)$, which is infinite dimensional. Second, the space of null-solutions for the L^p_1 Regularity Problem for the Bitsadze operator in the complement of the closed unit disk is infinite dimensional.

Proof of Theorem 8.1.7 This is justified by reasoning like in the proof of Theorem 8.1.3, now employing Theorem 8.1.4 in lieu of Theorem 8.1.1, and also bearing in mind (8.1.25). □

As a quick inspection of the proofs reveals, analogous results hold for other spaces of boundary data:

Remark 8.1.8 *Natural versions of Theorem 8.1.2, Theorem 8.1.3, Theorem 8.1.5, and Theorem 8.1.7 are valid for Muckenhoupt weighted Lebesgue spaces, Morrey spaces, vanishing Morrey spaces, block spaces, as well as for their Sobolev space counterparts.*

In the last portion of this section we indicate how our earlier results for the scalar Bitsadze's operator $\partial_{\bar z}^2$ may be relatively painlessly "lifted" to genuine systems. To set the stage, we first describe an abstract algebraic tool. Given any complex vector space X of complex-valued functions defined on a common fixed set, we agree to denote

$$\text{Pairs}\,[X] := \Big\{\big(\text{Re}\, f, \text{Im}\, f\big) + \mathrm{i}\big(\text{Re}\, g, \text{Im}\, g\big) : \ f, g \in X\Big\}. \qquad (8.1.131)$$

Then this is itself a complex vector space, and we have the linear isomorphism

$$\begin{aligned}&\mathrm{Pairs}\,[X] \cong X \oplus X \ \text{ via}\\ &\big(\mathrm{Re}\, f, \mathrm{Im}\, f\big) + \mathrm{i}\big(\mathrm{Re}\, g, \mathrm{Im}\, g\big) \mapsto f \oplus g.\end{aligned} \tag{8.1.132}$$

More generally, for any linear subspace Y of X we have the linear isomorphism

$$\frac{\mathrm{Pairs}\,[X]}{\mathrm{Pairs}\,[Y]} \cong \frac{X}{Y} \oplus \frac{X}{Y}. \tag{8.1.133}$$

Indeed, if $[\cdot]_{X/Y}$ denotes the equivalence class in X/Y, then the linear assignment

$$\big(\mathrm{Re}\, f, \mathrm{Im}\, f\big) + \mathrm{i}\big(\mathrm{Re}\, g, \mathrm{Im}\, g\big) \mapsto [f]_{X/Y} \oplus [g]_{X/Y} \tag{8.1.134}$$

maps $\mathrm{Pairs}\,[X]$ onto $(X/Y) \oplus (X/Y)$ and its kernel is precisely $\mathrm{Pairs}\,[Y]$, so (8.1.133) is a consequence of the First Group Isomorphism Theorem in algebra. Finally, we wish to observe that

$$\begin{aligned}&\text{if } X \text{ has the property that } \mathrm{Re}\, f \in X \text{ for each } f \in X, \text{ then}\\ &\text{in place of (8.1.132) one actually has } \mathrm{Pairs}\,[X] = X \oplus X.\end{aligned} \tag{8.1.135}$$

Consider now the second-order, homogeneous, real constant coefficient, 2×2 system

$$\mathbb{L}_B := \frac{1}{4}\begin{pmatrix} \partial_x^2 - \partial_y^2 & -2\partial_x\partial_y \\ 2\partial_x\partial_y & \partial_x^2 - \partial_y^2 \end{pmatrix}. \tag{8.1.136}$$

The characteristic matrix of the system L is given by

$$\mathbb{L}_B(\xi) = \frac{-1}{4}\begin{pmatrix} \xi_1^2 - \xi_2^2 & -2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_1^2 - \xi_2^2 \end{pmatrix} \ \text{ at each } \ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2. \tag{8.1.137}$$

Hence, at each $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}$ we have

$$\det\,[\mathbb{L}_B(\xi)] = \tfrac{1}{16}\big[(\xi_1^2 - \xi_2^2)^2 + (2\xi_1\xi_2)^2\big] = \tfrac{1}{16}(\xi_1^2 + \xi_2^2)^2 = \tfrac{1}{16}|\xi|^4 \neq 0, \tag{8.1.138}$$

which goes to show that

$$\text{the system } \mathbb{L}_B \text{ from (8.1.136) is weakly elliptic.} \tag{8.1.139}$$

The system $\mathbb{L}_B$ defined in (8.1.136) is closely related to Bitsadze's operator $L_B := \partial_{\bar z}^2$. Specifically, if we denote by $\pi_1, \pi_2 : \mathbb{C}^2 \to \mathbb{C}$ the canonical coordinate projections, defined as

$$\pi_1(z_1, z_2) := z_1 \ \text{ and } \ \pi_2(z_1, z_2) = z_2 \ \text{ for each } \ (z_1, z_2) \in \mathbb{C}^2, \tag{8.1.140}$$

then

$$\partial_{\bar z}^2(u_1 + \mathrm{i}u_2) = \pi_1 \mathbb{L}_B(u_1, u_2) + \mathrm{i}\pi_2 \mathbb{L}_B(u_1, u_2)$$
$$\text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any two} \tag{8.1.141}$$
$$\text{complex-valued functions } u_1, u_2 \in \mathscr{C}^2(\Omega).$$

In particular,

$$\mathbb{L}_B\big(\mathrm{Re}\, u, \mathrm{Im}\, u\big) = \big(\mathrm{Re}\,(\partial_{\bar z}^2 u), \mathrm{Im}\,(\partial_{\bar z}^2 u)\big)$$
$$\text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any} \tag{8.1.142}$$
$$\text{complex-valued function } u \in \mathscr{C}^2(\Omega).$$

To continue, fix an integrability exponent $p \in (1, \infty)$ and an aperture parameter $\kappa \in (0, \infty)$. We may then translate Theorem 8.1.2 using the above formalism and obtain the following characterization of the space of admissible boundary data for the L^p Dirichlet Problem for the system $\mathbb{L}_B$ defined in (8.1.136) in the unit disk in the plane:

$$\Big\{U\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} : U \in \big[\mathscr{C}^\infty(\mathbb{D})\big]^2,\ \mathbb{L}_B U = 0 \text{ in } \mathbb{D},\ \mathcal{N}_\kappa U \in L^p(\partial\mathbb{D}, \sigma), \tag{8.1.143}$$
$$\text{and } U\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\mathbb{D}\Big\} = \mathrm{Pairs}\Big[\mathcal{H}^p(\partial\mathbb{D}_+, \sigma) \oplus \langle \bar z\rangle\Big].$$

Also, from (8.1.72)-(8.1.73) we see that the space of null-solutions for the L^p Dirichlet Problem for the system $\mathbb{L}_B$ in the unit disk, i.e.,

$$\Big\{U \in \big[\mathscr{C}^\infty(\mathbb{D})\big]^2 : \mathbb{L}_B U = 0 \text{ in } \mathbb{D},\ \mathcal{N}_\kappa U \in L^p(\partial\mathbb{D}, \sigma),$$
$$\text{and } U\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} = 0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}\Big\}, \tag{8.1.144}$$

coincides with the infinite dimensional space $(1 - |z|)^2 \cdot \mathrm{Pairs}\,[X]$ where

$$X := \Big\{W \text{ holomorphic in } \mathbb{D} : \mathcal{N}_\kappa\big(\delta_{\partial\mathbb{D}} \cdot W\big) \in L^p(\partial\mathbb{D}, \sigma) \text{ and}$$
$$\big[\delta_{\partial\mathbb{D}} \cdot W\big]\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} = 0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}\Big\}. \tag{8.1.145}$$

Likewise, from Theorem 8.1.3 we see that the space of admissible boundary data for the L^p_1 Regularity Problem for the system $\mathbb{L}_B$ in the unit disk has the following description:

$$\Big\{U\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} : U \in \big[\mathscr{C}^\infty(\mathbb{D})\big]^2,\ \mathbb{L}_B U = 0 \text{ in } \mathbb{D},\ \mathcal{N}_\kappa(\nabla U) \in L^p(\partial\mathbb{D}, \sigma)\Big\} \tag{8.1.146}$$
$$= \mathrm{Pairs}\Big[\mathcal{H}^p_1(\partial\mathbb{D}_+, \sigma) \oplus \langle \bar z\rangle\Big],$$

while the space of null-solutions for the L^p_1 Regularity Problem for system $\mathbb{L}_B$ in the unit disk, i.e.,

$$\Big\{U \in \big[\mathscr{C}^\infty(\mathbb{D})\big]^2 : \ \mathbb{L}_B U = 0 \ \text{ in } \mathbb{D}, \ \mathcal{N}_\kappa(\nabla U) \in L^p(\partial\mathbb{D}, \sigma), \\ \text{and } U\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}\Big\}, \tag{8.1.147}$$

coincides with

$$\Big\{(1 - |z|^2)W : \ W \in \text{Pairs}\,\big[\mathcal{H}^p(\mathbb{D})\big]\Big\}. \tag{8.1.148}$$

Of course, similar results may be obtained in relation to Theorem 8.1.5 and Theorem 8.1.7. In addition, analogous results are valid for the transpose of the system $\mathbb{L}_B$, namely for

$$\mathbb{L}_B^\top = \frac{1}{4}\begin{pmatrix} \partial_x^2 - \partial_y^2 & 2\partial_x\partial_y \\ -2\partial_x\partial_y & \partial_x^2 - \partial_y^2 \end{pmatrix}. \tag{8.1.149}$$

Indeed, these are naturally derived from the corresponding results for $\mathbb{L}_B$ recorded above keeping in mind that

$$\mathbb{L}_B(u_1, u_2) = \Big(\pi_1\mathbb{L}_B^\top(u_1, -u_2), -\pi_2\mathbb{L}_B^\top(u_1, -u_2)\Big) \\ \text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any two} \\ \text{complex-valued functions } u_1, u_2 \in \mathscr{C}^2(\Omega); \tag{8.1.150}$$

in particular,

$$\mathbb{L}_B(u_1, u_2) = 0 \Longleftrightarrow \mathbb{L}_B^\top(u_1, -u_2) = 0 \\ \text{for any open set } \Omega \subseteq \mathbb{R}^2 \equiv \mathbb{C} \text{ and any} \\ \text{complex-valued functions } u_1, u_2 \in \mathscr{C}^2(\Omega). \tag{8.1.151}$$

Our final comment has to do with yet another manner of transferring results from the scalar Bitsadze operator to genuine systems. For example, we consider

$$\mathbf{L}_B := \begin{pmatrix} \partial_{\bar z}^2 & 0 \\ 0 & \partial_{\bar z}^2 \end{pmatrix} \tag{8.1.152}$$

then $\mathbf{L}_B$ is a second-order, homogeneous, complex constant coefficient, 2×2 weakly elliptic system. Keeping in mind the simple relationship this system bears to the scalar Bitsadze operator, we see from Theorem 8.1.2 that the space of admissible boundary data for the L^p Dirichlet Problem for the system $\mathbf{L}_B$ in the unit disk in the plane may be characterized as follows:

$$\Big\{U\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} : U \in \big[\mathscr{C}^\infty(\mathbb{D})\big]^2, \ \mathbf{L}_B U = 0 \ \text{ in } \mathbb{D}, \ \mathcal{N}_\kappa U \in L^p(\partial\mathbb{D}, \sigma), \\ \text{and } U\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\mathbb{D}\Big\} = \Big[\mathcal{H}^p(\partial\mathbb{D}_+, \sigma) \oplus \langle \bar z\rangle\Big]^2, \tag{8.1.153}$$

while the space of null-solutions for the L^p Dirichlet Problem for the system $\mathbf{L}_B$ in the unit disk, i.e.,

$$\Big\{U \in \big[\mathscr{C}^\infty(\mathbb{D})\big]^2 : \mathbf{L}_B U = 0 \text{ in } \mathbb{D},\ \mathcal{N}_\kappa U \in L^p(\partial\mathbb{D}, \sigma), \\ \text{and } U\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}\Big\}, \tag{8.1.154}$$

coincides with the infinite dimensional space $\big\{U = (u_1, u_2) : u_1, u_2 \in X\big\}$ where

$$X := \Big\{(1-|z|)^2 W : W \text{ holomorphic in } \mathbb{D} : \mathcal{N}_\kappa\big(\delta_{\partial\mathbb{D}} \cdot W\big) \in L^p(\partial\mathbb{D}, \sigma) \\ \text{and } \big[\delta_{\partial\mathbb{D}} \cdot W\big]\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}\Big\}. \tag{8.1.155}$$

Similarly, Theorem 8.1.3 readily implies that the space of admissible boundary data for the L_1^p Regularity Problem for the system $\mathbf{L}_B$ in the unit disk may be described as

$$\Big\{U\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} : U \in \big[\mathscr{C}^\infty(\mathbb{D})\big]^2,\ \mathbf{L}_B U = 0 \text{ in } \mathbb{D},\ \mathcal{N}_\kappa(\nabla U) \in L^p(\partial\mathbb{D}, \sigma)\Big\} \tag{8.1.156}$$
$$= \Big[\mathcal{H}_1^p(\partial\mathbb{D}_+, \sigma) \oplus \langle \bar z\rangle\Big]^2,$$

whereas the space of null-solutions for the L_1^p Regularity Problem for system $\mathbf{L}_B$ in the unit disk, i.e.,

$$\Big\{U \in \big[\mathscr{C}^\infty(\mathbb{D})\big]^2 : \mathbf{L}_B U = 0 \text{ in } \mathbb{D},\ \mathcal{N}_\kappa(\nabla U) \in L^p(\partial\mathbb{D}, \sigma), \\ \text{and } U\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}} = 0 \text{ at } \sigma\text{-a.e. point on } \partial\mathbb{D}\Big\}, \tag{8.1.157}$$

coincides with

$$\Big\{(1-|z|^2)W : W \in \big[\mathcal{H}^p(\mathbb{D})\big]^2\Big\}. \tag{8.1.158}$$

As expected, analogous results for the system $\mathbf{L}_B$ may be derived now starting from Theorem 8.1.5 and Theorem 8.1.7.

8.2 Null-Solutions and Boundary Traces for the Operator $\partial_{\bar z}^2 - \lambda^2 \partial_z^2$

Work in the complex plane. Bring in the Cauchy-Riemann operator $\partial_{\bar z} := \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$, and denote by $\partial_z := \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ its complex conjugate. We are interested in studying the family of scalar, homogeneous, constant (complex) coefficient, second-order operators

$$L_\lambda := \partial_{\bar z}^2 - \lambda^2\partial_z^2 = \big(\partial_{\bar z} + \lambda\partial_z\big)\big(\partial_{\bar z} - \lambda\partial_z\big), \qquad \forall\lambda\in\mathbb{C}. \tag{8.2.1}$$

In real variables we therefore have

$$L_\lambda = \tfrac{1}{4}\Big\{(1-\lambda^2)\partial_x^2 + 2\mathrm{i}(1+\lambda^2)\partial_x\partial_y - (1-\lambda^2)\partial_y^2\Big\}, \qquad \forall\lambda\in\mathbb{C}. \tag{8.2.2}$$

Hence, for each $\lambda\in\mathbb{C}$ we may express L_λ as

$$L_\lambda = \mathrm{div}A\nabla \ \text{ where } \ A := \frac{1}{4}\begin{pmatrix} 1-\lambda^2 & 2\mathrm{i}(1+\lambda^2) \\ 2\mathrm{i}(1+\lambda^2) & -(1-\lambda^2)\end{pmatrix}. \tag{8.2.3}$$

From [70, Example 1.4.25] we then see (bearing in mind [70, (1.4.178)]) that, for any given $\lambda\in\mathbb{C}$,

$$\begin{array}{l} L_\lambda \text{ does not satisfy the Legendre-Hadamard (strong) ellipticity condi-} \\ \text{tion, and } L_\lambda \text{ is weakly elliptic if and only if } \lambda\neq\pm\mathrm{i} \text{ and } \frac{\lambda^2+1}{\lambda^2-1}\in\mathbb{C}\setminus\mathrm{i}\mathbb{R}. \end{array} \tag{8.2.4}$$

Next, for each fixed $\lambda\in\mathbb{C}$ consider the transformations

$$\begin{array}{c} T_\lambda^{\pm} : \mathbb{C}\longrightarrow\mathbb{C} \ \text{ given by} \\ T_\lambda^{\pm}(z) := z\pm\lambda\bar z \ \text{ for each } \ z\in\mathbb{C}. \end{array} \tag{8.2.5}$$

Note that if $\zeta = z\pm\lambda\bar z$ then $-\zeta\pm\lambda\bar\zeta = (|\lambda|^2-1)z$. From this we conclude that

$$\begin{array}{c} T_\lambda^{\pm} \text{ is invertible if and only if } |\lambda|\neq 1, \text{ a scenario in which} \\ \big(T_\lambda^{\pm}\big)^{-1}(\zeta) = \frac{1}{1-|\lambda|^2}\zeta \mp \frac{\lambda}{1-|\lambda|^2}\bar\zeta \ \text{ for each } \ \zeta\in\mathbb{C}. \end{array} \tag{8.2.6}$$

Henceforth we agree to denote by $\mathscr{O}(\Omega)$ the collection of all holomorphic functions in a given open set $\Omega\subseteq\mathbb{C}$.

Lemma 8.2.1 *Suppose $\Omega\subseteq\mathbb{C}$ is an arbitrary open set and consider some function $w\in\mathscr{C}^1(\Omega)$. Also, pick some $\lambda\in\mathbb{C}$ with $|\lambda|\neq 1$. Then*

$$\big(\partial_{\bar z}\pm\lambda\partial_z\big)w = 0 \ \text{ in } \ \Omega \tag{8.2.7}$$

if and only if

$$w = \varphi\circ T_\lambda^{\mp} \ \text{ for some function } \ \varphi\in\mathscr{O}\big((T_\lambda^{\mp})^{-1}\Omega\big). \tag{8.2.8}$$

Proof Using the Chain Rule involving a holomorphic function (recalled in [70, (1.4.146)]) it may be checked without any difficulty that for each $\varphi\in\mathscr{O}\big((T_\lambda^{\mp})^{-1}\Omega\big)$ we have

$$\big(\partial_{\bar z}\pm\lambda\partial_z\big)\big[\varphi(z\mp\lambda\bar z)\big] = 0 \ \text{ for each } \ z\in\Omega. \tag{8.2.9}$$

This proves the left-pointing implication in the statement. In the opposite direction, given any $w\in\mathscr{C}^1(\Omega)$ satisfying (8.2.7), if for each $\zeta\in(T_\lambda^{\mp})^{-1}(\Omega)$ we define

$$\varphi(\zeta) := w\Big((T_\lambda^{\mp})^{-1}(\zeta)\Big) = w\Big(\frac{1}{1-|\lambda|^2}\zeta \pm \frac{\lambda}{1-|\lambda|^2}\overline{\zeta}\Big), \tag{8.2.10}$$

then a patient application of the ordinary Chain Rule shows that φ satisfies $\partial_{\bar\zeta}\big[\varphi(\zeta)\big] = 0$ for each $\zeta \in (T_\lambda^{\mp})^{-1}(\Omega)$. Hence $\varphi \in \mathscr{O}\big((T_\lambda^{\mp})^{-1}\Omega\big)$, as wanted. □

We are now prepared to prove the following "structure theorem" concerning the null-solutions of the operator L_λ from (8.2.1).

Theorem 8.2.2 *Let $\Omega \subseteq \mathbb{C}$ be a simply connected*[1] *set, and fix some $\lambda \in \mathbb{C}\setminus\{0\}$ with $|\lambda| \neq 1$. Also, consider a function $u \in \mathscr{C}^2(\Omega)$. Then $L_\lambda u = 0$ in Ω if and only if there exist two functions,*

$$\varphi \in \mathscr{O}\big((T_\lambda^{-})^{-1}\Omega\big) \text{ and } \psi \in \mathscr{O}\big((T_\lambda^{+})^{-1}\Omega\big), \tag{8.2.11}$$

such that

$$u = \varphi \circ T_\lambda^{-} + \psi \circ T_\lambda^{+} \text{ in } \Omega. \tag{8.2.12}$$

In addition, the above holomorphic functions φ, ψ are uniquely determined by u up to additive constants and, in fact,

$$\begin{aligned} \varphi' &= -\tfrac{1}{2\lambda}\big[\big(\partial_{\bar z} - \lambda\partial_z\big)u\big] \circ (T_\lambda^{-})^{-1}, \\ \psi' &= \tfrac{1}{2\lambda}\big[\big(\partial_{\bar z} + \lambda\partial_z\big)u\big] \circ (T_\lambda^{+})^{-1}. \end{aligned} \tag{8.2.13}$$

Proof In one direction, suppose $u \in \mathscr{C}^2(\Omega)$ has $L_\lambda u = 0$ in Ω. Then

$$w := \big(\partial_{\bar z} - \lambda\partial_z\big)u \in \mathscr{C}^1(\Omega) \tag{8.2.14}$$

satisfies

$$\big(\partial_{\bar z} + \lambda\partial_z\big)w = L_\lambda u = 0 \text{ in } \Omega. \tag{8.2.15}$$

Lemma 8.2.1 then guarantees the existence of a function $\Phi \in \mathscr{O}\big((T_\lambda^{-})^{-1}\Omega\big)$ such that

$$w = \Phi \circ T_\lambda^{-} \text{ in } \Omega. \tag{8.2.16}$$

Given that we are currently assuming $\lambda \neq 0$, and since $(T_\lambda^{-})^{-1}(\Omega)$ is a simply connected set, we may invoke the equivalence between items *(1)* and *(7)* in [68, Proposition 5.8.1] to conclude that there exists some

$$\varphi \in \mathscr{O}\big((T_\lambda^{-})^{-1}\Omega\big) \text{ satisfying } \varphi' = (-2\lambda)^{-1}\Phi \text{ in } (T_\lambda^{-})^{-1}(\Omega), \tag{8.2.17}$$

where 'prime' stands for the ordinary complex derivative of holomorphic functions. Granted this, an application of the Chain Rule involving a holomorphic function (recalled in [70, (1.4.146)]) presently gives

[1] hence, Ω is also open and connected

$$\begin{aligned}\big(\partial_{\bar z}-\lambda\partial_z\big)\big[\varphi(z-\lambda\bar z)\big]&=(-2\lambda)\varphi'(z-\lambda\bar z)=\Phi(z-\lambda\bar z)=w(z)\\&=\big[\big(\partial_{\bar z}-\lambda\partial_z\big)u\big](z)\ \text{ for each }\ z\in\Omega.\end{aligned}\tag{8.2.18}$$

Hence,

$$\big(\partial_{\bar z}-\lambda\partial_z\big)\big[u(z)-\varphi(z-\lambda\bar z)\big]=0\ \text{ for each }\ z\in\Omega,\tag{8.2.19}$$

so yet another application of Lemma 8.2.1 proves the existence of some function $\psi\in\mathscr{O}\big((T_\lambda^+)^{-1}\Omega\big)$ such that

$$u(z)-\varphi(z-\lambda\bar z)=\psi(z+\lambda\bar z)\ \text{ for each }\ z\in\Omega.\tag{8.2.20}$$

From this, (8.2.12) follows.

In the opposite direction, given any functions φ,ψ as in (8.2.11), Lemma 8.2.1 tells us that for each $z\in\Omega$ we have

$$\big(\partial_{\bar z}+\lambda\partial_z\big)\big[\varphi(z-\lambda\bar z)\big]=0\ \text{ and }\ \big(\partial_{\bar z}-\lambda\partial_z\big)\big[\psi(z+\lambda\bar z)\big]=0.\tag{8.2.21}$$

In view of this and the factorization in (8.2.1) we then conclude that

$$L_\lambda\big[\varphi(z-\lambda\bar z)+\psi(z+\lambda\bar z)\big]=0\ \text{ for each }\ z\in\Omega,\tag{8.2.22}$$

as wanted. There remains to prove the very last claim in the statement. In this regard we note that if (8.2.12) holds then

$$\begin{aligned}\big[\big(\partial_{\bar z}+\lambda\partial_z\big)u\big](z)&=\big(\partial_{\bar z}+\lambda\partial_z\big)\big[\varphi(z-\lambda\bar z)\big]+\big(\partial_{\bar z}+\lambda\partial_z\big)\big[\psi(z+\lambda\bar z)\big]\\&=2\lambda\psi'(z+\lambda\bar z)\ \text{ for each }\ z\in\Omega.\end{aligned}\tag{8.2.23}$$

Thus,

$$\psi'(z+\lambda\bar z)=\tfrac{1}{2\lambda}\big[\big(\partial_{\bar z}+\lambda\partial_z\big)u\big](z)\ \text{ for each }\ z\in\Omega\tag{8.2.24}$$

and, likewise,

$$\varphi'(z-\lambda\bar z)=\tfrac{-1}{2\lambda}\big[\big(\partial_{\bar z}-\lambda\partial_z\big)u\big](z)\ \text{ for each }\ z\in\Omega.\tag{8.2.25}$$

These formulas prove (8.2.13) and also make it clear that the holomorphic functions φ,ψ are uniquely determined by u up to additive constants. □

We are now in prepared to describe both the space of admissible boundary data and the space of null-solutions for the Homogeneous Regularity Problem for the operator L_λ in the upper half-plane.

Theorem 8.2.3 *Fix an aperture parameter $\kappa>0$ along with an integrability exponent $p\in(1,\infty)$ and define the homogeneous Hardy spaces with regularity* (*as in* (1.8.285) *for the upper half-plane*)

$$\dot{\mathscr{H}}^p_{1,\pm}(\mathbb{R},\mathcal{L}^1):=\Big\{w\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_\pm}\,:\,w\in\mathscr{O}(\mathbb{C}_\pm)\ \textit{with}\ \mathcal{N}_\kappa(\nabla w)\in L^p(\mathbb{R},\mathcal{L}^1)\Big\}.\tag{8.2.26}$$

Also, pick a real number

$$\lambda \in (-1, 1) \setminus \{0\}. \tag{8.2.27}$$

Then the space of admissible boundary data for the Homogeneous Regularity Problem for the operator L_λ in the upper half-plane, i.e., the subspace of $\dot{L}^p_1(\mathbb{R}, \mathcal{L}^1)$ described as

$$\dot{\mathscr{U}}^p_{1,\lambda} := \Big\{ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+} : \ u \in \mathscr{C}^\infty(\mathbb{C}_+), \ L_\lambda u = 0 \ \textit{in} \ \mathbb{C}_+, \ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, \mathcal{L}^1) \Big\}, \tag{8.2.28}$$

coincides with $\dot{\mathcal{H}}^p_{1,+}(\mathbb{R}, \mathcal{L}^1)$ (the "positive" homogeneous Hardy space with regularity on the real line). Moreover, the cokernel of $\dot{\mathscr{U}}^p_{1,\lambda}/\sim$ (i.e., the space of equivalence classes of function in $\dot{\mathscr{U}}^p_{1,\lambda}$ modulo constants) into the full space of boundary data modulo constants, $\dot{L}^p_1(\mathbb{R}, \mathcal{L}^1)/\sim$, is isomorphic to the "negative" homogeneous Hardy space with regularity modulo constants, $\dot{\mathcal{H}}^p_{1,-}(\mathbb{R}, \mathcal{L}^1)/\sim$, i.e.,

$$\frac{\dot{L}^p_1(\mathbb{R}, \mathcal{L}^1)/\sim}{\dot{\mathscr{U}}^p_{1,\lambda}/\sim} \cong \dot{\mathcal{H}}^p_{1,-}(\mathbb{R}, \mathcal{L}^1)/\sim . \tag{8.2.29}$$

In particular,

the codimension of the space of admissible boundary data, modulo constants, for the Homogeneous Regularity Problem for the operator L_λ in the upper half-plane (i.e., the space $\dot{\mathscr{U}}^p_{1,\lambda}/\sim$) into the full space of boundary data modulo constants, $\dot{L}^p_1(\mathbb{R}, \mathcal{L}^1)/\sim$, is $+\infty$. (8.2.30)

Finally, the space on null-solutions for the Homogeneous Regularity Problem for the operator L_λ in the upper half-plane, namely

$$\Big\{ u \in \mathscr{C}^\infty(\mathbb{C}_+) : \ L_\lambda u = 0 \ \textit{in} \ \mathbb{C}_+, \ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, \mathcal{L}^1), \ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+} = 0 \Big\}, \tag{8.2.31}$$

coincides with the infinite dimensional space

$$\Big\{ \phi\big((1+\lambda)(z - \lambda\bar z)\big) - \phi\big((1-\lambda)(z + \lambda\bar z)\big) : \phi \in \mathscr{O}(\mathbb{C}_+) \\ \textit{with} \ \mathcal{N}_\kappa(\nabla\phi) \in L^p(\mathbb{R}, \mathcal{L}^1) \Big\}. \tag{8.2.32}$$

The fact that the nontangential boundary trace exists in the context of (8.2.28) and that $\dot{\mathscr{U}}^p_{1,\lambda}$ is indeed a subspace of $\dot{L}^p_1(\mathbb{R}, \mathcal{L}^1)$ is seen from [69, Proposition 11.5.12]. Also, the space (8.2.32) is, as claimed, infinite dimensional. For example, this may be seen by considering the holomorphic functions

$$\phi_k(z) := (\mathrm{i} + z)^{-k} \ \text{ for each } \ z \in \mathbb{C}_+ \ \text{ with } \ k \in \mathbb{N} \ \text{ arbitrary}. \tag{8.2.33}$$

From [68, Lemma 8.3.7] we see that $\mathcal{N}_\kappa(\nabla\phi_k) \in L^p(\mathbb{R}, \mathcal{L}^1)$ for each $k \in \mathbb{N}$, and we claim that

$$\Big\{\phi_k\big((1+\lambda)(z-\lambda\overline{z})\big)-\phi_k\big((1-\lambda)(z+\lambda\overline{z})\big)\Big\}_{k\in\mathbb{N}} \tag{8.2.34}$$
is a family of linearly independent functions in $\mathbb{C}_+$.

Indeed, if for some integer $N\in\mathbb{N}$ and some coefficients $c_k\in\mathbb{C}$, with $1\leq k\leq N$, we have

$$\sum_{k=1}^{N} c_k\Big\{\phi_k\big((1+\lambda)(z-\lambda\overline{z})\big)-\phi_k\big((1-\lambda)(z+\lambda\overline{z})\big)\Big\}=0,\quad \forall z\in\mathbb{C}_+, \tag{8.2.35}$$

then applying $\partial_{\bar z}-\lambda\partial_z$ we obtain

$$\sum_{k=1}^{N}\frac{2\lambda(1+\lambda)k\,c_k}{\big[i+(1+\lambda)(z-\lambda\overline{z})\big]^{k+1}}=0,\quad \forall z\in\mathbb{C}_+, \tag{8.2.36}$$

hence also

$$\sum_{k=1}^{N}k\,c_k\big[i+(1+\lambda)(z-\lambda\overline{z})\big]^{N-k}=0,\quad \forall z\in\mathbb{C}_+. \tag{8.2.37}$$

Inspecting the coefficients of z (from highest to lowest) in this polynomial equation reveals that all c_k's vanish. As such, (8.2.34) follows, so the space (8.2.32) is infinite dimensional.

We now give the proof of Theorem 8.2.3.

Proof of Theorem 8.2.3 In one direction, given any $f\in\dot{\mathcal{H}}^p_{1,+}(\mathbb{R},\mathcal{L}^1)$ there exists a function

$$\begin{gathered} w\in\mathscr{O}(\mathbb{C}_+)\ \text{ with }\ \mathcal{N}_\kappa(\nabla w)\in L^p(\mathbb{R},\mathcal{L}^1)\ \text{ and}\\ \Big(w\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{C}_+}\Big)(z)=f(z)\ \text{ at }\mathcal{L}^1\text{-a.e. point }\ z\in\partial\mathbb{C}_+\equiv\mathbb{R}. \end{gathered} \tag{8.2.38}$$

If we define

$$u:\mathbb{C}_+\to\mathbb{C}_+,\quad u(z):=w\Big(\frac{z-\lambda\overline{z}}{1-\lambda}\Big)\ \text{ for each }\ z\in\mathbb{C}_+, \tag{8.2.39}$$

then $u\in\mathscr{C}^\infty(\mathbb{C}_+)$ and the Chain Rule involving a holomorphic function (recalled in [70, (1.4.146)]) gives

$$\big(\partial_{\bar z}\pm\lambda\partial_z\big)u=0\ \text{ in }\ \mathbb{C}_+. \tag{8.2.40}$$

In particular, $L_\lambda u=0$ in $\mathbb{C}_+$ (cf. (8.2.1)). Also, in view of the fact that

$$\mathbb{C}_+\ni z\mapsto\frac{z-\lambda\overline{z}}{1-\lambda}\in\mathbb{C}_+\ \text{ is a bi-Lipschitz homeomorphism,} \tag{8.2.41}$$

the properties listed in (8.2.38) together with [68, Lemma 8.1.7] and [68, Proposition 8.9.8] help us conclude that $\mathcal{N}_\kappa(\nabla u)\in L^p(\mathbb{R},\mathcal{L}^1)$ and

$$\Big(u\Big|_{\partial\mathbb{C}_+}^{\kappa-\text{n.t.}}\Big)(z) = \Big(w\Big|_{\partial\mathbb{C}_+}^{\kappa-\text{n.t.}}\Big)\Big(\frac{z-\lambda\bar z}{1-\lambda}\Big) = \Big(w\Big|_{\partial\mathbb{C}_+}^{\kappa-\text{n.t.}}\Big)(z)$$
$$= f(z) \text{ at } \mathcal{L}^1\text{-a.e. point } z \in \partial\mathbb{C}_+ \equiv \mathbb{R}, \tag{8.2.42}$$

since $\bar z = z$ on $\partial\mathbb{C}_+ \equiv \mathbb{R}$. All these ultimately put f in the space of admissible boundary data for the Homogeneous Regularity Problem for the operator L_λ in the upper half-plane, described in (8.2.28). The conclusion is that the space $\dot{\mathcal{H}}^p_{1,+}(\mathbb{R},\mathcal{L}^1)$ embeds into (8.2.28).

To prove the opposite inclusion, start with a function satisfying

$$u \in \mathscr{C}^\infty(\mathbb{C}_+), \quad L_\lambda u = 0 \text{ in } \mathbb{C}_+, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R},\mathcal{L}^1). \tag{8.2.43}$$

Based on these properties, Theorem 8.2.2 (used with $\Omega := \mathbb{C}_+$ and λ as in (8.2.27)), the fact that the transformations[2]

$$\begin{gathered} T_\lambda^\pm : \mathbb{C}_+ \to \mathbb{C}_+ \text{ given by } T_\lambda^\pm(z) := z \pm \lambda\bar z \text{ for all } z \in \mathbb{C}_+ \\ \text{are bi-Lipschitz homeomorphisms,} \end{gathered} \tag{8.2.44}$$

[68, Lemma 8.1.7], and [68, Proposition 8.4.1], we then conclude that there exist two functions

$$\varphi,\psi \in \mathscr{O}(\mathbb{C}_+) \text{ with } \mathcal{N}_\kappa(\nabla\varphi),\ \mathcal{N}_\kappa(\nabla\psi) \in L^p(\mathbb{R},\mathcal{L}^1), \tag{8.2.45}$$

and such that

$$u(z) = \varphi(z-\lambda\bar z) + \psi(z+\lambda\bar z) \text{ for each } z \in \mathbb{C}_+. \tag{8.2.46}$$

If we now define

$$\begin{gathered} \Phi,\Psi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+ \text{ given at each } z \in \mathbb{C}_+ \\ \text{by } \Phi(z) := \varphi\big((1-\lambda)z\big) \text{ and } \Psi(z) := \psi\big((1+\lambda)z\big), \end{gathered} \tag{8.2.47}$$

then

$$\Phi,\Psi \in \mathscr{O}(\mathbb{C}_+) \text{ and } \mathcal{N}_\kappa(\nabla\Phi),\ \mathcal{N}_\kappa(\nabla\Psi) \in L^p(\mathbb{R},\mathcal{L}^1). \tag{8.2.48}$$

In particular (cf. (8.2.26)),

$$\Phi\Big|_{\partial\mathbb{C}_+}^{\kappa-\text{n.t.}} \text{ and } \Psi\Big|_{\partial\mathbb{C}_+}^{\kappa-\text{n.t.}} \text{ belong to } \dot{\mathcal{H}}^p_{1,+}(\mathbb{R},\mathcal{L}^1). \tag{8.2.49}$$

Also, once again bearing in mind that $\bar z = z$ on $\partial\mathbb{C}_+ \equiv \mathbb{R}$, we see from (8.2.44)-(8.2.48), [68, Lemma 8.1.7], and [68, Proposition 8.9.8] that

[2] it is not difficult to see that the transformations $T_\lambda^\pm$ map the upper half-plane $\mathbb{C}_+$ bijectively onto itself if and only if $\lambda \in (-1, 1)$

$$\Big(u\Big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}}\Big)(z) = \Big(\varphi\Big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}}\Big)\big((1-\lambda)z\big) + \Big(\psi\Big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}}\Big)\big((1+\lambda)z\big)$$
$$= \Big(\Phi\Big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}}\Big)(z) + \Big(\Psi\Big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}}\Big)(z) \text{ at } \mathcal{L}^1\text{-a.e. point } z \in \partial\mathbb{C}_+ \equiv \mathbb{R} \tag{8.2.50}$$

which, in concert with (8.2.49), proves that

$$u\Big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}} = \Phi\Big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}} + \Psi\Big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}} \in \dot{\mathcal{H}}^p_{1,+}(\mathbb{R},\mathcal{L}^1). \tag{8.2.51}$$

From this we then conclude that the space (8.2.28) is contained in $\dot{\mathcal{H}}^p_{1,+}(\mathbb{R},\mathcal{L}^1)$. The coincidence of the space (8.2.28) with $\dot{\mathcal{H}}^p_{1,+}(\mathbb{R},\mathcal{L}^1)$ has therefore been established. Granted this, the isomorphism claimed in (8.2.29) becomes a consequence of (1.8.286). In turn, (8.2.29) readily implies (8.2.30) since the homogeneous Hardy space with regularity $\dot{\mathcal{H}}^p_{1,-}(\mathbb{R},\mathcal{L}^1)$ is infinite dimensional.

Next, given any $\phi \in \mathscr{O}(\mathbb{C}_+)$ with $\mathcal{N}_\kappa(\nabla\phi) \in L^p(\mathbb{R},\mathcal{L}^1)$ it follows from Theorem 8.2.2 and [68, Lemma 8.1.7] that the function

$$\begin{gathered} u : \mathbb{C}_+ \longrightarrow \mathbb{C} \text{ defined for each } z \in \mathbb{C}_+ \text{ by} \\ u(z) := \phi\big((1+\lambda)(z-\lambda\overline{z})\big) - \phi\big((1-\lambda)(z+\lambda\overline{z})\big) \end{gathered} \tag{8.2.52}$$

satisfies

$$u \in \mathscr{C}^\infty(\mathbb{C}_+), \quad L_\lambda u = 0 \text{ in } \mathbb{C}_+, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R},\mathcal{L}^1). \tag{8.2.53}$$

In addition, [69, Proposition 11.5.12] and [68, Lemma 8.1.7] allow us to compute

$$\big(u\big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}}\big)(z) = \big(\phi\big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}}\big)\big((1+\lambda)(z-\lambda\overline{z})\big) - \big(\phi\big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}}\big)\big((1-\lambda)(z+\lambda\overline{z})\big)$$
$$= \big(\phi\big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}}\big)\big((1+\lambda)(1-\lambda)z\big) - \big(\phi\big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}}\big)\big((1-\lambda)(1+\lambda)z\big)$$
$$= 0 \text{ at } \mathcal{L}^1\text{-a.e. point } z \in \mathbb{R} \equiv \partial\mathbb{C}_+, \tag{8.2.54}$$

since complex conjugation leaves points in $\mathbb{R} \equiv \partial\mathbb{C}_+$ invariant. Thus, the space described in (8.2.32) is contained in the space on null-solutions for the Homogeneous Regularity Problem for the operator L_λ in the upper half-plane, explicitly recorded in (8.2.31).

To establish the opposite inclusion, start with a function u satisfying

$$u \in \mathscr{C}^\infty(\mathbb{C}_+), \quad L_\lambda u = 0 \text{ in } \mathbb{C}_+, \quad \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R},\mathcal{L}^1), \quad u\Big|_{\partial\mathbb{C}_+}^{\kappa-\mathrm{n.t.}} = 0. \tag{8.2.55}$$

The structural result proved in Theorem 8.2.2 guarantees (also bearing in mind [68, Lemma 8.1.7] and [68, Proposition 8.4.1]) that

$$\begin{gathered} u(z) = \varphi(z-\lambda\overline{z}) + \psi(z+\lambda\overline{z}) \text{ for each } z \in \mathbb{C}_+ \text{ for some} \\ \varphi,\psi \in \mathscr{O}(\mathbb{C}_+) \text{ with } \mathcal{N}_\kappa(\nabla\varphi), \mathcal{N}_\kappa(\nabla\psi) \in L^p(\mathbb{R},\mathcal{L}^1). \end{gathered} \tag{8.2.56}$$

If we now introduce

$$\begin{gathered}\Phi, \Psi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+ \text{ given at each } z \in \mathbb{C}_+\\ \text{by } \Phi(z) := \varphi\big((1-\lambda)z\big) \text{ and } \Psi(z) := \psi\big((1+\lambda)z\big),\end{gathered} \tag{8.2.57}$$

then

$$\Phi, \Psi \in \mathscr{O}(\mathbb{C}_+) \text{ satisfy } \mathcal{N}_\kappa(\nabla\Phi), \mathcal{N}_\kappa(\nabla\Psi) \in L^p(\mathbb{R}, \mathcal{L}^1), \tag{8.2.58}$$

and [68, Lemma 8.1.7] permit us to compute, at $\mathcal{L}^1$-a.e. point $z \in \mathbb{R} \equiv \partial\mathbb{C}_+$,

$$\begin{aligned}0 = \big(u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+}\big)(z) &= \big(\varphi\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+}\big)(z-\lambda\bar z) + \big(\psi\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+}\big)(z+\lambda\bar z)\\ &= \big(\varphi\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+}\big)\big((1-\lambda)z\big) + \big(\psi\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+}\big)\big((1+\lambda)z\big)\\ &= \big(\Phi\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+}\big)(z) + \big(\Psi\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+}\big)(z),\end{aligned} \tag{8.2.59}$$

hence

$$(\Phi+\Psi)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+} = 0 \text{ at } \mathcal{L}^1\text{-a.e. point on } \mathbb{R} \equiv \partial\mathbb{C}_+. \tag{8.2.60}$$

From this and the integral representation formula (1.8.236) from Corollary 1.8.23 we first conclude that $\Phi+\Psi$ is a constant in $\mathbb{C}_+$, then another appeal to (8.2.60) gives that actually

$$\Phi + \Psi = 0 \text{ in } \mathbb{C}_+. \tag{8.2.61}$$

If we now define

$$\phi(z) := \Phi\Big(\frac{z}{1-\lambda^2}\Big) \text{ for each } z \in \mathbb{C}_+, \tag{8.2.62}$$

it follows from (8.2.62), (8.2.56), (8.2.57), (8.2.58), (8.2.61), [68, Lemma 8.1.7], and [68, Proposition 8.4.1] that $\phi \in \mathscr{O}(\mathbb{C}_+)$ has $\mathcal{N}_\kappa(\nabla\phi) \in L^p(\mathbb{R}, \mathcal{L}^1)$ and

$$u(z) = \phi\big((1+\lambda)(z-\lambda\bar z)\big) - \phi\big((1-\lambda)(z+\lambda\bar z)\big) \text{ for each } z \in \mathbb{C}_+. \tag{8.2.63}$$

This shows that (8.2.31) is indeed included in the space described in (8.2.32). □

We continue by presenting the following counterpart to Theorem 8.2.3, describing the space of admissible boundary data as well as the space of null-solutions for the Dirichlet Problem for the operator L_λ in the upper half-plane. The reader is reminded that the Hilbert transform on the real line is the singular integral operator acting on each function $f \in L^1\big(\mathbb{R}, \frac{dx}{1+|x|}\big)$ according to

$$Hf(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\pi} \int_{\substack{y\in\mathbb{R}\\ |x-y|>\varepsilon}} \frac{f(y)}{x-y}\,dy \text{ for } \mathcal{L}^1\text{-a.e. } x \in \mathbb{R}. \tag{8.2.64}$$

Theorem 8.2.4 *Pick some aperture parameter $\kappa > 0$ together with some integrability exponent $p \in (1,\infty)$ and recall the ordinary Hardy spaces on the real line*

$$\mathcal{H}^p_\pm(\mathbb{R},\mathcal{L}^1)=\Big\{w\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_\pm}:\ w\in\mathscr{O}(\mathbb{C}_\pm)\ \textit{with}\ \mathcal{N}_\kappa w\in L^p(\mathbb{R},\mathcal{L}^1)\Big\}$$
$$=\Big\{f\in L^p(\mathbb{R},\mathcal{L}^1):\ Hf=\mp\mathrm{i}f\Big\},\tag{8.2.65}$$

where H is the Hilbert transform on the real line (*cf.* (8.2.64)). *Finally, pick a real number*

$$\lambda\in(-1,1)\setminus\{0\}.\tag{8.2.66}$$

Then the space of admissible boundary data for the L^p Dirichlet Problem for the operator L_λ in the upper half-plane, i.e., the subspace of $L^p(\mathbb{R},\mathcal{L}^1)$ described as

$$\mathscr{U}^p_\lambda:=\Big\{u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+}:\ u\in\mathscr{C}^\infty(\mathbb{C}_+),\ L_\lambda u=0\ \textit{in}\ \mathbb{C}_+,$$
$$\mathcal{N}_\kappa u\in L^p(\mathbb{R},\mathcal{L}^1)\ \textit{and}\ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+}\ \textit{exists}\ \mathcal{L}^1\textit{-a.e. on}\ \mathbb{R}\Big\},\tag{8.2.67}$$

coincides with the "positive" Hardy space $\mathcal{H}^p_+(\mathbb{R},\mathcal{L}^1)$. Moreover, the cokernel of $\mathscr{U}^p_\lambda$ into the full space of boundary data $L^p(\mathbb{R},\mathcal{L}^1)$ is isomorphic to the "negative" Hardy space $\mathcal{H}^p_-(\mathbb{R},\mathcal{L}^1)$, i.e.,

$$\frac{L^p(\mathbb{R},\mathcal{L}^1)}{\mathscr{U}^p_\lambda}\cong\mathcal{H}^p_-(\mathbb{R},\mathcal{L}^1).\tag{8.2.68}$$

In particular,

the codimension of the space of admissible boundary data for the L^p Dirichlet Problem for the operator L_λ in the upper half-plane (*i.e., the space $\mathscr{U}^p_\lambda$*) *into the full space of boundary data $L^p(\mathbb{R},\mathcal{L}^1)$ is $+\infty$.* (8.2.69)

Finally, the space on null-solutions for the L^p Dirichlet Problem for the operator L_λ in the upper half-plane, namely

$$\Big\{u\in\mathscr{C}^\infty(\mathbb{C}_+):\ L_\lambda u=0\ \textit{in}\ \mathbb{C}_+,\ \mathcal{N}_\kappa(\nabla u)\in L^p(\mathbb{R},\mathcal{L}^1),\ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+}=0\Big\},\tag{8.2.70}$$

is infinite dimensional.

Proof Let u be a complex-valued function defined in the upper half-plane and satisfying

$$\begin{gathered}u\in\mathscr{C}^\infty(\mathbb{C}_+),\quad L_\lambda u=0\ \text{in}\ \mathbb{C}_+,\quad \mathcal{N}_\kappa u\in L^p(\mathbb{R},\mathcal{L}^1),\\ \text{and}\ u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{C}_+}\ \text{exists at}\ \mathcal{L}^1\text{-a.e. point on}\ \mathbb{R}.\end{gathered}\tag{8.2.71}$$

If we define

$$u_\varepsilon(z):=u(z+\mathrm{i}\varepsilon)\ \text{for each}\ z\in\mathbb{C}_+\tag{8.2.72}$$

then, based on (8.2.71)-(8.2.72) and interior estimates for null-solutions of the weakly elliptic operator L_λ (also bearing in mind [68, Proposition 8.4.1]), for each $\varepsilon>0$

the following properties are satisfied:

$$
\begin{gathered}
u_\varepsilon \in \mathscr{C}^\infty\big(\overline{\mathbb{C}_+}\big), \quad L_\lambda u_\varepsilon = 0 \text{ in } \mathbb{C}_+, \quad \mathcal{N}_\kappa(\nabla u_\varepsilon) \in L^p(\mathbb{R}, \mathcal{L}^1), \\
\text{and } \big|u_\varepsilon\big|_{\partial\mathbb{C}_+}\big| \leq \mathcal{N}_\kappa u \text{ at every point on } \mathbb{R}.
\end{gathered} \tag{8.2.73}
$$

From Lebesgue's Dominated Convergence Theorem we also see that

$$
f_\varepsilon := u_\varepsilon\big|_{\partial\mathbb{C}_+} \text{ converges in } L^p(\mathbb{R}, \mathcal{L}^1) \text{ to } u\big|_{\partial\mathbb{C}_+}^{\kappa-\text{n.t.}} \text{ as } \varepsilon \to 0^+. \tag{8.2.74}
$$

Note that each f_ε belongs to the space of admissible boundary data for the Homogeneous Regularity Problem for the operator L_λ in the upper half-plane, i.e., the space $\dot{\mathscr{U}}^p_{1,\lambda}$ described in (8.2.28). According to identification made in Theorem 8.2.3, each f_ε therefore belongs to the "positive" homogeneous Hardy space with regularity $\dot{\mathcal{H}}^p_{1,+}(\mathbb{R}, \mathcal{L}^1)$. In view of this and (1.8.288) we then conclude that for each $\varepsilon > 0$ there exists a constant $c_\varepsilon \in \mathbb{C}$ such that

$$
C_{\text{mod}} f_\varepsilon = \tfrac{1}{2} f_\varepsilon + c_\varepsilon. \tag{8.2.75}
$$

On the other hand, since $f_\varepsilon \in L^p(\mathbb{R}, \mathcal{L}^1)$, much as in (1.8.26) we have that $C_{\text{mod}} f_\varepsilon$ differs from $(i/2)Hf_\varepsilon$ by an additive constant. Ultimately, this implies (bearing in mind that H maps $L^p(\mathbb{R}, \mathcal{L}^1)$ into itself and any constant function in $L^p(\mathbb{R}, \mathcal{L}^1)$ is actually identically zero) that (8.2.75) simply reduces to

$$
Hf_\varepsilon = -\mathrm{i} f_\varepsilon \text{ for each } \varepsilon > 0. \tag{8.2.76}
$$

Passing to limit $\varepsilon \to 0^+$ and keeping in mind (8.2.74) plus the boundedness of H on $L^p(\mathbb{R}, \mathcal{L}^1)$, we then conclude that

$$
H\big(u\big|_{\partial\mathbb{C}_+}^{\kappa-\text{n.t.}}\big) = -\mathrm{i}\big(u\big|_{\partial\mathbb{C}_+}^{\kappa-\text{n.t.}}\big). \tag{8.2.77}
$$

In light of (8.2.65), this places the function $u\big|_{\partial\mathbb{C}_+}^{\kappa-\text{n.t.}}$ in the "positive" Hardy spaces $\mathcal{H}^p_+(\mathbb{R}, \mathcal{L}^1)$. In summary, $\mathscr{U}^p_\lambda \subseteq \mathcal{H}^p_+(\mathbb{R}, \mathcal{L}^1)$.

To justify the opposite inclusion, start with an arbitrary $f \in \mathcal{H}^p_+(\mathbb{R}, \mathcal{L}^1)$. This means that there exists a function

$$
\begin{gathered}
w \in \mathscr{O}(\mathbb{C}_+) \text{ with } \mathcal{N}_\kappa w \in L^p(\mathbb{R}, \mathcal{L}^1) \text{ and} \\
\Big(w\big|_{\partial\mathbb{C}_+}^{\kappa-\text{n.t.}}\Big)(z) = f(z) \text{ at } \mathcal{L}^1\text{-a.e. point } z \in \partial\mathbb{C}_+ \equiv \mathbb{R}.
\end{gathered} \tag{8.2.78}
$$

Use this to define

$$
u : \mathbb{C}_+ \to \mathbb{C}_+, \quad u(z) := w\Big(\frac{z - \lambda\bar{z}}{1-\lambda}\Big) \text{ for each } z \in \mathbb{C}_+. \tag{8.2.79}
$$

Then $u \in \mathscr{C}^\infty(\mathbb{C}_+)$ and the Chain Rule involving a holomorphic function (cf. [70, (1.4.146)]) presently gives

$$\big(\partial_{\bar z} \pm \lambda \partial_z\big)u = 0 \text{ in } \mathbb{C}_+. \tag{8.2.80}$$

Hence, $L_\lambda u = 0$ in $\mathbb{C}_+$ (cf. (8.2.1)). Moreover, in view of the fact that

$$\mathbb{C}_+ \ni z \mapsto \frac{z-\lambda\bar z}{1-\lambda} \in \mathbb{C}_+ \text{ is a bi-Lipschitz homeomorphism,} \tag{8.2.81}$$

the properties listed in (8.2.78) together with [68, Lemma 8.1.7] and [68, Proposition 8.9.8] permit us to conclude that $\mathcal{N}_\kappa u \in L^p(\mathbb{R}, \mathcal{L}^1)$ and

$$\begin{aligned}\Big(u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{C}_+}\Big)(z) &= \Big(w\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{C}_+}\Big)\Big(\frac{z-\lambda\bar z}{1-\lambda}\Big) = \Big(w\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{C}_+}\Big)(z)\\ &= f(z) \text{ at } \mathcal{L}^1\text{-a.e. point } z\in\partial\mathbb{C}_+\equiv\mathbb{R},\end{aligned} \tag{8.2.82}$$

since $\bar z = z$ on $\partial\mathbb{C}_+ \equiv \mathbb{R}$. As a result, f belongs to $\mathscr{U}^p_\lambda$, the space of admissible boundary data for the L^p Dirichlet Problem for the operator L_λ in the upper half-plane. The conclusion is that the space $\mathcal{H}^p_+(\mathbb{R}, \mathcal{L}^1)$ embeds into $\mathscr{U}^p_\lambda$.

The reasoning above proves that

$$\mathscr{U}^p_\lambda = \mathcal{H}^p_+(\mathbb{R}, \mathcal{L}^1). \tag{8.2.83}$$

Recall that the Hardy spaces $\mathcal{H}^p_\pm(\mathbb{R}, \mathcal{L}^1)$ are closed subspaces of $L^p(\mathbb{R}, \mathcal{L}^1)$ and we have the direct sum decomposition

$$L^p(\mathbb{R}, \mathcal{L}^1) = \mathcal{H}^p_+(\mathbb{R}, \mathcal{L}^1) \oplus \mathcal{H}^p_-(\mathbb{R}, \mathcal{L}^1). \tag{8.2.84}$$

Then (8.2.68) is a consequence of (8.2.83) and (8.2.84). In particular, since $\mathcal{H}^p_-(\mathbb{R}, \mathcal{L}^1)$ is infinite dimensional, the claim in (8.2.69) also follows.

Finally, that the space on null-solutions for the L^p Dirichlet Problem for the operator L_λ in the upper half-plane, described in (8.2.70), is infinite dimensional is seen from (8.2.34) (also bearing in mind [68, Lemma 8.1.7] and [68, Proposition 8.9.8]), upon noting that each function ϕ_k defined as in (8.2.33) satisfies $\mathcal{N}_\kappa \phi_k \in L^p(\mathbb{R}, \mathcal{L}^1)$ (cf. [68, Lemma 8.3.7]). □

Lastly, here is a result complementing Theorems 8.2.3-8.2.4, describing both the space of admissible boundary data and the space of null-solutions for the Inhomogeneous Regularity Problem for the operator L_λ in the upper half-plane.

Theorem 8.2.5 *Select some aperture parameter $\kappa > 0$ together with some integrability exponent $p \in (1,\infty)$ and define the inhomogeneous Hardy spaces with regularity*

$$\begin{aligned}\mathcal{H}^p_{1,\pm}(\mathbb{R}, \mathcal{L}^1) &:= \Big\{w\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{C}_\pm} : w \in \mathscr{O}(\mathbb{C}_\pm) \textit{ with } \mathcal{N}_\kappa w,\ \mathcal{N}_\kappa(\nabla w) \in L^p(\mathbb{R}, \mathcal{L}^1)\Big\}\\ &= \Big\{f \in L^p_1(\mathbb{R}, \mathcal{L}^1) : Hf = \mp i f\Big\},\end{aligned} \tag{8.2.85}$$

where H is the Hilbert transform on the real line (cf. (8.2.64)). Also, pick a real number

$$\lambda \in (-1, 1) \setminus \{0\}. \tag{8.2.86}$$

Then the space of admissible boundary data for the Inhomogeneous Regularity Problem for the operator L_λ in the upper half-plane, i.e., the subspace of $L_1^p(\mathbb{R}, \mathcal{L}^1)$ described as

$$\mathscr{U}^p_{1,\lambda} := \Big\{ u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{C}_+} : \ u \in \mathscr{C}^\infty(\mathbb{C}_+),\ L_\lambda u = 0 \ \text{in}\ \mathbb{C}_+,\ \mathcal{N}_\kappa u,\ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, \mathcal{L}^1) \Big\}, \tag{8.2.87}$$

coincides with $\mathcal{H}^p_{1,+}(\mathbb{R}, \mathcal{L}^1)$ (the "positive" inhomogeneous Hardy space with regularity on the real line). Furthermore, the cokernel of $\mathscr{U}^p_{1,\lambda}$ into the full space of boundary data $L_1^p(\mathbb{R}, \mathcal{L}^1)$ is isomorphic to the "negative" inhomogeneous Hardy space with regularity $\mathcal{H}^p_{1,-}(\mathbb{R}, \mathcal{L}^1)$, i.e.,

$$\frac{L_1^p(\mathbb{R}, \mathcal{L}^1)}{\mathscr{U}^p_{1,\lambda}} \cong \mathcal{H}^p_{1,-}(\mathbb{R}, \mathcal{L}^1). \tag{8.2.88}$$

As a consequence,

> *the codimension of the space of admissible boundary data for the Inhomogeneous Regularity Problem for the operator L_λ in the upper half-plane (i.e., the space $\mathscr{U}^p_{1,\lambda}$) into the full space of boundary data $L_1^p(\mathbb{R}, \mathcal{L}^1)$, is $+\infty$.* (8.2.89)

Finally, the space on null-solutions for the Inhomogeneous Regularity Problem for the operator L_λ in the upper half-plane, namely

$$\Big\{ u \in \mathscr{C}^\infty(\mathbb{C}_+) : \ L_\lambda u = 0 \ \text{in}\ \mathbb{C}_+,\ \mathcal{N}_\kappa u,\ \mathcal{N}_\kappa(\nabla u) \in L^p(\mathbb{R}, \mathcal{L}^1),\ u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{C}_+} = 0 \Big\}, \tag{8.2.90}$$

coincides with the infinite dimensional space

$$\Big\{ \phi\big((1+\lambda)(z - \lambda\bar z)\big) - \phi\big((1-\lambda)(z + \lambda\bar z)\big) : \phi \in \mathscr{O}(\mathbb{C}_+) \\ \text{with}\ \mathcal{N}_\kappa \phi,\ \mathcal{N}_\kappa(\nabla\phi) \in L^p(\mathbb{R}, \mathcal{L}^1) \Big\}. \tag{8.2.91}$$

Proof This follows from Theorems 8.2.3-8.2.4 and their proofs. □

We conclude with three remarks, the first of which ties up with the discussion in Remark 8.1.6.

Remark 8.2.6 *Similar results to those described in Theorems 8.2.3-8.2.5 are valid with the upper half-plane $\mathbb{C}_+$ replaced by lower half-plane $\mathbb{C}_-$. For example, the space of admissible boundary data for the L^p Dirichlet Problem for the operator*

L_λ in the lower half-plane turns out to be the "negative" Hardy space $\mathcal{H}^p_-(\mathbb{R},\mathcal{L}^1)$. Together with the identification made in Theorem 8.2.4, this shows that the spaces of admissible boundary data for the L^p Dirichlet Problem for the operator L_λ in $\mathbb{C}_\pm$ are, respectively, $\mathcal{H}^p_+(\mathbb{R},\mathcal{L}^1)$ and $\mathcal{H}^p_-(\mathbb{R},\mathcal{L}^1)$. Remarkably, the latter Hardy spaces sum up (directly) to the full space of boundary data $L^p(\mathbb{R},\mathcal{L}^1)$ (cf. (8.2.84)). Related considerations also hold in the case of the identifications made in Theorem 8.2.3 and Theorem 8.2.5.

Our second remark briefly elaborates on the nature of the operator L_λ considered in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane.

Remark 8.2.7 *Pick an arbitrary complex number $\lambda \in \mathbb{C}$. Then the space of null-solutions for the classical Dirichlet Problem for the operator L_λ in the unit disk, namely*

$$\Big\{u \in \mathscr{C}^\infty\big(\overline{\mathbb{D}}\big) : L_\lambda u = 0 \text{ in } \mathbb{D} \text{ and } u\big|_{\partial\mathbb{D}} = 0\Big\}, \tag{8.2.92}$$

is infinite dimensional. A key observation in this regard is that for any $\phi \in \mathcal{O}(\mathbb{C})$ the function $u : \overline{\mathbb{D}} \to \mathbb{C}$ given by

$$u(z) := \phi\big((z+\lambda\overline{z})^2 - 4\lambda\big) - \phi\big((z-\lambda\overline{z})^2\big) \text{ for each } z \in \overline{\mathbb{D}} \tag{8.2.93}$$

belongs to the space described in (8.2.92). Indeed, it is clear from this definition that $u \in \mathscr{C}^\infty\big(\overline{\mathbb{D}}\big)$. Next, that $L_\lambda u = 0$ in $\mathbb{D}$ is seen from the factorization in (8.2.2) plus the Chain Rule involving a holomorphic function recalled in [70, (1.4.146)]. *Finally, that $u\big|_{\partial\mathbb{D}} = 0$ follows from the observation that*

$$\begin{gathered}\text{if } z \in \partial\mathbb{D} \text{ then } \overline{z} = z^{-1} \text{ hence} \\ (z+\lambda\overline{z})^2 - 4\lambda = (z+\lambda/z)^2 - 4\lambda = (z-\lambda/z)^2 = (z-\lambda\overline{z})^2.\end{gathered} \tag{8.2.94}$$

Choosing $\phi(\zeta) := \zeta^k$ with $k \in \mathbb{N}$ arbitrary then yields an infinite family of functions in the space (8.2.92), which are linearly independent if $\lambda \neq 0$.

The final remark reads as follows:

Remark 8.2.8 *Our earlier results pertaining to the failure of Fredholm solvability for boundary value problems for the scalar operator L_λ may be "lifted" to genuine systems by considering*

$$\mathbf{L}_\lambda := \begin{pmatrix} L_\lambda & 0 \\ 0 & L_\lambda \end{pmatrix} \tag{8.2.95}$$

which, for each or each $\lambda \in (-1,1)\setminus\{0\}$, is a second-order, homogeneous, complex constant coefficient, 2×2 weakly elliptic system.

Appendix A
Terms and notation used in Volume IV

A

Ahlfors regular domain (cf. [68, Definition 5.9.15]):

$$\text{a nonempty open subset } \Omega \text{ of } \mathbb{R}^n \text{ such that } \partial\Omega \text{ is an Ahlfors regular set and } \mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0 \tag{A.0.1}$$

$A_\kappa(\partial\Omega)$, the κ-accessible (from within Ω) subset of $\partial\Omega$ (cf. [68, (8.8.2)]):

$$A_\kappa(\partial\Omega) := \left\{x \in \partial\Omega : \ x \in \overline{\Gamma_\kappa(x)}\right\} \tag{A.0.2}$$

$[w]_{A_p}$, the characteristic of the weight w on a space of homogeneous type (X, ρ, μ) (cf. [68, §7.7]):

$$[w]_{A_p} := \sup_{B \ \rho\text{-ball}} \left(⨍_B w \, \mathrm{d}\mu\right)\left(⨍_B w^{-1/(p-1)} \, \mathrm{d}\mu\right)^{p-1} \tag{A.0.3}$$

$A_p(X, \rho, \mu)$, the Muckenhoupt A_p-class on a space of homogeneous type (X, ρ, μ) (cf. [68, §7.7]):

$$A_p(X, \rho, \mu) := \left\{w \ \text{ weight function } : \ [w]_{A_p} < \infty\right\} \tag{A.0.4}$$

$A_\infty(X, \rho, \mu)$, the Muckenhoupt A_∞-class on a space of homogeneous type (X, ρ, μ) (cf. [68, §7.7]):

$$A_\infty(X, \rho, \mu) := \bigcup_{1 \leq p < \infty} A_p(X, \mu) \tag{A.0.5}$$

$V^\perp$, the annihilator $V^\perp$ of a subspace V of a Banach space X:

$$V^\perp := \left\{\Lambda \in X^* : \ \Lambda(x) = 0 \ \text{ for all } \ x \in V\right\} \tag{A.0.6}$$

${}^\perp W$, the annihilator of a subspace W of X^* (where X is Banach):

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$$^{\perp}W := \big\{x \in X :\ \Lambda(x) = 0 \ \text{ for all } \ \Lambda \in W\big\} \tag{A.0.7}$$

T^*, the adjoint of the operator $T \in \mathcal{L}(X \to Y)$:

$$T^* : Y^* \longrightarrow X^*, \quad T^*(\Lambda) := \Lambda \circ T \ \text{ for each } \ \Lambda \in Y^* \tag{A.0.8}$$

$\langle A \rangle_p$, the absolutely p-convex hull of a subset A of a vector space X, with $p \in (0, 1]$, defined as (cf. [68, §7.8.3]):

$$\langle A \rangle_p := \Big\{ \sum_{j=1}^{M} \lambda_j v_j : M \in \mathbb{N},\ \{v_j\}_{1 \le j \le M} \subseteq A, \\ \{\lambda_j\}_{1 \le j \le M} \subseteq \mathbb{C} \text{ with} \sum_{j=1}^{M} |\lambda_j|^p \le 1 \Big\} \tag{A.0.9}$$

$a \otimes b$, the tensor product of vectors $a = (a_j)_{1 \le j \le N} \in \mathbb{C}^N$ and $b = (b_k)_{1 \le k \le M} \in \mathbb{C}^M$, defined as the $N \times M$ matrix

$$a \otimes b := \big(a_j b_k\big)_{\substack{1 \le j \le N \\ 1 \le k \le M}} \tag{A.0.10}$$

$A^\top$, the transpose of a coefficient tensor $A := \big(a_{rs}^{\alpha\beta}\big)_{\substack{1 \le r,s \le n \\ 1 \le \alpha,\beta \le M}}$ (cf. [70, §2.7]):

$$\big(A^\top\big)_{sr}^{\beta\alpha} := a_{rs}^{\alpha\beta} \ \text{ for all } \ \alpha, \beta, r, s \tag{A.0.11}$$

$A\nabla u$, the action of the coefficient tensor A on the Jacobian matrix ∇u, defined if $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1 \le r,s \le n \\ 1 \le \alpha,\beta \le M}}$ $u = (u_\beta)_{1 \le \beta \le N} \in \big[\mathcal{D}'(\Omega)\big]^M$ as (cf. [70, §2.7]):

$$A\nabla u := \big(a_{rs}^{\alpha\beta} \partial_s u_\beta\big)_{\substack{1 \le \alpha \le M \\ 1 \le r \le n}} \tag{A.0.12}$$

$\big\langle A\cdot, \cdot\big\rangle$, the bilinear form associated with the coefficient tensor $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1 \le r,s \le n \\ 1 \le \alpha,\beta \le M}}$ (cf. [70, §2.7]):

$$\big\langle A\zeta, \eta\big\rangle := a_{rs}^{\alpha\beta} \zeta_s^\beta \eta_r^\alpha \ \text{ for all } \ \zeta := (\zeta_s^\beta)_{\beta,s} \in \mathbb{C}^{N \times n} \ \text{ and } \ \eta := (\eta_r^\alpha)_{\alpha,r} \in \mathbb{C}^{M \times n} \tag{A.0.13}$$

$\mathcal{A}_{q,\kappa}$, the L^q-based area-function in $\mathbb{R}^n \setminus \Sigma$, acting on each $u \in W^{1,1}_{\rm loc}(\mathbb{R}^n \setminus \Sigma)$ as (cf. [70]):

$$(\mathcal{A}_{q,\kappa} u)(x) := \Big(\int_{\Gamma_\kappa^{\Sigma^c}(x)} |(\nabla u)(y)|^q |x - y|^{q-n}\, dy \Big)^{1/q}, \quad x \in \Sigma \tag{A.0.14}$$

$\mathfrak{A}_{\text{WE}}(n, M)$, the class of weakly elliptic coefficient tensors, defined as the collection of all coefficient tensors $A = \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$ with complex entries, with the property that the $M \times M$ homogeneous second-order system L_A associated with A in $\mathbb{R}^n$ is weakly elliptic.

$\mathcal{A}_{\alpha,\beta}$, the boundary-to-domain integral operator acting on any given differential form $f \in L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta+1}$ at each $z \in \Omega$ as in (7.3.38):

$$\mathcal{A}_{\alpha,\beta} f(z) := -\int_{\partial\Omega} \big\langle f(\zeta), \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta, z)\big\rangle_{\mathbb{C}}\, \mathrm{d}\sigma(\zeta)$$

$A_{\alpha,\beta}$, the boundary-to-boundary integral operator acting on any given differential form $f \in L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta+1}$ at σ-a.e. $z \in \partial\Omega$ as in (7.3.39):

$$A_{\alpha,\beta} f(z) := -\lim_{\varepsilon\to 0^+} \int_{\substack{\zeta\in\partial\Omega\\ |z-\zeta|>\varepsilon}} \big\langle f(\zeta), \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta, z)\big\rangle_{\mathbb{C}}\, \mathrm{d}\sigma$$

$A_s^{p,q}(\Omega)$, the Besov/Triebel-Lizorkin space in the open set $\Omega \subseteq \mathbb{R}^n$ (with $A := B$ corresponding to Besov spaces, and with $A := F$ corresponding to Triebel-Lizorkin spaces), with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, defined as (cf. [69, §9.2]):

$$A_s^{p,q}(\Omega) := \big\{u \in \mathcal{D}'(\Omega) : \text{there exists } U \in A_s^{p,q}(\mathbb{R}^n) \text{ such that } U\big|_\Omega = u\big\} \quad \text{(A.0.15)}$$

and equipped with the quasi-norm

$$\|u\|_{A_s^{p,q}(\Omega)} := \inf\Big\{\|U\|_{A_s^{p,q}(\mathbb{R}^n)} : U \in A_s^{p,q}(\mathbb{R}^n),\ U\big|_\Omega = u\Big\} \quad \text{(A.0.16)}$$

$A_s^{p,q}(\Omega)_{\text{bdd}}$, the space of all distributions u in Ω such that $\big(\psi\big|_\Omega\big)u \in A_s^{p,q}(\Omega)$ for each cutoff function $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ (cf. [69, Convention 8.3.7] and (A.0.217)).

B

$B_\rho(x, r)$, the ρ-ball with center at $x \in X$ and radius $r > 0$ in the quasi-metric space (X, ρ) (cf. [68, §7.1]):

$$B_\rho(x, r) := \{y \in X : \rho(x, y) < r\} \quad \text{(A.0.17)}$$

$B_{n-1}(x', r) := \big\{y' \in \mathbb{R}^{n-1} : |y' - x'| < r\big\}$, the $(n-1)$-dimensional (open) ball in $\mathbb{R}^{n-1}$ centered at $x' \in \mathbb{R}^{n-1}$ and of radius $r \in (0, \infty)$

BMO_1, the BMO-based Sobolev spaces of order one (locally integrable functions with distributional first-order partial derivatives in BMO)

$\|f\|_*(\Delta)$, the local BMO norm of the function f on the surface ball Δ (cf. [68, §7.4]):

$$\|f\|_*(\Delta) := \sup_{\Delta'\subseteq\Delta} \fint_{\Delta'} |f - f_{\Delta'}|\,d\mu \tag{A.0.18}$$

$\|f\|_{\dot{\mathrm{BMO}}(X,\mu)}$, the homogeneous BMO semi-norm of the function f in the context of a space of homogeneous type (X,ρ,μ) (cf. [68, §7.4]):

$$\|f\|_{\dot{\mathrm{BMO}}(X,\mu)} := \sup_{x\in X,\, r>0} \fint_{B_\rho(x,r)} \big|f - f_{B_\rho(x,r)}\big|\,d\mu \tag{A.0.19}$$

$\|\cdot\|_{\mathrm{BMO}(X,\mu)}$, the inhomogeneous BMO "norm" in the context of a space of homogeneous type (X,ρ,μ) (cf. [68, §7.4]):

$$\|f\|_{\mathrm{BMO}(X,\mu)} := \begin{cases} \|f\|_{\dot{\mathrm{BMO}}(X,\mu)} & \text{if } X \text{ is unbounded} \\ \|f\|_{\dot{\mathrm{BMO}}(X,\mu)} + \Big|\int_X f\,d\mu\Big| & \text{if } X \text{ is bounded} \end{cases} \tag{A.0.20}$$

$\mathrm{BMO}\big(X,\mu\big)$, the space of functions of bounded mean oscillations for a space of homogeneous type (X,ρ,μ) (cf. [68, §7.4]):

$$\mathrm{BMO}\big(X,\mu\big) := \Big\{f \in L^1_{\mathrm{loc}}(X,\mu) :\ \|f\|_{\mathrm{BMO}(X,\mu)} < +\infty\Big\} \tag{A.0.21}$$

$\widetilde{\mathrm{BMO}}(X,\mu)$, the space BMO modulo constants for a space of homogeneous type (X,ρ,μ) (cf. [68, (7.4.96)]):

$$\widetilde{\mathrm{BMO}}(X,\mu) := \mathrm{BMO}(X,\mu)/\sim\, = \big\{[f] :\ f \in \mathrm{BMO}(X,\mu)\big\} \tag{A.0.22}$$

∂E, the topological boundary of the set E

$Borel_\tau(X)$, the Borelians of the topological space (X,τ)

$[A;B] := [A,B] := AB - BA$, the commutator of A and B

$\{A;B\} := AB + BA$, the anti-commutator of A and B

$\mathrm{Bd}(X \to Y)$, the space of linear and bounded operators from the quasi-normed vector space $\big(X,\|\cdot\|_X\big)$ into the quasi-normed vector space $\big(Y,\|\cdot\|_Y\big)$ (cf. [69, §1.2]):

$$\mathrm{Bd}(X \to Y) := \big\{T : X \to Y :\ T \text{ linear mapping with } \|T\|_{X\to Y} < +\infty\big\} \tag{A.0.23}$$

$\mathrm{Bd}(X)$, the space of linear and bounded operators from the quasi-normed vector space X into itself:

$$\mathrm{Bd}(X) := \mathrm{Bd}(X \to X) \tag{A.0.24}$$

$\mathcal{B}(X \to Y)$, the space of linear and bounded operators from X to Y, where X, Y are two linear topological spaces (cf. [69, §1.1]):

$$\mathscr{B}(X \to Y) := \{T : X \longrightarrow Y : T \ \text{ linear and bounded}\} \tag{A.0.25}$$

$\mathcal{B}^{q,\lambda}(\Sigma, \sigma)$, the block space on the Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, defined for $q \in (1, \infty)$ and $\lambda \in (0, n-1)$ as (cf. [69, §6.2]):

$$\begin{aligned} \mathcal{B}^{q,\lambda}(\Sigma, \sigma) := \Big\{ f \in \big(\mathrm{Lip}_c(\Sigma)\big)' : & \text{there exist a sequence } \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \text{ and} \\ & \text{a family } \{b_j\}_{j\in\mathbb{N}} \text{ of } \mathcal{B}^{q,\lambda}\text{-blocks on } \Sigma \text{ so that} \\ & f = \sum_{j=1}^{\infty} \lambda_j b_j \ \text{ with convergence in } \big(\mathrm{Lip}_c(\Sigma)\big)' \Big\} \end{aligned} \tag{A.0.26}$$

and equipped with the norm

$$\begin{aligned} \|f\|_{\mathcal{B}^{q,\lambda}(\Sigma,\sigma)} := \inf \Big\{ \sum_{j=1}^{\infty} |\lambda_j| : \ f = \sum_{j=1}^{\infty} \lambda_j b_j \ \text{ in } \big(\mathrm{Lip}_c(\Sigma)\big)' \ \text{ with} \\ \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \ \text{ and each } b_j \text{ a } \mathcal{B}^{q,\lambda}\text{-block on } \Sigma \Big\} \end{aligned} \tag{A.0.27}$$

$\dot{B}^{p,q}_s(\Sigma, \sigma)$, the homogeneous Besov space on the Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, defined (cf. [69, Definition 7.1.2]) for

$$\begin{aligned} & s \in (-1, 1), \qquad \max\left\{\tfrac{n-1}{n}, \tfrac{n-1}{n+s}\right\} < p \le \infty, \qquad 0 < q \le \infty, \\ & \max\left\{(s)_+, -s + (n-1)\Big(\tfrac{1}{p} - 1\Big)_+\right\} < \beta < 1, \\ & \max\left\{s - \tfrac{n-1}{p}, (n-1)\Big(\tfrac{1}{p} - 1\Big)_+, -s + (n-1)\Big(\tfrac{1}{p} - 1\Big)\right\} < \gamma < 1 \end{aligned} \tag{A.0.28}$$

as the collection of "distributions" f on Σ (specifically, functionals $f \in \big(\mathring{\mathscr{G}}^{\beta,\gamma}_0(\Sigma)\big)^*$) for which

$$\|f\|_{\dot{B}^{p,q}_s(\Sigma,\sigma)} := \left\{ \sum_{k\in\mathbb{Z}} \left[2^{ks}\|E_k f\|_{L^p(\Sigma,\sigma)}\right]^q \right\}^{1/q} < \infty \tag{A.0.29}$$

with the natural alterations when $p = \infty$ or $q = \infty$

$B^{p,q}_s(\Sigma, \sigma)$, the inhomogeneous Besov space on the Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, defined (cf. [69, §7.1]) for

$$\begin{aligned} & s \in (-1, 1), \qquad \max\left\{\tfrac{n-1}{n}, \tfrac{n-1}{n+s}\right\} < p \le \infty, \qquad 0 < q \le \infty, \\ & \max\left\{(s)_+, -s + (n-1)\Big(\tfrac{1}{p} - 1\Big)_+\right\} < \beta < 1, \quad (n-1)\Big(\tfrac{1}{p} - 1\Big)_+ < \gamma < 1, \end{aligned} \tag{A.0.30}$$

as the collection of all "distributions" f on Σ such that, if $\{E_k\}_{k\in\mathbb{Z},\,k\geq\widetilde{\kappa_\Sigma}}$ is the family of conditional expectation operators on the Ahlfors regular set Σ, then

$$\|f\|_{B^{p,q}_s(\Sigma,\sigma)} := \Big\{\sum_{\tau\in I_{\widetilde{\kappa_\Sigma}}}\sum_{\nu=1}^{N(\widetilde{\kappa_\Sigma},\tau)} \sigma(Q^{\widetilde{\kappa_\Sigma},\nu}_\tau)\Big[m_{Q^{\widetilde{\kappa_\Sigma},\nu}_\tau}\big(|E_{\widetilde{\kappa_\Sigma}}f|\big)\Big]^p\Big\}^{1/p} + \Big\{\sum_{\substack{k\in\mathbb{Z}\\ k\geq\widetilde{\kappa_\Sigma}+1}}\big[2^{ks}\|E_kf\|_{L^p(\Sigma,\sigma)}\big]^q\Big\}^{1/q} < \infty \tag{A.0.31}$$

$B^{p,q}_s(\mathbb{R}^n)$, the Besov space in $\mathbb{R}^n$ equipped with the quasi-norm $\|\cdot\|_{B^{p,q}_s(\mathbb{R}^n)}$ for $0<p,q\leq\infty$ and $s\in\mathbb{R}$ (cf. [69, §9.1])

$B^{p,q}_s(\Omega)$, the Besov space in the open set $\Omega\subseteq\mathbb{R}^n$ with $0<p,q\leq\infty$ and $s\in\mathbb{R}$, defined as the collection of $u\in\mathcal{D}'(\Omega)$ for which there exists $U\in B^{p,q}_s(\mathbb{R}^n)$ such that $U\big|_\Omega=u$, and equipped with the quasi-norm (cf. [69, (9.2.1)])

$$\|u\|_{B^{p,q}_s(\Omega)} := \inf\Big\{\|U\|_{B^{p,q}_s(\mathbb{R}^n)} : U\in B^{p,q}_s(\mathbb{R}^n),\ U\big|_\Omega=u\Big\}$$

$\mathrm{BMO}_{-1}(\partial\Omega,\sigma)$, the BMO-based negative Sobolev space of order minus one on $\partial\Omega$ (cf. [69, Definition 11.10.9]):

$$\mathrm{BMO}_{-1}(\partial\Omega,\sigma) := \Big(H_1^{\frac{n-1}{n-2},1}(\partial\Omega,\sigma)\Big)^* \tag{A.0.32}$$

$\mathcal{B}^{q,\lambda}_1(\partial\Omega,\sigma)$, the block-based Sobolev space of order one on $\partial\Omega$ (cf. [69, (11.7.20)]):

$$\mathcal{B}^{q,\lambda}_1(\partial\Omega,\sigma) := \Big\{f\in\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) : \text{ for each } j,k\in\{1,\dots,n\} \text{ one has } \partial_{\tau_{jk}}f\in\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\Big\} \tag{A.0.33}$$

equipped with the norm

$$\mathcal{B}^{q,\lambda}_1(\partial\Omega,\sigma)\ni f\longmapsto \|f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} + \sum_{j,k=1}^n\big\|\partial_{\tau_{jk}}f\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \tag{A.0.34}$$

$\mathcal{B}^{p,q,\lambda}_1(\partial\Omega,\sigma)$, the off-diagonal block-based Sobolev space of order one on $\partial\Omega$ (cf. [69, (11.7.18)-(11.7.19)]):

$$\mathcal{B}^{p,q,\lambda}_1(\partial\Omega,\sigma) := \Big\{f\in\mathcal{B}^{p,\lambda}(\partial\Omega,\sigma) : \text{ for each } j,k\in\{1,\dots,n\} \text{ one has } \partial_{\tau_{jk}}f\in\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)\Big\} \tag{A.0.35}$$

equipped with the norm

$$\mathcal{B}_1^{p,q,\lambda}(\partial\Omega,\sigma) \ni f \longmapsto \|f\|_{\mathcal{B}^{p,\lambda}(\partial\Omega,\sigma)} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \tag{A.0.36}$$

$\mathcal{B}_{-1}^{q,\lambda}(\partial\Omega,\sigma)$, the block-based negative Sobolev space of order minus one on $\partial\Omega$ (cf. [69, Definition 11.8.9]):

$$\mathcal{B}_{-1}^{q,\lambda}(\partial\Omega,\sigma) := \Big(\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)\Big)^* \tag{A.0.37}$$

if $p, q \in (1,\infty)$ are such that $1/p + 1/q = 1$ and $\lambda \in (0, n-1)$

$\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)$, the block-based homogeneous Sobolev space of order one on $\partial\Omega$, for $q \in (1,\infty)$ and $\lambda \in (0, n-1)$ (cf. [69, Definition 11.13.11]):

$$\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) \cap L^{q_\lambda}_{\mathrm{loc}}(\partial\Omega,\sigma) : \text{ for } 1 \le j,k \le n \text{ one has that } \partial_{\tau_{jk}} f \in \mathcal{B}^{q,\lambda}(\partial\Omega,\sigma) \Big\} \tag{A.0.38}$$

where

$$q_\lambda := \frac{q(n-1)}{n-1+\lambda(q-1)} \in (1,q), \tag{A.0.39}$$

and equipped with the semi-norm

$$\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma) \ni f \mapsto \|f\|_{\dot{\mathcal{B}}_1^{q,\lambda}(\partial\Omega,\sigma)} := \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \tag{A.0.40}$$

$\mathcal{B}_{0,\beta}$, the classical boundary-to-domain Bochner-Martinelli integral operator on $(0,\beta)$-forms associated with a bounded $\mathscr{C}^1$ domain $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$, whose action on a continuous $(0,\beta)$-form f on $\partial\Omega$ is defined as in (7.3.28):

$$\mathcal{B}_{0,\beta} f(z) := \int_{\partial\Omega} \iota_\zeta^*\big(f(\zeta) \wedge K_{n\beta}(\zeta,z)\big) = -\int_{\partial\Omega} \big\langle \nu(\zeta) \wedge f(\zeta), \bar{\partial}_\zeta \Gamma_{0,\beta}(\zeta,z)\big\rangle_{\mathbb{C}}\, d\sigma(\zeta)$$

for each $z \in \mathbb{C}^n \setminus \partial\Omega$, where $\iota : \partial\Omega \hookrightarrow \mathbb{C}^n$ is the canonical inclusion and ι_ζ^* indicates pull-back in the variable ζ

$\mathcal{B}_{\alpha,\beta}$, the higher degree boundary-to-domain Bochner-Martinelli integral operator in a given set of locally finite perimeter $\Omega \subseteq \mathbb{C}^n$, whose action on an arbitrary (α,β)-form $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta}$ at each $z \in \mathbb{C}^n \setminus \partial\Omega$ is as in (7.3.33):

$$\mathcal{B}_{\alpha,\beta} f(z) := -\frac{1}{2} \int_{\partial_* \Omega} \left\langle \nu^{0,1}(\zeta) \wedge f(\zeta), \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta, z) \right\rangle_{\mathbb{C}} \, d\sigma(\zeta)$$

$B_{\alpha,\beta}$, the boundary-to-boundary Bochner-Martinelli integral operator associated with in a given set of locally finite perimeter $\Omega \subseteq \mathbb{C}^n$, whose action on an arbitrary (α, β)-form $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta}$ at each σ-a.e. $z \in \partial\Omega$ is as in (7.3.68):

$$B_{\alpha,\beta} f(z) := -\frac{1}{2} \lim_{\varepsilon \to 0^+} \int_{\substack{\zeta \in \partial_* \Omega \\ |z-\zeta| > \varepsilon}} \left\langle \nu^{0,1}(\zeta) \wedge f(\zeta), \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta, z) \right\rangle_{\mathbb{C}} \, d\sigma(\zeta)$$

$\mathscr{B}^{p,q}_{\alpha}(\Omega; D)$, the (interior) Besov-Hardy space of null-solutions of the Dirac operator $D = \sum_{j=1}^{n} \mathbf{e}_j \odot \partial_j$ in an open set $\Omega \subseteq \mathbb{R}^n$, defined for any $p \in (0, \infty]$, any $q \in (0, \leq \infty]$, and any $\alpha \in \mathbb{R}$ as in (4.4.121):

$$\mathscr{B}^{p,q}_{\alpha}(\Omega; D) := \left\{ u \in B^{p,q}_{\alpha}(\Omega) \otimes \mathcal{C}\ell_n : \; Du = 0 \text{ in } \Omega \right\}$$

and equipped with the quasi-norm inherited from $B^{p,q}_{\alpha}(\Omega) \otimes \mathcal{C}\ell_n$

$\mathscr{B}^{p,q}_{s}(\partial\Omega; D)$, the boundary Besov-Hardy space associated with the Dirac operator $D = \sum_{j=1}^{n} \mathbf{e}_j \odot \partial_j$ in a bounded NTA domain $\Omega \subseteq \mathbb{R}^n$ with an Ahlfors regular boundary, defined for integrability exponents $p \in \big(\frac{n-1}{n}, \infty\big)$ and $q \in (0, \infty]$, and smoothness index $(n-1)\big(\frac{1}{p} - 1\big)_+ < s < 1$ as in (4.4.125):

$$\mathscr{B}^{p,q}_{s}(\partial\Omega; D) := \left\{ \operatorname{Tr}_{\Omega \to \partial\Omega} u : \; u \in \mathscr{B}^{p,q}_{s+\frac{1}{p}}(\Omega; D) \right\}$$

and equipped with the quasi-norm inherited from $B^{p,q}_{s}(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$

C

$\overline{U}$, the closure of the set $U \subseteq \mathbb{R}^n$

$\mathscr{C}^k(\overline{\Omega})$, the space of functions of class $\mathscr{C}^k$ in an open neighborhood of $\overline{\Omega}$

$\mathscr{C}^k_c(\Omega)$, the space of functions of class $\mathscr{C}^k$ with compact support in the open set Ω

$\mathscr{C}^k_b(\Omega)$, the space of bounded functions of class $\mathscr{C}^k$ in Ω

$\Big(\mathscr{C}^\infty_b(\Omega)\Big)^*$, the algebraic dual of $\mathscr{C}^\infty_b(\Omega)$

$\mathrm{CBM}(\Omega)$, the space of complex Borel measures in the open set $\Omega \subseteq \mathbb{R}^n$

$\mathcal{C}\ell_n$, the Clifford algebra $(\mathcal{C}\ell_n, +, \odot)$ generated by n imaginary units defined as the minimal enlargement of $\mathbb{R}^n$ to a unitary real algebra which is not generated (as an algebra) by any proper subspace of $\mathbb{R}^n$, and such that $x \odot x = -|x|^2$ for each $x \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n$ (cf. [68, §6.4])

$\|\cdot\|_{\dot{\mathscr{C}}^\alpha(U,\rho)}$, the homogeneous Hölder space semi-norm of order $\alpha > 0$ in the set $U \subseteq X$, in the context of a quasi-metric space (X, ρ), defined for each function $f : U \to \mathbb{R}$ as (cf. [68, (7.3.2)]):

$$\|f\|_{\dot{\mathscr{C}}^\alpha(U,\rho)} := \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|f(x) - f(y)|}{\rho(x, y)^\alpha} \tag{A.0.41}$$

$\dot{\mathscr{C}}^\alpha(U, \rho)$, the homogeneous Hölder space of order $\alpha > 0$ in the set $U \subseteq X$, defined in the context of a quasi-metric space (X, ρ) as (cf. [68, (7.3.1)]):

$$\dot{\mathscr{C}}^\alpha(U, \rho) := \Big\{f : U \to \mathbb{R} : \ \|f\|_{\dot{\mathscr{C}}^\alpha(U,\rho)} < +\infty\Big\} \tag{A.0.42}$$

$\dot{\mathscr{C}}^\alpha(U, \rho)/\sim$, the homogeneous Hölder space of order $\alpha > 0$ modulo constants, in the set $U \subseteq X$, defined in the context of a quasi-metric space (X, ρ) as (cf. [68, (7.3.6)]):

$$\dot{\mathscr{C}}^\alpha(U, \rho)/\sim := \big\{[f] : \ f \in \dot{\mathscr{C}}^\alpha(U, \rho)\big\} \tag{A.0.43}$$

$\dot{\mathscr{C}}^\alpha_{\rm loc}(U, \rho)$, the local homogeneous Hölder space of order $\alpha > 0$ in the set $U \subseteq X$, defined in the context of a quasi-metric space (X, ρ) as (cf. [68, (7.3.7)]):

$$\dot{\mathscr{C}}^\alpha_{\rm loc}(U, \rho) := \Big\{f : U \to \mathbb{C} : \ f\big|_{B_\rho(x,r)\cap U} \in \dot{\mathscr{C}}^\alpha\big(B_\rho(x, r) \cap U, \rho\big) \\ \text{for each } x \in U \text{ and } r \in (0, \infty)\Big\} \tag{A.0.44}$$

$\|\cdot\|_{\mathscr{C}^\alpha(U,\rho)}$, the inhomogeneous Hölder space norm of order $\alpha > 0$ in the set $U \subseteq X$, in the context of a quasi-metric space (X, ρ), defined for each function $f : U \to \mathbb{R}$ as (cf. [68, (7.3.20)]):

$$\|f\|_{\mathscr{C}^\alpha(U,\rho)} := \sup_U |f| + \|f\|_{\dot{\mathscr{C}}^\alpha(U,\rho)}, \qquad \forall f \in \mathscr{C}^\alpha(U, \rho) \tag{A.0.45}$$

$\mathscr{C}^\alpha(U, \rho)$, the inhomogeneous Hölder space of order $\alpha > 0$ in the set $U \subseteq X$, defined in the context of a quasi-metric space (X, ρ) as (cf. [68, (7.3.19)]):

$$\mathscr{C}^\alpha(U, \rho) := \big\{f \in \dot{\mathscr{C}}^\alpha(U, \rho) : f \text{ is bounded in } U\big\} \tag{A.0.46}$$

$\mathscr{C}^\alpha_c(U, \rho)$, the space of Hölder functions of order $\alpha > 0$ with ρ-bounded support in the set $U \subseteq X$, defined in the context of a quasi-metric space (X, ρ) as (cf. [68, (7.3.26), (7.3.27)]):

$$\mathscr{C}^\alpha_c(U, \rho) := \big\{f \in \dot{\mathscr{C}}^\alpha(U, \rho) : f \text{ vanishes outside of a } \rho\text{-bounded subset of } U\big\} \tag{A.0.47}$$

$\dot{\mathscr{C}}^{\gamma}_{\mathrm{van}}(\Sigma)$, the homogeneous "vanishing" Hölder space of order γ on the set Σ (cf. [69, (3.2.5)]):

$$\dot{\mathscr{C}}^{\gamma}_{\mathrm{van}}(\Sigma) := \Big\{ f \in \dot{\mathscr{C}}^{\gamma}(\Sigma) : \lim_{r\to 0^+} \Big(\sup_{x\in\Sigma} \|f\|_{\dot{\mathscr{C}}^{\gamma}(B(x,r)\cap\Sigma)} \Big) = 0 \Big\} \tag{A.0.48}$$

$\mathscr{C}^{\gamma}_{\mathrm{van}}(\Sigma)$, the inhomogeneous "vanishing" Hölder space of order γ on the set Σ (cf. [69, (3.2.8)]):

$$\mathscr{C}^{\gamma}_{\mathrm{van}}(\Sigma) := \Big\{ f \in \mathscr{C}^{\gamma}(\Sigma) : \lim_{r\to 0^+} \Big(\sup_{x\in\Sigma} \|f\|_{\dot{\mathscr{C}}^{\gamma}(B(x,r)\cap\Sigma)} \Big) = 0 \Big\} \tag{A.0.49}$$

$\mathrm{Cp}(X \to Y)$, the space of compact linear operators from the topological vector space X into the topological vector space Y:

$$\mathrm{Cp}(X \to Y) := \big\{ T : X \to Y : T \text{ linear compact mapping} \big\} \tag{A.0.50}$$

$\mathrm{Cp}(X)$, the space of compact linear operators from the topological vector space X into itself:

$$\mathrm{Cp}(X) := \mathrm{Cp}(X \to X) \tag{A.0.51}$$

$\mathrm{CMO}(\Sigma, \sigma)$, the Coifman-Weiss space (cf. [69, (4.6.11)]):

$\mathrm{CMO}(\Sigma, \sigma)$ is the closure in $\mathrm{BMO}(\Sigma, \sigma)$ of $\mathscr{C}^0_0(\Sigma)$, the space of all continuous functions on Σ which vanish at infinity (A.0.52)

C, the boundary-to-domain Cauchy-Clifford integral operator associated with a given set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, whose action on any given function f in $L^1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big) \otimes C\ell_n$ at each $x \in \mathring{\Omega}$ is

$$Cf(x) := \frac{1}{\omega_{n-1}} \int_{\partial_*\Omega} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, \mathrm{d}\sigma_*(y) \tag{A.0.53}$$

$\mathfrak{C}$, the boundary-to-boundary Cauchy-Clifford integral operator, acting on any function $f \in L^1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big) \otimes C\ell_n$ at σ_*-a.e. point $x \in \partial_*\Omega$ as:

$$\mathfrak{C}f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, \mathrm{d}\sigma_*(y) \tag{A.0.54}$$

$\mathfrak{C}^{\#}$, the transpose Cauchy-Clifford integral operator associated with a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, acting on functions $f \in L^1\big(\partial\Omega, \frac{\sigma(y)}{1+|y|^{n-1}}\big) \otimes C\ell_n$ at σ-a.e. point $x \in \partial_*\Omega$ as in (1.6.1):

$$\mathfrak{C}^{\#} f(x) := \lim_{\varepsilon\to 0^+} \frac{-1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \nu(x) \odot \frac{x-y}{|x-y|^n} \odot f(y)\,\mathrm{d}\sigma(y)$$

$\mathscr{C}_{\rm mod}$, the modified boundary-to-domain Cauchy integral operator in the plane, acting on $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^2}\big)$ at each $z \in \Omega$ as in (1.8.227):

$$(\mathscr{C}_{\rm mod})(z) := \frac{1}{2\pi \mathrm{i}} \int_{\partial_*\Omega} \Big\{ \frac{1}{\zeta - z} - \frac{1}{\zeta} \mathbf{1}_{\mathbb{C}\setminus\overline{B(0,1)}}(\zeta) \Big\} f(\zeta)\,\mathrm{d}\zeta$$

$\mathcal{C}_{\rm mod}$, the modified boundary-to-domain Cauchy-Clifford integral operator, acting on any function $f \in L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\big) \otimes \mathcal{C}\ell_n$ at each point $x \in \Omega$ as in (1.8.238):

$$\mathcal{C}_{\rm mod} f(x) := \frac{1}{\omega_{n-1}} \int\limits_{\partial_*\Omega} \Big\{ \frac{x-y}{|x-y|^n} + \frac{y}{|y|^n} \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,1)}}(y) \Big\} \odot \nu(y) \odot f(y)\,\mathrm{d}\sigma(y)$$

$\dot{C}^p_{q,\eta}(\Sigma,\sigma)$, the homogeneous Calderón space on the closed Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$ for $p \in [1,\infty]$, $q \in [1,\infty)$, and $\eta \in \mathbb{R}$, defined as in (3.1.10), (3.1.11):

$$\dot{C}^p_{q,\eta}(\Sigma,\sigma) := \Big\{ f \in L^1_{\rm loc}(\Sigma,\sigma) : \ f^{\#}_{q,\eta} \in L^p(\Sigma,\sigma) \Big\}$$

and equipped with the semi-norm

$$\|f\|_{\dot{C}^p_{q,\eta}(\Sigma,\sigma)} := \big\| f^{\#}_{q,\eta} \big\|_{L^p(\Sigma,\sigma)}, \qquad \forall f \in \dot{C}^p_{q,\eta}(\Sigma,\sigma)$$

where $f^{\#}_{q,\eta}(x) := \sup\limits_{R>0} \big\{ R^{-\eta} \big(⨍_{\Delta(x,R)} \big|f(y) - f_{\Delta(x,R)}\big|^q \,\mathrm{d}\sigma(y) \big)^{1/q} \big\}$ for all $x \in \Sigma$

$C^p_{q,\eta}(\Sigma,\sigma)$, the inhomogeneous Calderón space on the closed Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, defined for $p \in [1,\infty]$, $q \in [1,\infty)$, and $\eta \in \mathbb{R}$ as in (3.1.14), (3.1.15)

$$C^p_{q,\eta}(\Sigma,\sigma) := L^p(\Sigma,\sigma) \cap \dot{C}^p_{q,\eta}(\Sigma,\sigma) = \Big\{ f \in L^p(\Sigma,\sigma) : \ f^{\#}_{q,\eta} \in L^p(\Sigma,\sigma) \Big\}$$

and equip this space with the norm

$$\|f\|_{C^p_{q,\eta}(\Sigma,\sigma)} := \|f\|_{L^p(\Sigma,\sigma)} + \big\| f^{\#}_{q,\eta} \big\|_{L^p(\Sigma,\sigma)}, \qquad \forall f \in C^p_{q,\eta}(\Sigma,\sigma)$$

D

$u \cdot w = \langle u, w \rangle$, the dot product of two vectors $u, w \in \mathbb{R}^n$

$\mathrm{div}\vec{F}$, the divergence of the vector field $\vec{F}$

$\mathcal{D}'(\Omega)$, the space of distributions in the open set Ω

${}_{\mathcal{D}'(\Omega)}\langle \cdot, \cdot \rangle_{\mathcal{D}(\Omega)}$, the distributional pairing in the open set Ω

$\Delta := \partial_1^2 + \cdots + \partial_n^2$, the Laplace operator in $\mathbb{R}^n$

δ_x, the Dirac distribution with mass at x

D, the classical (homogeneous) Dirac operator in $\mathbb{R}^n$ defined as (cf. [68, (6.4.139)]):

$$D = \sum_{j=1}^{n} \mathbf{e}_j \odot \partial_j \tag{A.0.55}$$

$D = \Big(\sum_{j=1}^{n} a_j^{\alpha\beta} \partial_j + b^{\alpha\beta} \Big)_{\substack{1\le\alpha\le N \\ 1\le\beta\le N'}}$, a (generic) $N \times N'$ first-order system

$D^\top$, the (real) transpose of the first-order system D:

$$D^\top := \Big(- \sum_{j=1}^{n} a_j^{\alpha\beta} \partial_j + b^{\alpha\beta} \Big)_{\substack{1\le\beta\le N' \\ 1\le\alpha\le N}} \tag{A.0.56}$$

$\overline{D}$, the complex conjugate of the first-order system D

D^*, the Hermitian adjoint of the first-order system D

d, the exterior derivative operator acting on the differential form $u = \sum_J u_J \mathrm{d}x^J$ according to (cf. [68, (1.11.32)], and also [68, (6.4.140)-(6.4.141)] for the Clifford algebra context):

$$\mathrm{d}u = \sum_{j=1}^{n} \sum_{J} \frac{\partial u_J}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x^J \tag{A.0.57}$$

δ, the formal adjoint of the exterior derivative operator d on differential forms (see also [68, (6.4.142)] for the Clifford algebra context)

δ_{jk}, the Kronecker symbol, i.e., $\delta_{jk} := 1$ if $j = k$ and $\delta_{jk} := 0$ if $j \neq k$

$\delta_{\partial\Omega}(\cdot)$, the distance function to the boundary of Ω

$U \triangle V := (U \setminus V) \cup (V \setminus U)$, the symmetric difference of the sets U and V

$\Delta(x, r) := B(x, r) \cap \partial\Omega$, the surface ball on $\partial\Omega$ with center at $x \in \partial\Omega$ and radius $r > 0$

$\mathbf{D} := D_L$, the (homogeneous) Dirac operator in $\mathbb{R}^n$ acting from the left

$$\mathbf{D} := D_L := \sum_{j=1}^{n} \mathbf{e}_j \odot \partial_j \tag{A.0.58}$$

D_R, the (homogeneous) Dirac operator acting from the right on the Clifford algebra-valued function u according to

$$D_R u := \sum_{j=1}^{n} (\partial_j u) \odot \mathbf{e}_j \tag{A.0.59}$$

$\mathbb{D}_k(X)$, the k-th generation of dyadic cubes in the geometrically doubling quasi-metric space X, defined as in [68, Proposition 7.5.4]:

$$\mathbb{D}_k(X) := \{Q^k_\alpha\}_{\alpha \in I_k} \tag{A.0.60}$$

$\mathbb{D}(X)$, the dyadic grid on the geometrically doubling quasi-metric space X, defined as in [68, Proposition 7.5.4]:

$$\mathbb{D}(X) := \bigcup_{k\in\mathbb{Z},\, k\geq \kappa_X} \mathbb{D}_k(X) \tag{A.0.61}$$

$\dim X$, the dimension of a vector space X

$\mathrm{d}\zeta$, the complex arc-length, defined as:

$$\mathrm{d}\zeta := \mathrm{i}\nu(\zeta)\,\mathrm{d}\sigma(\zeta) \tag{A.0.62}$$

Def, the deformation tensor, acting on the vector-distribution $\vec{u} = (u_1, \dots, u_n)$ as:

$$\mathrm{Def}\,\vec{u} := \tfrac{1}{2}\big(\partial_j u_k + \partial_k u_j\big)_{1\leq j,k\leq n} \tag{A.0.63}$$

$\Delta + k^2$, the Helmholtz operator with wave number k

$\mathrm{d}V$, the volume element in $\mathbb{C}^n$, defined as in (7.1.47):

$$\mathrm{d}V := \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n = (-2\mathrm{i})^{-n}\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n$$

$\mathcal{D} = \mathcal{D}_A$, the boundary-to-domain double layer for a second-order, homogeneous, constant coefficient, $M \times M$ weakly elliptic system $L_A := \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ corresponding to a given coefficient tensor $A := \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$ (with canonical fundamental solution $E = \big(E_{\gamma\beta}\big)_{1\leq\gamma,\beta\leq M}$), whose action on any vector-valued function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ is defined at each point $x \in \Omega$ as in (1.3.18):

$$\mathcal{D}_A f(x) := \Big(-\int_{\partial_*\Omega} \nu_s(y) a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y) f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M}$$

$\mathcal{D}_{\text{mod}}$, the boundary-to-domain modified double layer potential operator for an $M\times M$ weakly elliptic system $L_A := \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ corresponding to the coefficient tensor $A := \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$ (with fundamental solution $E = \big(E_{\gamma\beta}\big)_{1\leq\gamma,\beta\leq M}$) whose action on $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\big]^M$ at each $x \in \Omega$ is defined as in (1.8.6):

$$\big(\mathcal{D}_{\text{mod}} f\big)(x) := \Big(-\int_{\partial_*\Omega} \nu_s(y) a^{\beta\alpha}_{rs}\big\{(\partial_r E_{\gamma\beta})(x-y) - k_1^{(r\gamma\beta)}(-y)\big\} f_\alpha(y)\,\mathrm{d}\sigma(y)\Big)_{1\leq\gamma\leq M}$$

where $k_1^{(r\gamma\beta)} := (\partial_r E_{\gamma\beta}) \cdot \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(0,1)}}$ for every r, γ, β

$\mathcal{D}_{\Delta,\mathrm{mod}}$, the boundary-to-domain modified harmonic double layer potential operator whose action on each $f \in L^1\big(\partial_*\Omega, \frac{\sigma(y)}{1+|y|^n}\big)$ is defined at each $x \in \Omega$ as in (1.8.50):

$$\mathcal{D}_{\Delta,\mathrm{mod}} f(x) := \frac{1}{\omega_{n-1}} \int_{\partial_*\Omega} \Big\{ \frac{\langle \nu(y), y-x \rangle}{|x-y|^n} - \frac{\langle \nu(y), y \rangle}{|y|^n} \cdot \mathbf{1}_{\mathbb{R}^n \setminus \overline{B(0,1)}}(y) \Big\} f(y)\, \mathrm{d}\sigma(y)$$

$\mathcal{D}_\lambda$, the boundary-to-domain double layer for the Stokes system associated with a given open set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter and with the coefficient tensor $A_\lambda := \big(\delta_{jk}\delta_{\alpha\beta} + \lambda\delta_{j\beta}\delta_{k\alpha}\big)_{\substack{1\leq\alpha,\beta\leq n\\ 1\leq j,k\leq n}}$ corresponding to any $\lambda \in \mathbb{C}$, whose action on each $\vec{f} = (f_j)_{1\leq j\leq n} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n$ at each $x \in \Omega$ as in (6.2.17), (6.2.18):

$$\begin{aligned}\mathcal{D}_\lambda \vec{f}(x) = \Bigg(\int_{\partial_*\Omega} \Bigg\{ &(\lambda-1)\frac{\delta_{j\gamma}}{2\omega_{n-1}} \frac{\langle x-y, \nu(y)\rangle}{|x-y|^n} \\ &- \frac{n(\lambda+1)}{2\omega_{n-1}} \frac{\langle x-y, \nu(y)\rangle (x_j-y_j)(x_\gamma-y_\gamma)}{|x-y|^{n+2}} \\ &+ \frac{\lambda-1}{2\omega_{n-1}} \frac{\nu_j(y)(x_\gamma-y_\gamma) - \nu_\gamma(y)(x_j-y_j)}{|x-y|^n} \Bigg\} f_j(y)\, \mathrm{d}\sigma(y) \Bigg)_{1\leq\gamma\leq n}\end{aligned}$$

$\mathcal{D}_{\alpha,\beta}$, the boundary-to-domain complex double layer potential operator associated with $\Omega \subseteq \mathbb{C}^n$, acting on any given (α,β)-form $f \in L^1\big(\partial_*\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big) \otimes \Lambda^{\alpha,\beta}$ at each $z \in \Omega$ as in (7.4.144):

$$\mathcal{D}_{\alpha,\beta} f(z) := \frac{1}{2} \int_{\partial_*\Omega} \Big\langle f(\zeta), \nu^{0,1}(\zeta) \wedge \vartheta_\zeta \Gamma_{\alpha,\beta}(\zeta,z) - \nu^{1,0}(\zeta) \vee \bar{\partial}_\zeta \Gamma_{\alpha,\beta}(\zeta,z) \Big\rangle_{\mathbb{C}} \mathrm{d}\sigma(\zeta)$$

${}_{\mathcal{D}'(\Omega)}\langle\langle \cdot, \cdot \rangle\rangle_{\mathcal{D}(\Omega)}$, the distributional Hermitian inner product for double forms (7.3.11)

E

(ε,δ)-domain: (cf. [68, Definition 5.11.8])

a nonempty, open, proper subset Ω of $\mathbb{R}^n$ with the property that for any $x, y \in \Omega$ with $|x-y| < \delta$ there exists a rectifiable curve $\gamma : [0,1] \longrightarrow \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$, as well as $\mathrm{length}(\gamma) \leq \frac{1}{\varepsilon}|x-y|$ and $\frac{|z-x|\,|z-y|}{|x-y|} \leq \frac{1}{\varepsilon}\mathrm{dist}(z, \partial\Omega)$ for each $z \in \gamma([0,1])$ (A.0.64)

$\mathbf{e}_j$, the unit vector in the j-th direction in $\mathbb{R}^n$, defined for each $j \in \{1,\dots,n\}$ as $\mathbf{e}_j := (\delta_{jk})_{1\leq k\leq n} \in \mathbb{R}^n$ where δ_{jk} is the Kronecker symbol

$\{\mathbf{e}_j\}_{1\leq j\leq n}$, the standard orthonormal basis in $\mathbb{R}^n$

E_Δ, the standard fundamental solution for the Laplacian, given for each $x \in \mathbb{R}^n \setminus \{0\}$ by

$$E_\Delta(x) := \begin{cases} \dfrac{1}{\omega_{n-1}(2-n)} \dfrac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\ \dfrac{1}{2\pi} \ln |x| & \text{if } n = 2 \end{cases} \tag{A.0.65}$$

$\mathscr{E}'(\Omega)$, the space of distributions compactly supported in the open set $\Omega \subseteq \mathbb{R}^n$

$[x]_{X/Y} := x + Y$, the equivalence class of the vector $x \in X$ in the quotient space X/Y

$\mathcal{E}_p(X)$, the p-envelope of a quasi-normed space X whose dual separates points, defined for $p \in (0,1]$ as the completion of X in the quasi-norm $|\!|\!| \cdot |\!|\!|_p$ (cf. [69, Definition 7.8.4])

$\mathrm{Ex}_{\partial\Omega\to\Omega}$, the extension operator from $\partial\Omega$ to Ω (cf. [69, Theorem 8.4.1])

$E = E_L = \big(E_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$, the canonical fundamental solution of the weakly elliptic $M \times M$ system L defined as in [70, Theorem 1.4.2]:

$$E(x) := \begin{cases} -\dfrac{\Delta_x^{(n-1)/2}}{4(2\pi \mathrm{i})^{n-1}} \left\{ \displaystyle\int_{S^{n-1}} \big|\langle x, \xi\rangle\big| \big[L(\xi)\big]^{-1} \, d\mathcal{H}^{n-1}(\xi) \right\} & \text{if } n \text{ is odd}, \\ \dfrac{\Delta_x^{(n-2)/2}}{(2\pi \mathrm{i})^{n}} \left\{ \displaystyle\int_{S^{n-1}} \ln\big|\langle x, \xi\rangle\big| \, \big[L(\xi)\big]^{-1} \, d\mathcal{H}^{n-1}(\xi) \right\} & \text{if } n \text{ is even}. \end{cases} \tag{A.0.66}$$

ε^A_B, the generalized Kronecker symbol defined for any two arrays A, B as (cf. [68, (6.4.116)]):

$$\varepsilon^A_B := \begin{cases} \det\big((\delta_{ab})_{a\in A, b\in B}\big) & \text{if } |A| = |B|, \\ 0 & \text{otherwise} \end{cases} \tag{A.0.67}$$

${}_{\mathcal{E}'(\Omega)}\langle\langle \cdot, \cdot \rangle\rangle_{\mathcal{E}(\Omega)}$, the (compact support) distributional Hermitian inner product for double forms (7.3.12)

$E_n(\cdot,\cdot)$, the fundamental solution for the complex Laplacian $\Box := \bar\partial\vartheta + \vartheta\bar\partial$ in $\mathbb{C}^n$ (7.3.18)

F

$\Phi(X \to Y)$, the collection of Fredholm operators from the linear topological space X into the linear topological space Y (cf. [69, Definition 2.2.1])

$\Phi_+(X \to Y)$, the collection of finite-dim kernel semi-Fredholm operators from the Banach space X into the Banach space Y (cf. [69, §2.1])

$\Phi_-(X \to Y)$, the collection of finite-dim cokernel semi-Fredholm operators from the Banach space X into the Banach space Y (cf. [69, §2.1])

$f^{\sharp}_{\gamma}$, the Fefferman-Stein grand maximal function (with parameter $\gamma \in (0,1)$), associating to each "distribution" $f \in \big(\mathrm{Lip}_c(\Sigma)\big)'$ on the Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$ the function defined at each point $x \in \Sigma$ by (cf. [69, (4.1.6)]):

$$f^{\sharp}_{\gamma}(x) := \sup_{\psi \in \mathcal{T}_{\gamma}(x)} \big|\langle f, \psi\rangle\big| \tag{A.0.68}$$

where $\mathcal{T}_{\gamma}(x) \subseteq \mathrm{Lip}_c(\Sigma)$ is the collection of all normalized (in $\dot{\mathscr{C}}^{\gamma}(\Sigma)$) bump functions centered at the point x

$\rho_{\mathrm{Fred}}(T;X)$, the Fredholm (or essential spectral) radius of $T \in \mathrm{Bd}\,(X)$ (cf. [69, Definition 2.2.5]):

$$\rho_{\mathrm{Fred}}(T;X) := \inf\big\{r > 0 :\ zI - T \in \Phi(X \to X)\ \text{ for each }\ z \in \mathbb{C} \setminus \overline{B(0,r)}\big\} \tag{A.0.69}$$

$[\vec{F}]_\infty$, the contribution of the vector field $\vec{F}$ at infinity defined (for a system of auxiliary functions, i.e., a family $\{\phi_R\}_{R>0}$ of smooth compactly supported functions in $\mathbb{R}^n$ which are globally bounded and progressively become pointwise equal to 1 on compact sets, in a uniform fashion) as in [68, Chapters 1 and 4]:

$$[\vec{F}]_\infty := -\lim_{R\to\infty} \int_{\Omega} \nabla\phi_R \cdot \vec{F}\,\mathrm{d}\mathcal{L}^n \tag{A.0.70}$$

$\dot{F}^{p,q}_s(\Sigma,\sigma)$, the homogeneous Triebel-Lizorkin space on the Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, defined (cf. [69, Definition 7.1.2]) for

$$\begin{gathered} s \in (-1,1), \quad \max\big\{\tfrac{n-1}{n}, \tfrac{n-1}{n+s}\big\} < p \leq \infty, \quad \max\big\{\tfrac{n-1}{n}, \tfrac{n-1}{n+s}\big\} < q \leq \infty, \\ \max\Big\{(s)_+, -s + (n-1)\Big(\tfrac{1}{p} - 1\Big)_+\Big\} < \beta < 1, \\ \max\Big\{s - \tfrac{n-1}{p}, (n-1)\Big(\tfrac{1}{p} - 1\Big)_+, -s + (n-1)\Big(\tfrac{1}{p} - 1\Big)\Big\} < \gamma < 1, \end{gathered} \tag{A.0.71}$$

as the collection of all "distributions" f on the closed set Σ (specifically, functionals in $\big(\mathring{\mathscr{G}}^{\beta,\gamma}_0(\Sigma)\big)^*$) with the property that

$$\|f\|_{\dot{F}^{p,q}_s(\Sigma,\sigma)} := \Bigg\|\bigg\{\sum_{k\in\mathbb{Z}} \big[2^{ks}|E_k f|\big]^q\bigg\}^{1/q}\Bigg\|_{L^p(\Sigma,\sigma)} < \infty \tag{A.0.72}$$

whenever $p < \infty$ (with natural alterations when $q = \infty$) and, corresponding to the case when $p = \infty$,

$$\|f\|_{\dot{F}^{\infty,q}_s(\Sigma,\sigma)} := \sup_{\ell\in\mathbb{Z}}\sup_{\tau\in I_\ell} \bigg[\fint_{Q^\ell_\tau} \sum_{k=\ell}^{\infty} \big[2^{ks}|E_k f|\big]^q\,\mathrm{d}\sigma\bigg]^{1/q} < \infty \tag{A.0.73}$$

again with natural alterations when $q = \infty$

$F^{p,q}_s(\Sigma, \sigma)$, the inhomogeneous Triebel-Lizorkin space on the Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, defined as the collection of all "distributions" f on Σ with the property that $\|f\|_{F^{p,q}_s(\Sigma,\sigma)} < +\infty$ (cf. [69, Definition 7.1.2])

$F^{p,q}_s(\mathbb{R}^n)$, the (inhomogeneous) Triebel-Lizorkin space in $\mathbb{R}^n$ for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, equipped with the quasi-norm $\|\cdot\|_{F^{p,q}_s(\mathbb{R}^n)}$ (cf. [69, (9.1.4)-(9.1.5)])

$F^{p,q}_s(\Omega) := \{u \in \mathcal{D}'(\Omega) : \text{there exists } U \in F^{p,q}_s(\mathbb{R}^n) \text{ such that } U\big|_\Omega = u\}$, the Triebel-Lizorkin space in the (arbitrary) open set $\Omega \subseteq \mathbb{R}^n$ for $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, equipped with the quasi-norm (cf. [69, (9.2.1)])

$$\|u\|_{F^{p,q}_s(\Omega)} := \inf\Big\{\|U\|_{F^{p,q}_s(\mathbb{R}^n)} : U \in F^{p,q}_s(\mathbb{R}^n),\ U\big|_\Omega = u\Big\}$$

$f_\Delta := ⨍_\Delta f \, d\sigma$, the integral average of f on the "surface ball" Δ

$\widehat{\phi}$, the Fourier transform of ϕ in $\mathbb{R}^n$:

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}^n} e^{-\mathrm{i}\langle x, \xi\rangle} \phi(x)\, dx, \qquad \xi \in \mathbb{R}^n \tag{A.0.74}$$

$\Phi_k(\cdot)$, the (unique) radiating fundamental solution of the Helmholtz operator $\Delta + k^2$ in $\mathbb{R}^n$, given at each $x \in \mathbb{R}^n \setminus \{0\}$ by the formula (cf. [70, (6.1.7)]):

$$\Phi_k(x) := \frac{1}{4i(2\pi)^{(n-2)/2}} k^{(n-2)/2} \frac{H^{(1)}_{(n-2)/2}(k|x|)}{|x|^{(n-2)/2}} \tag{A.0.75}$$

$|f|_{\mathbb{C}} := \sqrt{\langle f, f\rangle_{\mathbb{C}}} = \Big(2^{\alpha+\beta} \sum_{|I|=\alpha} \sum_{|J|=\beta} |f_{I,J}|^2\Big)^{1/2}$, the "complex" norm of the differential form $f = \sum_{|I|=\alpha, |J|=\beta} f_{I,J}\, dz^I \wedge d\bar{z}^J$ (7.1.46)

$f_{\tan,\mathbb{C}} := \frac{1}{2}\nu^{1,0} \vee (\nu^{0,1} \wedge f)$, the tangential complex part of a form f (7.1.68)

$f_{\mathrm{nor},\mathbb{C}} := \frac{1}{2}\nu^{0,1} \wedge (\nu^{1,0} \vee f)$, the normal complex part of a form f (7.1.68)

$\mathscr{F}^{p,q}_\alpha(\Omega; D)$, the Triebel-Lizorkin-Hardy space of null-solutions of the Dirac operator $D = \sum_{j=1}^{n} \mathbf{e}_j \odot \partial_j$ in an open set $\Omega \subseteq \mathbb{R}^n$ defined for $p \in (0, \infty)$, $q \in (0, \infty]$, and $\alpha \in \mathbb{R}$ as in (4.4.130):

$$\mathscr{F}^{p,q}_\alpha(\Omega; D) := \{u \in F^{p,q}_\alpha(\Omega) \otimes C\ell_n : Du = 0 \text{ in } \Omega\}$$

G

$g = \sum_{1 \leq j,k \leq n} g_{jk}\, dx_j \otimes dx_k$, the Riemannian metric tensor

∇u, the gradient (Jacobian matrix) of a $\mathbb{C}^M$-valued function $u = (u_\alpha)_{1\leq\alpha\leq M}$ defined in an open subset of $\mathbb{R}^n$, defined as:

$$\nabla u := \big(\partial_j u_\alpha\big)_{\substack{1\leq\alpha\leq M\\ 1\leq j\leq n}} = \begin{bmatrix} \partial_1 u_1 & \cdots & \partial_n u_1 \\ \vdots & \vdots & \vdots \\ \partial_1 u_M & \cdots & \partial_n u_M \end{bmatrix} \tag{A.0.76}$$

∇', the gradient operator in $\mathbb{R}^{n-1}$

$\Gamma_\kappa(x) = \Gamma_{\Omega,\kappa}(x)$, the ($\kappa$-)nontangential approach region with vertex at $x \in \partial\Omega$, defined as (cf. [68, (8.1.2)]):

$$\Gamma_\kappa(x) = \Gamma_{\Omega,\kappa}(x) := \big\{y \in \Omega : |x-y| < (1+\kappa)\delta_{\partial\Omega}(y)\big\}, \quad \forall x \in \partial\Omega \tag{A.0.77}$$

$\mathbb{X}$, a Generalized Banach Function Space on the measure space $(X, \mathfrak{M}, \mu)$ equipped with the norm $\|\cdot\|_{\mathbb{X}}$ (cf. [69, Definition 5.1.4])

$\mathring{\mathbb{X}} := \overline{L^\infty_{\rm comp}(X,\mu)}^{\|\cdot\|_{\mathbb{X}}}$, the closure of $L^\infty_{\rm comp}$ in the Generalized Banach Function Space $\big(\mathbb{X}, \|\cdot\|_{\mathbb{X}}\big)$ (cf. [69, Definition 5.2.6])

$\mathbb{X}$, a Generalized Banach Function Space on the measure space $(X, \mathfrak{M}, \mu)$ equipped with the norm $\|\cdot\|_{\mathbb{X}}$ (cf. [69, Definition 5.1.4])

$\mathring{\mathbb{X}} := \overline{L^\infty_{\rm comp}(X,\mu)}^{\|\cdot\|_{\mathbb{X}}}$, the closure of $L^\infty_{\rm comp}$ in the Generalized Banach Function Space $\big(\mathbb{X}, \|\cdot\|_{\mathbb{X}}\big)$ (cf. [69, Definition 5.2.6])

$\nabla_{\tan} f$, the tangential gradient of the function $f \in L^1_{1,\rm loc}(\partial_*\Omega, \sigma_*)$ (cf. [69, §11.4]):

$$\nabla_{\tan} f := \Big(\sum_{k=1}^n \nu_k \partial_{\tau_{kj}} f\Big)_{1\leq j\leq n} \quad \text{at } \sigma_*\text{-a.e. point on } \partial_*\Omega \tag{A.0.78}$$

$\nabla^A_{\tan}$, the tangential gradient of $u = (u_\beta)_\beta$ associated with the coefficient tensor $A := \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$, defined (with $\nabla_{\tan}$ denoting the tangential gradient acting on scalar functions along $\partial\Omega$) as

$$\nabla^A_{\tan} u := \Big(\nu_r a^{\alpha\beta}_{rs}\big(\nabla_{\tan}\big(u_\beta\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big)\big)_s\Big)_{1\leq\alpha\leq M}$$

$G_\Omega(\cdot,\cdot)$, the Green function for the Laplacian, where Ω is a bounded open set in $\mathbb{R}^n$

$\Gamma_{\alpha,\beta}$, the double form of type $\big((\alpha,\beta),(\beta,\alpha)\big)$, defined for $\alpha,\beta \in \{0,1,\dots,n\}$ as in (7.3.19):

$$\Gamma_{\alpha,\beta}(\zeta,z) := 2^{-\alpha-\beta} E_n(\zeta,z) \sum_{|J|=\alpha}\sum_{|I|=\beta} (d\zeta^J \wedge d\bar{\zeta}^I) \otimes (d\bar{z}^J \wedge dz^I)$$

H

$\mathcal{H}^{n-1}$, the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$

$\mathcal{H}^s$, the s-dimensional Hausdorff measure in $\mathbb{R}^n$

$H^p(\Sigma, \sigma)$, the Lebesgue-based Hardy space on the Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, defined for $p \in \big(\frac{n-1}{n}, \infty\big)$ and $\gamma \in \big((n-1)\big(\frac{1}{p}-1\big)_+, 1\big)$ as (cf. [69, Definition 4.2.1])

$$H^p(\Sigma, \sigma) := \Big\{ f \in \big(\mathrm{Lip}_c(\Sigma)\big)' : \ f^\sharp_\gamma \in L^p(\Sigma, \sigma) \Big\} \tag{A.0.79}$$

and equipped with the quasi-norm

$$\|f\|_{H^p(\Sigma,\sigma)} := \big\| f^\sharp_\gamma \big\|_{L^p(\Sigma,\sigma)}, \qquad \forall f \in H^p(\Sigma, \sigma) \tag{A.0.80}$$

$\dot{H}^p(\Sigma, \sigma)$, the homogeneous Hardy space on the Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, defined for each $p \in \big(\frac{n-1}{n}, \infty\big)$ as $\dot{H}^p(\Sigma, \sigma) := \big\{ f \in H^p(\Sigma, \sigma) : \ \langle f, 1 \rangle = 0 \big\}$ if Σ is bounded, and simply as $\dot{H}^p(\Sigma, \sigma) := H^p(\Sigma, \sigma)$ if Σ is unbounded [69, (4.2.12)]

$H^{p,q}(\Sigma, \sigma)$, the Lorentz-based Hardy space on the Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, defined for $p \in \big(\frac{n-1}{n}, \infty\big)$, $p \in (0, \infty]$, and $\gamma \in \big((n-1)\big(\frac{1}{p}-1\big)_+, 1\big)$ as (cf. [69, Definition 4.2.3])

$$H^{p,q}(\Sigma, \sigma) := \Big\{ f \in \big(\mathrm{Lip}_c(\Sigma)\big)' : \ f^\sharp_\gamma \in L^{p,q}(\Sigma, \sigma) \Big\} \tag{A.0.81}$$

and equipped with the quasi-norm

$$\|f\|_{H^{p,q}(\Sigma,\sigma)} := \big\| f^\sharp_\gamma \big\|_{L^{p,q}(\Sigma,\sigma)}, \qquad \forall f \in H^{p,q}(\Sigma, \sigma) \tag{A.0.82}$$

$H^{p,q}_{\mathrm{fin}}(\Sigma, \sigma)$, the vector space of all finite linear combinations of (p, q)-atoms on Σ equipped with the quasi-norm $\|f\|_{H^{p,q}_{\mathrm{fin}}(\Sigma,\sigma)}$ defined as the infimum of all $\Big(\sum\limits_{j=1}^{N} |\lambda_j|^p\Big)^{1/p}$ such that $f = \sum\limits_{j=1}^{N} \lambda_j a_j$ for $\{\lambda_j\}_{1 \leq j \leq N} \subseteq \mathbb{C}$ and (p, q)-atoms $\{a_j\}_{1 \leq j \leq N}$ (cf. [69, (4.4.113)])

$\mathfrak{H}$, the L^p-filtering operator, acting on each given distribution $f \in H^p(\Sigma, \sigma)$ with $\frac{n-1}{n} < p < \infty$ at each $x \in \Sigma$ according to (cf. [69, Theorem 4.9.1]):

$$(\mathfrak{H} f)(x) := \lim_{t \to 0^+} {}_{(H^p(\Sigma,\sigma))^*}\big\langle S_t(x, \cdot), f \big\rangle_{H^p(\Sigma,\sigma)} \tag{A.0.83}$$

where $\{S_t(\cdot, \cdot)\}_t$ are the integral kernels of a suitable approximation to the identity

$\mathscr{H}^{q,\lambda}(\Sigma, \sigma)$, the space defined for any given $q \in (1, \infty)$ and $\lambda \in (0, n-1)$ as (cf. [69, §6.1]):

$$\mathscr{H}^{q,\lambda}(\Sigma,\sigma) := \Big\{f \in \big(\mathrm{Lip}_c(\Sigma)\big)' : \text{there exist a sequence } \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \text{ and a family } \{a_j\}_{j\in\mathbb{N}} \text{ of } \mathscr{H}^{q,\lambda}\text{-atoms on } \Sigma \text{ so that } f = \sum_{j=1}^{\infty}\lambda_j a_j \text{ with convergence in } \big(\mathrm{Lip}_c(\Sigma)\big)'\Big\}, \tag{A.0.84}$$

and equipped with the norm

$$\|f\|_{\mathscr{H}^{q,\lambda}(\Sigma,\sigma)} := \inf\Big\{\sum_{j=1}^{\infty}|\lambda_j| : f = \sum_{j=1}^{\infty}\lambda_j a_j \text{ in } \big(\mathrm{Lip}_c(\Sigma)\big)' \text{ with } \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}) \text{ and each } a_j \text{ a } \mathscr{H}^{q,\lambda}\text{-atom on } \Sigma\Big\} \tag{A.0.85}$$

$H^s(\Omega)$, the L^2-based fractional Sobolev space of order $s \in \mathbb{R}$ in an open set $\Omega \subseteq \mathbb{R}^n$ (cf. [69, §9.2]):

$$H^s(\Omega) := \Big\{U\big|_{\Omega} : U \in H^s(\mathbb{R}^n)\Big\} = \Big\{U\big|_{\Omega} : U \in F^{2,2}_s(\mathbb{R}^n)\Big\}, \qquad \forall s \in \mathbb{R}, \tag{A.0.86}$$

and equipped with the norm

$$\begin{aligned}\|u\|_{H^s(\Omega)} &:= \inf\Big\{\|U\|_{H^s(\mathbb{R}^n)} : U \in H^s(\mathbb{R}^n) \text{ such that } u = U\big|_{\Omega}\Big\}\\ &\approx \inf\Big\{\|U\|_{F^{2,2}_s(\mathbb{R}^n)} : U \in F^{2,2}_s(\mathbb{R}^n) \text{ such that } u = U\big|_{\Omega}\Big\}\end{aligned} \tag{A.0.87}$$

$H^p_\kappa(\Omega)$, the Hardy space of harmonic functions u in the open set $\Omega \subseteq \mathbb{R}^n$ with a p-th power integral nontantangential maximal function:

$$H^p_\kappa(\Omega) := \big\{u \in \mathscr{C}^\infty(\Omega) : \Delta u = 0 \text{ in } \Omega, \text{ and } \mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma)\big\} \tag{A.0.88}$$

$H^{(1)}_\lambda(\cdot)$, the Hankel function of the first kind with index $\lambda \in \mathbb{R}$

$\dot{H}^p_1(\partial\Omega,\sigma)$, the Hardy-based homogeneous Sobolev space of order one on $\partial\Omega$, defined for each $p \in \big(\frac{n-1}{n},\infty\big)$ as (cf. [69, Definition 11.10.5]):

$$\dot{H}^p_1(\partial\Omega,\sigma) := \Big\{f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) : \partial_{\tau_{jk}} f \in H^p(\partial\Omega,\sigma) \text{ for each } j,k \in \{1,\dots,n\}\Big\} \tag{A.0.89}$$

and equipped with the semi-norm

$$\dot{H}^p_1(\partial\Omega,\sigma) \ni f \mapsto \|f\|_{\dot{H}^p_1(\partial\Omega,\sigma)} := \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{H^p(\partial\Omega,\sigma)} \tag{A.0.90}$$

$\dot{H}^p_1(\partial\Omega,\sigma)/\sim$, the quotient space of classes $[\cdot]$ of equivalence modulo constants of functions in $\dot{H}^p_1(\partial\Omega,\sigma)$ defined for $p \in \big(\frac{n-1}{n},\infty\big)$ as (cf. [69, (11.10.33)-(11.10.34)]):

$$\dot{H}^p_1(\partial\Omega,\sigma)/\sim \,:= \big\{[f] : f \in \dot{H}^p_1(\partial\Omega,\sigma)\big\} \tag{A.0.91}$$

equipped with the semi-quasinorm

$$\dot{H}^p_1(\partial\Omega,\sigma)/\sim\, \ni [f] \longmapsto \big\|[f]\big\|_{\dot{H}^p_1(\partial\Omega,\sigma)/\sim} := \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{H^p(\partial\Omega,\sigma)} \tag{A.0.92}$$

$H^{q,p}_1(\partial\Omega,\sigma)$, the Hardy-based inhomogeneous Sobolev space of order one on $\partial\Omega$, defined for $p \in \big(\frac{n-1}{n},\infty\big)$ and $q \in [1,\infty]$ as (cf. [69, Definition 11.10.6]):

$$\begin{aligned} H^{q,p}_1(\partial\Omega,\sigma) &:= L^q(\partial\Omega,\sigma) \cap \dot{H}^p_1(\partial\Omega,\sigma) \\ &= \Big\{f \in L^q(\partial\Omega,\sigma) : \partial_{\tau_{jk}} f \in H^p(\partial\Omega,\sigma) \text{ for } 1 \le j,k \le n\Big\} \end{aligned} \tag{A.0.93}$$

and equipped with the quasi-norm

$$\|f\|_{H^{q,p}_1(\partial\Omega,\sigma)} := \|f\|_{L^q(\partial\Omega,\sigma)} + \|f\|_{\dot{H}^p_1(\partial\Omega,\sigma)}, \qquad \forall f \in H^{q,p}_1(\partial\Omega,\sigma) \tag{A.0.94}$$

$\mathcal{H}^p(\Omega;D)$, the Hardy space in Ω associated with the first-order $N \times M$ system D, defined as

> $\mathcal{H}^p(\Omega;D)$ is the collection of all functions $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ satisfying $\mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma)$ and $Du = 0$ in Ω, and which also vanish at infinity (in the sense described in [70, Definition 1.6.3]) when Ω is an exterior domain (A.0.95)

and equipped with the quasi-norm

$$\|u\|_{\mathcal{H}^p(\Omega;D)} := \big\|\mathcal{N}_\kappa u\big\|_{L^p(\partial\Omega,\sigma)}, \qquad \forall u \in \mathcal{H}^p(\Omega;D) \tag{A.0.96}$$

$\mathscr{H}^p_\bullet(\partial\Omega;D)$, the "bullet" boundary Hardy space associated with the Dirac operator $D := \sum_{j=1}^n \mathbf{e}_j \odot \partial_j$ in the open set $\Omega \subseteq \mathbb{R}^n$, defined as in (A.0.97):

$$\mathscr{H}^p_\bullet(\partial\Omega;D) := \big\{\nu \circledbullet u : u \in \mathcal{H}^p(\Omega;D)\big\} \tag{A.0.97}$$

and equipped with the quasi-norm inherited from $H^p(\partial\Omega,\sigma) \otimes C\ell_n$

$*$, the Hodge star operator in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ whose action satisfies $u \wedge (*\bar{u}) = |u|_{\mathbb{C}}^2 \mathrm{d}V$ for $u \in \Lambda^{\alpha,\beta}\,\mathbb{C}^n$ and $\alpha, \beta \in \{0, 1, \dots, n\}$ (7.1.48)

$\dot{\mathcal{H}}^p_{1,\pm}(\partial\Omega, \sigma)$, the "positive/negative" homogeneous Hardy spaces with regularity on $\partial\Omega$, defined as in (1.8.285):

$$\dot{\mathcal{H}}^p_{1,\pm}(\partial\Omega, \sigma) := \Big\{ u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} : u \text{ holomorphic in } \Omega_\pm \text{ and } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma) \Big\}$$

where $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{C} \setminus \overline{\Omega}$

H_{mod}, the modified Hilbert transform on the real line, whose action on each function $f \in L^1\big(\mathbb{R}, \frac{\mathrm{d}x}{1+|x|^2}\big)$ at $\mathcal{L}^1$-a.e. point $x \in \mathbb{R}$ is defined as in (5.3.99):

$$\big(H_{\mathrm{mod}} f\big)(x) := \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} \Big\{ \frac{1}{x-y} \mathbf{1}_{\mathbb{R}\setminus[x-\varepsilon, x+\varepsilon]}(y) + \frac{1}{y} \mathbf{1}_{\mathbb{R}\setminus[-1,1]}(y) \Big\} f(y)\, \mathrm{d}y$$

$\mathcal{H}^p(\Omega)$, the (several variable) holomorphic Hardy space in the UR domain Ω in $\mathbb{R}^{2n} \equiv \mathbb{C}^n$ defined for each $p \in \big(\frac{2n-1}{2n}, \infty\big)$ as in (7.6.17):

$$\mathcal{H}^p(\Omega) := \Big\{ F : \Omega \to \mathbb{C} : F \text{ holomorphic, } \mathcal{N}_\kappa F \in L^p(\partial\Omega, \sigma), \text{ and } \lim_{|z|\to\infty} F(z) = 0 \text{ if } \Omega \text{ is an exterior domain} \Big\},$$

and equipped with the quasi-norm $\mathcal{H}^p(\Omega) \ni F \mapsto \|F\|_{\mathcal{H}^p(\Omega)} := \|\mathcal{N}_\kappa F\|_{L^p(\partial\Omega,\sigma)}$

$\langle \cdot, \cdot \rangle_{\mathbb{C}}$, the Hermitian inner product on the space of differential forms uniquely defined by the requirement that one has $\langle \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J, \mathrm{d}z^A \wedge \mathrm{d}\bar{z}^B \rangle_{\mathbb{C}} = 2^{|I|+|J|} \varepsilon^I_A \varepsilon^J_B$ for all arrays I, J, A, B (7.1.42)

$\langle\langle \cdot, \cdot \rangle\rangle_{\mathbb{C}}$, the pointwise Hermitian inner product for double forms

I

$\langle \cdot, \cdot \rangle$, the (real) inner product in $\mathbb{C}^M$ defined for any vectors $u = (u_k)_{1 \le k \le M} \in \mathbb{C}^M$ and $w = (w_k)_{1 \le k \le M} \in \mathbb{C}^M$ as:

$$\langle u, w \rangle := \sum_{k=1}^{M} u_k w_k \tag{A.0.98}$$

$\mathrm{i} := \sqrt{-1} \in \mathbb{C}$, the complex imaginary unit

ι^*, the pull-back map induced by the canonical inclusion ι

$\mathring{U}$, the interior of the set $U \subseteq \mathbb{R}^n$

$⨍_E f\, \mathrm{d}\mu := \frac{1}{\mu(E)} \int_E f\, \mathrm{d}\mu$, the integral average of the function f on the set $E \subseteq X$, in a measure space (X, μ)

$\fint_{B_\rho(x,r)}$, the integral average of f over the ρ-ball $B_\rho(x,r)$, in the context of a space of homogeneous type (X,ρ,μ), defined as (cf. [68, (7.4.9)]):

$$\fint_{B_\rho(x,r)} := \fint_{B_\rho(x,r)} f\,\mathrm{d}\mu := \frac{1}{\mu\big(B_\rho(x,r)\big)}\int_{B_\rho(x,r)} f(y)\,\mathrm{d}\mu(y) \tag{A.0.99}$$

$I_{E,\alpha}$, the fractional integral operator of order α on the set E contained in a metric space (X,ρ) equipped with an upper d-dimensional Borel measure μ on (X,τ_ρ), acting on functions $f \in L^1\big(E, \frac{\mu(x)}{1+\rho(x,x_0)^{d-\alpha}}\big)$ according to (cf. [68, (7.8.3)]):

$$I_{E,\alpha}f(x) := \int_E \frac{f(y)}{\rho(x,y)^{d-\alpha}}\,\mathrm{d}\mu(y) \quad \text{for } \mu\text{-a.e. } \ x \in E \tag{A.0.100}$$

$\mathrm{Im}\,(T : X \to Y) := \{Tx : \ x \in X\}$, the image (or range) of the operator $T : X \to Y$

$(X_0,X_1)_{\theta,q} := \big\{x \in X_0 + X_1 \ : \ \|x\|_{(X_0,X_1)_{\theta,q}} < +\infty\big\}$, the intermediate space for the real method of interpolation between the compatible pair of quasi-Banach spaces X_0 and X_1, equipped with the real interpolation quasi-norm $\|\cdot\|_{(X_0,X_1)_{\theta,q}}$ (cf. [69, §1.3])

$i(\Phi) := \sup\limits_{0<t<1} \frac{\ln h_\Phi(t)}{\ln t} = \lim\limits_{t\to 0^+} \frac{\ln h_\Phi(t)}{\ln t}$ where $h_\Phi(t) := \sup\limits_{s>0} \frac{\Phi(st)}{\Phi(s)}$ for $t \in (0,\infty)$, the lower dilation index of the Young function Φ (cf. [69, (5.3.14)])

$I(\Phi) := \inf\limits_{1<t<\infty} \frac{\ln h_\Phi(t)}{\ln t} = \lim\limits_{t\to\infty} \frac{\ln h_\Phi(t)}{\ln t}$ where $h_\Phi(t) := \sup\limits_{s>0} \frac{\Phi(st)}{\Phi(s)}$ for $t \in (0,\infty)$, the upper dilation index of the Young function Φ (cf. [69, (5.3.15)])

K

K_Δ, the boundary-to-boundary harmonic double layer potential, defined as (cf. [68, (1.1.32)]):

$$K_\Delta f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle \nu(y), y-x\rangle}{|x-y|^n} f(y)\,\mathrm{d}\mathcal{H}^{n-1}(y), \qquad x \in \partial\Omega \tag{A.0.101}$$

$K_\Delta^{\#}$, the transpose harmonic double layer potential, defined as (cf. [68, (1.1.33)]):

$$K_\Delta^{\#} f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle \nu(x), x-y\rangle}{|x-y|^n} f(y)\,\mathrm{d}\mathcal{H}^{n-1}(y), \qquad x \in \partial\Omega \tag{A.0.102}$$

$\mathrm{Ker}\,(T : X \to Y) := \{x \in X : \ Tx = 0\}$, the kernel (or null-space) of the operator T

$\mathrm{Ker}\,L := \Big\{u \in \big[\mathscr{C}^\infty(\Omega)\big]^M \ : \ Lu = 0 \ \text{ in } \ \Omega\Big\}$, the null-space of the $M \times M$ system L, in an open set Ω

$K = K_A$, the boundary-to-boundary double layer a given second-order $M \times M$ elliptic system written as $L_A := \big(a_{rs}^{\alpha\beta}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ corresponding to a choice of the coefficient tensor $A := \big(a_{rs}^{\alpha\beta}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$ (with canonical fundamental solution

$E = \big(E_{\gamma\beta}\big)_{1\leq\gamma,\beta\leq M}$) acting on any function $f = (f_\alpha)_{1\leq\alpha\leq M} \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ at σ-a.e. point $x \in \partial_*\Omega$ as in (1.3.68):

$$Kf(x) := \Big(-\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial_*\Omega\\|x-y|>\varepsilon}} \nu_s(y)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)f_\alpha(y)\,d\sigma(y)\Big)_{1\leq\gamma\leq M}$$

$K_{\rm mod}$, the modified boundary-to-boundary double layer potential operator for some second-order $M\times M$ weakly elliptic system written as $L_A := \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ corresponding to a choice of the coefficient tensor $A := \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq\alpha,\beta\leq M\\1\leq r,s\leq n}}$ (with fundamental solution $E = \big(E_{\gamma\beta}\big)_{1\leq\gamma,\beta\leq M}$) acting on any $f = (f_\alpha)_{1\leq\alpha\leq M}$ belonging to $\Big[L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^n}\big)\Big]^M$ at σ-a.e. point $x \in \partial\Omega$ as in (1.8.24):

$$K_{\rm mod}f(x) := \Big(-\lim_{\varepsilon\to 0^+}\int_{\partial_*\Omega} \nu_s(y)a^{\beta\alpha}_{rs}\{k^{(r\gamma\beta)}_\varepsilon(x-y) - k^{(r\gamma\beta)}_1(-y)\}f_\alpha(y)\,d\sigma(y)\Big)_{1\leq\gamma\leq M}$$

where $k^{(r\gamma\beta)}_\varepsilon := (\partial_r E_{\gamma\beta})\cdot\mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,\varepsilon)}}$ for each $\varepsilon > 0$

$K^\# = K^\#_A$, the transpose double layer for a given second-order $M\times M$ weakly elliptic system written as $L_A := \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ corresponding to a choice of the coefficient tensor $A := \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq\alpha,\beta\leq M\\1\leq r,s\leq n}}$ (with canonical fundamental solution $E = \big(E_{\gamma\beta}\big)_{1\leq\gamma,\beta\leq M}$) acting on any function $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ at σ-a.e. point $x \in \partial_*\Omega$ as in (1.3.72):

$$K^\# f(x) := \Big(\lim_{\varepsilon\to 0^+}\int\limits_{\substack{y\in\partial\Omega\\|x-y|>\varepsilon}} \nu_s(x)a^{\beta\alpha}_{rs}(\partial_r E_{\gamma\beta})(x-y)f_\gamma(y)\,d\sigma(y)\Big)_{1\leq\alpha\leq M}$$

Ker D, the null-space of a first-order $N\times M$ constant complex coefficient system $D = \Big(\sum_{j=1}^n a^{\alpha\beta}_j\partial_j\Big)_{\substack{1\leq\alpha\leq N\\1\leq\beta\leq M}}$ in $\mathbb{R}^n$ which is injectively elliptic (i.e., its symbol, given by the formula $\mathrm{Sym}(D;\xi) := \mathrm{i}\big(\sum_{j=1}^n a^{\alpha\beta}_j\xi_j\big)_{\substack{1\leq\alpha\leq N\\1\leq\beta\leq M}}$ is an injective mapping from $\mathbb{C}^M$ into $\mathbb{C}^N$, for each vector $\xi \in \mathbb{R}^n\setminus\{0\}$), defined as in (4.4.62):

$$\mathrm{Ker}\,D := \Big\{u \in \big[\mathscr{C}^\infty(\Omega)\big]^M : Du = 0 \text{ in } \Omega\Big\}$$

K_λ, the boundary-to-boundary double layer for the Stokes system associated with the coefficient tensor $A_\lambda := \big(\delta_{jk}\delta_{\alpha\beta} + \lambda\delta_{j\beta}\delta_{k\alpha}\big)_{\substack{1\leq\alpha,\beta\leq n\\1\leq j,k\leq n}}$ corresponding to any $\lambda \in \mathbb{C}$, acting on each $\vec{f} = (f_j)_{1\leq j\leq n} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n$ at σ-a.e. point $x \in \partial_*\Omega$ as in (6.2.95), (6.2.96):

$$K_\lambda \vec{f}(x) = \left(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \left\{ (\lambda-1)\frac{\delta_{j\gamma}}{2\omega_{n-1}} \frac{\langle x-y, \nu(y)\rangle}{|x-y|^n} \right.\right.$$
$$- \frac{n(\lambda+1)}{2\omega_{n-1}} \frac{\langle x-y,\nu(y)\rangle(x_j-y_j)(x_\gamma-y_\gamma)}{|x-y|^{n+2}}$$
$$\left.\left. + \frac{\lambda-1}{2\omega_{n-1}} \frac{\nu_j(y)(x_\gamma-y_\gamma)-\nu_\gamma(y)(x_j-y_j)}{|x-y|^n} \right\} f_j(y)\, \mathrm{d}\sigma(y) \right)_{1\leq\gamma\leq n}$$

$K_\lambda^\#$, the transpose double layer for the Stokes system associated with the coefficient tensor $A_\lambda := \big(\delta_{jk}\delta_{\alpha\beta} + \lambda\delta_{j\beta}\delta_{k\alpha}\big)_{\substack{1\leq\alpha,\beta\leq n \\ 1\leq j,k\leq n}}$ corresponding to any $\lambda \in \mathbb{C}$, acting on each $\vec{f} = (f_j)_{1\leq j\leq n} \in \Big[L^1\Big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)\Big]^n$ at σ-a.e. point $x \in \partial_*\Omega$ as in (6.2.98), (6.2.99):

$$K_\lambda^\# \vec{f}(x) = \left(\lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \left\{ -(\lambda-1)\frac{\delta_{j\gamma}}{2\omega_{n-1}} \frac{\langle x-y, \nu(x)\rangle}{|x-y|^n} \right.\right.$$
$$+ \frac{n(\lambda+1)}{2\omega_{n-1}} \frac{\langle x-y,\nu(x)\rangle(x_j-y_j)(x_\gamma-y_\gamma)}{|x-y|^{n+2}}$$
$$\left.\left. - \frac{\lambda-1}{2\omega_{n-1}} \frac{\nu_j(x)(x_\gamma-y_\gamma)-\nu_\gamma(x)(x_j-y_j)}{|x-y|^n} \right\} f_\gamma(y)\, \mathrm{d}\sigma(y) \right)_{1\leq j\leq n}$$

$K_{n\beta}(\cdot,\cdot)$, the Bochner-Martinelli kernel for $(0,\beta)$-forms in $\mathbb{C}^n$ with $\beta \in \{0, 1, \dots, n\}$, defined as in (7.3.27):

$$K_{n\beta}(\zeta, z) := - * \partial_\zeta \overline{\Gamma_{0,\beta}(\zeta, z)}$$

with the Hodge star isomorphism applied in ζ

k^{x_0}, the Poisson kernel for the Laplacian in the domain Ω, defined as

$$k^{x_0} := \frac{\mathrm{d}\omega^{x_0}}{\mathrm{d}\sigma}, \tag{A.0.103}$$

i.e., the Radon-Nikodym derivative of the harmonic measure ω^{x_0} with pole at $x_0 \in \Omega$ with respect to the surface measure $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ on the topological boundary $\partial\Omega$

L

local John condition, satisfied by an open set $\Omega \subseteq \mathbb{R}^n$ (cf. [68, Definition 5.11.7]):

> there exist $\theta \in (0,1)$, $M_o \in (1,\infty)$, and $R \in \big(0, \operatorname{diam}\partial\Omega\big]$ (the latter required to be ∞ if $\partial\Omega$ is unbounded) such that for every $x \in \partial\Omega$ and $r \in (0, R_*)$ one can find a point $x_r \in B(x,r)\cap\Omega$ with the property that $B(x_r, \theta r) \subseteq \Omega$ and for each $y \in \Delta(x,r)$ it is possible to find a rectifiable path $\gamma_y : [0,1] \to \overline{\Omega}$ whose length is $\leq M_o \cdot r$, which satisfies $\gamma_y(0) = y$, $\gamma_y(1) = x_r$, and such that $\operatorname{dist}\big(\gamma_y(t), \partial\Omega\big) > \theta \cdot |\gamma_y(t) - y|$ for every $t \in (0,1]$ (A.0.104)

$\mathcal{L}^n$, the (n-dimensional) Lebesgue measure in $\mathbb{R}^n$

$L^\infty_{\rm comp}$, the space of essentially bounded functions with compact support

$L^0(X,\mu)$, the space of measurable functions which are pointwise finite μ-a.e. on X

$L^p_{\rm bdd}(\Omega, \mathcal{L}^n)$, the space of functions which are p-th power integrable on bounded subsets of Ω (cf. [68, (4.2.4)]):

> $L^p_{\rm bdd}(\Omega, \mathcal{L}^n)$ be the collection of all $\mathcal{L}^n$-measurable functions defined in Ω which are p-th power absolutely integrable with respect to the Lebesgue measure on each bounded $\mathcal{L}^n$-measurable subset of the set Ω. (A.0.105)

$\|\cdot\|_{{\rm Lip}(X)}$, the natural semi-norm on ${\rm Lip}(X)$, defined in the context of a metric space $(\mathscr{X}, d)$ as (cf. [68, (3.7.1)]):

$$\|f\|_{{\rm Lip}(X)} := \sup_{x,y\in X,\, x\neq y} \frac{|f(x)-f(y)|}{d(x,y)} \tag{A.0.106}$$

${\rm Lip}(X)$, the space of Lipschitz functions on the (quasi-)metric space X, defined as (cf. [68, (3.7.2)]):

$${\rm Lip}(X) := \big\{ f : X \to \mathbb{C} : \ \|f\|_{{\rm Lip}(X)} < +\infty \big\} \tag{A.0.107}$$

${\rm Lip}_c(X)$, the space of Lipschitz functions with bounded support in the (quasi-)metric space X

$\big({\rm Lip}_c(\Sigma)\big)'$, the space distributions on a given set $\Sigma \subseteq \mathbb{R}^n$, defined as (cf. [68, (4.1.34)]):

$$\text{the topological dual of } \big({\rm Lip}_c(\Sigma), \tau_{\mathscr{D}}\big) \tag{A.0.108}$$

${}_{({\rm Lip}_c(\Sigma))'}\langle \cdot,\cdot \rangle_{{\rm Lip}_c(\Sigma)}$, or simply $\langle \cdot,\cdot\rangle$, the distributional pairing on the set Σ

$\|\cdot\|_{L^{p,q}(X,\mu)}$, the Lorentz space quasi-norm, defined as (cf. [68, (6.2.14)]):

$$\|f\|_{L^{p,q}(X,\mu)} := \begin{cases} \left(\displaystyle\int_0^\infty \big[t^{1/p} f^*_X(t)\big]^q \, \frac{dt}{t} \right)^{1/q} & \text{if } \ 0 < p, q < \infty, \\ \sup_{t>0} \big[t^{1/p} f^*_X(t)\big] & \text{if } \ 0 < p \leq \infty, \ q = \infty, \\ \|f\|_{L^\infty(X,\mu)} & \text{if } \ p = \infty, \ 0 < q \leq \infty \end{cases} \tag{A.0.109}$$

$L^{p,q}(X,\mu)$, the Lorentz space on X with respect to the measure μ defined as (cf. [68, (6.2.13)]):

$$L^{p,q}(X,\mu) := \Big\{ f : X \to \overline{\mathbb{R}} \ \mu\text{-measurable} : \|f\|_{L^{p,q}(X,\mu)} < +\infty \Big\} \tag{A.0.110}$$

$L^{p,q}_{\star}(\Omega,\mu)$, the maximal Lorentz space with respect to the Borel measure μ in the open set $\Omega \subseteq \mathbb{R}^n$, defined as (cf. [68, (6.6.41)]):

$$L^{p,q}_{\star}(\Omega,\mu) := \big\{ u : \Omega \to \mathbb{C} : u \text{ is } \mathcal{L}^n\text{-measurable and } u_{\star,\theta} \in L^{p,q}(\Omega,\mu) \big\} \tag{A.0.111}$$

$L^{p}_{\star}(\Omega,\mu)$, the maximal Lebesgue space with respect to the Borel measure μ in the open set $\Omega \subseteq \mathbb{R}^n$, defined as (cf. [68, (6.6.43)]):

$$\begin{aligned} L^{p}_{\star}(\Omega,\mu) &:= L^{p,p}_{\star}(\Omega,\mu) \\ &= \big\{ u : \Omega \to \mathbb{C} : u \text{ is } \mathcal{L}^n\text{-measurable and } u_{\star,\theta} \in L^{p}(\Omega,\mu) \big\} \end{aligned} \tag{A.0.112}$$

$\log_+$, the positive part of ln, defined for each $t \in [0,\infty)$ as (cf. [68, (7.6.68)]):

$$\log_+ t := \begin{cases} 0 & \text{if } \ t \in [0,1], \\ \ln t & \text{if } \ t \in [1,\infty) \end{cases} \tag{A.0.113}$$

$\mathcal{L}(X \to Y)$, the space of linear and continuous operators from X to Y

$L^{\Phi}(X,\mu)$, the Orlicz space on the sigma-finite measure space $(X,\mathfrak{M},\mu)$ associated with a Young function Φ (cf. [69, §5.3]):

$$L^{\Phi}(X,\mu) := \big\{ f \in \mathscr{M}(X,\mu) : \ \|f\|_{L^{\Phi}(X,\mu)} < \infty \big\} \tag{A.0.114}$$

where the Luxemburg norm $\|\cdot\|_{L^{\Phi}(X,\mu)}$ is defined as

$$\|f\|_{L^{\Phi}(X,\mu)} := \inf \Big\{ \lambda > 0 : \int_X \Phi(|f(x)|/\lambda)\, d\mu(x) \leq 1 \Big\} \tag{A.0.115}$$

$$\inf \Big\{ \lambda > 0 : \int_0^{\infty} \mu\Big(\big\{ x \in X : \ |f(x)| > \lambda \Phi^{-1}(t) \big\} \Big)\, dt \leq 1 \Big\} \tag{A.0.116}$$

$L^p(\log L)^{\alpha}(X,\mu) = \Big\{ f \in \mathscr{M}(X,\mu) : \ \int_X |f(x)|^p \big[\ln(e + |f(x)|) \big]^{\alpha}\, d\mu(x) < +\infty \Big\}$ for $p \in (1,\infty)$ and $\alpha \in \mathbb{R}$, Zygmund's space (cf. [69, §5.3])

$L^p(\Omega,\, w\mathcal{L}^n)$, the weighted L^p Lebesgue space on the set $\Omega \subseteq \mathbb{R}^n$, equipped with the natural norm $\|u\|_{L^p(\Omega,\, w\mathcal{L}^n)} := \Big(\int_\Omega |u|^p\, w\, d\mathcal{L}^n \Big)^{1/p}$ (cf. [69, §8.3])

$L^p_s(\Omega) := \Big\{ U\big|_{\Omega} : \ U \in L^p_s(\mathbb{R}^n) \Big\}$, the Bessel potential space in an open set $\Omega \subseteq \mathbb{R}^n$ for $p \in (1,\infty)$ and $s \in \mathbb{R}$, equipped with the norm (cf. [69, §9.2]):

$$\|u\|_{L^p_s(\Omega)} := \inf\Big\{\|U\|_{L^p_s(\mathbb{R}^n)} : U \in L^p_s(\mathbb{R}^n),\ u = U\big|_\Omega\Big\}$$

$\dot{\mathfrak{L}}^{p,\lambda}(\Sigma,\sigma)$, the homogeneous Morrey-Campanato space on a given Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, defined for each integrability exponent $p \in (1,\infty)$ and each parameter $\lambda \in (0, n-1)$ as (cf. [69, §6.1]):

$$\dot{\mathfrak{L}}^{p,\lambda}(\Sigma,\sigma) := \Big\{f \in L^1_{\rm loc}(\Sigma,\sigma) : \|f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\Sigma,\sigma)} < +\infty\Big\} \tag{A.0.117}$$

where, for each $f \in L^1_{\rm loc}(\Sigma,\sigma)$,

$$\|f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\Sigma,\sigma)} := \sup_{\substack{x\in\Sigma \text{ and} \\ 0<R<2\,{\rm diam}(\Sigma)}} \left\{R^{\frac{n-1-\lambda}{p}}\left(⨍_{\Sigma\cap B(x,R)} \big|f(y) - f_{\Delta(x,R)}\big|^p\, d\sigma(y)\right)^{\frac{1}{p}}\right\} \tag{A.0.118}$$

$\mathfrak{L}^{p,\lambda}(\Sigma,\sigma)$, the inhomogeneous Morrey-Campanato space equippe on a given Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, defined for each integrability exponent $p \in (1,\infty)$ and each parameter $\lambda \in (0, n-1)$ as (cf. [69, §6.1]):

$$\mathfrak{L}^{p,\lambda}(\Sigma,\sigma) := L^p(\Sigma,\sigma) \cap \dot{\mathfrak{L}}^{p,\lambda}(\Sigma,\sigma) = \Big\{f \in L^p(\Sigma,\sigma) : \|f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\Sigma,\sigma)} < +\infty\Big\} \tag{A.0.119}$$

where

$$\|f\|_{\mathfrak{L}^{p,\lambda}(\Sigma,\sigma)} := \|f\|_{L^p(\Sigma,\sigma)} + \|f\|_{\dot{\mathfrak{L}}^{p,\lambda}(\Sigma,\sigma)} \tag{A.0.120}$$

$L^p_1(\partial_*\Omega,\sigma_*)$, the L^p-based Sobolev space of order one on $\partial_*\Omega$, defined for each $p \in [1,\infty]$ as (cf. [69, Definition 11.1.2]):

$$L^p_1(\partial_*\Omega,\sigma_*) := \Big\{f \in L^p(\partial_*\Omega,\sigma_*) : \partial_{\tau_{jk}} f \text{ exists in } L^p(\partial_*\Omega,\sigma_*) \text{ for each } j,k \in \{1,\dots,n\}\Big\} \tag{A.0.121}$$

and equipped with the norm (cf. [69, Proposition 11.1.9]):

$$L^p_1(\partial_*\Omega,\sigma_*) \ni f \mapsto \|f\|_{L^p_1(\partial_*\Omega,\sigma_*)} := \|f\|_{L^p(\partial_*\Omega,\sigma_*)} + \sum_{1\le j,k\le n} \|\partial_{\tau_{jk}} f\|_{L^p(\partial_*\Omega,\sigma_*)} \tag{A.0.122}$$

$L^{p,q}_1(\partial_*\Omega,\sigma_*)$, the off-diagonal Sobolev space of order one on $\partial_*\Omega$, defined for each $p,q \in [1,\infty]$ as (cf. [69, Definition 11.1.2]):

$$L^{p,q}_1(\partial_*\Omega,\sigma_*) := \Big\{f \in L^p(\partial_*\Omega,\sigma_*) : \partial_{\tau_{jk}} f \text{ exists in } L^q(\partial_*\Omega,\sigma_*) \text{ for each } j,k \in \{1,\dots,n\}\Big\} \tag{A.0.123}$$

and equipped with the norm define for each $f \in L_1^{p,q}(\partial_*\Omega, \sigma_*)$ as (cf. [69, Proposition 11.1.9]):

$$\|f\|_{L_1^{p,q}(\partial_*\Omega,\sigma_*)} := \|f\|_{L^p(\partial_*\Omega,\sigma_*)} + \sum_{1\leq j,k\leq n} \|\partial_{\tau_{jk}} f\|_{L^q(\partial_*\Omega,\sigma_*)} \tag{A.0.124}$$

$L^{p,q}_{1,\mathrm{loc}}(\partial_*\Omega, \sigma_*)$, the local off-diagonal (boundary) Sobolev space defined for each pair of exponents $p, q \in [1, \infty]$ as (cf. [69, Definition 11.1.2]):

$$L^{p,q}_{1,\mathrm{loc}}(\partial_*\Omega, \sigma_*) := \Big\{f \in L^p_{\mathrm{loc}}(\partial_*\Omega, \sigma_*) : \partial_{\tau_{jk}} f \text{ exists in } L^q_{\mathrm{loc}}(\partial_*\Omega, \sigma_*) \text{ for each } j, k \in \{1, \dots, n\}\Big\} \tag{A.0.125}$$

$L^{p}_{1,\mathrm{loc}}(\partial_*\Omega, \sigma_*)$, the (boundary) Sobolev space defined for the exponent $p \in [1, \infty]$ as (cf. [69, Definition 11.1.2]):

$$L^{p}_{1,\mathrm{loc}}(\partial_*\Omega, \sigma_*) := L^{p,p}_{1,\mathrm{loc}}(\partial_*\Omega, \sigma_*) \tag{A.0.126}$$

$\dot{L}^p_1(\partial\Omega, \sigma)$, the L^p-based homogeneous Sobolev space of order one on $\partial\Omega$ as (cf. [69, Definition 11.5.3]):

$$\dot{L}^p_1(\partial\Omega, \sigma) := \Big\{f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\Big) : \partial_{\tau_{jk}} f \in L^p(\partial_*\Omega, \sigma) \text{ for each } j, k \in \{1, \dots, n\}\Big\} \tag{A.0.127}$$

and equipped with the semi-norm

$$\dot{L}^p_1(\partial\Omega, \sigma) \ni f \mapsto \|f\|_{\dot{L}^p_1(\partial\Omega,\sigma)} := \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{L^p(\partial_*\Omega,\sigma)} \tag{A.0.128}$$

$L^p_1(\partial_*\Omega, w\sigma_*)$, the weighted Sobolev space of order one on $\partial_*\Omega$, defined for an exponent $p \in [1, \infty]$ and a generic weight w on $\partial_*\Omega$ as (cf. [69, §11.7]):

$$L^p_1(\partial_*\Omega, w\sigma_*) := \big\{f \in L^p(\partial_*\Omega, w\sigma_*) : \partial_{\tau_{jk}} f \in L^p(\partial_*\Omega, w\sigma_*) \text{ for each } j, k \in \{1, \dots, n\}\big\} \tag{A.0.129}$$

equipped with the norm

$$L^p_1(\partial_*\Omega, w\sigma_*) \ni f \longmapsto \|f\|_{L^p(\partial_*\Omega,w\sigma_*)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{L^p(\partial_*\Omega,w\sigma_*)} \tag{A.0.130}$$

$L^p_1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^a}\big)$, the "power" weighted Sobolev space on $\partial_*\Omega$, defined for each $p \in [1,\infty]$ and $a \in [0,\infty)$ as (cf. [69, (11.7.6)]):

$$L^p_1\big(\partial_*\Omega, \tfrac{\sigma_*(x)}{1+|x|^a}\big) = \Big\{f \in L^p\big(\partial_*\Omega, \tfrac{\sigma_*(x)}{1+|x|^a}\big) : \partial_{\tau_{jk}} f \in L^p\big(\partial_*\Omega, \tfrac{\sigma_*(x)}{1+|x|^a}\big) \quad \text{for each } j,k \in \{1,\dots,n\}\Big\} \tag{A.0.131}$$

$L^p_1(\partial\Omega, w)$, the Muckenhoupt weighted Sobolev space of order one on $\partial\Omega$, defined for $p \in (1,\infty)$ and $w \in A_p(\partial\Omega,\sigma)$ as (cf. [69, §11.7]):

$$L^p_1(\partial\Omega, w) := \big\{f \in L^p(\partial\Omega, w) : \partial_{\tau_{jk}} f \in L^p(\partial\Omega, w),\ 1 \le j,k \le n\big\} \tag{A.0.132}$$

and equipped with the norm

$$L^p_1(\partial\Omega, w) \ni f \mapsto \|f\|_{L^p_1(\partial\Omega,w)} := \|f\|_{L^p(\partial\Omega,w)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{L^p(\partial\Omega,w)} \tag{A.0.133}$$

$L^{p,q}_1(\partial\Omega,\sigma)$, the Lorentz-based Sobolev space of order one on $\partial\Omega$, defined for $p \in (1,\infty)$ and $q \in (0,\infty]$ as (cf. [69, §11.7]):

$$L^{p,q}_1(\partial\Omega,\sigma) := \big\{f \in L^{p,q}(\partial\Omega,\sigma) : \partial_{\tau_{jk}} f \in L^{p,q}(\partial\Omega,\sigma),\ 1 \le j,k \le n\big\} \tag{A.0.134}$$

and equipped with the quasi-norm

$$L^{p,q}_1(\partial\Omega,\sigma) \ni f \mapsto \|f\|_{L^{p,q}_1(\partial\Omega,\sigma)} := \|f\|_{L^{p,q}(\partial\Omega,\sigma)} + \sum_{j,k=1}^{n} \|\partial_{\tau_{jk}} f\|_{L^{p,q}(\partial\Omega,\sigma)} \tag{A.0.135}$$

$L^p_{-1}(\partial_*\Omega,\sigma_*)$, the ($L^p$-based) negative Sobolev space of order minus one on $\partial_*\Omega$ defined for $p, p' \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ as (cf. [69, Definition 11.8.1]):

$$L^p_{-1}(\partial_*\Omega,\sigma_*) := \Big(L^{p'}_1(\partial_*\Omega,\sigma_*)\Big)^* \tag{A.0.136}$$

$L^{p,q}_{-1}(\partial_*\Omega,\sigma_*)$, the off-diagonal negative Sobolev space on $\partial_*\Omega$ defined for each two exponents $p, q \in (1,\infty)$ as (cf. [69, (11.8.28)]):

$$L^{p,q}_{-1}(\partial_*\Omega,\sigma_*) := \Big(L^{p',q'}_1(\partial_*\Omega,\sigma_*)\Big)^* \text{ where } \tfrac{1}{p} + \tfrac{1}{p'} = 1 \text{ and } \tfrac{1}{q} + \tfrac{1}{q'} = 1 \tag{A.0.137}$$

$L^p_{-1}(\partial\Omega, w)$, the Muckenhoupt weighted negative Sobolev space on $\partial\Omega$ defined for $p \in (1,\infty)$ and $w \in A_p(\partial\Omega,\sigma)$ as (cf. [69, Definition 11.8.7]):

$$L^p_{-1}(\partial\Omega, w) := \Big(L^{p'}_1(\partial\Omega, w')\Big)^* \tag{A.0.138}$$

where $p' := (1 - 1/p)^{-1} \in (1, \infty)$ is the Hölder conjugate exponent of the given p and where $w' := w^{1-p'} \in A_{p'}(\partial\Omega, \sigma)$ is the conjugate weight of w

$L = \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$, a homogeneous, second-order, constant (complex) coefficient, $M \times M$ system in $\mathbb{R}^n$

L_A, the second-order homogeneous constant coefficient system in $\mathbb{R}^n$ associated with a given coefficient tensor $A := \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha\leq M\\ 1\leq\beta\leq N}}$ as

$$L_A := \Big(a^{\alpha\beta}_{rs}\partial_r\partial_s\Big)_{\substack{1\leq\alpha\leq M\\ 1\leq\beta\leq N}} \tag{A.0.139}$$

$L(\xi)$, the characteristic matrix of the homogeneous (constant-coefficient) higher-order system $L = \sum_{|\alpha|=|\beta|=m} \partial^\alpha A_{\alpha\beta}\partial^\beta$, defined as (cf. [68, (6.5.39)]):

$$L(\xi) := (-1)^m \sum_{|\alpha|=|\beta|=m} \xi^{\alpha+\beta} A_{\alpha\beta}, \qquad \forall \xi \in \mathbb{R}^n \tag{A.0.140}$$

$L(\xi)$, the characteristic matrix of the homogeneous complex constant coefficient second-order $M \times M$ system $L = \big(a^{\alpha\beta}_{rs}\partial_r\partial_s\big)_{1\leq\alpha,\beta\leq M}$ in $\mathbb{R}^n$, defined for each vector $\xi = (\xi_r)_{1\leq r\leq n} \in \mathbb{R}^n$ as

$$L(\xi) := -\big(a^{\alpha\beta}_{rs}\xi_r\xi_s\big)_{1\leq\alpha,\beta\leq M} \tag{A.0.141}$$

$L_{\lambda,\mu}$, the complex Lamé system with Lamé moduli $\lambda, \mu \in \mathbb{C}$ defined as:

$$L_{\lambda,\mu} := \mu\Delta + (\lambda + \mu)\nabla\mathrm{div} \tag{A.0.142}$$

$L_{\lambda,\mu}(\xi) := -\mu|\xi|^2 I_{n\times n} - (\lambda + \mu)\xi \otimes \xi$ for each $\xi \in \mathbb{R}^n$, the characteristic matrix of the complex Lamé system $L_{\lambda,\mu} := \mu\Delta + (\lambda + \mu)\nabla\mathrm{div}$

$L^p_{A,z}(\partial\Omega, \sigma)$, the space defined for $p, p' \in (1, \infty)$ with $1/p + 1/p' = 1$ and $z \in \mathbb{C}$ as in (1.7.14):

$$L^p_{A,z}(\partial\Omega,\sigma) := \Big\{f \in \big[L^p(\partial\Omega,\sigma)\big]^M : \int_{\partial\Omega} \langle f, g\rangle\, d\sigma = 0 \ \text{ for each } g \in \big[L^{p'}(\partial\Omega,\sigma)\big]^M \ \text{ with } \ (zI + K_A)g = 0\Big\}$$

L_λ, the Stokes system in the special writing (for $\lambda \in \mathbb{C}$) as in (6.1.2):

$$L_\lambda := \big((\delta_{jk}\delta_{\alpha\beta} + \lambda\delta_{j\beta}\delta_{k\alpha})\partial_j\partial_k\big)_{1\leq\alpha,\beta\leq n}$$

L_B, the Bitsadze operator, defined as in (1.4.77):

$$L_B := \partial^2_{\bar z} = \tfrac{1}{4}(\partial_x + \mathrm{i}\partial_y)^2$$

$L^{p,1}_{\mathbb{C},\mathrm{loc}}(\partial_*\Omega,\sigma)$, the space defined for $p\in[1,\infty]$ as (7.1.7):

$$L^{p,1}_{\mathbb{C},\mathrm{loc}}(\partial_*\Omega,\sigma):=\Big\{f\in L^p_{\mathrm{loc}}(\partial_*\Omega,\sigma):\partial^{\mathbb{C}}_{\tau_{jk}}f\ \text{ belongs to }\ L^p_{\mathrm{loc}}(\partial_*\Omega,\sigma)\ \text{ for each }\ j,k\in\{1,\dots,n\}\Big\}$$

$L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)$, the complex boundary Sobolev space, defined for $p\in[1,\infty]$ as in (7.1.11):

$$L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma):=\Big\{f\in L^p(\partial_*\Omega,\sigma):\partial^{\mathbb{C}}_{\tau_{jk}}f\ \text{ belongs to }\ L^p(\partial_*\Omega,\sigma)\ \text{ for each }\ j,k\in\{1,\dots,n\}\Big\}$$

$\Lambda^{\alpha,\beta}\,\mathbb{C}^n$, the space of differential forms of (type) degree (α,β) with complex coefficients, defined for $\alpha,\beta\in\{0,1,\dots,n\}$ as in (7.1.36), namely the collection of all $u=\sum_{|I|=\alpha,\,|J|=\beta}u_{I,J}\,dz^I\wedge d\bar{z}^J$ where the sum is performed over strictly increasing arrays and $u_{I,J}\in\mathbb{C}$ for all I,J

$L^p(X,\mu)\otimes\Lambda^{\alpha,\beta}:=L^p(X,\mu)\otimes\Lambda^{\alpha,\beta}\,\mathbb{C}^n$ where (X,μ) is a given measure space, $\alpha,\beta\in\{0,1,\dots,n\}$ and p is an exponent in $(0,\infty]$, the space of differential forms of type (α,β) with coefficients from $L^p(X,\mu)$, equipped with the quasi-norm $\|f\|_{L^p(X,\mu)\otimes\Lambda^{\alpha,\beta}}:=\sum_{|I|=\alpha,\,|J|=\beta}\|f_{I,J}\|_{L^p(X,\mu)}$ if $f=\sum_{|I|=\alpha,\,|J|=\beta}f_{I,J}\,dz^I\wedge d\bar{z}^J$ (7.1.77), (7.1.78)

$L^p_{\tan,\mathbb{C}}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$, the complex tangential forms defined for $\alpha,\beta\in\{0,1,\dots,n\}$ and $p\in(0,\infty]$ as the collection of all $f\in L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ satisfying $\nu^{1,0}\vee f=0$ at σ-a.e. point on $\partial_*\Omega$ (7.1.79), (7.1.64)

$L^p_{\mathrm{nor},\mathbb{C}}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$, the complex normal forms defined for $\alpha,\beta\in\{0,1,\dots,n\}$ and $p\in(0,\infty]$ as the collection of all $f\in L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ satisfying $\nu^{0,1}\wedge f=0$ at σ-a.e. point on $\partial_*\Omega$ (7.1.80), (7.1.65)

$L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$, the space of differential forms of type (α,β) with coefficients from $L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)$ and equipped with the norm (cf. (7.1.84), (7.1.85))

$$\|f\|_{L^{p,1}_{\mathbb{C}}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}}:=\|f\|_{L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}}+\sum_{|K|=\alpha}\sum_{|I|=\beta}\sum_{j,k=1}^{n}\big\|\partial^{\mathbb{C}}_{\tau_{jk}}f_{K,I}\big\|_{L^p(\partial_*\Omega,\sigma)}$$

for each $f=\sum_{|K|=\alpha}\sum_{|I|=\beta}f_{K,I}\,dz^K\wedge d\bar{z}^I$

$L^{1,p,q}_{\mathbb{C}}(\partial_*\Omega,\sigma)$, the off-diagonal (partial) Sobolev space defined for $p,q\in[1,\infty]$ as the collection of all $f\in L^p(\partial_*\Omega,\sigma)$ satisfying $\partial^{\mathbb{C}}_{\tau_{jk}}f\in L^q(\partial_*\Omega,\sigma)$ for all $1\le j,k\le n$, equipped with the norm (cf. (7.5.27), (7.5.28))

$$\|f\|_{L^{1,p,q}_{\mathbb{C}}(\partial_*\Omega,\sigma)} := \|f\|_{L^p(\partial_*\Omega,\sigma)} + \sum_{1\leq j,k\leq n} \big\|\partial^{\mathbb{C}}_{\tau_{jk}} f\big\|_{L^q(\partial_*\Omega,\sigma)}$$

$L^{p,\bar\partial_b}_{\alpha,\beta}(\partial\Omega,\sigma) := L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial\Omega,\sigma) \cap L^p_{\tan,\mathbb{C}}(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ for $\alpha,\beta\in\{0,1,\dots,n\}$ and $p\in[1,\infty)$ (7.7.7)

$L^{p,\bar\partial_\tau}_{\alpha,\beta}(\partial_*\Omega,\sigma)$, the space defined for $p\in[1,\infty]$ as the collection of all forms f in $L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ with the property that $\bar\partial_\tau f\in L^p(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+2}$ (7.2.27)

M

$\mathcal{M}_{X,s,\alpha}$, the L^s-based fractional Hardy-Littlewood maximal operator of order α in the space of homogeneous type (X,ρ,μ), defined for each μ-measurable function f on X as (cf. [68, (7.6.1)]):

$$\mathcal{M}_{X,s,\alpha} f(x) := \sup_{r>0}\left[\mu(B_\rho(x,r))^\alpha\Big(⨍_{B_\rho(x,r)} |f|^s\,\mathrm{d}\mu\Big)^{\frac{1}{s}}\right],\quad \forall\, x\in X \tag{A.0.143}$$

$\mathcal{M}_{X,s}$, the L^s-based Hardy-Littlewood maximal operator in the space of homogeneous type (X,ρ,μ), defined for each μ-measurable function f on X as (cf. [68, (7.6.7)]):

$$\mathcal{M}_{X,s} f(x) := \sup_{r>0}\Big(⨍_{B_\rho(x,r)} |f|^s\,\mathrm{d}\mu\Big)^{\frac{1}{s}},\quad \forall\, x\in X \tag{A.0.144}$$

$\mathcal{M}_X$, the Hardy-Littlewood maximal operator on the space of homogeneous type (X,ρ,μ), defined for each μ-measurable function f on X as (cf. [68, (7.6.16)]):

$$\mathcal{M}_X f(x) := \sup_{r\in(0,\infty)} \frac{1}{\mu\big(B_\rho(x,\,r)\big)}\int_{B_\rho(x,r)} |f|\,\mathrm{d}\mu,\quad \forall\, x\in X \tag{A.0.145}$$

$u^{\max}_M$, the tangential maximal function of u (with exponent M), defined at each $x\in\partial\Omega$ as (cf. [68, (8.5.2)]):

$$\begin{gathered} u^{\max}_M : \partial\Omega \longrightarrow [0,+\infty] \ \text{ defined by} \\ u^{\max}_M(x) := \Big\| u(y)\Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M \Big\|_{L^\infty_y(\Omega,\mathcal{L}^n)} \ \text{ for each } \ x\in\partial\Omega \end{gathered} \tag{A.0.146}$$

$\mathscr{M}(X,\mu)$, the collection of μ-measurable functions defined on an arbitrary measure space $(X,\mathfrak{M},\mu)$

$\mathscr{M}_+(X,\mu)$, the collection of non-negative μ-measurable functions defined on an arbitrary measure space $(X,\mathfrak{M},\mu)$

$M^{p,\lambda}(\Sigma,\sigma)$, the Morrey space on the Ahlfors regular set $\Sigma\subseteq\mathbb{R}^n$, defined for each $p\in(0,\infty)$ and $\lambda\in(0,n-1)$ as (cf. [69, §6.2]):

$$M^{p,\lambda}(\Sigma,\sigma) := \Big\{ f : \Sigma \to \mathbb{C} : \ f \text{ is } \sigma\text{-measurable and } \|f\|_{M^{p,\lambda}(\Sigma,\sigma)} < +\infty \Big\} \tag{A.0.147}$$

where, for each σ-measurable function f on Σ,

$$\|f\|_{M^{p,\lambda}(\Sigma,\sigma)} := \sup_{\substack{x\in\Sigma \text{ and} \\ 0<R<2\,\mathrm{diam}(\Sigma)}} \left\{ R^{\frac{n-1-\lambda}{p}} \Big(⨍_{\Sigma\cap B(x,R)} |f|^p \, \mathrm{d}\sigma \Big)^{\frac{1}{p}} \right\} \tag{A.0.148}$$

$\mathring{M}^{p,\lambda}(\Sigma,\sigma)$, the vanishing Morrey space on the Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$ with $p \in (0,\infty)$ and $\lambda \in (0, n-1)$ (cf. [69, §6.2]):

$$\mathring{M}^{p,\lambda}(\Sigma,\sigma) := \text{the closure of } L^s(\Sigma,\sigma) \text{ with } s := \tfrac{p(n-1)}{n-1-\lambda} \text{ in } M^{p,\lambda}(\Sigma,\sigma) \tag{A.0.149}$$

$M_1^{p,\lambda}(\partial\Omega,\sigma)$, the Morrey-based Sobolev space of order one on $\partial\Omega$, defined for $p \in (1,\infty)$ and $\lambda \in (0, n-1)$ as (cf. [69, (11.7.12)-(11.7.13)]):

$$M_1^{p,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in M^{p,\lambda}(\partial\Omega,\sigma) : \text{ for each } j,k \in \{1,\dots,n\} \text{ one has } \partial_{\tau_{jk}} f \in M^{p,\lambda}(\partial\Omega,\sigma) \Big\} \tag{A.0.150}$$

equipped with the norm

$$M_1^{p,\lambda}(\partial\Omega,\sigma) \ni f \longmapsto \|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \tag{A.0.151}$$

$M_1^{p,q,\lambda}(\partial\Omega,\sigma)$, the off-diagonal Morrey-based Sobolev space of order one on $\partial\Omega$, defined for $p,q \in (1,\infty)$ and $\lambda \in (0, n-1)$ as (cf. [69, (11.7.14)-(11.7.15)]):

$$M_1^{p,q,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in M^{p,\lambda}(\partial\Omega,\sigma) : \text{ for each } j,k \in \{1,\dots,n\} \text{ one has } \partial_{\tau_{jk}} f \in M^{q,\lambda}(\partial\Omega,\sigma) \Big\}, \tag{A.0.152}$$

and endowed with the norm

$$M_1^{p,q,\lambda}(\partial\Omega,\sigma) \ni f \longmapsto \|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} + \sum_{j,k=1}^{n} \big\|\partial_{\tau_{jk}} f\big\|_{M^{q,\lambda}(\partial\Omega,\sigma)} \tag{A.0.153}$$

$\mathring{M}_1^{p,q,\lambda}(\partial\Omega,\sigma)$, the vanishing off-diagonal Morrey-based Sobolev space of order one on $\partial\Omega$, defined for $p,q \in (1,\infty)$ and $\lambda \in (0, n-1)$ as (cf. [69, (11.7.16)]):

$$\mathring{M}_1^{p,q,\lambda}(\partial\Omega,\sigma) := \Big\{ f \in \mathring{M}^{p,\lambda}(\partial\Omega,\sigma) : \text{ for each } j,k \in \{1,\dots,n\} \text{ one has } \partial_{\tau_{jk}} f \in \mathring{M}^{q,\lambda}(\partial\Omega,\sigma) \Big\} \tag{A.0.154}$$

$\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma)$, the vanishing Morrey-based Sobolev space of order one on $\partial\Omega$, defined for $p\in(1,\infty)$ and $\lambda\in(0,n-1)$ as (cf. [69, (11.7.17)]):

$$\mathring{M}_1^{p,\lambda}(\partial\Omega,\sigma) := \Big\{f\in\mathring{M}^{p,\lambda}(\partial\Omega,\sigma):\ \text{for each}\ \ j,k\in\{1,\dots,n\}\ \ \text{one has}\ \ \partial_{\tau_{jk}}f\in\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\Big\} \tag{A.0.155}$$

$M_{-1}^{p,\lambda}(\partial\Omega,\sigma)$, the Morrey-based negative Sobolev space on $\partial\Omega$, defined for any $p,q\in(1,\infty)$ with $1/p+1/q=1$ and $\lambda\in(0,n-1)$ as (cf. [69, Definition 11.8.9]):

$$M_{-1}^{p,\lambda}(\partial\Omega,\sigma) := \Big(\mathcal{B}_1^{q,\lambda}(\partial\Omega,\sigma)\Big)^* \tag{A.0.156}$$

$\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)$, the homogeneous Morrey-based Sobolev space of order one on $\partial\Omega$ defined for $p\in(1,\infty)$ and $\lambda\in(0,n-1)$ as (cf. [69, Definition 11.13.1]):

$$\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma) := \Big\{f\in L^1\Big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\Big)\cap L^p_{\text{loc}}(\partial\Omega,\sigma):\ \partial_{\tau_{jk}}f\in M^{p,\lambda}(\partial\Omega,\sigma)\ \ \text{for}\ \ 1\leq j,k\leq n\Big\} \tag{A.0.157}$$

and equipped with the semi-norm

$$\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)\ni f\mapsto \|f\|_{\dot{M}_1^{p,\lambda}(\partial\Omega,\sigma)} := \sum_{j,k=1}^n \|\partial_{\tau_{jk}}f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \tag{A.0.158}$$

$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)$, the homogeneous vanishing Morrey-based Sobolev space of order one on $\partial\Omega$, defined for $p\in(1,\infty)$ and $\lambda\in(0,n-1)$ as (cf. [69, Definition 11.13.15]):

$$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma) := \Big\{f\in L^1\Big(\partial\Omega,\frac{\sigma(x)}{1+|x|^n}\Big)\cap L^p_{\text{loc}}(\partial\Omega,\sigma):\ \partial_{\tau_{jk}}f\in\mathring{M}^{p,\lambda}(\partial\Omega,\sigma)\ \ \text{for}\ \ 1\leq j,k\leq n\Big\} \tag{A.0.159}$$

and equipped with the semi-norm

$$\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)\ni f\mapsto \|f\|_{\dot{\mathbb{M}}_1^{p,\lambda}(\partial\Omega,\sigma)} := \sum_{j,k=1}^n \|\partial_{\tau_{jk}}f\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \tag{A.0.160}$$

M_b, the operator of pointwise multiplication by the function $b\in L^\infty(\Sigma,\sigma)$

$[M_b, T]$, the commutator of M_b (the pointwise multiplication by b) with the operator T, defined by $[M_b, T]f := b(Tf) - T(bf)$

N

$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$

NTA domain: a nonempty open subset Ω of $\mathbb{R}^n$ satisfying a two-sided corkscrew condition as well as a Harnack chain condition (cf. [68, Definition 5.11.1])

two-sided NTA domain: a nonempty open subset Ω of $\mathbb{R}^n$ satisfying a two-sided corkscrew condition as well as a two-sided Harnack chain condition (cf. [68, Definition 5.11.1])

one-sided NTA domain (or interior NTA domain): a nonempty open subset Ω of $\mathbb{R}^n$ satisfying an interior corkscrew condition as well as a Harnack chain condition (cf. [68, Definition 5.11.1])

$\mathcal{N}_\kappa u$, the (κ-)nontangential maximal operator acting on the function measurable $u : \Omega \to \mathbb{R}^n$ according to (cf. [68, (8.2.1)]):

$$\mathcal{N}_\kappa u : \partial\Omega \longrightarrow [0, +\infty], \quad (\mathcal{N}_\kappa u)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x), \mathcal{L}^n)} \text{ for all } x \in \partial\Omega \tag{A.0.161}$$

$\mathcal{N}_\kappa^\varepsilon u$, the ($\kappa$-)nontangential maximal operator truncated at height $\varepsilon > 0$, acting on the function $u : \Omega \to \mathbb{R}^n$ according to (cf. [68, (1.5.5)]):

$$\mathcal{N}_\kappa^\varepsilon u := \mathcal{N}_\kappa(u \cdot \mathbf{1}_{O_\varepsilon}) \text{ where } O_\varepsilon := \big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\big\}. \tag{A.0.162}$$

$\mathcal{N}_\kappa^E u$, the restricted nontangential maximal function of $u : \Omega \to \mathbb{R}$, relative to the set E (cf. [68, (8.2.4)]):

$$\begin{aligned} \mathcal{N}_\kappa^E u : \partial\Omega \longrightarrow [0, +\infty] \ \text{ defined as} \\ (\mathcal{N}_\kappa^E u)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x)\cap E, \mathcal{L}^n)} \ \text{ for each } \ x \in \partial\Omega \end{aligned} \tag{A.0.163}$$

ν, the geometric measure theoretic outward unit normal to a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, defined via the requirement that (cf. [68, (5.6.2)-(5.6.3)]):

$$\begin{aligned} &\text{there exist a locally finite Borel-regular measure } \sigma_* \text{ in } \mathbb{R}^n \text{ and a} \\ &\text{vector-valued function } \nu \in \big[L^\infty(\mathbb{R}^n, \sigma_*)\big]^n \text{ satisfying } |\nu(x)| = 1 \text{ at} \\ &\sigma_*\text{-a.e. } x \in \mathbb{R}^n \text{ and with such that } \nabla \mathbf{1}_\Omega = -\nu\sigma_* \text{ in } \big[\mathcal{D}'(\mathbb{R}^n)\big]^n \end{aligned} \tag{A.0.164}$$

ν_g, the geometric measure theoretic outward unit normal induced by the metric tensor g

ν^{E}, the geometric measure theoretic outward unit normal induced by the standard Euclidean metric

$\nu \bullet \vec{F}$, the "bullet" product involving a vector field $\vec{F} \in \big[L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)\big]^n$ (where Ω is an arbitrary open subset of $\mathbb{R}^n$) whose divergence, considered in the sense of distributions in Ω, satisfies $\operatorname{div}\vec{F} \in L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)$, defined as a functional acting on each $\psi \in \mathrm{Lip}_c(\partial\Omega)$ according to (cf. [68, Proposition 4.2.3]):

$$\big\langle \nu \bullet \vec{F}, \psi \big\rangle := \int_\Omega \vec{F} \cdot \nabla\Psi \, \mathrm{d}\mathcal{L}^n + \int_\Omega (\mathrm{div}\vec{F})\Psi \, \mathrm{d}\mathcal{L}^n, \tag{A.0.165}$$

where Ψ is any complex-valued function satisfying

$$\begin{array}{l} \Psi \in \mathrm{Lip}(\overline{\Omega}), \Psi\big|_{\partial\Omega} = \psi, \text{and } \Psi \equiv 0 \text{ outside} \\ \text{of some compact subset of } \overline{\Omega} \end{array} \tag{A.0.166}$$

$\nu \circledbullet u$, the Clifford bullet product of ν with u (cf. [69, (10.2.100)]):

$$\nu \circledbullet u := (-\mathrm{i})\mathrm{Sym}(D; \nu) \bullet u \tag{A.0.167}$$

$N_\kappa^p(\Omega; \mu)$, the space of measurable functions in Ω with a p-th power integrable non-tangential maximal function on $\partial\Omega$ with respect to the measure μ (cf. [68, (8.3.31)]):

$$\begin{aligned} N_\kappa^p(\Omega; \mu) := \big\{ u : \Omega \to \mathbb{C} :\ u \ \text{ is } \mathcal{L}^n\text{-measurable, and} \\ \|u\|_{N_\kappa^p(\Omega;\mu)} := \|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,\mu)} < +\infty \big\} \end{aligned} \tag{A.0.168}$$

$\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x)$, the nontangential trace of the function $u : \Omega \to \mathbb{R}$ at the point $x \in \partial\Omega$ such that $x \in \overline{\Gamma_\kappa(x)}$, defined as (cf. [68, Definition 8.9.1]):

$$\begin{array}{l} \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) \text{ is the number } a \in \mathbb{R} \text{ with the property that for every} \\ \varepsilon > 0 \text{ there exists some } r > 0 \text{ such that } |u(y) - a| < \varepsilon \text{ for } \mathcal{L}^n\text{-a.e.} \\ \text{point } y \in \Gamma_\kappa(x) \cap B(x, r) \end{array} \tag{A.0.169}$$

$\|T\|_{X\to Y}$, the operator "norm" of a positively homogeneous mapping T acting from the quasi-normed vector space $\big(X, \|\cdot\|_X\big)$ with values in the quasi-normed vector space $\big(Y, \|\cdot\|_Y\big)$

$$\|T\|_{X\to Y} := \sup\big\{\|Tu\|_Y :\ u \in X,\ \|u\|_X = 1\big\} \in [0, +\infty] \tag{A.0.170}$$

$\|T\|_{\mathrm{Bd}(X\to Y)} := \|T\|_{X\to Y}$

$\|T\|_{\mathrm{Bd}(X)}$, the "norm" of a positively homogeneous mapping T acting from the quasi-normed vector space $\big(X, \|\cdot\|_X\big)$ into itself

$$\|T\|_{\mathrm{Bd}(X)} := \|T\|_{\mathrm{Bd}(X\to X)} = \sup_{x\in X,\ \|x\|_X=1} \|Tx\|_X \tag{A.0.171}$$

$\|T\|_{X\to Y}^{\mathrm{ess}}$, the essential norm of the operator $T \in \mathrm{Bd}(X \to Y)$, where X, Y are quasi-normed spaces (cf. [69, §1.2]):

$$\|T\|^{\rm ess}_{X\to Y} := \operatorname{dist}\big(T, \operatorname{Cp}(X \to Y)\big)$$
$$= \inf\big\{\|T-K\|_{X\to Y} : K \in \operatorname{Cp}(X\to Y)\big\} \qquad (A.0.172)$$

$\|\cdot\|_{(X_0,X_1)_{\theta,q}}$, the real interpolation quasi-norm

$\nu_{\mathbb{C}}$, the complex outward unit normal of a set of locally finite perimeter $\Omega \subseteq \mathbb{R}^{2n} \equiv \mathbb{C}^n$ defined as in (7.1.60):

$$\nu_{\mathbb{C}} := \big(\nu_1 + i\nu_2, \dots, \nu_{2n-1} + i\nu_{2n}\big) \in \mathbb{C}^n$$

if $\nu = (\nu_1, \nu_2, \dots, \nu_{2n-1}, \nu_{2n}) \in \mathbb{R}^{2n}$ is the geometric measure theoretic unit normal to Ω

O

$\mathbf{1}_E$, the characteristic function of a given set E

$\omega_{n-1} := \mathcal{H}^{n-1}(S^{n-1})$, the surface area of S^{n-1} (the $(n-1)$-dimensional sphere in $\mathbb{R}^n$)

$\mathcal{O}_\varepsilon := \big\{x \in \Omega : \delta_{\partial\Omega}(x) < \varepsilon\big\}$, the one-sided collar neighborhood of $\partial\Omega$ of "width" $\varepsilon > 0$

$\mathscr{O}(\Omega)$, the collection of all holomorphic functions in an open set $\Omega \subseteq \mathbb{C}$

P

$\mathfrak{P}$, the $\mathfrak{P}$-maximal operator acting on a Lebesgue measurable function $u : \Omega \to \mathbb{R}^n$ at the point $x \in \partial\Omega$ according to (cf. [69, §10.1]):

$$(\mathfrak{P}u)(x) := \sup_{0<r<2\operatorname{diam}(\partial\Omega)} \left\{ \frac{1}{\sigma\big(\partial\Omega \cap B(x,r)\big)} \int_{\Omega\cap B(x,r)} |u| \, d\mathcal{L}^n \right\} \qquad (A.0.173)$$

${}_{\mathscr{E}'(\Omega)}\big\langle\!\big\langle \cdot, \cdot \big\rangle\!\big\rangle_{\mathscr{E}(\Omega)}$, the pairing between a compactly supported distribution u in Ω and a smooth function $f \in \mathscr{C}^\infty(\operatorname{supp} u)$, say $f \in \mathscr{C}^\infty(\mathcal{O})$ with $\mathcal{O} \subseteq \Omega$ open set containing $\operatorname{supp} u$, defined for each $F \in \mathscr{C}^\infty(\Omega)$ with the property that $F = f$ near $\operatorname{supp} u$ as (cf. [68, (2.2.33)]):

$$ {}_{\mathscr{E}'(\Omega)}\big\langle\!\big\langle u, f \big\rangle\!\big\rangle_{\mathscr{E}(\Omega)} := {}_{\mathscr{E}'(\Omega)}\big\langle u, F \big\rangle_{\mathscr{E}(\Omega)} \qquad (A.0.174)$$

$\wedge$, the exterior product of differential forms

$\vee$, the interior product of differential forms

$\dot{\partial}_{\tau_{jk}} u$, the weak tangential derivative of a given function $u \in L^1_{\rm loc}(\Omega, \mathcal{L}^n)$ with the property that $\nabla u \in \big[L^1_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^n$, defined as (cf. [68, Example 4.2.4]):

$$\dot{\partial}_{\tau_{jk}} u := \nu \bullet \big((\partial_k u)\mathbf{e}_j - (\partial_j u)\mathbf{e}_k\big), \qquad (A.0.175)$$

i.e., the "bullet" product $\nu \bullet \vec{F}^u_{jk}$, where

$$\vec{F}^u_{jk} := (\partial_k u)\mathbf{e}_j - (\partial_j u)\mathbf{e}_k \tag{A.0.176}$$

$\partial_* E$, the measure theoretic boundary of a Lebesgue measurable set $E \subseteq \mathbb{R}^n$, defined as (cf. [68, (5.2.1)]):

$$\partial_* E := \Big\{ x \in \mathbb{R}^n : \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap E)}{r^n} > 0 \ \text{ and } \ \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\setminus E)}{r^n} > 0 \Big\} \tag{A.0.177}$$

$\partial^* E$, the reduced boundary of a set $E \subseteq \mathbb{R}^n$ of locally finite perimeter, defined as (cf. [68, (5.6.13)]):

$\partial^* E$ consists of all points $x \in \partial E$ satisfying the following three properties: $0 < \sigma_*(B(x,r)) < +\infty$ for each $r \in (0,\infty)$, formula $\lim\limits_{r\to 0^+} ⨍_{B(x,r)} \nu \, \mathrm{d}\sigma_* = \nu(x)$ is valid, and $|\nu(x)| = 1$. (A.0.178)

$\partial_{\text{nta}}\Omega$, the nontangentially accessible boundary of an open proper subset Ω of $\mathbb{R}^n$, defined as (cf. [68, Definition 8.8.5]):

$$\partial_{\text{nta}}\Omega := \bigcap_{\kappa>0} A_\kappa(\partial\Omega) = \big\{ x \in \partial\Omega : \ x \in \overline{\Gamma_\kappa(x)} \ \text{ for each } \ \kappa > 0 \big\} \tag{A.0.179}$$

$(a)_+ := \max\{a, 0\}$, the positive part of the number $a \in \mathbb{R}$

$\pi_\kappa(E) = \pi_{\Omega,\kappa}(E)$, the "shadow" (or projection) of a given set $E \subseteq \Omega$ onto $\partial\Omega$, defined as (cf. [68, (8.1.15)]):

$$\pi_\kappa(E) = \pi_{\Omega,\kappa}(E) := \big\{ x \in \partial\Omega : \ \Gamma_\kappa(x) \cap E \neq \varnothing \big\} \tag{A.0.180}$$

$p_{\mathbb{X}}$, the lower Boyd index of a rearrangement invariant Banach function space $\mathbb{X}$ on a non-atomic sigma-finite measure space $(X, \mathfrak{M}, \mu)$

$\partial^A_\nu(\cdot,\cdot)$, the conormal derivative operator with respect to the coefficient tensor A acting from weighted Sobolev spaces (cf. [69, Proposition 8.5.3]), and from Besov/Triebel-Lizorkin spaces (cf. [69, Proposition 9.5.2])

$\dot{\partial}^A_\nu u$, the weak conormal derivative of u with respect to the coefficient tensor A defined (cf. [69, Definition 10.2.18]) by first introducing

$$\vec{F}^\alpha := (A\nabla u)_\alpha = \big(a^{\alpha\beta}_{rs} \partial_s u_\beta \big)_{1\leq r\leq n} \ \text{ for each } \ \alpha \in \{1,\dots,M\} \tag{A.0.181}$$

and then setting

$$\dot{\partial}^A_\nu u := \big(\nu \bullet \vec{F}^\alpha \big)_{1\leq\alpha\leq M} \in \Big[\big(\mathrm{Lip}_c(\partial\Omega) \big)' \Big]^M \tag{A.0.182}$$

$\partial_{\tau_{jk}}\varphi$ with $\varphi \in \mathscr{C}^1(\mathcal{O})$, the pointwise tangential derivative operator (cf. [69, §11.1]):

$$\partial_{\tau_{jk}}\varphi := \nu_j\big[\partial_k\varphi\big]\big|_{\mathcal{O}\cap\partial_*\Omega} - \nu_k\big[\partial_j\varphi\big]\big|_{\mathcal{O}\cap\partial_*\Omega} \text{ at } \sigma_*\text{-a.e. point on } \mathcal{O}\cap\partial_*\Omega \quad \text{(A.0.183)}$$

$\partial_{\tau_{jk}}$, the tangential derivative operator defined in a weak, or distributional, sense (cf. [69, (11.1.17) and Definition 11.1.2], and [69, Definition 11.2.1])

$\overline{\partial} := \partial_{\bar z} := \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$, the Cauchy-Riemann operator

$\partial := \partial_z := \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$, the (complex) conjugate of the Cauchy-Riemann operator

∂^A_ν, the pointwise conormal derivative operator with respect to the coefficient tensor $A := \big(a^{\alpha\beta}_{rs}\big)_{\substack{1\leq\alpha,\beta\leq M\\ 1\leq r,s\leq n}}$ acting on some $u = (u_\beta)_{1\leq\beta\leq N} \in \big[\mathcal{D}'(\Omega)\big]^N$ (which is of function type near $\partial\Omega$) at $\mathcal{H}^{n-1}$-a.e. point on $\partial_*\Omega$ as (cf. [70, (1.7.9) in Definition 1.7.1]):

$$\partial^A_\nu u := \Big(\nu_r\big(a^{\alpha\beta}_{rs}\partial_s u_\beta\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)_{1\leq\alpha\leq M} \quad \text{(A.0.184)}$$

$\partial^{\widetilde{D},D}_\nu$, the conormal derivative operator associated with the factorization of the second-order system L as $\widetilde{D}D$, where $\widetilde{D}$ and D are homogeneous, constant complex coefficient, first-order systems in $\mathbb{R}^n$, acting on a given $u \in \big[\mathcal{D}'(\Omega)\big]^N$ (which is of function type near $\partial\Omega$) as:

$$\partial^{\widetilde{D},D}_\nu u := (-\mathrm{i})\mathrm{Sym}(\widetilde{D};\nu)(Du)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \quad \text{(A.0.185)}$$

$\mathrm{P.V.}\big(b\,k(x-\cdot)\big|_\Sigma$, the principal-value distribution associated with the smooth odd kernel k and the bounded function b on the countably rectifiable upper Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, acting on each test function $\phi \in \mathrm{Lip}_c(\Sigma)$ as (cf. [69, Proposition 11.9.1]):

$$\Big\langle \mathrm{P.V.}\big(b\,k(x-\cdot)\big|_\Sigma\big), \phi\Big\rangle := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\Sigma\\ |y-x|>\varepsilon}} b(y)k(x-y)\phi(y)\,\mathrm{d}\sigma(y)$$

Π_Ω, the Newtonian (volume) potential operator associated with the system L in the set Ω acting on a given function w as (cf. (1.3.4)):

$$\Pi_\Omega w := \int_\Omega E(\cdot - y)w(y)\,\mathrm{d}y$$

where E is the canonical fundamental solution of the system L

∂_τ, the tangential partial derivative in the two-dimensional setting, defined as in (1.6.40):

$$\partial_\tau := \partial_{\tau_{12}} = \nu_1\partial_2 - \nu_2\partial_1$$

$\partial_\nu^\lambda(\vec{u},\pi)$, the (pointwise) conormal derivative for the Stokes system associated with the coefficient tensor $A_\lambda := \big(\delta_{jk}\delta_{\alpha\beta} + \lambda\delta_{j\beta}\delta_{k\alpha}\big)_{\substack{1\le\alpha,\beta\le n\\ 1\le j,k\le n}}$ for each $\lambda \in \mathbb{C}$ at σ-a.e. point on $\partial_*\Omega$ as (cf. (6.1.6)):

$$\partial_\nu^\lambda(\vec{u},\pi) := \big[\nabla\vec{u} + \lambda(\nabla\vec{u})^\top\big]\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\nu - \Big(\pi\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)\nu$$

$\dot{\partial}_\nu^\lambda(\vec{u},\pi)$, the weak conormal derivative for the Stokes system associated with $\lambda \in \mathbb{C}$ for the coefficient tensor $A_\lambda := \big(\delta_{jk}\delta_{\alpha\beta} + \lambda\delta_{j\beta}\delta_{k\alpha}\big)_{\substack{1\le\alpha,\beta\le n\\ 1\le j,k\le n}}$ as in Definition 6.2.9, by first introducing the family of vector fields (cf. (6.2.239))

$$\vec{F}^\alpha := \nabla u_\alpha + \lambda\,\partial_\alpha\vec{u} - \pi\,\mathbf{e}_\alpha \ \text{ for each } \alpha \in \{1,\dots,n\}$$

and then defining (cf. (6.2.240)):

$$\dot{\partial}_\nu^\lambda(\vec{u},\pi) := \big(\nu \bullet \vec{F}^\alpha\big)_{1\le\alpha\le n} \in \Big[\big(\mathrm{Lip}_c(\partial\Omega)\big)'\Big]^n$$

$\partial^{\mathbb{C}}_{\tau_{jk}}$, the complex tangential derivative operator whose action on a complex-valued function ψ of class $\mathscr{C}^1$ in some open neighborhood of $\partial_*\Omega$ in $\mathbb{C}^n$ is defined as (cf. (7.1.5)):

$$\partial^{\mathbb{C}}_{\tau_{jk}}\psi := (\nu_{\mathbb{C}})_j\big[\partial_{\bar{z}_k}\psi\big]\big|_{\partial_*\Omega} - (\nu_{\mathbb{C}})_k\big[\partial_{\bar{z}_j}\psi\big]\big|_{\partial_*\Omega}$$

$\bar{\partial}$, the d-bar operator acting on a continuously differentiable complex-valued function f defined in an open subset of $\mathbb{C}^n$ as (cf. (7.1.70)):

$$\bar{\partial} f := \sum_{j=1}^n (\partial_{\bar{z}_j} f)\,\mathrm{d}\bar{z}_j$$

$\bar{\partial}_b$, the boundary d-bar operator from the space $L^{p,\bar{\partial}_b}_{\alpha,\beta}(\partial\Omega,\sigma)$ into the space $L^p_{\mathrm{tan},\mathbb{C}}(\partial\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+1}$ acting on each (α,β)-form $f \in L^{p,\bar{\partial}_b}_{\alpha,\beta}(\partial\Omega,\sigma)$ as in (7.7.8), (7.7.9), Definition 7.7.2:

$$\bar{\partial}_b f := \nu^{1,0} \vee \bar{\partial}_\tau f$$

$\bar{\partial}$, the standard d-bar operator acting on forms as (cf. (7.2.2)):

$$\bar{\partial} := \sum_{j=1}^n \partial_{\bar{z}_j}\mathrm{d}\bar{z}_j \wedge \cdot$$

∂, the complex conjugate of the standard d-bar operator acting on forms as (cf. (7.2.2)):

$$\partial := \sum_{j=1}^n \partial_{z_j}\mathrm{d}z_j \wedge \cdot$$

$\bar{\partial}_\tau$, the "partial" tangential derivative introduced in Definition 7.2.2: an (α,β)-form $f \in L^1_{\rm loc}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta}$ is said to have $\bar{\partial}_\tau f$ in the space $L^1_{\rm loc}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+2}$ provided there exists some form $g \in L^1_{\rm loc}(\partial_*\Omega,\sigma)\otimes\Lambda^{\alpha,\beta+2}$ with the property that $\int_{\partial_*\Omega}\langle \nu^{0,1}\wedge f, \vartheta\psi\rangle_{\mathbb{C}}\,{\rm d}\sigma = \int_{\partial_*\Omega}\langle g,\psi\rangle_{\mathbb{C}}\,{\rm d}\sigma$ for all $\psi\in\mathscr{C}^\infty_c(\mathbb{C}^n)\otimes\Lambda^{\alpha,\beta+2}$

$\mathcal{P}_\lambda$, the (double layer) pressure operator for the Stokes system acting on any function $\vec{f}\in\big[L^1\big(\partial_*\Omega,\frac{\sigma(x)}{1+|x|^n}\big)\big]^n$ at each point $x\in\Omega$ as (cf. (6.2.21)):

$$\mathcal{P}_\lambda\vec{f}(x) := -(1+\lambda)\int_{\partial_*\Omega}\nu_j(y)\langle(\partial_j\vec{q})(x-y),\vec{f}(y)\rangle\,{\rm d}\sigma(y)$$

where the pressure vector $\vec{q}$ is defined as in (6.2.2) by the formula $\vec{q}(x) := -\frac{1}{\omega_{n-1}}\,\frac{x}{|x|^n}$ for all points $x\in\mathbb{R}^n\setminus\{0\}$

$\Pi_{\alpha,\beta}$, the complex volume (Newtonian) potential operator acting on differential forms $U:\Omega\to\Lambda^{\alpha,\beta}\mathbb{C}^n$ at each point $z\in\Omega$ as (cf. (7.4.149)):

$$\Pi_{\alpha,\beta}U(z) := -\tfrac{1}{2}\int_\Omega\big\langle U(\zeta),\Gamma_{\alpha,\beta}(\zeta,z)\big\rangle_{\mathbb{C}}\,{\rm d}\mathcal{L}^{2n}(\zeta)$$

Q

X/Y, the quotient space of a vector space X and a linear subspace Y of X

$q_{\mathbb{X}}$, the upper Boyd index of a rearrangement invariant Banach function space $\mathbb{X}$ on a non-atomic sigma-finite measure space $(X,\mathfrak{M},\mu)$

$\mathcal{Q}$, the (single layer) pressure potential for the Stokes system acting on any function $\vec{f}\in\big[L^1\big(\partial\Omega,\frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^n$ at each point $x\in\Omega$ as in (6.2.14):

$$\mathcal{Q}\vec{f}(x) := \int_{\partial\Omega}\langle\vec{q}(x-y),\vec{f}(y)\rangle\,{\rm d}\sigma(y)$$

where the pressure vector $\vec{q}$ is defined as in (6.2.2) by the formula $\vec{q}(x) := -\frac{1}{\omega_{n-1}}\,\frac{x}{|x|^n}$ for each $x\in\mathbb{R}^n\setminus\{0\}$

R

$\mathbb{R}^n_+$, the (open) upper half-space in $\mathbb{R}^n$

$\mathbb{R}^n_-$, the (open) lower half-space in $\mathbb{R}^n$

${\rm rad}(\Omega)$, the number associated with any nonempty open set $\Omega\subseteq\mathbb{R}^n$ as (cf. [68, (5.11.31)]):

$$\begin{aligned}&{\rm rad}(\Omega) := \inf_j\,\inf_{x\in\Omega_j}\,\sup_{y\in\Omega_j}|x-y|,\quad\text{where}\\ &\{\Omega_j\}_j\quad\text{are the connected components of}\quad\Omega\end{aligned}\tag{A.0.186}$$

R_j, the j-th boundary-to-boundary Riesz transform, with $j\in\{1,\dots,n\}$, acting on $f\in L^1\big(\Sigma,\frac{\sigma}{1+|\cdot|^{n-1}}\big)$ at σ-a.e. $x\in\Sigma$ as:

$$R_j f(x) := \lim_{\varepsilon\to 0^+} \frac{2}{\omega_{n-1}} \int\limits_{\substack{|x-y|>\varepsilon \\ y\in\Sigma}} \frac{x_j - y_j}{|x-y|^n} f(y)\, \mathrm{d}\sigma(y) \tag{A.0.187}$$

$\mathcal{R}_j$, the j-th boundary-to-domain Riesz transform, with $j \in \{1,\dots,n\}$, acting on each $f \in L^1\big(\Sigma, \frac{\sigma}{1+|\cdot|^{n-1}}\big)$ at every point $x \in \mathbb{R}^n \setminus \Sigma$ as:

$$\mathcal{R}_j f(x) := \frac{2}{\omega_{n-1}} \int\limits_{\Sigma} \frac{x_j - y_j}{|x-y|^n} f(y)\, \mathrm{d}\sigma(y) \tag{A.0.188}$$

$\mathcal{R}_{jk}$, the boundary-to-domain integral operator, with $j,k \in \{1,\dots,n\}$, acting on each $f \in L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ at any point $x \in \Omega$ as:

$$\mathcal{R}_{jk} f(x) := \frac{-1}{\omega_{n-1}} \int\limits_{\partial_*\Omega} \frac{\nu_j(y)(x_k - y_k) - \nu_k(y)(x_j - y_j)}{|x-y|^n} f(y)\, \mathrm{d}\sigma_*(y) \tag{A.0.189}$$

R_{jk}, the principal-value singular integral operator, with $j,k \in \{1,\dots,n\}$, acting on each $f \in L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ at σ_*-a.e. point $x \in \partial_*\Omega$ as:

$$R_{jk} f(x) := \lim_{\varepsilon\to 0^+} \int\limits_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} \Big[\nu_j(y)(\partial_k E_\Delta)(y-x) - \nu_k(y)(\partial_j E_\Delta)(y-x)\Big] f(y)\, \mathrm{d}\sigma_*(y) \tag{A.0.190}$$

$(R_{jk})_{\max}$, the maximal operator, with $j,k \in \{1,\dots,n\}$, acting on each function $f \in L^1\big(\partial_*\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ at any point $x \in \partial_*\Omega$ as:

$$\begin{aligned} (R_{jk})_{\max} f(x) := \sup_{\varepsilon>0} \Bigg| \int\limits_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} &\big[\nu_j(y)(\partial_k E_\Delta)(y-x) \\ &- \nu_k(y)(\partial_j E_\Delta)(y-x)\big] f(y)\, \mathrm{d}\sigma_*(y)\Bigg| \end{aligned} \tag{A.0.191}$$

$R^{\#}_{jk}$, the principal-value singular integral operator on $\partial\Omega$, for $j,k \in \{1,\dots,n\}$, acting on any $f \in L^p(\partial\Omega,\sigma) \otimes C\ell_n$ at σ_*-a.e. point $x \in \partial_*\Omega$ as (cf. (1.6.16)):

$$R^{\#}_{jk} f(x) := \lim_{\varepsilon\to 0^+} \frac{-1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{\nu_j(x)(x_k-y_k)-\nu_k(x)(x_j-y_j)}{|x-y|^n} f(y)\, \mathrm{d}\sigma(y)$$

$\mathcal{R}_{C\ell}$, the boundary-to-domain Clifford-Riesz transform associated with a given set $\Omega \subset \mathbb{R}^n$, acting on any $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big) \otimes C\ell_n$ at each point $x \in \Omega$ as (cf. (2.1.77)):

$$\mathcal{R}_{C\ell} f(x) := \frac{2}{\omega_{n-1}} \int_{\partial\Omega} \frac{x-y}{|x-y|^n} \odot f(y)\, \mathrm{d}\sigma(y)$$

R_j^{mod}, the j-th modified Riesz transform (with $j \in \{1,\dots,n-1\}$) in the Euclidean space $\mathbb{R}^{n-1}$, acting on any given function $f \in L^1\big(\mathbb{R}^{n-1}, \frac{\mathrm{d}x'}{1+|x'|^n}\big)$ at $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$ as (cf. (5.3.53)):

$$\big(R_j^{\mathrm{mod}} f\big)(x) = \lim_{\varepsilon\to 0^+} \frac{2}{\omega_{n-1}} \int\limits_{\mathbb{R}^{n-1}} \Big\{ \tfrac{x_j-y_j}{|x'-y'|^n} \mathbf{1}_{\mathbb{R}^{n-1}\setminus \overline{B_{n-1}(x',\varepsilon)}}(y') - \tfrac{-y_j}{|-y'|^n} \mathbf{1}_{\mathbb{R}^{n-1}\setminus \overline{B_{n-1}(0',1)}}(y') \Big\} f(y')\, \mathrm{d}y'$$

where, generally speaking, $B_{n-1}(z',r)$ is the $(n-1)$-dimensional ball centered at $z' \in \mathbb{R}^{n-1}$ and of radius r

$\mathcal{R}_{\mathbb{C},j}$, the j-th boundary-to-domain complex Riesz transform (with $j \in \{1,\dots,n\}$), acting on any function $f \in L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big)$ at each point $z \in \Omega$ as (cf. (7.3.56)):

$$\mathcal{R}_{\mathbb{C},j} f(z) := \frac{2}{\omega_{2n-1}} \int\limits_{\partial\Omega} \frac{z_j-\zeta_j}{|z-\zeta|^{2n}} f(\zeta)\, \mathrm{d}\sigma(\zeta)$$

$R_{\mathbb{C},j}$, the j-th boundary-to-boundary complex Riesz transform (with $j \in \{1,\dots,n\}$), acting on any function $f \in L^1\big(\partial\Omega, \frac{\sigma(\zeta)}{1+|\zeta|^{2n-1}}\big)$ at σ-a.e. point $z \in \partial\Omega$ as (cf. (7.3.58)):

$$R_{\mathbb{C},j} f(x) := \lim_{\varepsilon\to 0^+} \tfrac{2}{\omega_{2n-1}} \int\limits_{\substack{|z-\zeta|>\varepsilon \\ \zeta\in\partial\Omega}} \tfrac{z_j-\zeta_j}{|z-\zeta|^{2n}} f(\zeta)\, \mathrm{d}\sigma(\zeta)$$

S

$\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, the surface measure on $\partial\Omega$

$\sigma_* := \mathcal{H}^{n-1}\lfloor\partial_*\Omega$, the surface measure on $\partial_*\Omega$

$\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$, the surface measure on the closed Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$

$\mathrm{Sym}(D;\cdot)$, the principal symbol of the first-order system $D = \Big(\sum_{j=1}^n a_j^{\alpha\beta}\partial_j\Big)_{\substack{1\le\alpha\le N\\1\le\beta\le M}}$ defined at each $\xi \in \mathbb{R}^n$ as:

$$\mathrm{Sym}(D;\xi) := \mathrm{i}\Big(\sum_{j=1}^n a_j^{\alpha\beta}\xi_j\Big)_{\substack{1\le\alpha\le N\\1\le\beta\le M}} \tag{A.0.192}$$

$*$, the Hodge star operator

$S^{n-1} := \partial B(0,1)$, the unit sphere in $\mathbb{R}^n$

$S^{n-1}_\pm := S^{n-1} \cap \mathbb{R}^n_\pm$, the upper and lower (open) hemispheres

$\operatorname{supp} f$, the support of the μ-measurable function f defined on a topological space (X,τ) (where μ is a Borel measure measure on (X,τ)), defined as (cf. [68, (3.8.7) in Definition 3.8.3]):

$$\operatorname{supp} f := \Big\{x \in X : \int_O |f|\,\mathrm{d}\mu > 0 \text{ for each open set } O \subseteq X \text{ with } x \in O\Big\} \tag{A.0.193}$$

$\mathscr{S}(\mathbb{R}^n)$, the space of (smooth, rapidly decreasing) Schwartz functions in $\mathbb{R}^n$

$\mathscr{S}'(\mathbb{R}^n)$, the space of tempered distributions in $\mathbb{R}^n$

$u_{\star,\theta}$, the solid maximal function of $u : \Omega \to \mathbb{C}$, defined at each point $x \in \Omega$ according to (cf. [68, (6.6.2)]):

$$u_{\star,\theta}(x) := \|u\|_{L^\infty(B(x,\theta\delta_{\partial\Omega}(x)),\mathcal{L}^n)} \in [0,\infty] \tag{A.0.194}$$

$f_p^\#$, the L^p-based Fefferman-Stein sharp maximal function of $f \in L^1_{\mathrm{loc}}(X,\mu)$, defined as (cf. [68, (7.4.110)]):

$$f_p^\#(x) := \sup_{r>0} \Big(⨍_{B_\rho(x,r)} \big|f(y) - f_{B_\rho(x,r)}\big|^p\,\mathrm{d}\mu(y)\Big)^{1/p}, \quad \forall x \in X \tag{A.0.195}$$

$\rho_{\mathrm{inv}}(T;X)$, the spectral radius of $T \in \mathrm{Bd}\,(X)$, with X a quasi-Banach space (cf. [69, Definition 2.2.5]):

$$\rho_{\mathrm{inv}}(T;X) := \inf\Big\{r > 0 : \ zI - T : X \to X \ \text{homeomorphism} \\ \text{for each point } z \in \mathbb{C} \setminus \overline{B(0,r)}\Big\} \tag{A.0.196}$$

$\mathscr{S}$, the boundary-to-domain single layer potential operator for an $M \times M$ second-order weakly elliptic system L (with fundamental solution E) acting on any given function $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\big)\big]^M$ at each point $x \in \Omega$ as in (1.3.6):

$$\mathscr{S} f(x) := \int_{\partial\Omega} E(x-y) f(y)\,\mathrm{d}\sigma(y)$$

$\mathscr{S}_{\mathrm{mod}}$, the boundary-to-domain modified single layer potential operator for an $M \times M$ second-order weakly elliptic system L (with fundamental solution E) acting on any $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\big)\big]^M$ at each point $x \in \Omega$ as in (1.5.50):

$$\mathscr{S}_{\mathrm{mod}} f(x) := \int_{\partial\Omega} \{E(x-y) - E(-y)\cdot \mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}(-y)\} f(y)\,\mathrm{d}\sigma(y)$$

$\mathscr{S}_{\Delta,\mathrm{mod}}$, the boundary-to-domain modified harmonic single layer potential operator acting on any $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ at each point $x \in \Omega$ as in (1.8.53):

$$\mathscr{S}_{\Delta,\mathrm{mod}} f(x) := \int_{\partial\Omega} \big\{E_\Delta(x-y) - E_\Delta(-y)\cdot \mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}(-y)\big\} f(y)\, d\sigma(y)$$

S, the boundary-to-boundary single layer for an $M\times M$ second-order weakly elliptic system L (with canonical fundamental solution E) acting on each given function $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\big)\big]^M$ (when $n \geq 3$) at σ-a.e. point $x \in \partial\Omega$ as in (1.3.62):

$$Sf(x) := \int_{\partial\Omega} E(x-y) f(y)\, d\sigma(y)$$

S_{mod}, the boundary-to-boundary modified single layer potential operator for an $M\times M$ second-order weakly elliptic system L (with fundamental solution E) acting on any given function $f \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)\big]^M$ at σ-a.e. point $x \in \partial\Omega$ as in (1.5.73):

$$S_{\mathrm{mod}} f(x) := \int_{\partial\Omega} \{E(x-y) - E(-y)\cdot \mathbf{1}_{\mathbb{R}^n\setminus B(0,1)}(-y)\} f(y)\, d\sigma(y)$$

$\mathscr{S}$, the boundary-to-domain single layer for the Stokes system acting on any given function $\vec{f} \in \big[L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-2}}\big)\big]^n$ at each point $x \in \Omega$ as in (6.2.12):

$$\mathscr{S}\vec{f}(x) := \int_{\partial\Omega} E(x-y)\vec{f}(y)\, d\sigma(y)$$

where E is the Kelvin matrix-valued fundamental solution of the Stokes system in $\mathbb{R}^n$ (cf. (6.2.1))

$\mathscr{S}_{\alpha,\beta}$, the boundary-to-domain complex single layer potential operator acting on a form $f : \partial_*\Omega \to \Lambda^{\alpha,\beta}\mathbb{C}^n$ at each $z \in \Omega$ as in (7.4.147):

$$\mathscr{S}_{\alpha,\beta} f(z) := \int_{\partial_*\Omega} \big\langle f(\zeta), \Gamma_{\alpha,\beta}(\zeta, z)\big\rangle_{\mathbb{C}}\, d\sigma(\zeta)$$

$S_{\alpha,\beta}$, the boundary-to-boundary complex single layer operator acting on a form $f : \partial_*\Omega \to \Lambda^{\alpha,\beta}\mathbb{C}^n$ at σ-a.e. point $z \in \partial\Omega$ as in (7.4.148):

$$S_{\alpha,\beta} f(z) := \int_{\partial_*\Omega} \big\langle f(\zeta), \Gamma_{\alpha,\beta}(\zeta, z)\big\rangle_{\mathbb{C}}\, d\sigma(\zeta)$$

T

$\mathscr{T}_\gamma(x)$, the family of "bump" (i.e., localized, and normalized in the Hölder norm) functions centered at $x \in \Sigma$ (cf. [69, §4.1])

$\mathrm{Tr}_{\mathbb{R}^n\to\Sigma}$, the trace operator from $\mathbb{R}^n$ to Σ defined for each $u \in W^{1,p}\big(\mathbb{R}^n, \delta_\Sigma^{ap}\mathcal{L}^n\big)$ as the limit (cf. [69, Theorem 8.1.1]):

$$(\mathrm{Tr}_{\mathbb{R}^n\to\Sigma}\, u)(x) := [u]_{\mathbb{R}^n}(x) := \lim_{r\to 0^+} \fint_{B(x,r)} u(y)\, dy \tag{A.0.197}$$

$\mathrm{Tr}_{\Omega\to\partial\Omega}$, the trace operator from the Euclidean space $\mathbb{R}^n$ into the set $\partial\Omega$ defined for each $u \in W_a^{1,p}(\Omega) := W^{1,p}\big(\Omega, \delta_{\partial\Omega}^{ap}\mathcal{L}^n\big)$ as the limit (cf. [69, Theorem 8.3.6]):

$$(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u)(x) := [u]_{\overline{\Omega}}(x) := \lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} u(y)\, dy \tag{A.0.198}$$

$\mathrm{Tr}_{\Omega\to\partial\Omega}$, the trace operator from the open set $\Omega \subseteq \mathbb{R}^n$ to its boundary $\partial\Omega$, acting on each given $u \in A_\alpha^{p,q}(\Omega)$ (and with $w \in A_\alpha^{p,q}(\mathbb{R}^n)$ such that $w\big|_\Omega = u$) according to (cf. [69, Theorem 9.4.5]):

$$\big(\mathrm{Tr}_{\Omega\to\partial\Omega}\, u\big)(x) := \lim_{r\to 0^+} \fint_{B(x,r)} w\, d\mathcal{L}^n \quad \text{at } \sigma\text{-a.e. } x \in \partial\Omega \tag{A.0.199}$$

T_ε, the truncated singular integral operator acting on each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ at any point $x \in \partial\Omega$ as:

$$(T_\varepsilon f)(x) := \int\limits_{y\in\partial\Omega,\ |x-y|>\varepsilon} k(x-y) f(y)\, d\sigma(y) \tag{A.0.200}$$

$T_{\max}$, the maximal operator acting on each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ at any point $x \in \partial\Omega$ as:

$$(T_{\max} f)(x) := \sup_{\varepsilon>0} \big|(T_\varepsilon f)(x)\big| \tag{A.0.201}$$

$T_{\rm mod}$, the modified principal-value singular integral operator acting on each function $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^n}\big)$ at σ-a.e. point in Σ as:

$$T_{\rm mod} f := \lim_{\varepsilon\to 0^+} \int\limits_\Sigma \big\{k_\varepsilon(\cdot - y) - k_1(-y)\big\} f(y)\, d\sigma(y) \tag{A.0.202}$$

where $k_\varepsilon := k \cdot \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(0,\varepsilon)}}$ for each $\varepsilon > 0$

$\mathcal{T}_{\rm mod}$, the boundary-to-domain modified integral operator, acting on any function $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^n}\big)$ at each point $x \in \Omega$ as:

$$(\mathcal{T}_{\rm mod} f)(x) := \int_{\partial\Omega} \big\{k(x-y) - k_1(-y)\big\} f(y)\, d\sigma(y) \tag{A.0.203}$$

where $k_1 := k \cdot \mathbf{1}_{\mathbb{R}^n\setminus\overline{B(1,0)}}$

$\vartheta := - * \partial *$ and $\bar{\vartheta} := - * \bar{\partial} *$ (cf. (7.2.10))

U

UR set (cf. [68, Definition 5.10.1]): a closed set $\Sigma \subset \mathbb{R}^n$ which is (upper) Ahlfors regular and has Big Pieces of Lipschitz Images (in a uniform, quantitative, scale-invariant fashion)

UR domain (cf. [68, Definition 5.10.6]): a nonempty open subset Ω of $\mathbb{R}^n$ such that $\partial\Omega$ is a UR set and

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0 \tag{A.0.204}$$

$\mathrm{UC}(X, \rho)$, the space of uniformly continuous functions on the metric space (X, ρ)

$[u]^A_\infty$, the contribution at infinity of a null-solution u for the vector-Helmholtz operator, associated with the coefficient tensor $A := \big(a^{JI}_{rs}\big)_{\substack{1\leq J,I\leq M\\ 1\leq r,s\leq n}}$ in the writing of the vector Laplacian as $\Delta = \mathrm{div} A\nabla$, defined at each $x \in \mathbb{R}^n$ as:

$$\begin{aligned}[u]^A_\infty := \lim_{R\to\infty}\Big(\int_{|y|=R} \widehat{y}_s\big\{a^{IJ}_{rs}(\partial_r\Phi_k)(x-y)u_J(y)\\ + a^{IJ}_{sr}\Phi_k(x-y)(\partial_r u_J)(y)\big\}\,\mathrm{d}\mathcal{H}^{n-1}(x)\Big)_{1\leq I\leq M}\end{aligned} \tag{A.0.205}$$

where Φ_k is the radiating fundamental solution of the Helmholtz operator $\Delta + k^2$

$u \otimes w$, the tensor product of differential forms (cf. (7.3.4))

$U \sqcup V$, the union of two disjoint sets U, V

V

$\mathrm{VMO}(X, \mu)$, the Sarason space of functions of vanishing mean oscillations on the measure metric space (X, ρ, μ) defined (with $\mathrm{UC}(X, \rho)$ denoting the space of uniformly continuous functions on (X, ρ)) as the space (cf. [69, §3.1]):

$$\mathrm{VMO}(X,\mu) := \text{the closure of } \mathrm{UC}(X,\rho)\cap\mathrm{BMO}(X,\mu) \text{ in } \mathrm{BMO}(X,\mu) \tag{A.0.206}$$

$\mathrm{VMO}_{-1}(\partial\Omega, \sigma)$, the VMO-based negative Sobolev space on $\partial\Omega$, defined for $n \geq 3$ as [69, Definition 11.10.9]):

$$\mathrm{VMO}_{-1}(\partial\Omega,\sigma) := \text{ the closure of } L^{n-1}(\partial\Omega,\sigma) \text{ in } \mathrm{BMO}_{-1}(\partial\Omega,\sigma) \tag{A.0.207}$$

$\mathrm{VMO}_{-1}(\partial\Omega, \sigma)$ in the two-dimensional setting defined as the closure of $L^q(\partial\Omega, \sigma)$ in $\mathrm{BMO}_{-1}(\partial\Omega, \sigma)$, for any $q \in (1, \infty)$ (2.3.134)

$\vee$, the interior product of differential forms (7.1.52)

W

weakly elliptic system: a system L whose characteristic matrix has the property that $\det[L(\xi)] \neq 0$ for each $\xi \in \mathbb{R}^n \setminus \{0\}$

weakly elliptic coefficient tensor: a coefficient tensor A with the property that the canonically associated second-order system L_A is weakly elliptic, i.e., such that $\det[L_A(\xi)] \neq 0$ for each $\xi \in \mathbb{R}^n \setminus \{0\}$

$W^{k,p}(\Omega)$, the L^p-based Sobolev space of order k in Ω (intrinsically defined)

$W^{k,p}_{\rm loc}(\Omega)$, the local L^p-based Sobolev space of order k in Ω

$W^{k,p}_{\rm bdd}(\Omega)$, the space of Sobolev functions on any bounded measurable subset of Ω (cf. [68, (3.0.4)]):

$$\begin{aligned}&W^{k,p}_{\rm bdd}(\Omega) \text{ denote the space of functions } u \in W^{k,p}_{\rm loc}(\Omega) \text{ with the property}\\ &\text{that } \partial^\alpha u \in L^p(\mathcal{O},\mathcal{L}^n) \text{ for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k \text{ and each bounded}\\ &\text{Lebesgue measurable subset } \mathcal{O} \text{ of } \Omega.\end{aligned} \tag{A.0.208}$$

$W^{k,p}(\mathbb{R}^n, w\mathcal{L}^n)$, the weighted Sobolev space in $\mathbb{R}^n$, defined for $p \in (0,\infty)$, $k \in \mathbb{N}_0$ and a generic weight w in $\mathbb{R}^n$ as (cf. [69, §8.1]):

$$W^{k,p}(\mathbb{R}^n, w\mathcal{L}^n) := \Big\{ f \in W^{k,1}_{\rm loc}(\mathbb{R}^n) : \|f\|_{W^{k,p}(\mathbb{R}^n, w\mathcal{L}^n)} < +\infty \Big\} \tag{A.0.209}$$

where

$$\|f\|_{W^{k,p}(\mathbb{R}^n, w\mathcal{L}^n)} := \sum_{|\beta|\leq k} \Big(\int_{\mathbb{R}^n} |\partial^\beta f|^p w \, d\mathcal{L}^n \Big)^{1/p} \tag{A.0.210}$$

$W^{k,p}_a(\Omega)$, the weighted Sobolev space in Ω defined as in [69, (8.3.5)] for the weight $w := \delta^{ap}_{\partial\Omega}$, with $k \in \mathbb{N}_0$, $p \in (0,\infty)$, $a \in \mathbb{R}$, and equipped with the quasi-norm (cf. [69, Definition 8.3.4]):

$$\|u\|_{W^{k,p}_a(\Omega)} := \sum_{|\alpha|\leq k} \Big(\int_\Omega |(\partial^\alpha u)(x)|^p \delta_{\partial\Omega}(x)^{ap} \, dx \Big)^{1/p} \tag{A.0.211}$$

$\mathring{W}^{1,p}_a(\Omega)$, the weighted Sobolev space in the open set Ω, defined for $p \in (1,\infty)$ and $a \in (-1/p, 1-1/p)$ as (cf. [69, (8.3.65)]):

$$\mathring{W}^{1,p}_a(\Omega) := \text{ the closure of } \mathscr{C}^\infty_c(\Omega) \text{ in } W^{1,p}_a(\Omega) \tag{A.0.212}$$

$W^{-1,p}_a(\Omega)$, the negative weighted Sobolev space in Ω, defined for $p \in (1,\infty)$ and $a \in (-1/p, 1-1/p)$ as (cf. [69, (8.5.1)-(8.5.2)]):

$$W^{-1,p}_a(\Omega) := \Big\{ f = f_0 + \sum_{j=1}^n \partial_j f_j \in \mathcal{D}'(\Omega) : f_j \in L^p\big(\Omega, \delta^{ap}_{\partial\Omega}\mathcal{L}^n\big), \ 0 \leq j \leq n \Big\} \tag{A.0.213}$$

equipped with the norm

$$W_a^{-1,p}(\Omega) \ni f \mapsto \inf\Big\{ \sum_{j=0}^{n+1} \|f_j\|_{L^p(\Omega,\delta_{\partial\Omega}^{ap}\mathcal{L}^n)} : f = f_0 + \sum_{j=1}^{n} \partial_j f_j, \qquad \text{(A.0.214)}$$
$$f_j \in L^p\big(\Omega, \delta_{\partial\Omega}^{ap}\mathcal{L}^n\big),\ 0 \le j \le n\Big\}$$

$W^{k,p}_{a,\star}(\Omega)$, the weighted maximal Sobolev space in Ω, defined for $p \in (0,\infty)$, $k \in \mathbb{N}_0$, and $a \in \mathbb{R}$ as (cf. [69, Definition 8.6.1]):

$$W^{k,p}_{a,\star}(\Omega) := \Big\{u \in W^{k,1}_{\rm loc}(\Omega) : \partial^\alpha u \in L^p_\star\big(\Omega, \delta^{ap}_{\partial\Omega}\mathcal{L}^n\big) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \le k\Big\} \tag{A.0.215}$$

equipped with the quasi-norm

$$\|u\|_{W^{k,p}_{a,\star}(\Omega)} := \sum_{|\alpha|\le k} \|\partial^\alpha u\|_{L^p_\star(\Omega,\delta^{ap}_{\partial\Omega}\mathcal{L}^n)}$$
$$\approx \sum_{|\alpha|\le k} \Big(\int_\Omega \big|(\partial^\alpha u)_{\star,\theta}\big|^p \delta^{ap}_{\partial\Omega}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}} \text{ for } \theta \in (0,1) \tag{A.0.216}$$

$\mathscr{W}^{1,p}_a(\Omega; D)$, the weighted Sobolev-Hardy space of null-solutions of the Dirac operator $D = \sum_{j=1}^{n} \mathbf{e}_j \odot \partial_j$ in an open set $\Omega \subseteq \mathbb{R}^n$ defined for $p \in (0,\infty)$ and $a \in \big(-\frac{1}{p}, 1-\frac{1}{p}\big)$ as in (4.4.131):

$$\mathscr{W}^{1,p}_a(\Omega; D) := \big\{u \in W^{1,p}_a(\Omega) \otimes C\ell_n : Du = 0 \text{ in } \Omega\big\}$$

X

${}_{X^*}\big(\cdot,\cdot\big)_X$, the duality pairing between a vector space X and its algebraic dual X^*

$\mathbb{X}'$, the associated space (aka Köthe dual) of the Generalized Banach Function Space $\mathbb{X}$, equipped with the norm $\|\cdot\|_{\mathbb{X}'}$ (cf. [69, Definition 5.1.11])

$|\!|\!| x |\!|\!|_p := |\!|\!| \cdot |\!|\!|_{X,p} := \inf\big\{\lambda > 0 : \lambda^{-1}x \in \big\langle B_X(0,1)\big\rangle_p\big\}$ for all $x \in X$, the Minkowski functional associated with the absolutely p-convex hull of the unit ball in X (cf. [69, (7.8.6)])

$\mathscr{X}_{\rm bdd}(\Omega)$, or $\mathscr{X}(\Omega)_{\rm bdd}$, the space of distributions in the open set $\Omega \subseteq \mathbb{R}^n$ defined as (cf. [69, Convention 8.3.7]):

$$\Big\{u \in \mathcal{D}'(\Omega) : \big(\psi\big|_\Omega\big)u \in \mathscr{X}(\Omega) \text{ for each } \psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)\Big\} \tag{A.0.217}$$

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