Developments in Mathematics

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Geometric Harmonic Analysis I

A Sharp Divergence Theorem with Nontangential Pointwise Traces

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Aims and Scope

The **Developments in Mathematics** (DEVM) book series is devoted to publishing well-written monographs within the broad spectrum of pure and applied mathematics. Ideally, each book should be self-contained and fairly comprehensive in treating a particular subject. Topics in the forefront of mathematical research that present new results and/or a unique and engaging approach with a potential relationship to other fields are most welcome. High-quality edited volumes conveying current state-of-the-art research will occasionally also be considered for publication. The DEVM series appeals to a variety of audiences including researchers, postdocs, and advanced graduate students.

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Dedicated with love to our parents

Prefacing the Full Series

The current work is part of a series, comprised of five volumes. In broad terms, the principal aim is to develop tools in Real and Harmonic Analysis, of geometric measure theoretic flavor, capable of treating a broad spectrum of boundary value problems formulated in rather general geometric and analytic settings.

In Volume I we establish a sharp version of Divergence Theorem (aka Fundamental Theorem of Calculus) which allows for an inclusive class of vector fields whose boundary trace is only assumed to exist in a nontangential pointwise sense.

Volume II is concerned with function spaces measuring size and/or smoothness, such as Hardy spaces, Besov spaces, Triebel–Lizorkin spaces, Sobolev spaces, Morrey spaces, Morrey–Campanato spaces, and spaces of functions of Bounded Mean Oscillations, in general geometric settings. Work here also highlights the close interplay between differentiability properties of functions and singular integral operators.

The topic of singular integral operators is properly considered in Volume III, where we develop a versatile Calderón–Zygmund theory for singular integral operators of convolution type (and with variable coefficient kernels) on uniformly rectifiable sets in the Euclidean ambient, and the setting of Riemannian manifolds. Applications to scattering by rough obstacles are also discussed in this volume.

In Volume IV we focus on singular integral operators of boundary layer type which enjoy more specialized properties (compared with generic, garden variety singular integral operators treated earlier in Volume III). Applications to Complex Analysis in several variables are subsequently presented, starting from the realizations that many natural integral operators in this setting, such as the Bochner–Martinelli operator, are actual particular cases of double layer potential operators associated with the complex Laplacian.

In Volume V, where everything comes together, finer estimates for a certain class of singular integral operators (of chord-dot-normal type) are produced in a manner which indicates how their size is affected by the (infinitesimal and global) flatness of the "surfaces" on which they are defined. Among the library of double layer potential operators associated with a given second-order system, we then identify those double layers which fall under this category of singular integral operators. It is precisely for this subclass of double layer potentials that Fredholm theory may then be implemented assuming the underlying domain has a compact boundary, which is sufficiently flat at infinitesimal scales. For domains with unbounded boundaries, this very category of double layer potentials may be outright inverted, using a Neumann series argument, assuming the "surface" in question is sufficiently flat globally. In turn, this opens the door for solving a large variety of boundary value problems for second-order systems (involving boundary data from Muckenhoupt weighted Lebesgue spaces, Lorentz spaces, Hardy spaces, Sobolev spaces, BMO, VMO, Morrey spaces, Hölder spaces, etc.) in a large class of domains which, for example, are allowed to have spiral singularities (hence more general than domains locally described as upper-graphs of functions). In the opposite direction, we show that the boundary value problems formulated for systems lacking such special layer potentials may fail to be Fredholm solvable even for really tame domains, like the upper half-space, or the unit disk. Save for the announcement [184], all principal results appear here in print for the first time.

We close with a short epilogue, attempting to place the work undertaken in this series into a broader picture. The main goal is to develop machinery of geometric harmonic analysis flavor capable of ultimately dealing with boundary value problems of a very general nature. One of the principal tools (indeed, the piecè de résistance) in this regard is a new and powerful version of the Divergence Theorem, devised in Volume I, whose very formulation has been motivated and shaped from the outset by its eventual applications to Harmonic Analysis, Partial Differential Equations, Potential Theory, and Complex Analysis. The fact that its footprints may be clearly recognized in the makeup of such a diverse body of results, as presented in Volumes II–V, serves as a testament to the versatility and potency of our brand of Divergence Theorem. Alas, our enterprise is multifaceted, so its success is crucially dependent on many other factors. For one thing, it is necessary to develop a robust Calderón–Zygmund theory for singular integrals of boundary layer type (as we do in Volumes III–IV), associated with generic weakly elliptic systems, capable of accommodating a large variety of function spaces of interest considered in rather inclusive geometric settings (of the sort discussed in Volume II). This renders these (boundary-to-domain) layer potentials useful mechanisms for generating lots of null-solutions for the given system of partial differential operators, whose format is compatible with the demands in the very formulation of the boundary value problem we seek to solve. Next, in order to be able to solve the boundary integral equation to which matters are reduced in this fashion, the success of employing Fredholm theory hinges on the ability to suitably estimate the essential norms of the (boundary-to-boundary) layer potentials. In this vein, we succeed in relating the distance from such layer potentials to the space of compact operators to the flatness of the boundary of the domain in question (measured in terms of infinitesimal mean oscillations of the unit normal) in a desirable manner which shows that, in a precise quantitative fashion, *the flatter the domain, the smaller the proximity to compact operators*. This subtle and powerful result, bridging between analysis and geometry, may be regarded as a far-reaching extension of the pioneering work of Radon and Carleman in the early 1900s.

Ultimately, our work aligns itself with the program stemming from A. P. Calderón's 1978 ICM plenary address in which he advocates the use of layer potentials "*for much more general elliptic systems* [*than the Laplacian*]", see [36, p. 90], and may be regarded as an optimal extension of the pioneering work of E. B. Fabes, M. Jodeit, and N. M. Rivière in [81] (where layer potential methods have been first used to solve boundary value problems for the Laplacian in bounded $\mathscr{C}^1$ domains). In this endeavor, we have been also motivated by the problem[1] posed by A. P. Calderón on [36, p. 95], asking to identify the function spaces on which singular integral operators (of boundary layer type) are well-defined and continuous. This is relevant since, as Calderón mentions, "*A clarification of this question would be very important in the study of boundary value problems for elliptic equations* [*in rough domains*]. *The methods employed so far seem to be insufficient for the treatment of these problems.*" We also wish to mention that our work is also in line with the issue raised as an open problem by C. Kenig in [147, Problem 3.2.2, pp. 116–117], where he asked whether operators of layer potential type may be inverted on appropriate Lebesgue and Sobolev spaces in suitable subclasses on NTA domains with compact Ahlfors regular boundaries.

The task of making geometry and analysis work in unison is fraught with difficulties, and only seldom can a two-way street be built on which to move between these two worlds without loss of information. Given this, it is actually surprising that in many instances we come very close to having optimal hypotheses, almost an accurate embodiment of the slogan *if it makes sense to write it, then it's true.*

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[1] In the last section of [36], simply titled "*Problems*," Calderón singles two directions for further study. The first one is the famous question whether the smallness condition on $\|a'\|_{L^\infty}$ (the Lipschitz constant of the curve $\{(x, a(x)) : x \in \mathbb{R}\}$ on which he proved the L^2-boundedness of the Cauchy operator) may be removed (as is well known, this has been solved in the affirmative by Coifman, McIntosh, and Meyer in [53]). We are referring here to the second (and final) problem formulated by Calderón on [36, p. 95].

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Description of Volume I

What sort of analysis can a certain geometric environment support[2]? What kind of geometry is required to ensure the veracity of a specific analytical result[3]? This series, comprised of five volumes, is a journey into Geometric Harmonic Analysis, a brand of harmonic analysis[4] of definite geometric flavor, whose ultimate goal is to build the necessary machinery capable of dealing with problems involving Partial Differential Equations in very general settings. The linchpin of this enterprise is a new, powerful and adaptable, higher-dimensional version of the Fundamental Theorem of Calculus.

If we were to summarize the key message of Volume I in just a few words, it would simply read:

it's time to redefine what the Divergence Theorem can do!

It seems faintly miraculous that after three centuries of being in the limelight this remains an active area of research, but the reality of the matter is that this is a demand-driven subject. Indeed, progress in a certain field often requires yet a new, and ever more potent, brand of Divergence Theorem, which can accommodate certain specific features.

For example, to deal with Plateau's problem, R. Caccioppoli and E. De Giorgi have introduced the class of sets of locally finite perimeter, an environment in which E. De Giorgi and H. Federer have subsequently produced a magnificent version of the Divergence Theorem. Alas, the class of vector fields to which the De Giorgi–Federer Divergence Theorem applies, smooth and compactly supported in the entire Euclidean space (in particular, completely unrelated to the original domain), is far too

[2] Is there a Hardy–Littlewood maximal inequality, a Poincaré inequality, a Fundamental Theorem of Calculus, a rich function space theory, etc.? Also, in a given setting, how can one measure smoothness of functions, what sort of operators are natural to consider, what type of boundary value problems are well-posed or Fredholm solvable, etc.?

[3] For example, one may seek geometrical conditions guaranteeing that certain singular integral operators (of boundary layer type) are bounded, or Fredholm, or invertible, on a variety of function spaces of interest.

[4] Classically understood as the breaking up of a whole into its parts as to elucidate their nature.

small for many other applications. For one thing, the very formulation of the Divergence Theorem in a given domain suggests it is unnatural to ask that the vector field in question is defined outside said domain. It is also apparent that vector fields used in the formulation of the Divergence Theorem should be allowed to have certain types of singularities. Since every college student who has completed the basic calculus sequence has learned about the Divergence Theorem, it is worth recalling a common experience in which the Divergence Theorem could not be applied directly due to a point-singularity for the given vector field. The only route available was to excise a small ball centered at the singularity, apply the Divergence Theorem in the resulting domain, then pass to limit as the radius of the ball shrinks to zero. However, this feels like reproving each time what the Divergence Theorem should do automatically, in the first place! Thus, as far as regularity is concerned, there is no compelling reason to assume continuity (as it turns out, not even in an almost everywhere sense) for the vector field involved in the formulation of the Divergence Theorem. Among other things, this renders the issue of defining the trace of the vector field on the boundary of the domain delicate.

In the first volume of this series, we produce a version of the Divergence Theorem for vector fields which may lack any type of continuity and for which the boundary trace is taken in a strong, nontangential pointwise fashion. The rationale for insisting on the latter feature is properly documented in the last part of Sect. 1.1, where this is traced back to classical results in Harmonic Analysis, Complex Analysis, Partial Differential Equations, and Potential Theory (specifically, the theory of Hardy spaces, Fatou-type theorems, boundary value problems, and Calderón–Zygmund theory for singular integral operators, among others). In turn, this innate affinity with the design of such basic results makes our brand of Divergence Theorem an effective tool in dealing with problems in these areas of mathematics. In short, the very formulation of our versions of the Divergence Theorem has been motivated and shaped from the outset by potential applications. Indeed, we have envisioned the versions of the Divergence Theorem described in Sect. 1 not as end-products, in and of themselves, but as effective tools to further progress in Harmonic Analysis, Partial Differential Equations, Potential Theory, and Complex Analysis. Subsequently, in Volumes II–V we elaborate on a wide spectrum of applications. The sheer magnitude of this portion of our work serves as a testament to the versatility of the Divergence Theorems established here[5]. Except for the announcement [184], all principal results (and proofs) appear here in print for the first time.

Volume I is made up of Chaps. 1–9, in addition to the present overview. Chapter 1 debuts by recalling, in Sect. 1.1, the De Giorgi–Federer version of the Divergence Theorem and a statement of the goals in Volume I. Our main results concerning the Divergence Theorem are given in Sects. 1.2–1.12. First, in progressively more generality, they treat the case when the divergence of the vector field is absolutely integrable (Sect. 1.2), the case when no decay conditions at infinity are imposed and

[5] Certain mathematical constructs reach rather far from their origins, and this is particularly true of the machinery surrounding the Fundamental Theorem of Calculus (recall the popular quip “when all else fails, integrate by parts”).

when the divergence of the vector field (considered in the sense of distributions) is actually a measure (Sect. 1.3), the scenario in which the vector field lacks local integrability on a compact subset of the domain and when no decay conditions are assumed (Sect. 1.4), the situation in which the surface measure of the domain may fail to be doubling and the vector field is, in our terminology, maximally singular (Sect. 1.5) and, finally, Divergence Formulas in domains whose boundaries may lack lower Ahlfors regularity (Sect. 1.6). We then proceed to discuss integration by parts formulas in open sets with Ahlfors regular boundaries, first for partial differential operators of first order in Sect. 1.7, then for those of higher order in Sect. 1.8. In Sects. 1.9–1.10 we present versions of the Divergence Theorem exhibiting other specialized features, such as the involvement of weak boundary traces, and the averaged nontangential maximal operator, respectively. The setting of Riemannian manifolds is treated in Sect. 1.11, where we state a sharp version of Stokes' formula, and in Sect. 1.12, where we prove a plethora of integration by parts formulas on boundaries of Ahlfors regular domains on manifolds.

Chapter 2 is largely reserved for discussing examples and counterexamples (Sects. 2.1–2.7), as well as other versions of the Divergence Theorem (Sect. 2.8). All details in the statements of our main results in Chap. 1 have their own significance, and the examples/counterexamples presented in Sects. 2.1–2.7 are designed to elucidate their specific nature. The picture that emerges is that our results are optimal[6] from a multitude of (geometric and analytic) points of view. In particular, in stark contrast with the classical De Giorgi–Federer Divergence Theorem, our simplest version of the Divergence Theorem already contains the sharp version of the Fundamental Theorem of Calculus, to which it precisely reduces in the one-dimensional setting.

In Chap. 3 we review useful basic notions of measure and topology, including the sigma-algebra of Borelian sets in a topological space, abstract measures and outer measure, inner, outer, and Borel regularity, the support of a measure, special classes of measures (Radon measures, complex Borel measures, separable measures), density and separability results for Lebesgue spaces, the topology on the space of measurable functions, and Riesz's Representation Theorem.

Chapter 4 contains a variety of selected topics from (or inspired by) distribution theory. Specifically, in Sect. 4.6 we study the algebraic dual of the space of functions which are simultaneously smooth and bounded in an open subset of the Euclidean ambient. In Sect. 4.1, we develop a brand of distribution theory on arbitrary subsets of $\mathbb{R}^n$, taking Lipschitz functions with bounded support as test functions. Next, in Sect. 4.2 we define and study what we call the "bullet product" which, in essence, is a weak version (modeled upon integration by parts) of the inner product of the normal vector to a domain with a given vector field satisfying only some very mild integrability properties in that domain. In Sect. 4.3 we provide a proof of Leibniz's product rule for weak derivatives, while in Sect. 4.4 we compare the divergence of

[6] One may argue that "optimality of a mathematical result" becomes a phrase of indefinite meaning in the absence of a concrete and clear identification of the sense in which said optimality is to be understood.

a differentiable vector field, taken in a pointwise sense, to its distributional divergence. Also, Sect. 4.5 contains a discussion pertaining to the issue of removability of singularities for distributional derivatives. Chapter 4 ends with a presentation in Sect. 4.7 of what we call the contribution at infinity of a vector field.

In Chap. 5 we then proceed to discuss basic results from Geometric Measure Theory, such as thick sets, the corkscrew condition, the geometric measure theoretic boundary, area and coarea formulas, countable rectifiability, approximate tangent planes, functions of bounded variation, sets of locally finite perimeter, Ahlfors regularity, uniformly rectifiable (UR) sets, and nontangentially accessible (NTA) domains. This body of facts is further augmented in Chap. 6, where we collect and develop tools from Harmonic Analysis which are relevant to our work. Concretely, in Sect. 6.1 we discuss the regularized distance function and Whitney's Extension Theorem, while Sect. 6.2 amounts to a brief survey of Lorentz spaces in generic measure spaces. Also, in Sect. 6.3 we introduce and study the fractional Hardy–Littlewood maximal operator in a very general, non-metric setting. Next, in Sect. 6.4 we review the setting of Clifford algebras. These are higher-dimensional versions of the field of complex numbers, that happen to be highly non-commutative, in which a brand of complex analysis may be developed (for example, there is a natural version in this setting of the classical Cauchy integral operator from the complex plane). We shall put the Clifford algebra machinery to good use later, in Volumes III–IV, to build a Calderón–Zygmund theory for singular integral operators on uniformly rectifiable sets. The discussion in Sect. 6.5 pertains to subaveraging functions, reverse Hölder estimates, and interior estimates. Finally, in Sect. 6.6 we introduce and study the solid maximal function introduced and maximal Lebesgue spaces.

Chapter 7 is concerned with analysis on quasi-metric spaces and on spaces of homogeneous type, such as a sharp metrization theorem, dyadic and Whitney decompositions, functions of bounded mean oscillations, a sharp version of Lebesgue's Differentiation Theorem, Muckenhoupt weights, and the Fractional Integration Theorem. The Hardy–Littlewood maximal operator, which has first been discussed in Sect. 6.3 in a non-metric setting, is also revisited in Sect. 7 in the framework of spaces of homogeneous type.

Chapter 8 contains the main technology behind our principal results pertaining to the Divergence Theorem from Sect. 1. Among the technical tools required in this endeavor, certain off-diagonal Carleson measure estimates of reverse Hölder type, which we formulate and prove in Sect. 8.6, feature prominently. In turn, these estimates use information on the nontangential maximal operator developed in Sects. 8.1–8.4 (as well as its version from Sect. 8.10), and the solid maximal function introduced and studied in Sect. 6.6. Other key players in this regard are the notion of nontangentially accessible boundary which we define and study in Sect. 8.8, and the pointwise nontangential boundary trace operator considered at length in Sect. 8.9.

Chapter 9 is exclusively reserved for presenting the proofs of the main results concerning the Divergence Theorem, formulated in Sect. 1.

Contents

Compendium of Notation Used in Volume I

A

$A_\kappa(\partial\Omega) := \big\{x \in \partial\Omega : x \in \overline{\Gamma_\kappa(x)}\big\}$ the subset of $\partial\Omega$ which is κ-accessible from within Ω (8.8.2)

$[w]_{A_p} := \sup\limits_{B\ \rho\text{-ball}} \Big(⨍_B w\,\mathrm{d}\mu\Big)\Big(⨍_B w^{-1/(p-1)}\,\mathrm{d}\mu\Big)^{p-1}$ the characteristic of the weight w on a space of homogeneous type (X, ρ, μ) (7.7.2)

$A_p(X, \rho, \mu) := \big\{w \text{ weight function} : [w]_{A_p} < \infty\big\}$ the Muckenhoupt A_p-class on a space of homogeneous type (X, ρ, μ) (7.7.2)

$A_\infty(X, \rho, \mu) := \bigcup\limits_{1\le p<\infty} A_p(X, \mu)$ the Muckenhoupt A_∞-class on a space of homogeneous type (X, ρ, μ) (7.7.9)

B

$B_\rho(x, r) := \{y \in X : \rho(x, y) < r\}$ the ρ-ball with center at $x \in X$ and radius $r > 0$ in the quasi-metric space (X, ρ) (7.1.5)

$B_{n-1}(x', r) := \big\{y' \in \mathbb{R}^{n-1} : |y' - x'| < r\big\}$ the $(n-1)$-dimensional ball in $\mathbb{R}^{n-1}$ centered at $x' \in \mathbb{R}^{n-1}$ and of radius $r \in (0, \infty)$

BMO_1 is the BMO-based Sobolev spaces of order one

$\|f\|_*(\Delta) := \sup\limits_{\Delta'\subseteq\Delta} ⨍_{\Delta'} |f - f_{\Delta'}|\,\mathrm{d}\mu$ local BMO norm of f on the surface ball Δ (7.4.60)

$\|f\|_{\dot{\mathrm{BMO}}(X,\mu)} := \sup\limits_{x\in X,\, r>0} ⨍_{B_\rho(x,r)} \big|f - f_{B_\rho(x,r)}\big|\,\mathrm{d}\mu$ homogeneous BMO semi-norm of f in the context of a space of homogeneous type (X, ρ, μ) (7.4.70)

$\|\cdot\|_{\mathrm{BMO}(X,\mu)}$ inhomogeneous BMO "norm" in the context of a space of homogeneous type (X, ρ, μ) (7.4.81)

$\mathrm{BMO}\big(X, \mu\big) := \big\{f \in L^1_{\mathrm{loc}}(X, \mu) : \|f\|_{\mathrm{BMO}(X,\mu)} < +\infty\big\}$ the space of functions of bounded mean oscillations for a space of homogeneous type (X, ρ, μ) (7.4.92)

$\widetilde{BMO}(X, \mu) := \mathrm{BMO}(X, \mu)\big/\sim = \big\{[f] : f \in \mathrm{BMO}(X, \mu)\big\}$ the space BMO modulo constants for a space of homogeneous type (X, ρ, μ) (7.4.96)

∂E the topological boundary of the set E

$Borel_\tau(X)$ Borelians of the topological space (X, τ)
$\mathrm{BL}(\Sigma)$ bounded Lipschitz functions defined on the set Σ
$\mathrm{BV}(\mathcal{O})$ the space of functions of bounded variation in the set $\mathcal{O}$ (5.5.5)
$\mathrm{BV}_{\mathrm{loc}}(\mathcal{O})$ the space of functions of locally bounded variation in $\mathcal{O}$ (5.5.6)
$[A; B] := [A, B] := AB - BA$ the commutator of A and B
$\{A; B\} := AB + BA$ the anti-commutator of A and B

C

$\mathscr{C}^k(\overline{\Omega})$ functions of class $\mathscr{C}^k$ in an open neighborhood of $\overline{\Omega}$
$\mathscr{C}^k_c(\Omega)$ functions of class $\mathscr{C}^k$ with compact support in the open set Ω
$\mathscr{C}^k_b(\Omega)$ bounded functions of class $\mathscr{C}^k$ in Ω
$\big(\mathscr{C}^\infty_b(\Omega)\big)^*$ the algebraic dual of $\mathscr{C}^\infty_b(\Omega)$
$\mathrm{CBM}(\Omega)$ complex Borel measures in the open set $\Omega \subseteq \mathbb{R}^n$
$\mathrm{CBM}(X, \tau)$ complex Borel measures in the topological space (X, τ)
$\overline{U}$ closure of the set $U \subseteq \mathbb{R}^n$
$\mathrm{C}_{\theta,b}(x, h) := \{y \in \mathbb{R}^n : \cos(\theta/2)\,|y - x| < (y - x)\cdot h < b\}$ cone with vertex at $x \in \mathbb{R}^n$, symmetry axis along $h \in S^{n-1}$ and full aperture $\theta \in (0, \pi)$ (5.6.93)
$C^{\mathrm{alt}}_{\max}$ the maximal "altered" Cauchy integral operator (5.9.27)
$\mathfrak{C}_{\max}$ the maximal Cauchy–Clifford integral operator (5.10.12)
$\mathfrak{C}_\varepsilon$ the truncated Cauchy–Clifford integral operator (5.10.13)
$\mathfrak{C}$ the boundary-to-boundary Cauchy–Clifford integral operator (5.10.14)
$\mathcal{C}\ell_n = (\mathcal{C}\ell_n, +, \odot)$ Clifford algebra generated by n imaginary units Sect. 6.4
C_ρ triangle inequality "penalty" constant associated with the quasi-distance ρ (7.1.3)
$\widetilde{C}_\rho$ symmetry "penalty" constant associated with the quasi-distance ρ (7.1.4)
$\dot{\mathscr{C}}^\alpha(U, \rho)$ homogeneous Hölder space of order $\alpha > 0$ in the set $U \subseteq X$, in the context of a quasi-metric space (X, ρ) (7.3.1)
$\|\cdot\|_{\dot{\mathscr{C}}^\alpha(U,\rho)}$ homogeneous Hölder space semi-norm of order $\alpha > 0$ in the set $U \subseteq X$, in the context of a quasi-metric space (X, ρ) (7.3.2)
$\dot{\mathscr{C}}^\alpha(U, \rho)/\sim$ homogeneous Hölder space of order $\alpha > 0$ modulo constants, in the set $U \subseteq X$, in the context of a quasi-metric space (X, ρ) (7.3.6)
$\dot{\mathscr{C}}^\alpha_{\mathrm{loc}}(U, \rho)$ local homogeneous Hölder space of order $\alpha > 0$ in the set $U \subseteq X$, in the context of a quasi-metric space (X, ρ) (7.3.7)
$\mathscr{C}^\alpha(U, \rho)$ inhomogeneous Hölder space of order $\alpha > 0$ in the set $U \subseteq X$, in the context of a quasi-metric space (X, ρ) (7.3.19)
$\|\cdot\|_{\mathscr{C}^\alpha(U,\rho)}$ inhomogeneous Hölder space norm of order $\alpha > 0$ in the set $U \subseteq X$, in the context of a quasi-metric space (X, ρ) (7.3.20)
$\mathscr{C}^\alpha_c(U, \rho)$ Hölder functions of order $\alpha > 0$ with ρ-bounded support in the set $U \subseteq X$, in the context of a quasi-metric space (X, ρ) (7.3.26)

D

$u \cdot w = \langle u, w\rangle$ dot product of two vectors $u, w \in \mathbb{R}^n$
$\mathrm{div}\vec{F}$ the divergence of the vector field $\vec{F}$
div_g differential geometric divergence (associated with the metric tensor g)
$\mathcal{D}'(\Omega)$ space of distributions in the open set Ω

${}_{\mathcal{D}'(\Omega)}\langle\cdot,\cdot\rangle_{\mathcal{D}(\Omega)}$ distributional pairing in Ω
$\Delta := \partial_1^2 + \cdots + \partial_n^2$ the Laplace operator in $\mathbb{R}^n$
δ_x Dirac distribution with mass at x
$D^\top$ (real) transpose of the first-order system D
$\overline{D}$ complex conjugate of the first-order system D
D^* Hermitian adjoint of the first-order system D
$d_g(x, y)$ geodesic distance (induced by the metric tensor g) between x and y
$\mathrm{d}u := \sum_{j=1}^n \sum_J \frac{\partial u_J}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x^J$ exterior derivative operator acting on the differential form $u = \sum_J u_J \mathrm{d}x^J$ (1.11.32) (see also (6.4.140)–(6.4.141) for the Clifford algebra context)
δ formal adjoint of the exterior derivative operator d on differential forms (see also (6.4.142) for the Clifford algebra context)
δ_{jk} Kronecker symbol, i.e., $\delta_{jk} := 1$ if $j = k$ and $\delta_{jk} := 0$ if $j \neq k$
Dist $[E, F]$ Pompeiu–Hausdorff distance between E and F (2.8.131)
$\delta_F(\cdot)$ distance function to the set F
$\delta_{\partial\Omega}(\cdot)$ distance function to the boundary of Ω
$U \triangle V := (U \setminus V) \cup (V \setminus U)$ the symmetric difference of the sets U and V
$\Delta(x, r) := B(x, r) \cap \partial\Omega$ surface ball on $\partial\Omega$ with center at $x \in \partial\Omega$ and radius $r > 0$
$\Delta(x, r) := B(x, r) \cap \Sigma$ surface ball on Σ with center at $x \in \Sigma$ and radius $r > 0$
$D := \sum_{j=1}^n \mathbf{e}_j \odot \partial_j$ the classical (homogeneous) Dirac operator in $\mathbb{R}^n$ (6.4.139)
$D_L u := \sum_{j=1}^n \mathbf{e}_j \odot (\partial_j u)$ Dirac operator acting from the left on the Clifford algebra-valued function u (6.4.48)
$D_R u := \sum_{j=1}^n (\partial_j u) \odot \mathbf{e}_j$ Dirac operator acting from the right on the Clifford algebra-valued function u (6.4.49)
$\mathrm{diam}_\rho(A) := \sup\{\rho(x, y) : x, y \in A\}$ the ρ-diameter of the set $A \subseteq X$, in the context of a quasi-metric space (X, ρ) (7.1.6)
$\mathbb{D}_k(X) := \{Q_\alpha^k\}_{\alpha \in I_k}$ the k-th generation of dyadic cubes in X (7.5.7)
$\mathbb{D}(X) := \bigcup_{k \in \mathbb{Z},\, k \geq \kappa_X} \mathbb{D}_k(X)$ dyadic grid on X (7.5.8)

E

$\mathbf{e}_j := (\delta_{jk})_{1 \leq k \leq n} \in \mathbb{R}^n$ where δ_{jk} is the Kronecker symbol
$\{\mathbf{e}_j\}_{1 \leq j \leq n}$ standard orthonormal basis in $\mathbb{R}^n$
E_Δ standard fundamental solution for the Laplacian (2.3.57)
$\mathrm{ext}_*(E)$ measure theoretic exterior of the set $E \subseteq \mathbb{R}^n$ (2.8.19)
$\mathcal{E}'_K(\Omega)$ distributions in Ω supported in the compact set $K \subset \Omega$
$\mathcal{E}'(\Omega)$ distributions compactly supported in the open set $\Omega \subseteq \mathbb{R}^n$
ε_B^A generalized Kronecker symbol (6.4.116)

F

$[\vec{F}]_\infty$ contribution of the vector field $\vec{F}$ at infinity (1.3.2)
f_E^* non-increasing rearrangement of $f : E \to \mathbb{R}$ (6.2.2)

G

$g=\sum_{1\le j,k\le n} g_{jk}\mathrm{d}x_j\otimes\mathrm{d}x_k$ Riemannian metric tensor
∇u gradient (Jacobian matrix) of u (3.0.8)
∇' gradient operator in $\mathbb{R}^{n-1}$
$\Gamma_\kappa(x)=\Gamma_{\Omega,\kappa}(x):=\big\{y\in\Omega:\ |x-y|<(1+\kappa)\delta_{\partial\Omega}(y)\big\}$ the (κ-)nontangential approach region with vertex at $x\in\partial\Omega$ (8.1.2)

H

$\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$
$\mathcal{H}^s$ the s-dimensional Hausdorff measure in $\mathbb{R}^n$
$\mathcal{H}^s_*$ the s-dimensional Hausdorff outer measure in $\mathbb{R}^n$
$\mathcal{H}^{n-1}_g$ the $(n-1)$-dimensional Hausdorff measure induced by the metric tensor g
$\widehat{\mu}$ the Cauchy–Clifford transform of the measure μ (6.4.96)

I

$\mathrm{i}:=\sqrt{-1}\in\mathbb{C}$ complex imaginary unit
ι^* pull-back map induced by the canonical inclusion ι
$\iota^*_\#$ sharp pull-back (1.11.58)
$\iota^*_{\mathrm{n.t.}}$ nontangential pull-back (1.11.68)
$\mathrm{int}_*(E)$ measure theoretic interior of the set $E\subseteq\mathbb{R}^n$ (2.8.18)
$\mathring{U}$ interior of the set $U\subseteq\mathbb{R}^n$
$⨍_E f\,\mathrm{d}\mu:=\frac{1}{\mu(E)}\int_E f\,\mathrm{d}\mu$, or $⨍_E f\,\mathrm{d}\mu:=\frac{1}{\mu(E)}\int_E f\,\mathrm{d}\mu$, integral average of the function f on the set $E\subseteq X$, in a measure space (X,μ)
$f_{B_\rho(x,r)}:=⨍_{B_\rho(x,r)} f\,\mathrm{d}\mu:=\frac{1}{\mu(B_\rho(x,r))}\int_{B_\rho(x,r)} f(y)\,\mathrm{d}\mu(y)$ integral average of f over the ρ-ball $B_\rho(x,r)$, in the context of a space of homogeneous type (X,ρ,μ) (7.4.9)
$I_{E,\alpha}f(x):=\int_E \frac{f(y)}{\rho(x,y)^{d-\alpha}}\,\mathrm{d}\mu(y)$ the fractional integral operator of order α on the set E contained in a metric space (X,ρ) equipped with upper d-dimensional Borel measure μ on (X,τ_ρ) (7.8.3)

K

K_Δ boundary-to-boundary harmonic double layer potential (1.1.32)
$K^\#_\Delta$ transpose harmonic double layer potential (1.1.33)

L

$\mathcal{L}^n$ Lebesgue measure in $\mathbb{R}^n$
L^∞_{comp} essentially bounded functions with compact support
$d\mathcal{L}^n_g:=\sqrt{g}\,\mathrm{d}\mathcal{L}^n$ Lebesgue measure induced by the metric tensor g
$\Lambda^\ell TM$ the ℓ-th exterior power of the vector bundle on the manifold M
$L^0(X,\mu)$ measurable functions which are pointwise finite μ-a.e. on X
$L^p_{\mathrm{bdd}}(\Omega,\mathcal{L}^n)$ functions p-th power integrable on bounded subsets of Ω (4.2.4)
$\mathrm{Lip}(X)$ Lipschitz functions on the (quasi-)metric space X (3.7.2)
$\|\cdot\|_{\mathrm{Lip}(X)}$ the natural semi-norm on $\mathrm{Lip}(X)$ (3.7.1)

$\mathrm{Lip}_c(X)$ the space of Lipschitz functions with bounded support in the (quasi-)metric space X
$\big(\mathrm{Lip}_c(\Sigma)\big)'$ the space distributions on the set Σ (4.1.34)
${}_{(\mathrm{Lip}_c(\Sigma))'}\langle\cdot,\cdot\rangle_{\mathrm{Lip}_c(\Sigma)}$ (or simply $\langle\cdot,\cdot\rangle$) distributional pairing on the set Σ
$L^{p,q}(X,\mu)$ Lorentz space on X with respect to the measure μ (6.2.13)
$L^r_{\mathrm{fin}}(X,\mu)$ the space of L^r-integrable functions on subsets of X of finite μ-measure (6.2.35)
$L^{p,q}_\star(\Omega,\mu) := \big\{u:\Omega\to\mathbb{C}: u$ is $\mathcal{L}^n$-measurable and $u_{\star,\theta}\in L^{p,q}(\Omega,\mu)\big\}$ maximal Lorentz space with respect to the Borel measure μ in the open set $\Omega\subseteq\mathbb{R}^n$ (6.6.41)
$L(\xi) := (-1)^m \sum_{|\alpha|=|\beta|=m} \xi^{\alpha+\beta}A_{\alpha\beta}$ characteristic matrix of the constant-coefficient system $L = \sum_{|\alpha|=|\beta|=m} \partial^\alpha A_{\alpha\beta}\partial^\beta$ (6.5.39)
$L^p_\star(\Omega,\mu) = \big\{u:\Omega\to\mathbb{C}: u$ is $\mathcal{L}^n$-measurable and $u_{\star,\theta}\in L^p(\Omega,\mu)\big\}$ maximal Lebesgue space with respect to the Borel measure μ in the open set $\Omega\subseteq\mathbb{R}^n$ (6.6.43)
$\log_+$ the positive part of ln (7.6.68)

M

$M^*_\gamma(F)$ upper γ-dimensional Minkowski content of the set F (4.5.1)
$m_E(\lambda,f) := \mu\big(\{x\in E: |f(x)|>\lambda\}\big)$ measure of the level set of f at height $\lambda>0$
$\mathcal{M}_{\mathcal{A},s,\alpha}$ fractional Hardy–Littlewood maximal operator (6.3.9)
$\mathcal{M}_{X,s,\alpha}f(x) := \sup_{r>0}\Big[\mu(B_\rho(x,r))^\alpha\Big(⨍_{B_\rho(x,r)}|f|^s\,d\mu\Big)^{\frac{1}{s}}\Big]$ the L^s-based fractional Hardy–Littlewood maximal operator of order α in the space of homogeneous type (X,ρ,μ) (7.6.1)
$\mathcal{M}_{X,s}f(x) := \sup_{r>0}\Big(⨍_{B_\rho(x,r)}|f|^s\,d\mu\Big)^{\frac{1}{s}}$ the L^s-based Hardy–Littlewood maximal operator in the space of homogeneous type (X,ρ,μ) (7.6.7)
$\mathcal{M}^R_{X,s}f(x) := \sup_{0<r\leq R}\Big(⨍_{B_\rho(x,r)}|f|^s\,d\mu\Big)^{\frac{1}{s}}$ the local L^s-based Hardy–Littlewood maximal operator in the space of homogeneous type (X,ρ,μ) (7.6.12)
$\mathcal{M}_X f(x) := \sup_{r\in(0,\infty)} ⨍_{B_\rho(x,r)}|f|\,d\mu$ the Hardy–Littlewood maximal operator of the function f on the space of homogeneous type (X,ρ,μ) (7.6.16)
$u^{\max}_M(x) := \Big\|u(y)\Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M\Big\|_{L^\infty_y(\Omega,\mathcal{L}^n)}$ the tangential maximal function of u (with exponent M), defined at $x\in\partial\Omega$ (8.5.2)

N

$\mathbb{N}_0 := \mathbb{N}\cup\{0\} = \{0,1,2,\dots\}$
$(\mathcal{N}_\kappa u)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x),\mathcal{L}^n)}$ the (κ-)nontangential maximal operator acting on the measurable function $u:\Omega\to\mathbb{R}^n$ at the point $x\in\partial\Omega$ (8.2.1)
$(\mathcal{N}^\varepsilon_\kappa u)(x) := \text{ess-sup}\big\{|u(y)|: x\in\Gamma_\kappa(x),\ \mathrm{dist}(y,\partial\Omega)<\varepsilon\big\}$ the (κ-)nontangential maximal operator truncated at height $\varepsilon>0$, acting on the function $u:\Omega\to\mathbb{R}^n$ at the point $x\in\partial\Omega$ (1.5.5)

$\mathcal{N}_\kappa^E u := \mathcal{N}_\kappa(u \cdot \mathbf{1}_E)$ restricted nontangential maximal function of $u : \Omega \to \mathbb{R}$ (relative to the set E) (8.2.4)
$(\widetilde{\mathcal{N}}_{\kappa,\theta,r} u)(x) := \Big\| \Gamma_\kappa(x) \ni y \mapsto \Big(⨍_{B(y,\theta\,\delta_{\partial\Omega}(y))} |u|^r \, d\mathcal{L}^n \Big)^{1/r} \Big\|_{L^\infty_y(\Gamma_\kappa(x),\mathcal{L}^n)}$ the averaged (κ-)nontangential maximal operator acting on the measurable function $u : \Omega \to \mathbb{R}$ at the point $x \in \partial\Omega$ (8.10.1)
ν geometric measure theoretic outward unit normal (5.6.2)–(5.6.3)
ν_g geometric measure theoretic outward unit normal induced by the metric tensor g
ν^{E} geometric measure theoretic outward unit normal induced by the standard Euclidean metric
$\nu \bullet \vec{F}$ the "bullet" product of ν with $\vec{F}$ (1.9.1), (4.2.12)
$N_\kappa^p(\Omega;\mu) := \big\{ u : \Omega \to \mathbb{C} : u \text{ is } \mathcal{L}^n\text{-measurable, and } \mathcal{N}_\kappa u \in L^p(\partial\Omega,\mu) \big\}$ the space of measurable functions with a p-th power integrable nontangential maximal function (8.3.31)
$\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \big)(x)$ nontangential trace of the function $u : \Omega \to \mathbb{R}$ at the point $x \in \partial\Omega$, is defined as the number $a \in \mathbb{R}$ with the property that for every $\varepsilon > 0$ there exists some $r > 0$ such that $|u(y) - a| < \varepsilon$ for $\mathcal{L}^n$-a.e. point $y \in \Gamma_\kappa(x) \cap B(x,r)$ (see Definition 8.9.1)

O

$\mathbf{1}_E$ characteristic function of E
$\omega_{n-1} := \mathcal{H}^{n-1}(S^{n-1})$ surface area of S^{n-1}
$\mathcal{O}_\varepsilon := \big\{ x \in \Omega : \delta_{\partial\Omega}(x) < \varepsilon \big\}$ one-sided collar neighborhood of $\partial\Omega$
$\mathrm{osc}_p(f;R) := \sup\limits_{x\in X,\, r\in(0,R)} \Big(⨍_{B_\rho(x,r)} \big| f(y) - f_{B_\rho(x,r)} \big|^p \, d\mu(y) \Big)^{\frac{1}{p}}$ the L^p-based mean oscillation of the function f at scales up to R, in a space of homogeneous type (X,ρ,μ) (7.4.107)

P

$(\mathfrak{P}u)(x) := \sup\limits_{0<r<2\,\mathrm{diam}(\partial\Omega)} \Big\{ \frac{1}{\sigma(\partial\Omega \cap B(x,r))} \int_{\Omega\cap B(x,r)} |u| \, d\mathcal{L}^n \Big\}$ the $\mathfrak{P}$-maximal operator acting on a Lebesgue measurable function $u : \Omega \to \mathbb{R}^n$ at the point $x \in \partial\Omega$ (1.9.3)
$\langle \cdot, \cdot \rangle_{\mathcal{E}}$ pointwise (real) pairing in the fibers of the Hermitian vector bundle $\mathcal{E}$
${}_{\mathscr{E}'(\Omega)}\big\langle \cdot, \cdot \big\rangle_{\mathscr{E}(\Omega)}$ pairing between compactly supported distributions in Ω and smooth functions in Ω (2.2.33)
$\langle \cdot, \cdot \rangle_{\Lambda^\ell TM}$ (real) pointwise pairing on $\Lambda^\ell TM$
$\wedge$ exterior product of differential forms
$\vee$ interior product of differential forms
$\partial_{\tau_{XY}}$ tangential derivative operator on manifolds (1.12.88)
$\dot{\partial}_{\tau_{jk}}$ weak tangential derivative (4.2.28)
$\partial_* E$ measure theoretic boundary of E (5.2.1)
$\partial^* E$ reduced boundary of E (5.6.13)
$\partial_T E$ points on $\partial_* E$ at which an approximate tangent plane exists (5.6.65)

$\partial^N E$ points at which E possesses a reasonable unit normal vector (5.6.69)
$\partial_{\rm lfp}\Omega$ the "locally finite perimeter" boundary of the set Ω (5.7.47)
$\partial_{\rm nta}\Omega := \bigcap_{\kappa>0} A_\kappa(\partial\Omega) = \big\{x \in \partial\Omega : x \in \overline{\Gamma_\kappa(x)}$ for each $\kappa > 0\big\}$ the nontangentially accessible boundary of Ω (8.8.47)
Π_m^ℓ projection map of differential forms of mixed degree m onto Λ_m^ℓ, the space of differential forms of degree ℓ (6.4.112)
$(a)_+ := \max\{a, 0\}$ the positive part of the number $a \in \mathbb{R}$
$\pi_\kappa(E) = \pi_{\Omega,\kappa}(E) := \big\{x \in \partial\Omega : \Gamma_\kappa(x) \cap E \neq \varnothing\big\}$ "shadow" (or projection) of a given set $E \subseteq \Omega$ onto $\partial\Omega$ (8.1.15)

R

$[w]_{RH_q} := \sup\limits_{B\ \rho\text{-ball}} \Big(⨍_B w^q \,d\mu\Big)^{\frac{1}{q}} \Big(⨍_B w \,d\mu\Big)^{-1}$ the L^q reverse Hölder constant of the weight function w on a space of homogeneous type (X, ρ, μ) (7.7.17)
$RH_q(X, \rho, \mu) := \big\{w$ weight function $: [w]_{RH_q} < \infty\big\}$ the L^q reverse Hölder class on a space of homogeneous type (X, ρ, μ) (7.7.17)
regsupp u smallest closed set outside of which the distribution u is a locally integrable function (1.5.4)
$\mathbb{R}^n_+$ (open) upper half-space in $\mathbb{R}^n$
$\mathbb{R}^n_-$ (open) lower half-space in $\mathbb{R}^n$
$\mathscr{R}_{\mathbb{R}^n\to\partial\Omega}$ restriction operator from $\mathbb{R}^n$ to $\partial\Omega$
$R_{j,\max}$ the maximal j-th Riesz transform (5.10.15)
$R_{j,\varepsilon}$ the truncated j-th Riesz transform (5.10.16)
R_j the boundary-to-boundary j-th Riesz transform (5.10.17)
$R_j^{\rm weak}$ the distributional j-th Riesz transform (5.10.18)
$\mathrm{rad}(\Omega) := \inf\big\{r \in (0,\infty] :$ there exists $x \in \Omega$ so that $\Omega \subseteq B(x,r)\big\}$, for any nonempty connected open set $\Omega \subseteq \mathbb{R}^n$, (5.11.31), (5.11.32)

S

$\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ surface measure on $\partial\Omega$
$\sigma_* := \mathcal{H}^{n-1}\lfloor\partial_*\Omega$ surface measure on $\partial_*\Omega$
σ_g surface measure induced by the metric tensor g
$\sigma^{\rm E}$ surface measure induced by the standard Euclidean metric
$\mathrm{Sym}(D;\xi)$ principal symbol of the first-order system D (1.7.16)
$*$ Hodge star operator
$\xi \mapsto \xi^\sharp$, $X \mapsto X^\flat$ musical isomorphisms (1.12.139)
$S^{n-1} := \partial B(0,1)$ unit sphere in $\mathbb{R}^n$
$S^{n-1}_\pm := S^{n-1} \cap \mathbb{R}^n_\pm$ upper and lower (open) hemispheres
$\mathscr{C}^k$-singsup u the smallest closed set outside of which the distribution u is of class $\mathscr{C}^k$
$S(X,\mu)$ simple functions on the measure space (X,μ)
$S_{\rm fin}(X,\mu)$ simple functions on (X,μ) with support of finite measure (3.1.10)
supp μ the support of the measure μ (3.8.1)

supp f the support of the measurable function f (3.8.7)
$\mathscr{S}(\mathbb{R}^n)$ Schwartz functions
$\mathscr{S}'(\mathbb{R}^n)$ tempered distributions
$u_{\rm scal}$ scalar part of the Clifford element $u \in \mathcal{C}\!\ell_n$ (6.4.26)
$u_{\star,\theta}(x) := \|u\|_{L^\infty(B(x,\theta\delta_{\partial\Omega}(x)),\mathcal{L}^n)}$ the solid maximal function of $u : \Omega \to \mathbb{C}$ at the point $x \in \Omega$ (6.6.2)
$u^E_{\star,\theta}$ local solid maximal function of $u : E \to \mathbb{C}$ in Ω (6.6.79)
ρ_{sym} the symmetrized version of the quasi-distance ρ for a quasi-metric space (X, ρ) (7.1.15)
$\rho_\#$ the regularized version of the quasi-distance ρ for a quasi-metric space (X, ρ) (7.1.17)
$f^\#_p(x) := \sup\limits_{r>0}\Big(⨍_{B_\rho(x,r)} \big|f(y) - f_{B_\rho(x,r)}\big|^p \,d\mu(y)\Big)^{1/p}$ the L^p-based Fefferman–Stein sharp maximal function of f (7.4.110)

T

τ_t dilation by a factor of t (4.5.36)
τ_ρ the topology induced by the quasi-distance ρ in a quasi-metric space (X, ρ) (7.1.7)

U

$U \sqcup V$ the union of two disjoint sets U, V

V

$dV_g := \sqrt{g}dx_1 \wedge \cdots \wedge dx_n$ volume element induced by the metric tensor g
Var F pointwise variation of F (2.6.12)
$\mathbf{V}(f; \mathcal{O})$ the variation of the function f in the set $\mathcal{O}$ (5.5.1)
$u_{\rm vect}$ vector part of the Clifford element $u \in \mathcal{C}\!\ell_n$ (6.4.25)

W

weakly elliptic system: a system L whose characteristic matrix satisfies the condition $\det[L(\xi)] \neq 0$ for each $\xi \in \mathbb{R}^n \setminus \{0\}$
$W^{k,p}(\Omega)$ the L^p-based Sobolev space of order k in Ω (intrinsically defined)
$W^{k,p}_{\rm loc}(\Omega)$ local L^p-based Sobolev space of order k in Ω
$W^{k,p}_{\rm bdd}(\Omega)$ Sobolev functions on any bounded measurable subset of Ω (3.0.4)

X

${}_{X^*}\big(\cdot, \cdot\big)_X$ the duality pairing between a vector space X and its algebraic dual X^*

Chapter 1
Statement of Main Results Concerning the Divergence Theorem

This chapter debuts with a brief overview of the Divergence Theorem, from its one-dimensional version (known as the Fundamental Theorem of Calculus) to the De Giorgi–Federer version involving sets of locally finite perimeter, in Sect. 1.1. This chapter also contains an outline of the main goals of the work undertaken in Volume I, as well as arguments pointing to the naturalness and suitability of these goals. Sections 1.2–1.12 comprise the main results concerning a new generation of Euclidean versions of the Divergence Theorem, and their generalizations to the setting of Riemannian manifolds.

1.1 The De Giorgi–Federer Version of the Divergence Theorem

The Fundamental Theorem of Calculus, one of the greatest mathematical achievements of all time, stands as beautiful, powerful, and relevant today as it did more than three centuries ago. Typically, Isaac Newton and Gottfried Leibniz are credited with fully developing the surrounding mathematical theory into a coherent calculus for infinitesimal quantities, a mathematical landscape within which the Fundamental Theorem of Calculus stands out as the crowning achievement. In its sharp one-dimensional version, involving the class $\mathrm{AC}\big([a,b]\big)$, of absolutely continuous functions on a finite interval $[a,b]$, the Fundamental Theorem of Calculus simply reads

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a) \quad \text{for every} \quad F \in \mathrm{AC}\big([a,b]\big). \tag{1.1.1}$$

It is a stark example of how local information, encoded in the instantaneous rate of change (aka derivative) F', can be pieced together via integration to derive conclusions of a global nature about the variation of F over $[a,b]$, a fundamental paradigm in calculus.

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Intriguingly, while (1.1.1) is essentially optimal, dealing with higher-dimensional versions of the Fundamental Theorem of Calculus remains an active area of research in contemporary mathematics. In its standard version, with the symbol $\mathcal{L}^n$ denoting the n-dimensional Lebesgue measure in $\mathbb{R}^n$ and $\mathscr{C}^k(\overline{\Omega})$ denoting functions of class $\mathscr{C}^k$ in an open neighborhood of $\overline{\Omega}$, the Divergence Theorem asserts that

$$\text{if } \Omega \subseteq \mathbb{R}^n \text{ is a bounded domain of class } \mathscr{C}^1\text{, with outward unit normal } \nu \text{ and surface measure } \sigma\text{, then } \int_\Omega \operatorname{div}\vec{F}\,\mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} \nu\cdot\big(\vec{F}\big|_{\partial\Omega}\big)\,\mathrm{d}\sigma \text{ for each vector field } \vec{F}\in\big[\mathscr{C}^1(\overline{\Omega})\big]^n. \tag{1.1.2}$$

Since the divergence of a continuously differentiable vector field $\vec{F}$ may be computed pointwise as $(\operatorname{div}\vec{F})(x) = \lim_{r\to 0^+} \mathcal{L}^n(B(x,r))^{-1} \int_{\partial B(x,r)} \big(\frac{y-x}{r}\big)\cdot\vec{F}(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)$ where, here and elsewhere, $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$, it follows that the quantity $(\operatorname{div}\vec{F})(x)$ is emblematic of the tendency of the vector field $\vec{F}$ to collect (sink effect) or disperse (source effect) at the point x. In view of this feature, the Divergence Formula in (1.1.2) may be regarded as a conservation law, asserting that the solid integral of all such sources and sinks associated with a given vector field is equal to the net flow of said vector field through the solid's boundary.[1] Ergo, in complete analogy to the Fundamental Theorem of Calculus mentioned earlier, the Divergence Theorem describes how the infinitesimal sink/source effects created by a vector field may be pieced together inside a given domain to produce a global, macroscopic effect, along the boundary.

The classical result recorded in (1.1.2) is usually associated with the names of J.-L. Lagrange who first established a special case of the Divergence Theorem in 1762 working on the propagation of sound waves (cf. [159]), C. F. Gauss who independently considered a particular case in 1813 (cf. [101]), M. V. Ostrogradsky who gave the first proof of the general theorem in 1826 (cf. [213]), G. Green who used a related formula in 1828 (cf. [109]), A. Cauchy who in 1846 first published, without proof, the nowadays familiar form of Green's Theorem (cf. [42]), B. Riemann who provided proof of Green's Formula in his 1851 inaugural dissertation (see [229]), Lord Kelvin who in 1850 discovered the special version of Stokes' theorem (in the three-dimensional setting, also known as the curl theorem), and É. Cartan who first published the general form of Stokes' theorem (in the language of differential forms on manifolds) in 1945, among others. However, a precise attribution is fraught with difficulty since the Divergence Theorem in its modern format has undergone successive waves of reformulations, generalizations, as well as more rigorous proofs, with inputs from a multitude of sources (general historical accounts may be found in [145, 246]).

Specializing the Divergence Formula in (1.1.2) to the case when $\vec{F}$ is the restriction to $\overline{\Omega}$ of vector fields from $\big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n$ (where $\mathscr{C}_c^\infty(\mathbb{R}^n)$ denotes the space of smooth, compactly supported functions in $\mathbb{R}^n$) yields the statement:

[1] For example, imagining $\vec{F}$ as the velocity field for an incompressible fluid flow (i.e., a given mass occupying a fixed region Ω), this informally states that "what goes in must come out".

$$\text{if } \Omega \subseteq \mathbb{R}^n \text{ is a bounded domain of class } \mathscr{C}^1, \text{ with outward unit normal } \nu \text{ and surface measure } \sigma, \text{ then } \nabla \mathbf{1}_\Omega = -\nu\sigma \text{ in } \big[\mathcal{D}'(\mathbb{R}^n)\big]^n, \tag{1.1.3}$$

where $\mathbf{1}_\Omega$ is the characteristic function of Ω and $\mathcal{D}'(\mathbb{R}^n)$ denotes the space of distributions in $\mathbb{R}^n$. Conversely, since both $\nabla \mathbf{1}_\Omega$ and $-\nu\sigma$ are vector distributions in $\mathbb{R}^n$ of order ≤ 1, their action canonically extends to vector fields from $\big[\mathscr{C}^1_c(\mathbb{R}^n)\big]^n$, in which scenario we precisely recover (1.1.2). Hence, (1.1.3) amounts to an equivalent reformulation of the classical Divergence Theorem (1.1.2), which has a purely geometric measure theoretic nature. In particular, (1.1.3) brings into focus the fact that the distributional gradient of the characteristic function of a bounded $\mathscr{C}^1$ domain is a locally finite Borel vector-valued measure in $\mathbb{R}^n$.

As far as the latter property is concerned, R. Caccioppoli, E. De Giorgi, and H. Federer registered a decisive leap forward by considering the largest class of Euclidean subsets enjoying the aforementioned property, i.e., the class of sets of locally finite perimeter.[2] It turns out that this consists of Lebesgue measurable subsets Ω of $\mathbb{R}^n$ with the property that $\mathbf{1}_\Omega$ is of locally bounded variation in $\mathbb{R}^n$, that is, $\mathbf{1}_\Omega \in BV_{loc}(\mathbb{R}^n)$ (cf. also (5.6.1)). In turn, membership to $BV_{loc}(\mathbb{R}^n)$ is conceived in such a way that the Riesz Representation Theorem (cf. Proposition 3.9.1) may be naturally applied to the functional

$$\Lambda(\vec{F}) := \int_\Omega \operatorname{div}\vec{F}\, d\mathcal{L}^n \quad \text{for all} \quad \vec{F} \in \big[\mathscr{C}^1_c(\mathbb{R}^n)\big]^n, \tag{1.1.4}$$

to conclude that there exist some locally finite Borel measure σ_* in $\mathbb{R}^n$, which is actually supported on $\partial\Omega$ (in other words, satisfying[3] $\sigma_*(\mathbb{R}^n \setminus \partial\Omega) = 0$), along with some σ_*-measurable vector-valued function $\nu : \mathbb{R}^n \to \mathbb{R}^n$ with the property that $|\nu| = 1$ at σ_*-a.e. point in $\mathbb{R}^n$ and

$$\Lambda(\vec{F}) = \int_{\mathbb{R}^n} \nu \cdot \vec{F}\, d\sigma_* \quad \text{for all} \quad \vec{F} \in \big[\mathscr{C}^1_c(\mathbb{R}^n)\big]^n. \tag{1.1.5}$$

The function ν is referred to as the geometric measure theoretic outward unit normal to Ω (see also (5.6.2)–(5.6.3)). Bearing in mind that σ_* is actually supported on the set $\partial\Omega$, from (1.1.4) and (1.1.5) the following version of the Divergence Theorem emerges:

$$\int_\Omega \operatorname{div}\vec{F}\, d\mathcal{L}^n = \int_{\partial\Omega} \nu \cdot \vec{F}\, d\sigma_* \quad \text{for each} \quad \vec{F} \in \big[\mathscr{C}^1_c(\mathbb{R}^n)\big]^n. \tag{1.1.6}$$

The real achievement of De Giorgi and Federer is further refining (1.1.6) by establishing that actually

$$\sigma_* = \mathcal{H}^{n-1} \lfloor \partial_*\Omega, \tag{1.1.7}$$

---

[2] Also referred to as *Caccioppoli sets*, in honor of Renato Caccioppoli who has first studied this class in [32].

[3] As is visible from (3.9.6) and the very format of Λ in (1.1.4).

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ and $\partial_*\Omega$ denotes the measure theoretic boundary of Ω (which, in principle, can be a much smaller set than the topological boundary $\partial\Omega$; see (5.2.1)). Substituting (1.1.7) back into (1.1.6) then yields the following result.

Theorem 1.1.1 (De Giorgi–Federer's version of the Divergence Theorem [72, 73, 84, 86]) *Suppose $\Omega \subseteq \mathbb{R}^n$ is a set of locally finite perimeter. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Then for each vector field $\vec{F} \in \big[\mathscr{C}_c^1(\mathbb{R}^n)\big]^n$, one has*

$$\int_\Omega \big(\operatorname{div}\vec{F}\big)\big|_\Omega \, d\mathcal{L}^n = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|_{\partial_*\Omega}\big)\, d\sigma. \tag{1.1.8}$$

In a nutshell, one of the key results of the De Giorgi–Federer theory is the identity

$$\begin{gathered} \nabla \mathbf{1}_\Omega = -\nu\mathcal{H}^{n-1}\lfloor\partial_*\Omega \text{ in the sense of distributions in } \mathbb{R}^n, \\ \text{whenever } \Omega \subseteq \mathbb{R}^n \text{ is a set of locally finite perimeter,} \end{gathered} \tag{1.1.9}$$

which may then readily be re-interpreted as the Divergence Formula (1.1.8) simply by untangling jargon. A timely exposition may be found in [80, Sect. 5.8, Theorem 1, p. 209]. For the original work, see [72, 73, 84, 86], as well as [46] for additional comments and references.

The nature of the Divergence Theorem is such that the smoother the category of vector fields considered, the rougher the class of domains which may be allowed in the formulation of this theorem. While the De Giorgi–Federer version of the Divergence Theorem applies to a large class of domains (i.e., sets of locally finite perimeter), the vector fields involved are assumed to have components in $\mathscr{C}_c^1(\mathbb{R}^n)$. Thus, the vector fields in the De Giorgi–Federer version of the Divergence Theorem belong to a very restrictive class, are exceedingly regular, as well as completely unrelated to the underlying domain. Moreover, when specialized to the case $n = 1$, for a finite interval of the real line, the De Giorgi–Federer version of the Divergence Theorem formulated in Theorem 1.1.1 *fails* to yield the sharp version of the Fundamental Theorem of Calculus, recorded in (1.1.1).

While formula (1.1.8) has been successfully used in many branches of mathematics, Theorem 1.1.1 is not adequate for a variety of problems in partial differential equations, scattering, and harmonic analysis, since in many fundamental instances $\vec{F}$ is *not* continuous up to and including the boundary, but rather the trace of $\vec{F}$ to $\partial\Omega$ is considered in a pointwise nontangential sense. As such, one needs a divergence formula for rough integrands and rough boundaries that can handle these cases. Of course, any significant weakening of the assumptions on the vector field $\vec{F}$ in Theorem 1.1.1 should be accompanied by a corresponding strengthening of the assumptions on Ω, the underlying domain.[4] Ad hoc techniques, based on approximating

[4] As such, the repertoire of divergence theorems is rather fluid. This has prompted some authors (see, e.g., [104, p. 17], [139, p. 31], [248, Theorem 2.3, p. 39]) to utilize the rather ambiguous label "*domain for which the Divergence Theorem holds*."

the original set Ω by a suitable sequence of subdomains $\Omega_j \nearrow \Omega$, have sufficed for continuous vector field in Lipschitz domains (cf. [63, 210, 257]), and also for a class of Reifenberg flat domains (cf. [150]), but to go beyond this one needs genuinely new techniques. Progress in this regard has been registered in [125] which treats a much larger class of domains than Lipschitz, without any flatness assumptions. However, the version of the Divergence Theorem established in [125] requires that the (nontangential) trace of the vector field $\vec{F}$ on the boundary is p-th power integrable for some $p > 1$. This requirement is an artifact of the proof, which relies on the boundedness of the Hardy–Littlewood maximal function on L^p with $p > 1$.

One of the main goals of this volume is to produce a brand of Divergence Theorem (that continues to imply the De Giorgi–Federer Divergence Formula stated in Theorem 1.1.1; see the discussion in the very last part of Sect. 1.3) exhibiting the following features (all of which are absent from De Giorgi–Federer's version of the Divergence Theorem recorded in Theorem 1.1.1):

- when $n = 1$ and Ω is a finite interval on the real line, our theorem reduces precisely to the sharp version of the Fundamental Theorem of Calculus formulated in (1.1.1);
- the vector field $\vec{F}$ is intrinsically defined in Ω, and may lack continuity, or even local boundedness;
- the divergence of $\vec{F}$ is computed in the sense of distributions and is allowed to exhibit certain types of singularities;
- the only quantitative aspect not directly associated with the ability of writing the two integrals making up the Divergence Formula in a meaningful way is an integrability condition imposed on the nontangential maximal function of the vector field $\vec{F}$;
- the trace of $\vec{F}$ on the boundary is considered in a pointwise nontangential sense (i.e., considering the limit of $\vec{F}$ from within certain nontangential approach regions with vertices at points on $\partial\Omega$).

Compared with the classical results of De Giorgi–Federer, our work brings into focus the role of the nontangential maximal operator and the nontangential boundary trace in the context of the Divergence Theorem. In relation to these aspects, we would like to mention that the idea of imposing an integrability condition on the nontangential maximal operator and then using this to prove the existence of nontangential boundary limits originates in the classical work of Fatou [83]. In particular, the class of functions for which such a nontangential boundary trace exists serves as a natural enlargement of the category of functions which are continuous up to, and including, the topological boundary of the underlying domain. In a broader perspective, describing the qualitative and quantitative boundary behavior of a function via its nontangential boundary trace and its nontangential maximal operator is a natural point of view which has been adopted in a multitude of branches of analysis. Here are concrete examples of this flavor, highlighting the adequacy and appropriateness of taking boundary traces in a nontangential pointwise sense, and imposing integrability conditions on the nontangential maximal operator.

(I) Hardy Space of Holomorphic Functions: Classically, if $0 < p \leq \infty$ then the Hardy space H^p in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane consists of holomorphic functions $F : \mathbb{D} \to \mathbb{C}$ satisfying

$$\sup_{0<r<1} \left(\int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta \right)^{1/p} < +\infty. \tag{1.1.10}$$

See, e.g., [97, Definition 2.17, pp. 35–36], [100, Appendix A, p. 435], [121, p. 39], [155], and [231, Definition 17.7, p. 330]. One of the central results of the theory associated with this brand of Hardy spaces is that for a holomorphic function F in $\mathbb{D}$ membership to H^p is equivalent to

$$\mathcal{N}_\kappa F \in L^p(\partial\mathbb{D}, \mathcal{H}^1) \tag{1.1.11}$$

for some, or all, $\kappa \in (0, \infty)$, where the nontangential maximal function $\mathcal{N}_\kappa F$ is defined as

$$(\mathcal{N}_\kappa F)(z) := \sup_{|\zeta - z| < (1+\kappa)(1-|\zeta|)} |F(\zeta)|, \qquad \forall z \in \partial\mathbb{D}. \tag{1.1.12}$$

Also, given any $\kappa \in (0, \infty)$, for each function F in H^p the nontangential boundary limit

$$\Big(F\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{D}}\Big)(z) := \lim_{\substack{|\zeta - z| < (1+\kappa)(1-|\zeta|) \\ \zeta \longrightarrow z}} F(\zeta) \ \text{ exists for } \mathcal{H}^1\text{-a.e. } z \in \partial\mathbb{D}, \tag{1.1.13}$$

and is actually independent of the aperture parameter κ. These considerations have perfectly natural analogues in the setting of the upper half-plane $\mathbb{R}^2_+$.

Turning to higher dimensions, the Hardy space H^p in the upper half-space $\mathbb{R}^n_+$, with $n \geq 2$ and $\frac{n-1}{n} < p < \infty$, as defined in Stein–Weiss and Fefferman–Stein (cf. [90, 240, 243–245]), is the collection of n-tuples $(u_j)_{1 \leq j \leq n}$ of functions satisfying the Moisil–Teodorescu system (cf. [202–204, 252]), or generalized Cauchy–Riemann equations,[5]

$$\sum_{j=1}^n \partial_j u_j = 0 \ \text{ and } \ \partial_j u_k = \partial_k u_j \ \text{ for } \ 1 \leq j, k \leq n, \tag{1.1.14}$$

in $\mathbb{R}^n_+$, subject to the uniform integrability condition

$$\sup_{t>0} \Big(\int_{\mathbb{R}^{n-1}} \sum_{j=1}^n |u_j(x', t)|^p \, dx' \Big)^{1/p} < +\infty. \tag{1.1.15}$$

[5] A piece of terminology used in [239, 244] (both of these papers cite [204]).

These conditions turn out to be equivalent with demanding that the Clifford algebra-valued function $F := \sum_{j=1}^n u_j \mathbf{e}_j$ is monogenic in $\mathbb{R}^n_+$, i.e., F is a null-solution in $\mathbb{R}^n_+$ of the Dirac operator

$$D := \sum_{j=1}^{n} \mathbf{e}_j \partial_j \tag{1.1.16}$$

(for more on this, see Sect. 6.4), and is uniformly L^p-integrable on hyperplanes parallel to $\partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}$. In fact, one of the main results in [90] is that, having fixed a background parameter $a \in (0, \infty)$, the latter uniform integrability condition is further equivalent to the demand that

$$\begin{aligned} &\mathcal{N}_a F \in L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \ \text{ where} \\ &(\mathcal{N}_a F)(x') := \sup\nolimits_{|y'-x'|<a\,t} |F(y',t)|, \quad x' \in \mathbb{R}^{n-1}. \end{aligned} \tag{1.1.17}$$

In addition, given any $a > 0$, the nontangential boundary limit

$$\Big(F\big|^{a-\text{n.t.}}_{\partial\mathbb{R}^n_+}\Big)(x') := \lim_{\substack{|y'-x'|<a\,t \\ (y',t)\longrightarrow(x',0)}} F(y',t) \ \text{ exists for } \mathcal{L}^{n-1}\text{-a.e. } \ x' \in \mathbb{R}^{n-1}, \tag{1.1.18}$$

and is in fact independent of the aperture parameter a.

One of the upshots of recasting (1.1.15) as (1.1.17) is making the theory more readily adaptable to domains with a more intricate geometry than the upper half-space. For example, a systematic study of Hardy spaces of holomorphic functions in Lipschitz domains in $\mathbb{C}$ has been taken up in [146], and a higher-dimensional theory for monogenic functions in Lipschitz domains in $\mathbb{R}^n$ (involving the Clifford algebra formalism) has been developed in [196]. See also [133] for Hardy spaces of holomorphic functions defined in terms of the nontangential maximal operator in chord-arc domains in the plane. Here, we only wish to mention that not only is the formulation of our main results concerning the Divergence Theorem sharing features intrinsic to the theory of Hardy spaces but, in turn, also having a Divergence Theorem that is compatible with the nature of these spaces leads to further progress in this theory (cf. the discussion in [186, Chap. 3]).

(II) `Singular Integral Operators of Calderón–Zygmund type :` Let us review some consequence of the classical Calderón–Zygmund theory of singular integral operators in the upper half-space $\mathbb{R}^n_+$. The starting point is fixing a kernel

$$\begin{aligned} &K \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\}) \ \text{ satisfying } \ K(-x) = -K(x) \ \text{ and} \\ &K(\lambda\, x) = \lambda^{-(n-1)} K(x) \ \text{ for all } \ \lambda > 0 \ \text{ and } \ x \in \mathbb{R}^n \setminus \{0\}, \end{aligned} \tag{1.1.19}$$

where $N = N(n)$ is a sufficiently large positive integer, depending solely on the dimension of the ambient space. Associated with this kernel, define the integral operator mapping functions defined on $\mathbb{R}^{n-1}$ into functions defined in $\mathbb{R}^n_+$ according to

$$\mathcal{T} f(x) := \int_{\mathbb{R}^{n-1}} K\big(x - (y', 0)\big) f(y')\,\mathrm{d}y', \qquad x \in \mathbb{R}^n_+. \tag{1.1.20}$$

Let us also consider its maximal version

$$T_{\max} f(x') := \sup_{\varepsilon>0} |(T_\varepsilon f)(x')|, \qquad x' \in \mathbb{R}^{n-1}, \quad \text{where} \tag{1.1.21}$$

$$(T_\varepsilon f)(x') := \int\limits_{\substack{y' \in \mathbb{R}^{n-1} \\ |x'-y'|>\varepsilon}} K(x'-y', 0) f(y')\,\mathrm{d}y', \qquad x' \in \mathbb{R}^{n-1}. \tag{1.1.22}$$

Then for each $p \in [1, \infty)$, there exists a constant $C \in (0, \infty)$ depending only on n and p such that for each $f \in L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ one has

$$\|T_{\max} f\|_{L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \leq C\|K|_{S^{n-1}}\|_{\mathscr{C}^N(S^{n-1})}\|f\|_{L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \text{ if } p > 1, \tag{1.1.23}$$

$$\|T_{\max} f\|_{L^{1,\infty}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \leq C\|K|_{S^{n-1}}\|_{\mathscr{C}^N(S^{n-1})}\|f\|_{L^1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \text{ if } p = 1. \tag{1.1.24}$$

Furthermore, for each $f \in L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ with $p \in [1, \infty)$ the limit

$$(Tf)(x') := \lim_{\varepsilon\to 0^+} (T_\varepsilon f)(x') \text{ exists for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \mathbb{R}^{n-1}, \tag{1.1.25}$$

and the induced principal-value singular integral operators

$$T : L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \longrightarrow L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \quad \text{if } p \in (1, \infty), \tag{1.1.26}$$

$$T : L^1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \longrightarrow L^{1,\infty}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \quad \text{if } p = 1 \tag{1.1.27}$$

are well-defined, linear, and bounded. In addition, for each fixed background parameter $a \in (0, \infty)$, the following pointwise Cotlar inequality for the nontangential maximal operator holds:

$$\begin{aligned} \mathcal{N}_a(\mathcal{T} f)(x') &:= \sup_{|x'-y'|<a\,t} |(\mathcal{T} f)(y', t)| \\ &\leq (T_{\max} f)(x') + C\big\|K\big|_{S^{n-1}}\big\|_{\mathscr{C}^1(S^{n-1})}\mathcal{M} f(x'), \qquad \forall x' \in \mathbb{R}^{n-1}, \end{aligned} \tag{1.1.28}$$

for some $C = C(n, p, a) \in (0, \infty)$, where $\mathcal{M}$ is the Hardy–Littlewood maximal operator in $\mathbb{R}^{n-1}$. As a consequence, for each $p \in [1, \infty)$ there exists a finite constant $C = C(n, p, a) > 0$ such that for each $f \in L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ one has the nontangential maximal function estimates

$$\big\|\mathcal{N}_a(\mathcal{T} f)\big\|_{L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \leq C\|K|_{S^{n-1}}\|_{\mathscr{C}^N(S^{n-1})}\|f\|_{L^p(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \text{ if } p > 1, \tag{1.1.29}$$

$$\big\|\mathcal{N}_a(\mathcal{T} f)\big\|_{L^{1,\infty}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \leq C\|K|_{S^{n-1}}\|_{\mathscr{C}^N(S^{n-1})}\|f\|_{L^1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})} \text{ if } p = 1. \tag{1.1.30}$$

Finally, for each $f \in L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ with $p \in [1, \infty)$, the nontangential trace of $\mathcal{T} f$ on $\partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}$ may be expressed as the jump-formula

$$\Big(\mathcal{T} f\Big|^{a-\text{n.t.}}_{\partial\mathbb{R}^n_+}\Big)(x') := \lim_{\substack{|y'-x'|<a\,t \\ (y',t)\longrightarrow(x',0)}} (\mathcal{T} f)(y', t) = \frac{1}{2\mathrm{i}}\widehat{K}(-\mathbf{e}_n) f(x') + T f(x') \tag{1.1.31}$$

at $\mathcal{L}^{n-1}$-a.e. point $x' \in \mathbb{R}^{n-1}$, where $\mathbf{e}_n := (0, \dots, 0, 1) \in S^{n-1}$ and $\widehat{K}$ is the Fourier transform of K.

Indeed, the estimates in (1.1.23)–(1.1.24) and the existence of the limit in (1.1.25) are part of the standard Calderón–Zygmund theory (as presented in, e.g., [179]). In turn, these imply that the operators in (1.1.26)–(1.1.27) are well-defined, linear and bounded. Going further, the pointwise Cotlar-type inequality for the nontangential maximal operator recorded in (1.1.28) is standard real-variable calculus. In concert with (1.1.23)–(1.1.24) and the boundedness of $\mathcal{M}$, this implies the nontangential maximal function estimates in (1.1.29)–(1.1.30). Lastly, the nontangential trace formula (1.1.31) is proved in [181, Corollary 4.81, p. 174] in the case when f is a Schwartz function, and then known real-variable techniques (relying on (1.1.29)–(1.1.30) and the density of the space of Schwartz functions in $L^p(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$) yield (1.1.31) as stated. Jump-formulas of this flavor in a much more general geometric setting have been proved in [125].

While the issue of boundedness of singular integral operators of Calderón–Zygmund type on the Lebesgue scale L^p with $p \in (1, \infty)$ is now largely understood, thanks to seminal work by G. David and S. Semmes which has brought to prominence the class of uniformly rectifiable sets (cf. (5.10.6)), more effort is required to clarify the behavior of basic singular integral operators, such as the harmonic double layer potential operator

$$K_\Delta f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle \nu(y), y - x\rangle}{|x - y|^n} f(y)\, d\mathcal{H}^{n-1}(y), \qquad x \in \partial\Omega, \tag{1.1.32}$$

and the transpose harmonic double layer potential operator

$$K^\#_\Delta f(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int_{\partial\Omega\setminus\overline{B(x,\varepsilon)}} \frac{\langle \nu(x), x - y\rangle}{|x - y|^n} f(y)\, d\mathcal{H}^{n-1}(y), \qquad x \in \partial\Omega, \tag{1.1.33}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set with a uniformly rectifiable boundary whose geometric measure theoretic outward unit normal ν is assumed to be defined $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, and ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$. For example, consider the issue as to whether $K^\#$ acts naturally on the scale of Hardy spaces $H^p(\partial\Omega, \mathcal{H}^{n-1})$ for $p \in \big(\frac{n-1}{n}, 1\big]$, defined in relation to $(\partial\Omega, \mathcal{H}^{n-1})$ viewed as a space of homogeneous type when equipped with the Euclidean distance. Since membership to this scale is subtly connected with cancelation properties, elucidating the aforementioned issue requires a considerably more sophisticated Divergence Theorem than anything currently available. Likewise, establishing that K preserves the scale of boundary

Sobolev spaces $L^p_1(\partial\Omega, \mathcal{H}^{n-1})$ with $1 < p < \infty$ (consisting of p-th power integrable functions on $\partial\Omega$ whose weak tangential derivatives are also p-th power integrable on $\partial\Omega$; cf. the discussion in [185, Chap. 11]) requires the ability to integrate by parts on the boundary and compute tangential derivatives of principal-value singular integral operators, a task which once again hinges on the availability of a potent version of the Divergence Theorem.

Our present work addresses this aspect in a satisfactory manner. Indeed, in [186, Chap. 2] we systematically pursue the goal of developing a Calderón–Zygmund theory for singular integral operators in geometrically inclusive classes of domains. In turn, this body of results becomes a collection of tools which are particularly well-suited for treating boundary value problems, via boundary layer potentials, of the sort described in the next item below.

(III) `Boundary Value Problems in Rough Domains :` Given some open subset Ω of $\mathbb{R}^n$, define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix a background parameter $\kappa \in (0, \infty)$. Then for each $p \in (1, \infty)$ the L^p Dirichlet problem for the Laplacian in Ω reads as follows:

$$\begin{cases} u \in \mathscr{C}^\infty(\Omega), \\ \Delta u = 0 \text{ in } \Omega, \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma), \\ u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} = f \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \end{cases} \tag{1.1.34}$$

for arbitrary data $f \in L^p(\partial\Omega, \sigma)$. Ergo, from the outset it is apparent that the nontangential maximal operator

$$\big(\mathcal{N}_\kappa u\big)(x) := \sup\big\{|u(y)| : y \in \Omega, \ |x-y| < (1+\kappa)\,\text{dist}(y, \partial\Omega)\big\}, \quad x \in \partial\Omega, \tag{1.1.35}$$

together with the nontangential boundary trace

$$\Big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) := \lim_{\substack{\Omega\ni y\longrightarrow x \\ |x-y|<(1+\kappa)\,\text{dist}(y,\partial\Omega)}} u(y), \qquad x \in \partial\Omega, \tag{1.1.36}$$

plays a crucial role in the very formulation of (1.1.34). Boundary value problems of this flavor have been considered by many authors, both in the Euclidean setting and in the context of manifolds. Some basic work in this regard is due to R. Brown, A. P. Calderón, B. E. Dahlberg, M. Dindoš, E. Fabes, S. Hofmann, D. Jerison, M. Jodeit, C. E. Kenig, J. Lewis, D. Mitrea, I. Mitrea, M. Mitrea, J. Pipher, N. Rivière, M. Taylor, G. Verchota, and Z. Shen, among others; see [28, 37, 62, 63, 74, 75, 81, 122, 125, 131, 133, 147, 189, 192, 194, 197, 198, 201, 237, 257], and the references therein.

While, as noted earlier, the nontangential maximal operator and the nontangential pointwise trace play a basic role in the formulation of the Dirichlet problem (1.1.34), the connection with a version of the Divergence Theorem amenable to such features is even more apparent when considering the Neumann boundary value problem for the Laplacian in an Ahlfors regular domain $\Omega \subset \mathbb{R}^n$ (cf. Definition 5.9.15):

$$\begin{cases} u \in \mathscr{C}^\infty(\Omega), \\ \Delta u = 0 \ \text{ in } \ \Omega, \\ \mathcal{N}_\kappa(\nabla u) \in L^1(\partial\Omega, \sigma), \\ \nu \cdot \Big((\nabla u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \Big) = f \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \end{cases} \tag{1.1.37}$$

Here ν is the geometric measure theoretic outward unit normal to Ω, which is defined a.e. on $\partial\Omega$ with respect to the measure $\sigma = \mathcal{H}^{n-1} \lfloor \partial\Omega$. Then a necessary condition for the solvability of (1.1.37) is that the boundary datum $f \in L^1(\partial\Omega, \sigma)$ satisfies

$$\int_{\partial\Omega} f \, d\sigma = 0. \tag{1.1.38}$$

In view of the last demand in (1.1.37), this may conceivably be proved by applying some suitable version of the Divergence Theorem to the divergence-free vector field $\vec{F} := \nabla u$. The aforementioned version of the Divergence Theorem should be sophisticated enough to be applicable to the vector field $\vec{F}$ which is defined strictly inside the Ahlfors regular domain Ω, satisfies $\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma)$, and whose boundary values exist only in the nontangential pointwise sense σ-a.e. on $\partial\Omega$. While Ω is a domain of locally finite perimeter, these weak hypotheses on $\vec{F}$ render the De Giorgi–Federer's version of the Divergence Theorem recalled in Theorem 1.1.1 (along with any other version presently available in the literature) hopelessly ineffective in this case.

An even more subtle and delicate aspect of the Neumann problem (1.1.37) directly affected by the Divergence Theorem is the fact that a necessary condition for its solvability is the membership of the boundary datum f to the *Hardy space* $H^1(\partial\Omega, \sigma)$ (associated with the space of homogeneous type $(\partial\Omega, |\cdot - \cdot|, \sigma)$ as in [57]). In this regard, see the definition and properties of the (co)normal derivative discussed in [185, Sect. 10.1].

Finally, we wish to note that similar considerations apply to other types of boundary value problems (involving mixed, transmission, Robin boundary conditions, etc., in the Euclidean setting as well as on Riemannian manifolds). In particular, formulating boundary value problems with more regular data requires developing a brand of Sobolev spaces on the boundaries of sets of locally finite perimeter, as we do in [185, Chap. 11].

(IV) <u>`Fatou−Type Results`</u> : Here is a classical result originating in Fatou's 1906 work [83]:

> with $\mathbb{D}$ denoting the unit disk in the complex plane, if the function $u : \mathbb{D} \to \mathbb{C}$ is holomorphic and bounded then for each aperture $\kappa > 0$ the nontangential boundary trace $\Big(u\Big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\Big)(e^{i\theta})$ exists for $\mathcal{L}^1$-a.e. angle $\theta \in [0, 2\pi)$.

(1.1.39)

While $u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}$ always belongs to $L^\infty(\partial\mathbb{D}, \mathcal{H}^1)$, in general this may exhibit jump-discontinuities even when such a nontangential limit exists at every point on $\partial\mathbb{D}$. A good example is offered by the function[6]

$$u : \mathbb{D} \longrightarrow \mathbb{C}, \quad u(z) := \exp\Big\{\frac{z+1}{z-1}\Big\} \text{ for each } z \in \mathbb{D}. \tag{1.1.40}$$

Then, by design, u is holomorphic in $\mathbb{D}$ and extends continuously to $\overline{\mathbb{D}} \setminus \{1\}$. Moreover, u is bounded since

$$|u(z)| = \exp\Big\{\frac{|z|^2-1}{|z-1|^2}\Big\} \leq 1, \qquad \forall z \in \overline{\mathbb{D}} \setminus \{1\}. \tag{1.1.41}$$

Let us now fix an arbitrary aperture parameter $\kappa > 0$. From the equality in (1.1.41) and the fact that u extends continuously to $\overline{\mathbb{D}} \setminus \{1\}$, it follows that the nontangential limit $\big(u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\big)(e^{i\theta})$ exists and satisfies $\Big|\big(u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\big)(e^{i\theta})\Big| = 1$ for each $\theta \in (0, 2\pi)$. This being said, we claim that we actually have

$$\big(u\big|_{\partial\mathbb{D}}^{\kappa-\text{n.t.}}\big)(1) = 0. \tag{1.1.42}$$

To justify (1.1.42) observe that if $z \in \mathbb{D}$ belongs to the nontangential approach region (or Stolz region) with apex at the point $1 \in \partial\mathbb{D}$ and aperture κ, i.e., if z satisfies $|z-1| < (1+\kappa)\,\mathrm{dist}(z, \partial\mathbb{D}) = (1+\kappa)(1-|z|)$, we have

$$\frac{|z|^2-1}{|z-1|^2} < -\frac{(1+\kappa)^{-1}}{|z-1|}. \tag{1.1.43}$$

Then (1.1.42) follows by observing that (1.1.41) and (1.1.43) permit us to estimate

$$0 \leq |u(z)| = \exp\Big\{\frac{|z|^2-1}{|z-1|^2}\Big\} < \exp\Big\{-\frac{(1+\kappa)^{-1}}{|z-1|}\Big\} \to 0 \text{ as } z \to 1. \tag{1.1.44}$$

In general, one cannot hope for a better conclusion in (1.1.39) since Lusin has proved in [166] that

for any Lebesgue measurable set $E \subseteq [0, 2\pi)$ with $\mathcal{L}^1(E) = 0$ there exists a bounded holomorphic function $u : \mathbb{D} \to \mathbb{C}$ whose radial limit $\lim\limits_{r\to 1^-} u(re^{i\theta})$ fails to exist for each $\theta \in E$. (1.1.45)

Also, insisting that the limit is taken from within nontangential approach regions is both natural and optimal in the context of Fatou's Theorem. Indeed, on the one hand, Lindelöf has shown in [163] that

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[6] Often referred to as the *atomic function*.

if $\gamma : [0,1) \to \mathbb{D}$ is a continuous curve such that $\lim\limits_{t\to 1^-} \gamma(t) = z \in \partial\mathbb{D}$ and u is a bounded holomorphic function in $\mathbb{D}$ with the property that $\zeta := \lim\limits_{t\to 1^-} u(\gamma(t))$ exists in $\mathbb{C}$ then actually the nontangential limit of u at the point z exists and equals ζ. (1.1.46)

On the other hand, Littlewood has given an example in [164] of a bounded holomorphic function in the unit disk which diverges almost everywhere along rotated copies of any given curve in the unit disk ending tangentially to the boundary. Specifically, the following result holds (see also [165] and [265, Theorem 7.44]):

if $\gamma_0 \subseteq \mathbb{D} \cup \{1\}$ is a simple closed curve which is tangent to the unit circle $\partial\mathbb{D}$ at the point 1, and for any angle $\theta \in [0, 2\pi)$ we denote by γ_θ the rotation of γ_0 by θ (i.e., set $\gamma_\theta := e^{i\theta}\gamma_0$), then there exists a bounded holomorphic function $u : \mathbb{D} \to \mathbb{C}$ (which may be taken to be a suitable Blaschke product) with the property that for $\mathcal{L}^1$-a.e. $\theta \in [0, 2\pi)$ the limit of u along γ_θ, i.e., $\lim\limits_{|z|\to 1,\, z\in\gamma_\theta} u(z)$, does *not* exist. (1.1.47)

In this vein, we wish to note that H. Aikawa has given in [4] a stronger rendition of (1.1.47) to the effect that

there exists a bounded harmonic function u in the unit disk $\mathbb{D}$ in the plane with the property that the limit $\lim\limits_{|z|\to 1,\, z\in\gamma_\theta} u(z)$ does not exist for *any* angle $\theta \in [0, 2\pi)$. (1.1.48)

Subsequently, in [5] H. Aikawa has established a higher-dimensional version of this result, of the following flavor:

if $n \in \mathbb{N}$ with $n \geq 2$, and if $\gamma \subseteq \mathbb{R}^n_+ \cup \{0\}$ is a simple closed curve which is tangent to the hyperplane $\partial\mathbb{R}^n_+$ at the origin, then there exists a bounded harmonic function u in the upper half-space $\mathbb{R}^n_+$ with the property that the limit $\lim_{x_n\to 0,\,(x',x_n)\in\gamma+(z',0)} u(x', x_n)$ does not exist for *any* vector $z' \in \mathbb{R}^{n-1}$. (1.1.49)

On the positive side, in 1950 A. P. Calderón has proved in [34] that if u is a harmonic function in $\mathbb{R}^n_+$ which is nontangentially bounded at every point of a measurable set $F \subseteq \partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}$ (in the sense that for each $x \in F$ there exist κ, h, M positive numbers such that $|u(y)| \leq M$ whenever $y \in B(x,h) \cap \mathbb{R}^n_+$ is such that $|y - x| < (1+\kappa)\,\mathrm{dist}(y, \partial\mathbb{R}^n_+)$) then u has a nontangential limit at $\mathcal{L}^{n-1}$-a.e. point in F. In 1962, L. Carleson [40] obtained the same conclusion, but with the hypothesis of nontangential boundedness replaced by nontangential boundedness from below. In 1977, B. Dahlberg has proved in [61] that non-negative harmonic functions in Lipschitz domains have nontangential limits at every boundary point except perhaps for a set of zero $(n-1)$-dimensional Hausdorff measure.

This body of work has been generalized by D. Jerison and C. Kenig in [132], whose stated goal was "to extend classical results on the boundary behavior of harmonic functions in [the upper half-space] to domains Ω of as general a type as possible." Specifically, the Fatou-type result in [132, Theorem 6.4, p. 112] in the setting of nontangentially accessible domains (aka NTA domains) asserts that

if $\Omega \subseteq \mathbb{R}^n$ is a bounded NTA domain, ω is the harmonic measure in Ω with pole at a fixed point in Ω, and u is a harmonic function in Ω which is nontangentially bounded from below (cf. [132, p. 110]) at each point belonging to some set $F \subseteq \partial\Omega$, then u has nontangential limits at ω-a.e. point on F. (1.1.50)

Again, the formulation of our main results regarding the Divergence Theorem exhibits features that are intrinsic to the Fatou-type results described above. In turn, the availability of such a brand of Divergence Theorem allows for a number of significant extensions and generalizations in this area, of the sort discussed in [186, Chap. 3].

(V) Integral Representation Formulas in Complex Analysis: The very path that complex analysis has carved for itself, while emerging early on as an independent branch of mathematics, has been strongly influenced by the two-dimensional version of the Fundamental Theorem of Calculus, aka Green's Theorem. One remarkable example of the deep and long-lasting impact that the latter theorem has had on the field as a whole is the following integral representation formula, valid for bounded $\mathscr{C}^1$ domains $\Omega \subseteq \mathbb{C}$ and complex-valued functions $u \in \mathscr{C}^1(\overline{\Omega})$:

$$u(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\, \mathrm{d}\zeta - \frac{1}{\pi} \int_{\Omega} \frac{(\bar{\partial} u)(\zeta)}{\zeta - z}\, \mathrm{d}\mathcal{L}^2(\zeta), \qquad \forall z \in \Omega. \tag{1.1.51}$$

This played a crucial role in the development of modern function theory in the twentieth century and continues to be most relevant today. For example, an elegant approach, adopted by Hörmander [126] and followed by many authors since (cf., e.g., [13, 16, 18, 119, 158, 227, 228]), is to derive the basic theorems of the theory of functions of one complex variable starting with the integral representation formula (1.1.51). The cornerstone of this approach is that formula (1.1.51) provides an integral solution operator for the inhomogeneous Cauchy–Riemann equation $\bar{\partial} u = f$ in the set $\Omega \subseteq \mathbb{C}$, that is, $u = Tf$ with

$$(Tf)(z) := \frac{1}{\pi} \int_{\Omega} \frac{f(\zeta)}{z - \zeta}\, \mathrm{d}\mathcal{L}^2(\zeta), \qquad z \in \Omega. \tag{1.1.52}$$

The idea, which appears to originate in the multidimensional case, is that many of these basic theorems reduce to solving $\bar{\partial}$ equations.

The integral representation formula (1.1.51) has been first published by Dimitrie Pompeiu in 1912 in a series of papers [221–223], and then revisited in 1913 in

[225]. In its original formulation, Pompeiu's integral representation formula reads as follows: *If* Ω *is a bounded piecewise* $\mathscr{C}^1$ *domain in the complex plane, then for each function* $u \in \mathscr{C}^0(\overline{\Omega})$ *with a continuous areolar derivative in a neighborhood of* $\overline{\Omega}$ *one has*

$$u(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{u(\zeta)}{\zeta - z}\, \mathrm{d}\zeta - \frac{1}{\pi} \int_{\Omega} \frac{Du}{D\omega}(\zeta) \frac{1}{\zeta - z}\, \mathrm{d}\mathcal{L}^2(\zeta), \qquad z \in \Omega. \tag{1.1.53}$$

Above, the areolar derivative $Du/D\omega$ (*dérivée aréolaire* in Pompeiu's original paper) is defined as the limit

$$\frac{Du}{D\omega}(\zeta) := \lim_{\mathcal{O} \downarrow \zeta} \frac{\int_{\partial\mathcal{O}} u(z)\, \mathrm{d}z}{2\mathrm{i} \cdot \mathrm{area}(\mathcal{O})}, \qquad \zeta \in \Omega, \tag{1.1.54}$$

where $\mathcal{O}$ is a piecewise $\mathscr{C}^1$ domain shrinking to the point ζ. This concept can be thought of as a quantitative way of measuring the failure of u to be holomorphic near the point ζ (indeed, if u were holomorphic in a neighborhood of ζ Morera's theorem would imply $(Du/D\omega)(\zeta) = 0$). It was introduced by Pompeiu in 1912, in [224], where he also noted that for continuously differentiable functions $D/D\omega = \overline{\partial}$, the Cauchy–Riemann operator. The integral representation formula (1.1.53) was the result of Pompeiu's efforts to construct an integration theory corresponding to this notion of areolar derivative. In his own words [222]:

"La considération de cette dérivée aréolaire conduit à une théorie de l'intégration tout à fait analogoue à la théorie classique de l'intégration d'une fonction réelle de variable réelle [$\cdots$]. *Dans ma derniére note des 'Rendiconti'* [cf. [225]] *j'ai donné la formule* [see (1.1.53)] *qui est tout à fait analogue à la formule classique*

$$F(x) = F(x_0) + \int_{x_0}^{x} F'(\xi)\, \mathrm{d}\xi \quad [\cdots]''. \tag{1.1.55}$$

For more information on the history of (1.1.53), the interested reader is referred to [200].

As is apparent from the above discussion, the very format of Green's Theorem available influences the nature of the integral representation formula presented earlier. In particular, Pompeiu's formula (1.1.53) brings into focus the usefulness of considering derivatives in a weak or generalized sense, rather than the standard strong pointwise sense. This aspect manifests itself in the manner we formulate our main results pertaining to the Divergence Theorem later in this chapter.

The higher-dimensional version of the Cauchy integral operator, tacitly appearing in (1.1.53), is the Bochner–Martinelli integral operator. Classically, the Bochner–Martinelli integral operator acting on a complex-valued function f defined on a $\mathscr{C}^1$-smooth submanifold Σ of $\mathbb{C}^n$ is given by

$$\mathcal{B}f(z) := \int_{\Sigma} f(\zeta) K(z, \zeta), \qquad \forall z \in \mathbb{C}^n \setminus \Sigma, \tag{1.1.56}$$

where, if $\mathrm{d}[\overline{\zeta}]_j := \mathrm{d}\overline{\zeta}_1 \wedge \cdots \wedge \mathrm{d}\overline{\zeta}_n$ with $\mathrm{d}\overline{\zeta}_j$ omitted,

$$K(z,\zeta) := c_n \sum_{j=1}^{n} (-1)^j \frac{\overline{\zeta}_j - \overline{z}_j}{|\zeta - z|^{2n}} \mathrm{d}[\overline{\zeta}]_j \wedge \mathrm{d}\zeta. \tag{1.1.57}$$

However, this commonly held point of view is no longer practical if Σ is lacking regularity. To find an alternative formula, we note that the pull-back of the differential form $\mathrm{d}[\overline{\zeta}]_j \wedge \mathrm{d}\zeta$ under the canonical embedding $\iota : \Sigma \hookrightarrow \mathbb{C}^n$ is

$$\iota^*\Big(\mathrm{d}[\overline{\zeta}]_j \wedge \mathrm{d}\zeta\Big) = c_n \sum_{j=1}^{n} (-1)^j (\nu_{\mathbb{C}})_j \,\mathrm{d}\sigma, \tag{1.1.58}$$

where, with $\nu = (\nu_1, \dots, \nu_{2n}) \in \mathbb{R}^{2n}$ denoting the (real) outward unit normal vector to Ω, we have denoted by $\nu_{\mathbb{C}} := (\nu_1 + \mathrm{i}\nu_2, \dots, \nu_{2n-1} + \mathrm{i}\nu_{2n}) \in \mathbb{C}^n$ the complex outward unit normal to Ω, and we have let σ stand for the surface measure on Σ. Thus, in some sense, the analysis implicit in (1.1.58) brings to light the geometry of Σ in a much more transparent fashion than (1.1.56) (admittedly, an elegant formula but which nonetheless obscures the geometric nature of Σ).

The true virtue of this seemingly mundane observation is that the concept of unit normal and surface measure makes sense in much greater generality (than that of a smooth surface) and, hence, it allows us to consider the Bochner–Martinelli integral operator in some very rough settings, and study it from the perspective of Calderón–Zygmund theory in a class of domains which is essentially optimal from the point of view of Geometric Measure Theory. We shall amply elaborate on this aspect later on. For now, we wish to point out that even when the underlying surface Σ is smooth, if the function f is merely p-th power integrable on Σ (with respect to the surface measure σ), then the size of $\mathcal{B}f$ is most naturally measured using the nontangential maximal operator, and the boundary trace of $\mathcal{B}f$ is most naturally described in terms of nontangential pointwise limits. The bottom line is that in order for the Bochner–Martinelli integral operator to maintain its central relevance within the theory of functions of several complex variables in rough settings, we need a Divergence Theorem which is the accommodating of such features.

(VI) `Potential Theory`: Centered around the study of harmonic functions, potential theory is a highly refined branch of mathematics within which the concepts of harmonic measure ω, Poisson kernel k, and Green function G play a key role. If Ω is a bounded domain of class $\mathscr{C}^\infty$ in $\mathbb{R}^n$, then boundary regularity shows that its associated Green function G is of class $\mathscr{C}^\infty$ in $\overline{\Omega} \times \overline{\Omega} \setminus \mathrm{diag}$. Also, the standard version of the Divergence Theorem gives that the Poisson kernel is the normal derivative of the Green function, $k = -\partial_\nu G$ where ν is the outward unit normal to Ω, and the harmonic measure ω^{x_o} with pole at some fixed point $x_o \in \Omega$ is related to the surface measure σ on $\partial\Omega$ via $\omega^{x_o} = k(\cdot, x_o)\sigma$. In particular, given that Hopf's Boundary Point Principle ensures that $\inf_{\partial\Omega}\big(-\nu \cdot \nabla G(\cdot, x_o)\big) \geq \varepsilon_o > 0$, it follows that in the smooth setting the harmonic measure ω^{x_o} behaves essentially like the surface measure σ.

This straightforward correlation between the smoothness of the domain and the regularity of the associated Green function remains valid up to the threshold $\mathscr{C}^{1,\alpha}$ for some $\alpha \in (0,1)$ (cf. [111, Theorem 3.5, p. 33]), though it breaks down at the end-point $\alpha = 0$. This being said, certain resilient features continue to survive well beyond this mark. For example, in the class of NTA domains with Ahlfors regular[7] boundaries the harmonic measure and the surface measure are mutually absolutely continuous, the gradient of the Green function has an integrable (truncated) maximal function, and the normal gradient of the Green function continues to make sense if interpreted as $\nu \cdot \big[(\nabla G)\big|^{\rm n.t.}_{\partial\Omega}\big]$. Elucidating the relationship of the normal derivative of G, thus interpreted, to the Poisson kernel requires a Divergence Theorem which is applicable to vector fields exhibiting similar behavior to ∇G. See the discussion in [186, Chap. 5] in this regard.

Our Divergence Theorem, several progressively more general versions of which are discussed in Sects. 1.2–1.12 (with certain specialized features presented in Sects. 1.9–1.10), then becomes a powerful and versatile tool in all these areas, as well as others. This point is amply illustrated by the applications discussed in subsequent volumes, which deal with topics in Function Space Theory, Complex Analysis (in one and several variables), Clifford Analysis, Potential Theory, Scattering, Calderón–Zygmund Theory, Partial Differential Equations, and Harmonic Analysis.

1.2 The Case When the Divergence Is Absolutely Integrable

Our first main result pertains to the Divergence Theorem in its standard format, as the equality between the solid integral of the divergence of the given vector field and the boundary integral of the inner product of the field in question with the geometric measure theoretic outward unit normal to the underlying domain. To facilitate its reading, here is a glossary of terms entering its formulation:

• The notion of lower Ahlfors regular set is introduced in part *(i)* of Definition 5.9.1. Also, it being a doubling measure is defined in (7.4.1).

• Given an open set $\Omega \subseteq \mathbb{R}^n$, the notation $\partial_{\rm nta}\Omega$ (cf. Definition 8.8.5) is reserved for what we call the nontangentially accessible boundary of Ω, discussed in detail in Sect. 8.8. In particular, part *(iii)* of Proposition 8.8.6 shows that if $\partial\Omega$ is a lower Ahlfors regular set and the measure $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is doubling, then $\sigma(\partial_*\Omega \setminus \partial_{\rm nta}\Omega) = 0$, i.e., $\partial_{\rm nta}\Omega$ covers $\partial_*\Omega$ up to a σ-nullset.

• For a given vector field $\vec{F}$ whose components are $\mathcal{L}^n$-measurable functions defined in an open set Ω, we shall denote by $\vec{F}\,\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ its nontangential boundary trace on $\partial\Omega$ (in the sense of Definition 8.9.1, in a componentwise fashion), i.e., the boundary limit

[7] Aka Ahlfors–David regular, or ADR for short.

of $\vec{F}$ taken from within nontangential approach regions, $\Gamma_\kappa(x)$, with apex at points $x \in \partial\Omega$ and fixed aperture parameter $\kappa \in (0, \infty)$ (described in (8.1.2)). Finally, the nontangentially maximal operator $\mathcal{N}_\kappa$ with aperture $\kappa > 0$ is formally introduced in (8.2.1) as the essential supremum of a given Lebesgue measurable function in Ω over such nontangential approach regions.

Here is the actual statement of the theorem alluded to earlier.

Theorem 1.2.1 *Pick $n \in \mathbb{N}$ and let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Fix $\kappa \in (0, \infty)$ and assume that the vector field $\vec{F} = (F_1, \dots, F_n) : \Omega \to \mathbb{C}^n$, with Lebesgue measurable components, has the property that*

$$\begin{gathered}\text{the nontangential trace } \vec{F}\Big|_{\partial\Omega}^{\kappa-n.t.} \text{ exists (in } \mathbb{C}^n\text{)}\ \sigma\text{-a.e. on } \partial_{nta}\Omega, \\ \mathcal{N}_\kappa\vec{F} \in L^1(\partial\Omega, \sigma), \text{ and } \operatorname{div}\vec{F} := \partial_1 F_1 + \cdots + \partial_n F_n \in L^1(\Omega, \mathcal{L}^n),\end{gathered} \tag{1.2.1}$$

where all partial derivatives are considered in the sense of distributions in Ω.

Then for any $\kappa' > 0$ the nontangential trace $\vec{F}\Big|_{\partial\Omega}^{\kappa'-n.t.}$ exists σ-a.e. on $\partial_{nta}\Omega$ and is actually independent of κ'. When regarding it as a function defined σ-a.e. on $\partial_*\Omega$ (which, up to a σ-nullset, is contained in $\partial_{nta}\Omega$), this nontangential trace belongs to $\big[L^1(\partial_*\Omega, \sigma)\big]^n$. Also, with the dependence on the parameter κ' dropped, one has

$$\int_\Omega \operatorname{div}\vec{F}\, d\mathcal{L}^n = \int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\Big|_{\partial\Omega}^{n.t.}\Big)\, d\sigma \tag{1.2.2}$$

when either Ω is bounded, or $\partial\Omega$ is unbounded. In the remaining case, i.e., when Ω is unbounded and $\partial\Omega$ is bounded, formula (1.2.2) *continues to hold under the additional assumption that there exists $\lambda \in (1, \infty)$ such that*

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |x \cdot \vec{F}(x)|\, d\mathcal{L}^n(x) = o(R^2) \ \text{ as } \ R \to \infty. \tag{1.2.3}$$

A few remarks, designed to elaborate on the nature and scope of the above theorem, are in order.

<u>Remark 1.</u> Assuming that the set Ω is open is natural from the perspective of being able to compute $\operatorname{div}\vec{F}$ in the sense of distributions. In relation to this, the membership of $\mathcal{N}_\kappa\vec{F}$ to $L^1(\partial\Omega, \sigma)$ forces $\vec{F}$ to be locally bounded, hence, locally integrable (cf. Lemma 8.3.1), so it is meaningful to consider $\operatorname{div}\vec{F}$ in $\mathcal{D}'(\Omega)$. The hypotheses made on the vector field $\vec{F}$ are just about strong enough to ensure that the two sides of the Divergence Formula (1.2.2) are meaningful (i.e., given by absolutely convergent integrals of measurable functions). Also, we work with the standard notion of integral (in the sense of Lebesgue) and the right-hand side of the Divergence Formula (1.2.2)

is defined in a logically independent way from the left-hand side of (1.2.2). These attributes are very desirable in applications.

Remark 2. The lower Ahlfors regularity condition imposed on $\partial\Omega$ is automatically satisfied if the set Ω is two-sided n-thick (cf. Definition 5.1.1 and Proposition 5.9.16), or if $n = 2$ and $\partial\Omega$ is a rectifiable curve (cf. Lemma 5.9.3). Also, it is clear from definitions that

all conclusions in Theorem 1.2.1 are valid in the case when the underlying domain Ω is an open nonempty proper subset of $\mathbb{R}^n$ with an Ahlfors regular boundary. (1.2.4)

Open sets with Ahlfors regular boundaries are rife in analysis, and a multitude of examples may be found in Sect. 5. Here we only wish to remark that the Ahlfors regularity of the boundary does not imply any regularity for the domain in question in a traditional sense (for example, it does not prevent the formation of inner or outer cusps) (Fig. 1.1).

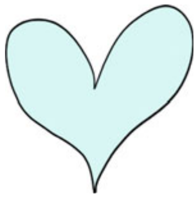


Fig. 1.1 An Ahlfors regular domain with cusps

Any reasonable definition of the class of piecewise Lipschitz domains should produce examples of open sets with lower Ahlfors regular boundaries. We further elaborate on the format of our brand of Divergence Theorem in the aforementioned class of domains in Corollary 1.2.4.

Remark 3. Since the geometric measure theoretic outward unit normal ν may be regarded as a function in $\big[L^\infty(\partial_*\Omega, \sigma)\big]^n$, being guaranteed that $\vec{F}\big|^{\rm n.t.}_{\partial\Omega}$ is a well-defined function σ-a.e. on $\partial_*\Omega$, which actually belongs to $\big[L^1(\partial_*\Omega, \sigma)\big]^n$, allows us to make sense of the right-hand side of (1.2.2) as an absolutely convergent integral.

Remark 4. Let us call an open subset Ω of $\mathbb{R}^n$ an exterior domain provided Ω is the complement of a compact subset of $\mathbb{R}^n$. Under the assumption $n \geq 2$, an open set $\Omega \subseteq \mathbb{R}^n$ is an exterior domain if and only if Ω is unbounded and $\partial\Omega$ is bounded (cf. Lemma 5.10.10). In this regard, we wish to observe that, as a simple application of Hölder's inequality shows,

if $n \geq 2$ and Ω is an exterior domain in $\mathbb{R}^n$, then (1.2.3) holds whenever $|\vec{F}|^{n/(n-1)}$ is Lebesgue integrable in a neighborhood of infinity in $\mathbb{R}^n$. (1.2.5)

More generally, condition (1.2.3) is satisfied whenever

$$\text{there exists a compact set } K_o \subseteq \mathbb{R}^n \text{ with the property that} \\ \vec{F} \in \big[L^p(\Omega \setminus K_o, \mathcal{L}^n)\big]^n \ \text{ for some finite } \ p \in \big[1, \tfrac{n}{n-1}\big]. \tag{1.2.6}$$

Indeed, in such a scenario we may use Hölder's inequality to estimate

$$R^{-1} \int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |\vec{F}| \, d\mathcal{L}^n \leq C R^{(n-1)-n/p} \Big(\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |\vec{F}|^p \, d\mathcal{L}^n \Big)^{1/p}, \tag{1.2.7}$$

for some constant $C = C_{n,\lambda} \in (0, \infty)$, and since $(n-1) - n/p \leq 0$ for $p \in \big[1, \frac{n}{n-1}\big]$, Lebesgue's Dominated Convergence Theorem (which uses (1.2.6) and the finiteness of p) may then be invoked to conclude that (1.2.3) holds.

In all dimensions $n \in \mathbb{N}$, if $\Omega \subseteq \mathbb{N}$ is an unbounded set then condition (1.2.3) is also implied by

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |\vec{F}(x)| \, d\mathcal{L}^n(x) = o(R) \ \text{ as } \ R \to \infty, \tag{1.2.8}$$

which, in turn, is implied by the following pointwise decay property:

$$\vec{F}(x) = o(|x|^{1-n}) \ \text{ as } \ x \in \Omega \ \text{ satisfies } \ |x| \to \infty. \tag{1.2.9}$$

In particular, condition (1.2.3) holds in such a setting provided there exists some number $\varepsilon > 0$ for which $\vec{F}(x) = O(|x|^{1-n-\varepsilon})$ as $x \in \Omega$ satisfies $|x| \to \infty$.

Finally, if $n \geq 2$ and Ω is an exterior domain in $\mathbb{R}^n$, then (1.2.3) may be equivalently recast as

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |x \cdot \vec{F}(x)| \, d\mathcal{L}^n(x) = o(R^{2-n}) \ \text{ as } \ R \to \infty \tag{1.2.10}$$

(where the barred integral stands for integral average), which holds whenever

$$⨍_{B(0,\lambda R)\setminus B(0,R)} |\vec{F}| \, d\mathcal{L}^n = o(R^{1-n}) \ \text{ as } \ R \to \infty. \tag{1.2.11}$$

Remark 5. Both the geometric hypotheses on the domain Ω and the analytic hypotheses on the vector field $\vec{F}$ made in Theorem 1.2.1 are stable under bi-Lipschitz changes of variables of the Euclidean ambient (see Lemma 8.1.7 and the transformational properties under bi-Lipschitz maps established in [124], in this regard).

Remark 6. It turns out (as may be seen with the help of Proposition 8.6.3) that the class of vector fields $\vec{F}$, with $\mathcal{L}^n$-measurable components in Ω, satisfying (1.2.1) and, whenever applicable, the growth condition (1.2.3), is a linear subspace of $\big[L^1_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^n$ which is stable under multiplication by scalar Lipschitz functions with bounded support. As regards the latter property, let us also note that if $\vec{F}$ is as above, and if $\varphi \in \mathrm{Lip}(\Omega)$ (the space of Lipschitz functions in Ω) is scalar-valued with bounded support, then the Divergence Formula (1.2.2) written for the vector field $\varphi\vec{F}$ yields the integration by parts formula

$$\int_{\Omega} \varphi \,\mathrm{div}\vec{F}\,\mathrm{d}\mathcal{L}^n = -\int_{\Omega} \nabla\varphi \cdot \vec{F}\,\mathrm{d}\mathcal{L}^n + \int_{\partial_*\Omega} \varphi\,\nu \cdot \big(\vec{F}\,\big|^{\mathrm{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma. \tag{1.2.12}$$

Remark 7. Of course, Theorem 1.2.1 contains (1.1.2) as a very special case. More generally, the scenario in which Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ and the vector field $\vec{F}$ belongs to $\big[\mathscr{C}^0(\overline{\Omega})\big]^n$, it is differentiable at every point in Ω, and the expression $\sum_{j=1}^{n} \partial_j F_j$ (where the partial derivatives are considered in a pointwise, classical sense) is continuous and absolutely integrable on Ω, is also covered by Theorem 1.2.1; cf. Proposition 2.8.11 for a more general result of this flavor.

Significantly, Theorem 1.2.1 contains (when $n = 1$) the sharp form of the Fundamental Theorem of Calculus recorded in (1.1.1) (see the discussion in Sect. 2.6 in this regard).

Remark 8. As regards the sharpness of Theorem 1.2.1, in Chap. 2 we provide a series of counterexamples which point to the fact that our hypotheses are in the nature of best possible, as far as the format of the conclusion is concerned. Concretely, the fact that the lower Ahlfors regularity of the boundary hypothesized in Theorem 1.2.1 may not be eliminated is visible from (2.4.9), or (2.4.18), while the naturalness of considering $\mathrm{div}\vec{F}$ in the sense of distributions is apparent from (2.5.13). Hypothesizing, as we do in the first line of (1.2.1), that the nontangential boundary trace $\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ exists at σ-a.e. point on $\partial_{\mathrm{nta}}\Omega$ cannot be relaxed to merely demanding that $\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ exists at σ-a.e. point on $\partial_*\Omega$, even though it is precisely the latter set that enters the formulation of the Divergence Formula (1.2.2). We justify this (by means of concrete counterexamples) in (2.1.17), or (2.1.47), or (2.1.64), or (2.1.87), or (2.1.102). The counterexample in (2.3.7) shows that, as far as the validity of the Divergence Formula (1.2.2) is concerned, the hypothesis that $\mathcal{N}_\kappa\vec{F}$ belongs to $L^1(\partial\Omega,\sigma)$ cannot be weakened to either $\mathcal{N}_\kappa\vec{F} \in L^{1,\infty}(\partial\Omega,\sigma)$, or $\mathcal{N}_\kappa\vec{F} \in L^1(\partial_*\Omega,\sigma)$. See also (2.3.64) in this regard. In (2.2.15) we note that replacing $o(R^2)$ by $O(R^2)$ in the formulation of the growth condition (1.2.3) may invalidate the Divergence Formula (1.2.2). Lastly, from (2.3.72) we learn that the Divergence Formula (1.2.2) may fail for open sets Ω with an unbounded lower Ahlfors regular boundary and a doubling "surface measure" if the assumption $\mathcal{N}_\kappa\vec{F} \in L^1(\partial\Omega,\sigma)$ is replaced by the weaker condition $\mathcal{N}_\kappa\vec{F} \in L^1_{\mathrm{loc}}(\partial\Omega,\sigma)$.

Remark 9. In relation to the hypothesis made in the first line of (1.2.1), we wish to note that the largest set on which it is meaningful to even contemplate the existence of the nontangential trace $\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ is $A_\kappa(\partial\Omega) := \big\{x \in \partial\Omega : x \in \overline{\Gamma_\kappa(x)}\big\}$. By design (cf. (8.8.46)), the set $A_\kappa(\partial\Omega)$ always contains the nontangentially accessible boundary $\partial_{\mathrm{nta}}\Omega$, but whenever σ is a doubling measure it turns out that $\sigma\big(A_\kappa(\partial\Omega) \setminus \partial_{\mathrm{nta}}\Omega\big) = 0$; see item *(ii)* in Proposition 8.8.6.

While it is natural to hypothesize, as we currently do, that the nontangential boundary trace of $\vec{F}$ exists σ-a.e. on $\partial_{\text{nta}}\Omega$, the format of the Divergence Formula (going back to its classical formulation in (1.1.8)) requires that said trace is integrated over $\partial_*\Omega$. For this to be possible we must therefore have that $\partial_{\text{nta}}\Omega$ covers, up to a σ-nullset, $\partial_*\Omega$. The current geometric hypotheses ensure that this is indeed the case, which is remarkable given that $\partial_{\text{nta}}\Omega$ is of a purely metric/topological nature while $\partial_*\Omega$ has a geometric measure theoretic character.

Remark 10. From Lemma 4.7.5 and (4.7.37), we see that Theorem 1.2.1 actually continues to hold in the limiting case $\Omega = \mathbb{R}^n$.

Remark 11. The proof of Theorem 1.2.1 as stated is presented in Sect. 9.1.

While we shall present several more refined versions of Theorem 1.2.1 in subsequent sections, our result here already improves upon [125, Theorem 2.8, p. 2587] where the set Ω was assumed to be bounded, $\partial_*\Omega$ was assumed to have full $\mathcal{H}^{n-1}$ measure in $\partial\Omega$, the topological boundary of the set Ω was assumed to be upper Ahlfors regular, the vector field $\vec{F}$ was assumed to be continuous in Ω, and its nontangential maximal function was assumed to satisfy the higher integrability condition $\mathcal{N}_\kappa\vec{F} \in L^p(\partial\Omega, \sigma)$ for some exponent $p \in (1, \infty)$. As apparent from the statement of Theorem 1.2.1, these features are unnecessarily strong demands for the validity of the Divergence Formula (1.2.2).

Theorem 1.2.1 yields new, nontrivial results even in ordinary geometric settings, such as the case when Ω is the upper half-space $\mathbb{R}^n_+$ with $n \geq 2$. In such a scenario, the specific geometry of the environment impacts the very formulation of the result. For one thing, the nontangential approach regions become genuine upright circular cones, of various fixed apertures, with vertices on the boundary of $\mathbb{R}^n_+$. Specifically, in this setting for each $\kappa \in (0, \infty)$ we now have

$$\Gamma_\kappa(x) = \Big\{y = (y', y_n) \in \mathbb{R}^n_+ : |x' - y'| < \big(\kappa^2 + 2\kappa\big)^{1/2} y_n\Big\}, \quad \forall x = (x', 0) \in \partial\mathbb{R}^n_+. \tag{1.2.13}$$

For another thing, there is a version of Theorem 1.2.1 in $\mathbb{R}^n_+$ which requires less when it comes to demanding the existence of the nontangential boundary trace in the first line of (1.2.1). Here is the actual statement of the result we have in mind.

Corollary 1.2.2 *Fix $n \in \mathbb{N}$ with $n \geq 2$ and pick some arbitrary $\kappa \in (0, \infty)$. Assume that the vector field $\vec{F} = (F_1, \dots, F_n) : \mathbb{R}^n_+ \to \mathbb{C}^n$, with Lebesgue measurable components, satisfies the following properties:*

$$\begin{aligned} &\text{the nontangential trace } F_n\Big|^{\kappa-n.t.}_{\partial\mathbb{R}^n_+} \text{ exists (in } \mathbb{C}) \ \mathcal{L}^{n-1}\text{-a.e. on } \partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}, \\ &\mathcal{N}_\kappa\vec{F} \in L^1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \ \text{and } \operatorname{div}\vec{F} := \partial_1 F_1 + \dots + \partial_n F_n \in L^1(\mathbb{R}^n_+, \mathcal{L}^n), \end{aligned} \tag{1.2.14}$$

where all partial derivatives are considered in the sense of distributions in $\mathbb{R}^n_+$.

Then for any other aperture parameter $\kappa' > 0$, the nontangential trace $F_n\Big|^{\kappa'-n.t.}_{\partial\mathbb{R}^n_+}$ exists $\mathcal{L}^{n-1}$-a.e. on $\mathbb{R}^{n-1}$ and is actually independent of κ'. When regarding it as a *function defined* $\mathcal{L}^{n-1}$*-a.e. on* $\mathbb{R}^{n-1}$*, this belongs to* $L^1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ *and, with the dependence on the parameter* κ' *dropped,*

$$\int_{\mathbb{R}^n_+} div\vec{F}\, d\mathcal{L}^n = -\int_{\mathbb{R}^{n-1}} \Big(F_n\Big|^{n.t.}_{\partial\mathbb{R}^n_+}\Big)\, d\mathcal{L}^{n-1}. \tag{1.2.15}$$

The crucial difference between Theorem 1.2.1 stated for $\Omega := \mathbb{R}^n_+$ and Corollary 1.2.2 is that in the formulation of the latter result we only demand that F_n, the n-th component of $\vec{F}$, has a nontangential trace at a.e. every boundary point (as opposed to asking that the entire vector field $\vec{F}$ has this property). For example, if $n \geq 2$ and $f_1, \dots, f_n \in \mathscr{C}^1_c(\mathbb{R}^n)$ then Theorem 1.2.1 may not be applicable (with $\Omega := \mathbb{R}^n_+$) to the vector field defined as

$$\begin{gathered}\vec{F}(x) := \big(\sin(1/x_n) f_1(x), f_2(x), \dots, f_n(x)\big)\\ \text{for each point } \ x = (x_1, \dots, x_n) \in \mathbb{R}^n_+,\end{gathered} \tag{1.2.16}$$

while Corollary 1.2.2 works just fine in this case. This being said, Theorem 1.2.1 is the key ingredient in the proof of Corollary 1.2.2, presented in Sect. 9.1.

The version of Corollary 1.2.2 corresponding to a vector field $\vec{F}$ with $F_n = 0$ already yields the significant, nontrivial formula $\int_{\mathbb{R}^n_+} \sum_{j=1}^{n-1} \partial_j F_j \, \mathrm{d}\mathcal{L}^n = 0$. This particular result is covered by Proposition 2.8.19, upon realizing that the membership imposed on the nontangential maximal function in (1.2.14) guarantees that all components of $\vec{F}$ belong to $L^{n/(n-1)}(\mathbb{R}^n_+, \mathcal{L}^n)$ (see (8.6.50) in this regard).

At a first glance, it may seem peculiar that in the formulation of Corollary 1.2.2 even though only the n-th component of $\vec{F}$ is required to have a nontangential boundary trace one actually demands nontangential maximal function control of the entire vector field $\vec{F}$. However, without some type of quantitative control for the entire vector field $\vec{F}$ the Divergence Formula (1.2.15) may fail. For example, this is the case in the two-dimensional setting (i.e., when $n = 2$) for the vector field

$$\vec{F}(x, y) := \big((\arctan x)\mathrm{e}^{-y}, 0\big) \ \text{ for each } \ (x, y) \in \mathbb{R}^2_+, \tag{1.2.17}$$

whose last component is identically zero and its divergence,

$$\big(\mathrm{div}\vec{F}\big)(x, y) = \frac{\mathrm{e}^{-y}}{1 + x^2} \ \text{ for each } \ (x, y) \in \mathbb{R}^2_+, \tag{1.2.18}$$

belongs to $L^1(\mathbb{R}^2_+, \mathcal{L}^2)$ but does not integrate to zero. The root of the failure of the Divergence Formula (1.2.15) in this situation lies in the lack of integrability of $\mathcal{N}_\kappa \vec{F}$ on the real line.

A result similar in flavor to Corollary 1.2.2 formulated for a ball, in place of a half-space, is presented in Proposition 2.8.21.

If in place of the first line in (1.2.14) one actually assumes that

$$\begin{array}{c}\text{the full nontangential trace } \ \vec{F}\big|_{\partial\mathbb{R}^n_+}^{\kappa-\text{n.t.}} \ \text{ exists (in } \mathbb{C}^n)\\ \text{at } \mathcal{L}^{n-1}\text{-a.e. point on } \ \partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1},\end{array} \tag{1.2.19}$$

then we may rephrase the conclusion in (1.2.15) more in line with the traditional formulation on the Divergence Formula, namely as

$$\int_{\mathbb{R}^n_+} \operatorname{div}\vec{F}\,\mathrm{d}\mathcal{L}^n = -\int_{\mathbb{R}^{n-1}} \Big(\vec{F}\big|_{\partial\mathbb{R}^n_+}^{\text{n.t.}}\Big)\cdot \mathbf{e}_n\,\mathrm{d}\mathcal{L}^{n-1}. \tag{1.2.20}$$

For mundane applications, it is worth stating a user-friendly version of the Divergence Theorem obtained by specializing Theorem 1.2.1 to a setting where some of the geometric measure theory jargon employed in the formulation of the latter theorem may be omitted by making more familiar, albeit stronger, assumptions on the vector field involved.

Corollary 1.2.3 *Let Ω be an open, bounded, nonempty subset of $\mathbb{R}^n$ (where $n \in \mathbb{N}$), with a lower Ahlfors regular boundary, and such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Denote by ν the geometric measure theoretic outward unit normal to Ω, and fix a σ-measurable set $N \subseteq \partial\Omega$ satisfying $\sigma(N) = 0$. Assume the vector field $\vec{F} \in \big[\mathscr{C}^0(\overline{\Omega}\setminus N)\big]^n$ has bounded components in Ω and its divergence, considered in the sense of distributions, satisfies $div\big(\vec{F}\big|_\Omega\big) \in L^1(\Omega, \mathcal{L}^n)$. Then*

$$\int_\Omega div\big(\vec{F}\big|_\Omega\big)\,d\mathcal{L}^n = \int_{\partial_*\Omega} \nu\cdot\big(\vec{F}\,\big|_{\partial\Omega}\big)\,d\sigma. \tag{1.2.21}$$

It should be noted that the above corollary cannot be derived directly from De Giorgi–Federer's version of the Divergence Theorem recorded in Theorem 1.1.1.

Here is another practical corollary of Theorem 1.2.1, in which all hypotheses are formulated in relation to the topological boundary of the underlying domain. This, nonetheless, is quite versatile. In particular, it applies to the class of piecewise smooth domains without cusps. See also Corollary 1.6.5 for a related version in the two-dimensional case.

Corollary 1.2.4 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) which is two-sided n-thick, in the sense that there exists $c \in (0,\infty)$ with the property that for each $x \in \partial\Omega$ one has*

$$\min\Big\{\mathcal{L}^n\big(B(x,r)\cap\Omega\big), \mathcal{L}^n\big(B(x,r)\setminus\Omega\big)\Big\} \geq cr^n, \quad \forall r \in \big(0, 2\,diam(\partial\Omega)\big). \tag{1.2.22}$$

Also, make the assumption that

$$\partial\Omega \subseteq \bigcup_{j=1}^N \Sigma_j \tag{1.2.23}$$

where $N \in \mathbb{N}$, each Σ_j is the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$, which has been rotated and translated (as a set in $\mathbb{R}^n$), and

$$\mathcal{H}^{n-1}(\Sigma_{j_1} \cap \Sigma_{j_2}) = 0 \ \textit{ for each } \ j_1, j_2 \in \{1, \dots, N\} \ \textit{ with } \ j_1 \neq j_2. \tag{1.2.24}$$

Use (1.2.23)–(1.2.24) to define the surface measure σ on $\partial\Omega$ and the outward unit normal ν to Ω at σ-a.e. point on $\partial\Omega$. Finally, suppose $\vec{F} = (F_1, \dots, F_n) : \Omega \to \mathbb{C}^n$ is a vector field with Lebesgue measurable components with the property that, for some $\kappa \in (0, \infty)$,

$$\begin{gathered} \textit{the nontangential trace } \ \vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{ exists } (\textit{in } \mathbb{C}^n)\ \sigma\textit{-a.e. on } \ \partial\Omega, \\ \mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma), \ \textit{ and } \operatorname{div}\vec{F} := \partial_1 F_1 + \cdots + \partial_n F_n \in L^1(\Omega, \mathcal{L}^n), \end{gathered} \tag{1.2.25}$$

where all partial derivatives are considered in the sense of distributions in Ω.

Then for any other aperture parameter $\kappa' > 0$ the nontangential trace $\vec{F}\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ exists σ-a.e. on $\partial\Omega$ and is actually independent of κ'. When regarding it as a vector-valued function defined σ-a.e. on $\partial\Omega$, this nontangential trace belongs to $\big[L^1(\partial\Omega, \sigma)\big]^n$ and, with the dependence on the parameter κ' dropped, one has

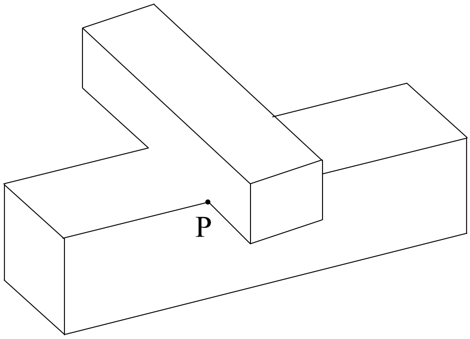
$$\int_\Omega \operatorname{div}\vec{F}\, d\mathcal{L}^n = \int_{\partial\Omega} \nu \cdot \big(\vec{F}\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)\, d\sigma \tag{1.2.26}$$

when either Ω is bounded, or $\partial\Omega$ is unbounded. In the case when Ω is an exterior domain (i.e., the complement of a compact set in $\mathbb{R}^n$), formula (1.2.26) continues to hold under the additional assumption that there exists some $\lambda \in (1, \infty)$ such that

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |x \cdot \vec{F}(x)|\, d\mathcal{L}^n(x) = o(R^2) \ \textit{ as } \ R \to \infty. \tag{1.2.27}$$

Concrete details as to what is meant when saying that (1.2.23)–(1.2.24) are used to define the surface measure on $\partial\Omega$ and the outward unit normal to Ω are given in the first part of the proof of Corollary 1.2.4 (itself provided in Sect. 9.1). Here we only wish to indicate that each Lipschitz graph Σ_j has a canonical surface measure (see (2.8.69)) and, thanks to (1.2.23)–(1.2.24), these may be pieced together to induce a surface measure on $\partial\Omega$. Also, according to the classical Rademacher Theorem, each Lipschitz function is differentiable almost everywhere. As such, each Lipschitz graph Σ_j has a tangent plane almost everywhere which, at points on $\partial\Omega$, we shall consider as being tangent planes to Ω itself. In view of (1.2.23)–(1.2.24) this is unambiguous, and ultimately yields an outward unit normal vector (cf. (2.8.68)) at almost every point on $\partial\Omega$ (chosen as to “point away” from the set Ω). Once again, this is made precise in the first part of the proof of Corollary 1.2.4.

It is clear that Corollary 1.2.4 applies to all Lipschitz domains (with compact boundaries, or upper-graph type), but the class of sets covered by this result extends beyond the category of Lipschitz domains. For example, the configuration known at the "two-brick" domain, depicted below



is a non-Lipschitz domain[8] for which, nonetheless, Corollary 1.2.4 does apply.

We close this section by specializing our main result so far to the two-dimensional setting. In this vein, recall the classical version of Green's Theorem asserting the validity of the formula

$$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y \tag{1.2.28}$$

whenever C is a positively oriented, piecewise smooth, simple closed curve in the plane, Ω is the two-dimensional open region enclosed by C, and $P, Q \in \mathscr{C}^1(\overline{\Omega})$. If $r(t) = (x(t), y(t))$ with $a \leq t \leq b$ is a piecewise smooth parametrization of the curve C, then the line integral in the left-hand side of (1.2.28) may be recast as

$$\begin{aligned}\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y &= \int_a^b \big\{P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\big\}\,\mathrm{d}t \\ &= \int_a^b \big(P(x(t), y(t)), Q(x(t), y(t))\big) \cdot \frac{r'(t)}{|r'(t)|}\sqrt{[x'(t)]^2 + [y'(t)]^2}\,\mathrm{d}t \\ &= \int_C (P, Q) \cdot \tau\,\mathrm{d}s, \end{aligned} \tag{1.2.29}$$

where (P, Q) is viewed as a vector field, $\tau(x(t), y(t)) := \frac{r'(t)}{|r'(t)|}$ is the unit tangent vector to the curve C, and $ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,\mathrm{d}t$ is the arc-length element on C. In this notation, (1.2.28) becomes

$$\int_C (P, Q) \cdot \tau\,\mathrm{d}s = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y. \tag{1.2.30}$$

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[8] Indeed, a moment's reflection shows that, near the point P, the boundary of the above domain is not the graph of any function (as it fails the vertical line test) in any system of coordinates isometric to the original one.

To make the connection between (1.2.30) and the two-dimensional version of the Divergence Formula even more apparent, it is convenient to consider the vector field $\vec{F} := (Q, -P)$ and observe that

$$(P, Q) \cdot \tau = \vec{F} \cdot \nu \quad \text{and} \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \mathrm{div}\vec{F}, \tag{1.2.31}$$

where, with (τ_1, τ_2) denoting the scalar components of the unit tangent vector τ to the curve C, we have set $\nu := (\tau_2, -\tau_1)$ (hence $\nu := -\mathrm{i}\tau$, under the identification $\mathbb{R}^2 \equiv \mathbb{C}$). This makes ν the outward unit normal vector to the domain Ω, and (1.2.30) now simply reads (after slight adjustments in notation)

$$\int_{\partial\Omega} \nu \cdot \big(\vec{F}\big|_{\partial\Omega}\big)\,\mathrm{d}s = \int_{\Omega} \mathrm{div}\vec{F}\,\mathrm{d}\mathcal{L}^2. \tag{1.2.32}$$

Since Theorem 1.2.1 provides a much more potent version of (1.2.32), we may reverse-engineer the route just taken from (1.2.28) to (1.2.32) in order to obtain a considerably more refined version of the classical Green formula recorded in (1.2.28).

Theorem 1.2.5 *Let Ω be an open nonempty proper subset of $\mathbb{R}^2$ with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal $\nu = (\nu_1, \nu_2)$ is defined σ-a.e. on $\partial_*\Omega$. Denote by $\tau := (-\nu_2, \nu_1)$ (or, equivalently, $\tau := \mathrm{i}\nu$ under the identification $\mathbb{R}^2 \equiv \mathbb{C}$) the positively oriented geometric measure theoretic unit tangent vector[9] to the boundary of Ω (which, once again, is defined σ-a.e. on $\partial_*\Omega$).*

Fix an aperture parameter $\kappa \in (0, \infty)$ and assume $P, Q : \Omega \to \mathbb{C}$ are two Lebesgue measurable functions with the following properties:

$$\begin{aligned} &\text{the nontangential traces } P\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ and } Q\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega, \\ &\mathcal{N}_\kappa P,\ \mathcal{N}_\kappa Q \in L^1(\partial\Omega, \sigma), \text{ and } \partial_x Q - \partial_y P \text{ belongs to } L^1(\Omega, \mathcal{L}^2), \end{aligned} \tag{1.2.33}$$

where all partial derivatives are considered in the sense of distributions in Ω.

Then for any $\kappa' > 0$ the nontangential traces $P\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}$ and $Q\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}$ exist σ-a.e. on $\partial_{\text{nta}}\Omega$ and are actually independent of κ'. When regarding them as functions defined σ-a.e. on $\partial_*\Omega$ (which, up to a σ-nullset, is contained in $\partial_{\text{nta}}\Omega$), these nontangential traces belong to $L^1(\partial_*\Omega, \sigma)$. Also, with the dependence on the parameter κ' suppressed, one has

$$\int_{\partial_*\Omega} \Big(P\Big|_{\partial\Omega}^{\text{n.t.}}, Q\Big|_{\partial\Omega}^{\text{n.t.}}\Big) \cdot \tau\,\mathrm{d}\sigma = \int_{\Omega} \big(\partial_x Q - \partial_y P\big)\,\mathrm{d}\mathcal{L}^2 \tag{1.2.34}$$

when either Ω is bounded, or $\partial\Omega$ is unbounded. In the remaining case, i.e., when Ω is unbounded and $\partial\Omega$ is bounded (in other words, when Ω is an exterior domain),

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[9] See (5.6.29)–(5.6.31) in this regard.

formula (1.2.34) *continues to hold under the additional assumption that there exists* $\lambda \in (1, \infty)$ *such that*

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |x\, Q(x, y) - y\, P(x, y)|\, \mathrm{d}x\, \mathrm{d}y = o(R^2) \ \ \textit{as} \ \ R \to \infty. \tag{1.2.35}$$

Moreover, if $\alpha, \beta \in [0, \pi]$ *denote the angles made by the positively oriented unit tangent vector* τ *with the coordinate axes in the plane, then in terms of the "differentials"*

$$\mathrm{d}x := \cos\alpha\mathcal{H}^1\lfloor\partial_*\Omega \ \ \textit{and} \ \ \mathrm{d}y := \cos\beta\mathcal{H}^1\lfloor\partial_*\Omega \tag{1.2.36}$$

formula (1.2.34) *may be recast as*

$$\int_{\partial_*\Omega} \big(P\big|^{\text{n.t.}}_{\partial\Omega}\big)\, \mathrm{d}x + \big(Q\big|^{\text{n.t.}}_{\partial\Omega}\big)\, \mathrm{d}y = \int_{\Omega} \big(\partial_x Q - \partial_y P\big)\, \mathrm{d}\mathcal{L}^2. \tag{1.2.37}$$

As noted earlier, (1.2.34) is implied by Theorem 1.2.1 currently employed with $\vec{F} := (Q, -P)$. Formula (1.2.34) refines (1.2.30), while (1.2.37) is a powerful version of the classical Green formula recalled in (1.2.28). Passing from (1.2.34) to (1.2.37) is done by writing (with $\mathbf{i}$ and $\mathbf{j}$ denoting the basic unit vectors along the coordinate axes in the plane)

$$\begin{aligned}\big(P\big|^{\text{n.t.}}_{\partial\Omega}, Q\big|^{\text{n.t.}}_{\partial\Omega}\big)\cdot\tau &= \Big\{\big(P\big|^{\text{n.t.}}_{\partial\Omega}\big)\mathbf{i} + \big(Q\big|^{\text{n.t.}}_{\partial\Omega}\big)\mathbf{j}\Big\}\cdot\tau = \big(P\big|^{\text{n.t.}}_{\partial\Omega}\big)\mathbf{i}\cdot\tau + \big(Q\big|^{\text{n.t.}}_{\partial\Omega}\big)\mathbf{j}\cdot\tau \\ &= \big(P\big|^{\text{n.t.}}_{\partial\Omega}\big)\cos\alpha + \big(Q\big|^{\text{n.t.}}_{\partial\Omega}\big)\cos\beta \ \ \text{at } \sigma\text{-a.e. point on } \partial_*\Omega.\end{aligned} \tag{1.2.38}$$

One key feature of (1.2.37) is the interpretation of the differentials $\mathrm{d}x$, $\mathrm{d}y$ as *Radon measures* on $\partial_*\Omega$, of the sort described in (1.2.36). It is of interest to single out a result in the spirit of Corollary 1.2.3, namely that

> Green's Formula (1.2.37) is valid when the set $\Omega \subset \mathbb{R}^2$, considered as in Theorem 1.2.5, is also assumed to be bounded, whenever the functions P, Q belong to $\mathscr{C}^0(\overline{\Omega}\setminus N)\cap L^\infty(\Omega, \mathcal{L}^2)$ for some σ-nullset $N \subseteq \partial\Omega$, and satisfy $\partial_x Q - \partial_y P \in L^1(\Omega, \mathcal{L}^2)$, with the partial derivative considered in the sense of distributions in Ω. (1.2.39)

Our result in (1.2.37) also frees Green's Formula from the traditional constraints that the topological boundary of the underlying domain Ω is connected and compact. As regards the latter feature, even the version of (1.2.37) corresponding to $\Omega := \mathbb{R}^2_+$ is new. Indeed, when specialized to such a scenario, Theorem 1.2.5 gives that

$$\int_{\mathbb{R}} \big(P\big|^{\text{n.t.}}_{\partial\mathbb{R}^2_+}\big)\, \mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}^2_+} \big(\partial_x Q - \partial_y P\big)\, \mathrm{d}\mathcal{L}^2 \tag{1.2.40}$$

whenever $P, Q : \mathbb{R}^2_+ \to \mathbb{C}$ are two Lebesgue measurable functions with the property that, for some aperture parameter $\kappa \in (0, \infty)$,

$$\begin{aligned} &\text{the nontangential traces } P\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \text{ and } Q\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^2_+} \text{ exist } \mathcal{L}^1\text{-a.e. on } \partial\mathbb{R}^2_+ \equiv \mathbb{R}, \\ &\mathcal{N}_\kappa P,\ \mathcal{N}_\kappa Q \in L^1(\mathbb{R}, \mathcal{L}^1), \text{ and } \partial_x Q - \partial_y P \text{ belongs to } L^1(\mathbb{R}^2_+, \mathcal{L}^2), \end{aligned} \tag{1.2.41}$$

where all partial derivatives are considered in the sense of distributions in $\mathbb{R}^2_+$.

1.3 The Case Without Decay and When the Divergence Is a Measure

Since absolutely integrable functions in an open subset Ω of $\mathbb{R}^n$ may be identified with complex Borel measures in Ω (the collection of which is henceforth denoted by $\mathrm{CBM}(\Omega)$; see Definition 3.5.4) via

$$L^1(\Omega, \mathcal{L}^n) \ni f \longmapsto \mu := f\mathcal{L}^n \in \mathrm{CBM}(\Omega), \tag{1.3.1}$$

adopting this point of view, we can make Theorem 1.2.1 be a special case of a more general result, stated in Theorem 1.3.1, in which the divergence of the vector field in question is allowed to be an arbitrary complex Borel measure. In addition, we are going to relax the assumption that the nontangential maximal function is absolutely integrable by merely assuming its local integrability plus the absolute integrability of the inner product of the normal with the nontangential trace of the given vector field on the geometric measure theoretic boundary. Finally, in contrast to Theorem 1.2.1, no decay conditions on the vector field are imposed to begin with. The price to pay is to incorporate an extra term in the formulation of the Divergence Formula which accounts for the behavior of the vector field at infinity.

To quantify the behavior of a vector field at infinity in a manner that suits the purposes we have in mind, we make the following definition. Given an open set $\Omega \subseteq \mathbb{R}^n$ along with a vector field $\vec{F}$ whose components are integrable on bounded open subsets of Ω, define the `contribution of` $\vec{F}$ `at infinity` as the (formal, for now) limit

$$[\vec{F}]_\infty := -\lim_{R\to\infty} \int_\Omega \nabla\phi_R \cdot \vec{F}\, \mathrm{d}\mathcal{L}^n \tag{1.3.2}$$

where the family $\{\phi_R\}_{R>0}$ (henceforth referred to as *a system of auxiliary functions*) consists of smooth compactly supported functions in $\mathbb{R}^n$ which are globally bounded and progressively become pointwise equal to 1 on compact sets in a uniform fashion, i.e.,

$$\begin{array}{c}\{\phi_R\}_{R>0} \subseteq \mathscr{C}_c^\infty(\mathbb{R}^n) \text{ such that } \sup_{R>0}\sup_{x\in\mathbb{R}^n}|\phi_R(x)| < +\infty \text{ and} \\ \text{for each compact set } K \subset \mathbb{R}^n \text{ there exists some } R_K \in (0,\infty) \text{ with} \\ \text{the property that } \phi_R(x) = 1 \text{ for each } x \in K \text{ whenever } R > R_K.\end{array} \tag{1.3.3}$$

We shall say that *the contribution of $\vec{F}$ at infinity is meaningfully and unambiguously defined* provided the limit in (1.3.2) exists (in $\mathbb{C}$) for each system of auxiliary functions $\{\phi_R\}_{R>0}$ and is independent of the actual choice of such a system of auxiliary functions. Whenever this is the case, we may, for example, take $\{\phi_R\}_{R>0}$ of the form

$$\begin{array}{l}\phi_R := \phi(\cdot/R) \text{ for some fixed } \phi \in \mathscr{C}_c^\infty(\mathbb{R}^n) \\ \text{satisfying } \phi \equiv 1 \text{ near the origin in } \mathbb{R}^n,\end{array} \tag{1.3.4}$$

in which scenario we have

$$[\vec{F}]_\infty = -\lim_{R\to\infty}\Big\{\frac{1}{R}\int_\Omega (\nabla\phi)(x/R)\cdot\vec{F}(x)\,\mathrm{d}\mathcal{L}^n(x)\Big\}. \tag{1.3.5}$$

As the name suggests, the contribution of $\vec{F}$ at infinity depends only on $\vec{F}\big|_{\Omega\setminus B(0,R)}$, for each $R > 0$. In particular, $[\vec{F}]_\infty = 0$ whenever Ω is bounded. For more on this topic, see Sect. 4.7.

We now turn to the formulation of the brand of Divergence Theorem advertised earlier.

Theorem 1.3.1 *Fix $n \in \mathbb{N}$ and let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$ (which, up to a σ-nullset, is contained in $\partial_{\text{nta}}\Omega$). Fix $\kappa \in (0,\infty)$ and assume that the vector field $\vec{F} = (F_1,\dots,F_n) : \Omega \to \mathbb{C}^n$, with $\mathcal{L}^n$-measurable components, has the following properties:*

$$\begin{array}{l}\textit{the nontangential trace } \vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists(in}\mathbb{C}^n) \ \sigma\textit{-a.e. on } \partial_{\text{nta}}\Omega, \\ \textit{its inner product with the normal } \nu\cdot\big(\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) \textit{ is in } L^1(\partial_*\Omega,\sigma), \\ \textit{the nontangential maximal function } \mathcal{N}_\kappa\vec{F} \textit{ belongs to } L^1_{\text{loc}}(\partial\Omega,\sigma),\end{array} \tag{1.3.6}$$

(with the last membership in (1.3.6) implying $\vec{F} \in \big[L^1_{\text{loc}}(\Omega,\mathcal{L}^n)\big]^n \subset \big[\mathcal{D}'(\Omega)\big]^n$) and, with all individual partial derivatives considered in the sense of distributions in Ω,

$$\begin{array}{c}\textit{the distribution } \operatorname{div}\vec{F} := \partial_1 F_1 + \cdots + \partial_n F_n \in \mathcal{D}'(\Omega) \textit{ extends} \\ \textit{to a complex Borel measure in } \Omega, \textit{ still denoted by } \operatorname{div}\vec{F}.\end{array} \tag{1.3.7}$$

Then for any $\kappa' > 0$ the nontangential trace $\vec{F}\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}$ exists σ-a.e. on $\partial_{\text{nta}}\Omega$ and is actually independent of κ'. Also, $[\vec{F}]_\infty$, the contribution of $\vec{F}$ at infinity, is meaningfully and unambiguously defined and, with the dependence on the parameter κ'

dropped, one has

$$\big(\operatorname{div}\vec{F}\big)(\Omega) = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\,\big|^{\rm n.t.}_{\partial\Omega}\big)\,{\rm d}\sigma + [\vec{F}]_\infty. \tag{1.3.8}$$

Moreover, formula (1.3.8) *reduces to*

$$\big(\operatorname{div}\vec{F}\big)(\Omega) = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\,\big|^{\rm n.t.}_{\partial\Omega}\big)\,{\rm d}\sigma \tag{1.3.9}$$

(*i.e., one has* $[\vec{F}]_\infty = 0$) *if either* Ω *is bounded, or when*

$$\partial\Omega \text{ is unbounded and } \mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega,\sigma), \tag{1.3.10}$$

or when there exists $\lambda \in (1,\infty)$ *such that*

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |x\cdot\vec{F}(x)|\,{\rm d}\mathcal{L}^n(x) = o(R^2) \quad as \quad R\to\infty. \tag{1.3.11}$$

We make several comments aimed at clarifying the context and scope of the above theorem.

Comment 1. Given any open set $\Omega \subseteq \mathbb{R}^n$, we say that *a distribution* $u \in \mathcal{D}'(\Omega)$ *extends to a complex Borel measure* μ *in* Ω provided

$$_{\mathcal{D}'(\Omega)}\langle u,\phi\rangle_{\mathcal{D}(\Omega)} = \int_\Omega \phi\,{\rm d}\mu, \qquad \forall\phi\in\mathscr{C}^\infty_c(\Omega), \tag{1.3.12}$$

where $_{\mathcal{D}'(\Omega)}\langle\cdot,\cdot\rangle_{\mathcal{D}(\Omega)}$ denotes the distributional pairing in the open set Ω. Recall that (cf. (3.5.18)–(3.5.19))

$$\text{every complex Borel measure in } \Omega \text{ is Radon}, \tag{1.3.13}$$

and, as seen from the classical Riesz–Markov–Kakutani Representation Theorem (cf. (3.5.17)),

any complex Radon measure in Ω is uniquely determined by its action (via integration) on $\mathscr{C}_0(\Omega)$, the collection of continuous functions on Ω vanishing at infinity (i.e., such that the set $\{x\in\Omega : |f(x)|\geq\varepsilon\}$ is compact for every $\varepsilon>0$). (1.3.14)

Then, since (as a standard mollifier argument shows)

for $\Omega\subseteq\mathbb{R}^n$ open, the collection of test functions $\mathscr{C}^\infty_c(\Omega)$ is dense in $\mathscr{C}_0(\Omega)$, equipped with the supremum norm, (1.3.15) the demand in (1.3.7) that the distribution $\operatorname{div}\vec{F} \in \mathcal{D}'(\Omega)$ extends to a complex Borel measure in Ω (in the sense of (1.3.12)) determines *uniquely* the measure in question. For example,

$$\text{if } \Omega \text{ is an open set in } \mathbb{R}^n\text{, then the Dirac distribution } \delta_{x_o} \in \mathcal{D}'(\Omega) \text{ with mass at } x_o \in \Omega \text{ extends uniquely (in the sense of (1.3.12)) to the complex Borel measure } \mu \text{ in } \Omega \text{ defined as } \mu(A) := \mathbf{1}_A(x_o) \text{ for each } A \subseteq \Omega\text{; this is a probability measure for which we shall henceforth retain the symbol } \delta_{x_o}. \tag{1.3.16}$$

In the context of Theorem 1.3.1, with $\operatorname{div}\vec{F}$ thus identified with a complex Radon measure in Ω (cf. (1.3.7)), it is therefore meaningful and unambiguous to speak of $\big(\operatorname{div}\vec{F}\big)(\Omega)$ in the left-hand side of (1.3.8).

<u>Comment 2.</u> As seen from (1.3.15) and the Riesz–Markov–Kakutani Representation Theorem mentioned earlier, a linear functional $\Lambda : \mathscr{C}_c^\infty(\Omega) \to \mathbb{C}$ (where $\Omega \subseteq \mathbb{R}^n$ is an arbitrary open set) is representable via integration against a complex Borel measure μ in Ω if and only if

$$\sup\Big\{\big|\Lambda(\phi)\big| : \phi \in \mathscr{C}_c^\infty(\Omega),\ \sup_{x\in\Omega}|\phi(x)| \leq 1\Big\} < +\infty. \tag{1.3.17}$$

Moreover, in such a case the complex measure μ is uniquely determined by Λ and, as seen from (3.5.14), the supremum in (1.3.17) is precisely $|\mu|(\Omega)$, where $|\mu|$ denotes the total variation of μ. Specializing these considerations to the scenario when

$$\Lambda : \mathscr{C}_c^\infty(\Omega) \to \mathbb{C} \ \text{ is given by } \ \Lambda(\phi) := -\int_\Omega \vec{F}\cdot\nabla\phi \,\mathrm{d}\mathcal{L}^n, \quad \forall \phi \in \mathscr{C}_c^\infty(\Omega), \tag{1.3.18}$$

shows that the demand in (1.3.7) may be equivalently rephrased as asking that the vector field $\vec{F}$ satisfies

$$\sup\Big\{\Big|\int_\Omega \vec{F}\cdot\nabla\phi\,\mathrm{d}\mathcal{L}^n\Big| : \phi \in \mathscr{C}_c^\infty(\Omega),\ \sup_{x\in\Omega}|\phi(x)| \leq 1\Big\} < +\infty. \tag{1.3.19}$$

Parenthetically, we note that the supremum in (1.3.19) is precisely $|\operatorname{div}\vec{F}|(\Omega)$, where the symbol $\operatorname{div}\vec{F}$ is presently used to denote the unique extension of the distributional divergence of $\vec{F}$ to a complex Radon measure in Ω.

<u>Comment 3.</u> Assume $\mathcal{O} \subset \mathbb{R}^n$ is a bounded domain of class $\mathscr{C}^1$, and consider a vector field $\vec{G}$ such that

$$\vec{G} \in \big[\mathscr{C}^0(\overline{\mathcal{O}}) \cap \mathscr{C}^1(\mathcal{O})\big]^n \ \text{ with } \ \operatorname{div}\vec{G} \in L^1(\mathcal{O}, \mathcal{L}^n). \tag{1.3.20}$$

Pick an open set $\Omega \subseteq \mathbb{R}^n$ such that $\overline{\mathcal{O}} \subset \Omega$, then introduce

$$\vec{F} := \begin{cases} \vec{G} & \text{in } \mathcal{O}, \\ 0 \in \mathbb{C}^n & \text{in } \Omega \setminus \overline{\mathcal{O}}. \end{cases} \tag{1.3.21}$$

Then the vector field $\vec{F}$ belongs to $\big[L^\infty(\Omega, \mathcal{L}^n)\big]^n$, vanishes near $\partial\Omega$, and its divergence, computed in the sense of distributions in Ω, is given by

$$\operatorname{div}\vec{F} = \widetilde{\operatorname{div}\vec{G}}\,\mathcal{L}^n - \Big(\nu_{\mathcal{O}} \cdot \vec{G}\big|_{\partial\mathcal{O}}\Big)\,\mathcal{H}^{n-1}\lfloor\partial\mathcal{O} \in \mathrm{CBM}(\Omega), \tag{1.3.22}$$

where tilde denotes the extension by zero to Ω, and $\nu_{\mathcal{O}}$ is the outward unit normal to $\mathcal{O}$.

This extension-by-zero procedure of reasonably smooth vector fields already points to the fact that there exist plenty of natural situations when one encounters Lebesgue measurable vector fields with distributional divergence a complex Borel measure. An amusing feature of it is that, in the case when Ω satisfies the background geometric assumptions in the statement of Theorem 1.3.1, the Divergence Formula (written as in (1.3.8)) for $\vec{F}$ in Ω reduces precisely to the ordinary Divergence Formula for the vector field $\vec{G}$ in the bounded $\mathscr{C}^1$ domain $\mathcal{O}$.

Comment 4. Other natural examples of vector fields $\vec{F} = (F_1, \dots, F_n)$ whose scalar components are locally integrable functions in an open set $\Omega \subseteq \mathbb{R}^n$ and whose distributional divergence is a complex Borel measure in Ω are provided by vector fields in $\big[\mathrm{BV}(\Omega)\big]^n$. More generally, if

$$\begin{gathered} \vec{F} = (F_j)_{1\le j\le n} \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^n \quad \text{with} \\ \mathbf{V}(F_j; \Omega) < +\infty \quad \text{for each} \quad j \in \{1, \dots, n\}, \end{gathered} \tag{1.3.23}$$

(where $\mathbf{V}(F_j; \Omega)$ denotes the variation of F_j in Ω, cf. (5.5.1)), then item *(v)* of Proposition 5.5.1 implies that $\operatorname{div}\vec{F} \in \mathrm{CBM}(\Omega)$.

Comment 5. The fact that $\mathcal{N}_\kappa\vec{F}$ belongs to $L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ is automatically satisfied if, e.g., $\vec{F}$ is essentially bounded in Ω, or if $\mathcal{N}_\kappa\vec{F}$ belongs to the Lorentz space $L^{p,q}(\partial\Omega, \sigma)$ for some $p \in (1, \infty)$ and $q \in (0, \infty]$.

Comment 6. In the context of Theorem 1.3.1 it is also instructive to remark that, under the additional assumption that

$$\vec{F} \text{ is continuous in } \Omega \setminus B(0, R) \text{ for some } R > 0, \tag{1.3.24}$$

Proposition 4.7.1 implies (keeping in mind (4.7.3)) that the contribution of $\vec{F}$ at infinity (originally defined in (1.3.4)–(1.3.5)) may be expressed as

$$\begin{gathered} [\vec{F}]_\infty = \lim_{R\to\infty} \Big\{ R^{-1} \int_{|x|=R} x \cdot \vec{F}(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) \Big\} \\ \text{whenever} \quad n \ge 2 \quad \text{and} \quad \mathbb{R}^n \setminus \Omega \quad \text{is bounded,} \end{gathered} \tag{1.3.25}$$

while Proposition 4.7.2 (together with (4.7.3)) gives

$$[\vec{F}]_\infty = \begin{cases} \vec{F}\Big|_{-\infty}^{+\infty} & \text{if } \Omega \text{ unbounded both from below and from above,} \\ \vec{F}(+\infty) & \text{if } \Omega \text{ bounded from below but unbounded from above,} \\ -\vec{F}(-\infty) & \text{if } \Omega \text{ unbounded from below but bounded from above,} \end{cases} \tag{1.3.26}$$

in the case when $n = 1$ and Ω is unbounded (since, in this setting, the local finiteness of the measure $\sigma = \mathcal{H}^0 \lfloor \partial\Omega$ implies that the set $\partial\Omega$ is of locally finite cardinality).

Comment 7. As in the case of Theorem 1.2.1, both the geometric hypotheses on the domain Ω as well as the analytic hypotheses on the vector field $\vec{F}$ stipulated in Theorem 1.3.1 are stable under bi-Lipschitz changes of variables of the Euclidean ambient. Also, the class of vector fields $\vec{F}$ satisfying the hypotheses of Theorem 1.3.1 is a linear subspace of $\big[L^1_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^n$ which is stable under multiplication by scalar Lipschitz functions with bounded support.

Moreover, if $\vec{F}$ is as in the statement of Theorem 1.3.1 and $\varphi \in \text{Lip}(\Omega)$ is scalar-valued with bounded support, then the Divergence Formula (1.3.8) written for the vector field $\varphi\vec{F}$ yields the integration by parts formula

$$\int_\Omega \nabla\varphi \cdot \vec{F}\, d\mathcal{L}^n = -\int_\Omega \varphi\, d(\text{div}\vec{F}) + \int_{\partial_*\Omega} \varphi\, \nu \cdot \Big(\vec{F}\,\Big|^{\text{n.t.}}_{\partial\Omega}\Big)\, d\sigma. \tag{1.3.27}$$

In turn, this implies that if

$$\vec{G} := \begin{cases} \vec{F} & \text{in } \Omega, \\ 0 \in \mathbb{C}^n & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.3.28}$$

then $\vec{G} \in \big[L^1_{\text{loc}}(\mathbb{R}^n, \mathcal{L}^n)\big]^n$ and, with the divergence taken in the sense of distributions in $\mathbb{R}^n$,

$$\text{div}\vec{G} = \text{div}\vec{F} - \nu \cdot \Big(\vec{F}\,\Big|^{\text{n.t.}}_{\partial\Omega}\Big)\, \mathcal{H}^{n-1} \lfloor \partial_*\Omega \in \text{CBM}(\mathbb{R}^n). \tag{1.3.29}$$

In light of Lemma 4.7.5, formula (1.3.29) is actually equivalent to the Divergence Formula (1.3.8).

Comment 8. Theorem 1.3.1 is sharp. Concretely, the role of the lower Ahlfors regular assumption on the boundary is brought into focus by (2.4.9) and (2.4.18), while the fact that it is natural to consider the divergence in the sense of distributions becomes apparent from (2.5.13). The hypothesis that $\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists at σ-a.e. point on $\partial_{\text{nta}}\Omega$, made in the first line of (1.3.6), cannot be weakened to simply asking that $\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists σ-a.e. on $\partial_*\Omega$, in spite of the fact that it is precisely the latter set that enters the formulation of the Divergence Formula (1.3.8). This is clear from the counterexamples provided in (2.1.17), or (2.1.47), or (2.1.64), or (2.1.87), or (2.1.102). Also, the counterexample in (2.3.7) shows that $\mathcal{N}_\kappa\vec{F} \in L^1_{\text{loc}}(\partial\Omega, \sigma)$ cannot be weakened to either $\mathcal{N}_\kappa\vec{F} \in L^{1,\infty}_{\text{loc}}(\partial\Omega, \sigma)$, or $\mathcal{N}_\kappa\vec{F} \in L^1_{\text{loc}}(\partial_*\Omega, \sigma)$. See also

(2.3.64) in this vein. Next, that replacing $o(R^2)$ by $O(R^2)$ in the formulation of the growth condition (1.3.11) may invalidate the Divergence Formula (1.3.9) is clear from (2.2.15). Lastly, the role of $[\vec{F}]_\infty$ in the context of (1.3.8) is highlighted by the manner in which the counterexample in (2.2.14) has been turned into the positive result in (2.2.18), the counterexample in (2.2.52) has been turned into the positive result in (2.2.59), and the counterexample in (2.2.14) has been turned into the positive result in (2.2.18).

Comment 9. Specializing Theorem 1.3.1 to the two-dimensional setting yields a version of Green's Formula in the spirit of (1.2.37) in which now $\partial_x Q - \partial_y P$ is a complex Borel measure in Ω, the integral in the right-hand side is replaced by its total mass, i.e., $(\partial_x Q - \partial_y P)(\Omega)$, while the left-hand side of (1.2.37) is augmented by including the contribution at infinity of the vector field defined as $\vec{F} := (Q, -P)$.

Comment 10. Lemma 4.7.5 may be regarded as the version of Theorem 1.3.1 corresponding to the limit case $\Omega = \mathbb{R}^n$.

Comment 11. It turns out that condition (1.3.11) is automatically satisfied when either Ω is bounded, or when

$$\partial\Omega \text{ is unbounded, } n \geq 2, \text{ and } \mathcal{N}_\kappa^{\Omega\setminus K}\vec{F} \in L^1(\partial\Omega, \sigma). \tag{1.3.30}$$

This is seen from the last part in Lemma 4.7.3 (used with $\Omega \setminus K$ in place of Ω) and (8.6.51) (used with $E := \Omega \setminus K$).

Comment 12. The proof of Theorem 1.3.1, as stated, is given in Sect. 9.1.

The next corollary contains a refinement of what Theorem 1.3.1 specialized to the case when the underlying set is the upper half-space would normally give.

Corollary 1.3.2 *Fix $n \in \mathbb{N}$ with $n \geq 2$ and pick some arbitrary $\kappa \in (0, \infty)$. Assume that the vector field $\vec{F} = (F_1, \dots, F_n) : \mathbb{R}^n_+ \to \mathbb{C}^n$, with Lebesgue measurable components, satisfies the following properties:*

$$\begin{aligned} & F_n\Big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}} \text{ exists } \mathcal{L}^{n-1}\text{-a.e. on } \partial\mathbb{R}^n_+ \text{ and } F_n\Big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}} \in L^1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \\ & \text{the nontangential maximal function } \mathcal{N}_\kappa\vec{F} \text{ belongs to } L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \\ & \operatorname{div}\vec{F} \in \mathcal{D}'(\mathbb{R}^n_+) \text{ extends to a complex Borel measure in } \mathbb{R}^n_+, \end{aligned} \tag{1.3.31}$$

$\big($still denoted by $\operatorname{div}\vec{F}\big)$.

Then for any other $\kappa' > 0$ the nontangential trace $F_n\Big|_{\partial\mathbb{R}^n_+}^{\kappa'-\mathrm{n.t.}}$ exists $\mathcal{L}^{n-1}$-a.e. on $\mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+$ and is actually independent of κ'. Also, the contribution of $\vec{F}$ at infinity is meaningfully and unambiguously defined and, with the dependence on the parameter κ' dropped, one has

$$\big(\operatorname{div}\vec{F}\big)(\mathbb{R}^n_+) = -\int_{\mathbb{R}^{n-1}} \Big(F_n\Big|_{\partial\mathbb{R}^n_+}^{\mathrm{n.t.}}\Big)\, d\mathcal{L}^{n-1} + [\vec{F}]_\infty. \tag{1.3.32}$$

Furthermore, formula (1.3.32) *reduces to*

$$\big(\mathrm{div}\vec{F}\big)(\mathbb{R}^n_+) = -\int_{\mathbb{R}^{n-1}} \Big(F_n\Big|^{\text{n.t.}}_{\partial\mathbb{R}^n_+}\Big)\, \mathrm{d}\mathcal{L}^{n-1} \tag{1.3.33}$$

(*i.e., one has* $[\vec{F}]_\infty = 0$) *whenever there exists* $\lambda \in (1,\infty)$ *such that*

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\mathbb{R}^n_+} |x\cdot\vec{F}(x)|\,\mathrm{d}\mathcal{L}^n(x) = o(R^2) \ \ \textit{as}\ \ R\to\infty, \tag{1.3.34}$$

a condition which is automatically satisfied when

$$\mathcal{N}_\kappa\vec{F} \in L^1(\mathbb{R}^{n-1},\mathcal{L}^{n-1}). \tag{1.3.35}$$

While one of the key features of Theorem 1.3.1 (compared with Theorem 1.1.1) is the ability of decreasing the regularity of the vector field provided the regularity of the underlying domain is (necessarily) increased, in and of itself, this does not tell the whole story. Indeed, the possibility of allowing far less regular vector fields than those considered in the De Giorgi–Federer Divergence Theorem is vastly useful, and can actually compensate for stipulating that the set in question is smoother than being of locally finite perimeter. For example, our version of the Divergence Formula, presented in Theorem 1.3.1, incorporates that of De Giorgi and Federer (recorded earlier in Theorem 1.1.1) in the manner indicated below.

How Theorem 1.3.1 *subsumes the De Giorgi–Federer Divergence Formula:*
Let $\Omega\subseteq\mathbb{R}^n$ be a set of locally finite perimeter, and fix a vector field $\vec{F}\in\big[\mathscr{C}^\infty_c(\mathbb{R}^n)\big]^n$. The idea is to select an open ball $B\subseteq\mathbb{R}^n$ containing the support of $\vec{F}$ and apply Theorem 1.3.1 to the (smooth, bounded) domain B and vector field

$$\vec{G} := \big(\mathbf{1}_\Omega\vec{F}\big)\Big|_B : B \longrightarrow \mathbb{C}^n. \tag{1.3.36}$$

Note that, by design,

$$\vec{G}\in\big[L^\infty_{\text{comp}}(B,\mathcal{L}^n)\big]^n. \tag{1.3.37}$$

In turn, having picked some arbitrary aperture parameter $\kappa\in(0,\infty)$, this membership trivially implies that $\mathcal{N}_\kappa\vec{G}\in L^\infty(\partial B,\mathcal{H}^{n-1})\subseteq L^1(\partial B,\mathcal{H}^{n-1})$. Also, it is clear that $\vec{G}\big|^{\kappa-\text{n.t.}}_{\partial B}=0$ at every point on ∂B since $\vec{G}$ vanishes identically near ∂B. Finally, with the divergence taken in the sense of distributions in B, we have (see (5.6.24))

$$\begin{aligned}\mathrm{div}\vec{G} &= \Big[\mathrm{div}\big(\mathbf{1}_\Omega\vec{F}\big)\Big]\Big|_B = \Big[(\nabla\mathbf{1}_\Omega)\cdot\vec{F}+\mathbf{1}_\Omega(\mathrm{div}\vec{F})\Big]\Big|_B \\ &= \Big[-(\nu\cdot\vec{F})\sigma_* + \mathbf{1}_\Omega(\mathrm{div}\vec{F})\Big]\Big|_B \end{aligned}\tag{1.3.38}$$

where $\sigma_* := \mathcal{H}^{n-1}\lfloor\partial_*\Omega$ and ν is the geometric measure theoretic outward unit normal to Ω. In relation to this, observe that

$$\begin{aligned}&\text{the total variation of the measure } -(\nu\cdot\vec{F})\sigma_* \text{ is}\\&\leq \big(\sup_{\mathbb{R}^n}|\vec{F}|\big)\mathcal{H}^{n-1}\big(\partial_*\Omega\cap\operatorname{supp}\vec{F}\big) < +\infty\end{aligned} \tag{1.3.39}$$

where the very last inequality is a consequence of the fact that Ω is a set of locally finite perimeter in $\mathbb{R}^n$ (cf. (5.6.35)). Since $\operatorname{div}\vec{F}\in L^1(\mathbb{R}^n,\mathcal{L}^n)$, we conclude from (1.3.38)–(1.3.39) that the distribution $\operatorname{div}\vec{G}$ extends to a complex Borel measure in Ω (as discussed in Comment 1 above; cf. (1.3.12) in particular). Thus,

$$\operatorname{div}\vec{G}\in \mathrm{CBM}(\Omega). \tag{1.3.40}$$

At this stage, all hypotheses of Theorem 1.3.1 have been verified by the set B and the vector field $\vec{G}$. As such, with N denoting the outward unit normal to the ball B, we may invoke the Divergence Formula (1.3.9) to write (bearing in mind that $\vec{G}\big|^{\kappa-\mathrm{n.t.}}_{\partial B}\equiv 0$ on ∂B and that $\vec{F}$ is supported in B)

$$\begin{aligned}0&=\int_{\partial_* B} N\cdot\big(\vec{G}\big|^{\kappa-\mathrm{n.t.}}_{\partial B}\big)\,d\mathcal{H}^{n-1}=\big(\operatorname{div}\vec{G}\big)(B)\\&=-\int_{\partial_*\Omega}\nu\cdot\vec{F}\,d\sigma_*+\int_\Omega \operatorname{div}\vec{F}\,d\mathcal{L}^n.\end{aligned} \tag{1.3.41}$$

From this, the De Giorgi–Federer Divergence Formula (1.1.8) for $\vec{F}\in\big[\mathscr{C}^\infty_c(\mathbb{R}^n)\big]^n$ readily follows. Finally, passing to arbitrary vector fields in $\big[\mathscr{C}^1_c(\mathbb{R}^n)\big]^n$ is just a standard mollifier argument. The transition from Theorems 1.3.1 to 1.1.1 is therefore complete.

1.4 The Divergence Theorem for Singular Vector Fields Without Decay

Theorem 1.4.1, stated below, further extends Theorem 1.3.1 in two important regards. First, we now allow the intervening vector field to be singular (again, without explicitly requiring any type of decay for said vector field at infinity, while permitting the nontangential maximal function to be only locally integrable). Second, the divergence of the intervening vector field, considered in the sense of distributions, is allowed to be the sum of a compactly supported distribution and a complex Borel measure.

The reader is alerted to the fact that the local version of the nontangential maximal operator $\mathcal{N}^E_\kappa$, relative to a measurable subset E of the underlying domain, is defined in (8.2.4). Also, $\mathscr{C}^\infty_b(\Omega)$ stands for the space of smooth and bounded functions in a given open set $\Omega\subseteq\mathbb{R}^n$ (in particular, the constant function 1 belongs to this space), while $\big(\mathscr{C}^\infty_b(\Omega)\big)^*$ denotes its algebraic dual; for more on this, see Chap. 4.

Theorem 1.4.1 *Fix $n \in \mathbb{N}$ and let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$ (which, up to a σ-nullset, is contained in $\partial_{\rm nta}\Omega$). Fix $\kappa \in (0,\infty)$ and assume that the vector field*

$$\vec{F} = (F_1, \dots, F_n) \in \big[\mathcal{D}'(\Omega)\big]^n \tag{1.4.1}$$

satisfies the following conditions:

$$\begin{gathered}\text{there exists a compact set } K \text{ contained in } \Omega \text{ such that}\\ \vec{F}\big|_{\Omega\setminus K} \in \big[L^1_{\rm loc}(\Omega\setminus K, \mathcal{L}^n)\big]^n \ \text{ and } \ \mathcal{N}_\kappa^{\Omega\setminus K}\vec{F} \in L^1_{\rm loc}(\partial\Omega,\sigma),\end{gathered} \tag{1.4.2}$$

the pointwise nontangential boundary trace

$$\begin{gathered}\vec{F}\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega} = \Big(F_1\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}, \dots, F_n\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) \text{ exists (in } \mathbb{C}^n) \ \sigma\text{-a.e. on } \ \partial_{\rm nta}\Omega \text{ and}\\ \text{has the property that } \ \nu\cdot\Big(\vec{F}\Big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big) \text{ belongs to the space } \ L^1(\partial_*\Omega,\sigma),\end{gathered} \tag{1.4.3}$$

and the divergence of $\vec{F}$, considered in the sense of distributions in Ω, is the sum (in $\mathcal{D}'(\Omega)$) of a compactly supported distribution in Ω and a complex Borel measure in Ω, i.e.,

$$\operatorname{div}\vec{F} \in \mathscr{E}'(\Omega) + \mathrm{CBM}(\Omega) \subseteq \big(\mathscr{C}_b^\infty(\Omega)\big)^*. \tag{1.4.4}$$

Then for any $\kappa' > 0$ the nontangential trace $\vec{F}\Big|^{\kappa'-{\rm n.t.}}_{\partial\Omega}$ exists σ-a.e. on $\partial_{\rm nta}\Omega$ and is actually independent of κ'. Also, the contribution of $\vec{F}$ at infinity (cf. (1.3.2)–(1.3.3)) is meaningfully and unambiguously defined and, with the dependence on the parameter κ' dropped, one has

$${}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_{\partial_*\Omega} \nu\cdot\Big(\vec{F}\Big|^{\rm n.t.}_{\partial\Omega}\Big)\,d\sigma + [\vec{F}]_\infty. \tag{1.4.5}$$

Moreover, the contribution at infinity vanishes, a scenario in which formula (1.4.5) reduces to

$${}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_{\partial_*\Omega} \nu\cdot\Big(\vec{F}\Big|^{\rm n.t.}_{\partial\Omega}\Big)\,d\sigma, \tag{1.4.6}$$

if either Ω is bounded, or when

$$\partial\Omega \text{ is unbounded and } \mathcal{N}_\kappa^{\Omega\setminus K}\vec{F} \in L^1(\partial\Omega,\sigma), \tag{1.4.7}$$

or when there exists $\lambda \in (1,\infty)$ such that

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |x\cdot\vec{F}(x)|\,\mathrm{d}\mathcal{L}^n(x) = o(R^2) \ \ as \ \ R\to\infty. \tag{1.4.8}$$

We make a number of notes clarifying some aspects of the above theorem.

Note 1. Under the additional assumption that

$$\begin{aligned}&\vec{F} \text{ is continuous in } \Omega\setminus B(0,R) \text{ for some } R\in(0,\infty), \text{ and}\\ &\text{either } \mathbb{R}^n\setminus\Omega \text{ is bounded, or } n=1 \text{ and } \Omega \text{ is unbounded,}\end{aligned} \tag{1.4.9}$$

it follows from Propositions 4.7.1 and 4.7.2 (while also keeping in mind (4.7.3)) that $[\vec{F}]_\infty$, the contribution of $\vec{F}$ at infinity, may be computed as in (1.3.25)–(1.3.26).

Note 2. The nontangential maximal operator in (1.4.2) is understood naturally, in the sense of (8.2.43)–(8.2.44).

Note 3. Since the contribution of the vector field $\vec{F}$ at infinity is meaningfully and unambiguously defined, it follows from (1.3.5) that

$$\vec{F}(x) = o(|x|^{1-n}) \ \text{ for } \ x\in\Omega \ \text{ with } \ |x|\to\infty \Longrightarrow [\vec{F}]_\infty = 0. \tag{1.4.10}$$

Note 4. Having a vector field $\vec{F}\in\big[\mathcal{D}'(\Omega)\big]^n$ with the property that there exists a compact set $K\subseteq\Omega$ such that $\vec{F}\big|_{\Omega\setminus K}\in\big[L^1_{\mathrm{loc}}(\Omega\setminus K,\mathcal{L}^n)\big]^n$ (see (1.4.1)–(1.4.2)) is simply equivalent to having

$$\vec{F}\in\big[\mathscr{E}'(\Omega)+L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n)\big]^n. \tag{1.4.11}$$

Note 5. As may be seen with the help of Proposition 8.6.3, the class of vector fields satisfying the hypotheses of Theorem 1.4.1 is a linear subspace of $\big[\mathcal{D}'(\Omega)\big]^n$ which is stable under multiplication by scalar functions in $\mathscr{C}^\infty$ with bounded support.

Note 6. As is apparent from (4.6.6),

$$\begin{aligned}&\text{all conclusions in Theorem 1.4.1 are valid when in place}\\ &\text{of (1.4.4) we now ask that } \operatorname{div}\vec{F}\in\mathscr{E}'(\Omega)+L^1(\Omega,\mathcal{L}^n).\end{aligned} \tag{1.4.12}$$

Note 7. For various applications it is useful to observe that the vector field $\vec{F}$ from (1.4.1) may be allowed to have components in $\mathcal{D}'(\Omega)\otimes\mathscr{V}$, where $\mathscr{V}$ is an arbitrary, fixed, finite-dimensional vector space. In such a scenario, the first condition in (1.4.2) now reads

$$\vec{F}\big|_{\Omega\setminus K}\in\big[L^1_{\mathrm{loc}}(\Omega\setminus K,\mathcal{L}^n)\otimes\mathscr{V}\big]^n, \tag{1.4.13}$$

in place of (1.4.4) we now demand

$$\operatorname{div}\vec{F}\in\mathscr{E}'(\Omega)\otimes\mathscr{V}+\mathrm{CBM}(\Omega)\otimes\mathscr{V}, \tag{1.4.14}$$

and (1.4.5) becomes the equality of two vectors in $\mathscr{V}$. A case in point, dealt with later, is the situation when the vector field $\vec{F}$ has Clifford algebra-valued components.

Note 8. Another version of Theorem 1.4.1 is demanding in place of (1.4.4) and (1.4.2) that

$$\begin{aligned}&\mathrm{div}\vec{F} \in \mathscr{E}'_K(\Omega) + L^1(\Omega, \mathcal{L}^n) \ \text{ for some compact set } \ K \subset \Omega\\&\qquad\text{with the property that } \ \mathcal{N}_\kappa^{\Omega\setminus K}\vec{F} \in L^1(\partial\Omega, \sigma),\end{aligned} \tag{1.4.15}$$

a scenario in which all conclusions in Theorem 1.4.1 continue to be valid.

Note 9. For similar reasons as in the case of Theorem 1.3.1, discussed under the heading Comment 8 at the end of Sect. 1.3, Theorem 1.4.1 is sharp.

Note 10. Specializing Theorem 1.4.1 to the two-dimensional setting yields a version of Green's Formula of the following sort

$$\int_{\partial_*\Omega} \big(P\big|^{\mathrm{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}x + \big(Q\big|^{\mathrm{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}y = {}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\partial_x Q - \partial_y P, 1\big)_{\mathscr{C}^\infty_b(\Omega)} - [(Q, -P)]_\infty, \tag{1.4.16}$$

which is valid under the assumption that Ω and the vector field $\vec{F} := (Q, -P)$ are as in Theorem 1.4.1 when $n = 2$.

Note 11. In the case when in place of (1.4.4) we simply have $\mathrm{div}\vec{F} \in \mathrm{CBM}(\Omega)$, formula (1.4.5) reduces to (1.3.8). Also, if in place of the membership in (1.4.4) we now have $\mathrm{div}\vec{F} \in \mathscr{E}'(\Omega)$, then formula (1.4.5) becomes

$${}_{\mathscr{E}'(\Omega)}\big\langle \mathrm{div}\vec{F}, 1\big\rangle_{\mathscr{E}(\Omega)} = \int_{\partial_*\Omega} \nu\cdot\big(\vec{F}\,\big|^{\mathrm{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma + [\vec{F}]_\infty, \tag{1.4.17}$$

where the bracket on the left-hand side is understood as the distributional pairing between the compactly supported distribution $\mathrm{div}\vec{F} \in \mathscr{E}'(\Omega)$ and the smooth function $1 \in \mathscr{E}(\Omega)$.

Note 12. As seen from the last part in Lemma 4.7.3 (used with $\Omega \setminus K$ in place of Ω) and (8.6.51) (used with $E := \Omega \setminus K$), condition (1.4.8) is automatically satisfied when either Ω is bounded, or when

$$\partial\Omega \text{ is unbounded, } n \geq 2, \text{ and } \mathcal{N}_\kappa^{\Omega\setminus K}\vec{F} \in L^1(\partial\Omega, \sigma). \tag{1.4.18}$$

Note 13. The proof of Theorem 1.4.1 may be found in Sect. 9.2.

In the next corollary, we present a refinement of what Theorem 1.4.1 specialized to the case when the underlying set is the upper half-space would ordinarily give.

Corollary 1.4.2 *Fix $n \in \mathbb{N}$ with $n \geq 2$ and pick some arbitrary $\kappa \in (0, \infty)$. Assume that the vector field*

$$\vec{F} = (F_1, \dots, F_n) \in \big[\mathcal{D}'(\mathbb{R}^n_+)\big]^n \tag{1.4.19}$$

satisfies the following conditions:

$$\begin{gathered}\text{there exists some compact set } K \text{ contained in } \mathbb{R}^n_+ \text{ such that}\\ \vec{F}\big|_{\mathbb{R}^n_+\setminus K} \in \big[L^1_{\rm loc}(\mathbb{R}^n_+ \setminus K, \mathcal{L}^n)\big]^n \ \text{ and } \ \mathcal{N}_\kappa^{\mathbb{R}^n_+\setminus K}\vec{F} \in L^1_{\rm loc}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\end{gathered} \tag{1.4.20}$$

the pointwise nontangential boundary trace

$$\begin{gathered}F_n\big|^{\kappa-\rm n.t.}_{\partial\mathbb{R}^n_+} \text{ exists at } \mathcal{L}^{n-1}\text{-a.e. point on } \partial\mathbb{R}^n_+\\ \text{and } \ F_n\big|^{\kappa-\rm n.t.}_{\partial\mathbb{R}^n_+} \in L^1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}),\end{gathered} \tag{1.4.21}$$

and the divergence of $\vec{F}$, considered in the sense of distributions in $\mathbb{R}^n_+$, is the sum (in $\mathcal{D}'(\mathbb{R}^n_+)$) of a compactly supported distribution in $\mathbb{R}^n_+$ and a complex Borel measure in $\mathbb{R}^n_+$, i.e.,

$$\operatorname{div}\vec{F} \in \mathscr{E}'(\mathbb{R}^n_+) + \mathrm{CBM}(\mathbb{R}^n_+). \tag{1.4.22}$$

Then for any other $\kappa' > 0$ the nontangential trace $F_n\big|^{\kappa'-\rm n.t.}_{\partial\mathbb{R}^n_+}$ exists $\mathcal{L}^{n-1}$-a.e. on $\mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+$ and is actually independent of κ'. Also, the contribution of $\vec{F}$ at infinity is meaningfully and unambiguously defined and, with the dependence on the parameter κ' dropped, one has

$$_{(\mathscr{C}^\infty_b(\mathbb{R}^n_+))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\mathbb{R}^n_+)} = -\int_{\mathbb{R}^{n-1}} \Big(F_n\big|^{\rm n.t.}_{\partial\mathbb{R}^n_+}\Big)\,\mathrm{d}\mathcal{L}^{n-1} + [\vec{F}]_\infty. \tag{1.4.23}$$

In addition, the contribution at infinity vanishes, a scenario in which formula (1.4.23) simply reduces to

$$_{(\mathscr{C}^\infty_b(\mathbb{R}^n_+))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\mathbb{R}^n_+)} = -\int_{\mathbb{R}^{n-1}} \Big(F_n\big|^{\rm n.t.}_{\partial\mathbb{R}^n_+}\Big)\,\mathrm{d}\mathcal{L}^{n-1}, \tag{1.4.24}$$

whenever there exists $\lambda \in (1, \infty)$ such that

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\mathbb{R}^n_+} |x \cdot \vec{F}(x)|\,\mathrm{d}\mathcal{L}^n(x) = o(R^2) \ \text{ as } \ R \to \infty. \tag{1.4.25}$$

In turn, condition (1.4.25) is automatically satisfied provided

$$\mathcal{N}_\kappa^{\mathbb{R}^n_+\setminus K}\vec{F} \in L^1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \tag{1.4.26}$$

(*a scenario in which the last condition in* (1.4.21) *is automatically guaranteed*).

Via a rotation and a translation, a similar result is valid in any half-space in $\mathbb{R}^n$, i.e., any set of the form $\Omega := \{x \in \mathbb{R}^n : (x - x_o)\cdot h > 0\}$ for some fixed $x_o \in \mathbb{R}^n$ and $h \in S^{n-1}$, a setting in which $h \cdot \vec{F}$ plays the role of the scalar component F_n.

Corollary 1.4.2 is in the nature of best possible. To illustrate this, work in the two-dimensional setting and identify $\mathbb{R}^2 \equiv \mathbb{C}$. Having fixed a point $z \in \mathbb{C}_+$, consider the vector field[10] defined for each $\zeta \in \mathbb{C}_+ \setminus \{z\}$ by

$$\vec{F}(\zeta) := \left(\frac{\mathrm{i}}{\zeta(\zeta - z)} + \frac{\mathrm{i}}{\overline{\zeta}(\overline{\zeta} - z)}, \frac{-1}{\zeta(\zeta - z)} + \frac{1}{\overline{\zeta}(\overline{\zeta} - z)} \right). \tag{1.4.27}$$

Then

$$\vec{F} \in \left[L^1_{\mathrm{loc}}\big(\mathbb{C}_+, \mathcal{L}^2\big) \cap \mathscr{C}^\infty\big(\overline{\mathbb{C}_+} \setminus \{0, z\}\big) \right]^2, \tag{1.4.28}$$

and a simple computation shows that

$$\mathrm{div} \vec{F} = \frac{2\pi \mathrm{i}}{z} \delta_z \quad \text{in} \quad \mathcal{D}'(\mathbb{C}_+). \tag{1.4.29}$$

In particular, $\mathrm{div}\vec{F}$ belongs to $\mathscr{E}'(\mathbb{C}_+)$. Also, it is apparent from (1.4.27) that the nontangential boundary trace $F_2\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+}$ vanishes on $\partial\mathbb{C}_+ \setminus \{0\} \equiv \mathbb{R} \setminus \{0\}$, hence

$$F_2\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^2_+} \text{ vanishes at } \mathcal{L}^1\text{-a.e. point on } \partial\mathbb{R}^2_+. \tag{1.4.30}$$

Finally, if $K := \overline{B\big(z, 2^{-1}\,\mathrm{Im}\, z\big)}$, which is a compact neighborhood of z contained in $\mathbb{C}_+$, we have

$$|\vec{F}(\zeta)| \leq C_K |\zeta|^{-2} \quad \text{for all} \quad \zeta \in \mathbb{C}_+ \setminus K, \tag{1.4.31}$$

(hence (1.4.25) is presently satisfied for any $\lambda \in (1, \infty)$), and from (1.4.27) and Lemma 8.3.7 we see that

$$\mathcal{N}_\kappa^{\mathbb{R}^2_+ \setminus K} \vec{F} \in L^{1,\infty}(\mathbb{R}, \mathcal{L}^1). \tag{1.4.32}$$

However, it is clear from (1.4.29) and (1.4.30) that the Divergence Formula (1.4.24) fails in this setting. The source of this failure is the lack of local integrability for $\mathcal{N}_\kappa^{\mathbb{R}^2_+ \setminus K} \vec{F}$. This analysis shows that, in the context of Corollary 1.4.2, the last membership in (1.4.20), namely having $\mathcal{N}_\kappa^{\mathbb{R}^n_+ \setminus K} \vec{F}$ locally integrable in $\mathbb{R}^{n-1}$ cannot, generally speaking, be replaced by the weaker property $\mathcal{N}_\kappa^{\mathbb{R}^n_+ \setminus K} \vec{F} \in L^{1,\infty}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ (cf. (1.4.32)) even if we now additionally assume that $F_n\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}$ belongs to $L^1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$.

It is also of interest to derive the following refinement of what Theorem 1.4.1, specialized to the case when the underlying set is the unit ball, would ordinarily give.

Corollary 1.4.3 *Fix $n \in \mathbb{N}$ and denote by $B(0, 1)$ the (open) unit ball centered at the origin in $\mathbb{R}^n$. Also, set $S^{n-1} = \partial B(0, 1)$ and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor S^{n-1}$. Having*

[10] Inspired by the counterexample to the Schwarz–Pompeiu formula in the upper-half space discussed in the last part of [186, Sect. 1.1].

picked an arbitrary aperture parameter $\kappa \in (0, \infty)$*, consider a vector field whose scalar components are distributions in* $B(0, 1)$*,*

$$\vec{F} = (F_1, \dots, F_n) \in \big[\mathcal{D}'\big(B(0, 1)\big)\big]^n, \tag{1.4.33}$$

satisfying

$$\begin{gathered}\text{there exists a compact set } K \text{ contained in } B(0,1) \text{ such that}\\ \vec{F}\big|_{B(0,1)\setminus K} \in \big[L^1_{\rm loc}\big(B(0, 1) \setminus K, \mathcal{L}^n\big)\big]^n \ \text{ and } \ \mathcal{N}_\kappa^{B(0,1)\setminus K}\vec{F} \in L^1(S^{n-1}, \sigma).\end{gathered} \tag{1.4.34}$$

Also, with

$$f(x) := x \cdot \vec{F}(x) = \sum_{j=1}^{n} x_j F_j(x) \ \text{ for each } \ x = (x_1, \dots, x_n) \in B(0, 1) \setminus K, \tag{1.4.35}$$

assume that

$$f\Big|_{\partial B(0,1)}^{\kappa-\text{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \ S^{n-1}. \tag{1.4.36}$$

Finally, suppose that the divergence of $\vec{F}$*, considered in the sense of distributions in* $B(0, 1)$*, is the sum (in* $\mathcal{D}'\big(B(0, 1)\big)$*) of a compactly supported distribution in* $B(0, 1)$ *and a complex Borel measure in* $B(0, 1)$*, i.e.,*

$$\text{div}\vec{F} \in \mathscr{E}'\big(B(0, 1)\big) + \text{CBM}\big(B(0, 1)\big) \subseteq \big(\mathscr{C}_b^\infty\big(B(0, 1)\big)\big)^*. \tag{1.4.37}$$

Then for any $\kappa' > 0$ *the nontangential trace* $f\Big|_{\partial B(0,1)}^{\kappa'-\text{n.t.}}$ *exists* σ*-a.e. on* S^{n-1} *and is actually independent of* κ'*. When regarding it as a function defined* σ*-a.e. on* S^{n-1}*, this belongs to* $L^1(S^{n-1}, \sigma)$ *and, with the dependence on the aperture parameter* κ' *dropped, one has*

$$_{(\mathscr{C}_b^\infty(B(0,1)))^*}\big(\text{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(B(0,1))} = \int_{S^{n-1}} \Big(f\Big|_{\partial B(0,1)}^{\text{n.t.}}\Big)\, d\sigma. \tag{1.4.38}$$

Of course, a natural version of Corollary 1.4.3 is valid in any (open) ball in $\mathbb{R}^n$. Corollary 1.4.3 is in fact optimal. To elaborate on this aspect, work in the two-dimensional setting and identify $\mathbb{R}^2 \equiv \mathbb{C}$. With $\mathbb{D} := B(0, 1)$ denoting the unit disk in the plane, fix an arbitrary point $z \in \mathbb{D} \setminus \{0\}$ and define the complex-valued functions

$$F_1(\zeta) := \frac{1}{2\pi}\left\{\frac{1-\zeta}{\zeta(1+\zeta)}\frac{\zeta+z}{\zeta-z} + \frac{1-\overline{\zeta}}{\overline{\zeta}(1+\overline{\zeta})}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\right\} \tag{1.4.39}$$

and

$$F_2(\zeta) := \frac{1}{2\pi}\left\{\frac{1-\zeta}{\zeta(1+\zeta)}\frac{\zeta+z}{\zeta-z} - \frac{1-\overline{\zeta}}{\overline{\zeta}(1+\overline{\zeta})}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\right\}, \tag{1.4.40}$$

for each $\zeta \in \mathbb{D} \setminus \{0, z\}$. Use these scalar-valued functions to define[11] the vector field (with complex-valued components)

$$\vec{F}(\zeta) := \big(F_1(\zeta), \mathrm{i}F_2(\zeta)\big) \ \text{ for } \ \zeta \in \mathbb{D} \setminus \{0, z\}. \tag{1.4.41}$$

Then

$$\vec{F} \in \big[L^1_{\mathrm{loc}}\big(\mathbb{D}, \mathcal{L}^2\big) \cap \mathscr{C}^\infty\big(\overline{\mathbb{D}} \setminus \{0, z, -1\}\big)\big]^2, \tag{1.4.42}$$

and a computation (taking into account the formula for the standard fundamental solution for the Cauchy–Riemann operator in the plane) shows that

$$\operatorname{div}\vec{F} = \frac{2(1-z)}{1+z}\delta_z \ \text{ in } \ \mathcal{D}'(\mathbb{D}), \tag{1.4.43}$$

hence $\operatorname{div}\vec{F}$ belongs to $\mathscr{E}'(\mathbb{D})$. Also, if $K := \overline{B(0, R)}$ for some fixed $R \in \big(|z|, 1\big)$ (a choice which renders K a compact subset of the unit disk such that both 0 and z are contained in its interior), then (1.4.39)–(1.4.41) and Lemma 8.3.7 allow us to conclude that

$$\mathcal{N}_\kappa^{\mathbb{D}\setminus K}\vec{F} \in L^{1,\infty}(\partial\mathbb{D}, \sigma). \tag{1.4.44}$$

In addition, for each $\zeta = x + \mathrm{i}y \in \mathbb{D} \setminus K$ we have

$$\begin{aligned} f(\zeta) := xF_1(\zeta) + \mathrm{i}yF_2(\zeta) &= \frac{1}{2\pi}\left\{\frac{1-\zeta}{\zeta(1+\zeta)}\frac{\zeta+z}{\zeta-z}\right\}\zeta + \frac{1}{2\pi}\left\{\frac{1-\overline{\zeta}}{\overline{\zeta}(1+\overline{\zeta})}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\right\}\overline{\zeta} \\ &= \frac{1}{\pi}\frac{\zeta+z}{\zeta-z}\mathrm{Re}\Big[\frac{1-\zeta}{1+\zeta}\Big] - \frac{1}{\pi}\frac{1-\overline{\zeta}}{1+\overline{\zeta}}\frac{z(1-|\zeta|^2)}{(1-z\overline{\zeta})(\zeta-z)}, \end{aligned} \tag{1.4.45}$$

where the first equality defines $f(\zeta)$, and where we have used (1.4.39)–(1.4.40) in the second equality. Due to the presence of the factors $\mathrm{Re}\Big[\frac{1-\zeta}{1+\zeta}\Big]$ and $1-|\zeta|^2$ in the last line in (1.4.45), it follows that

$$f\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \ \text{ vanishes at } \sigma\text{-a.e. point on } \ S^1 = \partial\mathbb{D}. \tag{1.4.46}$$

On account of the properties recorded in (1.4.43) and (1.4.46), we see that the Divergence Formula (1.4.38) presently fails. The source of this failure is that in place of $\mathcal{N}_\kappa^{B(0,1)\setminus K}\vec{F} \in L^1(S^1, \sigma)$ (as demanded in (1.4.34)) we currently only have

[11] Inspired by the counterexample to the Schwarz–Pompeiu formula in the unit disk discussed in the last part of [186, Sect. 1.1].

(1.4.44). Altogether, this analysis shows that the last membership in (1.4.34), namely having $\mathcal{N}_\kappa^{B(0,1)\setminus K}\vec{F} \in L^1(S^{n-1},\sigma)$ cannot, in general, be weakened to demanding that $\mathcal{N}_\kappa^{B(0,1)\setminus K}\vec{F} \in L^{1,\infty}(S^{n-1},\sigma)$, even if we now additionally assume that $f\big|^{\kappa-\mathrm{n.t.}}_{\partial B(0,1)}$ belongs to $L^1(S^{n-1},\sigma)$.

Finally, we present a companion result to Corollary 1.4.3, now dealing with a version of the Divergence Theorem in the complement of the closed unit ball in $\mathbb{R}^n$ which is a refinement of what Theorem 1.4.1 would give in such a setting.

Corollary 1.4.4 *Pick $n \in \mathbb{N}$ and let $B(0,1)$ stand for the (open) unit ball centered at the origin in $\mathbb{R}^n$; in particular, $S^{n-1} = \partial B(0,1)$ is the unit sphere in $\mathbb{R}^n$. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor S^{n-1}$ and choose some aperture parameter $\kappa \in (0,\infty)$. Consider a vector field whose scalar components are distributions in $\mathbb{R}^n \setminus \overline{B(0,1)}$,*

$$\vec{F} = (F_1, \dots, F_n) \in \big[\mathcal{D}'\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big)\big]^n, \tag{1.4.47}$$

satisfying

$$\begin{gathered} \text{there exists a compact set } K \text{ contained in } \mathbb{R}^n \setminus \overline{B(0,1)} \text{ such that} \\ \vec{F}\big|_{\left(\mathbb{R}^n\setminus\overline{B(0,1)}\right)\setminus K} \in \Big[L^1_{\mathrm{loc}}\Big(\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big)\setminus K, \mathcal{L}^n\Big)\Big]^n \\ \text{and } \mathcal{N}_\kappa^{(\mathbb{R}^n\setminus\overline{B(0,1)})\setminus K}\vec{F} \in L^1(S^{n-1},\sigma). \end{gathered} \tag{1.4.48}$$

In addition, having set

$$f(x) := x\cdot\vec{F}(x) = \sum_{j=1}^n x_j F_j(x) \ \text{ for each } \ x = (x_1,\dots,x_n) \in \big(\mathbb{R}^n \setminus \overline{B(0,1)}\big)\setminus K, \tag{1.4.49}$$

suppose

$$f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\left(\mathbb{R}^n\setminus\overline{B(0,1)}\right)} \ \text{ exists at } \sigma\text{-a.e. point on } \ S^{n-1}. \tag{1.4.50}$$

Finally, assume that the divergence of $\vec{F}$, considered in the sense of distributions in $\mathbb{R}^n \setminus \overline{B(0,1)}$, is the sum in $\mathcal{D}'\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big)$ of a compactly supported distribution in $\mathbb{R}^n \setminus \overline{B(0,1)}$ and a complex Borel measure in $\mathbb{R}^n \setminus \overline{B(0,1)}$, i.e.,

$$\operatorname{div}\vec{F} \in \mathscr{E}'\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big) + \mathrm{CBM}\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big) \subseteq \Big(\mathscr{C}^\infty_b\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big)\Big)^*. \tag{1.4.51}$$

Then for any $\kappa' > 0$ the nontangential trace $f\Big|^{\kappa'-\mathrm{n.t.}}_{\partial\left(\mathbb{R}^n\setminus\overline{B(0,1)}\right)}$ exists σ-a.e. on S^{n-1}, is actually independent of κ' and, when regarded as a function defined σ-a.e. on S^{n-1}, it belongs to $L^1(S^{n-1},\sigma)$. Also, the contribution of $\vec{F}$ at infinity is meaningfully and unambiguously defined and, with the dependence on the parameter κ' dropped, one has

$$
{}_{(\mathscr{C}_b^\infty(\mathbb{R}^n\setminus\overline{B(0,1)}))^*}\big(\mathrm{div}\vec{F},1\big)_{\mathscr{C}_b^\infty(\mathbb{R}^n\setminus\overline{B(0,1)})} = -\int_{S^{n-1}}\Big(f\big|^{\text{n.t.}}_{\partial(\mathbb{R}^n\setminus\overline{B(0,1)})}\Big)\,\mathrm{d}\sigma + [\vec{F}]_\infty. \tag{1.4.52}
$$

Finally, the contribution of $\vec{F}$ *at infinity vanishes if there exists* $\lambda\in(1,\infty)$ *such that*

$$
\int_{B(0,\lambda R)\setminus B(0,R)} |f|\,\mathrm{d}\mathcal{L}^n = o(R^2) \ \ as \ \ R\to\infty, \tag{1.4.53}
$$

a scenario in which the Divergence Formula (1.4.52) *reduces to*

$$
{}_{(\mathscr{C}_b^\infty(\mathbb{R}^n\setminus\overline{B(0,1)}))^*}\big(\mathrm{div}\vec{F},1\big)_{\mathscr{C}_b^\infty(\mathbb{R}^n\setminus\overline{B(0,1)})} = -\int_{S^{n-1}}\Big(f\big|^{\text{n.t.}}_{\partial(\mathbb{R}^n\setminus\overline{B(0,1)})}\Big)\,\mathrm{d}\sigma. \tag{1.4.54}
$$

We wish to note that Corollary 1.4.4 is in the nature of best possible. To be specific, we shall work in the two-dimensional setting and identify $\mathbb{R}^2\equiv\mathbb{C}$. Denote by $\mathbb{D}:=B(0,1)$ the (open) unit disk in the plane, and fix an arbitrary point $z\in\mathbb{C}\setminus\overline{\mathbb{D}}$. Consider the complex-valued functions

$$
F_1(\zeta) := \frac{1}{2\pi}\left\{\frac{1}{\zeta(1+\zeta)}\frac{\zeta+z}{\zeta-z} + \frac{1}{\overline{\zeta}(1+\overline{\zeta})}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\right\} \tag{1.4.55}
$$

and

$$
F_2(\zeta) := \frac{1}{2\pi}\left\{\frac{1}{\zeta(1+\zeta)}\frac{\zeta+z}{\zeta-z} - \frac{1}{\overline{\zeta}(1+\overline{\zeta})}\frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\right\}, \tag{1.4.56}
$$

for each $\zeta\in\big(\mathbb{C}\setminus\overline{\mathbb{D}}\big)\setminus\{z\}$. Use these scalar-valued functions to define[12] the vector field (with complex-valued components)

$$
\vec{F}(\zeta) := \big(F_1(\zeta), \mathrm{i}F_2(\zeta)\big) \ \text{ for } \ \zeta\in\big(\mathbb{C}\setminus\overline{\mathbb{D}}\big)\setminus\{z\}. \tag{1.4.57}
$$

It is then clear that

$$
\vec{F}\in\Big[L^1_{\mathrm{loc}}\big(\mathbb{C}\setminus\overline{\mathbb{D}},\mathcal{L}^2\big)\cap\mathscr{C}^\infty\big(\big(\mathbb{C}\setminus\mathbb{D}\big)\setminus\{z\}\big)\Big]^2, \tag{1.4.58}
$$

and a computation (taking into account the formula for the standard fundamental solution for the Cauchy–Riemann operator in the plane) shows that

$$
\mathrm{div}\vec{F} = \frac{2}{1+z}\delta_z \ \text{ in } \ \mathcal{D}'\big(\mathbb{C}\setminus\overline{\mathbb{D}}\big), \tag{1.4.59}
$$

hence $\mathrm{div}\vec{F}$ belongs to $\mathscr{E}'\big(\mathbb{C}\setminus\overline{\mathbb{D}}\big)$. Next, if K is a compact subset of $\mathbb{C}\setminus\overline{\mathbb{D}}$ whose interior contains z, then (1.4.55)–(1.4.57) and Lemma 8.3.7 allow us to conclude that

[12] Inspired by the counterexample to the Schwarz–Pompeiu formula in the complement of the unit disk discussed in the last part of [186, Sect. 1.1].

$$\mathcal{N}_\kappa^{(\mathbb{C}\setminus\overline{\mathbb{D}})\setminus K}\vec{F} \in L^{1,\infty}(\partial\mathbb{D}, \sigma). \tag{1.4.60}$$

Moreover, for each $\zeta = x + \mathrm{i}y \in \big(\mathbb{C} \setminus \mathbb{D}\big) \setminus K$ we have

$$\begin{aligned} f(\zeta) &:= xF_1(\zeta) + \mathrm{i}yF_2(\zeta) \\ &= \frac{1}{\pi}\frac{\zeta+z}{\zeta-z}\mathrm{Re}\Big[\frac{1}{1+\zeta}\Big] - \frac{1}{\pi}\frac{1}{1+\overline{\zeta}}\frac{z(1-|\zeta|^2)}{(1-z\overline{\zeta})(\zeta-z)}, \end{aligned} \tag{1.4.61}$$

where the first equality defines $f(\zeta)$, and the second equality uses (1.4.55)–(1.4.56). Consequently,

$$\Big(f\big|^{\kappa-\mathrm{n.t.}}_{\partial(\mathbb{C}\setminus\overline{\mathbb{D}})}\Big)(\zeta) = \frac{1}{2\pi}\frac{\zeta+z}{\zeta-z} \ \text{ for each } \ \zeta \in S^1 = \partial\big(\mathbb{C}\setminus\overline{\mathbb{D}}\big), \tag{1.4.62}$$

so a simple residue calculation gives

$$\int_{S^1}\Big(f\big|^{\mathrm{n.t.}}_{\partial(\mathbb{C}\setminus\overline{\mathbb{D}})}\Big)\,d\sigma = -\frac{1}{2\pi\mathrm{i}}\int_{S^1}\frac{\zeta+z}{\zeta(\zeta-z)}\,d\zeta = 1. \tag{1.4.63}$$

Finally, (1.4.61) implies

$$|f(\zeta)| \le C_K|\zeta|^{-1} \ \text{ for all } \ \zeta \in \big(\mathbb{C}\setminus\mathbb{D}\big)\setminus K, \tag{1.4.64}$$

hence (1.4.53) with $n = 2$ is satisfied for any $\lambda \in (1, \infty)$. From (1.4.59) and (1.4.63) we see that the Divergence Formula (1.4.54) presently fails. The source of this failure is that in place of $\mathcal{N}_\kappa^{(\mathbb{R}^n\setminus\overline{B(0,1)})\setminus K}\vec{F} \in L^1(S^{n-1}, \sigma)$ (as requested in (1.4.48)) we currently only have (1.4.60).

Altogether, this analysis shows that the last membership in (1.4.48), namely having $\mathcal{N}_\kappa^{(\mathbb{R}^n\setminus\overline{B(0,1)})\setminus K}\vec{F} \in L^1(S^{n-1}, \sigma)$ cannot, generally speaking, be weakened to demanding that $\mathcal{N}_\kappa^{(\mathbb{R}^n\setminus\overline{B(0,1)})\setminus K}\vec{F} \in L^{1,\infty}(S^{n-1}, \sigma)$, even if we now additionally assume that $f\big|^{\mathrm{n.t.}}_{\partial(\mathbb{R}^n\setminus\overline{B(0,1)})}$ belongs to $L^1(S^{n-1}, \sigma)$.

1.5 Non-doubling Surface Measures and Maximally Singular Vector Fields

There is also a version of Theorem 1.4.1 formulated in an open set $\Omega \subseteq \mathbb{R}^n$ *without* imposing the condition that the "surface measure" $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is doubling. Remarkably, there is only a relatively small price to pay in this scenario, namely the loss of flexibility in the choice of the aperture parameter $\kappa \in (0, \infty)$ used to define the nontangential approach regions entering the definition of the nontangential boundary trace $\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ and the nontangential maximal function $\mathcal{N}_\kappa \vec{F}$ for the given vector field $\vec{F}$. At the same time, we may further relax the demand made in (1.4.4) on the nature of the distribution $\mathrm{div}\vec{F}$ by now merely asking that this may be extended to a functional in the algebraic dual $\big(\mathscr{C}_b^\infty(\Omega)\big)^*$ exhibiting a mild, natural, continuity property (that is automatically satisfied when (1.4.4) holds).

To concretely state the theorem of the flavor just described, we make a couple of definitions. Given an open set $\Omega \subseteq \mathbb{R}^n$, call a functional $\Lambda \in \big(\mathscr{C}_b^\infty(\Omega)\big)^*$ `continuous` provided for each function $f \in \mathscr{C}_b^\infty(\Omega)$ one has

$$\lim_{j\to\infty} {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\Lambda, f_j\big)_{\mathscr{C}_b^\infty(\Omega)} = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\Lambda, f\big)_{\mathscr{C}_b^\infty(\Omega)} \tag{1.5.1}$$

whenever the sequence

$$\begin{array}{l}\{f_j\}_{j\in\mathbb{N}} \subset \mathscr{C}_b^\infty(\Omega) \text{ is such that } \sup_{j\in\mathbb{N}}\sup_{x\in\Omega}|f_j(x)| < +\infty \text{ and}\\ \text{for each compact set } K \subset \Omega \text{ there exists some } j_K \in \mathbb{N} \text{ with the}\\ \text{property that } f_j \equiv f \text{ on } K \text{ if the integer } j \in \mathbb{N} \text{ satisfies } j \geq j_K.\end{array} \tag{1.5.2}$$

Also, given $u \in \mathcal{D}'(\Omega)$ call a functional $\Lambda \in \big(\mathscr{C}_b^\infty(\Omega)\big)^*$ an `extension` of u provided

$$\Lambda\Big|_{\mathscr{C}_c^\infty(\Omega)} = u. \tag{1.5.3}$$

Define the `regular support` of a given distribution $u \in \mathcal{D}'(\Omega)$ to be

$$\begin{array}{l}\mathrm{regsupp}\, u := \text{the smallest relatively closed subset of } \Omega\\ \text{outside of which } u \text{ is a locally integrable function.}\end{array} \tag{1.5.4}$$

Finally, for each truncation parameter ε satisfying $0 < \varepsilon < \mathrm{dist}(\mathrm{regsupp}\, u, \partial\Omega)$, henceforth we shall abbreviate

$$\mathcal{N}_\kappa^\varepsilon u := \mathcal{N}_\kappa(u \cdot \mathbf{1}_{\mathcal{O}_\varepsilon}) \;\text{ where }\; \mathcal{O}_\varepsilon := \big\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\big\}. \tag{1.5.5}$$

In view of Corollary 8.9.9, Lemma 4.6.2, and Proposition 8.8.6, the following result generalizes Theorem 1.4.1.

Theorem 1.5.1 *Pick $n \in \mathbb{N}$ and let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a locally finite measure. The latter condition forces Ω to be a set of locally finite perimeter; in particular, its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. Then there exists $\kappa_\Omega \in (0, \infty)$ such that if $\kappa > \kappa_\Omega$ it follows that the accessibility set*

$$\begin{array}{c}A_\kappa(\partial\Omega) := \big\{x \in \partial\Omega : x \in \overline{\Gamma_\kappa(x)}\big\} \text{ is } \sigma\text{-measurable}\\ \text{and has the property that } \sigma\big(\partial_*\Omega \setminus A_\kappa(\partial\Omega)\big) = 0.\end{array} \tag{1.5.6}$$

Moreover, if the vector field

$$\vec{F} \in \big[\mathscr{E}'(\Omega) + L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^n \subset \big[\mathcal{D}'(\Omega)\big]^n \tag{1.5.7}$$

satisfies, for some $\kappa > \kappa_\Omega$,

$$\begin{gathered} \textit{for each } x \in \partial\Omega \textit{ there exist } r_x > 0 \textit{ and } \varepsilon_x \in \big(0, \mathrm{dist}(\mathrm{regsupp}\,\vec{F}, \partial\Omega)\big) \\ \textit{with the property that } \int_{B(x,r_x)\cap\partial\Omega} \mathcal{N}_\kappa^{\varepsilon_x}\vec{F}\,\mathrm{d}\sigma < +\infty \end{gathered} \tag{1.5.8}$$

$\big($*a condition automatically satisfied whenever* $\mathcal{N}_\kappa^{\varepsilon}\vec{F}$ *belongs to* $L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$, *for some* $0 < \varepsilon < \mathrm{dist}(\mathrm{regsupp}\,\vec{F}, \partial\Omega)\big)$, *as well as*

$$\begin{gathered} \vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \textit{ exists } \sigma\textit{-a.e. on } A_\kappa(\partial\Omega), \quad \nu \cdot \big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) \in L^1(\partial_*\Omega, \sigma), \quad \textit{and} \\ \mathrm{div}\vec{F} \in \mathcal{D}'(\Omega) \textit{ extends to a continuous functional in } \big(\mathscr{C}^\infty_b(\Omega)\big)^* \end{gathered} \tag{1.5.9}$$

(*with the agreement that the symbol* $\mathrm{div}\vec{F}$ *is retained for this extension*), *then the contribution of* $\vec{F}$ *at infinity is meaningfully and unambiguously defined and one has*

$${}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\Omega)} = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma + [\vec{F}]_\infty. \tag{1.5.10}$$

Furthermore, one has $[\vec{F}]_\infty = 0$, *a scenario in which* (1.5.10) *simply reduces to*

$${}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\Omega)} = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma, \tag{1.5.11}$$

if either Ω *is bounded, or when*

$$\begin{gathered} \partial\Omega \textit{ is unbounded and there exists some large } R_* > 0 \\ \textit{with } \mathrm{regsupp}\,\vec{F} \subseteq B(0, R_*) \textit{ and } \mathcal{N}_\kappa^{\Omega\setminus B(0,R_*)}\vec{F} \in L^1(\partial\Omega, \sigma), \end{gathered} \tag{1.5.12}$$

or when there exists $\lambda \in (1, \infty)$ *such that*

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |x \cdot \vec{F}(x)|\,\mathrm{d}\mathcal{L}^n(x) = o(R^2) \quad \textit{as} \quad R \to \infty. \tag{1.5.13}$$

We continue with a series of annotations designed to further clarify the scope and breadth of this theorem.

<u>Annotation 1.</u> From the last part of Lemma 4.6.1 it follows that if the distribution $\mathrm{div}\vec{F} \in \mathcal{D}'(\Omega)$ has an extension to a continuous functional $\Lambda \in \big(\mathscr{C}^\infty_b(\Omega)\big)^*$ then Λ is uniquely determined by these properties. An important consequence of this observation is that retaining the symbol $\mathrm{div}\vec{F}$ for the functional Λ is unambiguous. In particular, the Divergence Formula (1.5.10) has a clear meaning.

Annotation 2. As regards the nature of the hypotheses made in its statement, Theorem 1.5.1 is sharp in every respect. Specifically, in (2.3.30) we indicate (by means of a concrete counterexample) that the failure of the local integrability condition in the second line of (1.5.8), even at a single point $x \in \partial\Omega$, may invalidate the Divergence Formula recorded in (1.5.11), even when we assume that $\mathcal{N}_\kappa \vec{F} \in L^{1,\infty}(\partial\Omega, \sigma)$ and $\nu \cdot \big(\vec{F}\big|^{\text{n.t.}}_{\partial\Omega}\big) \in L^1(\partial_*\Omega, \sigma)$, or that $\vec{F}\big|^{\text{n.t.}}_{\partial\Omega} \in L^1(\partial\Omega, \sigma)$. Next, the requirement that $\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists at σ-a.e. point on $A_\kappa(\partial\Omega)$ made in (1.5.9) cannot be relaxed to merely asking that $\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists at σ-a.e. point on $\partial_*\Omega$. This is apparent from (2.1.17), or (2.1.47), or (2.1.64), or (2.1.87), or (2.1.102). Also, the role of the lower Ahlfors regularity assumption on the boundary, made in Theorem 1.5.1, is highlighted by (2.4.9) and (2.4.18), while the necessity of considering $\operatorname{div}\vec{F}$ in the sense of distributions has been justified in (2.5.13). Lastly, that $o(R^2)$ may not be replaced by $O(R^2)$ in the formulation of the growth condition (1.5.13) has been pointed out in (2.2.15).

Annotation 3. In the case when Ω is bounded, (1.5.11) implies a more general version of itself of the following sort:

$$ {}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\operatorname{div}\vec{F}, f\big)_{\mathscr{C}^\infty_b(\Omega)} = -\int_\Omega \nabla f \cdot \vec{F}\, d\mathcal{L}^n + \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\,\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big) f\, d\sigma \quad (1.5.14) $$

for each function $f \in \mathscr{C}^\infty_b(\Omega) \cap \mathscr{C}^0(\overline{\Omega})$ such that the distribution $\nabla f \cdot \vec{F}$ actually belongs to $L^1(\Omega, \mathcal{L}^n)$. Indeed, this may be seen by applying (1.5.11) to the vector field $f\vec{F}$, keeping in mind that its divergence (originally considered in the sense of distributions in Ω) may be extended to a continuous functional in $\big(\mathscr{C}^\infty_b(\Omega)\big)^*$ according to

$$ \begin{aligned} {}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\operatorname{div}(f\vec{F}), g\big)_{\mathscr{C}^\infty_b(\Omega)} &:= \int_\Omega (\nabla f \cdot \vec{F}) g\, d\mathcal{L}^n \\ &\quad + {}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\operatorname{div}\vec{F}, fg\big)_{\mathscr{C}^\infty_b(\Omega)} \end{aligned} \quad (1.5.15) $$

for each $g \in \mathscr{C}^\infty_b(\Omega)$.

Annotation 4. We also wish to note that the local integrability condition for the truncated nontangential maximal function recorded in (1.5.8) is automatically satisfied whenever $\vec{F}$ is locally bounded at boundary points, i.e., when for each $x \in \partial\Omega$ there exists some $r_x \in (0, \infty)$ such that $\vec{F} \in \big[L^\infty\big(B(x, r_x) \cap \Omega, \mathcal{L}^n\big)\big]^n$. In particular, this is the case when there exists some closed set $S \subset \Omega$ with the property that $\vec{F}\big|_{\Omega\setminus S}$ extends to $\big[\mathscr{C}^0(\overline{\Omega} \setminus S)\big]^n$.

Annotation 5. If σ is a doubling measure, then any $\kappa > 0$ will do, and $A_\kappa(\partial\Omega)$ may be replaced by $\partial_{\text{nta}}\Omega$.

Annotation 6. Reflecting back on the statements of Theorems 1.2.1, 1.3.1, 1.4.1, and 1.5.1, the emerging philosophy is that while the lower Ahlfors regularity of $\partial\Omega$ is a common assumption, having the surface measure $\sigma = \mathcal{H}^{n-1}\lfloor\partial\Omega$ doubling enables us to harmlessly change the aperture parameter κ in the nontangential approach regions used to define the nontangential boundary trace, but otherwise we may only request that σ is a locally finite measure without drastically affecting the format of the Divergence Formula.

Annotation 7. By specializing Theorem 1.5.1 to the two-dimensional setting we obtain a version of Green's Formula (1.4.16) in which now the measure $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ is no longer assumed to be doubling.

Annotation 8. It turns out that condition (1.5.13) is automatically satisfied when either Ω is bounded, or when

$$\begin{gathered}\partial\Omega \text{ is unbounded and there exists some large } R_* > 0\\ \text{with regsupp } \vec{F} \subseteq B(0, R_*) \text{ and } \mathcal{N}_\kappa^{\Omega\setminus B(0,R_*)}\vec{F} \in L^1(\partial\Omega, \sigma).\end{gathered} \tag{1.5.16}$$

Indeed, this follows from the last part in Lemma 4.7.3 (used with $\Omega \setminus B(0, R_o)$ in place of Ω) and (8.6.51) (used with $E := \Omega \setminus B(0, R_o)$).

Annotation 9. The proof of Theorem 1.5.1 is presented in Sect. 9.3.

In applications, it is useful to have a version of the Divergence Theorem in a context which assumes the "surface measure" to be doubling (as in Theorem 1.4.1) but allows the divergence of the vector field to be essentially "maximally singular" (as in Theorem 1.5.1). Compared with Theorem 1.5.1, the gain is the ability of considering arbitrary aperture parameters. Compared with Theorem 1.4.1, the gain is a less demanding hypothesis on the divergence of the vector field in question. Our next corollary accomplishes just that.

Corollary 1.5.2 *Fix $n \in \mathbb{N}$ and let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$ (which, up to a σ-nullset, is contained in $\partial_{\text{nta}}\Omega$). Fix $\kappa \in (0, \infty)$ arbitrary and assume that the vector field*

$$\vec{F} \in \big[\mathscr{E}'(\Omega) + L^1_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^n \subset \big[\mathcal{D}'(\Omega)\big]^n \tag{1.5.17}$$

satisfies

$$\begin{gathered}\mathcal{N}_\kappa^{\varepsilon}\vec{F} \in L^1(\partial\Omega,\sigma) \ \textit{ for some } \ 0 < \varepsilon < \operatorname{dist}(\text{regsupp}\,\vec{F}, \partial\Omega),\\ \textit{the nontangential trace } \ \vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \textit{ exists } \sigma\textit{-a.e. on } \ \partial_{\text{nta}}\Omega, \ \textit{ and}\\ \operatorname{div}\vec{F} \in \mathcal{D}'(\Omega) \ \textit{ extends to a continuous functional in } \ \big(\mathscr{C}_b^\infty(\Omega)\big)^*\\ \textit{(for which the symbol } \operatorname{div}\vec{F} \in \big(\mathscr{C}_b^\infty(\Omega)\big)^* \ \textit{ is henceforth retained).}\end{gathered} \tag{1.5.18}$$

Then for any $\kappa' > 0$ the nontangential trace $\vec{F}\Big|_{\partial\Omega}^{\kappa'-\text{n.t.}}$ exists σ-a.e. on $\partial_{\text{nta}}\Omega$ and is actually independent of κ'. Also, the contribution of $\vec{F}$ at infinity (cf. (1.3.2)–

(1.3.3)) *is meaningfully and unambiguously defined and, with the dependence on the parameter* κ' *dropped, one has*

$$ {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\,\big|^{\text{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma + [\vec{F}]_\infty. \tag{1.5.19} $$

Furthermore, one has $[\vec{F}]_\infty = 0$*, a scenario in which* (1.5.19) *simply reduces to*

$$ {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\,\big|^{\text{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma, \tag{1.5.20} $$

if either Ω *is bounded, or when*

$$ \begin{gathered} \partial\Omega \textit{ is unbounded and there exists some large } R_* > 0 \\ \textit{with } \operatorname{regsupp}\vec{F} \subseteq B(0,R_*) \textit{ and } \mathcal{N}_\kappa^{\Omega\setminus B(0,R_*)}\vec{F} \in L^1(\partial\Omega,\sigma), \end{gathered} \tag{1.5.21} $$

or when there exists $\lambda \in (1,\infty)$ *such that*

$$ \int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |x\cdot\vec{F}(x)|\,\mathrm{d}\mathcal{L}^n(x) = o(R^2) \quad \textit{as} \quad R \to \infty. \tag{1.5.22} $$

Finally,

$$ \begin{aligned} &\textit{all the above conclusions remain valid if the condition in the first line} \\ &\textit{of (1.5.18) is replaced by the demand that } \mathcal{N}_\kappa^\varepsilon\vec{F} \textit{ belongs to } L^1_{\mathrm{loc}}(\partial\Omega,\sigma) \\ &\textit{for some } 0<\varepsilon<\operatorname{dist}(\operatorname{regsupp}\vec{F},\partial\Omega) \textit{ together with the membership} \\ &\nu\cdot\big(\vec{F}\,\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big) \in L^1(\partial_*\Omega,\sigma). \end{aligned} \tag{1.5.23} $$

The proof of Corollary 1.5.2 is given in Sect. 9.3.

1.6 Divergence Formulas Without Lower Ahlfors Regularity

A permanency in the statements of our brands of Divergence Theorems so far has been the assumption that the domains involved have lower Ahlfors regular boundaries. The main goal of this section is to explore scenarios in which a version of the Divergence Theorem may be formulated in domains which may not have a lower Ahlfors regular boundary. Necessarily, stronger hypotheses should be imposed on the intervening vector fields. This being said, as in the past, we insist on having these vector fields defined exclusively inside the domains in question (again, in contrast to the classical

De Giorgi–Federer result recalled in Theorem 1.1.1, where the vector fields are completely unrelated to the underlying domain).

Here is our first result of this flavor (the notion of upper Minkowski content is reviewed in Definition 4.5.1).

Theorem 1.6.1 *Let $\widetilde{\Omega}$ be a nonempty bounded open subset of $\mathbb{R}^n$ (where $n \in \mathbb{N}$) with a lower Ahlfors regular boundary satisfying $\mathcal{H}^{n-1}(\partial\widetilde{\Omega}) < +\infty$. Suppose K is a compact subset of $\widetilde{\Omega}$ of finite upper $(n-d)$-dimensional Minkowski content for some $d > 1$ and define $\Omega := \widetilde{\Omega} \setminus K$. Also, fix a compact set $K_o \subseteq \Omega$. Next, consider a vector field*

$$\vec{F} \in \big[L^{\frac{d}{d-1}}(\Omega, \mathcal{L}^n) + \mathscr{E}'_{K_o}(\Omega)\big]^n \subset \big[\mathcal{D}'(\Omega)\big]^n \tag{1.6.1}$$

satisfying, for some sufficiently large aperture parameter $\kappa = \kappa_{\widetilde{\Omega}} \in (0, \infty)$,

$$\begin{aligned} &\textit{there exists some } \varepsilon \in \big(0, \operatorname{dist}(K_o, \partial\Omega)\big) \\ &\textit{with the property that } \int_{\partial\Omega} \mathcal{N}_\kappa^\varepsilon \vec{F}\, d\mathcal{H}^{n-1} < +\infty, \end{aligned} \tag{1.6.2}$$

as well as

$$\begin{aligned} &\textit{the boundary trace } \vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \textit{ exists } \mathcal{H}^{n-1}\textit{-a.e. on } A_\kappa(\partial\Omega) \setminus K \textit{ and} \\ &\operatorname{div}\vec{F}, \textit{ computed in } \mathcal{D}'(\Omega), \textit{ belongs to } L^1(\Omega, \mathcal{L}^n) + \mathscr{E}'_{K_o}(\Omega). \end{aligned} \tag{1.6.3}$$

Then Ω is a set of locally finite perimeter, the nontangential boundary trace $\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at $\mathcal{H}^{n-1}$-a.e. point on $\partial_*\Omega$ and actually belongs to $L^1(\partial_*\Omega, \mathcal{H}^{n-1})$ and, with ν denoting the geometric measure theoretic outward unit normal to Ω, one has

$$_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)\, d\mathcal{H}^{n-1}. \tag{1.6.4}$$

Furthermore, the same result is valid for $d = 1$ (with $L^{\frac{d}{d-1}}(\Omega, \mathcal{L}^n)$ in (1.6.1) now interpreted as $L^\infty(\Omega, \mathcal{L}^n)$) if the set K is assumed to have vanishing upper $(n-1)$-dimensional Minkowski content.

Note 1. Even though the auxiliary set $\widetilde{\Omega}$ is assumed to have a lower Ahlfors regular boundary, in general there is no reason to expect that $\Omega := \widetilde{\Omega} \setminus K$ has a lower Ahlfors regular boundary. To see this, consider $\widetilde{\Omega} := \{x \in \mathbb{R}^n : |x| < 1\}$, where $n \geq 2$, and $K := \{0\}$ which is a compact subset of $\widetilde{\Omega}$ having finite 0-dimensional upper Minkowski content. In such a scenario, the boundary of $\Omega := B(0, 1) \setminus \{0\}$ fails to satisfy the lower Ahlfors regularity condition at $0 \in \partial\Omega$. Hence, in contrast to our earlier results, the Divergence Formula (1.6.4) involves a set which may not have a lower Ahlfors regular boundary.

Note 2. Theorem 1.6.1 is sharp in the following precise sense. If $\widetilde{\Omega} := B(0, 1)$ is the unit ball in $\mathbb{R}^n$ with $n \geq 2$, and $K := \{0\}$, then $K \subset \widetilde{\Omega}$ is a compact set of finite upper

0-dimensional Minkowski content (cf. Definition 4.5.1). Taking $\vec{F}(x) := x/|x|^n$ for $\mathcal{L}^n$-a.e. $x \in \Omega = B(0,1) \setminus \{0\}$ then invalidates (1.6.4), even though (1.6.2)–(1.6.3) are satisfied (with, say, $K_o := \varnothing$). The issue is that the membership in (1.6.1) fails for $d := n$, even though $\vec{F} \in \big[L^{\frac{n}{n-1},\infty}(\Omega, \mathcal{L}^n)\big]^n$.

Note 3. The proof of Theorem 1.6.1 is given in Sect. 9.4.

Several consequences of Theorem 1.6.1 of independent interest are singled out in a series of corollaries below.

Corollary 1.6.2 *Let $\widetilde{\Omega}$ be a nonempty bounded open subset of $\mathbb{R}^n$ (for $n \in \mathbb{N}$, $n \geq 2$) with a lower Ahlfors regular boundary satisfying $\mathcal{H}^{n-1}(\partial\widetilde{\Omega}) < +\infty$. Also, suppose K is a compact subset of $\widetilde{\Omega}$ of finite upper $(n-d)$-dimensional Minkowski content for some $d > 1$ and such that $\mathcal{H}^{n-1}(K) < +\infty$. Define the set $\Omega := \widetilde{\Omega} \setminus K$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and denote by ν the geometric measure theoretic outward unit normal to Ω. In this context, consider a vector field*

$$\vec{F} \in \big[L^\infty(\Omega, \mathcal{L}^n)\big]^n \subset \big[\mathcal{D}'(\Omega)\big]^n \tag{1.6.5}$$

with the property that, for some sufficiently large $\kappa = \kappa_{\widetilde{\Omega}} \in (0, \infty)$,

$$\begin{aligned} &\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \text{ exists at } \sigma\text{-a.e. point on } \ A_\kappa(\partial\Omega) \setminus K, \ \text{ and} \\ &\mathrm{div}\vec{F}, \ \text{ computed in } \mathcal{D}'(\Omega), \ \text{ belongs to } \ L^1(\Omega, \mathcal{L}^n). \end{aligned} \tag{1.6.6}$$

Then the nontangential trace $\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial_*\Omega$ and, as a function, actually belongs to $L^1(\partial_*\Omega, \sigma)$. Also,

$$\int_\Omega \mathrm{div}\vec{F}\, \mathrm{d}\mathcal{L}^n = \int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)\, \mathrm{d}\sigma. \tag{1.6.7}$$

Furthermore, the same results are valid for $d = 1$ if the compact set K is now assumed to have vanishing upper $(n-1)$-dimensional Minkowski content.

The case when $K = \varnothing$ is already of interest. The proof of Corollary 1.6.2 is presented in Sect. 9.4.

Corollary 1.6.3 *Let $\widetilde{\Omega}$ be a nonempty bounded open subset of $\mathbb{R}^n$ (where $n \in \mathbb{N}$, $n \geq 2$) with a lower Ahlfors regular boundary satisfying $\mathcal{H}^{n-1}(\partial\widetilde{\Omega}) < +\infty$. Suppose K is a compact subset of $\widetilde{\Omega}$ of finite upper $(n-d)$-dimensional Minkowski content for some $d > 1$ and such that $\mathcal{H}^{n-1}(K) < +\infty$. Define $\Omega := \widetilde{\Omega} \setminus K$ and consider a vector field*

$$\begin{aligned} &\vec{F} \in \big[L^\infty(\Omega, \mathcal{L}^n)\big]^n \ \text{ with the property that} \\ &\mathrm{div}\vec{F}, \ \text{ computed in } \mathcal{D}'(\Omega), \text{ belongs to } \ L^1(\Omega, \mathcal{L}^n), \end{aligned} \tag{1.6.8}$$

such that

$$\vec{F}_{\mathrm{b}}(x) := \lim_{\Omega \ni y \to x} \vec{F}(y) \ \ \textit{exists at } \mathcal{H}^{n-1}\textit{-a.e.} \ \ x \in \partial\Omega \setminus K. \tag{1.6.9}$$

Then $\vec{F}_{\mathrm{b}}$ belongs to $L^1(\partial_*\Omega, \mathcal{H}^{n-1})$ and, with ν denoting the geometric measure theoretic outward unit normal to Ω, one has

$$\int_\Omega \operatorname{div}\vec{F}\, \mathrm{d}\mathcal{L}^n = \int_{\partial_*\Omega} \nu \cdot \vec{F}_{\mathrm{b}}\, \mathrm{d}\mathcal{H}^{n-1}. \tag{1.6.10}$$

Furthermore, the same result is valid for $d = 1$ if the compact set K is assumed to have vanishing upper $(n-1)$-dimensional Minkowski content.

Again, the case when $K = \varnothing$ is already significant. See Sect. 9.4 for a proof of Corollary 1.6.3.

Corollary 1.6.4 *Let Ω be a nonempty bounded open subset of $\mathbb{R}^2$ with the property that $\partial\Omega$ has finitely many connected components and $\mathcal{H}^1(\partial\Omega) < +\infty$. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Finally, consider a vector field*

$$\begin{gathered}\vec{F} \in \big[L^\infty(\Omega, \mathcal{L}^2)\big]^2 \ \ \textit{with the property that} \\ \operatorname{div}\vec{F}, \ \ \textit{computed in } \mathcal{D}'(\Omega)\textit{, belongs to } \ L^1(\Omega, \mathcal{L}^n),\end{gathered} \tag{1.6.11}$$

such that

$$\vec{F}_{\mathrm{b}}(x) := \lim_{\Omega \ni y \to x} \vec{F}(y) \ \ \textit{exists at } \sigma\textit{-a.e.} \ \ x \in \partial\Omega. \tag{1.6.12}$$

Then $\vec{F}_{\mathrm{b}}$ belongs to $L^1(\partial_*\Omega, \mathcal{H}^1)$ and one has

$$\int_\Omega \operatorname{div}\vec{F}\, \mathrm{d}\mathcal{L}^2 = \int_{\partial_*\Omega} \nu \cdot \vec{F}_{\mathrm{b}}\, \mathrm{d}\sigma. \tag{1.6.13}$$

The proof of Corollary 1.6.4 is given in Sect. 9.4.

Corollary 1.6.5 *Let Ω be a nonempty bounded open subset of $\mathbb{R}^2$ with the property that $\partial\Omega$ has finitely many connected components and $\mathcal{H}^1(\partial\Omega) < +\infty$. Abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Next, fix a compact set $K_o \subseteq \Omega$ and consider a vector field*

$$\vec{F} \in \big[L^2(\Omega, \mathcal{L}^2) + \mathscr{E}'_{K_o}(\Omega)\big]^2 \subset \big[\mathcal{D}'(\Omega)\big]^2 \tag{1.6.14}$$

satisfying, for some sufficiently large aperture parameter $\kappa = \kappa_\Omega \in (0, \infty)$ and some truncation parameter $\varepsilon \in \big(0, \operatorname{dist}(K_o, \partial\Omega)\big)$,

$$\int_{\partial\Omega} \mathcal{N}_\kappa^\varepsilon \vec{F}\, \mathrm{d}\sigma < +\infty, \tag{1.6.15}$$

as well as

$$\begin{aligned}&\text{the nontangential trace } \vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma\text{-a.e. on } A_\kappa(\partial\Omega) \text{ and}\\&\operatorname{div}\vec{F}, \text{ computed in } \mathcal{D}'(\Omega), \text{ belongs to } L^1(\Omega,\mathcal{L}^2)+\mathscr{E}'_{K_o}(\Omega).\end{aligned}\tag{1.6.16}$$

Then $\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ *exists at* σ*-a.e. point on* $\partial_*\Omega$ *and, as a function, actually belongs to* $L^1(\partial_*\Omega,\sigma)$*. Moreover,*

$$_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\operatorname{div}\vec{F},1\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_{\partial_*\Omega} \nu\cdot\big(\vec{F}\,\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\,d\sigma. \tag{1.6.17}$$

Again, see Sect. 9.4 for a proof of Corollary 1.6.5.

We conclude this section with the following refined version of the classical Green formula in the plane, accompanying our earlier result of this flavor in Theorem 1.2.5 (see Sect. 9.4 for a proof).

Corollary 1.6.6 *Let* Ω *be a nonempty bounded open subset of* $\mathbb{R}^2$ *with the property that* $\partial\Omega$ *has finitely many connected components and* $\mathcal{H}^1(\partial\Omega)<+\infty$*. Abbreviate* $\sigma:=\mathcal{H}^1\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω*. Also, under the identification* $\mathbb{R}^2\equiv\mathbb{C}$*, consider* $\tau:=i\nu$*, the positively oriented geometric measure theoretic unit tangent vector*[13] *to the boundary of* Ω*. Finally, fix a sufficiently large aperture parameter* $\kappa\in(0,\infty)$ *along with some small truncation parameter* $\varepsilon>0$ *and assume* P,Q *are two complex-valued functions defined in* Ω *with the following properties:*

$$\begin{aligned}&\text{the functions } P \text{ and } Q \text{ belong to the space } L^2(\Omega,\mathcal{L}^2),\\&\text{the boundary traces } P\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ and } Q\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exist } \sigma\text{-a.e. on } A_\kappa(\partial\Omega),\\&\mathcal{N}_\kappa^\varepsilon P,\ \mathcal{N}_\kappa^\varepsilon Q\in L^1(\partial\Omega,\sigma), \text{ and } \partial_x Q-\partial_y P \text{ belongs to } L^1(\Omega,\mathcal{L}^2),\end{aligned}\tag{1.6.18}$$

where all partial derivatives are considered in the sense of distributions in Ω*.*

Then the nontangential traces $P\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ *and* $Q\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ *exist* σ*-a.e. on* $\partial_*\Omega$ *belong to* $L^1(\partial_*\Omega,\sigma)$*, and*

$$\int_{\partial_*\Omega}\Big(P\big|_{\partial\Omega}^{\kappa-\text{n.t.}},\,Q\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)\cdot\tau\,d\sigma=\int_\Omega\big(\partial_x Q-\partial_y P\big)\,d\mathcal{L}^2. \tag{1.6.19}$$

Furthermore, if $\alpha,\beta\in[0,\pi]$ *denote the angles made by the positively oriented unit tangent vector* τ *with the coordinate axes in the plane, then in terms of the "differentials"*

$$dx:=\cos\alpha\,\mathcal{H}^1\lfloor\partial_*\Omega \ \text{ and } \ dy:=\cos\beta\,\mathcal{H}^1\lfloor\partial_*\Omega \tag{1.6.20}$$

[13] See (5.6.29)–(5.6.31) in this regard.

formula (1.6.19) *may be recast as*

$$\int_{\partial_*\Omega} \big(P\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\,\mathrm{d}x + \big(Q\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\,\mathrm{d}y = \int_{\Omega} \big(\partial_x Q - \partial_y P\big)\,\mathrm{d}\mathcal{L}^2. \tag{1.6.21}$$

1.7 Integration by Parts in Open Sets with Ahlfors Regular Boundaries

Let us specialize Theorem 1.2.1 to the case when the vector field involved has the particular form

$$\vec{F} = u\,w\,\mathbf{e}_j = (0,\dots,0,u\,w,0,\dots,0) : \Omega \longrightarrow \mathbb{C}^n \tag{1.7.1}$$

for some complex-valued functions u, w defined on Ω. In view of the Leibniz formula for weak derivatives from Proposition 4.3.1, this then produces the following integration by parts formula.

Theorem 1.7.1 *Pick $n \in \mathbb{N}$ and let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. This implies that Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal $\nu = (\nu_1,\dots,\nu_n)$ is defined σ-a.e. on $\partial_*\Omega$. Fix $\kappa, \kappa' > 0$ and suppose $u, w : \Omega \to \mathbb{C}$ are two $\mathcal{L}^n$-measurable functions with the property that*

$$\begin{gathered} \mathcal{N}_\kappa u < \infty \;\; \textit{and} \;\; \mathcal{N}_{\kappa'} w < \infty \;\; \textit{at } \sigma\textit{-a.e. point on} \;\; \partial\Omega, \\ \mathcal{N}_\kappa u \cdot \mathcal{N}_{\kappa'} w \;\; \textit{belongs to the space} \;\; L^1(\partial\Omega,\sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}},\;\; w\big|_{\partial\Omega}^{\kappa'-\text{n.t.}} \;\; \textit{exist at } \sigma\textit{-a.e. point on} \;\; \partial_{\text{nta}}\Omega, \end{gathered} \tag{1.7.2}$$

(*in particular, $u, w \in L^1_{\rm loc}(\Omega,\mathcal{L}^n)$; cf. Lemma 8.3.1*). *Also, pick $j \in \{1,\dots,n\}$ and assume, with all partial derivatives considered in the sense of distributions in Ω, that*

$$\partial_j u,\; \partial_j w \in L^1_{\rm loc}(\Omega,\mathcal{L}^n) \;\; \textit{and} \;\; u\,\partial_j w + w\partial_j u \in L^1(\Omega,\mathcal{L}^n). \tag{1.7.3}$$

Then, with the nontangential traces $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$, $w\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}$ regarded as functions defined σ-a.e. on $\partial_*\Omega$ (which, up to a σ-nullset, is contained in $\partial_{\text{nta}}\Omega$), their product belongs to $L^1(\partial_*\Omega,\sigma)$ and one has

$$\int_\Omega \big\{u\partial_j w + w\partial_j u\big\}\,\mathrm{d}\mathcal{L}^n = \int_{\partial_*\Omega} \nu_j\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\big(w\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big)\,\mathrm{d}\sigma \tag{1.7.4}$$

in the case when either Ω *is bounded, or* $\partial\Omega$ *is unbounded. Furthermore, formula* (1.7.4) *also holds if* Ω *is unbounded and* $\partial\Omega$ *is bounded provided there exists some* $\lambda \in (1, \infty)$ *such that*

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |u||w| \, \mathrm{d}\mathcal{L}^n = o(R) \quad \textit{as} \quad R \to \infty. \tag{1.7.5}$$

In the context of Theorem 1.7.1, the memberships

$$u, w \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \quad \text{and} \quad uw \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \tag{1.7.6}$$

are consequences of Lemma 8.3.1 and the assumptions in the first line of (1.7.2). In particular, it is meaningful to consider

$$\partial_j u, \partial_j w, \partial_j(uw) \in \mathcal{D}'(\Omega). \tag{1.7.7}$$

It should also be noted that, when the last condition in (1.7.3) is strengthened to

$$u\partial_j w \in L^1(\Omega, \mathcal{L}^n) \quad \text{and} \quad w\partial_j u \in L^1(\Omega, \mathcal{L}^n), \tag{1.7.8}$$

then in place of (1.7.4) we may write

$$\int_\Omega u\partial_j w \, \mathrm{d}\mathcal{L}^n = -\int_\Omega w\partial_j u \, \mathrm{d}\mathcal{L}^n + \int_{\partial_*\Omega} \nu_j \big(u\big|^{\text{n.t.}}_{\partial\Omega}\big)\big(w\big|^{\text{n.t.}}_{\partial\Omega}\big) \, \mathrm{d}\sigma, \tag{1.7.9}$$

which is more in line with the traditional format of an integration by parts formula.

Finally, we remark that the properties demanded in the first two lines of (1.7.2) are satisfied if, for example,

$$\begin{gathered} \mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma) \quad \text{and} \quad \mathcal{N}_{\kappa'} w \in L^{p'}(\partial\Omega, \sigma) \\ \text{for some} \quad p, p' \in [1, \infty] \quad \text{with} \quad 1/p + 1/p' = 1. \end{gathered} \tag{1.7.10}$$

Moreover, in such a scenario Proposition 8.9.8 guarantees that

$$\begin{gathered} \text{for each } \kappa'' \in (0, \infty) \text{ the nontangential traces } u\big|^{\kappa''-\text{n.t.}}_{\partial\Omega}, \ w\big|^{\kappa''-\text{n.t.}}_{\partial\Omega} \\ \text{exist } \sigma\text{-a.e. on the set } \partial_{\text{nta}}\Omega, \text{ and are actually independent of } \kappa''. \end{gathered} \tag{1.7.11}$$

One very useful extension of Theorem 1.7.1 is by allowing more general differential operators than individual first-order partial derivatives. Specifically, consider an $N \times N'$ first-order system (where $N, N' \in \mathbb{N}$ are arbitrary) with constant complex coefficients

$$D = \Big(\sum_{j=1}^n a_j^{\alpha\beta}\partial_j + b^{\alpha\beta}\Big)_{\substack{1\leq\alpha\leq N \\ 1\leq\beta\leq N'}}. \tag{1.7.12}$$

We shall refer to D as being homogeneous if all zero-th order terms vanish, i.e., if $b^{\alpha\beta} = 0$ for all α, β. Denote by $D^\top$ the real transpose of D, i.e., the $N' \times N$ first-order system given by

$$D^\top := \Big(-\sum_{j=1}^{n} a_j^{\alpha\beta} \partial_j + b^{\alpha\beta} \Big)_{\substack{1\le\beta\le N' \\ 1\le\alpha\le N}}, \tag{1.7.13}$$

by $\overline{D}$ the complex conjugate of D, i.e.,

$$\overline{D} := \Big(\sum_{j=1}^{n} \overline{a_j^{\alpha\beta}} \partial_j + \overline{b^{\alpha\beta}} \Big)_{\substack{1\le\alpha\le N \\ 1\le\beta\le N'}}, \tag{1.7.14}$$

and by D^* the Hermitian adjoint of D, i.e.,

$$D^* := \overline{(D^\top)} = (\overline{D})^\top = \Big(-\sum_{j=1}^{n} \overline{a_j^{\alpha\beta}} \partial_j + \overline{b^{\alpha\beta}} \Big)_{\substack{1\le\beta\le N' \\ 1\le\alpha\le N}}. \tag{1.7.15}$$

Also, define the principal symbol of D as the $N \times N'$ matrix

$$\mathrm{Sym}(D;\xi) := \mathrm{i}\Big(\sum_{j=1}^{n} a_j^{\alpha\beta} \xi_j \Big)_{\substack{1\le\alpha\le N \\ 1\le\beta\le N'}} \quad \text{for each } \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n. \tag{1.7.16}$$

In particular, the principal symbol satisfies the transposition, complex conjugation, Hermitian adjunction, and commutator laws:

$$\mathrm{Sym}\big(D^\top;\xi\big) = (-1)\,\mathrm{Sym}(D;\xi)^\top, \tag{1.7.17}$$

$$\mathrm{Sym}\big(\overline{D};\xi\big) = (-1)\,\overline{\mathrm{Sym}(D;\xi)}, \tag{1.7.18}$$

$$\mathrm{Sym}\big(D^*;\xi\big) = \mathrm{Sym}(D;\xi)^*, \tag{1.7.19}$$

$$(-\mathrm{i})\,\mathrm{Sym}(D;\nabla\varphi) = [D,\varphi], \tag{1.7.20}$$

for every real-valued function φ of class $\mathscr{C}^1$ (also identified with the operator of pointwise multiplication by φ) where, generally speaking, $[A, B]$ stands for the commutator $AB - BA$ of the operators A, B. It is also of interest to observe that

$$\begin{array}{l}\text{if } D \text{ is a homogeneous first-order system} \\ \text{in } \mathbb{R}^n \text{ then } D = \sum_{j=1}^{n} (-\mathrm{i})\mathrm{Sym}(D;\mathbf{e}_j)\partial_j.\end{array} \tag{1.7.21}$$

Finally, for each generic integer $M \in \mathbb{N}$, let us also agree to denote by $\langle\cdot,\cdot\rangle$ the (real) inner product in $\mathbb{C}^M$, namely

$$\langle a,b\rangle := \sum_{k=1}^{M} a_k b_k \quad \text{for every} \quad a=(a_k)_{1\le k\le M}\in\mathbb{C}^M \quad \text{and} \quad b=(b_k)_{1\le k\le M}\in\mathbb{C}^M. \tag{1.7.22}$$

Theorem 1.7.2 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$, where $n\in\mathbb{N}$, with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$.*

In this context, consider an $N\times N'$ first-order system D, with constant complex coefficients in $\mathbb{R}^n$, and suppose $u:\Omega\to\mathbb{C}^{N'}$ and $w:\Omega\to\mathbb{C}^N$ are two Lebesgue measurable vector-valued functions with the property that, for some $\kappa,\kappa'>0$,

$$\begin{aligned}
&\mathcal{N}_\kappa u<\infty \ \textit{ and } \ \mathcal{N}_{\kappa'}w<\infty \ \textit{ at } \sigma\textit{-a.e. point on } \ \partial\Omega,\\
&\mathcal{N}_\kappa u\cdot\mathcal{N}_{\kappa'}w \ \textit{ belongs to the space } \ L^1(\partial\Omega,\sigma),\\
&u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}},\ w\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}} \ \textit{ exist at } \sigma\textit{-a.e. point on } \ \partial_{\mathrm{nta}}\Omega,\\
&Du\in\big[L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n)\big]^N,\quad D^\top w\in\big[L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n)\big]^{N'},\\
&\textit{and } \ \big\langle Du,w\big\rangle-\big\langle u,D^\top w\big\rangle \ \textit{ belongs to } \ L^1(\Omega,\mathcal{L}^n).
\end{aligned}\tag{1.7.23}$$

Then, with $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$, $w\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ regarded as functions defined σ-a.e. on $\partial_*\Omega$ (which, up to a σ-nullset, is contained in $\partial_{\mathrm{nta}}\Omega$) the following formula, involving absolutely convergent integrals,

$$\int_\Omega\Big\{\big\langle Du,w\big\rangle-\big\langle u,D^\top w\big\rangle\Big\}\,\mathrm{d}\mathcal{L}^n=\int_{\partial_*\Omega}\Big\langle(-\mathrm{i})\mathrm{Sym}(D;\nu)\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big),\,w\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\Big\rangle\,\mathrm{d}\sigma, \tag{1.7.24}$$

holds in the case when either Ω is bounded, or $\partial\Omega$ is unbounded. Furthermore, formula (1.7.24) also holds if Ω is unbounded and $\partial\Omega$ is bounded provided there exists $\lambda\in(1,\infty)$ such that

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega}|u||w|\,\mathrm{d}\mathcal{L}^n=o(R)\ \ \textit{as}\ \ R\to\infty. \tag{1.7.25}$$

As a corollary, if the integrability condition in the last line of (1.7.23) is strengthened to

$$\big\langle Du,w\big\rangle\in L^1(\Omega,\mathcal{L}^n)\ \textit{ and }\ \big\langle u,D^\top w\big\rangle\in L^1(\Omega,\mathcal{L}^n) \tag{1.7.26}$$

then (1.7.24) may be refashion as the integration by parts formula

$$\int_\Omega\big\langle Du,w\big\rangle\,\mathrm{d}\mathcal{L}^n=\int_\Omega\big\langle u,D^\top w\big\rangle\,\mathrm{d}\mathcal{L}^n+\int_{\partial_*\Omega}\Big\langle(-\mathrm{i})\mathrm{Sym}(D;\nu)\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big),\,w\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\Big\rangle\,\mathrm{d}\sigma. \tag{1.7.27}$$

For example, the assumptions in the first two lines of (1.7.23) are satisfied if

$$\begin{array}{c}\mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma) \ \text{ and } \ \mathcal{N}_{\kappa'} w \in L^{p'}(\partial\Omega,\sigma)\\ \text{for some } p,\, p' \in [1,\infty] \text{ with } 1/p + 1/p' = 1.\end{array} \tag{1.7.28}$$

In such a scenario, Proposition 8.9.8 ensures that

$$\begin{array}{l}\text{for each } \kappa'' \in (0,\infty) \text{ the nontangential traces } u\big|_{\partial\Omega}^{\kappa''-\text{n.t.}},\\ w\big|_{\partial\Omega}^{\kappa''-\text{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega, \text{ and are independent of } \kappa''.\end{array} \tag{1.7.29}$$

While the main tool in the proof of the above theorem (presented in Sect. 9.5) is Theorem 1.2.1, we could have based the proof on Theorem 1.3.1 instead. In view of the format of the latter theorem, other versions of Theorem 1.7.2 then become possible. For example, in place of the last two lines in (1.7.23) we may demand that

$$\begin{array}{c} Du = (\mu_\alpha)_{1\le\alpha\le N} \in \big[\text{CBM}(\Omega)\big]^N, \quad D^\top w = (\lambda_\beta)_{1\le\beta\le N'} \in \big[\text{CBM}(\Omega)\big]^{N'},\\ w_\alpha \in L^1(\Omega, |\mu_\alpha|) \ \text{ for } \ \alpha \in \{1,\dots,N\}, \ \ u_\beta \in L^1(\Omega,|\lambda_\beta|) \ \text{ for } \ \beta \in \{1,\dots,N'\},\\ \text{and either } u = (u_\beta)_{1\le\beta\le N'} \text{ and } w = (w_\alpha)_{1\le\alpha\le N} \text{ are continuous in } \Omega,\\ \text{or } \ |\mu_\alpha| << \mathcal{L}^n \ \text{ for } \ \alpha \in \{1,\dots,N\} \ \text{ and } \ |\lambda_\beta| << \mathcal{L}^n \ \text{ for } \ \beta \in \{1,\dots,N'\}.\end{array} \tag{1.7.30}$$

In such a scenario, the integration by parts formula (1.7.27) takes the form

$$\sum_{1\le\alpha\le N} \int_\Omega w_\alpha \mathrm{d}\mu_\alpha = \sum_{1\le\beta\le N'} \int_\Omega u_\beta \,\mathrm{d}\lambda_\beta + \int_{\partial_*\Omega} \Big\langle (-\mathrm{i})\mathrm{Sym}(D;\nu)\big(u\big|_{\partial\Omega}^{\text{n.t.}}\big), w\big|_{\partial\Omega}^{\text{n.t.}}\Big\rangle \,\mathrm{d}\sigma. \tag{1.7.31}$$

Also, taking inspiration from (1.3.6), we may replace the second line in (1.7.23) by

$$\begin{array}{c}\mathcal{N}_\kappa u \cdot \mathcal{N}_{\kappa'} w \in L^1_{\text{loc}}(\partial\Omega,\sigma) \ \text{ and}\\ \Big\langle \mathrm{Sym}(D;\nu)\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big), w\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\Big\rangle \in L^1(\partial_*\Omega,\sigma),\end{array} \tag{1.7.32}$$

in which case (1.7.24) remains valid provided we also ask that there exists $\lambda \in (1,\infty)$ such that (1.7.25) holds.

As regards the optimality of Theorem 1.7.2, a glimpse is offered by considering the special case when $n = 1$, $\Omega = (a,b) \subset \mathbb{R}$ is a finite open interval, and $D = \mathrm{d}/\mathrm{d}x$ is the ordinary derivative on the real line. Then Theorem 1.7.2 asserts that for any functions $u, w \in L^\infty\big((a,b),\mathcal{L}^1\big)$ with weak derivatives u', w' in $L^1_{\text{loc}}\big((a,b),\mathcal{L}^1\big)$, such that $u'w$ and uw' are in $L^1\big((a,b),\mathcal{L}^1\big)$, and the limits $u(a^+) := \lim_{x\to a^+} u(x)$, $w(a^+) := \lim_{x\to a^+} w(x)$, $u(b^-) := \lim_{x\to b^-} u(x)$, $w(b^-) := \lim_{x\to b^-} w(x)$ exist, we have

$$\int_a^b u'w\,\mathrm{d}\mathcal{L}^1 = u(b^-)w(b^-) - u(a^+)w(a^+) - \int_a^b uw'\,\mathrm{d}\mathcal{L}^1. \tag{1.7.33}$$

The special case when $u \in \mathrm{AC}\big([a,b]\big)$ and $w = 1$ yields the sharp Fundamental Theorem of Calculus stated in (1.1.1).

Specializing Theorem 1.7.2 to the case when u, w are null-solutions of D and $D^\top$, respectively, yields the following remarkable corollary (which plays an important role in the context of Hardy spaces associated with first-order systems; cf. the discussion in [186, Sect. 3.2]).

Corollary 1.7.3 *Retain the assumptions made on the set $\Omega \subseteq \mathbb{R}^n$ and the $N \times N'$ first-order system D in Theorem 1.7.2, and suppose $u : \Omega \to \mathbb{C}^{N'}$ and $w : \Omega \to \mathbb{C}^N$ are two Lebesgue measurable functions with the property that, for some $\kappa, \kappa' > 0$,*

$$\begin{gathered} \mathcal{N}_\kappa u < \infty \;\text{ and }\; \mathcal{N}_{\kappa'} w < \infty \;\text{ at } \sigma\text{-a.e. point on } \partial\Omega, \\ \mathcal{N}_\kappa u \cdot \mathcal{N}_{\kappa'} w \;\text{ belongs to the space }\; L^1(\partial\Omega,\sigma), \\ \text{both }\; u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \;\text{ and }\; w\big|_{\partial\Omega}^{\kappa'-\text{n.t.}} \;\text{ exist at } \sigma\text{-a.e. point on }\; \partial_{\text{nta}}\Omega, \\ Du = 0 \;\text{ and }\; D^\top w = 0 \;\text{ in the sense of distributions in }\; \Omega. \end{gathered} \tag{1.7.34}$$

Then the cancelation property

$$\int_{\partial_*\Omega} \Big\langle (-\mathrm{i})\mathrm{Sym}(D;\nu)\Big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big), w\big|_{\partial\Omega}^{\kappa'-\text{n.t.}} \Big\rangle \,\mathrm{d}\sigma = 0 \tag{1.7.35}$$

is valid in the case when either Ω is bounded, or $\partial\Omega$ is unbounded. Moreover, formula (1.7.35) *is also valid if Ω is unbounded and $\partial\Omega$ is bounded provided there exists $\lambda \in (1,\infty)$ such that* (1.7.25) *holds.*

Specializing Theorem 1.7.2 to the case when the system D is homogeneous and w is chosen to be an arbitrary constant $\mathbb{C}^N$-valued function yields at once the following corollary.

Corollary 1.7.4 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$, where $n \in \mathbb{N}$, with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$. In this context, consider a homogeneous $N \times N'$ first-order system D, with constant complex coefficients in $\mathbb{R}^n$, and suppose $u : \Omega \to \mathbb{C}^{N'}$ is a Lebesgue measurable vector-valued function with the property that, for some $\kappa > 0$,*

$$\begin{gathered} \mathcal{N}_\kappa u \in L^1(\partial\Omega,\sigma), \quad Du \in \big[L^1(\Omega,\mathcal{L}^n)\big]^N, \\ \text{and }\; u\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \;\text{ exists at } \sigma\text{-a.e. point on }\; \partial_{\text{nta}}\Omega. \end{gathered} \tag{1.7.36}$$

Then, with $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ regarded as a function defined σ-a.e. on $\partial_*\Omega$ (which, up to a σ-nullset, is contained in $\partial_{\text{nta}}\Omega$), the following formula

$$\int_{\Omega} Du \, d\mathcal{L}^n = \int_{\partial_*\Omega} (-i)\mathrm{Sym}(D; \nu)\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big) \, d\sigma \tag{1.7.37}$$

holds in the case when either Ω *is bounded, or* $\partial\Omega$ *is unbounded. Furthermore, formula* (1.7.37) *also holds if* Ω *is unbounded and* $\partial\Omega$ *is bounded provided there exists* $\lambda \in (1, \infty)$ *such that*

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |u| \, d\mathcal{L}^n = o(R) \;\; as \;\; R \to \infty. \tag{1.7.38}$$

It is interesting to re-state Corollary 1.7.4 in the special case when $\Omega := \mathbb{R}^n_+$ with $n \geq 2$. Due to the specific nature of this geometric environment, we are able to require less when it comes to demanding the existence of the nontangential boundary trace in the second line of (1.7.36). Concretely, we have the following result, which is proved by relying on Corollary 1.7.4 and a regularization argument in precisely the same manner in which Corollary 1.2.2 has been deduced from Theorem 1.2.1 (cf. Sect. 9.1).

Corollary 1.7.5 *Fix* $n \in \mathbb{N}$ *with* $n \geq 2$*, and consider a homogeneous* $N \times N'$ *first-order system D, with constant complex coefficients in* $\mathbb{R}^n$*. Suppose* $u : \mathbb{R}^n_+ \to \mathbb{C}^{N'}$ *is a Lebesgue measurable vector-valued function with the property that, for some aperture parameter* $\kappa > 0$*,*

$$\begin{aligned} &\mathcal{N}_\kappa u \in L^1(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}), \quad Du \in \big[L^1(\mathbb{R}^n_+, \mathcal{L}^n)\big]^N, \;\; and \\ &\big[\mathrm{Sym}(D; \boldsymbol{e}_n)u\big]\big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}} \;\; exists\ \mathcal{L}^{n-1}\text{-a.e. on} \;\; \mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+. \end{aligned} \tag{1.7.39}$$

Then the following formula, involving absolutely convergent integrals, holds:

$$\int_{\mathbb{R}^n_+} Du \, d\mathcal{L}^n = \int_{\mathbb{R}^{n-1}} i\big[\mathrm{Sym}(D; \boldsymbol{e}_n)u\big]\big|_{\partial\mathbb{R}^n_+}^{\kappa-\mathrm{n.t.}} \, d\mathcal{L}^{n-1}. \tag{1.7.40}$$

For example, the choice $D := \mathrm{div}$ (which entails $N = 1$ and $N' = n$) in Corollary 1.7.5 yields precisely the version of the Divergence Theorem recorded in Corollary 1.2.2.

A suitable version of Theorem 1.7.2 holds when D is the standard Dirac operator in $\mathbb{R}^n$, and when the functions involved take values in the Clifford algebra $(\mathcal{C}\ell_n, +, \odot)$, canonically associated with $\mathbb{R}^n$ as discussed in Sect. 6.4.

Theorem 1.7.6 *Assume* Ω *is an open nonempty proper subset of* $\mathbb{R}^n$*, where* $n \in \mathbb{N}$*, with a lower Ahlfors regular boundary, such that* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *is a doubling measure on* $\partial\Omega$*; in particular,* Ω *is a set of locally finite perimeter. Canonically identify its geometric measure theoretic outward unit normal* $\nu = (\nu_1, \dots, \nu_n)$ *with the* $\mathcal{C}\ell_n$*-valued function* $\nu = \sum_{j=1}^n \nu_j \boldsymbol{e}_j$ *defined* σ*-a.e. on* $\partial_*\Omega$ *(cf.* (6.4.3)*).*

Suppose $u, w : \Omega \to \mathcal{C}\ell_n$ *are two Lebesgue measurable Clifford algebra-valued functions with the property that, for some* $\kappa, \kappa' > 0$*,*

$$\begin{gathered} \mathcal{N}_\kappa u < \infty \ \text{ and } \ \mathcal{N}_{\kappa'} w < \infty \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega, \\ \mathcal{N}_\kappa u \cdot \mathcal{N}_{\kappa'} w \ \text{ belongs to the space } \ L^1(\partial\Omega, \sigma), \\ u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}, \ w\big|_{\partial\Omega}^{\kappa'-\text{n.t.}} \ \text{ exist in } \mathcal{C}\ell_n \text{ at } \sigma\text{-a.e. point on } \ \partial_{\text{nta}}\Omega, \\ D_R u \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\ell_n, \ \ D_L w \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\ell_n, \\ (D_R u) \odot w + u \odot (D_L w) \ \text{ belongs to } \ L^1(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\ell_n, \end{gathered} \tag{1.7.41}$$

where D_L and D_R denote the action of the Dirac operator $D := \sum_{j=1}^n \boldsymbol{e}_j \odot \partial_j$ on a given $\mathcal{C}\ell_n$-valued distribution from the left and from the right, respectively (cf. (6.4.48)–(6.4.49)*).*

Then, with the nontangential traces $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$, $w\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}$ regarded as $\mathcal{C}\ell_n$-valued functions defined σ-a.e. on $\partial_*\Omega$ (which, up to a σ-nullset, is contained in $\partial_{\text{nta}}\Omega$), the following formula, involving absolutely convergent integrals,

$$\begin{aligned} \int_{\partial_*\Omega} \big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) \odot \nu \odot \big(w\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big)\, d\sigma & \\ = \int_\Omega \big\{(D_R u) \odot w + u \odot (D_L w)\big\}\, d\mathcal{L}^n & \end{aligned} \tag{1.7.42}$$

holds in the case when either Ω is bounded, or $\partial\Omega$ is unbounded. Moreover, formula (1.7.42) *also holds if Ω is unbounded and $\partial\Omega$ is bounded provided there exists $\lambda \in (1, \infty)$ such that*

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |u||w|\, d\mathcal{L}^n = o(R) \ \text{ as } \ R \to \infty. \tag{1.7.43}$$

In particular, if the integrability condition in the last line of (1.7.41) *is strengthened to*

$$(D_R u) \odot w \in L^1(\Omega, \mathcal{L}^n) \ \text{ and } \ u \odot (D_L w) \in L^1(\Omega, \mathcal{L}^n) \tag{1.7.44}$$

then (1.7.42) *may be reformulated as the integration by parts formula*

$$\begin{aligned} \int_\Omega (D_R u) \odot w\, d\mathcal{L}^n = - \int_\Omega u \odot (D_L w)\, d\mathcal{L}^n & \\ + \int_{\partial_*\Omega} \big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) \odot \nu \odot \big(w\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big)\, d\sigma. & \end{aligned} \tag{1.7.45}$$

The proof of Theorem 1.7.6 is given in Sect. 9.5. Once again, the assumptions in the first two lines of (1.7.41) are satisfied if we impose (1.7.28), in which scenario Proposition 8.9.8 ensures that (1.7.29) holds.

1.8 Higher-Order Integration by Parts

Here we are concerned with a higher-order version of the integration by parts formula proved in Theorem 1.7.1. To be able to state our first result of this nature, we introduce a brand of Sobolev spaces in relation to a specified arbitrary multi-index $\alpha \in \mathbb{N}_0^n$.

Definition 1.8.1 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. For each multi-index $\alpha \in \mathbb{N}_0^n$ and each exponent $p \in [1, \infty]$ define the "partial" Sobolev space $W^{\alpha,p}_{\rm loc}(\Omega)$ as*

$$W^{\alpha,p}_{\rm loc}(\Omega) := \Big\{u \in L^p_{\rm loc}(\Omega, \mathcal{L}^n) : \partial^\beta u \in L^p_{\rm loc}(\Omega, \mathcal{L}^n) \text{ for all } \beta \in \mathbb{N}_0^n,\ \beta \leq \alpha\Big\}. \tag{1.8.1}$$

The reader is also reminded about our convention of denoting by e_j, for each $j \in \{1, \dots, n\}$, the multi-index in $\mathbb{N}_0^n$ of length one with 1 on the j-th component. We are now ready to state our higher-order integration by parts formula alluded to earlier.

Theorem 1.8.2 *Let $n \in \mathbb{N}$ and assume Ω is an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω. Also, fix a multi-index $\alpha \in \mathbb{N}_0^n$ along with some aperture parameter $\kappa \in (0, \infty)$. In this setting, suppose $u, w \in W^{\alpha,1}_{\rm loc}(\Omega)$ are two scalar-valued functions such that*

$$(\partial^\alpha u) \cdot w - (-1)^{|\alpha|} u \cdot (\partial^\alpha w) \in L^1(\Omega, \mathcal{L}^n) \tag{1.8.2}$$

and, for all multi-indices $\beta, \gamma \in \mathbb{N}_0^n$ with $\beta + \gamma < \alpha$ and $|\beta| + |\gamma| = |\alpha| - 1$, the following properties hold:

(1) $\mathcal{N}_\kappa(\partial^\beta u)$, $\mathcal{N}_\kappa(\partial^\gamma w) < +\infty$ at σ-a.e. point on $\partial\Omega$ and $\mathcal{N}_\kappa(\partial^\beta u) \cdot \mathcal{N}_\kappa(\partial^\gamma w)$ belongs to $L^1(\partial\Omega, \sigma)$;

(2) the nontangential boundary traces $(\partial^\beta u)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ and $(\partial^\gamma w)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ exist at σ-a.e. point on $\partial_{\rm nta}\Omega$.

When Ω is unbounded and $\partial\Omega$ is bounded, make the additional assumption that there exists $\lambda \in (1, \infty)$ such that, with $A_{\lambda,R} := B(0, \lambda R) \setminus B(0, R)$, one has

$$\sum_{\substack{\beta,\gamma \in \mathbb{N}_0^n \\ j \in \{1,\dots,n\},\, \beta+\gamma+e_j=\alpha}} \int_{A_{\lambda,R}\cap\Omega} |\partial^\beta u|\,|\partial^\gamma w|\,{\rm d}\mathcal{L}^n = o(R) \quad \text{as} \quad R \to \infty. \tag{1.8.3}$$

Then

$$\int_\Omega \Big\{(\partial^\alpha u)\, w - (-1)^{|\alpha|} u(\partial^\alpha w)\Big\}\, \mathrm{d}\mathcal{L}^n \tag{1.8.4}$$
$$= \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^n \\ j\in\{1,\dots,n\},\beta+\gamma+e_j=\alpha \\ |\gamma|=\ell-1,\ |\beta|=|\alpha|-\ell}} (-1)^{\ell+1}\frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\beta!\gamma!}\times$$
$$\times \int_{\partial_*\Omega} \nu_j (\partial^\beta u)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \cdot (\partial^\gamma w)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\, \mathrm{d}\sigma.$$

We make some remarks aimed to shed further light on the nature and scope of this theorem. The first observation we make is that when (1.8.2) is strengthened to

$$(\partial^\alpha u)\cdot w \in L^1(\Omega,\mathcal{L}^n) \ \text{ and } \ u\cdot(\partial^\alpha w)\in L^1(\Omega,\mathcal{L}^n) \tag{1.8.5}$$

then in place of (1.8.4) we may write

$$\int_\Omega (\partial^\alpha u) w\, \mathrm{d}\mathcal{L}^n = (-1)^{|\alpha|}\int_\Omega u(\partial^\alpha w)\,\mathrm{d}\mathcal{L}^n \tag{1.8.6}$$
$$+ \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^n \\ j\in\{1,\dots,n\},\beta+\gamma+e_j=\alpha \\ |\gamma|=\ell-1,\ |\beta|=|\alpha|-\ell}} (-1)^{\ell+1}\frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\beta!\gamma!}\times$$
$$\times \int_{\partial_*\Omega} \nu_j (\partial^\beta u)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \cdot (\partial^\gamma w)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\, \mathrm{d}\sigma,$$

which is more in line with the traditional format of an integration by parts Formula.

Theorem 1.8.2 readily implies Theorem 1.7.1. Indeed, if some functions u, w satisfy the hypotheses of Theorem 1.7.1, then these functions also satisfy the hypothesis of Theorem 1.8.2 (with aperture equal to the largest of the apertures in the hypotheses of Theorem 1.7.1) corresponding to $\alpha = e_j$, $\beta=\gamma=0$. In particular, (1.8.4) becomes (1.7.4).

Finally, we note that the properties listed in *(1)* are satisfied if, for example,

$$\begin{gathered}\mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma) \ \text{ and } \ \mathcal{N}_\kappa w \in L^{p'}(\partial\Omega,\sigma) \\ \text{for some } \ p,p'\in[1,\infty] \ \text{ with } \ 1/p+1/p'=1.\end{gathered} \tag{1.8.7}$$

Moving on, let $m,n,M\in\mathbb{N}$. For each $\alpha,\beta\in\mathbb{N}_0^n$ such that $|\alpha|=|\beta|=m$, let $A_{\alpha\beta}$ be an $M\times M$ complex matrix and consider the $M\times M$ homogeneous constant coefficient system of order $2m$ in $\mathbb{R}^n$ given by

$$L := \sum_{|\alpha|=|\beta|=m} \partial^\alpha A_{\alpha\beta}\partial^\beta. \tag{1.8.8}$$

In what follows, we will call $A = (A_{\alpha\beta})_{|\alpha|=|\beta|=m}$ the tensor coefficient of L. We shall denote by $L^\top$ the transpose operator, that is, the $M \times M$ system whose tensor coefficient is

$$A^\top := (A^\top_{\beta\alpha})_{|\alpha|=|\beta|=m}, \tag{1.8.9}$$

where the superscript $\top$ indicates matrix transposition. In particular, we have

$$L^\top = \sum_{|\alpha|=|\beta|=m} \partial^\alpha A^\top_{\beta\alpha} \partial^\beta. \tag{1.8.10}$$

We are prepared to state a version of the "half" Green formula for the higher-order system L in a very general setting, described below.

Theorem 1.8.3 *Let $n \in \mathbb{N}$ and assume Ω is an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and fix some aperture parameter $\kappa \in (0, \infty)$. Also, with $m, M \in \mathbb{N}$ consider the higher-order system L as in (1.8.8).*

In this context, suppose $u \in [W^{2m,1}_{\text{loc}}(\Omega)]^M$ and $w \in [W^{m,1}_{\text{loc}}(\Omega)]^M$ are two vector-valued functions such that

$$\langle Lu, w\rangle \ \textit{belongs to} \ \ L^1(\Omega, \mathcal{L}^n), \tag{1.8.11}$$

and for each multi-index $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| = |\beta| = m$ the following properties hold:

(1) the product $|\partial^\alpha u| \cdot |\partial^\beta w|$ belongs to $L^1(\Omega, \mathcal{L}^n)$;
(2) for all multi-indices $\delta, \gamma \in \mathbb{N}_0^n$ and $j \in \{1, \dots, n\}$ satisfying $\delta + \gamma + e_j = \alpha$ one has

$$\begin{gathered} \mathcal{N}_\kappa(\partial^{\delta+\beta}u),\ \mathcal{N}_\kappa(\partial^\gamma w) < +\infty \ \ \textit{at } \sigma\textit{-a.e. point on} \ \ \partial\Omega, \\ \mathcal{N}_\kappa(\partial^{\delta+\beta}u) \cdot \mathcal{N}_\kappa(\partial^\gamma w) \in L^1(\partial\Omega, \sigma), \\ (\partial^{\delta+\beta}u)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \ \textit{and} \ \ (\partial^\gamma w)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \ \textit{exist at } \sigma\textit{-a.e. point on} \ \ \partial_{\text{nta}}\Omega, \end{gathered} \tag{1.8.12}$$

where the derivatives and the nontangential traces are taken componentwise.

Also, when Ω is unbounded and $\partial\Omega$ is bounded, make the additional assumption that there exists some $\lambda \in (1, \infty)$ such that, with $A_{\lambda,R} := B(0, \lambda R) \setminus B(0, R)$, one has

$$\sum_{\substack{\delta,\gamma\in\mathbb{N}_0^n \\ |\delta|+|\gamma|=m-1}} \int_{A_{\lambda,R}\cap\Omega} |\partial^{\delta+\beta}u|\,|\partial^\gamma w|\,\mathrm{d}\mathcal{L}^n = o(R) \ \ \textit{as} \ \ R \to \infty. \tag{1.8.13}$$

Then

$$\int_{\Omega} \langle Lu, w \rangle \, d\mathcal{L}^n = \sum_{|\alpha|=|\beta|=m} (-1)^m \int_{\Omega} \big\langle A_{\alpha\beta} \partial^{\beta} u, \partial^{\alpha} w \big\rangle \, d\mathcal{L}^n \tag{1.8.14}$$

$$+ \sum_{|\alpha|=|\beta|=m} \sum_{\ell=1}^{m} \sum_{\substack{\delta+\gamma+e_j=\alpha \\ |\gamma|=\ell-1,\, |\delta|=m-\ell}} (-1)^{\ell+1} \frac{\alpha!(m-\ell)!(\ell-1)!}{m!\delta!\gamma!} \times$$

$$\times \int_{\partial_* \Omega} \Big\langle \nu_j A_{\alpha\beta} (\partial^{\beta+\delta} u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}, (\partial^{\gamma} w)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \Big\rangle \, d\sigma.$$

Formula (1.8.14) suggests making the following definition.

Definition 1.8.4 *Let $n \in \mathbb{N}$ and assume Ω is an open nonempty proper subset of $\mathbb{R}^n$ of locally finite perimeter, set $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$, and let $\nu = (\nu_1, \dots, \nu_n)$ denote its geometric measure theoretic outward unit normal. Fix some aperture parameter $\kappa \in (0, \infty)$.*

Then, if $k \in \mathbb{N}$ and $u \in W^{k,1}_{\text{loc}}(\Omega)$ is such that for all $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq k$ the nontangential trace $\big(\partial^{\beta} u\big)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial_{\text{nta}}\Omega$, define the multi-trace operator

$$\text{Tr}_{\text{k}}^{\text{n.t.}}(u) := \Big((\partial^{\gamma} u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \Big)_{|\gamma| \leq k}. \tag{1.8.15}$$

Also, if $m, M \in \mathbb{N}$ and $A = (A_{\alpha\beta})_{|\alpha|=|\beta|=m}$ is a coefficient tensor with the property that $A_{\alpha\beta} \in \mathbb{C}^{M \times M}$ for each pair $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| = |\beta| = m$, given a function $u \in W^{2m-1,1}_{\text{loc}}(\Omega)$ for which the nontangential trace $\big(\partial^{\mu} u\big)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial_{\text{nta}}\Omega$, for every $\mu \in \mathbb{N}_0^n$ with $m \leq |\mu| \leq 2m-1$, define the higher-order conormal derivative of u associated with A as

$$\partial_{\nu}^{A} u := \big((\partial_{\nu}^{A} u)_{\gamma} \big)_{|\gamma| \leq m-1} \tag{1.8.16}$$

where

$$(\partial_{\nu}^{A} u)_{\gamma} := (-1)^{|\gamma|} \frac{|\gamma|!}{m!\gamma!} \sum_{\substack{|\alpha|=|\beta|=m \\ \alpha > \gamma}} \sum_{\substack{\delta,\gamma \in \mathbb{N}_0^n,\, j \in \{1,\dots,n\} \\ \delta+e_j=\alpha-\gamma \\ |\delta|=m-|\gamma|-1}} \frac{\alpha!|\delta|!}{\delta!} \nu_j A_{\alpha\beta} \big(\partial^{\beta+\delta} u\big) \Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \tag{1.8.17}$$

for each $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq m-1$.

Using Definition 1.8.4, we may then recast the "half" Green formula (1.8.14) as

$$\int_{\Omega} \langle Lu, w \rangle \, d\mathcal{L}^n = \sum_{|\alpha|=|\beta|=m} (-1)^m \int_{\Omega} \big\langle A_{\alpha\beta} \partial^{\beta} u, \partial^{\alpha} w \big\rangle \, d\mathcal{L}^n + \int_{\partial_* \Omega} \Big\langle \partial_{\nu}^{A} u, \text{Tr}_{\text{m}-1}^{\text{n.t.}}(w) \Big\rangle \, d\sigma. \tag{1.8.18}$$

We wish to augment this result by establishing the "full" Green formula for higher-order operators stated in the theorem below.

Theorem 1.8.5 *Let $n \in \mathbb{N}$ and assume Ω is an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Denote by $\nu = (\nu_1, \dots, \nu_n)$ the geometric measure theoretic outward unit normal to Ω and fix some aperture parameter $\kappa \in (0, \infty)$. Also, with $m, M \in \mathbb{N}$ consider the higher-order system L as in (1.8.8).*

In this context, suppose $u, w \in [W^{2m,1}_{\rm loc}(\Omega)]^M$ are two vector-valued functions such that

(1) $\langle Lu, w\rangle - \langle u, L^\top w\rangle \in L^1(\Omega, \mathcal{L}^n)$;
(2) *if $\theta, \mu \in \mathbb{N}_0^n$ with $|\theta| + |\mu| = 2m-1$ then $\mathcal{N}_\kappa(\partial^\theta u)$, $\mathcal{N}_\kappa(\partial^\mu w) < +\infty$ at σ-a.e. point on $\partial\Omega$ and*

$$\mathcal{N}_\kappa(\partial^\theta u)\cdot\mathcal{N}_\kappa(\partial^\mu w) \in L^1(\partial\Omega, \sigma); \tag{1.8.19}$$

(3) *the nontangential traces $(\partial^\gamma u)\big|^{\kappa-\rm n.t.}_{\partial\Omega}$ and $(\partial^\gamma w)\big|^{\kappa-\rm n.t.}_{\partial\Omega}$ exist at σ-a.e. point on $\partial_{\rm nta}\Omega$ for all $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \le 2m-1$.*

When Ω is unbounded and $\partial\Omega$ is bounded, make the additional assumption that there exists some $\lambda \in (1,\infty)$ such that, with $A_{\lambda,R} := B(0,\lambda R)\setminus B(0,R)$, one has

$$\sum_{\substack{\delta,\gamma\in\mathbb{N}_0^n\\ |\delta|+|\gamma|=m-1}} \int_{A_{\lambda,R}\cap\Omega} \big\{|\partial^{\delta+\beta}u|\,|\partial^\gamma w| + |\partial^\delta u|\,|\partial^{\alpha+\gamma}w|\big\}\,{\rm d}\mathcal{L}^n = o(R) \ \text{ as } \ R\to\infty. \tag{1.8.20}$$

Then

$$\begin{aligned}\int_\Omega \langle Lu, w\rangle\,{\rm d}\mathcal{L}^n - \int_\Omega \langle u, L^\top w\rangle\,{\rm d}\mathcal{L}^n \qquad\qquad (1.8.21)\\ = \int_{\partial_*\Omega}\big\langle \partial^A_\nu u, {\rm Tr}^{\rm n.t.}_{\rm m-1}(w)\big\rangle\,{\rm d}\sigma - \int_{\partial_*\Omega}\big\langle {\rm Tr}^{\rm n.t.}_{\rm m-1}(u), \partial^{A^\top}_\nu w\big\rangle\,{\rm d}\sigma.\end{aligned}$$

In relation to Theorem 1.8.5 we make two comments. First, note that when assumption *(1)* in Theorem 1.8.5 is strengthened by demanding $\langle Lu, w\rangle \in L^1(\Omega, \mathcal{L}^n)$ and $\langle u, L^\top w\rangle \in L^1(\Omega, \mathcal{L}^n)$ then in place of (1.8.21) we may write

$$\begin{aligned}\int_\Omega \langle Lu, w\rangle\,{\rm d}\mathcal{L}^n = \int_\Omega \langle u, L^\top w\rangle\,{\rm d}\mathcal{L}^n \qquad\qquad (1.8.22)\\ + \int_{\partial_*\Omega}\big\langle \partial^A_\nu u, {\rm Tr}^{\rm n.t.}_{\rm m-1}(w)\big\rangle\,{\rm d}\sigma - \int_{\partial_*\Omega}\big\langle {\rm Tr}^{\rm n.t.}_{\rm m-1}(u), \partial^{A^\top}_\nu w\big\rangle\,{\rm d}\sigma,\end{aligned}$$

which is more in line with the traditional format of Green's Formula.

Our second observation is that the assumptions in *(2)* are naturally satisfied when suitable conditions are imposed separately on the functions u and w. For example, this is the case if there exist two integrability exponents $p, p' \in [1,\infty]$ such that

$1/p + 1/p' = 1$ with the property that for each $\theta, \mu \in \mathbb{N}_0^n$ $|\theta| + |\mu| = 2m - 1$ one has $\mathcal{N}_\kappa(\partial^\theta u) \in L^p(\partial\Omega, \sigma)$ and $\mathcal{N}_\kappa(\partial^\mu w) \in L^{p'}(\partial\Omega, \sigma)$.

1.9 The Divergence Theorem with Weak Boundary Traces

We are interested in devising a version of the Divergence Formula (1.2.2) in the class of vector fields which may not necessarily possess nontangential pointwise traces on the boundary. In the absence of nontangential pointwise traces we are proposing to use a weaker notion of boundary trace, defined as the "bullet product," via a variational approach mimicking integration by parts.

To be specific, let Ω be an open subset of $\mathbb{R}^n$ and denote by $L^1_{\rm bdd}(\Omega, \mathcal{L}^n)$ the collection of all $\mathcal{L}^n$-measurable functions which are absolutely integrable on each bounded $\mathcal{L}^n$-measurable subset of the set Ω. Given any vector field $\vec{F} \in \big[L^1_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^n$ whose divergence, considered in the sense of distributions in the set Ω, satisfies ${\rm div}\vec{F} \in L^1_{\rm bdd}(\Omega, \mathcal{L}^n)$, we shall denote by $\nu \bullet \vec{F}$ (read "nu bullet F") the functional acting on each function[14] $\psi \in {\rm Lip}_c(\partial\Omega)$ according to

$$\big\langle \nu \bullet \vec{F}, \psi \big\rangle := \int_\Omega \vec{F} \cdot \nabla\Psi \, {\rm d}\mathcal{L}^n + \int_\Omega ({\rm div}\vec{F})\Psi \, {\rm d}\mathcal{L}^n, \tag{1.9.1}$$

where Ψ is any complex-valued function satisfying

$$\Psi \in {\rm Lip}(\overline{\Omega}),\ \Psi\big|_{\partial\Omega} = \psi, \text{ and } \Psi \equiv 0 \text{ outside of some compact subset of } \overline{\Omega}. \tag{1.9.2}$$

See Sect. 4.2 for more details.

We find it logistically convenient to start by presenting a version of the integration by parts formula (1.2.12) involving this weak boundary trace in Theorem 1.9.1 below, then use this to state a version of the Divergence Formula involving weak boundary traces in (1.9.11). To state Theorem 1.9.1, we make one more definition. Concretely, given any open subset Ω of $\mathbb{R}^n$, we define the action of the maximal operator $\mathfrak{P}$ on any $\mathcal{L}^n$-measurable function $u : \Omega \to \mathbb{C}$ as

$$(\mathfrak{P}u)(x) := \sup_{0<r<2\,{\rm diam}(\partial\Omega)} \left\{ \frac{1}{\sigma\big(\partial\Omega \cap B(x,r)\big)} \int_{\Omega\cap B(x,r)} |u| \, {\rm d}\mathcal{L}^n \right\} \in [0, \infty], \tag{1.9.3}$$

at every point $x \in \partial\Omega$. For more on this topic, the reader is referred to [185, Sect. 10.1].

Theorem 1.9.1 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \geq 2$) be an open set with an Ahlfors regular boundary and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider a vector field $\vec{F} : \Omega \to \mathbb{C}^n$ with*

[14] Here and elsewhere, ${\rm Lip}(E)$ is the space of complex-valued Lipschitz functions defined in a given set $E \subseteq \mathbb{R}^n$, and ${\rm Lip}_c(E)$ is subspace of ${\rm Lip}(E)$ consisting of functions with compact support.

$\mathcal{L}^n$-measurable components satisfying

$$\mathcal{N}_\kappa \vec{F} \in L^p(\partial\Omega, \sigma) \text{ for some } \kappa > 0 \text{ and } p \in [1, \infty]. \tag{1.9.4}$$

Also, with the divergence taken in the sense of distributions[15] *in Ω, assume that*

$$\mathrm{div}\vec{F} \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \quad \text{and} \quad \mathfrak{P}(\mathrm{div}\vec{F}) \in L^p(\partial\Omega, \sigma). \tag{1.9.5}$$

Then the following statements are true.

(i) There exists a unique function, denoted by $\nu \bullet \vec{F}$, which belongs to $L^p(\partial\Omega, \sigma)$ and for which the following integration by parts formula holds

$$\int_{\partial\Omega} (\nu \bullet \vec{F})\Psi \, \mathrm{d}\sigma = \int_\Omega \vec{F} \cdot \nabla\Psi \, \mathrm{d}\mathcal{L}^n + \int_\Omega (\mathrm{div}\vec{F})\Psi \, \mathrm{d}\mathcal{L}^n \quad \text{for all} \quad \Psi \in \mathrm{Lip}_c(\mathbb{R}^n). \tag{1.9.6}$$

(ii) There exists some constant $C = C(\Omega, n, \kappa, p) \in (0, \infty)$ independent of $\vec{F}$ such that

$$\big\|\nu \bullet \vec{F}\big\|_{L^p(\partial\Omega,\sigma)} \leq C\big\|\mathcal{N}_\kappa \vec{F}\big\|_{L^p(\partial\Omega,\sigma)} + C\big\|\mathfrak{P}(\mathrm{div}\vec{F})\big\|_{L^p(\partial\Omega,\sigma)}. \tag{1.9.7}$$

(iii) If $p = 1$, then actually $\nu \bullet \vec{F}$ belongs to the Hardy space $H^1(\partial\Omega, \sigma)$ and, for some constant $C = C(\Omega, n, \kappa) \in (0, \infty)$ independent of $\vec{F}$,

$$\big\|\nu \bullet \vec{F}\big\|_{H^1(\partial\Omega,\sigma)} \leq C\big\|\mathcal{N}_\kappa \vec{F}\big\|_{L^1(\partial\Omega,\sigma)} + C\big\|\mathfrak{P}(\mathrm{div}\vec{F})\big\|_{L^1(\partial\Omega,\sigma)}. \tag{1.9.8}$$

(iv) Under the additional assumptions that[16] *$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ and $\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ exists at σ-a.e. point on $\partial\Omega$, one has*

$$\nu \bullet \vec{F} = \nu \cdot \Big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big) \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega, \tag{1.9.9}$$

where ν is the geometric measure theoretic outward unit normal to Ω.

Here are a few observations designed to shed further light on the nature of Theorem 1.9.1.

Observation 1. Theorem 1.9.1 shows that there exists a mapping which is linear and bounded[17]

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[15] Lemma 8.3.1 ensures that $\vec{F} \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^n$.

[16] The demand that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ precludes $\partial\Omega$ from developing "too many" cusps and also prevents Ω from having "significant" cracks.

[17] When we consider the space in the left equipped with the natural norm $\vec{F} \mapsto \big\|\mathcal{N}_\kappa \vec{F}\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathfrak{P}(\mathrm{div}\vec{F})\big\|_{L^p(\partial\Omega,\sigma)}$ with $\kappa > 0$ fixed.

$$\{\vec{F} \in [L^1_{\rm loc}(\Omega, \mathcal{L}^n)]^n : \text{ satisfying (1.9.4)} - (1.9.5)\} \ni \vec{F} \mapsto \nu \bullet \vec{F} \in L^p(\partial\Omega, \sigma) \tag{1.9.10}$$

which renders the integration by parts formula (1.9.6) true. We shall refer to (1.9.10) as the weak normal trace of the vector field $\vec{F}$ on $\partial\Omega$.

Observation 2. In the case when Ω is bounded, we may choose $\Psi \in \mathrm{Lip}_c(\mathbb{R}^n)$ to be identically one in a neighborhood of $\overline{\Omega}$. Such a choice makes the integration by parts formula (1.9.6) look more in line with the traditional Divergence Formula, namely

$$\int_{\partial\Omega} \nu \bullet \vec{F}\, d\sigma = \int_{\Omega} \mathrm{div}\vec{F}\, d\mathcal{L}^n. \tag{1.9.11}$$

Observation 3. In the case when $p = \infty$, the condition that $\mathcal{N}_\kappa \vec{F} \in L^\infty(\partial\Omega, \sigma)$ in (1.9.4) may be alternatively reformulated as $\vec{F} \in L^\infty(\Omega, \mathcal{L}^n)$ (thanks to Lemma 8.3.1), while the hypothesis that $\mathfrak{P}(\mathrm{div}\vec{F}) \in L^\infty(\partial\Omega, \sigma)$ in (1.9.5) is equivalent to the demand that $(\mathrm{div}\vec{F})\mathcal{L}^n$ is a Carleson measure in Ω. In particular, the latter condition is satisfied if

$$\mathrm{div}\vec{F} \in L^n(\Omega, \mathcal{L}^n). \tag{1.9.12}$$

Observation 4. Generally speaking, the function $\nu \bullet \vec{F}$ is supported on the full topological boundary $\partial\Omega$, and not just on the measure theoretic boundary $\partial_*\Omega$. In addition, $\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ may fail to exist at σ-a.e. point on $\partial\Omega$. See (2.7.12) in this regard (cf. also (2.7.18)). We also wish to note that the hypothesis that $\partial\Omega$ is Ahlfors regular cannot be weakened to asking that $\partial\Omega$ is merely upper Ahlfors regular; see (2.7.27).

Observation 5. As is apparent from inspecting the main ingredients involved in its proof, a suitable version of Theorem 1.9.1 continues to be valid on $\mathscr{C}^1$ Riemannian manifolds.

Remarkably, Theorem 1.9.1 may be extended to the range $p \in \left(\frac{n-1}{n}, 1\right)$ provided Hardy spaces are used in place of Lebesgue spaces on $\partial\Omega$. The scale of Hardy spaces on Ahlfors regular sets is discussed at length in [185, Chap. 4].

Theorem 1.9.2 *Suppose $\Omega \subseteq \mathbb{R}^n$ (where $n \geq 2$) is an open set with an Ahlfors regular boundary and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Let $\vec{F} : \Omega \to \mathbb{C}^n$ be a vector field with $\mathcal{L}^n$-measurable components satisfying*

$$\mathcal{N}_\kappa \vec{F} \in L^p(\partial\Omega, \sigma) \text{ for some } \kappa > 0 \text{ and } p \in \left(\tfrac{n-1}{n}, 1\right). \tag{1.9.13}$$

In addition, with the divergence taken in the sense of distributions[18] *in Ω, assume that*

$$\mathrm{div}\vec{F} \in L^1_{\rm loc}(\Omega, \mathcal{L}^n) \text{ and } \mathfrak{P}(\mathrm{div}\vec{F}) \in L^p(\partial\Omega, \sigma). \tag{1.9.14}$$

[18] Lemma 8.3.1 ensures that $\vec{F} \in [L^1_{\rm loc}(\Omega, \mathcal{L}^n)]^n$.

Then there exists a unique distribution on $\partial\Omega$, denoted by $\nu \bullet \vec{F}$, which belongs to the Hardy space $H^p(\partial\Omega, \sigma)$ and with the property that for each $\Psi \in \mathrm{Lip}_c(\mathbb{R}^n)$ the following integration by parts formula is satisfied:

$$\int_\Omega \vec{F} \cdot \nabla\Psi \, \mathrm{d}\mathcal{L}^n + \int_\Omega (\mathrm{div}\vec{F})\Psi \, \mathrm{d}\mathcal{L}^n = \begin{cases} \big\langle \nu \bullet \vec{F}, \Psi|_{\partial\Omega} \big\rangle & \text{if } \partial\Omega \text{ bounded}, \\ \big\langle \nu \bullet \vec{F}, \big[\Psi|_{\partial\Omega}\big] \big\rangle & \text{if } \partial\Omega \text{ unbounded}, \end{cases} \tag{1.9.15}$$

with $\langle\cdot,\cdot\rangle$ denoting the duality bracket between the Hardy space $H^p(\partial\Omega, \sigma)$ and its dual,

$$\big(H^p(\partial\Omega,\sigma)\big)^* = \begin{cases} \mathscr{C}^\alpha(\partial\Omega) & \text{if } \partial\Omega \text{ bounded}, \\ \dot{\mathscr{C}}^\alpha(\partial\Omega)\big/\sim & \text{if } \partial\Omega \text{ unbounded}, \end{cases} \tag{1.9.16}$$

where $\alpha := (n-1)\big(\frac{1}{p} - 1\big) \in (0,1)$ (cf. the discussion in [185, Sect. 4.6]).

Moreover, there exists some constant $C = C(\Omega, n, \kappa, p) \in (0,\infty)$ independent of $\vec{F}$ such that

$$\big\|\nu \bullet \vec{F}\big\|_{H^p(\partial\Omega,\sigma)} \leq C\big\|\mathcal{N}_\kappa \vec{F}\big\|_{L^p(\partial\Omega,\sigma)} + C\big\|\mathfrak{P}(\mathrm{div}\vec{F})\big\|_{L^p(\partial\Omega,\sigma)}. \tag{1.9.17}$$

One of the salient features of Theorems 1.9.1–1.9.2 is the fact that the bullet product $\nu \bullet \vec{F}$ is intimately linked with the principal symbol of the divergence operator acting on vector fields. As such, it is desirable to expand upon these results by allowing first-order systems with constant complex coefficients which are more general than the divergence operator, say

$$D = \Big(\sum_{j=1}^n a_j^{\alpha\beta}\partial_j\Big)_{\substack{1\leq\alpha\leq N \\ 1\leq\beta\leq M}}. \tag{1.9.18}$$

If Ω is an open subset of $\mathbb{R}^n$ and $F \in \big[L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)\big]^M$ is a $\mathbb{C}^M$-valued function with the property that DF, considered in the sense of distributions in Ω, has components in $L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)$, we define the functional $\mathrm{Sym}(D;\nu) \bullet F$ acting on each $\psi \in \big[\mathrm{Lip}_c(\partial\Omega)\big]^N$ according to

$$\big\langle (-\mathrm{i})\mathrm{Sym}(D;\nu) \bullet F, \psi \big\rangle := \int_\Omega \langle DF, \Psi\rangle \, \mathrm{d}\mathcal{L}^n - \int_\Omega \langle F, D^\top\Psi\rangle \, \mathrm{d}\mathcal{L}^n, \tag{1.9.19}$$

whenever Ψ is a $\mathbb{C}^N$-valued function satisfying

$$\begin{array}{l} \Psi \in \big[\mathrm{Lip}(\overline{\Omega})\big]^N, \ \ \Psi\big|_{\partial\Omega} = \psi, \ \text{ and } \ \Psi \equiv 0 \\ \text{outside of some compact subset of } \overline{\Omega}. \end{array} \tag{1.9.20}$$

This brand of "bullet product," between the principal symbol of a first-order differential operator and a suitable vector field, is studied at length in [185, Sect. 10.2] where, among other things, it is shown that the functional $\mathrm{Sym}(D;\nu)\bullet F$ is meaningfully and unambiguously defined and, in fact, is a distribution on $\partial\Omega$, i.e., it belongs to $\Big[\big(\mathrm{Lip}_c(\partial\Omega)\big)'\Big]^M$. Here the focus is on the following version of the Divergence Theorem, whose formulation employs the bullet product introduced in (1.9.19).

Theorem 1.9.3 *Fix $n\in\mathbb{N}$ with $n\geq 2$, and suppose $\Omega\subseteq\mathbb{R}^n$ is an open set with an Ahlfors regular boundary. Abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, pick two integers $N,M\in\mathbb{N}$ and consider an arbitrary $N\times M$ homogeneous first-order system D with constant complex coefficients in $\mathbb{R}^n$, as in (1.7.12), and recall its (real) transpose $D^\top$, defined in (1.7.13).*

In this setting, consider a vector-valued function $F:\Omega\to\mathbb{C}^M$, whose components are $\mathcal{L}^n$-measurable, with the property that

$$\mathcal{N}_\kappa F\in L^p(\partial\Omega,\sigma)\text{ for some }\kappa\in(0,\infty)\text{ and }p\in\big(\tfrac{n-1}{n},\infty\big]. \tag{1.9.21}$$

In particular, $F\in\big[L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n)\big]^M$ (cf. Lemma 8.3.1), and one also assumes that DF, computed in the sense of distributions in Ω, has components in $L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n)$ and satisfies

$$\mathfrak{P}(DF)\in L^p(\partial\Omega,\sigma). \tag{1.9.22}$$

Then the following statements are valid.

(1) Corresponding to the regime $p\in[1,\infty]$, there exists a unique function, denoted by $\mathrm{Sym}(D;\nu)\bullet F$, which belongs to $\big[L^p(\partial\Omega,\sigma)\big]^N$ and such that for each $\Psi\in\big[\mathrm{Lip}_c(\mathbb{R}^n)\big]^N$ the following integration by parts formula is satisfied[19]:

$$\int_{\partial\Omega}\big\langle(-\mathrm{i})\mathrm{Sym}(D;\nu)\bullet F,\Psi\big\rangle\,d\sigma=\int_\Omega\langle DF,\Psi\rangle\,d\mathcal{L}^n-\int_\Omega\langle F,D^\top\Psi\rangle\,d\mathcal{L}^n. \tag{1.9.23}$$

In addition, there exists some constant $C=C(\Omega,D,\kappa,p)\in(0,\infty)$ independent of F such that

$$\Big\|\mathrm{Sym}(D;\nu)\bullet F\Big\|_{[L^p(\partial\Omega,\sigma)]^N}\leq C\big\|\mathcal{N}_\kappa F\big\|_{L^p(\partial\Omega,\sigma)}+C\|\mathfrak{P}(DF)\|_{L^p(\partial\Omega,\sigma)}. \tag{1.9.24}$$

Moreover, if $p=1$ then actually $\mathrm{Sym}(D;\nu)\bullet F$ belongs to the Hardy space $\big[H^1(\partial\Omega,\sigma)\big]^N$ and, for some constant $C=C(\Omega,D,\kappa)\in(0,\infty)$ independent of F,

[19] Compare with (1.7.24), which inspired the notation $\mathrm{Sym}(D;\nu)\bullet F$.

$$\left\|\mathrm{Sym}(D;\nu)\bullet F\right\|_{[H^1(\partial\Omega,\sigma)]^N}\leq C\left\|\mathcal{N}_\kappa F\right\|_{L^1(\partial\Omega,\sigma)}+C\|\mathfrak{P}(DF)\|_{L^1(\partial\Omega,\sigma)}. \tag{1.9.25}$$

Finally, under the additional assumptions that[20] $\mathcal{H}^{n-1}(\partial\Omega\setminus\partial_*\Omega)=0$ *and the nontangential trace* $F\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ *exists (in* $\mathbb{C}^M$*) at* σ*-a.e. point on* $\partial\Omega$*, one has*

$$\mathrm{Sym}(D;\nu)\bullet F=\mathrm{Sym}(D;\nu)\Big(F\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)\ \text{ at } \sigma\text{-a.e. point on } \partial\Omega, \tag{1.9.26}$$

where ν *is the geometric measure theoretic outward unit normal to* Ω*.*

(2) In the regime $p\in\left(\frac{n-1}{n},1\right)$*, there exists a unique distribution on* $\partial\Omega$*, denoted by the symbol* $\mathrm{Sym}(D;\nu)\bullet F$*, which belongs to the Hardy space* $\left[H^p(\partial\Omega,\sigma)\right]^N$ *and has the property that for each function* $\Psi\in\left[\mathrm{Lip}_c(\mathbb{R}^n)\right]^N$ *the following integration by parts formula is satisfied:*

$$\begin{aligned}\int_\Omega\langle DF,\Psi\rangle\,\mathrm{d}\mathcal{L}^n-\int_\Omega\langle F,D^\top\Psi\rangle\,\mathrm{d}\mathcal{L}^n&\\ =\begin{cases}\Big\langle(-\mathrm{i})\mathrm{Sym}(D;\nu)\bullet F,\Psi|_{\partial\Omega}\Big\rangle & \text{if } \partial\Omega \text{ bounded},\\ \Big\langle(-\mathrm{i})\mathrm{Sym}(D;\nu)\bullet F,\big[\Psi|_{\partial\Omega}\big]\Big\rangle & \text{if } \partial\Omega \text{ unbounded},\end{cases}&\end{aligned} \tag{1.9.27}$$

where the pairings under the integral sign are pointwise inner products, while $\langle\cdot,\cdot\rangle$ *appearing in the right-hand side of* (1.9.27) *denotes the duality bracket between the Hardy space* $\left[H^p(\partial\Omega,\sigma)\right]^N$ *and its dual space, identified analogously to* (1.9.16) *with the exponent* $\alpha:=(n-1)\left(\frac{1}{p}-1\right)\in(0,1)$*.*
Moreover, there exists some constant $C=C(\Omega,D,\kappa,p)\in(0,\infty)$ *independent of* F *such that*

$$\begin{aligned}&\left\|\mathrm{Sym}(D;\nu)\bullet F\right\|_{[H^p(\partial\Omega,\sigma)]^N}\\ &\qquad\leq C\left\|\mathcal{N}_\kappa F\right\|_{L^p(\partial\Omega,\sigma)}+C\left\|\mathfrak{P}(\mathrm{div}F)\right\|_{L^p(\partial\Omega,\sigma)}.\end{aligned} \tag{1.9.28}$$

We conclude this section by stating a version of the Divergence Theorem with weak traces valid in *arbitrary* open subsets of $\mathbb{R}^n$. Before stating it, the reader is reminded that, given a Lebesgue measurable set $\Omega\subseteq\mathbb{R}^n$, we let $L^1_{\text{bdd}}(\Omega,\mathcal{L}^n)$ stand for the space of (complex-valued) functions which are absolutely integrable on any bounded Lebesgue measurable subset of Ω.

[20] The demand that $\mathcal{H}^{n-1}(\partial\Omega\setminus\partial_*\Omega)=0$ precludes $\partial\Omega$ from developing “too many” cusps and also prevents Ω from having “significant” cracks.

Theorem 1.9.4 *Let Ω be an arbitrary open subset of $\mathbb{R}^n$ and consider a vector field*

$$\vec{F} \in \left[L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n) + \mathscr{E}'(\Omega)\right]^n \quad with \quad \operatorname{div}\vec{F} \in L^1(\Omega, \mathcal{L}^n) + \mathscr{E}'(\Omega), \tag{1.9.29}$$

where the divergence is taken in the sense of distributions in the set Ω. Also, suppose $\mathscr{F} := \{\phi_R\}_{R>0}$ is a system of auxiliary functions (as in (1.3.3)) for which the following limit exists:

$$[\vec{F}]_{\mathscr{F}} := -\lim_{R\to\infty} \int_\Omega \nabla\phi_R \cdot \vec{F}\, \mathrm{d}\mathcal{L}^n. \tag{1.9.30}$$

Then the limit

$$\left(\nu \bullet \vec{F}, 1\right)_{\mathscr{F}} := \lim_{R\to\infty} \left\langle \nu \bullet \vec{F}, \phi_R\big|_{\partial\Omega}\right\rangle \tag{1.9.31}$$

exists, where the pairing in the right-hand side is considered in the sense of Definition 4.2.6, and

$$_{(\mathscr{C}^\infty_b(\Omega))^*}\left(\operatorname{div}\vec{F}, 1\right)_{\mathscr{C}^\infty_b(\Omega)} = \left(\nu \bullet \vec{F}, 1\right)_{\mathscr{F}} + [\vec{F}]_{\mathscr{F}}. \tag{1.9.32}$$

A significant particular case of Theorem 1.9.4 is as follows. Assume that there exists $\lambda \in (1, \infty)$ such that

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |\vec{F}|\, \mathrm{d}\mathcal{L}^n = o(R) \quad \text{as} \quad R \to \infty. \tag{1.9.33}$$

Then taking

$$\mathscr{F} := \{\phi(\cdot/R)\}_{R>0} \text{ with } \phi \in \mathscr{C}^\infty_c\big(B(0,\lambda)\big) \text{ satisfying } \phi \equiv 1 \text{ on } B(0,1) \tag{1.9.34}$$

ensures that $[\vec{F}]_{\mathscr{F}} = 0$. Thus, in such a scenario, the Divergence Formula (1.9.32) reduces to

$$_{(\mathscr{C}^\infty_b(\Omega))^*}\left(\operatorname{div}\vec{F}, 1\right)_{\mathscr{C}^\infty_b(\Omega)} = \left(\nu \bullet \vec{F}, 1\right)_{\mathscr{F}}, \tag{1.9.35}$$

where the right-hand side is defined as in (1.9.31) with $\phi_R := \phi(\cdot/R)$ for each $R > 0$.

To close, we note that the proofs of Theorems 1.9.1–1.9.4 are presented in Sect. 9.7.

1.10 The Divergence Theorem Involving an Averaged Nontangential Maximal Operator

One key common characteristic shared by the versions of the Divergence Theorem recorded in Sects. 1.2–1.9 is the involvement of the nontangential maximal operator $\vec{F} \mapsto \mathcal{N}_\kappa \vec{F}$, as a means of quantifying information about the given vector field. It turns out that asking, as we do in Theorems 1.2.1 and 1.3.1, that $\mathcal{N}_\kappa \vec{F}$ is locally integrable on $\partial\Omega$ makes $\vec{F}$ locally bounded in the domain Ω. As such, the question arises whether a version of the Divergence Theorem in the spirit of those given in Theorems 1.2.1 and 1.3.1 exists for locally integrable vector fields which are not necessarily locally bounded.

Of course, as mentioned earlier, this presupposes that we alter the very definition of the nontangential maximal operator and work with a weakened variant which, nonetheless, continues to encode relevant information about the function it acts on even when the function in question is merely r-th power locally integrable for some exponent $r < \infty$. Having additionally fixed a scale parameter $\theta \in (0, 1)$, for each such function u and each point $x \in \partial\Omega$ we shall denote by $(\widetilde{\mathcal{N}}_{\kappa,\theta,r}u)(x)$ the supremum of[21]

$$\Gamma_\kappa(x) \ni y \mapsto \Big(⨍_{B(y,\theta\,\mathrm{dist}(y,\partial\Omega))} |u|^r \, d\mathcal{L}^n \Big)^{1/r}. \tag{1.10.1}$$

In relation to this "averaged" nontangential maximal operator, which is discussed at length in Sect. 8.10, we then have the following version of the Divergence Theorem.

Theorem 1.10.1 *Suppose $n \in \mathbb{N}$ satisfies $n \geq 2$. Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter, and its geometric measure theoretic outward unit normal ν is defined σ-a.e. on $\partial_*\Omega$(which, up to a σ-nullset, is contained in $\partial_{\mathrm{nta}}\Omega$). Fix $\kappa \in (0, \infty)$ along with $\theta \in (0, 1)$ and $r \in [1, \infty)$, and assume the vector field $\vec{F} = (F_1, \ldots, F_n) \in \big[L^r_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^n$, with complex-valued components, enjoys the following properties:*

$$\begin{gathered} \textit{the nontangential trace } \ \vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \textit{ exists (in } \mathbb{C}^n\textit{) } \sigma\textit{-a.e. on } \ \partial_{\mathrm{nta}}\Omega, \quad \textit{and} \\ \textit{the averaged nontangential maximal function } \ \widetilde{\mathcal{N}}_{\kappa,\theta,r}\vec{F} \ \textit{ is in } \ L^1(\partial\Omega, \sigma). \end{gathered} \tag{1.10.2}$$

Also, with all individual partial derivatives considered in the sense of distributions in Ω, assume

$$\begin{gathered} \textit{the distribution } \ \mathrm{div}\vec{F} := \partial_1 F_1 + \cdots + \partial_n F_n \in \mathcal{D}'(\Omega) \ \textit{ extends} \\ \textit{to a complex Borel measure in } \Omega\textit{, still denoted by } \ \mathrm{div}\vec{F}. \end{gathered} \tag{1.10.3}$$

Then the contribution of $\vec{F}$ at infinity is meaningfully and unambiguously defined and

$$\big(\mathrm{div}\vec{F}\big)(\Omega) = \int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\,\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)\, d\sigma + [\vec{F}]_\infty. \tag{1.10.4}$$

Moreover, $[\vec{F}]_\infty = 0$ in either of the following three cases:

$$\begin{aligned} &\textit{(i) } \Omega \textit{ is bounded;} \\ &\textit{(ii) } \partial\Omega \textit{ is unbounded and } r \geq \tfrac{n}{n-1}; \\ &\textit{(iii) } \vec{F} \textit{ satisfies (1.2.3) for some } \lambda \in (1, \infty). \end{aligned} \tag{1.10.5}$$

[21] The reader is reminded that $\Gamma_\kappa(x)$ is the nontangential approach region with apex at $x \in \partial\Omega$ and aperture parameter $\kappa \in (0, \infty)$, defined in (8.1.2).

Ergo, in either of these three scenarios, the Divergence Formula (1.10.4) *reduces simply to*

$$\big(\mathrm{div}\vec{F}\big)(\Omega) = \int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\,\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)\, d\sigma. \tag{1.10.6}$$

As with the other main results discussed so far, we make a number of remarks designed to provide additional context for the above theorem.

First Remark. The ordinary nontangential maximal operator $u \mapsto \mathcal{N}_\kappa u$ encodes more nuanced information about a given function than the averaged nontangential maximal operator $u \mapsto \widetilde{\mathcal{N}}_{\kappa,\theta,r} u$. Specifically, in Proposition 8.10.2 we shall show that if $\Omega \subset \mathbb{R}^n$ is an open set and σ is a doubling measure on $\partial\Omega$ then, in a quantitative sense,

$$\mathcal{N}_\kappa u \in L^1(\partial\Omega, \sigma) \Rightarrow \widetilde{\mathcal{N}}_{\kappa,\theta,r} u \in L^1(\partial\Omega, \sigma). \tag{1.10.7}$$

This points to the fact that asking $\widetilde{\mathcal{N}}_{\kappa,\theta,r}\vec{F} \in L^1(\partial\Omega, \sigma)$, as we do in the second line of (1.10.2), is *weaker* than the assumption that $\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma)$ made in (1.2.1) in Theorem 1.2.1. This being said, Theorem 1.2.1 and its proof self-improve and eventually yield Theorem 1.10.1, which is remarkable.

Second Remark. As far as the sharpness of Theorem 1.10.1 is concerned, there are counterexamples which show that the current requirement that $\widetilde{\mathcal{N}}_{\kappa,\theta,r}\vec{F} \in L^1(\partial\Omega, \sigma)$ cannot be weakened to either $\widetilde{\mathcal{N}}_{\kappa,\theta,r}\vec{F} \in L^{1,\infty}(\partial\Omega, \sigma)$, or $\widetilde{\mathcal{N}}_{\kappa,\theta,r}\vec{F} \in L^1(\partial_*\Omega, \sigma)$. Indeed, the counterexample in (2.3.7) does the job, since the vector field employed there has harmonic components, a scenario in which the ordinary and averaged nontangential maximal operators are essentially equivalent (cf. (8.10.14)).

Third Remark. It is instructive to give a concrete example of a vector field which is *not* locally bounded and yet the version of the Divergence Formula from Theorem 1.10.1 is valid for it. Concretely, work in $\mathbb{R}^n$ with $n \geq 2$ and take

$$\Omega := B(0,1) \;\text{ and }\; \vec{F}(x) := \frac{1}{\omega_{n-1}}\frac{x}{|x|^n} \;\text{ for } \mathcal{L}^n\text{-a.e. } x \in \Omega. \tag{1.10.8}$$

Then, having fixed $r \in \big[1, \frac{n}{n-1}\big)$, it follows that the vector field $\vec{F}$ actually belongs to $\big[L^r(\Omega, \mathcal{L}^n)\big]^n$ and satisfies $\mathrm{div}\vec{F} = \delta \in \mathrm{CBM}(\Omega)$, regarded as the *Dirac measure* acting according to $\delta(A) := \mathbf{1}_A(0)$ for each set $A \subseteq \Omega$ (cf. (1.3.16)). We also claim that for each $\kappa \in (0,\infty)$ and $\theta \in (0,1)$ we have

$$\widetilde{\mathcal{N}}_{\kappa,\theta,r}\vec{F} \in L^\infty(\partial\Omega, \sigma) \subseteq L^1(\partial\Omega, \sigma). \tag{1.10.9}$$

Indeed, given $y \in \Omega$ and $z \in B\big(y, \theta\,\mathrm{dist}(y, \partial\Omega)\big)$, if $|y| \geq \frac{3}{4}$ then $|z| \geq \frac{1}{2}$ hence $|\vec{F}(z)| \leq 2^{n-1}/\omega_{n-1}$, whereas if $|y| < 3/4$ then $\mathrm{dist}(y, \partial\Omega) \geq 1/4$ which, in turn, permits us to conclude that

$$\Big(\fint_{B(y,\theta\,\mathrm{dist}(y,\partial\Omega))} |\vec{F}|^r \, \mathrm{d}\mathcal{L}^n\Big)^{1/r} \leq C_{n,\theta,r} \cdot \|\vec{F}\|_{L^r(\Omega,\mathcal{L}^n)}. \tag{1.10.10}$$

Thus, the supremum of (1.10.1) is a finite number independent of $x \in \partial\Omega$, ultimately proving (1.10.9). In addition, we have $(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega})(x) = x/\omega_{n-1}$ at every point $x \in \partial\Omega = S^{n-1}$. Hence, all hypotheses of Theorem 1.10.1 are satisfied and, as the Divergence Formula (1.10.6) predicts, the numbers

$$\big(\mathrm{div}\vec{F}\big)(\Omega) = 1 \ \text{ and } \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\,\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} x \cdot x \, \mathrm{d}\mathcal{H}^{n-1}(x) = 1 \tag{1.10.11}$$

indeed match. By way of contrast, an attempt to justify this particular divergence formula via Theorem 1.2.1, or Theorem 1.3.1, runs into the insurmountable issue that

$$(\mathcal{N}_\kappa \vec{F})(x) = +\infty \ \text{ at every point } \ x \in \partial\Omega. \tag{1.10.12}$$

This being said, Theorems 1.4.1 and 1.5.1 (which are applicable to vector fields that are locally bounded only near $\partial\Omega$) are refined enough to handle the example given in (1.10.8). Nevertheless, it is possible to further modify this example so as to place it beyond the realm of Theorems 1.4.1 and 1.5.1, but still remain within the scope of Theorem 1.10.1. Concretely, with Ω and $\vec{F}$ as in (1.10.8), pick $\{x_j\}_{j\in\mathbb{N}} \subseteq \Omega$ which converges to some point $x_* \in S^{n-1}$ and, having fixed some $r \in \big[1, \frac{n}{n-1}\big)$ along with $\kappa \in (0,\infty)$ and $\theta \in (0,1)$, define

$$\begin{gathered}\vec{F}_* := \sum_{j=1}^{\infty} \frac{1}{C_j} \vec{F}(\cdot - x_j) \ \text{ at } \mathcal{L}^n\text{-a.e. point in } \ \Omega, \\ \text{where } \ C_j := 2^j \cdot \max\Big\{1, \big\|\mathcal{N}_{\kappa,\theta,r}\big(\vec{F}(\cdot - x_j)\big)\big\|_{L^1(\partial\Omega,\sigma)}\Big\}.\end{gathered} \tag{1.10.13}$$

Then $\vec{F}_* \in \big[L^r(\Omega,\mathcal{L}^n)\big]^n$ satisfies the hypotheses of Theorem 1.10.1 and fails to be locally bounded near $x_* \in \partial\Omega$ (hence, $\vec{F}_*$ lies outside the scope of Theorems 1.4.1–1.5.1).

Fourth Remark. The proof of Theorem 1.10.1 which, as alluded earlier, proceeds along the line of reasoning employed in the proof of Theorem 1.3.1, is given in Sect. 9.8.

1.11 The Manifold Setting and a Sharp Version of Stokes' Formula

In this section, the goal is to extend the scope of our earlier results by considering vector fields on subdomains of Riemannian manifolds. To set the stage, assume that

$$M \text{ is a connected, compact, boundaryless, oriented manifold of class } \mathscr{C}^1, \text{ of real dimension } n, \text{ equipped with a continuous Riemannian metric tensor } g = \sum_{1\le j,k\le n} g_{jk}\mathrm{d}x_j \otimes \mathrm{d}x_k. \tag{1.11.1}$$

Denote by $\mathrm{d}V_g$ the volume element on M induced by the metric tensor g and let $\mathcal{L}_g^n$ be the measure canonically associated with the n-form $\mathrm{d}V_g$. Hence, for each $f \in L^1(M, \mathcal{L}_g^n)$ we have

$$\int_M f\,\mathrm{d}\mathcal{L}_g^n = \int_M f\,\mathrm{d}V_g \tag{1.11.2}$$

with the right side interpreted as the integral of the n-form $f\,\mathrm{d}V_g$ on M. Since in local coordinates $(x_1, \dots, x_n)$ we have (cf. [189, (2.1.14), p. 50])

$$\mathrm{d}V_g = \sqrt{g}\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \quad \text{where} \quad \sqrt{g} := \sqrt{\det\big[(g_{jk})_{1\le j,k\le n}\big]}, \tag{1.11.3}$$

it follows that locally

$$\mathrm{d}\mathcal{L}_g^n = \sqrt{g}\,\mathrm{d}\mathcal{L}^n, \tag{1.11.4}$$

where $\mathcal{L}^n$ is the ordinary n-dimensional Lebesgue measure in $\mathbb{R}^n$.

Next, let TM and T^*M stand, respectively, for the tangent and cotangent bundles on M. In particular, the symbol ${}_{T^*M}\big(\cdot,\cdot\big)_{TM}$ is reserved for the metric pairing between cotangent vector and tangent vectors on M. For each $\ell \in \{0, 1, \dots, n\}$, the symbol $\langle\cdot,\cdot\rangle_{\Lambda^\ell TM}$ is used to denote the inner product canonically induced by the metric g on $\Lambda^\ell TM$, the ℓ-th exterior power of the vector bundle on M. Sections in $\Lambda^\ell TM$ are differential forms of degree ℓ (or, simply, ℓ-forms) on M.

By $\mathcal{H}_g^{n-1}$ we shall denote the $(n-1)$-dimensional Hausdorff measure associated with the intrinsic metric induced by g on M, aka the geodesic (inner) metric on M. The latter is defined (cf. [29, Proposition 3.103, p. 281], [140, pp. 15–16]) as

$$d_g(x,y) := \inf\Big\{\int_0^1 \|\dot{\gamma}(t)\|_{TM}\,\mathrm{d}t : \gamma : [0,1] \to M \ \text{ piecewise } \mathscr{C}^1 \text{ curve joining } x \text{ and } y\Big\}. \tag{1.11.5}$$

In particular, the metric topology generated by d_g on M coincides with the standard background (Hausdorff) topology on M. Also, (M, d_g) becomes a length metric space, and the length of any $\mathscr{C}^1$ curve in the metric space (M, d_g) is equal to its ordinary Riemannian length.[22]

Classes of Euclidean domains which are invariant under $\mathscr{C}^1$ diffeomorphisms (such as domains of locally finite perimeter) then extend naturally to the manifold

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[22] Recall that a metric space is called a length metric space if the intrinsic metric agrees with the original metric of the space. Metric spaces enjoying the stronger property that there exists a path achieving the infimum in the definition of the intrinsic metric between any two points are called geodesic metric spaces; see, e.g., [140, Theorem 1.6.1 on p. 31, and Theorem 1.7.1 on pp. 34–35] in this regard.

setting. See [124] for a discussion in this regard. Throughout, we agree to retain notation already introduced in the Euclidean setting whenever the corresponding object has a natural analogue in the context of manifolds.

If d denotes the exterior derivative operator on M, one may also define what it means for some given $\mathcal{L}^n_g$-measurable subset Ω of the manifold M to be of locally finite perimeter in an intrinsic manner, asking that (with $\mathbf{1}_\Omega$ denoting the characteristic function of Ω) we have[23]

$$\begin{array}{l}\mathrm{d}\mathbf{1}_\Omega = -\nu_g\,\sigma_g \text{ as distributions, for a locally finite Borel-regular measure } \sigma_g \\ \text{supported on } \partial_*\Omega, \text{ and for a } T^*M\text{-valued function } \nu_g \in L^\infty(\partial_*\Omega, \sigma_g) \otimes T^*M \\ \text{satisfying } |\nu_g|_{T^*M} = 1 \text{ at } \sigma_g\text{-a.e. point on } \partial_*\Omega.\end{array} \tag{1.11.6}$$

As in the Euclidean setting, σ_g and ν_g are uniquely determined by Ω.

It turns out that the quality of $\Omega \subseteq M$ being a set of locally finite perimeter is independent of the choice of the background Riemannian metric g. To elaborate on this topic, work near a boundary point $x_0 \in \partial\Omega$ and locally identify $\Omega \subseteq M$ with its Euclidean image under the corresponding coordinate chart (for which we retain the same notation). Under such an identification the quality of being a set of locally finite perimeter is preserved.

Next, in any local coordinates $(x_1, \dots, x_n)$ on M, we let

$$\begin{array}{l}\big(\nu_j^{\mathrm{E}}\big)_{1\leq j\leq n} \text{ denote the geometric measure theoretic outward unit normal} \\ \text{to the given set } \Omega \text{ with respect to the standard Euclidean metric tensor} \\ g^{\mathrm{E}} := \sum_{1\leq j\leq n} \mathrm{d}x_j \otimes \mathrm{d}x_j \text{ in } \mathbb{R}^n, \text{ and define } \sigma^{\mathrm{E}} := \mathcal{H}^{n-1}\lfloor\partial\Omega \text{ where} \\ \mathcal{H}^{n-1} \text{ is the usual } (n-1)\text{-dimensional Hausdorff measure in } \mathbb{R}^n.\end{array} \tag{1.11.7}$$

These are naturally related to one another. For one thing, in local coordinates we have the identification

$$\mathrm{d}\sigma_g = \rho\, \mathrm{d}\sigma^{\mathrm{E}} \;\text{ for some }\; \rho \in L^\infty(\partial\Omega, \sigma^{\mathrm{E}}) \;\text{ with }\; \rho^{-1} \in L^\infty(\partial\Omega, \sigma^{\mathrm{E}}). \tag{1.11.8}$$

Indeed, we claim that there exists a constant $C \in (1, \infty)$ with the property that

$$C^{-1}\sigma(A) \leq \sigma_g(A) \leq C\sigma(A) \;\text{ for each measurable set }\; A \subseteq \partial\Omega. \tag{1.11.9}$$

Keeping in mind that $\sigma_g = \mathcal{H}^{n-1}_g\lfloor\partial\Omega$ and $\sigma^{\mathrm{E}} = \mathcal{H}^{n-1}\lfloor\partial\Omega$, this can be seen from the very definition of the Hausdorff measure in metric spaces, by noting that subsets of a local coordinate patch on M have diameters relative to either the Euclidean metric or the Riemannian metric on M varying only by a bounded factor. Having established (1.11.9), we may then invoke the Radon–Nikodym theorem to conclude (1.11.8).

In fact, more can be said. Specifically, from [125, Proposition 5.7, p. 2774] (cf. also the discussion in [189, pp. 408–409]) it follows that we have the identifications

[23] This should be compared with the result recorded in Proposition 5.6.3.

$$\begin{array}{c} \sigma_g \lfloor \partial_*\Omega = \sqrt{g}\, \mathfrak{G}^{1/2}\, \sigma^{\rm E} \lfloor \partial_*\Omega \quad \text{and} \quad \nu_g = \mathfrak{G}^{-1/2} \sum_{j=1}^n \nu_j^{\rm E}\, {\rm d}x_j \\ \text{where} \quad \mathfrak{G} := \sum_{r,s=1}^n g^{rs} \nu_r^{\rm E} \nu_s^{\rm E} \quad \text{on} \quad \partial_*\Omega, \quad \text{and where we have set} \\ \sqrt{g} := \sqrt{\det\big[(g_{jk})_{1\leq j,k\leq n}\big]} \quad \text{with} \quad (g^{rs})_{1\leq r,s\leq n} := \big[(g_{jk})_{1\leq j,k\leq n}\big]^{-1}. \end{array} \tag{1.11.10}$$

We also agree to follow the convention in [199, Sect. 2] regarding the manner in which distributions on the manifold M may be locally identified with distributions in the Euclidean setting. Specifically,

if f is a locally integrable function on M and $X \in TM$ is a vector field which in local coordinates is written as $\sum_{j=1}^n X_j \partial_j$ then Xf, treated as a distribution on M, is identified in local coordinates with $\sum_{j=1}^n \sqrt{g}\, X_j \partial_j f$, where now f is regarded as a function in $\mathbb{R}^n$ and the partial derivatives are considered in the sense of distributions in $\mathbb{R}^n$. (1.11.11)

From (1.11.6) and (1.11.10)–(1.11.11) we then conclude that

the equation ${\rm d}\mathbf{1}_\Omega = -\nu_g\, \sigma_g$ in the sense of distributions on M becomes, in local Euclidean coordinates, $\nabla \mathbf{1}_\Omega = -\nu^{\rm E}\, \sigma^{\rm E}$. (1.11.12)

In particular, this goes to show that, indeed, being a set of locally finite perimeter is independent of the particular choice of the background Riemannian metric g.

Given a set $\Omega \subseteq M$ of locally finite perimeter, untangling what it means to have ${\rm d}\mathbf{1}_\Omega = -\nu_g\, \sigma_g$ in the sense of distributions on M readily leads to a divergence formula for vector fields $\vec{F}$ of class $\mathscr{C}^1$ on M. A slightly more general result is obtained by patching together the version of the Divergence Theorem for the Euclidean space equipped with a generic continuous Riemann metric from [125, Proposition 5.8, p. 2775], using a smooth partition of unity subordinate to a suitable finite open cover of the underlying domain with local coordinate patches (cf. [125, Remark, p. 2775]). This procedure yields the following extension of De Giorgi–Federer's version of the Gauss–Green Formula from Theorem 1.1.1 (cf. also (2.8.1)) to the manifold setting.

Theorem 1.11.1 *Let the Riemannian manifold* (M, g) *be as in* (1.11.1) *and denote by* $\mathcal{L}_g^n$ *the measure induced by the volume element* ${\rm d}V_g$ *on* M. *Consider a set* $\Omega \subseteq M$ *of locally finite perimeter, and define* $\sigma_g := \mathcal{H}_g^{n-1} \lfloor \partial\Omega$. *In particular, the geometric measure theoretic outward unit conormal to* Ω, *denoted by* $\nu_g : \partial_*\Omega \to T^*M$ *is defined* σ_g*-a.e. on* $\partial_*\Omega$. *In this setting, assume a vector field* $\vec{F} \in \mathscr{C}^0(M, TM)$ *has been given, with the property that* ${\rm div}_g \vec{F} \in L^1(M, \mathcal{L}_g^n)$, *where* ${\rm div}_g$ *is the (differential geometric) divergence taken in the sense of distributions on* M *(see, e.g.,* [189, (9.1.13), p. 373]*). Then*

$$\int_\Omega {\rm div}_g \vec{F}\, {\rm d}\mathcal{L}_g^n = \int_{\partial_*\Omega} {}_{T^*M}\big(\nu_g, \vec{F}\,\big|_{\partial_*\Omega}\big)_{TM}\, {\rm d}\sigma_g. \tag{1.11.13}$$

Remark 1.11.2 Regarding the hypotheses on the ambient manifold M, the compactness assumption placed on M is not essential, in the sense that natural versions of Theorem 1.11.1 continue to hold when M is merely an open (Riemannian, oriented) manifold. For example, this is the case if Ω is a relatively compact set of locally finite perimeter contained in the open manifold M. Indeed, in such a scenario matters may always be reduced to the compact setting by embedding $\overline{\Omega}$ isometrically in a compact, boundaryless Riemannian manifold $\mathcal{M}$ having the same dimension and regularity as the original M. Specifically, starting with a compact submanifold (with boundary) $\mathcal{O}$ of M whose interior contains $\overline{\Omega}$, one can take $\mathcal{M}$ to be the so-called geometric double of $\mathcal{O}$, manufactured by taking two replicas of $\mathcal{O}$ with opposite orientations and "gluing" them together by identifying boundary points (specifically, take $\mathcal{M} := \mathcal{O} \times \{0, 1\}\big/\sim$ where $(x, 0) \sim (x, 1)$ for every $x \in \partial\mathcal{O}$). Another favorable scenario is when the vector field $\vec{F}$ is also assumed to be compactly supported. Under this additional hypothesis, no relative compactness condition needs to be imposed on M or Ω (since we may once again reduce matters to working on a compact boundaryless manifold by arguing as before).

Very often, the vector fields arising in the applications of the Divergence Theorem are merely defined in Ω and may not extend to continuous vector fields on the entire manifold M. This renders Theorem 1.11.1 ill-suited for handling this basic case. Theorem 1.11.3, stated below, establishes a version of the Divergence Theorem which addresses this issue.

Theorem 1.11.3 *Assume the Riemannian manifold* (M, g) *is as in* (1.11.1). *Let* Ω *be a nonempty, open, proper subset of the manifold* M *such that* $\partial\Omega$ *is lower Ahlfors regular, and* $\sigma_g := \mathcal{H}_g^{n-1}\lfloor\partial\Omega$ *is a doubling measure on* $\partial\Omega$. *In particular,* Ω *is a set of finite perimeter and its geometric measure theoretic outward unit conormal* $\nu_g : \partial_*\Omega \to T^*M$ *is defined* σ_g*-a.e. on* $\partial_*\Omega$. *Denote by* $\mathcal{L}_g^n$ *the measure induced by the volume element* $\mathrm{d}V_g$ *on* M.

Fix $\kappa \in (0, \infty)$ *and assume that the vector field* $\vec{F} \in \mathcal{D}'(\Omega) \otimes TM$ *has the property that*

$$\begin{gathered}\text{there exists a compact set } K \subset \Omega \text{ such that}\\ \vec{F}\big|_{\Omega\setminus K} \in L^1_{\mathrm{loc}}(\Omega \setminus K, \mathcal{L}_g^n) \otimes TM \ \text{ and } \ \mathcal{N}_\kappa^{\Omega\setminus K}\big(\vec{F}\big|_{\Omega\setminus K}\big) \in L^1(\partial\Omega, \sigma_g),\end{gathered} \tag{1.11.14}$$

the pointwise nontangential boundary trace

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \text{ exists (in } TM\text{) at } \sigma_g\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega, \tag{1.11.15}$$

and its (*differential geometric*) *divergence, taken in the sense of distributions in* Ω, *is the sum* (*in* $\mathcal{D}'(\Omega)$) *of a compactly supported distribution in* Ω *and a complex Borel measure in* Ω, *i.e.,*

$$\mathrm{div}_g \vec{F} \in \mathscr{E}'(\Omega) + \mathrm{CBM}(\Omega). \tag{1.11.16}$$

Then for any aperture $\kappa' > 0$ *the nontangential trace* $\vec{F}\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ *exists* σ_g*-a.e. on* $\partial_{\mathrm{nta}}\Omega$ *and is actually independent of* κ'*. When regarding it as a function defined* σ_g*-a.e. on* $\partial_*\Omega$ *(which, up to a* σ_g*-nullset, is contained in* $\partial_{\mathrm{nta}}\Omega$*), this nontangential trace belongs to* $L^1(\partial_*\Omega, \sigma_g) \otimes TM$ *and, with the dependence on the parameter* κ' *dropped,*

$$ {}_{(\mathscr{C}_b^1(\Omega))^*}\big(\mathrm{div}_g \vec{F}, 1\big)_{\mathscr{C}_b^1(\Omega)} = \int_{\partial_*\Omega} {}_{T^*M}\big(\nu_g, \vec{F}\,\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)_{TM}\, d\sigma_g. \tag{1.11.17} $$

The proof of Theorem 1.11.3 is presented in Sect. 9.9.

Remark 1.11.4 *In applications, it is often useful to keep in mind that the vector field* $\vec{F}$ *from Theorem 1.11.3 may be allowed to take values in* $\mathscr{V} \otimes TM$*, where* $\mathscr{V}$ *is an arbitrary, fixed, finite-dimensional vector space. In such a scenario, in place of (1.11.16) we now demand*

$$ \mathrm{div}_g \vec{F} \in \mathscr{V} \otimes \mathscr{E}'(\Omega) + \mathscr{V} \otimes \mathrm{CBM}(\Omega), \tag{1.11.18} $$

and (1.11.17) becomes the equality of two vectors in $\mathscr{V}$*.*

A special case of Theorem 1.11.3 which is already quite general and practical, corresponding to a locally integrable vector field whose differential geometric divergence (considered in the sense of distributions) is actually a globally integrable function, reads as follows.

Corollary 1.11.5 *Let the Riemannian manifold* (M, g) *be as in (1.11.1). Assume* Ω *is a nonempty, open, proper subset of* M *such that* $\partial\Omega$ *is a lower Ahlfors regular set, and* $\sigma_g := \mathcal{H}_g^{n-1}\lfloor\partial\Omega$ *is a doubling measure on* $\partial\Omega$*. In particular,* Ω *is a set of finite perimeter, hence its geometric measure theoretic outward unit conormal* $\nu_g : \partial_*\Omega \to T^*M$ *is defined* σ_g*-a.e. on* $\partial_*\Omega$*. Denote by* $\mathcal{L}_g^n$ *the measure induced by the volume element on* M*, and fix some* $\kappa \in (0, \infty)$*.*

In this context, assume the vector field $\vec{F} \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}_g^n) \otimes TM$ *has the property that, with the differential geometric divergence taken in the sense of distributions in* Ω*,*

$$ \begin{gathered} \mathrm{div}_g \vec{F} \in L^1(\Omega, \mathcal{L}_g^n), \quad \mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma_g), \quad \textit{and} \\ \vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists (in } TM\textit{) at } \sigma_g\textit{-a.e. point on } \partial_{\mathrm{nta}}\Omega. \end{gathered} \tag{1.11.19} $$

Then, for any other aperture $\kappa' > 0$*, the nontangential trace* $\vec{F}\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ *exists* σ_g*-a.e. on* $\partial_{\mathrm{nta}}\Omega$ *and is actually independent of* κ'*. When regarding it as a function defined* σ_g*-a.e. on* $\partial_*\Omega$ *(which, up to a* σ_g*-nullset, is contained in* $\partial_{\mathrm{nta}}\Omega$*), this nontangential trace belongs to* $L^1(\partial_*\Omega, \sigma_g) \otimes TM$ *and, with the dependence on the parameter* κ' *dropped,*

$$ \int_\Omega \mathrm{div}_g \vec{F}\, d\mathcal{L}_g^n = \int_{\partial_*\Omega} {}_{T^*M}\big(\nu_g, \vec{F}\,\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)_{TM}\, d\sigma_g. \tag{1.11.20} $$

A consequence of Corollary 1.11.5 which is remarkably versatile in practice is contained in Theorem 1.11.6, stated a little later below. For now, we recall that given a first-order differential operator

$$D : \mathscr{C}^1(M, \mathcal{E}) \longrightarrow \mathscr{C}^0(M, \mathcal{F}) \tag{1.11.21}$$

acting between sections of two vector bundles $\mathcal{E}, \mathcal{F} \to M$, its principal symbol $\mathrm{Sym}(D; \xi)$ is defined for each $\xi \in T_x^*M$ (with $x \in M$ arbitrary) as a mapping in $\mathrm{Hom}(\mathcal{E}_x, \mathcal{F}_x)$ whose action on some $u \in \mathcal{E}_x$ is described as

$$\mathrm{Sym}(D; \xi)u := \mathrm{i}D\big(\psi\widetilde{u}\big)(x) \in \mathcal{F}_x. \tag{1.11.22}$$

Above, ψ is a scalar-valued function of class $\mathscr{C}^1$ near the point x with the property that $\psi(x) = 0$ and $(\mathrm{d}\psi)(x) = \xi$, and $\widetilde{u}$ is a smooth section in $\mathcal{E}$ with the property that $\widetilde{u}(x) = u$. See, e.g., [259, p. 115]. For example, if in a local coordinate chart $U \subseteq M$ we may express

$$D = \sum_{|\alpha|\leq 1} A_\alpha(x)\partial^\alpha \tag{1.11.23}$$

for some matrix-valued coefficients A_α then, for each $x \in U$ and each $\xi \in T_x^*M$,

$$\mathrm{Sym}(D; \xi) = \sum_{|\alpha|=1} (\mathrm{i}\xi)^\alpha A_\alpha(x). \tag{1.11.24}$$

In particular, this notion of principal symbol agrees with the one considered in the Euclidean setting in (1.7.16). For further use, let us also observe from (1.11.22) that if D is as in (1.11.21) then for every scalar-valued function ψ of class $\mathscr{C}^1$ on M (identified with the operator of pointwise multiplication by ψ) we have the commutator identity

$$[D, \psi] = (-\mathrm{i})\mathrm{Sym}(D; \mathrm{d}\psi) \tag{1.11.25}$$

where d is the exterior derivative operator on M.

Theorem 1.11.6 *Let the Riemannian manifold (M, g) be as in (1.11.1), and suppose $D : \mathcal{E} \to \mathcal{F}$ is a first-order differential operator, acting between the sections of two Hermitian vector bundles $\mathcal{E}, \mathcal{F} \to M$, whose top coefficients are of class $\mathscr{C}^1$ and the lower order coefficients are continuous.*

Also, let Ω be a nonempty, open, proper subset of M such that $\partial\Omega$ is a lower Ahlfors regular set, and $\sigma_g := \mathcal{H}_g^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of finite perimeter and its (geometric measure theoretic) outward unit conormal $\nu_g : \partial_*\Omega \to T^*M$ is defined σ_g-a.e. on $\partial_*\Omega$. Denote by $\mathcal{L}_g^n$ the measure induced by the volume element $\mathrm{d}V_g$ on M.

In this context, consider an $\mathcal{L}_g^n$-measurable section u of $\mathcal{E}$ on Ω, along with an $\mathcal{L}_g^n$-measurable section w of $\mathcal{F}$ on Ω which, for some $\kappa, \kappa' > 0$, satisfy

$$
\begin{array}{c}
\mathcal{N}_\kappa u < \infty \;\; and \;\; \mathcal{N}_{\kappa'} w < \infty \;\; at\ \sigma_g\text{-a.e. point on } \partial\Omega, \\
\mathcal{N}_\kappa u \cdot \mathcal{N}_{\kappa'} w \;\; belongs\ to\ the\ space \;\; L^1(\partial\Omega, \sigma_g), \\
u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}},\ w\Big|_{\partial\Omega}^{\kappa'-\text{n.t.}} \;\; exist\ at\ \sigma_g\text{-a.e. point on } \partial_{\text{nta}}\Omega, \\
Du \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n_g) \otimes \mathcal{F}, \quad D^\top w \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n_g) \otimes \mathcal{E}, \\
and \;\; \big\langle Du, w\big\rangle_{\mathcal{F}} - \big\langle u, D^\top w\big\rangle_{\mathcal{E}} \;\; belongs\ to \;\; L^1(\Omega, \mathcal{L}^n_g),
\end{array}
\tag{1.11.26}
$$

where all intervening differential operators are taken in the sense of distributions (with $D^\top$ denoting the real transpose of D), and where $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ are the real (i.e., complex bilinear) pointwise pairings in the fibers of $\mathcal{E}$ and $\mathcal{F}$, respectively.

Then, up to a σ-nullset, $\partial_*\Omega$ is contained in $\partial_{\text{nta}}\Omega$ and, with the nontangential traces $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$, $w\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}$ regarded as functions defined σ_g-a.e. on $\partial_*\Omega$, the following integration by parts formula holds:

$$
\begin{aligned}
\int_\Omega \Big\{\langle Du, w\rangle_{\mathcal{F}} - \langle u, D^\top w\rangle_{\mathcal{E}}\Big\}\, d\mathcal{L}^n_g &= \int_{\partial_*\Omega} \Big\langle (-i)\text{Sym}(D;\nu_g)\Big(u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big), w\Big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\Big\rangle_{\mathcal{E}}\, d\sigma_g \\
&= \int_{\partial_*\Omega} \Big\langle u\Big|_{\partial\Omega}^{\text{n.t.}}, i\text{Sym}(D^\top;\nu_g)\Big(w\Big|_{\partial\Omega}^{\text{n.t.}}\Big)\Big\rangle_{\mathcal{F}}\, d\sigma_g,
\end{aligned}
\tag{1.11.27}
$$

where $\text{Sym}(D;\xi)$ *and* $\text{Sym}(D^\top;\xi)$ *denote the principal symbols of D and $D^\top$, respectively, evaluated at a generic covector $\xi \in T^*M$.*

As a corollary, if the integrability condition in the last line of (1.11.26) is strengthened to

$$
\big\langle Du, w\big\rangle_{\mathcal{F}} \in L^1(\Omega, \mathcal{L}^n_g) \;\; and \;\; \big\langle u, D^\top w\big\rangle_{\mathcal{E}} \in L^1(\Omega, \mathcal{L}^n_g)
\tag{1.11.28}
$$

then (1.11.27) may be recast as the integration by parts formula

$$
\begin{aligned}
\int_\Omega \langle Du, w\rangle_{\mathcal{F}}\, d\mathcal{L}^n_g &= \int_\Omega \langle u, D^\top w\rangle_{\mathcal{E}}\, d\mathcal{L}^n_g \\
&\quad + \int_{\partial_*\Omega} \Big\langle (-i)\text{Sym}(D;\nu_g)\Big(u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big), w\Big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\Big\rangle_{\mathcal{E}}\, d\sigma_g.
\end{aligned}
\tag{1.11.29}
$$

For instance, the demands in the first two lines of (1.7.23) are satisfied if

$$
\begin{array}{l}
\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma_g) \;\; \text{and} \;\; \mathcal{N}_{\kappa'} w \in L^{p'}(\partial\Omega, \sigma_g) \\
\text{for some } p, p' \in [1,\infty] \text{ with } 1/p + 1/p' = 1.
\end{array}
\tag{1.11.30}
$$

In such a setting, Proposition 8.9.8 guarantees that

$$
\begin{array}{l}
\text{for each } \kappa'' \in (0,\infty) \text{ the nontangential traces } u\Big|_{\partial\Omega}^{\kappa''-\text{n.t.}},\ w\Big|_{\partial\Omega}^{\kappa''-\text{n.t.}} \\
\text{exist } \sigma_g\text{-a.e. on } \partial_{\text{nta}}\Omega, \text{ and are actually independent of } \kappa''.
\end{array}
\tag{1.11.31}
$$

The proof of Theorem 1.11.6 is given in Sect. 9.9. To offer an example of the many concrete embodiments of Theorem 1.11.6, we need some notation. Given a manifold M of class $\mathscr{C}^1$, denote by d the exterior derivative operator on M. In local coordinates, the action of d on a differential form locally expressed as $u = \sum_J u_J \, dx^J$ is

$$du = \sum_{j=1}^{n} \sum_{J} \frac{\partial u_J}{\partial x_j} dx_j \wedge dx^J, \tag{1.11.32}$$

where n is the dimension of M and "wedge" $\wedge$ denotes the exterior product of differential forms on M. Assuming that M is equipped with a $\mathscr{C}^1$ Riemannian metric tensor g, we also denote by δ the formal adjoint of d on M, and use the symbol $\vee$ for the interior product of differential forms on M. Let us also denote by $\Lambda^\ell TM$ the ℓ-th exterior power of the tangent plane to M (i.e., the vector bundle of differential forms of degree ℓ on M). The Riemannian metric tensor on M canonically induces a Hermitian structure on $\Lambda^\ell TM$ and we let $\langle \cdot, \cdot \rangle_{\Lambda^\ell TM}$ stand for the associated (real) bilinear pointwise pairing on $\Lambda^\ell TM$.

Corollary 1.11.7 *Let M be a connected, compact, boundaryless, oriented manifold of class $\mathscr{C}^1$, of real dimension n, equipped with a $\mathscr{C}^1$ Riemannian metric tensor g. Suppose Ω is a nonempty, open, proper subset of M such that $\partial\Omega$ is lower Ahlfors regular, and $\sigma_g := \mathcal{H}_g^{n-1} \lfloor \partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of finite perimeter and its (geometric measure theoretic) outward unit conormal $\nu_g : \partial_*\Omega \to T^*M \equiv \Lambda^1 TM$ is defined σ_g-a.e. on $\partial_*\Omega$. Denote by $\mathcal{L}_g^n$ the measure induced by the volume element dV_g on M.*

In this context, fix an arbitrary degree $\ell \in \{0, 1, \dots, n\}$ and suppose

$$u : \Omega \to \Lambda^\ell TM \quad \textit{and} \quad w : \Omega \to \Lambda^{\ell+1} TM \tag{1.11.33}$$

are two $\mathcal{L}_g^n$-measurable differential forms which, for some aperture parameters $\kappa, \kappa' \in (0, \infty)$, satisfy

$$\begin{gathered}
\mathcal{N}_\kappa u < \infty \;\; \textit{and} \;\; \mathcal{N}_{\kappa'} w < \infty \;\; \textit{at } \sigma_g\textit{-a.e. point on } \partial\Omega, \\
\mathcal{N}_\kappa u \cdot \mathcal{N}_{\kappa'} w \;\; \textit{belongs to the space} \;\; L^1(\partial\Omega, \sigma_g), \\
\textit{the traces} \;\; u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}, \;\; w\Big|_{\partial\Omega}^{\kappa'-\text{n.t.}} \;\; \textit{exist at } \sigma_g\textit{-a.e. point on } \partial_{\text{nta}}\Omega, \\
du \in L^1_{\text{loc}}(\Omega, \mathcal{L}_g^n) \otimes \Lambda^{\ell+1} TM, \quad \delta w \in L^1_{\text{loc}}(\Omega, \mathcal{L}_g^n) \otimes \Lambda^\ell TM, \\
\textit{and} \;\; \langle du, w \rangle_{\Lambda^{\ell+1} TM} - \langle u, \delta w \rangle_{\Lambda^\ell TM} \;\; \textit{belongs to} \;\; L^1(\Omega, \mathcal{L}_g^n),
\end{gathered} \tag{1.11.34}$$

where the exterior derivative operator d *and its (formal) transpose* δ *are considered in the sense of distributions.*

Then, up to a σ_g-nullset, $\partial_*\Omega$ is contained in $\partial_{\text{nta}}\Omega$ and, when the nontangential traces $u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}$, $w\Big|_{\partial\Omega}^{\kappa'-\text{n.t.}}$ are regarded as functions defined σ_g-a.e. on $\partial_*\Omega$, the following formula holds:

$$\int_{\Omega} \Big\{ \langle \mathrm{d}u, w \rangle_{\Lambda^{\ell+1}TM} - \langle u, \delta w \rangle_{\Lambda^{\ell}TM} \Big\} \, \mathrm{d}\mathcal{L}^n_g = \int_{\partial_*\Omega} \Big\langle \nu_g \wedge u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, w\Big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega} \Big\rangle \, \mathrm{d}\sigma_g$$
$$= \int_{\partial_*\Omega} \Big\langle u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, \nu_g \vee w\Big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega} \Big\rangle \, \mathrm{d}\sigma_g. \tag{1.11.35}$$

In particular, if the integrability condition in the last line of (1.11.34) *is strengthened to*

$$\langle \mathrm{d}u, w \rangle_{\Lambda^{\ell+1}TM} \in L^1(\Omega, \mathcal{L}^n_g) \ \text{ and } \ \langle u, \delta w \rangle_{\Lambda^{\ell}TM} \in L^1(\Omega, \mathcal{L}^n_g) \tag{1.11.36}$$

then (1.11.35) *may be reformulated as the integration by parts formula*

$$\int_{\Omega} \langle \mathrm{d}u, w \rangle_{\Lambda^{\ell+1}TM} \, \mathrm{d}\mathcal{L}^n_g = \int_{\Omega} \langle u, \delta w \rangle_{\Lambda^{\ell}TM} \, \mathrm{d}\mathcal{L}^n_g + \int_{\partial_*\Omega} \Big\langle \nu_g \wedge u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, w\Big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega} \Big\rangle \, \mathrm{d}\sigma_g. \tag{1.11.37}$$

This is implied by Theorem 1.11.6 in the particular case when $\mathcal{E} := \Lambda^{\ell}TM$, $\mathcal{F} := \Lambda^{\ell+1}TM$, and $D := \mathrm{d}$, the exterior derivative operator from (1.11.32). The latter choice entails $D^{\top} = \delta$, and it is well known that the principal symbols of the operators d and δ are, respectively, given by

$$\mathrm{Sym}(\mathrm{d}; \xi)u = \mathrm{i}\xi \wedge u \ \text{ and } \ \mathrm{Sym}(\delta; \xi)u = (-\mathrm{i})\xi \vee u \tag{1.11.38}$$

for each covector $\xi \in T^*M \equiv \Lambda^1 TM$ and each differential form u. Granted these, (1.11.35) follows on account of (1.11.27).

Other examples of first-order differential operators for which Theorem 1.11.6 yields integration by parts formulas of practical interest include

(1) $D := \mathrm{div}_g$, the differential geometric divergence on the Riemannian manifold M, with principal symbol $\mathrm{Sym}(\mathrm{div}_g; \xi)u = \mathrm{i}\xi(u) = \mathrm{i}_{TM}(u, \xi)_{T^*M}$ for any covector $\xi \in T^*M$ and any vector field $u \in TM$;
(2) $D := \nabla$, a connection on a Hermitian vector bundle $\mathcal{E} \to M$ (such as the Levi-Civita connection acting on differential forms), with principal symbol $\mathrm{Sym}(\nabla; \xi)u = \mathrm{i}\xi \otimes u$ for any covector $\xi \in T^*$ and any section $u \in \mathcal{E}$;
(3) $D := \nabla_X$, the covariant derivative associated with a connection ∇ on a Hermitian vector bundle $\mathcal{E} \to M$ and a vector field $X \in TM$, with principal symbol $\mathrm{Sym}(\nabla_X; \xi)u = \mathrm{i}\xi(X)u$ for each covector $\xi \in T^*$ and each section $u \in \mathcal{E}$;
(4) $D := \mathrm{Def}$, the deformation tensor, mapping any field $X \in TM$ into a symmetric tensor fields of type $(0, 2)$ according to

$$(\mathrm{Def}\, X)(Y, Z) := \tfrac{1}{2}\{\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle\}, \quad \forall X, Y, Z \in T^*M, \tag{1.11.39}$$

whose principal symbol is given by $\mathrm{Sym}(\mathrm{Def};\xi)u = \frac{\mathrm{i}}{2}(\xi \otimes u + u \otimes \xi)$ for each covector $\xi \in T^*M$ and each vector field $u \in TM$.

Corollary 1.11.5 can be a valuable tool even in the treatment of results formulated plainly in the Euclidean ambient. An example of this flavor is provided by the theorem below, dealing with an integration by parts formula on a patch of a $\mathscr{C}^1$ surface, whose proof (given in Sect. 9.9) proceeds by applying Corollary 1.11.5 to a suitable manifold and differential geometric vector field.

Theorem 1.11.8 *Fix $n \in \mathbb{N}$ with $n \geq 2$ and suppose $D \subseteq \mathbb{R}^n$ is a bounded $\mathscr{C}^1$ domain (consequently, ∂D may be regarded as an oriented, compact, boundaryless, $\mathscr{C}^1$ Riemannian manifold, equipped with the natural metric inherited from $\mathbb{R}^n$). Denote by $N = (N_1, \dots, N_n)$ the outward unit normal vector to D, and abbreviate $S := \mathcal{H}^{n-1}\lfloor\partial D$. Next, consider a relatively open subset Ω of ∂D with the property that there exists a constant $c \in (0,\infty)$ such that*

$$c\, r^{n-2} \leq \mathcal{H}^{n-2}\big(B(x,r) \cap \partial\Omega\big) \text{ for each } x \in \partial\Omega \text{ and } r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big). \tag{1.11.40}$$

Also, assume $\sigma := \mathcal{H}^{n-2}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of locally finite perimeter in the Riemannian manifold ∂D, hence its geometric measure theoretic outward unit normal $\nu = (\nu_1, \dots, \nu_n)$ is defined σ-a.e. on $\partial_*\Omega$ (the geometric measure theoretic boundary of Ω, defined relative to the Riemannian manifold ∂D as ambient). Going further, fix $M, M' \in \mathbb{N}$ and consider a coefficient tensor $A = \big(a^{\alpha\beta}_{jk}\big)_{\substack{1\leq j,k\leq n\\ 1\leq\alpha\leq M,\, 1\leq\beta\leq M'}}$ with complex entries. Also, with $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ denoting the space of all complex-valued functions which are locally Lipschitz in Ω, assume

$$u = (u_\beta)_{1\leq\beta\leq M'} \in \big[L^1_{1,\mathrm{loc}}(\Omega, S)\big]^M \text{ and } w = (w_\alpha)_{1\leq\alpha\leq M} \in \big[\mathrm{Lip}_{\mathrm{loc}}(\Omega)\big]^{M'} \tag{1.11.41}$$

are vector-valued functions for which

$$\sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a^{\alpha\beta}_{jk}\big(\partial_{\tau_{jk}}u_\beta\big)w_\alpha \text{ belongs to } L^1(\Omega, S) \tag{1.11.42}$$

and

$$\sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a^{\alpha\beta}_{jk}u_\beta\big(\partial_{\tau_{jk}}w_\alpha\big) \text{ belongs to } L^1(\Omega, S), \tag{1.11.43}$$

where $\big\{\partial_{\tau_{jk}}\big\}_{1\leq j,k\leq n}$ are the tangential derivative operators defined as in [185, Chap. 11] *on ∂D. In addition, for some aperture parameter $\kappa \in (0,\infty)$ assume that*

$$\begin{aligned}&\mathcal{N}_\kappa u < \infty \text{ and } \mathcal{N}_\kappa w < \infty, \quad \mathcal{N}_\kappa u \cdot \mathcal{N}_\kappa w \in L^1(\partial\Omega, \sigma),\\ &\text{the traces } u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ w\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exist at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega,\end{aligned} \tag{1.11.44}$$

where the nontangential maximal operators and traces of u, w, as well as the nontangentially accessible boundary of Ω are defined relative to the ambient ∂D (itself regarded as a Riemannian manifold).

Then, up to a σ-nullset, $\partial_*\Omega$ is contained in $\partial_{\text{nta}}\Omega$ and, regarding the nontangential traces $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$, $w\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ as functions defined at σ-a.e. point on $\partial_*\Omega$, one has the following integration by parts formula in Ω:

$$\int_\Omega \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a_{jk}^{\alpha\beta}\big(\partial_{\tau_{jk}}u_\beta\big)w_\alpha \,\mathrm{d}S \tag{1.11.45}$$
$$= -\int_\Omega \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a_{jk}^{\alpha\beta}u_\beta\big(\partial_{\tau_{jk}}w_\alpha\big)\,\mathrm{d}S$$
$$+\int_{\partial_*\Omega} \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a_{jk}^{\alpha\beta}(N_j\nu_k - N_k\nu_j)\big(u_\beta\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\big(w_\alpha\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\,\mathrm{d}\sigma.$$

A particular useful case of (1.11.45) is the formula

$$\int_\Omega \partial_{\tau_{jk}}u \,\mathrm{d}S = \int_{\partial_*\Omega}(N_j\nu_k - N_k\nu_j)\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\,\mathrm{d}\sigma \tag{1.11.46}$$

valid, in the same geometric context as in Theorem 1.11.8, for any $j,k\in\{1,\dots,n\}$ and any function $u\in L^1_1(\Omega,S)$ with the property that there exists $\kappa>0$ such that $\mathcal{N}_\kappa u\in L^1(\partial\Omega,\sigma)$ and $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial_{\text{nta}}\Omega$.

We also wish to remark that, in the three-dimensional setting, (1.11.45) contains as a particular case a sharp version of Stokes' classical formula to the effect that, with $\tau := N\times\nu$ regarded as a positively oriented unit tangent vector to $\partial\Omega$, we have

$$\int_\Omega N\cdot\big(\operatorname{curl}\vec{F}\big)\big|_\Omega \,\mathrm{d}S = \int_{\partial_*\Omega}\tau\cdot\big(\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\,\mathrm{d}\sigma \tag{1.11.47}$$

for any vector field $\vec{F}\in\big[\mathscr{C}^1(\mathcal{O})\big]^3$, where $\mathcal{O}$ is an open neighborhood in $\mathbb{R}^3$ of Ω, with the property that $N\cdot\big(\operatorname{curl}\vec{F}\big)\big|_\Omega\in L^1(\Omega,S)$ and for which one can find some $\kappa>0$ such that $\mathcal{N}_\kappa\vec{F}\in L^1(\partial\Omega,\sigma)$ and $\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exist σ-a.e. on $\partial_{\text{nta}}\Omega$. Moreover, if (P,Q,R) are the scalar components of $\vec{F}$ and if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote the basic unit vectors along the coordinate axes in $\mathbb{R}^3$, we may express the inner product under the integral in the right-hand side of (1.11.47) at σ-a.e. point on $\partial_*\Omega$ as

$$\begin{aligned}\tau \cdot \big(\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) &= \tau \cdot \big(P\big|_{\partial\Omega}^{\kappa-\text{n.t.}}, Q\big|_{\partial\Omega}^{\kappa-\text{n.t.}}, R\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\\ &= \tau \cdot \Big\{\big(P\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\mathbf{i} + \big(Q\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\mathbf{j} + \big(R\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\mathbf{k}\Big\}\\ &= \big(P\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\mathbf{i}\cdot\tau + \big(Q\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\mathbf{j}\cdot\tau + \big(R\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\mathbf{k}\cdot\tau\\ &= \big(P\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\cos\alpha + \big(Q\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\cos\beta + \big(R\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\cos\gamma, \qquad (1.11.48)\end{aligned}$$

where $\alpha, \beta, \gamma \in [0, \pi]$ are the angles made by the unit tangent vector τ with the coordinate axes in $\mathbb{R}^3$. Thus, if we interpret the "differentials" dx, dy, dz as the *Radon measures* on $\partial_*\Omega$ given by

$$dx := \cos\alpha\,\mathcal{H}^1\lfloor\partial_*\Omega, \quad dy := \cos\beta\,\mathcal{H}^1\lfloor\partial_*\Omega, \quad dz := \cos\gamma\,\mathcal{H}^1\lfloor\partial_*\Omega, \qquad (1.11.49)$$

we may recast formula (1.2.34) as[24]

$$\int_{\partial_*\Omega} \big(P\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\,dx + \big(Q\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\,dy + \big(R\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\,dz = \int_{\Omega} N\cdot\big(\operatorname{curl}\vec{F}\big)\big|_{\Omega}\,dS. \qquad (1.11.50)$$

The next goal is to present a version of Stokes' theorem on manifolds in the spirit of our results in Sects. 1.2–1.5. To set the stage, let M be an oriented manifold of class $\mathscr{C}^1$, of real dimension n. Given a domain $\Omega \subseteq M$ of class $\mathscr{C}^1$, the classical Stokes' formula reads

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \iota^*\omega, \qquad \forall\omega \in \mathscr{C}^1_c(M, \Lambda^{n-1}TM), \qquad (1.11.51)$$

where ι^* is the pull-back map induced by the canonical inclusion $\iota : \partial\Omega \hookrightarrow M$ of the $\mathscr{C}^1$ manifold $\partial\Omega$ into M. The goal is to find sharper geometrical and analytical conditions on the set Ω and the differential form ω under which Stokes' formula continues to hold.

A basic obstacle in this regard is that relaxing the smoothness assumptions on Ω to the point where $\partial\Omega$ is no longer a $\mathscr{C}^1$ submanifold of M calls into question the very meaning of the integral in the right-hand side of (1.11.51). This suggests that an alternative, more flexible, and accommodating route should be found for interpreting it. To accomplish this, equip the given manifold M with a continuous Riemannian metric tensor g (which is always possible). Subsequently, let dV_g denote the volume n-form induced by the Riemannian metric g on M, and denote by $\mathcal{H}^{n-1}_g$ the Hausdorff $(n-1)$-dimensional measure associated with the metric g on M. Continuing to assume that $\Omega \subseteq M$ is a domain of class $\mathscr{C}^1$, also denote by dS_g the volume $(n-1)$-form induced by the Riemannian metric g on the submanifold $\partial\Omega$ of M. Finally, let $\nu_g : \partial\Omega \to T^*M \equiv \Lambda^1 TM$ be the (differential geometric) outward unit conormal to Ω. Then, on the one hand, it is well known (cf., e.g., [250, (2.15),

[24] This is very much in line with the classical algebraic format of Stokes' theorem for the vector field $\vec{F} = (P, Q, R)$ and the "surface" Ω.

p. 127]) that for any 1-form a defined in a neighborhood of $\partial\Omega$ one has

$$\iota^*(a \vee \mathrm{d}V_g) = \big\langle a\big|_{\partial\Omega}, \nu_g\big\rangle_{T^*M}\,\mathrm{d}S_g \quad \text{on} \quad \partial\Omega. \tag{1.11.52}$$

On the other hand, if $*$ stands for the Hodge star operator canonically associated with the metric g on M (cf., e.g., [189, 250]), for any $(n-1)$-form ω defined in a subset of M one has

$$\omega = (-1)^{n-1}(*\omega) \vee (*1) = (-1)^{n-1}(*\omega) \vee \mathrm{d}V_g. \tag{1.11.53}$$

Above, the first equality is implied by [189, Lemma 2.2(2), p. 54], while the second equality is a consequence of [189, (2.1.17), p. 53]. From (1.11.52)–(1.11.53) we then conclude that

$$\begin{array}{l}\text{for every } (n-1)\text{-form } \omega \text{ defined in a neighborhood of } \partial\Omega\\ \text{on } M \text{ one has } \ \iota^*\omega = (-1)^{n-1}\big\langle(*\omega)\big|_{\partial\Omega}, \nu_g\big\rangle_{T^*M}\,\mathrm{d}S_g \quad \text{on} \quad \partial\Omega,\end{array} \tag{1.11.54}$$

where $\langle\cdot,\cdot\rangle_{T^*M}$ is the inner product in $T^*M \equiv \Lambda^1 TM$.

We wish to emphasize that the volume $(n-1)$-form $\mathrm{d}S_g$ on $\partial\Omega$ has no clear meaning if $\partial\Omega$ does not carry a manifold structure, hence $\iota^*\omega$ is lacking a clear interpretation in the latter case. This being said, with the help of [125, Proposition 5.7, p. 2774] one may check that

$$\begin{array}{l}\text{the measure canonically induced by the volume } (n-1)\text{-form } \mathrm{d}S_g\\ \text{on the submanifold } \partial\Omega \text{ is given by the formula } \sigma_g := \mathcal{H}_g^{n-1}\lfloor\partial\Omega.\end{array} \tag{1.11.55}$$

In view of (1.11.54)–(1.11.55), it follows that if Ω is, as before, a domain of class $\mathscr{C}^1$ in the manifold M, then the classical Stokes' formula (1.11.51) may be equivalently recast as

$$\begin{array}{c}\displaystyle\int_\Omega \mathrm{d}\omega = (-1)^{n-1}\int_{\partial\Omega}\big\langle(*\omega)\big|_{\partial\Omega}, \nu_g\big\rangle_{T^*M}\,\mathrm{d}\sigma_g,\\ \text{for each } (n-1)\text{-form } \ \omega \in \mathscr{C}_c^1(M, \Lambda^{n-1}TM),\end{array} \tag{1.11.56}$$

where g is a background continuous Riemannian metric tensor on M. Simply put,

$$\begin{array}{l}\text{passing from (1.11.51) to (1.11.56) is made possible by identifying,}\\ \text{for each fixed differential form } \omega \in \mathscr{C}_c^1(M, \Lambda^{n-1}TM), \text{ the } (n-1)\text{-}\\ \textit{form } \iota^*\omega \text{ on the } (n-1)\text{-dimensional } \mathscr{C}^1 \text{ manifold } \partial\Omega \text{ with the } \textit{Radon}\\ \textit{measure} \text{ given by } (-1)^{n-1}\big\langle(*\omega)\big|_{\partial\Omega}, \nu_g\big\rangle_{T^*M}\,\sigma_g \text{ on the set } \partial\Omega.\end{array} \tag{1.11.57}$$

The real upshot of reformulating Stokes' classical formula as in (1.11.56) is that in the latter scenario having a meaningfully defined right-hand side does not require $\partial\Omega$ to necessarily be a differentiable manifold. In stark contrast with (1.11.51), this actually opens the door for considering Stokes' formula on sets Ω which merely possess a reasonable notion of outward unit conormal. Of course, the class of sets which most naturally comes to mind is that of sets of locally finite perimeter. The veracity of the version of Stokes' formula recorded in (1.11.56) in such a setting is confirmed by the theorem below.

Theorem 1.11.9 *Let M be an oriented manifold of class $\mathscr{C}^1$, of real dimension n, and suppose $\Omega \subseteq M$ is a set of locally finite perimeter.*

In this setting, define a pull-back operator, mapping $(n-1)$-forms with continuous coefficients on M into Radon measures on the set $\partial_*\Omega$, of the following sort. As a preliminary step, bring in a continuous Riemannian metric tensor[25] *g on M, and consider the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}_g^{n-1}$ associated with the metric g on M. Also, denote by $\nu_g : \partial_*\Omega \to T^*M$ the geometric measure theoretic outward unit conormal to Ω, which is defined almost everywhere on $\partial_*\Omega$ with respect to the measure $\sigma_g := \mathcal{H}_g^{n-1} \lfloor \partial\Omega$. Finally, let $\langle \cdot, \cdot \rangle_{T^*M}$ be the inner product canonically induced by the metric g on $T^*M \equiv \Lambda^1 TM$, let $\mathrm{d}V_g$ be the volume n-form induced by the Riemannian metric g on M, and let $*$ stand for the Hodge star operator canonically associated with the metric g on M. With these in place, proceed to define the sharp pull-back to the set $\partial_*\Omega$ of any given $(n-1)$-form $\omega \in \mathscr{C}^0(M, \Lambda^{n-1}TM)$ as the Radon measure*

$$\iota_\#^* \omega := (-1)^{n-1} \big\langle (*\omega)\big|_{\partial_*\Omega}, \nu_g \big\rangle_{T^*M} \sigma_g \quad \text{on} \quad \partial_*\Omega. \tag{1.11.58}$$

Then the definition of the sharp pull-back in (1.11.58) is actually independent of the background Riemannian metric g, and the sharp pull-back of any $(n-1)$-form $\omega \in \mathscr{C}^0(M, \Lambda^{n-1}TM)$ may be alternatively expressed as

$$\begin{aligned} \iota_\#^* \omega &= \big\langle \omega\big|_{\partial_*\Omega}, *\nu_g \big\rangle_{\Lambda^{n-1}TM} \sigma_g = \big\langle \omega\big|_{\partial_*\Omega}, \nu_g \vee \mathrm{d}V_g \big\rangle_{\Lambda^{n-1}TM} \sigma_g \\ &= \big\langle \nu_g \wedge \big(\omega\big|_{\partial_*\Omega}\big), \mathrm{d}V_g \big\rangle_{\Lambda^n TM} \sigma_g = *\Big(\nu_g \wedge \big(\omega\big|_{\partial_*\Omega}\big)\Big)\sigma_g \\ &= (-1)^{n-1} \nu_g \vee \big(*\,\omega\big|_{\partial_*\Omega}\big)\sigma_g \quad \text{on} \quad \partial_*\Omega. \end{aligned} \tag{1.11.59}$$

Moreover, for each $(n-1)$-form $\omega \in \mathscr{C}_c^1(M, \Lambda^{n-1}TM)$ one has

$$\int_\Omega \mathrm{d}\omega = \int_{\partial_*\Omega} \iota_\#^* \omega. \tag{1.11.60}$$

Remark 1. From (1.11.54) to (1.11.56) and (1.11.58) it follows that (1.11.60) reduces precisely to the classical Stokes formula (1.11.51) in the case when Ω is actually a domain of class $\mathscr{C}^1$ in M.

[25] This may be constructed by locally transferring to M the Euclidean metric from $\mathbb{R}^n$, then fashioning a global Riemannian metric on M by gluing these local metrics using a suitable partition of unity.

Remark 2. In terms of the volume element $\mathrm{d}V_g$, the integrand in the left-hand side of (1.11.60) may be expressed (cf. [189, Lemma 2.2(1), p. 63] and [189, (2.1.29), p. 54]) as

$$\mathrm{d}\omega = \langle \mathrm{d}\omega, \mathrm{d}V_g \rangle_{\Lambda^n TM} \, \mathrm{d}V_g. \tag{1.11.61}$$

Remark 3. Fix $\ell \in \{0, 1, \dots, n\}$ and consider an ℓ-form $u \in \mathscr{C}_c^1(M, \Lambda^\ell TM)$ along with some $(\ell+1)$-form $w \in \mathscr{C}_c^1(M, \Lambda^{\ell+1} TM)$. Then Stokes' formula (1.11.60) written for the form $\omega := u \wedge (*w)$ becomes equivalent with the integration by parts formula for the exterior derivative operator (compare with Corollary 1.11.7)

$$\int_\Omega \langle \mathrm{d}u, w \rangle_{\Lambda^{\ell+1} TM} \, \mathrm{d}\mathcal{L}_g^n = \int_\Omega \langle u, \delta w \rangle_{\Lambda^\ell TM} \, \mathrm{d}\mathcal{L}_g^n + \int_{\partial_* \Omega} \langle \nu_g \wedge u\big|_{\partial_*\Omega}, w\big|_{\partial_*\Omega} \rangle \, \mathrm{d}\sigma_g. \tag{1.11.62}$$

Indeed, from [189, (2.1.27), p. 53], [189, Lemma 2.8(2), p. 63] we know that

$$\mathrm{d}u \wedge (*w) = \langle \mathrm{d}u, w \rangle_{\Lambda^{\ell+1} TM} \, \mathrm{d}V_g \tag{1.11.63}$$

and

$$(-1)^\ell u \wedge \mathrm{d}(*w) = -u \wedge *(\delta w) = -\langle u, \delta w \rangle_{\Lambda^\ell TM} \, \mathrm{d}V_g, \tag{1.11.64}$$

hence

$$\mathrm{d}\omega = \langle \mathrm{d}u, w \rangle_{\Lambda^{\ell+1} TM} \, \mathrm{d}V_g - \langle u, \delta w \rangle_{\Lambda^\ell TM} \, \mathrm{d}V_g. \tag{1.11.65}$$

Also, from the third equality in (1.11.59) together with [189, (2.1.29), p. 54] and [189, Lemma 2.2, p. 54] we see that

$$\begin{aligned}
\iota_\#^* \omega &= \langle \nu_g \wedge \big((u\big|_{\partial_*\Omega}) \wedge *(w\big|_{\partial_*\Omega}) \big), \mathrm{d}V_g \rangle_{\Lambda^n TM} \, \sigma_g \\
&= (-1)^{(\ell+1)(n-\ell-1)} \langle (* w\big|_{\partial_*\Omega}) \wedge (\nu_g \wedge u\big|_{\partial_*\Omega}), \mathrm{d}V_g \rangle_{\Lambda^n TM} \, \sigma_g \\
&= (-1)^{(\ell+1)(n-\ell-1)} \langle \nu_g \wedge u\big|_{\partial_*\Omega}, (* w\big|_{\partial_*\Omega}) \vee \mathrm{d}V_g \rangle_{\Lambda^{\ell+1} TM} \, \sigma_g \\
&= (-1)^{(\ell+1)(n-\ell-1)} \langle \nu_g \wedge u\big|_{\partial_*\Omega}, * * w\big|_{\partial_*\Omega} \rangle_{\Lambda^{\ell+1} TM} \, \sigma_g \\
&= \langle \nu_g \wedge u\big|_{\partial_*\Omega}, w\big|_{\partial_*\Omega} \rangle_{\Lambda^{\ell+1} TM} \, \sigma_g.
\end{aligned} \tag{1.11.66}$$

Then (1.11.62) is implied by (1.11.60), on account of (1.11.65) and (1.11.66).

The proof of Theorem 1.11.9 is contained in Sect. 9.9.

We complement the result presented in Theorem 1.11.9 with a version of Stokes' theorem in which the boundary integral involves what we call the nontangential pull-back to $\partial_*\Omega$, defined in a suitable sense, as a Radon measure.

Theorem 1.11.10 *Assume the Riemannian manifold (M, g) is as in (1.11.1). Let Ω be a nonempty, open, proper subset of M such that $\partial\Omega$ is a lower Ahlfors regular set, and $\sigma_g := \mathcal{H}_g^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. In particular, Ω is a set of finite perimeter and its (geometric measure theoretic) outward unit conormal $\nu_g : \partial_*\Omega \to T^*M$ is defined σ_g-a.e. on $\partial_*\Omega$. Denote by $\mathcal{L}_g^n$ the measure induced by the volume element $\mathrm{d}V_g$ on M, and let $*$ stand for the Hodge star operator canonically associated with the metric g on M. In this context, fix $\kappa \in (0, \infty)$ and consider an $(n-1)$-form $\omega \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}_g^n) \otimes \Lambda^{n-1}TM$ satisfying the following properties:*

$$\begin{aligned} &\textit{the nontangential trace } \ \omega\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \textit{exists } \sigma_g\textit{-a.e. on } \ \partial_{\mathrm{nta}}\Omega, \\ &\mathcal{N}_\kappa\omega \in L^1(\partial\Omega, \sigma_g), \ \ \textit{and } \ \mathrm{d}\omega \in L^1(\Omega, \mathcal{L}_g^n) \otimes \Lambda^n TM, \end{aligned} \tag{1.11.67}$$

where the action of the exterior derivative operator is considered in the sense of distributions in Ω.

Then for any $\kappa' > 0$ the nontangential trace $\omega\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ exists σ_g-a.e. on $\partial_{\mathrm{nta}}\Omega$ and is actually independent of κ'. Moreover, if one defines (with the dependence on the parameter κ' dropped) the nontangential pull-back of ω to $\partial_*\Omega$ as the Radon measure

$$\iota^*_{\mathrm{n.t.}}\,\omega := (-1)^{n-1}\big\langle (*\omega)\big|_{\partial\Omega}^{\mathrm{n.t.}}, \nu_g\big\rangle_{T^*M}\sigma_g \ \ \textit{on} \ \ \partial_*\Omega, \tag{1.11.68}$$

then this is actually independent of the Riemannian metric tensor g, it may be alternatively expressed as

$$\begin{aligned} \iota^*_{\mathrm{n.t.}}\,\omega &= \big\langle \omega\big|_{\partial\Omega}^{\mathrm{n.t.}}, *\nu_g\big\rangle_{\Lambda^{n-1}TM}\sigma_g = \big\langle \omega\big|_{\partial\Omega}^{\mathrm{n.t.}}, \nu_g \vee \mathrm{d}V_g\big\rangle_{\Lambda^{n-1}TM}\sigma_g \\ &= \big\langle \nu_g \wedge \big(\omega\big|_{\partial\Omega}^{\mathrm{n.t.}}\big), \mathrm{d}V_g\big\rangle_{\Lambda^n TM}\sigma_g = *\Big(\nu_g \wedge \big(\omega\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)\Big)\sigma_g \\ &= (-1)^{n-1}\nu_g \vee \big(*\,\omega\big|_{\partial\Omega}^{\mathrm{n.t.}}\big)\sigma_g \ \ \textit{on} \ \ \partial_*\Omega, \end{aligned} \tag{1.11.69}$$

and one has

$$\int_\Omega \mathrm{d}\omega = \int_{\partial_*\Omega} \iota^*_{\mathrm{n.t.}}\,\omega. \tag{1.11.70}$$

The proof of Theorem 1.11.10 is presented in Sect. 9.9.

While Theorem 1.11.10 establishes Stokes' formula in its traditional format under rather general assumptions on the domain Ω and the differential form ω, its proof relies on a particular case of Corollary 1.11.5, applied to a vector field that suitably fashioned out the given $(n-1)$-form ω. As such, it is significant to note that by making use of the full force of Theorem 1.11.3 one may further extend the scope of Theorem 1.11.10 by allowing the $(n-1)$-form ω to actually be singular on a compact subset of Ω, a scenario in which the left-hand side of (1.11.70) should be suitably interpreted. Specifically, we have the following theorem.

Theorem 1.11.11 *Retain the geometric hypotheses made on the Riemannian manifold (M, g) and the set $\Omega \subseteq M$ made in Theorem 1.11.10, and fix some $\kappa > 0$. In this setting, assume the $(n-1)$-form $\omega \in \mathcal{D}'(\Omega) \otimes \Lambda^{n-1}TM$ has the property that*

$$\begin{gathered}\textit{there exists a compact set } K \subset \Omega \textit{ such that}\\ \omega\big|_{\Omega\setminus K} \in L^1_{\rm loc}(\Omega \setminus K, \mathcal{L}^n_g) \otimes \Lambda^{n-1}TM\\ \textit{and } \mathcal{N}_\kappa^{\Omega\setminus K}\big(\omega\big|_{\Omega\setminus K}\big) \in L^1(\partial\Omega, \sigma_g),\end{gathered} \tag{1.11.71}$$

the pointwise nontangential boundary trace

$$\omega\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \quad \textit{exists at } \sigma_g\textit{-a.e. point on } \partial_{\rm nta}\Omega, \tag{1.11.72}$$

and

$$* \,{\rm d}\omega \in \mathscr{E}'(\Omega) + {\rm CBM}(\Omega). \tag{1.11.73}$$

Then for any other aperture parameter $\kappa' > 0$ the nontangential trace $\omega\Big|_{\partial\Omega}^{\kappa'-{\rm n.t.}}$ exists σ_g-a.e. on $\partial_{\rm nta}\Omega$ and is actually independent of κ'. Moreover, with the nontangential pull-back of ω to $\partial_*\Omega$ defined (*with the dependence on the parameter κ' dropped*) *as the Radon measure* (1.11.68), *one has*

$$_{(\mathscr{C}^1_b(\Omega))^*}\big(* \,{\rm d}\omega, 1\big)_{\mathscr{C}^1_b(\Omega)} = \int_{\partial_*\Omega} \iota^*_{\rm n.t.}\, \omega. \tag{1.11.74}$$

For a proof of Theorem 1.11.11, see Sect. 9.9.

1.12 Integrating by Parts on Boundaries of Ahlfors Regular Domains on Manifolds

The main aim here is to produce a versatile integration by parts formula on the boundary of an Ahlfors regular subdomains on a Riemannian manifold, involving general first-order (tangential) differential operators. This is accomplished later, in (1.12.91). We begin building in this direction by first proving the following key result.

Theorem 1.12.1 *Let M be a $\mathscr{C}^2$ manifold of dimension n, equipped with a Riemannian metric g, and denote by $\mathcal{L}^n_g$ the Lebesgue measure induced by the volume element on M. Next, consider three Hermitian vector bundles, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$, over M, of class $\mathscr{C}^2$. Assume all metrics involved are of class $\mathscr{C}^1$. Going further, suppose*

$$P : \mathscr{C}^1(M, \mathcal{E}) \longrightarrow \mathscr{C}^0(M, \mathcal{H}) \;\; \textit{and} \;\; Q : \mathscr{C}^1(M, \mathcal{H}) \longrightarrow \mathscr{C}^0(M, \mathcal{F}) \tag{1.12.1}$$

are two first-order differential operators with $\mathscr{C}^1$ coefficients for the top part and $\mathscr{C}^0$ coefficients for the zero-th order part, having the property that their principal symbols satisfy the cancelation condition

$$\mathrm{Sym}(Q;\xi)\mathrm{Sym}(P;\xi) = 0, \qquad \forall \xi \in T^*M. \tag{1.12.2}$$

Denote by $P^\top : \mathscr{C}^1(M, \mathcal{H}) \to \mathscr{C}^0(M, \mathcal{E})$ and $Q^\top : \mathscr{C}^1(M, \mathcal{F}) \to \mathscr{C}^0(M, \mathcal{H})$ the (real) transpose of P, Q (considered in the usual sense on the manifold M).

Next, let Ω be a relatively compact open subset of M, with a lower Ahlfors regular boundary, such that $\sigma_g := \mathcal{H}_g^{n-1}\lfloor\partial\Omega$ is a doubling measure. Denote by ν_g the geometric measure theoretic outward unit conormal to Ω and fix an aperture parameter $\kappa > 0$. Finally, pick two Lebesgue measurable sections, $u : \Omega \to \mathcal{E}$ and $w : \Omega \to \mathcal{F}$, satisfying

$$\begin{gathered} Pu \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n_g) \otimes \mathcal{H}, \quad \mathcal{N}_\kappa(Pu),\ \mathcal{N}_\kappa u < \infty \ \textit{ at } \sigma_g\textit{-a.e. point on } \ \partial\Omega, \\ \mathcal{N}_\kappa(Pu)\cdot\mathcal{N}_\kappa u \ \textit{ belongs to the space } \ L^1(\partial\Omega, \sigma_g), \\ \textit{the nontangential traces } u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ (Pu)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \textit{exist } \sigma_g\textit{-a.e. on } \partial_{\mathrm{nta}}\Omega, \end{gathered} \tag{1.12.3}$$

and

$$\begin{gathered} Q^\top w \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n_g) \otimes \mathcal{H}, \quad \mathcal{N}_\kappa(Q^\top w),\ \mathcal{N}_\kappa w < \infty \ \textit{ at } \sigma_g\textit{-a.e. point on } \ \partial\Omega, \\ \mathcal{N}_\kappa(Q^\top w)\cdot\mathcal{N}_\kappa w \ \textit{ belongs to the space } \ L^1(\partial\Omega, \sigma_g), \\ \textit{the nontangential traces } w\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ (Q^\top w)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \textit{exist } \sigma_g\textit{-a.e. on } \partial_{\mathrm{nta}}\Omega. \end{gathered} \tag{1.12.4}$$

Then

$$\begin{gathered} QP : \mathscr{C}^1(M, \mathcal{E}) \longrightarrow \mathscr{C}^0(M, \mathcal{F}) \ \textit{ and } \\ P^\top Q^\top : \mathscr{C}^1(M, \mathcal{F}) \longrightarrow \mathscr{C}^0(M, \mathcal{E}) \\ \textit{are first-order differential operators}, \end{gathered} \tag{1.12.5}$$

and, with all principal symbols taken in the sense of first-order differential operators,

$$\begin{aligned} &\int_{\partial_*\Omega} \Big\langle i\mathrm{Sym}(Q;\nu_g)(Pu)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\, w\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{F}}\, d\sigma_g \\ &= \int_{\partial_*\Omega} \Big\langle u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\, i\mathrm{Sym}\big(P^\top;\nu_g\big)(Q^\top w)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{E}}\, d\sigma_g \\ &\quad + \int_{\partial_*\Omega} \Big\langle i\mathrm{Sym}\big(QP;\nu_g\big)u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\, w\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{F}}\, d\sigma_g \\ &= \int_{\partial_*\Omega} \Big\langle u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\, i\mathrm{Sym}\big(P^\top;\nu_g\big)(Q^\top w)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - i\mathrm{Sym}\big(P^\top Q^\top;\nu_g\big)w\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{E}}\, d\sigma_g. \end{aligned} \tag{1.12.6}$$

Moreover, if the original assumptions on the sections u, w are strengthened to $u \in \mathscr{C}^1(M, \mathcal{E})$ and $w \in \mathscr{C}^1_c(M, \mathcal{F})$, then formula (1.12.6) is valid provided Ω is merely a set of locally finite perimeter in M.

Proof First, (1.12.5) is a direct consequence of (1.12.2). Second, from Lemma 8.3.1 and (1.12.3)–(1.12.4) we obtain

$$\begin{aligned} u \in L^\infty_{\rm loc}(\Omega, \mathcal{L}^n_g) \otimes \mathcal{E} \ \text{ and } \ Pu \in L^\infty_{\rm loc}(\Omega, \mathcal{L}^n_g) \otimes \mathcal{H}, \\ w \in L^\infty_{\rm loc}(\Omega, \mathcal{L}^n_g) \otimes \mathcal{F} \ \text{ and } \ Q^\top w \in L^\infty_{\rm loc}(\Omega, \mathcal{L}^n_g) \otimes \mathcal{H}. \end{aligned} \tag{1.12.7}$$

Consider now the vector field $\vec{F} : \Omega \to TM$ defined by asking that

$$\begin{aligned} {}_{T^*M}\big(\xi, \vec{F}\big)_{TM} = \Big\langle \mathrm{iSym}(Q; \xi)(Pu), w \Big\rangle_{\mathcal{F}} - \Big\langle u, \mathrm{iSym}(P^\top; \xi)(Q^\top w)\Big\rangle_{\mathcal{E}} \\ - \Big\langle \mathrm{iSym}(QP; \xi)u, w \Big\rangle_{\mathcal{F}} \end{aligned} \tag{1.12.8}$$

for $\mathcal{L}^n_g$-a.e. point in Ω, and each covector ξ in the corresponding fiber in T^*M. The linearity of the right-hand side in ξ ensures that this is a well-defined object. Also, from (1.12.7) we conclude that $\vec{F} \in L^1_{\rm loc}(\Omega, \mathcal{L}^n_g) \otimes TM$. We now claim that

$$\mathrm{div}_g \vec{F} = 0 \ \text{ in } \ \mathscr{D}'(\Omega). \tag{1.12.9}$$

To justify this claim, fix a scalar-valued function $\psi \in \mathscr{C}^2_c(\Omega)$ and compute the distributional pairing

$$\begin{aligned} {}_{\mathscr{D}'(\Omega)}\big(\mathrm{div}_g \vec{F}, \psi\big)_{\mathscr{D}(\Omega)} &= -{}_{\mathscr{D}'(\Omega)}\big(\vec{F}, \mathrm{grad}\, \psi\big)_{\mathscr{D}(\Omega)} \\ &= -\int_\Omega {}_{TM}\langle \mathrm{grad}\, \psi, \vec{F}\rangle_{TM} \, d\mathcal{L}^n_g \\ &= -\int_\Omega {}_{T^*M}\big(d\psi, \vec{F}\big)_{TM} \, d\mathcal{L}^n_g \\ &= -\int_\Omega \Big\langle \mathrm{iSym}\big(Q; d\psi\big) Pu, w \Big\rangle_{\mathcal{F}} \, d\mathcal{L}^n_g \\ &\quad + \int_\Omega \Big\langle u, \mathrm{iSym}\big(P^\top; d\psi\big) Q^\top w \Big\rangle_{\mathcal{E}} \, d\mathcal{L}^n_g \\ &\quad + \int_\Omega \Big\langle \mathrm{iSym}\big(QP; d\psi\big) u, w \Big\rangle_{\mathcal{F}} \, d\mathcal{L}^n_g \\ &= I + II + III, \end{aligned} \tag{1.12.10}$$

where, in view of (1.12.5) and (1.11.25), we may take

$$
\begin{aligned}
I &:= \int_\Omega \Big\langle \big[Q, \psi\big] Pu, w \Big\rangle_{\mathcal{F}} \, \mathrm{d}\mathcal{L}_g^n, \\
II &:= - \int_\Omega \Big\langle u, \big[P^\top, \psi\big] Q^\top w \Big\rangle_{\mathcal{E}} \, \mathrm{d}\mathcal{L}_g^n, \\
III &:= - \int_\Omega \Big\langle \big[QP, \psi\big] u, w \Big\rangle_{\mathcal{F}} \, \mathrm{d}\mathcal{L}_g^n.
\end{aligned}
\tag{1.12.11}
$$

Thanks to (1.12.7), we may employ a local mollifier procedure to produce a sequence

$$
\{w_j\}_{j\in\mathbb{N}} \subseteq \mathscr{C}^2(\Omega, \mathcal{F}) \tag{1.12.12}
$$

such that

$$
\begin{aligned}
& w_j \to w \ \text{ and } \ Q^\top w_j \to Q^\top w \text{ pointwise } \mathcal{L}_g^n\text{-a.e. in } \Omega \text{ as } \ j \to \infty, \\
& \sup_{j\in\mathbb{N}} \sup_{\operatorname{supp}\psi} |w_j|_{\mathcal{F}} < \infty \ \text{ and } \ \sup_{j\in\mathbb{N}} \sup_{\operatorname{supp}\psi} |Q^\top w_j|_{\mathcal{F}} < \infty.
\end{aligned}
\tag{1.12.13}
$$

In particular, if for each $j \in \mathbb{N}$ we denote by I_j, II_j, III_j the versions of I, II, III with w replaced by w_j, then (1.12.7) and (1.12.13) ensure that

$$
I_j \to I, \quad II_j \to II, \quad \text{and} \quad III_j \to III, \quad \text{as} \quad j \to \infty. \tag{1.12.14}
$$

For each fixed $j \in \mathbb{N}$, we may integrate by parts (without boundary terms, since ψ is compactly supported in Ω) to obtain

$$
\begin{aligned}
I_j &= \int_\Omega \Big\langle u, P^\top \big[Q, \psi\big]^\top w_j \Big\rangle_{\mathcal{E}} \, \mathrm{d}\mathcal{L}_g^n \\
&= - \int_\Omega \Big\langle u, P^\top \big[Q^\top, \psi\big] w_j \Big\rangle_{\mathcal{E}} \, \mathrm{d}\mathcal{L}_g^n,
\end{aligned}
\tag{1.12.15}
$$

where the last equality is a consequence of the fact that, in general,

$$
[A, B]^\top = -\big[A^\top, B^\top\big]. \tag{1.12.16}
$$

Another application of (1.12.16) gives

$$
III_j = \int_\Omega \Big\langle u, \big[P^\top Q^\top, \psi\big] w_j \Big\rangle_{\mathcal{E}} \, \mathrm{d}\mathcal{L}_g^n. \tag{1.12.17}
$$

For each $j \in \mathbb{N}$ we also have

$$
-P^\top \big[Q^\top, \psi\big] w_j - \big[P^\top, \psi\big] Q^\top w_j + \big[P^\top Q^\top, \psi\big] w_j = 0 \ \text{ in } \ \Omega. \tag{1.12.18}
$$

This may be verified by writing out all commutators and canceling like-terms, keeping in mind that $w_j \in \mathscr{C}^2(\Omega, \mathcal{F})$. From (1.12.10)–(1.12.18) we may then conclude that (1.12.9) holds.

Moving on, (1.12.8) and the original assumptions on u, w imply that the nontangential trace $\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists σ_g-a.e. on $\partial_{\mathrm{nta}}\Omega$ and, moreover,

$$\begin{aligned} {}_{T^*M}\Big(\nu_g, \vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)_{TM} &= \Big\langle \mathrm{iSym}(Q;\nu_g)(Pu)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, w\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\rangle_{\mathcal{F}} \\ &\quad - \Big\langle u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \mathrm{iSym}\big(P^\top;\nu_g\big)(Q^\top w)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\rangle_{\mathcal{E}} \\ &\quad - \Big\langle \mathrm{iSym}\big(QP;\nu_g\big)u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, w\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\rangle_{\mathcal{F}}. \end{aligned} \tag{1.12.19}$$

In addition, (1.12.8) also gives that

$$\mathcal{N}_\kappa \vec{F} \leq C\Big\{\mathcal{N}_\kappa u \cdot \mathcal{N}_\kappa w + \mathcal{N}_\kappa(Pu)\cdot\mathcal{N}_\kappa w + \mathcal{N}_\kappa u \cdot \mathcal{N}_\kappa(Q^\top w)\Big\} \ \text{ on } \ \partial\Omega. \tag{1.12.20}$$

Granted the original assumptions on u, w, this shows that $\mathcal{N}_\kappa \vec{F}$ belongs to $L^1(\partial\Omega, \sigma_g)$. With this in hand, the first equality in formula (1.12.6) now follows from Corollary 1.11.5, bearing in mind (1.12.19) and (1.12.9). The second equality in formula (1.12.6) is a consequence of what we have just proved and (1.7.17).

Lastly, the very last claim in the statement of the theorem (pertaining to the validity of (1.12.6) when $u \in \mathscr{C}^1(M, \mathcal{E})$, $w \in \mathscr{C}^1_c(M, \mathcal{F})$, and $\Omega \subseteq M$ is a set of locally finite perimeter) is proved in a similar fashion, the main difference being that we now make use of Theorem 1.11.1 in place of Corollary 1.11.5. □

We next discuss an integral identity which may be regarded as a far-reaching generalization of the classical Stokes' formula in the three-dimensional setting (see the comments following the statement of Theorem 1.12.2 in this regard).

Theorem 1.12.2 *Let $\widetilde{M}$ be a $\mathscr{C}^2$ manifold of dimension $n+1$, equipped with a Riemannian metric g, and consider an open, oriented, $\mathscr{C}^1$ submanifold M of $\widetilde{M}$, of dimension n. Denote by $\mathcal{L}^n_g$ the Lebesgue measure induced by the intrinsic volume element on M (regarded as a Riemannian manifold in its own right), and also let $N_g : M \to T^*\widetilde{M}$ stand for the unit conormal to M.*

Next, consider three Hermitian vector bundles, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$, over $\widetilde{M}$, of class $\mathscr{C}^2$. Assume all metrics involved are of class $\mathscr{C}^1$. Going further, suppose

$$P : \mathscr{C}^1(\widetilde{M}, \mathcal{E}) \longrightarrow \mathscr{C}^0(\widetilde{M}, \mathcal{H}) \ \text{ and } \ Q : \mathscr{C}^1(\widetilde{M}, \mathcal{H}) \longrightarrow \mathscr{C}^0(\widetilde{M}, \mathcal{F}) \tag{1.12.21}$$

are two first-order differential operators with $\mathscr{C}^1$ coefficients for the top part and $\mathscr{C}^0$ coefficients for the zero-th order part, having the property that their principal symbols satisfy the cancelation condition

$$\mathrm{Sym}(Q;\widetilde{\xi})\,\mathrm{Sym}(P;\widetilde{\xi})=0,\qquad \forall\widetilde{\xi}\in T^*\widetilde{M}. \tag{1.12.22}$$

Denote by $P^\top:\mathscr{C}^1(\widetilde{M},\mathcal{H})\to\mathscr{C}^0(\widetilde{M},\mathcal{E})$ *and* $Q^\top:\mathscr{C}^1(\widetilde{M},\mathcal{F})\to\mathscr{C}^0(\widetilde{M},\mathcal{H})$ *the* (*real*) *transpose of* P, Q, *considered in the usual sense on the manifold* $\widetilde{M}$.

Next, let Ω *be a relatively compact open subset of* M, *with a lower Ahlfors regular boundary, such that* $\sigma_g:=\mathcal{H}_g^{n-1}\lfloor\partial\Omega$ *is a doubling measure* (*all relative to the manifold* M, *as ambient*). *Denote by* ν_g *the geometric measure theoretic outward unit conormal to* Ω *and fix an aperture parameter* $\kappa>0$. *Finally, consider an open neighborhood* $\mathcal{O}$ *of* Ω *in* $\widetilde{M}$ *and pick two sections,* $u\in\mathscr{C}^1(\mathcal{O},\mathcal{E})$ *and* $w\in\mathscr{C}^1(\mathcal{O},\mathcal{F})$, *satisfying* (*with the nontangential maximal operator and nontangential traces taken from within* Ω)

$$\begin{gathered}\mathcal{N}_\kappa\big(u\big|_\Omega\big)<\infty\ \text{ and }\ \mathcal{N}_\kappa\big(w\big|_\Omega\big)<\infty\ \text{ at }\sigma_g\text{-a.e. point on }\ \partial\Omega,\\ \mathcal{N}_\kappa\big(u\big|_\Omega\big)\cdot\mathcal{N}_\kappa\big(w\big|_\Omega\big)\ \text{ belongs to the space }\ L^1(\partial\Omega,\sigma_g),\\ (u|_\Omega)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\ \text{and}\ (w|_\Omega)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\ \text{exist at }\sigma_g\text{-a.e. point on }\partial_{\mathrm{nta}}\Omega,\end{gathered} \tag{1.12.23}$$

as well as

$$|Pu||w|+|u||Q^\top w|+|u||w|\in L^1(\Omega,\mathcal{L}_g^n). \tag{1.12.24}$$

Then

$$\begin{gathered}QP:\mathscr{C}^1(\widetilde{M},\mathcal{E})\longrightarrow\mathscr{C}^0(\widetilde{M},\mathcal{F})\ \text{ and}\\ P^\top Q^\top:\mathscr{C}^1(\widetilde{M},\mathcal{F})\longrightarrow\mathscr{C}^0(\widetilde{M},\mathcal{E})\\ \text{are first-order differential operators},\end{gathered} \tag{1.12.25}$$

and, with all principal symbols taken in the sense of first-order differential operators,

$$\begin{aligned}&\int_\Omega\Big\langle i\mathrm{Sym}(Q;N_g)(Pu)\big|_\Omega,w\big|_\Omega\Big\rangle_{\mathcal{F}}\,d\mathcal{L}_g^n \qquad (1.12.26)\\ &\quad=\int_\Omega\Big\langle u\big|_\Omega,i\mathrm{Sym}\big(P^\top;N_g\big)(Q^\top w)\big|_\Omega-i\mathrm{Sym}\big(P^\top Q^\top;N_g\big)w\big|_\Omega\Big\rangle_{\mathcal{E}}\,d\mathcal{L}_g^n\\ &\qquad+\int_{\partial_*\Omega}\Big\langle\mathrm{Sym}\big(Q;N_g\big)\,\mathrm{Sym}\big(P;\widetilde{\nu}_g\big)(u|_\Omega)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}},(w|_\Omega)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big\rangle_{\mathcal{E}}\,d\sigma_g,\end{aligned}$$

where $\widetilde{\nu}_g$ *is the unique extension of* $\nu_g\in T^*M$ *to a linear functional* $\widetilde{\nu}_g\in T^*\widetilde{M}$ *such that* $\widetilde{\nu}_g(N_g)=0$.

Moreover, if the original assumptions on the sections u,w *are strengthened to* $u\in\mathscr{C}^1(\widetilde{M},\mathcal{E})$ *and* $w\in\mathscr{C}_c^1(\widetilde{M},\mathcal{F})$, *then formula* (1.12.26) *is valid provided* Ω *is merely a set of locally finite perimeter in* M.

Note that if $QP=0$ on $\widetilde{M}$ then (1.12.22) is automatically satisfied; in this case we also have $P^\top Q^\top=0$ on $\widetilde{M}$, hence $\mathrm{Sym}\big(P^\top Q^\top;N_g\big)=0$ (so (1.12.26) takes a simpler form in this situation). To see a concrete example of this sort, pick some

$\ell \in \{1, \dots, n, n+1\}$ (recall that $\dim \widetilde{M} = n+1$) and, with $d_{\widetilde{M}}$ denoting the exterior derivative operator on $\widetilde{M}$, define

$$\begin{gathered} \mathcal{E} := \Lambda^{\ell-1} T\widetilde{M}, \quad \mathcal{H} := \Lambda^{\ell} T\widetilde{M}, \quad \mathcal{F} := \Lambda^{\ell+1} T\widetilde{M}, \\ P := d_{\widetilde{M}} \text{ on } \ell\text{-forms on } \widetilde{M},\ Q := d_{\widetilde{M}} \text{ on } (\ell+1)\text{-forms on } \widetilde{M}. \end{gathered} \tag{1.12.27}$$

In such a scenario, with $\delta_{\widetilde{M}} := (d_{\widetilde{M}})^{\top}$ denoting the transpose of $d_{\widetilde{M}}$ on $\widetilde{M}$, formula (1.12.26) becomes

$$\begin{aligned} -\int_{\Omega} \big\langle N_g \wedge (d_{\widetilde{M}} u)\big|_{\Omega}, w\big|_{\Omega}\big\rangle_{\Lambda^{\ell+1} T\widetilde{M}} \, d\mathcal{L}_g^n &= \int_{\Omega} \big\langle u\big|_{\Omega}, N_g \vee (\delta_{\widetilde{M}} w)\big|_{\Omega}\big\rangle_{\Lambda^{\ell-1} T\widetilde{M}} \, d\mathcal{L}_g^n \\ &\quad - \int_{\partial_*\Omega} \Big\langle N_g \wedge \Big(\widetilde{\nu}_g \wedge (u|_{\Omega})\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big), (w|_{\Omega})\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big\rangle_{\Lambda^{\ell-1} T\widetilde{M}} \, d\sigma_g, \end{aligned} \tag{1.12.28}$$

for any $(\ell-1)$-form u and $(\ell+1)$-form w which are of class $\mathscr{C}^1$ in an open neighborhood of Ω in $\widetilde{M}$ and satisfy (1.12.23) as well as

$$|d_{\widetilde{M}} u||w| + |u||\delta_{\widetilde{M}} w| + |u||w| \in L^1(\Omega, \mathcal{L}_g^n). \tag{1.12.29}$$

Further specialize these considerations to following scenario. Suppose $\dim \widetilde{M} = 3$ (i.e., $n = \dim M = 2$) and recall that, in this context, the curl operator and cross-product are defined as follows (with $*$ denoting the Hodge star operator on $\widetilde{M}$):

$$\begin{gathered} \operatorname{curl} \omega := *(d_{\widetilde{M}} \omega), \quad \alpha \times \beta := *(\alpha \wedge \beta) \\ \text{for any 1-forms } \omega, \alpha, \beta \text{ on } \widetilde{M}. \end{gathered} \tag{1.12.30}$$

Take u to be a 1-form in an open neighborhood of Ω in $\widetilde{M}$ satisfying

$$\begin{gathered} \mathcal{N}_\kappa\big(u\big|_{\Omega}\big) \in L^p(\partial\Omega, \sigma_g), \quad |\operatorname{curl} u| + |u| \in L^1(\Omega, \mathcal{L}_g^2) \\ \text{and } (u|_{\Omega})\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \sigma_g\text{-a.e. point on } \partial_{\text{nta}}\Omega, \end{gathered} \tag{1.12.31}$$

and consider the 2-form $w := *1$ on $\widetilde{M}$. Then (1.12.28) becomes

$$\int_{\Omega} \big\langle N_g, (\operatorname{curl} u)\big|_{\Omega}\big\rangle_{\Lambda^1 T\widetilde{M}} \, d\mathcal{L}_g^2 = \int_{\partial_*\Omega} \Big\langle (u|_{\Omega})\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}, N_g \times \widetilde{\nu}_g\Big\rangle_{\Lambda^1 T\widetilde{M}} \, d\sigma_g. \tag{1.12.32}$$

Interpreting $\tau_g := N_g \times \widetilde{\nu}_g$ as the tangent covector to the "curve" $\partial_*\Omega$, we may rephrase (1.12.32) as

$$\int_{\Omega} \big\langle N_g, (\operatorname{curl} u)\big|_{\Omega}\big\rangle_{\Lambda^1 T\widetilde{M}} \, d\mathcal{L}_g^2 = \int_{\partial_*\Omega} \Big\langle \tau_g, (u|_{\Omega})\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big\rangle_{\Lambda^1 T\widetilde{M}} \, d\sigma_g. \tag{1.12.33}$$

This is very much akin to the classical Stokes' formula in the three-dimensional setting, with the added bonus of providing very general geometric and analytic conditions ensuring its actual validity.

Proof (*Proof of Theorem* 1.12.2) We start by defining the vector field $\vec{F} : \Omega \to T\widetilde{M}$ in an implicit fashion, via the demand that

$$ {}_{T^*\widetilde{M}}\big(\widetilde{\xi}, \vec{F}\big)_{T\widetilde{M}} = \Big\langle \mathrm{Sym}(Q; N_g)\,\mathrm{Sym}(P; \widetilde{\xi})u, w\Big\rangle_{\mathcal{F}} \tag{1.12.34} $$

for each point in Ω and each covector $\widetilde{\xi}$ in the corresponding fiber in $T^*\widetilde{M}$. The linearity of the right-hand side in $\widetilde{\xi}$ ensures that this is indeed well defined. From (1.12.34) and (1.12.22) we see that

$$ {}_{T^*\widetilde{M}}\big(N_g, \vec{F}\big)_{T\widetilde{M}} = \Big\langle \mathrm{Sym}(Q; N_g)\,\mathrm{Sym}(P; N_g)u, w\Big\rangle_{\mathcal{F}} = 0, \tag{1.12.35} $$

hence $\vec{F}$ is tangent to M at points in Ω. As such, we may regard $\vec{F}$ as an intrinsic vector field to the manifold M, i.e., $\vec{F} : \Omega \to TM$. Viewed as such, it follows that, in fact,

$$ \vec{F} \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n_g) \otimes TM. \tag{1.12.36} $$

Let div_M denote the differential geometric divergence operator on the manifold M (equipped with the Riemann metric inherited from the ambient $\widetilde{M}$). We claim that, in the sense of distributions in Ω, we have

$$ \begin{aligned} \mathrm{div}_M \vec{F} = {} & \Big\langle i\mathrm{Sym}(Q; N_g)(Pu)\big|_{\Omega}, w\big|_{\Omega}\Big\rangle_{\mathcal{F}} \\ & - \Big\langle u\big|_{\Omega}, i\mathrm{Sym}\big(P^\top; N_g\big)(Q^\top w)\big|_{\Omega} - i\mathrm{Sym}\big(P^\top Q^\top; N_g\big)w\big|_{\Omega}\Big\rangle_{\mathcal{E}}. \end{aligned} \tag{1.12.37} $$

To justify this claim, fix an arbitrary scalar-valued function $\psi \in \mathscr{C}^1_c(\widetilde{M})$ with the property that $\mathrm{supp}\,\psi \cap M \subseteq \Omega$. With grad_M denoting the gradient operator on M, and d denoting the exterior derivative operator on $\widetilde{M}$, we may then compute (bearing in mind (1.12.35))

$$ \begin{aligned} {}_{\mathscr{D}'(\Omega)}\big(\mathrm{div}_M \vec{F}, \psi|_{\Omega}\big)_{\mathscr{D}(\Omega)} &= -{}_{\mathscr{D}'(\Omega)}\big(\vec{F}, \mathrm{grad}_M(\psi|_{\Omega})\big)_{\mathscr{D}(\Omega)} \\ &= -\int_{\Omega} {}_{TM}\langle \mathrm{grad}_M(\psi|_{\Omega}), \vec{F}\rangle_{TM}\, d\mathcal{L}^n_g \\ &= -\int_{\Omega} {}_{T^*\widetilde{M}}\big(d\psi, \vec{F}\big)_{T\widetilde{M}}\, d\mathcal{L}^n_g \\ &= -\int_{\Omega} \Big\langle \mathrm{Sym}\big(Q; N_g\big)\mathrm{Sym}\big(P; d\psi\big)u, w\Big\rangle_{\mathcal{F}}\, d\mathcal{L}^n_g \\ &= I - II, \end{aligned} \tag{1.12.38} $$

where, in view of (1.12.25) and (1.11.25), we may take

$$I := -\int_{\Omega} \Big\langle \mathrm{iSym}\big(Q; N_g\big) P(\psi u), w \Big\rangle_{\mathcal{F}} \, \mathrm{d}\mathcal{L}^n_g, \tag{1.12.39}$$

$$II := -\int_{\Omega} \Big\langle \mathrm{iSym}\big(Q; N_g\big)(Pu), w \Big\rangle_{\mathcal{F}} \psi \, \mathrm{d}\mathcal{L}^n_g. \tag{1.12.40}$$

Given the nature of our hypotheses and the format of the conclusion we seek, there is no loss of generality in assuming that there exists a relatively compact $\mathscr{C}^1$ domain $\widetilde{\Omega}$ of $\widetilde{M}$ whose closure is contained in the open neighborhood $\mathcal{O}$ and with the property that $\operatorname{supp}\psi \cap M = \operatorname{supp}\psi \cap \partial\widetilde{\Omega}$, or, equivalently,

$$\operatorname{supp}\psi \cap M \subseteq \partial\widetilde{\Omega} \quad \text{and} \quad \operatorname{supp}\psi \cap \partial\widetilde{\Omega} \subseteq M. \tag{1.12.41}$$

In such a scenario, N_g agrees with the outward unit conormal to $\partial\widetilde{\Omega}$ on $\operatorname{supp}\psi \cap \Omega$. Keeping this in mind, denoting by σ_g the surface measure on $\partial\widetilde{\Omega}$, and making use of (1.12.6) (with M, Ω replaced by $\widetilde{M}$, $\widetilde{\Omega}$), we may then express

$$\begin{aligned} I &= -\int_{\partial\widetilde{\Omega}} \Big\langle \mathrm{iSym}\big(Q; N_g\big) P(\psi u), w \Big\rangle_{\mathcal{F}} \, \mathrm{d}\widetilde{\sigma}_g \\ &= -\int_{\partial\widetilde{\Omega}} \Big\langle \psi u, \mathrm{iSym}\big(P^{\top}; N_g\big)(Q^{\top} w) - \mathrm{iSym}\big(P^{\top} Q^{\top}; N_g\big) w \Big\rangle_{\mathcal{E}} \, \mathrm{d}\widetilde{\sigma}_g \\ &= -\int_{\Omega} \Big\langle u, \mathrm{iSym}\big(P^{\top}; N_g\big)(Q^{\top} w) - \mathrm{iSym}\big(P^{\top} Q^{\top}; N_g\big) w \Big\rangle_{\mathcal{E}} \psi \, \mathrm{d}\mathcal{L}^n_g. \end{aligned} \tag{1.12.42}$$

From (1.12.38)–(1.12.40), and (1.12.42), formula (1.12.37) now follows. In concert with (1.12.24), this implies

$$\operatorname{div}_M \vec{F} \in L^1(\Omega, \mathcal{L}^n_g). \tag{1.12.43}$$

Next, (1.12.34) and the original assumptions on u, w imply that the nontangential trace $\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ exists at σ_g-a.e. point on $\partial_{\mathrm{nta}}\Omega$ (when considered from within the ambient $\Omega \subset M$) and

$$\begin{aligned} {}_{T^*M}\big(\nu_g, \vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)_{TM} &= {}_{T^*\widetilde{M}}\big(\widetilde{\nu}_g, \vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)_{T\widetilde{M}} \\ &= \Big\langle \mathrm{Sym}(Q; N_g)\mathrm{Sym}(P; \widetilde{\nu}_g)\big((u|_{\Omega})\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big), \big((w|_{\Omega})\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\Big\rangle_{\mathcal{F}}. \end{aligned} \tag{1.12.44}$$

In addition, (1.12.34) also gives that

$$\mathcal{N}_\kappa \vec{F} \leq C \mathcal{N}_\kappa u \cdot \mathcal{N}_\kappa w \quad \text{on} \quad \partial\Omega. \tag{1.12.45}$$

Granted the original assumptions on u, w, this shows that $\mathcal{N}_\kappa \vec{F}$ belongs to $L^1(\partial\Omega, \sigma_g)$. With this in hand, formula (1.12.6) now follows from Corollary 1.11.5, bearing in mind (1.12.44) and (1.12.37).

Lastly, the very last claim in the statement of the theorem (pertaining to the validity of (1.12.26) when $u \in \mathscr{C}^1(\widetilde{M}, \mathcal{E})$, $w \in \mathscr{C}^1_c(\widetilde{M}, \mathcal{F})$, and $\Omega \subseteq M$ is a set of locally finite perimeter) is proved in a similar fashion, the main difference being that we now make use of Theorem 1.11.1 in place of Corollary 1.11.5. □

One significant instance when the cancelation condition (1.12.2) is automatically satisfied is singled out next.

Corollary 1.12.3 *Let M be a $\mathscr{C}^2$ manifold of dimension n, equipped with a Riemannian metric g, and denote by $\mathcal{L}^n_g$ the Lebesgue measure induced by the volume element on M. Let $\mathcal{E}$, $\mathcal{F}$ be Hermitian vector bundles over M, of class $\mathscr{C}^2$. Assume all metrics involved are of class $\mathscr{C}^1$. Suppose*

$$P : \mathscr{C}^1(M, \mathcal{E}) \longrightarrow \mathscr{C}^0(M, \mathcal{F}) \tag{1.12.46}$$

is a first-order differential operator with $\mathscr{C}^1$ coefficients for the top part and $\mathscr{C}^0$ coefficients for the zero-th order part. Let $P^\top : \mathscr{C}^1(M, \mathcal{F}) \to \mathscr{C}^0(M, \mathcal{E})$ denote its transpose.

Next, let Ω be a relatively compact open subset of M, with a lower Ahlfors regular boundary, such that $\sigma_g := \mathcal{H}^{n-1}_g \lfloor \partial\Omega$ is a doubling measure. Denote by ν_g the geometric measure theoretic outward unit conormal to Ω and select some aperture parameter $\kappa > 0$. Finally, pick two Lebesgue measurable sections, $u : \Omega \to \mathcal{E}$ and $w : \Omega \to \mathcal{F}$, satisfying

$$\begin{gathered} \nabla u \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n_g) \otimes (T^*M \otimes \mathcal{E}), \\ \mathcal{N}_\kappa u,\ \mathcal{N}_\kappa(\nabla u) < \infty \ \text{ at } \sigma_g\text{-a.e. point on } \ \partial\Omega, \\ u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ and } (\nabla u)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ exist at } \sigma_g\text{-a.e. point on } \partial_{\text{nta}}\Omega, \end{gathered} \tag{1.12.47}$$

as well as

$$\begin{gathered} \nabla w \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n_g) \otimes (T^*M \otimes \mathcal{F}), \\ \mathcal{N}_\kappa w,\ \mathcal{N}_\kappa(\nabla w) < \infty \ \text{ at } \sigma_g\text{-a.e. point on } \ \partial\Omega, \\ w\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ and } (\nabla u)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ exist } \sigma_g\text{-a.e. on } \partial_{\text{nta}}\Omega, \end{gathered} \tag{1.12.48}$$

and

$$\mathcal{N}_\kappa u \cdot \mathcal{N}_\kappa w,\ \mathcal{N}_\kappa u \cdot \mathcal{N}_\kappa(\nabla w),\ \mathcal{N}_\kappa(\nabla u) \cdot \mathcal{N}_\kappa w \in L^1(\partial\Omega, \sigma_g). \tag{1.12.49}$$

Lastly, fix a $\mathscr{C}^1$-vector field X on M and denote by $\nabla^{\mathcal{E}}_X$, $\nabla^{\mathcal{F}}_X$, the covariant derivatives along X for sections of $\mathcal{E}$ and $\mathcal{F}$, respectively.

Then

$$P\nabla^{\mathcal{E}}_X - \nabla^{\mathcal{F}}_X P : \mathscr{C}^2(M,\mathcal{E}) \longrightarrow \mathscr{C}^0(M,\mathcal{F}) \tag{1.12.50}$$

is a first-order differential operator and

$$\begin{aligned}\int_{\partial_*\Omega} \Big\langle \nu_g(X)(Pu)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} + \mathrm{i}\mathrm{Sym}(P;\nu_g)(\nabla^{\mathcal{E}}_X u)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, w\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{F}}\, d\sigma_g \\ = \int_{\partial_*\Omega} \Big\langle u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, \nu_g(X)\big(P^\top w\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} - \mathrm{i}\mathrm{Sym}\big(P^\top;\nu_g\big)\big((\nabla^{\mathcal{F}}_X)^\top w\big)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \\ + \mathrm{i}\mathrm{Sym}\Big(P\nabla^{\mathcal{E}}_X - \nabla^{\mathcal{F}}_X P;\nu_g\Big)^\top w\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{E}}\, d\sigma_g,\end{aligned} \tag{1.12.51}$$

where all principal symbols are taken in the sense of first-order differential operators.

In addition, if the original assumptions on the sections u, w *are strengthened to* $u \in \mathscr{C}^1(M,\mathcal{E})$ *and* $w \in \mathscr{C}^1_c(M,\mathcal{F})$*, then formula (1.12.51) is valid provided* Ω *is simply a set of locally finite perimeter in* M.

Proof Recall that for each vector field $X = X_j\partial_j \in TM$ the covariant derivative ∇_X associated with a connection ∇ on a vector bundle is a first-order differential operator whose principal symbol is given by

$$\mathrm{Sym}(\nabla_X;\xi) = \mathrm{i}\xi(X)\, I = \mathrm{i}X_j\xi_j\, I, \qquad \forall \xi = \xi_j dx_j \in T^*M, \tag{1.12.52}$$

where I is the identity operator (cf., e.g., [189, Proposition 9.3, pp. 375–377]). To proceed in earnest, with $\mathcal{E}, \mathcal{F}, P, Q$ as in the statement, consider the Hermitian vector bundle $\mathcal{H} := \mathcal{E} \oplus \mathcal{F}$, and define the first-order differential operators

$$\begin{aligned}\widetilde{P} : \mathscr{C}^1(M,\mathcal{E}) \longrightarrow \mathscr{C}^0(M,\mathcal{H}), \\ \widetilde{P}\phi := \big(\nabla^{\mathcal{E}}_X\phi, -P\phi\big), \quad \forall \phi \in \mathscr{C}^1(M,\mathcal{E}),\end{aligned} \tag{1.12.53}$$

and

$$\begin{aligned}\widetilde{Q} : \mathscr{C}^1(M,\mathcal{H}) \longrightarrow \mathscr{C}^0(M,\mathcal{F}), \\ \widetilde{Q}(\varphi,\psi) := P\varphi + \nabla^{\mathcal{F}}_X\psi, \quad \forall (\varphi,\psi) \in \mathscr{C}^1(M,\mathcal{H}).\end{aligned} \tag{1.12.54}$$

Their transposes are given, respectively, by

$$\begin{aligned}\widetilde{P}^\top : \mathscr{C}^1(M,\mathcal{H}) \longrightarrow \mathscr{C}^0(M,\mathcal{E}), \\ \widetilde{P}^\top(\varphi,\psi) = \big(\nabla^{\mathcal{E}}_X\big)^\top\varphi - P^\top\psi, \quad \forall (\varphi,\psi) \in \mathscr{C}^1(M,\mathcal{H}),\end{aligned} \tag{1.12.55}$$

and

$$\begin{aligned}\widetilde{Q}^\top : \mathscr{C}^1(M,\mathcal{F}) \longrightarrow \mathscr{C}^0(M,\mathcal{H}), \\ \widetilde{Q}^\top\eta = \big(P^\top\eta, (\nabla^{\mathcal{F}}_X)^\top\eta\big), \quad \forall \eta \in \mathscr{C}^1(M,\mathcal{F}).\end{aligned} \tag{1.12.56}$$

Observe that, by (1.12.53)–(1.12.54) and (1.12.52), for every section $\phi \in \mathcal{E}$ we have

$$\begin{aligned}\mathrm{Sym}(\widetilde{Q};\xi)\mathrm{Sym}(\widetilde{P};\xi)\phi &= \mathrm{Sym}(\widetilde{Q};\xi)\big(\mathrm{Sym}(\nabla_X^{\mathcal{E}};\xi)\phi, -\mathrm{Sym}(P;\xi)\phi\big)\\ &= \mathrm{Sym}(P;\xi)\mathrm{Sym}(\nabla_X^{\mathcal{E}};\xi)\phi - \mathrm{Sym}(\nabla_X^{\mathcal{F}};\xi)\mathrm{Sym}(P;\xi)\phi\\ &= \mathrm{i}\xi(X)\mathrm{Sym}(P;\xi)\phi - \mathrm{i}\xi(X)\mathrm{Sym}(P;\xi)\phi\\ &= 0 \ \text{ for all } \ \xi \in T^*M. \end{aligned} \tag{1.12.57}$$

Also, for each section $\eta \in \mathcal{F}$,

$$\begin{aligned}\widetilde{P}^\top \widetilde{Q}^\top \eta = \widetilde{P}^\top\big(P^\top\eta, (\nabla_X^{\mathcal{F}})^\top\eta\big) &= \big(\nabla_X^{\mathcal{E}}\big)^\top P^\top\eta - P^\top(\nabla_X^{\mathcal{F}})^\top\eta\\ &= \big(P\nabla_X^{\mathcal{E}} - \nabla_X^{\mathcal{F}}P\big)^\top\eta. \end{aligned} \tag{1.12.58}$$

Granted these identifications, all desired conclusions are now seen directly from Theorem 1.12.1 applied to the vector bundles $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$, and differential operators $\widetilde{P}$, $\widetilde{Q}$. □

Appropriately specializing Theorem 1.12.1 yields the following useful result.

Corollary 1.12.4 *Let M be a $\mathscr{C}^2$ manifold of dimension n, equipped with a Riemannian metric g, and denote by $\mathcal{L}_g^n$ the Lebesgue measure induced by the volume element on M. Let $\mathcal{E}$ be a Hermitian vector bundle over M, of class $\mathscr{C}^2$. Assume all metrics involved are of class $\mathscr{C}^1$. Suppose $P, Q : \mathscr{C}^1(M,\mathcal{E}) \to \mathscr{C}^0(M,\mathcal{E})$ are first-order differential operators with $\mathscr{C}^1$ coefficients having the property that their principal symbols commute, in the sense that*

$$\mathrm{Sym}(P;\xi)\mathrm{Sym}(Q;\xi) = \mathrm{Sym}(Q;\xi)\,\mathrm{Sym}(P;\xi), \qquad \forall \xi \in T^*M. \tag{1.12.59}$$

Denote by $P^\top, Q^\top : \mathscr{C}^1(M,\mathcal{E}) \to \mathscr{C}^0(M,\mathcal{E})$ the transposes of P, Q (considered in the usual sense on the manifold M).

Next, let Ω be a relatively compact open subset of M, with a lower Ahlfors regular boundary, such that $\sigma_g := \mathcal{H}_g^{n-1}\lfloor\partial\Omega$ is a doubling measure. Denote by ν_g the geometric measure theoretic outward unit conormal to Ω and select an aperture parameter $\kappa > 0$. Finally, pick two Lebesgue measurable sections $u, w : \Omega \to \mathcal{E}$ satisfying

$$\begin{aligned}&Pu \ \text{ and } \ Qu \ \text{ belong to the space } \ L^1_{\mathrm{loc}}(\Omega,\mathcal{L}_g^n)\otimes\mathcal{E},\\ &\mathcal{N}_\kappa u,\ \mathcal{N}_\kappa(Pu),\ \mathcal{N}_\kappa(Qu) < \infty \ \text{ at } \sigma_g\text{-a.e. point on } \ \partial\Omega,\\ &u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ (Pu)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ (Qu)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \text{ exist } \sigma_g\text{-a.e. on } \partial_{\mathrm{nta}}\Omega, \end{aligned} \tag{1.12.60}$$

as well as

$$\begin{aligned}&P^\top w \ \text{ and } \ Q^\top w \ \text{ belong to the space } \ L^1_{\mathrm{loc}}(\Omega,\mathcal{L}_g^n)\otimes\mathcal{E},\\ &\mathcal{N}_\kappa w,\ \mathcal{N}_\kappa(P^\top w),\ \mathcal{N}_\kappa(Q^\top w) < \infty \ \text{ at } \sigma_g\text{-a.e. point on } \ \partial\Omega,\\ &w\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ (P^\top w)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\ (Q^\top w)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \ \text{ exist } \sigma_g\text{-a.e. on } \partial_{\mathrm{nta}}\Omega, \end{aligned} \tag{1.12.61}$$

and

$$\begin{aligned} &\big[\mathcal{N}_\kappa u + \mathcal{N}_\kappa(Pu) + \mathcal{N}_\kappa(Qu)\big] \cdot \mathcal{N}_\kappa w \in L^1(\partial\Omega, \sigma_g), \\ &\big[\mathcal{N}_\kappa w + \mathcal{N}_\kappa(P^\top w) + \mathcal{N}_\kappa(Q^\top w)\big] \cdot \mathcal{N}_\kappa u \in L^1(\partial\Omega, \sigma_g). \end{aligned} \tag{1.12.62}$$

Then the commutator $[P, Q]$ *is a first-order differential operator acting on sections of* $\mathcal{E}$ *and*

$$\begin{aligned} &\int_{\partial_*\Omega} \Big\langle \mathrm{iSym}(P;\nu_g)(Qu)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - \mathrm{iSym}(Q;\nu_g)(Pu)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \Big\rangle_{\mathcal{E}} \,\mathrm{d}\sigma_g \\ &\quad = -\int_{\partial_*\Omega} \Big\langle u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \mathrm{iSym}\big(P^\top;\nu_g\big)(Q^\top w)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} - \mathrm{iSym}\big(Q^\top;\nu_g\big)(P^\top w)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \Big\rangle_{\mathcal{E}} \,\mathrm{d}\sigma_g \\ &\quad\quad + \int_{\partial_*\Omega} \Big\langle \mathrm{iSym}\big([P,Q];\nu_g\big)u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \Big\rangle_{\mathcal{E}} \,\mathrm{d}\sigma_g, \end{aligned} \tag{1.12.63}$$

where all principal symbols are taken in the sense of first-order differential operators.

Furthermore, if the original assumptions on the sections u, w *are strengthened to* $u \in \mathscr{C}^1(M, \mathcal{E})$ *and* $w \in \mathscr{C}^1_c(M, \mathcal{F})$*, then formula* (1.12.63) *remains valid if* Ω *is simply a set of locally finite perimeter in* M*.*

Proof With $\mathcal{E}, P, Q$ as in the statement of the corollary, consider the Hermitian vector bundles $\mathcal{F} := \mathcal{E}$ and $\mathcal{H} := \mathcal{E} \oplus \mathcal{E}$. Also, define the first-order differential operators

$$\begin{aligned} &\widetilde{P} : \mathscr{C}^1(M, \mathcal{E}) \longrightarrow \mathscr{C}^0(M, \mathcal{H}), \\ &\widetilde{P}\phi := (Q\phi, -P\phi), \quad \forall \phi \in \mathscr{C}^1(M, \mathcal{E}), \end{aligned} \tag{1.12.64}$$

and

$$\begin{aligned} &\widetilde{Q} : \mathscr{C}^1(M, \mathcal{H}) \longrightarrow \mathscr{C}^0(M, \mathcal{F}), \\ &\widetilde{Q}(\varphi, \psi) := P\varphi + Q\psi, \quad \forall (\varphi, \psi) \in \mathscr{C}^1(M, \mathcal{H}). \end{aligned} \tag{1.12.65}$$

Their transposes are given, respectively, by

$$\begin{aligned} &\widetilde{P}^\top : \mathscr{C}^1(M, \mathcal{H}) \longrightarrow \mathscr{C}^0(M, \mathcal{E}), \\ &\widetilde{P}^\top(\varphi, \psi) = Q^\top\varphi - P^\top\psi, \quad \forall (\varphi, \psi) \in \mathscr{C}^1(M, \mathcal{H}), \end{aligned} \tag{1.12.66}$$

and

$$\begin{aligned} &\widetilde{Q}^\top : \mathscr{C}^1(M, \mathcal{F}) \longrightarrow \mathscr{C}^0(M, \mathcal{H}), \\ &\widetilde{Q}^\top\eta = (P^\top\eta, Q^\top\eta), \quad \forall \eta \in \mathscr{C}^1(M, \mathcal{F}). \end{aligned} \tag{1.12.67}$$

Note that, by design and (1.12.59), for every section $\theta \in \mathcal{E}$ we have

$$\begin{aligned} \mathrm{Sym}(\widetilde{Q};\xi)\mathrm{Sym}(\widetilde{P};\xi)\theta &= \mathrm{Sym}(\widetilde{Q};\xi)\big(\mathrm{Sym}(Q;\xi)\theta, -\mathrm{Sym}(P;\xi)\theta\big) \\ &= \mathrm{Sym}(P;\xi)\mathrm{Sym}(Q;\xi)\theta - \mathrm{Sym}(Q;\xi)\mathrm{Sym}(P;\xi)\theta \\ &= 0 \ \text{ for all } \ \xi \in T^*M, \end{aligned} \tag{1.12.68}$$

and

$$\widetilde{Q}\widetilde{P}\theta = \widetilde{Q}(Q\theta, -P\theta) = PQ\theta - QP\theta = [P, Q]\theta, \tag{1.12.69}$$

hence

$$\widetilde{P}^{\top}\widetilde{Q}^{\top} = \big(\widetilde{Q}\widetilde{P}\big)^{\top} = [P, Q]^{\top}. \tag{1.12.70}$$

In light of these identifications, the desired conclusions are now seen directly from Theorem 1.12.1 applied to the vector bundles $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$, and differential operators $\widetilde{P}$, $\widetilde{Q}$. □

A useful consequence of Theorem 1.12.2 is recorded below.

Corollary 1.12.5 *Let $\widetilde{M}$ be a $\mathscr{C}^2$ manifold of dimension $n+1$, equipped with a Riemannian metric g, and consider an open, oriented, $\mathscr{C}^1$ submanifold M of $\widetilde{M}$, of dimension n. Denote by $\mathcal{L}^n_g$ the Lebesgue measure induced by the intrinsic volume element on the Riemannian manifold M, and let $N_g : M \to T^*\widetilde{M}$ stand for the unit conormal to M. Also, consider a Hermitian vector bundle $\mathcal{E}$, of class $\mathscr{C}^2$ over the manifold $\widetilde{M}$. Assume all metrics involved are of class $\mathscr{C}^1$. Going further, suppose*

$$P, Q : \mathscr{C}^1(\widetilde{M}, \mathcal{E}) \longrightarrow \mathscr{C}^0(\widetilde{M}, \mathcal{E}) \tag{1.12.71}$$

are two first-order differential operators with $\mathscr{C}^1$ coefficients for the top part and $\mathscr{C}^0$ coefficients for the zero-th order part, having the property that their principal symbols commute, in the sense that

$$\mathrm{Sym}(P;\widetilde{\xi})\mathrm{Sym}(Q;\widetilde{\xi}) = \mathrm{Sym}(Q;\widetilde{\xi})\mathrm{Sym}(P;\widetilde{\xi}), \qquad \forall \widetilde{\xi} \in T^*\widetilde{M}. \tag{1.12.72}$$

Denote by $P^{\top}$, $Q^{\top} : \mathscr{C}^1(\widetilde{M}, \mathcal{E}) \to \mathscr{C}^0(\widetilde{M}, \mathcal{E})$ the (real) transpose of P, Q, considered in the usual sense on the manifold $\widetilde{M}$.

Next, let Ω be a relatively compact open subset of M, with a lower Ahlfors regular boundary, such that $\sigma_g := \mathcal{H}^{n-1}_g \lfloor \partial\Omega$ is a doubling measure (all relative to the manifold M, as ambient). Denote by ν_g the geometric measure theoretic outward unit conormal to Ω and select an aperture parameter $\kappa > 0$. Lastly, consider an open neighborhood $\mathcal{O}$ of Ω in $\widetilde{M}$ and pick two sections, $u, w \in \mathscr{C}^1(\mathcal{O}, \mathcal{E})$, satisfying (with the nontangential maximal operator and nontangential traces taken from within Ω)

$$\begin{aligned} &\mathcal{N}_\kappa\big(u\big|_{\Omega}\big) < \infty \ \text{ and } \ \mathcal{N}_\kappa\big(w\big|_{\Omega}\big) < \infty \ \text{ at } \sigma_g\text{-a.e. point on } \partial\Omega, \\ &\mathcal{N}_\kappa\big(u\big|_{\Omega}\big)\cdot\mathcal{N}_\kappa\big(w\big|_{\Omega}\big) \ \text{ belongs to the space } \ L^1(\partial\Omega, \sigma_g), \\ &(u|_{\Omega})\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \text{ and } \ (w|_{\Omega})\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \ \text{ exist at } \sigma_g\text{-a.e. point on } \partial_{\text{nta}}\Omega, \end{aligned} \tag{1.12.73}$$

as well as

$$\big(|Pu| + |Qu|\big)|w| + |u|\big(|P^\top w| + |Q^\top w|\big) + |u||w| \in L^1(\Omega, \mathcal{L}^n_g). \tag{1.12.74}$$

Then the commutator $[P, Q]$ *is a first-order differential operator on sections of* $\mathcal{E}$ *and, with all principal symbols taken in the sense of first-order differential operators,*

$$\begin{aligned}
&\int_\Omega \Big\langle \mathrm{iSym}(P; N_g)(Qu)\big|_\Omega - \mathrm{iSym}(Q; N_g)(Pu)\big|_\Omega, w\big|_\Omega\Big\rangle_{\mathcal{E}} \, \mathrm{d}\mathcal{L}^n_g \\
&\qquad = -\int_\Omega \Big\langle u\big|_\Omega, \mathrm{iSym}\big(P^\top; N_g\big)(Q^\top w)\big|_\Omega - \mathrm{iSym}\big(Q^\top; N_g\big)(P^\top w)\big|_\Omega\Big\rangle_{\mathcal{E}} \, \mathrm{d}\mathcal{L}^n_g \\
&\qquad\quad + \int_\Omega \Big\langle \mathrm{iSym}\big([P, Q]; N_g\big)u\big|_\Omega, w\big|_\Omega\Big\rangle_{\mathcal{E}} \, \mathrm{d}\mathcal{L}^n_g \\
&\qquad\quad - \int_{\partial_*\Omega} \Big\langle \mathrm{Sym}\big(Q; N_g\big)\,\mathrm{Sym}\big(P; \widetilde{\nu}_g\big)(u|_\Omega)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, (w|_\Omega)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{E}} \, \mathrm{d}\sigma_g \\
&\qquad\quad + \int_{\partial_*\Omega} \Big\langle \mathrm{Sym}\big(P; N_g\big)\,\mathrm{Sym}\big(Q; \widetilde{\nu}_g\big)(u|_\Omega)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}, (w|_\Omega)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{E}} \, \mathrm{d}\sigma_g,
\end{aligned} \tag{1.12.75}$$

where $\widetilde{\nu}_g$ *is the unique extension of* $\nu_g \in T^*M$ *to a linear functional* $\widetilde{\nu}_g \in T^*\widetilde{M}$ *such that* $\widetilde{\nu}_g(N_g) = 0$.

Moreover, if the original assumptions on the sections u, w *are strengthened to* $u \in \mathscr{C}^1(\widetilde{M}, \mathcal{E})$ *and* $w \in \mathscr{C}^1_c(\widetilde{M}, \mathcal{E})$*, then formula* (1.12.75) *is valid provided* Ω *is merely a set of locally finite perimeter in* M.

Proof The same purely algebraic argument which has produced Corollary 1.12.4 by applying Theorem 1.12.1 to the differential operators $\widetilde{P}$, $\widetilde{Q}$ from (1.12.64)–(1.12.65) proves the present corollary if in place of Theorem 1.12.1 we now employ Theorem 1.12.2. □

A version of Corollary 1.12.4 which is particularly useful in the development of our brand of boundary Sobolev spaces on manifolds is established in the proposition below. To facilitate its statement, we introduce some basic notation. The context is that of a $\mathscr{C}^2$ manifold M of dimension n, equipped with a Riemannian metric g. Consider a set of locally finite perimeter $\Omega \subset M$ and denote by $\nu : \partial^*\Omega \to T^*M$ its geometric measure theoretic outward unit conormal. Finally, let $\mathcal{E}$ be a Hermitian vector bundle over M. Assume all metrics involved are of class $\mathscr{C}^1$. In this setting, given two continuous vector fields $X, Y \in TM$, for every section $\varphi \in \mathscr{C}^1_c(M, \mathcal{E})$ we define

$$\partial_{\tau_{XY}}\varphi := \nu(X)\big(\nabla_Y \varphi\big)\Big|_{\partial_*\Omega} - \nu(Y)\big(\nabla_X \varphi\big)\Big|_{\partial_*\Omega} \quad \text{on } \partial_*\Omega, \tag{1.12.76}$$

where ∇ is a connection (or covariant derivative) on $\mathcal{E}$, with continuous connection coefficients. In particular, if in local coordinates the outward unit conormal ν is expressed as $\nu = \sum_{\ell=1}^n \nu_\ell \mathrm{d}x_\ell$, then we agree to re-denote $\partial_{\tau_{XY}}$ defined as in (1.12.76) corresponding to the choice $X := \partial_j$ and $Y := \partial_k$ simply as $\partial_{\tau_{jk}}$. That is, for every section $\varphi \in \mathscr{C}^1_c(M, \mathcal{E})$ we define

$$\partial_{\tau_{jk}}\varphi := \nu_j\big(\nabla_{\partial_k}\varphi\big)\Big|_{\partial_*\Omega} - \nu_k\big(\nabla_{\partial_j}\varphi\big)\Big|_{\partial_*\Omega} \ \text{ locally on } \partial_*\Omega. \tag{1.12.77}$$

We are now ready to present the result alluded to the above.

Proposition 1.12.6 *Let M be a $\mathscr{C}^2$ manifold of dimension n, equipped with a Riemannian metric g, and denote by $\mathcal{L}^n_g$ the Lebesgue measure induced by the volume element on M. Let $\mathcal{E}$ be a Hermitian vector bundle over M, of class $\mathscr{C}^2$. Assume all metrics involved are of class $\mathscr{C}^1$. Next, consider a set of locally finite perimeter $\Omega \subset M$. Define $\sigma_g := \mathcal{H}^{n-1}_g \lfloor \partial\Omega$, and denote by ν_g the geometric measure theoretic outward unit conormal to Ω.*

Then for every pair of sections $\varphi \in \mathscr{C}^1(M, \mathcal{E})$, $\psi \in \mathscr{C}^1_c(M, \mathcal{E})$ and every pair of $\mathscr{C}^1$ vector fields X, Y on M there holds

$$\begin{aligned}\int_{\partial_*\Omega}\big\langle\partial_{\tau_{XY}}\varphi, \psi\big\rangle_{\mathcal{E}}\,\mathrm{d}\sigma_g &= -\int_{\partial_*\Omega}\big\langle\varphi, \partial_{\tau_{XY}}\psi\big\rangle_{\mathcal{E}}\,\mathrm{d}\sigma_g \\ &\quad - \int_{\partial_*\Omega}\big\langle\varphi, \psi\big\rangle_{\mathcal{E}}\big(\nu_g(X)\,\mathrm{div}_g Y - \nu_g(Y)\,\mathrm{div}_g X + \nu_g([X, Y])\big)\,\mathrm{d}\sigma_g,\end{aligned} \tag{1.12.78}$$

where the tangential derivative operator $\partial_{\tau_{XY}}$ is defined as in (1.12.76) relative to a metric connection ∇ on $\mathcal{E}$.

In particular, if ψ is supported in a local chart with coordinates $(x_1, \dots, x_n)$ relative to which the outward unit normal is expressed as $\nu_g = \sum_{r=1}^n \nu_r \mathrm{d}x_r$, then for every $j, k \in \{1, \dots, n\}$ one has (with $\partial_{\tau_{jk}}$ defined as in (1.12.77))

$$\begin{aligned}\int_{\partial_*\Omega}\big\langle\partial_{\tau_{jk}}\varphi, \psi\big\rangle_{\mathcal{E}}\,\mathrm{d}\sigma_g &= -\int_{\partial\Omega}\big\langle\varphi, \partial_{\tau_{jk}}\psi\big\rangle_{\mathcal{E}}\,\mathrm{d}\sigma_g \\ &\quad - \sum_{\ell=1}^n \int_{\partial_*\Omega}\big\langle\varphi, \psi\big\rangle_{\mathcal{E}}\big(\nu_j\Gamma^\ell_{k\ell} - \nu_k\Gamma^\ell_{j\ell}\big)\,\mathrm{d}\sigma_g,\end{aligned} \tag{1.12.79}$$

where Γ^s_{rt} are the Christoffel symbols associated with the metric g.

Proof With ∇ denoting a metric connection on the vector bundle $\mathcal{E}$, consider the first-order differential operators

$$P := \nabla_X, \quad Q := \nabla_Y, \quad P, Q : \mathscr{C}^1(M, \mathcal{E}) \to \mathscr{C}^0(M, \mathcal{E}). \tag{1.12.80}$$

Bearing in mind (1.12.52), it follows that such a choice satisfies (1.12.59). In addition, in such a scenario we have

$$\begin{aligned}
&\mathrm{iSym}(P;\nu_g)(Q\varphi)\big|_{\partial_*\Omega} - \mathrm{iSym}(Q;\nu_g)(P\varphi)\big|_{\partial_*\Omega} \\
&\qquad = \mathrm{iSym}\big(\nabla_X;\nu_g\big)\big(\nabla_Y\varphi\big)\big|_{\partial_*\Omega} - \mathrm{iSym}\big(\nabla_Y;\nu_g\big)\big(\nabla_X\varphi\big)\big|_{\partial_*\Omega} \\
&\qquad = -\nu_g(X)\big(\nabla_Y\varphi\big)\big|_{\partial_*\Omega} + \nu_g(Y)\big(\nabla_X\varphi\big)\big|_{\partial_*\Omega} \\
&\qquad = -\partial_{\tau_{XY}}\varphi \quad \text{at } \sigma_g\text{-a.e. point on } \ \partial_*\Omega,
\end{aligned} \tag{1.12.81}$$

by virtue of (1.12.52) and (1.12.76). Since the connection ∇ on the vector bundle $\mathcal{E}$ is metric, we have (cf. [189, (9.1.55), p. 377])

$$(\nabla_Z)^\top = -\nabla_Z - \mathrm{div}_g Z, \qquad \forall Z \in \mathscr{C}^1(M, TM). \tag{1.12.82}$$

In turn, from (1.12.82), (1.12.52), and (1.12.76) we deduce that

$$\begin{aligned}
&\mathrm{iSym}\big(P^\top;\nu_g\big)(Q^\top\psi)\big|_{\partial_*\Omega} - \mathrm{iSym}\big(Q^\top;\nu_g\big)(P^\top\psi)\big|_{\partial_*\Omega} \qquad (1.12.83)\\
&\quad = \mathrm{iSym}\big(\nabla_X;\nu_g\big)\Big(\big(\nabla_Y\psi\big)\big|_{\partial_*\Omega} + \big((\mathrm{div}_g Y)\psi\big)\big|_{\partial_*\Omega}\Big) \\
&\qquad - \mathrm{iSym}\big(\nabla_Y;\nu_g\big)\Big(\big(\nabla_X\psi\big)\big|_{\partial_*\Omega} + \big((\mathrm{div}_g X)\psi\big)\big|_{\partial_*\Omega}\Big) \\
&\quad = -\nu_g(X)\Big(\big(\nabla_Y\psi\big)\big|_{\partial_*\Omega} + \big((\mathrm{div}_g Y)\psi\big)\big|_{\partial_*\Omega}\Big) \\
&\qquad + \nu_g(Y)\Big(\big(\nabla_X\psi\big)\big|_{\partial_*\Omega} + \big((\mathrm{div}_g X)\psi\big)\big|_{\partial_*\Omega}\Big) \\
&\quad = -\partial_{\tau_{XY}}\psi + \Big(\nu_g(Y)(\mathrm{div}_g X)\big|_{\partial_*\Omega} - \nu_g(X)(\mathrm{div}_g Y)\big|_{\partial_*\Omega}\Big)\big(\psi\big|_{\partial_*\Omega}\big)
\end{aligned}$$

at σ_g-a.e. point on $\partial_*\Omega$. Lastly, with I denoting the identity operator, we have

$$\begin{aligned}
\mathrm{iSym}\big([P,Q];\nu_g\big) &= \mathrm{iSym}\big(\big[\nabla_X,\nabla_Y\big];\nu_g\big) \\
&= \mathrm{iSym}\big(\nabla_{[X,Y]} + R(X,Y);\nu_g\big) \\
&= \mathrm{iSym}\big(\nabla_{[X,Y]};\nu_g\big) = -\nu_g\big([X,Y]\big)\, I,
\end{aligned} \tag{1.12.84}$$

thanks to (1.12.52) and the fact that the curvature

$$R(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = \big[\nabla_X,\nabla_Y\big] - \nabla_{[X,Y]} \tag{1.12.85}$$

is actually a zero-th order operator. Granted these, formula (1.12.63) (used with φ, ψ in place of u, w) then yields (1.12.78), keeping in mind the very last comment in the statement of Corollary 1.12.4.

Finally, (1.12.79) is obtained by particularizing (1.12.78) to the case when $X = \partial_j$, $Y = \partial_k$, keeping in mind (1.12.77) and the identity

$$\mathrm{div}_g \partial_r = \sum_{\ell=1}^{n} \Gamma^{\ell}_{r\ell} \quad \text{for every} \quad r \in \{1, \dots, n\}, \tag{1.12.86}$$

valid in any local coordinate system on M. □

In the geometric context considered in the proposition above, we shall say that a function $f \in L^p(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}$, with $1 \leq p \leq \infty$, belongs to the (global, boundary) Sobolev space $L^p_1(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}$ if for any two $\mathscr{C}^1$ vector fields $X, Y \in TM$ there exists some function $h_{XY} \in L^p(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}$ with the property that for each section $\varphi \in \mathscr{C}^1_c(M, \mathcal{E})$ one has

$$\int_{\partial_*\Omega} \langle f, \partial_{\tau_{XY}} \varphi \rangle_{\mathcal{E}} \, d\sigma_g = \int_{\partial_*\Omega} \langle h_{XY}, \varphi\big|_{\partial_*\Omega} \rangle_{\mathcal{E}} \, d\sigma_g. \tag{1.12.87}$$

It turns out that the function $h_{XY} \in L^p(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}$ doing the job in (1.12.87) is unique (see the discussion in [185, Sect. 11.6]). Comparing (1.12.87) with (1.12.78) then suggests making the following definition.

Definition 1.12.7 *Suppose M is a $\mathscr{C}^2$ manifold equipped with a Riemannian metric g, and consider a Hermitian vector bundle $\mathcal{E}$ over M of class $\mathscr{C}^2$. Assume all metrics involved are of class $\mathscr{C}^1$. Also, let $\Omega \subset M$ be a set of locally finite perimeter. Introduce $\sigma_g := \mathcal{H}^{n-1}_g \lfloor \partial\Omega$, and denote by ν_g the geometric measure theoretic outward unit conormal to Ω.*

Then for any function $f \in L^p_1(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}$ with $1 \leq p \leq \infty$, and any two compactly supported $\mathscr{C}^1$ vector fields X, Y on M, define

$$\partial_{\tau_{XY}} f := -h_{XY} - \big(\nu_g(X)\,\mathrm{div}_g Y - \nu_g(Y)\,\mathrm{div}_g X + \nu_g([X, Y])\big) f \tag{1.12.88}$$

at σ_g-a.e. point on $\partial_*\Omega$, where the function $h_{XY} \in L^p(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}$ is uniquely associated with f and X, Y as in (1.12.87) relative to a metric connection ∇ on $\mathcal{E}$.

Thanks to Corollary 1.12.6, this definition is then consistent with that made in (1.12.77) for functions in $\mathscr{C}^1(M, \mathcal{E})$. Other basic properties of the tangential differential operators $\partial_{\tau_{XY}}$ considered in (1.12.88) in the context of boundary Sobolev spaces are contained in the proposition below.

Proposition 1.12.8 *Let M be a $\mathscr{C}^2$ manifold equipped with a Riemannian metric g, and consider a Hermitian vector bundle $\mathcal{E}$ over M of class $\mathscr{C}^2$. Assume all metrics involved are of class $\mathscr{C}^1$. Also, let $\Omega \subset M$ be a set of locally finite perimeter. Introduce $\sigma_g := \mathcal{H}^{n-1}_g \lfloor \partial\Omega$, and denote by ν_g the geometric measure theoretic outward unit conormal to Ω.*

Then, given any compactly supported $\mathscr{C}^1$ vector fields X, Y on M,

$$\begin{aligned} &\partial_{\tau_{XY}} : L^p_1(\partial_*\Omega, \sigma_g) \otimes \mathcal{E} \longrightarrow L^p(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}, \ \ 1 \leq p \leq \infty, \\ &\qquad \text{is a well-defined, linear and bounded operator,} \end{aligned} \tag{1.12.89}$$

and

$$\partial_{\tau_{YX}} f = -\partial_{\tau_{XY}} f, \qquad \forall f \in L^p_1(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}. \tag{1.12.90}$$

Moreover, given any function $f \in L^p_1(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}$ *with* $1 \leq p \leq \infty$, *for every section* $\varphi \in \mathscr{C}^1_c(M, \mathcal{E})$ *and every compactly supported* $\mathscr{C}^1$ *vector fields* X, Y *on* M *one has*

$$\begin{aligned}\int_{\partial_*\Omega} \big\langle \partial_{\tau_{XY}} f, \varphi \big\rangle_{\mathcal{E}} \, \mathrm{d}\sigma_g &= -\int_{\partial_*\Omega} \big\langle f, \partial_{\tau_{XY}} \varphi \big\rangle_{\mathcal{E}} \, \mathrm{d}\sigma_g \\ &\quad - \int_{\partial_*\Omega} \big\langle f, \varphi \big\rangle_{\mathcal{E}} \big(\nu_g(X) \operatorname{div}_g Y - \nu_g(Y) \operatorname{div}_g X + \nu_g([X, Y]) \big) \, \mathrm{d}\sigma_g.\end{aligned} \tag{1.12.91}$$

In particular, if φ *is supported in a local patch on the manifold* M, *with local coordinates* $(x_1, \dots, x_n)$ *relative to which the outward unit normal is expressed as* $\nu_g = \sum_{r=1}^n \nu_r \mathrm{d}x_r$, *then for every index* $j, k \in \{1, \dots, n\}$ *one has*

$$\begin{aligned}\int_{\partial_*\Omega} \big\langle \partial_{\tau_{jk}} f, \varphi \big\rangle_{\mathcal{E}} \, \mathrm{d}\sigma_g &= -\int_{\partial_*\Omega} \big\langle f, \partial_{\tau_{jk}} \varphi \big\rangle_{\mathcal{E}} \, \mathrm{d}\sigma_g \\ &\quad - \sum_{\ell=1}^n \int_{\partial_*\Omega} \big\langle f, \varphi \big\rangle_{\mathcal{E}} \big(\nu_j \Gamma^\ell_{k\ell} - \nu_k \Gamma^\ell_{j\ell} \big) \, \mathrm{d}\sigma_g.\end{aligned} \tag{1.12.92}$$

Also, for each $p \in [1, \infty]$,

$$\big\{ \varphi\big|_{\partial_*\Omega} : \varphi \in \mathscr{C}^1_c(M, \mathcal{E}) \big\} \subset L^p_1(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}, \tag{1.12.93}$$

and for each $\varphi \in \mathscr{C}^1_c(M, \mathcal{E})$ *the following compatibility property holds for any compactly supported* $\mathscr{C}^1$ *vector fields* X, Y *on* M*:*

$$\begin{aligned}&\partial_{\tau_{XY}}\big(\varphi\big|_{\partial_*\Omega}\big), \text{ considered in the sense of (1.12.88) by viewing } \varphi\big|_{\partial_*\Omega} \\ &\text{in the Sobolev space } L^p_1(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}, \text{ agrees } \sigma_g\text{-a.e. on } \partial_*\Omega \text{ with} \\ &\nu_g(X)\big(\nabla_Y \varphi\big)\big|_{\partial_*\Omega} - \nu_g(Y)\big(\nabla_X \varphi\big)\big|_{\partial_*\Omega}.\end{aligned} \tag{1.12.94}$$

Lastly, for any function $f \in L^p_1(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}$ *with* $1 \leq p \leq \infty$, *any compactly supported* $\mathscr{C}^1$ *vector fields* X, Y *on* M, *any scalar-valued function* $\psi \in \mathscr{C}^1(M)$, *one has* $\psi f \in L^p_1(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}$, *while at* σ_g*-a.e. point on* $\partial_*\Omega$ *one has*

$$\partial_{\tau_{XY}}(\psi f) - \psi \partial_{\tau_{XY}} f = \big(\nu_g(X) Y(\psi) - \nu_g(Y) X(\psi) \big) f. \tag{1.12.95}$$

Proof The claims in (1.12.89)–(1.12.90) follow from definitions, while (1.12.91) follows from (1.12.88) and (1.12.78). The inclusion in (1.12.93) is seen from (1.12.78) and the discussion pertaining to (1.12.87). Next, the claim in (1.12.94) is a consequence of (1.12.78) and (1.12.88). Finally, given an arbitrary function $f \in L^p_1(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}$ with $1 \leq p \leq \infty$, along with two compactly supported $\mathscr{C}^1$ vector fields X, Y on M and a scalar-valued function $\psi \in \mathscr{C}^1(M)$, consider the task of proving (1.12.95). To this end, fix an arbitrary section $\varphi \in \mathscr{C}^1_c(M, \mathcal{E})$ and observe that

$$
\begin{aligned}
&\partial_{\tau_{XY}}(\psi\varphi) - \psi\partial_{\tau_{XY}}\varphi \\
&\qquad = \nu_g(X)\nabla_Y(\psi\varphi) - \nu_g(Y)\nabla_X(\psi\varphi) - \nu_g(X)\psi\nabla_Y\varphi - \nu_g(Y)\psi\nabla_X\varphi \\
&\qquad = \big(\nu_g(X)Y(\psi) - \nu_g(Y)X(\psi)\big)\varphi \quad \text{at } \sigma_g\text{-a.e. point on } \partial_*\Omega.
\end{aligned}
\tag{1.12.96}
$$

Next, based on (1.12.91) and (1.12.96) we compute

$$
\begin{aligned}
&\int_{\partial_*\Omega} \langle \psi f, \partial_{\tau_{XY}}\varphi\rangle_{\mathcal{E}}\, d\sigma_g \\
&\quad = \int_{\partial_*\Omega} \langle f, \psi\partial_{\tau_{XY}}\varphi\rangle_{\mathcal{E}}\, d\sigma_g \\
&\quad = \int_{\partial_*\Omega} \langle f, \partial_{\tau_{XY}}(\psi\varphi)\rangle_{\mathcal{E}}\, d\sigma_g \\
&\qquad - \int_{\partial_*\Omega} \big\langle f, \big(\nu_g(X)Y(\psi) - \nu_g(Y)X(\psi)\big)\varphi\big\rangle_{\mathcal{E}}\, d\sigma_g \\
&\quad = -\int_{\partial_*\Omega} \langle \partial_{\tau_{XY}} f, \psi\varphi\rangle_{\mathcal{E}}\, d\sigma_g \\
&\qquad - \int_{\partial_*\Omega} \langle f, \varphi\rangle_{\mathcal{E}}\big(\nu_g(X)\,\mathrm{div}_g Y - \nu_g(Y)\,\mathrm{div}_g X + \nu_g([X,Y])\big)\, d\sigma_g \\
&\qquad - \int_{\partial_*\Omega} \big\langle \big(\nu_g(X)Y(\psi) - \nu_g(Y)X(\psi)\big) f, \varphi\big\rangle_{\mathcal{E}}\, d\sigma_g.
\end{aligned}
\tag{1.12.97}
$$

From this and the discussion pertaining to (1.12.87) we then conclude that ψf belongs to $L^p_1(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}$. In addition, from (1.12.97), (1.12.87), and (1.12.88) we see that

$$
\partial_{\tau_{XY}}(\psi f) = \psi\partial_{\tau_{XY}} f + \big(\nu_g(X)Y(\psi) - \nu_g(Y)X(\psi)\big) f \tag{1.12.98}
$$

which establishes (1.12.95). □

We are now in a position to elaborate on the manner in which weak tangential derivatives interact with pointwise nontangential traces, in the manifold setting.

Proposition 1.12.9 *Let M be a $\mathscr{C}^2$ manifold equipped with a Riemannian metric g, and consider a Hermitian vector bundle $\mathcal{E}$ over M of class $\mathscr{C}^2$. Assume all metrics involved are of class $\mathscr{C}^1$. Also, let Ω be a relatively compact open subset of M, with* *a lower Ahlfors regular boundary, such that $\sigma_g := \mathcal{H}_g^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Denote by ν_g the geometric measure theoretic outward unit conormal to Ω. Pick some $p \in [1,\infty]$ and suppose $u : \Omega \to \mathcal{E}$ is a Lebesgue measurable section satisfying (with the dependence on the aperture parameter suppressed)*

$$\begin{gathered}\mathcal{N}u \in L^p(\partial\Omega,\sigma_g), \quad \mathcal{N}(\nabla u) \in L^p(\partial\Omega,\sigma_g),\\ \textit{and } u\big|_{\partial\Omega}^{\text{n.t.}}, (\nabla u)\big|_{\partial\Omega}^{\text{n.t.}} \textit{ exist } \sigma_g\textit{-a.e. on } \partial_{\text{nta}}\Omega.\end{gathered} \tag{1.12.99}$$

Then

$$\begin{gathered}\textit{the function } u\big|_{\partial\Omega}^{\text{n.t.}} \textit{ belongs to } L_1^p(\partial_*\Omega,\sigma_g)\otimes\mathcal{E} \textit{ and}\\ \Big\|u\big|_{\partial\Omega}^{\text{n.t.}}\Big\|_{L_1^p(\partial_*\Omega,\sigma_g)\otimes\mathcal{E}} \leq C\Big(\|\mathcal{N}u\|_{L^p(\partial\Omega,\sigma_g)} + \|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega,\sigma_g)}\Big)\end{gathered} \tag{1.12.100}$$

for some finite constant $C > 0$ independent of u. Moreover, for every two $\mathscr{C}^1$ vector fields X, Y, at σ_g-a.e. point on $\partial_*\Omega$ one has

$$\partial_{\tau_{XY}}\Big(u\big|_{\partial\Omega}^{\text{n.t.}}\Big) = \nu_g(X)\big(\nabla_Y u\big)\big|_{\partial\Omega}^{\text{n.t.}} - \nu_g(Y)\big(\nabla_X u\big)\big|_{\partial\Omega}^{\text{n.t.}}. \tag{1.12.101}$$

In particular, for every $j,k \in \{1,\dots,n\}$, one locally has

$$\partial_{\tau_{jk}}\Big(u\big|_{\partial\Omega}^{\text{n.t.}}\Big) = \nu_j\big(\nabla_{\partial_k} u\big)\big|_{\partial\Omega}^{\text{n.t.}} - \nu_k\big(\nabla_{\partial_j} u\big)\big|_{\partial\Omega}^{\text{n.t.}} \ \textit{ at } \sigma_g\textit{-a.e. point on } \partial_*\Omega. \tag{1.12.102}$$

Proof Clearly,

$$u\big|_{\partial\Omega}^{\text{n.t.}} \in L^p(\partial_*\Omega,\sigma_g)\otimes\mathcal{E}, \quad (\nabla u)\big|_{\partial\Omega}^{\text{n.t.}} \in L^p(\partial_*\Omega,\sigma_g)\otimes(T^*M\otimes\mathcal{E}), \tag{1.12.103}$$

and

$$\begin{aligned}\Big\|u\big|_{\partial\Omega}^{\text{n.t.}}\Big\|_{L^p(\partial_*\Omega,\sigma_g)\otimes\mathcal{E}} &+ \Big\|(\nabla u)\big|_{\partial\Omega}^{\text{n.t.}}\Big\|_{L^p(\partial_*\Omega,\sigma_g)\otimes(T^*M\otimes\mathcal{E})}\\ &\leq C\Big(\|\mathcal{N}u\|_{L^p(\partial\Omega,\sigma_g)} + \|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega,\sigma_g)}\Big),\end{aligned} \tag{1.12.104}$$

for some finite constant $C > 0$ independent of u. To proceed, fix an arbitrary function $\varphi \in \mathscr{C}_c^1(M,\mathcal{E})$ along with two $\mathscr{C}^1$ vector fields X, Y. Then, granted the assumptions on u, from formula (1.12.63) used with $P = \nabla_X$, $Q = \nabla_Y$, and (1.12.81)–(1.12.83) we obtain

$$\begin{aligned}&\int_{\partial_*\Omega}\Big\langle u\big|^{\text{n.t.}}_{\partial\Omega},\partial_{\tau_{XY}}\varphi\Big\rangle_{\mathcal{E}}\,d\sigma_g \qquad (1.12.105)\\ &\quad=-\int_{\partial_*\Omega}\Big\langle \nu_g(X)\big(\nabla_Y u\big)\big|^{\text{n.t.}}_{\partial\Omega}-\nu_g(Y)\big(\nabla_X u\big)\big|^{\text{n.t.}}_{\partial\Omega},\varphi\big|_{\partial_*\Omega}\Big\rangle_{\mathcal{E}}\,d\sigma_g\\ &\qquad-\int_{\partial_*\Omega}\Big\langle u\big|^{\text{n.t.}}_{\partial\Omega},\varphi\big|_{\partial_*\Omega}\Big\rangle_{\mathcal{E}}\big(\nu_g(X)\,\mathrm{div}_g Y-\nu_g(Y)\,\mathrm{div}_g X+\nu_g([X,Y])\big)\,d\sigma_g.\end{aligned}$$

From this, it follows that the function

$$\begin{aligned}h_{XY}&:=-\nu_g(X)\big(\nabla_Y u\big)\big|^{\text{n.t.}}_{\partial\Omega}+\nu_g(Y)\big(\nabla_X u\big)\big|^{\text{n.t.}}_{\partial\Omega} \qquad (1.12.106)\\ &\quad-\big(\nu_g(X)\,\mathrm{div}_g Y-\nu_g(Y)\,\mathrm{div}_g X+\nu_g([X,Y])\big)\big(u\big|^{\text{n.t.}}_{\partial\Omega}\big)\in L^p(\partial_*\Omega,\sigma_g)\otimes\mathcal{E}\end{aligned}$$

does the job in (1.12.87). Together with the discussion pertaining to (1.12.87), this shows that $u\big|^{\text{n.t.}}_{\partial\Omega}$ belongs to the Sobolev space $L^p_1(\partial_*\Omega,\sigma_g)\otimes\mathcal{E}$. Having established this, formula (1.12.88) from Definition 1.12.7 then gives that

$$\begin{aligned}\partial_{\tau_{XY}}\big(u\big|^{\text{n.t.}}_{\partial\Omega}\big)&=-h_{XY}-\big(\nu_g(X)\,\mathrm{div}_g Y-\nu_g(Y)\,\mathrm{div}_g X+\nu_g([X,Y])\big)\big(u\big|^{\text{n.t.}}_{\partial\Omega}\big)\\ &=\nu_g(X)\big(\nabla_Y u\big)\big|^{\text{n.t.}}_{\partial\Omega}-\nu_g(Y)\big(\nabla_X u\big)\big|^{\text{n.t.}}_{\partial\Omega}. \qquad (1.12.107)\end{aligned}$$

This justifies (1.12.101). Lastly, the estimate in the second line of (1.12.100) is implicit in what we have proved so far. □

We next present a basic integration by parts formula along the geometric measure theoretic boundary of an open set in a Riemannian manifold.

Corollary 1.12.10 *Let M be a $\mathscr{C}^2$ manifold equipped with a Riemannian metric g, and consider a Hermitian vector bundle $\mathcal{E}$ over M of class $\mathscr{C}^2$. Assume all metrics involved are of class $\mathscr{C}^1$. Also, let Ω be a relatively compact open subset of M, with a lower Ahlfors regular boundary, such that $\sigma_g:=\mathcal{H}^{n-1}_g\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Denote by ν_g the geometric measure theoretic outward unit conormal to Ω. Suppose $u,w:\Omega\to\mathcal{E}$ are two Lebesgue measurable sections which, for some $p,p'\in[1,\infty]$ with $1/p+1/p'=1$, satisfy* (*with the dependence on the aperture parameter suppressed*)

$$\begin{aligned}&\mathcal{N}u\in L^p(\partial\Omega,\sigma_g),\quad \mathcal{N}(\nabla u)\in L^p(\partial\Omega,\sigma_g),\\ &\mathcal{N}w\in L^{p'}(\partial\Omega,\sigma_g),\quad \mathcal{N}(\nabla w)\in L^{p'}(\partial\Omega,\sigma_g), \qquad (1.12.108)\\ &u\big|^{\text{n.t.}}_{\partial\Omega},\ w\big|^{\text{n.t.}}_{\partial\Omega},\ (\nabla u)\big|^{\text{n.t.}}_{\partial\Omega},\ (\nabla w)\big|^{\text{n.t.}}_{\partial\Omega}\ \textit{exist } \sigma_g\textit{-a.e. on } \partial_{\text{nta}}\Omega.\end{aligned}$$

Then

$$u\big|^{\text{n.t.}}_{\partial\Omega}\in L^p_1(\partial_*\Omega,\sigma_g)\otimes\mathcal{E},\quad w\big|^{\text{n.t.}}_{\partial\Omega}\in L^{p'}_1(\partial_*\Omega,\sigma_g)\otimes\mathcal{E}, \qquad (1.12.109)$$

and for each two $\mathscr{C}^1$ vector fields X, Y on M one has the integration by parts formula on the geometric measure theoretic boundary:

$$\int_{\partial_*\Omega} \Big\langle \partial_{\tau_{XY}}\big(u\big|^{\text{n.t.}}_{\partial\Omega}\big), w\big|^{\text{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{E}} \, d\sigma_g = -\int_{\partial_*\Omega} \Big\langle u\big|^{\text{n.t.}}_{\partial\Omega}, \partial_{\tau_{XY}}\big(w\big|^{\text{n.t.}}_{\partial\Omega}\big)\Big\rangle_{\mathcal{E}} \, d\sigma_g \tag{1.12.110}$$
$$- \int_{\partial_*\Omega} \Big\langle u\big|^{\text{n.t.}}_{\partial\Omega}, w\big|^{\text{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{E}} \big(\nu_g(X)\,\mathrm{div}_g Y - \nu_g(Y)\,\mathrm{div}_g X + \nu_g([X,Y])\big)\, d\sigma_g.$$

In particular, if one of the sections u, w is supported in a coordinate patch on M then, in local coordinates, for each $j, k \in \{1, \dots, n\}$ one has

$$\int_{\partial_*\Omega} \Big\langle \partial_{\tau_{jk}}\big(u\big|^{\text{n.t.}}_{\partial\Omega}\big), w\big|^{\text{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{E}} \, d\sigma_g = -\int_{\partial_*\Omega} \Big\langle u\big|^{\text{n.t.}}_{\partial\Omega}, \partial_{\tau_{jk}}\big(w\big|^{\text{n.t.}}_{\partial\Omega}\big)\Big\rangle_{\mathcal{E}} \, d\sigma_g \tag{1.12.111}$$
$$- \int_{\partial_*\Omega} \Big\langle u\big|^{\text{n.t.}}_{\partial\Omega}, w\big|^{\text{n.t.}}_{\partial\Omega}\Big\rangle_{\mathcal{E}} \big(\nu_j \Gamma^{\ell}_{k\ell} - \nu_k \Gamma^{\ell}_{j\ell}\big)\, d\sigma_g.$$

Proof The memberships in (1.12.109) are direct consequences of (1.12.100) and assumptions. Formula (1.12.110) is implied by (1.12.63) used with P, Q as in (1.12.80), reasoning much as in (1.12.81)–(1.12.84). Finally, (1.12.111) is a consequence of (1.12.110) and (1.12.86). □

Regarding the results presented so far in this section, there is a more general phenomenon at play, which we would like to describe. To set things up, let the Riemannian manifold (M, g), the Hermitian vector bundles $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$, and the first-order differential operators P, Q be as in Theorem 1.12.1. That is,

$$\begin{aligned} P : \mathscr{C}^1(M, \mathcal{E}) \longrightarrow \mathscr{C}^0(M, \mathcal{H}) \ \text{ and} \\ Q : \mathscr{C}^1(M, \mathcal{H}) \longrightarrow \mathscr{C}^0(M, \mathcal{F}) \end{aligned} \tag{1.12.112}$$

are two first-order differential operators with $\mathscr{C}^1$ coefficients for the top part and $\mathscr{C}^0$ coefficients for the zero-th order part, having the property that their principal symbols satisfy the cancelation condition

$$\mathrm{Sym}(Q;\xi)\,\mathrm{Sym}(P;\xi) = 0, \qquad \forall \xi \in T^*M. \tag{1.12.113}$$

Next, consider a relatively compact set of locally finite perimeter $\Omega \subset M$ and define $\sigma_g := \mathcal{H}^{n-1}_g \lfloor \partial\Omega$. Also, denote by ν_g the geometric measure theoretic outward unit conormal to Ω. Work in local coordinates $(x_1, \dots, x_n)$ and express

$$P = A_j \nabla_{\partial_j} + A_0, \qquad Q = B_k \nabla_{\partial_k} + B_0 \tag{1.12.114}$$

for some matrices A_j, B_k. As such, the cancelation property (1.12.113) reads

$$\xi_j \xi_k B_k A_j = 0 \ \text{ for every cotangent vector } \ \xi = \xi_\ell \, dx_\ell. \tag{1.12.115}$$

In turn, this is equivalent to having

$$B_k A_j = -B_j A_k \quad \text{for every} \quad j, k \in \{1, \dots, n\}. \tag{1.12.116}$$

Granted this, if we locally express $\nu_g = \nu_r \mathrm{d}x_r$, we may then write

$$\begin{aligned} \mathrm{iSym}(Q; \nu_g)P &= -B_k A_j \nu_k \nabla_{\partial_j} - \nu_\ell B_\ell A_0 \\ &= \tfrac{1}{2}\big(B_k A_j \nu_j \nabla_{\partial_k} - B_k A_j \nu_k \nabla_{\partial_j}\big) - \nu_\ell B_\ell A_0 \\ &= \tfrac{1}{2} B_k A_j \partial_{\tau_{jk}} - \nu_\ell B_\ell A_0. \end{aligned} \tag{1.12.117}$$

Bearing (1.12.89) in mind, for each $p \in [1, \infty]$ we then proceed to define the bounded linear mapping

$$\begin{aligned} \partial_\tau^{Q,P} : L_1^p(\partial_*\Omega, \sigma_g) \otimes \mathcal{E} &\longrightarrow L^p(\partial_*\Omega, \sigma_g) \otimes \mathcal{F}, \\ \partial_\tau^{Q,P} f &:= \tfrac{1}{2} B_k A_j \partial_{\tau_{jk}} f - \nu_\ell B_\ell A_0 f. \end{aligned} \tag{1.12.118}$$

From (1.12.118), (1.12.117), and (1.12.93)–(1.12.94) we may then conclude that, under the embedding (1.12.93),

$$\partial_\tau^{Q,P}\big(\psi\big|_{\partial_*\Omega}\big) = \mathrm{iSym}(Q; \nu_g)(P\psi)\big|_{\partial_*\Omega}, \qquad \forall \psi \in \mathscr{C}_c^1(M, \mathcal{E}). \tag{1.12.119}$$

Let us also note that since the (ordered pair of) differential operators $Q^\top$, $P^\top$ satisfy analogous properties as the original P, Q, we may define

$$\partial_\tau^{P^\top, Q^\top} : L_1^p(\partial_*\Omega, \sigma_g) \otimes \mathcal{F} \longrightarrow L^p(\partial_*\Omega, \sigma_g) \otimes \mathcal{E}, \qquad p \in [1, \infty] \tag{1.12.120}$$

in a similar fashion to (1.12.118) and conclude that

$$\partial_\tau^{P^\top, Q^\top}\big(\varphi\big|_{\partial_*\Omega}\big) = \mathrm{iSym}(P^\top; \nu_g)(Q^\top\varphi)\big|_{\partial_*\Omega}, \qquad \forall \varphi \in \mathscr{C}_c^1(M, \mathcal{F}). \tag{1.12.121}$$

Having made these definitions, we are ready to state and prove the following versatile boundary integration by parts formula.

Theorem 1.12.11 *Let M be a $\mathscr{C}^2$ manifold equipped with a Riemannian metric g, and consider three Hermitian vector bundles, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$, over M, of class $\mathscr{C}^2$. Suppose all metrics involved are of class $\mathscr{C}^1$. Assume*

$$\begin{aligned} P : \mathscr{C}^1(M, \mathcal{E}) \longrightarrow \mathscr{C}^0(M, \mathcal{H}) \ \ \textit{and} \\ Q : \mathscr{C}^1(M, \mathcal{H}) \longrightarrow \mathscr{C}^0(M, \mathcal{F}) \end{aligned} \tag{1.12.122}$$

are two first-order differential operators with $\mathscr{C}^1$ coefficients for the top part and $\mathscr{C}^0$ coefficients for the zero-th order part, having the property that their principal symbols satisfy

$$\mathrm{Sym}(Q;\xi)\,\mathrm{Sym}(P;\xi)=0,\qquad \forall\xi\in T^*M. \tag{1.12.123}$$

Denote by $P^\top:\mathscr{C}^1(M,\mathcal{H})\to\mathscr{C}^0(M,\mathcal{E})$ *and* $Q^\top:\mathscr{C}^1(M,\mathcal{F})\to\mathscr{C}^0(M,\mathcal{H})$ *the (real) transposes of* P, Q *(considered in the usual sense on the manifold* M*). Finally, given a set of locally finite perimeter* $\Omega\subset M$*, introduce* $\sigma_g:=\mathcal{H}_g^{n-1}\lfloor\partial\Omega$*, and denote by* ν_g *the geometric measure theoretic outward unit conormal to* Ω*.*

Then for any $f\in L_1^p(\partial_*\Omega,\sigma_g)\otimes\mathcal{E}$ *with* $1\le p\le\infty$*, and any* $\varphi\in\mathscr{C}_c^1(M,\mathcal{F})$ *one has*

$$\begin{aligned}
&\int_{\partial_*\Omega}\Big\langle\partial_\tau^{Q,P}f,\varphi\big|_{\partial_*\Omega}\Big\rangle_{\mathcal{F}}\,d\sigma_g \qquad (1.12.124)\\
&\quad=\int_{\partial_*\Omega}\Big\langle f,\partial_\tau^{P^\top,Q^\top}\big(\varphi\big|_{\partial_*\Omega}\big)\Big\rangle_{\mathcal{E}}\,d\sigma_g+\int_{\partial_*\Omega}\Big\langle i\mathrm{Sym}\big(QP;\nu_g\big)f,\varphi\big|_{\partial_*\Omega}\Big\rangle_{\mathcal{E}}\,d\sigma_g\\
&\quad=\int_{\partial_*\Omega}\Big\langle f,\partial_\tau^{P^\top,Q^\top}\big(\varphi\big|_{\partial_*\Omega}\big)-i\mathrm{Sym}\big(P^\top Q^\top;\nu_g\big)\varphi\big|_{\partial_*\Omega}\Big\rangle_{\mathcal{E}}\,d\sigma_g\\
&\quad=\int_{\partial_*\Omega}\Big\langle f,i\mathrm{Sym}(P^\top;\nu_g)(Q^\top\varphi)\big|_{\partial_*\Omega}-i\mathrm{Sym}\big(P^\top Q^\top;\nu_g\big)\varphi\big|_{\partial_*\Omega}\Big\rangle_{\mathcal{E}}\,d\sigma_g,
\end{aligned}$$

where all principal symbols are taken in the sense of first-order differential operators.

Proof Fix $\varphi\in\mathscr{C}_c^1(M,\mathcal{F})$ along with $f\in L_1^p(\partial_*\Omega,\sigma_g)\otimes\mathcal{E}$ for some $p\in[1,\infty]$. There is no loss of generality in assuming that φ is supported in a coordinate patch. Working locally, by making use of (1.12.118) and (1.12.92) we may write

$$\begin{aligned}
\int_{\partial_*\Omega}\langle\partial_\tau^{Q,P}f,\varphi\big|_{\partial_*\Omega}\rangle_{\mathcal{F}}\,d\sigma_g&=\int_{\partial_*\Omega}\langle\tfrac12 B_kA_j\partial_{\tau_{jk}}f-\nu_\ell B_\ell A_0f,\varphi\big|_{\partial_*\Omega}\rangle_{\mathcal{F}}\,d\sigma_g\\
&=-\int_{\partial_*\Omega}\langle f,\tfrac12\partial_{\tau_{jk}}\big(A_j^\top B_k^\top\varphi\big|_{\partial_*\Omega}\big)\rangle_{\mathcal{E}}\,d\sigma_g\\
&\quad-\sum_{\ell=1}^n\int_{\partial_*\Omega}\langle f,\tfrac12A_j^\top B_k^\top\varphi\big|_{\partial_*\Omega}\rangle_{\mathcal{E}}\big(\nu_j\Gamma_{k\ell}^\ell-\nu_k\Gamma_{j\ell}^\ell\big)\,d\sigma_g\\
&\quad-\int_{\partial_*\Omega}\langle f,\nu_\ell A_0^\top B_\ell^\top\varphi\big|_{\partial_*\Omega}\rangle_{\mathcal{E}}\,d\sigma_g.
\end{aligned}\tag{1.12.125}$$

Given the goal we have in mind, in view of (1.12.121) it remains to show that

$$\begin{aligned}
&i\mathrm{Sym}(P^\top;\nu_g)(Q^\top\varphi)\big|_{\partial_*\Omega}-i\mathrm{Sym}\big(P^\top Q^\top;\nu_g\big)\varphi\big|_{\partial_*\Omega}\\
&\qquad=-\tfrac12\partial_{\tau_{jk}}\big(A_j^\top B_k^\top\varphi\big|_{\partial_*\Omega}\big)\\
&\qquad\quad-\sum_{\ell=1}^n\big(\nu_j\Gamma_{k\ell}^\ell-\nu_k\Gamma_{j\ell}^\ell\big)\tfrac12A_j^\top B_k^\top\varphi\big|_{\partial_*\Omega}\\
&\qquad\quad-\nu_\ell A_0^\top B_\ell^\top\varphi\big|_{\partial_*\Omega}
\end{aligned}\tag{1.12.126}$$

at σ_g-a.e. point on $\partial_*\Omega$. Thanks to Proposition 3.7.2, it suffices to check that the two sides of (1.12.126) match when paired (in an integral sense) with the restriction to $\partial_*\Omega$ of an arbitrary section $\psi \in \mathscr{C}^1_c(M,\mathcal{E})$. To this end, fix such a ψ and write

$$\begin{aligned}
&-\int_{\partial_*\Omega}\big\langle\psi\big|_{\partial_*\Omega}, \tfrac{1}{2}\partial_{\tau_{jk}}\big(A_j^\top B_k^\top\varphi\big|_{\partial_*\Omega}\big)\big\rangle_{\mathcal{E}}\,d\sigma_g\\
&-\sum_{\ell=1}^n\int_{\partial_*\Omega}\big\langle\psi\big|_{\partial_*\Omega}, \tfrac{1}{2}A_j^\top B_k^\top\varphi\big|_{\partial_*\Omega}\big\rangle_{\mathcal{E}}\big(\nu_j\Gamma^\ell_{k\ell}-\nu_k\Gamma^\ell_{j\ell}\big)\,d\sigma_g\\
&-\int_{\partial_*\Omega}\big\langle\psi\big|_{\partial_*\Omega}, \nu_\ell A_0^\top B_\ell^\top\varphi\big|_{\partial_*\Omega}\big\rangle_{\mathcal{E}}\,d\sigma_g\\
&\qquad=\int_{\partial_*\Omega}\big\langle\tfrac{1}{2}B_kA_j\partial_{\tau_{jk}}\big(\psi\big|_{\partial_*\Omega}\big)-\nu_\ell B_\ell A_0\psi\big|_{\partial_*\Omega}, \varphi\big|_{\partial_*\Omega}\big\rangle_{\mathcal{F}}\,d\sigma_g\\
&\qquad=\int_{\partial_*\Omega}\big\langle\partial_\tau^{Q,P}\big(\psi\big|_{\partial_*\Omega}\big), \varphi\big|_{\partial_*\Omega}\big\rangle_{\mathcal{F}}\,d\sigma_g\\
&\qquad=\int_{\partial_*\Omega}\big\langle\mathrm{iSym}(Q;\nu_g)(P\psi)\big|_{\partial_*\Omega}, \varphi\big|_{\partial_*\Omega}\big\rangle_{\mathcal{F}}\,d\sigma_g\\
&\qquad=\int_{\partial_*\Omega}\big\langle\psi\big|_{\partial_*\Omega}, \mathrm{iSym}(P^\top;\nu_g)(Q^\top\varphi)\big|_{\partial_*\Omega}\big\rangle_{\mathcal{F}}\,d\sigma_g\\
&\qquad\quad-\int_{\partial_*\Omega}\Big\langle\psi\big|_{\partial_*\Omega}, \mathrm{iSym}\big(P^\top Q^\top;\nu_g\big)\varphi\big|_{\partial_*\Omega}\Big\rangle_{\mathcal{F}}\,d\sigma_g.
\end{aligned}\tag{1.12.127}$$

Above, the first equality is implied by (1.12.79), the second equality follows from (1.12.118), the third equality is seen from (1.12.119), and the last equality is a consequence of the very last claim in Theorem 1.12.1. Having proved (1.12.127), we conclude that (1.12.126) holds, and this finishes the proof of the theorem. □

Here is a special case of Theorem 1.12.11 when the cancelation condition (1.12.123) is automatically satisfied.

Corollary 1.12.12 *Let M be a $\mathscr{C}^2$ manifold equipped with a Riemannian metric, and let $\mathcal{E}$, $\mathcal{F}$ be Hermitian vector bundles, over M, of class $\mathscr{C}^2$. Suppose all metrics involved are of class $\mathscr{C}^1$. Assume*

$$P:\mathcal{F}\longrightarrow\mathcal{E}\tag{1.12.128}$$

is a first-order differential operator with $\mathscr{C}^1$ coefficients for the top part and $\mathscr{C}^0$ coefficients for the zero-th order part, and denote by $P^\top:\mathcal{E}\to\mathcal{F}$ its transposed (considered in the usual sense on the manifold M).

Next, fix a $\mathscr{C}^1$-vector field $X\in TM$ and denote by $\nabla_X^{\mathcal{E}}$, $\nabla_X^{\mathcal{F}}$, the covariant derivatives along X for sections of $\mathcal{E}$ and $\mathcal{F}$, respectively. Also, introduce the first-order differential operators

$$\begin{aligned}&\mathbf{P}:\mathcal{E}\to\mathcal{F}\oplus\mathcal{E}, &&\mathbf{P}u:=\big(P^\top u,-\nabla_X^{\mathcal{E}}u\big),\\ &\mathbf{Q}:\mathcal{F}\oplus\mathcal{E}\to\mathcal{F}, &&\mathbf{Q}(v,w):=\nabla_X^{\mathcal{F}}v+P^\top w.\end{aligned} \tag{1.12.129}$$

Consider a set $\Omega\subset M$ of locally finite perimeter, let $\sigma:=\mathcal{H}_g^{n-1}\lfloor\partial\Omega$, and denote by ν_g the geometric measure theoretic outward unit conormal to Ω.

Then the operators $\mathbf{P}$, $\mathbf{Q}$ satisfy

$$\mathrm{Sym}(\mathbf{Q};\xi)\,\mathrm{Sym}(\mathbf{P};\xi)=0,\qquad\forall\xi\in T^*M. \tag{1.12.130}$$

Consequently, for each $p\in[1,\infty]$, the mapping

$$\partial_\tau^{\mathbf{Q},\mathbf{P}}:L_1^p(\partial\Omega,\sigma_g)\otimes\mathcal{E}\longrightarrow L^p(\partial\Omega,\sigma_g)\otimes\mathcal{F}, \tag{1.12.131}$$

associated with $\mathbf{P}$, $\mathbf{Q}$ as in (1.12.118), is well defined, linear, and continuous. Moreover, for any two given functions, $f\in L_1^p(\partial\Omega,\sigma_g)\otimes\mathcal{E}$ with $1\leq p\leq\infty$, and $\varphi\in\mathscr{C}_c^1(M)\otimes\mathcal{F}$, one has

$$\begin{aligned}&\int_{\partial_*\Omega}\big\langle\partial_\tau^{\mathbf{Q},\mathbf{P}}f,\varphi\big|_{\partial_*\Omega}\big\rangle_{\mathcal{F}}\,\mathrm{d}\sigma_g\\ &\quad=\int_{\partial_*\Omega}\Big\langle f,\mathrm{i}\mathrm{Sym}(P;\nu_g)\big((\nabla_X^{\mathcal{F}})^*\varphi\big)\big|_{\partial_*\Omega}-\nu_g(X)\big(P\varphi\big)\big|_{\partial_*\Omega}\\ &\qquad\qquad+\mathrm{i}\mathrm{Sym}\Big(\nabla_X^{\mathcal{F}}P^\top-P^\top\nabla_X^{\mathcal{E}};\nu_g\Big)^\top\varphi\big|_{\partial_*\Omega}\Big\rangle_{\mathcal{E}}\,\mathrm{d}\sigma_g,\end{aligned} \tag{1.12.132}$$

where all principal symbols are taken in the sense of first-order differential operators.

Proof The covariant derivative $\nabla_X^{\mathcal{F}}$ associated with the connection $\nabla^{\mathcal{F}}$ on $\mathcal{F}$ and the vector field $X=X_j\partial_j\in TM$ is a first-order differential operator whose principal symbol is given by (see, e.g., [189, Proposition 9.3, pp. 375–377])

$$\begin{aligned}\mathrm{Sym}(\nabla_X^{\mathcal{F}};\xi)&=\mathrm{i}\xi(X)\,I=\mathrm{i}X_j\xi_j\,I\\ &=\mathrm{i}\langle X^\flat,\xi\rangle\,I=\mathrm{i}\langle X,\xi^\sharp\rangle\,I,\qquad\forall\xi=\xi_j\mathrm{d}x_j\in T^*M,\end{aligned} \tag{1.12.133}$$

where $\langle\cdot,\cdot\rangle$ denotes the pointwise inner product in T^*M, and I is the identity operator. Keeping this in mind, (1.12.132) follows from Theorem 1.12.11 applied to the operators $\mathbf{P}$, $\mathbf{Q}$. □

With the definition (1.12.118) of the tangential differential operators $\partial_\tau^{Q,P}$ in place, it is now possible to prove a generalization of Proposition 1.12.9 of the sort discussed below.

Theorem 1.12.13 *Let M be a $\mathscr{C}^2$ manifold equipped with a Riemannian metric g, and consider three Hermitian vector bundles, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{H}$, over M, of class $\mathscr{C}^2$. Assume all metrics involved are of class $\mathscr{C}^1$. Also, let Ω be a relatively compact open subset* *of M, with a lower Ahlfors regular boundary, and such that $\sigma_g := \mathcal{H}_g^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Denote by ν_g the geometric measure theoretic outward unit conormal to Ω. Pick some $p \in [1,\infty]$ and suppose $u : \Omega \to \mathcal{E}$ is a Lebesgue measurable section satisfying (with the dependence on the aperture parameter suppressed)*

$$\begin{aligned}&\mathcal{N}u \in L^p(\partial\Omega,\sigma_g), \quad \mathcal{N}(\nabla u) \in L^p(\partial\Omega,\sigma_g),\\&\text{and } u\big|_{\partial\Omega}^{\text{n.t.}}, (\nabla u)\big|_{\partial\Omega}^{\text{n.t.}} \text{ exist } \sigma_g\text{-a.e. on } \partial_{\text{nta}}\Omega.\end{aligned}\tag{1.12.134}$$

In addition, consider two first-order differential operators

$$\begin{aligned}P : \mathscr{C}^1(M,\mathcal{E}) \longrightarrow \mathscr{C}^0(M,\mathcal{H}) \text{ and}\\Q : \mathscr{C}^1(M,\mathcal{H}) \longrightarrow \mathscr{C}^0(M,\mathcal{F})\end{aligned}\tag{1.12.135}$$

with $\mathscr{C}^1$ coefficients for the top part and $\mathscr{C}^0$ coefficients for the zero-th order part, having the property that their principal symbols satisfy the cancelation condition

$$\mathrm{Sym}(Q;\xi)\,\mathrm{Sym}(P;\xi) = 0, \qquad \forall \xi \in T^*M. \tag{1.12.136}$$

Then the function $u\big|_{\partial\Omega}^{\text{n.t.}}$ belongs to the boundary Sobolev space $L_1^p(\partial_*\Omega,\sigma_g)\otimes\mathcal{E}$, and at σ_g-a.e. point on $\partial_*\Omega$ one has

$$\partial_\tau^{Q,P}\big(u\big|_{\partial\Omega}^{\text{n.t.}}\big) = i\mathrm{Sym}(Q;\nu_g)(Pu)\big|_{\partial\Omega}^{\text{n.t.}}. \tag{1.12.137}$$

Proof That $f := u\big|_{\partial\Omega}^{\text{n.t.}}$ belongs to $L_1^p(\partial_*\Omega,\sigma_g)\otimes\mathcal{E}$ has been already proved in Proposition 1.12.9. To justify the identity in (1.12.137), pick an arbitrary section $\varphi \in \mathscr{C}_c^1(M,\mathcal{F})$. In addition, as before, denote by $P^\top : \mathscr{C}^1(M,\mathcal{H}) \to \mathscr{C}^0(M,\mathcal{E})$ and $Q^\top : \mathscr{C}^1(M,\mathcal{F}) \to \mathscr{C}^0(M,\mathcal{H})$ the (real) transposes of P, Q (considered in the usual sense on the manifold M). Then, on account of (1.12.124), (1.12.121), and (1.12.6), we may write (with all principal symbols taken in the sense of first-order differential operators)

$$\begin{aligned}\int_{\partial_*\Omega}\big\langle\partial_\tau^{Q,P}\big(u\big|_{\partial\Omega}^{\text{n.t.}}\big),\varphi\big|_{\partial_*\Omega}\big\rangle_{\mathcal{F}}\,d\sigma_g &= \int_{\partial_*\Omega}\big\langle u\big|_{\partial\Omega}^{\text{n.t.}},\partial_\tau^{P^\top,Q^\top}\big(\varphi\big|_{\partial_*\Omega}\big)\big\rangle_{\mathcal{E}}\,d\sigma_g\\&\quad-\int_{\partial_*\Omega}\big\langle u\big|_{\partial\Omega}^{\text{n.t.}}, i\mathrm{Sym}\big(P^\top Q^\top;\nu_g\big)\varphi\big|_{\partial_*\Omega}\big\rangle_{\mathcal{E}}\,d\sigma_g\\&=\int_{\partial_*\Omega}\big\langle u\big|_{\partial\Omega}^{\text{n.t.}}, i\mathrm{Sym}(P^\top;\nu_g)(Q^\top\varphi)\big|_{\partial_*\Omega}\big\rangle_{\mathcal{E}}\,d\sigma_g\\&\quad-\int_{\partial_*\Omega}\big\langle u\big|_{\partial\Omega}^{\text{n.t.}}, i\mathrm{Sym}\big(P^\top Q^\top;\nu_g\big)\varphi\big|_{\partial_*\Omega}\big\rangle_{\mathcal{E}}\,d\sigma_g\\&=\int_{\partial_*\Omega}\Big\langle i\mathrm{Sym}(Q;\nu_g)(Pu)\big|_{\partial\Omega}^{\text{n.t.}},\varphi\big|_{\partial_*\Omega}\Big\rangle_{\mathcal{F}}\,d\sigma_g.\end{aligned}\tag{1.12.138}$$

Having established this, Proposition 3.7.2 applies and yields (1.12.137). □

To proceed, the reader is reminded that, given a Riemannian manifold (M, g) of real dimension n, the `musical isomorphisms` (aka metric identifications) between tangent and cotangent vectors are given by

$$\begin{aligned} TM \ni X = X_j \partial_j &\longmapsto X^\flat := g_{jk} X_k \mathrm{d}x_j \in T^*M, \\ T^*M \ni \xi = \xi_j \mathrm{d}x_j &\longmapsto \xi^\sharp := g^{jk} \xi_j \partial_k \in TM. \end{aligned} \tag{1.12.139}$$

These satisfy, for each $X, Y \in TM$, $\xi, \eta \in T^*M$, and $i, j \in \{1, \dots, n\}$,

$$\begin{aligned} &\langle \xi^\sharp, X \rangle = \xi(X) = \langle \xi, X^\flat \rangle, \quad X^\flat(Y) = \langle X, Y \rangle = \langle X^\flat, Y^\flat \rangle, \\ &(\xi^\sharp)^\flat = \xi, \quad \langle \xi^\sharp, \eta^\sharp \rangle = \langle \xi, \eta \rangle = \xi(\eta^\sharp), \quad (X^\flat)^\sharp = X, \\ &(\partial_j)^\flat = g_{jk}\, \mathrm{d}x_k, \quad (\mathrm{d}x_j)^\sharp = g^{jk}\, \partial_k, \quad \partial_j = g_{jk} (\mathrm{d}x_k)^\sharp, \\ &\mathrm{d}x_j = g^{jk} (\partial_k)^\flat, \quad \langle \mathrm{d}x_j, (\partial_i)^\flat \rangle = \delta_{ij}, \quad \langle (\mathrm{d}x_j)^\sharp, \partial_i \rangle = \delta_{ij}. \end{aligned} \tag{1.12.140}$$

We conclude this section with an integration by parts formula on an open patch Ω of a codimension one submanifold M on an ambient Riemannian manifold $\widetilde{M}$.

Theorem 1.12.14 *Let $\widetilde{M}$ be a $\mathscr{C}^2$ manifold of dimension $n+1$, equipped with a Riemannian metric g, and consider a relatively compact $\mathscr{C}^1$ domain $D \subset \widetilde{M}$. Denote by $\mathcal{L}_g^n$ the surface measure on $\partial\Omega$, and let $N_g : \partial D \to T^*\widetilde{M}$ stand for the unit conormal to D. In particular, $M := \partial D$ is a compact, oriented, submanifold of $\widetilde{M}$, of dimension n, which may be naturally equipped with the Riemannian metric inherited from $\widetilde{M}$. Denote by $\mathcal{H}_g^{n-1}$ the $(n-1)$-dimensional Hausdorff measure induced on M by said metric. In addition, consider a Hermitian vector bundle $\mathcal{E}$, of class $\mathscr{C}^2$ over the manifold $\widetilde{M}$. Assume all metrics involved are of class $\mathscr{C}^1$.*

Next, let Ω be a relatively open subset of M, with a lower Ahlfors regular boundary, such that $\sigma_g := \mathcal{H}_g^{n-1} \lfloor \partial\Omega$ is a doubling measure (all relative to the manifold M, as ambient). Denote by $\nu_g : \partial_*\Omega \to T^*M$ the geometric measure theoretic outward unit conormal to Ω and select two integrability exponents $p, p' \in [1, \infty]$ such that $1/p + 1/p' = 1$. Also, consider an open neighborhood $\mathcal{O}$ of Ω in $\widetilde{M}$ and pick two sections[26]

$$u \in W^{1,1}_{\mathrm{loc}}(\Omega) \otimes \mathcal{E}, \quad w \in \mathscr{C}^1(\mathcal{O}, \mathcal{E}), \tag{1.12.141}$$

satisfying (with the nontangential maximal operator and nontangential traces taken from within Ω, and with the dependence on the aperture parameter suppressed throughout)

[26] Where $W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local Sobolev space in Ω, with respect to the measure $\mathcal{L}_g^n$.

$$\mathcal{N}u \in L^p(\partial\Omega, \sigma_g), \quad \mathcal{N}\big(w\big|_{\Omega}\big) \in L^{p'}(\partial\Omega, \sigma_g), \\ u\Big|_{\partial\Omega}^{\text{n.t.}} \text{ and } (w|_{\Omega})\Big|_{\partial\Omega}^{\text{n.t.}} \text{ exist at } \sigma_g\text{-a.e. point on } \partial_{\text{nta}}\Omega, \tag{1.12.142}$$

as well as

$$|\partial_{\tau_{XY}} u||w| + |u||\partial_{\tau_{XY}} w| + |u||w| \in L^1(\Omega, \mathcal{L}_g^n). \tag{1.12.143}$$

Lastly, fix two arbitrary $\mathscr{C}^1$ vector fields X, Y on $\widetilde{M}$ and denote by $\partial_{\tau_{XY}}$ the tangential derivative operator along $M = \partial D$. (cf. Definition 1.12.7). Then the following integration by parts formula in Ω holds:

$$\begin{aligned} \int_{\Omega} \big\langle \partial_{\tau_{XY}} u, w \big\rangle_{\mathcal{E}} \, d\mathcal{L}_g^n &= -\int_{\Omega} \big\langle u, \partial_{\tau_{XY}}(w|_{\Omega}) \big\rangle_{\mathcal{E}} \, d\mathcal{L}_g^n \\ &\quad - \int_{\Omega} \langle u, w \rangle_{\mathcal{E}} \big(N_g(X)\, \mathrm{div}_g Y - N_g(Y)\, \mathrm{div}_g X + N_g([X, Y]) \big) \, d\mathcal{L}_g^n \\ &\quad + \int_{\partial_*\Omega} \Big\langle \big(N_g(X)\widetilde{\nu}_g(Y) - N_g(Y)\widetilde{\nu}_g(X)\big)\big(u\big|_{\partial\Omega}^{\text{n.t.}}\big), \big((w|_{\Omega})\big|_{\partial\Omega}^{\text{n.t.}}\big) \Big\rangle_{\mathcal{E}} \, d\sigma_g \end{aligned} \tag{1.12.144}$$

where $\widetilde{\nu}_g$ is the unique extension of $\nu_g \in T^*M$ to a linear functional $\widetilde{\nu}_g \in T^*\widetilde{M}$ such that $\widetilde{\nu}_g(N_g) = 0$.

Proof By design,

$$|N_g| = 1 \ \text{ and } \ N_g(Z) = 0 \ \text{ for each } \ Z \in TM. \tag{1.12.145}$$

With the help of the musical isomorphism on the Riemannian manifold $(\widetilde{M}, g)$,

$$T^*\widetilde{M} \ni \eta = \sum_j \eta_j dx_j \longmapsto \eta^\sharp := \sum_{j,k} g^{jk} \eta_j \partial_k \in T\widetilde{M}, \tag{1.12.146}$$

we can consider $N_g^\sharp \in T\widetilde{M}$ which, thanks to (1.12.145) and (1.12.140), satisfies

$$|N_g^\sharp| = 1, \quad \langle N_g^\sharp, Z\rangle = N_g(Z) \ \text{ for each } \ Z \in T\widetilde{M}, \ \text{ and} \\ N_g^\sharp \text{ is orthogonal to } TM \text{ with respect to the inner product in } T\widetilde{M}. \tag{1.12.147}$$

Granted these properties, we may then express the orthogonal projection operator of $T\widetilde{M}$ onto its subspace TM as

$$T\widetilde{M} \ni Z \longmapsto Z - N_g(Z) N_g^\sharp \in TM. \tag{1.12.148}$$

In turn, this allows us to introduce the assignment

$$\begin{aligned} T^*M \ni \xi &\longmapsto \widetilde{\xi} \in T^*\widetilde{M} \\ \widetilde{\xi}(Z) &:= \xi(Z - N_g(Z)N_g^\sharp), \quad \forall Z \in T\widetilde{M}, \end{aligned} \tag{1.12.149}$$

lifting covectors on M to covectors on $\widetilde{M}$. With this mechanism in place, we then proceed to define the vector field $\vec{F} : \Omega \to TM$ by asking that

$${}_{T^*M}\big(\xi, \vec{F}\big)_{TM} = \Big\langle \big(N_g(X)\widetilde{\xi}(Y) - N_g(Y)\widetilde{\xi}(X)\big)u, w\Big\rangle_{\mathcal{E}} \tag{1.12.150}$$

for $\mathcal{L}_g^n$-a.e. point in Ω, and each covector ξ in the corresponding fiber in T^*M. The linearity of the right-hand side in ξ ensures that this is a well-defined object. Also, from (1.12.7) we conclude that $\vec{F} \in L^1_{\rm loc}(\Omega, \mathcal{L}_g^n) \otimes TM$. We now claim that in the sense of distributions in Ω we have

$$\begin{aligned} \operatorname{div}_M \vec{F} &= \langle \partial_{\tau_{XY}} u, w\rangle_{\mathcal{E}} + \langle u, \partial_{\tau_{XY}} w\rangle_{\mathcal{E}} \\ &\quad + \langle u, w\rangle_{\mathcal{E}} \big(N_g(X)\operatorname{div}_g Y - N_g(Y)\operatorname{div}_g X + N_g([X,Y])\big). \end{aligned} \tag{1.12.151}$$

To justify the above claim, fix some scalar-valued function $\psi \in \mathscr{C}_c^1(\widetilde{M})$ with the additional property that $\operatorname{supp}\psi \cap M = \operatorname{supp}\psi \cap \Omega$. If div_M and grad_M stand, respectively, for the differential geometric divergence operator and gradient operator on the manifold M, then (1.12.150) allows us to compute

$$\begin{aligned} &{}_{\mathscr{D}'(\Omega)}\big(\operatorname{div}_M \vec{F}, \psi|_\Omega\big)_{\mathscr{D}(\Omega)} \\ &\quad = -{}_{\mathscr{D}'(\Omega)}\big(\vec{F}, \operatorname{grad}_M(\psi|_\Omega)\big)_{\mathscr{D}(\Omega)} \\ &\quad = -\int_\Omega {}_{TM}\langle \operatorname{grad}_M(\psi|_\Omega), \vec{F}\rangle_{TM}\, d\mathcal{L}_g^n \\ &\quad = -\int_\Omega {}_{T^*M}\big(d_M(\psi|_\Omega), \vec{F}\big)_{TM}\, d\mathcal{L}_g^n \\ &\quad = -\int_\Omega \Big\langle \Big(N_g(X)\widetilde{d_M(\psi|_\Omega)}(Y) - N_g(Y)\widetilde{d_M(\psi|_\Omega)}(X)\Big)u, w\Big\rangle_{\mathcal{E}}\, d\mathcal{L}_g^n \end{aligned} \tag{1.12.152}$$

where d_M denotes the intrinsic exterior derivative operator on the differentiable manifold M. Recall that if $d_{\widetilde{M}}$ stands for the exterior derivative on $\widetilde{M}$ and $\iota : M \hookrightarrow \widetilde{M}$ is the canonical inclusion of M into $\widetilde{M}$, then

$$\iota^*\big(d_{\widetilde{M}}\psi\big) = d_M(\psi|_M) \text{ on } M. \tag{1.12.153}$$

Based on this and (1.12.149), for each $Z \in T\widetilde{M}$ we may then write

$$\begin{aligned} \widetilde{d_M(\psi|_\Omega)}(Z) &= \big(d_M(\psi|_\Omega)\big)(Z - N_g(Z)N_g^\sharp) = \big(d_{\widetilde{M}}\psi\big)(Z - N_g(Z)N_g^\sharp) \\ &= Z(\psi) - N_g(Z)\big(d_{\widetilde{M}}\psi\big)(N_g^\sharp) \text{ on } \Omega. \end{aligned} \tag{1.12.154}$$

Consequently, at points in Ω we have

$$\begin{aligned}\Big(N_g(X)\mathrm{d}_M\widetilde{(\psi|_\Omega)}(Y) - N_g(Y)\mathrm{d}_M\widetilde{(\psi|_\Omega)}(X)\Big)u \\ &= \big\{N_g(X)Y(\psi) - N_g(X)N_g(Y)\big(\mathrm{d}_{\widetilde{M}}\psi\big)(N_g^\sharp)\big\}u \\ &\quad - \big\{N_g(Y)X(\psi) - N_g(Y)N_g(X)\big(\mathrm{d}_{\widetilde{M}}\psi\big)(N_g^\sharp)\big\}u \\ &= N_g(X)Y(\psi)u - N_g(Y)X(\psi)u \\ &= \partial_{\tau_{XY}}(\psi u) - \psi\,\partial_{\tau_{XY}}u, \end{aligned} \tag{1.12.155}$$

where the last equality is provided by (1.12.95). Using this back in (1.12.152) and then integrating by parts as in (1.12.91) permits us to further compute

$$\begin{aligned}{}_{\mathscr{D}'(\Omega)}\big(\mathrm{div}_M\vec{F},\psi|_\Omega\big)_{\mathscr{D}(\Omega)} &= \int_\Omega \big\langle \psi\,\partial_{\tau_{XY}}u - \partial_{\tau_{XY}}(\psi\,u), w\big\rangle_{\mathcal{E}} \\ &= \int_\Omega \big\langle \partial_{\tau_{XY}}u, w\big\rangle_{\mathcal{E}}\psi\,\mathrm{d}\mathcal{L}_g^n - \int_M \big\langle \partial_{\tau_{XY}}(\psi\,u), w\big\rangle_{\mathcal{E}}\,\mathrm{d}\mathcal{L}_g^n \\ &= \int_\Omega \big\langle \partial_{\tau_{XY}}u, w\big\rangle_{\mathcal{E}}\psi\,\mathrm{d}\mathcal{L}_g^n + \int_M \big\langle \psi\,u, \partial_{\tau_{XY}}w\big\rangle_{\mathcal{E}}\,\mathrm{d}\mathcal{L}_g^n \\ &\quad + \int_M \big\langle \psi\,u, w\big\rangle_{\mathcal{E}}\big(N_g(X)\,\mathrm{div}_g Y - N_g(Y)\,\mathrm{div}_g X + N_g([X,Y])\big)\,\mathrm{d}\mathcal{L}_g^n. \end{aligned} \tag{1.12.156}$$

At this stage, the claim in (1.12.151) follows from (1.12.156).

In turn, from (1.12.151) and (1.12.143) we deduce that

$$\mathrm{div}_M\vec{F} \in L^1(\Omega, \mathcal{L}_g^n). \tag{1.12.157}$$

Also, from (1.12.150) and the assumptions on u, w we see that the nontangential trace $\vec{F}\big|^{\rm n.t.}_{\partial\Omega}$ (considered from within the ambient $\Omega \subset M$) exists at σ_g-a.e. point on $\partial_{\rm nta}\Omega$ and, in fact,

$${}_{T^*M}\big(\nu_g, \vec{F}\,\big|^{\rm n.t.}_{\partial\Omega}\big)_{TM} = \Big\langle\big(N_g(X)\widetilde{\nu}_g(Y) - N_g(Y)\widetilde{\nu}_g(X)\big)\big(u\big|^{\rm n.t.}_{\partial\Omega}\big), \big((w|_\Omega)\big|^{\rm n.t.}_{\partial\Omega}\big)\Big\rangle_{\mathcal{E}}. \tag{1.12.158}$$

Moreover, (1.12.34) also implies that $\mathcal{N}(\vec{F}) \leq C\mathcal{N}u\cdot\mathcal{N}(w|_\Omega)$ on $\partial\Omega$, hence

$$\mathcal{N}(\vec{F}) \in L^1(\partial\Omega, \sigma_g), \tag{1.12.159}$$

given the original assumptions on u, w. At this point, the integration by parts formula (1.12.144) is implied by the Divergence Formula established in Corollary 1.11.5, taking into account (1.12.151) and (1.12.158). □

Chapter 2
Examples, Counterexamples, and Additional Perspectives

This chapter is largely reserved for discussing examples and counterexamples (Sects. 2.1–2.7), as well as other versions of the Divergence Theorem (Sect. 2.8). All details in the statements of our main results in Chap. 1 have their own significance, and the examples/counterexamples presented in Sects. 2.1–2.7 are designed to elucidate their specific nature. The picture that emerges is that our results are optimal[1] from a multitude of (geometric and analytic) points of view. In particular, in stark contrast with the classical De Giorgi–Federer Divergence Theorem, our simplest version of the Divergence Theorem already contains the sharp version of the Fundamental Theorem of Calculus, to which it precisely reduces in the one-dimensional setting.

As a cursory search of the literature reveals, there are no significant counterexamples of the classical formulation of the Divergence Theorem,[2] to the effect that

$$\begin{array}{l}\text{given a bounded domain } \Omega \text{ of class } \mathscr{C}^1 \text{ in } \mathbb{R}^n, \text{ with outward unit}\\ \text{normal } \nu \text{ and surface measure } \sigma, \text{ for each vector field } \vec{F} \text{ belonging}\\ \text{to } \big[\mathscr{C}^1(\overline{\Omega})\big]^n \text{ it follows that } \int_\Omega \operatorname{div}\vec{F}\,d\mathcal{L}^n = \int_{\partial\Omega} \nu\cdot\big(\vec{F}\big|_{\partial\Omega}\big)\,d\sigma.\end{array} \tag{2.0.1}$$

Indeed, one typically looks for counterexamples as a justification as to why a certain result can no longer be improved, and it has long been understood that the statement of the Divergence Theorem given in (2.0.1) is far from sharp. For instance, one may allow the domain Ω to only be piecewise smooth and ask that the vector field $\vec{F}$ belongs to $\big[\mathscr{C}^0(\overline{\Omega})\cap\mathscr{C}^1(\Omega)\big]^n$ and has an absolutely integrable divergence in Ω.

[1] One may argue that "optimality of a mathematical result" becomes a phrase of indefinite meaning in the absence of a concrete and clear identification of the sense in which said optimality is to be understood.

[2] Save, perhaps, for the one-dimensional case, corresponding to the Fundamental Theorem of Calculus.

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Of course, this version is not sharp either, and so the search for counterexamples is further postponed.

In contrast to this state of affairs, it is possible to produce counterexamples illustrating the necessity of the hypotheses made in our main results, formulated in Chap. 1, and the goal in this chapter is to do just that. Often, such counterexamples are slightly tweaked to produce positive results, thus yielding relevant examples for the range and scope of the theorems stated in Chap. 1. The aforementioned fine-tuning also provides a better understanding of how wide or narrow the gap between positive and negative results (embodied by the counterexamples discussed here and their accompanying examples) really is. Ultimately, this analysis reveals that there is a razor-thin margin between what holds and what fails. Remarkably, on one side of that margin, even simple counterexamples abound, while on the other side of the margin there is a vastly powerful and general array of positive results (of the sort described in Chap. 1).

The close relationship of our theorems from Chap. 1 with classical results pertaining to the formulation of the Fundamental Theorem of Calculus in the one-dimensional setting is highlighted in Sect. 2.6. The penultimate section of the present chapter (Sect. 2.8) includes other variants of the Divergence Theorem, and concludes with a brief survey of additional work concerning the Divergence Theorem. Finally, the last section in this chapter (Sect. 4.7) is reserved for introducing what we call the "contribution at infinity" of vector fields, and for an in-depth discussion of this notion.

Some notation used throughout is as follows. We let $S^{n-1} := \partial B(0,1)$ stand for the origin-centered unit sphere in $\mathbb{R}^n$, and denote by $\omega_{n-1} := \mathcal{H}^{n-1}(S^{n-1})$ its area. By $\{\mathbf{e}_j\}_{1\leq j\leq n}$ we denote the standard orthonormal basis in $\mathbb{R}^n$, i.e., $\mathbf{e}_j = (\delta_{jk})_{1\leq k\leq n}$ for each $j \in \{1,\dots,n\}$, where δ_{jk} is the Kronecker symbol (i.e., $\delta_{jk} := 1$ if $j = k$ and $\delta_{jk} := 0$ if $j \neq k$). The upper and lower half-spaces $\mathbb{R}^n_\pm$ are defined as

$$\mathbb{R}^n_\pm := \big\{x \in \mathbb{R}^n : \pm\langle x, \mathbf{e}_n\rangle > 0\big\}, \tag{2.0.2}$$

and we denote the upper and lower hemispheres of S^{n-1} by

$$S^{n-1}_\pm := S^{n-1} \cap \mathbb{R}^n_\pm. \tag{2.0.3}$$

Going further, denote by $0'$ the origin in $\mathbb{R}^{n-1}$ and let

$$B_{n-1}(x', r) := \big\{y' \in \mathbb{R}^{n-1} : |y' - x'| < r\big\} \tag{2.0.4}$$

stand for the $(n-1)$-dimensional ball of radius $r \in (0,\infty)$ centered at $x' \in \mathbb{R}^{n-1}$.

Finally, we make the following agreement (sometimes tacitly employed):

Convention: We shall canonically identify (in a two-way fashion) scalar-valued functions $F : \Omega \to \mathbb{R}$, where Ω is an open subset of the real line $(-\infty,+\infty)$, with one-dimensional vector fields $\vec{F}$ defined on Ω simply by thinking of the number 1 as a unit vector (spanning $\mathbb{R}$, regarded as a one-dimensional vector space). (2.0.5)

2.1 Failure of Hypotheses on the Nontangential Boundary Trace

Consistently, in Theorems 1.2.1, 1.3.1, and 1.4.1 we have assumed that the vector field in question has a nontangential boundary limit at $\mathcal{H}^{n-1}$-a.e. point belonging to the nontangentially accessible boundary of the underlying domain. Since in the class of domains considered there, the latter set is, up to a $\mathcal{H}^{n-1}$-nullset, larger than the geometric measure theoretic boundary, at a superficial glance this might seem too strong of an assumption, particularly in light of the way the Divergence Formula is written in the aforementioned theorems. The main goal of this section is to dispel this perceived disparity between the strength of the assumption and the manner in which the conclusion is formulated.

Counterexamples/Examples Part 2.1A: Fix $n \in \mathbb{N}$ with $n \geq 2$ and consider the slit unit ball in $\mathbb{R}^n$ given by (Fig. 2.1)

$$\Omega := B(0,1) \setminus \big\{(x',0) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}\big\}. \tag{2.1.1}$$

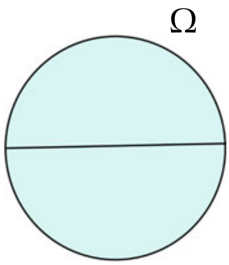

Fig. 2.1 Ω as in (2.1.1)

Also, define the piecewise constant vector field

$$\vec{F} := \begin{cases} +\mathbf{e}_n & \text{in } \Omega \cap \mathbb{R}^n_+, \\ -\mathbf{e}_n & \text{in } \Omega \cap \mathbb{R}^n_-. \end{cases} \tag{2.1.2}$$

Then, introducing

$$\Sigma := \big\{(x',0) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1} \text{ with } |x'| \leq 1\big\}, \tag{2.1.3}$$

$$\Sigma_0 := \big\{(x',0) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1} \text{ with } |x'| < 1\big\}, \tag{2.1.4}$$

$$\Upsilon := \Sigma \setminus \Sigma_0 = \big\{(x',0) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1} \text{ with } |x'| = 1\big\}, \tag{2.1.5}$$

we have that

$$\Omega \text{ is a bounded, nonempty, open set in } \mathbb{R}^n\text{, with an Ahlfors regular boundary, and } \partial\Omega = S^{n-1} \cup \Sigma_0. \tag{2.1.6}$$

Consequently, Ω has finite perimeter and

$$\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega \text{ is a doubling finite measure on } \partial\Omega. \tag{2.1.7}$$

Moreover, it is apparent from definitions that

$$\partial_*\Omega = S^{n-1}, \qquad \partial_{\text{nta}}\Omega = \partial\Omega \setminus \Upsilon, \qquad \partial_{\text{nta}}\Omega \setminus \partial_*\Omega = \Sigma_0, \tag{2.1.8}$$

and if ν denotes the geometric measure theoretic outward unit normal to Ω, then

$$\nu(x) = x \quad \text{for each} \quad x \in \partial_*\Omega = S^{n-1}. \tag{2.1.9}$$

Since $\vec{F}$ is locally constant in Ω, we have

$$\vec{F} \in \big[\mathscr{C}^\infty(\Omega)\big]^n \quad \text{and} \quad \operatorname{div}\vec{F} = 0 \quad \text{in} \quad \Omega. \tag{2.1.10}$$

Also, having fixed $\kappa \in (0, \infty)$ arbitrary, it follows that

$$\mathcal{N}_\kappa \vec{F} \in L^\infty(\partial\Omega, \sigma) \subset L^1(\partial\Omega, \sigma). \tag{2.1.11}$$

As regards the nontangential boundary behavior of $\vec{F}$, it is clear that

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \pm\mathbf{e}_n \quad \text{at every point on} \quad S^{n-1}_\pm. \tag{2.1.12}$$

Hence, on the one hand we have (with ν as in (2.1.9) and Σ_0 as in (2.1.4))

$$\begin{aligned} \int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)\, d\sigma &= \int_{S^{n-1}_+} \nu \cdot \mathbf{e}_n \, d\mathcal{H}^{n-1} - \int_{S^{n-1}_-} \nu \cdot \mathbf{e}_n \, d\mathcal{H}^{n-1} \\ &= 2\int_{\Sigma_0} \mathbf{e}_n \cdot \mathbf{e}_n \, d\mathcal{H}^{n-1} = 2\mathcal{H}^{n-1}(\Sigma_0) \neq 0, \end{aligned} \tag{2.1.13}$$

where the fact that $\int_{S^{n-1}_\pm} \nu \cdot \mathbf{e}_n \, d\mathcal{H}^{n-1} = \pm \int_{\Sigma_0} \mathbf{e}_n \cdot \mathbf{e}_n \, d\mathcal{H}^{n-1}$ (used in the second step) uses the Divergence Theorem in a half-ball for the constant vector field $\mathbf{e}_n$. On the other hand, (2.1.10) gives

$$\int_\Omega \operatorname{div}\vec{F} \, d\mathcal{L}^n = 0. \tag{2.1.14}$$

In turn, from (2.1.13)–(2.1.14) we see that

$$\text{\textit{the Divergence Formula} (1.2.2) \textit{fails for the set } } \Omega \text{ \textit{as in} (2.1.1), \textit{and for the vector field} } \vec{F} \text{ \textit{given in} (2.1.2).} \tag{2.1.15}$$

A direct comparison with the statement of Theorem 1.2.1 points to the first condition in (1.2.1) as the only possible source of this failure. To check that this is indeed the case, observe that

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ does \textit{not} exist at any point on } \Sigma_0 = \partial_{\text{nta}}\Omega \setminus \partial_*\Omega \tag{2.1.16}$$

and, since $\sigma(\Sigma_0) > 0$, it follows that the first condition in (1.2.1) is presently violated. In particular,

$$\text{\textit{the counterexample from} (2.1.15) \textit{highlights the necessity of demanding that the nontangential trace} } \vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ \textit{exists} } \sigma\text{\textit{-a.e. on} } \partial_{\text{nta}}\Omega, \text{ \textit{and not just on the (potentially smaller) set} } \partial_*\Omega, \text{ \textit{even though it is the latter which appears in the very formulation of the Divergence Formula} (1.2.2).} \tag{2.1.17}$$

This being said, the gap between failure and validity is rather thin. Let us indicate how the counterexample described in (2.1.15) may be slightly altered as to produce an example which is relevant as far as the scope of Theorem 1.3.1 is concerned. The idea is to eliminate the slit inside the domain, then consider the vector field to be only defined $\mathcal{L}^n$-a.e. in the domain thus enlarged. This procedure affects the divergence of the latter vector field and, ultimately, it is this which ensures the validity of the Divergence Formula (1.3.8). Specifically, if instead of (2.1.1)–(2.1.2) we now take

$$\Omega := B(0,1) \quad \text{and} \quad \vec{F} := \begin{cases} +\mathbf{e}_n & \text{in } \Omega \cap \mathbb{R}^n_+, \\ -\mathbf{e}_n & \text{in } \Omega \cap \mathbb{R}^n_-, \end{cases} \tag{2.1.18}$$

then $\vec{F}$ is a well-defined vector field in $\big[L^\infty(\Omega, \mathcal{L}^n)\big]^n$. Moreover, a simple computation reveals that its divergence (taken in the sense of distributions in Ω) is given by

$$\operatorname{div}\vec{F} = 2\,\mathcal{H}^{n-1}\lfloor\Sigma_0 \in \operatorname{CBM}(\Omega). \tag{2.1.19}$$

Hence, on the one hand,

$$\big(\operatorname{div}\vec{F}\big)(\Omega) = 2\,\mathcal{H}^{n-1}(\Sigma_0). \tag{2.1.20}$$

On the other hand, a computation similar to (2.1.13) gives that

$$\int_{\partial_*\Omega} \nu \cdot \left(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\right) \mathrm{d}\sigma = \int_{S^{n-1}} \nu \cdot \left(\mathbf{1}_{S^{n-1}_+}\mathbf{e}_n - \mathbf{1}_{S^{n-1}_-}\mathbf{e}_n\right) \mathrm{d}\mathcal{H}^{n-1} = 2\mathcal{H}^{n-1}(\Sigma_0), \tag{2.1.21}$$

which is in agreement with (2.1.20). Consequently, this discussion shows that

all hypotheses of Theorem 1.3.1 are satisfied by the domain Ω *and vector field* $\vec{F}$ *as in* (2.1.18), *and the Divergence Formula* (1.3.8) *is valid for this choice.* (2.1.22)

Counterexamples/Examples Part 2.1B: Fix $n \in \mathbb{N}$ satisfying $n \geq 2$. Consider a real-valued function

$$\phi : \overline{B_{n-1}(0', 1/2)} \longrightarrow (-1/2, 1/2), \quad \text{of class } \mathscr{C}^1, \tag{2.1.23}$$

and denote by Σ its graph, i.e.,

$$\Sigma := \left\{(x', \phi(x')) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\ |x'| \leq 1/2\right\} \subset B(0, 1). \tag{2.1.24}$$

Then there is a unique choice of a unit normal vector field N_Σ on the $\mathscr{C}^1$ surface Σ with the property that $\mathbf{e}_n \cdot N_\Sigma < 0$ at every point on Σ, namely

$$N_\Sigma\big(x', \phi(x')\big) = \frac{\big((\nabla'\phi)(x'), -1\big)}{\sqrt{|(\nabla'\phi)(x')|^2 + 1}}, \qquad \forall x' \in \overline{B_{n-1}(0', 1/2)}, \tag{2.1.25}$$

where ∇' denotes the gradient in $\mathbb{R}^{n-1}$.

If we now consider the crack domain (Fig. 2.2)

$$\Omega := B(0, 1) \setminus \Sigma \subset \mathbb{R}^n, \tag{2.1.26}$$

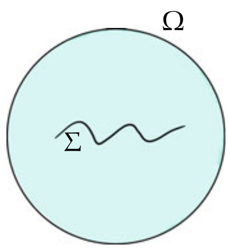

Fig. 2.2 Ω as in (2.1.26)

then

$$\Omega \text{ is a bounded, connected, nonempty, open set in } \mathbb{R}^n\text{, possessing an Ahlfors regular boundary.} \tag{2.1.27}$$

In particular, Ω has finite perimeter, and

$$\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega \;\text{ is a doubling measure on }\; \partial\Omega = S^{n-1} \cup \Sigma. \tag{2.1.28}$$

In addition, it is clear from definitions that

$$\partial_*\Omega = S^{n-1}, \qquad \partial_{\text{nta}}\Omega = \partial\Omega, \qquad \partial_{\text{nta}}\Omega \setminus \partial_*\Omega = \Sigma. \tag{2.1.29}$$

Also, if ν denotes the geometric measure theoretic outward unit normal to Ω, then

$$\nu(x) = x \;\text{ for each }\; x \in \partial_*\Omega = S^{n-1}. \tag{2.1.30}$$

Regarding the nature of the nontangential approach regions for Ω, with apexes at points on Σ, we note the following. Fix $\kappa \in (0, \infty)$ along with $r \in (0, 1/4)$ and, for every $x \in B(0, 1/4) \cap \Sigma$, introduce

$$\begin{aligned} \Gamma_\kappa^{(+)}(x) &:= \big\{y = (y', y_n) \in \Gamma_\kappa(x) \cap B(x, r) :\; y_n > \phi(y')\big\}, \\ \Gamma_\kappa^{(-)}(x) &:= \big\{y = (y', y_n) \in \Gamma_\kappa(x) \cap B(x, r) :\; y_n < \phi(y')\big\}. \end{aligned} \tag{2.1.31}$$

Then we have the decomposition

$$\begin{gathered} \Gamma_\kappa(x) \cap B(x, r) = \Gamma_\kappa^{(+)}(x) \cup \Gamma_\kappa^{(-)}(x), \quad \text{disjoint union,} \\ \text{for every point } x \text{ belonging to } \; B(0, 1/4) \cap \Sigma. \end{gathered} \tag{2.1.32}$$

Going forward, pick a complex-valued function satisfying

$$f \in \mathscr{C}^0(\Sigma), \quad \operatorname{supp} f \subseteq \Sigma \cap B(0, 1/4), \quad \int_\Sigma f \, d\mathcal{H}^{n-1} \neq 0, \tag{2.1.33}$$

and use it to define the vector field $\vec{F} : \Omega \to \mathbb{C}^n$ by setting

$$\vec{F}(x) := \frac{1}{\omega_{n-1}} \int_\Sigma \frac{x - y}{|x - y|^n} f(y) \, d\mathcal{H}^{n-1}(y), \qquad \forall x \in \Omega. \tag{2.1.34}$$

Then, by design, we have

$$\begin{gathered} \vec{F} \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \quad \operatorname{div}\vec{F} = 0 \;\text{ in }\; \Omega, \;\text{ and} \\ \vec{F} \text{ may be extended continuously across } S^{n-1}. \end{gathered} \tag{2.1.35}$$

In particular, the latter quality of $\vec{F}$ implies that the nontangential boundary trace $\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at every point on $S^{n-1} = \partial_*\Omega$ and, in fact,

$$\Big(\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) = \frac{1}{\omega_{n-1}} \int_\Sigma \frac{x-y}{|x-y|^n} f(y)\,\mathrm{d}\mathcal{H}^{n-1}(y), \qquad \forall x \in S^{n-1} = \partial_*\Omega. \tag{2.1.36}$$

Also, given that $f \in L^2(\Sigma, \mathcal{H}^{n-1}\lfloor\Sigma)$, the Calderón–Zygmund theory ensures that $\mathcal{N}_\kappa \vec{F}$ belongs to $L^2(\partial\Omega, \sigma)$. Hence, since $\sigma(\partial\Omega) < \infty$, we have

$$\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma). \tag{2.1.37}$$

Going further, based on (2.1.36) and Fubini's theorem we may write

$$\begin{aligned} \int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)\,\mathrm{d}\sigma &= \int_{S^{n-1}} x \cdot \Big(\frac{1}{\omega_{n-1}} \int_\Sigma \frac{x-y}{|x-y|^n} f(y)\,\mathrm{d}\mathcal{H}^{n-1}(y)\Big)\,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &= \int_\Sigma f(y)\Big(\frac{1}{\omega_{n-1}} \int_{S^{n-1}} x \cdot \frac{x-y}{|x-y|^n}\,\mathrm{d}\mathcal{H}^{n-1}(x)\Big)\,\mathrm{d}\mathcal{H}^{n-1}(y). \end{aligned} \tag{2.1.38}$$

To proceed, for each $y \in \Sigma \subset B(0,1)$ fixed introduce the auxiliary vector field

$$\vec{G}_y(x) := \frac{1}{\omega_{n-1}} \frac{x-y}{|x-y|^n}, \qquad \forall x \in \mathbb{R}^n \setminus \{y\}, \tag{2.1.39}$$

and note that $(\mathrm{div}\vec{G}_y)(x) = 0$ for each $x \in \mathbb{R}^n \setminus \{y\}$. Keeping this in mind, an application of the classical Divergence Theorem for the smooth, divergence-free, vector field $\vec{G}_y$ considered in the smooth domain $B(0,1) \setminus \overline{B(y,\varepsilon)}$, with $\varepsilon > 0$ sufficiently small, permits us to re-write the inner integral in (2.1.38) as

$$\begin{aligned} \int_{S^{n-1}} x \cdot \vec{G}_y(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) &= \int_{\partial B(y,\varepsilon)} \frac{x-y}{\varepsilon} \cdot \vec{G}_y(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &= \frac{1}{\omega_{n-1}} \int_{\partial B(y,\varepsilon)} \frac{x-y}{\varepsilon} \cdot \frac{x-y}{|x-y|^n}\,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &= 1. \end{aligned} \tag{2.1.40}$$

Plugging this back in (2.1.38) then proves that, on the one hand,

$$\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)\,\mathrm{d}\sigma = \int_\Sigma f(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) \neq 0, \tag{2.1.41}$$

on account of the last property in (2.1.33). On the other hand, thanks to the second property in (2.1.35), we trivially have

$$\int_{\Omega} \operatorname{div}\vec{F}\, d\mathcal{L}^n = 0. \tag{2.1.42}$$

The bottom line (apparent from (2.1.41)–(2.1.42)) is that

$$\text{the Divergence Formula (1.2.2) fails for the set } \Omega \text{ as in (2.1.26), and for the vector field } \vec{F} \text{ given in (2.1.34).} \tag{2.1.43}$$

As such, one of the hypotheses in Theorem 1.2.1 must be violated by the present choice of domain and vector field. A glance at the statement of Theorem 1.2.1 then points to the first condition in (1.2.1) as the source of this failure. Since the nontangential trace $\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ is known to exist everywhere on $S^{n-1} = \partial_*\Omega$ (cf. (2.1.36)), this warrants taking a closer look at the nontangential trace $\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ at points on $\partial_{\mathrm{nta}}\Omega \setminus S^{n-1} = \Sigma$. To this end, according to classical jump-formulas (for the gradient of the harmonic single layer across $\mathscr{C}^1$ surfaces), at $\mathcal{H}^{n-1}$-a.e. point $x \in B(0, 1/4) \cap \Sigma$ we have

$$\lim_{\Gamma^{(\pm)}_{\kappa}(x)\ni z\to x} \vec{F}(z) = \mp\frac{1}{2}N_{\Sigma}(x) f(x) + \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\Sigma\\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} f(y)\, d\mathcal{H}^{n-1}(y), \tag{2.1.44}$$

where $\Gamma^{(\pm)}_{\kappa}(x)$ are as in (2.1.31), and the unit vector field N_Σ is as in (2.1.25). In turn, from (2.1.44) and (2.1.32) we deduce that for each $x \in \Sigma$ the nontangential boundary trace

$$\Big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) = \lim_{\Gamma_{\kappa}(x)\ni y\to x} \vec{F}(y) \ \text{ does not exist if } x \in \Sigma \text{ is such that } f(x) \neq 0. \tag{2.1.45}$$

The assumption $\int_\Sigma f\, d\mathcal{H}^{n-1} \neq 0$ ensures that $\mathcal{H}^{n-1}\big(\{x \in \Sigma :\ f(x) \neq 0\}\big) > 0$ which, in light of (2.1.45), forces

$$\sigma\bigg(\Big\{x \in \partial_{\mathrm{nta}}\Omega : \Big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)(x) \ \text{ does not exist}\Big\}\bigg) > 0. \tag{2.1.46}$$

This makes it clear that our example violates the first condition in (1.2.1). In particular,

$$\begin{aligned}&\textit{the counterexample in (2.1.43) points to the necessity of hypothesizing}\\ &\textit{the } \sigma\textit{-a.e. existence of the nontangential trace } \vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \textit{ on the entire } \partial_{\mathrm{nta}}\Omega,\\ &\textit{and not just on the (potentially smaller) set } \partial_*\Omega, \textit{ even though it is the}\\ &\textit{latter set which appears in the Divergence Formula (1.2.2).}\end{aligned} \tag{2.1.47}$$

Next, we shall indicate how to modify the counterexample described in (2.1.43) in order to turn it into an actual example. The idea is to eliminate the crack inside the domain while now considering the vector field to be only defined $\mathcal{L}^n$-a.e. in the domain thus enlarged. This affects the divergence of the new vector field which, in turn, is what eventually renders the Divergence Formula (1.4.5) valid. In the execution of this plan, we adopt a more general point of view. Specifically, in place of (2.1.26) we now take

$$\begin{array}{l}\Omega \text{ an arbitrary bounded open subset of } \mathbb{R}^n\\ \text{possessing an Ahlfors regular boundary.}\end{array} \tag{2.1.48}$$

Also, consider an arbitrary compact subset Σ of Ω satisfying $\mathcal{H}^{n-1}(\Sigma) < +\infty$, and pick an arbitrary function $f \in L^1(\Sigma, \mathcal{H}^{n-1}\lfloor\Sigma)$. These choices imply that the measure

$$\mu := f\,\mathcal{H}^{n-1}\lfloor\Sigma \text{ belongs to CBM}(\Omega) \text{ and has compact support.} \tag{2.1.49}$$

Hence, if we define

$$\vec{F}(x) := \frac{1}{\omega_{n-1}} \int_\Sigma \frac{x-y}{|x-y|^n} f(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) \ \text{ for } \mathcal{L}^n\text{-a.e. } x \in \Omega, \tag{2.1.50}$$

then Lemma 3.5.6 ensures that

$$\vec{F} \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^n \ \text{ and } \ \operatorname{div}\vec{F} = \mu \ \text{ in } \ \mathcal{D}'(\Omega). \tag{2.1.51}$$

Thus, bearing (4.6.24) in mind, on the one hand we have

$${}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\Omega)} = \big(\operatorname{div}\vec{F}\big)(\Omega) = \int_\Sigma f\,\mathrm{d}\mathcal{H}^{n-1}. \tag{2.1.52}$$

On the other hand, if $\mathcal{O}$ is an open subset of Ω such that $\Sigma \subset \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$ then $\overline{\Omega} \setminus \mathcal{O}$ is a compact set and

$$\vec{F}\big|_{\Omega\setminus\mathcal{O}} \in \big[\mathscr{C}^0(\overline{\Omega} \setminus \mathcal{O}, \mathcal{L}^n)\big]^n. \tag{2.1.53}$$

Hence conditions (1.4.2)–(1.4.4) are presently satisfied and having fixed some $\kappa > 0$ then, much as in the case of (2.1.41), we have

$$\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)\,\mathrm{d}\sigma = \int_\Sigma f\,\mathrm{d}\mathcal{H}^{n-1}, \tag{2.1.54}$$

which agrees with (2.1.52). We may therefore conclude that *all hypotheses of Theorem 1.4.1 are satisfied by the domain* Ω *as in* (2.1.48) *and the vector field* $\vec{F}$ *as in* (2.1.50), *and the Divergence Formula* (1.4.5) *is valid for these choices.* (2.1.55)

Counterexamples/Examples Part 2.1**C:** Consider the open subset of $\mathbb{R}^n$ described as (Fig. 2.3)

$$\Omega := B(0,2) \setminus S^{n-1} \tag{2.1.56}$$

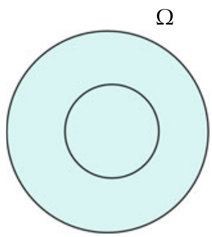

Fig. 2.3 Ω as in (2.1.56)

and introduce the vector field

$$\vec{F}(x) := \begin{cases} \dfrac{x}{|x|^n} & \text{for } x \in B(0,2) \setminus \overline{B(0,1)}, \\ 0 \in \mathbb{C}^n & \text{for } x \in B(0,1). \end{cases} \tag{2.1.57}$$

Then, by design,

$$\begin{aligned} &\partial\Omega = \partial B(0,2) \cup S^{n-1} \text{ is Ahlfors regular, and we have} \\ &\partial_*\Omega = \partial B(0,2), \quad \partial_{\text{nta}}\Omega = \partial\Omega, \quad \partial_{\text{nta}}\Omega \setminus \partial_*\Omega = S^{n-1}. \end{aligned} \tag{2.1.58}$$

In particular, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling finite measure on $\partial\Omega$. Also,

$$\begin{aligned} &\vec{F} \in \big[\mathscr{C}^\infty(\Omega) \cap L^\infty(\Omega, \mathcal{L}^n)\big]^n, \quad \operatorname{div}\vec{F} = 0 \text{ in } \Omega, \text{ and} \\ &\vec{F} \text{ may be extended continuously across } \partial B(0,2). \end{aligned} \tag{2.1.59}$$

As such, on the one hand we have

$$\int_\Omega \operatorname{div}\vec{F}\, d\mathcal{L}^n = 0. \tag{2.1.60}$$

On the other hand, with ν denoting the geometric measure theoretic outward unit normal to Ω, and having fixed some aperture parameter $\kappa > 0$,

$$\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \Big)\, d\sigma = \int_{\partial B(0,2)} \frac{x}{2} \cdot \frac{x}{|x|^n}\, d\mathcal{H}^{n-1}(x) = \omega_{n-1} \neq 0. \tag{2.1.61}$$

Thus, from (2.1.60)–(2.1.61) we conclude that

$$\text{the Divergence Formula (1.2.2) fails for the set } \Omega \text{ as in (2.1.56), and for the vector field } \vec{F} \text{ given in (2.1.57).} \tag{2.1.62}$$

Contrasting the present setting with that in Theorem 1.2.1 identifies the first condition in (1.2.1) as the only possible source of this failure. To see that this is indeed the case, note that

$$\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ does } \textit{not} \text{ exist at any point on } S^{n-1} = \partial_{\mathrm{nta}}\Omega \setminus \partial_*\Omega \tag{2.1.63}$$

and, since $\sigma(S^{n-1}) = \omega_{n-1} > 0$, it follows that the first condition in (1.2.1) is currently violated. Consequently,

$$\text{the counterexample described in (2.1.62) shows the necessity of having } \vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \text{ exist } \sigma\text{-a.e. on } \partial_{\mathrm{nta}}\Omega, \text{ and not just on } \partial_*\Omega. \tag{2.1.64}$$

The counterexample just presented may be altered to yield an actual example for the Divergence Formula (1.3.8) by eliminating the cut inside the domain. Specifically, let us now take

$$\Omega := B(0,2) \ \text{ and } \ \vec{F}(x) := \begin{cases} \dfrac{x}{|x|^n} & \text{for } \ x \in B(0,2) \setminus \overline{B(0,1)}, \\ 0 \in \mathbb{C}^n & \text{for } \ x \in B(0,1). \end{cases} \tag{2.1.65}$$

This time, we have (compare with (1.3.20)–(1.3.22))

$$\begin{gathered} \vec{F} \in \big[L^\infty(\Omega, \mathcal{L}^n)\big]^n, \quad \operatorname{div}\vec{F} = \mathcal{H}^{n-1}\lfloor S^{n-1} \ \text{ in } \ \mathcal{D}'(\Omega), \\ \text{and } \vec{F} \text{ may be extended continuously across } \partial\Omega. \end{gathered} \tag{2.1.66}$$

Hence, on the one hand, $\operatorname{div}\vec{F}$ is a complex Borel measure in Ω with total mass

$$\big(\operatorname{div}\vec{F}\big)(\Omega) = \mathcal{H}^{n-1}(S^{n-1}) = \omega_{n-1}, \tag{2.1.67}$$

while on the other hand,

$$\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \Big)\, d\sigma = \int_{\partial B(0,2)} \frac{x}{2} \cdot \frac{x}{|x|^n}\, d\mathcal{H}^{n-1}(x) = \omega_{n-1}, \tag{2.1.68}$$

in agreement with (2.1.67). Thus,

all hypotheses of Theorem 1.3.1 are satisfied by the domain Ω *and vector field* $\vec{F}$ *as in* (2.1.65), *and the Divergence Formula* (1.3.8) *holds for such a choice*. (2.1.69)

Counterexamples/Examples Part 2.1D: Start by considering the open subset of $\mathbb{C} \equiv \mathbb{R}^2$ given by (Fig. 2.4)

$$\Omega := \big\{z \in \mathbb{C} : |z| < 1 \text{ and } z \notin [0, 1)\big\}. \tag{2.1.70}$$

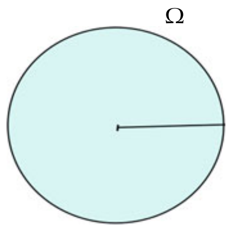

Fig. 2.4 Ω as in (2.1.70)

Hence, by design,

$$\begin{gathered}\partial\Omega = \partial B(0, 1) \cup \big([0, 1) \times \{0\}\big) \text{ is Ahlfors regular, and we have} \\ \partial_*\Omega = \partial B(0, 1), \quad \partial_{\text{nta}}\Omega = \partial\Omega \setminus \{(1, 0)\}, \quad \partial_{\text{nta}}\Omega \setminus \partial_*\Omega = [0, 1) \times \{0\}.\end{gathered} \tag{2.1.71}$$

In particular, $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ is a doubling finite measure on $\partial\Omega$. Next, observe that the function

$$\begin{gathered}f : (0, 1) \times (0, 2\pi) \to \Omega, \quad f(r, \theta) := r\mathrm{e}^{\mathrm{i}\theta} = r\cos\theta + ir\sin\theta, \\ \text{for each } r \in (0, 1) \text{ and each } \theta \in (0, 2\pi),\end{gathered} \tag{2.1.72}$$

is a bijection of class $\mathscr{C}^\infty$, whose Jacobian matrix Df satisfies $\det(Df)(r, \theta) = r > 0$ for each pair $(r, \theta) \in (0, 1) \times (0, 2\pi)$. The Inverse Function Theorem then ensures that f is a $\mathscr{C}^\infty$-diffeomorphism and, if (u, w) are the real-components of f^{-1}, we have

$$\begin{gathered}u : \Omega \to (0, 1), \; w : \Omega \to (0, 2\pi) \text{ are of class } \mathscr{C}^\infty \\ \text{and for each pair } (r, \theta) \in (0, 1) \times (0, 2\pi) \text{ we have} \\ \begin{pmatrix} (\partial_x u)(r\mathrm{e}^{\mathrm{i}\theta}) & (\partial_y u)(r\mathrm{e}^{\mathrm{i}\theta}) \\ (\partial_x w)(r\mathrm{e}^{\mathrm{i}\theta}) & (\partial_y w)(r\mathrm{e}^{\mathrm{i}\theta}) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.\end{gathered} \tag{2.1.73}$$

In particular, this shows that

$$(\partial_y w)(r\cos\theta, r\sin\theta) = \frac{\cos\theta}{r} = \frac{r\cos\theta}{r^2}, \quad \forall (r,\theta) \in (0,1)\times(0,2\pi), \tag{2.1.74}$$

which further entails

$$(\partial_y w)(x,y) = \frac{x}{x^2+y^2}, \quad \forall (x,y) \in \Omega. \tag{2.1.75}$$

Let us also observe that since similar considerations imply that for each $R \in (0,\infty)$ the function

$$\begin{aligned} f_R : (0,R)\times(0,2\pi) \to \Omega_R := \big\{z \in \mathbb{C} : \ |z| < R \ \text{ and } \ z \notin [0,R)\big\}, \\ f_R(r,\theta) := r\mathrm{e}^{\mathrm{i}\theta} \ \text{ for each pair } \ (r,\theta) \in (0,R)\times(0,2\pi), \end{aligned} \tag{2.1.76}$$

is also a $\mathscr{C}^\infty$-diffeomorphism, it follows that

$$\begin{aligned} w \in \mathscr{C}^\infty\big(\overline{\Omega} \setminus [0,1]\times\{0\}\big), \quad \text{and} \quad w(r\cos\theta, r\sin\theta) = \theta \\ \text{for each } \ r \in (0,1] \ \text{ and each } \ \theta \in (0,2\pi). \end{aligned} \tag{2.1.77}$$

To proceed, introduce the vector field

$$\vec{F} : \Omega \to \mathbb{R}^2, \qquad \vec{F}(x,y) := \big(0, w(x,y)\big) \ \text{ for each } \ (x,y) \in \Omega. \tag{2.1.78}$$

From (2.1.73) and (2.1.77) we see that

$$\begin{aligned} \vec{F} \in \big[\mathscr{C}^\infty(\Omega) \cap L^\infty(\Omega, \mathcal{L}^2)\big]^2 \ \text{ and} \\ \vec{F} \ \text{ extends continuously to } \ \overline{\Omega} \setminus [0,1]\times\{0\}. \end{aligned} \tag{2.1.79}$$

In particular, if $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$ (which is a doubling finite measure on $\partial\Omega$), then for each $\kappa > 0$ we have that

$$\begin{aligned} \mathcal{N}_\kappa \vec{F} \in L^\infty(\partial\Omega, \sigma) \subset L^1(\partial\Omega, \sigma) \ \text{ and} \\ \vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{ exists everywhere on } \ \partial_*\Omega = \partial\Omega \setminus [0,1)\times\{0\}. \end{aligned} \tag{2.1.80}$$

In fact, as is apparent from (2.1.77),

$$\Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(\cos\theta, \sin\theta) = (0,\theta) \ \text{ for each } \ \theta \in (0,2\pi). \tag{2.1.81}$$

Also,

$$(\mathrm{div}\vec{F})(x,y) = (\partial_y w)(x,y) = \frac{x}{x^2+y^2}, \qquad \forall (x,y) \in \Omega, \tag{2.1.82}$$

has a weak singularity at the origin, and is an odd function in $\Omega \setminus \big(\mathbb{R}\times\{0\}\big)$. Hence, on the one hand,

$$\mathrm{div}\vec{F} \in L^1(\Omega, \mathcal{L}^2) \quad \text{and} \quad \int_\Omega \mathrm{div}\vec{F}\, \mathrm{d}\mathcal{L}^2 = 0. \tag{2.1.83}$$

On the other hand, if ν is the geometric measure theoretic outward unit normal to Ω then (2.1.81) gives

$$\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)\, \mathrm{d}\sigma = \int_0^{2\pi} \theta\, \sin\theta\, \mathrm{d}\theta = -2\pi \neq 0. \tag{2.1.84}$$

Collectively, (2.1.83)–(2.1.84) prove that

$$\text{\textit{the Divergence Formula} (1.2.2) \textit{fails for the set} } \Omega \text{ \textit{as in} (2.1.70), \textit{and for the vector field} } \vec{F} \text{ \textit{given in} (2.1.78).} \tag{2.1.85}$$

A direct comparison of the present setting with that of Theorem 1.2.1 points to the first condition in (1.2.1) as the only possible source of this failure. That the first condition in (1.2.1) is indeed violated is clear upon noting that (2.1.77)–(2.1.78) imply

$$\begin{array}{l} \vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ does \textit{not} exist at any point on } [0,1)\times\{0\} = \partial_{\mathrm{nta}}\Omega \setminus \partial_*\Omega \\ \text{(as } \vec{F} \text{ has a jump-discontinuity of } 2\pi \text{ across the slit } (0,1)\times\{0\}), \\ \text{and } \sigma\big([0,1)\times\{0\}\big) = \mathcal{H}^1\big([0,1)\times\{0\}\big) = \mathcal{L}^1\big([0,1)\big) = 1 > 0. \end{array} \tag{2.1.86}$$

In summary,

$$\begin{array}{l} \textit{the counterexample described in (2.1.85) highlights the necessity of hav-} \\ \textit{ing } \vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \textit{ exist } \sigma\textit{-a.e. on } \partial_{\mathrm{nta}}\Omega \textit{ and shows that, without it, the Divergence} \\ \textit{Formula (1.2.2) may fail even when } \Omega \textit{ is a bounded simply connected} \\ \textit{open set with an Ahlfors regular boundary, and } \vec{F} \textit{ is a bounded vector} \\ \textit{field of class } \mathscr{C}^\infty \textit{ with an absolutely integrable divergence in } \Omega \textit{ and such} \\ \textit{that } \vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \textit{ exists everywhere on } \partial_*\Omega. \end{array} \tag{2.1.87}$$

The issue of $\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ failing to exist σ-a.e. on $\partial_{\mathrm{nta}}\Omega$ may be avoided if, to begin with, the vector field $\vec{F}$ is restricted to a suitably smaller domain. Specifically, with Ω and $\vec{F}$ as above (cf. (2.1.70) and (2.1.78), respectively), for an angle $\alpha \in (0, \pi)$ consider (Fig. 2.5)

$$\Omega_\alpha := \big\{z \in \Omega : \alpha < \arg z < 2\pi - \alpha\big\} \quad \text{and} \quad \vec{F}_\alpha := \vec{F}\big|_{\Omega_\alpha}. \tag{2.1.88}$$

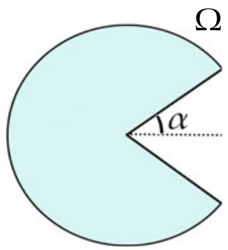

Fig. 2.5 Ω as in (2.1.88)

Then Ω_α is a bounded open subset of $\mathbb{R}^2$ with an Ahlfors regular boundary. In particular, $\sigma_\alpha := \mathcal{H}^{n-1}\lfloor\partial\Omega_\alpha$ is a finite doubling measure. Also, for each fixed $\kappa > 0$ we have

$$\mathcal{N}_\kappa \vec{F}_\alpha \in L^\infty(\partial\Omega_\alpha, \sigma_\alpha) \;\text{ and }\; \vec{F}_\alpha\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega_\alpha} \text{ exists everywhere on } \partial\Omega_\alpha \setminus \{0\}. \tag{2.1.89}$$

Actually, from the definitions of Ω_α, $\vec{F}_\alpha$, and (2.1.77) we see that

$$\Big(\vec{F}_\alpha\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega_\alpha}\Big)(z) = \begin{cases} (0,\theta) & \text{if } z = \mathrm{e}^{\mathrm{i}\theta} \text{ with } \theta \in (\alpha, 2\pi-\alpha), \\ (0,\alpha) & \text{if } z = r\mathrm{e}^{\mathrm{i}\alpha} \text{ with } r \in (0,1], \\ (0, 2\pi-\alpha) & \text{if } z = r\mathrm{e}^{\mathrm{i}(2\pi-\alpha)} \text{ with } r \in (0,1], \end{cases} \tag{2.1.90}$$

hence, if ν_α is the geometric measure theoretic outward unit normal to Ω_α, then

$$\begin{aligned} \int_{\partial_*\Omega_\alpha} \nu_\alpha \cdot \Big(\vec{F}_\alpha\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega_\alpha}\Big)\, d\sigma_\alpha &= \int_\alpha^{2\pi-\alpha} \theta\, \sin\theta\, d\theta - \int_0^1 \alpha\cos\alpha\, dr + \int_0^1 (2\pi-\alpha)\cos\alpha\, dr \\ &= -2\sin\alpha. \end{aligned} \tag{2.1.91}$$

Lastly, it is clear from (2.1.82) that $\operatorname{div}\vec{F}_\alpha \in L^1(\Omega_\alpha, \mathcal{L}^2)$ and

$$\begin{aligned} \int_{\Omega_\alpha} \operatorname{div}\vec{F}_\alpha\, d\mathcal{L}^2 &= \int_{\Omega_\alpha} \frac{x}{x^2+y^2}\, dx\,dy = 2\int_{\{z=x+iy\in\Omega_\alpha:\, \alpha<\arg z<\pi\}} \frac{x}{x^2+y^2}\, dx\,dy \\ &= 2\int_{\{z=x+iy\in\Omega_\alpha:\, \pi-\alpha<\arg z<\pi\}} \frac{x}{x^2+y^2}\, dx\,dy \\ &= 2\int_{\pi-\alpha}^{\pi}\int_0^1 \Big(\frac{r\cos\theta}{r^2}\Big) r\, dr\, d\theta \\ &= 2\int_{\pi-\alpha}^{\pi} \cos\theta\, d\theta = -2\sin\alpha, \end{aligned} \tag{2.1.92}$$

in agreement with (2.1.91). Hence, as predicted by Theorem 1.2.1, the Divergence Formula (1.2.2) holds for the domain Ω_α and vector field $\vec{F}_\alpha$ defined in (2.1.88).

To close, we wish to remark that even though the Divergence Formula (1.2.2) is verified by such a choice of domain Ω_α and vector field $\vec{F}_\alpha$ with $\alpha \in (0, \pi)$ arbitrary, the aforementioned formula fails for the limiting value $\alpha = 0$, which corresponds precisely to the counterexample presented in (2.1.85).

Counterexamples/Examples Part 2.1E: Work in the one-dimensional setting (i.e., $n = 1$). Take the bounded open subset of $\mathbb{R}$ given by (Fig. 2.6)

$$\Omega := (-1, 0) \cup (0, 1) \tag{2.1.93}$$

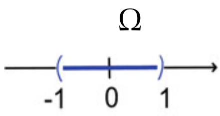

Fig. 2.6 Ω as in (2.1.93)

and, for two fixed numbers $a, b \in \mathbb{R}$, consider $F : \Omega \to \mathbb{R}$ defined at each $x \in \Omega$ as

$$F(x) := \begin{cases} a & \text{if } x \in (-1, 0), \\ b & \text{if } x \in (0, 1). \end{cases} \tag{2.1.94}$$

Then, by design,

$$\begin{aligned} &\partial\Omega = \{-1, 0, 1\} \text{ is an Ahlfors regular subset of } \mathbb{R}, \text{ and} \\ &\partial_*\Omega = \partial^*\Omega = \{-1, 1\}, \quad \partial_{\text{nta}}\Omega = \partial\Omega, \quad \partial_{\text{nta}}\Omega \setminus \partial_*\Omega = \{0\}. \end{aligned} \tag{2.1.95}$$

Also, $\sigma := \mathcal{H}^0 \lfloor \partial\Omega$ is the counting measure (thus, finite and doubling) on $\partial\Omega$. Next,

$$F \in \mathscr{C}^\infty(\Omega) \cap L^\infty(\Omega, \mathcal{L}^1) \text{ and } F' = 0 \text{ in } \Omega, \tag{2.1.96}$$

hence, on the one hand,

$$\int_\Omega F' \, d\mathcal{L}^1 = 0. \tag{2.1.97}$$

On the other hand, for each $\kappa > 0$ we have

$$\begin{aligned} &F\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists everywhere on } \partial_*\Omega, \text{ specifically} \\ &\big(F\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(-1) = a \text{ and } \big(F\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(1) = b, \end{aligned} \tag{2.1.98}$$

and, with ν denoting the geometric measure theoretic outward unit normal to Ω, we have

$$\int_{\partial_*\Omega} \nu \cdot \Big(F\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \Big)\, d\sigma = b - a. \tag{2.1.99}$$

From (2.1.97)–(2.1.99) we conclude that

$$\text{if } a \neq b \text{ then the Divergence Formula (1.2.2) fails for } \Omega \text{ as in (2.1.93) and } F \text{ as in (2.1.94).} \tag{2.1.100}$$

Comparing the present setting with that in Theorem 1.2.1 points to the first condition in (1.2.1) as the only possible source of this failure. To see that this is indeed the case, note that

$$\text{if } a \neq b \text{ then } F\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ does } \textit{not} \text{ exist at } 0 \in \partial_{\text{nta}}\Omega \setminus \partial_*\Omega \tag{2.1.101}$$

and, since $\sigma(\{0\}) = 1 > 0$, it follows that the first condition in (1.2.1) is currently violated. Consequently,

$$\text{the counterexample described in (2.1.100) shows the necessity of having } F\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exist } \sigma\text{-a.e. on } \partial_{\text{nta}}\Omega, \text{ and not just on } \partial_*\Omega. \tag{2.1.102}$$

Of course, the counterexample just presented may be altered to yield an actual example for the Divergence Formula (1.3.8) simply by taking $a = b$.

2.2 Failure of Hypotheses on Behavior at Infinity

Theorems 1.2.1, 1.3.1, 1.4.1, and 1.5.1 take into account the behavior of the vector field $\vec{F}$ at infinity either directly, through the incorporation of the contribution of $[\vec{F}]_\infty$ in the Divergence Formula, or indirectly by imposing suitable decay conditions on $\vec{F}$. The aim of this section is to study the extent to which these actions are appropriate.

Counterexamples/Examples Part 2.2A: A simple yet revealing counterexample to the Divergence Formula (1.2.2) in the one-dimensional setting (i.e., when $n = 1$) is to take[3]

$$\Omega := (0, +\infty) \text{ and } F : \Omega \to \mathbb{R} \text{ given by } F(x) := 1 \text{ for each } x \in \Omega. \tag{2.2.1}$$

Ω is an open nonempty proper subset of $\mathbb{R}$ with $\partial\Omega = \{0\}$, with $\sigma := \mathcal{H}^0 \lfloor \partial\Omega$ given by $\sigma(\{0\}) = 1$ and $\sigma(\varnothing) = 0$. This makes $\partial\Omega$ an Ahlfors regular set (given that $n = 1$), and σ a doubling measure on $\partial\Omega$. Also, $\partial_*\Omega = \partial^*\Omega = \partial\Omega = \{0\}$, and the geometric measure theoretic outward unit normal to Ω at the point 0 is $\nu(0) = -1$. Next, F is Lebesgue measurable and $F' = 0$ in Ω, so trivially $F' \in L^1(\Omega, \mathcal{L}^1)$. In

[3] Recall the convention made in (2.0.5).

addition, having fixed an aperture parameter $\kappa \in (0, \infty)$, we see that $\big(F\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(0) = 1$ and since $(\mathcal{N}_\kappa F)(0) = 1 < +\infty$, it follows that $\mathcal{N}_\kappa F \in L^1(\partial\Omega, \sigma)$. And yet

$$\int_\Omega F' \, d\mathcal{L}^1 = 0, \quad \text{while} \quad \int_{\partial_*\Omega} \nu \cdot \big(F\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big) \, d\sigma = -1. \tag{2.2.2}$$

This shows that

$$\textit{the Divergence Formula (1.2.2) fails if } n = 1 \textit{ and } \Omega, F \textit{ are as in (2.2.1).} \tag{2.2.3}$$

When comparing the present setting with that of Theorem 1.2.1, it becomes apparent that the source for this failure is the behavior of F at infinity. Specifically, the issue is that Ω is unbounded with $\partial\Omega$ bounded, a scenario in which according to Theorem 1.2.1 we should also impose (1.2.3) to guarantee the validity of the Divergence Formula (1.2.2). Let us look further and zoom in on this issue. Fix a dilation parameter $\lambda \in (1, \infty)$ and, bearing in mind that we are presently working with $n = 1$, consider the "annulus"

$$A_{\lambda,R} := (-\lambda R, \lambda R) \setminus (-R, R) \quad \text{for each} \quad R > 0. \tag{2.2.4}$$

Then for Ω and F as in (2.2.1) we have

$$\int_{A_{\lambda,R}\cap\Omega} |x \cdot F(x)| \, d\mathcal{L}^1(x) = \int_R^{\lambda R} x \, dx = \tfrac{1}{2}(\lambda^2 - 1)R^2 = O(R^2) \quad \text{as} \quad R \to \infty, \tag{2.2.5}$$

whereas the integral condition (1.2.3) imposed in Theorem 1.2.1 actually requires $o(R^2)$ in place of $O(R^2)$. In summary,

the counterexample in (2.2.1) *points to the fact that the behavior of F at infinity is an essential attribute, strongly influencing the correctness of the Divergence Formula* (1.2.2); *in particular,* $o(R^2)$ *may not be replaced by* $O(R^2)$ *in the formulation of the growth condition* (1.2.3). (2.2.6)

Let us also note that $F(x) = O(1)$ as $x \to \infty$, whereas the pointwise decay condition (1.2.9) written for $n = 1$ stipulates that we should have $F(x) = o(1)$ as $x \to \infty$ in order for the Divergence Formula (1.2.2) to hold. Consequently, at least in dimension one (see (2.2.16) for higher dimensions),

the counterexample in (2.2.1) *shows that in the class of unbounded sets* $\Omega \subseteq \mathbb{R}^n$, *the pointwise decay condition* (1.2.9) (*to the effect that* $\vec{F}(x) = o(|x|^{1-n})$ *at infinity*) *may not be weakened to asking* $\vec{F}(x) = O(|x|^{1-n})$ *at infinity.* (2.2.7)

The counterexample presented in (2.2.3) may be converted into a positive result if we include information about the behavior of F at infinity in the very formulation of the Divergence Formula, as we have done in Theorem 1.3.1 through the consideration of the contribution of F at infinity. Specifically, in the present case Proposition 4.7.2 gives

$$[F]_\infty = F(+\infty) = 1, \tag{2.2.8}$$

which, in view of (2.2.2), ultimately shows that

all hypotheses of Theorem 1.3.1 are satisfied when $n = 1$ *and* Ω, F *are as in* (2.2.1), *and the Divergence Formula* (1.3.8) *is valid in this setting*. (2.2.9)

Counterexamples/Examples Part 2.2B: Here we discuss higher-dimensional versions of the counterexamples/examples from Part 2.2A above. To set the stage, assume $n \in \mathbb{N}$ satisfies $n \geq 2$. Consider the exterior domain $\Omega \subset \mathbb{R}^n$, and the vector field $\vec{F}$ on Ω, given by (Fig. 2.7)

$$\Omega := \mathbb{R}^n \setminus \overline{B(0,1)} \quad \text{and} \quad \vec{F}(x) := \frac{x}{|x|^n} \quad \text{for each} \quad x \in \Omega. \tag{2.2.10}$$

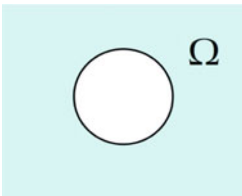

Fig. 2.7 Ω as in (2.2.10)

Then

$$\begin{gathered} \partial\Omega = \partial_*\Omega = S^{n-1}, \quad \vec{F} \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \quad \operatorname{div}\vec{F} = 0 \ \text{ in } \ \Omega, \\ \text{and} \ \ \vec{F} \text{ extends continuously to } \overline{\Omega}. \end{gathered} \tag{2.2.11}$$

Hence,

$$\int_\Omega \operatorname{div}\vec{F}\, \mathrm{d}\mathcal{L}^n = 0. \tag{2.2.12}$$

Also, having fixed an aperture parameter $\kappa > 0$, the nontangential boundary trace $\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists pointwise everywhere on $\partial\Omega$, and

$$\int_{\partial_*\Omega} \nu \cdot \left(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \right) \mathrm{d}\sigma = \int_{S^{n-1}} (-x) \cdot \frac{x}{|x|^n} \, \mathrm{d}\mathcal{H}^{n-1}(x) = -\omega_{n-1} \neq 0. \tag{2.2.13}$$

Thus, as is apparent from (2.2.12)–(2.2.13),

$$\text{\textit{the Divergence Formula} (1.2.2) \textit{fails for the set } } \Omega \text{ \textit{and the vector field} } \vec{F}\text{, \textit{as defined in} (2.2.10).} \tag{2.2.14}$$

When comparing the present setting with that of Theorem 1.2.1, it becomes apparent that the source for this failure is the fact that condition (1.2.3) is not presently satisfied. To see that this is indeed the case, note that $|x \cdot \vec{F}(x)| = |x|^{2-n}$ for each $x \in \Omega$, and $\int_{A_{\lambda\,R}\cap\Omega} |x|^{2-n}\,\mathrm{d}x = c_{n,\lambda} R^2$ for each $R > 0$, where $c_{n,\lambda} \in (0, \infty)$. In the bigger picture,

the counterexample in (2.2.14) *points to the fact that the behavior of* $\vec{F}$ *at infinity can be a pivotal factor, strongly affecting the veracity of the Divergence Formula* (1.2.2); *in particular*, $o(R^2)$ *may not be replaced by* $O(R^2)$ *in the formulation of the growth condition* (1.2.3). (2.2.15)

We also wish to observe that $\vec{F}(x) = O(|x|^{1-n})$ as $|x| \to \infty$, while the pointwise decay condition (1.2.9) actually requires that $\vec{F}(x) = o(|x|^{1-n})$ as $|x| \to \infty$ for the Divergence Formula (1.2.2) to hold. Therefore, in dimensions $n \geq 2$,

the counterexample in (2.2.14) *shows that in the class of unbounded sets* $\Omega \subseteq \mathbb{R}^n$, *the pointwise decay condition* (1.2.9) (*to the effect that* $\vec{F}(x) = o(|x|^{1-n})$ *at infinity*) *may not be weakened to asking* $\vec{F}(x) = O(|x|^{1-n})$ *at infinity.* (2.2.16)

The counterexample in (2.2.14) may be turned into a positive result if one incorporates information about the behavior of $\vec{F}$ at infinity in the very formulation of the Divergence Formula, as done in Theorem 1.3.1 through the consideration of the contribution of $\vec{F}$ at infinity. Concretely, in the present case we have (cf. (1.3.25))

$$[\vec{F}]_\infty = \lim_{R\to\infty} \int_{|x|=R} \frac{x}{|x|} \cdot \frac{x}{|x|^n} \, \mathrm{d}\mathcal{H}^{n-1}(x) = \omega_{n-1} \tag{2.2.17}$$

which, in light of (2.2.12) and (2.2.13), proves that

all hypotheses of Theorem 1.3.1 are satisfied by the set Ω *and vector field* $\vec{F}$ *as in* (2.2.10), *and the Divergence Formula* (1.3.8) *is valid for this choice.* (2.2.18)

Counterexamples/Examples Part 2.2C: Here we discuss a more revealing family of counterexamples and examples for the Divergence Formula than the ones in Part 2.2B above. To get started, suppose $n \in \mathbb{N}$ satisfies $n \geq 2$. Consider the exterior domain (Fig. 2.8)

$$\Omega := \mathbb{R}^n \setminus \overline{B(0,1)} \tag{2.2.19}$$

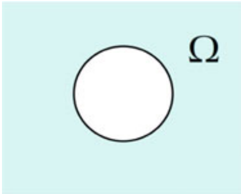

Fig. 2.8 Ω as in (2.2.19)

and, for some $\mu \in \mathrm{CBM}\big(B(0,1)\big)$ which is compactly supported in $B(0,1)$, define the vector field $\vec{F}$ as the Riesz transform of the measure μ in Ω, i.e., set

$$\vec{F}(x) := \frac{1}{\omega_{n-1}} \int_{B(0,1)} \frac{x-y}{|x-y|^n}\, d\mu(y) \quad \text{for each} \quad x \in \Omega. \tag{2.2.20}$$

Then, by design,

$$\vec{F} \in \big[\mathscr{C}^\infty(\overline{\Omega})\big]^n \quad \text{and} \quad \operatorname{div}\vec{F} = 0 \quad \text{in} \quad \Omega. \tag{2.2.21}$$

In particular, on the one hand,

$$\int_\Omega \operatorname{div}\vec{F}\, d\mathcal{L}^n = 0. \tag{2.2.22}$$

On the other hand, having fixed an aperture parameter $\kappa > 0$, the nontangential boundary trace $\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists pointwise everywhere on $\partial\Omega$ and based on Fubini's theorem we may write

$$\begin{aligned}
\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)\, d\sigma &= \int_{S^{n-1}} (-x) \cdot \bigg(\frac{1}{\omega_{n-1}} \int_{B(0,1)} \frac{x-y}{|x-y|^n}\, d\mu(y)\bigg)\, d\mathcal{H}^{n-1}(x) \\
&= -\int_{B(0,1)} \bigg(\frac{1}{\omega_{n-1}} \int_{S^{n-1}} x \cdot \frac{x-y}{|x-y|^n}\, d\mathcal{H}^{n-1}(x)\bigg)\, d\mu(y) \\
&= -\int_{B(0,1)} 1\, d\mu(y) = -\mu\big(B(0,1)\big),
\end{aligned} \tag{2.2.23}$$

where the penultimate equality uses (2.1.39)–(2.1.40). From (2.2.22)–(2.2.23) we conclude that

$$\text{if } \mu\big(B(0,1)\big) \neq 0, \text{ the Divergence Formula (1.2.2) fails for the domain } \Omega \text{ as in (2.2.19) and the vector field } \vec{F} \text{ as in (2.2.20).} \tag{2.2.24}$$

When $\mu\big(B(0,1)\big) \neq 0$, the source of failure of formula (1.2.2) in the present setting is the violation of the growth condition in (1.2.3). This is most apparent from the following equivalence:

$$(1.2.3) \text{ holds} \iff \mu\big(B(0,1)\big) = 0. \tag{2.2.25}$$

To prove (2.2.25), for any compact set $K \subset \mathbb{R}^n$ use the Mean Value Theorem to write

$$\frac{x-y}{|x-y|^n} - \frac{x}{|x|^n} = O\big(|x|^{-n}\big) \text{ as } |x| \to \infty, \text{ uniformly for } y \in K. \tag{2.2.26}$$

In turn, this shows that $\vec{F}$ has the following asymptotic behavior at infinity

$$\vec{F}(x) = \frac{\mu\big(B(0,1)\big)}{\omega_{n-1}} \frac{x}{|x|^n} + O\big(|x|^{-n}\big) \text{ as } |x| \to \infty, \tag{2.2.27}$$

which further implies

$$\big|x \cdot \vec{F}(x)\big| = \frac{\big|\mu\big(B(0,1)\big)\big|}{\omega_{n-1}} |x|^{2-n} + O\big(|x|^{1-n}\big) \text{ as } |x| \to \infty. \tag{2.2.28}$$

Hence, if $\lambda \in (1,\infty)$ is fixed and $A_{\lambda,R} := B(0,\lambda R) \setminus B(0,R)$ for each $R > 0$, there exists a constant $c_{n,\lambda} \in (0,\infty)$ for which

$$\int_{A_{\lambda,R}\cap\Omega} \big|x \cdot \vec{F}(x)\big| \, d\mathcal{L}^n(x) = c_{n,\lambda}\big|\mu\big(B(0,1)\big)\big| R^2 + O(R) \text{ as } R \to \infty. \tag{2.2.29}$$

From this, (2.2.25) is now clear.

Moving on, we may turn the counterexample presented in (2.2.24) into a positive result by either requiring that $\mu\big(B(0,1)\big) = 0$, or employing the version of the Divergence Formula recorded in (1.4.5), which also takes into account the contribution of $\vec{F}$ at infinity. As regards the latter scenario, observe that the asymptotic formula (2.2.27) permits us to identify the contribution of $\vec{F}$ at infinity (cf. (1.3.25)) as

$$[\vec{F}]_\infty = \lim_{R\to\infty} \int_{|x|=R} \frac{x}{|x|} \cdot \vec{F}(x) \, d\mathcal{H}^{n-1}(x) = \mu\big(B(0,1)\big). \tag{2.2.30}$$

In view of (2.2.22)–(2.2.23) and (2.2.30), we may summarize our findings as follows:

$$\text{\textit{all hypotheses of Theorem 1.4.1 are satisfied by the domain Ω as in (2.2.19) and the vector field $\vec{F}$ as in (2.2.20), and the Divergence Formula (1.4.5) holds for these choices.}} \tag{2.2.31}$$

It is worth remarking that the vector field defined in (2.2.20) reduces precisely to the vector field in (2.2.10) when μ is the measure canonically induced on $B(0, 1)$ by Dirac's delta distribution with mass at zero (i.e., $\mu(A) := \mathbf{1}_A(0)$ for every Borel set $A \subseteq B(0, 1)$).

Counterexamples/Examples Part 2.2D: Here we present a counterexample, together with a naturally accompanying example, in the spirit of those discussed in Part 2.2C above, but now working with a vector field whose divergence is an arbitrary compactly supported distribution. The construction of such a vector field, presented in Proposition 2.2.2, requires the functional analytic result described in Lemma 2.2.1 below. Before stating the latter, we make a definition.

Specifically, given an open set $\Omega \subseteq \mathbb{R}^n$ and a distribution in Ω with compact support, call it $u \in \mathscr{E}'(\Omega)$, we agree to extend the action of u to the family of functions

$$\mathscr{C}^\infty(\operatorname{supp} u) := \bigcup_{\substack{O \text{ open subset of } \Omega \\ \text{with } \operatorname{supp} u \subset O}} \mathscr{C}^\infty(O), \tag{2.2.32}$$

which is strictly larger than $\mathscr{C}^\infty(\Omega)$, in the following natural manner. For each function $f \in \mathscr{C}^\infty(\operatorname{supp} u)$, say $f \in \mathscr{C}^\infty(O)$ with $O \subseteq \Omega$ open set containing $\operatorname{supp} u$, define

$$ {}_{\mathscr{E}'(\Omega)}\langle\!\langle u, f \rangle\!\rangle_{\mathscr{E}(\Omega)} := {}_{\mathscr{E}'(\Omega)}\langle u, F \rangle_{\mathscr{E}(\Omega)} \tag{2.2.33}$$

where F is any function in $\mathscr{C}^\infty(\Omega)$ with the property that $F = f$ near $\operatorname{supp} u$. It is then clear that the above definition is meaningful and unambiguous (i.e., the right-hand side of (2.2.33) does not depend on the choice of F). With this piece of notation, we then have the following result.

Lemma 2.2.1 *Let $f \in \mathcal{D}'(\mathbb{R}^n)$ be such that $f\big|_{\mathbb{R}^n \setminus \{0\}} \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$, and fix an arbitrary $u \in \mathscr{E}'(\mathbb{R}^n)$. Then the convolution $w := f * u \in \mathcal{D}'(\mathbb{R}^n)$ is of function type on $\mathbb{R}^n \setminus \operatorname{supp} u$. In fact,*

$$w \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \operatorname{supp} u), \tag{2.2.34}$$

and for each $x \in \mathbb{R}^n \setminus \operatorname{supp} u$ one has

$$w(x) = {}_{\mathscr{E}'(\mathbb{R}^n)}\langle\!\langle u, f(x - \cdot) \rangle\!\rangle_{\mathscr{E}(\mathbb{R}^n)} \tag{2.2.35}$$

where the right-hand side is interpreted in the sense of (2.2.33), viewing $f(x - \cdot)$ as a function in $\mathscr{C}^\infty(\mathbb{R}^n \setminus \{x\})$.

Proof Abbreviate $K := \operatorname{supp} u$ and consider the family of compact sets

$$K_\varepsilon := \big\{x \in \mathbb{R}^n : \operatorname{dist}(x, K) \le \varepsilon\big\}, \quad \text{for each } \ \varepsilon > 0. \tag{2.2.36}$$

Also, for each $\varepsilon > 0$ pick $\eta_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n)$ satisfying $\eta_\varepsilon \equiv 1$ on $\mathbb{R}^n \setminus B(0, \varepsilon/2)$ and $\eta_\varepsilon \equiv 0$ on $B(0, \varepsilon/4)$. In particular, $\eta_\varepsilon f \in \mathscr{C}^\infty(\mathbb{R}^n)$ for each $\varepsilon > 0$.

To proceed, fix $\varepsilon > 0$ and consider an arbitrary $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^n \setminus K_\varepsilon)$. Denote by $\widetilde{\varphi}$ the extension of φ by zero to the entire $\mathbb{R}^n$, and select a function $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which is identically one on the set

$$\big\{(x, y) : \ x \in K \ \text{ and } \ y \in \mathbb{R}^n \text{ such that } x + y \in \operatorname{supp} \varphi\big\}. \tag{2.2.37}$$

Bearing in mind that $\eta_\varepsilon \equiv 1$ on the algebraic difference $(\mathbb{R}^n \setminus K_\varepsilon) - K$ and that, in a natural sense we have $\mathscr{E}'(\mathbb{R}^n) * \mathscr{C}^\infty(\mathbb{R}^n) \subseteq \mathscr{C}^\infty(\mathbb{R}^n)$ (cf. [181, Exercise 2.103, p. 74]), we may then compute

$$\begin{aligned}
&{}_{\mathcal{D}'(\mathbb{R}^n\setminus K_\varepsilon)}\big\langle w\big|_{\mathbb{R}^n\setminus K_\varepsilon}, \varphi \big\rangle_{\mathcal{D}(\mathbb{R}^n\setminus K_\varepsilon)} \qquad (2.2.38)\\
&\quad = {}_{\mathcal{D}'(\mathbb{R}^n)}\big\langle w, \widetilde{\varphi} \big\rangle_{\mathcal{D}(\mathbb{R}^n)} = {}_{\mathcal{D}'(\mathbb{R}^n)}\big\langle u * f, \widetilde{\varphi} \big\rangle_{\mathcal{D}(\mathbb{R}^n)}\\
&\quad = {}_{\mathcal{D}'(\mathbb{R}^n)}\Big\langle u(x), {}_{\mathcal{D}'(\mathbb{R}^n)}\big\langle f(y), \psi(x,y)\widetilde{\varphi}(x+y) \big\rangle_{\mathcal{D}(\mathbb{R}^n)}\Big\rangle_{\mathcal{D}(\mathbb{R}^n)}\\
&\quad = {}_{\mathcal{D}'(\mathbb{R}^n)}\Big\langle u(x), {}_{\mathcal{D}'(\mathbb{R}^n)}\big\langle (\eta_\varepsilon f)(y), \psi(x,y)\widetilde{\varphi}(x+y) \big\rangle_{\mathcal{D}(\mathbb{R}^n)}\Big\rangle_{\mathcal{D}(\mathbb{R}^n)}\\
&\quad = {}_{\mathcal{D}'(\mathbb{R}^n)}\big\langle u * (\eta_\varepsilon f), \widetilde{\varphi} \big\rangle_{\mathcal{D}(\mathbb{R}^n)}\\
&\quad = \int_{\mathbb{R}^n} \Big({}_{\mathscr{E}'(\mathbb{R}^n)}\big\langle u(y), (\eta_\varepsilon f)(x-y) \big\rangle_{\mathscr{E}(\mathbb{R}^n)}\Big)\widetilde{\varphi}(x)\,\mathrm{d}x\\
&\quad = \int_{\mathbb{R}^n\setminus K_\varepsilon} \Big({}_{\mathscr{E}'(\mathbb{R}^n)}\big\langle u(y), (\eta_\varepsilon f)(x-y) \big\rangle_{\mathscr{E}(\mathbb{R}^n)}\Big)\varphi(x)\,\mathrm{d}x.
\end{aligned}$$

This proves that the distribution $w\big|_{\mathbb{R}^n\setminus K_\varepsilon}$ is of function type and, in fact, for each $x \in \mathbb{R}^n \setminus K_\varepsilon$ we have

$$\begin{aligned}
\Big(w\big|_{\mathbb{R}^n\setminus K_\varepsilon}\Big)(x) &= {}_{\mathscr{E}'(\mathbb{R}^n)}\big\langle u, (\eta_\varepsilon f)(x-\cdot) \big\rangle_{\mathscr{E}(\mathbb{R}^n)}\\
&= {}_{\mathscr{E}'(\mathbb{R}^n)}\big\langle u, \eta_\varepsilon(x-\cdot) f(x-\cdot) \big\rangle_{\mathscr{E}(\mathbb{R}^n)}\\
&= {}_{\mathscr{E}'(\mathbb{R}^n)}\langle\!\langle u, f(x-\cdot)\rangle\!\rangle_{\mathscr{E}(\mathbb{R}^n)}
\end{aligned} \tag{2.2.39}$$

where the last equality uses the fact that $\eta_\varepsilon(x-\cdot) \equiv 1$ near K for each fixed point $x \in \mathbb{R}^n \setminus K_\varepsilon$. Since $\varepsilon > 0$ was arbitrary, all desired conclusions follow from this. □

In turn, Lemma 2.2.1 is an important ingredient in the proof of Proposition 2.2.2 below. To facilitate stating the latter result, we introduce one piece of notation. Concretely, given $n \in \mathbb{N}$ define the vector field $\vec{R} = (R_1, \dots, R_n)$ by setting

$$\vec{R}(x) := \frac{1}{\omega_{n-1}} \frac{x}{|x|^n} \quad \text{for each} \ \ x \in \mathbb{R}^n \setminus \{0\}. \tag{2.2.40}$$

Proposition 2.2.2 *Let $\mathcal{O} \subseteq \mathbb{R}^n$ be a nonempty open set and suppose $u \in \mathcal{E}'(\mathcal{O})$ is arbitrary. Then the vector field defined as*

$$\vec{F} := \big(\vec{R} * u\big)\big|_{\mathcal{O}} \in \big[\mathcal{D}'(\mathcal{O})\big]^n \tag{2.2.41}$$

satisfies

$$\operatorname{div}\vec{F} = u \ \ \textit{in} \ \ \mathcal{D}'(\mathcal{O}), \qquad \vec{F}\big|_{\mathcal{O}\setminus \operatorname{supp} u} \in \big[\mathscr{C}^\infty(\mathcal{O} \setminus \operatorname{supp} u)\big]^n, \tag{2.2.42}$$

and, at each point $x \in \mathcal{O} \setminus \operatorname{supp} u$,

$$\vec{F}(x) = {}_{\mathcal{E}'(\mathbb{R}^n)}\big\langle u, \vec{R}(x - \cdot)\big\rangle_{\mathcal{E}(\mathbb{R}^n)} := \sum_{j=1}^{n} {}_{\mathcal{E}'(\mathbb{R}^n)}\big\langle u, R_j(x - \cdot)\big\rangle_{\mathcal{E}(\mathbb{R}^n)} \boldsymbol{e}_j. \tag{2.2.43}$$

Proof To get started, observe that, by design,

$$\begin{aligned} &\vec{R} \in \big[\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{L}^n)\big]^n \subset \big[\mathcal{D}'(\mathbb{R}^n)\big]^n, \\ &\text{and} \ \ \operatorname{div}\vec{R} = \delta \ \ \text{in the sense of distributions in} \ \ \mathbb{R}^n. \end{aligned} \tag{2.2.44}$$

In particular, from (2.2.44) and the fact that distributional derivatives commute with the operator of restriction to open sets, we conclude that

$$\begin{aligned} \operatorname{div}\vec{F} &= \operatorname{div}\Big[\big(\vec{R} * u\big)\big|_{\mathcal{O}}\Big] = \big((\operatorname{div}\vec{R}) * u\big)\big|_{\mathcal{O}} \\ &= (\delta * u)\big|_{\mathcal{O}} = u\big|_{\mathcal{O}} = u \ \ \text{in} \ \ \mathcal{D}'(\mathcal{O}). \end{aligned} \tag{2.2.45}$$

This justifies the first equality in (2.2.42). Finally, all remaining claims in (2.2.42)–(2.2.43) follow from Lemma 2.2.1 and (2.2.44). □

After this detour, we are ready to deal with the main item of business. Bring in the exterior domain (Fig. 2.9)

$$\Omega := \mathbb{R}^n \setminus \overline{B(0,1)} \tag{2.2.46}$$

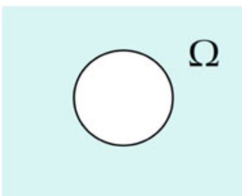

Fig. 2.9 Ω as in (2.2.46)

and, for some given distribution $u \in \mathscr{E}'(\mathbb{R}^n)$ with compact support contained in $B(0,1)$, define the vector field

$$\vec{F} := \big(\vec{R} * u\big)\big|_{\Omega} \in \big[\mathcal{D}'(\Omega)\big]^n. \tag{2.2.47}$$

From Proposition 2.2.2 (used with $\mathcal{O} = \mathbb{R}^n$) and the fact that distributional derivatives commute with restriction to open sets, we see that

$$\operatorname{div}\vec{F} = \operatorname{div}\Big[\big(\vec{R} * u\big)\big|_{\Omega}\Big] = u\big|_{\Omega} = 0 \ \text{ in } \ \mathcal{D}'(\Omega), \tag{2.2.48}$$

and $\vec{F} \in \big[\mathscr{C}^\infty(\overline{\Omega})\big]^n$ is given at every point $x \in \overline{\Omega}$ by

$$\vec{F}(x) = {}_{\mathscr{E}'(\mathbb{R}^n)}\langle\!\langle u, \vec{R}(x-\cdot)\rangle\!\rangle_{\mathscr{E}(\mathbb{R}^n)}. \tag{2.2.49}$$

As a result, for each aperture parameter $\kappa > 0$ it follows that the nontangential boundary trace $\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists pointwise everywhere on $\partial\Omega$ and

$$\begin{aligned}\int_{\partial_*\Omega} \nu\cdot\Big(\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)\,d\sigma &= \int_{S^{n-1}} (-x)\cdot\Big({}_{\mathscr{E}'(\mathbb{R}^n)}\langle\!\langle u, \vec{R}(x-\cdot)\rangle\!\rangle_{\mathscr{E}(\mathbb{R}^n)}\Big)\,d\mathcal{H}^{n-1}(x)\\ &= -\,{}_{\mathscr{E}'(\mathbb{R}^n)}\Big\langle\!\!\Big\langle u(y), \frac{1}{\omega_{n-1}}\int_{S^{n-1}} x\cdot\frac{x-y}{|x-y|^n}\,d\mathcal{H}^{n-1}(x)\Big\rangle\!\!\Big\rangle_{\mathscr{E}(\mathbb{R}^n)}\\ &= -\,{}_{\mathscr{E}'(\mathbb{R}^n)}\langle\!\langle u, 1\rangle\!\rangle_{\mathscr{E}(\mathbb{R}^n)} = -\,{}_{\mathscr{E}'(\mathbb{R}^n)}\langle u, 1\rangle_{\mathscr{E}(\mathbb{R}^n)}\end{aligned} \tag{2.2.50}$$

where the third equality is based on (2.1.39)–(2.1.40). Also, as seen from (2.2.48), we have

$${}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\Omega)} = 0. \tag{2.2.51}$$

Together, (2.2.50) and (2.2.51) prove that

if ${}_{\mathscr{E}'(\mathbb{R}^n)}\langle u, 1\rangle_{\mathscr{E}(\mathbb{R}^n)} \neq 0$, *the Divergence Formula* (1.4.6) *fails for the domain* Ω *as in* (2.2.46) *and the vector field* $\vec{F}$ *as in* (2.2.47). (2.2.52)

In the case when ${}_{\mathscr{E}'(\mathbb{R}^n)}\langle u, 1\rangle_{\mathscr{E}(\mathbb{R}^n)} \neq 0$, the reason for the failure of formula (1.4.6) in the current setting is the fact that the growth condition (1.4.8) is not satisfied. Remarkably, the following equivalence actually holds:

$$(1.4.8) \ \text{ holds } \iff {}_{\mathscr{E}'(\mathbb{R}^n)}\langle u, 1\rangle_{\mathscr{E}(\mathbb{R}^n)} = 0. \tag{2.2.53}$$

To justify (2.2.53) recall first (cf., e.g., [181, Fact 2.63, p. 43]) that since u is a compactly supported distribution in $\mathbb{R}^n$, there exist a compact set $K \subset \mathbb{R}^n$, an integer

$N \in \mathbb{N}_0$, and a constant $C \in (0, \infty)$, with the property that

$$\Big| {}_{\mathscr{E}'(\mathbb{R}^n)}\langle u, \varphi \rangle_{\mathscr{E}(\mathbb{R}^n)} \Big| \le C \sup_{\substack{\alpha \in \mathbb{N}_0^n,\, |\alpha| \le N \\ x \in K}} |(\partial^\alpha \varphi)(x)|, \qquad \forall \varphi \in \mathscr{C}^\infty(\mathbb{R}^n). \tag{2.2.54}$$

Also, observe that, thanks to (2.2.26), for each multi-index $\alpha \in \mathbb{N}_0^n$ we have

$$\partial_y^\alpha \left\{ \frac{x-y}{|x-y|^n} - \frac{x}{|x|^n} \right\} = O\big(|x|^{-n}\big) \text{ as } |x| \to \infty, \text{ uniformly for } y \in K. \tag{2.2.55}$$

Collectively, (2.2.54) and (2.2.55) prove that

$${}_{\mathscr{E}'(\mathbb{R}^n)}\big\langle\!\big\langle u, \vec{R}(x - \cdot) \big\rangle\!\big\rangle_{\mathscr{E}(\mathbb{R}^n)} = {}_{\mathscr{E}'(\mathbb{R}^n)}\langle u, 1 \rangle_{\mathscr{E}(\mathbb{R}^n)} \vec{R}(x) + O\big(|x|^{-n}\big) \text{ as } |x| \to \infty, \tag{2.2.56}$$

which, in turn, goes to show that $\vec{F}$ defined in (2.2.49) has the following asymptotic behavior at infinity:

$$\vec{F}(x) = \frac{{}_{\mathscr{E}'(\mathbb{R}^n)}\langle u, 1 \rangle_{\mathscr{E}(\mathbb{R}^n)}}{\omega_{n-1}} \frac{x}{|x|^n} + O\big(|x|^{-n}\big) \text{ as } |x| \to \infty. \tag{2.2.57}$$

With this in hand, (2.2.53) is then proved much like the equivalence in (2.2.25).

The counterexample in (2.2.52) may be turned into a positive result by either requiring that ${}_{\mathscr{E}'(\mathbb{R}^n)}\langle u, 1 \rangle_{\mathscr{E}(\mathbb{R}^n)} = 0$, or employing the version of the Divergence Formula recorded in (1.4.5). The latter also takes into account the contribution of $\vec{F}$ at infinity which, in view of the asymptotic formula (2.2.57), may be concretely identified as

$$[\vec{F}]_\infty = \lim_{R \to \infty} \int_{|x|=R} \frac{x}{|x|} \cdot \vec{F}(x) \, d\mathcal{H}^{n-1}(x) = {}_{\mathscr{E}'(\mathbb{R}^n)}\langle u, 1 \rangle_{\mathscr{E}(\mathbb{R}^n)}. \tag{2.2.58}$$

Keeping in mind (2.2.50)–(2.2.51) and (2.2.58), we may therefore conclude that

all hypotheses of Theorem 1.4.1 *are satisfied by the domain* Ω *as in* (2.2.46) *and the vector field* $\vec{F}$ *as in* (2.2.47), *and the Divergence Formula* (1.4.5) *holds for these choices.* (2.2.59)

In closing we wish to note that the domain and vector field from (2.2.46)–(2.2.47) also constitute, for the same reasons as above, a counterexample for the version of the Divergence Formula recorded in Theorem 1.5.1.

2.3 Failure of Hypotheses on the Nontangential Maximal Function

Invariably, in Theorems 1.2.1, 1.3.1, 1.4.1, and 1.5.1 some integrability condition (of a global or local nature) has been imposed on the nontangential maximal function of the given vector field. Since the nontangential maximal function does enter (directly) the formulation of our Divergence Formulas, it is therefore important to clarify the role of such a quantitative assumption, and the goal of this section is to do just that.

Counterexamples/Examples Part 2.3A: Assume $n \geq 2$ and consider the open subset of $\mathbb{R}^n$ given by (Fig. 2.10)

$$\Omega := B(0,1) \setminus \Sigma, \quad \text{where} \quad \Sigma := \overline{B_{n-1}(0',1/2)} \times \{0\}. \tag{2.3.1}$$

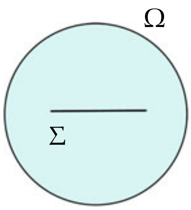

Fig. 2.10 Ω as in (2.3.1)

Also, bring in the vector field

$$\vec{F}(x) := \frac{x}{|x|^n} \quad \text{for each} \quad x \in \Omega. \tag{2.3.2}$$

These definitions ensure that

$$\begin{aligned} &\partial\Omega = S^{n-1} \cup \Sigma \quad \text{is Ahlfors regular,} \\ &\partial_*\Omega = S^{n-1}, \quad \text{and} \quad \partial\Omega \setminus \partial_*\Omega = \Sigma. \end{aligned} \tag{2.3.3}$$

In particular, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling finite measure on $\partial\Omega$. Furthermore,

$$\begin{aligned} &\vec{F} \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \quad \operatorname{div}\vec{F} = 0 \quad \text{in} \quad \Omega, \quad \text{and} \\ &\vec{F} \text{ extends continuously to } \overline{\Omega} \setminus \{0\}. \end{aligned} \tag{2.3.4}$$

Fix $\kappa > 0$. Since $\mathcal{H}^{n-1}(\{0\}) = 0$ (given that we are assuming $n \geq 2$), the last property above shows that $\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists σ-a.e. on $\partial\Omega$. As far as the Divergence Formula (1.2.2) is concerned, we therefore have

$$\int_{\Omega} \operatorname{div}\vec{F}\,\mathrm{d}\mathcal{L}^n = 0, \tag{2.3.5}$$

whereas with ν denoting the geometric measure theoretic outward unit normal to Ω we have

$$\begin{aligned}\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)\,\mathrm{d}\sigma &= \int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|_{\partial\Omega}\Big)\,\mathrm{d}\sigma \\ &= \int_{S^{n-1}} x \cdot \frac{x}{|x|^n}\,\mathrm{d}\mathcal{H}^{n-1}(x) = \omega_{n-1} \neq 0. \end{aligned} \tag{2.3.6}$$

Hence, as is apparent from (2.3.5)–(2.3.6),

the Divergence Formula (1.2.2) *fails for the set* Ω *as in* (2.3.1), *and for the vector field* $\vec{F}$ *as in* (2.3.2). (2.3.7)

A direct comparison of the present setting with that in Theorem 1.2.1 points to the second condition in (1.2.1), namely the membership of $\mathcal{N}_\kappa \vec{F}$ to $L^1(\partial\Omega, \sigma)$, as the only possible source of this failure. To check that this is the case, observe that for each $\kappa > 0$ fixed we have

$$\big(\mathcal{N}_\kappa \vec{F}\big)(x) \approx |x'|^{-(n-1)} \text{ uniformly for } x = (x', 0) \in \Sigma, \tag{2.3.8}$$

and since $\int_{B_{n-1}(0',1/2)} |x'|^{-(n-1)}\,\mathrm{d}x' = +\infty$, we conclude that

$$\mathcal{N}_\kappa \vec{F} \notin L^1(\partial\Omega, \sigma). \tag{2.3.9}$$

Consequently, the second condition in (1.2.1) is indeed violated in the present context, even though

$$\mathcal{N}_\kappa \vec{F} \in L^{1,\infty}(\partial\Omega, \sigma) \text{ and } \mathcal{N}_\kappa \vec{F} \in L^1(\partial_*\Omega, \sigma). \tag{2.3.10}$$

In summary,

the counterexample in (2.3.7) *shows that, as far as the validity of the Divergence Formula* (1.2.2) *is concerned, the hypothesis that* $\mathcal{N}_\kappa \vec{F}$ *belongs to* $L^1(\partial\Omega, \sigma)$ *cannot be weakened to either* $\mathcal{N}_\kappa \vec{F} \in L^{1,\infty}(\partial\Omega, \sigma)$, *or* $\mathcal{N}_\kappa \vec{F} \in L^1(\partial_*\Omega, \sigma)$, *or even both.* (2.3.11)

In other words, as far as the integrability properties of the nontangential maximal function $\mathcal{N}_\kappa \vec{F}$ are concerned, the space $L^1(\partial\Omega, \sigma)$ cannot be enlarged to its weak version, nor can we alter the set over which this Lebesgue space is defined by considering $L^1(\partial_*\Omega, \sigma)$ instead, even though the Divergence Formula (1.2.2) takes into account only the behavior of $\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ on $\partial_*\Omega$.

It is worth noting that (2.3.7) is also a counterexample for the version of the Divergence Formula recorded in Theorem 1.5.1.

Going forward, it is instructive to slightly alter the counterexample in (2.3.7) in order to produce an actual example for the applicability of the Divergence Formula (1.4.6). The idea is to eliminate the cut inside the domain and interpret the vector field as now being defined only $\mathcal{L}^n$-a.e. in this larger environment. This interpretation alters the divergence of the vector field, and it is precisely this change which ultimately produces a valid Divergence Formula. Specifically, in place of (2.3.1)–(2.3.2) let us now take

$$\Omega := B(0,1) \text{ and } \vec{F}(x) := \frac{x}{|x|^n} \text{ for } \mathcal{L}^n\text{-a.e. } x \in \Omega. \tag{2.3.12}$$

Then, of course, Ω is a bounded $\mathscr{C}^\infty$ domain, and with δ denoting the Dirac distribution with mass at the origin in Ω, we have

$$\begin{gathered} \vec{F} \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n, \quad \operatorname{div}\vec{F} = \omega_{n-1}\delta \text{ in } \mathcal{D}'(\Omega), \\ \text{and } \vec{F} \text{ extends continuously to } \overline{\Omega}\setminus\{0\}. \end{gathered} \tag{2.3.13}$$

Since $\delta \in \mathscr{E}'(\Omega)$, it follows that Ω and $\vec{F}$ satisfy all hypotheses of Theorem 1.4.1. We therefore expect the Divergence Formula (1.4.6) to be valid and, indeed, (4.6.21) gives

$$_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\Omega)} = \omega_{n-1}\,{}_{\mathscr{E}'(\Omega)}\langle\delta, 1\rangle_{\mathscr{E}(\Omega)} = \omega_{n-1} \tag{2.3.14}$$

and from (2.3.12) we see that for each $\kappa > 0$ we have

$$\int_{\partial_*\Omega} \nu\cdot\Big(\vec{F}\,\Big|^{\kappa-\rm n.t.}_{\partial\Omega}\Big)\,d\sigma = \int_{S^{n-1}} x\cdot\frac{x}{|x|^n}\,d\mathcal{H}^{n-1}(x) = \omega_{n-1}, \tag{2.3.15}$$

in agreement with (2.3.14). This discussion proves that

$$\text{\textit{all hypotheses of Theorem 1.4.1 are satisfied by the domain Ω and vector field $\vec{F}$ as in} (2.3.12), \textit{and the Divergence Formula} (1.4.6) \textit{holds for this choice.}} \tag{2.3.16}$$

Counterexamples/Examples Part 2.3B: Suppose $n \in \mathbb{N}$ satisfies $n \geq 2$, and let (Fig. 2.11)

$$\Omega := B(0,1) \cap \mathbb{R}^n_+ \text{ and } \vec{F}(x) := \frac{x}{|x|^n} \text{ for each } x \in \Omega. \tag{2.3.17}$$

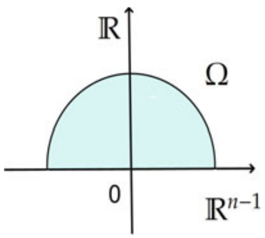

Fig. 2.11 Ω as in (2.3.17)

Introducing $\Sigma := \overline{B_{n-1}(0', 1)} \times \{0\}$ then allows us to express

$$\partial\Omega = \partial_*\Omega = S^{n-1}_+ \cup \Sigma, \quad \text{disjoint union.} \tag{2.3.18}$$

In particular, $\partial\Omega$ is compact and Ahlfors regular, hence $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling finite measure on $\partial\Omega$. Also,

$$\begin{gathered}\vec{F} \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \quad \operatorname{div}\vec{F} = 0 \ \text{ in } \ \Omega, \\ \text{and } \ \vec{F} \text{ extends continuously to } \overline{\Omega} \setminus \{0\}.\end{gathered} \tag{2.3.19}$$

As such,

$$\int_\Omega \operatorname{div}\vec{F} \, d\mathcal{L}^n = 0, \tag{2.3.20}$$

and for each $\kappa > 0$ the nontangential pointwise boundary trace $\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists σ-a.e. on $\partial\Omega$. Since the geometric measure theoretic outward unit normal to Ω is given by $\nu(x) = x$ for each $x \in S^{n-1}_+$ and $\nu(x) = -\mathbf{e}_n$ for σ-a.e. $x \in \Sigma$, it follows that

$$\Big(\nu \cdot \vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) = \begin{cases} 1 & \text{for each } \ x \in S^{n-1}_+, \\ 0 & \text{for } \sigma\text{-a.e. } \ x \in \Sigma. \end{cases} \tag{2.3.21}$$

As a consequence,

$$\nu \cdot \Big(\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) \in L^1(\partial\Omega, \sigma), \tag{2.3.22}$$

and

$$\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) \, d\sigma = \int_{S^{n-1}_+} 1 \, d\mathcal{H}^{n-1}(x) = \tfrac{1}{2}\omega_{n-1} \neq 0. \tag{2.3.23}$$

At this stage, it becomes apparent from (2.3.20) and (2.3.23) that

$$\text{\textit{the Divergence Formula} (1.2.2) \textit{fails for the domain} } \Omega \text{ \textit{and vector field} } \vec{F} \text{ \textit{as in} (2.3.17).} \tag{2.3.24}$$

As for the source of this failure, a direct comparison of the present setting with the hypotheses of Theorem 1.2.1 points to the second condition in (1.2.1), namely the membership of $\mathcal{N}_\kappa\vec{F}$ to $L^1(\partial\Omega, \sigma)$. To see that this, indeed, does not materialize, for each $\kappa > 0$ fixed we observe that there exists a constant $C \in (1, \infty)$, depending only on κ and n, with the property that

$$\text{for each number } \varepsilon > 0 \text{ one has the estimates} \\ C^{-1}|x'|^{-(n-1)} \leq \big(\mathcal{N}_\kappa^\varepsilon \vec{F}\big)(x) \leq C|x'|^{-(n-1)}, \\ \text{at all points } x = (x', 0) \text{ belonging to } \Sigma \setminus \{0\}. \tag{2.3.25}$$

Given that $\int_{B_{n-1}(0',r)} |x'|^{-(n-1)}\,dx' = +\infty$ for every $r > 0$, we conclude from (2.3.25) and the second line in (2.3.19) that

$$\int_{B(0,r)\cap\partial\Omega} \mathcal{N}_\kappa^\varepsilon \vec{F}\,d\sigma = +\infty \;\text{ for each } \varepsilon > 0 \text{ and } r > 0, \text{ whereas} \\ \forall x \in \partial\Omega \setminus \{0\}, \exists\, \varepsilon_x, r_x > 0 \text{ with } \int_{B(x,r_x)\cap\partial\Omega} \mathcal{N}_\kappa^{\varepsilon_x} \vec{F}\,d\sigma < +\infty. \tag{2.3.26}$$

In particular, as anticipated,

$$\mathcal{N}_\kappa \vec{F} \notin L^1(\partial\Omega, \sigma). \tag{2.3.27}$$

This being said, the failure in (2.3.27) is minimal, in the sense that

$$\mathcal{N}_\kappa \vec{F} \;\text{ belongs to the weak Lebesgue space }\; L^{1,\infty}(\partial\Omega, \sigma). \tag{2.3.28}$$

In summary, the counterexample in (2.3.24) shows that

as far as the validity of the Divergence Formula (1.2.2) *is concerned, the hypothesis $\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma)$ cannot be weakened by demanding instead $\mathcal{N}_\kappa \vec{F} \in L^{1,\infty}(\partial\Omega, \sigma)$ and $\nu \cdot \Big(\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big) \in L^1(\partial_*\Omega, \sigma)$,* (2.3.29)

even though it is precisely the latter condition which is most directly connected with the very formulation of the Divergence Formula (1.2.2).

In view of (2.3.26) the counterexample in (2.3.24) also shows that

in the context of Theorem 1.5.1, the failure of the local integrability condition in the second line of (1.5.8), *even at a single point x_0 on $\partial\Omega$, may invalidate the Divergence Formula recorded in* (1.5.11) *even if one additionally assumes that Ω is bounded, that $\vec{F}$ belongs to $\big[\mathscr{C}^\infty(\overline{\Omega} \setminus \{x_0\})\big]^n$, and that $\mathcal{N}_\kappa \vec{F}$ belongs to $L^{1,\infty}(\partial\Omega, \sigma)$.* (2.3.30)

The above counterexample may be further generalized to conical domains of arbitrary aperture as follows. Assume $n \in \mathbb{N}$ satisfies $n \geq 2$ and, for each $\theta \in (0, \pi)$, consider the open set in $\mathbb{R}^n$ described as (Fig. 2.12)

$$\Omega_\theta := \big\{x \in B(0,1) : \langle x, \mathbf{e}_n\rangle > |x| \cos\theta\big\}, \tag{2.3.31}$$

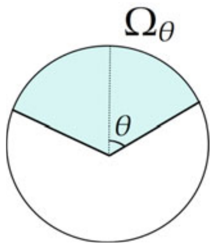

Fig. 2.12 Ω_θ as in (2.3.31)

along with the vector field

$$\vec{F}(x) := \frac{x}{|x|^n} \quad \text{for each} \quad x \in \Omega_\theta. \tag{2.3.32}$$

Then $\vec{F} \in \big[\mathscr{C}^\infty(\Omega_\theta)\big]^n$, $\operatorname{div}\vec{F} = 0$ in Ω_θ, and $\vec{F}$ extends continuously to $\overline{\Omega_\theta} \setminus \{0\}$. Also, $\partial\Omega$ is an Ahlfors regular set, and if we introduce the sets

$$\begin{aligned} \Sigma_\theta &:= \big\{x \in \overline{B(0,1)} : \langle x, \mathbf{e}_n\rangle = |x|\cos\theta\big\}, \\ S^{n-1}_\theta &:= \big\{x \in S^{n-1} : \langle x, \mathbf{e}_n\rangle > \cos\theta\big\}, \end{aligned} \tag{2.3.33}$$

then

$$\partial_*\Omega_\theta = \partial\Omega_\theta = S^{n-1}_\theta \cup \Sigma_\theta, \quad \text{disjoint union.} \tag{2.3.34}$$

Having fixed $\kappa > 0$ and with ν_θ denoting the geometric measure theoretic outward unit normal to Ω_θ, we therefore have

$$\nu_\theta \cdot \Big(\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega_\theta}\Big) = \begin{cases} 1 & \text{everywhere on } S^{n-1}_\theta, \\ 0 & \mathcal{H}^{n-1}\text{-a.e. on } \Sigma_\theta. \end{cases} \tag{2.3.35}$$

Thus, if $\sigma_\theta := \mathcal{H}^{n-1}\lfloor\partial\Omega_\theta$, it follows that $\nu_\theta \cdot \Big(\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega_\theta}\Big) \in L^1(\partial\Omega_\theta, \sigma_\theta)$ and

$$\begin{aligned} \int_{\partial_*\Omega_\theta} \nu_\theta \cdot \Big(\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega_\theta}\Big)\, d\sigma_\theta &= \int_{S^{n-1}_\theta} 1\, d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}\big(S^{n-1}_\theta\big) \\ &= \frac{\theta}{\pi}\omega_{n-1} \neq 0 = \int_{\Omega_\theta} \operatorname{div}\vec{F}\, d\mathcal{L}^n. \end{aligned} \tag{2.3.36}$$

The conclusion from (2.3.36) is that

the Divergence Formula (1.2.2) *fails for the set* Ω_θ *in* (2.3.31) *and the vector field* $\vec{F}$ *defined as in* (2.3.32). (2.3.37)

The source of failure of formula (1.2.2) in the present setting is the violation of the second condition in (1.2.1). Concretely, for each $\kappa > 0$ fixed we have

$$\big(\mathcal{N}_\kappa^\varepsilon \vec{F}\big)(x) \approx |x|^{-(n-1)}, \quad \text{uniformly for} \;\; x \in \Sigma_\theta \setminus \{0\} \;\; \text{and} \;\; \varepsilon > 0. \tag{2.3.38}$$

This ultimately implies that for each $\varepsilon > 0$ the truncated nontangential maximal function $\mathcal{N}_\kappa^\varepsilon \vec{F}$ is not locally integrable near the origin. In particular, $\mathcal{N}_\kappa \vec{F} \notin L^1(\partial\Omega_\theta, \sigma_\theta)$, even though

$$\mathcal{N}_\kappa \vec{F} \in L^{1,\infty}(\partial\Omega_\theta, \sigma_\theta) \;\; \text{and} \;\; \nu_\theta \cdot \Big(\vec{F}\Big|^{\kappa-\text{n.t.}}_{\partial\Omega_\theta}\Big) \in L^1(\partial\Omega_\theta, \sigma_\theta). \tag{2.3.39}$$

To turn the counterexample mentioned in (2.3.37) into a positive result, we retain the domain Ω_θ as in (2.3.31) and alter the vector field by taking, in place of (2.3.32),

$$\vec{F}_\varepsilon(x) := \frac{x}{|x|^{n-\varepsilon}} \;\; \text{for each} \;\; x \in \Omega_\theta, \tag{2.3.40}$$

where $\varepsilon > 0$ is arbitrary and fixed. Then, as before, $\vec{F}_\varepsilon \in \big[\mathscr{C}^\infty(\Omega_\theta)\big]^n$, and $\vec{F}_\varepsilon$ extends continuously to $\overline{\Omega_\theta} \setminus \{0\}$. This time, however, a direct computation gives that

$$(\operatorname{div}\vec{F}_\varepsilon)(x) = \frac{\varepsilon}{|x|^{n-\varepsilon}} \;\; \text{for each} \;\; x \in \Omega_\theta. \tag{2.3.41}$$

Hence,

$$\operatorname{div}\vec{F}_\varepsilon \in L^1(\Omega_\theta, \mathcal{L}^n) \tag{2.3.42}$$

and

$$\begin{aligned} \int_{\Omega_\theta} \operatorname{div}\vec{F}_\varepsilon \, d\mathcal{L}^n &= \int_{\Omega_\theta} \frac{\varepsilon}{|x|^{n-\varepsilon}} \, dx = \mathcal{H}^{n-1}\big(S^{n-1}_\theta\big) \int_0^1 \varepsilon \rho^{\varepsilon-1} \, d\rho \\ &= \frac{\theta}{\pi} \omega_{n-1}. \end{aligned} \tag{2.3.43}$$

Another aspect affected by the change in the definition of the vector field is the size of the nontangential maximal function since, in lieu of (2.3.38), we now have

$$\big(\mathcal{N}_\kappa \vec{F}\big)(x) \approx |x|^{-(n-1-\varepsilon)}, \quad \text{uniformly for} \;\; x \in \Sigma_\theta. \tag{2.3.44}$$

In turn, this ultimately implies that

$$\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega_\theta, \sigma_\theta). \tag{2.3.45}$$

Since, much as in (2.3.35),

$$\nu_\theta \cdot \Big(\vec{F}_\varepsilon\Big|^{\kappa-\text{n.t.}}_{\partial\Omega_\theta}\Big) = \mathbf{1}_{S^{n-1}_\theta} \;\; \text{at } \mathcal{H}^{n-1}\text{-a.e. point on} \;\; \partial_*\Omega_\theta, \tag{2.3.46}$$

we see that

$$\int_{\partial_*\Omega_\theta} \nu_\theta \cdot \Big(\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega_\theta}\Big)\,\mathrm{d}\sigma_\theta = \mathcal{H}^{n-1}\big(S^{n-1}_\theta\big) = \frac{\theta}{\pi}\omega_{n-1}, \tag{2.3.47}$$

which is in agreement with (2.3.43). All in all, this discussion shows that

all hypotheses of Theorem 1.2.1 are satisfied by the domain Ω_θ *as in* (2.3.31) *and the vector field* $\vec{F}_\varepsilon$ *as in* (2.3.40), *and the Divergence Formula* (1.2.2) *holds for these choices.* (2.3.48)

Counterexamples/Examples Part 2.3C: Fix $n \in \mathbb{N}$ satisfying $n \geq 2$, and pick

$$\begin{aligned} &\varphi \in \mathscr{C}^\infty_c(\mathbb{R}^n) \ \text{ real-valued even function, satisfying} \\ &\varphi \equiv 1 \ \text{ on } \ B(0,1) \ \text{ and } \ \varphi \equiv 0 \ \text{ on } \ \mathbb{R}^n \setminus B(0,2). \end{aligned} \tag{2.3.49}$$

Define

$$\Omega := \mathbb{R}^n_+ \ \text{ and } \ \vec{F}(x) := \varphi(x)\frac{x}{|x|^n} \ \text{ for each } \ x \in \Omega. \tag{2.3.50}$$

Then Ω is an open set with an unbounded Ahlfors regular boundary satisfying $\partial_*\Omega = \partial\Omega \equiv \mathbb{R}^{n-1}$, and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ may be canonically identified with $\mathcal{L}^{n-1}$ (in particular, σ is doubling). Also,

$$\begin{gathered} \vec{F} \in \big[\mathscr{C}^\infty(\Omega)\big]^n, \quad \vec{F} \equiv 0 \ \text{ on } \ \mathbb{R}^n_+ \setminus B(0,2), \\ (\mathrm{div}\vec{F})(x) = (\nabla\varphi)(x) \cdot \tfrac{x}{|x|^n} \ \text{ at each point } \ x \in \Omega, \\ \text{and } \ \vec{F} \text{ extends continuously to } \overline{\Omega} \setminus \{0\}. \end{gathered} \tag{2.3.51}$$

Since $\vec{F}$ vanishes identically outside of a bounded set, it trivially follows that

$$\text{the growth condition (1.2.3) is satisfied and } [\vec{F}]_\infty = 0. \tag{2.3.52}$$

In addition, $\mathrm{div}\vec{F}$ vanishes outside the bounded set $B^+(0,2) := B(0,2) \cap \mathbb{R}^n_+$ and belongs to $\mathscr{C}^\infty(\overline{\Omega})$, so

$$\mathrm{div}\vec{F} \in L^1(\Omega, \mathcal{L}^n), \tag{2.3.53}$$

and for each $\kappa > 0$ the nontangential pointwise boundary trace $\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ exists σ-a.e. on $\partial\Omega$. Given that the geometric measure theoretic outward unit normal to Ω is $\nu = -\mathbf{e}_n$ at each point on $\partial\Omega$, it follows that $\nu(x) \cdot x = 0$ for each $x \in \partial\Omega$, hence

$$\nu \cdot \Big(\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big) = 0 \ \text{ at } \sigma\text{-a.e. point on } \ \partial\Omega. \tag{2.3.54}$$

Thus, trivially,

$$\nu \cdot \Big(\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big) \in L^1(\partial_*\Omega, \sigma), \tag{2.3.55}$$

so, on the one hand,

$$\int_{\partial_*\Omega} \nu \cdot \left(\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \right) \mathrm{d}\sigma = 0. \tag{2.3.56}$$

To compute the solid integral featuring in the divergence formula, bring in the standard fundamental solution for Δ, the Laplacian in $\mathbb{R}^n$, i.e.,

$$E_\Delta(x) := \begin{cases} \dfrac{1}{\omega_{n-1}(2-n)} \dfrac{1}{|x|^{n-2}} & \text{if } n \geq 3, \\[2ex] \dfrac{1}{2\pi} \ln |x| & \text{if } n = 2, \end{cases} \tag{2.3.57}$$

for each $x \in \mathbb{R}^n \setminus \{0\}$, and note that, in all dimensions $n \geq 2$,

$$(\nabla E_\Delta)(x) = \frac{1}{\omega_{n-1}} \cdot \frac{x}{|x|^n} \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}. \tag{2.3.58}$$

Bearing in mind that φ is a smooth, compactly supported, even function, with the property that $\varphi(0) = 1$, we may then compute

$$\begin{aligned} \int_\Omega \operatorname{div}\vec{F}\, \mathrm{d}\mathcal{L}^n &= \int_{\mathbb{R}^n_+} (\nabla\varphi)(x) \cdot \frac{x}{|x|^n}\, \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}^n} (\nabla\varphi)(x) \cdot \frac{x}{|x|^n}\, \mathrm{d}x \\ &= \frac{\omega_{n-1}}{2} \big\langle \nabla E_\Delta, \nabla\varphi \big\rangle = -\frac{\omega_{n-1}}{2} \big\langle \operatorname{div}\nabla E_\Delta, \varphi \big\rangle \\ &= -\frac{\omega_{n-1}}{2} \big\langle \Delta E_\Delta, \varphi \big\rangle = -\frac{\omega_{n-1}}{2} \big\langle \delta, \varphi \big\rangle \\ &= -\frac{\omega_{n-1}}{2} \varphi(0) = -\frac{\omega_{n-1}}{2}, \end{aligned} \tag{2.3.59}$$

where the angled brackets stand for pairing in the sense of distributions in $\mathbb{R}^n$, and δ is the Dirac distribution in $\mathbb{R}^n$. Thus, on the other hand,

$$\int_\Omega \operatorname{div}\vec{F}\, \mathrm{d}\mathcal{L}^n = -\frac{\omega_{n-1}}{2}. \tag{2.3.60}$$

It is then apparent from (2.3.56) and (2.3.60) that

the Divergence Formulas (1.2.2) *and* (1.3.8) *fail for the domain* Ω *and the vector field* $\vec{F}$ *as in* (2.3.49)–(2.3.50). (2.3.61)

As to what causes this failure, a direct comparison of the present setting with the hypotheses of Theorems 1.2.1 and 1.3.1 points to the second condition in (1.2.1), and the last condition in (1.3.6). Indeed, for each $\kappa > 0$ fixed, the membership of $\mathcal{N}_\kappa \vec{F}$

to $L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ does not materialize. To see that this is the case, observe that there exists a constant $C \in (1, \infty)$, depending only on κ and n, with the property that

$$
\begin{gathered}
C^{-1}|x'|^{-(n-1)} \leq \big(\mathcal{N}_\kappa \vec{F}\big)(x) \leq C|x'|^{-(n-1)} \\
\text{at all points } x = (x', 0) \text{ belonging to } \partial\Omega \cap B(0,1).
\end{gathered}
\tag{2.3.62}
$$

Given that $\mathcal{N}_\kappa \vec{F}$ vanishes outside of a bounded subset of $\partial\Omega$, we ultimately conclude that

$$
\begin{gathered}
\mathcal{N}_\kappa \vec{F} \text{ belongs to the weak Lebesgue space } L^{1,\infty}(\partial\Omega, \sigma) \\
\text{but } \mathcal{N}_\kappa \vec{F} \text{ fails to be in } L^1_{\mathrm{loc}}(\partial\Omega, \sigma).
\end{gathered}
\tag{2.3.63}
$$

In this sense, the failure described in (2.3.61) is minimal, so there is a razor-thin margin between what's valid and what fails. To summarize, the counterexample in (2.3.61) shows (compare with (2.3.29) in the case of bounded domains) that

for the open set with unbounded boundary Ω *and the vector field* $\vec{F}$ *as in* (2.3.49)-(2.3.50), *the Divergence Formula* (1.2.2) *and the Divergence Formula* (1.3.8) *fail if in place of the hypothesis* $\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma)$, *respectively, in place of the hypothesis* $\mathcal{N}_\kappa \vec{F} \in L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$, *one now only demands* $\mathcal{N}_\kappa \vec{F} \in L^{1,\infty}(\partial\Omega, \sigma)$ *and* $\nu \cdot \Big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big) \in L^1(\partial_*\Omega, \sigma)$. (2.3.64)

Counterexamples/Examples Part 2.3D: Work in the one-dimensional setting (i.e., when $n = 1$), and consider the following subset of the real line:

$$
\Omega := \Big(\bigcup_{j=1}^{\infty} \big(-2j, -2j+1\big)\Big) \bigcup (0, +\infty).
\tag{2.3.65}
$$

Also, define

$$
F : \Omega \to \mathbb{R}, \quad F(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad x \in \Omega.
\tag{2.3.66}
$$

Then Ω is an open unbounded set in $\mathbb{R}$, with unbounded boundary $\partial\Omega = \mathbb{Z} \setminus \mathbb{N}$ which has no (finite) accumulation points. Consequently, $\sigma := \mathcal{H}^0 \lfloor \partial\Omega$ is the counting measure on $\partial\Omega$, which makes $\partial\Omega$ lower Ahlfors regular (bearing in mind that $n = 1$). Also, σ is locally finite. In fact, an elementary analysis shows that σ is a doubling measure on $\partial\Omega$ (though $\partial\Omega$ is *not* upper Ahlfors regular).

To proceed, fix an aperture parameter $\kappa > 0$. Observe that for each $x \in \partial\Omega$ the nontangential approach region $\Gamma_\kappa(x)$ contains $(-x/\kappa, +\infty)$, which is an open subset of $(0, +\infty)$ where F is identically 1. Thus,

$$
(\mathcal{N}_\kappa F)(x) = 1 \quad \text{for each } x \in \partial\Omega.
\tag{2.3.67}
$$

As such,

$$\mathcal{N}_\kappa F \in L^1_{\rm loc}(\partial\Omega,\sigma) \text{ yet } \mathcal{N}_\kappa F \text{ fails to be in } L^1(\partial\Omega,\sigma). \tag{2.3.68}$$

Next, $\partial_{\rm nta}\Omega = \partial\Omega$ and the nontangential boundary trace $F\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}$ exists at each point in $\partial\Omega$. Specifically, for each $x \in \partial\Omega$ we have

$$\big(F\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\big)(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases} \tag{2.3.69}$$

Consequently,

$$\int_{\partial_*\Omega} \nu \cdot \Big(F\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\Big)\, d\mathcal{L}^1 = -1. \tag{2.3.70}$$

Finally, $F' = 0$ on Ω, so

$$\int_\Omega F'\, d\mathcal{L}^1 = 0. \tag{2.3.71}$$

Collectively, (2.3.70)–(2.3.71) prove that the Divergence Formula (1.2.2) fails for Ω and F as in (2.3.65)–(2.3.66). In view of this analysis, we conclude that

> the Divergence Formula (1.2.2) may fail for open sets Ω with an unbounded lower Ahlfors regular boundary and a doubling "surface measure" if the assumption $\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega,\sigma)$ is replaced by the weaker condition $\mathcal{N}_\kappa \vec{F} \in L^1_{\rm loc}(\partial\Omega,\sigma)$. (2.3.72)

Lastly, we remark that the counterexample discussed in (2.3.72) may be converted into a positive result if information about the behavior of F at infinity is included in the very formulation of the Divergence Formula, as we have done in Theorem 1.3.1 through the consideration of the contribution of F at infinity. Concretely, in the present case Proposition 4.7.2 gives

$$[F]_\infty = F\Big|_{-\infty}^{+\infty} = 1 - 0 = 1, \tag{2.3.73}$$

which, in light of (2.3.70)–(2.3.71), ultimately shows that

> *all hypotheses of Theorem 1.3.1 are satisfied when $n = 1$ and Ω, F are as in* (2.3.65)–(2.3.66), *and the Divergence Formula* (1.3.8) *is valid in this setting.* (2.3.74)

Counterexamples/Examples Part 2.3E: Consider

$$\Omega := B(0,1), \quad \text{the unit disk in } \mathbb{R}^2 \equiv \mathbb{C}, \tag{2.3.75}$$

and abbreviate $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$. In this setting, fix an arbitrary point $z \in \Omega$ and introduce the vector field $\vec{F}_z \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^2)\big]^2$ defined for each $\zeta \in \Omega \setminus \{z\}$ according to

$$\vec{F}_z(\zeta) := \left(\frac{\exp\left\{\frac{1+\zeta}{1-\zeta}\right\}}{\zeta - z}, \mathrm{i}\frac{\exp\left\{\frac{1+\zeta}{1-\zeta}\right\}}{\zeta - z}\right) \in \mathbb{C}^2, \tag{2.3.76}$$

where $\mathrm{i} := \sqrt{-1} \in \mathbb{C}$. Then, taking the divergence in the sense of distributions in the variable $\zeta = (\zeta_1, \zeta_2) \in \Omega$ yields (bearing in mind [181, Theorem 7.43, p. 289])

$$\begin{aligned} \operatorname{div}\vec{F}_z(\zeta) &= \partial_{\zeta_1}\left[\frac{\exp\left\{\frac{1+\zeta}{1-\zeta}\right\}}{\zeta - z}\right] + \mathrm{i}\partial_{\zeta_2}\left[\frac{\exp\left\{\frac{1+\zeta}{1-\zeta}\right\}}{\zeta - z}\right] \\ &= 2\partial_{\overline{\zeta}}\left[\frac{\exp\left\{\frac{1+\zeta}{1-\zeta}\right\}}{\zeta - z}\right] = 2\pi \exp\left\{\frac{1+\zeta}{1-\zeta}\right\}\delta_z(\zeta) \\ &= 2\pi \exp\left\{\frac{1+z}{1-z}\right\}\delta_z(\zeta) \in \mathscr{E}'(\Omega), \end{aligned} \tag{2.3.77}$$

where $\partial_{\overline{\zeta}} := \frac{1}{2}(\partial_{\zeta_1} + \mathrm{i}\partial_{\zeta_2})$ is the Cauchy–Riemann operator in the variable ζ, and δ_z is the Dirac distribution with mass at z in Ω. Also, as is apparent from (2.3.76),

$$\vec{F}_z \in \big[\mathscr{C}^0\big(\overline{\Omega} \setminus \{1, z\}\big)\big]^2 \tag{2.3.78}$$

and since

$$\exp\left\{\frac{1+e^{\mathrm{i}\theta}}{1-e^{\mathrm{i}\theta}}\right\} = \exp\left\{\frac{\mathrm{i}\sin\theta}{1-\cos\theta}\right\} \in \mathbb{T} := \partial B(0, 1), \qquad \forall\theta \in (0, 2\pi), \tag{2.3.79}$$

it follows that, for each fixed $\kappa \in (0, \infty)$,

$$\begin{aligned} &\vec{F}_z\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ exists everywhere on } \partial\Omega \setminus \{1\} \text{ (ergo, } \sigma\text{-a.e. on } \partial\Omega), \\ &\text{and } \vec{F}_z\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ belongs to } L^\infty(\partial\Omega, \sigma), \text{ hence also to } L^1(\partial\Omega, \sigma). \end{aligned} \tag{2.3.80}$$

On the other hand, an elementary computation shows that

$$\lim_{\theta\to 0^+} (e^{\mathrm{i}\theta} - (1-\theta))/\theta = 1 + \mathrm{i}. \tag{2.3.81}$$

Thus, given any aperture parameter $\kappa > \sqrt{2} - 1$, we have

$$\begin{aligned} &|e^{\mathrm{i}\theta} - (1-\theta)| < (1+\kappa)\,\theta \text{ provided} \\ &\theta \in (0, 1) \text{ is sufficiently small,} \end{aligned} \tag{2.3.82}$$

hence

$$1-\theta \in \Gamma_\kappa(e^{i\theta}) \quad \text{for every} \quad \theta \in (0,1) \quad \text{small.} \tag{2.3.83}$$

As such,

$$\big(\mathcal{N}_\kappa \vec{F}_z\big)(e^{i\theta}) \geq |\vec{F}_z(1-\theta)| \approx e^{2/\theta}, \quad \text{uniformly for } \theta \in (0,1) \text{ small.} \tag{2.3.84}$$

This goes to show that

$$\mathcal{N}_\kappa \vec{F}_z \notin L^1(\partial\Omega, \sigma). \tag{2.3.85}$$

If the Divergence Formula (1.4.6) were to hold for the vector field $\vec{F}_z$ in Ω, it would give (keeping in mind that $d\zeta = i\nu(\zeta)\, d\sigma(\zeta)$ on $\partial\Omega = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$)

$$\begin{aligned} 2\pi \exp\Big\{\frac{1+z}{1-z}\Big\} &= {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\operatorname{div}\vec{F}_z, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_{\partial\Omega} \Big\langle \nu, \big(\vec{F}_z\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)\Big\rangle\, d\sigma \\ &= \int_{\partial\Omega} \frac{\exp\big\{\frac{1+\zeta}{1-\zeta}\big\}}{\zeta - z}\nu(\zeta)\, d\sigma(\zeta) = \frac{1}{i}\int_{|\zeta|=1} \frac{\exp\big\{\frac{1+\zeta}{1-\zeta}\big\}}{\zeta - z}\, d\zeta, \end{aligned} \tag{2.3.86}$$

which would then ultimately lead to the conclusion that the following Cauchy integral representation formula holds:

$$\exp\Big\{\frac{1+z}{1-z}\Big\} = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\exp\big\{\frac{1+\zeta}{1-\zeta}\big\}}{\zeta - z}\, d\zeta, \qquad \forall z \in B(0,1). \tag{2.3.87}$$

Introducing

$$u : B(0,1) \to \mathbb{C}, \quad u(z) := \exp\Big\{\frac{1+z}{1-z}\Big\}, \quad \forall z \in B(0,1), \tag{2.3.88}$$

and

$$f(\zeta) := \exp\Big\{\frac{1+\zeta}{1-\zeta}\Big\}, \qquad \forall \zeta \in \partial B(0,1), \tag{2.3.89}$$

we may refashion (2.3.87) as

$$u(z) = (Cf)(z), \qquad \forall z \in B(0,1), \tag{2.3.90}$$

where C denotes the classical Cauchy integral operator in the unit disk. Note that, thanks to (2.3.79),

$$f \in L^\infty\big(\partial B(0,1), \sigma\big) \subseteq \bigcap_{1<p<\infty} L^p\big(\partial B(0,1), \sigma\big). \tag{2.3.91}$$

In turn, from (2.3.90), (2.3.91), and standard Hardy space theory in the unit disk we would then be able to conclude that, for each $\kappa > 0$,

$$\mathcal{N}_\kappa u = \mathcal{N}_\kappa(Cf) \in \bigcap_{1<p<\infty} L^p\big(\partial B(0,1),\sigma\big) \subseteq L^1\big(\partial B(0,1),\sigma\big). \tag{2.3.92}$$

This being said, if $\kappa > \sqrt{2} - 1$ then (2.3.83) holds which, in turn, permits us to estimate

$$\big(\mathcal{N}_\kappa u\big)(e^{i\theta}) \geq |u(1-\theta)| \approx e^{2/\theta}, \quad \text{uniformly for } \theta \in (0,1) \text{ small}, \tag{2.3.93}$$

thus arriving at the conclusion that

$$\mathcal{N}_\kappa u \notin L^1\big(\partial B(0,1),\sigma\big). \tag{2.3.94}$$

The fact that (2.3.92) and (2.3.94) are mutually exclusive is a contradiction which proves that

$$\begin{array}{l} \textit{there exists a number } z \in B(0,1) \textit{ with the property that the} \\ \textit{Divergence Formula } (1.4.6) \textit{ fails for the domain } \Omega \textit{ as in} \\ (2.3.75) \textit{ and the vector field } \vec{F}_z \textit{ as in } (2.3.76), \end{array} \tag{2.3.95}$$

even though Ω is a bounded domain of class $\mathscr{C}^\infty$, and for each $z \in B(0,1)$ the vector field $\vec{F}_z$ satisfies

$$\begin{array}{l} \vec{F}_z \in \big[L^1_{\rm loc}(\Omega,\mathcal{L}^2) \cap \mathscr{C}^0\big(\overline{\Omega} \setminus \{1,z\}\big)\big]^2, \ \operatorname{div}\vec{F}_z \in \mathscr{E}'(\Omega), \\ \text{and, for each } \ \kappa > 0, \ \text{ the nontangential boundary trace} \\ \vec{F}_z\big|^{\kappa-\rm n.t.}_{\partial\Omega} \ \text{ exists } \sigma\text{-a.e. on } \partial\Omega \text{ and belongs to } \ L^1(\partial\Omega,\sigma). \end{array} \tag{2.3.96}$$

Comparing these properties with the hypotheses of Theorem 1.4.1 reveals that the only possible source of failure for formula (1.4.6) in the present setting is for the second membership in (1.4.2) not to hold. That, indeed, the second membership in (1.4.2) is violated can be seen directly from (2.3.84).

In summary, the counterexample (2.3.95) to the Divergence Formula (1.4.6) shows that

$$\begin{array}{l} \textit{as far as the validity of the Divergence Formula } (1.4.6) \textit{ is concerned,} \\ \textit{the hypothesis } \mathcal{N}_\kappa^{\Omega\setminus K}\big(\vec{F}\big|_{\Omega\setminus K}\big) \in L^1_{\rm loc}(\partial\Omega,\sigma), \textit{ for some compact set} \\ K \subset \Omega, \textit{ cannot be weakened to } \vec{F}\big|^{\kappa-\rm n.t.}_{\partial\Omega} \in L^1(\partial\Omega,\sigma), \end{array} \tag{2.3.97}$$

even though the latter condition would, otherwise, suggest itself as a natural hypothesis, given the very formulation of the Divergence Formula (1.4.6).

Lastly, we wish to point out that the domain and vector field from (2.3.75)–(2.3.76) also serve as a counterexample for the version of the Divergence Formula found in Theorem 1.5.1.

2.4 Failure of Hypotheses of Geometric Measure Theoretic Nature

Here we examine the lower Ahlfors regular hypothesis made on the boundary of the domain in Theorems 1.2.1, 1.3.1, 1.4.1, and 1.5.1.

Counterexamples/Examples Part 2.4A: Assume $n \geq 2$ and let

$$\Omega := B(0,1) \setminus \{0\} \quad \text{and} \quad \vec{F}(x) := \frac{x}{|x|^n} \quad \text{for each} \quad x \in \Omega. \tag{2.4.1}$$

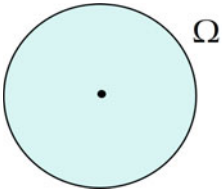

Fig. 2.13 Ω as in (2.4.1)

Then, by design, Ω is a bounded open set in $\mathbb{R}^n$, with an upper Ahlfors regular boundary, satisfying (Fig. 2.13)

$$\partial\Omega = S^{n-1} \cup \{0\}, \quad \partial_*\Omega = S^{n-1}, \quad \partial_{\text{nta}}\Omega = \partial\Omega. \tag{2.4.2}$$

In particular,

$$\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega \text{ is a locally finite measure on } \partial\Omega \tag{2.4.3}$$

(albeit not doubling in the sense of (7.4.1), since $\sigma\big(B(0,r) \cap \partial\Omega\big) = 0$ whenever $0 < r < 1$). Let us also note that since $n \geq 2$, we have

$$\sigma(\partial\Omega \setminus \partial_*\Omega) = \sigma(\{0\}) = 0. \tag{2.4.4}$$

As regards the vector field $\vec{F}$, for each fixed $\kappa > 0$ we have

$$\vec{F} \in \big[\mathscr{C}^\infty(\overline{\Omega} \setminus \{0\})\big]^n \quad \text{and} \quad \mathcal{N}_\kappa \vec{F} \in L^\infty(\partial\Omega, \sigma) \subset L^1(\partial\Omega, \sigma). \tag{2.4.5}$$

Also, the nontangential trace $\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists everywhere on S^{n-1}, hence σ-a.e. on $\partial\Omega$. Moreover, $\operatorname{div}\vec{F} = 0$ in Ω, which implies

$$
{}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = 0. \tag{2.4.6}
$$

Finally, with ν denoting the geometric measure theoretic outward unit normal to Ω, we have

$$
\int_{\partial_*\Omega} \nu \cdot \left(\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\right) \mathrm{d}\sigma = \int_{S^{n-1}} x \cdot \frac{x}{|x|^n}\, \mathrm{d}\mathcal{H}^{n-1}(x) = \omega_{n-1} \neq 0. \tag{2.4.7}
$$

Thus, from (2.4.6)–(2.4.7) we conclude that

the Divergence Formula (1.5.11) *fails for the domain* Ω *and vector field* $\vec{F}$ *as in* (2.4.1). (2.4.8)

Contrasting the current setting with that of Theorem 1.5.1 reveals that only the condition that $\partial\Omega$ is lower Ahlfors regular fails to be satisfied (due to the fact that there is not sufficient mass on $\partial\Omega$ near the point $0 \in \partial\Omega$). In summary, the counterexample (2.4.8) shows that

as far as the validity of the Divergence Formula (1.5.11) *is concerned, the hypothesis that* $\partial\Omega$ *is lower Ahlfors regular cannot be replaced by asking that* $\partial\Omega$ *is upper Ahlfors regular.* (2.4.9)

The issue just mentioned may be eliminated by considering

$$
\Omega := B(0,1) \ \text{ and } \ \vec{F}(x) := \frac{x}{|x|^n} \ \text{ for } \mathcal{L}^n\text{-a.e. } \ x \in \Omega \tag{2.4.10}
$$

in place of (2.4.1). Indeed, while for each $\kappa > 0$ we now have

$$
(\mathcal{N}_\kappa \vec{F})(x) = +\infty \ \text{ at every point } \ x \in \partial\Omega, \tag{2.4.11}
$$

such a choice ensures that all hypotheses of Theorem 1.4.1 (and also Theorem 1.5.1) are satisfied and, since we now have $\mathrm{div}\vec{F} = \omega_{n-1}\delta$ in $\mathcal{D}'(\Omega)$, the Divergence Formula (1.4.6) presently holds.

Counterexamples/Examples Part 2.4B: The example/counterexample in Part 2.4A may be generalized as follows. Work in $\mathbb{R}^n$ with $n \geq 2$ and let $\Sigma \subseteq B(0,1)$ be a compact set with $\mathcal{H}^{n-1}(\Sigma) = 0$. Fix a measure $\mu \in \mathrm{CBM}\big(B(0,1)\big)$ supported on Σ and satisfying $\mu(\Sigma) \neq 0$. Lastly, define

$$
\vec{F}(x) := \frac{1}{\omega_{n-1}} \int_\Sigma \frac{x-y}{|x-y|^n}\, \mathrm{d}\mu(y), \qquad \forall x \in \Omega := B(0,1) \setminus \Sigma. \tag{2.4.12}
$$

Then Ω is a bounded open subset of $\mathbb{R}^n$, satisfying $\partial\Omega = S^{n-1} \cup \Sigma$ and $\partial_*\Omega = S^{n-1}$. In particular, $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = \mathcal{H}^{n-1}(\Sigma) = 0$. Moreover, Ω is upper

Ahlfors regular and has finite perimeter. As in the past, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Going further, observe that for each fixed aperture parameter $\kappa > 0$ we have

$$\begin{gathered}\vec{F} \in \big[\mathscr{C}^\infty(\Omega)\big]^n,\ \operatorname{div}\vec{F} = 0 \ \text{ in } \ \Omega,\ \mathcal{N}_\kappa \vec{F} \text{ is bounded on } \ S^{n-1}, \\ \text{and } \vec{F} \text{ may be extended continuously across } S^{n-1}.\end{gathered} \tag{2.4.13}$$

As such,

$$\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma), \tag{2.4.14}$$

and the nontangential boundary trace $\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at every point on $S^{n-1} = \partial_*\Omega$, ergo σ-a.e. on $\partial\Omega$. In fact,

$$\Big(\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = \frac{1}{\omega_{n-1}}\int_\Sigma \frac{x-y}{|x-y|^n}\,\mathrm{d}\mu(y), \qquad \forall x \in S^{n-1} = \partial_*\Omega. \tag{2.4.15}$$

Arguing as in (2.1.38)–(2.1.41) (and bearing in mind the assumptions on μ), we then obtain

$$\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)\,\mathrm{d}\sigma = \mu(\Sigma) \neq 0. \tag{2.4.16}$$

Since $\vec{F}$ is divergence-free in Ω, this ultimately permits us to conclude that

the Divergence Formula (1.5.11) *fails for the domain* Ω *and vector field* $\vec{F}$ *as in* (2.4.12). (2.4.17)

Comparing the present context with that of Theorem 1.5.1 shows that only the condition that $\partial\Omega$ is lower Ahlfors regular fails to be presently satisfied (as there is not enough mass on $\partial\Omega$ near points Σ). To summarize, the counterexample (2.4.17) shows that

the Divergence Formula (1.5.11) *is not expected to hold if the hypothesis that* $\partial\Omega$ *is lower Ahlfors regular is replaced by the demand that* $\partial\Omega$ *is upper Ahlfors regular.* (2.4.18)

The above issue is rendered moot by considering in place of (2.4.12)

$$\vec{F}(x) := \frac{1}{\omega_{n-1}}\int_\Sigma \frac{x-y}{|x-y|^n}\,\mathrm{d}\mu(y) \ \text{ for } \mathcal{L}^n\text{-a.e. } \ x \in \Omega := B(0,1). \tag{2.4.19}$$

Specifically, $\partial_*\Omega = \partial\Omega = S^{n-1}$ so as before we have

$$\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \Big)\, d\sigma = \mu(\Sigma), \tag{2.4.20}$$

while part *(ii)* in Lemma 3.5.6 gives that

$$\vec{F} \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^n \ \text{ and } \ \mathrm{div}\vec{F} = \mu \ \text{ in } \ \mathcal{D}'(\Omega). \tag{2.4.21}$$

Then all hypotheses of Theorem 1.5.1 are satisfied, and the Divergence Formula (1.5.11) now holds, as visible from (2.4.20)–(2.4.21).

2.5 Failure of Hypotheses on the Nature of the Divergence of the Vector Field

In the standard formulation of the Divergence Formula, there are several competing points of view pertaining to the manner in which the actual divergence of the given vector field $\vec{F}$ should be interpreted. For example, while in Theorems 1.2.1, 1.3.1, 1.4.1, and 1.5.1 we have consistently considered $\mathrm{div}\vec{F}$ in the sense of distributions, one may wonder whether the Divergence Formula holds assuming that $\vec{F}$ has components which are differentiable in a classical sense almost everywhere and $\mathrm{div}\vec{F}$, considered in this pointwise sense, is an absolutely integrable function. The counterexamples in this section dispel this notion.

Counterexamples/Examples Part 2.5A: Let $S_C \subseteq (0,1)$ be Cantor's ternary set (obtained by recurrently removing middle thirds), and denote by $f_C : [0,1] \to \mathbb{R}$ the associated Cantor's ternary function (aka the Devil's staircase); see, e.g., [161, Example 1.43 and Exercise 1.44, pp. 30–32], [249], as well as [76] and the references therein. Then, with

$$\alpha_C := \frac{\ln 2}{\ln 3} \in (0,1), \tag{2.5.1}$$

it is well known that the function f_C enjoys the following properties:

> $f_C \in \mathscr{C}^{\alpha_C}([0,1])$, f_C is non-decreasing on $[0,1]$, $f_C(0) = 0$, $f_C(1) = 1$, f_C is *not* absolutely continuous on $[0,1]$, the graph of f_C is a rectifiable curve of arc-length 2, f_C is differentiable $\mathcal{L}^1$-a.e. in $(0,1)$ and its pointwise derivative satisfies $(f_C)'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in (0,1)$ (i.e., f_C is a singular function, in the sense of [161, Definition 3.71, p. 107]). (2.5.2)

Moreover,

> $\mu_C := \mathcal{H}^{\alpha_C} \lfloor S_C$ is a Borel-regular probability measure on $(0,1)$ and $f_C(x) = \mu_C\big((0,x)\big)$ for every $x \in [0,1]$; hence, with the derivative of f_C taken in the sense of distributions, we have $(f_C)' = \mu_C$ in $\mathcal{D}'\big((0,1)\big)$. (2.5.3)

Recall that for a continuously differentiable function in an open subset of $\mathbb{R}^n$, its pointwise derivative and its distributional derivative coincide $\mathcal{L}^n$-a.e. The fact that f_C is differentiable $\mathcal{L}^1$-a.e. in $(0, 1)$ with $(f_C)'(x) = 0$ for $\mathcal{L}^1$-a.e. $x \in (0, 1)$ and yet $(f_C)' = \mu_C$ in $\mathcal{D}'\big((0, 1)\big)$ points to the fact that this correspondence breaks down if the function in question is merely a.e. differentiable even if the derivative is absolutely integrable.

Next, having fixed $n \in \mathbb{N}$, consider the set

$$\Omega := (0, 1)^n = (0, 1) \times \cdots \times (0, 1) \subseteq \mathbb{R}^n \tag{2.5.4}$$

and define the vector field

$$\vec{F} : \Omega \to \mathbb{R}^n, \;\; \vec{F}(x) := \big(f_C(x_1), 0, \dots, 0\big) \text{ for each } x = (x_1, \dots, x_n) \in \Omega. \tag{2.5.5}$$

Note that Ω is a bounded Lipschitz domain. In particular, Ω is a bounded open set with an Ahlfors regular boundary, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure, and $\partial_*\Omega = \partial\Omega$. Also,

$$\partial^*\Omega = \bigcup_{i\in\{0,1\}} \bigcup_{j=1}^{n} S_{ij} \tag{2.5.6}$$

where the S_{ij}'s are mutually disjoint sets given for each $i \in \{0, 1\}$ and $j \in \{1, \dots, n\}$ by

$$S_{ij} := (0, 1) \times \cdots \times (0, 1) \times \{i\} \times (0, 1) \times \cdots \times (0, 1) \tag{2.5.7}$$

with n factors in the Cartesian product, the singleton occupying the j-th slot. Also, the geometric measure theoretic outward unit normal ν to Ω is identified as

$$\nu = (-1)^{i+1}\mathbf{e}_j \;\; \text{on each} \;\; S_{ij}. \tag{2.5.8}$$

As regards the vector field $\vec{F}$ defined in (2.5.5), from (2.5.2) and (2.5.5) it follows that for each $\kappa > 0$ we have

$$\begin{gathered}\vec{F} \in \big[\mathscr{C}^{\alpha_C}(\overline{\Omega})\big]^n, \quad \mathcal{N}_\kappa\vec{F} \in L^\infty(\partial\Omega, \sigma) \subset L^1(\partial\Omega, \sigma), \\ \text{and } \; \vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists everywhere on } \; \partial\Omega.\end{gathered} \tag{2.5.9}$$

Finally,

the components of $\vec{F}$ are differentiable $\mathcal{L}^n$-a.e. in Ω and the divergence of $\vec{F}$ computed in a pointwise sense satisfies $(\operatorname{div}\vec{F})(x) = 0$ for $\mathcal{L}^n$-a.e. $x \in \Omega$. (2.5.10)

Consequently, on the one hand we have $\int_\Omega (\mathrm{div}\vec{F})(x)\,\mathrm{d}x = 0$ where the divergence of $\vec{F}$ is considered in a pointwise sense. On the other hand, (5.6.21), (2.5.2), and (2.5.4)–(2.5.9) imply

$$\begin{aligned}\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)\,\mathrm{d}\sigma &= \int_{\partial^*\Omega} \nu \cdot \Big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big)\,\mathrm{d}\sigma \\ &= \sum_{i\in\{0,1\}} (-1)^{i+1} \int_{\{i\}\times(0,1)^{n-1}} f_C(x_1)\,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &= f_C(1) - f_C(0) = 1 - 0 = 1. \qquad (2.5.11)\end{aligned}$$

This analysis shows that

given any $n \in \mathbb{N}$, *the Divergence Formula* (1.2.2) *fails for the domain* Ω *as in* (2.5.4) *and the vector field* $\vec{F}$ *as in* (2.5.5) *if the divergence of* $\vec{F}$ *is considered in a pointwise sense.* (2.5.12)

Incidentally, the case $n = 1$ corresponds to saying that the (one-variable) Fundamental Theorem of Calculus $\int_0^1 f'(x)\,\mathrm{d}x = f(1) - f(0)$ in which the derivative is taken in a pointwise sense at $\mathcal{L}^1$-a.e. $x \in (0, 1)$ fails for Cantor's ternary function f_C.

Ultimately, the counterexample (2.5.12) proves that

the Divergence Formula (1.2.2) *may fail even when* $\Omega \subset \mathbb{R}^n$ *is a bounded Lipschitz domain if* $\vec{F} \in \big[\mathscr{C}^0(\overline{\Omega})\big]^n$ *is a vector field whose components are only assumed to be differentiable at* $\mathcal{L}^n$*-a.e. point in* Ω, *and whose divergence, computed in a pointwise sense, is assumed to belong to* $L^1(\Omega, \mathcal{L}^n)$. (2.5.13)

By way of contrast, Theorem 1.2.1 requires that $\mathrm{div}\vec{F}$, taken in the sense of distributions, belongs to $L^1(\Omega, \mathcal{L}^n)$. Hence, the specific manner in which the divergence is considered (pointwise sense versus distributional sense) drastically affects the veracity of the Divergence Formula (1.2.2).

According to Theorem 1.3.1, the remedy to (2.5.12) is to consider the divergence of $\vec{F}$ in the sense of distributions in Ω, provided the latter turns out to be a complex Borel measure in Ω. To this end, consider an arbitrary test function $\varphi \in \mathscr{C}^\infty_c(\Omega)$ and use (2.5.3) together with (1.3.12) to write

$$\begin{aligned}\int_\Omega \vec{F}(x)\cdot\nabla\varphi(x)\,\mathrm{d}x &= \int_\Omega f_C(x_1)(\partial_1\varphi)(x)\,\mathrm{d}x \\ &= \int_{(0,1)^{n-1}} \Big(\int_0^1 f_C(x_1)(\partial_1\varphi)(x_1, x_2, \dots, x_n)\,\mathrm{d}x_1\Big)\mathrm{d}x_2\cdots\mathrm{d}x_n \\ &= -\int_{(0,1)^{n-1}} \Big(\int_0^1 \varphi(x_1, x_2, \dots, x_n)\,\mathrm{d}\mu_C(x_1)\Big)\mathrm{d}x_2\cdots\mathrm{d}x_n\end{aligned}$$

$$= - \int_{\Omega} \varphi \, \mathrm{d}(\mu_C \otimes \mathcal{L}^{n-1}). \tag{2.5.14}$$

In view of the arbitrariness of φ, this yields

$$\operatorname{div}\vec{F} = \mu_C \otimes \big(\mathcal{L}^{n-1} \lfloor (0,1)^{n-1}\big) \ \text{ in } \ \mathcal{D}'(\Omega). \tag{2.5.15}$$

Thus, the distributional divergence of $\vec{F}$ satisfies $\operatorname{div}\vec{F} \in \mathrm{CBM}(\Omega)$ and

$$\begin{aligned} (\operatorname{div}\vec{F})(\Omega) &= \Big(\mu_C \otimes \big(\mathcal{L}^{n-1} \lfloor (0,1)^{n-1}\big)\Big)\big((0,1) \times (0,1)^{n-1}\big) \\ &= \mu_C\big((0,1)\big) \cdot \mathcal{L}^{n-1}\big((0,1)^{n-1}\big) = 1 \cdot 1 = 1. \end{aligned} \tag{2.5.16}$$

The latter value agrees with (2.5.11), as indeed predicted by Theorem 1.3.1.

2.6 Relationship with Classical Results in the One-Dimensional Setting

Recall that the classical formulation of the one-variable Fundamental Theorem of Calculus, involving the notion of Riemann integral, reads as follows.

Theorem 2.6.1 *Suppose $-\infty < a < b < +\infty$ and let $F : [a,b] \to \mathbb{C}$ be a continuous function which is differentiable at each point in (a,b) and such that F' is Riemann integrable on $[a,b]$. Then the Riemann integral of F' on $[a,b]$ equals $F(b) - F(a)$.*

Since any Riemann integrable function is bounded, it follows that any function F as in Theorem 2.6.1 is Lipschitz on $[a,b]$ hence, in particular, absolutely continuous on $[a,b]$. Within the latter class of functions (which, generally speaking, are only differentiable $\mathcal{L}^1$-a.e. in (a,b)) there is a version of the Fundamental Theorem of Calculus which employs the notion of Lebesgue integral (cf., e.g., [91, Theorem 3.35, p. 106], or [161, Theorem 3.30, p. 85], among a plethora of texts on real analysis).

Theorem 2.6.2 *Suppose $-\infty < a < b < +\infty$ and let $F : [a,b] \to \mathbb{C}$ be an absolutely continuous function. Then F is differentiable $\mathcal{L}^1$-a.e. in (a,b), the pointwise derivative F' belongs to $L^1\big((a,b), \mathcal{L}^1\big)$ and agrees with the distributional derivative of F in (a,b), and*

$$\int_a^b F'(x) \, \mathrm{d}x = F(b) - F(a). \tag{2.6.1}$$

Before going any further we wish to note that, given a bounded interval $[a,b]$, there exist absolutely continuous functions on $[a,b]$ satisfying all hypotheses of Theorem 2.6.1 except the Riemann integrability of the derivative. A striking example was given in 1906 by D. Pompeiu[4] (cf. [220]). Pompeiu's construction (discussed in detail in, e.g., [253, pp. 402–421]) yields a

$$\begin{array}{l}\text{strictly increasing, Lipschitz function } f_P : [a,b] \to \mathbb{R}, \text{ which} \\ \text{is differentiable everywhere on } (a,b), \text{ and whose derivative} \\ \text{is bounded and vanishes on a dense subset of } (a,b).\end{array} \tag{2.6.2}$$

Note that being Lipschitz makes f_P absolutely continuous on $[a, b]$, hence the version of the Fundamental Theorem of Calculus from Theorem 2.6.2 applies to Pompeiu's function f_P, though this is not the case for Theorem 2.6.1. Indeed, we claim that

$$(f_P)' \text{ is not Riemann integrable on any subinterval of } [a,b]. \tag{2.6.3}$$

To justify this, observe that if $(f_P)'$ were to be Riemann integrable on some interval $[c, d] \subseteq [a, b]$ then, using Riemann sums with intermediate points selected from the set where $(f_P)' = 0$, we conclude that the Riemann integral of $(f_P)'$ on $[c, d]$ vanishes, while on the other hand Theorem 2.6.1 gives that the Riemann integral of $(f_P)'$ on $[c, d]$ equals $f_P(d) - f_P(c) > 0$, bearing in mind the fact that f_P is strictly increasing. This contradiction proves (2.6.3).

In summary, using the Lebesgue notion of integrability in place of Riemann integrability leads to a more inclusive brand of the Fundamental Theorem of Calculus, as recorded in Theorem 2.6.2.

We wish to compare the latter result with the one-dimensional version of Theorem 1.2.1 stated for a finite interval of the real line. To this end, assume $n = 1$, fix $a, b \in \mathbb{R}$ with $a < b$, and consider $\Omega := (a, b)$. Then Ω is a set of locally finite perimeter in $\mathbb{R}$, with $\partial\Omega = \{a, b\}$, $\partial_*\Omega = \partial^*\Omega = \partial\Omega$, and since $\mathcal{H}^0$ is the counting measure, we also have $\mathcal{H}^0\big(B(x,r) \cap \partial\Omega\big) \approx 1 = r^0$ for every $x \in \partial\Omega$ and every $r \in (0, \infty)$. Hence, $\partial\Omega$ is an Ahlfors regular set in $\mathbb{R}^1$. Also, if $\kappa \in (0, \infty)$ is fixed, then the nontangential approach regions $\Gamma_\kappa(a)$, $\Gamma_\kappa(b)$ are as in (8.1.28).

Turning to the conditions on $F : (a, b) \to \mathbb{C}$ stated in Theorem 1.2.1, requiring that $\mathcal{N}_\kappa F \in L^1(\partial\Omega, \mathcal{H}^0)$ is equivalent with $(\mathcal{N}_\kappa F)(a) < +\infty$ and $(\mathcal{N}_\kappa F)(b) < +\infty$ which, in light of (8.1.29), is further equivalent to having $F \in L^\infty\big((a, b), \mathcal{L}^1\big)$. Thus,

$$\mathcal{N}_\kappa F \in L^1(\{a,b\}, \mathcal{H}^0) \iff F \in L^\infty\big((a,b), \mathcal{L}^1\big). \tag{2.6.4}$$

Consider next the requirement that F', taken in the distributional sense in (a, b), belongs to the space $L^1\big((a, b), \mathcal{L}^1\big)$. If we define $G(x) := \int_a^x F' \,\mathrm{d}\mathcal{L}^1$ for each x in $[a, b]$, it follows that G is absolutely continuous on $[a, b]$ and $G' = F'$ in $\mathcal{D}'\big((a, b)\big)$. Consequently, there exists some constant $c \in \mathbb{C}$ with the property that $F = G + c$ at $\mathcal{L}^1$-a.e. point in (a, b) (cf., e.g., [181, Proposition 2.47(2), p. 35]).

[4] D. Pompeiu was a student of H. Poincaré who, ironically, was unjustifiably biased against unconventional mathematical objects such as Pompeiu's function f_P, as apparent from his well-known quote: "*In the old days when people invented a new function they had something useful in mind. Now, they invent them deliberately just to invalidate our ancestors' reasoning, and that is all they are ever going to get out of them.*"

Hence, the fact that F has an integrable distributional derivative on (a, b) is equivalent to F agreeing at $\mathcal{L}^1$-a.e. point in (a, b) with an absolutely continuous function $\widetilde{F}$ defined on $[a, b]$. Note that whenever such a coincidence takes place, the nontangential trace of F exists on $\partial\Omega = \{a, b\}$ since

$$\big(F\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(a) = \widetilde{F}(a) \quad \text{and} \quad \big(F\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(b) = \widetilde{F}(b). \tag{2.6.5}$$

In summary,

$F : (a, b) \to \mathbb{C}$ satisfies the hypotheses of Theorem 1.2.1 (stated for $n = 1$ and $\Omega := (a, b)$ where $a, b \in \mathbb{R}$ with $a < b$) if and only if there exists an absolutely continuous function $\widetilde{F} : [a, b] \to \mathbb{C}$ which agrees with F at $\mathcal{L}^1$-a.e. point in (a, b) (in which case we have $F' = \widetilde{F}'$ in $\mathcal{D}'\big((a, b)\big)$ and the formulas in (2.6.5) hold). (2.6.6)

On the other hand, it is well known (cf. [161, Exercise 3.7 on p. 75 and Corollary 7.14 on p. 223]) that, for any numbers $a, b \in \mathbb{R}$ with $a < b$, we have (with $W^{1,1}\big((a, b)\big)$ denoting the standard L^1-based Sobolev space of order one in the interval (a, b))

if $F : (a, b) \to \mathbb{C}$ is a given function, then $F \in W^{1,1}\big((a, b)\big)$ if and only if there exists an absolutely continuous function $\widetilde{F} : [a, b] \to \mathbb{C}$ which agrees with the given F at $\mathcal{L}^1$-a.e. point in the interval (a, b). (2.6.7)

Given that the Divergence Formula (1.2.2) in Theorem 1.2.1 stated for $n = 1$ and $\Omega := (a, b)$ becomes

$$\int_a^b F' \, \mathrm{d}\mathcal{L}^1 = \big(F\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(b) - \big(F\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(a), \tag{2.6.8}$$

from (2.6.6)–(2.6.8) we may then conclude that

the one-dimensional version of our Theorem 1.2.1 stated for a bounded interval (a, b) of the real line is an extension of the classical Fundamental Theorem of Calculus stated in Theorem 2.6.2 from absolutely continuous functions on $[a, b]$ to functions in the Sobolev space $W^{1,1}\big((a, b)\big)$ (alternatively, an extension to the class of functions on (a, b) which become absolutely continuous on $[a, b]$ after eventually being redefined on an $\mathcal{L}^1$-nullset). (2.6.9)

After some further streamlining, the one-dimensional version of Theorem 1.2.1 for a bounded interval on the real line may ultimately be stated as follows.

Theorem 2.6.3 *Suppose $-\infty < a < b < +\infty$ and let $F \in L^1_{\mathrm{loc}}\big((a, b), \mathcal{L}^1\big)$ be a function with the property that its distributional derivative $F' \in \mathcal{D}'\big((a, b)\big)$ actually belongs to the space $L^1\big((a, b), \mathcal{L}^1\big)$ (in particular, this is the case if $F \in W^{1,1}\big((a, b)\big)$ to begin with). Then there exists an $\mathcal{L}^1$-nullset $N \subseteq (a, b)$ such that the limits*

$$F(a) := \lim_{(a,b)\setminus N \ni x \to a} F(x) \quad \textit{and} \quad F(b) := \lim_{(a,b)\setminus N \ni x \to b} F(x) \quad \textit{exist in } \mathbb{C} \tag{2.6.10}$$

and, with the values of F at end-points interpreted as such,

$$\int_a^b F' \,\mathrm{d}\mathcal{L}^1 = F(b) - F(a). \tag{2.6.11}$$

Let us remark that, in contrast to the classical result recorded in Theorem 2.6.2, the version of the Fundamental Theorem of Calculus presented above is stable to altering the function F on an $\mathcal{L}^1$-nullset which, for a multitude of purposes, is a desirable feature.

Proof of Theorem 2.6.3 In view of (2.6.9), it suffices to show that if F is a distribution on (a, b) such that $F' \in L^1\big((a, b), \mathcal{L}^1\big)$, then $F \in W^{1,1}\big((a, b)\big)$. To this end, consider the function $G : (a, b) \to \mathbb{C}$ given by $G(x) := \int_a^x F' \,\mathrm{d}\mathcal{L}^1$ for each x in (a, b). Then G is locally integrable and bounded in (a, b), and also satisfies $G' = F'$ in $\mathcal{D}'\big((a, b)\big)$. Hence, the distribution $F - G$ satisfies $(F - G)' = 0$ in $\mathcal{D}'\big((a, b)\big)$ which forces it to be a constant (cf., e.g., [181, Proposition 2.47(2), p. 35]). This proves that $F = c + G$ for some $c \in \mathbb{C}$, thus $F \in L^\infty\big((a, b), \mathcal{L}^1\big) \subset L^1\big((a, b), \mathcal{L}^1\big)$. The desired conclusion follows.

We can rephrase Theorem 2.6.2 in a manner which points more transparently to the fact that the class of absolutely continuous functions is the largest environment in which the Fundamental Theorem of Calculus may be formulated in the context of Lebesgue integration. Specifically, for a complex-valued function F defined on a compact interval $[a, b]$ the following two conditions are equivalent:

(1) F is absolutely continuous on $[a, b]$;
(2) F is differentiable $\mathcal{L}^1$-a.e. on (a, b), and the (pointwise) derivative F' belongs to $L^1\big((a, b), \mathcal{L}^1\big)$, and $\int_a^x F' \,\mathrm{d}\mathcal{L}^1 = F(x) - F(a)$ for each $x \in [a, b]$.

This being said, a suitable version of the Fundamental Theorem of Calculus does hold outside the scope of Lebesgue integration. This involves the class of functions $F : (a, b) \to \mathbb{C}$ whose `pointwise variation`

$$\operatorname{Var} F\Big|_a^b := \sup\Big\{ \sum_{j=1}^N |F(x_j) - F(x_{j-1})| : N \in \mathbb{N} \text{ and } a < x_0 < \cdots < x_N < b \Big\} \in [0, +\infty] \tag{2.6.12}$$

is finite. In such a scenario, the distributional derivative of F turns out to be a complex Borel measure μ, called the `Lebesgue–Stieltjes measure generated by F on` (a, b) and, on the left-hand side of (2.6.1), the total mass of μ, i.e., $\mu\big((a, b)\big)$, now plays the role of the Lebesgue integral of the pointwise derivative of F. As is apparent from Theorem 2.6.4 below (which is essentially a classical result; cf., e.g., [161, Theorem 5.13, p. 162], or [91, Theorem 3.29, p. 105]), this only affects the manner in which the Fundamental Theorem of Calculus is written. Moreover, since absolutely continuous functions are precisely those functions of finite pointwise variation whose distributional derivatives happen to be absolutely continuous (as measures) with respect to $\mathcal{L}^1$ (cf. (2.6.23)), Theorem 2.6.4 constitutes a natural generalization of Theorem 2.6.2.

Theorem 2.6.4 *Suppose $-\infty < a < b < +\infty$ and let $F : (a, b) \to \mathbb{C}$ have finite pointwise variation. Then the one-sided limits*

$$F(a^+) := \lim_{x \searrow a} F(x) \quad \text{and} \quad F(b^-) := \lim_{x \nearrow b} F(x) \quad \text{exist in } \mathbb{C}, \tag{2.6.13}$$

the function F is $\mathcal{L}^1$-measurable and bounded on (a, b) (in particular, F is locally integrable on (a, b)), and there exists a unique measure $\mu \in \mathrm{CBM}\big((a, b)\big)$ with the property that the distributional derivative of F satisfies $F' = \mu$ in $\mathcal{D}'\big((a, b)\big)$, and

$$\mu\big((a, b)\big) = F(b^-) - F(a^+). \tag{2.6.14}$$

Proof We find it convenient to extend F by zero to the entire real line by setting

$$\widetilde{F}(x) := \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus (a, b), \\ F(x) & \text{if } a < x < b, \end{cases} \qquad \forall x \in \mathbb{R}. \tag{2.6.15}$$

Since for every $c \in (a, b)$ we obviously have

$$\sup_{x \in (a,b)} |F(x)| \leq |F(c)| + \mathrm{Var}\, F\Big|_a^b \tag{2.6.16}$$

it follows that F is bounded. In turn, this readily implies that the function $\widetilde{F} : \mathbb{R} \to \mathbb{C}$ has finite pointwise variation, i.e.,

$$\mathrm{Var}\, \widetilde{F}\Big|_{-\infty}^{+\infty} < +\infty. \tag{2.6.17}$$

Granted this, the Jordan decomposition theorem (cf., e.g., [161, Theorem 2.18, p. 46]) implies that both the real and the imaginary part of $\widetilde{F}$ may be written as a difference of two real-valued bounded non-decreasing functions defined on $\mathbb{R}$. This has several notable consequences. First, $\widetilde{F}$ (hence also F) is $\mathcal{L}^1$-measurable. Second, as noted in [161, Corollary 2.23, p. 47], for each $x \in [-\infty, +\infty)$ and $y \in (-\infty, +\infty]$ the one-sided limits,

$$\widetilde{F}(x^+) := \lim_{z \searrow x} \widetilde{F}(z), \qquad \widetilde{F}(y^-) := \lim_{z \nearrow y} \widetilde{F}(z), \tag{2.6.18}$$

exist in $\mathbb{C}$. In particular, this takes care of the claim made about the one-sided limits in (2.6.13). Third, [161, Theorem 5.3, p. 157] and [161, Remark 5.6, p. 159] ensure the existence of a complex Borel measure $\widetilde{\mu}$ on $\mathbb{R}$ with the property that

$$\widetilde{\mu}\big((x, y)\big) = \widetilde{F}(y^-) - \widetilde{F}(x^+) \quad \text{whenever} \quad -\infty < x < y < +\infty, \tag{2.6.19}$$

and

$$\widetilde{\mu}\big(\{x\}\big) = \widetilde{F}(x^+) - \widetilde{F}(x^-) \quad \text{for each} \quad x \in \mathbb{R}. \tag{2.6.20}$$

Furthermore, according to [161, Corollary 5.41, p. 183] this measure turns out to be the distributional derivative of $\widetilde{F}$ on $\mathbb{R}$. In particular, if $\mu := \widetilde{\mu}\lfloor(a, b)$ then

$$\mu \in \mathrm{CBM}\big((a, b)\big) \quad \text{and} \quad F' = \mu \quad \text{in} \quad \mathcal{D}'\big((a, b)\big). \tag{2.6.21}$$

Lastly, there remains to notice that

$$\mu\big((a, b)\big) = \widetilde{\mu}\big((a, b)\big) = \widetilde{F}(b^-) - \widetilde{F}(a^+) = F(b^-) - F(a^+), \tag{2.6.22}$$

since, as seen from (2.6.15), we have $\widetilde{F}(b^-) = F(b^-)$ and $\widetilde{F}(a^+) = F(a^+)$. □

A couple of comments pertaining to the nature of Theorem 2.6.4 are in order. First, if F is as in the statement of this theorem, [161, Corollary 2.23, pp. 47–48] implies that F is differentiable at $\mathcal{L}^1$-a.e. point in (a, b) and the pointwise derivative of F belongs to $L^1\big((a, b), \mathcal{L}^1\big)$. However, even though this pointwise derivative is absolutely integrable with respect to the one-dimensional Lebesgue measure, the (one-variable) Divergence Formula $\int_a^b F'(x)\,\mathrm{d}x = F(b^-) - F(a^+)$ may fail even when F has a continuous extension to $[a, b]$, with Cantor's ternary function f_C serving as a counterexample. In this vein, observe that the Lebesgue–Stieltjes measure generated by f_C on $(0, 1)$ is $\mu_C = \mathcal{H}^{\alpha_C}\lfloor S_C$, which is singular with respect to the one-dimensional Lebesgue measure since $\mathcal{L}^1(S_C) = 0$. In general, we have the following result (compare with [161, Theorem 5.19, p. 166]), establishing a bridge between Theorems 2.6.2 and 2.6.4:

> if (a, b) is a bounded interval of the real line and $F : (a, b) \to \mathbb{C}$ has finite pointwise variation, then F extends to an absolutely continuous function on $[a, b]$ (a scenario in which formula (2.6.1) does hold; cf. Theorem 2.6.2) if and only if the Lebesgue–Stieltjes measure generated by F on (a, b) is absolutely continuous with respect to $\mathcal{L}^1$. (2.6.23)

To justify (2.6.23), suppose $-\infty < a < b < +\infty$, the function $F : (a, b) \to \mathbb{C}$ has finite pointwise variation, and its Lebesgue–Stieltjes measure $\mu_F \in \mathrm{CBM}\big((a, b)\big)$ satisfies $\mu_F \prec \mathcal{L}^1$ on (a, b). In concert with (2.6.20) the latter property eventually forces F to be continuous on (a, b). This also implies that the Radon–Nikodym derivative $f := \mathrm{d}\mu_F/\mathrm{d}\mathcal{L}^1 \in L^1\big((a, b), \mathcal{L}^1\big)$. Bearing these in mind and having fixed some $x_0 \in (a,b)$, formula (2.6.14) allows us to write $F(x) = F(x_0) + \int_{x_0}^{x} f \, d\mathcal{L}^1$ for every $x \in (a,b)$ which finally shows that F canonically extends to an absolutely continuous function on $[a,b]$. The opposite implication follows easily from Theorem 2.6.2, finishing the proof of (2.6.23).

Our second comment elaborates on the relationship between functions of finite pointwise variation in the sense of (2.6.12) and the one-dimensional version of the class BV, the space of functions of bounded variation in a distributional sense (see (5.5.5) for a definition). In one direction, given a bounded interval (a,b) of $\mathbb{R}$ and a function $F : (a,b) \to \mathbb{C}$, we have (as seen from [161, Theorem 7.2, p. 216] and also taking into account the one-dimensional version of item *(v)* in Proposition 5.5.1)

$$\operatorname{Var} F\Big|_a^b < +\infty \implies F \in \mathrm{BV}\big((a,b)\big) \text{ and } \mathbf{V}\big(F;(a,b)\big) \leq \operatorname{Var} F\Big|_a^b. \tag{2.6.24}$$

In the converse direction we have that, given any interval (a,b) of $\mathbb{R}$,

$$F \in \mathrm{BV}\big((a,b)\big) \implies \begin{cases} \text{there exists some } \widetilde{F} : (a,b) \to \mathbb{C} \text{ such that} \\ \widetilde{F} \text{ is right-continuous, agrees with } F \text{ at } \mathcal{L}^1\text{-a.e.} \\ \text{point in } (a,b), \text{ and } \operatorname{Var}\widetilde{F}\Big|_a^b = \mathbf{V}\big(F;(a,b)\big). \end{cases} \tag{2.6.25}$$

As such, given a bounded interval (a,b) of the real line, the space $\mathrm{BV}\big((a,b)\big)$ of functions of bounded variation in the open set (a,b) may be viewed as the space of $\mathcal{L}^1$-measurable functions having a representative of finite pointwise variation on (a,b) (i.e., which can be redefined on an $\mathcal{L}^1$-nullset in order to have finite pointwise variation in the sense of (2.6.12)). Also, from (2.6.25) and the fact that functions of pointwise finite variation are bounded we conclude that

$$\mathrm{BV}\big((a,b)\big) \subset L^\infty\big((a,b),\mathcal{L}^1\big) \text{ whenever } -\infty \leq a < b \leq +\infty. \tag{2.6.26}$$

Lastly, we note that up to $\mathcal{L}^1$-a.e. identification of functions and with the derivative understood in the sense of distributions,

$$\mathrm{BV}\big((a,b)\big) \equiv \big\{F \in L^1\big((a,b),\mathcal{L}^1\big) : F' \in \mathrm{CBM}\big((a,b)\big)\big\} \text{ if } -\infty < a < b < +\infty. \tag{2.6.27}$$

Indeed, we have a genuine left-to-right inclusion, as a consequence of (2.6.25) and Theorem 2.6.4, while the right-to-left inclusion (with the caveat about $\mathcal{L}^1$-a.e. identification of functions in effect) is seen from [161, Theorem 7.8, p. 220].

Moving on, we wish to compare the one-dimensional version of Theorem 1.3.1 with Theorem 2.6.4. In this vein, observe that, collectively, (2.6.4), (2.6.25), (2.6.27), and Theorem 2.6.4 lead to the following conclusion:

a function $F : (a, b) \to \mathbb{C}$ satisfies the hypotheses of Theorem 1.3.1 (stated for $n = 1$ and $\Omega := (a, b)$ where $a, b \in \mathbb{R}$ with $a < b$) if and only if there exists a function $\widetilde{F} : (a, b) \to \mathbb{C}$ having finite pointwise variation in the sense of (2.6.12) (i.e., such that $\operatorname{Var} F\big|_a^b < +\infty$) and which agrees with the given function F at $\mathcal{L}^1$-a.e. point in (a, b); moreover, in such a scenario we have $F' = \widetilde{F}'$ in $\mathcal{D}'\big((a, b)\big)$, as well as $\big(F\big|^{\text{n.t.}}_{\partial\Omega}\big)(a) = \widetilde{F}(a^+)$ and $\big(F\big|^{\text{n.t.}}_{\partial\Omega}\big)(b) = \widetilde{F}(b^-)$. (2.6.28)

Since the Divergence Formula (1.3.8) presently reads (interpreting F' as a measure on $\Omega = (a, b)$)

$$F'\big((a, b)\big) = \big(F\big|^{\text{n.t.}}_{\partial\Omega}\big)(b) - \big(F\big|^{\text{n.t.}}_{\partial\Omega}\big)(a), \tag{2.6.29}$$

from (2.6.28)–(2.6.29) we may then conclude that

the one-dimensional version of our Theorem 1.3.1 stated for a bounded interval (a, b) of the real line is an extension of Theorem 2.6.4 which, in place of functions of finite bounded variation on (a, b), allows the consideration of functions from $\mathrm{BV}\big((a, b)\big)$ *(or, alternatively, an extension to the class of functions which acquire a finite pointwise variation on (a, b) after eventually being redefined on an $\mathcal{L}^1$-nullset).* (2.6.30)

After further fine-tuning this result, the one-dimensional version of Theorem 1.3.1 for a bounded interval on the real line finally reads as follows.

Theorem 2.6.5 *Let $-\infty < a < b < +\infty$ and consider a complex-valued function $F \in L^1_{\mathrm{loc}}\big((a, b), \mathcal{L}^1\big)$ with the property that $\boldsymbol{V}\big(F; (a, b)\big) < +\infty$ (which is the case if $F \in \mathrm{BV}\big((a, b)\big)$ to begin with). Then the distributional derivative $F' \in \mathcal{D}'\big((a, b)\big)$ actually belongs to $\mathrm{CBM}\big((a, b)\big)$, there exists some $\mathcal{L}^1$-nullset $N \subseteq (a, b)$ such that the limits*

$$F(a) := \lim_{(a,b)\setminus N \ni x \to a} F(x) \ \ \textit{and} \ \ F(b) := \lim_{(a,b)\setminus N \ni x \to b} F(x) \ \ \textit{exist in} \ \mathbb{C} \tag{2.6.31}$$

and, with the values of F at end-points considered as above and interpreting F' as a measure on (a, b), one has

$$F'\big((a, b)\big) = F(b) - F(a). \tag{2.6.32}$$

Proof In light of (2.6.30), we only need to check that if F is as in the statement of the theorem then actually F belongs to $\mathrm{BV}\big((a, b)\big)$. With this goal in mind, pick two monotonic sequences $\{a_j\}_{j\in\mathbb{N}} \subseteq \big(a, (a + b)/2\big)$ and $\{b_j\}_{j\in\mathbb{N}} \subseteq \big((a + b)/2, b\big)$ such that $a_j \to a$ and $b_j \to b$ as $j \to \infty$. Since for each $j \in \mathbb{N}$ the assumptions on F imply that $F \in \mathrm{BV}\big((a_j, b_j)\big)$, from (2.6.25) and (5.5.3) we deduce that there exists some $\widetilde{F}_j : (a_j, b_j) \to \mathbb{C}$ which agrees with F at $\mathcal{L}^1$-a.e. point in (a_j, b_j) and satisfies

$$\mathrm{Var}\widetilde{F}_j\Big|_{a_j}^{b_j} = \mathbf{V}\big(F; (a_j, b_j)\big) \leq \mathbf{V}\big(F; (a, b)\big). \tag{2.6.33}$$

Note that it is possible to pick $c \in \bigcap_{j\in\mathbb{N}} (a_j, b_j)$ with the property that $|F(c)| < +\infty$ and such that $\widetilde{F}_j(c) = F(c)$ for each $j \in \mathbb{N}$. Using this, and keeping in mind (2.6.16) as well as (5.5.3), for each $j \in \mathbb{N}$ we may then estimate

$$\begin{aligned} \|F\|_{L^\infty((a_j,b_j),\mathcal{L}^1)} &\leq \sup_{x\in(a_j,b_j)} |\widetilde{F}_j(x)| \leq |\widetilde{F}_j(c)| + \mathrm{Var}\widetilde{F}_j\Big|_{a_j}^{b_j} \\ &= |F(c)| + \mathbf{V}\big(F; (a_j, b_j)\big) \leq |F(c)| + \mathbf{V}\big(F; (a, b)\big). \end{aligned} \tag{2.6.34}$$

In turn, this readily implies

$$\|F\|_{L^\infty((a,b),\mathcal{L}^1)} \leq |F(c)| + \mathbf{V}\big(F; (a, b)\big) < +\infty, \tag{2.6.35}$$

hence $F \in L^\infty\big((a, b), \mathcal{L}^1\big) \subset L^1\big((a, b), \mathcal{L}^1\big)$. In view of (5.5.5) and the assumptions on F, this ultimately proves that $F \in \mathrm{BV}\big((a, b)\big)$, as wanted. □

In closing, we wish to note that, in contrast to Theorem 2.6.4, the version of the Fundamental Theorem of Calculus presented above is stable to altering the function F on an $\mathcal{L}^1$-nullset.

2.7 Examples and Counterexamples Pertaining to Weak Traces

The aim of this section is to shed further light on the nature of the results in Sect. 1.9, centered around the Divergence Theorem with weak boundary traces formulated in Theorem 1.9.1.

Counterexamples/Examples Part 2.7A: Having picked $n \in \mathbb{N}$ with $n \geq 2$, consider

$$\begin{aligned} &\phi : \mathbb{R}^{n-1} \to \mathbb{R} \ \text{ Lipschitz function, and} \\ &\Sigma := \big\{(x', \phi(x')) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}\big\}. \end{aligned} \tag{2.7.1}$$

Also, set (Fig. 2.14)

$$\Omega := \mathbb{R}^n \setminus \Sigma, \quad \Omega_\pm := \big\{(x' \pm t, \phi(x')) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, \ t \in (0, \infty)\big\}. \tag{2.7.2}$$

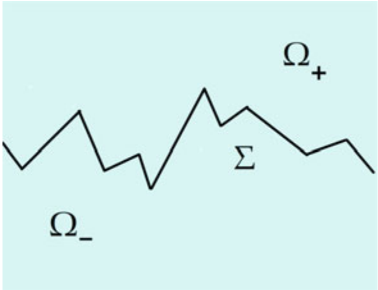

Fig. 2.14 Ω as in (2.7.2)

Then

$$\Omega_{\pm} \text{ are Ahlfors regular domains sharing a common boundary, } \partial\Omega = \partial\Omega_{+} = \partial\Omega_{-} = \Sigma, \text{ satisfying } \Omega = \Omega_{+} \cup \Omega_{-} \text{ and } \Omega_{+} \cap \Omega_{-} = \varnothing. \tag{2.7.3}$$

Moreover,

$$\Omega_{\pm} \text{ are actually (graph) Lipschitz domains, and if } N \text{ is the outward unit normal to } \Omega_{+}, \text{ then the outward unit normal to } \Omega_{-} \text{ is } -N. \tag{2.7.4}$$

Introduce $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$ and, having fixed some $p \in (1, \infty)$ along with some complex-valued function $f \in L^p(\Sigma, \sigma)$, define the vector field $\vec{F} : \Omega \to \mathbb{C}^n$ by setting

$$\vec{F}(x) := \frac{1}{\omega_{n-1}} \int_{\Sigma} \frac{x-y}{|x-y|^n} f(y)\, d\sigma(y), \qquad \forall x \in \Omega. \tag{2.7.5}$$

Note that, by design,

$$\vec{F} \in \big[\mathscr{C}^\infty(\Omega)\big]^n \ \text{ and } \ \operatorname{div}\vec{F} = 0 \ \text{ in } \ \Omega. \tag{2.7.6}$$

In addition, for each fixed $\kappa > 0$, the Calderón–Zygmund theory ensures that

$$\mathcal{N}_\kappa \vec{F} \in L^p(\partial\Omega, \sigma) \tag{2.7.7}$$

and that at $\mathcal{H}^{n-1}$-a.e. point $x \in \Sigma$ we have

$$\Big(\vec{F}\Big|^{\kappa-\text{n.t.}}_{\partial(\Omega_{\pm})}\Big)(x) = \mp\frac{1}{2}N(x)f(x) + \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\Sigma \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} f(y)\, d\sigma(y). \tag{2.7.8}$$

In particular,

$$N \cdot \Big(\vec{F}\Big|^{\kappa-\text{n.t.}}_{\partial(\Omega_{+})}\Big) - N \cdot \Big(\vec{F}\Big|^{\kappa-\text{n.t.}}_{\partial(\Omega_{-})}\Big) = -f \ \text{ at } \sigma\text{-a.e. point on } \ \Sigma. \tag{2.7.9}$$

Then, based on (2.7.3)–(2.7.4), (2.7.6)–(2.7.9), and the Divergence Theorem in Theorem 1.2.1, for each $\Psi \in \mathrm{Lip}_c(\mathbb{R}^n)$ we may write

$$\begin{aligned}\int_{\Omega} \vec{F} \cdot \nabla\Psi \, d\mathcal{L}^n &= \int_{\Omega_+} \vec{F} \cdot \nabla\Psi \, d\mathcal{L}^n + \int_{\Omega_-} \vec{F} \cdot \nabla\Psi \, d\mathcal{L}^n \\ &= \int_{\partial(\Omega_+)} N \cdot \Big(\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial(\Omega_+)}\Big)\Psi \, d\sigma - \int_{\partial(\Omega_-)} N \cdot \Big(\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial(\Omega_-)}\Big)\Psi \, d\sigma \\ &= -\int_{\partial\Omega} f \, \Psi \, d\sigma. \end{aligned} \tag{2.7.10}$$

In turn, from (2.7.10), (2.7.6), (1.9.6), and Corollary 3.7.3 we conclude that

$$\nu \bullet \vec{F} = -f \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{2.7.11}$$

As a consequence, we have

The weak normal trace $\nu \bullet \vec{F}$ of the vector field $\vec{F}$ defined as in (2.7.5) *on the boundary of the domain Ω introduced in* (2.7.2),*may have full support in the topological boundary $\partial\Omega$. At the same time, $\vec{F}\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ may fail to exist at σ-a.e. point on $\partial\Omega$* (*as seen from* (2.7.8)). (2.7.12)

This stands in sharp contrast with Theorem 1.2.1, whose formulation involves the nontangential pointwise normal trace of the vector field in an essential fashion (see the integrand on the right-hand side of the Divergence Formula (1.2.2)). Due to the presence of the geometric measure theoretic outward unit normal, the latter trace could only make sense at σ-a.e. point on the geometric measure theoretic boundary of the underlying domain.

Counterexamples/Examples Part 2.7B Fix $n \in \mathbb{N}$ with $n \geq 2$ and consider the slit unit ball in $\mathbb{R}^n$ given by (Fig. 2.15)

$$\Omega := B(0,1) \setminus \Sigma \quad \text{where} \quad \Sigma := \big\{(x', 0) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1} \quad \text{with} \quad |x'| < 1\big\} \tag{2.7.13}$$

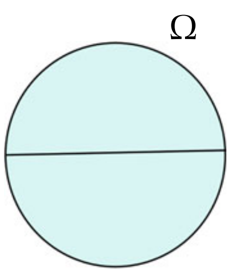

Fig. 2.15 Ω as in (2.1.1)

along with the piecewise constant vector field

$$\vec{F} := \begin{cases} +\mathbf{e}_n & \text{in } \Omega \cap \mathbb{R}^n_+, \\ -\mathbf{e}_n & \text{in } \Omega \cap \mathbb{R}^n_-. \end{cases} \tag{2.7.14}$$

Then

$$\begin{array}{l} \Omega \text{ is a bounded, nonempty, open set in } \mathbb{R}^n, \text{ with} \\ \text{an Ahlfors regular boundary, } \partial\Omega = S^{n-1} \cup \Sigma. \end{array} \tag{2.7.15}$$

Also, since $\vec{F}$ is locally constant in Ω we have

$$\vec{F} \in \big[\mathscr{C}^\infty(\Omega) \cap L^\infty(\Omega, \mathcal{L}^n)\big]^n \text{ and } \operatorname{div}\vec{F} = 0 \text{ in } \Omega. \tag{2.7.16}$$

Then, with $\nu(x) = x$ for each $x \in \partial^*\Omega = S^{n-1}$, a direct computation based on (4.2.12) shows that the distribution $\nu \bullet \vec{F} \in \big(\mathrm{Lip}_c(\partial\Omega)\big)'$ is given by a locally integrable function (in the sense of Proposition 4.1.4), namely

$$\nu \bullet \vec{F} = (\nu \cdot \mathbf{e}_n)\mathbf{1}_{S^{n-1}_+} - (\nu \cdot \mathbf{e}_n)\mathbf{1}_{S^{n-1}_-} - 2\mathbf{1}_\Sigma \text{ on } \partial\Omega. \tag{2.7.17}$$

Hence, for each fixed background aperture parameter $\kappa \in (0, \infty)$,

$$\begin{array}{l} \textit{the weak normal trace } \nu \bullet \vec{F} \textit{ of the vector field } \vec{F} \textit{ defined as in (2.7.14)} \\ \textit{on the boundary of the domain } \Omega \textit{ introduced in (2.7.13) has full support} \\ \textit{in the topological boundary } \partial\Omega, \textit{ while the nontangential trace } \vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \\ \textit{fails to exist at each point on } \Sigma := \partial\Omega \setminus \partial_*\Omega. \end{array} \tag{2.7.18}$$

Given that $\mathcal{H}^{n-1}(\Sigma) > 0$, this once again points to the fact that the weak normal boundary trace and the pointwise nontangential trace of a vector field can have rather distinct natures.

Counterexamples/Examples Part 2.7C: Fix $n \in \mathbb{N}$ with $n \geq 2$ and define (Fig. 2.16)

$$\Omega := B(0,1) \setminus \{0\} \text{ and } \vec{F}(x) := \frac{x}{|x|^n} \text{ for each } x \in \Omega. \tag{2.7.19}$$

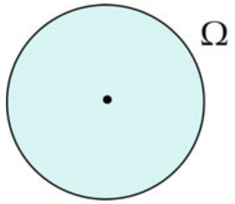

Fig. 2.16 Ω as in (2.7.19)

Then, by design,

$$\Omega \text{ is a bounded open set in } \mathbb{R}^n\text{, with an upper Ahlfors regular boundary, and } \partial\Omega = S^{n-1} \cup \{0\},\ \partial_*\Omega = S^{n-1}. \tag{2.7.20}$$

Consequently,

$$\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega \text{ becomes a locally finite measure on the topological boundary } \partial\Omega\text{, satisfying } \sigma(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{2.7.21}$$

Also, Ω is a set of finite perimeter whose geometric measure theoretic outward unit normal is given by $\nu(x) = x$ at σ-a.e. $x \in \partial\Omega$. As regards the vector field $\vec{F}$, for each fixed $\kappa > 0$ we have

$$\begin{gathered}\vec{F} \in \left[\mathscr{C}^\infty(\overline{\Omega} \setminus \{0\}) \cap L^1_{\text{bdd}}(\Omega, \mathcal{L}^n)\right]^n, \\ \mathcal{N}_\kappa \vec{F} \in L^\infty(\partial\Omega, \sigma), \quad \text{and} \quad \operatorname{div}\vec{F} = 0 \quad \text{in} \quad \Omega.\end{gathered} \tag{2.7.22}$$

Granted these properties, Theorem 1.2.1 implies that for each $\Psi \in \operatorname{Lip}_c(\mathbb{R}^n)$ we have

$$\begin{aligned}\int_\Omega \vec{F} \cdot \nabla\Psi \, d\mathcal{L}^n &= \int_\Omega \operatorname{div}(\Psi\vec{F}) \, d\mathcal{L}^n \\ &= \int_{\partial\Omega} \nu \cdot \left(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\right)\Psi \, d\sigma = \int_{\partial\Omega} \Psi \, d\sigma.\end{aligned} \tag{2.7.23}$$

From (2.7.23), Propositions 4.2.3 and 4.1.4, we then deduce that

$$\nu \bullet \vec{F} = 1 \quad \text{at } \sigma\text{-a.e. point on} \quad \partial\Omega. \tag{2.7.24}$$

Choosing $\Psi \in \operatorname{Lip}_c(\mathbb{R}^n)$ so that $\Psi \equiv 1$ near $\overline{B(0,1)}$ then forces (in light of (2.7.24) and (2.7.22))

$$\int_{\partial\Omega} (\nu \bullet \vec{F})\Psi \, d\sigma = 1 \quad \text{and} \quad \int_\Omega \vec{F} \cdot \nabla\Psi \, d\mathcal{L}^n + \int_\Omega (\operatorname{div}\vec{F})\Psi \, d\mathcal{L}^n = 0. \tag{2.7.25}$$

Finally, from (2.7.25) we conclude that

the integration by parts formula (1.9.6) *fails for the domain* Ω *and vector field* $\vec{F}$ *as in* (2.7.19). (2.7.26)

Comparing the current context with that of Theorem 1.9.1 shows that only the condition that $\partial\Omega$ is lower Ahlfors regular fails to be satisfied (given that there is not sufficient mass on $\partial\Omega$ near the point $0 \in \partial\Omega$). Ultimately, the counterexample (2.7.26) shows that *regarding the validity of the integration by parts formula* (1.9.6), *the hypothesis that* $\partial\Omega$ *is Ahlfors regular cannot be weakened to asking that* $\partial\Omega$ *is merely an upper Ahlfors regular set.* (2.7.27)

2.8 Other Versions of the Gauss–Green Formula

An easy extension of Theorem 1.1.1, based on a standard mollifier argument (as in [125, Proposition 2.6]), states that

if $\Omega \subseteq \mathbb{R}^n$ is a set of locally finite perimeter, then Gauss–Green's Formula (1.1.8) holds for each vector field $\vec{F} \in \big[\mathscr{C}_c^0(\mathbb{R}^n)\big]^n$ such that $\operatorname{div}\vec{F} \in L^1(\mathbb{R}^n, \mathcal{L}^n)$. (2.8.1)

It seems reasonable to try to establish a version of the De Giorgi–Federer's version of the Divergence Theorem in which the underlying set is asked to be of the locally finite perimeter (in an appropriate sense) *only near the support of the given vector field*. In the theorem below we accomplish this goal, using terminology and results developed in Chap. 5.

Theorem 2.8.1 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an* $\mathcal{L}^n$*-measurable set. Denote by* N_Ω *its geometric outward unit normal* (*defined at points in* $\partial^N\Omega$ *as in item (i) of Lemma 5.6.12*)*, and recall the definition of* $\partial_{\text{lfp}}\Omega$ *from* (5.7.47)*. Then for each vector field* $\vec{F} \in \big[\mathscr{C}_c^1(\mathbb{R}^n)\big]^n$ *with the property that*

$$\big(\operatorname{supp}\vec{F}\big) \cap \partial\Omega \subseteq \partial_{\text{lfp}}\Omega \tag{2.8.2}$$

one has

$$\int_\Omega \operatorname{div}\vec{F}\,d\mathcal{L}^n = \int_{\partial_*\Omega} N_\Omega \cdot \vec{F}\,d\mathcal{H}^{n-1}. \tag{2.8.3}$$

We wish to note that if Ω is a set of locally finite perimeter to begin with, then $\partial_{\text{lfp}}\Omega = \partial\Omega$ (cf. (5.7.51)), a scenario in which condition (2.8.2) is trivially satisfied and N_Ω coincides with the geometric measure theoretic outward unit normal ν to Ω up to a $\mathcal{H}^{n-1}$-nullset (cf. item *(ii)* of Lemma 5.6.12). As such, Theorem 2.8.1 extends Theorem 1.1.1 by allowing a more general class of sets.

Let us also remark that, thanks to (2.8.2), (5.2.3), and (5.7.55), we have (using the convention made in (5.7.38))

$$\big(\operatorname{supp}\vec{F}\big) \cap \partial_*\Omega \subseteq \partial_*\Omega \cap \partial_{\text{lfp}}\Omega \subseteq \partial^N\Omega \quad \text{modulo} \quad \mathcal{H}^{n-1}. \tag{2.8.4}$$

Hence, N_Ω is defined $\mathcal{H}^{n-1}$-a.e. on $\big(\operatorname{supp}\vec{F}\big) \cap \partial_*\Omega$ which, in turn, ensures that the integral on the right-hand side of (2.8.3) is meaningful.

Proof of Theorem 2.8.1 From (2.8.2) it follows that for each $x \in \big(\operatorname{supp} \vec{F}\big) \cap \partial\Omega$ there exists some $r_x > 0$ with the property that $B(x, r_x) \cap \Omega$ is a set of locally finite perimeter. Next, using the fact that $\big(\operatorname{supp} \vec{F}\big) \cap \partial\Omega$ is a compact set, it is possible to find a finite collection of points $\{x_j\}_{1 \le j \le M} \subseteq \big(\operatorname{supp} \vec{F}\big) \cap \partial\Omega$ such that, if $r_j := r_{x_j} \in (0, \infty)$ for $j \in \{1, \dots, M\}$, then

$$\big(\operatorname{supp} \vec{F}\big) \cap \partial\Omega \subseteq \bigcup_{j=1}^{M} B(x_j, r_j). \tag{2.8.5}$$

Subordinate to this finite open cover of a compact set, bring in a smooth partition of unity. That is, select a family of functions $\{\psi_j\}_{1 \le j \le M}$ with $\psi_j \in \mathscr{C}_c^\infty\big(B(x_j, r_j)\big)$ for each $j \in \{1, \dots, M\}$ and satisfying

$$\psi := \sum_{1 \le j \le M} \psi_j = 1 \;\text{ near }\; \big(\operatorname{supp} \vec{F}\big) \cap \partial\Omega. \tag{2.8.6}$$

See, e.g., [181, Theorem 14.37, p. 562]. Then for each $j \in \{1, \dots, M\}$ we may invoke Theorem 1.1.1 for the set of locally finite perimeter $B(x_j, r_j) \cap \Omega$ and the vector field $\psi_j \vec{F} \in \big[\mathscr{C}_c^1(\mathbb{R}^n)\big]^n$ to write

$$\int_{B(x_j, r_j) \cap \Omega} \operatorname{div}(\psi_j \vec{F})\, d\mathcal{L}^n = \int_{\partial_*(B(x_j, r_j) \cap \Omega)} \nu_j \cdot (\psi_j \vec{F})\, d\mathcal{H}^{n-1}, \tag{2.8.7}$$

where ν_j is the geometric measure theoretic outward unit normal to $B(x_j, r_j) \cap \Omega$. Note that since $\operatorname{supp}(\psi_j \vec{F}) \subseteq \operatorname{supp} \psi_j \subseteq B(x_j, r_j)$, we may re-write the integral on the right-hand side above as

$$\begin{aligned} \int_{\partial_*(B(x_j, r_j) \cap \Omega)} \nu_j \cdot (\psi_j \vec{F})\, d\mathcal{H}^{n-1} &= \int_{B(x_j, r_j) \cap \partial_*(B(x_j, r_j) \cap \Omega)} \nu_j \cdot (\psi_j \vec{F})\, d\mathcal{H}^{n-1} \\ &= \int_{B(x_j, r_j) \cap \partial_*\Omega} \nu_j \cdot (\psi_j \vec{F})\, d\mathcal{H}^{n-1} \\ &= \int_{B(x_j, r_j) \cap \partial_*\Omega} N_\Omega \cdot (\psi_j \vec{F})\, d\mathcal{H}^{n-1} \\ &= \int_{\partial_*\Omega} N_\Omega \cdot (\psi_j \vec{F})\, d\mathcal{H}^{n-1}, \end{aligned} \tag{2.8.8}$$

where we have also used Proposition 5.2.3 in the second equality, and (5.7.54) in the third equality. Based on (2.8.8) and (2.8.6) (and keeping in mind that $\partial_*\Omega \subseteq \partial\Omega$) we therefore obtain

$$\sum_{j=1}^{M}\int_{\partial_*(B(x_j,r_j)\cap\Omega)}\nu_j\cdot(\psi_j\vec{F})\,\mathrm{d}\mathcal{H}^{n-1}=\sum_{j=1}^{M}\int_{\partial_*\Omega}N_\Omega\cdot(\psi_j\vec{F})\,\mathrm{d}\mathcal{H}^{n-1}$$
$$=\int_{\partial_*\Omega}\Big(\sum_{j=1}^{M}\psi_j\Big)N_\Omega\cdot\vec{F}\,\mathrm{d}\mathcal{H}^{n-1}$$
$$=\int_{\partial_*\Omega}N_\Omega\cdot\vec{F}\,\mathrm{d}\mathcal{H}^{n-1}. \tag{2.8.9}$$

On the other hand, since for each $j\in\{1,\dots,M\}$ we have

$$\int_{B(x_j,r_j)\cap\Omega}\mathrm{div}(\psi_j\vec{F})\,\mathrm{d}\mathcal{L}^n=\int_\Omega\mathrm{div}(\psi_j\vec{F})\,\mathrm{d}\mathcal{L}^n, \tag{2.8.10}$$

it follows that

$$\sum_{j=1}^{M}\int_{B(x_j,r_j)\cap\Omega}\mathrm{div}(\psi_j\vec{F})\,\mathrm{d}\mathcal{L}^n=\int_\Omega\mathrm{div}\Big(\sum_{j=1}^{M}\psi_j\vec{F}\Big)\,\mathrm{d}\mathcal{L}^n=\int_\Omega\mathrm{div}(\psi\vec{F})\,\mathrm{d}\mathcal{L}^n$$
$$=\int_\Omega\mathrm{div}\vec{F}\,\mathrm{d}\mathcal{L}^n+\int_\Omega\mathrm{div}\big((\psi-1)\vec{F}\big)\,\mathrm{d}\mathcal{L}^n. \tag{2.8.11}$$

If we now introduce

$$\vec{G}:=\begin{cases}(\psi-1)\vec{F} & \text{in } \Omega,\\ 0 & \text{in } \mathbb{R}^n\setminus\Omega,\end{cases} \tag{2.8.12}$$

then property (2.8.6) ensures that $\vec{G}\in\big[\mathscr{C}^1_c(\mathbb{R}^n)\big]^n$, hence

$$\int_\Omega\mathrm{div}\big((\psi-1)\vec{F}\big)\,\mathrm{d}\mathcal{L}^n=\int_{\mathbb{R}^n}\mathrm{div}\vec{G}\,\mathrm{d}\mathcal{L}^n=0 \tag{2.8.13}$$

by the standard version of the Divergence Theorem in a sufficiently large ball, containing the support of $\vec{G}$. Collectively, (2.8.7), (2.8.9), (2.8.11), and (2.8.13) now imply (2.8.3).

A consequence of Theorem 2.8.1 worth stating separately is recorded in the corollary below.

Corollary 2.8.2 *Suppose $\Omega\subseteq\mathbb{R}^n$ is an $\mathcal{L}^n$-measurable set and denote by N_Ω its geometric outward unit normal (defined at points in $\partial^N\Omega$ as in item (i) of Lemma 5.6.12). Also, let $\mathcal{O}\subseteq\mathbb{R}^n$ be an open set with the property that $\mathcal{O}\cap\Omega$ is a set of locally finite perimeter. Then for each vector field $\vec{F}\in\big[\mathscr{C}^1_c(\mathbb{R}^n)\big]^n$ satisfying*

$$\big(\mathrm{supp}\,\vec{F}\big)\cap\partial\Omega\subseteq\mathcal{O} \tag{2.8.14}$$

there holds

$$\int_{\Omega} \operatorname{div}\vec{F}\, d\mathcal{L}^n = \int_{\partial_*\Omega} N_\Omega \cdot \vec{F}\, d\mathcal{H}^{n-1}. \tag{2.8.15}$$

Proof The fact that $\mathcal{O} \subseteq \mathbb{R}^n$ is open and $\mathcal{O} \cap \Omega$ has locally finite perimeter implies (with the help of (5.7.47) and Lemma 5.6.5) that

$$\mathcal{O} \cap \partial\Omega \subseteq \partial_{\text{lfp}}\Omega. \tag{2.8.16}$$

Consequently,

$$\big(\operatorname{supp}\vec{F}\big) \cap \partial\Omega \subseteq \mathcal{O} \cap \partial\Omega \subseteq \partial_{\text{lfp}}\Omega, \tag{2.8.17}$$

so (2.8.15) is provided by Theorem 2.8.1. □

In Proposition 2.8.6 below we present a version of Theorem 1.1.1 in which the vector field is allowed to be singular. To set the stage, we begin by making a couple of definitions and recall some background results. Our first definition introduces the measure theoretic interior and exterior of Euclidean sets.

Definition 2.8.3 *Given a Lebesgue measurable set $E \subseteq \mathbb{R}^n$, its* `measure theoretic interior` *is defined as*

$$\begin{aligned}\operatorname{int}_*(E) &:= \Big\{x \in \mathbb{R}^n : \lim_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\setminus E\big)}{\mathcal{L}^n\big(B(x,r)\big)} = 0\Big\} \\ &= \Big\{x \in \mathbb{R}^n : \lim_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\cap E\big)}{\mathcal{L}^n\big(B(x,r)\big)} = 1\Big\},\end{aligned} \tag{2.8.18}$$

and its `measure theoretic exterior` *is defined as*

$$\begin{aligned}\operatorname{ext}_*(E) &:= \operatorname{int}_*(\mathbb{R}^n \setminus E) \\ &= \Big\{x \in \mathbb{R}^n : \lim_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\cap E\big)}{\mathcal{L}^n\big(B(x,r)\big)} = 0\Big\}.\end{aligned} \tag{2.8.19}$$

Clearly, for each Lebesgue measurable set $E \subseteq \mathbb{R}^n$ we have

$$\mathring{E} \subseteq \operatorname{int}_*(E) \subseteq \overline{E}, \qquad \mathbb{R}^n \setminus \overline{E} \subseteq \operatorname{ext}_*(E) \subseteq \mathbb{R}^n \setminus \mathring{E}, \tag{2.8.20}$$

and (recall (5.2.2))

$$\mathbb{R}^n = \partial_* E \sqcup \operatorname{int}_*(E) \sqcup \operatorname{ext}_*(E). \tag{2.8.21}$$

Also, [80, Lemma 2(i), p. 222], Lebesgue's Differentiation Theorem and the above definitions imply that given any Lebesgue measurable set $E \subseteq \mathbb{R}^n$, it follows that $\mathrm{int}_*(E)$ and $\mathrm{ext}_*(E)$ are Borel-measurable sets with $\mathcal{L}^n\big(\mathrm{int}_*(E) \setminus E\big) = 0$, $\mathcal{L}^n\big(E \setminus \mathrm{int}_*(E)\big) = 0$, $\mathcal{L}^n\big(\mathrm{ext}_*(E) \cap E\big) = 0$, $\mathcal{L}^n\big(\mathbb{R}^n \setminus \big(E \cup \mathrm{ext}_*(E)\big)\big) = 0$.

(2.8.22)

A variety of other useful properties enjoyed by the measure theoretic interiors and exteriors may be found in [216, pp. 49–58].

Lemma 2.8.4 *Let $\theta \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n, \mathcal{L}^n)$ be a non-negative radial function satisfying $\int_{\mathbb{R}^n} \theta \, \mathrm{d}\mathcal{L}^n = 1$ and $\operatorname{supp}\theta \subseteq B(0,1)$. For each $\varepsilon > 0$, define $\theta_\varepsilon := \varepsilon^{-n}\theta(\cdot/\varepsilon)$ in $\mathbb{R}^n$. Then for every set $E \subseteq \mathbb{R}^n$ of locally finite perimeter one has*

$$\lim_{\varepsilon \to 0^+} \big(\theta_\varepsilon * \boldsymbol{1}_E\big)(x) = \begin{cases} 1 & \text{if } x \in \mathrm{int}_*(E), \\ \frac{1}{2} & \text{if } x \in \partial^* E, \\ 0 & \text{if } x \in \mathrm{ext}_*(E), \end{cases} \tag{2.8.23}$$

for each

$$x \in \partial^* E \sqcup \mathrm{int}_*(E) \sqcup \mathrm{ext}_*(E). \tag{2.8.24}$$

Proof Assume first that $x \in \mathrm{ext}_*(E)$. Then, on account of the properties of θ and (2.8.19), we may write

$$\begin{aligned} 0 &\leq \limsup_{\varepsilon \to 0^+} \big(\theta_\varepsilon * \mathbf{1}_E\big)(x) = \limsup_{\varepsilon \to 0^+} \int_E \theta_\varepsilon(x-y)\,\mathrm{d}y \\ &= \limsup_{\varepsilon \to 0^+} \varepsilon^{-n} \int_{E \cap B(x,\varepsilon)} \theta\big((x-y)/\varepsilon\big)\,\mathrm{d}y \\ &\leq \frac{\omega_{n-1}}{n} \|\theta\|_{L^\infty(\mathbb{R}^n,\mathcal{L}^n)} \limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^n\big(B(x,\varepsilon) \cap E\big)}{\mathcal{L}^n\big(B(x,\varepsilon)\big)} = 0. \end{aligned} \tag{2.8.25}$$

This is in agreement with (2.8.23). Consider next the case when $x \in \mathrm{int}_*(E)$. First, making use of the properties of θ and (2.8.18) we may estimate

$$\begin{aligned} 0 &\leq \limsup_{\varepsilon \to 0^+} \int_{B(x,\varepsilon)\setminus E} \theta_\varepsilon(x-y)\,\mathrm{d}y = \limsup_{\varepsilon \to 0^+} \varepsilon^{-n} \int_{B(x,\varepsilon)\setminus E} \theta\big((x-y)/\varepsilon\big)\,\mathrm{d}y \\ &\leq \frac{\omega_{n-1}}{n} \|\theta\|_{L^\infty(\mathbb{R}^n,\mathcal{L}^n)} \limsup_{\varepsilon \to 0^+} \frac{\mathcal{L}^n\big(B(x,\varepsilon) \setminus E\big)}{\mathcal{L}^n\big(B(x,\varepsilon)\big)} = 0. \end{aligned} \tag{2.8.26}$$

This ultimately proves that in this case

$$\lim_{\varepsilon \to 0^+} \int_{B(x,\varepsilon)\setminus E} \theta_\varepsilon(x-y)\,\mathrm{d}y = 0. \tag{2.8.27}$$

Consequently,

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+}\big(\theta_\varepsilon * \mathbf{1}_E\big)(x) &= \lim_{\varepsilon\to 0^+}\int_E \theta_\varepsilon(x-y)\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0^+}\varepsilon^{-n}\int_{E\cap B(x,\varepsilon)}\theta\big((x-y)/\varepsilon\big)\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0^+}\varepsilon^{-n}\int_{B(x,\varepsilon)}\theta\big((x-y)/\varepsilon\big)\,\mathrm{d}y \\
&\quad - \lim_{\varepsilon\to 0^+}\varepsilon^{-n}\int_{B(x,\varepsilon)\setminus E}\theta\big((x-y)/\varepsilon\big)\,\mathrm{d}y \\
&= 1-0=1, \qquad (2.8.28)
\end{aligned}
$$

which once again is in agreement with (2.8.23). There remains to consider the case when $x\in\partial^* E$. In such a scenario, we shall employ a blow-up argument. Specifically, with $\nu(x)$ denoting the geometric measure theoretic outward unit normal to E at the point x, bring in the half-space

$$
H_x := \big\{y\in\mathbb{R}^n : \langle\nu(x), y\rangle \le 0\big\} \qquad (2.8.29)
$$

and, for each $\varepsilon>0$, define

$$
E_\varepsilon := \big\{y\in\mathbb{R}^n : \varepsilon(y-x)+x\in E\big\}. \qquad (2.8.30)
$$

Next, making the change of variables $z=(y-x)/\varepsilon$ and also observing that we have the equality $(E-x)/\varepsilon = E_\varepsilon - x$, we may compute

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+}\big(\theta_\varepsilon * \mathbf{1}_E\big)(x) &= \lim_{\varepsilon\to 0^+}\int_E \theta_\varepsilon(x-y)\,\mathrm{d}y = \lim_{\varepsilon\to 0^+}\int_E \theta_\varepsilon(y-x)\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0^+}\varepsilon^{-n}\int_E \theta\big((y-x)/\varepsilon\big)\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0^+}\int_{(E-x)/\varepsilon}\theta(z)\,\mathrm{d}z = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}\theta\mathbf{1}_{E_\varepsilon - x}\,\mathrm{d}\mathcal{L}^n \\
&= \int_{\mathbb{R}^n}\theta\mathbf{1}_{H_x}\,\mathrm{d}\mathcal{L}^n, \qquad (2.8.31)
\end{aligned}
$$

where the last equality is supplied by [80, Theorem 1, p. 199]. However, θ is radial so working in polar coordinates allows us to conclude that

$$
\int_{\mathbb{R}^n}\theta\mathbf{1}_{H_x}\,\mathrm{d}\mathcal{L}^n = \int_{H_x}\theta\,\mathrm{d}\mathcal{L}^n = \frac{1}{2}\int_{\mathbb{R}^n}\theta\,\mathrm{d}\mathcal{L}^n = \frac{1}{2}. \qquad (2.8.32)
$$

Together, (2.8.31) and (2.8.32) then finish the proof of (2.8.23)–(2.8.24). □

Finally, we define the $\mathscr{C}^k$-singular support of a distribution.

Definition 2.8.5 *Let $k \in \mathbb{N}_0$ be an arbitrary number and let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary open set. Given a distribution $u \in \mathcal{D}'(\Omega)$, define its $\mathscr{C}^k$-*`singular support`*, denoted by $\mathscr{C}^k$-singsup u, as the smallest relatively closed subset K of Ω with the property that $u\big|_{\Omega\setminus K} \in \mathscr{C}^k(\Omega \setminus K)$.*

Here is the version of the Divergence Theorem advertised earlier.

Proposition 2.8.6 *Given an open set of locally finite perimeter $\Omega \subseteq \mathbb{R}^n$, denote by ν its geometric measure theoretic outward unit normal and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Suppose $\mathcal{O} \subseteq \mathbb{R}^n$ is an open set containing $\overline{\Omega}$ and consider a vector field $\vec{F}$ satisfying the following properties:*

$$\begin{gathered}\vec{F} \in \big[\mathscr{E}'(\mathcal{O})\big]^n, \quad \mathscr{C}^0\text{- singsupp}\, \vec{F} \ \text{ is a compact subset of } \ \Omega \\ \text{and } \ \operatorname{div}\vec{F} = \mu + u \ \text{ in } \ \mathcal{D}'(\mathcal{O}) \ \text{ where} \\ \mu \in \mathrm{CBM}(\mathcal{O}) \ \text{ and } \ u \in \mathscr{E}'(\mathcal{O}) \ \text{ with } \ \operatorname{supp} u \subseteq \Omega.\end{gathered} \tag{2.8.33}$$

Then

$$\big(\operatorname{div}\vec{F}\big)\big|_{\Omega} \in \mathrm{CBM}(\Omega) + \mathscr{E}'(\Omega) \hookrightarrow \big(\mathscr{C}_b^{\infty}(\Omega)\big)^* \tag{2.8.34}$$

and

$$\begin{aligned}{}_{(\mathscr{C}_b^{\infty}(\Omega))^*}\Big(\big(\operatorname{div}\vec{F}\big)\big|_{\Omega}, 1\Big)_{\mathscr{C}_b^{\infty}(\Omega)} &= \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|_{\partial_*\Omega}\big)\, d\sigma \\ &\quad - \tfrac{1}{2}\mu(\partial^*\Omega) - \mu\big(\mathrm{int}_*(\Omega) \setminus \Omega\big).\end{aligned} \tag{2.8.35}$$

In particular, with $L^1(\mathcal{O}, \mathcal{L}^n)$ canonically identified as a subspace of CBM$(\mathcal{O})$,

$$\begin{gathered}\text{if actually } \mu \in L^1(\mathcal{O}, \mathcal{L}^n) \text{ then } (2.8.35) \text{ simply becomes} \\ {}_{(\mathscr{C}_b^{\infty}(\Omega))^*}\Big(\big(\operatorname{div}\vec{F}\big)\big|_{\Omega}, 1\Big)_{\mathscr{C}_b^{\infty}(\Omega)} = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|_{\partial_*\Omega}\big)\, d\sigma.\end{gathered} \tag{2.8.36}$$

Since both $\partial^*\Omega$ and $\mathrm{int}_*(\Omega) \setminus \Omega$ are contained in $\partial\Omega$ (cf. (5.6.21) and (2.8.20)), another natural scenario when (2.8.35) reduces to the Divergence Formula recorded in (2.8.36) is when $|\mu|(\partial\Omega) = 0$.

Proof of Proposition 2.8.6 The fact that $\vec{F}$ belongs to $\big[\mathscr{E}'(\mathcal{O})\big]^n$ implies that μ has compact support in $\mathcal{O}$. Observing that $\mu|_{\Omega} \in \mathrm{CBM}(\Omega)$ and $u|_{\Omega} \in \mathscr{E}'(\Omega)$ then yields (2.8.34). Next, let K be the union of $\mathscr{C}^0$-singsupp $\vec{F}$ and supp u. Then K is a compact subset of Ω, which makes it possible to select a scalar function $\varphi \in \mathscr{C}_c^{\infty}(\Omega)$ with $\varphi \equiv 1$ near K. Split

$$\vec{F} = \vec{G} + \vec{H} \ \text{ in } \ \big[\mathcal{D}'(\mathcal{O})\big]^n \ \text{ where } \ \vec{G} := (1-\varphi)\vec{F} \ \text{ and } \ \vec{H} := \varphi\vec{F}. \tag{2.8.37}$$

Note that since the support of $\vec{H} \in \big[\mathcal{D}'(\mathcal{O})\big]^n$ is a compact subset of Ω, we have

$$\vec{H}\big|_{\Omega} \in \big[\mathscr{E}'(\Omega)\big]^n \ \text{ hence also } \ \big(\mathrm{div}\vec{H}\big)\big|_{\Omega} = \mathrm{div}\big(\vec{H}\big|_{\Omega}\big) \in \mathscr{E}'(\Omega). \tag{2.8.38}$$

By virtue of the compatibility condition (4.6.21), this permits us to compute

$$\begin{aligned}
{}_{(\mathscr{C}_b^\infty(\Omega))^*}\Big(\big(\mathrm{div}\vec{H}\big)\big|_{\Omega}, 1\Big)_{\mathscr{C}_b^\infty(\Omega)} &= {}_{\mathscr{E}'(\Omega)}\Big\langle\big(\mathrm{div}\vec{H}\big)\big|_{\Omega}, 1\Big\rangle_{\mathscr{E}(\Omega)} \\
&= {}_{\mathscr{E}'(\Omega)}\Big\langle\mathrm{div}\big(\vec{H}\big|_{\Omega}\big), 1\Big\rangle_{\mathscr{E}(\Omega)} \\
&= -{}_{[\mathscr{E}'(\Omega)]^n}\Big\langle\vec{H}\big|_{\Omega}, \nabla 1\Big\rangle_{[\mathscr{E}(\Omega)]^n} = 0.
\end{aligned} \tag{2.8.39}$$

Also, since $(1-\varphi)u = 0$, we have

$$\begin{gathered}
\vec{G} \in \big[\mathscr{C}_c^0(\mathcal{O})\big]^n \ \text{ and } \ \mathrm{div}\vec{G} = f + \lambda \ \text{ in } \ \mathcal{D}'(\mathcal{O}) \ \text{ where} \\
f := -(\nabla\varphi)\cdot\vec{F} \in \mathscr{C}_c^0(\Omega) \ \text{ and } \ \lambda := (1-\varphi)\mu \in \mathrm{CBM}(\mathcal{O}).
\end{gathered} \tag{2.8.40}$$

In addition, with $|\mathrm{div}\vec{G}|$ denoting the total variation of the measure $\mathrm{div}\vec{G}$, from [44, Proposition 3.1, p. 101] we conclude that

$$\begin{gathered}
|\mathrm{div}\vec{G}|(A) = 0 \ \text{ whenever } \ A \subseteq \mathcal{O} \ \text{ is a Borel} \\
\text{measurable set satisfying } \mathcal{H}^{n-1}(A) = 0.
\end{gathered} \tag{2.8.41}$$

In view of (2.8.40), from (2.8.41) we further deduce that

$$\begin{gathered}
|\lambda|(A) = 0 \text{ whenever } A \subseteq \mathcal{O} \text{ is a Borel set} \\
\text{with the property that } \mathcal{H}^{n-1}(A) = 0.
\end{gathered} \tag{2.8.42}$$

Next, pick a non-negative radial function $\theta \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \theta\, d\mathcal{L}^n = 1$ as well as $\mathrm{supp}\,\theta \subseteq B(0,1)$ and, for each $\varepsilon > 0$, introduce $\theta_\varepsilon := \varepsilon^{-n}\theta(\cdot/\varepsilon)$ in $\mathbb{R}^n$. Lastly, fix some $\varepsilon_o > 0$ sufficiently small and, whenever $0 < \varepsilon < \varepsilon_o$, define

$$\vec{G}_\varepsilon := \theta_\varepsilon * \vec{G} \in \big[\mathscr{C}_c^\infty(\mathcal{O})\big]^n. \tag{2.8.43}$$

Note that

$$\begin{gathered}
\textstyle\bigcup_{0<\varepsilon<\varepsilon_o} \mathrm{supp}\,\vec{G}_\varepsilon \ \text{ is a relatively compact subset of } \ \mathcal{O} \\
\text{and } \ \sup_{\mathcal{O}}\big|\vec{G} - \vec{G}_\varepsilon\big| \longrightarrow 0 \ \text{ as } \ \varepsilon \to 0^+.
\end{gathered} \tag{2.8.44}$$

Granted these, Theorem 1.1.1 applies and, together with (2.8.44), permits us to write

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_{\Omega}\big(\mathrm{div}\vec{G}_\varepsilon\big)\big|_{\Omega}\, d\mathcal{L}^n &= \lim_{\varepsilon\to 0^+}\int_{\partial_*\Omega} \nu\cdot\big(\vec{G}_\varepsilon\big|_{\partial_*\Omega}\big)\, d\sigma = \int_{\partial_*\Omega}\nu\cdot\big(\vec{G}\big|_{\partial_*\Omega}\big)\, d\sigma \\
&= \int_{\partial_*\Omega}\nu\cdot\big(((1-\varphi)\vec{F})\big|_{\partial_*\Omega}\big)\, d\sigma
\end{aligned}$$

$$= \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|_{\partial_*\Omega}\big)\,\mathrm{d}\sigma, \tag{2.8.45}$$

where the last equality uses the fact that, by design, $1 - \varphi \equiv 1$ near $\partial\Omega \supseteq \partial_*\Omega$. The decomposition of $\mathrm{div}\vec{G}$ in (2.8.40) implies

$$\mathrm{div}\vec{G}_\varepsilon = \theta_\varepsilon * \big(\mathrm{div}\vec{G}\big) = \theta_\varepsilon * f + \theta_\varepsilon * \lambda \ \text{ in } \ \mathcal{D}'(\mathcal{O}) \tag{2.8.46}$$

for each $\varepsilon \in \big(0, \varepsilon_o\big)$. Then, thanks to Lebesgue's Dominated Convergence Theorem and (4.6.19), we have

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \int_\Omega \big(\theta_\varepsilon * f\big)\big|_\Omega \,\mathrm{d}\mathcal{L}^n &= \int_\Omega \big(f\big|_\Omega\big)\,\mathrm{d}\mathcal{L}^n = -\int_\Omega (\nabla\varphi)\cdot\vec{F}\,\mathrm{d}\mathcal{L}^n \\ &= {}_{(\mathscr{C}_b^\infty(\Omega))^*}\Big(-\big((\nabla\varphi)\cdot\vec{F}\big)\big|_\Omega, 1\Big)_{\mathscr{C}_b^\infty(\Omega)}. \end{aligned} \tag{2.8.47}$$

Since $\mathcal{H}^{n-1}\big(\partial_*\Omega \setminus \partial^*\Omega\big) = 0$ (cf. (5.6.21)) and since $\mathcal{H}^{n-1}\lfloor\partial\Omega$ is a Borel-regular measure (cf. Lemma 3.6.4) it follows that there exists a Borel set $A \subseteq \partial\Omega$ such that

$$\partial_*\Omega \setminus \partial^*\Omega \subseteq A \ \text{ and } \ \mathcal{H}^{n-1}(A) = 0. \tag{2.8.48}$$

In concert with (2.8.42) (and (3.1.1)), the last property above further implies

$$|\lambda|(A) = 0. \tag{2.8.49}$$

In view of Lemma 2.8.4, (2.8.48), and (2.8.49), Fubini's theorem together with Lebesgue's Dominated Convergence Theorem and (4.6.19) permits us to compute

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \int_\Omega \big(\theta_\varepsilon * \lambda\big)\big|_\Omega \,\mathrm{d}\mathcal{L}^n \qquad (2.8.50)\\ &\quad = \lim_{\varepsilon\to 0^+} \int_\Omega \int_{\mathcal{O}} \theta_\varepsilon(x-y)\,\mathrm{d}\lambda(y)\,\mathrm{d}\mathcal{L}^n(x) \\ &\quad = \lim_{\varepsilon\to 0^+} \int_{\mathcal{O}} \Big(\int_\Omega \theta_\varepsilon(x-y)\,\mathrm{d}\mathcal{L}^n(x)\Big)\,\mathrm{d}\lambda(y) \\ &\quad = \lim_{\varepsilon\to 0^+} \int_{\mathcal{O}} \big(\theta_\varepsilon * \mathbf{1}_\Omega\big)\big|_{\mathcal{O}}\,\mathrm{d}\lambda = \int_{\mathcal{O}} \Big\{\tfrac{1}{2}\mathbf{1}_{\partial^*\Omega} + \mathbf{1}_{\mathrm{int}_*(\Omega)}\Big\}\,\mathrm{d}\lambda \\ &\quad = \tfrac{1}{2}\lambda(\partial^*\Omega) + \lambda\big(\mathrm{int}_*(\Omega)\setminus\Omega\big) + \lambda(\Omega) \\ &\quad = \tfrac{1}{2}\mu(\partial^*\Omega) + \mu\big(\mathrm{int}_*(\Omega)\setminus\Omega\big) + \int_\Omega (1-\varphi)\,\mathrm{d}\mu \\ &\quad = \tfrac{1}{2}\mu(\partial^*\Omega) + \mu\big(\mathrm{int}_*(\Omega)\setminus\Omega\big) + {}_{(\mathscr{C}_b^\infty(\Omega))^*}\Big(\big((1-\varphi)\mu\big)\big|_\Omega, 1\Big)_{\mathscr{C}_b^\infty(\Omega)}. \end{aligned}$$

Above, we have also used the fact that since $1 - \varphi \equiv 1$ near $\partial\Omega$ it follows that λ coincides with μ on Borel subsets of $\partial\Omega$. Collectively, (2.8.46), (2.8.47), (2.8.50), and (2.8.40) imply

$$\lim_{\varepsilon\to 0^+}\int_\Omega \big(\mathrm{div}\vec{G}_\varepsilon\big)\big|_\Omega \,\mathrm{d}\mathcal{L}^n = \tfrac{1}{2}\mu(\partial^*\Omega) + \mu\big(\mathrm{int}_*(\Omega)\setminus\Omega\big) \\ + {}_{(\mathscr{C}_b^\infty(\Omega))^*}\Big(\big(\mathrm{div}\vec{G}\big)\big|_\Omega, 1\Big)_{\mathscr{C}_b^\infty(\Omega)}. \tag{2.8.51}$$

At this stage, (2.8.35) follows from (2.8.37), (2.8.39), (2.8.45), and (2.8.51).

To finish the proof of the proposition, there remains to justify the claim in (2.8.36). In this regard, recall from (2.8.22) that $\mathcal{L}^n\big(\mathrm{int}_*(\Omega)\setminus\Omega\big) = 0$. Also, we know from Lemma 5.2.1 that $\mathcal{L}^n(\partial_*\Omega) = 0$. Hence, in the case when $\mu \in L^1(\mathcal{O}, \mathcal{L}^n)$ we have $\mu\big(\mathrm{int}_*(\Omega)\setminus\Omega\big) = 0$ and $\mu(\partial^*\Omega) = 0$, so (2.8.35) reduces to (2.8.36). □

Corollary 2.8.7 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set of locally finite perimeter. Denote by ν its geometric measure theoretic outward unit normal and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Then, given an open set $\mathcal{O} \subseteq \mathbb{R}^n$ containing $\overline{\Omega}$, for each vector field*

$$\vec{F} \in \big[\mathscr{C}_c^0(\mathcal{O})\big]^n \ \ \textit{with} \ \ \mathrm{div}\vec{F} \in \mathrm{CBM}(\mathcal{O}) \tag{2.8.52}$$

one has

$$\int_{\partial_*\Omega} \nu\cdot\big(\vec{F}\big|_{\partial_*\Omega}\big)\,\mathrm{d}\sigma = \big(\mathrm{div}\vec{F}\big)\big(\mathrm{int}_*(\Omega)\big) + \tfrac{1}{2}\big(\mathrm{div}\vec{F}\big)(\partial^*\Omega). \tag{2.8.53}$$

Moreover, if Ω is also bounded, then the compact support assumption on $\vec{F}$ may be dropped.

Proof Formula (2.8.53) is a direct consequence of (2.8.35) in Proposition 2.8.6 (corresponding to the case when $u = 0$). The last claim is justified simply working with $\varphi\vec{F}$ in place of $\vec{F}$ where $\varphi \in \mathscr{C}_c^\infty(\mathcal{O})$ satisfying $\varphi \equiv 1$ near $\overline{\Omega}$. □

In the context of Corollary 2.8.7, if $\mathrm{div}\vec{F} = \mu_a + \mu_s$ is the Lebesgue decomposition (cf., e.g., [231, Theorem 6.10, p. 121]) of the complex measure $\mathrm{div}\vec{F}$ into an absolutely continuous part, μ_a, and a singular part, μ_s, with respect to the Lebesgue measure $\mathcal{L}^n$ in $\mathcal{O}$, arguing as in the proof of (2.8.36) we may recast (2.8.53) as

$$\int_{\partial_*\Omega} \nu\cdot\big(\vec{F}\big|_{\partial_*\Omega}\big)\,\mathrm{d}\sigma = \big(\mathrm{div}\vec{F}\big)(\Omega) + \mu_s\big(\mathrm{int}_*(\Omega)\setminus\Omega\big) + \tfrac{1}{2}\mu_s(\partial^*\Omega). \tag{2.8.54}$$

Comparing (2.8.54) with the Divergence Formula (1.3.9), we conclude that given an open set $\mathcal{O} \subseteq \mathbb{R}^n$ along with a vector field $\vec{F} \in \big[\mathscr{C}^0_c(\mathcal{O})\big]^n$ such that $\operatorname{div}\vec{F} \in \mathrm{CBM}(\mathcal{O})$ it follows that for every open set Ω with $\overline{\Omega} \subseteq \mathcal{O}$, having a lower Ahlfors regular boundary and such that $\mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure (which is the case if, e.g., $\partial\Omega$ is Ahlfors regular), we necessarily have $\mu_s\big(\mathrm{int}_*(\Omega) \setminus \Omega\big) + \frac{1}{2}\mu_s(\partial^*\Omega) = 0$. (2.8.55)

This points to the fact that the singular parts (with respect to the Lebesgue measure) of complex Borel measures arising as (distributional) divergences of continuous vector fields have rather subtle vanishing properties. For more on this topic, see [217, 226], and the references therein.

Corollary 2.8.8 *Assume $\Omega \subseteq \mathbb{R}^n$ is an open set of locally finite perimeter. Denote by ν its geometric measure theoretic outward unit normal and let $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Then for each vector field $\vec{F} \in \big[\mathscr{E}'(\mathbb{R}^n)\big]^n$ whose $\mathscr{C}^1$-singular support is a compact subset of Ω one has (with the divergence taken in the sense of distributions in $\mathbb{R}^n$)*

$$\big(\operatorname{div}\vec{F}\big)\big|_\Omega \in L^1(\Omega, \mathcal{L}^n) + \mathscr{E}'(\Omega) \hookrightarrow \big(\mathscr{C}^\infty_b(\Omega)\big)^* \tag{2.8.56}$$

and

$$ {}_{(\mathscr{C}^\infty_b(\Omega))^*}\Big(\big(\operatorname{div}\vec{F}\big)\big|_\Omega, 1\Big)_{\mathscr{C}^\infty_b(\Omega)} = \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|_{\partial_*\Omega}\big)\,d\sigma. \tag{2.8.57}$$

Proof This is a corollary of Proposition 2.8.6 specialized to $\mathcal{O} := \mathbb{R}^n$. Specifically, the present assumptions on $\vec{F}$ imply that $\operatorname{div}\vec{F} = f + u$ where $f \in \mathscr{C}^0_c(\mathbb{R}^n)$ and $u \in \mathscr{E}'(\mathbb{R}^n)$ with $\operatorname{supp} u \subseteq \Omega$. Hence, the measure $\mu := f\mathcal{L}^n \in \mathrm{CBM}(\mathbb{R}^n)$ belongs to $L^1(\mathcal{O}, \mathcal{L}^n)$, so (2.8.36) applies and the desired conclusion follows. □

We make a couple of comments on the nature of Corollary 2.8.8. First, asking that $\Omega \subseteq \mathbb{R}^n$ is an open set of locally finite perimeter is natural in light of the conclusions in (2.8.56)–(2.8.57). Second, in the case when the $\mathscr{C}^1$-singular support of the given vector field $\vec{F}$ is the empty set, it follows that $\vec{F} \in \big[\mathscr{C}^1_c(\mathbb{R}^n)\big]^n$ and (2.8.57) reduces precisely to the Divergence Formula (1.1.8). Third, given any $\vec{F} \in \big[\mathscr{E}'(\mathbb{R}^n)\big]^n$ it follows that $\mathscr{C}^1\text{-singsupp}\,\vec{F}$ is a compact subset of $\mathbb{R}^n$. That having $\mathscr{C}^1\text{-singsupp}\,\vec{F}$ actually contained in Ω is necessary in the context of Corollary 2.8.8 may be seen by considering the following counterexample to (2.8.56)–(2.8.57). Work in $\mathbb{R}^n$ with $n \in \mathbb{N}$, $n \geq 2$. Pick a scalar-valued cutoff function $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ satisfying $\psi \equiv 1$ on $B(0, 2)$, then define

$$\Omega := B(0, 1) \setminus \{0\} \quad\text{and}\quad \vec{F}(x) := \psi(x)\frac{x}{|x|^n} \quad\text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n. \tag{2.8.58}$$

It follows that $\Omega \subseteq \mathbb{R}^n$ is an open set of locally finite perimeter, and

$$\vec{F} \in \big[L^1_{\mathrm{comp}}(\mathbb{R}^n, \mathcal{L}^n)\big]^n \subset \big[\mathscr{E}'(\mathbb{R}^n)\big]^n \quad\text{has}\quad \mathscr{C}^1\text{-singsupp}\,\vec{F} = \{0\}. \tag{2.8.59}$$

Also, in the sense of distributions in $\mathbb{R}^n$,

$$\begin{aligned} &\operatorname{div}\vec{F} = f + \omega_{n-1}\delta, \quad \text{where} \quad f \in L^1_{\text{comp}}(\mathbb{R}^n, \mathcal{L}^n) \subset \mathscr{E}'(\mathbb{R}^n) \\ &\quad\text{is given by} \quad f(x) = \frac{x \cdot (\nabla\psi)(x)}{|x|^n} \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{R}^n. \end{aligned} \tag{2.8.60}$$

In particular, granted the choice of Ω and ψ, we have $\big(\operatorname{div}\vec{F}\big)\big|_{\Omega} = 0$, so (2.8.56) is trivially satisfied. However, (2.8.57) fails since its right-hand side presently becomes $\omega_{n-1} \neq 0$. The source of this failure is the fact that the $\mathscr{C}^1$-singular support of $\vec{F}$ is not a compact subset of Ω.

In many practical situations, given an open set $\Omega \subseteq \mathbb{R}^n$, one deals with functions defined only on $\overline{\Omega}$, and one would like to avoid assuming they have extensions to $\mathbb{R}^n$ with nice properties. To describe a result for such functions, following [125] we shall say that Ω has a `tame interior approximation` if there exists a family $\{\Omega_j\}_{j\in\mathbb{N}}$ of open subsets of $\mathbb{R}^n$ satisfying

$$\overline{\Omega}_j \subseteq \Omega \ \text{ and } \ \Omega_j \subseteq \Omega_{j+1} \ \text{ for each } \ j \in \mathbb{N}, \qquad \Omega = \bigcup_{j\in\mathbb{N}} \Omega_j, \tag{2.8.61}$$

as well as

$$\sup_{j\in\mathbb{N}} \|\nabla \mathbf{1}_{\Omega_j}\|_{\mathrm{TV}(B(0,R))} < +\infty, \qquad \forall R \in (0,\infty), \tag{2.8.62}$$

where TV stands for the total variation norm of a vector measure. In such a scenario, we shall call $\{\Omega_j\}_{j\in\mathbb{N}}$ a `tame interior approximation` to Ω. The following result, appearing in [125, Proposition 2.7, p. 2583], is a partial extension of (2.8.1):

if the set $\Omega \subseteq \mathbb{R}^n$ has locally finite perimeter and a tame interior approximation, then Gauss–Green's Formula (1.1.8) holds for each $\vec{F} \in \big[\mathscr{C}^0(\overline{\Omega})\big]^n$ with bounded support satisfying $\operatorname{div}\vec{F} \in L^1(\Omega, \mathcal{L}^n)$. (2.8.63)

This should be compared with the following result, which is a slight version of the one given by Federer in [85, p. 314]:

if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set such that $\mathcal{H}^{n-1}(\partial\Omega) < +\infty$ (so that, in particular, Ω has finite perimeter), then for each $j \in \{1, \dots, n\}$ we have $\int_\Omega \partial_j f \, d\mathcal{L}^n = \int_{\partial_*\Omega} \nu_j \, f \, d\mathcal{H}^{n-1}$ where ν_j is the j-th component of the geometric measure theoretic outward unit normal to Ω, and where $f \in \mathscr{C}^0(\overline{\Omega})$ is some scalar-valued function with the property that $\partial_j f$ belongs to the space $L^1(\Omega, \mathcal{L}^n)$. (2.8.64)

In turn, Federer's result described in (2.8.64) yields a version of Gauss–Green's Formula (1.1.8) for vector fields $\vec{F} = (F_1, \dots, F_n) \in \big[\mathscr{C}^0(\overline{\Omega})\big]^n$ with the property that each individual term $\partial_j F_j$ in $\operatorname{div}\vec{F}$ belongs to $L^1(\Omega, \mathcal{L}^n)$. However, the vector fields arising in the applications of Gauss–Green's Formula need not have this additional structure, so the result just mentioned is not effective.

The class of open nonempty proper subsets Ω of $\mathbb{R}^n$ such that

$$\begin{aligned}&\Omega \text{ has locally finite perimeter, satisfies } \partial\Omega = \partial(\overline{\Omega}) \text{, has a tame interior} \\ &\text{approximation, and } \sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega \text{ is a Borel measure which is locally} \\ &\text{finite, complete, and Borel-regular (hence a complete Radon measure),} \\ &\text{with the property that } \sigma(\partial\Omega \setminus \partial_*\Omega) = \sigma(\partial\Omega \setminus \partial^*\Omega) = 0,\end{aligned} \tag{2.8.65}$$

is rather large. For example, it includes the upper-graph

$$\Omega := \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, x_n > \phi(x')\big\} \tag{2.8.66}$$

of any real-valued function

$$\phi \in \mathscr{C}^0(\mathbb{R}^{n-1}) \;\text{ such that }\; \nabla'\phi \in \big[L^1_{\rm loc}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1} \tag{2.8.67}$$

(with ∇' denoting the gradient in $\mathbb{R}^{n-1}$), as well as open sets which locally coincide, up to a rigid transformation of the space, with upper-graphs as in (2.8.66). Moreover, in such a scenario, the geometric measure theoretic outward unit normal of Ω is given by

$$\nu\big(x', \phi(x')\big) = \frac{\big(\nabla'\phi(x'), -1\big)}{\sqrt{1 + |(\nabla'\phi)(x')|^2}} \quad \text{for } \mathcal{L}^{n-1}\text{-a.e. } \; x' \in \mathbb{R}^{n-1}, \tag{2.8.68}$$

and for each $\mathcal{L}^{n-1}$-measurable set $\mathcal{O}' \subseteq \mathbb{R}^{n-1}$ we have

$$\begin{aligned}\sigma\big(\big\{(x', \phi(x')) : \, x' \in \mathcal{O}'\big\}\big) &= \mathcal{H}^{n-1}\big(\big\{(x', \phi(x')) : \, x' \in \mathcal{O}'\big\}\big) \\ &= \int_{\mathcal{O}'} \sqrt{1 + |(\nabla'\phi)(x')|^2}\, dx'.\end{aligned} \tag{2.8.69}$$

All these properties follow from [125, Proposition 2.3, p. 2578], [125, Proposition 2.4, p. 2581], [125, Proposition 2.5, p. 2582], and [125, p. 2583] where it was noted that the family

$$\Omega_j := \big\{(x', x_n) \in \mathbb{R}^n : \, x_n > \phi(x') + j^{-1}\big\}, \qquad \forall\, j \in \mathbb{N}, \tag{2.8.70}$$

is a tame interior approximation to Ω. In particular, as seen from (2.8.63),

$$\begin{aligned}&\text{if } \Omega \text{ is as in (2.8.66)–(2.8.67), Gauss–Green's Formula (1.1.8) holds} \\ &\text{for all vector fields } \vec{F} \in \big[\mathscr{C}^0(\overline{\Omega})\big]^n \text{ with bounded support satisfying} \\ &\operatorname{div}\vec{F} \in L^1(\Omega, \mathcal{L}^n).\end{aligned} \tag{2.8.71}$$

In this vein, let us also remark that if (2.8.67) is strengthened to

$$\phi \in L^1_{\text{loc}}(\mathbb{R}^{n-1}) \text{ with } \nabla'\phi \in \big[\text{BMO}(\mathbb{R}^{n-1})\big]^{n-1}, \tag{2.8.72}$$

then, in fact, the upper-graph Ω in (2.8.66) has an Ahlfors regular boundary (see [125, Proposition 2.25, p. 2616]). Furthermore, from [125, Proposition 3.15, p. 2637] and [132, Proposition 3.6, p. 94] it follows that actually Ω is an NTA domain (cf. Definition 5.11.1). In particular, Ω satisfies a two-sided corkscrew condition (cf. Definition 5.1.3), hence $\partial_*\Omega = \partial\Omega$ in this case (compare with (2.8.65)).

It is natural to attempt to consider the closure of the space of vector fields used in the formulation of the De Giorgi–Federer Divergence Theorem in a norm which takes into account the quantitative aspects of the Divergence Formula recorded in (1.1.8). A result in this spirit is presented next, in Proposition 2.8.9. To facilitate its statement, given an arbitrary closed set $E \subseteq \mathbb{R}^n$, we agree to denote

$$\mathscr{C}^\infty_c(E) := \big\{\phi|_E : \phi \in \mathscr{C}^\infty_c(\mathbb{R}^n)\big\}. \tag{2.8.73}$$

Proposition 2.8.9 *Let $\Omega \subseteq \mathbb{R}^n$ be a set of locally finite perimeter and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider a vector field $\vec{F} : \Omega \to \mathbb{C}^n$ whose components are absolutely integrable in each bounded open subset of Ω and fix some linear functional $\Lambda : \mathscr{C}^\infty_c(\overline{\Omega}) \to \mathbb{C}$. Assume that these are related via the existence of a sequence $\{\vec{\varphi}_j\}_{j\in\mathbb{N}} \subset \big[\mathscr{C}^\infty_c(\overline{\Omega})\big]^n$ satisfying*

$$\sup_{j\in\mathbb{N}} \sup_{x\in\partial_*\Omega} |\vec{\varphi}_j(x)| < +\infty, \tag{2.8.74}$$

$$\lim_{j\to\infty} \int_{\Omega\cap B(0,R)} \big|\vec{\varphi}_j - \vec{F}\big| \,\mathrm{d}\mathcal{L}^n = 0 \ \text{ for each } \ R > 0, \tag{2.8.75}$$

$$\lim_{j\to\infty} \int_{\Omega} \psi \,\mathrm{div}\vec{\varphi}_j \,\mathrm{d}\mathcal{L}^n = \Lambda(\psi) \ \text{ for each } \ \psi \in \mathscr{C}^\infty_c(\overline{\Omega}). \tag{2.8.76}$$

Then there exists a unique function $f \in L^\infty(\partial_*\Omega, \sigma)$ such that

$$\Lambda(\psi) = -\int_\Omega \nabla\psi \cdot \vec{F} \,\mathrm{d}\mathcal{L}^n + \int_{\partial_*\Omega} f\,\psi \,\mathrm{d}\sigma \ \text{ for every } \ \psi \in \mathscr{C}^\infty_c(\overline{\Omega}). \tag{2.8.77}$$

Moreover, f depends linearly on the pair $(\vec{F}, \Lambda)$, and

$$\|f\|_{L^\infty(\partial_*\Omega,\sigma)} \le \sup_{j\in\mathbb{N}} \sup_{\partial_*\Omega} |\vec{\varphi}_j|. \tag{2.8.78}$$

Proof With ν denoting the geometric measure theoretic outward unit normal to Ω, for each function $\psi \in \mathscr{C}^\infty_c(\overline{\Omega})$ we may write

$$\begin{aligned}\Lambda(\psi)+\int_{\Omega}\nabla\psi\cdot\vec{F}\,\mathrm{d}\mathcal{L}^n &= \lim_{j\to\infty}\left\{\int_{\Omega}\psi\,\mathrm{div}\vec{\varphi}_j\,\mathrm{d}\mathcal{L}^n+\int_{\Omega}\nabla\psi\cdot\vec{\varphi}_j\,\mathrm{d}\mathcal{L}^n\right\}\\ &= \lim_{j\to\infty}\int_{\Omega}\mathrm{div}(\psi\,\vec{\varphi}_j)\,\mathrm{d}\mathcal{L}^n\\ &= \lim_{j\to\infty}\int_{\partial_*\Omega}\psi\,\nu\cdot\vec{\varphi}_j\,\mathrm{d}\sigma, \end{aligned}\tag{2.8.79}$$

thanks to assumptions and Theorem 1.1.1. In particular,

$$\Lambda(\psi)+\int_{\Omega}\nabla\psi\cdot\vec{F}\,\mathrm{d}\mathcal{L}^n=0\ \text{ whenever }\ \psi\in\mathscr{C}_c^\infty(\overline{\Omega})\ \text{ has }\ \psi\big|_{\partial_*\Omega}=0.\tag{2.8.80}$$

Let us also observe that Corollary 3.7.3 (whose applicability in the present setting is ensured by hypotheses, (5.2.6), (3.0.3), and (5.6.35)) gives that

$$\mathcal{V}:=\left\{\psi|_{\partial_*\Omega}:\ \psi\in\mathscr{C}_c^\infty(\overline{\Omega})\right\}\ \text{ is dense in }\ L^1(\partial_*\Omega,\sigma).\tag{2.8.81}$$

If we now introduce the functional $L:\mathcal{V}\to\mathbb{C}$ by setting

$$L\big(\psi|_{\partial_*\Omega}\big):=\Lambda(\psi)+\int_{\Omega}\nabla\psi\cdot\vec{F}\,\mathrm{d}\mathcal{L}^n\ \text{ for each }\ \psi\in\mathscr{C}_c^\infty(\overline{\Omega}),\tag{2.8.82}$$

then (2.8.80) implies that this definition is unambiguous, while (2.8.79) shows that

$$L\big(\psi|_{\partial_*\Omega}\big)=\lim_{j\to\infty}\int_{\partial_*\Omega}\psi\,\nu\cdot\vec{\varphi}_j\,\mathrm{d}\sigma\ \text{ for each }\ \psi\in\mathscr{C}_c^\infty(\overline{\Omega}).\tag{2.8.83}$$

Hence, for each $\psi\in\mathscr{C}_c^\infty(\overline{\Omega})$ we may estimate

$$\begin{aligned}\big|L\big(\psi|_{\partial_*\Omega}\big)\big| &\le \limsup_{j\to\infty}\left|\int_{\partial_*\Omega}\psi\,\nu\cdot\vec{\varphi}_j\,\mathrm{d}\sigma\right|\\ &\le \big\|\psi|_{\partial_*\Omega}\big\|_{L^1(\partial_*\Omega,\sigma)}\sup_{j\in\mathbb{N}}\sup_{\partial_*\Omega}|\vec{\varphi}_j|. \end{aligned}\tag{2.8.84}$$

In light of (2.8.74) this goes to show that $L:\mathcal{V}\to\mathbb{C}$ is continuous when $\mathcal{V}$ is equipped with the norm inherited from $L^1(\partial_*\Omega,\sigma)$. In concert with (2.8.81), this further implies that L extends uniquely to a continuous linear functional on $L^1(\partial_*\Omega,\sigma)$. Since $\big(L^1(\partial_*\Omega,\sigma)\big)^*=L^\infty(\partial_*\Omega,\sigma)$ by Riesz' Representation Theorem, it follows that there exists a unique function $f\in L^\infty(\partial_*\Omega,\sigma)$ with the property that (2.8.78) holds and such that

$$L\big(\psi|_{\partial_*\Omega}\big)=\int_{\partial_*\Omega}f\,\psi\,\mathrm{d}\sigma\ \text{ for all }\ \psi\in\mathscr{C}_c^\infty(\overline{\Omega}).\tag{2.8.85}$$

In turn, (2.8.82) and (2.8.85) prove (2.8.77). □

As an application of Proposition 2.8.9, in Corollary 2.8.10 below we present a result of the flavor of [45, Theorem 2, p. 257]. While this result extends (2.8.1), it does not imply (2.8.63), nor any of the results given in the earlier sections of this volume. Further results related to (2.8.1) and (2.8.63) can be found in [46, 205].

Corollary 2.8.10 *Assume $\Omega \subseteq \mathbb{R}^n$ is a bounded set of locally finite perimeter. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, consider an open set $\mathcal{O} \subseteq \mathbb{R}^n$ containing $\overline{\Omega}$ and fix*

$$\vec{F} \in \big[L^\infty_{\rm loc}(\mathcal{O},\mathcal{L}^n)\big]^n \ \ \textit{with} \ \ {\rm div}\vec{F} = u + \mu \ \ \textit{in} \ \ \mathcal{D}'(\mathcal{O}), \tag{2.8.86}$$

where the distribution $u \in \mathcal{D}'(\mathcal{O})$ is compactly supported in $\mathring{\Omega}$, the interior of Ω, and μ is a Borel measure of locally finite total variation in $\mathcal{O}$. Then there exists a unique function $f \in L^\infty(\partial_*\Omega,\sigma)$ such that

$$\begin{gathered} {}_{\mathscr{E}'(\mathring{\Omega})}\langle u,\psi\rangle_{\mathscr{E}(\mathring{\Omega})} + \int_\Omega \psi\,{\rm d}\mu = -\int_\Omega \nabla\psi\cdot\vec{F}\,{\rm d}\mathcal{L}^n + \int_{\partial_*\Omega} f\,\psi\,{\rm d}\sigma \\ \textit{for every function} \ \ \psi \in \mathscr{C}^\infty_c(\overline{\Omega}). \end{gathered} \tag{2.8.87}$$

Moreover, f depends linearly on $\vec{F}$, one has

$$\begin{gathered} \|f\|_{L^\infty(\partial_*\Omega,\sigma)} \leq \lim_{r\to 0^+} \big\|\vec{F}\big\|_{[L^\infty(K_r,\mathcal{L}^n)]^n} \ \ \textit{where} \\ K_r := \{x\in\mathcal{O} :\ {\rm dist}(x,\partial\Omega) \leq r\} \ \ \textit{for each} \ \ r>0, \end{gathered} \tag{2.8.88}$$

and

$$f = \nu\cdot\big(\vec{F}\big|_{\partial_*\Omega}\big) \ \ \textit{if } \vec{F} \textit{ is actually continuous in a neighborhood of } \partial\Omega. \tag{2.8.89}$$

Proof Consider a real-valued, non-negative function $\theta \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ (the space of compactly supported functions from $\mathscr{C}^\infty(\mathbb{R}^n)$) satisfying $\theta \equiv 1$ on $B(0,1)$ as well as $\int_{\mathbb{R}^n}\theta\,{\rm d}\mathcal{L}^n = 1$, then set $\theta_j(x) := j^n\theta(jx)$ for each $j\in\mathbb{N}$ and each $x\in\mathbb{R}^n$. Next, having fixed some compact neighborhood K of $\overline{\Omega}$ contained in $\mathcal{O}$, introduce

$$\vec{G} \in \big[L^\infty_{\rm comp}(\mathbb{R}^n,\mathcal{L}^n)\big]^n \ \ \text{by setting} \ \ \vec{G} := \begin{cases} \vec{F} & \text{in } K, \\ \vec{0} & \text{in } \mathbb{R}^n\setminus K, \end{cases} \tag{2.8.90}$$

then define

$$\vec{\varphi}_j := \big(\vec{G}*\theta_j\big)\Big|_{\overline{\Omega}} \in \big[\mathscr{C}^\infty_c(\overline{\Omega})\big]^n \ \ \text{for each} \ \ j\in\mathbb{N}. \tag{2.8.91}$$

In particular, standard properties of mollifiers ensure that

$$\sup_{j\in\mathbb{N}}\sup_{x\in\overline{\Omega}}|\vec{\varphi}_j(x)| \leq \big\|\vec{F}\big\|_{[L^\infty(K,\mathcal{L}^n)]^n} \ \ \text{for each} \ \ r>0, \tag{2.8.92}$$

and

$$\lim_{j\to\infty}\int_\Omega \big|\vec{\varphi}_j-\vec{F}\big|\,\mathrm{d}\mathcal{L}^n=0. \tag{2.8.93}$$

Moreover, for each $j\in\mathbb{N}$ large enough we may split

$$\mathrm{div}\vec{\varphi}_j=A_j+B_j \ \text{ in } \ \Omega, \tag{2.8.94}$$

where

$$A_j(x):=\int_O \theta_j(x-y)\,\mathrm{d}\mu(y) \ \text{ for each } \ x\in\Omega, \tag{2.8.95}$$

and

$$B_j(x):={}_{\mathscr{E}'(O)}\langle u,\theta_j(x-\cdot)\rangle_{\mathscr{E}(O)} \ \text{ for each } \ x\in\Omega. \tag{2.8.96}$$

As such, if $\psi\in\mathscr{C}_c^\infty(\overline{\Omega})$ and $\widetilde{\psi}$ denotes the extension of $\psi|_\Omega$ by zero to $\mathbb{R}^n$, we may compute

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega \psi\, A_j\,\mathrm{d}\mathcal{L}^n&=\lim_{j\to\infty}\int_\Omega \psi(x)\Big(\int_O\theta_j(x-y)\,\mathrm{d}\mu(y)\Big)\,\mathrm{d}\mathcal{L}^n(x)\\&=\lim_{j\to\infty}\int_O\Big(\int_{\mathbb{R}^n}\widetilde{\psi}(x)\theta_j(x-y)\,\mathrm{d}\mathcal{L}^n(x)\Big)\,\mathrm{d}\mu(y)\\&=\int_O\widetilde{\psi}(y)\,\mathrm{d}\mu(y)=\int_\Omega\psi\,\mathrm{d}\mu,\end{aligned} \tag{2.8.97}$$

based on a simple application of Fubini's theorem (bearing in mind that μ is sigma-finite) and Lebesgue's Dominated Convergence Theorem. Also, if $\Psi\in\mathscr{C}_c^\infty(\mathbb{R}^n)$ is such that $\Psi\big|_{\overline{\Omega}}=\psi$, then keeping in mind that u is compactly supported in $\mathring{\Omega}$, we may compute

$$\begin{aligned}\lim_{j\to\infty}\int_\Omega \psi\, B_j\,\mathrm{d}\mathcal{L}^n&=\lim_{j\to\infty}\int_\Omega \psi(x)\Big({}_{\mathscr{E}'(O)}\langle u,\theta_j(x-\cdot)\rangle_{\mathscr{E}(O)}\Big)\,\mathrm{d}\mathcal{L}^n(x)\\&=\lim_{j\to\infty}{}_{\mathscr{E}'(O)}\Big\langle u\,,\int_\Omega\psi(x)\theta_j(x-\cdot)\,\mathrm{d}\mathcal{L}^n(x)\Big\rangle_{\mathscr{E}(O)}\\&=\lim_{j\to\infty}{}_{\mathscr{E}'(O)}\Big\langle u\,,\int_{\mathbb{R}^n}\Psi(x)\theta_j(x-\cdot)\,\mathrm{d}\mathcal{L}^n(x)\Big\rangle_{\mathscr{E}(O)}\\&={}_{\mathscr{E}'(O)}\big\langle u,\Psi|_O\big\rangle_{\mathscr{E}(O)}={}_{\mathscr{E}'(\mathring{\Omega})}\langle u,\psi\rangle_{\mathscr{E}(\mathring{\Omega})}.\end{aligned} \tag{2.8.98}$$

Hence, if we define the linear functional $\Lambda:\mathscr{C}_c^\infty(\overline{\Omega})\to\mathbb{C}$ by setting

$$\Lambda(\psi):={}_{\mathscr{E}'(\mathring{\Omega})}\langle u,\psi\rangle_{\mathscr{E}(\mathring{\Omega})}+\int_\Omega\psi\,\mathrm{d}\mu \ \text{ for every } \ \psi\in\mathscr{C}_c^\infty(\overline{\Omega}), \tag{2.8.99}$$

it follows from (2.8.92)–(2.8.99) that conditions (2.8.74)–(2.8.76) are satisfied. Granted these, the existence of a unique function $f \in L^\infty(\partial_*\Omega, \sigma)$ satisfying (2.8.87)–(2.8.88) and which depends linearly on $\vec{F}$ follows from Proposition 2.8.9.

Finally, in the case when $\vec{F}$ is also assumed to be continuous in a neighborhood of $\partial\Omega$, from (2.8.90)–(2.8.91) we conclude that

$$\lim_{j\to\infty} \sup_{x\in\partial\Omega} \left|\vec{\varphi}_j(x) - \vec{F}(x)\right| = 0. \tag{2.8.100}$$

Together with (2.8.79) and the current choice of Λ (made in (2.8.99)) this allows us to conclude that

$$\begin{gathered} {}_{\mathscr{E}'(\mathring{\Omega})}\langle u, \psi\rangle_{\mathscr{E}(\mathring{\Omega})} + \int_\Omega \psi \, \mathrm{d}\mu = -\int_\Omega \nabla\psi \cdot \vec{F} \, \mathrm{d}\mathcal{L}^n + \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|_{\partial_*\Omega}\big)\, \psi \, \mathrm{d}\sigma \\ \text{for every function } \ \psi \in \mathscr{C}_c^\infty(\overline{\Omega}). \end{gathered} \tag{2.8.101}$$

Comparing (2.8.101) with (2.8.87) and invoking (3.7.23) from Corollary 3.7.3 ultimately proves that, in this case, we actually have $f = \nu \cdot \big(\vec{F}\big|_{\partial_*\Omega}\big)$ at σ-a.e. point on $\partial_*\Omega$. This finishes the proof of Corollary 2.8.10. □

A version of the Divergence Theorem for differentiable vector fields whose pointwise divergence happens to be a continuous function is described next.

Proposition 2.8.11 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with an Ahlfors regular boundary, and suppose $\vec{F} = (F_j)_{1\leq j\leq n}$ is a vector field satisfying the following properties:*

$$\begin{aligned} &\vec{F} \textit{ is continuous on } \overline{\Omega} \textit{ and differentiable at every point in } \Omega, \\ &\textit{and } \sum_{j=1}^n \partial_j F_j \textit{ is continuous and absolutely integrable on } \Omega, \end{aligned} \tag{2.8.102}$$

where the partial derivatives are considered in a pointwise, classical sense.

Then, if ν denotes the geometric measure theoretic outward unit normal to Ω, there holds

$$\int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|_{\partial_*\Omega}\big)\, \mathrm{d}\mathcal{H}^{n-1} = \int_\Omega \Big(\sum_{j=1}^n \partial_j F_j\Big)\, \mathrm{d}\mathcal{L}^n. \tag{2.8.103}$$

We make a couple of comments regarding the nature of Proposition 2.8.11. First, if $n = 1$, then of course the conditions in (2.8.102) imply $\vec{F} \in \mathscr{C}^1(\Omega)$. However, if $n \geq 2$, there are vector fields satisfying the hypotheses made in (2.8.102) and which are *not* of class $\mathscr{C}^1$ in Ω. An example is as follows. Consider the case when $n = 2$ and take $\vec{F}(x, y) := \big(f(x-y), f(x-y)\big)$ for each $(x, y) \in \mathbb{R}^2$ where $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with the property that f' is *not* continuous (for example, $f(t) := t^2 \sin(1/t)$ if $t \in \mathbb{R} \setminus \{0\}$ and $f(0) := 0$ will do). Then $\vec{F} = (F_1, F_2)$ is differentiable in $\mathbb{R}^2$, its pointwise divergence $\partial_x F_1 + \partial_y F_2$ is zero at each point in $\mathbb{R}^2$, yet $\vec{F}$ fails to be of class $\mathscr{C}^1$ in any neighborhood of the origin in $\mathbb{R}^2$.

Second, the counterexample in (2.5.13) shows that weakening the assumption in (2.8.102) to asking that the vector field $\vec{F} \in \big[\mathscr{C}^0(\overline{\Omega})\big]^n$ is only differentiable at $\mathcal{L}^n$-a.e. point in Ω and whose divergence, computed in a pointwise sense, belongs to $L^1(\Omega, \mathcal{L}^n)$ no longer guarantees the validity of the Divergence Formula (2.8.103) (even when Ω is very nice, say a bounded Lipschitz domain in $\mathbb{R}^n$).

Finally, we wish to note that variants of Proposition 2.8.11 may be found in [142] where a notion of absolute continuity for differential forms (hence also for vector fields) was introduced and used to formulate suitable versions of Stokes' theorem (both in the Euclidean setting and on manifolds).

After this preamble, we are ready to present the proof of Proposition 2.8.11.

Proof of Proposition 2.8.11 Fix $\kappa > 0$. Since the current hypotheses imply that $\vec{F}$ is bounded in Ω, we have $\mathcal{N}_\kappa \vec{F} \in L^\infty(\partial\Omega, \mathcal{H}^{n-1}) \subseteq L^1(\partial\Omega, \mathcal{H}^{n-1})$. Also, the nontangential trace $\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists and matches the ordinary restriction $\vec{F}\big|_{\partial\Omega}$ at $\mathcal{H}^{n-1}$-a.e. point on $\partial_*\Omega$, thanks to item *(iii)* in Proposition 8.8.6. In addition, Proposition 4.4.2 (presently used with $\mathcal{O} := \Omega$) guarantees that $\operatorname{div}\vec{F}$, considered in sense of distributions in Ω, is equal to the pointwise divergence $\sum_{j=1}^n \partial_j F_j$ which, in turn, is assumed to belong to $L^1(\Omega, \mathcal{L}^n)$. Granted these properties, Theorem 1.2.1 applies and yields (2.8.103). □

We continue by giving the formal definition of the category of Lipschitz domains, as well as Lyapunov domains of order α (or, domains of class $\mathscr{C}^{1,\alpha}$), where $\alpha \in (0, 1]$.

Definition 2.8.12 *Let Ω be a nonempty, proper, open subset of $\mathbb{R}^n$. Also, fix $x_0 \in \partial\Omega$. Call Ω a* `Lipschitz domain near` x_0 *if there exist two finite parameters $r, c > 0$ with the following significance. There exist an $(n-1)$-dimensional plane $H \subseteq \mathbb{R}^n$ passing through the point x_0, a choice N of the unit normal to H, and an open cylinder*

$$C_{r,c} := C(x_0, H, N, r, c) := \big\{x' + tN : x' \in H,\ |x' - x_0| < r,\ \ |t| < c\big\} \tag{2.8.104}$$

(*called coordinate cylinder near x_0*) *such that*

$$C_{r,c} \cap \Omega = C_{r,c} \cap \{x' + tN : x' \in H \ \text{ and } \ t > \varphi(x')\}, \tag{2.8.105}$$

for some Lipschitz function $\varphi : H \to \mathbb{R}$, called the defining function for $\partial\Omega$ near x_0, satisfying

$$\varphi(x_0) = 0 \ \text{ and } \ |\varphi(x')| < c \ \text{ if } \ |x' - x_0| \leq r. \tag{2.8.106}$$

Collectively, the pair $(C_{r,c}, \varphi)$ will be referred to as a local chart near x_0, whose geometrical characteristics consist of r, c and the Lipschitz constant of φ.

Moreover, call Ω a `locally Lipschitz domain` *if it is a Lipschitz domain near every point $x \in \partial\Omega$. Finally, Ω is simply called a* `Lipschitz domain` *if it is locally Lipschitz and such that the geometrical characteristics of the local charts associated with each boundary point* (*making up what occasionally is referred to as the Lipschitz character of the domain*) *are independent of the point in question. In this scenario, call a family of local charts covering $\partial\Omega$ an* `atlas`.

The categories of $\mathscr{C}^{1,\alpha}$ *domains with* $\alpha \in (0, 1]$ (*occasionally referred to as Lyapunov domains of order* α), *as well as their local versions, are defined analogously, requiring that the defining functions* φ *have first-order directional derivatives* (*along vectors parallel to the hyperplane* H) *which are of class* $\mathscr{C}^{\alpha}$ (*the Hölder space of order* α).

A few useful observations related to the property of an open subset of $\mathbb{R}^n$ being a Lipschitz domain near one of its boundary points are collected below. The reader is reminded that the superscript c is the operation of taking the complement of a set, relative to the ambient $\mathbb{R}^n$.

Lemma 2.8.13 *Assume that* Ω *is a nonempty, proper, open subset of* $\mathbb{R}^n$, *and fix* $x_0 \in \partial\Omega$.

(i) If Ω *is a Lipschitz domain near* x_0 *and if* $(C_{r,c}, \varphi)$ *is a local chart near* x_0 (*in the sense of Definition 2.8.12*) *then, in addition to* (2.8.105), *one also has*

$$C_{r,c} \cap \partial\Omega = C_{r,c} \cap \{x' + tN : x' \in H,\ t = \varphi(x')\}, \tag{2.8.107}$$

$$C_{r,c} \cap (\overline{\Omega})^c = C_{r,c} \cap \{x' + tN : x' \in H,\ t < \varphi(x')\}. \tag{2.8.108}$$

Furthermore,

$$C_{r,c} \cap \overline{\Omega} = C_{r,c} \cap \{x' + tN : x' \in H,\ t \geq \varphi(x')\}, \tag{2.8.109}$$

$$C_{r,c} \cap \mathring{\overline{\Omega}} = C_{r,c} \cap \{x' + tN : x' \in H,\ t > \varphi(x')\}, \tag{2.8.110}$$

and, consequently,

$$E \cap \partial\Omega = E \cap \partial(\overline{\Omega}), \qquad \forall E \subseteq C_{r,c}. \tag{2.8.111}$$

(ii) Assume that there exist an $(n-1)$*-dimensional plane* $H \subseteq \mathbb{R}^n$ *passing through* x_0, *a choice* N *of the unit normal to* H, *an open cylinder*

$$C_{r,c} := \{x' + tN : x' \in H,\ |x' - x_0| < r,\ |t| < c\}, \tag{2.8.112}$$

and a Lipschitz function $\varphi : H \to \mathbb{R}$ *satisfying* (2.8.106) *such that* (2.8.107) *holds. Then, if* $x_0 \notin \mathring{\overline{\Omega}}$, *it follows that* Ω *is a Lipschitz domain near* x_0.

See [9, Proposition 2.8] for a proof. From Definition 2.8.12 and Lemma 2.8.13 it follows that[5]

[5] Recall that the strict epigraph or strict supergraph of a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is the set of points lying strictly above its graph, i.e., Sepigraph $f := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > f(x')\}$.

if Ω is a Lipschitz domain in $\mathbb{R}^n$ with compact boundary then $\partial\Omega$ is an Ahlfors regular set, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling Borel-regular measure, $\partial_*\Omega = \partial\Omega$, and $\partial(\overline{\Omega}) = \partial\Omega$; moreover, in any local chart where Ω may be locally identified with the strict epigraph of some Lipschitz functions $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, formulas (2.8.68)–(2.8.69), detailing the nature of the surface measure σ and the outward unit normal vector ν to Ω, are valid. (2.8.113)

Here is a utilitarian version of the Divergence Theorem in bounded Lipschitz domains.

Proposition 2.8.14 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Denote by ν its outward unit normal and by σ its surface measure. Suppose $\vec{F} \in \big[\mathscr{C}^1(\Omega)\big]^n$ is a vector field which, for some aperture parameter $\kappa \in (0, \infty)$, satisfies*

$$\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma), \quad \vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists } \sigma\text{-a.e. on } \partial\Omega, \\ \text{and } \operatorname{div}\vec{F} \text{ belongs to the space } L^1(\Omega, \mathcal{L}^n). \tag{2.8.114}$$

Then

$$\int_\Omega \operatorname{div}\vec{F}\, d\mathcal{L}^n = \int_{\partial\Omega} \nu \cdot \Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)\, d\sigma. \tag{2.8.115}$$

While, in view of (2.8.113), Proposition 2.8.14 is a special case of Theorem 1.2.1, its perceived versatility and usefulness in applications prompted us to single it out. There is also a version of Proposition 2.8.14 involving the strict epigraph of a real-valued Lipschitz function, which is an unbounded Lipschitz domain. Specifically, Theorem 1.2.1 implies the following result:

Suppose $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a given Lipschitz function and consider the set $\Omega := \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \phi(x')\big\}$. Then, the Divergence Formula (2.8.115) holds for each vector field $\vec{F} \in \big[\mathscr{C}^1(\Omega)\big]^n$ satisfying the conditions in (2.8.114) for some aperture parameter $\kappa \in (0, \infty)$. (2.8.116)

Moreover, according to [194, Proposition 2.2, p. 25] we may replace the nontangential approach regions $\Gamma_\kappa(x)$, with $x \in \partial\Omega$, used in the definition of the nontangential maximal operator $\mathcal{N}_\kappa$ (cf. (8.2.1)) by any family of the form $x + \Gamma$, with $x \in \partial\Omega$, where Γ is a fixed genuine open, one-component, circular cone, whose symmetry axis is in the vertical direction and whose aperture is sufficiently small (depending on the Lipschitz constant of ϕ). We also wish to note that Theorem 1.2.1 implies (bearing in mind (5.9.21)–(5.9.22)) that

the result stated in (2.8.116) continues to hold if ϕ is assumed to belong to the larger category of BMO_1 functions, i.e., assuming that $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})$ has distributional first-order partial derivatives $\partial_j\phi$, with $j \in \{1, \dots, n-1\}$, belonging to the space $\mathrm{BMO}(\mathbb{R}^{n-1})$. (2.8.117)

Finally, we remark that the version of the Divergence Theorem from (2.8.117) may be adapted to the class of bounded BMO_1-domains, i.e., nonempty open bounded subsets of $\mathbb{R}^n$ which may be locally described (up to a rigid transformation) as strict epigraphs of real-valued BMO_1 functions defined in $\mathbb{R}^{n-1}$.

Moving on, we shall give Proposition 2.8.14 two proofs, the first of which is the reasonably self-contained argument presented below.

Proof of Proposition 2.8.14 For starters, the fact that $\vec{F}$ has an absolutely integrable nontangential maximal function implies that $\vec{F} \in \big[L^1_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^n$ (cf. (8.6.50)). Bearing this in mind and using a smooth, finite, partition of unity, matters may be localized to the case when Ω is the strict epigraph of some Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ and $\vec{F} \in \big[\mathscr{C}^1(\Omega)\big]^n$ is as in (2.8.114) with the additional property that it vanishes identically outside of a bounded subset of Ω. Fix an arbitrary $\varepsilon > 0$ and define

$$\vec{F}^\varepsilon := \vec{F}(\cdot + \varepsilon \mathbf{e}_n) \in \big[\mathscr{C}^1(\overline{\Omega})\big]^n. \tag{2.8.118}$$

Next, we temporarily digress and make the claim that, if Ω is the strict epigraph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, then for each scalar-valued $f \in \mathscr{C}^1(\overline{\Omega})$ which vanishes outside of a compact subset of $\overline{\Omega}$ we have

$$\int_\Omega \xi \cdot \nabla f \, \mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} (\nu \cdot \xi) f \, \mathrm{d}\sigma \quad \text{each vector } \ \xi \in \mathbb{R}^n. \tag{2.8.119}$$

In the proof of this claim we follow [251, Proposition 1.2, p. 310], with some additional clarifications. Consider first the case when $\xi = \mathbf{e}_n$, in which scenario we may write

$$\begin{aligned} \int_\Omega \mathbf{e}_n \cdot \nabla f \, \mathrm{d}\mathcal{L}^n &= \int_\Omega \partial_n f \, \mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^{n-1}} \Big(\int_{\varphi(x')}^\infty (\partial_n f)(x', x_n) \, \mathrm{d}x_n \Big) \, \mathrm{d}x' \\ &= - \int_{\mathbb{R}^{n-1}} f\big(x', \varphi(x')\big) \, \mathrm{d}x' = \int_{\partial\Omega} \nu_n f \, \mathrm{d}\sigma \\ &= \int_{\partial\Omega} (\nu \cdot \mathbf{e}_n) f \, \mathrm{d}\sigma. \end{aligned} \tag{2.8.120}$$

Above, the second equality is a consequence of Fubini's theorem, the third equality is implied by the Fundamental Theorem of Calculus (applied in the variable x_n, keeping in mind that, for each x' fixed, $f(x', x_n)$ vanishes for x_n sufficiently large), and the fourth equality is a consequence of (2.8.66)–(2.8.69) (cf. also Proposition 5.6.17). Allowing other vectors $\xi \in \mathbb{R}^n$ is done as follows. According to (the proof of) Corollary 5.6.23, having Ω the strict epigraph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$ is a quality preserved in any other system of coordinates obtained by a rotation in $\mathbb{R}^n$ sufficiently close to the identity. As such, the argument in (2.8.120) (carried out in this new, rotated coordinate system of axes, in place of the "standard" one) shows that the integral identity

$$\int_{\Omega} \mathbf{e}_n \cdot \nabla f \, d\mathcal{L}^n = \int_{\partial\Omega} (\nu \cdot \mathbf{e}_n) f \, d\sigma \tag{2.8.121}$$

remains valid if $\mathbf{e}_n$ is replaced by $\mathscr{R}\mathbf{e}_n$ where $\mathscr{R}$ is any rotation in $\mathbb{R}^n$ sufficiently close to the identity. Hence, there exists some small $\delta > 0$ with the property that for each $j \in \{1, \ldots, n\}$ we have

$$\int_{\Omega} (\mathbf{e}_n + \delta \mathbf{e}_j) \cdot \nabla f \, d\mathcal{L}^n = \int_{\partial\Omega} \nu \cdot (\mathbf{e}_n + \delta \mathbf{e}_j) f \, d\sigma. \tag{2.8.122}$$

Subtracting (2.8.121) from (2.8.122) then dividing by δ proves the integral identity in (2.8.119) for each vector $\xi \in \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. By linearity, this establishes (2.8.119) as stated.

Returning to the mainstream discussion, writing formula (2.8.119) for $\xi := \mathbf{e}_j$ and $f := F_j^{\varepsilon}$, the j-th component of the vector field from (2.8.118), then summing up over $j \in \{1, \ldots, n\}$ leads to the conclusion that

$$\int_{\Omega} \operatorname{div} \vec{F}^{\varepsilon} \, d\mathcal{L}^n = \int_{\partial\Omega} \nu \cdot \vec{F}^{\varepsilon} \, d\sigma. \tag{2.8.123}$$

Note that, thanks to the first two hypotheses in (2.8.114) and Lebesgue's Dominated Convergence Theorem,

$$\vec{F}^{\varepsilon}\big|_{\partial\Omega} \longrightarrow \vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \text{ in } \ L^1(\partial\Omega, \sigma) \ \text{ as } \ \varepsilon \to 0^+. \tag{2.8.124}$$

Also,

$$\begin{aligned} \int_{\Omega} \operatorname{div} \vec{F}^{\varepsilon} \, d\mathcal{L}^n &= \int_{\Omega} (\operatorname{div} \vec{F})(\cdot + \varepsilon \mathbf{e}_n) \, d\mathcal{L}^n = \int_{\Omega + \varepsilon \mathbf{e}_n} \operatorname{div} \vec{F} \, d\mathcal{L}^n \\ &= \int_{\Omega} \mathbf{1}_{\Omega + \varepsilon \mathbf{e}_n} \operatorname{div} \vec{F} \, d\mathcal{L}^n \longrightarrow \int_{\Omega} \operatorname{div} \vec{F} \, d\mathcal{L}^n \ \text{ as } \ \varepsilon \to 0^+, \end{aligned} \tag{2.8.125}$$

by the last hypothesis in (2.8.114) and Lebesgue's Dominated Convergence Theorem. At this stage, (2.8.115) follows from (2.8.123)–(2.8.125).

In relation to the proof of Proposition 2.8.14 it is worth pointing out that we could have established (2.8.123) directly, albeit via an argument that is a little less elementary. Concretely, use the bi-Lipschitz change of variables

$$\mathbb{R}^{n-1} \times (0, t) \ni (x', t) \longmapsto \big(x', \varphi(x') + t\big) \in \Omega \tag{2.8.126}$$

whose Jacobian is 1 almost everywhere (cf., e.g., [80, Theorem 2, p. 99]), then rely on Fubini's theorem to write

$$\int_\Omega \operatorname{div}\vec{F}^\varepsilon \,\mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^{n-1}}\int_0^\infty \sum_{j=1}^n (\partial_j F_j^\varepsilon)\big(x',\varphi(x')+t\big)\,\mathrm{d}t\,\mathrm{d}x'. \tag{2.8.127}$$

Observe that, if $1 \le j \le n-1$, then for each $t \in (0,\infty)$ and $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned}(\partial_j F_j^\varepsilon)\big(x',\varphi(x')+t\big) =&\partial_j\big[F_j^\varepsilon\big(x',\varphi(x')+t\big)\big]\\ &-\frac{\mathrm{d}}{\mathrm{d}t}\big[(\partial_j\varphi)(x')F_j^\varepsilon\big(x',\varphi(x')+t\big)\big]\end{aligned} \tag{2.8.128}$$

and, corresponding to $j=n$,

$$(\partial_n F_n^\varepsilon)\big(x',\varphi(x')+t\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big[F_n^\varepsilon\big(x',\varphi(x')+t\big)\big]. \tag{2.8.129}$$

Since for each fixed $t \in (0,\infty)$ the function $\partial_j\big[F_j^\varepsilon\big(x',\varphi(x')+t\big)\big]$ integrates to zero in the variable $x' \in \mathbb{R}^{n-1}$ given that F_j^ε has compact support, from (2.8.127)–(2.8.129) and the Fundamental Theorem of Calculus we therefore obtain

$$\begin{aligned}\int_\Omega \operatorname{div}\vec{F}^\varepsilon\,\mathrm{d}\mathcal{L}^n &= \int_{\mathbb{R}^{n-1}}\int_0^\infty \sum_{j=1}^n (\partial_j F_j^\varepsilon)\big(x',\varphi(x')+t\big)\,\mathrm{d}t\,\mathrm{d}x'\\ &= \int_{\mathbb{R}^{n-1}} \Big\{\Big(\sum_{j=1}^{n-1}(\partial_j\varphi)(x')F_j^\varepsilon\big(x',\varphi(x')\big)\Big) - F_n^\varepsilon\big(x',\varphi(x')\big)\Big\}\,\mathrm{d}x'\\ &= \int_{\mathbb{R}^{n-1}} \frac{\big((\nabla'\varphi)(x'),-1\big)}{\sqrt{\big|(\nabla'\varphi)(x')\big|^2+1}}\cdot \vec{F}^\varepsilon\big(x',\varphi(x')\big)\sqrt{\big|(\nabla'\varphi)(x')\big|^2+1}\,\mathrm{d}x'\\ &= \int_{\partial\Omega} \nu\cdot\vec{F}^\varepsilon\,\mathrm{d}\sigma,\end{aligned} \tag{2.8.130}$$

again, bearing in mind (2.8.66)–(2.8.69). This proves (2.8.123).

There is yet another proof of Proposition 2.8.14 in which the idea is to approximate Ω by smooth subdomains $\Omega_j \nearrow \Omega$ in an appropriate sense as $j\to\infty$, then pass to the limit in the Divergence Formula written for $\vec{F}$ restricted to each Ω_j to obtain the desired Divergence Formula in the original domain Ω. Such an approximation theorem, of wider interest, is proved in [183]. To state it, we agree to let $\operatorname{Dist}[E,F]$ denote the Pompeiu–Hausdorff distance[6] between arbitrary nonempty subsets E, F

[6] What we here call the Pompeiu–Hausdorff distance has been typically referred to in the literature simply as the Hausdorff distance. For historical accuracy, it is significant to note that D. Pompeiu was the first to introduce (a slight version of) this concept in his thesis (written under the supervision of H. Poincaré). Pompeiu's thesis has appeared in print in [219], published in 1905, where Pompeiu calls this notion *écart (mutuel)* between two sets. Subsequently, F. Hausdorff has revisited this topic

of $\mathbb{R}^n$, defined as

$$\mathrm{Dist}\,[E,F] := \max\Big\{\sup_{x\in E}\inf_{y\in F}|x-y|, \sup_{y\in F}\inf_{x\in E}|x-y|\Big\}. \tag{2.8.131}$$

Theorem 2.8.15 *Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain with compact boundary. Then there exist two sequences of open subsets of $\mathbb{R}^n$, denoted by $\{\Omega_j^\pm\}_{j\in\mathbb{N}}$, which satisfy the following properties.*

(1) For each $j \in \mathbb{N}$, the sets $\Omega_j^\pm$ are $\mathscr{C}^{1,1}$ domains with compact boundaries, and

$$\overline{\Omega_j^+} \subseteq \Omega_{j+1}^+ \subseteq \Omega \subseteq \overline{\Omega} \subseteq \Omega_{j+1}^- \subseteq \overline{\Omega_j^-} \quad \textit{for all} \quad j \in \mathbb{N}, \tag{2.8.132}$$

$$\lim_{j\to\infty}\mathrm{Dist}\big[\partial\Omega, \partial(\Omega_j^+)\big] = 0 = \lim_{j\to\infty}\mathrm{Dist}\big[\partial\Omega, \partial(\Omega_j^-)\big]. \tag{2.8.133}$$

(2) For each $j \in \mathbb{N}$, both Ω_j^+ and Ω_j^- are Lipschitz domains and the Lipschitz character of $\Omega_j^\pm$ is controlled by that of Ω independently of $j \in \mathbb{N}$.

(3) There exist a vector field $\vec{h} \in \big[\mathscr{C}^\infty(\mathbb{R}^n)\big]^n$ and a constant $c > 0$ such that the following transversality conditions hold for every $j \in \mathbb{N}$:

$$\vec{h}\cdot\nu_j^\pm \geq c \quad \textit{on} \quad \partial(\Omega_j^\pm) \quad \textit{and} \quad \vec{h}\cdot\nu \geq c \quad \textit{at } \mathcal{H}^{n-1}\textit{-a.e. point on} \quad \partial\Omega, \tag{2.8.134}$$

where ν and $\nu_j^\pm$ are, respectively, the outward unit normals to Ω and $\Omega_j^\pm$.

(4) There exists a covering of $\partial\Omega$ with finitely many local coordinate cylinders (cf. Definition 2.8.12), say,

$$\partial\Omega \subseteq \bigcup_{1\leq k\leq K} C_k, \qquad C_k = C(x_k, H_k, N_k, r_k, c_k), \tag{2.8.135}$$

which, along with their concentric doubles, are also local coordinate cylinders for $\partial(\Omega_j^\pm)$ for each $j \in \mathbb{N}$. Also, if for $1 \leq k \leq K$ one denotes the middle cross-section of the cylinder C_k by $\Delta_{C_k} := \{x' \in H_k : |x' - x_k| < r_k\}$, and if

$$\varphi_k : H_k \longrightarrow \mathbb{R}, \qquad \varphi_{k,j}^\pm : H_k \longrightarrow \mathbb{R}, \quad j \in \mathbb{N}, \tag{2.8.136}$$

denote the Lipschitz functions whose graphs inside C_k coincide, respectively, with $\partial\Omega$ and $\partial(\Omega_j^\pm)$, then

$$\sup_{j\in\mathbb{N}}\Big(\max_{1\leq k\leq K}\|\nabla\varphi_{k,j}^\pm\|_{L^\infty(H_k,\mathcal{H}^{n-1})}\Big) < +\infty, \tag{2.8.137}$$

and, for each $k \in \{1,\dots,K\}$,

---

in 1914, and on p. 463 of his book [117] he correctly attributes the introduction of this notion to Pompeiu.

$$\nabla\varphi^{\pm}_{k,j} \longrightarrow \nabla\varphi_k \ \text{ at } \mathcal{H}^{n-1}\text{-a.e. point in } \Delta_{C_k} \text{ as } j \to \infty. \tag{2.8.138}$$

(5) For each $j \in \mathbb{N}$ there exist bi-Lipschitz homeomorphisms (with constants independent of j)

$$\Lambda^{\pm}_j : \partial\Omega \longrightarrow \partial(\Omega^{\pm}_j) \tag{2.8.139}$$

with the property that there exists an aperture parameter $\kappa > 0$ such that

$$\Lambda^{\pm}_j(x) \in \Gamma_\kappa(x) \ \text{ for each } j \in \mathbb{N}, \text{ and each } x \in \partial\Omega, \tag{2.8.140}$$

$$\Lambda^{\pm}_j(x) \to x \text{ as } j \to \infty, \text{ for each fixed point } x \in \partial\Omega, \tag{2.8.141}$$

$$\nu^{\pm}_j \circ \Lambda^{\pm}_j \to \nu \text{ at } \mathcal{H}^{n-1}\text{-a.e. point on } \partial\Omega \text{ as } j \to \infty. \tag{2.8.142}$$

Furthermore, there exist two constants, $C_0 \in (0,1)$ and $C_1 \in (1,\infty)$, along with a sequence of $\mathcal{H}^{n-1}$-measurable functions $\omega^{\pm}_j : \partial\Omega \to [C_0, C_1]$ for $j \in \mathbb{N}$, such that

$$\lim_{j\to\infty} \omega^{\pm}_j(x) = 1 \ \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega, \tag{2.8.143}$$

and with the property that for each $j \in \mathbb{N}$ and each $f \in L^1\big(\partial(\Omega^{\pm}_j), \mathcal{H}^{n-1}\big)$ the following change of variable formula holds:

$$\int_{\partial(\Omega^{\pm}_j)} f \, d\mathcal{H}^{n-1} = \int_{\partial\Omega} f \circ \Lambda^{\pm}_j \, \omega^{\pm}_j \, d\mathcal{H}^{n-1}. \tag{2.8.144}$$

(6) If, in addition, Ω satisfies a uniform exterior ball condition with radius $R > 0$, then there exists a (typically small) constant $c > 0$, which depends only on the Lipschitz character of Ω, and which has the property that for each $j \in \mathbb{N}$ both Ω^+_j and Ω^-_j satisfy a uniform exterior ball condition with radius $c \cdot R$.
Furthermore, in this scenario, there exists a constant $C \in \mathbb{R}$ which depends only on the Lipschitz character of Ω with the property that if $W^{\pm}_j$ are the Weingarten matrices (i.e., second fundamental forms) of $\partial(\Omega^{\pm}_j)$ then, for every $j \in \mathbb{N}$,

$$W^{\pm}_j \geq C/R \ \text{ at } \mathcal{H}^{n-1}\text{-a.e. point on } \ \partial(\Omega^{\pm}_j). \tag{2.8.145}$$

In particular, if $\mathcal{G}^{\pm}_j$ are the mean curvatures of $\partial(\Omega^{\pm}_j)$ then, for every $j \in \mathbb{N}$,

$$\mathcal{G}^{\pm}_j \geq C/R \ \text{ at } \mathcal{H}^{n-1}\text{-a.e. point on } \ \partial(\Omega^{\pm}_j). \tag{2.8.146}$$

(7) If Ω is a convex set then both Ω^+_j and Ω^-_j are also convex sets for each $j \in \mathbb{N}$.

Approximation results similar in spirit, of various degrees of generality and inclusiveness, have been previously proved by a number of authors, including J. Nečas [210], P. Grisvard [110], A.P. Calderón [37], G. Verchota [257], C. Kenig and T. Toro [150], V. Adofsson [3], S. Hofmann, M. Mitrea, and M. Taylor [124].

Here is how Theorem 2.8.15 may be used to produce an alternative proof of Proposition 2.8.14.

Second Proof of Proposition 2.8.14 Consider a sequence $\{\Omega_j^+\}_{j\in\mathbb{N}}$ of bounded $\mathscr{C}^{1,1}$ subdomains of Ω, which exhaust Ω in the manner described in Theorem 2.8.15. Given that for each fixed $j \in \mathbb{N}$ we have $\vec{F}\big|_{\overline{\Omega_j^+}} \in \big[\mathscr{C}^1(\overline{\Omega_j^+})\big]^n$, we may rely on the classical Divergence Theorem (for smooth vector fields in smooth bounded domains) to write

$$\int_{\Omega_j^+} \operatorname{div}\vec{F}\, d\mathcal{L}^n = \int_{\partial(\Omega_j^+)} \nu_j^+ \cdot \big(\vec{F}\big|_{\partial\Omega_j^+}\big)\, d\mathcal{H}^{n-1}, \tag{2.8.147}$$

where ν_j^+ is the outward unit normal to Ω_j^+. In addition, for each fixed $j \in \mathbb{N}$ the change of variable formula (2.8.144) gives

$$\int_{\partial(\Omega_j^+)} \nu_j^+ \cdot \big(\vec{F}\big|_{\partial\Omega_j^+}\big)\, d\mathcal{H}^{n-1} = \int_{\partial\Omega} (\nu_j^+ \circ \Lambda_j^+) \cdot \big(\vec{F} \circ \Lambda_j^+\big)\, \omega_j^+\, d\mathcal{H}^{n-1}, \tag{2.8.148}$$

where the bi-Lipschitz homeomorphism $\Lambda_j^+ : \partial\Omega \to \partial(\Omega_j^+)$ and the Jacobian ω_j^+ are as in item *(5)* of Theorem 2.8.15. In particular, there exists $\kappa_o \in (0,\infty)$ such that

$$\Lambda_j^+(x) \in \Gamma_{\kappa_o}(x) \quad \text{for each } j \in \mathbb{N}, \text{ and each } x \in \partial\Omega, \tag{2.8.149}$$

$$\Lambda_j^+(x) \longrightarrow x \text{ as } j \to \infty, \text{ for each fixed point } x \in \partial\Omega, \tag{2.8.150}$$

$$\nu_j^+ \circ \Lambda_j^+ \longrightarrow \nu \text{ at } \mathcal{H}^{n-1}\text{-a.e. point on } \partial\Omega \text{ as } j \to \infty, \tag{2.8.151}$$

$$\text{there exists } C \in (0,\infty) \textit{with} \sup_{j\in\mathbb{N}} \|\omega_j^+\|_{L^\infty(\partial\Omega,\sigma)} \le C, \tag{2.8.152}$$

$$\text{and } \lim_{j\to\infty} \omega_j^+(x) = 1 \text{ for } \mathcal{H}^{n-1}\text{-a.e. point } x \in \partial\Omega. \tag{2.8.153}$$

The properties recorded in (2.8.149)–(2.8.150) imply that

$$\begin{gathered}\big(\vec{F} \circ \Lambda_j^+\big)(x) \longrightarrow \big(\vec{F}\big|^{\kappa_o-\text{n.t.}}_{\partial\Omega}\big)(x) \quad \text{as} \quad j \to \infty, \\ \text{for each } x \in \partial\Omega \text{ where the nontangential limit exists,}\end{gathered} \tag{2.8.154}$$

and

$$\big|\vec{F} \circ \Lambda_j^+\big| \le \mathcal{N}_{\kappa_o}\vec{F} \quad \text{on} \quad \partial\Omega, \quad \text{for each} \quad j \in \mathbb{N}. \tag{2.8.155}$$

From (2.8.154), (2.8.155), (2.8.114), Propositions 8.4.1 and 8.9.8 we then conclude that

$$\vec{F} \circ \Lambda_j^+ \longrightarrow \vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \quad \text{in} \quad L^1(\partial\Omega,\sigma) \quad \text{as} \quad j \to \infty. \tag{2.8.156}$$

At this stage we may pass to the limit $j \to \infty$ in (2.8.147) and, on account of (2.8.148), (2.8.151), (2.8.156), (2.8.152), (2.8.153), the fact that $\operatorname{div}\vec{F} \in L^1(\Omega, \mathcal{L}^n)$, and Lebesgue's Dominated Convergence Theorem, conclude that (2.8.115) holds. □

Moving on, let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary open set, and pick a sequence $\{K_j\}_{j\in\mathbb{N}}$ of compact subsets of Ω such that $K_j \nearrow \Omega$. Then, as is well known,

$$d(f,g) := \sum_{j=1}^{\infty} \frac{\|f-g\|_{L^1(K_j,\mathcal{L}^n)}}{1+\|f-g\|_{L^1(K_j,\mathcal{L}^n)}}, \qquad \forall\, f,g \in L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n), \tag{2.8.157}$$

is a distance, and

$$\big(L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n),\mathrm{d}\big) \text{ is a complete metric space.} \tag{2.8.158}$$

Proposition 2.8.16 *Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set of locally finite perimeter. Denote by ν the geometric measure theoretic outward unit normal to Ω and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, assume $|||\cdot|||$ is a norm on the space*

$$\big[\mathscr{C}^1_c(\mathbb{R}^n)\big|_{\Omega}\big]^n := \Big\{\vec{G}\big|_{\Omega} : \vec{G} \in \big[\mathscr{C}^1_c(\mathbb{R}^n)\big]^n\Big\} \tag{2.8.159}$$

satisfying the properties:

(i) the canonical embedding $\Big(\big[\mathscr{C}^1_c(\mathbb{R}^n)\big|_{\Omega}\big]^n, |||\cdot|||\Big) \hookrightarrow \big[L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n)\big]^n$, where the latter space is equipped with the topology induced by the distance function (2.8.157)*, is continuous;*

(ii) there exists a constant $C \in (0,\infty)$ such that, for every $\vec{F} \in \big[\mathscr{C}^1_c(\mathbb{R}^n)\big|_{\Omega}\big]^n$,

$$\|\mathrm{div}\vec{F}\|_{L^1(\Omega,\mathcal{L}^n)} + \big\|\vec{F}\big|_{\partial_*\Omega}\big\|_{[L^1(\partial_*\Omega,\sigma)]^n} \leq C|||\vec{F}|||. \tag{2.8.160}$$

Define

$$\mathcal{V}(\Omega) := \text{ the closure of } \big[\mathscr{C}^1_c(\mathbb{R}^n)\big|_{\Omega}\big]^n \text{ in } |||\cdot|||. \tag{2.8.161}$$

Then the following conclusions are true:

(a) $\mathcal{V}(\Omega) \hookrightarrow \big[L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n)\big]^n$ continuously;

(b) if $\vec{F} \in \mathcal{V}(\Omega)$ then $\mathrm{div}\vec{F}$, taken in the sense of distributions in $\mathcal{D}'(\Omega)$ belongs to $L^1(\Omega,\mathcal{L}^n)$;

(c) there exists a linear and bounded mapping

$$\mathrm{Tr} : \mathcal{V}(\Omega) \longrightarrow \big[L^1(\partial_*\Omega,\sigma)\big]^n \tag{2.8.162}$$

such that

$$\int_{\Omega} \mathrm{div}\vec{F}\,\mathrm{d}\mathcal{L}^n = \int_{\partial_*\Omega} \nu\cdot\mathrm{Tr}\,\vec{F}\,\mathrm{d}\sigma \quad \text{for every } \vec{F} \in \mathcal{V}(\Omega). \tag{2.8.163}$$

Proof We begin by noting that the continuity of the embedding from item *(i)* implies that for every $\varepsilon \in (0, \infty)$ there exists $\delta \in (0, \infty)$ with the property that the ball $B_{|||\cdot|||}(0, \delta)$ considered in $\Big(\big[\mathscr{C}_c^1(\mathbb{R}^n)\big|_\Omega\big]^n, |||\cdot|||\Big)$ is contained in the pre-image under the aforementioned embedding of the ball $B_d(0, \varepsilon)$ considered in the metric space $\Big(\big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n, d\Big)$, where d is the distance defined in (2.8.157). As such, if $\vec{F}$ in $\big[\mathscr{C}_c^1(\mathbb{R}^n)\big|_\Omega\big]^n$ satisfies $|||\vec{F}||| < \delta$ then necessarily $d(\vec{F}, 0) < \varepsilon$. In particular,

$$
\begin{aligned}
&\text{if } \{\vec{F}_j\}_{j\in\mathbb{N}} \text{ is a Cauchy sequence in } \Big(\big[\mathscr{C}_c^1(\mathbb{R}^n)\big|_\Omega\big]^n, |||\cdot|||\Big) \text{ then} \\
&\{\vec{F}_j\}_{j\in\mathbb{N}} \text{ is a Cauchy sequence in } \Big(\big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n, d\Big).
\end{aligned}
\tag{2.8.164}
$$

Then (2.8.164) and (2.8.158) guarantee that

$$
\begin{aligned}
&\text{for each Cauchy sequence } \{\vec{F}_j\}_{j\in\mathbb{N}} \text{ in } \Big(\big[\mathscr{C}_c^1(\mathbb{R}^n)\big|_\Omega\big]^n, |||\cdot|||\Big) \\
&\text{there exists a unique } \vec{F} \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n \text{ such that } \vec{F}_j \to \vec{F} \text{ in} \\
&\big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n \text{ as } j \to \infty.
\end{aligned}
\tag{2.8.165}
$$

Within the space of Cauchy sequences in $\Big(\big[\mathscr{C}_c^1(\mathbb{R}^n)\big|_\Omega\big]^n, |||\cdot|||\Big)$, use the notation $\{\vec{F}_j\}_{j\in\mathbb{N}} \sim \{\vec{G}_j\}_{j\in\mathbb{N}}$ if interlacing $\{\vec{F}_j\}_{j\in\mathbb{N}}$ and $\{\vec{G}_j\}_{j\in\mathbb{N}}$ yields a Cauchy sequence in $\Big(\big[\mathscr{C}_c^1(\mathbb{R}^n)\big|_\Omega\big]^n, |||\cdot|||\Big)$. Then $\sim$ is an equivalence relation on the space of Cauchy sequences in $\Big(\big[\mathscr{C}_c^1(\mathbb{R}^n)\big|_\Omega\big]^n, |||\cdot|||\Big)$ and $\mathcal{V}(\Omega)$ may be realized as the space of Cauchy sequences in $\Big(\big[\mathscr{C}_c^1(\mathbb{R}^n)\big|_\Omega\big]^n, |||\cdot|||\Big)$ modulo $\sim$, i.e.,

$$
\mathcal{V}(\Omega) = \Big\{\big[\{\vec{F}_j\}_{j\in\mathbb{N}}\big] : \{\vec{F}_j\}_{j\in\mathbb{N}} \text{ Cauchy sequence in } \Big(\big[\mathscr{C}_c^1(\mathbb{R}^n)\big|_\Omega\big]^n, |||\cdot|||\Big)\Big\},
\tag{2.8.166}
$$

where $[\cdot]$ denotes the equivalence class modulo $\sim$. The mapping

$$
\mathcal{V}(\Omega) \ni \big[\{\vec{F}_j\}_{j\in\mathbb{N}}\big] \longmapsto \vec{F} \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n
\tag{2.8.167}
$$

where $\vec{F}$ is associated with $\{\vec{F}_j\}_{j\in\mathbb{N}}$ as in (2.8.165) then yields a continuous embedding of $\mathcal{V}(\Omega)$ into the space $\big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n$. This proves *(a)*.

To prove *(b)*, let $\vec{F} \in \mathcal{V}(\Omega)$ be arbitrary. What we proved so far ensures the existence of a sequence $\{\vec{F}_j\}_{j\in\mathbb{N}}$ from $\big[\mathscr{C}_c^1(\mathbb{R}^n)\big|_\Omega\big]^n$ which is Cauchy with respect to the norm $|||\cdot|||$ and such that $\vec{F}_j \to \vec{F}$ as $j \to \infty$ in $\big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n$. By *(ii)* it follows that

$$
\{\mathrm{div}\vec{F}_j\}_{j\in\mathbb{N}} \text{ is a Cauchy sequence in } L^1(\Omega, \mathcal{L}^n)
\tag{2.8.168}
$$

and

$$\big\{\vec{F}_j\big|_{\partial_*\Omega}\big\}_{j\in\mathbb{N}} \text{ is a Cauchy sequence in } L^1(\partial_*\Omega,\sigma). \tag{2.8.169}$$

Hence, (2.8.168) guarantees the existence of some function $u \in L^1(\Omega, \mathcal{L}^n)$ such that

$$\operatorname{div}\vec{F}_j \longrightarrow u \ \text{ in } \ L^1(\Omega,\mathcal{L}^n) \ \text{ as } \ j \to \infty. \tag{2.8.170}$$

Then for each test function $\psi \in \mathscr{C}_c^\infty(\Omega)$ we may write (with $\langle\cdot,\cdot\rangle$ denoting the distributional pairing in Ω)

$$\begin{aligned}\langle\vec{F},\nabla\psi\rangle &= \int_\Omega \vec{F}\cdot\nabla\psi \,\mathrm{d}\mathcal{L}^n = \lim_{j\to\infty}\int_\Omega \vec{F}_j\cdot\nabla\psi\,\mathrm{d}\mathcal{L}^n \\ &= -\lim_{j\to\infty}\int_\Omega(\operatorname{div}\vec{F}_j)\psi\,\mathrm{d}\mathcal{L}^n = -\int_\Omega u\psi\,\mathrm{d}\mathcal{L}^n,\end{aligned} \tag{2.8.171}$$

where the last equality uses (2.8.170). Thus, $\operatorname{div}\vec{F} = u \in L^1(\Omega,\mathcal{L}^n)$, proving *(b)*.

From (2.8.169) we also see that there exists some $\vec{f} \in \big[L^1(\partial_*\Omega,\sigma)\big]^n$ such that $\vec{F}_j\big|_{\partial_*\Omega} \to \vec{f}$ in $\big[L^1(\partial_*\Omega,\sigma)\big]^n$ as $j\to\infty$. By interlacing sequences we see that $\vec{f}$ is uniquely determined by $\vec{F}$. Moreover, (2.8.160) implies $\|\vec{f}\|_{[L^1(\partial_*\Omega,\sigma)]^n} \leq C|\!|\!|\vec{F}|\!|\!|$. Denoting $\operatorname{Tr}\vec{F} := \vec{f}$ then yields a mapping as in (2.8.162) which is well-defined, linear, and bounded. In addition, the De Giorgi–Federer version of the Divergence Theorem (cf. Theorem 1.1.1) gives

$$\int_\Omega \operatorname{div}\vec{F}_j\,\mathrm{d}\mathcal{L}^n = \int_{\partial_*\Omega}\nu\cdot\big(\vec{F}_j\big|_{\partial_*\Omega}\big)\,\mathrm{d}\sigma \quad \text{for each } \ j\in\mathbb{N}. \tag{2.8.172}$$

Passing to the limit as $j\to\infty$ in the latter equality then establishes (2.8.163). This completes the proof of *(c)*. □

A special case of Proposition 2.8.16 worth singling out is when Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. In that scenario, if we take

$$\begin{aligned}|\!|\!|\vec{F}|\!|\!| := \|\vec{F}\|_{[L^1(\Omega,\mathcal{L}^n)]^n} + \textstyle\sum_{j=1}^n \|\partial_j\vec{F}\|_{[L^1(\Omega,\mathcal{L}^n)]^n} \\ \text{for each } \ \vec{F} \in \big[\mathscr{C}_c^1(\mathbb{R}^n)\big|_\Omega\big]^n,\end{aligned} \tag{2.8.173}$$

then conditions *(i)–(ii)* in Proposition 2.8.16 are satisfied. Also, corresponding to this choice of $|\!|\!|\cdot|\!|\!|$ the space defined in (2.8.161) becomes $\mathcal{V}(\Omega) = \big[W^{1,1}(\Omega)\big]^n$ and the mapping (2.8.162) is the Sobolev trace operator

$$\operatorname{Tr}: \big[W^{1,1}(\Omega)\big]^n \longrightarrow \big[L^1(\partial\Omega,\sigma)\big]^n. \tag{2.8.174}$$

As a consequence of (2.8.163), whenever Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ we have the Divergence Formula (with the trace understood in the sense of (2.8.174))

$$\int_{\Omega} \operatorname{div}\vec{F}\, d\mathcal{L}^n = \int_{\partial\Omega} \nu \cdot \operatorname{Tr}\vec{F}\, d\sigma \quad \text{for every } \vec{F} \in \big[W^{1,1}(\Omega)\big]^n. \tag{2.8.175}$$

Our next proposition contains a version of the Divergence Theorem for NTA domains with Ahlfors regular boundaries, and vectors fields in Sobolev spaces with bounded support possessing a nontangential pointwise trace.

Proposition 2.8.17 *Let* $\Omega \subseteq \mathbb{R}^n$ (*where* $n \in \mathbb{N}$, $n \geq 2$) *be an* NTA *domain with the property that* $\partial\Omega$ *is an Ahlfors regular set. Denote by* ν *the geometric measure theoretic outward unit normal to* Ω *and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Then there exists an aperture parameter* $\kappa = \kappa(\Omega) \in (0,\infty)$ *with the property that for each vector field* $\vec{F} \in \big[W^{1,p}(\Omega)\big]^n$, *with* $p \in (1,\infty)$, *which vanishes outside of a bounded subset of* Ω *and whose nontangential boundary trace* $\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ *exists at* σ*-a.e. point on* $\partial\Omega$ *one has*

$$\int_{\Omega} \operatorname{div}\vec{F}\, d\mathcal{L}^n = \int_{\partial\Omega} \nu \cdot \Big(\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)\, d\sigma. \tag{2.8.176}$$

Proof For each $z \in \partial\Omega$, denote by z_ρ the corkscrew point relative to z at scale $\rho \in (0,\infty)$ in Ω, then abbreviate $\mathcal{U}_\rho(z) := B(z_\rho, \rho/C)$ where $C = C(\Omega) \in (1,\infty)$ is some fixed, sufficiently large constant. In particular, there exist $c \in (0,\infty)$ and $\rho_o \in (0,\infty)$ such that

$$\begin{gathered} \mathcal{U}_\rho(z) \subseteq \Omega \cap B(z,\rho) \quad \text{and} \quad \mathcal{L}^n\big(\mathcal{U}_\rho(z)\big) \geq c\,\rho^n, \\ \text{for all} \quad z \in \partial\Omega \quad \text{and all} \quad \rho \in (0,\rho_o). \end{gathered} \tag{2.8.177}$$

Also, since $\operatorname{dist}(y,\partial\Omega) \approx \rho \approx |y-z|$ uniformly for $y \in \mathcal{U}_\rho(z)$, there exists some aperture parameter $\kappa \in (0,\infty)$ such that

$$\mathcal{U}_\rho(z) \subseteq \Gamma_\kappa(z) \quad \text{for each} \quad z \in \partial\Omega. \tag{2.8.178}$$

To proceed, fix a vector field $\vec{F} \in \big[W^{1,p}(\Omega)\big]^n$, with $p \in (1,\infty)$, whose nontangential boundary trace $\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$, and such that $\vec{F}$ vanishes outside $\Omega \cap K$ where $K \subseteq \mathbb{R}^n$ is a compact set. Assume that for each point $x \in \partial\Omega \cap K$ it is possible to find a radius $r_x \in (0,\infty)$ with the property that for each vector field $\vec{G} \in \big[W^{1,p}(\Omega)\big]^n$, with $p \in (1,\infty)$, whose nontangential boundary trace $\vec{G}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$, and such that $\vec{G}$ vanishes outside $\Omega \cap B(x,r_x)$ we have

$$\int_{\Omega} \operatorname{div}\vec{G}\, d\mathcal{L}^n = \int_{\partial\Omega} \nu \cdot \Big(\vec{G}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)\, d\sigma. \tag{2.8.179}$$

We then claim that (2.8.176) holds. This follows from a partition of unity argument. Specifically, the open cover $\big\{B(x,r_x)\big\}_{x\in\partial\Omega\cap K}$ of the compact set $\partial\Omega \cap K$ may be refined to a finite cover, say $\big\{B(x_i,r_{x_i})\big\}_{1\leq i\leq N}$. Bring in a family of functions $\xi_i \in \mathscr{C}_c^\infty\big(B(x_i,r_{x_i})\big)$ with $1 \leq i \leq N$, satisfying $\sum_{i=1}^N \xi_i \equiv 1$ near $\partial\Omega \cap K$, and decompose $\vec{F} = \sum_{i=0}^{N} \vec{G}_i$ where $\vec{G}_0 := \big(1 - \sum_{i=1}^{N} \xi_i\big)\vec{F}$ and $\vec{G}_i := \xi_i \vec{F}$ for $1 \leq i \leq N$. Then for each $i \in \{1, \dots, N\}$ the vector field $\vec{G}_i$ has all the qualities required in relation to (2.8.179). As such, for each $i \in \{1, \dots, N\}$ we have

$$\int_\Omega \mathrm{div}\vec{G}_i \, \mathrm{d}\mathscr{L}^n = \int_{\partial\Omega} \nu \cdot \big(\vec{G}_i\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)\, \mathrm{d}\sigma. \tag{2.8.180}$$

Also, since $\vec{G}_0 \in \big[W^{1,p}(\Omega)\big]^n$ is compactly supported in Ω, ordinary integration by parts gives

$$\int_\Omega \mathrm{div}\vec{G}_0 \, \mathrm{d}\mathscr{L}^n = \int_{\partial\Omega} \nu \cdot \big(\vec{G}_0\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)\, \mathrm{d}\sigma. \tag{2.8.181}$$

Summing up formulas (2.8.180)–(2.8.181) then yields (2.8.176), as claimed.

Recall from Proposition 5.11.15 that there exist $r, R \in (0, \infty)$ with $r < R$ such that for each $x \in \partial\Omega$ one may find an NTA domain $\Omega_x \subseteq \mathbb{R}^n$ such that $\Omega_x \subseteq \Omega$ and $\Omega \cap B(x, r) \subseteq \Omega_x \subseteq \Omega \cap B(x, R)$. In view of the fact that (2.8.179) implies (2.8.176) it is therefore enough to show that for each $x \in \partial\Omega \cap K$ and each vector field $\vec{G} \in \big[W^{1,p}(\Omega)\big]^n$, with $p \in (1, \infty)$, whose nontangential boundary trace $\vec{G}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$, and such that $\vec{G}$ vanishes outside $\Omega \cap B(x, r)$, formula (2.8.179) holds. To justify (2.8.179), observe that $\vec{G}\big|_{\Omega_x} \in \big[W^{1,p}(\Omega_x)\big]^n$. On the other hand, Ω_x is a bounded NTA domain, hence an (ε, δ)-domain (cf. (5.11.66)). As such, P. Jones' result recorded in (5.11.30) ensures that there exists

$$\vec{H} \in \big[W^{1,p}(\mathbb{R}^n)\big]^n \text{ with compact support, such that } \vec{H}\big|_{\Omega_x} = \vec{G}\big|_{\Omega_x}. \tag{2.8.182}$$

We may then find a large number $R \in (0, \infty)$ and a sequence $\{\vec{H}_j\}_{j\in\mathbb{N}} \subseteq \big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n$ such that each $\vec{H}_j$ is supported in $B(0, R)$ and $\vec{H}_j \to \vec{H}$ in $\big[W^{1,p}(\mathbb{R}^n)\big]^n$ as $j \to \infty$. Also, as a special case of general results regarding traces on Ahlfors regular closed subsets of $\mathbb{R}^n$ proved by A. Jonsson and H. Wallin in [138], we know that for each $\omega \in W^{1,p}(\mathbb{R}^n)$ the limit

$$\big(\mathscr{R}_{\mathbb{R}^n\to\partial\Omega}\, \omega\big)(x) := \lim_{r\to 0^+} \fint_{B(x,r)} \omega \, \mathrm{d}\mathscr{L}^n \text{ exists at } \sigma\text{-a.e. } x \in \partial\Omega, \tag{2.8.183}$$

and this trace operator induces a well-defined, linear, and continuous mapping

$$\mathscr{R}_{\mathbb{R}^n\to\partial\Omega} : W^{1,p}(\mathbb{R}^n) \longrightarrow L^p(\partial\Omega, \sigma). \tag{2.8.184}$$

We may then write

$$
\begin{aligned}
\int_{\Omega} \operatorname{div}\vec{G}\, d\mathcal{L}^n &= \int_{\Omega_x} \operatorname{div}\vec{G}\, d\mathcal{L}^n = \int_{\Omega_x} \operatorname{div}\vec{H}\, d\mathcal{L}^n \\
&= \lim_{j\to\infty} \int_{\Omega_x} \operatorname{div}\vec{H}_j\, d\mathcal{L}^n = \lim_{j\to\infty} \int_{\Omega} \operatorname{div}\vec{H}_j\, d\mathcal{L}^n \\
&= \lim_{j\to\infty} \int_{\partial\Omega} \nu\cdot\big(\vec{H}_j\big|_{\partial\Omega}\big)\, d\sigma = \lim_{j\to\infty} \int_{\partial\Omega} \nu\cdot\big(\mathscr{R}_{\mathbb{R}^n\to\partial\Omega}\, \vec{H}_j\big)\, d\sigma \\
&= \int_{\partial\Omega} \nu\cdot\big(\mathscr{R}_{\mathbb{R}^n\to\partial\Omega}\, \vec{H}\big)\, d\sigma. \qquad (2.8.185)
\end{aligned}
$$

At this stage, there remains to show that

$$
\mathscr{R}_{\mathbb{R}^n\to\partial\Omega}\, \vec{H} = \vec{G}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \quad \text{at } \sigma\text{-a.e. point on } \ \partial\Omega\cap B(x,r). \qquad (2.8.186)
$$

With the goal of proving (2.8.186), denote by A the collection of all $z\in\partial\Omega\cap B(x,r)$ with the property that both $\big(\mathscr{R}_{\mathbb{R}^n\to\partial\Omega}\, \vec{H}\big)(z) = \lim_{\rho\to 0^+} \fint_{B(z,\rho)} \vec{H}\, d\mathcal{L}^n$ and $\big(\vec{G}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(z)$ exist. In such a scenario, we then have

$$
\lim_{\rho\to 0^+} \fint_{\mathcal{U}_\rho(z)} \vec{H}\, d\mathcal{L}^n = \lim_{\rho\to 0^+} \fint_{\mathcal{U}_\rho(z)} \vec{G}\, d\mathcal{L}^n = \big(\vec{G}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(z) \qquad (2.8.187)
$$

by (2.8.182). Note that $\sigma\big(\partial\Omega\cap B(x,r)\setminus A\big) = 0$ by (2.8.183) and assumptions. For each $\lambda > 0$ fixed, let us now introduce

$$
E_\lambda := \left\{ z\in A : \left| \lim_{\rho\to 0^+} \fint_{B(z,\rho)} \vec{H}\, d\mathcal{L}^n - \lim_{\rho\to 0^+} \fint_{\mathcal{U}_\rho(z)} \vec{H}\, d\mathcal{L}^n \right| > \lambda \right\}. \qquad (2.8.188)
$$

Then, with $\mathcal{M}_{\mathbb{R}^n}$ denoting the classical Hardy–Littlewood maximal operator in $\mathbb{R}^n$, for every $z\in E_\lambda$ and each $\vec{\varphi}\in\big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n$ we have

$$
\begin{aligned}
\lambda &< \left| \lim_{\rho\to 0^+} \fint_{B(z,\rho)} \vec{H}\, d\mathcal{L}^n - \lim_{\rho\to 0^+} \fint_{\mathcal{U}_\rho(z)} \vec{H}\, d\mathcal{L}^n \right| \\
&= \left| \lim_{\rho\to 0^+} \fint_{B(z,\rho)} (\vec{H}-\vec{\varphi})\, d\mathcal{L}^n - \lim_{\rho\to 0^+} \fint_{\mathcal{U}_\rho(z)} (\vec{H}-\vec{\varphi})\, d\mathcal{L}^n \right| \\
&\leq C\mathcal{M}_{\mathbb{R}^n}(\vec{H}-\vec{\varphi})(z), \qquad (2.8.189)
\end{aligned}
$$

thanks to (2.8.177). To continue, we shall make two claims asserting that, for any $\omega\in W^{1,p}(\mathbb{R}^n)$,

$$
\big(\mathcal{M}_{\mathbb{R}^n}\omega\big)(z) \leq \lim_{t\to 0^+} \fint_{B(z,t)} \mathcal{M}_{\mathbb{R}^n}\omega\, d\mathcal{L}^n \quad \text{for } \sigma\text{-a.e. } \ z\in\partial\Omega, \qquad (2.8.190)
$$

and

$$\left\| \mathcal{M}_{\mathbb{R}^n}\omega \right\|_{W^{1,p}(\mathbb{R}^n)} \leq C(n,p)\|\omega\|_{W^{1,p}(\mathbb{R}^n)}, \tag{2.8.191}$$

where $C(n,p) > 0$ is a finite constant independent of ω. Assuming these for the time being, we then deduce from (2.8.189)–(2.8.190) that

$$\begin{aligned} &\lambda < C \lim_{t\to 0^+} ⨍_{B(z,t)} \mathcal{M}_{\mathbb{R}^n}(\vec{H}-\vec{\varphi})\, d\mathcal{L}^n = C\mathscr{R}_{\mathbb{R}^n\to\partial\Omega}\big(\mathcal{M}_{\mathbb{R}^n}(\vec{H}-\vec{\varphi})\big)(z) \\ &\qquad \text{for } \sigma\text{-a.e. point } z\in E_\lambda \text{ and each } \vec{\varphi}\in\big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^M. \end{aligned} \tag{2.8.192}$$

Thus, on account of (2.8.191)–(2.8.192), Chebyshev's inequality, and (2.8.184), for every vector field $\vec{\phi}\in\big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n$ we may estimate

$$\begin{aligned} \lambda^p\sigma(E_\lambda) &\leq C\int_{\partial\Omega}\big|\mathscr{R}_{\mathbb{R}^n\to\partial\Omega}\big(\mathcal{M}_{\mathbb{R}^n}(\vec{H}-\vec{\varphi})\big)\big|^p\, d\sigma \\ &\leq C\big\|\mathcal{M}_{\mathbb{R}^n}(\vec{H}-\vec{\varphi})\big\|^p_{W^{1,p}(\mathbb{R}^n)} \leq C\big\|\vec{H}-\vec{\varphi}\big\|^p_{[W^{1,p}(\mathbb{R}^n)]^n}, \end{aligned} \tag{2.8.193}$$

where $C\in(0,\infty)$ is a constant independent of $\vec{\varphi}$. Upon recalling that we have $\mathscr{C}_c^\infty(\mathbb{R}^n)\hookrightarrow W^{1,p}(\mathbb{R}^n)$ densely, we may conclude from (2.8.193) that $\sigma(E_\lambda)=0$ for every $\lambda>0$. In view of (2.8.188) and (2.8.178), this further entails

$$\begin{aligned} (\mathscr{R}_{\mathbb{R}^n\to\partial\Omega}\vec{H})(z) &= \lim_{\rho\to 0^+}⨍_{B(z,\rho)}\vec{H}\, d\mathcal{L}^n = \lim_{\rho\to 0^+}⨍_{\mathcal{U}_\rho(z)}\vec{H}\, d\mathcal{L}^n \\ &= \lim_{\rho\to 0^+}⨍_{\mathcal{U}_\rho(z)}\vec{G}\, d\mathcal{L}^n \\ &= \Big(\vec{G}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(z) \quad \text{for } \sigma\text{-a.e. } z\in\partial\Omega\cap B(x,r), \end{aligned} \tag{2.8.194}$$

so (2.8.186) follows from (2.8.194). This finishes the proof of (2.8.186), modulo the justification of (2.8.190)–(2.8.191).

Estimate (2.8.191) has been established in [152]. As regards (2.8.190), fix $\omega\in W^{1,p}(\mathbb{R}^n)$ and denote by S the collection of points $z\in\partial\Omega$ where the limit $\lim\limits_{t\to 0^+}⨍_{B(z,t)}\mathcal{M}_{\mathbb{R}^n}\omega\, d\mathcal{L}^n$ exists. Since by (2.8.191) the function $\mathcal{M}_{\mathbb{R}^n}\omega$ belongs to $W^{1,p}(\mathbb{R}^n)$, it follows from (2.8.183) that, on the one hand,

$$\sigma(\partial\Omega\setminus S)=0. \tag{2.8.195}$$

On the other hand, for every $z\in S$ and every $R>0$ we may write

$$
\begin{aligned}
&\lim_{t\to 0^+} ⨍_{B(z,t)} \big(\mathcal{M}_{\mathbb{R}^n}\omega\big)(y)\,\mathrm{d}y \\
&\geq \lim_{t\to 0^+}\left[\frac{1}{\mathcal{L}^n(B(z,t))}\int_{B(z,t)}\frac{1}{\mathcal{L}^n(B(y,R))}\int_{B(y,R)}|\omega(x)|\,\mathrm{d}x\,\mathrm{d}y\right] \\
&= \lim_{t\to 0^+}\left[\frac{1}{\mathcal{L}^n(B(z,t))\,\mathcal{L}^n(B(0,R))}\int_{B(z,t+R)}|\omega(x)|\,\mathcal{L}^n\big(B(z,t)\cap B(x,R)\big)\,\mathrm{d}x\right] \\
&\geq \lim_{t\to 0^+}\left[⨍_{B(z,R)}|\omega(x)|\,\frac{\mathcal{L}^n\big(B(z,t)\cap B(x,R)\big)}{\mathcal{L}^n(B(z,t))}\,\mathrm{d}x\right] \\
&= ⨍_{B(z,R)}|\omega(x)|\,\mathrm{d}x, \qquad (2.8.196)
\end{aligned}
$$

where the first equality in (2.8.196) uses Fubini's theorem, and the very last equality in (2.8.196) follows from an application of Lebesgue's Dominated Convergence Theorem. In this regard, it helps to observe that if $x \in B(z, R)$ then $z \in B(x, R)$, hence $B(z,t) \subseteq B(x,R)$ for $t > 0$ sufficiently small. On account of (2.8.195), the estimate in (2.8.190) now follows by taking the supremum over $R > 0$ of the most extreme sides in (2.8.196). This justifies (2.8.190) and finishes the proof of the proposition. □

If a compactly supported vector distribution is such that its divergence is a complex Borel measure, then the total mass of this measure is zero.

Lemma 2.8.18 *Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary open set and consider*

$$\vec{F} \in \big[\mathscr{E}'(\Omega)\big]^n \ \ \textit{with} \ \ \operatorname{div}\vec{F} \in \mathrm{CBM}(\Omega). \qquad (2.8.197)$$

Then

$$(\operatorname{div}\vec{F})(\Omega) = 0. \qquad (2.8.198)$$

In particular, with $L^1(\Omega, \mathcal{L}^n)$ canonically identified as a subspace of CBM(Ω)*,*

$$\textit{if actually } \operatorname{div}\vec{F} \in L^1(\Omega, \mathcal{L}^n) \textit{ then } \int_\Omega \operatorname{div}\vec{F}\,\mathrm{d}\mathcal{L}^n = 0. \qquad (2.8.199)$$

Proof The current assumptions imply that $\operatorname{div}\vec{F}$ belongs to $\mathscr{E}'(\Omega) \cap \mathrm{CBM}(\Omega)$. As such, we may invoke (4.6.25) to write

$$(\operatorname{div}\vec{F})(\Omega) = {}_{\mathscr{E}'(\Omega)}\langle \operatorname{div}\vec{F}, 1\rangle_{\mathscr{E}(\Omega)} = -{}_{[\mathscr{E}'(\Omega)]^n}\big\langle \vec{F}, \nabla 1\big\rangle_{[\mathscr{E}(\Omega)]^n} = 0, \qquad (2.8.200)$$

as wanted. □

In turn, Lemma 2.8.18 is an ingredient in the proof of the following result.

Proposition 2.8.19 *Fix $n \in \mathbb{N}$ with $n \geq 2$. Then for each family of functions*

$$f_1, \dots, f_{n-1} \in L^{n/(n-1)}(\mathbb{R}^n_+, \mathcal{L}^n) \;\; \textit{with} \;\; \sum_{j=1}^{n-1} \partial_j f_j \in L^1(\mathbb{R}^n_+, \mathcal{L}^n) \tag{2.8.201}$$

one has

$$\int_{\mathbb{R}^n_+} \sum_{j=1}^{n-1} \partial_j f_j \, \mathrm{d}\mathcal{L}^n = 0. \tag{2.8.202}$$

Proof Pick some function $\theta \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ with the property that $\theta \equiv 1$ near the origin in $\mathbb{R}^n$, and for each $R > 0$ define $\theta_R(x) := \theta(x/R)$ for all $x \in \mathbb{R}^n$. Then, in the sense of distributions in $\mathbb{R}^n_+$, for each $j \in \{1, \dots, n-1\}$ we have

$$\partial_j(\theta_R f_j) = \theta_R(\partial_j f_j) + (\partial_j \theta_R) f_j \quad \text{for each} \quad R > 0. \tag{2.8.203}$$

Note that there exists $C \in (0, \infty)$ with the property that for each $j \in \{1, \dots, n-1\}$ and each $R > 0$ we may write

$$\begin{aligned} \int_{\mathbb{R}^n_+} |\partial_j \theta_R||f_j| \, \mathrm{d}\mathcal{L}^n &\leq \frac{C}{R} \int_{\substack{|x| \approx R \\ x \in \mathbb{R}^n_+}} |f_j(x)| \, \mathrm{d}x \\ &\leq C \Big(\int_{\substack{|x| \approx R \\ x \in \mathbb{R}^n_+}} |f_j(x)|^{\frac{n}{n-1}} \, \mathrm{d}x \Big)^{\frac{n-1}{n}}. \end{aligned} \tag{2.8.204}$$

Since the first condition in (2.8.201) and Lebesgue's Dominated Convergence Theorem entail

$$\lim_{R \to \infty} \Big(\int_{\substack{|x| \approx R \\ x \in \mathbb{R}^n_+}} |f_j(x)|^{\frac{n}{n-1}} \, \mathrm{d}x \Big)^{\frac{n-1}{n}} = 0 \quad \text{for each} \quad j \in \{1, \dots, n-1\}, \tag{2.8.205}$$

we conclude that

$$\lim_{R \to \infty} \int_{\mathbb{R}^n_+} (\partial_j \theta_R) f_j \, \mathrm{d}\mathcal{L}^n = 0 \quad \text{for each} \quad j \in \{1, \dots, n-1\}. \tag{2.8.206}$$

Also, the last condition in (2.8.201) and Lebesgue's Dominated Convergence Theorem give

$$\lim_{R \to \infty} \int_{\mathbb{R}^n_+} \theta_R \sum_{j=1}^{n-1} \partial_j f_j \, \mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^n_+} \sum_{j=1}^{n-1} \partial_j f_j \, \mathrm{d}\mathcal{L}^n. \tag{2.8.207}$$

Collectively, (2.8.203), (2.8.206), and (2.8.207) prove that

$$\int_{\mathbb{R}^n_+} \sum_{j=1}^{n-1} \partial_j f_j \, \mathrm{d}\mathcal{L}^n = \lim_{R \to \infty} \int_{\mathbb{R}^n_+} \sum_{j=1}^{n-1} \partial_j(\theta_R f_j) \, \mathrm{d}\mathcal{L}^n. \tag{2.8.208}$$

Henceforth, fix a number $R \in (0, \infty)$. Bring in a function $\eta \in \mathscr{C}^\infty(\mathbb{R})$ with the property that $\eta \equiv 0$ on $(-1, 1)$ and $\eta \equiv 1$ on $\mathbb{R} \setminus (-2, 2)$. Also, for each $\varepsilon > 0$ define $\eta_\varepsilon(x) := \eta(x_n/\varepsilon)$ for all $x = (x', x_n) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. Then

$$\begin{aligned}\int_{\mathbb{R}^n_+} \sum_{j=1}^{n-1} \partial_j(\theta_R f_j)\,\mathrm{d}\mathcal{L}^n &= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n_+} \eta_\varepsilon \sum_{j=1}^{n-1} \partial_j(\theta_R f_j)\,\mathrm{d}\mathcal{L}^n \\ &= \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n_+} \sum_{j=1}^{n-1} \partial_j(\eta_\varepsilon\theta_R f_j)\,\mathrm{d}\mathcal{L}^n = 0, \end{aligned} \tag{2.8.209}$$

with the first equality a consequence of Lebesgue's Dominated Convergence Theorem, the second equality implied by the fact that $\partial_j \eta_\varepsilon = 0$ for $j \in \{1, \dots, n-1\}$, and the final equality provided by Lemma 2.8.18 (bearing in mind that each function $\eta_\varepsilon\theta_R f_j$ has compact support in $\mathbb{R}^n_+$).

Then (2.8.202) follows from (2.8.208) and (2.8.209). □

Our next result generalizes Proposition 2.8.19, which corresponds precisely to the choice

$$D(f_1, \dots, f_{n-1}) := \sum_{j=1}^{n-1} \partial_j f_j. \tag{2.8.210}$$

Proposition 2.8.20 *Consider a homogeneous first-order $M \times N$ system D with constant (complex) coefficients in $\mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, so that* $\mathrm{Sym}(D; \boldsymbol{e}_n) = 0$. *Then for any*

$$u \in \big[L^{n/(n-1)}(\mathbb{R}^n_+, \mathcal{L}^n)\big]^N \quad \textit{with} \quad Du \in \big[L^1(\mathbb{R}^n_+, \mathcal{L}^n)\big]^M \tag{2.8.211}$$

one has

$$\int_{\mathbb{R}^n_+} Du \,\mathrm{d}\mathcal{L}^n = 0. \tag{2.8.212}$$

Proof Having $\mathrm{Sym}(D; \mathbf{e}_n) = 0$ guarantees the absence of the partial derivative ∂_{x_n} in the writing of D, that is, $D = \sum_{j=1}^{n-1} A_j \partial_j$ where each A_j is an $M \times N$ matrix with complex entries. Keeping this in mind, the desired result follows by suitably invoking Proposition 2.8.19. □

We have seen in Corollary 1.2.2 that in special circumstances (having to do with a very particular geometry of the underlying domain Ω), we may further relax the hypotheses made on the existence of the nontangential boundary trace of the vector field $\vec{F}$ in Theorem 1.2.1. Here is a result of a similar flavor to Corollary 1.2.2, corresponding to the case when Ω is a ball.

Proposition 2.8.21 *Fix $n \in \mathbb{N}$ and pick some aperture parameter $\kappa \in (0, \infty)$. Consider a vector field $\vec{F} = (F_1, \dots, F_n) : B(0, 1) \to \mathbb{C}^n$ with Lebesgue measurable components and define*

$$f(x) := x \cdot \vec{F}(x) = \sum_{j=1}^{n} x_j F_j(x) \;\; \textit{for each} \;\; x = (x_1, \dots, x_n) \in B(0,1). \tag{2.8.213}$$

With $S^{n-1} = \partial B(0,1)$*, abbreviate* $\sigma := \mathcal{H}^{n-1} \lfloor S^{n-1}$*, and assume that the following properties are satisfied:*

$$\begin{aligned} & f\big|^{\kappa-\mathrm{n.t.}}_{\partial B(0,1)} \;\; \textit{exists at } \sigma\textit{-a.e. point on} \;\; S^{n-1}, \quad \mathcal{N}_\kappa \vec{F} \in L^1(S^{n-1}, \sigma), \quad \textit{and} \\ & \mathrm{div}\vec{F} \in \mathcal{D}'\big(B(0,1)\big) \;\textit{extends to a complex Borel measure on}\; B(0,1) \end{aligned} \tag{2.8.214}$$

(*still denoted by* $\mathrm{div}\vec{F}$).

Then for any other aperture parameter $\kappa' > 0$ *the nontangential trace* $f\big|^{\kappa'-\mathrm{n.t.}}_{\partial B(0,1)}$ *exists* σ*-a.e. on* S^{n-1} *and is actually independent of* κ'*. When regarding it as a function defined* σ*-a.e. on* S^{n-1}*, this belongs to* $L^1(S^{n-1}, \sigma)$ *and, with the dependence on the parameter* κ' *dropped,*

$$(\mathrm{div}\vec{F})\big(B(0,1)\big) = \int_{S^{n-1}} \Big(f\big|^{\mathrm{n.t.}}_{\partial B(0,1)} \Big) \, \mathrm{d}\sigma. \tag{2.8.215}$$

In particular, if in place of the second line in (2.8.214) *one now assumes that*

$$\mathrm{div}\vec{F} := \partial_1 F_1 + \cdots + \partial_n F_n \in L^1\big(B(0,1), \mathcal{L}^n\big), \tag{2.8.216}$$

where all partial derivatives are considered in the sense of distributions in $B(0,1)$*, then* (2.8.215) *becomes*

$$\int_{B(0,1)} \mathrm{div}\vec{F} \, \mathrm{d}\mathcal{L}^n = \int_{S^{n-1}} \Big(f\big|^{\mathrm{n.t.}}_{\partial B(0,1)} \Big) \, \mathrm{d}\sigma. \tag{2.8.217}$$

For example, if $\vec{F} : B(0,1) \to \mathbb{C}^n$ has Lebesgue measurable components, and satisfies (for some aperture parameter $\kappa > 0$)

$$\begin{aligned} & \mathcal{N}_\kappa \vec{F} \in L^1(S^{n-1}, \sigma), \quad \mathrm{div}\vec{F} \in L^1\big(B(0,1), \mathcal{L}^n\big), \\ & \text{and} \;\; x \cdot \vec{F}(x) = 0 \;\; \text{for } \mathcal{L}^n\text{-a.e. point} \;\; x \in B(0,1), \end{aligned} \tag{2.8.218}$$

then Proposition 2.8.21 guarantees that

$$\int_{B(0,1)} \mathrm{div}\vec{F} \, \mathrm{d}\mathcal{L}^n = 0. \tag{2.8.219}$$

In relation to this special case, there is a more general phenomenon at play here, namely

$$\text{if } \vec{F} \in \big[\mathcal{D}'\big(B(0,1)\big)\big]^n \text{ has } x \cdot \vec{F}(x) = 0 \text{ in } \mathcal{D}'\big(B(0,1) \setminus \overline{B(0,r)}\big) \text{ for an } r \in (0,1), \text{ and } \operatorname{div}\vec{F} \in \mathrm{CBM}\big(B(0,1)\big), \text{ then } (\operatorname{div}\vec{F})\big(B(0,1)\big) = 0. \tag{2.8.220}$$

A direct justification of (2.8.220), which also sheds some light on the role of the condition stipulating that $x \cdot \vec{F}(x) = 0$ as a distribution near the boundary of the unit ball, goes as follows. Choose $\eta \in \mathscr{C}^\infty(\mathbb{R})$ with the property that $\eta \equiv 0$ on $(-1, 1)$ and $\eta \equiv 1$ on $\mathbb{R} \setminus (-2, 2)$. For each $\varepsilon > 0$ set $\eta_\varepsilon(x) := \eta\big((1 - |x|^2)/\varepsilon\big)$ for all points $x \in B(0, 1)$. Also, introduce $\vec{G}_\varepsilon := \eta_\varepsilon \vec{F} \in \big[\mathscr{E}'\big(B(0,1)\big)\big]^n$ for each $\varepsilon \in (0, 1)$. Since

$$\begin{gathered} \operatorname{div}\vec{G}_\varepsilon = \eta_\varepsilon \operatorname{div}\vec{F} + (\nabla\eta_\varepsilon)\cdot\vec{F} \text{ in } \mathcal{D}'\big(B(0,1)\big) \text{ and} \\ (\nabla\eta_\varepsilon)(x)\cdot\vec{F}(x) = -2\varepsilon^{-1}\eta'\big((1-|x|^2)/\varepsilon\big)x\cdot\vec{F}(x) = 0 \\ \text{in } \mathcal{D}'\big(B(0,1)\setminus\overline{B(0,r)}\big), \end{gathered} \tag{2.8.221}$$

we have $\operatorname{div}\vec{G}_\varepsilon = \eta_\varepsilon \operatorname{div}\vec{F} \in \mathscr{E}'\big(B(0,1)\big)$. Thus, with $\mu := \operatorname{div}\vec{F} \in \mathrm{CBM}\big(B(0,1)\big)$, we may write (using Lebesgue's Dominated Convergence Theorem, (1.3.12), and [181, Proposition 2.72, p. 47])

$$\begin{aligned} (\operatorname{div}\vec{F})\big(B(0,1)\big) &= \mu\big(B(0,1)\big) = \lim_{\varepsilon\to0^+}\int_{B(0,1)} \eta_\varepsilon \,\mathrm{d}\mu \\ &= \lim_{\varepsilon\to0^+} {}_{\mathcal{D}'(\Omega)}\big\langle \operatorname{div}\vec{F}, \eta_\varepsilon\big\rangle_{\mathcal{D}(\Omega)} = \lim_{\varepsilon\to0^+} {}_{\mathscr{E}'(\Omega)}\big\langle \eta_\varepsilon \operatorname{div}\vec{F}, 1\big\rangle_{\mathscr{E}(\Omega)} \\ &= \lim_{\varepsilon\to0^+} {}_{\mathscr{E}'(\Omega)}\big\langle \operatorname{div}\vec{G}_\varepsilon, 1\big\rangle_{\mathscr{E}(\Omega)} = -\lim_{\varepsilon\to0^+} {}_{[\mathscr{E}'(\Omega)]^n}\big\langle \vec{G}_\varepsilon, \nabla 1\big\rangle_{[\mathscr{E}(\Omega)]^n} \\ &= 0. \end{aligned} \tag{2.8.222}$$

Thus, (2.8.220) is established. As a consequence,

$$\text{if } \vec{F} \in \big[\mathcal{D}'\big(B(0,1)\big)\big]^n \text{ satisfies } x\cdot\vec{F}(x) = 0 \text{ in } \mathcal{D}'\big(B(0,1)\big) \text{ and has } \operatorname{div}\vec{F} \in \mathrm{CBM}\big(B(0,1)\big) \text{ then } (\operatorname{div}\vec{F})\big(B(0,1)\big) = 0. \tag{2.8.223}$$

Here is the proof of Proposition 2.8.21.

Proof of Proposition 2.8.21 In view of the assumption that $\mathcal{N}_\kappa \vec{F} \in L^1(S^{n-1}, \sigma)$, Lemma 8.3.1 implies

$$\vec{F} \in \big[L^\infty_{\mathrm{loc}}\big(B(0,1), \mathcal{L}^n\big)\big]^n. \tag{2.8.224}$$

As such, it is meaningful to consider $\operatorname{div}\vec{F}$ in the sense of distributions in $B(0, 1)$. We next make the claim that there exists a small constant $c = c(\kappa) \in \big(0, 1/2\big)$ such that

$$\begin{gathered} B\big((1+\varepsilon)^{-1}x, c\,\varepsilon\big) \subseteq \Gamma_\kappa(z) \text{ for each } \varepsilon \in (0,1), \\ \text{each } z \in S^{n-1}, \text{ and each } x \in \overline{\Gamma_\kappa(z)}. \end{gathered} \tag{2.8.225}$$

Indeed, given any $z \in S^{n-1}$, $x \in \overline{\Gamma_\kappa(z)}$, $\varepsilon \in (0,1)$, and $y \in B\big((1+\varepsilon)^{-1}x, c\,\varepsilon\big)$, we have

$$|y| < c\,\varepsilon + (1+\varepsilon)^{-1}|x| \tag{2.8.226}$$

hence $|y| < \frac{\varepsilon}{2} + (1+\varepsilon)^{-1} < 1$ and

$$\mathrm{dist}\big(y, S^{n-1}\big) = 1 - |y| > 1 - c\,\varepsilon - (1+\varepsilon)^{-1}|x|. \tag{2.8.227}$$

Since we also have

$$\begin{aligned} |y-z| &\leq \big|y-(1+\varepsilon)^{-1}x\big| + (1+\varepsilon)^{-1}|x-z| + \big(\tfrac{\varepsilon}{1+\varepsilon}\big)|z| \\ &\leq c\,\varepsilon + (1+\varepsilon)^{-1}(1+\kappa)\,\mathrm{dist}\big(x, S^{n-1}\big) + \tfrac{\varepsilon}{1+\varepsilon} \\ &= \varepsilon\big(c+(1+\varepsilon)^{-1}\big) + (1+\varepsilon)^{-1}(1+\kappa)(1-|x|), \end{aligned} \tag{2.8.228}$$

we then conclude that $|y-z| < (1+\kappa)\,\mathrm{dist}\big(y, S^{n-1}\big)$ provided

$$\varepsilon\big(c+(1+\varepsilon)^{-1}\big) + (1+\varepsilon)^{-1}(1+\kappa)(1-|x|) < (1+\kappa)\big(1-c\,\varepsilon-(1+\varepsilon)^{-1}|x|\big) \tag{2.8.229}$$

or, equivalently, $c < \big(\frac{\kappa}{2+\kappa}\big)(1+\varepsilon)^{-1}$. Thus, any $c \in \big(0, \frac{\kappa}{4+2\kappa}\big)$ will do, as far as the claim in (2.8.225) is concerned.

Going further, pick a non-negative function $\theta \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\theta \subseteq B(0,c)$ such that $\int_{\mathbb{R}^n}\theta\,\mathrm{d}\mathcal{L}^n = 1$. For each $\varepsilon \in (0,1)$ set $\theta_\varepsilon(x) := \varepsilon^{-n}\theta(x/\varepsilon)$ for all $x \in \mathbb{R}^n$, then define

$$\begin{aligned} \vec{G}_\varepsilon(x) &:= \int_{\mathbb{R}^n} \vec{F}\big((1+\varepsilon)^{-1}(x-y)\big)\theta_\varepsilon(y)\,\mathrm{d}y \\ &= (1+\varepsilon)^n \int_{B(0,1)} \vec{F}(z)\theta_\varepsilon\big(x-(1+\varepsilon)z\big)\,\mathrm{d}z \\ &= (1+\varepsilon)^n \Big({}_{\mathcal{D}'(B(0,1))}\Big\langle F_j, \theta_\varepsilon\big(x-(1+\varepsilon)\,\cdot\big)\Big\rangle_{\mathcal{D}(B(0,1))}\Big)_{1\leq j\leq n} \end{aligned} \tag{2.8.230}$$

for each $x \in \overline{B(0,1)}$. From (2.8.230) and (2.8.224) we see that $\vec{G}_\varepsilon$ is well-defined and, in fact,

$$\vec{G}_\varepsilon \in \big[\mathscr{C}^\infty(\overline{B(0,1)})\big]^n. \tag{2.8.231}$$

Also, if we abbreviate

$$\mu := \operatorname{div}\vec{F} \in \mathrm{CBM}\big(B(0,1)\big), \tag{2.8.232}$$

then for each $\varepsilon \in (0,1)$ and each $x \in B(0,1)$ we may write

$$\begin{aligned}(\mathrm{div}\vec{G}_\varepsilon)(x) &= (1+\varepsilon)^n \sum_{j=1}^{n} {}_{\mathcal{D}'(B(0,1))}\Big\langle F_j, (\partial_j\theta_\varepsilon)\big(x-(1+\varepsilon)\cdot\big)\Big\rangle_{\mathcal{D}(B(0,1))} \\ &= -(1+\varepsilon)^{n-1} \sum_{j=1}^{n} {}_{\mathcal{D}'(B(0,1))}\Big\langle F_j, \partial_j\big[\theta_\varepsilon\big(x-(1+\varepsilon)\cdot\big)\big]\Big\rangle_{\mathcal{D}(B(0,1))} \\ &= (1+\varepsilon)^{n-1} \sum_{j=1}^{n} {}_{\mathcal{D}'(B(0,1))}\big\langle \partial_j F_j, \theta_\varepsilon\big(x-(1+\varepsilon)\cdot\big)\big\rangle_{\mathcal{D}(B(0,1))} \\ &= (1+\varepsilon)^{n-1}\, {}_{\mathcal{D}'(B(0,1))}\Big\langle \mathrm{div}\vec{F}, \theta_\varepsilon\big(x-(1+\varepsilon)\cdot\big)\Big\rangle_{\mathcal{D}(B(0,1))} \\ &= (1+\varepsilon)^{n-1}\int_{B(0,1)} \theta_\varepsilon\big(x-(1+\varepsilon)y\big)\,\mathrm{d}\mu(y), \qquad (2.8.233)\end{aligned}$$

where we have used (2.8.230) and (1.3.12). For every $\varepsilon\in(0,1)$ define

$$f_\varepsilon(y) := \int_{B(0,1)} \theta_\varepsilon\big(x-(1+\varepsilon)y\big)\,\mathrm{d}x \quad \text{for each} \quad y\in B(0,1), \qquad (2.8.234)$$

and note that, for each $y\in B(0,1)$ fixed, the change of variables $z := x-(1+\varepsilon)y$ places the new variable z in $B(0,1)-(1+\varepsilon)y = B\big(-(1+\varepsilon)y, 1\big)$. Thus,

$$f_\varepsilon(y) = \int_{B(-(1+\varepsilon)y,1)} \theta_\varepsilon(z)\,\mathrm{d}z \quad \text{for each} \quad y\in B(0,1). \qquad (2.8.235)$$

Since for each given point $y\in B(0,1)$ there exists a small threshold $\varepsilon_y\in(0,1)$ with the property that the inclusion $B(0,c\,\varepsilon)\subseteq B\big(-(1+\varepsilon)y,1\big)$ holds whenever $\varepsilon\in(0,\varepsilon_y)$, we conclude from (2.8.235) and the properties of θ that

$$\lim_{\varepsilon\to 0^+} f_\varepsilon(y) = 1 \quad \text{for each} \quad y\in B(0,1). \qquad (2.8.236)$$

Also, as seen from (2.8.234), for every $\varepsilon\in(0,1)$ we have

$$|f_\varepsilon(y)| \le \int_{\mathbb{R}^n} \theta_\varepsilon\big(x-(1+\varepsilon)y\big)\,\mathrm{d}x = 1 \quad \text{for each} \quad y\in B(0,1). \qquad (2.8.237)$$

Since (2.8.233), Fubini's theorem, and (2.8.234) imply that for each $\varepsilon\in(0,1)$ we have

$$\begin{aligned}\int_{B(0,1)} (\mathrm{div}\vec{G}_\varepsilon)(x)\,\mathrm{d}x &= (1+\varepsilon)^{n-1}\int_{B(0,1)}\Big(\int_{B(0,1)} \theta_\varepsilon\big(x-(1+\varepsilon)y\big)\,\mathrm{d}\mu(y)\Big)\,\mathrm{d}x \\ &= (1+\varepsilon)^{n-1}\int_{B(0,1)}\Big(\int_{B(0,1)} \theta_\varepsilon\big(x-(1+\varepsilon)y\big)\,\mathrm{d}x\Big)\,\mathrm{d}\mu(y) \\ &= (1+\varepsilon)^{n-1}\int_{B(0,1)} f_\varepsilon(y)\,\mathrm{d}\mu(y), \qquad (2.8.238)\end{aligned}$$

we deduce from (2.8.238), (2.8.236), (2.8.237), and Lebesgue's Dominated Convergence Theorem (bearing in mind (2.8.232)) that

$$\lim_{\varepsilon\to 0^+}\int_{B(0,1)} \operatorname{div}\vec{G}_\varepsilon \,\mathrm{d}\mathcal{L}^n = \int_{B(0,1)} 1\,\mathrm{d}\mu = \mu\big(B(0,1)\big) = (\operatorname{div}\vec{F})\big(B(0,1)\big). \tag{2.8.239}$$

Moving on, we claim that for each $\varepsilon \in (0,1)$ we have

$$|\vec{G}_\varepsilon| \le \mathcal{N}_\kappa \vec{F} \quad\text{at each point on}\quad S^{n-1} = \partial B(0,1). \tag{2.8.240}$$

Indeed, (2.8.225) implies that for each $x \in S^{n-1}$ and each $y \in \operatorname{supp}\theta_\varepsilon \subseteq B(0, c\,\varepsilon)$ we have $(1+\varepsilon)^{-1}(x-y) \in B\big((1+\varepsilon)^{-1}x, c\,\varepsilon\big) \subseteq \Gamma_\kappa(x)$. Keeping this in mind, we conclude from (2.8.230) that (2.8.240) holds.

Next, observe that for each $\varepsilon \in (0,1)$, each $y \in \operatorname{supp}\theta_\varepsilon \subseteq B(0, c\,\varepsilon)$, and each $x \in S^{n-1}$, we have

$$\begin{aligned} &(1+\varepsilon)^{-1}(x-y) \in B\big((1+\varepsilon)^{-1}x, c\,\varepsilon\big) \subseteq \Gamma_\kappa(x) \\ \text{and}\quad &\big|(1+\varepsilon)^{-1}(x-y) - x\big| \le \varepsilon(1+\varepsilon)^{-1}(1+c), \end{aligned} \tag{2.8.241}$$

thanks to (2.8.225). Since for each $\varepsilon \in (0,1)$ we may write

$$x\cdot\vec{G}_\varepsilon(x) = \int_{\mathbb{R}^n} x\cdot\vec{F}\big((1+\varepsilon)^{-1}(x-y)\big)\theta_\varepsilon(y)\,\mathrm{d}y \quad\text{for each}\quad x \in S^{n-1}, \tag{2.8.242}$$

it follows that for σ-a.e. point $x \in S^{n-1}$ we have

$$\begin{aligned} &\Big|x\cdot\vec{G}_\varepsilon(x) - \Big(f\Big|^{\kappa-\mathrm{n.t.}}_{\partial B(0,1)}\Big)(x)\Big| \\ &\qquad\le \int_{\mathbb{R}^n}\Big|x\cdot\vec{F}\big((1+\varepsilon)^{-1}(x-y)\big) - \Big(f\Big|^{\kappa-\mathrm{n.t.}}_{\partial B(0,1)}\Big)(x)\Big|\theta_\varepsilon(y)\,\mathrm{d}y \\ &\qquad\le \mathrm{I}_\varepsilon(x) + \mathrm{II}_\varepsilon(x), \end{aligned} \tag{2.8.243}$$

where

$$\begin{aligned} \mathrm{I}_\varepsilon(x) &:= \int_{\mathbb{R}^n}\Big|x\cdot\vec{F}\big(\tfrac{x-y}{1+\varepsilon}\big) - \tfrac{x-y}{1+\varepsilon}\cdot\vec{F}\big(\tfrac{x-y}{1+\varepsilon}\big)\Big|\theta_\varepsilon(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n}\Big|\big(x - \tfrac{x-y}{1+\varepsilon}\big)\cdot\vec{F}\big(\tfrac{x-y}{1+\varepsilon}\big)\Big|\theta_\varepsilon(y)\,\mathrm{d}y, \end{aligned} \tag{2.8.244}$$

and

$$\begin{aligned}\mathrm{II}_\varepsilon(x) &:= \int_{\mathbb{R}^n} \Big| \tfrac{x-y}{1+\varepsilon} \cdot \vec{F}\big(\tfrac{x-y}{1+\varepsilon}\big) - \Big(f\big|^{\kappa-\mathrm{n.t.}}_{\partial B(0,1)}\Big)(x)\Big| \theta_\varepsilon(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \Big| f\big(\tfrac{x-y}{1+\varepsilon}\big) - \Big(f\big|^{\kappa-\mathrm{n.t.}}_{\partial B(0,1)}\Big)(x)\Big| \theta_\varepsilon(y)\,\mathrm{d}y, \end{aligned} \tag{2.8.245}$$

with the second equality provided by (2.8.213). Note that (2.8.241) implies

$$\mathrm{I}_\varepsilon(x) \le \frac{\varepsilon(1+c)}{1+\varepsilon} \mathcal{N}_\kappa \vec{F}(x) \text{ for each } x \in S^{n-1} \text{ and each } \varepsilon \in (0,1). \tag{2.8.246}$$

From (2.8.241), the first line in (2.8.214), and Definition 8.9.1 we also see that

$$\lim_{\varepsilon \to 0^+} \mathrm{II}_\varepsilon(x) = 0 \text{ for } \sigma\text{-a.e. point } x \in S^{n-1}. \tag{2.8.247}$$

Collectively, (2.8.243), (2.8.246), and (2.8.247) imply that

$$\lim_{\varepsilon \to 0^+} x \cdot \vec{G}_\varepsilon(x) = \big(f\big|^{\kappa-\mathrm{n.t.}}_{\partial B(0,1)}\big)(x) \text{ at } \sigma\text{-a.e. } x \in S^{n-1} = \partial B(0,1). \tag{2.8.248}$$

In turn, as a consequence of (2.8.240), (2.8.248), the first membership in (2.8.214), and Lebesgue's Dominated Convergence Theorem we have

$$\begin{array}{l}\text{the family of functions } S^{n-1} \ni x \mapsto x \cdot \vec{G}_\varepsilon(x), \text{ indexed by} \\ \varepsilon \in (0,1), \text{ converges to } f\big|^{\kappa-\mathrm{n.t.}}_{\partial B(0,1)} \text{ in } L^1(S^{n-1},\sigma) \text{ as } \varepsilon \to 0^+.\end{array} \tag{2.8.249}$$

Granted (2.8.231), we may invoke the classical version of the Divergence Theorem (i.e., (1.1.2) with $\Omega := B(0,1)$, so that $\nu(x) = x$ for each $x \in S^{n-1} = \partial B(0,1)$) and conclude that

$$\int_{B(0,1)} \mathrm{div}\vec{G}_\varepsilon \, \mathrm{d}\mathcal{L}^n = \int_{S^{n-1}} x \cdot \vec{G}_\varepsilon(x)\,\mathrm{d}\sigma(x) \text{ for each } \varepsilon \in (0,1). \tag{2.8.250}$$

Sending $\varepsilon \to 0^+$ in (2.8.250) then yields (2.8.215), in view of (2.8.239) and (2.8.249). Lastly, that for any other aperture parameter $\kappa' > 0$ the nontangential trace $f\big|^{\kappa'-\mathrm{n.t.}}_{\partial B(0,1)}$ exists σ-a.e. on S^{n-1}, is actually independent of κ', and belongs to $L^1(S^{n-1},\sigma)$ are consequences of assumptions, Proposition 8.9.8, Corollary 8.9.6, and (8.9.8).

Parenthetically, we wish to remark that if the stronger assumption (2.8.216) is adopted, then in place of (2.8.239) we actually have

$$\lim_{\varepsilon \to 0^+} \mathrm{div}\vec{G}_\varepsilon = \mathrm{div}\vec{F} \text{ in } L^1\big(B(0,1),\mathcal{L}^n\big). \tag{2.8.251}$$

To justify this claim, observe that (2.8.233) presently implies that for each point $x \in B(0,1)$ we have

$$
\begin{aligned}
(\mathrm{div}\vec{G}_\varepsilon)(x) &= (1+\varepsilon)^{n-1}\int_{B(0,1)}(\mathrm{div}\vec{F})(z)\theta_\varepsilon\big(x-(1+\varepsilon)z\big)\,\mathrm{d}z \\
&= (1+\varepsilon)^{-1}\int_{\mathbb{R}^n}(\mathrm{div}\vec{F})\big((1+\varepsilon)^{-1}(x-y)\big)\theta_\varepsilon(y)\,\mathrm{d}y, \qquad (2.8.252)
\end{aligned}
$$

also using (2.8.230) and a change of variables (also keeping in mind the inclusion $\operatorname{supp}\theta_\varepsilon \subseteq B(0,c\,\varepsilon)$). To proceed, pick an arbitrary threshold $\delta > 0$ and select a function $\phi \in \mathscr{C}_c^\infty\big(B(0,1)\big)$ such that

$$
\big\|(\mathrm{div}\vec{F}) - \phi\big\|_{L^1(B(0,1),\mathscr{L}^n)} \leq \delta. \qquad (2.8.253)
$$

We may then estimate

$$
\begin{aligned}
&\int_{B(0,1)}\Big(\int_{\mathbb{R}^n}\Big|(\mathrm{div}\vec{F})\big(\tfrac{x-y}{1+\varepsilon}\big) - \phi\big(\tfrac{x-y}{1+\varepsilon}\big)\Big|\theta_\varepsilon(y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^n}\theta_\varepsilon(y)\Big(\int_{B(0,1)}\Big|(\mathrm{div}\vec{F})\big(\tfrac{x-y}{1+\varepsilon}\big) - \phi\big(\tfrac{x-y}{1+\varepsilon}\big)\Big|\,\mathrm{d}x\Big)\,\mathrm{d}y \\
&\quad = (1+\varepsilon)^n\int_{\mathbb{R}^n}\theta_\varepsilon(y)\Big(\int_{(1+\varepsilon)^{-1}(B(0,1)+B(0,c\,\varepsilon))}\big|(\mathrm{div}\vec{F})(z) - \phi(z)\big|\,\mathrm{d}z\Big)\,\mathrm{d}y \\
&\quad \leq (1+\varepsilon)^n\int_{\mathbb{R}^n}\theta_\varepsilon(y)\Big(\int_{B(0,1)}\big|(\mathrm{div}\vec{F})(z) - \phi(z)\big|\,\mathrm{d}z\Big)\,\mathrm{d}y \\
&\quad \leq (1+\varepsilon)^n\big\|(\mathrm{div}\vec{F}) - \phi\big\|_{L^1(B(0,1),\mathscr{L}^n)} \leq (1+\varepsilon)^n\delta. \qquad (2.8.254)
\end{aligned}
$$

In addition, using the Mean Value Theorem we may estimate

$$
\begin{aligned}
&\int_{B(0,1)}\Big(\int_{\mathbb{R}^n}\big|\phi\big((1+\varepsilon)^{-1}(x-y)\big) - \phi(x)\big|\theta_\varepsilon(y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&\qquad \leq C\varepsilon\cdot\Big(\sup_{B(0,1)}|\nabla\phi|\Big)\int_{\mathbb{R}^n}\theta_\varepsilon(y)\,\mathrm{d}y = C\varepsilon\cdot\Big(\sup_{B(0,1)}|\nabla\phi|\Big), \qquad (2.8.255)
\end{aligned}
$$

for some universal constant $C \in (0,\infty)$. Together, (2.8.252), (2.8.254), (2.8.255), and (2.8.253) allow us to write

$$
\begin{aligned}
&\big\|\mathrm{div}\vec{G}_\varepsilon - (1+\varepsilon)^{-1}\mathrm{div}\vec{F}\big\|_{L^1(B(0,1),\mathscr{L}^n)} \\
&\quad \leq (1+\varepsilon)^{-1}\int_{B(0,1)}\Big(\int_{\mathbb{R}^n}\big|(\mathrm{div}\vec{F})\big(\tfrac{x-y}{1+\varepsilon}\big) - \phi\big(\tfrac{x-y}{1+\varepsilon}\big)\big|\theta_\varepsilon(y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&\qquad + (1+\varepsilon)^{-1}\int_{B(0,1)}\Big(\int_{\mathbb{R}^n}\big|\phi\big(\tfrac{x-y}{1+\varepsilon}\big) - \phi(x)\big|\theta_\varepsilon(y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&\qquad + (1+\varepsilon)^{-1}\|(\mathrm{div}\vec{F}) - \phi\|_{L^1(B(0,1),\mathscr{L}^n)}
\end{aligned}
$$

$$\leq \delta(1+\varepsilon)^{n-1} + C\varepsilon(1+\varepsilon)^{-1} \cdot \Big(\sup_{B(0,1)} |\nabla\phi| \Big) + \delta(1+\varepsilon)^{-1}. \tag{2.8.256}$$

Consequently,

$$\limsup_{\varepsilon\to 0^+} \big\| \mathrm{div}\vec{G}_\varepsilon - (1+\varepsilon)^{-1}\mathrm{div}\vec{F} \big\|_{L^1(B(0,1),\mathcal{L}^n)} \leq 2\delta \tag{2.8.257}$$

which, in view of the arbitrariness of $\delta > 0$, ultimately establishes (2.8.251). □

It is also of interest to prove a version of Proposition 2.8.21 for the complement of the closed unit ball in $\mathbb{R}^n$, of the sort presented below.

Proposition 2.8.22 *Fix $n \in \mathbb{N}$ and pick some aperture parameter $\kappa \in (0,\infty)$. Consider a vector field $\vec{F} = (F_1, \dots, F_n) : \mathbb{R}^n \setminus \overline{B(0,1)} \to \mathbb{C}^n$ with Lebesgue measurable components and define*

$$f(x) := x \cdot \vec{F}(x) = \sum_{j=1}^{n} x_j F_j(x) \ \textit{ for each } \ x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \overline{B(0,1)}. \tag{2.8.258}$$

With $S^{n-1} = \partial B(0,1)$, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor S^{n-1}$, and assume that the following properties are satisfied:

$$\begin{gathered} f\Big|^{\kappa-\mathrm{n.t.}}_{\partial\left(\mathbb{R}^n\setminus\overline{B(0,1)}\right)} \ \textit{exists at } \sigma\textit{-a.e. point on } \ S^{n-1}, \\ \mathcal{N}_\kappa \vec{F} \in L^1(S^{n-1},\sigma), \ \ \textit{and} \ \ \mathrm{div}\vec{F} \in \mathcal{D}'\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big) \\ \textit{extends to a complex Borel measure in } \mathbb{R}^n \setminus \overline{B(0,1)} \end{gathered} \tag{2.8.259}$$

(still denoted by $\mathrm{div}\vec{F}$*).*

Then $[\vec{F}]_\infty$, the contribution of $\vec{F}$ at infinity, is meaningfully and unambiguously defined. Also, for any other aperture parameter $\kappa' > 0$ the nontangential trace $f\Big|^{\kappa'-\mathrm{n.t.}}_{\partial\left(\mathbb{R}^n\setminus\overline{B(0,1)}\right)}$ exists σ-a.e. on S^{n-1} and is actually independent of κ'. When the latter is regarded as a function defined σ-a.e. on S^{n-1}, it belongs to $L^1(S^{n-1},\sigma)$ and, with the dependence on the parameter κ' dropped, one has

$$(\mathrm{div}\vec{F})\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big) = -\int_{S^{n-1}} \Big(f\Big|^{\mathrm{n.t.}}_{\partial\left(\mathbb{R}^n\setminus\overline{B(0,1)}\right)} \Big)\, d\sigma + [\vec{F}]_\infty. \tag{2.8.260}$$

Furthermore, the contribution of $\vec{F}$ at infinity vanishes if there exists $\lambda \in (1,\infty)$ such that

$$\int_{B(0,\lambda R)\setminus B(0,R)} |f|\, d\mathcal{L}^n = o(R^2) \ \ \textit{as} \ \ R \to \infty, \tag{2.8.261}$$

a scenario in which the Divergence Formula (2.8.260) *becomes*

$$(\operatorname{div}\vec{F})\big(\mathbb{R}^n\setminus\overline{B(0,1)}\big) = -\int_{S^{n-1}}\Big(f\big|^{\text{n.t.}}_{\partial(\mathbb{R}^n\setminus\overline{B(0,1)})}\Big)\,d\sigma. \tag{2.8.262}$$

Proof To get started, as a consequence of assumptions, Proposition 8.9.8, Corollary 8.9.6, and (8.9.8), it follows that for any other aperture parameter $\kappa' > 0$ the nontangential trace $f\big|^{\kappa'-\text{n.t.}}_{\partial(\mathbb{R}^n\setminus\overline{B(0,1)})}$ exists σ-a.e. on S^{n-1}, is actually independent of κ', and belongs to $L^1(S^{n-1},\sigma)$. Next, invoke Lemma 8.3.1 to conclude that

$$\vec{F}\in\big[L^\infty_{\text{loc}}\big(\mathbb{R}^n\setminus\overline{B(0,1)},\mathcal{L}^n\big)\big]^n. \tag{2.8.263}$$

Pick a test function $\varphi\in\mathscr{C}^\infty_c(\mathbb{R}^n)$ satisfying $\varphi\equiv 1$ in $\overline{B(0,2)}$ and use this to decompose

$$\vec{F}=\vec{F}_{\text{near}}+\vec{F}_{\text{far}} \quad\text{where}\quad \vec{F}_{\text{near}}:=\varphi\vec{F} \quad\text{and}\quad \vec{F}_{\text{far}}:=(1-\varphi)\vec{F}. \tag{2.8.264}$$

From (2.8.263)–(2.8.264) and assumptions we then see that

$$\begin{gathered}
\vec{F}_{\text{near}},\vec{F}_{\text{far}} \text{ belong to the space } \big[L^\infty_{\text{loc}}\big(\mathbb{R}^n\setminus\overline{B(0,1)},\mathcal{L}^n\big)\big]^n,\\
\vec{F}_{\text{near}} \text{ vanishes at infinity and coincides with } \vec{F} \text{ in } B(0,2)\setminus\overline{B(0,1)},\\
\vec{F}_{\text{far}} \text{ coincides with } \vec{F} \text{ near infinity and vanishes in } B(0,2)\setminus\overline{B(0,1)},\\
\operatorname{div}\vec{F}_{\text{near}}=\varphi\operatorname{div}\vec{F}+(\nabla\varphi)\cdot\vec{F}\in\text{CBM}\big(\big(\mathbb{R}^n\setminus\overline{B(0,1)}\big),\\
\operatorname{div}\vec{F}_{\text{far}}=(1-\varphi)\operatorname{div}\vec{F}-(\nabla\varphi)\cdot\vec{F}\in\text{CBM}\big(\big(\mathbb{R}^n\setminus\overline{B(0,1)}\big),\\
\mathcal{N}_\kappa\vec{F}_{\text{near}}\in L^1(S^{n-1},\sigma) \text{ and } \mathcal{N}_\kappa\vec{F}_{\text{far}}\in L^1(S^{n-1},\sigma).
\end{gathered} \tag{2.8.265}$$

Granted these properties, Theorem 1.3.1 (used with $\Omega:=\mathbb{R}^n\setminus\overline{B(0,1)}$) guarantees that $[\vec{F}_{\text{far}}]_\infty$, the contribution of $\vec{F}_{\text{far}}$ at infinity, is meaningfully and unambiguously defined, and (1.3.8) presently gives

$$\big(\operatorname{div}\vec{F}_{\text{far}}\big)\big(\mathbb{R}^n\setminus\overline{B(0,1)}\big)=[\vec{F}_{\text{far}}]_\infty. \tag{2.8.266}$$

Since the contribution at infinity only depends on the behavior of the vector field in question in a neighborhood of infinity (cf. (1.3.5)), from this and the third line in (2.8.265) we conclude that $[\vec{F}]_\infty$, the contribution of $\vec{F}$ at infinity, is meaningfully and unambiguously defined, and in fact

$$[\vec{F}]_\infty=[\vec{F}_{\text{far}}]_\infty. \tag{2.8.267}$$

Together, (2.8.264)–(2.8.267) imply

$$\begin{aligned}
\big(\operatorname{div}\vec{F}\big)\big(\mathbb{R}^n\setminus\overline{B(0,1)}\big)&=\big(\operatorname{div}\vec{F}_{\text{near}}\big)\big(\mathbb{R}^n\setminus\overline{B(0,1)}\big)+\big(\operatorname{div}\vec{F}_{\text{far}}\big)\big(\mathbb{R}^n\setminus\overline{B(0,1)}\big)\\
&=\big(\operatorname{div}\vec{F}_{\text{near}}\big)\big(\mathbb{R}^n\setminus\overline{B(0,1)}\big)+[\vec{F}]_\infty.
\end{aligned} \tag{2.8.268}$$

Hence, as far as (2.8.260) is concerned, there remains to show that

$$\big(\mathrm{div}\vec{F}_{\text{near}}\big)\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big) = -\int_{S^{n-1}} \Big(f\Big|^{\text{n.t.}}_{\partial\left(\mathbb{R}^n\setminus\overline{B(0,1)}\right)}\Big)\,\mathrm{d}\sigma. \tag{2.8.269}$$

We shall accomplish this using a regularization procedure akin to the one used in the proof of Proposition 2.8.21. As a preliminary step, we fix $c \in \big(0, \frac{\kappa}{4+2\kappa}\big)$ and claim that this implies that

$$B\big((1-\varepsilon)^{-1}x, 2c\,\varepsilon\big) \subseteq \Gamma_\kappa(x) \quad \text{for each} \quad x \in S^{n-1} \quad \text{and each} \quad \varepsilon \in (0,1). \tag{2.8.270}$$

To justify this, pick $x \in S^{n-1}$, $\varepsilon \in (0,1)$, and $y \in B\big((1-\varepsilon)^{-1}x, 2c\,\varepsilon\big)$. Then

$$\begin{aligned}|y| &\geq \big|(1-\varepsilon)^{-1}x\big| - \big|(1-\varepsilon)^{-1}x - y\big| > (1-\varepsilon)^{-1} - 2c\,\varepsilon\\ &= 1 + \varepsilon\big(\tfrac{1}{1-\varepsilon} - 2c\big) > 1 + \varepsilon(1-2c) > 1,\end{aligned} \tag{2.8.271}$$

since $c < 1/2$. Thus, y belongs to $\mathbb{R}^n \setminus \overline{B(0,1)}$. In addition,

$$\mathrm{dist}\Big(y, \partial\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big)\Big) = \mathrm{dist}\big(y, S^{n-1}\big) = |y| - 1 > \varepsilon\big(\tfrac{1}{1-\varepsilon} - 2c\big). \tag{2.8.272}$$

On the other hand,

$$|y-x| \leq \big|y - (1-\varepsilon)^{-1}x\big| + \big|(1-\varepsilon)^{-1}x - x\big| < 2c\,\varepsilon + \tfrac{\varepsilon}{1-\varepsilon} \tag{2.8.273}$$

so in order to have $|y-x| < (1+\kappa)\,\mathrm{dist}\big(y, S^{n-1}\big)$ it suffices that

$$2c\,\varepsilon + \tfrac{\varepsilon}{1-\varepsilon} < (1+\kappa)\varepsilon\big(\tfrac{1}{1-\varepsilon} - 2c\big) \tag{2.8.274}$$

or, equivalently, $4c < \kappa\big(\frac{1}{1-\varepsilon} - 2c\big)$. Since $\frac{1}{1-\varepsilon} > 1$, this inequality holds whenever we have $4c < \kappa(1-2c)$ which, in turn, is true by virtue of the initial choice of c. The above reasoning shows that $y \in \Gamma_\kappa(x)$, finishing the proof of the claim made in (2.8.270).

Next, with c as before, bring in a non-negative function $\theta \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ satisfying $\mathrm{supp}\,\theta \subseteq B(0,c)$ and $\int_{\mathbb{R}^n}\theta\,\mathrm{d}\mathcal{L}^n = 1$. For each $\varepsilon \in (0,1)$ set $\theta_\varepsilon(x) := \varepsilon^{-n}\theta(x/\varepsilon)$ for all $x \in \mathbb{R}^n$. Introduce the vector field defined for each $x \in \mathbb{R}^n \setminus B(0,1)$ as

$$\begin{aligned}\vec{G}_\varepsilon(x) &:= \int_{\mathbb{R}^n} \vec{F}_{\text{near}}\big((1-\varepsilon)^{-1}(x-y)\big)\theta_\varepsilon(y)\,\mathrm{d}y \\ &= (1-\varepsilon)^n \int_{\mathbb{R}^n\setminus\overline{B(0,1)}} \vec{F}_{\text{near}}(z)\theta_\varepsilon\big(x-(1-\varepsilon)z\big)\,\mathrm{d}z \\ &= (1-\varepsilon)^n \Big({}_{\mathcal{D}'(\mathbb{R}^n\setminus\overline{B(0,1)})}\Big\langle\big(\vec{F}_{\text{near}}\big)_j, \theta_\varepsilon\big(x-(1-\varepsilon)\,\cdot\,\big)\Big\rangle_{\mathcal{D}(\mathbb{R}^n\setminus\overline{B(0,1)})}\Big)_{1\leq j\leq n}.\end{aligned} \tag{2.8.275}$$

From (2.8.275) and the first two lines in (2.8.265) we see that $\vec{G}_\varepsilon$ is well-defined and, in fact,

$$\vec{G}_\varepsilon \in \big[\mathscr{C}^\infty\big(\mathbb{R}^n \setminus B(0,1)\big)\big]^n \text{ and } \vec{G}_\varepsilon \text{ vanishes identically in a neighborhood of infinity.} \tag{2.8.276}$$

Also, if we abbreviate

$$\mu := \operatorname{div}\vec{F}_{\text{near}} \in \mathrm{CBM}\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big), \tag{2.8.277}$$

then reasoning much as in (2.8.233) from (2.8.275) and (1.3.12) we see that for each $\varepsilon \in (0,1)$ and each $x \in \mathbb{R}^n \setminus \overline{B(0,1)}$ we have

$$(\operatorname{div}\vec{G}_\varepsilon)(x) = (1-\varepsilon)^{n-1}\int_{\mathbb{R}^n\setminus\overline{B(0,1)}} \theta_\varepsilon\big(x-(1-\varepsilon)y\big)\,\mathrm{d}\mu(y). \tag{2.8.278}$$

To proceed, for every $\varepsilon \in (0,1)$ define

$$f_\varepsilon(y) := \int_{\mathbb{R}^n\setminus\overline{B(0,1)}} \theta_\varepsilon\big(x-(1-\varepsilon)y\big)\,\mathrm{d}x \quad \text{for each } \ y \in \mathbb{R}^n \setminus \overline{B(0,1)}, \tag{2.8.279}$$

and note that, for each $y \in \mathbb{R}^n \setminus \overline{B(0,1)}$ fixed, the change of variables $z := x-(1-\varepsilon)y$ places the new variable z in $\mathbb{R}^n \setminus \overline{B(0,1)} - (1-\varepsilon)y = \mathbb{R}^n \setminus \overline{B\big(-(1-\varepsilon)y, 1\big)}$. Thus,

$$f_\varepsilon(y) = \int_{\mathbb{R}^n\setminus\overline{B(-(1-\varepsilon)y,1)}} \theta_\varepsilon(z)\,\mathrm{d}z \quad \text{for each } \ y \in \mathbb{R}^n \setminus \overline{B(0,1)}. \tag{2.8.280}$$

We claim that for each given point $y \in \mathbb{R}^n \setminus \overline{B(0,1)}$ there exists a small threshold $\varepsilon_y \in (0,1)$ with the property that

$$B(0, c\,\varepsilon) \subseteq \mathbb{R}^n \setminus \overline{B\big(-(1-\varepsilon)y, 1\big)} \quad \text{for each } \ \varepsilon \in (0, \varepsilon_y). \tag{2.8.281}$$

Indeed, for each $y \in \mathbb{R}^n \setminus \overline{B(0,1)}$ and $\xi \in B(0, c\,\varepsilon)$ we have

$$|\xi + (1-\varepsilon)y| \geq (1-\varepsilon)|y| - |\xi| > (1-\varepsilon)|y| - c\,\varepsilon > 1, \tag{2.8.282}$$

with the very last inequality valid provided $0 < \varepsilon < (|y|-1)/(|y|+c) =: \varepsilon_y$. Having established (2.8.281), we conclude from (2.8.280) and the properties of θ that

$$\lim_{\varepsilon\to 0^+} f_\varepsilon(y) = 1 \quad \text{for each } \ y \in \mathbb{R}^n \setminus \overline{B(0,1)}. \tag{2.8.283}$$

Also, as seen from (2.8.279), for every $\varepsilon \in (0,1)$ we have

$$|f_\varepsilon(y)| \leq \int_{\mathbb{R}^n} \theta_\varepsilon\big(x-(1-\varepsilon)y\big)\,\mathrm{d}x = 1 \quad \text{for each} \quad y \in \mathbb{R}^n \setminus \overline{B(0,1)}. \tag{2.8.284}$$

Going further, (2.8.278), Fubini's theorem, and (2.8.279) imply that for each given $\varepsilon \in (0,1)$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus\overline{B(0,1)}} (\operatorname{div}\vec{G}_\varepsilon)(x)\,\mathrm{d}x \\
&\qquad = (1-\varepsilon)^{n-1}\int_{\mathbb{R}^n\setminus\overline{B(0,1)}}\Big(\int_{\mathbb{R}^n\setminus\overline{B(0,1)}}\theta_\varepsilon\big(x-(1-\varepsilon)y\big)\,\mathrm{d}\mu(y)\Big)\,\mathrm{d}x \\
&\qquad = (1-\varepsilon)^{n-1}\int_{\mathbb{R}^n\setminus\overline{B(0,1)}}\Big(\int_{\mathbb{R}^n\setminus\overline{B(0,1)}}\theta_\varepsilon\big(x-(1-\varepsilon)y\big)\,\mathrm{d}x\Big)\,\mathrm{d}\mu(y) \\
&\qquad = (1-\varepsilon)^{n-1}\int_{\mathbb{R}^n\setminus\overline{B(0,1)}} f_\varepsilon(y)\,\mathrm{d}\mu(y).
\end{aligned} \tag{2.8.285}$$

In turn, from (2.8.285), (2.8.283), (2.8.284), and Lebesgue's Dominated Convergence Theorem (bearing in mind (2.8.277)) we deduce that

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n\setminus\overline{B(0,1)}} \operatorname{div}\vec{G}_\varepsilon\,\mathrm{d}\mathcal{L}^n &= \int_{\mathbb{R}^n\setminus\overline{B(0,1)}} 1\,\mathrm{d}\mu = \mu\big(\mathbb{R}^n\setminus\overline{B(0,1)}\big) \\
&= \big(\operatorname{div}\vec{F}_{\text{near}}\big)\big(\mathbb{R}^n\setminus\overline{B(0,1)}\big).
\end{aligned} \tag{2.8.286}$$

Pressing on, we claim that

$$\begin{gathered}
|\vec{G}_\varepsilon(x)| \leq \mathcal{N}_\kappa\vec{F}_{\text{near}}(x) \quad \text{for each} \quad \varepsilon\in(0,1/2) \\
\text{and each} \quad x \in S^{n-1} = \partial\big(\mathbb{R}^n\setminus\overline{B(0,1)}\big).
\end{gathered} \tag{2.8.287}$$

To justify this, fix some $\varepsilon\in(0,1/2)$. The key observation is that for each point $x\in S^{n-1}$ and each point $y\in\operatorname{supp}\theta_\varepsilon\subseteq B(0,c\,\varepsilon)$ we have

$$(1-\varepsilon)^{-1}(x-y)\in B\big((1-\varepsilon)^{-1}x, 2c\,\varepsilon\big)\subseteq\Gamma_\kappa(x), \tag{2.8.288}$$

with the veracity of the membership ensured by the assumption $\varepsilon\in(0,1/2)$, and the subsequent inclusion coming from (2.8.270). From (2.8.288) and (2.8.275) we then conclude that (2.8.287) holds.

Next, observe that for each $\varepsilon\in(0,1/2)$, each $y\in\operatorname{supp}\theta_\varepsilon\subseteq B(0,c\,\varepsilon)$, and each $x\in S^{n-1}$ we have

$$\begin{aligned}
&(1-\varepsilon)^{-1}(x-y)\in B\big((1-\varepsilon)^{-1}x, 2c\,\varepsilon\big)\subseteq\Gamma_\kappa(x) \\
\text{and}\quad &\big|(1-\varepsilon)^{-1}(x-y)-x\big|\leq\varepsilon(1-\varepsilon)^{-1}(1+c),
\end{aligned} \tag{2.8.289}$$

thanks to (2.8.288). Henceforth, restrict

$$0 < \varepsilon < \min\left\{\frac{1}{2},\ \frac{1}{1+c}\right\} \tag{2.8.290}$$

and write

$$x \cdot \vec{G}_\varepsilon(x) = \int_{\mathbb{R}^n} x \cdot \vec{F}_{\text{near}}\big((1-\varepsilon)^{-1}(x-y)\big)\theta_\varepsilon(y)\,\mathrm{d}y \quad \text{for each} \quad x \in S^{n-1}. \tag{2.8.291}$$

For σ-a.e. point $x \in S^{n-1} = \partial\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big)$, use (2.8.291) to estimate

$$\begin{aligned}
&\Big| x \cdot \vec{G}_\varepsilon(x) - \Big(f\big|^{\kappa-\text{n.t.}}_{\partial(\mathbb{R}^n\setminus\overline{B(0,1)})}\Big)(x)\Big| \\
&\qquad \leq \int_{\mathbb{R}^n} \Big| x \cdot \vec{F}_{\text{near}}\big((1-\varepsilon)^{-1}(x-y)\big) - \Big(f\big|^{\kappa-\text{n.t.}}_{\partial(\mathbb{R}^n\setminus\overline{B(0,1)})}\Big)(x)\Big|\theta_\varepsilon(y)\,\mathrm{d}y \\
&\qquad \leq \mathrm{I}_\varepsilon(x) + \mathrm{II}_\varepsilon(x),
\end{aligned} \tag{2.8.292}$$

where

$$\begin{aligned}
\mathrm{I}_\varepsilon(x) &:= \int_{\mathbb{R}^n} \Big| x \cdot \vec{F}_{\text{near}}\big(\tfrac{x-y}{1+\varepsilon}\big) - \tfrac{x-y}{1+\varepsilon} \cdot \vec{F}_{\text{near}}\big(\tfrac{x-y}{1+\varepsilon}\big)\Big|\theta_\varepsilon(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \Big| \big(x - \tfrac{x-y}{1+\varepsilon}\big) \cdot \vec{F}_{\text{near}}\big(\tfrac{x-y}{1+\varepsilon}\big)\Big|\theta_\varepsilon(y)\,\mathrm{d}y,
\end{aligned} \tag{2.8.293}$$

and

$$\begin{aligned}
\mathrm{II}_\varepsilon(x) &:= \int_{\mathbb{R}^n} \Big| \tfrac{x-y}{1+\varepsilon} \cdot \vec{F}_{\text{near}}\big(\tfrac{x-y}{1+\varepsilon}\big) - \Big(f\big|^{\kappa-\text{n.t.}}_{\partial(\mathbb{R}^n\setminus\overline{B(0,1)})}\Big)(x)\Big|\theta_\varepsilon(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \Big| f\big(\tfrac{x-y}{1+\varepsilon}\big) - \Big(f\big|^{\kappa-\text{n.t.}}_{\partial(\mathbb{R}^n\setminus\overline{B(0,1)})}\Big)(x)\Big|\theta_\varepsilon(y)\,\mathrm{d}y.
\end{aligned} \tag{2.8.294}$$

In (2.8.294), the second equality is provided by (2.8.258), the second line in (2.8.265), and the observation that (2.8.290) forces $(1-\varepsilon)^{-1}(x-y)$ to be in $B(0,2) \setminus \overline{B(0,1)}$ whenever $x \in S^{n-1}$ and $y \in \operatorname{supp}\theta_\varepsilon$. Together, (2.8.293) and (2.8.289) imply

$$\mathrm{I}_\varepsilon(x) \leq \varepsilon(1-\varepsilon)^{-1}(1+c)\mathcal{N}_\kappa \vec{F}_{\text{near}}(x) \quad \text{for each} \quad x \in S^{n-1}. \tag{2.8.295}$$

From (2.8.289), the first property in the first line of (2.8.259), and Definition 8.9.1 we also see that

$$\lim_{\varepsilon\to 0^+} \mathrm{II}_\varepsilon(x) = 0 \quad \text{for } \sigma\text{-a.e. point} \quad x \in S^{n-1}. \tag{2.8.296}$$

Collectively, (2.8.292), (2.8.295), and (2.8.296) imply

$$\lim_{\varepsilon\to 0^+} x \cdot \vec{G}_\varepsilon(x) = \Big(f\big|^{\kappa-\mathrm{n.t.}}_{\partial(\mathbb{R}^n\setminus\overline{B(0,1)})}\Big)(x) \quad \text{at } \sigma\text{-a.e. } x \in S^{n-1} = \partial\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big). \tag{2.8.297}$$

In turn, as a consequence of (2.8.287), (2.8.297), the first membership in (2.8.259), and Lebesgue's Dominated Convergence Theorem we obtain

$$\begin{aligned}&\text{the family of functions } S^{n-1} \ni x \mapsto x \cdot \vec{G}_\varepsilon(x), \text{ indexed by } \varepsilon \text{ as in}\\ &\text{(2.8.290), converges to } f\big|^{\kappa-\mathrm{n.t.}}_{\partial(\mathbb{R}^n\setminus\overline{B(0,1)})} \text{ in } L^1(S^{n-1},\sigma) \text{ as } \varepsilon \to 0^+.\end{aligned} \tag{2.8.298}$$

Granted (2.8.276), we may invoke the ordinary version of the Divergence Theorem for the domain $\mathbb{R}^n \setminus \overline{B(0,1)}$, whose outward unit normal is $\nu(x) = -x$ for each $x \in S^{n-1} = \partial\big(\mathbb{R}^n \setminus \overline{B(0,1)}\big)$, and the smooth vector field $\vec{G}_\varepsilon$ as in (2.8.276), and conclude that

$$\int_{\mathbb{R}^n\setminus\overline{B(0,1)}} \mathrm{div}\vec{G}_\varepsilon \, d\mathcal{L}^n = -\int_{S^{n-1}} x \cdot \vec{G}_\varepsilon(x)\, d\sigma(x) \quad \text{for each } \varepsilon \in (0,1). \tag{2.8.299}$$

Upon letting $\varepsilon \to 0^+$ in (2.8.299) we arrive at (2.8.269), in light of (2.8.286) and (2.8.298). This finishes the proof of (2.8.260).

Finally, that $[\vec{F}]_\infty$ is actually zero if (2.8.261) holds for some $\lambda \in (1,\infty)$ is a direct consequence of the definition made in (2.8.258), Lemma 4.7.3, and what we have proved already. □

We conclude this section by providing a brief survey of additional work concerning the Divergence Theorem. Various versions of the Divergence Theorem involving sets with fractal boundaries appear in [115], where the authors establish a version of Stokes's Theorem by introducing a certain brand of surface integral for smooth forms over what they called chainlets (a class of sets general enough to contain fractals), and [167] where the classical Gauss–Green Theorem is extended to certain fractal domains (by showing that the boundary of a Hölder domain is a geometric rough path). See also [114] in this regard, where versions of Gauss', Green's, and Stokes' Theorems are produced on regions that may not be locally Euclidean and have no tangent vectors defined anywhere, by replacing the parametrization of a domain with suitable polyhedral approximations.

There is also a considerable body of work pertaining to the Divergence Theorem whose formulation employs the Henstock–Kurzweil integral; cf., e.g., [130, 143, 157, 168, 212, 215], and the references therein.

Within the framework of Lebesgue integration, the monograph [216] starts by establishing a variety of Divergence Theorem via a combinatorial argument involving dyadic cubes, which is subsequently adapted to sets of locally finite perimeter and a certain class of bounded vector fields which are admissible, in a suitable sense, relative to the underlying domain (cf. [216, Definition 2.3.1, p. 27]). Earlier references to the Divergence Theorem in the context of Lebesgue integration and emphasis on weaker assumptions on the vector fields involved include [24, 51, 235, 236].

Finally, we wish to mention the version of the Gauss–Green Theorem for BV functions with integrable rough traces, as presented in [178, Theorem, p. 506] (cf. also [80, Theorem 1, p. 177] and [216, Theorem 7.6.1, p. 179] for versions of this result involving Lipschitz domains).

Chapter 3
Measure Theoretical and Topological Rudiments

The material in this chapter is centered around the notions of measure and topology, including sigma-algebras, Borelians, ordinary measures, outer measures, Borel regularity, inner and outer regularity, separable measures, the support of a measure, Radon measures, complex Borel measures, Lebesgue spaces (separability, density results), the topology on the space of measurable functions, and Riesz's Representation Theorem.

Before beginning in earnest, we first describe some standard notation, and elaborate on conventions frequently employed. Throughout, we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In the absence of any other specifications, it is understood that $n \in \mathbb{N}$. For $s \geq 0$, by $\mathcal{H}^s_*$ we denote the s-dimensional Hausdorff outer measure in $\mathbb{R}^n$, with the usual normalization. Specifically, for each set $A \subseteq \mathbb{R}^n$ one defines

$$\mathcal{H}^s_*(A) := \limsup_{\varepsilon \to 0^+} \mathcal{H}^s_\varepsilon(A), \tag{3.0.1}$$

where, for each $\varepsilon > 0$,

$$\mathcal{H}^s_\varepsilon(A) := \inf\Big\{\frac{\pi^{s/2}}{\Gamma(1+s/2)}\sum_{j=1}^{\infty}\Big(\frac{\operatorname{diam}(A_j)}{2}\Big)^s\Big\} \tag{3.0.2}$$

where $\Gamma(t) := \int_0^\infty e^{-x} x^{t-1}\,dx$ with $0 < t < \infty$ is the usual Gamma function, and where the infimum is taken over all countable families $\{A_j\}_{j\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$ having diameters $\leq \varepsilon$ and with the property that $A \subseteq \bigcup_{j\in\mathbb{N}} A_j$. Finally, for each $s \in [0, \infty)$, we then denote by $\mathcal{H}^s$ the s-dimensional Hausdorff measure in $\mathbb{R}^n$ (i.e., the restriction of $\mathcal{H}^s_*$ to the sigma-algebra of its measurable sets, in the sense of Carathéodory; cf., e.g., [80, p. 2], [91, p. 29]). As is well known (cf., e.g., [80, Theorem 1, p. 61])

$$\mathcal{H}^s \text{ is a Borel-regular measure in } \mathbb{R}^n, \text{ for each } s \in [0, \infty). \tag{3.0.3}$$

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Corresponding to $s = 0$, it is clear that $\mathcal{H}^0$ is the counting measure. We also let $\mathcal{L}^n$ stand for the Lebesgue measure in $\mathbb{R}^n$. The interior, closure, and (topological) boundary of an arbitrary set $U \subseteq \mathbb{R}^n$ are denoted by $\mathring{U}$, $\overline{U}$, and ∂U, respectively. We also let $U \triangle V$ denote the symmetric difference $(U \setminus V) \cup (V \setminus U)$ of any two given sets U, V, and write $U \sqcup V$ for the union of two disjoint sets U, V. The symbol $\mathbf{1}_E$ is used to denote the characteristic function of a given set E. We let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ denote the standard orthonormal basis in $\mathbb{R}^n$. Given an arbitrary open set $\Omega \subseteq \mathbb{R}^n$ along with $p \in [1, \infty]$ and $k \in \mathbb{N}$, denote by $W^{k,p}(\Omega)$ the standard L^p-based Sobolev space of order k in Ω, consisting of locally integrable functions (with respect to the Lebesgue measure) in Ω whose partial derivatives of order $\leq k$ (considered in the sense of distributions in Ω) are p-th power integrable functions in Ω. Also, we denote by $W^{k,p}_{\mathrm{loc}}(\Omega)$ the local version of this space, and let

$W^{k,p}_{\mathrm{bdd}}(\Omega)$ denote the space of functions $u \in W^{k,p}_{\mathrm{loc}}(\Omega)$ with the property that $\partial^\alpha u \in L^p(\mathcal{O}, \mathcal{L}^n)$ for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ and each bounded Lebesgue measurable subset $\mathcal{O}$ of Ω. (3.0.4)

In particular, classical embeddings ensure that

$$W^{1,p}_{\mathrm{loc}}(\Omega) \hookrightarrow \mathscr{C}^0_{\mathrm{loc}}(\Omega) \ \text{ if } \ p > n, \tag{3.0.5}$$

and

$$W^{1,\infty}_{\mathrm{loc}}(\Omega) = \mathrm{Lip}_{\mathrm{loc}}(\Omega), \tag{3.0.6}$$

the space of locally Lipschitz functions in Ω. Throughout, we shall adopt the following convention. For a function u belonging to $W^{k,1}_{\mathrm{loc}}$ (with $k \in \mathbb{N}$) in an open subset of $\mathbb{R}^n$, we agree to abbreviate

$$|\nabla^k u| := \sum_{\gamma \in \mathbb{N}_0^n,\ |\gamma| = k} |\partial^\gamma u|. \tag{3.0.7}$$

The Jacobian matrix of a differentiable $\mathbb{C}^M$-valued function $u = (u_\alpha)_{1 \leq \alpha \leq M}$ defined in an open subset of $\mathbb{R}^n$ is the $\mathbb{C}^{M \cdot n}$-valued function

$$\nabla u := \big(\partial_j u_\alpha\big)_{\substack{1 \leq \alpha \leq M \\ 1 \leq j \leq n}} = \begin{bmatrix} \partial_1 u_1 & \cdots & \partial_n u_1 \\ \vdots & \vdots & \vdots \\ \partial_1 u_M & \cdots & \partial_n u_M \end{bmatrix}. \tag{3.0.8}$$

Finally, we shall retain the same symbol ∇u when the components of u are actually distributions.

3.1 Sigma-Algebras, Measures, Lebesgue Spaces

Recall that a sigma-algebra of subsets of an arbitrary set X is a family that contains the empty set and which is closed under taking complements relative to X, as well as countable unions (hence also countable intersections).

Let (X, τ) be now a topological space. We agree to denote by $Borel_\tau(X)$ the Borelians of (X, τ), i.e., the smallest sigma-algebra of X containing τ. Also, given an arbitrary set $A \subseteq X$, we shall denote by $\tau|_A$ the (relative) topology induced by τ on A. Of course, $(A, \tau|_A)$ then becomes a topological space itself. In this connection, it is useful to remark that for any $A \subseteq X$

$$\big\{A \cap B : \ B \in Borel_\tau(X)\big\} = Borel_{\tau|_A}(A). \tag{3.1.1}$$

Indeed, if we consider

$$\mathcal{F} := \big\{A \cap B \colon B \in Borel_\tau(X)\big\}, \quad \mathcal{G} := \big\{B \subseteq X \colon B \cap A \in Borel_{\tau|_A}(A)\big\}, \tag{3.1.2}$$

then it is easily checked that $\mathcal{F}$ is a sigma-algebra of subsets of A which contains the open subsets of $(A, \tau|_A)$, whereas $\mathcal{G}$ is a sigma-algebra of subsets of X which contains the open subsets of (X, τ). Consequently, $Borel_{\tau|_A}(A) \subseteq \mathcal{F}$ and $Borel_\tau(X) \subseteq \mathcal{G}$. Now, the first of these two inclusions yields the right-to-left inclusion in (3.1.1), while the second one gives the left-to-right inclusion in (3.1.1). Hence, (3.1.1) follows.

Assume X is an arbitrary, fixed set. Denote by 2^X the collection of all subsets of X. A measure space structure on X is a triplet $(X, \mathfrak{M}, \mu)$ in which $\mathfrak{M} \subseteq 2^X$ is a sigma-algebra of sets and the measure μ is an extended real-valued, non-negative function $\mu : \mathfrak{M} \to [0, +\infty]$ satisfying

$$\mu(\varnothing) = 0 \text{ and } \mu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) = \sum_{j\in\mathbb{N}} \mu(A_j) \text{ if } A_j \in \mathfrak{M},\ j \in \mathbb{N}, \text{ are mutually disjoint.} \tag{3.1.3}$$

Whenever $(X, \mathfrak{M}, \mu)$ is a measure space, we shall refer to $(\mathfrak{M}, \mu)$ as being a measure on X.

As is well known, in any measure space $(X, \mathfrak{M}, \mu)$, one has

$$\mu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) \le \sum_{j\in\mathbb{N}} \mu(A_j) \text{ if } A_j \in \mathfrak{M},\ j \in \mathbb{N}, \tag{3.1.4}$$

$$\mu\Big(\bigcap_{j\in\mathbb{N}} A_j\Big) = \lim_{j\to+\infty} \mu(A_j) \text{ if } A_j \in \mathfrak{M},\ A_{j+1} \subseteq A_j,\ j \in \mathbb{N}, \text{ and } \mu(A_1) < +\infty, \tag{3.1.5}$$

$$\mu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) = \lim_{j\to+\infty} \mu(A_j) \text{ if } A_j \in \mathfrak{M},\ A_j \subseteq A_{j+1},\ j \in \mathbb{N}. \tag{3.1.6}$$

Sets in the family $\mathfrak{M}$ will be referred to as being μ-measurable. A given function $f : X \to [-\infty, +\infty]$ is called μ-measurable (or $\mathfrak{M}$-measurable) if $f^{-1}(I) \in \mathfrak{M}$ for any interval I of any (hence all) of the following types: $\big\{[-\infty, \alpha] : \alpha \in \mathbb{R}\big\}$, $\big\{[-\infty, \alpha) : \alpha \in \mathbb{R}\big\}$, $\big\{[\alpha, +\infty] : \alpha \in \mathbb{R}\big\}$, $\big\{(\alpha, +\infty) : \alpha \in \mathbb{R}\big\}$.

Given an arbitrary measure space $(X, \mathfrak{M}, \mu)$ along with an arbitrary integrability exponent $p \in (0, \infty)$, the Lebesgue space $L^p(X, \mu) = L^p(X, \mathfrak{M}, \mu)$ consists of equivalence classes[1] of scalar-valued μ-measurable functions f on X satisfying $\|f\|_{L^p(X,\mu)} := \Big(\int_X |f|^p \,\mathrm{d}\mu\Big)^{1/p} < +\infty$. Corresponding to the end-point case $p = \infty$, the space $L^\infty(X, \mu)$ consists of all equivalence classes of scalar-valued μ-measurable functions f on X for which

$$\|f\|_{L^\infty(X,\mu)} := \inf \big\{\lambda > 0 : \ \mu(\{x \in X : \ |f(x)| > \lambda\}) = 0\big\} < +\infty. \tag{3.1.7}$$

As is well known, for each $p \in (0, \infty]$, the space $L^p(X, \mu)$ is quasi-Banach. Moreover, for each $p \in (0, 1]$, the space $L^p(X, \mu)$ is a p-Banach space since

$$L^p(X, \mu) \ni f, g \mapsto \|f - g\|^p_{L^p(X,\mu)} \in [0, \infty) \ \text{ is a metric} \tag{3.1.8}$$

with respect to which this space is complete, and $L^p(X, \mu)$ is a Banach space if $p \in [1, \infty]$.

Given a measure space $(X, \mathfrak{M}, \mu)$, denote by $S(X, \mu)$ the space of all `simple functions` on X (i.e., μ-measurable real-valued functions defined on X whose range has finite cardinality). More specifically, $S(X, \mu)$ consists of all functions which may be expressed as $\sum_{i=1}^N a_i \mathbf{1}_{A_i}$, where $N \in \mathbb{N}$, the a_i's are nonzero real numbers, the A_i's are mutually disjoint sets in $\mathfrak{M}$. Such a writing is unique. It is known (cf., e.g., [91, Theorem 6.8(e), p. 184]) that

$$S(X, \mu) \hookrightarrow L^\infty(X, \mu) \ \text{ densely.} \tag{3.1.9}$$

Consider the subspace of $S(X, \mu)$ consisting of functions $s = \sum_{i=1}^N a_i \mathbf{1}_{A_i}$, where N, a_i, A_i are as before, with the additional property that $\mu(A_i) < \infty$ for each index $i \in \{1, \dots, N\}$. If we denote the aforementioned space by $S_{\mathrm{fin}}(X, \mu)$, then

$$S_{\mathrm{fin}}(X, \mu) = \big\{s \in S(X, \mu) : \ \mu(\{s \neq 0\}) < \infty\big\}, \tag{3.1.10}$$

and

$$S_{\mathrm{fin}}(X, \mu) \hookrightarrow L^p(X, \mu) \ \text{ densely, for each } \ p \in (0, \infty). \tag{3.1.11}$$

Ultimately, (3.1.11) is a consequence of Lebesgue's Monotone Convergence Theorem and the fact that if f is a non-negative, μ-measurable, function on X then [231, Theorem 1.17, p. 15] guarantees the existence of a sequence of simple functions $\{s_j\}_{j \in \mathbb{N}} \subset S(X, \mu)$ with the property that

[1] identifying functions which are equal μ-a.e. on X.

$$\begin{aligned}&0 \le s_j \le s_{j+1} \le f \ \text{ everywhere on } X, \ \text{for each } j \in \mathbb{N},\\&\text{and such that } \lim_{j\to\infty} s_j(x) = f(x) \text{ for every point } x \in X.\end{aligned} \tag{3.1.12}$$

Indeed, if f also belongs to $L^p(X, \mu)$ then the pointwise inequality $0 \le s_j \le f$ on X implies that each s_j actually belongs to $S_{\text{fin}}(X, \mu)$. As a corollary of (3.1.11), we deduce that

$$L^p(X, \mu) \cap L^q(X, \mu) \hookrightarrow L^p(X, \mu) \ \text{ densely, for each } \ p, q \in (0, \infty), \tag{3.1.13}$$

and that

$$\begin{aligned}&\text{given any function } f \in L^p(X, \mu) \cap L^q(X, \mu) \text{ with } p, q \in (0, \infty), \text{ there}\\&\text{exists some sequence } \{s_j\}_{j\in\mathbb{N}} \subset S_{\text{fin}}(X, \mu) \text{ which converges to } f \text{ both}\\&\text{in } L^p(X, \mu) \text{ and in } L^q(X, \mu).\end{aligned} \tag{3.1.14}$$

For further use, let us also point out that

$$\begin{aligned}&\text{if } (X, \mathfrak{M}, \mu) \text{ is a sigma-finite measure space, then given any non-negative}\\&\mu\text{-measurable function } f \text{ defined on } X, \text{ there exists a sequence } \{s_j\}_{j\in\mathbb{N}}\\&\text{of functions in } S_{\text{fin}}(X, \mu) \text{ satisfying all properties listed in (3.1.12).}\end{aligned} \tag{3.1.15}$$

Specifically, assume $X = \bigcup_{i=1}^{\infty} O_i$ with each $O_i \in \mathfrak{M}$ satisfying $\mu(O_i) < +\infty$, and set $W_j := \bigcup_{i=1}^{j} O_i$ for each $j \in \mathbb{N}$. If $\{s_j\}_{j\in\mathbb{N}} \subset S(X, \mu)$ are as in (3.1.12), then the sequence $\big\{s_j \cdot \mathbf{1}_{W_j}\big\}_{j\in\mathbb{N}} \subset S_{\text{fin}}(X, \mu)$ does the job.

In the proposition below, we introduce some generalized local-L^p spaces and study their topologies.

Proposition 3.1.1 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and suppose $\mathscr{F} := \{\Delta_j\}_{j\in\mathbb{N}}$ is a countable subfamily of $\mathfrak{M}$ with the property that $X = \bigcup_{j\in\mathbb{N}} \Delta_j$. For each integrability exponent $p \in (0, \infty]$, consider the linear space*

$$L^p_{\mathscr{F}}(X, \mu) := \Big\{f : X \to \mathbb{C} : f \text{ is } \mu\text{-measurable and } \|f\|_{L^p(\Delta_j, \mu)} < +\infty \ \text{ for each } \ j \in \mathbb{N}\Big\} \tag{3.1.16}$$

then define $\tau_{\mathscr{F}}$ to be the collection of all subsets O of $L^p_{\mathscr{F}}(X, \mu)$ enjoying the following property:

$$\begin{aligned}&\textit{for each given } f \in O \textit{ there exist some number } \varepsilon > 0 \textit{ and}\\&\textit{some finite set } J \subseteq \mathbb{N} \textit{ such that any } g \in L^p_{\mathscr{F}}(X, \mu) \textit{ satisfying}\\&\max\nolimits_{j\in J} \|f - g\|_{L^p(\Delta_j, \mu)} < \varepsilon \textit{ necessarily belongs to the set } O.\end{aligned} \tag{3.1.17}$$

Then $\tau_{\mathscr{F}}$ is a topology on $L^p_{\mathscr{F}}(X, \mu)$, which is actually locally convex whenever $p \in [1, \infty]$, and $\big(L^p_{\mathscr{F}}(X, \mu), \tau_{\mathscr{F}}\big)$ is a topological vector space which is metrizable via a translation invariant metric which renders $L^p_{\mathscr{F}}(X, \mu)$ a complete metric space.

Proof That $\tau_{\mathscr{F}}$ is indeed a topology on $L^p_{\mathscr{F}}(X, \mu)$ (which is actually locally convex if $p \in [1, \infty]$), and $\big(L^p_{\mathscr{F}}(X, \mu), \tau_{\mathscr{F}}\big)$ is a topological vector space, may be seen from definitions in a straightforward fashion. To show that the topology $\tau_{\mathscr{F}}$ is metrizable, starting from the observation that the function $\Phi : [0, \infty] \to [0, 1]$, defined for each $t \in [0, \infty]$ as $\Phi(t) := t/(1 + t)$, is a strictly increasing homeomorphism which is subadditive (i.e., satisfies $\Phi(t_1 + t_2) \leq \Phi(t_1) + \Phi(t_2)$ for all $t_1, t_2 \in [0, \infty]$), we may check without difficulty that the assignment

$$L^p_{\mathscr{F}}(X, \mu) \ni f, g \longmapsto d_p(f, g) := \sum_{j=1}^{\infty} 2^{-j} \Phi\Big(\|f - g\|^{\min\{1,p\}}_{L^p(\Delta_j, \mu)}\Big) \tag{3.1.18}$$

defines a complete, translation invariant metric on $L^p_{\mathscr{F}}(X, \mu)$ which induces the same topology as $\tau_{\mathscr{F}}$ on this space. □

Going forward, given a measure space $(X, \mathfrak{M}, \mu)$ and an arbitrary set $E \in \mathfrak{M}$, the `restriction of` μ `to` E is the measure $\mu\lfloor E$ defined on the sigma-algebra

$$\mathfrak{M}\lfloor E := \{A \in \mathfrak{M} : A \subseteq E\} = \{B \cap E : B \in \mathfrak{M}\} \tag{3.1.19}$$

of subsets of E via

$$\mu\lfloor E : \mathfrak{M}\lfloor E \longrightarrow [0, +\infty], \quad \big(\mu\lfloor E\big)(B) := \mu(B), \quad \forall B \in \mathfrak{M}\lfloor E. \tag{3.1.20}$$

Hence, for each $E \in \mathfrak{M}$, the triplet $\big(E, \mathfrak{M}\lfloor E, \mu\lfloor E\big)$ is a measure space. We shall also say that

$$\begin{array}{c}\text{the measure } \mu \text{ is } \textit{concentrated} \text{ on } E \text{ provided}\\ \mu(A) = \mu(A \cap E) \text{ for every } A \in \mathfrak{M}.\end{array} \tag{3.1.21}$$

Henceforth, we shall tacitly adopt the following convention:

> *Convention:* given any measure space $(X, \mathfrak{M}, \mu)$ with the property that μ is concentrated on $E \in \mathfrak{M}$, we shall identify the original measure μ with $\mu\lfloor E$, its restriction to the set E. (3.1.22)

We next briefly review the concept of complete measure. Let $(X, \mathfrak{M}, \mu)$ be a measure space. Recall that the measure μ is called `complete` if any subset of one of its null sets is μ-measurable. That is

$$\begin{array}{c}A \subseteq B \text{ where } B \text{ is } \mu\text{-measurable and } \mu(B) = 0\\ \text{implies } A \text{ is } \mu\text{-measurable and } \mu(A) = 0.\end{array} \tag{3.1.23}$$

There is a simple procedure which associates to μ a complete measure $\overline{\mu}$ on X. Specifically

$$\overline{\mathfrak{M}} := \{A \cup E : A \in \mathfrak{M}, \; E \subseteq B \in \mathfrak{M}, \; \mu(B) = 0\}, \tag{3.1.24}$$

is a sigma-algebra on X which contains $\mathfrak{M}$, and $\overline{\mu} : \overline{\mathfrak{M}} \to [0, +\infty]$ defined by

$$\overline{\mu}(A \cup E) := \mu(A) \;\text{ whenever } A \in \mathfrak{M} \text{ and } E \subseteq B \in \mathfrak{M} \text{ are such that } \mu(B) = 0, \tag{3.1.25}$$

is a well-defined measure which is complete and extends μ. It is then easy to check that the null sets for $\overline{\mu}$ are precisely all subsets of null sets of μ, i.e.

$$\{A \in \overline{\mathfrak{M}} : \overline{\mu}(A) = 0\} = \{E \subseteq X : \text{ there exists } B \in \mathfrak{M} \text{ with } E \subseteq B \text{ and } \mu(B) = 0\}. \tag{3.1.26}$$

Of course

given a measure space $(X, \mathfrak{M}, \mu)$, it follows that $\overline{\mu}$, μ coincide on $\mathfrak{M}$, and the measure μ is complete if and only if one has $\overline{\mathfrak{M}} = \mathfrak{M}$. (3.1.27)

One may also easily see from definitions that completeness is hereditary, in the sense that

completeness is preserved when restricting a complete measure to any measurable subset of the ambient. (3.1.28)

As indicated in the remark below, completeness is equivalent to a number of desirable properties in a general measure space.

Remark 3.1.2 *Generally speaking, for a measure μ on a set X, the following properties are equivalent* (*cf., e.g.,* [91, Proposition 2.11, p. 47])*:*

(i) the measure μ is complete;
(ii) whenever $f, g : X \to \mathbb{R}$ are two functions satisfying $f = g$ at μ-a.e. point on X and the function f is μ-measurable, then g is also μ-measurable;
(iii) if $(f_j)_{j \in \mathbb{N}}$ is a sequence of real-valued functions defined on X, each of which is μ-measurable, and if the pointwise limit $f := \lim_{j \to \infty} f_j$ exists μ-a.e. on X then f is also μ-measurable.

It is worth noting that passing from a measure μ to its completion $\overline{\mu}$ (described above) preserves the space of measurable functions on X in the following precise sense:

if f is μ-measurable on X then f is $\overline{\mu}$-measurable on X and, in the converse direction, if f is $\overline{\mu}$-measurable on X, then there exists a unique (up to μ-a.e. coincidence) function g which is μ-measurable on X with the property that $f = g$ at μ-a.e. (or $\overline{\mu}$-a.e.) point in X. (3.1.29)

Indeed, this can be checked first at the level of simple functions and then passing to the limit using (3.1.12) (cf. also [91, Proposition 2.12, p. 48]). From (3.1.29) and Remark 3.1.2, it follows that

whenever $(f_j)_{j\in\mathbb{N}}$ is some sequence of μ-measurable real-valued functions defined on X with the property that the pointwise limit $f := \lim_{j\to\infty} f_j$ exists at μ-a.e. point in X, then it is possible to redefine f on a μ-nullset as to become itself μ-measurable. (3.1.30)

Another consequence of (3.1.29) is that completing a given measure μ on X preserves the Lebesgue scale L^p, $0 < p \leq \infty$, in the sense that if $f \in L^p(X,\mu)$, then $f \in L^p(X,\overline{\mu})$ and, conversely, whenever $f \in L^p(X,\overline{\mu})$ there exists a unique (up to μ-a.e. coincidence) function $g \in L^p(X,\mu)$ with the property that $f = g$ at μ-a.e. point in X.

In fact, there is a more general phenomenon at work here that we now wish to describe. First, we make a definition

Definition 3.1.3 *Let $(X,\mathfrak{M},\mu)$ be a measure space and assume that $\mathfrak{M}_0$ is a sigma-algebra of subsets of X contained in $\mathfrak{M}$. Then the measure μ is called $\mathfrak{M}_0$-*`regular` *provided*

$$\begin{aligned} &\textit{for each } A \in \mathfrak{M} \textit{ there exists } B \in \mathfrak{M}_0 \\ &\textit{such that } A \subseteq B \textit{ and } \mu(A) = \mu(B). \end{aligned} \tag{3.1.31}$$

In order to link this regularity concept to the notion of completion of a measure, it is worth noting that, as is trivially checked, if $(X,\mathfrak{M},\mu)$ is a measure space and if $(X,\overline{\mathfrak{M}},\overline{\mu})$ is the measure space obtained by completing the original measure, then $\overline{\mu}$ is $\mathfrak{M}$-regular. Now, the phenomenon alluded to above may be stated as follows:

if $(X,\mathfrak{M},\mu)$ is a measure space and if the measure μ is $\mathfrak{M}_0$-regular for some sigma-algebra $\mathfrak{M}_0 \subseteq \mathfrak{M}$, then the μ-measurable functions on X may be canonically identified with μ_0-measurable functions on X, where $\mu_0 := \mu\big|_{\mathfrak{M}_0}$. (3.1.32)

Indeed, this is proved much as in the particular case of a measure and its completion, described above. As a consequence, granted the above assumptions, for any integrability exponent $p \in (0,\infty)$ there is a natural identification (understood as before)

$$L^p(X,\mu) \equiv L^p(X,\mu_0), \quad \text{where} \quad \mu_0 := \mu\big|_{\mathfrak{M}_0}. \tag{3.1.33}$$

3.2 The Topology on the Space of Measurable Functions

Given a measure space $(X,\mathfrak{M},\mu)$, denote by $L^0(X,\mu)$ the linear space consisting of (equivalence classes of) μ-measurable functions which are finite μ-a.e. on X.

Consider now a measure space $(X, \mathfrak{M}, \mu)$ with the property that there exists a function

$$\omega \in L^0(X, \mu) \text{ such that } \omega(x) > 0 \text{ for } \mu\text{-a.e. } x \in X, \text{ and } \int_X \omega \, \mathrm{d}\mu < +\infty. \tag{3.2.1}$$

In this context, one may define a translation invariant distance on $L^0(X, \mu)$ according to

$$L^0(X, \mu) \times L^0(X, \mu) \ni (f, g) \mapsto \int_X \frac{|f - g|}{1 + |f - g|} \omega \, \mathrm{d}\mu \in [0, \infty). \tag{3.2.2}$$

In this conjunction, we have the following basic result describing the nature of the topology of the space of measurable functions (compare with [30, Theorem 1.2.1, p. 4]).

Lemma 3.2.1 *Let $(X, \mathfrak{M}, \mu)$ be a sigma-finite measure space. Then the following properties hold:*

(i) A function ω with the properties specified in (3.2.1) always exists in this setting.

(ii) For each ω as in (3.2.1), the Lévy distance introduced in (3.2.2) defines on $L^0(X, \mu)$ a Hausdorff topology of a separable linear topological space, which is independent of the choice of the function ω.

(iii) Let $\{A_j\}_{j\in\mathbb{N}}$ be a nested, increasing family of sets in $\mathfrak{M}$, of finite measure, which exhausts X. Then a fundamental system of neighborhoods for the zero function in the topology described above is given by

$$\mathcal{V}_{\varepsilon,j} := \Big\{ f \in L^0(X, \mu) : \mu\big(\{x \in A_j : |f(x)| > \varepsilon\}\big) < \varepsilon \Big\}, \quad \varepsilon > 0, \ j \in \mathbb{N}. \tag{3.2.3}$$

(iv) For each choice of the function ω as in (3.2.1), the space $L^0(X, \mu)$ is complete in the metric (3.2.2) provided the measure μ is complete. In fact, whenever μ is complete, $L^0(X, \mu)$ becomes an F-space though this is not, in general, locally bounded, nor is it locally convex.

(v) For each choice of ω as in (3.2.1), the convergence of a sequence $(f_j)_{j\in\mathbb{N}}$ from $L^0(X, \mu)$ to some $f \in L^0(X, \mu)$ in the metric (3.2.2) is equivalent to convergence in measure on sets of finite measure, i.e.

$$\begin{aligned} &\forall A \in \mathfrak{M} \text{ with } \mu(A) < +\infty, \ \forall \varepsilon > 0 \Longrightarrow \\ &\lim_{j\to\infty} \mu\Big(\big\{x \in A : |f_j(x) - f(x)| > \varepsilon\big\}\Big) = 0. \end{aligned} \tag{3.2.4}$$

As a consequence of this and Chebytchev's inequality

$$L^p(X, \mu) \hookrightarrow L^0(X, \mu) \ \text{ continuously, for each } \ p \in (0, \infty). \tag{3.2.5}$$

3.3 Outer Measures

Let X be a fixed, arbitrary set. An outer measure μ^* on X is an extended real-valued, non-negative function $\mu^* : 2^X \to [0, +\infty]$ (recall that 2^X denotes the collection of all subsets of X) satisfying

$$\mu^*(\varnothing) = 0 \text{ and } \mu^*(A) \le \sum_{j\in\mathbb{N}} \mu^*(A_j) \text{ if } A, A_j \subseteq X,\ j \in \mathbb{N}, \text{ with } A \subseteq \bigcup_{j\in\mathbb{N}} A_j. \tag{3.3.1}$$

In this context, a set $A \subseteq X$ is said to be μ^*-measurable provided

$$\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y \setminus A) \text{ for each } Y \subseteq X. \tag{3.3.2}$$

Note that the left-pointing inequality in (3.3.2) is always true (thanks to (3.3.1)), so the crux of the matter is the validity of the right-pointing inequality in (3.3.2). We set

$$\mathfrak{M}_{\mu^*} := \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}. \tag{3.3.3}$$

Lemma 3.3.1 *Consider an outer measure μ^* on an arbitrary set X. Then any nullset for μ^* belongs to $\mathfrak{M}_{\mu^*}$ (i.e., is μ^*-measurable).*

Proof Let $A \subseteq X$ be such that $\mu^*(A) = 0$. Given that, by the monotonicity of the outer measure, any subset of A is a nullset for μ^*, verifying the right-pointing inequality in (3.3.2) presently reduces to checking that for every $Y \subseteq X$ we have $\mu^*(Y) \ge \mu^*(Y \setminus A)$. This, however, is clear from the monotonicity of μ^*. □

In fact, according to Carathéodory's classical theorem (cf., e.g., [91, Theorem 1.11, p. 29]), if μ^* is an outer measure on X, then

$$\begin{aligned} &\mathfrak{M}_{\mu^*} \text{ is a sigma-algebra of subsets of } X, \text{ and} \\ &\mu^*\big|_{\mathfrak{M}_{\mu^*}} \text{ is a complete measure on } X. \end{aligned} \tag{3.3.4}$$

Given an arbitrary set $E \subseteq X$, the restriction of the outer measure μ^* to E is the outer measure $\mu^*\lfloor E$ on E defined by

$$\mu^*\lfloor E := \mu^*\Big|_{2^E}. \tag{3.3.5}$$

Then, as is easily verified, for any $E \subseteq X$

$$\mu^*\lfloor E \text{ is an outer measure on } E, \text{ and } \{A \cap E : A \in \mathfrak{M}_{\mu^*}\} \subseteq \mathfrak{M}_{\mu^*\lfloor E}. \tag{3.3.6}$$

Remark 3.3.2 *Assume that X is a given set, $E \subseteq X$ and μ^* is an outer measure on E. In this context, define $(\mu^*)^X$, the* lifting *of μ^* from E to X, by setting*

$$(\mu^*)^X(A) := \mu^*(A \cap E), \qquad \forall A \subseteq X. \tag{3.3.7}$$

Then the following properties can be verified based on definitions:

(i) $(\mu^*)^X$ *is an outer measure on* X*;*
(ii) $\{A \subseteq X : A \cap E \in \mathfrak{M}_{\mu^*}\} \subseteq \mathfrak{M}_{(\mu^*)^X}$.

The next lemma indicates when the measure associated with the restriction of an outer measure to a set coincides with the restriction to that set of the measure associated with the given outer measure.

Lemma 3.3.3 *If* μ^* *is an outer measure on* X*, then*

$$\big(\mu^* \lfloor A\big)\Big|_{\mathfrak{M}_{(\mu^*\lfloor A)}} = \big(\mu^*\big|_{\mathfrak{M}_{\mu^*}}\big)\lfloor A, \qquad \forall A \in \mathfrak{M}_{\mu^*}. \tag{3.3.8}$$

Proof Fix some $A \in \mathfrak{M}_{\mu^*}$. Since both sides of (3.3.8) act in a compatible fashion with μ^* on their domains of definition, it suffices to show that the two measures in (3.3.8) act on the same sigma-algebra. Unraveling definitions, this comes down to checking that

$$\{C \subseteq A : \mu^*(Y) = \mu^*(Y \cap C) + \mu^*(Y \setminus C), \ \forall Y \subseteq A\} = \{B \in \mathfrak{M}_{\mu^*} : B \subseteq A\}. \tag{3.3.9}$$

The right-to-left inclusion in (3.3.9) is clear from the definition of $\mathfrak{M}_{\mu^*}$, so we shall focus on the opposite one. To this end, assume that $C \subseteq A$ satisfies

$$\mu^*(Y) = \mu^*(Y \cap C) + \mu^*(Y \setminus C), \qquad \forall Y \subseteq A, \tag{3.3.10}$$

and fix an arbitrary set $Z \subseteq X$. Then

$$\begin{aligned}
\mu^*(Z) &= \mu^*(Z \setminus A) + \mu^*(Z \cap A) \\
&= \mu^*(Z \setminus A) + \mu^*\big((Z \cap A) \setminus C\big) + \mu^*\big((Z \cap A) \cap C\big) \\
&= \mu^*(Z \setminus A) + \mu^*\big((Z \cap A) \setminus C\big) + \mu^*(Z \cap C) \\
&\geq \mu^*(Z \setminus C) + \mu^*(Z \cap C),
\end{aligned} \tag{3.3.11}$$

where the first equality is due to the fact that $A \in \mathfrak{M}_{\mu^*}$, the second equality follows from (3.3.10) used with $Y := Z \cap A \subseteq A$, the third equality uses $C \subseteq A$, while the subsequent inequality is a consequence of the subadditivity of μ^* and the readily checked identity

$$(Z \setminus A) \cup \big((Z \cap A) \setminus C\big) = Z \setminus C. \tag{3.3.12}$$

In turn, (3.3.11) and the subadditivity of μ^* give that $\mu^*(Z) = \mu^*(Z \setminus C) + \mu^*(Z \cap C)$. Since the set $Z \subseteq X$ has been arbitrarily chosen, this proves that C belongs to the right-hand side of (3.3.9). Hence, the left-to-right inclusion in (3.3.9) holds as well, finishing the proof of this equality. As noted earlier, this concludes the proof of the lemma. □

To any given measure space $(X, \mathfrak{M}, \mu)$ one can associate an outer measure μ^* by setting

$$\mu^*(A) := \inf\Big\{\sum_{j\in\mathbb{N}} \mu(A_j) : A_j \in \mathfrak{M},\ j \in \mathbb{N},\ A \subseteq \bigcup_{j\in\mathbb{N}} A_j\Big\}$$
$$= \inf\Big\{\mu(E) : A \subseteq E \in \mathfrak{M}\Big\} \text{ for each } A \subseteq X. \tag{3.3.13}$$

Then, it is apparent from this definition that

$$\mu^*(A) = \mu(A), \qquad \forall A \in \mathfrak{M}, \tag{3.3.14}$$

and (see, e.g., [251, # 9, p. 68]),

$$\left.\begin{array}{l}\text{if } (X, \mathfrak{M}, \mu) \text{ is a measure space and } \mu^* \text{ is the}\\ \text{outer measure associated with } \mu \text{ as in (3.3.13)}\end{array}\right\} \Longrightarrow \left\{\begin{array}{l}\mathfrak{M}_{\mu^*} = \overline{\mathfrak{M}} \text{ and}\\ \mu^*\big|_{\mathfrak{M}_{\mu^*}} = \overline{\mu},\end{array}\right. \tag{3.3.15}$$

i.e., the measure $\mu^*\big|_{\mathfrak{M}_{\mu^*}}$ becomes the completion $\overline{\mu}$ of μ (given in (3.1.25)). Furthermore, from the last statement of [230, Proposition 6, p. 293], we know that

whenever $(X, \mathfrak{M}, \mu)$ is a measure space and the outer measure μ^* is associated with μ as in (3.3.13), it follows that for every $A \subseteq X$ there exists some set $B \in \mathfrak{M}$ with the property that $A \subseteq B$ and $\mu^*(A) = \mu(B)$. (3.3.16)

We now introduce the following piece of terminology, pertaining to the regularity of outer measures, in the spirit of Definition 3.1.3.

Definition 3.3.4 *Let X be an arbitrary set and assume that μ^* is an outer measure on X. Furthermore, suppose that $\mathfrak{M}$ is a sigma-algebra of subsets of X. Then the outer measure μ^* is called $\mathfrak{M}$-*`regular` *provided $\mathfrak{M} \subseteq \mathfrak{M}_{\mu^*}$ and*

$$\forall A \subseteq X\ \exists B \in \mathfrak{M} \textit{ such that } A \subseteq B \textit{ and } \mu^*(A) = \mu^*(B). \tag{3.3.17}$$

There is a close relationship between the concept of regularity for genuine measures from Definition 3.1.3 and the notion of regularity for outer measures from Definition 3.3.4. For example, it is immediate from these definitions that

$$\left.\begin{array}{r}\text{if } \mathfrak{M} \text{ is a sigma-algebra on } X, \text{ and}\\ \mu^* \text{ is a } \mathfrak{M}\text{-regular outer measure on } X\end{array}\right\} \Longrightarrow \mu^*\big|_{\mathfrak{M}_{\mu^*}} \text{ is a } \mathfrak{M}\text{-regular measure on } X. \tag{3.3.18}$$

In the same spirit, we also wish to note the following result.

Lemma 3.3.5 *Let $(X, \mathfrak{M}, \mu)$ be a measure space and assume that $\mathfrak{M}_0$ is a sigma-algebra of subsets of X contained in $\mathfrak{M}$. Denote by μ^* the outer measure associated* *with μ as in (3.3.13). Then*

$$\begin{aligned}&\textit{the measure } \mu \textit{ is } \mathfrak{M}_0\textit{-regular (in the sense of Definition 3.1.3)} \iff \\ &\textit{the outer measure } \mu^* \textit{ is } \mathfrak{M}_0\textit{-regular (in the sense of Definition 3.3.4)}.\end{aligned} \tag{3.3.19}$$

As a corollary

$$\textit{the outer measure } \Big(\mu\big|_{\mathfrak{M}_0}\Big)^* \textit{ is } \mathfrak{M}_0\textit{-regular.} \tag{3.3.20}$$

Proof The left-pointing implication in (3.3.19) is a simple consequence of (3.3.16) and definitions, so we shall focus on the right-pointing implication in (3.3.19). To this end, fix an arbitrary set $A \subseteq X$. If $\mu^*(A) = +\infty$, then $\mu^*(X) = +\infty$ and $A \subseteq X \in \mathfrak{M}_0$. There remains to treat the case when $\mu^*(A) < +\infty$. In this scenario, from (3.3.13), we know that for each $j \in \mathbb{N}$, there exists $E_j \in \mathfrak{M}$ with the property that $A \subseteq E_j$ and $\mu(E_j) < \mu^*(A) + 1/j$. Next, using the fact that the measure μ is $\mathfrak{M}_0$-regular, for each $j \in \mathbb{N}$ it is possible to find $F_j \in \mathfrak{M}_0$ such that $E_j \subseteq F_j$ and $\mu(E_j) = \mu(F_j)$. Consequently

$$A \subseteq F_j \in \mathfrak{M}_0 \text{ and } \mu^*(A) \leq \mu(F_j) < \mu^*(A) + 1/j \quad \text{for very} \quad j \in \mathbb{N}. \tag{3.3.21}$$

Hence, if we set $F := \bigcap_{j\in\mathbb{N}} F_j$, it follows from (3.3.14) and (3.3.21) that

$$A \subseteq F \in \mathfrak{M}_0 \text{ and } \mu^*(A) \leq \mu^*(F) = \mu(F) \leq \mu(F_j) < \mu^*(A) + 1/j \tag{3.3.22}$$

for very $j \in \mathbb{N}$. Note that the above double inequality forces $\mu^*(A) = \mu(F)$. Since $F \in \mathfrak{M}$, this ultimately permits us to conclude that $\mu^*(A) = \mu^*(F)$ (cf. (3.3.14)). All in all, the above reasoning shows that μ^* is a $\mathfrak{M}_0$-regular outer measure. Finally, (3.3.20) is a direct consequence of (3.3.19) and the obvious fact that the measure $\mu\big|_{\mathfrak{M}_0}$ is $\mathfrak{M}_0$-regular. □

Lemma 3.3.6 *Assume X is an arbitrary set and μ^* is an outer measure on X. Also, suppose $\mathfrak{M}$ is a sigma-algebra of subsets of X with the property that $\mathfrak{M} \subseteq \mathfrak{M}_{\mu^*}$. Then*

$$\textit{the outer measure } \mu^* \textit{ is } \mathfrak{M}\textit{-regular} \iff \mu^* = \Big(\mu^*\big|_{\mathfrak{M}}\Big)^*. \tag{3.3.23}$$

Proof To prove the right-pointing implication in (3.3.23), observe that if $A \subseteq X$ is arbitrary then, on the one hand,

$$\Big(\mu^*\big|_{\mathfrak{M}}\Big)^*(A) = \inf\Big\{\mu^*(E) : A \subseteq E \in \mathfrak{M}\Big\} \geq \mu^*(A), \tag{3.3.24}$$

by the monotonicity of μ^*. On the other hand, given that the outer measure μ^* is $\mathfrak{M}$-regular, it follows that there exists $B \in \mathfrak{M}$ such that $A \subseteq B$ and $\mu^*(A) = \mu^*(B)$. In turn, this shows that the opposite of the inequality in (3.3.24) also holds, and hence $\left(\mu^*\big|_{\mathfrak{M}}\right)^*(A) = \mu^*(A)$. Since the left-pointing implication in (3.3.23) is a direct consequence of (3.3.20) in Lemma 3.3.5, the desired conclusion follows. □

3.4 Borel-Regular Measure and Outer Measures

To get started, recall the class of Borel-regular measures on a topological space.

Definition 3.4.1 *Given a topological space (X, τ), call a measure $\mu : \mathfrak{M} \to [0, +\infty]$ (where $\mathfrak{M}$ is a sigma-algebra of subsets of X) a* `Borel measure` *(or, simply,* `Borel`*) provided $Borel_\tau(X) \subseteq \mathfrak{M}$. In the same context as above, call a given measure $\mu : \mathfrak{M} \to [0, +\infty]$* `Borel-regular` *(on X) provided μ is a Borel measure (on X) and*

$$\begin{array}{l}\textit{for every } E \in \mathfrak{M} \textit{ there exists } B \in Borel_\tau(X) \textit{ with} \\ \textit{the property that } E \subseteq B \textit{ as well as } \mu(E) = \mu(B).\end{array} \tag{3.4.1}$$

Tautologically, in a given topological space (X, τ)

$$\text{any measure } \mu : Borel_\tau(X) \to [0, +\infty] \text{ is Borel-regular.} \tag{3.4.2}$$

Lemma 3.4.2 *Let μ be a locally finite Borel-regular measure on a sigma-compact topological space (X, τ). Also, pick an arbitrary μ-measurable non-negative function f on X. Then $\widetilde{\mu} := f\mu$ is a Borel-regular measure on (X, τ).*

Proof By design, $\widetilde{\mu}$ is a Borel measure on (X, τ), defined on the sigma-algebra $\mathfrak{M}_{\widetilde{\mu}} = \mathfrak{M}_\mu$. To prove that $\widetilde{\mu}$ is, in fact, Borel-regular, let $\{K_N\}_{N\in\mathbb{N}}$ be a sequence of compact sets in X such that $K_N \nearrow X$ as $N \to \infty$. Fix some set $A \in \mathfrak{M}_{\widetilde{\mu}}$ and define $A_N := A \cap K_N \in \mathfrak{M}_\mu$ for each $N \in \mathbb{N}$. In particular, $A_N \nearrow A$ as $N \to \infty$. Given that μ is Borel-regular, for each $N \in \mathbb{N}$ there exists a Borel set B_N such that $A_N \subseteq B_N$ and $\mu(B_N) = \mu(A_N) \leq \mu(K_N) < \infty$. Hence, $\mu(B_N \setminus A_N) = 0$, which implies $f\mathbf{1}_{A_N} = f\mathbf{1}_{B_N}$ at μ-a.e. point in X. This further entails $\int_{A_N} f\,d\mu = \int_{B_N} f\,d\mu$ which, in turn, gives

$$\widetilde{\mu}(A_N) = \widetilde{\mu}(B_N) \quad \text{for each} \quad N \in \mathbb{N}. \tag{3.4.3}$$

We find it useful to replace the B_N's with a nested family of sets enjoying similar properties. Specifically, consider the Borel set $C_N := B_1 \cup B_2 \cup \cdots \cup B_N$ for each $N \in \mathbb{N}$. Then

$$C := \bigcup_{N\in\mathbb{N}} B_N \text{ is a Borel set and } C_N \nearrow C \text{ as } N \to \infty. \tag{3.4.4}$$

From definitions, it is also clear that $A \subseteq C$. In addition, for each $N \in \mathbb{N}$, we may express

$$C_N = B_N \sqcup \Big(\bigcup_{j=1}^{N-1} (B_j \setminus B_N) \Big). \tag{3.4.5}$$

Note that whenever $1 \leq j \leq N-1$, we have $B_j \setminus B_N \subseteq B_j \setminus A_j$, hence

$$\mu(B_j \setminus B_N) \leq \mu(B_j \setminus A_j) = 0 \text{ for } j \in \{1, \dots, N-1\}. \tag{3.4.6}$$

The latter combined with (3.4.5) forces

$$\widetilde{\mu}(A_N) = \widetilde{\mu}(C_N) \text{ for each } N \in \mathbb{N}. \tag{3.4.7}$$

The definition of C, the fact that $A_N \nearrow A$, properties (3.4.4) and (3.4.7), and Lebesgue's Monotone Convergence Theorem ultimately yield $\widetilde{\mu}(A) = \widetilde{\mu}(C)$. Hence, the measure $\widetilde{\mu}$ is Borel-regular. □

A weaker notion of regularity than the one considered in Definition 3.4.1 has been introduced in [11, Definition 3.9, pp. 87-88]. As a preamble, the reader is reminded that $A \triangle B$ stands for the symmetric difference of the sets A and B, in other words, $A \triangle B := (A \setminus B) \cup (B \setminus A)$.

Definition 3.4.3 *If (X, τ) is a topological space and $\mathfrak{M}$ is a sigma-algebra of subsets of X, call a measure $\mu : \mathfrak{M} \to [0, +\infty]$* `Borel-semiregular` *on (X, τ) (or simply on X if the topology is understood) provided μ is Borel (i.e., $Borel_\tau(X) \subseteq \mathfrak{M}$), and*

$$\begin{array}{c} \text{for every } E \in \mathfrak{M} \text{ with } \mu(E) < +\infty, \text{ there exists} \\ B \in Borel_\tau(X) \text{ with the property that } \mu(E \triangle B) = 0. \end{array} \tag{3.4.8}$$

A moment's reflection shows that any Borel-regular measure is Borel-semiregular. A related definition may be considered by demanding, in place of (3.4.8), that for every $E \in \mathfrak{M}$ there exists $B \in Borel_\tau(X)$ such that $\mu(E \triangle B) = 0$. Under the background assumption that X is sigma-finite (relative to $\mathfrak{M}$), this definition becomes equivalent to Definition 3.4.3.

It turns out that for a given Borel measure μ, the quality of being Borel-semiregular hinges upon the ability to express characteristic functions of μ-measurable sets as limits, pointwise μ-almost everywhere, of sequences of Borel-measurable functions. The following result of this nature is proved in [11, Lemma 3.10, p. 88].

Lemma 3.4.4 *Assume (X, τ) is a topological space and $\mu : \mathfrak{M} \to [0, +\infty]$ is a Borel measure. In this context, consider a set $E \in \mathfrak{M}$ which has the property that there exists a sequence $\{f_j\}_{j \in \mathbb{N}}$ of real-valued Borel-measurable functions defined on X such that $f_j \to \mathbf{1}_E$ pointwise μ-almost everywhere on X as $j \to \infty$. Then there exists $B \in Borel_\tau(X)$ satisfying $\mu(E \triangle B) = 0$.*

The notion of Borel-regularity may be naturally adapted to outer measures. Based on the general blueprint employed in Definition 3.3.4, we introduce this concept as follows.

Definition 3.4.5 *Let (X, τ) be a topological space and let μ^* be an outer measure on X. Call μ^* a* `Borel outer measure` *on X if $Borel_\tau(X) \subseteq \mathfrak{M}_{\mu^*}$. Furthermore, call μ^* a* `Borel-regular outer measure` *if μ^* is a Borel outer measure on X and*

$$\begin{array}{c}\textit{for each } A \subseteq X \textit{ there exists some } B \in Borel_\tau(X) \\ \textit{with the property that } A \subseteq B \textit{ and } \mu^*(A) = \mu^*(B).\end{array} \tag{3.4.9}$$

Remark 3.4.6 *Let (X, τ) be a topological space, $E \subseteq X$ arbitrary, and assume that μ^* is a Borel outer measure on E. Lift μ^* to an outer measure $(\mu^*)^X$ on X as in Remark 3.3.2. It follows then from property (ii) in Remark 3.3.2 and (3.1.1) that if μ^* is a Borel outer measure on E then $(\mu^*)^X$ is a Borel outer measure on X.*

Given a topological space (X, τ) and a Borel outer measure μ^* on X, Lemma 3.3.6 implies that

$$\mu^* \text{ Borel-regular outer measure} \Longleftrightarrow \mu^* = \Big(\mu^*\big|_{Borel_\tau(X)}\Big)^*. \tag{3.4.10}$$

Also, if μ is a Borel measure on (X, τ) and μ^* denotes the outer measure associated with μ as in (3.3.13), then Lemma 3.3.5 guarantees that

$$\mu \text{ is a Borel-regular measure } \Longleftrightarrow \mu^* \text{ is a Borel-regular outer measure.} \tag{3.4.11}$$

Lemma 3.4.7 *Let (X, τ) be a topological space and assume μ is a Borel measure on X. Then, employing the notation introduced in (3.3.13), it follows that*

$$\Big(\mu\big|_{Borel_\tau(X)}\Big)^* \textit{ is a Borel-regular outer measure on } X. \tag{3.4.12}$$

Moreover, with the outer measure μ^* associated with μ as in (3.3.13)

$$\mu^* \textit{ is a Borel-regular outer measure} \Longleftrightarrow \mu^* = \Big(\mu\big|_{Borel_\tau(X)}\Big)^*. \tag{3.4.13}$$

Proof The claim in (3.4.12) follows from definitions and the second part in Lemma 3.3.5, whereas (3.4.13) is easily seen from (3.4.12) and Lemma 3.3.6. □

Given a topological space (X, τ) along with a Borel outer measure μ^* on X, in the lemma below, we identify some useful features retained by restricting μ^* to an arbitrary set $A \subseteq X$.

Lemma 3.4.8 *Let (X, τ) be a topological space and assume that μ^* is a Borel outer measure on X. Then for every $A \subseteq X$, it follows that $\mu^* \lfloor A$ is a Borel outer measure on $(A, \tau|_A)$.*

If actually the outer measure μ^* is Borel-regular on (X, τ), then for every set $A \subseteq X$, the outer measure $\mu^* \lfloor A$ is Borel-regular on $(A, \tau|_A)$.

Proof To deal with the claim made in the first part of the statement of the lemma, fix an arbitrary set $A \subseteq X$. From (3.3.6), we know that $\mu^* \lfloor A$ is an outer measure on A and, by (3.1.1) and the fact that $Borel_\tau(X) \subseteq \mathfrak{M}_{\mu^*}$,

$$\begin{aligned} Borel_{\tau|_A}(A) &= \big\{A \cap B : B \in Borel_\tau(X)\big\} \\ &\subseteq \big\{A \cap B : B \in \mathfrak{M}_{\mu^*}\big\} \subseteq \mathfrak{M}_{\mu^* \lfloor A}, \end{aligned} \tag{3.4.14}$$

where we have also used the second part in (3.3.6). This allows us to conclude that $\mu^* \lfloor A$ is a Borel outer measure on A.

There remains to settle the regularity issue from the second part of the statement of the lemma. To this end, assume that μ^* is a Borel regular outer measure on X and let $E \subseteq A$ be arbitrary. Then there exists $B \in Borel_\tau(X)$ such that $E \subseteq B$ and $\mu^*(E) = \mu^*(B)$. Then if we set $B_o := B \cap A$, we have $E \subseteq B_o \subseteq A$ and $B_o \in Borel_{\tau|_A}(A)$ by (3.1.1). By the monotonicity of the outer measures μ^* and $\mu^* \lfloor A$, and keeping in mind these inclusions, we may write

$$(\mu^* \lfloor A)(B_o) = \mu^*(B_o) \leq \mu^*(B) = \mu^*(E) = (\mu^* \lfloor A)(E) \leq (\mu^* \lfloor A)(B_o). \tag{3.4.15}$$

Hence, $(\mu^* \lfloor A)(E) = (\mu^* \lfloor A)(B_o)$ which, given what we have shown already, proves that $\mu^* \lfloor A$ is a Borel-regular outer measure on A. □

Remark 3.4.9 *Given a topological space (X, τ), the quality of being Borel-regular is hereditary, in the precise sense that for each $A \subseteq X$, one has*

$$\begin{aligned} &\mu^* \textit{ Borel-regular outer measure on } (X, \tau) \\ &\qquad \Rightarrow \mu^* \lfloor A \textit{ Borel-regular outer measure on } (A, \tau|_A) \\ &\qquad \Rightarrow (\mu^* \lfloor A)\Big|_{\mathfrak{M}_{(\mu^* \lfloor A)}} \textit{ Borel-regular measure on } (A, \tau|_A), \end{aligned} \tag{3.4.16}$$

thanks to Lemma 3.4.8 and (3.3.18). In concert with Lemma 3.3.3, the last implication in (3.4.16) further proves that

$$\begin{aligned} &\mu^* \textit{ Borel-regular outer measure on } (X, \tau) \textit{ and } A \in \mathfrak{M}_{\mu^*} \\ &\qquad \Rightarrow \big(\mu^*\big|_{\mathfrak{M}_{\mu^*}}\big) \lfloor A \textit{ is a Borel-regular measure on } (A, \tau|_A). \end{aligned} \tag{3.4.17}$$

For a measure, the quality of being Borel-regular is preserved under completion. Here is a more general result of this flavor.

Lemma 3.4.10 *Let (X, τ) be a topological space and let $(X, \mathfrak{M}, \mu)$ be a measure space such that μ is a Borel-regular measure on (X, τ). Let $(\overline{\mathfrak{M}}, \overline{\mu})$ be the completion of this measure. Then for each $E \in \overline{\mathfrak{M}}$, it follows that $\overline{\mu} \lfloor E$ is a Borel-regular measure on $\big(E, \tau\big|_E\big)$.*

As a corollary, if (X, τ) is a topological space and $(X, \mathfrak{M}, \mu)$ is a measure space such that μ is a complete Borel-regular measure on (X, τ), then $\mu\lfloor E$ is a Borel-regular measure on $\big(E, \tau\big|_E\big)$ for each $E \in \mathfrak{M}$.

Proof The first claim is a consequence of (3.4.11), (3.4.17), and (3.3.15). The second claim is implied by the first and (3.1.27). □

Given a measure space $(X, \mathfrak{M}, \mu)$, for each fixed set $E \in \mathfrak{M}$ define

$$\mu_E : \mathfrak{M} \longrightarrow [0, +\infty], \quad \mu_E(B) := \mu(B \cap E), \quad \forall B \in \mathfrak{M}. \tag{3.4.18}$$

Then $\mu_E = \mathbf{1}_E \cdot \mu$, and it is clear from definitions (cf. (3.1.21)) that

$$\mu_E \text{ is a measure on } X \text{ which is concentrated on } E. \tag{3.4.19}$$

Since $\mu_E\lfloor E = \mu\lfloor E$, it follows from (3.4.19) and (3.1.22) that

$$\text{the measure } \mu_E \text{ may be canonically identified with } \mu\lfloor E. \tag{3.4.20}$$

Lemma 3.4.11 *Let (X, τ) be a topological space and let $(X, \mathfrak{M}, \mu)$ be a measure space such that μ is a Borel-regular measure on (X, τ). Then for each $E \in Borel_\tau(X)$ it follows that μ_E (defined as in (3.4.18)) is a Borel-regular measure on (X, τ) which is concentrated on E.*

Proof Pick an arbitrary set $A \in \mathfrak{M}$. Since μ is Borel, $Borel_\tau(X) \subseteq \mathfrak{M}$, hence $A \cap E \in \mathfrak{M}$. Next, the fact that μ is a Borel-regular measure on (X, τ) ensures that we may find some $B \in Borel_\tau(X)$ such that

$$A \cap E \subseteq B \ \text{ and } \ \mu(A \cap E) = \mu(B). \tag{3.4.21}$$

Define $C := B \cup (X \setminus E) \in Borel_\tau(X)$. Then $A = (A \cap E) \cup (A \setminus E) \subseteq B \cup (X \setminus E) = C$. As such, $A \subseteq C$ so, on the one hand, we have $\mu_E(A) \leq \mu_E(C)$. On the other hand,

$$\mu_E(C) = \mu(C \cap E) = \mu(B \cap E) \leq \mu(B) = \mu(A \cap E) = \mu_E(A), \tag{3.4.22}$$

thanks to (3.4.18), the definition of C, and (3.4.21). Ultimately, this proves that $C \in Borel_\tau(X)$ satisfies $\mu_E(C) = \mu_E(A)$, hence μ_E is a Borel-regular measure on (X, τ). Finally, that μ_E is concentrated on E has been already noted in (3.4.19). □

For the next lemma, see also [80, Lemma 1, p. 6] for the case $X = \mathbb{R}^n$ and [88, Theorem 2.2.2, p. 60] for the case when X is a metric space.

Lemma 3.4.12 *Assume (X, τ) is a topological space with the property that*

$$\begin{array}{c}\textit{any open set (in the topology } \tau\textit{) can be written as}\\ \textit{a countable union of closed sets (in the topology } \tau\textit{).}\end{array} \tag{3.4.23}$$

Also, suppose μ^* is a Borel outer measure on X. Then

$$B \in Borel_\tau(X) \text{ and } \mu^*(B) < +\infty \Rightarrow \mu^*(B) = \sup_{\substack{C \text{ closed} \\ C \subseteq B}} \mu^*(C). \tag{3.4.24}$$

Condition (3.4.23) is satisfied in a variety of situations of interest. For example, the following result is proved in [11, Lemma 2.2, p. 39].

Lemma 3.4.13 *Assume (X, ρ) is a quasi-metric space. Equip X with the topology τ_ρ canonically induced by the quasi-distance ρ (cf. (7.1.7)). Then any open subset of X may be written as a countable union of closed sets of X.*

Proof of Lemma 3.4.12 Fix a set $B \in Borel_\tau(X)$ for which $\mu^*(B) < +\infty$ and let $\nu := \mu^*\lfloor_B$. Then ν is a finite Borel outer measure on B. Define the set

$$\begin{aligned} \mathcal{F} := \{A \subseteq X : \ & A \text{ is } \mu\text{-measurable and } \forall \varepsilon > 0 \text{ there exists} \\ & C \subseteq A \text{ closed set such that } \nu(A \setminus C) < \varepsilon\}. \end{aligned} \tag{3.4.25}$$

Then clearly all closed sets in X belong to $\mathcal{F}$. We next claim that

$$\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcap_{i\in\mathbb{N}} A_i \in \mathcal{F} \text{ and } \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}. \tag{3.4.26}$$

To prove (3.4.26), assume that $A_i \in \mathcal{F}$ for each $i \in \mathbb{N}$ and fix an arbitrary $\varepsilon > 0$. Then, for each $i \in \mathbb{N}$, there exists a closed set $C_i \subseteq A_i$ such that $\nu(A_i \setminus C_i) < \varepsilon/2^i$. Consequently, $\bigcap_{i\in\mathbb{N}} C_i$ is a closed set contained in $\bigcap_{i\in\mathbb{N}} A_i$, and we have

$$\nu\Big(\bigcap_{i\in\mathbb{N}} A_i \setminus \bigcap_{i\in\mathbb{N}} C_i\Big) \le \nu\Big(\bigcup_{i\in\mathbb{N}} (A_i \setminus C_i)\Big) \le \sum_{i\in\mathbb{N}} \nu(A_i \setminus C_i) < \sum_{i\in\mathbb{N}} 2^{-i}\varepsilon = \varepsilon, \tag{3.4.27}$$

proving that $\bigcap_{i\in\mathbb{N}} A_i \in \mathcal{F}$. Also, since ν is finite, we can apply (3.1.5) (note that, by (3.3.4), $\mu^*\Big|_{\mathfrak{M}_{\mu^*}}$ is a measure on X) to write

$$\begin{aligned} \lim_{N\to\infty} \nu\Big(\bigcup_{i\in\mathbb{N}} A_i \setminus \bigcup_{i=1}^{N} C_i\Big) &= \nu\Big(\bigcup_{i\in\mathbb{N}} A_i \setminus \bigcup_{i\in\mathbb{N}} C_i\Big) \le \nu\Big(\bigcup_{i\in\mathbb{N}} (A_i \setminus C_i)\Big) \\ &\le \sum_{i\in\mathbb{N}} \nu(A_i \setminus C_i) < \sum_{i\in\mathbb{N}} 2^{-i}\varepsilon = \varepsilon. \end{aligned} \tag{3.4.28}$$

Hence, there exists $N_o \in \mathbb{N}$ such that $\nu\Big(\bigcup_{i\in\mathbb{N}} A_i \setminus \bigcup_{i=1}^{N_o} C_i\Big) < \varepsilon$. The latter, together with the fact that $\bigcup_{i=1}^{N_o} C_i$ is closed proves that $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$. This completes the proof of (3.4.26).

In light of (3.4.23), what we proved so far also implies that all open sets in X are contained in $\mathcal{F}$. Consider next the set

$$\mathcal{G} := \{A \in \mathcal{F} : X \setminus A \in \mathcal{F}\}. \tag{3.4.29}$$

Trivially, if $A \in \mathcal{G}$, then $X \setminus A \in \mathcal{G}$, so $\mathcal{G}$ is closed under taking complements. Since we proved that $\mathcal{F}$ contains all open and closed sets of X, it follows that $\mathcal{G}$ also contains all open and closed sets of X. Moreover, $\mathcal{G}$ is closed under taking countable unions. Indeed, if $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{G}$, then by definition $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$ and $\{X \setminus A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{F}$, so that by (3.4.26), we have $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{F}$ and $X \setminus \bigcup_{i\in\mathbb{N}} A_i = \bigcap_{i\in\mathbb{N}}(X \setminus A_i) \in \mathcal{F}$. This proves that $\bigcup_{i\in\mathbb{N}} A_i \in \mathcal{G}$, as desired. Summing up, we have proved that $\mathcal{G}$ is a sigma-algebra containing all open sets of X. Hence, $\mathcal{G}$ also contains $Borel_\tau(X)$. In particular, $B \in \mathcal{G}$. The latter implies that $B \in \mathcal{F}$ which ultimately shows that (3.4.24) holds. □

It turns out that any Borel outer measure on a topological space satisfying a few milder assumptions enjoys a certain type of inner and outer regularity as described in the lemma below.

Lemma 3.4.14 *Assume (X, τ) is a topological space which satisfies (3.4.23) and suppose μ^* is a Borel outer measure on X with the property that*

$$\text{there exists a sequence } \{O_j\}_{j\in\mathbb{N}} \text{ of open sets in } X \text{ such that} \\ X = \bigcup_{j\in\mathbb{N}} O_j \ \text{ and } \ \mu^*(O_j) < +\infty \ \text{ for each } \ j \in \mathbb{N}. \tag{3.4.30}$$

Then

$$\text{for each } B \in Borel_\tau(X) \text{ and each } \varepsilon > 0 \text{ one can find} \\ \text{an open set } O \subseteq X \text{ with } B \subseteq O \text{ and } \mu^*(O \setminus B) < \varepsilon. \tag{3.4.31}$$

Moreover,

$$\text{if } \mu^* \text{ is a Borel-regular outer measure on } X \text{ satisfying} \\ \text{(3.4.30) then } \mu^*(A) = \inf_{\substack{O \text{ open} \\ A\subseteq O}} \mu^*(O) \text{ for every } A \subseteq X. \tag{3.4.32}$$

Proof Introduce

$$U_i := \bigcup_{1\le j\le i} O_j, \qquad \forall i \in \mathbb{N}, \tag{3.4.33}$$

so that

$$X = \bigcup_{i\in\mathbb{N}} U_i, \quad \text{and } U_i \text{ is open}, \quad \mu^*(U_i) < +\infty, \ U_i \subseteq U_{i+1}, \quad \text{for all} \quad i \in \mathbb{N}. \tag{3.4.34}$$

Also, fix some set $B \in Borel_\tau(X)$ along with an arbitrary number $\varepsilon > 0$. Then for each $i \in \mathbb{N}$, we have that $U_i \setminus B \in Borel_\tau(X)$ and $\mu(U_i \setminus B) < +\infty$. Consequently, we may invoke Lemma 3.4.12 and deduce that there exists a closed set $C_i \subseteq X$ with the property that $C_i \subseteq U_i \setminus B$ and

$$\mu^*\big((U_i \setminus B) \setminus C_i\big) < 2^{-i}\varepsilon. \tag{3.4.35}$$

Note that

$$O := \bigcup_{i\in\mathbb{N}} (U_i \setminus C_i) \quad \text{is an open set}. \tag{3.4.36}$$

Since for each $i \in \mathbb{N}$, we have $C_i \subseteq X \setminus B$, it follows that $U_i \cap B \subseteq U_i \setminus C_i$, so that

$$B = \bigcup_{i\in\mathbb{N}} (U_i \cap B) \subseteq \bigcup_{i\in\mathbb{N}} (U_i \setminus C_i) = O. \tag{3.4.37}$$

Furthermore, by (3.4.36), (3.4.35) and the subadditivity of the outer measure μ^*

$$\begin{aligned}\mu^*(O \setminus B) &= \mu^*\Big(\bigcup_{i\in\mathbb{N}} \big((U_i \setminus C_i) \setminus B\big)\Big) \leq \sum_{i\in\mathbb{N}} \mu^*\big((U_i \setminus C_i) \setminus B\big) \\ &= \sum_{i\in\mathbb{N}} \mu^*\big((U_i \setminus B) \setminus C_i\big) \leq \sum_{i\in\mathbb{N}} 2^{-i}\varepsilon = \varepsilon.\end{aligned} \tag{3.4.38}$$

Now, (3.4.31) follows from (3.4.36), (3.4.37) and (3.4.38).

As far as (3.4.32) is concerned, assume that μ^* is a Borel-regular outer measure on X and let $A \subseteq X$ be arbitrary. If $\mu^*(A) = +\infty$, there is nothing to prove, so there is no loss of generality in assuming in what follows that $\mu^*(A) < +\infty$. Also, fix an arbitrary $\varepsilon > 0$. Given that μ^* is a Borel-regular outer measure, there exists $B \in Borel_\tau(X)$ with the property that $A \subseteq B$ and $\mu^*(B) = \mu^*(A)$. Going further, by (3.4.31), one may find some open set $O \subseteq X$ such that $B \subseteq O$ and $\mu^*(O \setminus B) < \varepsilon$. This entails $A \subseteq O$ and since $O = (O \setminus B) \cup B$, the subadditivity of μ^* gives

$$\mu^*(O) \leq \mu^*(O \setminus B) + \mu^*(B) < \varepsilon + \mu^*(A). \tag{3.4.39}$$

Since $\varepsilon > 0$ is arbitrary, this shows that

$$\mu^*(A) \geq \inf_{\substack{O \text{ open} \\ A\subseteq O}} \mu^*(O). \tag{3.4.40}$$

The opposite inequality in (3.4.40) is clear from the monotonicity of μ^* and this finishes the proof of (3.4.32). □

Our next proposition shows that any Borel measure (on a topological space satisfying an additional mild condition) automatically possesses some type of inner and outer regularity.

Proposition 3.4.15 *Let (X, τ) be a topological space and assume $\mu : \mathfrak{M} \to [0, +\infty]$ is a Borel measure on X. Then the following statements are true:*

(1) If (X, τ) satisfies (3.4.23) then

$$\mu(B) = \sup_{\substack{C \text{ closed} \\ C \subseteq B}} \mu(C) \tag{3.4.41}$$

for any Borel set B of the form

$$B = \bigcup_{j=1}^{\infty} B_j \ \textit{ with } \ B_j \in \textit{Borel}_\tau(X) \ \textit{ and } \ \mu(B_j) < +\infty \ \textit{ for each } \ j \in \mathbb{N}. \tag{3.4.42}$$

In particular, (3.4.41) holds for every $B \in \textit{Borel}_\tau(X)$ whenever the background measure space satisfies

$$\begin{aligned} &\textit{there exist } \ \{X_j\}_{j\in\mathbb{N}} \subseteq \textit{Borel}_\tau(X) \ \textit{ such that} \\ &X = \bigcup_{j\in\mathbb{N}} X_j \ \textit{ and } \ \mu(X_j) < +\infty \ \textit{ for each } \ j \in \mathbb{N}. \end{aligned} \tag{3.4.43}$$

(2) If (X, τ) satisfies (3.4.23) and

$$\begin{aligned} &\textit{there exists a sequence } \{O_j\}_{j\in\mathbb{N}} \textit{ of open subsets of } X \textit{ such that} \\ &X = \bigcup_{j\in\mathbb{N}} O_j \ \textit{ and } \ \mu(O_j) < +\infty \ \textit{ for each } \ j \in \mathbb{N}. \end{aligned} \tag{3.4.44}$$

then

$$\begin{aligned} &\textit{for each } B \in \textit{Borel}_\tau(X) \textit{ and each } \varepsilon > 0 \textit{ there exists} \\ &\textit{an open set } O \subseteq X \textit{ with } B \subseteq O \textit{ and } \mu(O \setminus B) < \varepsilon. \end{aligned} \tag{3.4.45}$$

(3) If (X, τ) satisfies (3.4.23) and the measure μ is Borel-regular and satisfies (3.4.44), then μ also satisfies the outer-regularity condition

$$\mu(E) = \inf_{\substack{O \text{ open} \\ E \subseteq O}} \mu(O), \qquad \forall E \in \mathfrak{M}, \tag{3.4.46}$$

as well as the inner-regularity condition

$$\mu(E) = \sup_{\substack{C \text{ closed} \\ C \subseteq E}} \mu(C), \qquad \forall E \in \mathfrak{M}. \tag{3.4.47}$$

Proof Let μ^* be the outer measure associated with μ as in (3.3.13). From (3.3.15), we know that μ^* is a Borel outer measure. If (X, τ) satisfies (3.4.23) then Lemma 3.4.12 gives (keeping (3.3.13) in mind) that

$$\forall B \in Borel_\tau(X) \text{ with } \mu(B) < +\infty \Rightarrow \mu(B) = \sup_{\substack{C \text{ closed} \\ C \subseteq B}} \mu(C). \tag{3.4.48}$$

Consider now the case when B is as in (3.4.42) and has the additional property that $\mu(B) = +\infty$. Without loss of generality, we may assume that the sets $B_j \in Borel_\tau(X)$ are pairwise disjoint to begin with. For each integer $j \in \mathbb{N}$, use (3.4.48) to find a closed set $C_j \subseteq X$ such that $C_j \subseteq B_j$ and $\mu(B_j) < \mu(C_j) + 2^{-j}$. If for each $N \in \mathbb{N}$, we now define $C^N := \bigcup_{j=1}^{N} C_j$, then C^N is closed and $C^N \subseteq B$. Moreover, as $N \to \infty$, we have

$$\mu\big(C^N\big) = \sum_{j=1}^{N} \mu(C_j) \geq \sum_{j=1}^{N} \big[\mu(B_j) - 2^{-j}\big] \to \Big(\sum_{j=1}^{\infty} \mu(B_j)\Big) - 1 = \mu(B) - 1 = +\infty, \tag{3.4.49}$$

which goes to show that $\sup\big\{\mu(C) : C \text{ closed}, \ C \subseteq B\big\} = +\infty$ in this case. Since we are currently assuming $\mu(B) = +\infty$, the desired conclusion follows. This concludes the treatment of item *(1)*.

Moving on, the claims in item *(2)* as well as (3.4.46) in item *(3)* in the statement of the proposition are direct consequence of (3.4.31) and (3.4.32) in Lemma 3.4.14, respectively, given that μ and μ^* agree on $\mathfrak{M}$ which, in turn, contains $Borel_\tau(X)$.

At this point, there remains to prove (3.4.47) under the assumption that (X, τ) satisfies (3.4.23) and μ is a Borel-regular measure which satisfies (3.4.44). To this end, fix $E \in \mathfrak{M}$ and note that, obviously

$$\mu(E) \geq \sup_{\substack{C \text{ closed} \\ C \subseteq E}} \mu(C). \tag{3.4.50}$$

To prove the opposite inequality, assume first that

$$\mu(E) < +\infty \tag{3.4.51}$$

and fix an arbitrary $\varepsilon > 0$. Since μ is a Borel-regular measure, there exists

$$B \in Borel_\tau(X) \text{ with the property that } E \subseteq B \text{ and } \mu(E) = \mu(B). \tag{3.4.52}$$

In particular, thanks to (3.4.51), $\mu(B) < +\infty$ so (3.4.48) applies and yields

$$C \subseteq X \text{ closed with the property that } C \subseteq B \text{ and } \mu(B) < \mu(C) + \varepsilon/2. \tag{3.4.53}$$

On the other hand, from (3.4.46) applied to the set $B \setminus E \in \mathfrak{M}$, we know that there exists

$$\text{an open set } O \subseteq X \text{ such that } B \setminus E \subseteq O \text{ and } \mu(O) < \mu(B \setminus E) + \varepsilon/2. \quad (3.4.54)$$

At this stage, define $C_\varepsilon := C \setminus O \subseteq X$ and observe that, since $B \setminus E \subseteq O$ and $C \subseteq B$, we necessarily have $C_\varepsilon \subseteq C \setminus (B \setminus E) = C \cap E$. Hence

$$C_\varepsilon \text{ is closed and } C_\varepsilon \subseteq E. \quad (3.4.55)$$

Furthermore, since $E \setminus C \subseteq B \setminus C$ and $E \cap O = O \setminus (B \setminus E)$, we have

$$E \setminus C_\varepsilon = (E \setminus C) \cup (E \cap O) \subseteq (B \setminus C) \cup [O \setminus (B \setminus E)]. \quad (3.4.56)$$

Consequently, from (3.4.56), (3.4.53) and (3.4.54), we obtain

$$\mu(E \setminus C_\varepsilon) \leq \mu(B \setminus C) + \mu(O \setminus (B \setminus E)) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad (3.4.57)$$

Thus, $\mu(E) < \mu(C_\varepsilon) + \varepsilon$ which, when used in concert with (3.4.55), justifies the opposite inequality in (3.4.50), completing the proof of (3.4.47) under the additional hypothesis that (3.4.51) holds.

Finally, there remains to prove (3.4.47) as stated. To this end, assume that $E \in \mathfrak{M}$ is such that $\mu(E) = +\infty$ and recall the sequence $\{O_j\}_{j \in \mathbb{N}}$ from (3.4.44). Furthermore, let the U_i's retain the same significance as in (3.4.33), so that

$$X = \bigcup_{i \in \mathbb{N}} U_i \quad \text{and } U_i \text{ is open}, \quad \mu(U_i) < +\infty, , \quad U_i \subseteq U_{i+1}, \quad \text{for all } \; i \in \mathbb{N}. \quad (3.4.58)$$

Then $E \cap U_i \in \mathfrak{M}$ and $\mu(E \cap U_i) < +\infty$ for each $i \in \mathbb{N}$, so what we have proved up to this point in relation to (3.4.47) applies and gives that for each $i \in \mathbb{N}$ there exists some closed set $C_i \subseteq X$ with $C_i \subseteq E \cap U_i$ and $\mu(C_i) + 1/i > \mu(E \cap U_i)$. Hence,

$$\lim_{i \to \infty} \mu(C_i) \geq \lim_{i \to \infty} \big(\mu(E \cap U_i) - 1/i\big) = \mu(E) = +\infty, \quad (3.4.59)$$

which proves that there are closed subsets of E of arbitrarily large measure. As a result, (3.4.47) also holds in the case when $E \in \mathfrak{M}$ satisfies $\mu(E) = +\infty$. This finishes the proof of the proposition. □

Remark 3.4.16 *Let (X, τ) be a locally compact, Hausdorff topological space with the property that every open set in X is sigma-compact, i.e.,*

$$\begin{gathered} \textit{for each open set } O \subseteq X \textit{ there exist compact sets} \\ K_j \subseteq X, \; \textit{with } j \in \mathbb{N}, \; \textit{such that} \;\; O = \bigcup_{j \in \mathbb{N}} K_j \end{gathered} \quad (3.4.60)$$

(*parenthetically, note that any separable, locally compact, topological space whose topology is induced by a metric satisfies* (3.4.60)).

Then, if μ *is a Borel measure on* X *such that* $\mu(K) < +\infty$ *for every compact set* $K \subseteq X$, *it follows that conditions* (3.4.23) *and* (3.4.44) *hold. Likewise, if* μ^* *is a Borel outer measure on* X, *which is finite on compact subsets of* X, *then conditions* (3.4.23) *and* (3.4.30) *hold as well.*

3.5 Radon Measures

To set the stage, we formally recall the class of Radon measures.

Definition 3.5.1 *Let* (X, τ) *be a topological space, and let* $\mathfrak{M}$ *be a sigma-algebra of subsets of* X. *Call a measure* $\mu : \mathfrak{M} \to [0, +\infty]$ `Radon` *provided* $Borel_\tau(X) \subseteq \mathfrak{M}$ (*i.e.,* μ *is Borel*), μ *is locally finite* (*i.e.,* $\mu(K) < +\infty$ *for every compact set* $K \subseteq X$), *every open set is inner-regular, i.e.,*

$$\mu(O) = \sup_{\substack{K \text{ compact} \\ K \subseteq O}} \mu(K), \quad \textit{for each open set} \quad O \subseteq X, \tag{3.5.1}$$

and every Borel set is outer-regular, i.e.

$$\mu(E) = \inf_{\substack{O \text{ open} \\ E \subseteq O}} \mu(O), \qquad \forall E \in Borel_\tau(X). \tag{3.5.2}$$

It turns out (cf. [91, Proposition 7.5, p. 216]) that in any topological space,

$$\text{any Radon measure is inner-regular on all sets which may be written as countable unions of Borel sets of finite measure.} \tag{3.5.3}$$

In addition, the following regularity result holds (compare with [91, Theorem 7.8, p. 217]).

Proposition 3.5.2 *Let* (X, τ) *be a locally compact Hausdorff topological space in which every open set is sigma-compact* (*recall that the latter condition automatically holds if* (X, τ) *is second countable hence, in particular, if* (X, τ) *is metrizable and separable*). *Also, let* $\mu : \mathfrak{M} \to [0, +\infty]$ *be a locally finite Borel measure on* X (*i.e.,* $Borel_\tau(X) \subseteq \mathfrak{M}$ *and* $\mu(K) < +\infty$ *for every compact set* $K \subseteq X$). *Then every set in* $Borel_\tau(X)$ *is both inner-regular and outer-regular with respect to* μ, *i.e.*

$$\mu(E) = \sup_{\substack{K \text{ compact} \\ K \subseteq E}} \mu(K) = \inf_{\substack{O \text{ open} \\ E \subseteq O}} \mu(O), \qquad \forall E \in Borel_\tau(X). \tag{3.5.4}$$

In particular, μ *is a Radon measure.*

As a corollary, (3.5.4) *is valid for any locally finite Borel measure* μ *on a locally compact, separable metric space* X (*equipped with the topology* τ *canonically induced by the metric*).

Proof The outer-regularity formula in (3.5.4) is a consequence of part *(2)* in Proposition 3.4.15 and Remark 3.4.16. As far as the inner-regularity formula in (3.5.4) is concerned, let us first treat the case when $\mu(E) < +\infty$. In this scenario, thanks to part *(1)* in Proposition 3.4.15 and Remark 3.4.16, it suffices to observe that if

$$\bigcup_{j\in\mathbb{N}} K_j = X \text{ with } K_j \subseteq X \text{ compact and } K_j \subseteq K_{j+1} \text{ for every } j \in \mathbb{N}, \tag{3.5.5}$$

then for every closed set $C \subseteq X$, we have $\mu(C \cap K_j) \to \mu(C)$ as $j \to \infty$, and each $C \cap K_j$ is a compact set (since (X, τ) is a Hausdorff topological space). In the situation when $\mu(E) = +\infty$, consider the pairwise disjoint Borel sets $D_j := K_{j+1} \setminus K_j$, $j \in \mathbb{N}$, and note that since $E = \cup_{j\in\mathbb{N}}(D_j \cap E)$, it follows that $+\infty = \mu(E) = \sum_{j\in\mathbb{N}} \mu(D_j \cap E)$. On the other hand, since $D_j \cap E$ is a Borel set of finite measure, what we have proved already gives that, for each $j \in \mathbb{N}$, one can find a compact set $C_j \subseteq D_j \cap E$ with the property that $\mu(C_j) \geq \mu(D_j \cap E) - 2^{-j}$. Then, since the C_j's are disjoint, we obtain

$$\lim_{k\to\infty} \mu\Big(\bigcup_{j=1}^{k} C_j\Big) = \sum_{j=1}^{\infty} \mu(C_j) \geq \sum_{j=1}^{\infty} \big(\mu(D_j \cap E) - 2^{-j}\big) = +\infty. \tag{3.5.6}$$

Since for each fixed k the set $\bigcup_{j=1}^{k} C_j$ is compact and contained in E, it follows that the inner-regularity formula in (3.5.4) is valid in the case when $\mu(E) = +\infty$ as well.

Finally, the very last claim in the statement of the proposition is a corollary of what has just been proved, given that any separable metric space is Lindelöf. □

Corollary 3.5.3 *Assume* (X, τ) *is a locally compact Hausdorff topological space in which every open set is sigma-compact* (*recall that the latter condition automatically holds if* (X, τ) *is second countable hence, in particular, if* (X, τ) *is metrizable and separable*) *and suppose* $\mu : \mathfrak{M} \to [0, +\infty]$ *is a locally finite Borel-regular measure on* X.

Then μ *is a Radon measure which is both outer-regular and inner-regular, i.e.,*

$$\mu(E) = \inf_{\substack{O \text{ open} \\ E\subseteq O}} \mu(O), \qquad \forall E \in \mathfrak{M}, \tag{3.5.7}$$

and

$$\mu(E) = \sup_{\substack{K \text{ compact} \\ K\subseteq E}} \mu(K), \qquad \forall E \in \mathfrak{M}. \tag{3.5.8}$$

Proof All claims in the statement follow from Proposition 3.5.2, bearing in mind (the second part of) Definition 3.4.3. □

Moving on, we record the following basic definition:

Definition 3.5.4 *(i) Let X be an arbitrary set. Call μ a* `complex measure` *on X provided μ is a function mapping a sigma-algebra $\mathfrak{M}$ (of subsets of X) into the field of complex numbers $\mathbb{C}$ with the property that*

$$\text{for each countable family } \{E_j\}_{j\in\mathbb{N}} \subseteq \mathfrak{M} \text{ of mutually disjoint sets it follows that } \sum_{j=1}^{\infty} |\mu(E_j)| < +\infty \text{ and } \mu\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \sum_{j=1}^{\infty} \mu(E_j). \tag{3.5.9}$$

(ii) Let (X, τ) be a topological space. Call μ a `complex Borel measure` *on X provided μ is a complex measure on X with the property that the sigma-algebra $\mathfrak{M}$ on which μ is defined contains $Borel_\tau(X)$. Henceforth, the family of all complex Borel measures in a topological space (X, τ) is going to be denoted by* $\mathrm{CBM}(X, \tau)$ *(or simply* $\mathrm{CBM}(X)$ *if the topology τ is understood from the specific context).*

It is worth stressing that a complex measure cannot take infinite values, so an "ordinary" positive measure is a complex measure if and only if it is finite. Standard references are [231, Chapter 6], [91, Sect. 3.3]. The class of signed/complex Radon measures is formally introduced next.

Definition 3.5.5 *Let (X, τ) be a topological space. A* `signed Radon measure` *on X is a signed Borel measure on X whose positive and negative variations (given by the Jordan Decomposition Theorem; cf.* [91, Theorem 3.4, p. 87]) *are positive Radon measures.*

Also, a `complex Radon measure` *on X is a complex Borel measure μ on X whose real and imaginary parts are signed Radon measures on X.*

It turns out (see [91, Proposition 7.16, p. 222]) that for any topological space (X, τ) and any complex Borel measure μ on X,

$$\mu \text{ is a complex Radon measure on } X \text{ if and only if } |\mu| \text{ is a (positive, finite) Radon measure on } X. \tag{3.5.10}$$

Here and elsewhere, if μ is a complex measure on an arbitrary measurable space X, we let $|\mu|$ denote its total variation, i.e.

$$|\mu|(E) := \sup \sum_{j=1}^{\infty} |\mu(E_j)|, \qquad \forall E \in \mathfrak{M}, \tag{3.5.11}$$

where $\mathfrak{M}$ is the sigma-algebra on which μ is defined and the supremum is taken over all partitions $\{E_j\}_{j\in\mathbb{N}}$ of E (i.e., countable families of mutually disjoint sets in $\mathfrak{M}$ whose union in E; see [231, p. 116]). As is well known (cf., e.g., [231, Theorem 6.2, p. 117] and [231, Theorem 6.4, p. 118])

$$\text{the total variation } |\mu| \text{ of the complex measure } \mu \text{ is a finite (positive) measure on } X. \tag{3.5.12}$$

Let us also recall (cf. [91, pp. 94-95]) that the action of $|\mu|$ may alternatively be described as

$$|\mu|(E) = \sup \left\{ \left| \int_E f \, d\mu \right| : f \ \mu\text{-measurable, with } |f| \leq 1 \mu\text{-a.e. on } X \right\} \tag{3.5.13}$$

for each μ-measurable set $E \subseteq X$. If the ambient set X is an open subset Ω of $\mathbb{R}^n$ and if the given measure μ is actually a complex Radon measure on Ω, then Lusin's theorem (cf. [91, Theorem 7.10, p. 217]) plus a standard mollifier argument permit us to express

$$|\mu|(\Omega) = \sup \left\{ \left| \int_\Omega \phi \, d\mu \right| : \phi \in \mathscr{C}_c^\infty(\Omega), \ \sup_{x \in \Omega} |\phi(x)| \leq 1 \right\}. \tag{3.5.14}$$

Given a topological space (X, τ), denote by $\text{CRM}(X, \tau)$ the collection of all complex Radon measures on (X, τ). This is a linear space and

$$\text{CRM}(X, \tau) \ni \mu \longmapsto \|\mu\| := |\mu|(X) \in [0, \infty) \tag{3.5.15}$$

is a norm on it. The classical Riesz–Markov–Kakutani Representation Theorem isometrically identifies this normed space with the topological dual of continuous functions vanishing at infinity. Concretely, if (X, τ) is a locally compact Hausdorff topological space, denote by $\mathscr{C}^0(X)$ the space of continuous functions on X, and define

$$\begin{aligned} \mathscr{C}_o(X) &:= \left\{ f \in \mathscr{C}^0(X) : \forall \varepsilon > 0 \ \exists \, K \subseteq X \ \text{ compact, such that } \sup_{x \in X \setminus K} |f(x)| < \varepsilon \right\} \\ &= \left\{ f \in \mathscr{C}^0(X) : \{x \in X : |f(x)| \geq \varepsilon\} \ \text{ is compact for each } \ \varepsilon > 0 \right\}, \end{aligned} \tag{3.5.16}$$

which is a Banach space when equipped with the supremum norm. Then the mapping

$$\begin{gathered} \text{CRM}(X, \tau) \ni \mu \longmapsto \Lambda_\mu \in \big(\mathscr{C}_o(X)\big)^*, \\ \Lambda_\mu(f) := \textstyle\int_X f \, d\mu \ \text{ for all } \ f \in \mathscr{C}_o(X), \\ \text{constitutes an isometric isomorphism.} \end{gathered} \tag{3.5.17}$$

Cf., e.g., [91, Theorem 7.17, p. 223] for a proof.

For future references, let us also agree to denote by $\text{CBM}(X, \tau)$ the collection of all complex Borel measures on a given topological space (X, τ). By design $\text{CRM}(X, \tau) \subseteq \text{CBM}(X, \tau)$, and we also have (cf. [91, p. 222])

$$\text{CRM}(X, \tau) = \text{CBM}(X, \tau) \text{ whenever } (X, \tau) \text{ is a second countable locally compact Hausdorff topological space.} \tag{3.5.18}$$

In particular, since any given open subset Ω of $\mathbb{R}^n$ becomes a second countable locally compact Hausdorff topological space when equipped with the relative topology induced by the Euclidean ambient, we have (after a slight adjustment in notation)

$$\text{CRM}(\Omega) = \text{CBM}(\Omega). \tag{3.5.19}$$

In the next lemma, we study the vector Riesz transforms of certain Borel measures in open subsets of $\mathbb{R}^n$.

Lemma 3.5.6 *Let Ω be an open subset of $\mathbb{R}^n$.*

(i) Suppose μ is a positive Borel measure on Ω with the property that, for some $d \in [0, n)$, satisfies

$$\int_\Omega \frac{1}{1 + |y|^d} \, d\mu(y) < +\infty. \tag{3.5.20}$$

Then the function

$$\mathbb{R}^n \ni x \longmapsto \int_\Omega \frac{1}{|x - y|^d} \, d\mu(y) \in [0, +\infty] \tag{3.5.21}$$

belongs to $L^p_{\text{loc}}(\mathbb{R}^n, \mathcal{L}^n)$ whenever $1 \le p < n/d$.

(ii) If either μ is a positive measure as in item (i) corresponding to $d := n - 1$, or $\mu \in \text{CBM}(\Omega)$, then the vector Riesz transform of the measure μ, i.e., the vector field

$$\vec{F}(x) := \frac{1}{\omega_{n-1}} \int_\Omega \frac{x - y}{|x - y|^n} \, d\mu(y) \ \text{ for } \mathcal{L}^n\text{-a.e. } x \in \Omega, \tag{3.5.22}$$

is well defined as an element in $\left[L^p_{\text{loc}}(\Omega, \mathcal{L}^n)\right]^n$ whenever $1 \le p < n/(n - 1)$, and satisfies

$$\text{div}\vec{F} = \mu \ \text{ in } \ \mathcal{D}'(\Omega). \tag{3.5.23}$$

Proof First note that, given any $m \in [0, n)$, there exists a constant $C_{n,m} \in (0, \infty)$ such that for each $R \in (0, \infty)$, we have

$$\int_{B(0,R)} \frac{dx}{|x - y|^m} \le C_{n,m} \cdot \frac{R^n}{R^m + |y|^m} \ \text{ for each } \ y \in \mathbb{R}^n. \tag{3.5.24}$$

Indeed, since $m < n$, it follows that if $y \in B(0, 2R)$, then

$$\int_{B(0,R)} \frac{dx}{|x - y|^m} \le \int_{B(0,3R)} \frac{dz}{|z|^m} \le C_{n,m} R^{n-m}, \tag{3.5.25}$$

while since $m \geq 0$, if $y \in \mathbb{R}^n \setminus B(0, 2R)$, we have

$$\int_{B(0,R)} \frac{\mathrm{d}x}{|x-y|^m} \leq \int_{B(0,R)} \frac{2^m}{|y|^m}\,\mathrm{d}x \leq C_{n,m} R^n |y|^{-m}. \tag{3.5.26}$$

Collectively, (3.5.25)–(3.5.26) prove (3.5.24).

To proceed, select some $d \in [0, n)$ along with some $p \in [1, n/d)$. Then for each $R \in (0, \infty)$ Minkowski's inequality and (3.5.20) permit us to estimate

$$\begin{aligned}\left(\int_{B(0,R)} \left(\int_\Omega \frac{\mathrm{d}\mu(y)}{|x-y|^d}\right)^p \mathrm{d}x\right)^{1/p} &\leq \int_\Omega \left(\int_{B(0,R)} \frac{\mathrm{d}x}{|x-y|^{pd}}\right)^{1/p} \mathrm{d}\mu(y)\\ &\leq \int_\Omega \left(C_{n,p,d} \cdot \frac{R^n}{R^{pd}+|y|^{pd}}\right)^{1/p} \mathrm{d}\mu(y)\\ &\leq C_{n,p,d,R} \int_\Omega \frac{1}{1+|y|^d}\,\mathrm{d}\mu(y) < +\infty, \end{aligned} \tag{3.5.27}$$

where the second inequality is implied by (3.5.24) used with $m := pd \in [0, n)$. Bearing in mind that the function in (3.5.21) is also $\mathcal{L}^n$-measurable (as seen from Fubini–Tonelli's Theorem, which is presently applicable given that (3.5.20) ensures that the measure μ is sigma-finite), this finishes the proof of the claim in item *(i)*.

In turn, this readily implies the first claim in item *(ii)*. As regards the second claim in item *(ii)*, pick an arbitrary test function $\varphi \in \mathscr{C}_c^\infty(\Omega)$ and compute

$$\begin{aligned} {}_{\mathcal{D}'(\Omega)}\langle \mathrm{div}\vec{F}, \varphi\rangle_{\mathcal{D}(\Omega)} &= -_{[\mathcal{D}'(\Omega)]^n}\langle \vec{F}, \nabla\varphi\rangle_{[\mathcal{D}(\Omega)]^n} = -\int_\Omega \langle \vec{F}, \nabla\varphi\rangle\,\mathrm{d}\mathcal{L}^n\\ &= -\frac{1}{\omega_{n-1}} \int_\Omega \Big\langle \int_\Omega \frac{x-y}{|x-y|^n}\,\mathrm{d}\mu(y), \nabla\varphi(x)\Big\rangle \mathrm{d}x\\ &= \int_\Omega \left(-\frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{\langle x-y, \nabla\varphi(x)\rangle}{|x-y|^n}\,\mathrm{d}x\right) \mathrm{d}\mu(y)\\ &= \int_\Omega \varphi(y)\,\mathrm{d}\mu(y) = {}_{\mathcal{D}'(\Omega)}\langle \mu, \varphi\rangle_{\mathcal{D}(\Omega)} \end{aligned} \tag{3.5.28}$$

where we have used the fact that $\vec{F} \in \left[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\right]^n$ in the second equality, Fubini's Theorem in the fourth equality, and the well-known fact that the distributional Laplacian of the classical Newtonian potential is Dirac's distribution in the fifth equality. □

The following result is useful, among other things, in establishing embeddings of Lorentz spaces into weighted Lebesgue spaces (see Lemma 6.2.9).

Lemma 3.5.7 *Suppose $n \in \mathbb{N}$, $m \in [0, n)$, and consider a Lebesgue measurable set $\Omega \subseteq \mathbb{R}^n$ along with a non-negative Lebesgue measurable function w defined in Ω. Then the following statements are equivalent:*

(1) The function w belongs to the space $L^1\big(\Omega, \frac{dy}{1+|y|^m}\big)$*, i.e.*

$$\int_\Omega \frac{w(y)}{1+|y|^m}\,dy < +\infty. \tag{3.5.29}$$

(2) There exists a point $x \in \mathbb{R}^n$ *with the property that*

$$\int_\Omega \frac{w(y)}{|x-y|^m}\,dy < +\infty. \tag{3.5.30}$$

(3) For $\mathcal{L}^n$*-a.e. point* $x \in \mathbb{R}^n$*, one has*

$$\int_\Omega \frac{w(y)}{|x-y|^m}\,dy < +\infty. \tag{3.5.31}$$

Proof To show that *(1)⇒(3)*, for each $R \in (0,\infty)$ use Fubini–Tonelli's Theorem and (3.5.24) to write

$$\begin{aligned}\int_{B(0,R)}\Big(\int_\Omega \frac{w(y)}{|x-y|^m}\,dy\Big)\,dx &= \int_\Omega w(y)\Big(\int_{B(0,R)} \frac{dx}{|x-y|^m}\Big)\,dy \\ &\leq C\int_\Omega \frac{w(y)}{1+|y|^m}\,dy < +\infty, \end{aligned} \tag{3.5.32}$$

for some constant $C = C(n,m,R) \in (0,\infty)$. In turn, (3.5.32) implies that (3.5.31) holds for $\mathcal{L}^n$-a.e. point $x \in B(0,R)$, and the claim in item *(3)* follows on account of the arbitrariness of R. Next, the implication *(3)⇒(2)* is obvious, while the implication *(2)⇒(1)* is justified by observing that for each fixed $x \in \mathbb{R}^n$, there exists $C_x \in (0,\infty)$ such that $|x-y|^m \leq C_x(1+|y|^m)$ for each $y \in \mathbb{R}^n$. □

It is also of interest to have a version of Lemma 3.5.7 for a logarithmic weight, of the sort described below.

Lemma 3.5.8 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a Lebesgue measurable set and consider a non-negative Lebesgue measurable function w defined in* Ω*. Then the following statements are equivalent:*

(1) One has

$$\int_\Omega w(y)\ln(2+|y|)\,dy < +\infty. \tag{3.5.33}$$

(2) There exists a point $x \in \mathbb{R}^n$ *with the property that*

$$\int_\Omega w(y)\big(1+\big|\ln|x-y|\big|\big)\,dy < +\infty. \tag{3.5.34}$$

(3) For $\mathscr{L}^n$-a.e. point $x \in \mathbb{R}^n$, one has

$$\int_\Omega w(y)\big(1 + \big|\ln|x-y|\big|\big)\,\mathrm{d}y < +\infty. \tag{3.5.35}$$

Proof We reason similarly to the proof of Lemma 3.5.7, this time noting that for each $R \in (0, \infty)$ there exists a constant $C_R \in (0, \infty)$ such that

$$\int_{B(0,R)} \big(1 + \big|\ln|x-y|\big|\big)\,\mathrm{d}x \leq C_R \ln(2+|y|) \quad \text{for each} \quad y \in \mathbb{R}^n. \tag{3.5.36}$$

In turn, estimate (3.5.36) is established by analyzing two cases, namely the situations when $y \in B(0, 2R)$ and, $y \in \mathbb{R}^n \setminus B(0, 2R)$, respectively. □

3.6 Separable Measures

Let $(X, \mathfrak{M}, \mu)$ be an arbitrary measure space. Consider the equivalence relation on $\mathfrak{M}$ given by

$$A \sim B \overset{\text{def}}{\Longleftrightarrow} \mu(A \triangle B) = 0, \tag{3.6.1}$$

where "△" denotes the set theoretic symmetric difference, and denote by $[A]$ the equivalence class of a generic set $A \in \mathfrak{M}$. Then (cf., e.g., [188, Corollary 5.19, p. 342])

$$\begin{gathered}\big\{[A] : A \in \mathfrak{M},\ \mu(A) < +\infty\big\} \text{ equipped with the distance} \\ \big([A],[B]\big) \longmapsto \mu(A \triangle B) \text{ is a complete metric space.}\end{gathered} \tag{3.6.2}$$

Definition 3.6.1 *Given a measure space $(\Sigma, \mathfrak{M}, \mu)$, call the measure μ* `separable` *provided the metric space (3.6.2) is separable.*

Then [188, Theorem 5.5, p. 300] implies that

if $(X, \mathfrak{M}, \mu)$ is a sigma-finite measure space and $p \in (0, \infty)$, then the Lebesgue space $L^p(X, \mu)$ is separable whenever the measure μ is separable (in the sense of Definition 3.6.1). (3.6.3)

In fact, an inspection of the proof of [17, Theorem 5.5, p. 27] shows that the separability of the measure μ is actually also a necessary condition for the separability of the space $L^p(X, \mu)$.

Lemma 3.6.2 *Let (X, τ) be a second-countable*[2] *topological space and consider a Borel measure $\mu : \mathfrak{M} \to [0, \infty]$ on X with the property that*

[2] A topological space is called second countable if there exists a countable family $\mathscr{U}$ of open sets with the property that any given open set may be written as the union of the sets belonging to a subfamily of $\mathscr{U}$.

$$\text{there exist } \{X_j\}_{j\in\mathbb{N}} \subseteq Borel_\tau(X) \text{ such that } X = \bigcup_{j\in\mathbb{N}} X_j \text{ and } \mu(X_j) < +\infty \text{ for each } j \in \mathbb{N}. \tag{3.6.4}$$

Then the measure $\mu\big|_{Borel_\tau(X)}$ *is separable.*

Proof In a first stage, strengthen the hypothesis made in (3.6.4) by asking that μ is actually finite. Let $\mathscr{U} := \{O_j\}_{j\in\mathbb{N}}$ be a sequence of open subsets of X with the property that any open set in X may be written as a union of sets in $\mathscr{U}$. If we then define

$$\mathscr{F}_1(X) := \Big\{A \subseteq X : \text{ either } A = \varnothing, \text{ or } A \in \mathscr{U}, \text{ or } X \setminus A \in \mathscr{U}\Big\} \tag{3.6.5}$$

$$\mathscr{F}_2(X) := \Big\{\bigcap_{j=1}^{N} E_j : N \in \mathbb{N} \text{ and } E_j \in \mathscr{F}_1(X) \text{ for } 1 \leq j \leq N\Big\}, \tag{3.6.6}$$

$$\mathscr{F}_3(X) := \Big\{\bigcup_{j=1}^{M} F_j : M \in \mathbb{N} \text{ and } F_j \in \mathscr{F}_2(X) \text{ for } 1 \leq j \leq M\Big\}, \tag{3.6.7}$$

it follows (cf. [25, p. 5] for the simple argument) that the collection $\mathscr{F}_3(X)$ is an algebra of subsets of X, i.e., it satisfies

$$\begin{aligned} &\varnothing, X \in \mathscr{F}_3(X), \text{ and for each } A, B \in \mathscr{F}_3(X) \text{ one has} \\ &A \cap B \in \mathscr{F}_3(X), A \cup B \in \mathscr{F}_3(X), A \setminus B \in \mathscr{F}_3(X). \end{aligned} \tag{3.6.8}$$

Moreover, it is apparent from the above definitions that

$$\begin{aligned} &\mathscr{F}_3(X) \text{ is a countable family of subsets of } X \text{ with} \\ &\text{the property that } \mathscr{U} \subseteq \mathscr{F}_3(X) \subseteq Borel_\tau(X). \end{aligned} \tag{3.6.9}$$

In particular, the double inclusion in (3.6.9) implies that

$$\text{the sigma-algebra generated by } \mathscr{F}_3(X) \text{ is precisely } Borel_\tau(X). \tag{3.6.10}$$

In relation to this, we claim that if

$$\begin{aligned} \mathcal{A} := \Big\{E \subseteq X : &\text{ for each } \varepsilon > 0 \text{ there exists} \\ &F \in \mathscr{F}_3(X) \text{ such that } \mu(E \Delta F) < \varepsilon\Big\} \end{aligned} \tag{3.6.11}$$

then

$$\mathcal{A} \text{ is a sigma-algebra of subsets of } X, \text{ containing } Borel_\tau(X). \tag{3.6.12}$$

To justify this claim, first observe that $\varnothing, X \in \mathcal{A}$. Also, if $E \in \mathcal{A}$, $\varepsilon > 0$, and $F \in \mathscr{F}_3(X)$ are such that $\mu(E\Delta F) < \varepsilon$, then $X \setminus F$ belongs to $\mathscr{F}_3(X)$ and satisfies $\mu\big((X \setminus E)\Delta(X \setminus F)\big) = \mu(E\Delta F) < \varepsilon$. This goes to show that $\mathcal{A}$ is stable under taking complements. To prove that $\mathcal{A}$ is a sigma-algebra, there remains to show that if $\{E_j\}_{j\in\mathbb{N}} \subseteq \mathcal{A}$, then $E := \bigcup_{j\in\mathbb{N}} E_j$ belongs to $\mathcal{A}$. To this end, fix some $\varepsilon > 0$ and note that (3.6.11) guarantees that

$$\begin{aligned}&\text{for each } j \in \mathbb{N} \text{ there exists some } F_j \in \mathscr{F}_3(X)\\ &\text{with the property that } \mu(E_j\Delta F_j) < \varepsilon \cdot 2^{-j-1}.\end{aligned} \tag{3.6.13}$$

Upon recalling that we are assuming the measure μ to be finite, it is possible to

$$\text{pick } N \in \mathbb{N} \text{ such that } \mu\Big(E \setminus \bigcup_{j=1}^{N} E_j\Big) < \varepsilon/2. \tag{3.6.14}$$

Then (3.6.8) implies that $F := \bigcup_{j=1}^{N} F_j$ belongs to $\mathscr{F}_3(X)$ and (3.6.13)–(3.6.14) permit us to estimate

$$\mu(E\Delta F) \le \mu\Big(E \setminus \bigcup_{j=1}^{N} E_j\Big) + \sum_{j=1}^{N} \mu(E_j\Delta F_j) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{3.6.15}$$

This proves that $E \in \mathcal{A}$, hence $\mathcal{A}$ is indeed a sigma-algebra. Since, by design, $\mathcal{A}$ contains $\mathscr{F}_3(X)$, we conclude from (3.6.10) that the sigma-algebra $\mathcal{A}$ actually contains $Borel_\tau(X)$. This finishes the proof of (3.6.12).

In turn, from (3.6.11)–(3.6.12), we see that, given any $E \in Borel_\tau(X)$, for each $\varepsilon > 0$ there exists $F \in \mathscr{F}_3(X)$ such that $\mu(E\Delta F) < \varepsilon$. In view of the first property in (3.6.9) and Definition 3.6.1, it follows that the measure $\mu\big|_{Borel_\tau(X)}$ is indeed separable.

At this stage, there remains to relax the assumption that the measure μ is finite to (3.6.4). In the latter scenario, for each $j \in \mathbb{N}$ define τ_j to be the relative topology induced by X on X_j, and consider $\mu_j := \mu\lfloor X_j$. Then each (X_j, τ_j) continues to be a second-countable topological space (since this property is hereditary) and, thanks to (3.1.1), we have

$$\begin{aligned} Borel_{\tau_j}(X_j) &= \big\{B \cap X_j : B \in Borel_\tau(X)\big\}\\ &= \big\{B \in Borel_\tau(X) : B \subseteq X_j\big\}. \end{aligned} \tag{3.6.16}$$

Since μ_j is defined on the sigma-algebra

$$\mathfrak{M}_j := \big\{E \cap X_j :\ E \in \mathfrak{M}\big\} = \big\{E \in \mathfrak{M} :\ E \subseteq X_j\big\}, \tag{3.6.17}$$

it follows that each μ_j is a finite Borel measure on (X_j, τ_j). Granted this, it follows from the first part of the proof that for each $j \in \mathbb{N}$, the measure $\mu_j\big|_{Borel_{\tau_j}(X_j)}$ is separable. More specifically, what we have proved so far implies that

$$\text{for each } j \in \mathbb{N}, \text{each } E_j \in Borel_{\tau_j}(X_j), \text{and each } \varepsilon > 0, \text{there exists some set } F_j \in \mathscr{F}_3(X_j) \text{ such that } \mu_j(E_j \Delta F_j) < \varepsilon. \tag{3.6.18}$$

Then

$$\mathscr{F} := \bigcup_{j \in \mathbb{N}} \mathscr{F}_3(X_j) \quad \text{is a countable subset of } \; Borel_\tau(X). \tag{3.6.19}$$

Fix now $B \in Borel_\tau(X)$ along with $\varepsilon > 0$ arbitrary. It follows from (3.6.18) that for each $j \in \mathbb{N}$, we have $B \cap X_j \in Borel_{\tau_j}(X_j)$ so there exists $F_j \in \mathscr{F}_3(X_j)$ such that $\mu_j\big((B \cap X_j)\Delta F_j\big) < \varepsilon \cdot 2^{-j}$. If we now define $F := \bigcup_{j \in \mathbb{N}} F_j$, then $F \in \mathscr{F}$ and

$$\mu(B \Delta F) \leq \sum_{j=1}^{\infty} \mu\big((B \cap X_j)\Delta F_j\big) = \sum_{j=1}^{\infty} \mu_j\big((B \cap X_j)\Delta F_j\big) \tag{3.6.20}$$

$$< \sum_{j=1}^{\infty} \varepsilon \cdot 2^{-j} = \varepsilon. \tag{3.6.21}$$

This and (3.6.19) then imply that the measure $\mu\big|_{Borel_\tau(X)}$ is separable. □

Separability is particularly important from the point of view of the Sequential Banach–Alaoglu Theorem (cf. [232, Theorem 3.17, p. 70]), which asserts that

given a separable topological vector space X, if $\{\Lambda_j\}_{j\in\mathbb{N}} \subseteq X^*$ has the property that there exists a neighborhood V of the origin in X for which $\sup_{x\in V} \sup_{j\in\mathbb{N}} |\Lambda_j x| < +\infty$ then there exists a subsequence $\{\Lambda_{j_k}\}_{k\in\mathbb{N}}$ of $\{\Lambda_j\}_{j\in\mathbb{N}}$ which is weak-$*$ convergent to a $\Lambda \in X^*$, i.e., $\Lambda_{j_k} x \to \Lambda x$ in $\mathbb{C}$ as $k \to \infty$ for each fixed $x \in X$. (3.6.22)

As a corollary[3],

if V is a reflexive normed space such that V^* is separable then the closed unit ball in V is sequentially weakly compact. (3.6.23)

Here is a versatile criterion guaranteeing the separability of Lebesgue spaces in a fairly general setting.

[3] In relation to (3.6.23), generally speaking, the closed unit ball in the reflexive normed space V is merely weakly compact if the separability assumption on V^* is dropped.

Proposition 3.6.3 *Let (X, τ) be a second-countable topological space and consider a sigma-finite Borel-regular measure $\mu : \mathfrak{M} \to [0, +\infty]$ on X. Then the measure μ is separable, and for each integrability exponent $p \in (0, \infty)$ the Lebesgue space $L^p(X, \mu)$ is separable.*

Proof Since μ is sigma-finite, there exist $\{E_j\}_{j\in\mathbb{N}} \subseteq \mathfrak{M}$ such that $X = \bigcup_{j\in\mathbb{N}} E_j$ and $\mu(E_j) < +\infty$ for each $j \in \mathbb{N}$. Then (3.4.1) implies that for each $j \in \mathbb{N}$, there exists $X_j \in Borel_\tau(X)$ with the property that $E_j \subseteq X_j$ and $\mu(X_j) = \mu(E_j) < +\infty$. Thus, condition (3.6.4) holds for the family $\{X_j\}_{j\in\mathbb{N}}$. Granted this, Lemma 3.6.2 applies and gives that the measure $\mu\big|_{Borel_\tau(X)}$ is separable. More specifically, from the proof of Lemma 3.6.2, we know that there exists a countable subset $\mathscr{F}$ of $Borel_\tau(X)$ with the property that

$$\begin{aligned}&\text{for each } B \in Borel_\tau(X) \text{ and each } \varepsilon > 0 \text{ there}\\&\text{exists some set } F \in \mathscr{F} \text{ such that } \mu(B\Delta F) < \varepsilon.\end{aligned} \tag{3.6.24}$$

Fix now an arbitrary set $E \in \mathfrak{M}$ with $\mu(E) < +\infty$, and pick some arbitrary $\varepsilon > 0$. Then (3.4.1) guarantees that there exists $B \in Borel_\tau(X)$ satisfying $E \subseteq B$ and $\mu(E) = \mu(B)$. Consequently, the set $F \in \mathscr{F}$ associated with the current B, ε as in (3.6.24) satisfies $\mu(E\Delta F) = \mu(B\Delta F) < \varepsilon$ (since the fact that E has finite measure forces $\mu(B \setminus E) = 0$).

Hence, $\big\{[F] : F \in \mathscr{F}, \ \mu(F) < +\infty\big\}$ is a dense countable subset of the metric space defined in (3.6.2). Thus, this metric space is separable, which proves that the measure μ is separable. In concert with (3.6.3), this shows that each Lebesgue space $L^p(X, \mu)$ with $p \in (0, \infty)$ is separable. □

When restricted to measurable sets of locally finite mass, the Hausdorff measure enjoys a wealth of useful properties, described in the following lemma.

Lemma 3.6.4 *Consider $s \in [0, \infty)$ and let*

$$\begin{aligned}&X \subseteq \mathbb{R}^n \textit{ be some } \mathcal{H}^s\textit{-measurable set satisfying}\\&\mathcal{H}^s(X \cap K) < +\infty \textit{ for every compact } K \subset \mathbb{R}^n.\end{aligned} \tag{3.6.25}$$

If $\tau_{\mathbb{R}^n}\big|_X$ denotes the topology induced by the ambient Euclidean space $\mathbb{R}^n$ on the set X, then

$$\begin{aligned}&\mathcal{H}^s \lfloor X \textit{ is a complete, locally finite,}\\&\textit{Borel-regular measure on } \big(X, \tau_{\mathbb{R}^n}\big|_X\big).\end{aligned} \tag{3.6.26}$$

Moreover,

$$\begin{aligned}&\textit{the measure } \mathcal{H}^s \lfloor X \textit{ is separable, and for each exponent}\\&p \textit{ in } (0, \infty) \textit{ the Lebesgue space } L^p\big(X, \mathcal{H}^s \lfloor X\big) \textit{ is separable.}\end{aligned} \tag{3.6.27}$$

Proof First, it is well known (cf. [80, Theorem 1, p. 61]) that the s-dimensional Hausdorff outer measure is a Borel-regular outer measure in $\mathbb{R}^n$. Second, the measure induced by an arbitrary outer measure (as in Carathéodory's theorem) is automatically complete (cf. (3.3.4)), hence

$$\mathcal{H}^s \text{ is a complete Borel-regular measure in } \mathbb{R}^n. \tag{3.6.28}$$

Third, the quality of being Borel-regular is hereditary, in the precise described in (3.4.17). Fourth, completeness is also hereditary, as remarked in (3.1.28). In concert with (3.6.25), these considerations imply that $\mathcal{H}^s\lfloor X$ is a complete locally finite Borel-regular measure on $\big(X, \tau_{\mathbb{R}^n}\big|_X\big)$.

Having established this, Proposition 3.6.3 applies (upon observing that, as a topological space, $\mathbb{R}^n$ is second countable, and this property is hereditary) and gives (3.6.27). □

3.7 Density Results for Lebesgue Spaces

In the proposition below, we identify an optimal geometric measure theoretic context in which the associated Lebesgue spaces enjoy useful density properties. To set the stage, for a given subset X of an ambient metric space $(\mathscr{X}, d)$, consider the seminorm

$$\|f\|_{\mathrm{Lip}(X)} := \sup_{x,y\in X,\, x\neq y} \frac{|f(x)-f(y)|}{d(x,y)} \tag{3.7.1}$$

for each (scalar-valued) function f defined on X, and introduce the space of complex-valued Lipschitz functions defined on X as

$$\mathrm{Lip}(X) := \big\{ f : X \to \mathbb{C} : \ \|f\|_{\mathrm{Lip}(X)} < +\infty \big\} \tag{3.7.2}$$

Also, denote by $\mathrm{Lip}_c(X)$ the space of complex-valued Lipschitz functions defined on X which vanish identically outside of a bounded subset of X. Classical work of McShane and Whitney (cf., e.g., the discussion in [10], [11], and [188, Sect. 4.2, pp. 156-164]) gives

$$\mathrm{Lip}_c(X) = \big\{ f\big|_X : \ f \in \mathrm{Lip}_c(\mathbb{R}^n) \big\}, \qquad \forall X \subseteq \mathbb{R}^n. \tag{3.7.3}$$

Proposition 3.7.1 *Assume $X \subseteq \mathbb{R}^n$ is an arbitrary set, and denote by $\tau_{\mathbb{R}^n}\big|_X$ the topology induced by the Euclidean ambient on X. Also, let μ be a locally finite Borel measure on $\big(X, \tau_{\mathbb{R}^n}\big|_X\big)$. Then the following are equivalent:*

(1) the measure μ is Borel-semiregular;
(2) for some, or every, $p \in (0, \infty)$ the natural inclusion

$$\mathrm{Lip}_c(X) \hookrightarrow L^p(X, \mu) \ \ \textit{has dense range}; \tag{3.7.4}$$

(3) for some, or every, $p \in (0, \infty)$ the natural inclusion

$$\{\phi\big|_X : \phi \in \mathscr{C}_c^\infty(\mathbb{R}^n)\} \hookrightarrow L^p(X, \mu) \text{ has dense range.} \tag{3.7.5}$$

Proof In a first stage, assume the measure μ is Borel-semiregular and fix an arbitrary exponent $p \in (0, \infty)$. The goal is to approximate arbitrarily well in $L^p(X, \mu)$ a given function $f \in L^p(X, \mu)$ with functions from $\mathrm{Lip}_c(X)$. Since simple functions are dense in $L^p(X, \mu)$ (cf. (3.1.11)), there is no loss of generality in assuming that $f = \mathbf{1}_E$ where $E \subseteq X$ is μ-measurable and $\mu(E) < \infty$. Because μ is a Borel-semiregular measure, there exists $B \in Borel_\tau(X)$ satisfying $\mu(B \Delta E) = 0$. This forces $\mathbf{1}_E = \mathbf{1}_B$ pointwise μ-almost everywhere on X, hence $\mathbf{1}_E = \mathbf{1}_B$ when regarded as functions in $L^p(X, \mu)$. As such, matters have been reduced to approximating $\mathbf{1}_B$ arbitrarily well in $L^p(X, \mu)$ with functions from $\mathrm{Lip}_c(X)$, for any $B \in Borel_\tau(X)$ with $\mu(B) < \infty$.

With this goal in mind, we first claim that it may be assumed that the Borel set B is actually bounded. Indeed, if we consider $B_j := B \cap B(0, j)$ for each $j \in \mathbb{N}$, then each B_j is a bounded Borel subset of X and $\mathbf{1}_{B_j} \to \mathbf{1}_B$ in $L^p(X, \mu)$ as $j \to \infty$. Hence, approximating $\mathbf{1}_B$ in the desired manner is implied by the ability of approximating each $\mathbf{1}_{B_j}$ in a similar fashion. This concludes the proof of the claim at the beginning of the paragraph.

Moving on, in the scenario when B is a bounded Borel set, item *(1)* in Proposition 3.4.15 applies (since (3.4.23) holds in the current setting thanks to Lemma 3.4.13, and since $\mu(B) < \infty$) and (3.4.41) gives

$$\mu(B) = \sup_{\substack{C \text{ relatively closed in } X \\ C \text{ bounded, } C \subseteq B}} \mu(C). \tag{3.7.6}$$

From (3.7.6), we can find a sequence of sets $\{C_i\}_{i\in\mathbb{N}} \subseteq B$ such that $\mu(C_i) \nearrow \mu(B)$ as $i \to \infty$ where each C_i is a bounded, relatively closed subset of X. In particular, this implies $\mathbf{1}_{C_i} \to \mathbf{1}_B$ in $L^p(X, \mu)$ as $i \to \infty$. Hence, ultimately it suffices to approximate each $\mathbf{1}_{C_i}$ in $L^p(X, \mu)$ with functions from $\mathrm{Lip}_c(X)$. At this point, for each fixed $i \in \mathbb{N}$, we may invoke [188, Lemma 4.14, p. 166] in order to find a sequence $\{\phi_j^{(i)}\}_{j\in\mathbb{N}} \subset \mathrm{Lip}_c(X)$ along with a number $R \in (0, \infty)$ satisfying

$$\begin{aligned}
&\mathrm{supp}\ \phi_j^{(i)} \subseteq B(0, R) \quad \text{for each} \quad j \in \mathbb{N}, \\
&0 \le \phi_j^{(i)} \le 1 \quad \text{on} \quad X \quad \text{for each} \quad j \in \mathbb{N}, \\
&\text{and} \quad \phi_j^{(i)} \searrow \mathbf{1}_{C_i} \quad \text{pointwise on } X \text{ as} \quad j \to \infty.
\end{aligned} \tag{3.7.7}$$

In particular, $\phi_j^{(i)} \to \mathbf{1}_{C_i}$ in $L^p(X, \mu)$ as $j \to \infty$, finishing the proof of (3.7.4). Bearing in mind (3.7.3), a standard mollifier argument then allows us to deduce (3.7.5) from (3.7.4).

Finally, that (3.7.5) forces μ to be a Borel-semiregular measure follows from Lemma 3.4.4. $\square$

We augment Proposition 3.7.1 with the following criterion ensuring that a locally integrable function vanishes almost everywhere.

Proposition 3.7.2 *Let $X \subseteq \mathbb{R}^n$ be an arbitrary set, and denote by $\tau_{\mathbb{R}^n}\big|_X$ the topology induced on X by the ambient Euclidean topology. Also, let μ be a locally finite Borel-regular measure on $\big(X, \tau_{\mathbb{R}^n}\big|_X\big)$. Then for every function $f \in L^1_{\rm loc}(X, \mu)$, one has*

$$f = 0 \text{ at } \mu\text{-a.e. point on } X \iff \int_X f\phi\,\mathrm{d}\mu = 0 \text{ for every } \phi \in \mathscr{C}^\infty_c(\mathbb{R}^n). \tag{3.7.8}$$

Moreover, for every open set $O \subseteq \mathbb{R}^n$ and every $f \in L^1_{\rm loc}(O \cap X, \mu)$ one has

$$f = 0 \text{ at } \mu\text{-a.e. point on } O \cap X \iff \int_{O\cap X} f\phi\,\mathrm{d}\mu = 0 \text{ for every } \phi \in \mathscr{C}^\infty_c(O). \tag{3.7.9}$$

Proof Consider an arbitrary $f \in L^1_{\rm loc}(X, \mu)$ satisfying the cancellation condition recorded in the right-hand side of (3.7.8). The first observation is that, thanks to (3.7.3) and a standard mollifier argument, this self-improves to

$$\int_X f\phi\,\mathrm{d}\mu = 0 \quad \text{for every} \quad \phi \in \mathrm{Lip}_c(X). \tag{3.7.10}$$

To proceed, fix an arbitrary bounded and relatively closed subset K of X, and again invoke [188, Lemma 4.14, p. 166] in order to find a sequence $\{\phi_j\}_{j\in\mathbb{N}} \subset \mathrm{Lip}_c(X)$ along with a number $R \in (0, \infty)$ satisfying

$$\begin{aligned}
&\mathrm{supp}\,\phi_j \subseteq B(0, R) \quad \text{for each} \quad j \in \mathbb{N},\\
&0 \le \phi_j \le 1 \quad \text{on} \quad \partial\Omega \quad \text{for each} \quad j \in \mathbb{N},\\
&\text{and} \quad \phi_j \searrow \mathbf{1}_K \quad \text{pointwise as} \quad j \to \infty.
\end{aligned} \tag{3.7.11}$$

Granted these, Lebesgue's Dominated Convergence Theorem and (3.7.10) permit us to conclude that

$$\int_K f\,\mathrm{d}\mu = \lim_{j\to\infty} \int_X f\phi_j\,\mathrm{d}\mu = 0. \tag{3.7.12}$$

Next, if for every $r \in (0, \infty)$, we introduce

$$A_r^{\pm} := \big\{x \in \overline{B(0, r)} \cap X : \pm f(x) \ge 0\big\}, \tag{3.7.13}$$

then item *(3)* in Proposition 3.4.15 applies and gives that for each $r > 0$, we have

$$\mu(A_r^{\pm}) = \sup\big\{\mu(K) : K \subseteq A_r^{\pm}, K \text{ bounded, relatively closed subset of } X\big\}. \tag{3.7.14}$$

Fix $r > 0$ and $\varepsilon > 0$ arbitrary. Since $\mathbf{1}_{\overline{B(0,r)}\cap X}|f|\,\mathrm{d}\mu$ is a finite measure which is absolutely continuous with respect to μ, it follows that there exists $\theta > 0$ with the property that

$$\int_E |f|\,\mathrm{d}\mu < \varepsilon \ \text{ for each } \mu\text{-measurable set } E \subseteq \overline{B(0,r)} \cap X \text{ with } \mu(E) < \theta. \tag{3.7.15}$$

For this θ, use (3.7.14) to find two bounded relatively closed subsets $K_r^{\pm}$ of X such that

$$K_r^{\pm} \subseteq A_r^{\pm} \ \text{ and } \ \mu\big(A_r^{\pm} \setminus K_r^{\pm}\big) < \theta. \tag{3.7.16}$$

Then, thanks to (3.7.12), we may write

$$\int_{\overline{B(0,r)}\cap X} |f|\,\mathrm{d}\mu = \int_{A_r^+} f\,\mathrm{d}\mu - \int_{A_r^-} f\,\mathrm{d}\mu = \int_{A_r^+\setminus K_r^+} f\,\mathrm{d}\mu - \int_{A^-\setminus K_r^-} f\,\mathrm{d}\mu, \tag{3.7.17}$$

and then rely on (3.7.15) and (3.7.16) to estimate

$$\left|\int_{A_r^{\pm}\setminus K_r^{\pm}} f\,\mathrm{d}\mu\right| \leq \int_{A_r^{\pm}\setminus K_r^{\pm}} |f|\,\mathrm{d}\mu < \varepsilon. \tag{3.7.18}$$

The bottom line is that $\int_{\overline{B(0,r)}\cap X} |f|\,\mathrm{d}\mu < 2\varepsilon$ for every $r > 0$ and $\varepsilon > 0$. Ultimately, this forces $f = 0$ at μ-a.e. point on X, finishing the left-pointing implication in (3.7.8). Since the opposite implication is trivial, this concludes the proof of (3.7.8).

As regards the last claim in the statement of the proposition, pick an open set $\mathcal{O} \subseteq \mathbb{R}^n$ and assume $f \in L^1_{\mathrm{loc}}(\mathcal{O} \cap X, \mu)$ satisfies the cancellation condition in the right-hand side of (3.7.9). Pick a sequence of functions $\{\psi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{C}_c^\infty(\mathcal{O})$ with the property that

$$\lim_{j\to\infty} \psi_j(x) = 1 \ \text{ for every } \ x \in \mathcal{O}. \tag{3.7.19}$$

For each $j \in \mathbb{N}$, introduce $f_j := \widetilde{\psi_j f}$, where tilde denotes the extension by zero outside of $\mathcal{O}$ to the entire X. Then for each $j \in \mathbb{N}$, we have $f_j \in L^1(X, \mu)$ and

$$\int_X f_j \phi\,\mathrm{d}\mu = \int_{\mathcal{O}\cap X} f(\psi_j\phi)\,\mathrm{d}\mu = 0 \ \text{ for every } \ \phi \in \mathscr{C}_c^\infty(\mathbb{R}^n). \tag{3.7.20}$$

Thanks to (3.7.8), this proves that for each $j \in \mathbb{N}$ we have $f_j = 0$ at μ-a.e. point on X. In light of (3.7.19), this ultimately implies that

$$f = 0 \ \text{ at } \mu\text{-a.e. point on } \ \mathcal{O} \cap X, \tag{3.7.21}$$

finishing the proof of the proposition. □

Corresponding to the Hausdorff measure, our earlier work yields the following result.

Corollary 3.7.3 *Pick $s \in (0, \infty)$ and consider a $\mathcal{H}^s$-measurable set $X \subseteq \mathbb{R}^n$ with the property that $\mathcal{H}^s(X \cap K) < \infty$ for every compact $K \subset \mathbb{R}^n$. Define $\mu := \mathcal{H}^s \lfloor X$. Then for every $p \in (0, \infty)$, the natural inclusion*

$$\big\{\phi\big|_X : \phi \in \mathscr{C}_c^\infty(\mathbb{R}^n)\big\} \hookrightarrow L^p(X, \mu) \ \text{ has dense range}, \tag{3.7.22}$$

and for every function $f \in L^1_{\mathrm{loc}}(X, \mu)$ and every open set $\mathcal{O} \subseteq \mathbb{R}^n$, one has

$$f = 0 \ \mu\text{-a.e. on } \ \mathcal{O} \cap X \iff \int_{\mathcal{O}\cap X} f\phi \, \mathrm{d}\mu = 0 \ \text{ for every } \ \phi \in \mathscr{C}_c^\infty(\mathcal{O}). \tag{3.7.23}$$

Proof All claims follow by combining Lemma 3.6.4, Proposition 3.7.1, and Proposition 3.7.2. □

3.8 The Support of a Measure

The notion of the support of a measure is formally defined below.

Definition 3.8.1 *Given a topological space (X, τ) along with some (non-negative) Borel measure μ on X, define the support of μ to be*

$$\operatorname{supp}\mu := \big\{x \in X : \mu(O) > 0 \text{ for each } O \subseteq X \text{ open such that } x \in O\big\}. \tag{3.8.1}$$

In the proposition below, we collect a number of basic properties of the brand of support introduced in Definition 3.8.1. Before stating this, we wish to clarify some terminology. Call a topological space (X, τ) `Lindelöf` if every open cover of X has a countable sub-cover. Next, a `strongly Lindelöf` space is a topological space such that every open set is itself Lindelöf (with the topology inherited from the ambient). Lastly, a topological space (X, τ) is said to be `second-countable` if there exists a countable family $\mathscr{U}$ of open subsets of X with the property that any open set $O \subseteq X$ may be written as a union of sets in $\mathscr{U}$. In particular, being second countable is a hereditary property that implies separability. Second countability also implies that any collection of mutually disjoint open sets is at most countable. It is well known that any second-countable space is a strongly Lindelöf space. Also, a quasi-metric space is strongly Lindelöf if and only if it is separable.

Proposition 3.8.2 *Suppose (X, τ) is a topological space and that μ is a (non-negative) Borel measure on X. Then the following statements are true:*

(1) Given $x \in X$, one has $x \in X \setminus \operatorname{supp}\mu$ if and only if there exists an open set $O \subseteq X$ such that $x \in O$ and $\mu(O) = 0$.
(2) One has

$$X \setminus \operatorname{supp}\mu = \bigcup_{\substack{O \text{ open} \\ \mu(O)=0}} O. \tag{3.8.2}$$

(3) *The set* $\operatorname{supp}\mu$ *is closed in* (X, τ).
(4) *One may describe* $\operatorname{supp}\mu$ *as the largest (with respect to inclusion) closed set* C *in* (X, τ) *with the property that whenever* $O \subseteq X$ *is an open set satisfying* $O \cap C \neq \varnothing$ *one necessarily has* $\mu(O) > 0$.
(5) *If* $A \subseteq X$ *is a* μ*-measurable set with the property that*

$$\begin{gathered} \mu \textit{ is concentrated on } A, \textit{ in the sense that } \mu(E) = \mu(E \cap A) \\ \textit{for every } \mu\textit{-measurable set } E \subseteq X \,(\textit{cf. } (3.1.21)), \end{gathered} \tag{3.8.3}$$

then $\operatorname{supp}\mu \subseteq \overline{A}$, *the closure of* A *in the topological space* (X, τ).
(6) *If* λ *is another (non-negative) Borel measure on* X *such that* $\mu << \lambda$ *then* $\operatorname{supp}\mu \subseteq \operatorname{supp}\lambda$.
(7) *Given any another (non-negative) Borel measure* λ *on* X, *one has*

$$\operatorname{supp}(\mu + \lambda) = (\operatorname{supp}\mu) \cup (\operatorname{supp}\lambda). \tag{3.8.4}$$

(8) *Under the additional assumption that* (X, τ) *is strongly Lindelöf (which, e.g., is the case if* (X, τ) *is second countable), it follows that* $X \setminus \operatorname{supp}\mu$ *is a* μ*-nullset. In particular,* $\mu(A) = 0$ *whenever* $A \subseteq X \setminus \operatorname{supp}\mu$ *is a* μ*-measurable set, and*

$$\int_X f\,d\mu = \int_{\operatorname{supp}\mu} f\,d\mu, \qquad \forall f \in L^1(X, \mu). \tag{3.8.5}$$

Also, in such a scenario, $X \setminus \operatorname{supp}\mu$ *is the largest (with respect to inclusion) open* μ*-nullset in the topological space* (X, τ).

Proof The claim in item *(1)* is a direct consequence of Definition 3.8.1, and the claim in item *(2)* is readily implied by *(1)*. Clearly, (3.8.2) implies the claim made in item *(3)*. Also, the claim in item *(4)* follows by untangling definitions, while the claim in item *(5)* is straightforward. To prove the claim in item *(6)*, recall that if $\mu << \lambda$ then any λ-nullset is a μ-nullset. Bearing this in mind, (3.8.2) permits us to write

$$X \setminus \operatorname{supp}\lambda = \bigcup_{\substack{O \text{ open} \\ \lambda(O)=0}} O \subseteq \bigcup_{\substack{O \text{ open} \\ \mu(O)=0}} O = X \setminus \operatorname{supp}\mu. \tag{3.8.6}$$

Passing to complements then gives $\operatorname{supp}\mu \subseteq \operatorname{supp}\lambda$, as wanted. Next, the left-to-right inclusion in (3.8.4) is seen from item *(6)* upon observing that $\mu << \mu + \lambda$ and $\lambda << \mu + \lambda$. To prove the opposite inclusion, suppose $x \in X \setminus (\operatorname{supp}\mu \cup \operatorname{supp}\lambda)$. The fact that $x \notin \operatorname{supp}\mu$ implies the existence of some open set $O \subseteq$ such that $x \in O$ and $\mu(O) = 0$, while having $x \notin \operatorname{supp}\lambda$ guarantees the existence of some open set $\widetilde{O} \subseteq X$ such that $x \in \widetilde{O}$ and $\lambda(\widetilde{O}) = 0$. Then $O \cap \widetilde{O}$ is an open set satisfying

$x \in O \cap \widetilde{O}$ and $(\mu + \lambda)(O \cap \widetilde{O}) = 0$ which, according to Definition 3.8.1, goes to show that $x \notin \mathrm{supp}(\mu + \lambda)$. This finishes the proof of (3.8.4). Finally, the main claim in item *(8)* is a consequence of (3.8.2) and assumptions. □

The definition below elaborates on the notion of support for a given measurable function, relative to the background measure.

Definition 3.8.3 *Let* (X, τ) *be a topological space and let* μ *be a (non-negative) Borel measure on* X. *Given any scalar-valued* μ*-measurable function* f *on* X, *define the* `support` of f *(relative to the measure* μ*) to be the support of the measure* $|f|\mu$, *i.e.,*

$$\mathrm{supp} f := \Big\{x \in X : \int_O |f| \,\mathrm{d}\mu > 0 \textit{ for each } O \subseteq X \textit{ open with } x \in O\Big\}. \tag{3.8.7}$$

Some of the most basic properties of the support of measurable functions are collected in the lemma below which, to a large extent, parallels Proposition 3.8.2 dealing with the support of measures.

Lemma 3.8.4 *Let* (X, τ) *be a topological space and assume* μ *is a (non-negative) Borel measure on* X. *Also, let* f, g *be two scalar-valued* μ*-measurable functions on* X. *Then the following properties hold:*

(1) *The set* $\mathrm{supp}\, f$ *is closed in* (X, τ).
(2) *If* $f = g$ *at* μ*-a.e. point on* X *then* $\mathrm{supp}\, f = \mathrm{supp}\, g$.
(3) *One has*

$$X \setminus \mathrm{supp}\, f = \bigcup_{\substack{O \text{ open such that} \\ f=0\ \mu\text{-a.e. on } O}} O. \tag{3.8.8}$$

(4) *Given* $x \in X$, *one has* $x \in X \setminus \mathrm{supp}\, f$ *if and only if there exists an open set* $O \subseteq X$ *such that* $x \in O$ *and* $f = 0$ *at* μ*-a.e. point in* O.
(5) *If* $O \subseteq X$ *is an open set and* $f = 0$ *at* μ*-a.e. point in* O, *then* $O \cap \mathrm{supp}\, f = \varnothing$.
(6) *For any* μ*-measurable subset* E *of* X, *one has* $\mathrm{supp}\, \mathbf{1}_E \subseteq \overline{E}$, *the closure of* E *in* (X, τ). *Moreover,*

$$\begin{array}{l}\textit{if } \mu(O) > 0 \textit{ for each nonempty open subset } O \textit{ of } X, \textit{ then} \\ \textit{for each open set } E \subseteq X \textit{ one has } \mathrm{supp}\, \mathbf{1}_E = \overline{E}.\end{array} \tag{3.8.9}$$

(7) *If* A *is some* μ*-measurable subset of* X *with the property that* $f = 0$ *at* μ*-a.e. point in* A, *then* $\mathrm{supp}\, f \subseteq \mathrm{supp}\, \mathbf{1}_{X\setminus A} \subseteq \overline{X \setminus A}$.
(8) *Assuming that* f, g *are finite* μ*-a.e., one has*

$$\mathrm{supp}(fg) \subseteq \big(\mathrm{supp}\, f\big) \cap \big(\mathrm{supp}\, g\big), \tag{3.8.10}$$

$$\mathrm{supp}(f+g) \subseteq \big(\mathrm{supp}\, f\big) \cup \big(\mathrm{supp}\, g\big). \tag{3.8.11}$$

(9) *Under the additional assumption that* (X, τ) *is strongly Lindelöf (which, e.g., is the case if* (X, τ) *is second countable), it follows that* $f = 0$ *at* μ*-a.e. point on* $X \setminus \operatorname{supp} f$.

(10) *One has* $\operatorname{supp} f \subseteq \overline{\{x \in X : f(x) \neq 0\}}$*, where the closure is taken in* (X, τ)*. Moreover, one actually has equality if* μ *has the property that* $\mu(O) > 0$ *for every nonempty open subset* O *of* X*, and if the function* f *is continuous on* (X, τ).

Proof Part *(1)* follows from Definition 3.8.3 and item *(1)* in Proposition 3.8.2. Part *(2)* is implied directly by Definition 3.8.3 upon noting that having $f = g$ at μ-a.e. point on X forces $|f|\mu = |g|\mu$ as measures. Formula (3.8.8) is clear from (3.8.2) and Definition 3.8.3, while parts *(4)-(5)* are immediate consequences of (3.8.8). The first claim in part *(6)* is readily seen from Definition 3.8.3. As regards (3.8.9), suppose the measure μ has the property that $\mu(O) > 0$ for each nonempty open subset O of X. Fix an arbitrary open set $E \subseteq X$. From what we have proved so far, $\operatorname{supp} \mathbf{1}_E \subseteq \overline{E}$. To justify the opposite inclusion, start with some arbitrary point $x \in X \setminus \operatorname{supp} \mathbf{1}_E$. Thanks to item *(4)*, this implies that there exists an open set $O \subseteq X$ and a nullset $N \subseteq X$ for μ such that $x \in O$ and $\mathbf{1}_E = 0$ at each point in $O \setminus N$. The latter condition then forces $O \setminus N \subseteq X \setminus E$ hence, further, $O \cap E \subseteq N$. As a consequence, $O \cap E$ is an open set with the property that $\mu(O \cap E) = 0$. Thus, $O \cap E = \varnothing$ given the current assumptions on μ. In turn, this permits us to conclude that $x \notin \overline{E}$ hence, ultimately, $\overline{E} \subseteq \operatorname{supp} \mathbf{1}_E$. Via double inclusion, (3.8.9) is now established.

To deal with the claim in part *(7)*, observe that if A is a μ-measurable subset of X such that $f = 0$ at μ-a.e. point in A, then $|f| << \mathbf{1}_{X \setminus A}\, \mu$. Thanks to item *(6)* in Proposition 3.8.2 and Definition 3.8.3, this gives $\operatorname{supp} f \subseteq \operatorname{supp} \mathbf{1}_{X \setminus A} \subseteq \overline{X \setminus A}$, with the last inclusion provided by part *(6)*.

As regards part *(8)*, since $|fg|\mu << |f|\mu$ and $|fg|\mu << |g|\mu$, item *(6)* in Proposition 3.8.2 and Definition 3.8.3 combined imply that $\operatorname{supp}(fg) \subseteq \operatorname{supp} f$ and $\operatorname{supp}(fg) \subseteq \operatorname{supp} g$. Collectively, these establish (3.8.10). To prove (3.8.11), start by noting that $|f + g|\mu << |f|\mu + |g|\mu$, hence

$$\begin{aligned} \operatorname{supp}(f + g) &= \operatorname{supp}\big(|f + g|\mu\big) \subseteq \operatorname{supp}\big(|f|\mu + |g|\mu\big) \\ &= \big(\operatorname{supp}(|f|\mu)\big) \cup \big(\operatorname{supp}(|g|\mu)\big) = \big(\operatorname{supp} f\big) \cup \big(\operatorname{supp} g\big), \end{aligned} \tag{3.8.12}$$

by Definition 3.8.3 and items *(6)-(7)* in Proposition 3.8.2. Going further, the claim in part *(9)* is readily implied by (3.8.8) and assumptions. There remains to deal with the claims in part *(10)*. First, if $x \in X \setminus \overline{\{f \neq 0\}}$, then there exists an open set $O \subseteq X$ such that $x \in O$ and $f = 0$ everywhere on O. In light of (3.8.8), this places x in $X \setminus \operatorname{supp} f$, proving that $\operatorname{supp} f \subseteq \overline{\{f \neq 0\}}$. Let us now prove the opposite inclusion under the additional assumptions that f is continuous and μ is strictly positive on nonempty open sets. With this goal in mind, pick an arbitrary point $x \in X \setminus \operatorname{supp} f$. In view of part *(4)*, there exist an open set $O \subseteq X$ and a μ-measurable set $N \subseteq O$ with the property that $\mu(N) = 0$ such that $x \in O$ and $f = 0$ everywhere on $O \setminus N$. We claim that

$$O \subseteq \overline{O \setminus N}. \tag{3.8.13}$$

If not, there exists $y \in O$ and an open set $U \subseteq X$ such that $y \in U$ and $U \cap (O \setminus N) = \varnothing$. The latter property forces $\widetilde{O} := O \cap U$ to be contained in N which, in turn, would make the open set $\widetilde{O}$ be a μ-nullset, hence necessarily empty. However, $y \in \widetilde{O}$, a contradiction which proves (3.8.13). Next, recall that $O \setminus N \subseteq f^{-1}(\{0\})$ and observe that $f^{-1}(\{0\})$ is a closed set given that f is continuous. In concert with (3.8.13), this permits us to write

$$O \subseteq \overline{O \setminus N} \subseteq \overline{f^{-1}(\{0\})} = f^{-1}(\{0\}). \tag{3.8.14}$$

As such, the function f vanishes everywhere on O which (bearing in mind that x belongs to the open set O) goes to show that $x \notin \overline{\{f \neq 0\}}$. Since $x \in X \setminus \operatorname{supp} f$ has been arbitrarily chosen, this ultimately proves that $\overline{\{f \neq 0\}} \subseteq \operatorname{supp} f$, as desired. □

3.9 The Riesz Representation Theorem

We discuss the following version of the Riesz Representation Theorem describing the nature of linear continuous functionals on smooth compactly supported vector fields.

Proposition 3.9.1 *Fix $n, m \in \mathbb{N}$. Assume $\Omega \subseteq \mathbb{R}^n$ is an open set and consider a linear functional*

$$\Lambda : \big[\mathscr{C}_c^\infty(\Omega)\big]^m \longrightarrow \mathbb{R} \tag{3.9.1}$$

with the property that for each compact set $K \subset \Omega$, one has

$$C(K) := \sup\Big\{|\Lambda\vec{\phi}| : \vec{\phi} \in \big[\mathscr{C}_c^\infty(\Omega)\big]^m,\ \operatorname{supp}\vec{\phi} \subseteq K,\ \sup_K |\vec{\phi}| \leq 1\Big\} < +\infty. \tag{3.9.2}$$

Then Λ extends uniquely to a linear functional

$$\widetilde{\Lambda} : \big[\mathscr{C}_c^0(\Omega)\big]^m \longrightarrow \mathbb{R} \tag{3.9.3}$$

with the property that for each compact set $K \subset \Omega$, one has

$$\sup\Big\{|\widetilde{\Lambda}\vec{\psi}| :\ \vec{\psi} \in \big[\mathscr{C}_c^0(\Omega)\big]^m,\ \operatorname{supp}\vec{\psi} \subseteq K,\ \sup_K |\vec{\psi}| \leq 1\Big\} < +\infty, \tag{3.9.4}$$

and there exist some locally finite Borel-regular measure μ on Ω, together with some μ-measurable vector-valued function $N : \Omega \to \mathbb{R}^m$, satisfying

$$\begin{aligned} &|N| = 1 \ \ \textit{at } \mu\textit{-a.e. point belonging to } \Omega, \ \textit{and} \\ &\widetilde{\Lambda}\vec{\psi} = \int_\Omega N \cdot \vec{\psi}\, \mathrm{d}\mu \ \ \textit{for each} \ \ \vec{\psi} \in \big[\mathscr{C}_c^0(\Omega)\big]^m. \end{aligned} \tag{3.9.5}$$

Moreover, for each open set $W \subseteq \Omega$, one has

$$\mu(W) = \sup\Big\{|\widetilde{\Lambda}\vec{\psi}| : \vec{\psi} \in \big[\mathscr{C}_c^0(\Omega)\big]^m,\ \operatorname{supp}\vec{\psi} \subseteq W,\ \sup_{\Omega}|\vec{\psi}| \leq 1\Big\}$$
$$= \sup\Big\{|\Lambda\vec{\phi}| : \vec{\phi} \in \big[\mathscr{C}_c^\infty(\Omega)\big]^m,\ \operatorname{supp}\vec{\phi} \subseteq W,\ \sup_{\Omega}|\vec{\phi}| \leq 1\Big\}. \tag{3.9.6}$$

Proof Fix an arbitrary compact set $K \subset \Omega$ and choose an open set O such that $K \subset O$ and $\overline{O}$ is a compact subset of Ω. Given a vector field $\vec{\psi} \in \big[\mathscr{C}_c^0(\Omega)\big]^m$ with $\operatorname{supp}\vec{\psi} \subseteq K$, use a mollifier to produce a sequence

$$\begin{gathered} \{\vec{\phi}_j\}_{j\in\mathbb{N}} \subseteq \big[\mathscr{C}_c^\infty(O)\big]^m \ \text{ such that} \\ \vec{\phi}_j \longrightarrow \vec{\psi} \ \text{ uniformly on } \ \Omega \ \text{ as } \ j \to \infty. \end{gathered} \tag{3.9.7}$$

Since (3.9.2) entails

$$|\Lambda\vec{\phi}| \leq C(\overline{O}) \sup_{\overline{O}}|\vec{\phi}| \ \text{ for each } \ \vec{\phi} \in \big[\mathscr{C}_c^\infty(O)\big]^m, \tag{3.9.8}$$

it follows $\{\Lambda\vec{\phi}\}_{j\in\mathbb{N}}$ is a Cauchy sequence of real numbers, hence convergent. Consequently, the following limit exists:

$$\widetilde{\Lambda}\vec{\psi} := \lim_{j\to\infty} \Lambda\vec{\phi}_j. \tag{3.9.9}$$

A well-known argument based on interlacing sequences also shows that said limit is independent of the sequence $\{\vec{\phi}_j\}_{j\in\mathbb{N}}$ as in (3.9.7). Hence, Λ extends uniquely to a linear functional $\widetilde{\Lambda}$ as in (3.9.3) with the property that (3.9.4) holds for each compact set $K \subset \Omega$. Granted this, a version of the Riesz Representation Theorem discussed in [80, Theorem 1, p. 49] applies and yields the existence of a locally finite Borel-regular measure μ on Ω and a μ-measurable vector-valued function $N : \Omega \to \mathbb{R}^m$ satisfying (3.9.5) as well as the first equality in (3.9.6) (see [80, item 1 in the proof of Theorem 1, p. 49]).

There remains to prove the second equality in (3.9.6). We proceed by double inequality. Since $\widetilde{\Lambda}$ is an extension of Λ, one such inequality is immediate. To establish the remaining inequality, fix an arbitrary open set $W \subseteq \Omega$ and pick an arbitrary $\vec{\psi} \in \big[\mathscr{C}_c^0(\Omega)\big]^m$ with $\operatorname{supp}\vec{\psi} \subseteq W$ and $\sup_\Omega |\vec{\psi}| \leq 1$. From the first part of the proof, we know that there exists a sequence

$$\begin{gathered} \{\vec{\phi}_j\}_{j\in\mathbb{N}} \subseteq \big[\mathscr{C}_c^\infty(\Omega)\big]^m \text{ with } \operatorname{supp}\vec{\phi}_j \subseteq W \text{ for each } j \in \mathbb{N}, \text{ such that} \\ \vec{\phi}_j \to \vec{\psi} \ \text{ uniformly on } \ \Omega \ \text{ as } \ j \to \infty, \ \text{ and } \ \widetilde{\Lambda}\vec{\psi} = \lim_{j\to\infty} \Lambda\vec{\phi}_j. \end{gathered} \tag{3.9.10}$$

Note that for each $j \in \mathbb{N}$, we have

$$\sup_\Omega |\vec{\phi}_j| \leq \sup_\Omega |\vec{\phi}_j - \vec{\psi}| + \sup_\Omega |\vec{\psi}| \leq \sup_\Omega |\vec{\phi}_j - \vec{\psi}| + 1. \tag{3.9.11}$$

As such, given any $\varepsilon > 0$, after eventually excluding finitely many terms in the sequence $\{\vec{\phi}_j\}_{j\in\mathbb{N}}$ there is no loss of generality in assuming that

$$\sup_{\Omega} |\vec{\phi}_j| \leq 1+\varepsilon \quad \text{for each} \quad j \in \mathbb{N}. \tag{3.9.12}$$

In concert with (3.9.10), this permits us to estimate

$$\begin{aligned} (1+\varepsilon)^{-1}\big|\widetilde{\Lambda}\vec{\psi}\big| &= \lim_{j\to\infty} \Big|\Lambda\big(\vec{\phi}_j/(1+\varepsilon)\big)\Big| \\ &\leq \sup\Big\{|\Lambda\vec{\phi}| : \vec{\phi} \in \big[\mathscr{C}_c^\infty(\Omega)\big]^m,\ \operatorname{supp}\vec{\phi} \subseteq W,\ \sup_{\Omega}|\vec{\phi}| \leq 1\Big\}. \end{aligned} \tag{3.9.13}$$

After sending $\varepsilon \to 0^+$ this ultimately yields

$$\big|\widetilde{\Lambda}\vec{\psi}\big| \leq \sup\Big\{|\Lambda\vec{\phi}| : \vec{\phi} \in \big[\mathscr{C}_c^\infty(\Omega)\big]^m,\ \operatorname{supp}\vec{\phi} \subseteq W,\ \sup_{\Omega}|\vec{\phi}| \leq 1\Big\}. \tag{3.9.14}$$

which now readily implies the missing inequality in the justification of the second equality in (3.9.6). This concludes the proof of Proposition 3.9.1. □

Chapter 4
Selected Topics in Distribution Theory

The topics treated in this chapter are as follows. First we develop a brand of distribution theory on arbitrary subsets of $\mathbb{R}^n$, taking Lipschitz functions with bounded support as test functions. Second, we define and study what we call the "bullet product" which, in essence, is a weak version (modeled upon integration by parts) of the inner product of the normal vector to a domain with a given vector field satisfying only some very mild integrability properties in that domain. Third, we provide a proof of the product rule for weak derivatives. Fourth, we compare the divergence of a differentiable vector field, taken in a pointwise sense, to its distributional divergence. Fifth, we study the algebraic dual of the space of functions which are simultaneously smooth and bounded in an open subset of the Euclidean ambient. Finally, we introduce and study the notion of contribution at infinity of a given vector field.

4.1 Distribution Theory on Arbitrary Sets

In this section, we discuss a certain brand of distribution theory on arbitrary subsets of the Euclidean ambient. Let Σ be a subset of $\mathbb{R}^n$. Denote by $\mathrm{Lip}(\Sigma)$ the space of complex-valued Lipschitz functions defined on Σ, equipped with the semi-norm

$$\|\phi\|_{\mathrm{Lip}(\Sigma)} := \sup_{\substack{x,y\in\Sigma\\ x\neq y}} \frac{|\phi(x)-\phi(y)|}{|x-y|}, \qquad \forall \phi \in \mathrm{Lip}(\Sigma). \tag{4.1.1}$$

Consider the space of bounded Lipschitz functions on Σ, i.e.,

$$\mathrm{BL}(\Sigma) := \big\{\phi \in \mathrm{Lip}(\Sigma) : \phi \text{ bounded on } \Sigma\big\}. \tag{4.1.2}$$

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This becomes a Banach space when equipped with the norm

$$\|\phi\| := \sup_{x\in\Sigma} |\phi(x)| + \|\phi\|_{\mathrm{Lip}(\Sigma)}, \qquad \forall \phi \in \mathrm{BL}(\Sigma). \tag{4.1.3}$$

Recall that $\mathrm{Lip}_c(\Sigma)$ denotes the subspace of $\mathrm{Lip}(\Sigma)$ consisting of functions which vanish identically outside a bounded subset of Σ. We define the class of `test functions` on Σ as

$$\big(\mathrm{Lip}_c(\Sigma), \tau_{\mathscr{D}}\big) \tag{4.1.4}$$

i.e., the space of complex-valued Lipschitz functions defined on Σ which vanish identically outside a bounded subset of Σ, equipped with a certain topology, $\tau_{\mathscr{D}}$. To define this topology on $\mathrm{Lip}_c(\Sigma)$, fix a reference point $x_0 \in \Sigma$ and pick

$$\begin{array}{l}\{K_m\}_{m\in\mathbb{N}}, \text{ bounded sets in } \mathbb{R}^n, \text{ with the property that } K_m \subseteq K_{m+1} \\ \text{for each } m \in \mathbb{N}, \text{ and for every } r > 0 \text{ there exists } m \in \mathbb{N} \text{ such that} \\ B(x_0, r) \subseteq K_m. \end{array} \tag{4.1.5}$$

For each $m \in \mathbb{N}$, denote

$$\mathscr{V}_m(\Sigma) := \big\{\phi \in \mathrm{Lip}(\Sigma) : \phi \equiv 0 \ \text{ on } \ \Sigma \setminus K_m\big\}. \tag{4.1.6}$$

This becomes a Banach space when equipped with the norm from (4.1.3), and we have

$$\mathscr{V}_m(\Sigma) \subseteq \mathscr{V}_{m+1}(\Sigma) \ \text{ for every } \ m \in \mathbb{N}, \quad \text{and } \ \mathrm{Lip}_c(\Sigma) = \bigcup_{m\in\mathbb{N}} \mathscr{V}_m(\Sigma). \tag{4.1.7}$$

Denote by τ_m the topology induced on $\mathscr{V}_m(\Sigma)$ by the norm from (4.1.3). Then

$$\tau_m \text{ is the topology induced by } \mathrm{BL}(\Sigma) \text{ on } \mathscr{V}_m(\Sigma). \tag{4.1.8}$$

To justify this, consider an arbitrary set $\mathcal{O} \subseteq \mathscr{V}_m(\Sigma)$ which is open in τ_m. Then for each $\phi \in \mathcal{O}$ there exists $r_\phi > 0$ with the property that $B(\phi, r_\phi)$, the ball in $\mathrm{BL}(\Sigma)$ centered at ϕ with radius r_ϕ, satisfies $B(\phi, r_\phi) \cap \mathscr{V}_m(\Sigma) \subseteq \mathcal{O}$. If we now define $\mathcal{U} := \bigcup_{\phi\in\mathcal{O}} B(\phi, r_\phi)$, then $\mathcal{U}$ is open in $\mathrm{BL}(\Sigma)$ and $\mathcal{U} \cap \mathscr{V}_m(\Sigma) = \mathcal{O}$. Conversely, it is clear that whenever $\mathcal{U}$ is open in $\mathrm{BL}(\Sigma)$ the set $\mathcal{U} \cap \mathscr{V}_m(\Sigma)$ is open in τ_m, finishing the proof of (4.1.8).

In particular, (4.1.8) implies that for each $m \in \mathbb{N}$ the topology induced by τ_{m+1} on $\mathscr{V}_m(\Sigma)$ coincides with τ_m. As such, we may define the topological space (4.1.4) as the strict inductive limit of the family of topological spaces $\big\{(\mathscr{V}_m(\Sigma), \tau_m)\big\}_{m\in\mathbb{N}}$ (cf., e.g., the discussion in [181, p. 546]). That is,

$$\begin{array}{l}\tau_{\mathscr{D}} \text{ is the finest topology on the space } \mathrm{Lip}_c(\Sigma) \text{ making} \\ \text{each inclusion } \big(\mathscr{V}_m(\Sigma), \tau_m\big) \hookrightarrow (\mathrm{Lip}_c(\Sigma), \tau_{\mathscr{D}}) \text{ a continuous} \\ \text{map.}\end{array} \tag{4.1.9}$$

More specifically,[1]

$$\mathcal{W} := \Big\{ W \subseteq \mathrm{Lip}_c(\Sigma) : W \text{ is balanced, convex, and} \\ W \cap \mathscr{V}_m(\Sigma) \text{ is open in } \tau_m \text{ for each } m \in \mathbb{N} \Big\} \tag{4.1.10}$$

is a local base in the topology $\tau_{\mathscr{D}}$ on $\mathrm{Lip}_c(\Sigma)$, i.e.,

$$\begin{array}{l} \tau_{\mathscr{D}} \text{ is the collection of all unions of sets of the} \\ \text{form } \phi + W, \text{ with } \phi \in \mathrm{Lip}_c(\Sigma) \text{ and } W \in \mathcal{W}. \end{array} \tag{4.1.11}$$

Proposition 4.1.1 *Let Σ be a subset of $\mathbb{R}^n$. Then the space of test functions introduced in* (4.1.4) *satisfies the following properties.*

(1) The topology $\tau_{\mathscr{D}}$ is independent of the particular choice of the family $\{K_m\}_{m\in\mathbb{N}}$ as in (4.1.5).
(2) The pair $(\mathrm{Lip}_c(\Sigma), \tau_{\mathscr{D}})$ is a Hausdorff, locally convex, topological vector space. Also, for each $m \in \mathbb{N}$, the topology induced by $\tau_{\mathscr{D}}$ on $\mathscr{V}_m(\Sigma)$ coincides with τ_m.
(3) A convex and balanced subset O of $\mathrm{Lip}_c(\Sigma)$ is open in $\tau_{\mathscr{D}}$ if and only if $O \cap \mathscr{V}_m(\Sigma)$ is open in τ_m for every $m \in \mathbb{N}$, i.e., if and only if

$$\begin{array}{c} \textit{for each } m \in \mathbb{N} \textit{ there exists some } \varepsilon > 0 \textit{ with the property that} \\ \big\{ \phi \in \mathrm{Lip}(\Sigma) : \phi \equiv 0 \textit{ on } \Sigma \setminus K_m \textit{ and } \sup_\Sigma |\phi| + \|\phi\|_{\mathrm{Lip}(\Sigma)} < \varepsilon \big\} \subseteq O. \end{array} \tag{4.1.12}$$

(4) One has

$$\begin{array}{l} \{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathrm{Lip}_c(\Sigma) \textit{ converges to zero in the topology } \tau_{\mathscr{D}} \textit{ if and only} \\ \textit{if there exists } m \in \mathbb{N} \textit{ such that } \phi_j \equiv 0 \textit{ on } \Sigma \setminus K_m \textit{ for all } j \in \mathbb{N} \textit{ and} \\ \lim\limits_{j\to\infty} \big[\sup_\Sigma |\phi_j| + \|\phi_j\|_{\mathrm{Lip}(\Sigma)} \big] = 0. \end{array} \tag{4.1.13}$$

(5) A given sequence $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathrm{Lip}_c(\Sigma)$ is Cauchy (in the sense of topological vector spaces) if and only if there exists $m \in \mathbb{N}$ with the property that $\phi_j \equiv 0$ on $\Sigma \setminus K_m$ for every $j \in \mathbb{N}$ and $\sup_\Sigma |\phi_j - \phi_k| + \|\phi_j - \phi_k\|_{\mathrm{Lip}(\Sigma)} \to 0$ as $j, k \to \infty$.
(6) The topological space $\big(\mathrm{Lip}_c(\Sigma), \tau_{\mathscr{D}}\big)$ is sequentially complete, in the sense that any Cauchy sequence in $\big(\mathrm{Lip}_c(\Sigma), \tau_{\mathscr{D}}\big)$ converges to a (unique) function from $\mathrm{Lip}_c(\Sigma)$ in the topology $\tau_{\mathscr{D}}$.
(7) A set $\mathscr{B} \subseteq \mathrm{Lip}_c(\Sigma)$ is bounded in $\tau_{\mathscr{D}}$ (i.e., any neighborhood of the origin in this topology contains a positive dilate of $\mathscr{B}$) if and only if there exists $m \in \mathbb{N}$ with the property that

$$\begin{array}{l} \phi \equiv 0 \;\; \textit{on} \;\; \Sigma \setminus K_m \;\; \textit{for each} \;\; \phi \in \mathscr{B}, \;\; \textit{and} \\ \sup \big\{ \sup_\Sigma |\phi| + \|\phi\|_{\mathrm{Lip}(\Sigma)} : \phi \in \mathscr{B} \big\} < \infty. \end{array} \tag{4.1.14}$$

[1] Recall that a set W is said to be balanced if $\lambda W \subseteq W$ for all scalars λ with $|\lambda| \leq 1$.

Proof To deal with the claim in item *(1)*, fix two families $\{K_m\}_{m\in\mathbb{N}}$ and $\{\widetilde{K}_m\}_{m\in\mathbb{N}}$ as in (4.1.5). We agree to decorate by tilde objects associated with the second family, much as those associated with the former family in the build-up to the statement of Proposition 4.1.1. In this regard, we shall prove that

$$\text{if } \mathcal{O} \subseteq \mathrm{Lip}_c(\Sigma) \text{ has the property that } \mathcal{O} \cap \mathscr{V}_m(\Sigma) \text{ is open in } \tau_m \text{ for each } m \in \mathbb{N} \text{ then } \mathcal{O} \cap \widetilde{\mathscr{V}}_m(\Sigma) \text{ is also open in } \widetilde{\tau}_m \text{ for each } m \in \mathbb{N}. \tag{4.1.15}$$

To this end, assume the set $\mathcal{O}$ satisfies the hypotheses in (4.1.15) and fix some number $m_0 \in \mathbb{N}$. Also, pick an arbitrary function $\phi \in \mathcal{O} \cap \widetilde{\mathscr{V}}_{m_0}(\Sigma)$. Then (4.1.5) implies that there exists $m_1 \in \mathbb{N}$ such that $\widetilde{K}_{m_0} \subseteq K_{m_1}$. This ensures that $\phi \in \mathcal{O} \cap \mathscr{V}_{m_1}(\Sigma)$. Since the latter set is open in τ_{m_1}, there exists some $r > 0$ such that if $B(\phi, r)$ denotes the ball in $\mathrm{BL}(\Sigma)$ centered at ϕ and of radius r then $B(\phi, r) \cap \mathscr{V}_{m_1}(\Sigma) \subseteq \mathcal{O} \cap \mathscr{V}_{m_1}(\Sigma)$. The key observation is that since $\widetilde{\mathscr{V}}_{m_0}(\Sigma) \subseteq \mathscr{V}_{m_1}(\Sigma)$, intersecting both sides with $\widetilde{\mathscr{V}}_{m_0}(\Sigma)$ yields $B(\phi, r) \cap \widetilde{\mathscr{V}}_{m_0}(\Sigma) \subseteq \mathcal{O} \cap \widetilde{\mathscr{V}}_{m_0}(\Sigma)$. Since ϕ has been chose arbitrarily, this goes to show that $\mathcal{O} \cap \widetilde{\mathscr{V}}_{m_0}(\Sigma)$ is open in $\widetilde{\tau}_{m_0}$. The proof of (4.1.15) is therefore complete. In turn, from (4.1.15) and (4.1.11) we conclude that the set $\widetilde{\tau}_{\mathscr{D}} = \tau_{\mathscr{D}}$, finishing the treatment of item *(1)*.

For the remainder of the proof we reason along the lines of the classical setting considered in [232, Theorems 6.4–6.5, pp. 152–153]. Let us first show that

$$\tau_{\mathscr{D}} \text{ is a topology on } \mathrm{Lip}_c(\Sigma) \text{ and the family } \mathcal{W} \text{ is a local base for } \tau_{\mathscr{D}}. \tag{4.1.16}$$

This follows as soon as we prove that, whenever $\mathcal{O}_1, \mathcal{O}_2$ are open sets in $\tau_{\mathscr{D}}$ and $\phi \in \mathcal{O}_1 \cap \mathcal{O}_2$, there exists $W \in \mathcal{W}$ such that

$$\phi + W \subseteq \mathcal{O}_1 \cap \mathcal{O}_2. \tag{4.1.17}$$

To find such a set W, recall from (4.1.11) that there exist $\phi_1, \phi_2 \in \mathrm{Lip}_c(\Sigma)$ and $W_1, W_2 \in \mathcal{W}$ such that

$$\phi \in \phi_1 + W_1 \subseteq \mathcal{O}_1 \quad \text{and} \quad \phi \in \phi_2 + W_2 \subseteq \mathcal{O}_2. \tag{4.1.18}$$

Select $m \in \mathbb{N}$ large enough so that $\phi, \phi_1, \phi_2 \in \mathscr{V}_m(\Sigma)$. Given that $W_1 \cap \mathscr{V}_m(\Sigma)$ and $W_2 \cap \mathscr{V}_m(\Sigma)$ are open in $\mathscr{V}_m(\Sigma)$ (cf. (4.1.10)), it follows that we may find $\theta_1, \theta_2 \in (0, 1)$ such that, for $j \in \{1, 2\}$,

$$(1 - \theta_1)^{-1}(\phi - \phi_j) \in W_j \cap \mathscr{V}_m(\Sigma), \quad \text{hence} \quad \phi - \phi_j \in (1 - \theta_j)W_j. \tag{4.1.19}$$

Upon recalling that W_1, W_2 are convex, we then conclude that, for $j \in \{1, 2\}$,

$$\phi - \phi_j + \theta_j W_j \in (1 - \theta_j)W_j + \theta_j W_j = W_j, \tag{4.1.20}$$

hence

$$\phi + \theta_j W_j \in \phi_j + W_j \subseteq \mathcal{O}_j. \tag{4.1.21}$$

This shows that (4.1.17) holds with $W := (\theta_1 W_1) \cap (\theta_2 W_2)$, completing the proof of (4.1.16).

Next, given two distinct functions $\phi_1, \phi_2 \in \mathrm{Lip}_c(\Sigma)$, introduce $r := \|\phi_1 - \phi_2\| > 0$ and define $W_0 := \{\phi \in \mathrm{Lip}_c(\Sigma) : \|\phi\| < r/2\}$. Then $W_0 \in \mathcal{W}$ and we also have $(\phi_1 + W_0) \cap (\phi_2 + W_0) = \varnothing$, proving that $\tau_{\mathscr{D}}$ is a Hausdorff topology on $\mathrm{Lip}_c(\Sigma)$.

Let us now show that $(\mathrm{Lip}_c(\Sigma), \tau_{\mathscr{D}})$ is a topological vector space. Note that for any two functions $\phi_1, \phi_2 \in \mathrm{Lip}_c(\Sigma)$ and any set $W \in \mathcal{W}$, the convexity of W implies

$$(\phi_1 + \tfrac{1}{2}W) + (\phi_2 + \tfrac{1}{2}W) = (\phi_1 + \phi_2) + W, \tag{4.1.22}$$

which goes to show that the addition is continuous in the product topology induced by $\tau_{\mathscr{D}}$ on $\mathrm{Lip}_c(\Sigma) \times \mathrm{Lip}_c(\Sigma)$. As regards multiplication by scalars, fix some function $\phi_0 \in \mathrm{Lip}_c(\Sigma)$ along with some number $\lambda_0 \in \mathbb{C}$. Also, pick some $W \in \mathcal{W}$. Then there exists $\varepsilon > 0$ small enough so that $\varepsilon\phi_0 \in \frac{1}{2}W$. If we now choose $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_0| < \varepsilon$ and $\phi \in \phi_0 + \delta W$ where $\delta := [2(|\lambda_0| + \varepsilon)]^{-1}$, then, since W is convex and balanced,

$$\lambda\phi - \lambda_0\phi_0 = \lambda(\phi - \phi_0) + (\lambda - \lambda_0)\phi_0 \in \tfrac{1}{2}W + \tfrac{1}{2}W = W. \tag{4.1.23}$$

This proves that multiplication by scalars is a continuous mapping. The reasoning so far shows that $(\mathrm{Lip}_c(\Sigma), \tau_{\mathscr{D}})$ is a Hausdorff, locally convex, topological vector space, which is the first claim in item *(2)*. For further use, let us also observe here that

$$\text{each set } W \in \mathcal{W} \text{ is an open neighborhood of the origin } 0 \in \mathrm{Lip}_c(\Sigma) \text{ in the topology } \tau_{\mathscr{D}}. \tag{4.1.24}$$

Indeed, given $W \in \mathcal{W}$, the fact that W is balanced entails $0 \in W$, while from (4.1.11) it is clear that W is open in $\tau_{\mathscr{D}}$, hence W is an open neighborhood of 0 in the topology $\tau_{\mathscr{D}}$.

Pressing on, we claim that

$$\text{for each } m \in \mathbb{N}, \text{ and each } \mathcal{O} \subseteq \mathrm{Lip}_c(\Sigma) \text{ which is open in } \tau_{\mathscr{D}}, \text{ the set } \mathcal{O} \cap \mathscr{V}_m(\Sigma) \text{ is open in } \tau_m. \tag{4.1.25}$$

To prove this, fix a number $m \in \mathbb{N}$ along with a set $\mathcal{O} \subseteq \mathrm{Lip}_c(\Sigma)$ which is open in $\tau_{\mathscr{D}}$. Also, pick $\phi \in \mathcal{O} \cap \mathscr{V}_m(\Sigma)$ arbitrary. Then (4.1.11) guarantees the existence of some $W \in \mathcal{W}$ such that $\phi + W \subseteq \mathcal{O}$. Consequently, $\phi + (W \cap \mathscr{V}_m(\Sigma)) \subseteq \mathcal{O} \cap \mathscr{V}_m(\Sigma)$ and since from (4.1.10) we know that $W \cap \mathscr{V}_m(\Sigma)$ is open in τ_m, we conclude that $\mathcal{O} \cap \mathscr{V}_m(\Sigma)$ is a neighborhood of ϕ in τ_m. Then the arbitrariness of $\phi \in \mathcal{O} \cap \mathscr{V}_m(\Sigma)$ establishes (4.1.25).

Collectively, (4.1.24)–(4.1.25) prove (keeping in mind (4.1.10)) the claim in item *(3)*.

Another significant consequence of (4.1.25) is the fact that, for each $m \in \mathbb{N}$, the topology induced by $\tau_{\mathscr{D}}$ on $\mathscr{V}_m(\Sigma)$ is contained in τ_m. For the opposite inclusion, pick an arbitrary $r > 0$ and recall that $B(0,r)$ denotes the ball in the normed space $\mathrm{BL}(\Sigma)$ centered at $0 \in \mathrm{BL}(\Sigma)$ and of radius r. Then the set $W_r := B(0,r) \cap \mathrm{Lip}_c(\Sigma)$ is balanced, convex, and satisfies $W_r \cap \mathscr{V}_m(\Sigma) = B(0,r) \cap \mathscr{V}_m(\Sigma)$ for each number $m \in \mathbb{N}$. Given that $B(0,r)$ is open in $\mathrm{BL}(\Sigma)$, from this and (4.1.8) we conclude that the set $W_r \cap \mathscr{V}_m(\Sigma)$ is open in τ_m for each $m \in \mathbb{N}$. Collectively, the properties just established guarantee that

$$W_r = B(0,r) \cap \mathrm{Lip}_c(\Sigma) \in \mathcal{W} \text{ for each } r > 0. \tag{4.1.26}$$

To proceed, fix $m \in \mathbb{N}$ and pick some $\mathcal{U} \subseteq \mathscr{V}_m(\Sigma)$ which is open in τ_m. The desired conclusion (i.e., that the topology induced by $\tau_{\mathscr{D}}$ on $\mathscr{V}_m(\Sigma)$ contains τ_m) follows as soon as we find a set $\mathcal{O} \subseteq \mathrm{Lip}_c(\Sigma)$ which is open in $\tau_{\mathscr{D}}$ and satisfies

$$\mathcal{U} = \mathcal{O} \cap \mathscr{V}_m(\Sigma). \tag{4.1.27}$$

With this goal in mind, consider an arbitrary $\phi \in \mathcal{U}$. Since $\mathcal{U}$ is open in τ_m, it follows that there exists $r_\phi > 0$ with the property that if $B(\phi, r_\phi)$ is the ball in $\mathrm{BL}(\Sigma)$ centered at ϕ with radius r_ϕ then $B(\phi, r_\phi) \cap \mathscr{V}_m(\Sigma) \subseteq \mathcal{U}$. Consequently, with W_{r_ϕ} defined as in (4.1.26) with $r := r_\phi$, we have

$$\mathscr{V}_m(\Sigma) \cap \big(\phi + W_{r_\phi}\big) = \mathscr{V}_m(\Sigma) \cap B(\phi, r_\phi) \subseteq \mathcal{U}. \tag{4.1.28}$$

If at this stage we define $\mathcal{O} := \bigcup_{\phi \in \mathcal{U}} \big(\phi + W_{r_\phi}\big)$, then (4.1.11) ensures that the set $\mathcal{O}$ is open in $\tau_{\mathscr{D}}$ and that $\mathcal{O} \cap \mathscr{V}_m(\Sigma) = \mathcal{U}$ (here, (4.1.28) and (4.1.26) are used). This proves (4.1.27) which, in turn, completes the proof of the fact that for each $m \in \mathbb{N}$ the topology induced by $\tau_{\mathscr{D}}$ on $\mathscr{V}_m(\Sigma)$ coincides with τ_m. This is the second claim in item *(2)*, hence the treatment of item *(2)* is complete.

Jumping on to item *(7)*, suppose next that $\mathscr{B} \subseteq \mathrm{Lip}_c(\Sigma)$ is not contained in any $\mathscr{V}_m(\Sigma)$. Then there exist a sequence of functions $\{\phi_m\}_{m \in \mathbb{N}} \subseteq \mathscr{B}$ together with a sequence of points $\{x_m\}_{m \in \mathbb{N}} \subseteq \Sigma$ such that

$$x_m \in \Sigma \setminus K_m \quad \text{and} \quad \phi_m(x_m) \neq 0 \quad \text{for each} \quad m \in \mathbb{N}. \tag{4.1.29}$$

In particular, (4.1.5) and (4.1.29) force $\lim_{m \to \infty} |x_m - x_0| = +\infty$, hence

$$\begin{array}{l}\text{for each } m \in \mathbb{N}, \text{ the set } K_m \text{ contains only finitely} \\ \text{many points from the sequence } \{x_m\}_{m \in \mathbb{N}}.\end{array} \tag{4.1.30}$$

We next claim that if

$$W := \big\{\phi \in \mathrm{Lip}_c(\Sigma) : \ |\phi(x_m)| < m^{-1} \cdot |\phi_m(x_m)| \quad \text{for each} \quad m \in \mathbb{N}\big\} \tag{4.1.31}$$

then

$$W \cap \mathscr{V}_m(\Sigma) \text{ is open in } \tau_m, \text{ for each } m \in \mathbb{N}. \tag{4.1.32}$$

To justify this claim, fix an integer $m_0 \in \mathbb{N}$ and define $I_0 := \{m \in \mathbb{N} : x_m \in K_{m_0}\}$. Then (4.1.30) implies that I_0 is a finite set of positive integers. Pick now an arbitrary $\phi \in W \cap \mathscr{V}_{m_0}(\Sigma)$ and set $\varepsilon := \min\{m^{-1} \cdot |\phi_m(x_m)| - |\phi(x_m)| : m \in I_0\} > 0$. It is then clear from (4.1.3) that for each $\psi \in \mathscr{V}_{m_0}(\Sigma)$ with the property that $\|\psi - \phi\| < \varepsilon$ we necessarily have

$$\begin{aligned} |\psi(x_m)| &\leq |\psi(x_m) - \phi(x_m)| + |\phi(x_m)| < \|\psi - \phi\| + |\phi(x_m)| \\ &< \varepsilon + |\phi(x_m)| \leq m^{-1} \cdot |\phi_m(x_m)| \quad \text{for each } \ m \in I_0. \end{aligned} \tag{4.1.33}$$

Since $\psi(x_m) = 0$ for each $m \in \mathbb{N} \setminus I_0$, we conclude that $\psi \in W \cap \mathscr{V}_{m_0}(\Sigma)$. This further proves that $B(\phi, \varepsilon) \cap \mathscr{V}_{m_0}(\Sigma) \subseteq W \cap \mathscr{V}_{m_0}(\Sigma)$ which ultimately establishes the claim made in (4.1.32). Since, as is apparent from (4.1.31), the set W is also balanced and convex, we deduce from (4.1.32) and (4.1.10) that actually $W \in \mathcal{W}$. However, since by design $\phi_m \notin m \cdot W$, it follows that no scalar multiple of W contains $\mathscr{B}$. Thus, $\mathscr{B}$ cannot be bounded in the topology $\tau_{\mathscr{D}}$. This proves that, given a set $\mathscr{B} \subseteq \mathrm{Lip}_c(\Sigma)$ which is bounded in $\tau_{\mathscr{D}}$, there exists $m \in \mathbb{N}$ such that $\mathscr{B} \subseteq \mathscr{V}_m(\Sigma)$. Thanks to the last claim in item *(2)*, $\mathscr{B}$ is then bounded in τ_m. In view of (4.1.8), this finishes the proof of the claim made in item *(7)*.

Given that any Cauchy sequence is bounded (cf., e.g., [232, Sect. 1.29]), item *(7)* implies that any sequence $\{\phi_j\}_{j \in \mathbb{N}} \subseteq \mathrm{Lip}_c(\Sigma)$ which is Cauchy in $\tau_{\mathscr{D}}$ necessarily lies in some $\mathscr{V}_m(\Sigma)$. By the last claim in item *(2)* and (4.1.8), this also implies that this sequence is Cauchy in $\|\cdot\|$. This proves the claim in item *(5)*. In turn, item *(5)* implies the claim made in item *(4)* by interlacing the given convergent sequence with 0's and writing what it means for this new sequence to be Cauchy.

Finally, the claim in item *(6)* is a consequence of item *(2)*, item *(5)*, and the fact that each $\mathcal{V}_m(\Sigma)$ is complete. □

Having talked about "test" functions, our next goal is to introduce distributions in this general setting. Specifically, we agree to let

the `space of distributions` $(\mathrm{Lip}_c(\Sigma))'$ on a set $\Sigma \subseteq \mathbb{R}^n$ be the topological dual of $(\mathrm{Lip}_c(\Sigma), \tau_{\mathscr{D}})$. (4.1.34)

Also, we agree to

denote by $\langle \cdot, \cdot \rangle$, or more precisely, ${}_{(\mathrm{Lip}_c(\Sigma))'}\langle \cdot, \cdot \rangle_{\mathrm{Lip}_c(\Sigma)}$, the natural duality pairing between distributions in $(\mathrm{Lip}_c(\Sigma))'$ and test functions in $\mathrm{Lip}_c(\Sigma)$. (4.1.35)

Here are various characterizations of continuity in the spirit of [232, Theorem 6.6 on p. 155, and Theorem 6.8 on p. 156].

Proposition 4.1.2 *Let $\Sigma \subseteq \mathbb{R}^n$ be an arbitrary set, and consider a linear mapping $\Lambda : \mathrm{Lip}_c(\Sigma) \to \mathbb{C}$. Then the following conditions are equivalent.*

(1) The mapping Λ belongs to $\big(\mathrm{Lip}_c(\Sigma)\big)'$.
(2) The mapping Λ is bounded, i.e., it sends bounded subsets of the topological vector space $(\mathrm{Lip}_c(\Sigma), \tau_{\mathscr{D}})$ into bounded subsets of $\mathbb{C}$.
(3) If a sequence of functions $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathrm{Lip}_c(\Sigma)$ converges to zero in the topology $\tau_{\mathscr{D}}$ then $\Lambda(\phi_j) \to 0$ in $\mathbb{C}$ as $j \to \infty$.
(4) For each $m \in \mathbb{N}$, the restriction of Λ to $(\mathscr{V}_m(\Sigma), \tau_m)$ is continuous.
(5) For each $m \in \mathbb{N}$ there exists $C_m \in (0, \infty)$ with the property that

$$\big|\Lambda(\phi)\big| \le C_m\Big(\sup_{\Sigma}|\phi| + \|\phi\|_{\mathrm{Lip}(\Sigma)}\Big), \qquad \forall \phi \in \mathscr{V}_m(\Sigma). \tag{4.1.36}$$

Proof The implication *(1)*$\Rightarrow$*(2)* follows from the first part of item *(2)* in Proposition 4.1.1 and [232, Theorem 1.32(a)$\Rightarrow$(b), p. 24]. To prove that *(2)*$\Rightarrow$*(3)*, assume Λ is bounded and pick a sequence $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathrm{Lip}_c(\Sigma)$ convergent to zero in $\tau_{\mathscr{D}}$. Thanks to item *(4)* in Proposition 4.1.1, there exists $m \in \mathbb{N}$ such that $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathcal{V}_m(\Sigma)$ and $\{\phi_j\}_{j\in\mathbb{N}}$ convergent to zero in τ_m. Also, from (4.1.9) and [232, Theorem 1.32(a)$\Rightarrow$(b), p. 24] (applied to the inclusion map) we see that $\Lambda : \big(\mathcal{V}_m(\Sigma), \tau_m\big) \to \mathbb{C}$ is bounded. Since τ_m is obviously metrizable, [232, Theorem 1.32(b)$\Rightarrow$(d), p. 24] may be invoked to conclude that $\Lambda(\phi_j) \to 0$ in $\mathbb{C}$ as $j \to \infty$. This establishes that *(2)*$\Rightarrow$*(3)*.

Going further, assume that *(3)* holds, and pick some $m \in \mathbb{N}$ along with a sequence of functions $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathcal{V}_m(\Sigma)$ convergent to zero in the topology τ_m. The second part of item *(2)* in Proposition 4.1.1 then implies that $\{\phi_j\}_{j\in\mathbb{N}}$ convergent to zero in the topology $\tau_{\mathscr{D}}$. As such, the current hypotheses imply that $\Lambda(\phi_j) \to 0$ in $\mathbb{C}$ as $j \to \infty$. Since, as already noted, τ_m is metrizable, this proves that the restriction of Λ to $(\mathscr{V}_m(\Sigma), \tau_m)$ is continuous. Hence, *(3)*$\Rightarrow$*(4)*.

Next, we shall show that *(4)*$\Rightarrow$*(1)*. Let O be a convex balanced neighborhood of $0 \in \mathbb{C}$ and define $\mathcal{U} := \Lambda^{-1}(O)$. Then $\mathcal{U}$ is obviously convex and balanced. From item *(3)* in Proposition 4.1.1 it follows that $\mathcal{U}$ is open in $\tau_{\mathscr{D}}$ if and only if $\mathcal{U} \cap \mathcal{V}_m(\Sigma)$ is open in τ_m for every $m \in \mathbb{N}$. This concludes the proof *(4)*$\Rightarrow$*(1)*. Hence, at this stage, we have shown the equivalence of *(1)*, *(2)*, *(3)*, and *(4)*.

Finally, the fact that *(1)* is also equivalent to *(5)* is a consequence of the equivalence between *(1)* and *(4)*, bearing in mind the description of the topology τ_m in terms of the norm $\|\cdot\|$. □

Given an arbitrary set $\Sigma \subseteq \mathbb{R}^n$, it follows that $\big(\mathrm{Lip}_c(\Sigma)\big)'$ has a natural vector space structure. We shall turn this into a locally convex topological vector space by equipping it with the weak-$*$ topology, which we shall denote by $\tau_{\mathscr{D}'}$. Hence, $\tau_{\mathscr{D}'}$ is

the topology induced by the family of semi-norms $\{p_\phi\}_{\phi\in\mathrm{Lip}_c(\Sigma)}$ on $\big(\mathrm{Lip}_c(\Sigma)\big)'$, where $p_\phi(\Lambda) := |\langle\Lambda, \phi\rangle|$ for each $\phi \in \mathrm{Lip}_c(\Sigma)$ and each $\Lambda \in \big(\mathrm{Lip}_c(\Sigma)\big)'$. (4.1.37)

Thus, for a sequence $\{\Lambda_j\}_{j\in\mathbb{N}} \subseteq \big(\mathrm{Lip}_c(\Sigma)\big)'$ and a distribution $\Lambda \in \big(\mathrm{Lip}_c(\Sigma)\big)'$,

$$\lim_{j\to\infty} \Lambda_j = \Lambda \text{ in } \tau_{\mathscr{D}'} \iff \lim_{j\to\infty} \langle \Lambda_j, \phi\rangle = \langle \Lambda, \phi\rangle \text{ in } \mathbb{C} \text{ for each } \phi \in \mathrm{Lip}_c(\Sigma). \tag{4.1.38}$$

We also wish to explicitly mention that

$$\text{the topology } \tau_{\mathscr{D}'} \text{ is Hausdorff.} \tag{4.1.39}$$

Indeed, given any distinct $\Lambda_1, \Lambda_2 \in \big(\mathrm{Lip}_c(\Sigma)\big)'$ it follows that there exists some $\phi_o \in \mathrm{Lip}_c(\Sigma)$ such that $r := \big|\langle \Lambda_1, \phi_o\rangle - \langle \Lambda_2, \phi_o\rangle\big| > 0$. Then

$$O_j := \big\{\Lambda \in \big(\mathrm{Lip}_c(\Sigma)\big)' : \ p_{\phi_o}(\Lambda - \Lambda_j) < r/2\big\}, \qquad j = 1, 2, \tag{4.1.40}$$

are disjoint open neighborhoods in the topology $\tau_{\mathscr{D}'}$ of Λ_1 and, respectively, Λ_2. This proves (4.1.39).

As in the standard Euclidean setting (cf. [127, Theorems 2.1.8, pp. 38–39]), the space of distributions on an arbitrary set is sequentially complete, in the sense made precise below.

Proposition 4.1.3 *Suppose Σ is a set in $\mathbb{R}^n$. If a sequence $\{\Lambda_j\}_{j\in\mathbb{N}} \subseteq \big(\mathrm{Lip}_c(\Sigma)\big)'$ has the property that*

$$\lim_{j\to\infty} \langle \Lambda_j, \phi\rangle \text{ exists in } \mathbb{C} \text{ for each } \phi \in \mathrm{Lip}_c(\Sigma), \tag{4.1.41}$$

then the functional which associates to each test function $\phi \in \mathrm{Lip}_c(\Sigma)$ the number defined as the limit in (4.1.41) is a distribution $\Lambda \in \big(\mathrm{Lip}_c(\Sigma)\big)'$, and the following additional properties are satisfied:

(1) one has $\lim_{j\to\infty} \Lambda_j = \Lambda$ in $\tau_{\mathscr{D}'}$;

(2) for every $m \in \mathbb{N}$ there exists $C_m \in (0, \infty)$ such that

$$\big|\langle \Lambda_j, \phi\rangle\big| \le C_m\Big(\sup_{\Sigma} |\phi| + \|\phi\|_{\mathrm{Lip}(\Sigma)}\Big) \text{ for all } \phi \in \mathscr{V}_m(\Sigma) \text{ and all } j \in \mathbb{N}; \tag{4.1.42}$$

(3) for every sequence $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathrm{Lip}_c(\Sigma)$ converging in the topology $\tau_{\mathscr{D}}$ to a limit $\phi \in \mathrm{Lip}_c(\Sigma)$ one has $\lim_{j\to\infty} \langle \Lambda_j, \phi_j\rangle = \langle \Lambda, \phi\rangle$.

Proof Recall (4.1.5), (4.1.6). For each $m \in \mathbb{N}$, we know that $\big(\mathcal{V}_m(\Sigma), \|\cdot\|\big)$ is a Banach space and each Λ_j maps this space linearly and boundedly into $\mathbb{C}$ (cf. Proposition 4.1.2). The hypothesis in (4.1.41) ensures that for each $\phi \in \mathcal{V}_m(\Sigma)$ the sequence $\big\{\langle \Lambda_j, \phi\rangle\big\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{C}$. As such, the Banach–Steinhaus principle of uniform boundedness applies. This guarantees the existence of a constant C_m in $(0, \infty)$ such that (4.1.42) holds. Passing to the limit $j \to \infty$ then proves that Λ satisfies (4.1.36). Thanks to the equivalence *(1)* $\Leftrightarrow$ *(5)* in Proposition 4.1.2, this ultimately shows that Λ belongs to $\big(\mathrm{Lip}_c(\Sigma)\big)'$. Having established this, (4.1.38) then gives that $\lim_{j\to\infty} \Lambda_j = \Lambda$ in $\tau_{\mathscr{D}'}$. Finally, if $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathrm{Lip}_c(\Sigma)$ converges in $\tau_{\mathscr{D}}$ to some $\phi \in \mathrm{Lip}_c(\Sigma)$, then item *(4)* in Proposition 4.1.1 guarantees the existence of some integer $m \in \mathbb{N}$ with the property that $\phi, \phi_j \in \mathcal{V}_m(\Sigma)$ for all $j \in \mathbb{N}$ and such that $\sup_\Sigma |\phi - \phi_j| + \|\phi - \phi_j\|_{\mathrm{Lip}(\Sigma)} \to 0$ as $j \to \infty$. Thanks to (4.1.7) we may actually assume that m is large enough so that (4.1.42) holds. When used with ϕ replaced by $\phi - \phi_j$, the latter estimate implies that $\lim_{j\to\infty} \langle \Lambda_j, \phi - \phi_j\rangle = 0$ in $\mathbb{C}$. In concert with the current item *(1)* and (4.1.38), this finally yields $\lim_{j\to\infty} \langle \Lambda_j, \phi_j\rangle = \langle \Lambda, \phi\rangle$, as wanted. □

A variety of other features enjoyed by the classical notion of distribution have natural counterparts in the present, general setting. As an example, given an arbitrary set $\Sigma \subseteq \mathbb{R}^n$, we may multiply distributions $\Lambda \in \big(\mathrm{Lip}_c(\Sigma)\big)'$ by a given fixed function $\psi \in \mathrm{Lip}(\Sigma)$ according to

$$\psi\Lambda : \mathrm{Lip}_c(\Sigma) \to \mathbb{C}, \quad (\psi\Lambda)(\phi) := \langle \Lambda, \psi \cdot \phi\rangle, \quad \forall \phi \in \mathrm{Lip}_c(\Sigma). \tag{4.1.43}$$

Then (7.3.23) and the equivalence *(1)*⇔*(5)* in Proposition 4.1.2 imply that this definition is meaningful and that, in fact, $\psi\Lambda \in \big(\mathrm{Lip}_c(\Sigma)\big)'$. Moreover, from (4.1.38) we see that the multiplication operator

$$\begin{gathered}\big(\mathrm{Lip}_c(\Sigma)\big)' \ni \Lambda \longmapsto \psi\Lambda \in \big(\mathrm{Lip}_c(\Sigma)\big)' \\ \text{is well defined, linear and sequentially continuous,}\end{gathered} \tag{4.1.44}$$

when $\big(\mathrm{Lip}_c(\Sigma)\big)'$ is equipped with the topology $\tau_{\mathscr{D}'}$.

The proposition below elaborates on the manner in which locally integrable functions induce distributions via integration against test functions.

Proposition 4.1.4 *Let $\Sigma \subseteq \mathbb{R}^n$ be arbitrary and let μ be a locally finite Borel-regular measure on Σ. Then for each $f \in L^1_{\mathrm{loc}}(\Sigma, \mu)$ the functional $\Lambda_f : \mathrm{Lip}_c(\Sigma) \to \mathbb{C}$ given by*

$$\Lambda_f(\phi) := \int_\Sigma f\phi \,\mathrm{d}\mu \ \ \textit{for every} \ \ \phi \in \mathrm{Lip}_c(\Sigma) \tag{4.1.45}$$

belongs to $\big(\mathrm{Lip}_c(\Sigma)\big)'$ and the mapping

$$L^1_{\mathrm{loc}}(\Sigma, \mu) \ni f \longmapsto \Lambda_f \in \big(\mathrm{Lip}_c(\Sigma)\big)' \tag{4.1.46}$$

is linear, injective, and continuous.

As a consequence, one may naturally identify $L^1_{\mathrm{loc}}(\Sigma, \mu)$ with a linear subspace of $\big(\mathrm{Lip}_c(\Sigma)\big)'$ via the embedding (4.1.46). As such, by simply writing f in place of Λ_f, one has

$$\langle f, \phi\rangle = \int_\Sigma f\phi \,\mathrm{d}\mu \ \ \textit{for every} \ \ f \in L^1_{\mathrm{loc}}(\Sigma, \mu) \ \ \textit{and} \ \ \phi \in \mathrm{Lip}_c(\Sigma), \tag{4.1.47}$$

where $\langle \cdot, \cdot \rangle$ is the distributional pairing on Σ (cf. (4.1.35)).

Proof The fact that for each $f \in L^1_{\rm loc}(\Sigma, \mu)$ we have $\Lambda_f \in \big(\mathrm{Lip}_c(\Sigma)\big)'$ is clear from (4.1.45) and the equivalence *(1)* $\Leftrightarrow$ *(5)* in Proposition 4.1.2. That the mapping in (4.1.46) is linear is obvious, while its injectivity is a consequence of Proposition 3.7.2. Finally, the continuity of the mapping in (4.1.46) is a consequence of [232, Theorem 1.32(d) $\Rightarrow$ (a), p. 24] and (4.1.38), bearing in mind that the topology on $L^1_{\rm loc}(\Sigma, \mu)$ is metrizable (cf. Proposition 3.1.1). $\square$

In the same spirit, given an arbitrary set $\Sigma \subseteq \mathbb{R}^n$, any locally finite Borel-regular measure μ on Σ induces a distribution Λ_μ on Σ via

$$\langle \Lambda_\mu, \phi\rangle := \int_\Sigma \phi \, {\rm d}\mu \quad \text{for every} \quad \phi \in \mathrm{Lip}_c(\Sigma), \tag{4.1.48}$$

and the mapping which associates to each such μ the distribution $\Lambda_\mu \in \big(\mathrm{Lip}_c(\Sigma)\big)'$ is linear and injective. We also wish to note that for each fixed point $x_o \in \Sigma$,

$$\text{the Dirac distribution with mass at } x_o \text{ is the functional } \delta_{x_o} \in \big(\mathrm{Lip}_c(\Sigma)\big)' \\ \text{acting on each test function } \phi \in \mathrm{Lip}_c(\Sigma) \text{ according to } \langle \delta_{x_o}, \phi\rangle := \phi(x_o). \tag{4.1.49}$$

We conclude by discussing a criterion guaranteeing the coincidence of a distribution with a function in a Lebesgue space.

Proposition 4.1.5 *Let $\Sigma \subseteq \mathbb{R}^n$ be an arbitrary set, and let μ be a locally finite Borel-regular measure on Σ. Also, fix $p \in (1, \infty]$ and $p' \in [1, \infty)$ such that $1/p + 1/p' = 1$, and if $p = \infty$ make the additional assumption that the measure μ is sigma-finite. Finally, assume $\Lambda \in \big(\mathrm{Lip}_c(\Sigma)\big)'$ is a distribution with the property that there exists $C \in (0, \infty)$ such that*

$$\big|\langle \Lambda, \phi\rangle\big| \leq C \|\phi\|_{L^{p'}(\Sigma, \mu)} \quad \textit{for every} \quad \phi \in \mathrm{Lip}_c(\Sigma). \tag{4.1.50}$$

Then there exists a unique function $f \in L^p(\Sigma, \mu)$ such that

$$\langle \Lambda, \phi\rangle = \int_\Sigma f\phi \, {\rm d}\mu \quad \textit{for every} \quad \phi \in \mathrm{Lip}_c(\Sigma). \tag{4.1.51}$$

In addition, $\|f\|_{L^p(\Sigma, \mu)} \leq C$, where C is as in (4.1.50).

Proof Since $\mathrm{Lip}_c(\Sigma)$ is dense in the space $L^{p'}(\Sigma, \mu)$ (cf. Proposition 3.7.1) and since the mapping $\mathrm{Lip}_c(\Sigma) \ni \phi \mapsto \langle \Lambda, \phi\rangle \in \mathbb{C}$ is linear and bounded with respect to the norm in $L^{p'}(\Sigma, \mu)$ it follows that this mapping extends (by density) to a unique functional $\Theta \in \big(L^{p'}(\Sigma, \mu)\big)^*$ of norm $\leq C$, the constant appearing in (4.1.50). Given that the current hypotheses guarantee that $\big(L^{p'}(\Sigma, \mu)\big)^* = L^p(\Sigma, \mu)$ quantitatively (cf., e.g., [91, Theorem 6.15, p. 190]), we conclude that there exists $f \in L^p(\Sigma, \mu)$ satisfying $\|f\|_{L^p(\Sigma, \mu)} \leq C$ and such that $\Theta(g) = \int_\Sigma fg \, {\rm d}\mu$ for each $g \in L^{p'}(\Sigma, \mu)$. In concert with the fact that $\Theta(\phi) = \langle \Lambda, \phi\rangle$ for each $\phi \in \mathrm{Lip}_c(\Sigma)$, this establishes (4.1.51). The uniqueness of f is then a consequence of Proposition 3.7.2. $\square$

4.2 The Bullet Product

The ultimate goal here is to define a weak notion of inner product between the outward unit normal ν to a given domain and a vector field $\vec{F}$ satisfying some mild integrability properties in said domain. It is worth noting that this notion is so general that it does not actually require the outward unit normal to actually exist. We shall denote the resulting object $\nu \bullet \vec{F}$ and call it the *bullet product* (of ν and $\vec{F}$), in order to distinguish it from the ordinary dot product. In general, this distinction is self-evident, as $\nu \bullet \vec{F}$ typically makes sense only as a distribution on the boundary. These matters are elucidated in Proposition 4.2.3 where the bullet product is first introduced, via a variational formula mimicking integration by parts. As a preamble, we deal with some preparatory results.

Lemma 4.2.1 *Let Ω be an open subset of $\mathbb{R}^n$ and suppose $\vec{F} \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n$ is a vector field whose divergence, considered in the sense of distributions in Ω, satisfies* $\operatorname{div}\vec{F} \in L^1_{\rm loc}(\Omega, \mathcal{L}^n)$. *Then for every scalar-valued function* $\varphi \in \operatorname{Lip}(\Omega)$ *with compact support in Ω one has*

$$\int_\Omega \vec{F} \cdot \nabla\varphi \, d\mathcal{L}^n = -\int_\Omega (\operatorname{div}\vec{F})\varphi \, d\mathcal{L}^n. \tag{4.2.1}$$

Proof A standard mollifier argument yields a sequence $\{\varphi_\varepsilon\}_{0<\varepsilon<\varepsilon_*} \subset \mathscr{C}^\infty_c(\Omega)$ (where $\varepsilon_* > 0$ depends only on the distance from $\operatorname{supp}\varphi$ to $\partial\Omega$) satisfying, for some compact subset K of Ω,

$$\begin{aligned}
&\operatorname{supp}\varphi_\varepsilon \subseteq K \quad \text{for each} \quad \varepsilon \in (0, \varepsilon_*),\\
&\sup\nolimits_{x\in\Omega} |\varphi_\varepsilon(x) - \varphi(x)| \xrightarrow[\varepsilon\to 0^+]{} 0,\\
&\nabla\varphi_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} \nabla\varphi \quad \text{at } \mathcal{L}^n\text{-a.e. point in} \quad \Omega,\\
&\sup\nolimits_{\varepsilon\in(0,\varepsilon_*)} \sup\nolimits_{x\in\Omega} \big\{|\varphi_\varepsilon(x)| + |(\nabla\varphi_\varepsilon)(x)|\big\} < +\infty.
\end{aligned} \tag{4.2.2}$$

Then, based on this and Lebesgue's Dominated Convergence Theorem we may write

$$\begin{aligned}
\int_\Omega \vec{F} \cdot \nabla\varphi \, d\mathcal{L}^n &= \lim_{\varepsilon\to 0^+} \int_\Omega \vec{F} \cdot \nabla\varphi_\varepsilon \, d\mathcal{L}^n = -\lim_{\varepsilon\to 0^+} \int_\Omega (\operatorname{div}\vec{F})\varphi_\varepsilon \, d\mathcal{L}^n\\
&= -\int_\Omega (\operatorname{div}\vec{F})\varphi \, d\mathcal{L}^n,
\end{aligned} \tag{4.2.3}$$

proving (4.2.1). □

We momentarily digress for the purpose of introducing a useful piece of notation, frequently used in the sequel. Specifically, given an $\mathcal{L}^n$-measurable set $\Omega \subseteq \mathbb{R}^n$ and some $p \in (0, \infty)$, we let

$$L^p_{\rm bdd}(\Omega, \mathcal{L}^n) \text{ be the space consisting of all } \mathcal{L}^n\text{-measurable functions which are } p\text{-th power absolutely integrable on each bounded } \mathcal{L}^n\text{-measurable subset of the set } \Omega. \tag{4.2.4}$$

Equivalently,

$$L^p_{\rm bdd}(\Omega, \mathcal{L}^n) \text{ consists of all functions } f : \Omega \to \mathbb{C} \text{ which are } \mathcal{L}^n\text{-measurable and } \int_{B(0,R)\cap\Omega} |f|^p \, d\mathcal{L}^n < \infty \text{ for each } R > 0. \tag{4.2.5}$$

By Hölder's inequality, $L^p_{\rm bdd}(\Omega, \mathcal{L}^n) \subseteq L^q_{\rm bdd}(\Omega, \mathcal{L}^n)$ whenever $0 < q \leq p < \infty$, and if Ω is open then obviously $L^p_{\rm bdd}(\Omega, \mathcal{L}^n) \subset L^p_{\rm loc}(\Omega, \mathcal{L}^n)$ for each $p \in (0, \infty)$.

Our next result shows that, under slightly stronger hypotheses on the vector field involved, we may relax the assumptions made on the Lipschitz function intervening in Lemma 4.2.1.

Lemma 4.2.2 *Let Ω be an open subset of $\mathbb{R}^n$ and suppose $\vec{F} \in \big[L^1_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^n$ is a vector field whose divergence, considered in the sense of distributions in Ω, satisfies* $\operatorname{div}\vec{F} \in L^1_{\rm bdd}(\Omega, \mathcal{L}^n)$. *Then for every scalar-valued function $\varphi \in \operatorname{Lip}(\overline{\Omega})$ which vanishes outside of a compact subset of $\overline{\Omega}$ and satisfies $\varphi\big|_{\partial\Omega} = 0$ one has*

$$\int_\Omega \vec{F} \cdot \nabla\varphi \, d\mathcal{L}^n = -\int_\Omega (\operatorname{div}\vec{F})\varphi \, d\mathcal{L}^n. \tag{4.2.6}$$

Proof Bring in the family of functions $\{\Phi_\varepsilon\}_{\varepsilon>0}$ constricted as in Lemma 6.1.2 in relation to the set Ω. Availing ourselves of Lebesgue's Dominated Convergence Theorem and then invoking Lemma 4.2.1 (bearing in mind that, for each $\varepsilon > 0$, the function $\Phi_\varepsilon\varphi \in \operatorname{Lip}(\Omega)$ has compact support in Ω) we may write

$$\begin{aligned}\int_\Omega (\operatorname{div}\vec{F})\varphi \, d\mathcal{L}^n &= \lim_{\varepsilon\to 0^+} \int_\Omega (\operatorname{div}\vec{F})(\Phi_\varepsilon\varphi) \, d\mathcal{L}^n \\ &= -\lim_{\varepsilon\to 0^+} \int_\Omega \vec{F} \cdot \nabla(\Phi_\varepsilon\varphi) \, d\mathcal{L}^n = -\lim_{\varepsilon\to 0^+} \mathrm{I}_\varepsilon - \lim_{\varepsilon\to 0^+} \mathrm{II}_\varepsilon \end{aligned} \tag{4.2.7}$$

where, for each $\varepsilon > 0$, we have set

$$\mathrm{I}_\varepsilon := \int_\Omega \vec{F} \cdot (\nabla\Phi_\varepsilon)\varphi \, d\mathcal{L}^n \quad\text{and}\quad \mathrm{II}_\varepsilon := \int_\Omega \vec{F} \cdot (\nabla\varphi)\Phi_\varepsilon \, d\mathcal{L}^n. \tag{4.2.8}$$

Next fix $\varepsilon > 0$ and consider an arbitrary $x \in O_\varepsilon$. If $x_* \in \partial\Omega$ is selected such that $|x - x_*| = \operatorname{dist}(x, \partial\Omega)$ then, using the fact that φ vanishes on the boundary of Ω and that φ is Lipschitz (with, say, Lipschitz constant $M \in [0, \infty)$), we may write

$|\varphi(x)| = |\varphi(x) - \varphi(x_*)| \leq M|x - x_*| \leq M\varepsilon$. In turn, for each fixed $\varepsilon > 0$ this permits us to estimate

$$\begin{aligned} \big|\mathrm{I}_\varepsilon\big| &\leq \int_\Omega |\vec{F}||\nabla\Phi_\varepsilon||\varphi| \,\mathrm{d}\mathcal{L}^n \leq C\varepsilon^{-1} \int_{O_\varepsilon \cap \operatorname{supp}\varphi} |\vec{F}(x)||\varphi(x)| \,\mathrm{d}x \\ &\leq C \int_{O_\varepsilon \cap \operatorname{supp}\varphi} |\vec{F}| \,\mathrm{d}\mathcal{L}^n. \end{aligned} \tag{4.2.9}$$

Having proved this, Lebesgue's Dominated Convergence Theorem then shows that

$$\lim_{\varepsilon\to 0^+} \mathrm{I}_\varepsilon = 0. \tag{4.2.10}$$

Finally, by once again appealing to Lebesgue's Dominated Convergence Theorem we see that

$$\lim_{\varepsilon\to 0^+} \mathrm{II}_\varepsilon = \int_\Omega \vec{F} \cdot (\nabla\varphi) \,\mathrm{d}\mathcal{L}^n. \tag{4.2.11}$$

At this stage, (4.2.6) is clear from (4.2.7) and (4.2.10)–(4.2.11). □

In the proposition below we introduce a notion of normal component of a vector field in an arbitrary open set, considered in a weak sense, as a functional on the space of compactly supported Lipschitz functions on the boundary of the given set. The actual definition, given in (4.2.12) below, is of variation nature and is inspired by the integration by parts formula (1.2.12).

Proposition 4.2.3 *Let Ω be an open subset of $\mathbb{R}^n$ and suppose $\vec{F} \in \big[L^1_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^n$ is a vector field whose divergence, considered in the sense of distributions in Ω, satisfies $\operatorname{div}\vec{F} \in L^1_{\rm bdd}(\Omega, \mathcal{L}^n)$. Consider a functional, denoted by $\nu \bullet \vec{F}$ (read "nu bullet F"), which acts on each $\psi \in \mathrm{Lip}_c(\partial\Omega)$ according to*

$$\big\langle \nu \bullet \vec{F}, \psi \big\rangle := \int_\Omega \vec{F} \cdot \nabla\Psi \,\mathrm{d}\mathcal{L}^n + \int_\Omega (\operatorname{div}\vec{F})\Psi \,\mathrm{d}\mathcal{L}^n, \tag{4.2.12}$$

where Ψ is any complex-valued function satisfying

$$\begin{gathered} \Psi \in \mathrm{Lip}(\overline{\Omega}), \quad \Psi\big|_{\partial\Omega} = \psi, \ \text{ and} \\ \Psi \equiv 0 \text{ outside of some compact subset of } \overline{\Omega}. \end{gathered} \tag{4.2.13}$$

Then $\nu \bullet \vec{F}$ is meaningfully and unambiguously defined, and actually belongs to $\big(\mathrm{Lip}_c(\partial\Omega)\big)'$. Moreover, for each given scalar-valued function $\varphi \in \mathrm{Lip}(\overline{\Omega})$ it follows that

$$\begin{gathered} \varphi\vec{F} \in \big[L^1_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^n, \quad \operatorname{div}(\varphi\vec{F}) \in L^1_{\rm bdd}(\Omega, \mathcal{L}^n), \\ \text{and } \ \nu \bullet (\varphi\vec{F}) = \big(\varphi\big|_{\partial\Omega}\big)(\nu \bullet \vec{F}) \ \text{ in } \ \big(\mathrm{Lip}_c(\partial\Omega)\big)' \end{gathered} \tag{4.2.14}$$

where the distribution in the right-hand side of the last line in (4.2.14) *is considered in the sense of* (4.1.43)*. Also,*

$$\nu \bullet \vec{F} = 0 \ \ in \ \ \big(\mathrm{Lip}_c(\partial\Omega)\big)' \ \ \textit{if there exists } \varepsilon > 0 \\ \textit{such that } \vec{F} \textit{ vanishes } \mathcal{L}^n\textit{-a.e. in the set} \\ O_\varepsilon := \big\{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) < \varepsilon\big\}. \tag{4.2.15}$$

Finally, the bullet product mapping

$$\Big\{\vec{F} \in \big[L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)\big]^n : \ \mathrm{div}\vec{F} \in L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)\Big\} \ni \vec{F} \mapsto \nu \bullet \vec{F} \in \big(\mathrm{Lip}_c(\partial\Omega)\big)' \tag{4.2.16}$$

is a well-defined, linear, and continuous mapping (*assuming that the space on the left is equipped with the topology canonically induced by the family of semi-norms* $p_j(\vec{F}) := \int_{\Omega\cap B(0,j)} \big\{|\vec{F}| + |\mathrm{div}\vec{F}|\big\}\,\mathrm{d}\mathcal{L}^n$ *indexed by* $j \in \mathbb{N}$).

We wish to stress that the symbol ν above is purely formal, simply part of the formalism associated with the bullet product. In particular, ν does *not* represent, in and of itself, the actual outward unit normal to Ω, which generally does not exist in any conventional sense given that Ω is merely an arbitrary open set.

Proof of Proposition 4.2.3 First, observe that for each $\psi \in \mathrm{Lip}_c(\partial\Omega)$ there exists a function Ψ as in (4.2.13). Indeed, we may take Ψ as the Whitney extension of ψ to $\mathbb{R}^n$ (cf. Theorem 6.1.3), suitably truncated and restricted to $\overline{\Omega}$. Second, note that for each Ψ as in (4.2.13) the integrals in the right-hand side of (4.2.12) are absolutely convergent, thanks to the assumptions on $\vec{F}$ and the properties of Ψ. Third, if $\Psi_1, \Psi_2 \in \mathrm{Lip}(\overline{\Omega})$ vanish outside a compact subset of $\overline{\Omega}$ and $\Psi_1\big|_{\partial\Omega} = \Psi_2\big|_{\partial\Omega}$, then since the difference $\Phi := \Psi_1 - \Psi_2$ belongs to $\mathrm{Lip}(\overline{\Omega})$, vanishes outside a compact subset of $\overline{\Omega}$, and satisfies $\Phi\big|_{\partial\Omega} = 0$, Lemma 4.2.2 applied to this particular function Φ presently yields

$$\int_\Omega \vec{F} \cdot \nabla\Psi_1 \,\mathrm{d}\mathcal{L}^n + \int_\Omega (\mathrm{div}\vec{F})\Psi_1 \,\mathrm{d}\mathcal{L}^n = \int_\Omega \vec{F} \cdot \nabla\Psi_2 \,\mathrm{d}\mathcal{L}^n + \int_\Omega (\mathrm{div}\vec{F})\Psi_2 \,\mathrm{d}\mathcal{L}^n. \tag{4.2.17}$$

Collectively, these observations prove that $\nu \bullet \vec{F}$ is meaningfully and unambiguously defined. In turn, these qualities further imply that $\nu \bullet \vec{F}$ is linear on $\mathrm{Lip}_c(\partial\Omega)$.

As regards the continuity of the linear functional $\nu \bullet \vec{F}$, fix $x_0 \in \partial\Omega$ along with some $r > 0$. Also, select a cutoff function $\theta \in \mathscr{C}^\infty_c\big(B(x_0, 2r)\big)$ satisfying $\theta \equiv 1$ on $B(x_0, r)$ and $|\nabla\theta| \le C_n/r$. If for $\psi \in \mathrm{Lip}_c(\partial\Omega)$ with $\mathrm{supp}\,\psi \subseteq \partial\Omega \cap B(x_0, r)$ we now let $\mathcal{E}\psi$ be the Whitney extension of ψ to $\mathbb{R}^n$ (as described in Theorem 6.1.3) and set $\Psi := \big(\theta\mathcal{E}\psi\big)\big|_{\overline{\Omega}}$, then Ψ satisfies

$$\begin{aligned}&\Psi \in \mathrm{Lip}(\overline{\Omega}) \cap \mathscr{C}^\infty(\Omega), \quad \Psi\big|_{\partial\Omega} = \psi, \quad \Psi \equiv 0 \text{ on } \overline{\Omega} \setminus B(x_0, 2r),\\ &\sup_{\Omega} |\nabla\Psi| \le C_n \Big\{ \frac{1}{r} \cdot \sup_{\partial\Omega} |\psi| + \sup_{\substack{x,y\in\partial\Omega\\ x\neq y}} \frac{|\psi(x)-\psi(y)|}{|x-y|} \Big\}, \\ &\text{as well as } \sup_{\Omega} |\Psi| \le C_n \cdot \sup_{\partial\Omega} |\psi|,\end{aligned} \tag{4.2.18}$$

for some purely dimensional constant $C_n \in (0, \infty)$. In concert with (4.2.12), this implies

$$\begin{aligned}\big|\langle \nu \bullet \vec{F}, \psi\rangle\big| &\le C_n \Big(\frac{1}{r} \cdot \sup_{\partial\Omega} |\psi| + \sup_{\substack{x,y\in\partial\Omega\\ x\neq y}} \frac{|\psi(x)-\psi(y)|}{|x-y|} \Big) \int_{\Omega\cap B(x_0,2r)} |\vec{F}| \, \mathrm{d}\mathcal{L}^n \\ &\quad + C_n \cdot \Big(\sup_{\partial\Omega} |\psi| \Big) \int_{\Omega\cap B(x_0,2r)} |\mathrm{div}\vec{F}| \, \mathrm{d}\mathcal{L}^n,\end{aligned} \tag{4.2.19}$$

from which we ultimately conclude that $\nu \bullet \vec{F}$ belongs to $\big(\mathrm{Lip}_c(\partial\Omega)\big)'$, on account of the equivalence *(1)* $\Leftrightarrow$ *(5)* in Proposition 4.1.2.

Going further, fix a scalar-valued function $\varphi \in \mathrm{Lip}(\overline{\Omega})$. Then we clearly have $\varphi\vec{F} \in \big[L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)\big]^n$. The fact that $\mathrm{div}(\varphi\vec{F}) \in L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)$ also follows as soon as we show that

$$\mathrm{div}(\varphi\vec{F}) = \varphi(\mathrm{div}\vec{F}) + \vec{F} \cdot \nabla\varphi \text{ in } \mathcal{D}'(\Omega). \tag{4.2.20}$$

With this goal in mind, fix some arbitrary scalar-valued function $\phi \in \mathscr{C}^\infty_c(\Omega)$ and set $K := \mathrm{supp}\,\phi$. Then a standard mollifier argument allows us to construct

$$\begin{aligned}&\{\varphi_\varepsilon\}_{\varepsilon>0} \subset \mathscr{C}^\infty(\Omega) \text{ with } \lim_{\varepsilon\to 0^+} \varphi_\varepsilon = \varphi \text{ uniformly on } K, \ \lim_{\varepsilon\to 0^+} \nabla\varphi_\varepsilon = \nabla\varphi \\ &\text{at } \mathcal{L}^n\text{-a.e. point in } K, \text{ and such that } \sup_{\varepsilon>0} \|\nabla\varphi_\varepsilon\|_{[L^\infty(K,\mathcal{L}^n)]^n} < \infty.\end{aligned} \tag{4.2.21}$$

Granted this, we may write

$$\begin{aligned}\int_\Omega \varphi\vec{F} \cdot \nabla\phi \, \mathrm{d}\mathcal{L}^n &= \lim_{\varepsilon\to 0^+} \int_\Omega \vec{F} \cdot (\varphi_\varepsilon \nabla\phi) \, \mathrm{d}\mathcal{L}^n \\ &= \lim_{\varepsilon\to 0^+} \int_\Omega \vec{F} \cdot \nabla(\varphi_\varepsilon\phi) \, \mathrm{d}\mathcal{L}^n - \lim_{\varepsilon\to 0^+} \int_\Omega \phi\vec{F} \cdot \nabla\varphi_\varepsilon \, \mathrm{d}\mathcal{L}^n \\ &= -\lim_{\varepsilon\to 0^+} \int_\Omega (\mathrm{div}\vec{F})\varphi_\varepsilon\phi \, \mathrm{d}\mathcal{L}^n - \lim_{\varepsilon\to 0^+} \int_\Omega \phi\vec{F} \cdot \nabla\varphi_\varepsilon \, \mathrm{d}\mathcal{L}^n \\ &= -\int_\Omega (\mathrm{div}\vec{F})\varphi\,\phi \, \mathrm{d}\mathcal{L}^n - \int_\Omega \phi\vec{F} \cdot \nabla\varphi \, \mathrm{d}\mathcal{L}^n\end{aligned} \tag{4.2.22}$$

which, in view of the arbitrariness of ϕ, establishes (4.2.20). Consider now an arbitrary function $\psi \in \mathrm{Lip}_c(\partial\Omega)$, and pick Ψ as in (4.2.13). Then

$$\begin{aligned}&\varphi\Psi\in \mathrm{Lip}(\overline{\Omega}),\ \ (\varphi\Psi)\big|_{\partial\Omega}=\big(\varphi\big|_{\partial\Omega}\big)\psi,\ \text{ and also}\\&\varphi\Psi\equiv 0 \text{ outside of some compact subset of } \overline{\Omega}.\end{aligned}\tag{4.2.23}$$

As such, (4.1.43) and (4.2.12) permit us to compute

$$\begin{aligned}\big\langle\big(\varphi\big|_{\partial\Omega}\big)\nu\bullet\vec{F},\psi\big\rangle&=\big\langle\nu\bullet\vec{F},\big(\varphi\big|_{\partial\Omega}\big)\psi\big\rangle\\&=\int_\Omega\vec{F}\cdot\nabla(\varphi\Psi)\,\mathrm{d}\mathcal{L}^n+\int_\Omega(\mathrm{div}\vec{F})\varphi\Psi\,\mathrm{d}\mathcal{L}^n\\&=\int_\Omega\varphi\vec{F}\cdot\nabla\Psi\,\mathrm{d}\mathcal{L}^n+\int_\Omega\Psi\vec{F}\cdot\nabla\varphi\,\mathrm{d}\mathcal{L}^n+\int_\Omega(\mathrm{div}\vec{F})\varphi\Psi\,\mathrm{d}\mathcal{L}^n\\&=\int_\Omega(\varphi\vec{F})\cdot\nabla\Psi\,\mathrm{d}\mathcal{L}^n+\int_\Omega\mathrm{div}(\varphi\vec{F})\Psi\,\mathrm{d}\mathcal{L}^n\\&=\big\langle\nu\bullet(\varphi\vec{F}),\psi\big\rangle,\end{aligned}\tag{4.2.24}$$

where the next-to-last equality makes use of (4.2.20). At this stage, the formula on the second line of (4.2.14) follows from (4.2.24).

Consider next the task of proving the claim made in (4.2.15). To set the stage, denote by C_0, C_1 the constants appearing in (6.1.2) with $F:=\partial\Omega$, pick some number $N>C_1/C_0$ then choose a function $\theta\in\mathscr{C}^\infty(\mathbb{R})$ satisfying $0\le\theta\le 1$, as well as $\theta\equiv 1$ on $(-\infty, C_1/N)$ and $\theta\equiv 0$ on (C_0,∞). For $\varepsilon>0$ as in (4.2.15) define $\Psi_\varepsilon(x):=\theta\Big(\frac{\delta^{\mathrm{reg}}_{\partial\Omega}(x)}{\varepsilon/2}\Big)$ for all $x\in\mathbb{R}^n$, where $\delta^{\mathrm{reg}}_{\partial\Omega}$ denotes the regularized distance to $\partial\Omega$ (cf. Proposition 6.1.1). Then

$$\begin{aligned}&\Psi_\varepsilon\in\mathscr{C}^\infty(\mathbb{R}^n),\quad \Psi_\varepsilon\equiv 0 \text{ in } \Omega\setminus\mathcal{O}_{\varepsilon/2},\quad \Psi_\varepsilon\equiv 1 \text{ near } \partial\Omega,\\&\text{and}\ \sup_{x\in\mathbb{R}^n}\big|(\partial^\alpha\Psi_\varepsilon)(x)\big|<+\infty\ \text{ for each }\ \alpha\in\mathbb{N}_0^n.\end{aligned}\tag{4.2.25}$$

Given any $\psi\in\mathrm{Lip}_c(\partial\Omega)$ along with Ψ as in (4.2.13), it follows that

$$\begin{aligned}&\Psi_\varepsilon\Psi\in\mathrm{Lip}(\overline{\Omega}),\quad (\Psi_\varepsilon\Psi)\big|_{\partial\Omega}=\psi,\ \text{ and}\\&\Psi_\varepsilon\Psi\equiv 0\ \text{ outside of some compact subset of }\ \overline{\Omega}.\end{aligned}\tag{4.2.26}$$

Granted this, we may now write (4.2.12) with Ψ replaced by $\Psi_\varepsilon\Psi$ and conclude that $\big\langle\nu\bullet\vec{F},\psi\big\rangle=0$ since $\Psi_\varepsilon\Psi\equiv 0$ in $\Omega\setminus\mathcal{O}_{\varepsilon/2}$ and $\vec{F}$ vanishes in $\mathcal{O}_\varepsilon$. This finishes the proof of (4.2.15).

The argument in the first part of the proof shows that the bullet product mapping (4.2.16) is well defined and linear. To prove its continuity, observe that since the countable family of semi-norms $\{p_j\}_{j\in\mathbb{N}}$ is separating (in the sense that $p_j(\vec{F})=0$ for each j forces $\vec{F}=0$), the topology induced by this family on the space of functions $\Big\{\vec{F}\in\big[L^1_{\mathrm{bdd}}(\Omega,\mathcal{L}^n)\big]^n : \mathrm{div}\vec{F}\in L^1_{\mathrm{bdd}}(\Omega,\mathcal{L}^n)\Big\}$ is metrizable (cf., e.g., [181, pp. 545–546]). Bearing this in mind, it follows from [232, Theorem 1.32(d)$\Rightarrow$(a), p. 24] that we only need to check the sequential continuity of the bullet product mapping (4.2.16) at the origin. However, this is clear from (4.1.38) and (4.2.19). □

Example 4.2.4 *Assume $\Omega \subseteq \mathbb{R}^n$ is an open set and consider a scalar function $u \in L^1_{\rm loc}(\Omega, \mathcal{L}^n)$ with the property that $\nabla u \in \big[L^1_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^n$. For any two indices $j, k \in \{1, \dots, n\}$ define the divergence-free vector field*

$$\vec{F}^u_{jk} := (\partial_k u)\mathbf{e}_j - (\partial_j u)\mathbf{e}_k \in \big[L^1_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^n. \tag{4.2.27}$$

Then Proposition 4.2.3 implies that the `weak tangential derivative`

$$\overset{\bullet}{\partial}_{\tau_{jk}} u := \nu \bullet \vec{F}^u_{jk} = \nu \bullet \big((\partial_k u)\mathbf{e}_j - (\partial_j u)\mathbf{e}_k\big) \tag{4.2.28}$$

is a well-defined distribution on $\partial\Omega$, i.e., belongs to $\big(\mathrm{Lip}_c(\partial\Omega)\big)'$, and

$${}_{(\mathrm{Lip}_c(\partial\Omega))'}\big\langle \overset{\bullet}{\partial}_{\tau_{jk}} u, \Psi\big|_{\partial\Omega}\big\rangle_{\mathrm{Lip}_c(\partial\Omega)} = \int_\Omega \big\{(\partial_k u)(\partial_j \Psi) - (\partial_j u)(\partial_k \Psi)\big\}\, d\mathcal{L}^n \tag{4.2.29}$$

for each $\Psi \in \mathrm{Lip}_c(\mathbb{R}^n)$.

Here is another example casting light on the notion of bullet product, introduced earlier.

Example 4.2.5 *Let Σ be a closed* UR *set in $\mathbb{R}^n$ (in the sense of Definition 5.10.1). Consider the open set $\Omega := \mathbb{R}^n \setminus \Sigma$ with $\partial\Omega = \Sigma$, and abbreviate $\sigma := \mathcal{H}^1 \lfloor \Sigma$. Having picked an arbitrary function $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ define the vector field by setting*

$$\vec{F}(x) := \frac{1}{\omega_{n-1}} \int_\Sigma \frac{x-y}{|x-y|^n} f(y)\, d\sigma(y) \quad \text{for each} \quad x \in \Omega. \tag{4.2.30}$$

Then

$$\vec{F} \in \big[L^1_{\rm bdd}(\Omega, \mathcal{L}^n) \cap \mathscr{C}^\infty(\Omega)\big]^n \quad \text{with} \quad \mathrm{div}\vec{F} = 0 \quad \text{in} \quad \Omega, \tag{4.2.31}$$

and

$$\nu \bullet \vec{F} = -f \quad \text{in} \quad \big(\mathrm{Lip}_c(\partial\Omega)\big)'. \tag{4.2.32}$$

Let us prove the claims made in Example 4.2.5. First, by design $\vec{F}$ is a divergence-free vector field, with smooth components in Ω. If we pick an arbitrary aperture parameter $\kappa > 0$, then Calderón-Zygmund theory (cf. [186, Sect. 2.4]) tells us that

$$\mathcal{N}^{\Sigma^c}_\kappa \vec{F} \in L^{1,\infty}_{\rm loc}(\Sigma, \sigma), \tag{4.2.33}$$

with the nontangential maximal operator in (4.2.33) being associated with the open set $\Sigma^c := \mathbb{R}^n \setminus \Sigma = \Omega$. In concert with Lemma 6.2.4, this implies

$$\mathcal{N}^{\Sigma^c}_\kappa \vec{F} \in L^p_{\rm loc}(\Sigma, \sigma) \quad \text{for each} \quad p \in (0, 1), \tag{4.2.34}$$

which together with Proposition 8.6.3 further gives

$$\vec{F} \in \big[L^q_{\rm bdd}(\Omega, \mathcal{L}^n)\big]^n \quad \text{for each} \quad q \in \big(0, \tfrac{n}{n-1}\big). \tag{4.2.35}$$

Thus, all claims in (4.2.31) are justified.

To verify (4.2.32), select an arbitrary $\psi \in \mathrm{Lip}_c(\partial\Omega)$ and let $\Psi \in \mathrm{Lip}_c(\mathbb{R}^n)$ be any complex-valued function satisfying $\Psi\big|_{\partial\Omega} = \psi$. In view of (4.2.12) and the properties of $\vec{F}$ we may write

$$\begin{aligned}\big\langle \nu \bullet \vec{F}, \psi\big\rangle &= \int_\Omega \vec{F} \cdot \nabla\Psi \, \mathrm{d}\mathcal{L}^n \\ &= \frac{1}{\omega_{n-1}} \int_\Sigma \Big(\int_\Omega \frac{x-y}{|x-y|^n} \cdot (\nabla\Psi)(x)\, \mathrm{d}x \Big) f(y)\, \mathrm{d}\sigma(y).\end{aligned} \tag{4.2.36}$$

The last equality above is a consequence of Fubini's Theorem. To check that this is applicable in the present setting, write

$$\begin{aligned}&\int_\Sigma \Big(\int_\Omega \frac{1}{|x-y|^{n-1}} |(\nabla\Psi)(x)|\, \mathrm{d}x \Big) |f(y)|\, \mathrm{d}\sigma(y) \\ &\quad \le \|\nabla\Psi\|_{[L^\infty(\mathbb{R}^n, \mathcal{L}^n)]^n} \int_\Sigma \Big(\int_{\mathrm{supp}\,\Psi} \frac{\mathrm{d}x}{|x-y|^{n-1}} \Big) |f(y)|\, \mathrm{d}\sigma(y) \\ &\quad \le C \int_\Sigma \frac{|f(y)|}{1+|y|^{n-1}}\, \mathrm{d}\sigma(y) < +\infty,\end{aligned} \tag{4.2.37}$$

where the first equality is based on the fact that $\Psi \in \mathrm{Lip}_c(\mathbb{R}^n)$, the second inequality uses (3.5.24) (with $m := n-1$), and the last inequality is ensured by the membership of f to $L^1\big(\Sigma, \frac{\sigma(y)}{1+|y|^{n-1}}\big)$.

Thus, (4.2.36) is established. Let us also observe that for each fixed $y \in \Sigma$ we have

$$\begin{aligned}\frac{1}{\omega_{n-1}} \int_\Omega \frac{x-y}{|x-y|^n} \cdot (\nabla\Psi)(x)\, \mathrm{d}x &= \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n} \cdot (\nabla\Psi)(x)\, \mathrm{d}x \\ &= -\Psi(y) = -\psi(y),\end{aligned} \tag{4.2.38}$$

given that $\mathcal{L}^n(\mathbb{R}^n \setminus \Omega) = \mathcal{L}^n(\Sigma) = 0$ since Σ is upper Ahlfors regular, and the distributional divergence of the vector field $\mathbb{R}^n \ni x \mapsto (x-y)/|x-y|^n$ is $\omega_{n-1}\delta_y$. Collectively, (4.2.36) and (4.2.38) imply

$$\big\langle \nu \bullet \vec{F}, \psi\big\rangle = -\int_\Sigma f(y)\psi(y)\, \mathrm{d}\sigma(y), \tag{4.2.39}$$

so (4.2.32) follows from this, in view of the arbitrariness of $\psi \in \mathrm{Lip}_c(\partial\Omega)$. This completes the verification of all claims made in Example 4.2.5.

Our last remark is that, having dealt with Proposition 4.2.3, it is now possible to extend the definition of the bullet product to vector fields which are allowed to be singular. Specifically, we make the following definition:

Definition 4.2.6 *Let Ω be an arbitrary open subset of $\mathbb{R}^n$ and consider a vector field*

$$\vec{F} \in \big[L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n) + \mathscr{E}'(\Omega)\big]^n \quad \textit{with} \quad \mathrm{div}\vec{F} \in L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n) + \mathscr{E}'(\Omega), \tag{4.2.40}$$

where the divergence is taken in the sense of distributions in Ω.

Define the distribution $\nu \bullet \vec{F} \in \big(\mathrm{Lip}_c(\partial\Omega)\big)'$ as being $\nu \bullet \big((1-\eta)\vec{F}\big)$, interpreted in the sense of Proposition 4.2.3, where

$$\begin{aligned}&\eta \in \mathscr{C}^\infty_c(\Omega) \textit{ is a function satisfying } \eta \equiv 1 \textit{ near } K, \textit{ with } K \subseteq \Omega\\ &\textit{some compact set such that } \vec{F} \in \big[L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n) + \mathscr{E}'_K(\Omega)\big]^n \textit{ and}\\ &\mathrm{div}\vec{F} \in L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n) + \mathscr{E}'_K(\Omega).\end{aligned} \tag{4.2.41}$$

In relation to this definition, we wish to make a couple of comments. First, since for any function η as in (4.2.41) we have $(1-\eta)\vec{F} \in \big[L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)\big]^n$ and

$$\mathrm{div}\big((1-\eta)\vec{F}\big) = (1-\eta)\,\mathrm{div}\vec{F} - \nabla\eta \cdot \vec{F} \in L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n), \tag{4.2.42}$$

it is meaningful to consider $\nu \bullet \big((1-\eta)\vec{F}\big) \in \big(\mathrm{Lip}_c(\partial\Omega)\big)'$ in the sense of Proposition 4.2.3. Second, (4.2.15) ensures that the extension of the bullet product in Definition 4.2.6 is unambiguous (i.e., is independent of the choice of the function η as in (4.2.41)).

4.3 The Product Rule for Weak Derivatives

A suitable version of Leibniz's product formula holds for weak derivatives. Such a result appears to be folklore, but since a reference does not seem to be easy to find, we include a complete proof.

Proposition 4.3.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set and fix some arbitrary $j \in \{1, \dots, n\}$. Suppose u, w are two scalar-valued functions on Ω satisfying*

$$\begin{aligned}u, w \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n), \quad \partial_j u,\ \partial_j w \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n),\\ u \cdot w \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n), \quad u\,\partial_j w + w\partial_j u \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n).\end{aligned} \tag{4.3.1}$$

Then

$$\partial_j(u \cdot w) = u\,\partial_j w + w\partial_j u \quad \textit{in} \quad \mathcal{D}'(\Omega). \tag{4.3.2}$$

Proof As a preamble, we remark that a standard mollifier argument gives the following approximation result.

$$\begin{array}{l}\text{for each } \phi \in L^\infty_{\rm loc}(\Omega,\mathcal{L}^n) \text{ with the property that } \partial_j\phi \in L^1_{\rm loc}(\Omega,\mathcal{L}^n),\\ \text{there exists a sequence } \{\phi_\alpha\}_{\alpha\in\mathbb{N}} \subset \mathscr{C}^\infty(\Omega) \text{ such that both } \lim\limits_{\alpha\to\infty}\phi_\alpha = \phi\\ \text{and } \lim\limits_{\alpha\to\infty}\partial_j\phi_\alpha = \partial_j\phi \text{ in } L^1_{\rm loc}(\Omega,\mathcal{L}^n), \text{ and } \sup_{\alpha\in\mathbb{N}}\|\phi_\alpha\|_{L^\infty(K,\mathcal{L}^n)} < +\infty\\ \text{for each fixed compact set } K \subset \Omega.\end{array} \tag{4.3.3}$$

The remainder of the proof is divided into five steps.

Step I. *Formula* (4.3.2) *holds whenever*

$$\begin{aligned} &u, w \in L^\infty_{\rm loc}(\Omega,\mathcal{L}^n), \quad \partial_j u,\ \partial_j w \in L^1_{\rm loc}(\Omega,\mathcal{L}^n),\\ &u\cdot w \in L^1_{\rm loc}(\Omega,\mathcal{L}^n), \quad u\,\partial_j w + w\partial_j u \in L^1_{\rm loc}(\Omega,\mathcal{L}^n).\end{aligned} \tag{4.3.4}$$

Consider u, w as in (4.3.4) and fix $\varphi \in \mathscr{C}^\infty_c(\Omega)$. Apply (4.3.3) to u and w to produce two approximating sequences $\{u_\alpha\}_{\alpha\in\mathbb{N}}$ and $\{w_\alpha\}_{\alpha\in\mathbb{N}}$ enjoying the conditions specified there. Note that

$$\begin{aligned}\langle\partial_j(uw),\varphi\rangle &= -\langle uw,\partial_j\varphi\rangle = -\int_\Omega uw\partial_j\varphi\,d\mathcal{L}^n\\ &= -\lim_{\alpha\to\infty}\int_\Omega u_\alpha w_\alpha\partial_j\varphi\,d\mathcal{L}^n = \lim_{\alpha\to\infty}\int_\Omega(u_\alpha\partial_j w_\alpha + w_\alpha\partial_j u_\alpha)\varphi\,d\mathcal{L}^n\\ &= \int_\Omega(u\partial_j w + w\partial_j u)\varphi\,d\mathcal{L}^n = \langle u\partial_j w + w\partial_j u,\varphi\rangle.\end{aligned} \tag{4.3.5}$$

As regards the third equality above, note that for each $\alpha \in \mathbb{N}$

$$\int_\Omega u_\alpha w_\alpha\partial_j\varphi\,d\mathcal{L}^n = \mathrm{I}_\alpha + \mathrm{II}_\alpha + \int_\Omega uw\partial_j\varphi\,d\mathcal{L}^n \tag{4.3.6}$$

where

$$\mathrm{I}_\alpha := \int_\Omega(u_\alpha - u)w_\alpha\partial_j\varphi\,d\mathcal{L}^n \ \text{ and } \ \mathrm{II}_\alpha := \int_\Omega u(w_\alpha - w)\partial_j\varphi\,d\mathcal{L}^n. \tag{4.3.7}$$

Since the approximation properties from (4.3.3) imply

$$\begin{aligned}\limsup_{\alpha\to\infty}|\mathrm{I}_\alpha| \leq &\Big(\sup_{\alpha\in\mathbb{N}}\|w_\alpha\|_{L^\infty(\operatorname{supp}\varphi,\mathcal{L}^n)}\Big)\|\partial_j\varphi\|_{L^\infty(\Omega,\mathcal{L}^n)}\times\\ &\times\limsup_{\alpha\to\infty}\|u_\alpha - u\|_{L^1(\operatorname{supp}\varphi,\mathcal{L}^n)} = 0,\end{aligned} \tag{4.3.8}$$

we have $\lim\limits_{\alpha\to\infty}\mathrm{I}_\alpha = 0$. Similarly, $\lim\limits_{\alpha\to\infty}\mathrm{II}_\alpha = 0$, so the third equality in (4.3.5) is justified.

As regards the fifth equality in (4.3.5), for each $\alpha \in \mathbb{N}$ re-write

$$(u_\alpha \partial_j w_\alpha + w_\alpha \partial_j u_\alpha)\varphi \\ = \underbrace{u_\alpha(\partial_j w_\alpha - \partial_j w)\varphi}_{\mathrm{III}_\alpha} + u_\alpha(\partial_j w)\varphi + \underbrace{w_\alpha(\partial_j u_\alpha - \partial_j u)\varphi}_{\mathrm{IV}_\alpha} + w_\alpha(\partial_j u)\varphi. \tag{4.3.9}$$

Thanks to the approximation properties of the scheme discussed in (4.3.3), much as before, the integrals of the terms III_α and IV_α above vanish after passing to limit $\alpha \to \infty$. As such,

$$\lim_{\alpha\to\infty} \int_\Omega (u_\alpha \partial_j w_\alpha + w_\alpha \partial_j u_\alpha)\varphi \, \mathrm{d}\mathcal{L}^n = \lim_{\alpha\to\infty} \int_\Omega (u_\alpha \partial_j w + w_\alpha \partial_j u)\varphi \, \mathrm{d}\mathcal{L}^n, \tag{4.3.10}$$

assuming the second limit exists. To see that this is indeed the case, split the integrand

$$(u_\alpha \partial_j w + w_\alpha \partial_j u)\varphi = (u_\alpha - u)(\partial_j w)\varphi + (w_\alpha - w)(\partial_j u)\varphi + (u\partial_j w + w\partial_j u)\varphi \\ =: \mathrm{V}_\alpha + \mathrm{VI}_\alpha + \mathrm{VII} \tag{4.3.11}$$

and observe that Lebesgue's Dominated Convergence Theorem gives that both the integral of V_α and the integral of VI_α converge to zero as $\alpha \to \infty$. Therefore,

$$\lim_{\alpha\to\infty} \int_\Omega (u_\alpha \partial_j w + w_\alpha \partial_j u)\varphi \, \mathrm{d}\mathcal{L}^n = \int_\Omega (u\partial_j w + w\partial_j u)\varphi \, \mathrm{d}\mathcal{L}^n, \tag{4.3.12}$$

finishing the proof of (4.3.5). In turn, in view of the arbitrariness of $\varphi \in \mathscr{C}_c^\infty(\Omega)$, formula (4.3.5) implies (4.3.2).

`Step II.` *Formula* (4.3.2) *holds whenever*

$$u \in L^\infty_{\mathrm{loc}}(\Omega, \mathcal{L}^n), \quad w \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n), \quad \partial_j u,\ \partial_j w \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n), \\ u \cdot w \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n), \quad u\,\partial_j w + w\partial_j u \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n). \tag{4.3.13}$$

To justify this, fix u, w as in (4.3.13). For each $k \in \mathbb{N}$, define the piecewise smooth function $f_k \in \mathscr{C}^0(\mathbb{R})$ by, setting for each $t \in \mathbb{R}$,

$$f_k(t) := \begin{cases} k, & \text{if } t > k, \\ t, & \text{if } |t| \leq k, \\ -k, & \text{if } t < -k. \end{cases} \tag{4.3.14}$$

Then for each $k \in \mathbb{N}$ we have $f_k \circ w \in L^\infty(\Omega, \mathcal{L}^n)$ and an inspection of the proof of the Chain Rule Formula given in [104, Theorem 7.8, p. 153] shows that the function $\partial_j(f_k \circ w)$ belongs to $L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)$ and, in fact,

$$\partial_j(f_k \circ w) = (\partial_j w) \cdot \mathbf{1}_{|w|<k} \ \text{ at} \mathcal{L}^n\text{-a.e. point in } \ \Omega. \tag{4.3.15}$$

In view of the fact that $|f_k \circ w| \leq |w|$ on Ω and $uw \in L^1_{\rm loc}(\Omega, \mathscr{L}^n)$, we have the membership $u \cdot (f_k \circ w) \in L^1_{\rm loc}(\Omega, \mathscr{L}^n)$. In addition,

$$\underbrace{u}_{L^\infty_{\rm loc}(\Omega,\mathscr{L}^n)} \cdot \underbrace{\partial_j(f_k \circ w)}_{L^1_{\rm loc}(\Omega,\mathscr{L}^n)} + \underbrace{(f_k \circ w)}_{L^\infty(\Omega,\mathscr{L}^n)} \cdot \underbrace{\partial_j u}_{L^1_{\rm loc}(\Omega,\mathscr{L}^n)} \in L^1_{\rm loc}(\Omega, \mathscr{L}^n). \tag{4.3.16}$$

As such, for each fixed $k \in \mathbb{N}$, the functions u and $f_k \circ w$ satisfy all hypotheses in (4.3.4). Having fixed an arbitrary $\varphi \in \mathscr{C}^\infty_c(\Omega)$, Step I then implies that for each $k \in \mathbb{N}$ we may write

$$-\int_\Omega u(f_k \circ w)\partial_j\varphi \, \mathrm{d}\mathscr{L}^n = \int_\Omega \big\{u\partial_j(f_k \circ w) + (f_k \circ w)\partial_j u\big\}\, \varphi \, \mathrm{d}\mathscr{L}^n. \tag{4.3.17}$$

Observe that $f_k \circ w \to w$ pointwise $\mathscr{L}^n$-a.e. in Ω as $k \to \infty$ and note that we have $|f_k \circ w| \leq |w| \in L^1_{\rm loc}(\Omega, \mathscr{L}^n)$. Also, recall that $uw \in L^1_{\rm loc}(\Omega, \mathscr{L}^n)$. These permit us to invoke Lebesgue's Dominated Convergence Theorem to conclude that

$$\lim_{k\to\infty} \int_\Omega u(f_k \circ w)\partial_j\varphi \, \mathrm{d}\mathscr{L}^n = \int_\Omega uw\partial_j\varphi \, \mathrm{d}\mathscr{L}^n. \tag{4.3.18}$$

Moreover, (4.3.15) implies that at $\mathscr{L}^n$-a.e. point in Ω we have $|\partial_j(f_k \circ w)| \leq |\partial_j w|$ for each $k \in \mathbb{N}$, and

$$\partial_j(f_k \circ w) \to \partial_j w \quad \text{as} \quad k \to \infty. \tag{4.3.19}$$

In addition, the fact that we are currently assuming $u \in L^\infty_{\rm loc}(\Omega, \mathscr{L}^n)$ permits us to estimate

$$|\varphi \, u \, \partial_j(f_k \circ w)| \leq \underbrace{|u\varphi|}_{L^\infty_{\rm comp}(\Omega,\mathscr{L}^n)} \cdot \underbrace{|\partial_j w|}_{L^1_{\rm loc}(\Omega,\mathscr{L}^n)} \in L^1(\Omega, \mathscr{L}^n). \tag{4.3.20}$$

We also remark that

$$\begin{aligned} |\varphi(f_k \circ w)\partial_j u| &= |\varphi||f_k \circ w||\partial_j u| \leq |\varphi||w||\partial_j u| = |\varphi||w\partial_j u| \qquad (4.3.21) \\ &\leq \underbrace{|\varphi|}_{L^\infty_{\rm comp}(\Omega,\mathscr{L}^n)} \cdot \underbrace{|u\partial_j w + w\partial_j u|}_{L^1_{\rm loc}(\Omega,\mathscr{L}^n)} + \underbrace{|\varphi u|}_{L^\infty_{\rm comp}(\Omega,\mathscr{L}^n)} \cdot \underbrace{|\partial_j w|}_{L^1_{\rm loc}(\Omega,\mathscr{L}^n)} \in L^1(\Omega, \mathscr{L}^n). \end{aligned}$$

Granted (4.3.19), (4.3.20), and (4.3.21), we may once again rely on Lebesgue's Dominated Convergence Theorem to conclude that

$$\lim_{k\to\infty} \int_\Omega \{u\partial_j(f_k \circ w) + (f_k \circ w)\partial_j u\}\varphi \, \mathrm{d}\mathscr{L}^n = \int_\Omega \big(u\partial_j w + w\partial_j u\big)\, \varphi \, \mathrm{d}\mathscr{L}^n. \tag{4.3.22}$$

Combining now (4.3.17), (4.3.18), and (4.3.22) yields (4.3.2).

As an intermission, for any real-valued function f define

$$f^+ := \max\{f, 0\} = \frac{|f| + f}{2} = \begin{cases} f & \text{if } f \geq 0, \\ 0 & \text{if } f < 0, \end{cases} \tag{4.3.23}$$

and

$$f^- := \max\{-f, 0\} = \frac{|f| - f}{2} = \begin{cases} -f & \text{if } f < 0, \\ 0 & \text{if } f \geq 0. \end{cases} \tag{4.3.24}$$

In particular, $f^\pm \geq 0$ and $f = f^+ - f^-$. From the proof of [104, Lemma 7.6, p. 152] we know that if $f \in L^1_{\rm loc}(\Omega, \mathcal{L}^n)$ has $\partial_j f \in L^1_{\rm loc}(\Omega, \mathcal{L}^n)$, then $f^\pm$ have similar properties and, in fact,

$$\partial_j f^+ = \begin{cases} \partial_j f & \text{if } f > 0, \\ 0 & \text{if } f \leq 0, \end{cases} \qquad \partial_j f^- = \begin{cases} 0 & \text{if } f \geq 0, \\ -\partial_j f & \text{if } f < 0, \end{cases} \tag{4.3.25}$$

hence

$$\partial_j f^+ = (\partial_j f) \cdot \mathbf{1}_{f>0} \ \text{ and } \ \partial_j f^- = -(\partial_j f) \cdot \mathbf{1}_{f<0}. \tag{4.3.26}$$

`Step III.` *Formula* (4.3.2) *holds whenever*

$$\begin{gathered} u \geq 0 \text{ and } w \geq 1 \text{ at } \mathcal{L}^n\text{-a.e. point in } \Omega, \\ u, w \in L^1_{\rm loc}(\Omega, \mathcal{L}^n), \quad \partial_j u,\ \partial_j w \in L^1_{\rm loc}(\Omega, \mathcal{L}^n), \\ u \cdot w \in L^1_{\rm loc}(\Omega, \mathcal{L}^n), \quad u\, \partial_j w + w\, \partial_j u \in L^1_{\rm loc}(\Omega, \mathcal{L}^n). \end{gathered} \tag{4.3.27}$$

Consider u, w as in (4.3.27) and define $u_k := \min\{u, k/w\}$ for each integer $k \in \mathbb{N}$. Then

$$0 \leq u_k \leq u \in L^1_{\rm loc}(\Omega, \mathcal{L}^n), \quad \text{and} \tag{4.3.28}$$

$$0 \leq u_k \leq \frac{k}{w} \leq k \in L^\infty(\Omega, \mathcal{L}^n). \tag{4.3.29}$$

In particular, $u_k \in L^\infty(\Omega, \mathcal{L}^n)$ for each $k \in \mathbb{N}$. Upon recalling (4.3.23), for each $k \in \mathbb{N}$ we may express

$$u_k = u + \min\left\{0, \frac{k}{w} - u\right\} = u - \max\left\{u - \frac{k}{w}, 0\right\} = u - \left(u - \frac{k}{w}\right)^+. \tag{4.3.30}$$

For each $k \in \mathbb{N}$ we also have $k/w \in L^\infty(\Omega, \mathcal{L}^n) \subset L^1_{\rm loc}(\Omega, \mathcal{L}^n)$, and the Chain Rule in the spirit of [104, Lemma 7.5, p. 151] (applied with $f \in \mathscr{C}^1(\mathbb{R})$ such that $f(t) = k/t$ for $t \geq 1$) further yields

$$\partial_j \left(\frac{k}{w}\right) = -\frac{k\partial_j w}{w^2} \in L^1_{\rm loc}(\Omega, \mathcal{L}^n). \tag{4.3.31}$$

Using (4.3.31) and (4.3.25) we have that

$$\partial_j u_k = \begin{cases} -k\dfrac{\partial_j w}{w^2} & \text{if } uw > k, \\ \partial_j u & \text{if } uw \leq k, \end{cases} \tag{4.3.32}$$

thus $\partial_j u_k \in L^1_{\rm loc}(\Omega, \mathcal{L}^n)$. Furthermore,

$$0 \leq u_k \cdot w \leq \frac{k}{w} \cdot w = k \in L^\infty(\Omega, \mathcal{L}^n) \subset L^1_{\rm loc}(\Omega, \mathcal{L}^n). \tag{4.3.33}$$

From the definition for u_k and (4.3.32) we obtain

$$u_k\partial_j w + w\,\partial_j u_k = \begin{cases} \dfrac{k}{w}\partial_j w - k\dfrac{\partial_j w}{w} = 0 & \text{if } uw > k, \\ u\partial_j w + w\,\partial_j u & \text{if } uw \leq k. \end{cases} \tag{4.3.34}$$

In particular, $u_k\partial_j w + w\,\partial_j u_k \in L^1_{\rm loc}(\Omega, \mathcal{L}^n)$. All together, the functions u_k and w satisfy the hypotheses in (4.3.13). As such, Step II applies and, for each $\varphi \in \mathscr{C}^\infty_c(\Omega)$, permits us to write

$$-\int_\Omega u_k w \partial_j \varphi \, d\mathcal{L}^n = \int_\Omega \{u_k\partial_j w + w\,\partial_j u_k\}\varphi \, d\mathcal{L}^n, \qquad \forall k \in \mathbb{N}. \tag{4.3.35}$$

As $k \to \infty$, it is clear that $u_k \to u$ at $\mathcal{L}^n$-a.e. point in Ω. In addition, we have $|u_k w| \leq |uw| \in L^1_{\rm loc}(\Omega, \mathcal{L}^n)$. Hence, Lebesgue's Dominated Convergence Theorem applies and gives

$$\lim_{k\to\infty} \int_\Omega u_k w\,\partial_j \varphi \, d\mathcal{L}^n = \int_\Omega uw\,\partial_j \varphi \, d\mathcal{L}^n. \tag{4.3.36}$$

In addition, from (4.3.34), the last condition in (4.3.27), and Lebesgue's Dominated Convergence Theorem we conclude that

$$\lim_{k\to\infty} \int_\Omega \{u_k\partial_j w + w\,\partial_j u_k\}\varphi \, d\mathcal{L}^n = \int_\Omega \{u\partial_j w + w\,\partial_j u\}\varphi \, d\mathcal{L}^n. \tag{4.3.37}$$

Formula (4.3.2) now follows from (4.3.35)–(4.3.37).

`Step IV.` *Formula (4.3.2) holds whenever*

$$\begin{aligned} & u \geq 0 \text{ and } w \geq 0 \text{ at } \mathcal{L}^n\text{-a.e. point in } \Omega, \\ & u, w \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n), \quad \partial_j u,\ \partial_j w \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n), \\ & u \cdot w \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n), \quad u\,\partial_j w + w\,\partial_j u \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n). \end{aligned} \tag{4.3.38}$$

Take u, w as in (4.3.38) and define $\widetilde{w} := w + 1 \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n)$. Since $u, \widetilde{w}$ satisfy (4.3.27), the result in Step III applies and allows us to write

$$\begin{aligned} \partial_j(uw) &= \partial_j\big(u(\widetilde{w} - 1)\big) = \partial_j(u\widetilde{w}) - \partial_j u = u\partial_j\widetilde{w} + \widetilde{w}\,\partial_j u - \partial_j u \\ &= u\partial_j(w+1) + (w+1)\partial_j u - \partial_j u = u\partial_j w + w\,\partial_j u, \end{aligned} \tag{4.3.39}$$

proving (4.3.2) in this case.

Step V. *Formula* (4.3.2) *holds whenever* u, w *are as in* (4.3.1). To see that this is the case, consider

$$0 \leq u^{\pm} := \frac{|u| \pm u}{2} \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n), \ \ 0 \leq w^{\pm} := \frac{|w| \pm w}{2} \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n). \tag{4.3.40}$$

Then $|u^{\pm} \cdot w^{\pm}| \leq |uw| \in L^1_{\text{loc}}(\Omega, \mathcal{L}^n)$, and (4.3.25) implies that

$$\partial_j(u^+) = \begin{cases} \partial_j u & \text{if } u > 0, \\ 0 & \text{if } u \leq 0, \end{cases} \quad \text{and} \quad \partial_j(u^-) = \begin{cases} 0 & \text{if } u \geq 0, \\ -\partial_j u & \text{if } u < 0, \end{cases} \tag{4.3.41}$$

plus similar formulas for $\partial_j(w^{\pm})$. These entail

$$u^+\partial_j w^+ + w^+\partial_j u^+ = \begin{cases} u\partial_j w + w\,\partial_j u & \text{if } u > 0 \text{ and } w > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3.42}$$

$$u^+\partial_j w^- + w^-\partial_j u^+ = \begin{cases} -(u\partial_j w + w\,\partial_j u) & \text{if } u > 0 \text{ and } w < 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3.43}$$

$$u^-\partial_j w^+ + w^+\partial_j u^- = \begin{cases} -(u\partial_j w + w\,\partial_j u) & \text{if } u < 0 \text{ and } w > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.3.44}$$

and

$$u^-\partial_j w^- + w^-\partial_j u^- = \begin{cases} u\partial_j w + w\,\partial_j u & \text{if } u < 0 \text{ and } w < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3.45}$$

In particular, from (4.3.42)–(4.3.45) and the last hypothesis in (4.3.1) we conclude that

$$\begin{aligned} &u^+\partial_j w^+ + w^+\partial_j u^+ \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n), \quad u^+\partial_j w^- + w^-\partial_j u^+ \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n), \\ &u^-\partial_j w^+ + w^+\partial_j u^- \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n), \quad u^-\partial_j w^- + w^-\partial_j u^- \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n). \end{aligned} \tag{4.3.46}$$

Consequently, any of the pairs (u^+, w^+), (u^+, w^-), (u^-, w^+), (u^-, w^-) satisfy the hypotheses in (4.3.38). Having established this, Step IV applies and, for each $\varphi \in \mathscr{C}_c^\infty(\Omega)$, allows us to compute

$$\begin{aligned} -\int_\Omega uw\,\partial_j\varphi\,\mathrm{d}\mathcal{L}^n \\ &= -\int_\Omega (u^+ - u^-)(w^+ - w^-)\partial_j\varphi\,\mathrm{d}\mathcal{L}^n \\ &= -\int_\Omega u^+w^+\partial_j\varphi\,\mathrm{d}\mathcal{L}^n + \int_\Omega u^+w^-\partial_j\varphi\,\mathrm{d}\mathcal{L}^n \\ &\quad + \int_\Omega u^-w^+\partial_j\varphi\,\mathrm{d}\mathcal{L}^n - \int_\Omega u^-w^-\partial_j\varphi\,\mathrm{d}\mathcal{L}^n \\ &= \int_\Omega \{u^+\partial_j w^+ + w^+\partial_j u^+\}\varphi\,\mathrm{d}\mathcal{L}^n - \int_\Omega \{u^+\partial_j w^- + w^-\partial_j u^+\}\varphi\,\mathrm{d}\mathcal{L}^n \\ &\quad - \int_\Omega \{u^-\partial_j w^+ + w^+\partial_j u^-\}\varphi\,\mathrm{d}\mathcal{L}^n + \int_\Omega \{u^-\partial_j w^- + w^-\partial_j u^-\}\varphi\,\mathrm{d}\mathcal{L}^n \\ &= \int_\Omega \big\{(u^+ - u^-)\partial_j(w^+ - w^-) + (w^+ - w^-)\partial_j(u^+ - u^-)\big\}\varphi\,\mathrm{d}\mathcal{L}^n \\ &= \int_\Omega \{u\partial_j w + w\,\partial_j u\}\varphi\,\mathrm{d}\mathcal{L}^n. \end{aligned} \tag{4.3.47}$$

In view of the arbitrariness of $\varphi \in \mathscr{C}_c^\infty(\Omega)$, formula (4.3.2) follows. This concludes the proof of Proposition 4.3.1. □

4.4 Pointwise Divergence Versus Distributional Divergence

We first establish a Divergence Formula in rectangles, for differentiable vector fields whose divergence, considered in a pointwise sense, happens to continuous.

Lemma 4.4.1 *Let O be an open set in $\mathbb{R}^n$ and suppose R is an open rectangle in $\mathbb{R}^n$ such that $\overline{R} \subseteq O$. Suppose $\vec{F} = (F_i)_{1\leq i\leq n} : O \to \mathbb{C}^n$ is a vector field with the property that*

$$\vec{F} \text{ is differentiable at every point in } \overline{R} \text{ and } \sum_{i=1}^{n} \partial_i F_i \text{ is continuous on } \overline{R}, \tag{4.4.1}$$

where the partial derivatives are considered in a pointwise, classical sense.

Then, with ν *denoting the outward unit normal to* R*, one has*

$$\int_{\partial R} \nu \cdot \vec{F}\, \mathrm{d}\mathcal{H}^{n-1} = \int_R \Big(\sum_{i=1}^{n} \partial_i F_i \Big)\, \mathrm{d}\mathcal{L}^n. \tag{4.4.2}$$

Proof For any sub-rectangle $Q \subseteq R$ we agree to denote

$$\Phi(Q) := \int_{\partial Q} \nu \cdot \vec{F}\, \mathrm{d}\mathcal{H}^{n-1} - \int_Q \Big(\sum_{i=1}^{n} \partial_i F_i \Big)\, \mathrm{d}\mathcal{L}^n \tag{4.4.3}$$

where ν denotes the outward unit normal to Q. Seeking a contradiction, suppose (4.4.2) fails. Hence, there exists $\varepsilon > 0$ such that

$$|\Phi(R)| \geq \varepsilon. \tag{4.4.4}$$

By dividing each side of the rectangle R into two equal parts we generate 2^n congruent sub-rectangles, call them $\{R_1^k\}_{1 \leq k \leq 2^n}$, of R with disjoint interiors. Given the fact the union of the closures of these sub-rectangles is $\overline{R}$, and that $\Phi(R) = \sum_{k=1}^{2^n} \Phi(R_1^k)$ (since for any two such sub-rectangles sharing a common face their outward unit normals on that face have opposite orientations), from (4.4.4) it follows that we may select one sub-rectangle, call it $R_1 \in \{R_1^k\}_{1 \leq k \leq 2^n}$, with the property that

$$|\Phi(R_1)| \geq \frac{\varepsilon}{2^n}. \tag{4.4.5}$$

Proceeding inductively, we obtain a sequence of rectangles $\{R_j\}_{j \in \mathbb{N}}$ satisfying

$$\begin{gathered} R_{j+1} \subseteq R_j \subseteq R \text{ for } j \in \mathbb{N}, \text{ diam}\, R_j \to 0 \text{ as } j \to \infty, \text{ and} \\ \Big| \int_{\partial R_j} \nu_j \cdot \vec{F}\, \mathrm{d}\mathcal{H}^{n-1} - \int_{R_j} \Big(\sum_{i=1}^{n} \partial_i F_i \Big)\, \mathrm{d}\mathcal{L}^n \Big| \geq \frac{\varepsilon}{2^{nj}} \text{ for } j \in \mathbb{N}, \end{gathered} \tag{4.4.6}$$

where ν_j is the outward unit normal to R_j. Then there exists $x^* = (x_1^*, \dots, x_n^*) \in \overline{R}$ with $\bigcap_{j \in \mathbb{N}} \overline{R_j} = \{x^*\}$ and, using the continuity property from (4.4.1), we have

$$\frac{1}{\mathcal{L}^n(R_j)} \int_{R_j} \Big(\sum_{i=1}^{n} \partial_i F_i \Big)\, \mathrm{d}\mathcal{L}^n \longrightarrow \sum_{i=1}^{n} (\partial_i F_i)(x^*) \text{ as } j \to \infty. \tag{4.4.7}$$

In addition, for $x = (x_1, \dots, x_n)$ near x^*, the differentiability property from (4.4.1) implies

$$\vec{F}(x) = \vec{F}(x^*) + \sum_{i=1}^{n}(x_i - x_i^*)(\partial_i \vec{F})(x^*) + o(|x - x^*|). \tag{4.4.8}$$

Thus, if we define $\vec{G}(x) := \vec{F}(x^*) + \sum_{i=1}^{n}(x_i - x_i^*)(\partial_i \vec{F})(x^*)$ for each $x \in \mathbb{R}^n$, then $\vec{G}$ is a smooth vector field whose divergence is constant, $\mathrm{div}\vec{G} = \sum_{i=1}^{n}(\partial_i F_i)(x^*)$ in $\mathbb{R}^n$. The Divergence Theorem applied to $\vec{G}$ on each rectangle R_j then gives

$$\int_{\partial R_j} \nu_j \cdot \vec{G}\, \mathrm{d}\mathcal{H}^{n-1} = \int_{R_j} \mathrm{div}\vec{G}\, \mathrm{d}\mathcal{L}^n = \mathcal{L}^n(R_j)\sum_{i=1}^{n}(\partial_i F_i)(x^*), \;\; \forall j \in \mathbb{N}. \tag{4.4.9}$$

Moreover, from (4.4.8) and the definition of $\vec{G}$ it follows that

$$\frac{1}{\mathcal{L}^n(R_j)}\int_{\partial R_j} \nu_j \cdot \big(\vec{F} - \vec{G}\big)\, \mathrm{d}\mathcal{H}^{n-1} \longrightarrow 0 \;\text{ as }\; j \to \infty. \tag{4.4.10}$$

In concert, (4.4.9)–(4.4.10) yield

$$\frac{1}{\mathcal{L}^n(R_j)}\int_{\partial R_j} \nu_j \cdot \vec{F}\, \mathrm{d}\mathcal{H}^{n-1} \longrightarrow \sum_{i=1}^{n}(\partial_i F_i)(x^*) \;\text{ as }\; j \to \infty. \tag{4.4.11}$$

From (4.4.7) and (4.4.11), we conclude that

$$2^{nj}\Big|\int_{\partial R_j} \nu_j \cdot \vec{F}\, \mathrm{d}\mathcal{H}^{n-1} - \int_{R_j}\Big(\sum_{i=1}^{n}\partial_i F_i\Big)\mathrm{d}\mathcal{L}^n\Big| \longrightarrow 0 \;\text{ as }\; j \to \infty. \tag{4.4.12}$$

The conclusion in (4.4.12) contradicts the condition in the second line of (4.4.6). This contradiction shows that in fact (4.4.2) holds true, completing the proof of the lemma. □

In turn, Lemma 4.4.1 is the main ingredient in the proof of the following criterion guaranteeing the coincidence between the divergence of a vector field taken in the sense of distributions and the divergence of the vector field in question computed in a classical pointwise sense.

Proposition 4.4.2 *Let $\mathcal{O}$ be an open set in $\mathbb{R}^n$ and suppose $\vec{F} = (F_i)_{1\leq i\leq n} : \mathcal{O} \to \mathbb{C}^n$ is a vector field which is differentiable at every point in $\mathcal{O}$ and $\sum_{i=1}^{n}\partial_i F_i$ is continuous on $\mathcal{O}$, where the partial derivatives are considered in a pointwise, classical sense. Then $\mathrm{div}\vec{F}$ taken in sense of distributions in $\mathcal{D}'(\mathcal{O})$ is equal to $\sum_{i=1}^{n}\partial_i F_i$.*

Proof Let $\varphi \in \mathscr{C}_c^\infty(\mathcal{O})$ be arbitrary. Then $\varphi\vec{F}$ is differentiable in $\mathcal{O}$ and the product rule gives

$$\operatorname{div}(\varphi\vec{F}) = \nabla\varphi\cdot\vec{F} + \varphi\sum_{i=1}^{n}\partial_i F_i \ \text{ pointwise in } \mathcal{O}. \tag{4.4.13}$$

In concert with the hypotheses on $\vec{F}$, this allows us to conclude that, for every open rectangle R with $\overline{R}\subseteq\mathcal{O}$, the vector field $\varphi\vec{F}$ satisfies the assumptions in Lemma 4.4.1. As such, for every open rectangle R with $\overline{R}\subseteq\mathcal{O}$ we may write

$$\int_{\partial R}\nu\cdot(\varphi\vec{F})\,\mathrm{d}\mathcal{H}^{n-1} = \int_R\Big(\nabla\varphi\cdot\vec{F}+\varphi\sum_{i=1}^{n}\partial_i F_i\Big)\,\mathrm{d}\mathcal{L}^n \tag{4.4.14}$$

where ν is the outward unit normal to R.

Next, fix a scale $0<r<\operatorname{dist}\big(\operatorname{supp}\varphi,\partial\mathcal{O}\big)/\sqrt{n}$ and consider the standard grid $\mathcal{Q}_r$ of closed cubes in $\mathbb{R}^n$ of side-length r. Then the polyhedron

$$\mathcal{U} := \bigcup_{Q\in\mathcal{Q}_r,\ Q\cap\operatorname{supp}\varphi\neq\varnothing} Q \tag{4.4.15}$$

has the property that $\mathcal{U}\subseteq\mathcal{O}$ and $\operatorname{supp}\varphi\subseteq\mathring{\mathcal{U}}$. Since for each $Q\in\mathcal{Q}_r$ with the property that $Q\cap\operatorname{supp}\varphi\neq\varnothing$ formula (4.4.14) holds for each $R:=\mathring{Q}$, summing up over all such Q's yields

$$\int_{\partial\mathcal{U}}\nu\cdot(\varphi\vec{F})\,\mathrm{d}\mathcal{H}^{n-1} = \int_{\mathcal{U}}\Big(\nabla\varphi\cdot\vec{F}+\varphi\sum_{i=1}^{n}\partial_i F_i\Big)\,\mathrm{d}\mathcal{L}^n. \tag{4.4.16}$$

From (4.4.16) and the fact that $\operatorname{supp}\varphi\subseteq\mathring{\mathcal{U}}$ it follows that the integral in the left-hand side of (4.4.16) is zero, which further implies

$$\int_{\mathcal{O}}\nabla\varphi\cdot\vec{F}\,\mathrm{d}\mathcal{L}^n = -\int_{\mathcal{O}}\varphi\Big(\sum_{i=1}^{n}\partial_i F_i\Big)\,\mathrm{d}\mathcal{L}^n. \tag{4.4.17}$$

Since $\varphi\in\mathscr{C}_c^\infty(\mathcal{O})$ is arbitrary, the desired conclusion follows. □

4.5 Removability of Singularities for Distributional Derivatives

The main result in this section is Proposition 4.5.2. To set the stage, we first review the upper Minkowski content in the definition below.

Definition 4.5.1 *Given a nonempty set $F\subseteq\mathbb{R}^n$ and some number $\gamma\in\mathbb{R}$, the upper γ-dimensional Minkowski content of F is defined as*

$$M_\gamma^*(F) := \limsup_{r\to 0^+} \frac{\mathcal{L}^n\big(\{x \in \mathbb{R}^n : \operatorname{dist}(x, F) < r\}\big)}{r^{n-\gamma}} \in [0, +\infty]. \tag{4.5.1}$$

Refer to F as having `finite upper` γ`-dimensional Minkowski content` *if $M_\gamma^*(F) < +\infty$, and say that F has* `vanishing upper` γ`-dimensional Minkowski content` *provided $M_\gamma^*(F) = 0$.*

Moreover, say that F has `locally finite upper` γ`-dimensional Minkowski content` *if $M_\gamma^*(F \cap K) < +\infty$ for each compact set $K \subseteq \mathbb{R}^n$ with $F \cap K \neq \varnothing$, and say that F has* `locally vanishing upper` γ`-dimensional Minkowski content` *provided $M_\gamma^*(F \cap K) = 0$ for each compact set $K \subseteq \mathbb{R}^n$ with $F \cap K \neq \varnothing$.*

Obviously, any singleton has finite upper 0-dimensional Minkowski content, and so does any set of finite cardinality. Also, for each $\gamma \in \mathbb{R}$ we have

$$M_\gamma^*(E) \leq M_\gamma^*(F) \quad \text{whenever} \quad \varnothing \neq E \subseteq F \subseteq \mathbb{R}^n. \tag{4.5.2}$$

Given $m \in \mathbb{N}$ with $m \leq n$, for any Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}^n$ and any nonempty compact set $K \subset \mathbb{R}^m$ the set $F := f(K)$ has $M_m^*(F) = \mathcal{H}^m(F)$; see [88, p. 275]. Consequently, in this case $M_m^*(F) < +\infty$ whenever $\mathcal{H}^m(F) < +\infty$ (hence, in particular, when $\mathcal{L}^m(K) < +\infty$; cf. (5.3.1)). As a consequence,

$$\text{any rectifiable curve in } \mathbb{R}^n \text{ has finite upper 1-dimensional Minkowski content.} \tag{4.5.3}$$

Later on, in (8.6.88), we shall prove a result which readily implies that

$$\text{any nonempty closed Ahlfors regular set in } \mathbb{R}^n \text{ has locally finite upper } (n-1)\text{-dimensional Minkowski content.} \tag{4.5.4}$$

The notion of upper Minkowski content plays a crucial role in the following criterion for removability of singularities for distributional derivatives.

Proposition 4.5.2 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and consider a relatively closed subset K of Ω having locally finite upper $(n-d)$-dimensional Minkowski content for some $d > 1$. Also, fix a compact set $K_o \subseteq \Omega \setminus K$. Suppose D is a homogeneous constant (complex) coefficient first-order $M \times N$ system (for some $M, N \in \mathbb{N}$), and assume*

$$u \in \Big[L^{\frac{d}{d-1}}_{\mathrm{loc}}(\Omega, \mathcal{L}^n) + \mathscr{E}'_{K_o}(\Omega)\Big]^N \quad \text{and} \quad f \in \Big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) + \mathscr{E}'_{K_o}(\Omega)\Big]^M \tag{4.5.5}$$

are such that

$$D\big(u\big|_{\Omega\setminus K}\big) = f\big|_{\Omega\setminus K} \quad \text{in} \quad \big[\mathcal{D}'(\Omega \setminus K)\big]^M. \tag{4.5.6}$$

Then

$$Du = f \quad \text{in} \quad \big[\mathcal{D}'(\Omega)\big]^M. \tag{4.5.7}$$

Moreover, the same result is valid for $d = 1$ (with the space $L^{\frac{d}{d-1}}_{\rm loc}(\Omega, \mathcal{L}^n)$ now interpreted as $L^{\infty}_{\rm loc}(\Omega, \mathcal{L}^n)$) if the set K is assumed to have locally vanishing upper $(n-1)$-dimensional Minkowski content.

For example, if K is a singleton, say $K = \{x_o\}$ for some $x_o \in \Omega$, it follows that

$$\begin{array}{l}\text{whenever } u \in \Big[L^{\frac{n}{n-1}}_{\rm loc}(\Omega, \mathcal{L}^n)\Big]^N \text{ and } f \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^M \text{ are such that} \\ D\big(u\big|_{\Omega\setminus\{x_o\}}\big) = f\big|_{\Omega\setminus\{x_o\}} \text{ in } \big[\mathcal{D}'(\Omega\setminus\{x_o\})\big]^M \text{ then } Du = f \text{ in } \big[\mathcal{D}'(\Omega)\big]^M.\end{array} \tag{4.5.8}$$

The exponent $\frac{n}{n-1}$ in (4.5.8) is sharp. To see this consider $\Omega := B(0,1)$, the open unit ball in $\mathbb{R}^n$, $x_o := 0$, $u(x) := x/|x|^n$ for $\mathcal{L}^n$-a.e. $x \in \Omega$, $f := 0$, and $D := \operatorname{div}$ (hence $N := n$ and $M := 1$). The source of failure is that $u \notin \Big[L^{\frac{n}{n-1}}_{\rm loc}(\Omega, \mathcal{L}^n)\Big]^n$, even though we do have $u \in \Big[L^{\frac{n}{n-1},\infty}(\Omega, \mathcal{L}^n)\Big]^n$ as well as $u \in \big[L^p(\Omega, \mathcal{L}^n)\big]^n$ for each $p < \frac{n}{n-1}$.

When further specialized to $n := 2$ and $D := \bar\partial$, the Cauchy–Riemann operator in $\mathbb{R}^2 \equiv \mathbb{C}$, the result in (4.5.8) becomes a criterion for removability of isolated singularities for holomorphic functions. Typically, the function u is required to be bounded in a neighborhood of the singularity, and (4.5.8) shows that boundedness may be relaxed to square-integrability. This being said, we do stress that the first-order system D in the statement of Proposition 4.5.2 is *not* required to be elliptic.

Finally, we wish to note that related removability results (and additional bibliographical references) may be found in [116].

Here is the proof of Proposition 4.5.2.

Proof of Proposition 4.5.2 We divide the argument into two steps, the first of which deals with the special case when K is assumed to be compact, while the second treats the general case.

`Step I:` *Assume K is actually a compact subset of* Ω. For each given number $\varepsilon \in \big(0, \operatorname{dist}(K, \partial\Omega)\big)$ define $K_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, K) < \varepsilon\}$. Hence, $K \subseteq K_\varepsilon$ and there exists $\varepsilon_o > 0$ such that

$$\mathcal{L}^n(K) \le \mathcal{L}^n(K_\varepsilon) \le \big(1 + M^*_{n-d}(K)\big) \cdot \varepsilon^d \quad \text{for each} \quad \varepsilon \in (0, \varepsilon_o), \tag{4.5.9}$$

thanks to our assumption on K. In particular, (4.5.9) implies that

$$\mathcal{L}^n(K) = 0. \tag{4.5.10}$$

Let us also note that for any $\varepsilon \in \big(0, \operatorname{dist}(K, \partial\Omega)\big)$ we have $\operatorname{dist}(\mathbb{R}^n \setminus K_{2\varepsilon}, K_\varepsilon) \ge \varepsilon$. This separation property permits us to select a real-valued function $\psi_\varepsilon \in \mathscr{C}^\infty_c(\Omega)$ (the space of compactly supported functions from $\mathscr{C}^\infty(\Omega)$) satisfying, for some purely dimensional constant $C_n \in (0, \infty)$ (see [181, Proposition 14.33, p. 561]),

$$\operatorname{supp}\psi_\varepsilon \subseteq K_{2\varepsilon}, \quad 0 \le \psi_\varepsilon \le 1, \quad |\nabla\psi_\varepsilon| \le C_n/\varepsilon, \quad \text{and} \quad \psi_\varepsilon \equiv 1 \ \text{ on } \ K_\varepsilon. \tag{4.5.11}$$

In concert with (4.5.10), these ensure that

$$\lim_{\varepsilon\to 0^+}\psi_\varepsilon(x)=0 \ \text{ for } \mathcal{L}^n\text{-a.e. } \ x\in\Omega. \tag{4.5.12}$$

To proceed, pick $\varphi\in\big[\mathscr{C}_c^\infty(\Omega)\big]^M$ arbitrary and for each $\varepsilon\in\big(0,\operatorname{dist}(K,\partial\Omega)\big)$ introduce

$$\eta_\varepsilon:=[(1-\psi_\varepsilon)\varphi]\big|_{\Omega\setminus K}\in\big[\mathscr{C}_c^\infty(\Omega\setminus K)\big]^M. \tag{4.5.13}$$

In particular, if tilde denotes the extension by zero from $\Omega\setminus K$ to Ω, we have

$$\widetilde{\eta}_\varepsilon=(1-\psi_\varepsilon)\varphi. \tag{4.5.14}$$

We may then compute

$$\begin{aligned}
{}_{[\mathcal{D}'(\Omega)]^M}\Big\langle Du,\varphi\Big\rangle_{[\mathcal{D}(\Omega)]^M}&=-\,{}_{[\mathcal{D}'(\Omega)]^N}\Big\langle u,D^\top\varphi\Big\rangle_{[\mathcal{D}(\Omega)]^N}\\
&=-\,{}_{[\mathcal{D}'(\Omega)]^N}\Big\langle u,D^\top[\psi_\varepsilon\varphi]\Big\rangle_{[\mathcal{D}(\Omega)]^N}\\
&\quad-\,{}_{[\mathcal{D}'(\Omega)]^N}\Big\langle u,D^\top[(1-\psi_\varepsilon)\varphi]\Big\rangle_{[\mathcal{D}(\Omega)]^N}\\
&=:\mathrm{I}_\varepsilon+\mathrm{II}_\varepsilon.
\end{aligned} \tag{4.5.15}$$

Note that if $\varepsilon>0$ is sufficiently small (say, $0<\varepsilon<\operatorname{dist}(K,K_o)$) we may compute

$$\begin{aligned}
\mathrm{II}_\varepsilon&=-\,{}_{[\mathcal{D}'(\Omega)]^N}\big\langle u,\widetilde{D^\top\eta_\varepsilon}\big\rangle_{[\mathcal{D}(\Omega)]^N}\\
&=-\,{}_{[\mathcal{D}'(\Omega\setminus K)]^N}\Big\langle u\big|_{\Omega\setminus K},D^\top\eta_\varepsilon\Big\rangle_{[\mathcal{D}(\Omega\setminus K)]^N}\\
&={}_{[\mathcal{D}'(\Omega\setminus K)]^M}\Big\langle D\big(u\big|_{\Omega\setminus K}\big),\eta_\varepsilon\Big\rangle_{[\mathcal{D}(\Omega\setminus K)]^M}\\
&={}_{[\mathcal{D}'(\Omega\setminus K)]^M}\Big\langle f\big|_{\Omega\setminus K},\eta_\varepsilon\Big\rangle_{[\mathcal{D}(\Omega\setminus K)]^M}\\
&={}_{[\mathcal{D}'(\Omega)]^M}\Big\langle f,\widetilde{\eta}_\varepsilon\Big\rangle_{[\mathcal{D}(\Omega)]^M}={}_{[\mathcal{D}'(\Omega)]^M}\Big\langle f,(1-\psi_\varepsilon)\varphi\Big\rangle_{[\mathcal{D}(\Omega)]^M}\\
&={}_{[\mathcal{D}'(\Omega)]^M}\Big\langle f,\varphi\Big\rangle_{[\mathcal{D}(\Omega)]^M}-{}_{[\mathcal{D}'(\Omega)]^M}\Big\langle f,\psi_\varepsilon\varphi\Big\rangle_{[\mathcal{D}(\Omega)]^M}\\
&={}_{[\mathcal{D}'(\Omega)]^M}\Big\langle f,\varphi\Big\rangle_{[\mathcal{D}(\Omega)]^M}-\int_\Omega\langle f,\varphi\rangle\psi_\varepsilon\,\mathrm{d}\mathcal{L}^n.
\end{aligned} \tag{4.5.16}$$

Since, thanks to the second membership in (4.5.5), (4.5.12), and Lebesgue's Dominated Convergence Theorem,

$$\lim_{\varepsilon\to 0^+}\int_\Omega\langle f,\varphi\rangle\psi_\varepsilon\,\mathrm{d}\mathcal{L}^n=0, \tag{4.5.17}$$

we conclude that

$$\lim_{\varepsilon\to 0^+} \mathrm{II}_\varepsilon = {}_{[\mathcal{D}'(\Omega)]^M}\Big\langle f, \varphi\Big\rangle_{[\mathcal{D}(\Omega)]^M}. \tag{4.5.18}$$

As regards I_ε in (4.5.15), use the commutator law (1.7.20) to decompose

$$\mathrm{I}_\varepsilon = \mathrm{I}_\varepsilon^{(1)} + \mathrm{I}_\varepsilon^{(2)}, \tag{4.5.19}$$

where

$$\mathrm{I}_\varepsilon^{(1)} := -\, {}_{[\mathcal{D}'(\Omega)]^N}\Big\langle u, (-\mathrm{i})\mathrm{Sym}(D^\top; \nabla\psi_\varepsilon)\varphi\Big\rangle_{[\mathcal{D}(\Omega)]^N} \tag{4.5.20}$$

and

$$\mathrm{I}_\varepsilon^{(2)} := -\, {}_{[\mathcal{D}'(\Omega)]^N}\Big\langle u, \psi_\varepsilon D^\top\varphi\Big\rangle_{[\mathcal{D}(\Omega)]^N}. \tag{4.5.21}$$

If $0 < \varepsilon < \mathrm{dist}(K, K_o)$ we actually have

$$\mathrm{I}_\varepsilon^{(1)} = -\int_\Omega \big\langle u, (-\mathrm{i})\mathrm{Sym}(D^\top; \nabla\psi_\varepsilon)\varphi\big\rangle \,\mathrm{d}\mathcal{L}^n \tag{4.5.22}$$

and

$$\mathrm{I}_\varepsilon^{(2)} = -\int_\Omega \big\langle u, \psi_\varepsilon D^\top\varphi\big\rangle \,\mathrm{d}\mathcal{L}^n. \tag{4.5.23}$$

From the first membership in (4.5.5), (4.5.12), and Lebesgue's Dominated Convergence Theorem we see that

$$\lim_{\varepsilon\to 0^+} \int_\Omega \big\langle u, \psi_\varepsilon D^\top\varphi\big\rangle \,\mathrm{d}\mathcal{L}^n = 0, \tag{4.5.24}$$

hence

$$\lim_{\varepsilon\to 0^+} \mathrm{I}_\varepsilon^{(2)} = 0. \tag{4.5.25}$$

Moreover, based on (4.5.11) and Hölder's inequality we may estimate

$$\begin{aligned}
&\Big| -\int_\Omega \big\langle u\,, (-\mathrm{i})\mathrm{Sym}(D^\top; \nabla\psi_\varepsilon)\varphi\big\rangle \,\mathrm{d}\mathcal{L}^n \Big| \\
&\qquad \leq \frac{C}{\varepsilon}\int_{K_{2\varepsilon}\setminus K_\varepsilon} |u| \,\mathrm{d}\mathcal{L}^n \\
&\qquad \leq \frac{C}{\varepsilon}\Big(\int_{K_{2\varepsilon}\setminus K_\varepsilon} |u|^{\frac{d}{d-1}} \,\mathrm{d}\mathcal{L}^n\Big)^{\frac{d-1}{d}} \Big(\int_{K_{2\varepsilon}\setminus K_\varepsilon} 1 \,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{d}} \\
&\qquad = \frac{C}{\varepsilon}\Big(\int_{K_{2\varepsilon}\setminus K_\varepsilon} |u|^{\frac{d}{d-1}} \,\mathrm{d}\mathcal{L}^n\Big)^{\frac{d-1}{d}} \cdot \mathcal{L}^n\big(K_{2\varepsilon}\setminus K_\varepsilon\big)^{\frac{1}{d}}
\end{aligned} \tag{4.5.26}$$

which, in concert with (4.5.20) and (4.5.9), goes to show that

$$\begin{aligned}\big|\mathrm{I}_\varepsilon^{(1)}\big| &\leq \frac{C}{\varepsilon}\Big(\int_{K_{2\varepsilon}\setminus K_\varepsilon} |u|^{\frac{d}{d-1}}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{d-1}{d}} \cdot \big(1+M^*_{n-d}(K)\big)^{\frac{1}{d}} \cdot (2\varepsilon)\\ &= 2C\big(1+M^*_{n-d}(K)\big)^{\frac{1}{d}}\Big(\int_{K_{2\varepsilon}\setminus K_\varepsilon} |u|^{\frac{d}{d-1}}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{d-1}{d}}, \qquad (4.5.27)\end{aligned}$$

for each $\varepsilon \in (0, \varepsilon_o/2)$. From this, (4.5.10), and Lebesgue's Dominated Convergence Theorem we finally conclude that

$$\lim_{\varepsilon\to 0^+} \mathrm{I}_\varepsilon^{(1)} = 0. \qquad (4.5.28)$$

At this stage, combining (4.5.15), (4.5.18), (4.5.19), (4.5.25), and (4.5.28) yields (4.5.7), in view of the arbitrariness of $\varphi \in \big[\mathscr{C}_c^\infty(\Omega)\big]^M$.

The last claim in the statement, corresponding to $d = 1$, is proved in a similar fashion, in place of (4.5.26) now writing

$$\begin{aligned}\Big| - \int_\Omega \big\langle u\,,\,(-\mathrm{i})\mathrm{Sym}(D^\top;\nabla\psi_\varepsilon)\varphi\big\rangle\,\mathrm{d}\mathcal{L}^n\Big| &\leq \frac{C}{\varepsilon}\int_{K_{2\varepsilon}\setminus K_\varepsilon} |u|\,\mathrm{d}\mathcal{L}^n\\ &\leq \frac{C}{\varepsilon}\|u\|_{L^\infty(K_{2\varepsilon}\setminus K_\varepsilon,\mathcal{L}^n)} \cdot \mathcal{L}^n\big(K_{2\varepsilon}\setminus K_\varepsilon\big) \qquad (4.5.29)\end{aligned}$$

and using the fact that, since we are presently assuming that K has vanishing upper $(n-1)$-dimensional Minkowski content, in place of (4.5.9) we have

for each $\delta > 0$ there exists some $\varepsilon_\delta > 0$ such that $\mathcal{L}^n(K_\varepsilon) \leq \delta\cdot\varepsilon$ whenever $\varepsilon \in (0, \varepsilon_\delta)$. (4.5.30)

Then from (4.5.22), (4.5.29), and (4.5.30) we once again deduce that (4.5.28) holds, and the desired conclusion follows.

`Step II:` *Assume K is merely a relatively closed subset of* Ω. Fix an arbitrary point $x_* \in \Omega$ and select $r \in (0,\infty)$ such that $B(x_*, r) \subseteq \Omega$. In a first stage, the goal is to show that

$$D\big(u\big|_{B(x_*,r)}\big) = f\big|_{B(x_*,r)} \ \text{ in } \ \big[\mathcal{D}'\big(B(x_*,r)\big)\big]^M. \qquad (4.5.31)$$

To this end, pick a scalar-valued function $\phi \in \mathscr{C}_c^\infty(\Omega)$ with $\phi \equiv 1$ on $B(x_*, r)$, and introduce

$$\begin{aligned} w &:= \phi u \in \Big[L^{\frac{d}{d-1}}(\Omega,\mathcal{L}^n) + \mathscr{E}'_{K_o}(\Omega)\Big]^N,\\ g &:= \phi f + (-\mathrm{i})\mathrm{Sym}(D;\nabla\phi)u \in \Big[L^1(\Omega,\mathcal{L}^n) + \mathscr{E}'_{K_o}(\Omega)\Big]^M. \end{aligned} \qquad (4.5.32)$$

In addition, consider the compact set

$$K_* := K \cap \operatorname{supp} \phi \subseteq \Omega, \tag{4.5.33}$$

and note that $K_o \subseteq \Omega \setminus K_*$. From (4.5.6) and (4.5.32)–(4.5.33) we see that

$$D\big(w\big|_{\Omega\setminus K_*}\big) = g\big|_{\Omega\setminus K_*} \;\text{ in }\; \big[\mathcal{D}'(\Omega \setminus K_*)\big]^M. \tag{4.5.34}$$

If $K_* = \varnothing$ then (4.5.34) readily implies (4.5.31) by further restricting to $B(x_*, r)$ and keeping in mind that ϕ is identically one there. Henceforth assume $K_* \neq \varnothing$. In this scenario, K_* has finite upper $(n-d)$-dimensional Minkowski content. Moreover, if K is assumed to have locally vanishing upper $(n-1)$-dimensional Minkowski content, then K_* has vanishing upper $(n-d)$-dimensional Minkowski content. As such, we may use the result proved in Step I (with u, f, K replaced by w, g, K_*) to conclude that

$$Dw = g \;\text{ in }\; \big[\mathcal{D}'(\Omega)\big]^M. \tag{4.5.35}$$

From this, (4.5.31) once again readily follows. Thus, (4.5.31) is established in all cases. With this in hand, we may then conclude that (4.5.7) holds since, as is well known, the equality of distributions has a local character (cf., e.g., [181, Proposition 2.52, p. 38]). □

Moving on, recall that $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ denote, respectively, the space of Schwartz functions and the space of tempered distributions in $\mathbb{R}^n$.

For each $t \in (0, \infty)$ define the operator of dilation by a factor of t, i.e.,

$$\tau_t : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n), \quad (\tau_t\varphi)(x) := \varphi(tx), \quad \forall \varphi \in \mathscr{S}(\mathbb{R}^n),\ \forall x \in \mathbb{R}^n, \tag{4.5.36}$$

and

$$\begin{aligned} \tau_t : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n), \quad \langle \tau_t u, \varphi\rangle := t^{-n}\langle u, \tau_{\frac{1}{t}}\varphi\rangle, \\ \text{for all } \; u \in \mathscr{S}'(\mathbb{R}^n) \;\text{ and }\; \varphi \in \mathscr{S}(\mathbb{R}^n). \end{aligned} \tag{4.5.37}$$

Then (see, e.g., [181, Exercise 4.46, p. 141]) one has

$$\mathcal{F}(\tau_t u) = t^{-n}\tau_{\frac{1}{t}}\mathcal{F}(u) \;\text{ in }\; \mathcal{S}'(\mathbb{R}^n), \quad \forall u \in \mathcal{S}'(\mathbb{R}^n), \tag{4.5.38}$$

where $\mathcal{F}$ denotes the Fourier transform acting on tempered distributions in $\mathbb{R}^n$.

Call a tempered distribution $u \in \mathscr{S}'(\mathbb{R}^n)$ `positive homogeneous of degree` $m \in \mathbb{R}$ provided

$$\tau_t u = t^m u \;\text{ in }\; \mathscr{S}'(\mathbb{R}^n) \;\text{ for every }\; t \in (0, \infty). \tag{4.5.39}$$

For example,

$$\text{the Dirac delta function } \delta \in \mathscr{S}'(\mathbb{R}^n) \text{ is a positive homogeneous distribution of degree } -n. \tag{4.5.40}$$

It is also clear from definitions that

if $u \in \mathscr{S}'(\mathbb{R}^n)$ is a positive homogeneous distribution having the degree $m \in \mathbb{R}$, then for each multi-index $\alpha \in \mathbb{N}_0^n$ it follows that $\partial^\alpha u$ is a positive homogeneous distribution of degree $m - |\alpha|$ (4.5.41)

and that

if $u \in \mathscr{S}'(\mathbb{R}^n)$ is a positive homogeneous distribution of degree $m \in \mathbb{R}$, then its Fourier transform, $\mathcal{F}u$, is a positive homogeneous distribution of degree $-m-n$. (4.5.42)

Lemma 4.5.3 *Suppose $u_1, u_2, \dots, u_N \in \mathscr{S}'(\mathbb{R}^n)$ are tempered distributions such that u_j is positive homogeneous of degree m_j for each index $j \in \{1, \dots, N\}$, where $-\infty < m_1 < m_2 < \cdots < m_N < +\infty$. In addition, assume $u_1 + \cdots + u_N = 0$ in $\mathscr{S}'(\mathbb{R}^n)$. Then $u_1 = u_2 = \cdots = u_N = 0$ in $\mathscr{S}'(\mathbb{R}^n)$.*

Proof Fix a Schwartz function $\varphi \in \mathscr{S}(\mathbb{R}^n)$ along with some $t \in (0, \infty)$. Then

$$\begin{aligned} 0 &= t^{-n}\Big\langle \sum_{j=1}^N u_j, \tau_{\frac{1}{t}}\varphi \Big\rangle = \Big\langle \tau_t\Big(\sum_{j=1}^N u_j\Big), \varphi \Big\rangle \\ &= \sum_{j=1}^N \big\langle \tau_t u_j, \varphi \big\rangle = \sum_{j=1}^N t^{m_j}\big\langle u_j, \varphi \big\rangle. \end{aligned} \tag{4.5.43}$$

The fact the latter expression vanishes for each $t \in (0, \infty)$ forces (in view of the fact that we are assuming $-\infty < m_1 < m_2 < \cdots < m_N < +\infty$) that $\langle u_j, \varphi \rangle = 0$ for each $j \in \{1, \dots, N\}$. Hence, $u_1 = u_2 = \cdots = u_N = 0$ in $\mathscr{S}'(\mathbb{R}^n)$, as wanted. □

We may now use Lemma 4.5.3 to establish the following brand of linear independence result for tempered distributions.

Proposition 4.5.4 *Suppose $u_1, u_2, \dots, u_N \in \mathscr{S}'(\mathbb{R}^n)$ are tempered distributions such that each u_j is positive homogeneous of degree m_j for $1 \leq j \leq N$, where $-n < m_1 < m_2 < \cdots < m_N < +\infty$. If* $\operatorname{supp}(u_1 + \cdots + u_N) \subseteq \{0\}$ *then necessarily $u_1 = u_2 = \cdots = u_N = 0$ in $\mathscr{S}'(\mathbb{R}^n)$.*

Proof The fact that $\operatorname{supp}(u_1 + \cdots + u_N) \subseteq \{0\}$ implies that $u_1 + \cdots + u_N$ is a linear combination of derivatives of the Dirac distribution δ, say

$$u_1 + \cdots + u_N = \sum_{|\alpha| \leq M} c_\alpha \partial^\alpha \delta \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^n), \tag{4.5.44}$$

for some $M \in \mathbb{N}_0$ and some coefficients $c_\alpha \in \mathbb{C}$ with $|\alpha| \leq M$. Consequently,

$$u_1 + \cdots + u_N - \sum_{|\alpha| \leq M} c_\alpha \partial^\alpha \delta = 0 \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^n), \tag{4.5.45}$$

and since $m_j > -n$ for each $j \in \{1, \dots, N\}$, Lemma 4.5.3 applies (thanks to assumptions and (4.5.40)–(4.5.41)) and gives that, in fact, $u_1 = u_2 = \dots = u_N = 0$ in $\mathscr{S}'(\mathbb{R}^n)$. □

Here is a useful consequence of Proposition 4.5.4.

Lemma 4.5.5 *Suppose $f \in L^1_{\rm loc}(\mathbb{R}^n, \mathcal{L}^n)$ is positive homogeneous of degree k for some $k \in \mathbb{R}$ with $k > -n$ and there exists $C \in (0, \infty)$ such that $|f(x)| \leq C|x|^k$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Also, suppose $u \in \mathscr{S}'(\mathbb{R}^n)$ is positive homogeneous of degree k and satisfies $u\big|_{\mathbb{R}^n\setminus\{0\}} = f$. Then integration of Schwartz functions against f in $\mathbb{R}^n$ induces a tempered distribution u_f, and $u = u_f$ in $\mathscr{S}'(\mathbb{R}^n)$.*

Proof That integration against f induces a tempered distribution u_f is clear from the properties of f. Moreover, the tempered distribution u_f is positive homogeneous of degree k. Since $u\big|_{\mathbb{R}^n\setminus\{0\}} = u_f\big|_{\mathbb{R}^n\setminus\{0\}}$ in $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$, we have $\operatorname{supp}(u - u_f) \subseteq \{0\}$. Then Proposition 4.5.4 applies and gives that $u - u_f = 0$ in $\mathscr{S}'(\mathbb{R}^n)$. □

We conclude by computing the divergence, in the sense of distributions, of vector-valued functions which are smooth and positive homogeneous of degree $1 - n$ in $\mathbb{R}^n \setminus \{0\}$.

Proposition 4.5.6 *Consider a vector-valued function $\vec{k} \in \big[\mathscr{C}^1(\mathbb{R}^n \setminus \{0\})\big]^n$ which is positive homogeneous of degree $1 - n$.*

Then $\vec{k}$ induces a tempered distribution via integration against Schwartz functions in $\mathbb{R}^n$. Also, $\operatorname{div}\vec{k}$ is a continuous function which is positive homogeneous of degree $-n$ in $\mathbb{R}^n \setminus \{0\}$, and integrates to zero on S^{n-1}; in particular, it induces a principal-value distribution $\mathrm{P.V.}\big(\operatorname{div}\vec{k}\big)$ in $\mathbb{R}^n$ (see [181, (4.4.2), p. 148] *and* [181, Example 4.71, p. 51]*). Finally, with δ denoting Dirac's distribution with mass at the origin in $\mathbb{R}^n$, one has*

$$\operatorname{div}\vec{k} = \Big(\int_{S^{n-1}} \big\langle \omega, \vec{k}(\omega)\big\rangle \, d\mathcal{H}^{n-1}(\omega)\Big)\delta + \mathrm{P.V.}\big(\operatorname{div}\vec{k}\big) \quad in \quad \mathscr{S}'(\mathbb{R}^n). \tag{4.5.46}$$

As a corollary, if $k \in \mathscr{C}^1(\mathbb{R}^n \setminus \{0\})$ is a scalar-valued function which is positive homogeneous of degree $-n$, then for each $j \in \{1, \dots, n\}$ the function $x_j k(x)$ induces a tempered distribution via integration against Schwartz functions in $\mathbb{R}^n$ and

$$\sum_{j=1}^{n} \partial_j\big[x_j k(x)\big] = \Big(\int_{S^{n-1}} k \, d\mathcal{H}^{n-1}\Big)\delta \quad in \quad \mathscr{S}'(\mathbb{R}^n). \tag{4.5.47}$$

Proof That the vector-valued function $\vec{k}$ defines a tempered distribution is a well-known fact; see [181, Exercise 4.54, p. 142]. Also, [181, Theorem 4.27, p. 152] yields (4.5.46). As regards the final portion in the statement, it suffices to observe that since k is positive homogeneous of degree $-n$ in $\mathbb{R}^n \setminus \{0\}$, Euler's formula gives

$$x \cdot (\nabla k)(x) = -nk(x), \quad \text{for all} \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{4.5.48}$$

hence

$$\sum_{j=1}^{n} \partial_j\big[x_j k(x)\big] = nk(x) + \sum_{j=1}^{n} x_j (\partial_j k)(x) = 0 \ \text{ pointwise in } \ \mathbb{R}^n \setminus \{0\}. \qquad (4.5.49)$$

With this in hand, (4.5.47) follows from (4.5.46) applied to $\vec{k}(x) := xk(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$. □

4.6 The Algebraic Dual of the Space of Smooth and Bounded Functions

Consider a nonempty open set $\Omega \subseteq \mathbb{R}^n$. Throughout, we let $\mathcal{D}'(\Omega)$ denote the space of distributions in the set Ω. For some $M \in \mathbb{N}$, when working with vector distributions $u = (u_1, \dots, u_M) \in [\mathcal{D}'(\Omega)]^M$, the distributional pairing with a vector-valued test function $\varphi = (\varphi_1, \dots, \varphi_n) \in [\mathscr{C}_c^\infty(\Omega)]^M$ is defined as

$$_{[\mathcal{D}'(\Omega)]^M}\langle u, \varphi\rangle_{[\mathcal{D}(\Omega)]^M} := {}_{\mathcal{D}'(\Omega)}\langle u_1, \varphi_1\rangle_{\mathcal{D}(\Omega)} + \cdots + {}_{\mathcal{D}'(\Omega)}\langle u_M, \varphi_M\rangle_{\mathcal{D}(\Omega)}. \qquad (4.6.1)$$

As is customary, the notation $\mathscr{E}(\Omega)$ is reserved for the space $\mathscr{C}^\infty(\Omega)$ equipped with the topology of uniform convergence on compact subsets of Ω for partial derivatives of any given order. This becomes a Frechét space, and we denote by $\mathscr{E}'(\Omega)$ its topological dual, endowed with the weak-$*$ topology. Then $\mathscr{E}'(\Omega)$ may be identified with the subspace of $\mathcal{D}'(\Omega)$ consisting of distributions which are compactly supported in Ω. In particular, if for each compact set $K \subset \Omega$ we define

$$\mathscr{E}'_K(\Omega) := \big\{u \in \mathcal{D}'(\Omega) : \operatorname{supp} u \subseteq K\big\}, \qquad (4.6.2)$$

then

$$\mathscr{E}'(\Omega) = \bigcup_{\substack{K \text{ compact} \\ \text{subset of } \Omega}} \mathscr{E}'_K(\Omega). \qquad (4.6.3)$$

Given a nonempty open set $\Omega \subseteq \mathbb{R}^n$, we agree to denote by $\mathrm{CBM}(\Omega)$) the collection of all complex Borel measures on the set Ω (equipped with the relative topology induced by the Euclidean ambient). In this vein, we wish to note that

within $\mathrm{CBM}(\Omega)$, viewed as a subspace of $\mathcal{D}'(\Omega)$, the restrictions to an open subset $\mathcal{O}$ of Ω taken, respectively, in a distributional sense and in the sense of measures, actually coincide. (4.6.4)

In other words, given any $\mu \in \mathrm{CBM}(\Omega) \subset \mathcal{D}'(\Omega)$, the distributional restriction $\mu\big|_{\mathcal{O}}$ coincides with the measure $\mu\lfloor\mathcal{O}$. Indeed, for every $\varphi \in \mathscr{C}_c^\infty(\mathcal{O})$, if $\widetilde{\varphi} \in \mathscr{C}_c^\infty(\Omega)$ denotes the extension of φ by zero outside $\mathcal{O}$, we have

$$
\begin{aligned}
{}_{\mathcal{D}'(O)}\big\langle \mu\big|_{O}, \varphi\big\rangle_{\mathcal{D}(O)} &= {}_{\mathcal{D}'(\Omega)}\big\langle \mu, \widetilde{\varphi}\big\rangle_{\mathcal{D}(\Omega)} = \int_{\Omega} \widetilde{\varphi}\,\mathrm{d}\mu = \int_{O} \varphi\,\mathrm{d}\mu \\
&= \int_{O} \varphi\,\mathrm{d}\big(\mu\lfloor O\big) = {}_{\mathcal{D}'(O)}\big\langle \mu\lfloor O, \varphi\big\rangle_{\mathcal{D}(O)}
\end{aligned}
\tag{4.6.5}
$$

where the penultimate equality is justified by observing that it holds when φ is a simple function, then using a standard density argument.

We continue to assume that $\Omega \subseteq \mathbb{R}^n$ is a nonempty open set. Also, as in the past, $\mathrm{CBM}(\Omega)$ denotes the collection of all complex Borel measures on Ω. We then have the injective embeddings

$$
\mathscr{E}'(\Omega) \hookrightarrow \mathcal{D}'(\Omega) \quad\text{and}\quad L^1(\Omega, \mathcal{L}^n) \hookrightarrow \mathrm{CBM}(\Omega) \hookrightarrow \mathcal{D}'(\Omega). \tag{4.6.6}
$$

In view of these embeddings, it makes sense to consider the subspace of $\mathcal{D}'(\Omega)$ defined as follows:

$$
\begin{aligned}
\mathscr{E}'(\Omega) + \mathrm{CBM}(\Omega) := \big\{ u \in \mathcal{D}'(\Omega) :\ &\text{there exist } w \in \mathscr{E}'(\Omega) \text{ and } \mu \in \mathrm{CBM}(\Omega) \\
&\text{such that } u = w + \mu \text{ in } \mathcal{D}'(\Omega)\big\}.
\end{aligned}
\tag{4.6.7}
$$

Also, introduce the space of smooth, bounded, complex-valued functions in Ω, i.e.,

$$
\mathscr{C}_b^\infty(\Omega) := \big\{ f \in \mathscr{C}^\infty(\Omega) : f \text{ bounded in } \Omega \big\}, \tag{4.6.8}
$$

and denote by $\big(\mathscr{C}_b^\infty(\Omega)\big)^*$ the algebraic dual of this linear space. Throughout, we agree to use ${}_{X^*}\big(\cdot,\cdot\big)_X$ to denote the duality pairing between a linear space X and its algebraic dual X^*.

Lemma 4.6.1 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and consider $u \in \mathcal{D}'(\Omega)$. Suppose $\Lambda \in \big(\mathscr{C}_b^\infty(\Omega)\big)^*$ is a functional which is continuous in the sense of* (1.5.1)–(1.5.2) *and which extends the distribution u, in the sense that*

$$
\Lambda\big|_{\mathscr{C}_c^\infty(\Omega)} = u. \tag{4.6.9}
$$

Then

$$
\begin{aligned}
&{}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\Lambda, f\big)_{\mathscr{C}_b^\infty(\Omega)} = 0 \ \text{ for each function} \\
&f \in \mathscr{C}_b^\infty(\Omega) \ \text{ satisfying } \ f \equiv 0 \ \text{ near } \ \mathrm{supp}\, u.
\end{aligned}
\tag{4.6.10}
$$

In particular, corresponding to the case when $u = 0$, one has

$$
\left.\begin{aligned}
&\Lambda \in \big(\mathscr{C}_b^\infty(\Omega)\big)^* \ \text{ continuous} \\
&\text{and such that } \ \Lambda\big|_{\mathscr{C}_c^\infty(\Omega)} = 0
\end{aligned}\right\} \Longrightarrow \Lambda \equiv 0 \ \text{ as a functional on } \ \mathscr{C}_b^\infty(\Omega).
\tag{4.6.11}
$$

Proof Bring in the sequence $\{\Phi_\varepsilon\}_{\varepsilon>0}$ of functions constructed in Lemma 6.1.2 for the set Ω. Also, fix some $\Psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ with $\Psi \equiv 1$ near the origin in $\mathbb{R}^n$ and, for each $\varepsilon > 0$, define $\Psi_\varepsilon(x) := \Psi(x/\varepsilon)$ for every $x \in \mathbb{R}^n$. Then for each $f \in \mathscr{C}_b^\infty(\Omega)$ satisfying $f \equiv 0$ near $\operatorname{supp} u$ we may write

$$
\begin{aligned}
{}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\Lambda, f\big)_{\mathscr{C}_b^\infty(\Omega)} &= \lim_{\varepsilon\to 0^+} {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\Lambda, \Phi_\varepsilon\Psi_\varepsilon f\big)_{\mathscr{C}_b^\infty(\Omega)} \\
&= \lim_{\varepsilon\to 0^+} {}_{\mathcal{D}'(\Omega)}\big\langle u, \Phi_\varepsilon\Psi_\varepsilon f\big\rangle_{\mathcal{D}(\Omega)} = 0, \qquad (4.6.12)
\end{aligned}
$$

thanks to the continuity of Λ, the extension condition recorded in (4.6.9), and the fact that, for each $\varepsilon > 0$, the function $\Phi_\varepsilon\Psi_\varepsilon f$ belongs to $\mathscr{C}_c^\infty(\Omega)$ and vanishes near the support of u. □

For an arbitrary open set $\Omega \subseteq \mathbb{R}^n$, compactly supported distributions in Ω along with complex Borel measures in Ω are prime examples of functionals in the dual space $(\mathscr{C}_b^\infty(\Omega))^*$.

Lemma 4.6.2 *Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary nonempty open set. Given an arbitrary distribution $u \in \mathscr{E}'(\Omega) + \mathrm{CBM}(\Omega) \subset \mathcal{D}'(\Omega)$, pick $w \in \mathscr{E}'(\Omega)$ and $\mu \in \mathrm{CBM}(\Omega)$ such that $u = w + \mu$ in $\mathcal{D}'(\Omega)$, and define the functional $\Lambda_u \in \big(\mathscr{C}_b^\infty(\Omega)\big)^*$ by*

$$
{}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\Lambda_u, f\big)_{\mathscr{C}_b^\infty(\Omega)} := {}_{\mathscr{E}'(\Omega)}\langle w, f\rangle_{\mathscr{E}(\Omega)} + \int_\Omega f \,\mathrm{d}\mu, \quad \forall f \in \mathscr{C}_b^\infty(\Omega). \qquad (4.6.13)
$$

Then

$$
\begin{gathered}
\mathscr{E}'(\Omega) + \mathrm{CBM}(\Omega) \ni u \longmapsto \Lambda_u \in \big(\mathscr{C}_b^\infty(\Omega)\big)^* \\
\text{is an unambiguous assignment}
\end{gathered} \qquad (4.6.14)
$$

which induces an injective embedding

$$
\mathscr{E}'(\Omega) + \mathrm{CBM}(\Omega) \hookrightarrow \big(\mathscr{C}_b^\infty(\Omega)\big)^*. \qquad (4.6.15)
$$

Moreover, Λ_u is the unique extension of the original distribution u to a continuous functional in $\big(\mathscr{C}_b^\infty(\Omega)\big)^*$ (in the sense of (1.5.1)–(1.5.3)).

Proof The fact that the assignment $u \mapsto \Lambda_u$ just defined is unambiguous follows by observing that whenever $w \in \mathscr{E}'_K(\Omega)$ for some compact set $K \subset \Omega$ and $\mu \in \mathrm{CBM}(\Omega)$ are such that $w + \mu = 0$ in $\mathcal{D}'(\Omega)$ then

$$
{}_{\mathscr{E}'(\Omega)}\langle w, f\rangle_{\mathscr{E}(\Omega)} + \int_\Omega f \,\mathrm{d}\mu = 0, \qquad \forall f \in \mathscr{C}_b^\infty(\Omega). \qquad (4.6.16)
$$

Indeed, the hypotheses on w, μ imply that $\mu\big|_{\Omega\setminus K} = -w\big|_{\Omega\setminus K} = 0$ in $\mathcal{D}'(\Omega \setminus K)$. Bearing in mind (4.6.4), we then conclude that the measure μ is supported in K. Consequently, if $\varphi \in \mathscr{C}_c^\infty(\Omega)$ is such that $\varphi \equiv 1$ near K then $(1-\varphi)\mu = 0$ as a complex Borel measure in Ω. Since $w = -\mu$ in $\mathcal{D}'(\Omega)$, for each $f \in \mathscr{C}_b^\infty(\mathbb{R}^n)$ we may then write

$$ {}_{\mathscr{E}'(\Omega)}\langle w, f\rangle_{\mathscr{E}(\Omega)} = {}_{\mathcal{D}'(\Omega)}\big\langle w, \varphi f\big\rangle_{\mathcal{D}(\Omega)} = -\int_\Omega \varphi f \, \mathrm{d}\mu = -\int_\Omega f \, \mathrm{d}\mu, \tag{4.6.17} $$

from which the claim in (4.6.16) follows. Having established the well-definiteness of the assignment $u \mapsto \Lambda_u$, the injectivity of the mapping

$$ \mathscr{E}'(\Omega) + \mathrm{CBM}(\Omega) \ni u \mapsto \Lambda_u \in \big(\mathscr{C}_b^\infty(\Omega)\big)^* \tag{4.6.18} $$

follows from (4.6.13), (4.6.6), and the inclusion $\mathscr{C}_c^\infty(\Omega) \subset \mathscr{C}_b^\infty(\Omega)$.

Going further, the continuity of the functional $\Lambda_u \in \big(\mathscr{C}_b^\infty(\Omega)\big)^*$ (in the sense of (1.5.1)–(1.5.2)) is clear from (4.6.13), the manner in which the topology on $\mathscr{E}(\Omega)$ is defined (cf., e.g., [181, Sect. 14.1.0.1, p. 549]), and Lebesgue's Dominated Convergence Theorem. Lastly, that $\Lambda_u \in \big(\mathscr{C}_b^\infty(\Omega)\big)^*$ is an extension of the original distribution u (in the sense of (1.5.3)) is readily implied by (4.6.13) and (1.3.12), while its uniqueness among all functionals in $\big(\mathscr{C}_b^\infty(\Omega)\big)^*$ with these qualities is seen from Lemma 4.6.1 and linearity. □

In the context of Lemma 4.6.2, since the assignment $u \mapsto \Lambda_u$ is injective, in the sequel we may identify $u \equiv \Lambda_u$ and, in place of (4.6.13), simply write, in an unambiguous fashion,

$$ \begin{gathered} {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(u, f\big)_{\mathscr{C}_b^\infty(\Omega)} = {}_{\mathscr{E}'(\Omega)}\langle w, f\rangle_{\mathscr{E}(\Omega)} + \int_\Omega f \, \mathrm{d}\mu, \ \text{ for every } f \in \mathscr{C}_b^\infty(\Omega), \\ \text{if } u \in \mathcal{D}'(\Omega),\, w \in \mathscr{E}'(\Omega), \text{ and } \mu \in \mathrm{CBM}(\Omega) \text{ satisfy } u = w + \mu \text{ in } \mathcal{D}'(\Omega). \end{gathered} \tag{4.6.19} $$

This implies that we have the injective embeddings

$$ \mathscr{E}'(\Omega) + L^1(\Omega, \mathcal{L}^n) \hookrightarrow \mathscr{E}'(\Omega) + \mathrm{CBM}(\Omega) \hookrightarrow \big(\mathscr{C}_b^\infty(\Omega)\big)^* \tag{4.6.20} $$

as well as the compatibility properties

$$ \begin{gathered} {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(u, \varphi\big)_{\mathscr{C}_b^\infty(\Omega)} = {}_{\mathscr{E}'(\Omega)}\langle u, \varphi\rangle_{\mathscr{E}(\Omega)} \ \text{ whenever} \\ u \in \mathscr{E}'(\Omega) \subset \big(\mathscr{C}_b^\infty(\Omega)\big)^* \ \text{ and } \ \varphi \in \mathscr{C}_b^\infty(\Omega) \subset \mathscr{E}(\Omega), \end{gathered} \tag{4.6.21} $$

and

$$ \begin{gathered} {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mu, \varphi\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_\Omega \varphi \, \mathrm{d}\mu \ \text{ whenever} \\ \mu \in \mathrm{CBM}(\Omega) \subset \big(\mathscr{C}_b^\infty(\Omega)\big)^* \ \text{ and } \ \varphi \in \mathscr{C}_b^\infty(\Omega) \subset L^\infty(\Omega, \mu). \end{gathered} \tag{4.6.22} $$

In particular,

$$
\begin{aligned}
&{}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(f,\varphi\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_\Omega f\varphi\,\mathrm{d}\mathcal{L}^n \quad \text{provided} \quad \varphi \in \mathscr{C}_b^\infty(\Omega) \\
&\quad \text{and} \quad L^1(\Omega,\mathcal{L}^n) \ni f \equiv f\mathcal{L}^n \in \mathrm{CBM}(\Omega) \subset \big(\mathscr{C}_b^\infty(\Omega)\big)^*,
\end{aligned}
\tag{4.6.23}
$$

and

$$
{}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mu,1\big)_{\mathscr{C}_b^\infty(\Omega)} = \mu(\Omega) \quad \text{if} \quad \mu \in \mathrm{CBM}(\Omega) \subset \big(\mathscr{C}_b^\infty(\Omega)\big)^*. \tag{4.6.24}
$$

Finally, as a consequence of (4.6.16) (or (4.6.21) and (4.6.24)), we have

$$
{}_{\mathscr{E}'(\Omega)}\langle \omega, 1\rangle_{\mathscr{E}(\Omega)} = \omega(\Omega) \quad \text{for each} \quad \omega \in \mathscr{E}'(\Omega) \cap \mathrm{CBM}(\Omega). \tag{4.6.25}
$$

4.7 The Contribution at Infinity of a Vector Field

Fix $n \in \mathbb{N}$. Let Ω be an open subset of $\mathbb{R}^n$ and consider a vector field $\vec{F} \in \big[\mathcal{D}'(\Omega)\big]^n$ with the property that there exists $R_o \in (0,\infty)$ such that

$$
\vec{F}\big|_{\Omega\cap[B(0,R)\setminus B(0,R_o)]} \in \big[L^1\big(\Omega \cap [B(0,R)\setminus B(0,R_o)],\mathcal{L}^n\big)\big]^n \quad \text{for all} \quad R > R_o. \tag{4.7.1}
$$

In this setting, recall from (1.3.2)–(1.3.3) that, whenever meaningfully and unambiguously defined, the contribution of $\vec{F}$ at infinity is given by

$$
\begin{aligned}
&[\vec{F}]_\infty = -\lim_{R\to\infty}\int_\Omega \nabla\phi_R \cdot \vec{F}\,\mathrm{d}\mathcal{L}^n \quad \text{for any} \\
&\text{system of auxiliary functions } \{\phi_R\}_{R>0} \text{ as in (1.3.3).}
\end{aligned}
\tag{4.7.2}
$$

Note that, by design, the contribution at infinity is entirely determined by how the vector field behaves in the complement of an arbitrarily large ball, i.e.,

$$
[\vec{F}]_\infty = \Big[\vec{F}\big|_{\Omega\setminus B(0,R)}\Big]_\infty \quad \text{for each} \quad R > R_o, \tag{4.7.3}
$$

in the sense that whenever one side happens to be meaningfully and unambiguously defined then so is the other side and the two quantities are equal.

Our first goal here is to offer alternative descriptions of the contribution at infinity in the case when the vector in question is also assumed to be continuous. A concrete result in this regard, formulated for domains with a bounded complement, reads as follows.

Proposition 4.7.1 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set with bounded complement. Consider a vector field $\vec{F} \in \big[\mathscr{C}^0(\Omega)\big]^n$ such that* $\mathrm{div}\vec{F} \in \mathrm{CBM}(\Omega)$. *Lastly, consider a family $\{D_j\}_{j\in\mathbb{N}}$ of bounded closed subsets of $\mathbb{R}^n$ of locally finite perimeter with the property that*

$$\begin{aligned}&\textit{for each compact set } K\subset\mathbb{R}^n \textit{ there exists}\\ &j_K\in\mathbb{N} \textit{ such that } \; K\subseteq D_j \;\; \textit{whenever} \;\; j>j_K.\end{aligned}\tag{4.7.4}$$

Then the contribution of $\vec{F}$ at infinity is meaningfully and unambiguously defined, and may actually be expressed as the limit

$$[\vec{F}]_\infty=\lim_{j\to\infty}\int_{\partial_* D_j}\nu_j\cdot\big(\vec{F}\big|_{\partial_* D_j}\big)\,\mathrm{d}\mathcal{H}^{n-1}\tag{4.7.5}$$

where ν_j is the geometric measure theoretic outward unit normal to D_j.

In the context of Proposition 4.7.1,

> if there exists a compact set $K_0\subseteq\mathbb{R}^n$ such that $\operatorname{div}\vec{F}=0$ in $\Omega\setminus K_0$, then there exists some $j_0\in\mathbb{N}$ such that $\int_{\partial_* D_j}\nu_j\cdot\big(\vec{F}\big|_{\partial_* D_j}\big)\,\mathrm{d}\mathcal{H}^{n-1}$ is independent of $j\geq j_0$. (4.7.6)

This follows from a suitable application of Corollary 2.8.7. Specifically, pick an origin centered open ball B_0 containing $K_0\cup(\mathbb{R}^n\setminus\Omega)$ and assume $j\in\mathbb{N}$ is large enough so that $B_0\subseteq D_j$. Also, select an origin centered open ball B_j containing D_j, and introduce $\mathcal{O}:=\Omega\setminus K_0$. Then Corollary 2.8.7 applied first with $\Omega:=B_j\setminus D_j$ yields $\int_{\partial_* D_j}\nu_j\cdot\big(\vec{F}\big|_{\partial_* D_j}\big)\,\mathrm{d}\mathcal{H}^{n-1}=\int_{\partial B_j}(x/|x|)\cdot\vec{F}(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)$. A second application of Corollary 2.8.7 with $\Omega:=B_j\setminus\overline{B_0}$ implies

$$\int_{\partial B_j}(x/|x|)\cdot\vec{F}(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)=\int_{\partial B_0}(x/|x|)\cdot\vec{F}(x)\,\mathrm{d}\mathcal{H}^{n-1}(x).\tag{4.7.7}$$

Ultimately,

$$\int_{\partial_* D_j}\nu_j\cdot\big(\vec{F}\big|_{\partial_* D_j}\big)\,\mathrm{d}\mathcal{H}^{n-1}=\int_{\partial B_0}(x/|x|)\cdot\vec{F}(x)\,\mathrm{d}\mathcal{H}^{n-1}(x)\tag{4.7.8}$$

and the desired conclusion follows.

Proof of Proposition 4.7.1 In light of the conclusions we presently seek, there is no loss of generality in assuming that

$$\vec{F}\text{ vanishes identically near }\partial\Omega.\tag{4.7.9}$$

Indeed, thanks to (4.7.3), this may be arranged by considering $(1-\psi)\vec{F}$ in place of $\vec{F}$, where $\psi\in\mathscr{C}_c^\infty(\mathbb{R}^n)$ is such that $\psi\equiv 1$ in a large ball containing $\partial\Omega$.

To begin in earnest, fix an arbitrary system of auxiliary functions, that is, a family of scalar functions $\{\phi_R\}_{R>0}\subseteq\mathscr{C}_c^\infty(\mathbb{R}^n)$ satisfying

$$\begin{aligned}&\sup_{R>0}\sup_{x\in\mathbb{R}^n}|\phi_R(x)|<+\infty \text{ and for each compact set } K\subset\mathbb{R}^n\\ &\text{there exists } R_K>0 \text{ such that } \phi_R\equiv 1 \text{ on } K \text{ whenever } R>R_K.\end{aligned} \tag{4.7.10}$$

Fix $\rho\in(0,\infty)$ so that $\partial\Omega\subseteq B(0,\rho)$ and pick $j_\rho\in\mathbb{N}$ large enough so that $B(0,\rho)\subseteq D_j$ for each $j\geq j_\rho$. Next, for each $j\geq j_\rho$ select $R_j\in(0,\infty)$ so that $\phi_R\equiv 1$ near D_j whenever $R>R_j$. Abbreviate

$$\mu_{\vec F}:=\mathrm{div}\vec F\in \mathrm{CBM}(\Omega) \tag{4.7.11}$$

and, having fixed $j\geq j_\rho$, for $R\in(R_j,\infty)$ approaching infinity we may compute

$$\begin{aligned}\lim_{R\to\infty}\int_\Omega\nabla\phi_R\cdot\vec F\,\mathrm{d}\mathcal{L}^n&=\lim_{R\to\infty}\int_{\Omega\setminus D_j}\nabla\phi_R\cdot\vec F\,\mathrm{d}\mathcal{L}^n\\ &=\lim_{R\to\infty}{}_{(\mathscr{C}_b^\infty(\Omega\setminus D_j))^*}\big(\nabla\phi_R\cdot\vec F,1\big)_{\mathscr{C}_b^\infty(\Omega\setminus D_j)}=\mathrm{I}+\mathrm{II},\end{aligned} \tag{4.7.12}$$

where

$$\mathrm{I}:=\lim_{R\to\infty}{}_{(\mathscr{C}_b^\infty(\Omega\setminus D_j))^*}\Big(\mathrm{div}(\phi_R\,\vec F)\big|_{\Omega\setminus D_j},1\Big)_{\mathscr{C}_b^\infty(\Omega\setminus D_j)}, \tag{4.7.13}$$

$$\mathrm{II}:=-\lim_{R\to\infty}{}_{(\mathscr{C}_b^\infty(\Omega\setminus D_j))^*}\Big(\big(\phi_R\,\mathrm{div}\vec F\big)\big|_{\Omega\setminus D_j},1\Big)_{\mathscr{C}_b^\infty(\Omega\setminus D_j)}. \tag{4.7.14}$$

Above, the first equality in (4.7.12) uses the fact that $\nabla\phi_R\equiv 0$ near D_j, since $R>R_j$. The second equality in (4.7.12) is a consequence of the compatibility condition (4.6.21), while the third equality in (4.7.12) relies on the product rule.

The idea is now to invoke (2.8.35) in Proposition 2.8.6 applied with Ω replaced by $\mathbb{R}^n\setminus D_j$, the ambient set O replaced by the current Ω, and with the vector field $\vec F$ replaced by $\phi_R\,\vec F$ regarded as a vector field in $\big[\mathscr{C}_c^0(\Omega)\big]^n$ (cf. (4.7.9)). Since $\mathrm{div}(\phi_R\,\vec F)=\phi_R\,\mathrm{div}\vec F+\nabla\phi_R\cdot\vec F$, the role of the complex Borel measure μ in Proposition 2.8.6 is presently played by $\phi_R\,\mu_{\vec F}$, with $\mu_{\vec F}$ as in (4.7.11). Bearing this in mind, we may now write

$$\begin{aligned}\mathrm{I}&=-\lim_{R\to\infty}\int_{\partial_*D_j}\nu_j\cdot\big(\vec F\big|_{\partial_*D_j}\big)\phi_R\,\mathrm{d}\mathcal{H}^{n-1}\\ &\quad-\lim_{R\to\infty}\tfrac12\big(\phi_R\,\mu_{\vec F}\big)\big(\partial^*(\mathbb{R}^n\setminus D_j)\big)-\lim_{R\to\infty}\big(\phi_R\,\mu_{\vec F}\big)\big(\mathrm{int}_*(\mathbb{R}^n\setminus D_j)\setminus(\mathbb{R}^n\setminus D_j)\big)\\ &=-\int_{\partial_*D_j}\nu_j\cdot\big(\vec F\big|_{\partial_*D_j}\big)\,\mathrm{d}\mathcal{H}^{n-1}-\tfrac12\mu_{\vec F}\big(\partial^*D_j\big)-\mu_{\vec F}\big(\mathrm{ext}_*(D_j)\cap D_j\big)\\ &=-\int_{\partial_*D_j}\nu_j\cdot\big(\vec F\big|_{\partial_*D_j}\big)\,\mathrm{d}\mathcal{H}^{n-1}-\tfrac12\int_\Omega\mathbf{1}_{\partial^*D_j}\,\mathrm{d}\mu_{\vec F}-\int_\Omega\mathbf{1}_{\mathrm{ext}_*(D_j)\cap D_j}\,\mathrm{d}\mu_{\vec F}.\end{aligned} \tag{4.7.15}$$

The second equality in (4.7.15) follows by observing that $\phi_R\equiv 1$ on ∂D_j since $R>R_j$. Here we have also used (5.6.16) and the fact that

$$\text{both } \partial^* D_j \text{ and } \operatorname{ext}_*(D_j) \cap D_j \text{ are contained in } \partial D_j \tag{4.7.16}$$

(cf. (2.8.20)). The final equality in (4.7.15) is obvious. Also, based on Lebesgue's Dominated Convergence Theorem (whose current applicability is guaranteed by (4.7.10) and (4.7.11)) we may express

$$\mathrm{II} = -\lim_{R\to\infty} \int_{\Omega\setminus D_j} \phi_R \,\mathrm{d}\mu_{\vec F} = -\int_{\Omega\setminus D_j} 1\,\mathrm{d}\mu_{\vec F} = \int_{\Omega} \mathbf{1}_{\Omega\setminus D_j}\,\mathrm{d}\mu_{\vec F}. \tag{4.7.17}$$

Let us record our progress so far. From (4.7.12)–(4.7.17), we conclude that for each $j \geq j_\rho$ we have

$$\begin{aligned} \lim_{R\to\infty} \int_\Omega \nabla\phi_R \cdot \vec F \,\mathrm{d}\mathcal{L}^n = &-\int_{\partial_* D_j} \nu_j \cdot \big(\vec F\big|_{\partial_* D_j}\big)\,\mathrm{d}\mathcal{H}^{n-1} \\ &- \tfrac{1}{2}\int_\Omega \mathbf{1}_{\partial^* D_j}\,\mathrm{d}\mu_{\vec F} - \int_\Omega \mathbf{1}_{\operatorname{ext}_*(D_j)\cap D_j}\,\mathrm{d}\mu_{\vec F} - \int_\Omega \mathbf{1}_{\Omega\setminus D_j}\,\mathrm{d}\mu_{\vec F}. \end{aligned} \tag{4.7.18}$$

Let us also observe that, in view of (4.7.4) and (4.7.16), Lebesgue's Dominated Convergence Theorem gives

$$\lim_{j\to\infty} \int_\Omega \mathbf{1}_{\partial^* D_j}\,\mathrm{d}\mu_{\vec F} = 0, \quad \lim_{j\to\infty} \int_\Omega \mathbf{1}_{\operatorname{ext}_*(D_j)\cap D_j}\,\mathrm{d}\mu_{\vec F} = 0, \quad \lim_{j\to\infty} \int_\Omega \mathbf{1}_{\Omega\setminus D_j}\,\mathrm{d}\mu_{\vec F} = 0. \tag{4.7.19}$$

Granted this, we may now conclude that the contribution of $\vec F$ at infinity is meaningfully and unambiguously defined, and that formula (4.7.5) holds, by combining (4.7.2), (4.7.18), and (4.7.19). □

Corresponding to $n = 1$ we have the result described in the proposition below.

Proposition 4.7.2 *Let $\Omega \subseteq (-\infty, +\infty)$ be an unbounded open set of locally finite perimeter such that $\partial\Omega$ is locally of finite cardinality, and abbreviate $\sigma := \mathcal{H}^0\lfloor\partial\Omega$. In this context, assume $F : \Omega \to \mathbb{R}$ is a continuous function with the property that*[2] *$F' \in \mathrm{CBM}(\Omega)$, and such that $F\big|^{\text{n.t.}}_{\partial\Omega}$ exists at each point on $\partial\Omega$ and its restriction to $\partial_*\Omega$ belongs to the space $L^1(\partial_*\Omega, \sigma)$. Then the contribution of F at infinity (cf. (1.3.4)–(1.3.5)) may be computed as*

$$[F]_\infty = \begin{cases} F\Big|_{-\infty}^{+\infty} & \text{if } \Omega \text{ unbounded both from below and from above,} \\ F(+\infty) & \text{if } \Omega \text{ bounded from below but unbounded from above,} \\ -F(-\infty) & \text{if } \Omega \text{ unbounded from below but bounded from above,} \end{cases} \tag{4.7.20}$$

where $F(\pm\infty)$ stand for the limits of F at $\pm\infty$ (taken from within Ω), and

[2] With the derivative taken in the sense of distributions.

$$F\Big|_{-\infty}^{+\infty} := \lim_{j\to\infty}\big[F(\beta_j) - F(\alpha_j)\big] \textit{ for any two sequences}$$
$$\{\alpha_j\}_{j\in\mathbb{N}}, \{\beta_j\}_{j\in\mathbb{N}} \subset \Omega \textit{ with } \lim_{j\to\infty}\alpha_j = -\infty \textit{ and } \lim_{j\to\infty}\beta_j = +\infty. \tag{4.7.21}$$

Before proving this proposition, we make some preliminary observations. First, we note that $\partial\Omega$ is an at most countable set and σ is simply the counting measure on $\partial\Omega$. The fact that $\partial\Omega$ is locally of finite cardinality is then equivalent to saying that σ is a locally finite measure. Also, since Ω is open, it follows that Ω is Lebesgue measurable and for each $R \in (0,\infty)$ we have

$$\sigma\big(\partial_*\Omega \cap (-R,R)\big) \le \sigma\big(\partial\Omega \cap (-R,R)\big) < +\infty, \tag{4.7.22}$$

with the last inequality a consequence of the fact that $\partial\Omega$ is locally of finite cardinality. From (4.7.22) and (5.6.35) we then see that Ω is a set of locally finite perimeter. Finally, we wish to note that, thanks to Proposition 8.8.10 we have $\partial_{\text{nta}}\Omega = \partial\Omega$. As such, in view of Definition 8.9.1 it is meaningful to assume that the nontangential limit $F\big|_{\partial\Omega}^{\text{n.t.}}$ exists at every point on $\partial\Omega$.

We are ready to present the proof of Proposition 4.7.2.

Proof of Proposition 4.7.2 For starters, it is useful to observe that the assumptions on F ensure that this function extends continuously to $\overline{\Omega}$; in particular $F \in L^1_{\text{bdd}}(\Omega, \mathcal{L}^1)$ (which allows us to begin considering the contribution of F at infinity). Indeed, $\overline{\Omega} \setminus \Omega = \partial\Omega = \partial_{\text{nta}}\Omega$ so the function

$$\widetilde{F} : \overline{\Omega} \longrightarrow \mathbb{R} \tag{4.7.23}$$

given at each $x \in \overline{\Omega}$ by

$$\widetilde{F}(x) := \begin{cases} F(x) & \text{if } x \in \Omega, \\ \big(F\big|_{\partial\Omega}^{\text{n.t.}}\big)(x) & \text{if } x \in \partial\Omega, \end{cases} \tag{4.7.24}$$

is well defined and satisfies

$$\widetilde{F} \in \mathscr{C}^0\big(\overline{\Omega}\big) \text{ and } \widetilde{F}\big|_{\Omega} = F. \tag{4.7.25}$$

Going further, the fact that Ω is unbounded forces Ω to be either unbounded from below, or unbounded from above. To fix ideas, suppose first that Ω is unbounded both from below and from above. Select two sequences

$$\{\alpha_j\}_{j\in\mathbb{N}}, \{\beta_j\}_{j\in\mathbb{N}} \subset \Omega \text{ such that } \lim_{j\to\infty}\alpha_j = -\infty \text{ and } \lim_{j\to\infty}\beta_j = +\infty \tag{4.7.26}$$

and consider the compact intervals

$$D_j := [\alpha_j, \beta_j] \text{ for each } j \in \mathbb{N}. \tag{4.7.27}$$

Next, fix an arbitrary system of auxiliary functions $\{\phi_R\}_{R>0}$ (i.e., a subfamily of $\mathscr{C}_c^\infty(\mathbb{R})$ satisfying (4.7.10) with $n = 1$) and abbreviate $\mu_F := F' \in \mathrm{CBM}(\Omega)$. For each $j \in \mathbb{N}$, if ν_j and ν denote the geometric measure theoretic outward unit normals to $\Omega \setminus D_j$, and to Ω, respectively, we may then compute

$$\begin{aligned}\lim_{R\to\infty}\int_\Omega \phi_R'\, F\, \mathrm{d}\mathcal{L}^1 &= \lim_{R\to\infty}\int_{\Omega\setminus D_j} \phi_R'\, F\, \mathrm{d}\mathcal{L}^1 = \lim_{R\to\infty}(\phi_R'\, F)\big(\Omega\setminus D_j\big) \\ &= \lim_{R\to\infty}(\phi_R\, F)'\big(\Omega\setminus D_j\big) - \lim_{R\to\infty}(\phi_R\ F')\big(\Omega\setminus D_j\big) \\ &= \lim_{R\to\infty}\int_{\partial_*(\Omega\setminus D_j)} \nu_j\, F\,\phi_R\, \mathrm{d}\mathcal{H}^0 - \lim_{R\to\infty}\int_{\Omega\setminus D_j}\phi_R\, \mathrm{d}\mu_F \\ &= \int_{\partial_*\Omega\setminus D_j} \nu\, F\, \mathrm{d}\mathcal{H}^0 - F(\beta_j) + F(\alpha_j) - \int_{\Omega\setminus D_j} 1\, \mathrm{d}\mu_F.\end{aligned} \tag{4.7.28}$$

Above, the first equality relies on the fact that $\phi_R' \equiv 0$ near D_j if R is large. The second equality is simply the result of identifying $\phi_R'\, F$ with the measure $\phi_R'\, F\, \mathrm{d}\mathcal{L}^1$. The third equality relies on the product rule to express the measure $\phi_R'\, F$ as the difference of measures $(\phi_R\, F)' - \phi_R\, F'$. The fourth equality uses Theorem 2.6.5 applied to each connected component of $(I_R \cap \Omega) \setminus D_j$ (which contains $I_R \cap (\Omega \setminus D_j)$) intersecting the support of ϕ_R, where I_R is a large open finite interval containing the support of ϕ_R. The assumption that $\partial\Omega$ is locally of finite cardinality guarantees that there are only finitely many such components.

Bearing (4.7.25) in mind, it follows from (2.6.27) that Theorem 2.6.5 is indeed applicable in the current context. Finally, the last equality in (4.7.28) takes into account that $\phi_R \equiv 1$ on ∂D_j if R is large, that since $\partial\Omega$ is locally of finite cardinality we have

$$\partial_*(\Omega\setminus D_j) = \big(\partial_*\Omega\setminus D_j\big)\cup\{\alpha_j, \beta_j\} \quad \text{where the union is disjoint,} \tag{4.7.29}$$

as well as Lebesgue's Dominated Convergence Theorem. Since, given the present assumptions, Lebesgue's Dominated Convergence Theorem also gives

$$\int_{\partial_*\Omega\setminus D_j} \nu\, F\, \mathrm{d}\mathcal{H}^0 \to 0 \quad \text{and} \quad \int_{\Omega\setminus D_j} 1\, \mathrm{d}\mu_F \to 0 \quad \text{as} \quad j\to\infty, \tag{4.7.30}$$

from (4.7.2), (4.7.28), and (4.7.30) we conclude that, in the present setting, the contribution of F at infinity is meaningfully and unambiguously defined, and

$$[F]_\infty = F\Big|_{-\infty}^{+\infty} \tag{4.7.31}$$

which is in agreement with (4.7.20).

Consider next the scenario when Ω is unbounded from above but is bounded from below. In this case, pick $a \in \mathbb{R}$ satisfying $a < \inf\Omega$. Augmenting $\widetilde{\Omega}:=(-\infty, a)\cup\Omega$

yields an open set of the sort just treated above. If we also denote by $\widetilde{F}$ the extension of the original F to $\widetilde{\Omega}$ by setting it zero on $(-\infty, a)$, then (4.7.31) is applicable to $\widetilde{F}$ and $\widetilde{\Omega}$ and presently yields

$$[F]_\infty = \big[\widetilde{F}\,\big]_\infty = \widetilde{F}\Big|_{-\infty}^{+\infty} = F(+\infty), \tag{4.7.32}$$

which once again is in agreement with (4.7.20). Lastly, the case when Ω is unbounded from below but is bounded from above is handled similarly, finishing the proof of the proposition. □

We continue by identifying intrinsic conditions guaranteeing that the contribution at infinity of a vector field is actually zero.

Lemma 4.7.3 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$, be an open set and consider a vector field $\vec{F}$ whose components are absolutely integrable functions on bounded measurable subsets of Ω, and whose contribution at infinity is meaningfully and unambiguously defined. In addition, suppose that there exists some number $\lambda \in (1, \infty)$ such that*

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |x \cdot \vec{F}(x)|\,\mathrm{d}\mathcal{L}^n(x) = o(R^2) \;\; \textit{as} \;\; R \to \infty. \tag{4.7.33}$$

Then, in fact, $[\vec{F}]_\infty = 0$. As a corollary, if $n \geq 2$ and in place of (4.7.33) one assumes

$$\vec{F} \in \big[L^{\frac{n}{n-1}}(\Omega, \mathcal{L}^n)\big]^n \tag{4.7.34}$$

then (4.7.33) holds, hence once again $[\vec{F}]_\infty = 0$.

Proof With λ as in the statement of the lemma, pick a real-valued function θ in $\mathscr{C}_c^\infty(\mathbb{R})$ satisfying $\theta \equiv 1$ on the interval $(-1, 1)$ as well as $\theta \equiv 0$ on $\mathbb{R} \setminus (-\lambda^2, \lambda^2)$, and define

$$\phi : \mathbb{R}^n \to \mathbb{R}, \quad \phi(x) := \theta(|x|^2), \quad \forall x \in \mathbb{R}^n. \tag{4.7.35}$$

Consequently,

$$\begin{gathered} \phi \in \mathscr{C}_c^\infty(\mathbb{R}^n), \;\; \phi \equiv 1 \;\text{ on }\; B(0,1), \;\; \phi \equiv 0 \;\text{ on }\; \mathbb{R}^n \setminus B(0,\lambda), \\ \operatorname{supp}(\nabla\phi) \subseteq \overline{B(0,\lambda)} \setminus B(0,1), \\ \text{and } \;(\nabla\phi)(x) = 2\theta'(|x|^2)x \;\text{ for all }\; x \in \mathbb{R}^n. \end{gathered} \tag{4.7.36}$$

Thanks to (4.7.36) and the independence of $[\vec{F}]_\infty$ in (1.3.5) on the choice of the function ϕ as in (1.3.4) we have

$$\begin{aligned}\big|[\vec{F}]_\infty\big| &\leq \limsup_{R\to\infty}\left|\int_\Omega R^{-1}(\nabla\phi)(x/R)\cdot\vec{F}(x)\,\mathrm{d}\mathcal{L}^n(x)\right| \\ &\leq \limsup_{R\to\infty}\left\{2R^{-2}\big(\sup_{\mathbb{R}}|\theta'|\big)\cdot\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega}|x\cdot\vec{F}(x)|\,\mathrm{d}\mathcal{L}^n(x)\right\}.\end{aligned} \tag{4.7.37}$$

From this it is clear that $[\vec{F}]_\infty = 0$ if the integral growth condition (4.7.33) holds.

Finally, consider the case when $n \geq 2$ and in place of (4.7.33) we now assume (4.7.34). In such a scenario, fix some $\lambda \in (1,\infty)$ and for each $R \in (0,\infty)$ use Cauchy–Schwarz' inequality and Hölder's inequality to estimate

$$\begin{aligned}R^{-2}&\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega}|x\cdot\vec{F}(x)|\,\mathrm{d}\mathcal{L}^n(x) \\ &\leq \lambda R^{-1}\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega}|\vec{F}|\,\mathrm{d}\mathcal{L}^n \\ &\leq \lambda R^{-1}\Big(\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega}|\vec{F}|^{\frac{n}{n-1}}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{n}}\Big(\int_{B(0,\lambda R)}1\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{n}} \\ &\leq C\Big(\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega}|\vec{F}|^{\frac{n}{n-1}}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{n}},\end{aligned} \tag{4.7.38}$$

for some constant $C = C_{n,\lambda} \in (0,\infty)$. In view of (4.7.34), Lebesgue's Dominated Convergence Theorem applies and proves that (4.7.33) holds in this case. As noted earlier, this integral growth condition forces $[\vec{F}]_\infty = 0$. □

We augment the result from the previous lemma with the following criterion guaranteeing the vanishing of the contribution at infinity in the one-dimensional setting.

Proposition 4.7.4 *Suppose $\Omega \subseteq (-\infty,+\infty)$ is an open nonempty proper subset of the real line, with the property that $\partial\Omega$ is an unbounded set without (finite) accumulation points. Abbreviate $\sigma := \mathcal{H}^0\lfloor\partial\Omega$ and fix an aperture parameter $\kappa > 0$. In this context, assume $F : \Omega \to \mathbb{R}$ is a Lebesgue measurable function with the property that $\mathcal{N}_\kappa F \in L^1(\partial\Omega,\sigma)$, and such that the contribution of F at infinity is meaningfully and unambiguously defined (as introduced in (1.3.4)–(1.3.5)).*

Then the contribution of F at infinity actually vanishes, i.e., $[F]_\infty = 0$.

Proof Since $\partial\Omega$ is an at most countable set which is unbounded and without (finite) accumulation points, it follows that $\partial\Omega$ has infinite cardinality. Then the family of connected components of Ω is countable and has infinite cardinality. Call this family

$\{I_j\}_{j\in\mathbb{N}}$, where the I_j's are mutually disjoint open intervals of the form $I_j = (a_j, b_j)$ with $-\infty \le a_j < b_j \le +\infty$ and $\min\{|a_j|, |b_j|\} < \infty$ for each $j \in \mathbb{N}$, such that

$$\Omega = \bigcup_{j=1}^{\infty} I_j. \tag{4.7.39}$$

Moreover, as in the proof of Proposition 8.8.10, we have

$$\partial\Omega = \overline{\bigcup_{j\in\mathbb{N}} \partial I_j} = \bigcup_{j\in\mathbb{N}} \partial I_j, \tag{4.7.40}$$

while from (8.3.29) we see that

$$\sum_{j=1}^{\infty} \|F\|_{L^\infty(I_j,\mathcal{L}^1)} \le \sum_{j=1}^{\infty} \|\mathcal{N}_\kappa F\|_{L^1(\partial I_j,\sigma)} \le \|\mathcal{N}_\kappa F\|_{L^1(\partial\Omega,\sigma)}. \tag{4.7.41}$$

Since we are assuming $\|\mathcal{N}_\kappa F\|_{L^1(\partial\Omega,\sigma)} < +\infty$, this further implies

$$\lim_{j\to\infty} \|F\|_{L^\infty(I_j,\mathcal{L}^1)} = 0. \tag{4.7.42}$$

We shall put this to good use shortly. For now, we wish to remark that from Lemma 8.3.4 and current assumptions we also know that

$$\|F\|_{L^\infty(\Omega,\mathcal{L}^1)} \le \|\mathcal{N}_\kappa F\|_{L^1(\partial\Omega,\sigma)} < +\infty, \tag{4.7.43}$$

hence

$$F \in L^\infty(\Omega, \mathcal{L}^1) \subseteq L^1_{\rm bdd}(\Omega, \mathcal{L}^1). \tag{4.7.44}$$

In particular, the latter property ensures that it is possible to consider the contribution of F at infinity as in (1.3.4)–(1.3.5). By assumption, this contribution of F at infinity is meaningfully and unambiguously defined. To show that we actually have $[F]_\infty = 0$, we shall consider a special system of auxiliary functions. Specifically, we shall construct a subfamily $\{\phi_R\}_{R>0}$ of $\mathscr{C}^\infty_c(\mathbb{R})$ satisfying (1.3.3) with $n = 1$, depending on the nature of Ω, as discussed in the cases below.

`Case I:` *Assume there exists $j_o \in \mathbb{N}$ such that $I_{j_o} = (-\infty, a)$ for some $a \in \mathbb{R}$.* In such a scenario, in light of the original assumptions on Ω, we may relabel the intervals $I_j = (a_j, b_j)$ with $j \in \mathbb{N} \setminus \{j_o\}$, now taking $j \in \mathbb{N}$ arbitrary, so that

$$-\infty < a \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_j < b_j \le a_{j+1} < b_{j+1} \le \cdots < +\infty. \tag{4.7.45}$$

Moreover, the fact that $\partial\Omega$ is an infinite set without (finite) accumulation points implies that

$$\lim_{j\to\infty} a_j = +\infty \quad\text{and}\quad \lim_{j\to\infty} b_j = +\infty. \tag{4.7.46}$$

For each $R \in (0, \infty)$ introduce

$$j_R := \min\big\{j \in \mathbb{N} : a_j > R\big\} \tag{4.7.47}$$

and note that, as seen from (4.7.46),

$$\{j_R\}_{R>0} \text{ is a well-defined sequence of integers convergent to } +\infty\text{, as } R \text{ tends to infinity.} \tag{4.7.48}$$

For each $R \in (0, \infty)$ it is possible to construct $\phi_R \in \mathscr{C}_c^\infty(\mathbb{R})$ such that

$$\begin{aligned} &0 \le \phi_R \le 1,\ \ \phi_R \equiv 1 \text{ on } \big[a - R, a_{j_R}\big],\ \ \phi_R \equiv 0 \text{ on } \mathbb{R}\setminus\big[a - 2R, b_{j_R}\big],\\ &|\phi_R'| \le C/R \text{ on } \big[a - 2R, a - R\big],\ \ |\phi_R'| \le C/(b_{j_R} - a_{j_R}) \text{ on } \big[a_{j_R}, b_{j_R}\big], \end{aligned} \tag{4.7.49}$$

for some absolute constant $C \in (0, \infty)$. Then $\{\phi_R\}_{R>0}$ is a family in $\mathscr{C}_c^\infty(\mathbb{R})$ which satisfies (1.3.3) with $n = 1$. Also, the one-dimensional version of (4.7.2) becomes

$$\begin{aligned} [F]_\infty &= -\lim_{R\to\infty}\int_\Omega \phi_R' \, F \,\mathrm{d}\mathcal{L}^1\\ &= -\lim_{R\to\infty}\Big\{\int_{[a-2R,a-R]}\phi_R' \, F\,\mathrm{d}\mathcal{L}^1 + \int_{[a_{j_R},b_{j_R}]}\phi_R'\, F\,\mathrm{d}\mathcal{L}^1\Big\}, \end{aligned} \tag{4.7.50}$$

in view of the support properties for ϕ_R' implicit in (4.7.49). Note that thanks to the estimates in (4.7.49), (4.7.42), and (4.7.48),

$$\lim_{R\to\infty}\Big|\int_{[a_{j_R},b_{j_R}]}\phi_R'\, F\,\mathrm{d}\mathcal{L}^1\Big| \le C\lim_{R\to\infty}\|F\|_{L^\infty(I_{j_R},\mathcal{L}^1)} = 0, \tag{4.7.51}$$

which means that (4.7.50) reduces to

$$[F]_\infty = -\lim_{R\to\infty}\int_{[a-2R,a-R]}\phi_R'\, F\,\mathrm{d}\mathcal{L}^1. \tag{4.7.52}$$

To proceed, observe that for each $j \in \mathbb{N}$ we have

$$\Big(-\infty,\ \min\big\{a,\ [(1+\kappa)a - a_j]/\kappa\big\}\Big) \subseteq \Gamma_\kappa(a_j). \tag{4.7.53}$$

Indeed, given any x in the interval above, it follows that $x \in (-\infty, a)$ (hence also $x \in \Omega$), and $\operatorname{dist}(x, \partial\Omega) = a - x$. Consequently, having $|a_j - x| < (1+\kappa)\operatorname{dist}(x, \partial\Omega)$ becomes equivalent to $x < [(1+\kappa)a - a_j]/\kappa$, which is true given the membership of x to the interval in (4.7.53). In light of (8.1.1)–(8.1.2), this ultimately shows that $x \in \Gamma_\kappa(a_j)$, finishing the proof of (4.7.53).

For each $j \in \mathbb{N}$ fixed, select $R > 0$ large enough so that

$$[a - 2R, a - R] \subseteq \Big(-\infty, \ \min\big\{a, \ [(1+\kappa)a - a_j]/\kappa\big\}\Big). \tag{4.7.54}$$

Then the first estimate in the second line of (4.7.49) together with (8.2.1) allow to write

$$\Big| \int_{[a-2R,a-R]} \phi_R' \, F \, \mathrm{d}\mathcal{L}^1 \Big| \leq C \|F\|_{L^\infty([a-2R,a-R],\mathcal{L}^1)} \leq C(\mathcal{N}_\kappa F)(a_j) \tag{4.7.55}$$

given that $[a - 2R, a - R] \subseteq \Gamma_\kappa(a_j)$, as seen from (4.7.54) and (4.7.53). Thus, on the one hand,

$$\limsup_{R\to\infty} \Big| \int_{[a-2R,a-R]} \phi_R' \, F \, \mathrm{d}\mathcal{L}^1 \Big| \leq C(\mathcal{N}_\kappa F)(a_j) \ \text{ for each } \ j \in \mathbb{N}. \tag{4.7.56}$$

Since $\sigma = \mathcal{H}^0 \lfloor \partial\Omega$ is the counting measure on $\partial\Omega$, the last estimate in (4.7.41) may also be recast as

$$\sum_{j=1}^{\infty} \big\{ (\mathcal{N}_\kappa F)(a_j) + (\mathcal{N}_\kappa F)(b_j) \big\} \leq \|\mathcal{N}_\kappa F\|_{L^1(\partial\Omega,\sigma)}. \tag{4.7.57}$$

Given we are assuming $\|\mathcal{N}_\kappa F\|_{L^1(\partial\Omega,\sigma)} < +\infty$, this permits us to conclude that, on the other hand,

$$\lim_{j\to\infty} (\mathcal{N}_\kappa F)(a_j) = 0. \tag{4.7.58}$$

In concert, (4.7.56) and (4.7.58) show that

$$\lim_{R\to\infty} \int_{[a-2R,a-R]} \phi_R' \, F \, \mathrm{d}\mathcal{L}^1 = 0 \tag{4.7.59}$$

which, together with (4.7.52), ultimately proves that $[F]_\infty = 0$.

`Case II:` *Assume there exists* $j_o \in \mathbb{N}$ *such that* $I_{j_o} = (a, +\infty)$ *for some* $a \in \mathbb{R}$. Reasoning similarly as we have done in Case I, we once again have $[F]_\infty = 0$ in this scenario.

`Case III:` *Assume all* I_j*'s are finite intervals*. If either Ω is bounded from above, or is bounded from below, then much as in Cases I-II we obtain $[F]_\infty = 0$. To be more specific, assume that Ω is bounded from below (the case when Ω is bounded from above is treated similarly). Pick $a \in \mathbb{R}$ with $a < \inf \Omega$ then augment $\widetilde{\Omega} := (-\infty, a) \cup \Omega$ and extend F by zero to a function $\widetilde{F}$ defined on $\widetilde{\Omega}$. Then $\widetilde{F}$ continues to be Lebesgue measurable. Also, since for each $x \in \partial\Omega$ we have

$$\Gamma_{\Omega,\kappa}(x) \subseteq \Gamma_{\widetilde{\Omega},\kappa}(x) \text{ and } \Gamma_{\widetilde{\Omega},\kappa}(x) \setminus \Gamma_{\Omega,\kappa}(x) \subseteq (-\infty, a) \tag{4.7.60}$$

(where the first inclusion comes from Lemma 8.1.3), we see that

$$\mathcal{N}_\kappa \widetilde{F} = \mathcal{N}_\kappa F \text{ on } \partial\Omega. \tag{4.7.61}$$

In addition, (4.7.43) gives

$$0 \le (\mathcal{N}_\kappa \widetilde{F})(a) \le \|\widetilde{F}\|_{L^\infty(\widetilde{\Omega},\mathcal{L}^1)} = \|F\|_{L^\infty(\Omega,\mathcal{L}^1)} \le \|\mathcal{N}_\kappa F\|_{L^1(\partial\Omega,\sigma)}. \tag{4.7.62}$$

If $\widetilde{\sigma} := \mathcal{H}^0 \lfloor \partial\widetilde{\Omega}$, then (4.7.61)–(4.7.62) collectively prove that

$$\begin{aligned} \|\mathcal{N}_\kappa \widetilde{F}\|_{L^1(\partial\widetilde{\Omega},\widetilde{\sigma})} &\le (\mathcal{N}_\kappa \widetilde{F})(a) + \|\mathcal{N}_\kappa F\|_{L^1(\partial\Omega,\sigma)} \\ &\le 2\|\mathcal{N}_\kappa F\|_{L^1(\partial\Omega,\sigma)} < +\infty, \end{aligned} \tag{4.7.63}$$

since $\mathcal{N}_\kappa F$ is assumed to be in $L^1(\partial\Omega, \sigma)$. Hence, $\mathcal{N}_\kappa \widetilde{F} \in L^1(\partial\widetilde{\Omega}, \widetilde{\sigma})$. Finally, it is clear from definitions and assumptions that the contribution of $\widetilde{F}$ at infinity is meaningfully and unambiguously defined and, in fact, $[\widetilde{F}]_\infty = [F]_\infty$. Having checked these properties, what we have proved in Case I gives $[\widetilde{F}]_\infty = 0$, ergo $[F]_\infty = 0$.

There remains to consider the situation when Ω is unbounded both from above and below. Given that $\partial\Omega$ is an infinite set without (finite) accumulation points, it follows that there exist two infinite subfamilies

$$\begin{gathered} \{I_j^+ = (a_j^+, b_j^+)\}_{j\in\mathbb{N}} \text{ and } \{I_j^- = (a_j^-, b_j^-)\}_{j\in\mathbb{N}} \text{ of } \{I_j\}_{j\in\mathbb{N}} \text{ with} \\ \lim_{j\to\infty} a_j^+ = \lim_{j\to\infty} b_j^+ = +\infty \text{ and } \lim_{j\to\infty} a_j^- = \lim_{j\to\infty} b_j^- = -\infty. \end{gathered} \tag{4.7.64}$$

Next, for each $R \in (0, \infty)$ introduce

$$j_R^+ := \min\big\{j \in \mathbb{N} : a_j^+ > R\big\} \text{ and } j_R^- := \min\big\{j \in \mathbb{N} : a_j^- < -R\big\}. \tag{4.7.65}$$

Bearing in mind (4.7.64), we then conclude that

$$\begin{gathered} \{j_R^\pm\}_{R>0} \text{ are two well-defined sequences of} \\ \text{integers convergent to } +\infty, \text{ as } R \to \infty. \end{gathered} \tag{4.7.66}$$

The stage is now set to consider, for each $R \in (0, \infty)$, a function $\phi_R \in \mathscr{C}_c^\infty(\mathbb{R})$ satisfying

$$\begin{gathered} 0 \le \phi_R \le 1, \ \phi_R \equiv 1 \text{ on } \big[b_{j_R^-}^-, a_{j_R^+}^+\big], \ \phi_R \equiv 0 \text{ on } \mathbb{R} \setminus \big[a_{j_R^-}^-, b_{j_R^+}^+\big], \\ |\phi_R'| \le C/(b_{j_R^-}^- - a_{j_R^-}^-) \text{ on } \big[a_{j_R^-}^-, b_{j_R^-}^-\big], \text{ and} \\ |\phi_R'| \le C/(b_{j_R^+}^+ - a_{j_R^+}^+) \text{ on } \big[a_{j_R^+}^+, b_{j_R^+}^+\big], \end{gathered} \tag{4.7.67}$$

for some absolute constant $C \in (0, \infty)$. Then $\{\phi_R\}_{R>0}$ is a family in $\mathscr{C}_c^\infty(\mathbb{R})$ which satisfies the one-dimensional version of (1.3.3). Then the same type of argument as in Case I gives that

$$\begin{aligned}[F]_\infty &= -\lim_{R\to\infty} \int_\Omega \phi_R' \, F \, \mathrm{d}\mathcal{L}^1 \\ &= -\lim_{R\to\infty} \Big\{ \int_{[a^-_{j^-_R}, b^-_{j^-_R}]} \phi_R' \, F \, \mathrm{d}\mathcal{L}^1 + \int_{[a^+_{j^+_R}, b^+_{j^+_R}]} \phi_R' \, F \, \mathrm{d}\mathcal{L}^1 \Big\} = 0, \end{aligned} \tag{4.7.68}$$

thanks to (4.7.67) and (4.7.42) (keeping in mind (4.7.66)). This finishes the proof of Proposition 4.7.4. □

Moving on, we shall change focus. Specifically, in a consistent fashion with (4.7.2) we shall now define, whenever meaningful and unambiguous, the contribution at infinity of a vector field $\vec{F} \in \big[\mathcal{D}'(\mathbb{R}^n)\big]^n$ by setting (with $\langle \cdot, \cdot \rangle$ denoting the distributional pairing in $\mathbb{R}^n$)

$$\begin{gathered}[\vec{F}]_\infty := -\lim_{R\to\infty} \big\langle \vec{F}, \nabla\phi_R \big\rangle \ \text{ for any} \\ \text{system of auxiliary functions } \{\phi_R\}_{R>0} \ (cf.\ (1.3.3)). \end{gathered} \tag{4.7.69}$$

Lemma 4.7.5 *Suppose $\vec{F} \in \big[\mathcal{D}'(\mathbb{R}^n)\big]^n$ has the property that the distribution* $\mathrm{div}\vec{F}$ *extends to a complex Borel measure in $\mathbb{R}^n$ (in the sense of (1.3.12)). Then the contribution of $\vec{F}$ at infinity is meaningfully and unambiguously defined and, retaining the notation* $\mathrm{div}\vec{F}$ *for the aforementioned measure, one has*

$$\big(\mathrm{div}\vec{F}\big)(\mathbb{R}^n) = [\vec{F}]_\infty. \tag{4.7.70}$$

More generally, if $\vec{F} \in \big[\mathcal{D}'(\mathbb{R}^n)\big]^n$ is such that

$$\mathrm{div}\vec{F} \in \mathscr{E}'(\mathbb{R}^n) + \mathrm{CBM}(\mathbb{R}^n) \tag{4.7.71}$$

then the contribution of $\vec{F}$ at infinity is meaningfully and unambiguously defined, and (with $\mathscr{C}_b^\infty(\mathbb{R}^n)$ defined as in (4.6.8)) one has

$${}_{(\mathscr{C}_b^\infty(\mathbb{R}^n))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\mathbb{R}^n)} = [\vec{F}]_\infty. \tag{4.7.72}$$

Formula (4.7.72) may be thought of as the limiting case $\Omega = \mathbb{R}^n$ of (1.4.5).

Proof of Lemma 4.7.5 Consider $\vec{F}$ as in the first part of the statement and, for ease of notation, abbreviate

$$\mu := \mathrm{div}\vec{F} \in \mathrm{CBM}(\mathbb{R}^n). \tag{4.7.73}$$

Then, having fixed an arbitrary family of functions $\{\phi_R\}_{R>0}$ as in (1.3.3), based on (1.3.12) and Lebesgue's Dominated Convergence Theorem we may write

$$
\begin{aligned}
-\lim_{R\to\infty}\big\langle \vec{F}, \nabla\phi_R\big\rangle = \lim_{R\to\infty}\big\langle \mathrm{div}\vec{F}, \phi_R\big\rangle &= \lim_{R\to\infty}\int_{\mathbb{R}^n}\phi_R\,\mathrm{d}\mu \\
&= \int_{\mathbb{R}^n} 1\,\mathrm{d}\mu = \mu(\mathbb{R}^n). \qquad (4.7.74)
\end{aligned}
$$

On account of (4.7.73), (4.7.74), and (4.7.69) we then conclude that the contribution of $\vec{F}$ at infinity is meaningfully and unambiguously defined and that (4.7.70) holds.

Under the assumption that $\vec{F} \in \big[\mathcal{D}'(\mathbb{R}^n)\big]^n$ satisfies (4.7.71), the fact that the contribution of $\vec{F}$ at infinity is meaningfully and unambiguously defined and that (4.7.72) is valid are justified in a similar manner, making use of (4.6.19). □

Chapter 5
Sets of Locally Finite Perimeter and Other Categories of Euclidean Sets

Here we discuss basic results from Geometric Measure Theory, including thick sets, the corkscrew condition, the geometric measure theoretic boundary, area and coarea formulas, countable rectifiability, approximate tangent planes, functions of bounded variation, sets of locally finite perimeter, Ahlfors regularity, uniformly rectifiable (UR) sets, the local John condition, (ε, δ)-domains, and nontangentially accessible (NTA) domains.

There are several excellent accounts on the topics dealt with in this chapter, including the monographs by L. Evans and R. Gariepy [80], F. Lin and X. Yang [162], H. Federer [88], P. Mattila [177], L. Simon [238], W. Ziemer [263]. Here we shall review material that is relevant to the current goals from these (and other) sources, and also further build, refine, and develop results and tools that are necessary in subsequent work. Although here we shall work exclusively in Euclidean spaces, a great deal of the material in this chapter translates routinely to the manifold setting.

5.1 Thick Sets and Corkscrew Conditions

We begin by making the following definition.

Definition 5.1.1 *An $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$ is said to be n-*`thick` *provided there exists a number $c \in (0, \infty)$ such that for each $x \in \partial E$ one has*

$$\mathcal{L}^n\big(B(x,r)\cap E\big) \geq cr^n, \qquad \forall r \in \big(0, 2\,\mathrm{diam}(\partial E)\big). \tag{5.1.1}$$

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Also, call an $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$ `two-sided` n-`thick` *if both E and $\mathbb{R}^n \setminus E$ are n-thick.*

A quick inspection shows that in place of $r \in \big(0, 2\,\mathrm{diam}(\partial E)\big)$ we may ask that $r \in (0, 2\,\mathrm{diam}\,E)$. This is equivalent to the condition in (5.1.1), up to a change in the choice of the constant $c \in (0, \infty)$.

It is also easy to see that for any given $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$, the quality of being n-thick is equivalent to the demand that (5.1.1) holds for every $x \in E$ (hence for every $x \in \overline{E}$). Moreover, since the closure of an arbitrary set is bigger than the original set but has a smaller boundary, it follows that

$$\text{if an } \mathcal{L}^n\text{-measurable set } E \subseteq \mathbb{R}^n \text{ is } n\text{-thick, then } \overline{E} \text{ is also } n\text{-thick.} \tag{5.1.2}$$

Lemma 5.1.2 *If $\Omega \subseteq \mathbb{R}^n$ is an $\mathcal{L}^n$-measurable set which is n-thick then we have $\mathcal{L}^n\big(\overline{\Omega} \setminus \Omega\big) = 0$. In particular, any open set $\Omega \subseteq \mathbb{R}^n$ which is n-thick has the property that $\mathcal{L}^n(\partial\Omega) = 0$.*

Proof Assume $\Omega \subseteq \mathbb{R}^n$ is an $\mathcal{L}^n$-measurable set which is n-thick. From the earlier discussion, it follows that there exists a constant $c > 0$ with the property that

$$\mathcal{L}^n\big(B(x, r) \cap \Omega\big) \geq cr^n, \qquad \forall x \in \overline{\Omega}, \quad \forall r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big). \tag{5.1.3}$$

On the other hand, Lebesgue's Differentiation Theorem applied to the locally integrable function $\mathbf{1}_{\mathbb{R}^n \setminus \Omega}$ implies that there exists an $\mathcal{L}^n$-measurable set $N \subseteq \mathbb{R}^n$ such that $\mathcal{L}^n(N) = 0$ and

$$\lim_{r \to 0^+} \frac{\mathcal{L}^n\big(B(x, r) \cap \Omega\big)}{\mathcal{L}^n\big(B(x, r)\big)} = 0 \quad \text{for each} \quad x \in (\mathbb{R}^n \setminus \Omega) \setminus N. \tag{5.1.4}$$

In concert, (5.1.3)–(5.1.4) imply $(\mathbb{R}^n \setminus \Omega) \setminus N \subseteq \mathbb{R}^n \setminus \overline{\Omega}$ which, after taking complements, allows us to conclude that $\overline{\Omega} \subseteq \Omega \cup N$. Hence, $\overline{\Omega} \setminus \Omega \subseteq N$ which forces $\mathcal{L}^n\big(\overline{\Omega} \setminus \Omega\big) = 0$. If Ω is open, then $\overline{\Omega} \setminus \Omega = \partial\Omega$, so this further implies $\mathcal{L}^n(\partial\Omega) = 0$, as wanted. □

The following is a slight variation of concepts introduced by D. Jerison and C. Kenig in [132].

Definition 5.1.3 *Consider an arbitrary set $D \subseteq \mathbb{R}^n$.*

(i) The set D is said to satisfy an `interior corkscrew condition` *if there exists some constant $\theta \in (0, 1)$ such that for each $x \in \partial D$ and each $r \in \big(0, 2\,\mathrm{diam}(\partial D)\big)$ one can find*

$$\begin{aligned} &z_r(x) \in D, \text{ called corkscrew point relative to } x, \\ &\text{with the property that } B\big(z_r(x), \theta r\big) \subseteq B(x, r) \cap D. \end{aligned} \tag{5.1.5}$$

(ii) The set D is said to satisfy an `exterior corkscrew condition`[1] *if $\mathbb{R}^n \setminus D$ satisfies an interior corkscrew condition.*

(iii) Say that the set D satisfies a `two-sided corkscrew condition` *if D satisfies both an interior and exterior corkscrew condition.*

(iv) Say that the set D satisfies an `interior corkscrew condition up to scale` *$R \in \big(0, 2\,\mathrm{diam}(\partial D)\big)$ if (5.1.5) holds for each $x \in \partial D$ and each $r \in (0, R)$. Likewise, one defines the exterior and the two-sided corkscrew conditions up to scale $R \in \big(0, 2\,\mathrm{diam}(\partial D)\big)$.*

As far as item *(i)* is concerned, in place of $r \in \big(0, 2\,\mathrm{diam}(\partial D)\big)$ we may demand that $r \in (0, 2\,\mathrm{diam}\, D)$, as this is equivalent to the condition in (5.1.5) up to a change in the choice of the constant $\theta \in (0, 1)$. This comment is also relevant in items *(ii)–(iii)*. Likewise, in item *(iv)*, in place of $R \in \big(0, 2\,\mathrm{diam}(\partial D)\big)$ we take $R \in (0, 2\,\mathrm{diam}\, D)$, with the same effect.

Obviously,

$$\begin{array}{l}\text{if an } \mathcal{L}^n\text{-measurable set } E \subseteq \mathbb{R}^n \text{ satisfies an interior (resp., exterior)}\\ \text{corkscrew condition then } E \text{ (resp., } \mathbb{R}^n \setminus E) \text{ is } n\text{-thick,}\end{array} \tag{5.1.6}$$

and

$$\begin{array}{l}\text{if an } \mathcal{L}^n\text{-measurable set } E \subseteq \mathbb{R}^n \text{ satisfies a two-sided}\\ \text{corkscrew condition then } E \text{ is also two-sided } n\text{-thick.}\end{array} \tag{5.1.7}$$

For example, the von Koch snowflake is a bounded, simply connected, open subset of $\mathbb{R}^2$, satisfying a two-sided corkscrew condition (in fact it is an NTA domain; cf. Definition 5.11.1). In particular, Lemma 5.1.2 and (5.1.6) imply that von Koch snowflake's topological boundary has zero $\mathcal{L}^2$-measure.

Next, let us note that given any set $D \subseteq \mathbb{R}^n$ we always have

$$\mathring{D} \subseteq D \subseteq \overline{D}, \quad \partial(\mathring{D}) \subseteq \partial D, \quad \text{and} \quad \partial(\overline{D}) \subseteq \partial D. \tag{5.1.8}$$

Hence, if D is assumed to satisfy an interior corkscrew condition, then it is clear from Definition 5.1.3 that both $\mathring{D}$ and $\overline{D}$ also satisfy an interior corkscrew condition (with the same constant as D). In addition, any point $x \in \partial D$ is the limit of $z_r(x) \in \mathring{D}$ as $r \to 0^+$. This implies that $\partial D \subseteq \overline{(\mathring{D})}$ hence, further, $\partial D \subseteq \overline{(\mathring{D})} \setminus \mathring{D} = \partial(\mathring{D})$. Bearing in mind (5.1.8), these considerations prove that

[1] Note that $\partial(\mathbb{R}^n \setminus D) = \partial D$, and that having $B\big(z_r(x), \theta r\big) \subseteq B(x, r) \setminus D$ is equivalent to having $B\big(z_r(x), \theta r\big)$ contained in the interior of $B(x, r) \cap (\mathbb{R}^n \setminus D)$, i.e., in $B(x, r) \cap (\mathbb{R}^n \setminus D)^\circ = B(x, r) \setminus \overline{D}$. Hence, having D satisfy an exterior corkscrew condition is equivalent to the existence of some $\theta \in (0, 1)$ such that for each $x \in \partial D$ and each $r \in \big(0, 2\,\mathrm{diam}(\partial D)\big)$ one can find $z_r(x) \in \mathbb{R}^n \setminus D$, corkscrew point relative to x, with the property that $B\big(z_r(x), \theta r\big) \subseteq B(x, r) \setminus \overline{D}$. The latter equivalent reformulation is how actually the exterior corkscrew condition is often defined in the literature.

$$\text{if the subset } D \text{ of } \mathbb{R}^n \text{ satisfies an interior corkscrew condition, then both } \mathring{D} \text{ and } \overline{D} \text{ satisfy an interior corkscrew condition (with the same constant as } D\text{) and } \partial D = \partial(\mathring{D}), \text{ hence also } \partial(\overline{D}) = \partial\big(\overline{\mathring{D}}\big). \tag{5.1.9}$$

Note that condition (5.1.5) may be viewed as a quantitative, scale-invariant, version of the topological property $\partial D = \partial(\mathring{D})$ featuring in (5.1.9). Passing to complements in (5.1.9) also proves that

$$\text{if the subset } D \text{ of } \mathbb{R}^n \text{ satisfies an exterior corkscrew condition, then both } \overline{D} \text{ and } \mathring{D} \text{ satisfy an exterior corkscrew condition (with the same constant as } D\text{) and } \partial D = \partial(\overline{D}), \text{ hence also } \partial(\mathring{D}) = \partial\big(\overline{\mathring{D}}\big). \tag{5.1.10}$$

We conclude with a companion result to Lemma 5.1.2.

Lemma 5.1.4 *Let $\Omega \subseteq \mathbb{R}^n$ be a Lebesgue measurable set which satisfies either an interior corkscrew condition, or an interior corkscrew condition. Then $\mathcal{L}^n(\partial\Omega) = 0$.*

Proof Since the function $\mathbf{1}_{\partial\Omega}$ is Borel measurable and bounded, it follows that $\mathbf{1}_{\partial\Omega} \in L^1_{\rm loc}(\mathbb{R}^n, \mathcal{L}^n)$. Granted this, Lebesgue's Differential Theorem ensures the existence of a Lebesgue measurable set $N \subseteq \mathbb{R}^n$ with $\mathcal{L}^n(N) = 0$ with the property that

$$\mathbf{1}_{\partial\Omega}(x) = \lim_{r\to 0^+} \fint_{B(x,r)} \mathbf{1}_{\partial\Omega}\, d\mathcal{L}^n \quad \text{for each} \quad x \in \mathbb{R}^n \setminus N. \tag{5.1.11}$$

In particular,

$$1 = \lim_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\cap\partial\Omega\big)}{\mathcal{L}^n\big(B(x,r)\big)} \quad \text{for each} \quad x \in \partial\Omega \setminus N. \tag{5.1.12}$$

To fix ideas, suppose Ω satisfies the interior corkscrew condition. This guarantees the existence of some $\theta \in (0,1)$ with the following significance: having fixed $x \in \partial\Omega$ and $r \in \big(0, 2\,{\rm diam}(\partial\Omega)\big)$, we can find $z_r(x) \in \Omega$ with the property that $B\big(z_r(x), \theta r\big)$ is included in $B(x,r)\cap\Omega$. The latter self-improves to $B\big(z_r(x), \theta r\big) \subseteq B(x,r)\cap\mathring{\Omega}$. Bearing in mind that $\partial\Omega$ and $\mathring{\Omega}$ are Lebesgue measurable and disjoint, we may then write

$$\begin{aligned}
\mathcal{L}^n\big(B(x,r)\big) &\geq \mathcal{L}^n\big(B(x,r)\cap\partial\Omega\big) + \mathcal{L}^n\big(B(x,r)\cap\mathring{\Omega}\big) \\
&\geq \mathcal{L}^n\big(B(x,r)\cap\partial\Omega\big) + \mathcal{L}^n\Big(B\big(z_r(x), \theta r\big)\Big) \\
&= \mathcal{L}^n\big(B(x,r)\cap\partial\Omega\big) + \theta^n \cdot \mathcal{L}^n\big(B(x,r)\big).
\end{aligned} \tag{5.1.13}$$

As a consequence,

$$(1-\theta^n)\mathcal{L}^n\big(B(x,r)\big) \geq \mathcal{L}^n\big(B(x,r)\cap\partial\Omega\big) \tag{5.1.14}$$

which, when used back in (5.1.12), leads to the conclusion that $1 < 1 - \theta^n$ if $\partial\Omega \setminus N$ is a nonempty set. Thus, we necessarily have $\partial\Omega \subseteq N$, which ultimately shows that $\mathcal{L}^n(\partial\Omega) = 0$.

Finally, when Ω satisfies an exterior corkscrew condition, the same type of reasoning applied to $\mathbb{R}^n \setminus \Omega$ once again leads to the conclusion that $\mathcal{L}^n(\partial\Omega) = 0$, since Ω and $\mathbb{R}^n \setminus \Omega$ share the same boundary. □

5.2 The Geometric Measure Theoretic Boundary

Given an $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$, its `geometric measure theoretic boundary`, $\partial_* E$, is defined (see, e.g., [80, Definition p. 208]) as

$$\partial_* E := \Big\{ x \in \mathbb{R}^n : \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap E)}{r^n} > 0 \ \text{ and } \ \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \setminus E)}{r^n} > 0 \Big\}. \tag{5.2.1}$$

Hence, near points in $\partial_* E$ there is enough mass both in E and in $\mathbb{R}^n \setminus E$ (relative to the scale). Alternatively,

$$\partial_* E = \Big\{ x \in \mathbb{R}^n : \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} > 0 \ \text{ and } \ \liminf_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} < 1 \Big\}. \tag{5.2.2}$$

It is then clear from this definition that for each $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$ we have

$$\partial_* E \subseteq \partial E \ \text{ and } \ \partial_* E = \partial_*(\mathbb{R}^n \setminus E), \tag{5.2.3}$$

and

$\partial_* E = \partial E$ if E is two-sided n-thick (hence, in particular, if E satisfies a two-sided corkscrew condition). (5.2.4)

Also,

for any $\mathcal{L}^n$-measurable sets $E_1, E_2 \subseteq \mathbb{R}^n$ satisfying $\mathcal{L}^n(E_1 \,\Delta\, E_2) = 0$ we necessarily have $\partial_* E_1 = \partial_* E_2$. (5.2.5)

From [80, Lemma 2, p. 222] we know that

any $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$ has the property that $\partial_* E$ is a Borel-measurable subset of $\mathbb{R}^n$. (5.2.6)

Lemma 5.2.1 *For each $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$ one has $\mathcal{L}^n(\partial_* E) = 0$.*

Proof For starters, (5.2.6) guarantees that $\partial_* E$ is a Borel-measurable subset of $\mathbb{R}^n$. In particular, $\partial_* E$ is $\mathcal{L}^n$-measurable. Since we have $E \cap \partial_* E \subseteq E \setminus \mathrm{int}_*(E)$ (cf. (2.8.21)), it follows that

$$\mathcal{L}^n\big(E \cap \partial_* E\big) \leq \mathcal{L}^n\big(E \setminus \mathrm{int}_*(E)\big) = 0, \tag{5.2.7}$$

with the equality provided by (2.8.22). Hence,

$$\mathcal{L}^n\big(E \cap \partial_* E\big) = 0 \quad \text{for every } \mathcal{L}^n\text{-measurable set } \ E \subseteq \mathbb{R}^n. \tag{5.2.8}$$

Writing this for $\mathbb{R}^n \setminus E$ in place of E then gives (bearing in mind (5.2.3))

$$0 = \mathcal{L}^n\big((\mathbb{R}^n \setminus E) \cap \partial_*(\mathbb{R}^n \setminus E)\big) = \mathcal{L}^n\big((\mathbb{R}^n \setminus E) \cap \partial_* E\big), \tag{5.2.9}$$

which together with (5.2.8) ultimately proves that $\mathcal{L}^n(\partial_* E) = 0$. □

We next consider how the geometric measure theoretic boundary behaves under set-theoretic operations. Related properties may be found in [216, Sect. 4.2, pp. 51–53].

Lemma 5.2.2 *For any $\mathcal{L}^n$-measurable sets $E, F \subseteq \mathbb{R}^n$ one has*

$$\partial_*(E \cap F) \subseteq \big[(\partial_* E) \cap \overline{F}\,\big] \cup \big[(\partial_* F) \cap \overline{E}\,\big], \tag{5.2.10}$$

$$\partial_*(E \cup F) \subseteq \big[(\partial_* E) \setminus \mathring{F}\big] \cup \big[(\partial_* F) \setminus \mathring{E}\big], \tag{5.2.11}$$

$$\partial_*(E \cup F) \cup \partial_*(E \cap F) \cup \partial_*(E \setminus F) \cup \partial_*(F \setminus E) \subseteq \partial_* E \cup \partial_* F, \tag{5.2.12}$$

$$\partial_*(E \cup F) \cap \partial_*(E \cap F) \subseteq \partial_* E \cap \partial_* F. \tag{5.2.13}$$

Proof Fix two arbitrary $\mathcal{L}^n$-measurable sets $E, F \subseteq \mathbb{R}^n$. In a first stage, we claim that

$$\partial_*(E \cap F) \subseteq \partial_* E \cup \partial_* F. \tag{5.2.14}$$

To justify this, reason by contradiction and assume that there exists $x \in \partial_*(E \cap F)$ such that $x \notin \partial_* E$ and $x \notin \partial_* F$. The lack of membership of x to $\partial_* E$ implies that either

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap E)}{r^n} = 0, \tag{5.2.15}$$

or (with $E^c := \mathbb{R}^n \setminus E$ denoting the complement of E in $\mathbb{R}^n$)

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap E^c)}{r^n} = 0. \tag{5.2.16}$$

Note that the eventuality (5.2.15) cannot materialize since this would imply

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\cap(E\cap F)\big)}{r^n} = 0 \tag{5.2.17}$$

which, in turn, would prevent x from belonging to $\partial_*(E\cap F)$. Likewise, the failure of membership of x to $\partial_* F$ implies that either

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap F)}{r^n} = 0, \tag{5.2.18}$$

or

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap F^c)}{r^n} = 0. \tag{5.2.19}$$

Once again, (5.2.18) cannot happen since this would prevent x from belonging to $\partial_*(E\cap F)$. As such, we conclude that both (5.2.16) and (5.2.19) hold. This further implies

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\setminus(E\cap F)\big)}{r^n} = \limsup_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\cap(E^c\cup F^c)\big)}{r^n} = 0 \tag{5.2.20}$$

which, in turn, conflicts with the membership of x to $\partial_*(E\cap F)$. This contradiction establishes the claim made in (5.2.14).

In a second stage we observe that (thanks to the first formula in (5.2.3) and simple topology)

$$\partial_*(E\cap F)\subseteq\partial(E\cap F)\subseteq\overline{E}\cap\overline{F}. \tag{5.2.21}$$

Then (5.2.10) follows by combining (5.2.14) with (5.2.21) (keeping in mind that $\partial_* E\subseteq\partial E\subseteq\overline{E}$, plus similar inclusions for F). Next, (5.2.11) is obtained by writing (5.2.10) for E^c, F^c in place of E, F and bearing in mind the second formula in (5.2.3) (as well as the fact that taking complements intertwines the closure with the interior of an arbitrary set in the Euclidean setting). Going further, formula (5.2.12) is a direct consequence of (5.2.10)–(5.2.11) (used also with E and/or F replaced by their complements).

Lastly, as far as (5.2.13) is concerned, due to the symmetric role played by E and F in the left-hand side of (5.2.13), it suffices to show that

$$\partial_*(E\cup F)\cap\partial_*(E\cap F)\subseteq\partial_* E. \tag{5.2.22}$$

This, however, may be justified reasoning by contradiction (much as in the first part of the current proof). □

The geometric measure theoretic boundary also satisfies a locality property of the sort described in the lemma below.

Proposition 5.2.3 *If $E \subseteq \mathbb{R}^n$ is an $\mathcal{L}^n$-measurable set and $\mathcal{O} \subseteq \mathbb{R}^n$ is an open set, then*

$$\mathcal{O} \cap \partial_* E = \mathcal{O} \cap \partial_*(\mathcal{O} \cap E). \tag{5.2.23}$$

Proof Pick an arbitrary point $x \in \mathcal{O} \cap \partial_* E$. Since for each $r > 0$ sufficiently small we have (again, with the superscript "c" denoting the complement relative to $\mathbb{R}^n$)

$$\begin{aligned} B(x,r) \cap \big(\mathcal{O} \cap E\big) = B(x,r) \cap E, \quad \text{as well as} \\ B(x,r) \cap \big(\mathcal{O} \cap E\big)^c = B(x,r) \cap \big(\mathcal{O}^c \cup E^c\big) = B(x,r) \cap E^c, \end{aligned} \tag{5.2.24}$$

it follows that

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^n\Big(B(x,r) \cap \big(\mathcal{O} \cap E\big)\Big)}{r^n} = \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap E)}{r^n} > 0 \tag{5.2.25}$$

and

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^n\Big(B(x,r) \cap \big(\mathcal{O} \cap E\big)^c\Big)}{r^n} = \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r) \cap E^c)}{r^n} > 0. \tag{5.2.26}$$

Collectively, (5.2.25) and (5.2.26) prove that $x \in \partial_*(\mathcal{O} \cap E)$ hence, ultimately,

$$\mathcal{O} \cap \partial_* E \subseteq \partial_*(\mathcal{O} \cap E). \tag{5.2.27}$$

On the other hand, from Lemma 5.2.2 and (5.2.3) we see that

$$\partial_*(\mathcal{O} \cap E) \subseteq \partial_*\mathcal{O} \cup \partial_* E \subseteq \partial\mathcal{O} \cup \partial_* E. \tag{5.2.28}$$

Intersecting the most extreme sides with $\mathcal{O}$ and bearing in mind that $\mathcal{O} \cap \partial\mathcal{O} = \varnothing$ (since $\mathcal{O}$ is open) then yields

$$\mathcal{O} \cap \partial_*(\mathcal{O} \cap E) \subseteq \mathcal{O} \cap \partial_* E. \tag{5.2.29}$$

At this stage, (5.2.23) readily follows from (5.2.27) and (5.2.29). □

5.3 Area/Coarea Formulas, and Countable Rectifiability

Vector-valued Lipschitz functions map sets of zero Hausdorff measure from the Euclidean space containing the domain of the function into sets of zero Hausdorff measure in the target Euclidean ambient. In fact, if $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ is Lipschitz, then for each $s \in [0, \infty)$ and $E \subseteq A$ we have[2]

$$\mathcal{H}^s_*\big(f(E)\big) \leq \Bigg[\sup_{\substack{x,y\in A\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}\Bigg]^s \mathcal{H}^s_*(E) \tag{5.3.1}$$

(compare with [162, Theorem 3.1.2, p. 61]). We continue by recording a basic approximation result of Lipschitz functions (cf. [80, Theorem 1, p. 251]).

Proposition 5.3.1 *Fix $n \in \mathbb{N}$. Then there exists a dimensional constant $C_n \in (0, \infty)$ with the property that for each Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and each $\varepsilon > 0$ there exists a Lipschitz function $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ which is also continuously differentiable and satisfies*

$$\mathcal{L}^n\Big(\big\{x \in \mathbb{R}^n : \ f_\varepsilon(x) \neq f(x) \ \text{ or } \ \nabla f_\varepsilon(x) \neq \nabla f(x)\big\}\Big) < \varepsilon, \tag{5.3.2}$$

as well as

$$\sup_{z\in\mathbb{R}^n} |(\nabla f_\varepsilon)(z)| \leq C_n \cdot \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}. \tag{5.3.3}$$

Next, we recall some basic change of variable formulas (see, e.g., [88], [80, Theorem 2 on p. 99, and Theorem 2 on p. 117], [162, Theorem 4.1.2 on p. 106, and Theorem 4.2.1 on p. 111]).

Proposition 5.3.2 *Fix $n, m \in \mathbb{N}$ and assume $f = (f_1, \dots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued Lipschitz function. If $Df := (\partial f_j/\partial x_k)_{\substack{1\leq j\leq m\\ 1\leq k\leq n}}$ is the Jacobian $m \times n$ matrix of first-order partial derivatives of the scalar components of f, define the Jacobian determinant Jf of f as*

$$Jf := \begin{cases} \sqrt{\det\,[(Df)^\top(Df)]} & \text{if } \ n \leq m, \\ \sqrt{\det\,[(Df)(Df)^\top]} & \text{if } \ n \geq m. \end{cases} \tag{5.3.4}$$

Also, fix an $\mathcal{L}^n$-measurable set $A \subseteq \mathbb{R}^n$ along with a function $g \in L^1(A, \mathcal{L}^n)$.

(i) If $n \leq m$ then

[2] Recall that for each $s \geq 0$ we have agreed to denote by $\mathcal{H}^s_*$ the s-dimensional Hausdorff outer-measure in $\mathbb{R}^n$.

for $\mathcal{H}^n$-a.e. point $y \in \mathbb{R}^m$ the set $f^{-1}(\{y\})$ is at most countable and $\sum_{x\in A\cap f^{-1}(\{y\})} |g(x)| < \infty$, (5.3.5)

and the following change of variable formula holds:

$$\int_A g(x)(Jf)(x)\,\mathrm{d}\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \Big(\sum_{x\in A\cap f^{-1}(\{y\})} g(x)\Big)\,\mathrm{d}\mathcal{H}^n(y). \tag{5.3.6}$$

(ii) If $n \geq m$ then

$$g\Big|_{A\cap f^{-1}(\{y\})} \text{ is } \mathcal{H}^{n-m}\text{-summable for } \mathcal{L}^m\text{-a.e. } y \in \mathbb{R}^m \tag{5.3.7}$$

and the following change of variable formula holds:

$$\int_A g(x)(Jf)(x)\,\mathrm{d}\mathcal{L}^n(x) = \int_{\mathbb{R}^m} \Big(\int_{A\cap f^{-1}(\{y\})} g(x)\,\mathrm{d}\mathcal{H}^{n-m}(x)\Big)\,\mathrm{d}\mathcal{L}^m(y). \tag{5.3.8}$$

In the particular case when $g \equiv 1$, the identities in (5.3.6) and (5.3.8) are typically referred to as the Area Formula and the Coarea Formula, respectively. Specializing Proposition 5.3.2 to the case when $A := \{x \in \mathbb{R}^n : Jf(x) = 0\}$ and $g \equiv 1$ yields variants of the Morse–Sard Theorem. Specifically, the Coarea Formula implies that

if $n, m \in \mathbb{N}$ satisfy $n \geq m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz then for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$ one has $\mathcal{H}^{n-m}\big(\{x \in f^{-1}(\{y\}) : Jf(x) = 0\}\big) = 0$, (5.3.9)

while the Area Formula gives that

if $n, m \in \mathbb{N}$ satisfy $n \leq m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function then one has $\mathcal{H}^n\Big(f\big(\{x \in \mathbb{R}^n : Jf(x) = 0\}\big)\Big) = 0$. (5.3.10)

The fact that $\{x \in \mathbb{R}^n : Jf(x) = 0\}$ is unambiguously defined (thanks to the classical Rademacher Theorem) only up to a $\mathcal{L}^n$-nullset does not create any issues in (5.3.9) or (5.3.10). Indeed, in the case of (5.3.9) this follows by noting that the Coarea Formula gives

if $n, m \in \mathbb{N}$ satisfy $n \geq m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz function then for any $\mathcal{L}^n$-nullset $A \subseteq \mathbb{R}^n$ one has $\mathcal{H}^{n-m}\big(A \cap f^{-1}(\{y\})\big) = 0$ for $\mathcal{L}^m$-a.e. point $y \in \mathbb{R}^m$, (5.3.11)

while in the case of (5.3.10) this is seen directly from (5.3.1).

In relation to (5.3.10), we also wish to make the following observation. Recall that in general if $n, m \in \mathbb{N}$ satisfy $n \leq m$ and M is a real $m \times n$ matrix, then $\sqrt{\det(M^\top M)}$ is the $\mathcal{H}^n$-measure of the parallelepiped spanned in $\mathbb{R}^m$ by the n columns of M, (5.3.12)

which further implies that

$$\text{if } n, m \in \mathbb{N} \text{ satisfy } n \leq m \text{ and } M \text{ is a real } m \times n \text{ matrix, then } \det(M^\top M) = 0 \text{ if and only if } \operatorname{rank} M < n. \tag{5.3.13}$$

Using (5.3.13) with $M := Df$ in the context of (5.3.10) then gives

$$\text{if } n, m \in \mathbb{N} \text{ satisfy } n \leq m \text{ and } f : \mathbb{R}^n \to \mathbb{R}^m \text{ is Lipschitz, one has } \mathcal{H}^n\Big(f\big(\{x \in \mathbb{R}^n : \operatorname{rank}(Df)(x) < n\}\big)\Big) = 0. \tag{5.3.14}$$

Moving on, we discuss the notion of countable rectifiability. Recall that $\Sigma \subset \mathbb{R}^2$ is a rectifiable curve provided Σ is the image of a continuous function γ defined on an interval $[a, b]$ which has bounded total variation, i.e.,

$$\text{length}\,\gamma := \sup \sum_{j=1}^{N} |\gamma(t_j) - \gamma(t_{j-1})| < +\infty, \tag{5.3.15}$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_N = b$ of the interval $[a, b]$. It turns out that $\Sigma \subset \mathbb{R}^2$ is a rectifiable curve if and only if $\Sigma = f(I)$ for some bounded interval $I \subset \mathbb{R}$ and some Lipschitz map $f : I \to \mathbb{R}^2$ (this is a result originally established in [258]; see also the discussion in [8, Theorem 4.4, p. 49]). The higher-dimensional version of this notion is as follows. A $\mathcal{H}^{n-1}$-measurable set $\Sigma \subset \mathbb{R}^n$ is called `countably rectifiable` (of dimension $n - 1$) provided

$$\Sigma \subseteq \Big(\bigcup_{k=1}^{\infty} f_k(\mathbb{R}^{n-1})\Big) \cup N, \tag{5.3.16}$$

where $\mathcal{H}^{n-1}(N) = 0$ and each $f_k : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is a Lipschitz function. See, e.g., [170, p. 96], as well as [162, Definition 3.3.1, p. 71] and the subsequent comment; compare also with [177, Definition 15.3, p. 204].

By McShane's Extension Theorem and the regularity properties of Radon measures, it turns out that a $\mathcal{H}^{n-1}$-measurable set $\Sigma \subset \mathbb{R}^n$ is countably rectifiable (of dimension $n - 1$) provided

$$\Sigma = \Big(\bigcup_{k=1}^{\infty} f_k(F_k)\Big) \cup N, \tag{5.3.17}$$

where $\mathcal{H}^{n-1}(N) = 0$ and for each $k \in \mathbb{N}$, the function $f_k : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is Lipschitz and the set $F_k \subseteq \mathbb{R}^{n-1}$ is Borel. See, e.g., [170, (10.4), p. 97]. To summarize, any given $\mathcal{H}^{n-1}$-measurable set $\Sigma \subset \mathbb{R}^n$ is countably rectifiable (of dimension $n - 1$) if (and only if) it may be written as a countable union

$$\Sigma = \Big(\bigcup_{k=1}^{\infty} L_k\Big) \cup N, \tag{5.3.18}$$

where $\mathcal{H}^{n-1}(N) = 0$ and each L_k is the image of a Borel subset of $\mathbb{R}^{n-1}$ under a $\mathbb{R}^n$-valued Lipschitz map. In fact (cf., e.g., [162, Lemma 3.3.2, p. 71], or [238, Lemma 11.1, p. 59]), (5.3.18) self-improves to

$$\Sigma = \Big(\bigcup_{k\in\mathbb{N}} M_k\Big) \cup N \tag{5.3.19}$$

where the residual set $N \subseteq \mathbb{R}^n$ satisfies $\mathcal{H}^{n-1}(N) = 0$ and each M_k is a Borel subset of an $(n-1)$-dimensional $\mathscr{C}^1$ submanifold[3] of $\mathbb{R}^n$. In Proposition 5.3.3 below we discuss a more precise version of the structure theorem for countably rectifiable sets than the one recorded in (5.3.19). To facilitate stating this proposition, we first make a definition. Concretely, given a function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ and some $i \in \{1, \dots, n\}$, we agree to define the graph of ϕ relative to the i-th coordinate axis as being the set $\mathscr{G}_\phi^{(i)} \subseteq \mathbb{R}^n$ described as

$$\mathscr{G}_\phi^{(i)} := \big\{(x_1, \dots, x_{i-1}, \phi(x'), x_{i+1}, \dots, x_n) : \\ x' = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}\big\}. \tag{5.3.20}$$

Proposition 5.3.3 *Let $\Sigma \subset \mathbb{R}^n$ be a $\mathcal{H}^{n-1}$-measurable set which is countably rectifiable (of dimension $n-1$). Then there exist a sequence of real-valued functions $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{C}_c^1(\mathbb{R}^{n-1})$, a sequence of integers $\{i_j\}_{j\in\mathbb{N}} \subseteq \{1, \dots, n\}$, and a set $N \subseteq \mathbb{R}^n$ with $\mathcal{H}^{n-1}(N) = 0$, such that*

$$\Sigma \subseteq \Big(\bigcup_{j=1}^{\infty} \mathscr{G}_{\phi_j}^{(i_j)}\Big) \cup N. \tag{5.3.21}$$

Proof Thanks to (5.3.18), it suffices to show that if $F : \mathbb{R}^{n-1} \to \mathbb{R}^n$ is a Lipschitz function then there exist $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{C}_c^1(\mathbb{R}^{n-1})$ along with $\{i_j\}_{j\in\mathbb{N}} \subseteq \{1, \dots, n\}$ and $N \subseteq \mathbb{R}^n$ with $\mathcal{H}^{n-1}(N) = 0$ such that

$$F(\mathbb{R}^{n-1}) \subseteq \Big(\bigcup_{j=1}^{\infty} \mathscr{G}_{\phi_j}^{(i_j)}\Big) \cup N. \tag{5.3.22}$$

[3] Relatively open.

With this goal in mind observe that, for each natural number j, Proposition 5.3.1 guarantees the existence of a Lipschitz function $f_j : \mathbb{R}^{n-1} \to \mathbb{R}^n$ which is also continuously differentiable and satisfies

$$\mathcal{L}^{n-1}\Big(\big\{x' \in \mathbb{R}^{n-1} : f_j(x') \neq F(x')\big\}\Big) < j^{-1}. \tag{5.3.23}$$

Then

$$\mathcal{L}^{n-1}\Big(\bigcap_{j\in\mathbb{N}} \big\{x' \in \mathbb{R}^{n-1} : f_j(x') \neq F(x')\big\}\Big) = 0 \tag{5.3.24}$$

which, in concert with (5.3.1), implies that

$$F\Big(\bigcap_{j\in\mathbb{N}} \big\{x' \in \mathbb{R}^{n-1} : f_j(x') \neq F(x')\big\}\Big) \text{ is an } \mathcal{H}^{n-1}\text{-nullset in } \mathbb{R}^n. \tag{5.3.25}$$

Since the above set contains

$$A := F(\mathbb{R}^{n-1}) \setminus \Big(\bigcup_{j\in\mathbb{N}} f_j(\mathbb{R}^{n-1})\Big), \tag{5.3.26}$$

we conclude that

$$F(\mathbb{R}^{n-1}) \subseteq \Big(\bigcup_{j\in\mathbb{N}} f_j(\mathbb{R}^{n-1})\Big) \cup A, \quad \text{with } \mathcal{H}^{n-1}(A) = 0. \tag{5.3.27}$$

To proceed, for each $j \in \mathbb{N}$ define the closed set

$$Z_j := \big\{x' \in \mathbb{R}^{n-1} : \operatorname{rank}(Df_j)(x') < n-1\big\} \tag{5.3.28}$$

and observe that, thanks to (5.3.14), we have

$$\mathcal{H}^{n-1}\big(f_j(Z_j)\big) = 0 \text{ for each } j \in \mathbb{N}. \tag{5.3.29}$$

Let us temporarily fix $j_* \in \mathbb{N}$ and $x'_* \in \mathbb{R}^{n-1} \setminus Z_{j_*}$. If we write $(f^{(1)}_{j_*}, \dots, f^{(n)}_{j_*})$ for the scalar components of f_{j_*} then the latter membership ensures the existence of some $i_{j_*, x'_*} \in \{1, \dots, n\}$ such that

$$\det\left(\frac{D\big(f^{(1)}_{j_*}, \dots, f^{(i_{j_*,x'_*}-1)}_{j_*}, f^{(i_{j_*,x'_*}+1)}_{j_*}, \dots, f^{(n)}_{j_*}\big)}{D(x'_1, \dots, x'_{n-1})}\right)(x'_*) \neq 0. \tag{5.3.30}$$

Let $\pi' : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the coordinate projection mapping defined by

$$\pi'(x) := \big(x_1, \dots, x_{i_{j_*,x'_*}-1}, x_{i_{j_*,x'_*}+1}, \dots, x_n\big), \ \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{5.3.31}$$

and consider the function

$$f'_{j_*} := \pi' \circ f_{j_*} = \big(f_{j_*}^{(1)}, \dots, f_{j_*}^{(i_{j_*,x'_*}-1)}, f_{j_*}^{(i_{j_*,x'_*}+1)}, \dots, f_{j_*}^{(n)}\big) : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}. \tag{5.3.32}$$

Given that f_{j_*} is of class $\mathscr{C}^1$, it follows that f'_{j_*} is of class $\mathscr{C}^1$ as well. In fact, by virtue of the Inverse Function Theorem (whose applicability is ensured by (5.3.30)),

$$\text{there exists an open set } O \subseteq \mathbb{R}^{n-1} \setminus Z_{j_*} \text{ such that } x'_* \in O, \text{ the set } f'_{j_*}(O) \text{ is an open neighborhood of } \pi'\big(f_{j_*}(x'_*)\big) \text{ in } \mathbb{R}^{n-1}, \text{ and the function } f'_{j_*} : O \longrightarrow f'_{j_*}(O) \text{ is a } \mathscr{C}^1 \text{ diffeomorphism.} \tag{5.3.33}$$

Then

$$\big(f'_{j_*}\big)^{-1} : f'_{j_*}(O) \longrightarrow O \quad \text{is a function of class } \mathscr{C}^1, \tag{5.3.34}$$

hence

$$f_{j_*} \circ \big(f'_{j_*}\big)^{-1} : f'_{j_*}(O) \longrightarrow f_{j_*}(O) \quad \text{is well defined.} \tag{5.3.35}$$

We claim that

$$\text{the mapping (5.3.35) is bijective and its inverse is the function } \pi' : f_{j_*}(O) \longrightarrow f'_{j_*}(O). \tag{5.3.36}$$

To justify this, note that $\pi' \circ f_{j_*} \circ \big(f'_{j_*}\big)^{-1} = f'_{j_*} \circ \big(f'_{j_*}\big)^{-1}$ is the identity on $f'_{j_*}(O)$, whereas if $x = f_{j_*}(x') \in f_{j_*}(O)$ for some $x' \in O$, then

$$\begin{aligned}\Big(f_{j_*} \circ \big(f'_{j_*}\big)^{-1}\Big)\big(\pi'(x)\big) &= \Big(f_{j_*} \circ \big(f'_{j_*}\big)^{-1}\Big)\big(\pi' \circ f_{j_*}\big)(x') \\ &= \Big(f_{j_*} \circ \big(f'_{j_*}\big)^{-1} \circ f'_{j_*}\Big)(x') = f_{j_*}(x') = x,\end{aligned} \tag{5.3.37}$$

proving the claim in (5.3.36). Define next

$$\psi := f_{j_*}^{(i_{j_*,x'_*})} \circ \big(f'_{j_*}\big)^{-1} : f'_{j_*}(O) \longrightarrow \mathbb{R}. \tag{5.3.38}$$

Then ψ is a function of class $\mathscr{C}^1$, and (5.3.36) implies that

$$\begin{gathered}\Big(f_{j_*} \circ \big(f'_{j_*}\big)^{-1}\Big)(x') = \big(x_1, \dots, x_{i_{j_*,x'_*}-1}, \psi(x'), x_{i_{j_*,x'_*}+1}, \dots, x_n\big), \\ \text{for each point } \; x' = (x_1, \dots, x_{i_{j_*,x'_*}-1}, x_{i_{j_*,x'_*}+1}, x_n) \in f'_{j_*}(O).\end{gathered} \tag{5.3.39}$$

In turn, from (5.3.39) we conclude that

$$\begin{aligned}f_{j_*}(O) = \Big\{\big(x_1, \dots, x_{i_{j_*,x'_*}-1}, \psi(x'), x_{i_{j_*,x'_*}+1}, \dots, x_n\big) : \\ x' = (x_1, \dots, x_{i_{j_*,x'_*}-1}, x_{i_{j_*,x'_*}+1}, x_n) \in f'_{j_*}(O)\Big\}.\end{aligned} \tag{5.3.40}$$

Having picked a cutoff function $\xi \in \mathscr{C}_c^1\big(f'_{j_*}(O)\big)$ which is identically one near $\pi'\big(f_{j_*}(x'_*)\big)$, let ϕ_{j_*,x'_*} be the function defined as the extension of $\xi\psi$ by zero outside of $f'_{j_*}(O)$ to the entire $\mathbb{R}^{n-1}$. Then $\phi_{j_*,x'_*} \in \mathscr{C}_c^1(\mathbb{R}^{n-1})$ is real-valued and coincides with ψ near $\pi'\big(f_{j_*}(x'_*)\big)$. In particular, by further shrinking O to an open the neighborhood O_{j_*,x'_*} of x'_* in $\mathbb{R}^{n-1}$ it follows from (5.3.40) that

$$f_{j_*}(O_{j_*,x'_*}) \subseteq \mathscr{G}_{\phi_{j_*,x'_*}}^{(i_{j_*,x'_*})}. \tag{5.3.41}$$

To summarize,

$$\begin{array}{l}\text{for each number } j_* \in \mathbb{N} \text{ and each point } x'_* \in \mathbb{R}^{n-1} \setminus Z_{j_*}, \text{ we have}\\ \text{identified some real-valued function } \phi_{j_*,x'_*} \in \mathscr{C}_c^1(\mathbb{R}^{n-1}), \text{ along with an}\\ \text{integer } i_{j_*,x'_*} \in \{1,\dots,n\} \text{ and an open neighborhood } O_{j_*,x'_*} \text{ of } x'_* \text{ con-}\\ \text{tained in } \mathbb{R}^{n-1} \setminus Z_{j_*}, \text{ with the property that (5.3.41) holds.}\end{array} \tag{5.3.42}$$

Tautologically, for each $j_* \in \mathbb{N}$ fixed it follows that

$$\big\{O_{j_*,x'_*}\big\}_{x'_* \in \mathbb{R}^{n-1}\setminus Z_{j_*}} \quad \text{is an open cover of } \ \mathbb{R}^{n-1} \setminus Z_{j_*}. \tag{5.3.43}$$

Since any second-countable space is a strongly Lindelöf space, this open cover has a countable sub-cover. Hence, there exists $\big\{x'_{j_*,k}\big\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^{n-1} \setminus Z_{j_*}$ such that

$$\mathbb{R}^{n-1} \setminus Z_{j_*} = \bigcup_{k\in\mathbb{N}} O_{j_*,x'_{j_*,k}}. \tag{5.3.44}$$

Then for each $j_* \in \mathbb{N}$ fixed, based on (5.3.41) and (5.3.44) we may write

$$\begin{aligned} f_{j_*}(\mathbb{R}^{n-1}) &\subseteq f_{j_*}(\mathbb{R}^{n-1} \setminus Z_{j_*}) \cup f_{j_*}(Z_{j_*}) \\ &\subseteq \Big(\bigcup_{k\in\mathbb{N}} f_{j_*}\big(O_{j_*,x'_{j_*,k}}\big)\Big) \cup f_{j_*}(Z_{j_*}) \\ &\subseteq \Big(\bigcup_{k\in\mathbb{N}} \mathscr{G}_{\phi_{j_*,x'_{j_*,k}}}^{(i_{j_*,x'_{j_*,k}})}\Big) \cup f_{j_*}(Z_{j_*}). \end{aligned} \tag{5.3.45}$$

At this stage, (5.3.22) follows from (5.3.27), (5.3.29), and (5.3.45). □

It is easy to see (cf., e.g., [177, Lemma 15.4, p. 204]) that

(1) every countably rectifiable set of dimension $n-1$ has sigma-finite $\mathcal{H}^{n-1}$ measure;
(2) any subset of a countably rectifiable set (of dimension $n-1$) is itself a countably rectifiable set (of dimension $n-1$);
(3) any countable union of countably rectifiable sets (of dimension $n-1$) is itself a countably rectifiable set (of dimension $n-1$);

(4) if $\Sigma \subseteq \mathbb{R}^n$ is a countably rectifiable set (of dimension $n-1$) then there exists $B \subseteq \mathbb{R}^n$, a countably rectifiable set (of dimension $n-1$) which is also Borel, such that $\Sigma \subseteq B$ and $\mathcal{H}^{n-1}(\Sigma) = \mathcal{H}^{n-1}(B)$.

From (5.3.19) (or (5.3.21)) it follows that

$$\text{if } \Sigma \subseteq \mathbb{R}^n \text{ is a countably rectifiable set (of dimension } n-1\text{) then either its Hausdorff dimension is } n-1\text{, or } \mathcal{H}^{n-1}(\Sigma) = 0. \tag{5.3.46}$$

A result of X. Tolsa (cf. [254]) gives that if an $\mathcal{H}^{n-1}$-measurable set $\Sigma \subseteq \mathbb{R}^n$ has $\mathcal{H}^{n-1}(\Sigma) < +\infty$ then

Σ is countably rectifiable (of dimension $n-1$) if and only if for each index $j \in \{1, \dots, n\}$ the principal-value limit

$$\lim_{\varepsilon \to 0^+} \int_{\substack{y \in \Sigma \\ |y-x|>\varepsilon}} \frac{x_j - y_j}{|x-y|^n} \, d\mathcal{H}^{n-1}(y) \tag{5.3.47}$$

exists (in $\mathbb{R}$) at $\mathcal{H}^{n-1}$-a.e. point x belonging to the set Σ.

A more general result of similar flavor is contained in the proposition below.

Proposition 5.3.4 *Suppose $k \in \mathscr{C}^2(\mathbb{R}^n \setminus \{0\})$ is an odd complex-valued function with the property that there exists a constant $C \in (0, \infty)$ such that for each point $x \in \mathbb{R}^n \setminus \{0\}$ one has*

$$|k(x)| \le C|x|^{1-n}, \quad |(\nabla k)(x)| \le C|x|^{-n}, \quad |(\nabla^2 k)(x)| \le C|x|^{-1-n}. \tag{5.3.48}$$

Also, assume μ is a signed (or complex) Radon measure in $\mathbb{R}^n$ (cf. Definition 3.5.5) whose total variation, denoted by $|\mu|$, satisfies

$$\int_{\mathbb{R}^n} \frac{d|\mu|(x)}{1+|x|^{n-1}} < +\infty. \tag{5.3.49}$$

Finally, let $\Sigma \subseteq \mathbb{R}^n$ be a $\mathcal{H}^{n-1}$-measurable set which is countably rectifiable (of dimension $n-1$) and satisfies $\mathcal{H}^{n-1}(B \cap \Sigma) < +\infty$ for each open ball $B \subseteq \mathbb{R}^n$. Then the limit

$$\lim_{\varepsilon \to 0^+} \int_{\substack{y \in \mathbb{R}^n \\ |x-y|>\varepsilon}} k(x-y) \, d\mu(y) \quad \textit{exists for } \mathcal{H}^{n-1}\textit{-a.e. } x \in \Sigma. \tag{5.3.50}$$

Before presenting the proof of Proposition 5.3.4 we make two comments. First, each truncated integral in (5.3.50) is absolutely convergent, thanks to (5.3.49) and the first estimate in (5.3.48). Second, in [177, Theorem 20.27, p. 302] a similar result is established under more restrictive conditions. Specifically, one now demands that μ is a complex Radon measure in $\mathbb{R}^n$ (cf. (3.5.10)), that $\mathcal{H}^{n-1}(\Sigma) < +\infty$, the odd function k now belongs to $\mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ and, in place of (5.3.48), one now asks that for each $j \in \mathbb{N}_0$ there exists $C_j \in (0, \infty)$ such that

$$|(\nabla^j k)(x)| \leq C_j |x|^{1-j-n} \quad \text{for all} \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{5.3.51}$$

Here is the proof of Proposition 5.3.4.

Proof of Proposition 5.3.4 Fix an even function $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ with the property that $\operatorname{supp}\psi \subseteq B(0, 2)$ and $\psi \equiv 1$ on $B(0, 1)$. Use this to decompose $k = k_0 + k_1$ where $k_0 := \psi k$ and $k_1 := (1 - \psi)k$ in $\mathbb{R}^n \setminus \{0\}$. In particular, the function $k_0 \in \mathscr{C}^2(\mathbb{R}^n \setminus \{0\})$ is odd and there exists some $C \in (0, \infty)$ such that for each $x \in \mathbb{R}^n \setminus \{0\}$ we have

$$|k_0(x)| \leq C|x|^{1-n}, \quad |(\nabla k_0)(x)| \leq C|x|^{-n}, \quad |(\nabla^2 k_0)(x)| \leq C|x|^{-1-n}. \tag{5.3.52}$$

Also,

$$\begin{aligned} k_1 \in \mathscr{C}^2(\mathbb{R}^n), \quad k_1 \equiv 0 \ \text{ on } \ \overline{B(0, 1)} \ \text{ and} \\ |k_1(x)| \leq C\big(1 + |x|^{n-1}\big)^{-1} \ \text{ for all } \ x \in \mathbb{R}^n. \end{aligned} \tag{5.3.53}$$

To proceed, pick an arbitrary $R \in (0, \infty)$ and choose some $\varepsilon \in (0, 1)$. Keeping in mind the first property in (5.3.53), for each point $x \in B(0, R) \cap \Sigma$ we may then write

$$\int\limits_{\substack{y\in\mathbb{R}^n \\ |x-y|>\varepsilon}} k(x - y)\, \mathrm{d}\mu(y) = \int\limits_{\substack{y\in\mathbb{R}^n \\ 1>|x-y|>\varepsilon}} k_0(x - y)\, \mathrm{d}\mu(y) + \int\limits_{\substack{y\in\mathbb{R}^n \\ |x-y|\geq 1}} k(x - y)\, \mathrm{d}\mu(y). \tag{5.3.54}$$

Thanks to the first estimate in (5.3.48) and the first estimate in (5.3.52), all integrals above are absolutely convergent. Also, for each $y \in \mathbb{R}^n$ with $1 > |x - y|$ we have $|y| \leq |x - y| + |x| < R + 1$ which places y in $B(0, R + 1)$. Note that

$$\begin{aligned} &\text{the set } E := \Sigma \cap B(0, R + 1) \text{ is } \mathcal{H}^{n-1}\text{-measurable, count-} \\ &\text{ably rectifiable (of dimension } n - 1\text{), and } \mathcal{H}^{n-1}(E) < +\infty. \end{aligned} \tag{5.3.55}$$

From (5.3.49) we also know that $\mu_R := \mu\lfloor B(0, R + 1)$ is a complex Radon measure in $\mathbb{R}^n$ (of finite total variation). Decompose

$$\begin{aligned} \int\limits_{\substack{y\in\mathbb{R}^n \\ 1>|x-y|>\varepsilon}} k_0(x - y)\, \mathrm{d}\mu(y) &= \int\limits_{\substack{y\in\mathbb{R}^n \\ 1>|x-y|>\varepsilon}} k_0(x - y)\, \mathrm{d}\mu_R(y) \\ &= \int\limits_{\substack{y\in\mathbb{R}^n \\ |x-y|>\varepsilon}} k_0(x - y)\, \mathrm{d}\mu_R(y) - \int\limits_{\substack{y\in\mathbb{R}^n \\ |x-y|\geq 1}} k_0(x - y)\, \mathrm{d}\mu_R(y), \end{aligned} \tag{5.3.56}$$

and invoke [174, Corollary 1.6] (cf. also [177, Theorem 20.27, p. 302]) to conclude that

$$\lim_{r\to 0^+} \int_{\substack{y\in\mathbb{R}^n \\ |z-y|>r}} k_0(z-y)\,\mathrm{d}\mu_R(y) \;\text{ exists at } \mathcal{H}^{n-1}\text{-a.e. } z \in E. \tag{5.3.57}$$

From (5.3.54), (5.3.56), and (5.3.57) we ultimately see that the limit

$$\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\mathbb{R}^n \\ |x-y|>\varepsilon}} k(x-y)\,\mathrm{d}\mu(y) \;\text{ exists for } \mathcal{H}^{n-1}\text{-a.e. } x \in B(0,R)\cap\Sigma. \tag{5.3.58}$$

With this in hand, the claim in (5.3.50) now follows in view of the arbitrariness of $R \in (0,\infty)$. □

We continue by presenting two corollaries to Proposition 5.3.4. The first such corollary reads as follows:

Corollary 5.3.5 *Suppose $k \in \mathscr{C}^2(\mathbb{R}^n \setminus \{0\})$ is an odd complex-valued function with the property that there exists a constant $C \in (0,\infty)$ such that for each $x \in \mathbb{R}^n \setminus \{0\}$ one has*

$$|k(x)| \le C|x|^{1-n}, \quad |(\nabla k)(x)| \le C|x|^{-n}, \quad |(\nabla^2 k)(x)| \le C|x|^{-1-n}. \tag{5.3.59}$$

Next, assume $\Sigma \subseteq \mathbb{R}^n$ is an $\mathcal{H}^{n-1}$-measurable set which is countably rectifiable (of dimension $n-1$) and satisfies $\mathcal{H}^{n-1}(K\cap\Sigma) < +\infty$ for each compact set $K \subseteq \mathbb{R}^n$. Finally, suppose μ is a Borel-regular measure in $\mathbb{R}^n$ which is locally finite and is concentrated on Σ (cf. (3.1.21) and (3.1.22)).

Then for each function $f \in L^1\big(\Sigma, \frac{\mu(x)}{1+|x|^{n-1}}\big)$ the limit

$$\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\Sigma \\ |x-y|>\varepsilon}} k(x-y) f(y)\,\mathrm{d}\mu(y) \;\text{ exists for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma. \tag{5.3.60}$$

Moreover, if for $\mathcal{H}^{n-1}$-a.e. $x \in \Sigma$ one denotes by $F(x)$ the value of the limit in (5.3.60), then the function F, thus defined at $\mathcal{H}^{n-1}$-a.e. point on Σ, is actually $\mathcal{H}^{n-1}$-measurable.

Proof Fix a function $f \in L^1\big(\Sigma, \frac{\mu(x)}{1+|x|^{n-1}}\big)$. Without loss of generality we may assume that f is real-valued and non-negative. If for each μ-measurable set $A \subseteq \mathbb{R}^n$ we define

$$\widetilde{\mu}(A) := \int_{A\cap\Sigma} f\,\mathrm{d}\mu, \tag{5.3.61}$$

then Lemma 3.4.2 gives that $\widetilde{\mu}$ is a positive Borel-regular measure in $\mathbb{R}^n$ which is concentrated on Σ (cf. (3.1.21)). In addition,

$$\int_{\mathbb{R}^n} \frac{\mathrm{d}\widetilde{\mu}(x)}{1+|x|^{n-1}} = \int_{\Sigma} \frac{f(x)}{1+|x|^{n-1}}\, \mathrm{d}\mu(x) < +\infty. \tag{5.3.62}$$

Let us also note that, for each compact set $K \subseteq \mathbb{R}^n$, we have

$$\widetilde{\mu}(K) = \int_{K\cap\Sigma} f\, \mathrm{d}\mu \leq C_K \int_{\Sigma} \frac{f(x)}{1+|x|^{n-1}}\, \mathrm{d}\mu(x) < +\infty, \tag{5.3.63}$$

with $C_K := \sup_{x\in K}(1+|x|^{n-1}) \in (0,\infty)$. This proves that $\widetilde{\mu}$ is locally finite in $\mathbb{R}^n$. We may therefore invoke Corollary 3.5.3 to conclude that $\widetilde{\mu}$ is a (positive) Radon measure in $\mathbb{R}^n$. Having established this, Proposition 5.3.4 applies and gives that the limit

$$\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\mathbb{R}^n\\ |x-y|>\varepsilon}} k(x-y)\, \mathrm{d}\widetilde{\mu}(y) \text{ exists for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Sigma. \tag{5.3.64}$$

Upon observing that for each $x \in \mathbb{R}^n$ and each $\varepsilon > 0$ we have

$$\int_{\substack{y\in\mathbb{R}^n\\ |x-y|>\varepsilon}} k(x-y)\, \mathrm{d}\widetilde{\mu}(y) = \int_{\substack{y\in\Sigma\\ |x-y|>\varepsilon}} k(x-y) f(y)\, \mathrm{d}\mu(y), \tag{5.3.65}$$

the claim in (5.3.60) follows from (5.3.64)–(5.3.65).

Finally, if for each fixed $\varepsilon > 0$ we define

$$F_\varepsilon(x) := \int_{\substack{y\in\Sigma\\ |x-y|>\varepsilon}} k(x-y) f(y)\, \mathrm{d}\mu(y) \text{ for every } x \in \Sigma, \tag{5.3.66}$$

then each F_ε is continuous on Σ (thanks to the continuity of k in $\mathbb{R}^n \setminus \{0\}$, the first estimate in (5.3.59), the membership of f to $L^1\big(\Sigma, \frac{\mu(x)}{1+|x|^{n-1}}\big)$, and Lebesgue's Dominated Convergence Theorem). Since in the first part of the proof we have already seen that $F(x) = \lim_{\varepsilon\to 0^+} F_\varepsilon(x)$ for $\mathcal{H}^{n-1}$-a.e. point $x \in \Sigma$, we may now invoke Remark 3.1.2 (bearing in mind (3.6.28)) to conclude that F is indeed $\mathcal{H}^{n-1}$-measurable. □

Here is the second corollary to Proposition 5.3.4 alluded to above; this is particularly useful in applications.

Corollary 5.3.6 *Let $k \in \mathscr{C}^2(\mathbb{R}^n \setminus \{0\})$ be an odd complex-valued function for which there exists a constant $C \in (0,\infty)$ such that for each $x \in \mathbb{R}^n \setminus \{0\}$ one has*

$$|k(x)| \leq C|x|^{1-n}, \quad |(\nabla k)(x)| \leq C|x|^{-n}, \quad |(\nabla^2 k)(x)| \leq C|x|^{-1-n}. \tag{5.3.67}$$

Also, suppose $\Sigma \subseteq \mathbb{R}^n$ is a Borel set which is countably rectifiable (of dimension $n-1$) and satisfies $\mathcal{H}^{n-1}(K\cap\Sigma) < +\infty$ for each compact set $K \subseteq \mathbb{R}^n$. Abbreviate

$\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$ *and pick an arbitrary function* $f \in L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$. *Then the limit*

$$\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\Sigma \\ |x-y|>\varepsilon}} k(x-y)f(y)\,\mathrm{d}\sigma(y) \;\; \textit{exists for } \sigma\textit{-a.e. } x \in \Sigma. \tag{5.3.68}$$

Furthermore, if for σ*-a.e.* $x \in \Sigma$ *one denotes by* $F(x)$ *the value of the limit in* (5.3.68)*, then the function* F*, thus defined at* σ*-a.e. point on* Σ*, is actually* σ*-measurable.*

Finally, the limit

$$\mathscr{K}(x) := \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\Sigma \\ 1>|x-y|>\varepsilon}} k(x-y)\,\mathrm{d}\sigma(y) \;\; \textit{exists for } \sigma\textit{-a.e. } \; x \in \Sigma, \tag{5.3.69}$$

and the function $\mathscr{K}$ *defined at* σ*-a.e. point on* $\partial\Omega$ *as in* (5.3.69) *is* σ*-measurable.*

Proof Since $\mathcal{H}^{n-1}$ is a Borel measure, it follows that Σ is an $\mathcal{H}^{n-1}$-measurable set. Consider $\mu := \mathcal{H}^{n-1}_{\Sigma}$ which, according to (3.4.18)–(3.4.19) is a measure in $\mathbb{R}^n$ concentrated on Σ. In addition, Lemma 3.4.11 guarantees that μ is a Borel-regular measure in $\mathbb{R}^n$. Finally, $\mu(K) = \mathcal{H}^{n-1}(K \cap \Sigma) < +\infty$ for each compact set $K \subseteq \mathbb{R}^n$, so μ is also locally finite. Since, according to (3.4.20), we may naturally identify $\mu \equiv \sigma$, we may invoke Corollary 5.3.5 to conclude that both (5.3.68) and the subsequent measurability claim are true.

As far as the claims regarding (5.3.69) are concerned, pick a reference point $x_0 \in \partial\Omega$ along with an arbitrary radius $r \in (1, \infty)$, and define $f := \mathbf{1}_{B(x_0,r)\cap\Sigma}$. Since Σ has locally finite $\mathcal{H}^{n-1}$ measure, it follows that f belongs to $L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$. Given any $\varepsilon \in (0, 1)$, for each $x \in B(x_0, r-1) \cap \Sigma$ may write

$$\begin{aligned} \int_{\substack{y\in\Sigma \\ 1>|x-y|>\varepsilon}} k(x-y)\,\mathrm{d}\sigma(y) &= \int_{\substack{y\in\Sigma \\ 1>|x-y|>\varepsilon}} k(x-y)f(y)\,\mathrm{d}\sigma(y) \\ &= \int_{\substack{y\in\Sigma \\ |x-y|>\varepsilon}} k(x-y)f(y)\,\mathrm{d}\sigma(y) \\ &\quad - \int_{\substack{y\in\Sigma \\ |x-y|\geq 1}} k(x-y)f(y)\,\mathrm{d}\sigma(y). \end{aligned} \tag{5.3.70}$$

Granted this, we may invoke (5.3.68) and the subsequent measurability claim to conclude that the limit in (5.3.69) exists at σ-a.e. point $x \in B(x_0, r-1) \cap \Sigma$ and that the function defined as such on $B(x_0, r-1) \cap \Sigma$ is σ-measurable. In view of the arbitrariness of r in $(1, \infty)$, we ultimately conclude that the function $\mathscr{K}$ from (5.3.69) is well defined and σ-measurable. $\square$

Examples of countably rectifiable sets are offered by level sets of real-valued Lipschitz functions. Concretely, according to [88, 3.2.15],

$$\text{if } f : \mathbb{R}^n \to \mathbb{R} \text{ is Lipschitz then } f^{-1}(\{y\}) \text{ is a countably rectifiable set (of dimension } n-1) \text{ for } \mathcal{L}^1\text{-a.e. } y \in \mathbb{R}. \tag{5.3.71}$$

Rectifiable curves are prime examples of countably rectifiable sets (of dimension 1) in $\mathbb{R}^n$. In this vein, it is worth recalling the following classical result (see, e.g., [82, Exercise 3.5]) characterizing the images of rectifiable curves:

a set $E \subseteq \mathbb{R}^n$ is (the image of) a rectifiable curve (possibly self-intersecting and/or re-tracing itself) if and only if E is compact, connected, and $\mathcal{H}^1(E) < +\infty$. (5.3.72)

This points to the fact that countable rectifiability of dimension 1 for subsets of $\mathbb{R}^n$ is much more straightforward to characterize, at least for connected compact sets of finite one-dimensional Hausdorff measure. Indeed, a more precise version of (5.3.72) (which appears in [69, Theorem 1.8, p. 6]) asserts that

every connected compact set $K \subseteq \mathbb{R}^n$ with $\mathcal{H}^1(K) < +\infty$ is of the form $f(I)$ where $I \subseteq \mathbb{R}$ is a compact interval of length comparable to $\mathcal{H}^1(K)$ and $f : I \to \mathbb{R}^n$ is a Lipschitz function satisfying $|f'(t)| = 1$ for $\mathcal{L}^1$-a.e. point $t \in \mathbb{R}$; (5.3.73)

See also [8, Theorem 4.4, p. 49] which shows that, for any given continuum,[4] the upper comparability constant referred to above can always be taken to be 2.

5.4 Approximate Tangent Planes

Suppose $\Sigma \subseteq \mathbb{R}^n$, where $n \geq 2$, has the property that $\mathcal{H}_*^{n-1}(\Sigma \cap K) < +\infty$ for each compact set $K \subseteq \mathbb{R}^n$. Given a point $x_0 \in \Sigma$, an $(n-1)$-plane $\pi \subset \mathbb{R}^n$ passing through x_0 is called an `approximate tangent` $(n-1)$`-plane to` Σ `at` x_0 provided

$$\limsup_{r \to 0^+} \frac{\mathcal{H}_*^{n-1}\big(\Sigma \cap B(x_0, r)\big)}{r^{n-1}} > 0 \tag{5.4.1}$$

and, for each $s \in (0, 1)$,

$$\lim_{r \to 0^+} \frac{\mathcal{H}_*^{n-1}\big(\{x \in \Sigma \cap B(x_0, r) : \operatorname{dist}(x, \pi) > s|x - x_0|\}\big)}{r^{n-1}} = 0. \tag{5.4.2}$$

[4] I.e., a compact connected subset of $\mathbb{R}^n$.

Note that $C_{\pi,s} := \{x \in \mathbb{R}^n : \operatorname{dist}(x,\pi) > s|x - x_0|\}$ is a double cone with vertex at x_0, axis perpendicular to the $(n-1)$-plane π, and aperture angle $2\arccos s \in (0,\pi)$. As a consequence, if $\pi_1 \neq \pi_2$ are two distinct $(n-1)$-planes in $\mathbb{R}^n$ passing through the point x_0 then, collectively, the families $\{C_{\pi_1,j^{-1}}\}_{j\in\mathbb{N}}$ and $\{C_{\pi_2,j^{-1}}\}_{j\in\mathbb{N}}$ cover $\mathbb{R}^n \setminus (\pi_1 \cap \pi_2)$. Hence, such families cover $\mathbb{R}^n$ up to a $\mathcal{H}^{n-1}$-nullset. In concert with conditions (5.4.1)–(5.4.2), this implies that

$$\begin{array}{l}\text{if an approximate tangent } (n-1)\text{-plane } \pi\\ \text{to the set } \Sigma \text{ at } x_0 \text{ exists, then it is unique.}\end{array} \tag{5.4.3}$$

It turns out that countable rectifiability (of dimension $n-1$) may actually be characterized in terms of the existence of approximate tangent planes. Concretely, the following result is contained in [88, Theorem 3.2.19]; see also [69, Theorem 1.5, p. 5], [177, p. 214].

Proposition 5.4.1 *Assume $\Sigma \subset \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is $\mathcal{H}^{n-1}$-measurable and satisfies $\mathcal{H}^{n-1}(\Sigma \cap K) < +\infty$ for each compact set $K \subseteq \mathbb{R}^n$. Then Σ is a countably rectifiable set (of dimension $n-1$) if and only if there exists an approximate tangent $(n-1)$-plane to Σ at $\mathcal{H}^{n-1}$-a.e. point in Σ.*

5.5 Functions of Bounded Variation

Given an open set $\mathcal{O} \subseteq \mathbb{R}^n$ and a function $f \in L^1_{\rm loc}(\mathcal{O},\mathcal{L}^n)$, recall that the `variation of` f `in` $\mathcal{O}$ is defined as

$$\mathbf{V}(f;\mathcal{O}) := \sup\Big\{\Big|\int_{\mathcal{O}} f\,\mathrm{div}\vec{\varphi}\,\,\mathrm{d}\mathcal{L}^n\Big| : \vec{\varphi}\in\big[\mathscr{C}^1_c(\mathcal{O})\big]^n \text{ with } \sup_{\mathcal{O}}|\vec{\varphi}| \leq 1\Big\} \in [0,+\infty]. \tag{5.5.1}$$

Reasoning as in the proof of Proposition 3.9.1 we see that we may also express the variation of f in $\mathcal{O}$ as

$$\mathbf{V}(f;\mathcal{O}) = \sup\Big\{\Big|\int_{\mathcal{O}} f\,\mathrm{div}\vec{\varphi}\,\,\mathrm{d}\mathcal{L}^n\Big| : \vec{\varphi} \in \big[\mathscr{C}^\infty_c(\mathcal{O})\big]^n \text{ with } \sup_{\mathcal{O}}|\vec{\varphi}| \leq 1\Big\}. \tag{5.5.2}$$

Clearly, the variation is monotonic with respect to the underlying open set, in the sense that

$$\mathbf{V}(f;\mathcal{U}) \leq \mathbf{V}(f;\mathcal{O}) \text{ for every open set } \mathcal{U} \subseteq \mathcal{O}. \tag{5.5.3}$$

Also, it follows directly from (5.5.1) that the variation is lower-semicontinuous in the sense that

$$\begin{array}{l}\text{for any sequence } \{f_j\}_{j\in\mathbb{N}} \subset L^1_{\rm loc}(\mathcal{O},\mathcal{L}^n) \text{ that converges to}\\ f \text{ in } L^1_{\rm loc}(\mathcal{O},\mathcal{L}^n) \text{ we have } \mathbf{V}(f;\mathcal{O}) \leq \liminf\limits_{j\to\infty} \mathbf{V}(f_j;\mathcal{O}).\end{array} \tag{5.5.4}$$

Whenever $\mathbf{V}(f;\mathcal{O}) < +\infty$ we shall say that f `has finite variation in` $\mathcal{O}$. Let us also define

$$\mathrm{BV}(\mathcal{O}) := \big\{ f \in L^1(\mathcal{O},\mathcal{L}^n) : \mathbf{V}(f;\mathcal{O}) < +\infty \big\}, \tag{5.5.5}$$

called the `space of functions of bounded variation in` $\mathcal{O}$, and its local version

$$\mathrm{BV}_{\mathrm{loc}}(\mathcal{O}) := \big\{ f \in L^1_{\mathrm{loc}}(\mathcal{O},\mathcal{L}^n) : \mathbf{V}(f;U) < +\infty \ \text{ for each open set } U \text{ in } \mathbb{R}^n \text{ with } \overline{U} \text{ compact subset of } \mathcal{O} \big\}, \tag{5.5.6}$$

the `space of functions of locally bounded variation in` $\mathcal{O}$. In particular, for each open set $\mathcal{O} \subseteq \mathbb{R}^n$ we have

$$W^{1,1}(\mathcal{O}) \subseteq \mathrm{BV}(\mathcal{O}) \ \text{ and } \ W^{1,1}_{\mathrm{loc}}(\mathcal{O}) \subseteq \mathrm{BV}_{\mathrm{loc}}(\mathcal{O}). \tag{5.5.7}$$

Proposition 5.5.1 *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let $f \in \mathrm{BV}_{\mathrm{loc}}(\mathcal{O})$. Then there exist a locally finite Borel-regular measure μ in $\mathcal{O}$ along with some μ-measurable vector-valued function $\vec{h} : \mathcal{O} \to \mathbb{R}^n$ satisfying the following properties:*

(i) For μ-a.e. $x \in \mathcal{O}$ one has $|\vec{h}(x)| = 1$.
(ii) For each $\vec{\varphi} \in \big[\mathscr{C}^\infty_c(\mathcal{O})\big]^n$ there holds $\int_{\mathcal{O}} f \,\mathrm{div}\vec{\varphi}\, \mathrm{d}\mathcal{L}^n = -\int_{\mathcal{O}} \vec{h}\cdot\vec{\varphi}\, \mathrm{d}\mu$.
(iii) For each open set $\mathcal{U}$ in $\mathbb{R}^n$ with $\overline{\mathcal{U}}$ compact subset of $\mathcal{O}$ one has

$$\nabla f\big|_{\mathcal{U}} = \big(\vec{h}\big|_{\mathcal{U}}\big)\big(\mu\lfloor\mathcal{U}\big) \ \text{ in } \ \big[\mathcal{D}'(\mathcal{U})\big]^n, \tag{5.5.8}$$

hence the vector distribution ∇f may be locally described as a locally finite vector Borel-regular measure whose total variation in each set $\mathcal{U}$ as above is $\mu\lfloor\mathcal{U}$.
(iv) The measure μ and the function $\vec{h}$ are uniquely determined by f.
(v) For every open set $\mathcal{U} \subseteq \mathcal{O}$ one has $\mu(\mathcal{U}) = \mathbf{V}(f;\mathcal{U})$ (in particular, μ is a finite measure in $\mathcal{O}$ if and only if the function f has finite variation in $\mathcal{O}$).
(vi) For each open set $\mathcal{U}$ in $\mathbb{R}^n$ with $\overline{\mathcal{U}}$ compact subset of $\mathcal{O}$ the total variation norm of the vector measure $(\nabla f)\big|_{\mathcal{U}}$ satisfies $\big\|(\nabla f)\big|_{\mathcal{U}}\big\|_{\mathrm{TV}(\mathcal{U})} = \mathbf{V}(f;\mathcal{U})$.

Henceforth, in the context of Proposition 5.5.1 we shall say[5] that

$$\mu \text{ is the total variation measure of } \nabla f, \text{ and simply write } \mu = |\nabla f|. \tag{5.5.9}$$

[5] Slightly abusing notation and terminology, since ∇f is not a genuine measure.

Proof of Proposition 5.5.1 The claims in items *(i)–(ii)* make up the Structure Theorem for functions of locally bounded variation as presented in [80, Theorem 1, p. 167]. The proof of the latter result is based on the Riesz Representation Theorem applied to the functional $\Lambda : \big[\mathscr{C}_c^\infty(\mathcal{O})\big]^n \to \mathbb{R}$ defined as

$$\Lambda(\vec{\varphi}) := \int_{\mathcal{O}} f \operatorname{div}\vec{\varphi} \, \mathrm{d}\mathcal{L}^n \quad \text{for each } \vec{\varphi} \in \big[\mathscr{C}_c^\infty(\mathcal{O})\big]^n. \tag{5.5.10}$$

In the format given in Proposition 3.9.1, this also gives (see (3.9.6))

$$\mu(\mathcal{U}) = \mathbf{V}(f;\mathcal{U}) \quad \text{for every open subset } \mathcal{U} \text{ of } \mathcal{O}, \tag{5.5.11}$$

which takes care of item *(v)*. Going further, the claims in *(iii)* are implicit in *(i)–(ii)*. In particular, the last property in item *(iii)* also shows that μ is uniquely determined by f. Since for each open set $\mathcal{U}$ in $\mathbb{R}^n$ with $\overline{\mathcal{U}}$ compact subset of $\mathcal{O}$ we also have

$$\vec{h}\big|_{\mathcal{U}} = \frac{\mathrm{d}\big(\nabla f\big|_{\mathcal{U}}\big)}{\mathrm{d}\big(\mu\lfloor\mathcal{U}\big)} = \frac{\mathrm{d}\big(\nabla f\big|_{\mathcal{U}}\big)}{\mathrm{d}\big(|\nabla f|_{\mathcal{U}}|\big)} \tag{5.5.12}$$

it follows that $\vec{h}$ is uniquely determined by f. This finishes the proof of item *(iv)*.

Finally, for every open set $\mathcal{U} \subseteq \mathcal{O}$ we may use what we have proved in items *(iii)* and *(v)* in order to write $\|\nabla f\|_{\mathrm{TV}(\mathcal{U})} = \mu(\mathcal{U}) = \mathbf{V}(f;\mathcal{U})$. □

5.6 Sets of Locally Finite Perimeter

Here the focus is on the concept of set of locally finite perimeter as initially introduced by R. Caccioppoli and E. De Giorgi (in connection with Plateau's problem), as well as H. Federer. This is basically the largest category of domains possessing a reasonable outward unit normal and boundary surface measure. Turning to details, call a subset Ω of $\mathbb{R}^n$ a set of `locally finite perimeter` provided Ω is $\mathcal{L}^n$-measurable and its characteristic function has finite variation in every bounded open subset of $\mathbb{R}^n$. The latter property amounts to

$$\begin{gathered}\mathbf{1}_\Omega \in BV_{\mathrm{loc}}(\mathbb{R}^n), \text{ i.e., for each relatively compact open subset } \mathcal{O} \text{ of } \mathbb{R}^n \text{ one has} \\ \mathbf{V}\big(\mathbf{1}_\Omega;\mathcal{O}\big) = \sup\Big\{\Big|\int_\Omega \operatorname{div}\vec{\varphi}\,\mathrm{d}\mathcal{L}^n\Big| : \vec{\varphi} \in \big[\mathscr{C}_c^\infty(\mathcal{O})\big]^n \text{ with } \sup_{\mathbb{R}^n}|\vec{\varphi}| \leq 1\Big\} < +\infty.\end{gathered} \tag{5.6.1}$$

Equivalently, an $\mathcal{L}^n$-measurable set $\Omega \subseteq \mathbb{R}^n$ has locally finite perimeter if and only if there exist

$$\begin{gathered}\text{a locally finite Borel-regular measure } \sigma_* \text{ in } \mathbb{R}^n \text{ and a vector-valued} \\ \text{function } \nu \in \big[L^\infty(\mathbb{R}^n,\sigma_*)\big]^n \text{ satisfying } |\nu(x)| = 1 \text{ at } \sigma_*\text{-a.e. } x \in \mathbb{R}^n\end{gathered} \tag{5.6.2}$$

with the property that[6]

$$\nabla \mathbf{1}_\Omega = -\nu\sigma_* \ \text{ in } \ \big[\mathcal{D}'(\mathbb{R}^n)\big]^n. \tag{5.6.3}$$

Indeed, if Ω has locally finite perimeter, then the existence of σ_* and ν as above is guaranteed by Proposition 5.5.1 applied to the function $\mathbf{1}_\Omega$. For the converse implication, observe that if (5.6.3) holds then for any relatively compact open subset $\mathcal{O}$ of $\mathbb{R}^n$ and any $\vec{\varphi} \in \big[\mathscr{C}_c^\infty(\mathcal{O})\big]^n$ with $\sup_{\mathbb{R}^n} |\vec{\varphi}| \leq 1$ we have

$$\begin{aligned} \Big| \int_\Omega \operatorname{div}\vec{\varphi}\, d\mathcal{L}^n \Big| &= \Big| -\langle \nabla \mathbf{1}_\Omega, \vec{\varphi} \rangle \Big| = \Big| \langle \nu\sigma_*, \vec{\varphi} \rangle \Big| \\ &= \Big| \int_{\mathbb{R}^n} \nu \cdot \vec{\varphi}\, d\sigma_* \Big| \leq \int_{\mathcal{O}} |\vec{\varphi}|\, d\sigma_* \leq \sigma_*(\mathcal{O}). \end{aligned} \tag{5.6.4}$$

This readily implies that $\mathbf{V}\big(\mathbf{1}_\Omega; \mathcal{O}\big) \leq \sigma_*(\mathcal{O}) < +\infty$, hence Ω has locally finite perimeter.

Furthermore, Proposition 5.5.1 ensures that, for an arbitrary set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, the measure σ_* together with the vector-valued function ν from (5.6.2)–(5.6.3) holds are uniquely determined[7] by Ω. In this vein, it is also worth recalling from Corollary 3.5.3 that, given any set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter,

$$\text{σ_* is a Radon measure (in the sense of Definition (3.5.1)) in $\mathbb{R}^n$, both outer-regular and inner-regular in the sense of (3.5.7)–(3.5.8).} \tag{5.6.5}$$

Moreover, from item *(v)* in Proposition 5.5.1 we see that

$$\sigma_*(\mathcal{O}) = \mathbf{V}(\mathbf{1}_\Omega; \mathcal{O}) \ \text{ for every open set } \ \mathcal{O} \subseteq \mathbb{R}^n. \tag{5.6.6}$$

Remark 5.6.1 *In spite of* (5.6.3)*, for a given set* $\Omega \subseteq \mathbb{R}^n$ *of locally finite perimeter the distribution* $\nabla \mathbf{1}_\Omega$ *is not, generally speaking, a vector-valued measure in* $\mathbb{R}^n$*. Indeed, multiplying the locally finite Borel-regular measure* σ_* *with the vector-valued function* ν*, which is locally* (*yet not necessarily globally*) *integrable with respect to* σ_**, while meaningful in the sense of distributions in* $\mathbb{R}^n$*, does not yield a measure in a traditional sense. However, restricting the distribution* $\nabla \mathbf{1}_\Omega$ *to any relatively compact open set* $\mathcal{O} \subseteq \mathbb{R}^n$ *does yield a genuine measure, namely*

$$\nabla \mathbf{1}_\Omega\big|_{\mathcal{O}} = -\big(\nu\big|_{\mathcal{O}}\big)\big(\sigma_* \lfloor \mathcal{O}\big) \ \text{ in } \ \big[\mathcal{D}'(\mathcal{O})\big]^n, \tag{5.6.7}$$

[6] Here, it is useful to observe that the product between a locally finite Borel measure ω in an open set $\mathcal{O} \subset \mathbb{R}^n$ and a vector-valued function $\vec{\psi} \in \big[L^1_{\rm loc}(\mathcal{O}, \omega)\big]^n$ is well defined as the vector distribution $\vec{\psi}\,\omega \in \big[\mathcal{D}'(\mathcal{O})\big]^n$, acting on each vector-valued test function $\vec{\varphi} \in \big[\mathscr{C}_c^\infty(\mathcal{O})\big]^n$ according to $\langle \vec{\psi}\,\omega, \vec{\varphi} \rangle := \int_{\mathcal{O}} \vec{\psi} \cdot \vec{\varphi}\, d\omega$.

[7] For a given set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, σ_* can be thought of as the total variation measure of $\nabla \mathbf{1}_\Omega$, i.e., $\sigma_* = \big|\nabla \mathbf{1}_\Omega\big|$ in the sense of (5.5.9).

with the right-hand side of the equality in (5.6.7) naturally interpreted as the measure obtained by multiplying the measure $\sigma_*\lfloor\mathcal{O}$ by the integrable function $-\nu\big|_{\mathcal{O}}$ (with respect to $\sigma_*\lfloor\mathcal{O}$).

In view of (5.6.7), it makes sense to think of the vector distribution $\nabla\mathbf{1}_\Omega$ as being locally a vector measure.

Prefiguring more refined Divergence Theorems discussed elsewhere in this volume, we wish to note here that by simply eliminating the distribution theory jargon implicit in the interpretation of (5.6.3) one already arrives at the formula

$$\int_\Omega \operatorname{div}\vec{F}\,d\mathcal{L}^n = \int_{\partial\Omega} \nu\cdot\big(\vec{F}\big|_{\partial\Omega}\big)\,d\sigma_* \quad \text{for each vector field } \vec{F}\in\big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n. \tag{5.6.8}$$

In fact, via a straightforward limiting argument involving a mollifier, the Divergence Formula (5.6.8) readily extends to arbitrary vector fields $\vec{F}\in\big[\mathscr{C}_c^1(\mathbb{R}^n)\big]^n$. More work goes into elucidating the nature of the measure σ_*, a topic discussed further below (see (5.6.20)–(5.6.22) and (5.6.34) in this regard). For now we note the following criteria for local finite perimeter.

Proposition 5.6.2 *For any $\mathcal{L}^n$-measurable set $\Omega\subseteq\mathbb{R}^n$ the following statements are equivalent:*

(1) There exists a positive locally finite Borel measure λ in $\mathbb{R}^n$ with the property that for each vector field $\vec{F}\in\big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n$ one has

$$\Big|\int_\Omega \operatorname{div}\vec{F}\,d\mathcal{L}^n\Big| \le \int_{\mathbb{R}^n}|\vec{F}|\,d\lambda. \tag{5.6.9}$$

(2) The set Ω has locally finite perimeter.

Proof It is apparent from definitions that *(1)* implies $\mathbf{1}_\Omega\in BV_{loc}(\mathbb{R}^n)$, hence Ω has locally finite perimeter. Also, the fact that *(2)*$\Rightarrow$*(1)* is seen from (5.6.8). □

Continuing our discussion pertaining to a set $\Omega\subseteq\mathbb{R}^n$ of locally finite perimeter, we shall refer to ν in (5.6.2)–(5.6.3) as the `geometric measure theoretic outward unit normal` to Ω. In particular, from (5.6.3) we see that the scalar components $(\nu_1,\dots,\nu_n)$ of ν satisfy

$$\partial_j\mathbf{1}_\Omega = -\nu_j\,\sigma_* \ \text{ in } \ \mathcal{D}'(\mathbb{R}^n) \ \text{ for each } \ j\in\{1,\dots,n\}. \tag{5.6.10}$$

Also, bearing in mind the convention in the last part of Remark 5.6.1,

$$\partial_\nu\mathbf{1}_\Omega := \nu\cdot\nabla\mathbf{1}_\Omega = -\sigma_* \ \text{ locally, as measures,} \tag{5.6.11}$$

while the Lebesgue–Besicovitch Differentiation Theorem (cf., e.g., [80, Theorem 1, p. 43]) implies that

$$\text{at } \sigma_*\text{-a.e. } x \in \mathbb{R}^n \text{ one has } 0 < \sigma_*(B(x,r)) < \infty \text{ for each } r \in (0,\infty) \\ \text{and } \lim_{r\to 0^+} \frac{-\nabla \mathbf{1}_\Omega\big(B(x,r)\big)}{|\nabla \mathbf{1}_\Omega|\big(B(x,r)\big)} = \lim_{r\to 0^+} \fint_{B(x,r)} \nu(y)\,\mathrm{d}\sigma_*(y) = \nu(x), \tag{5.6.12}$$

where the barred integral indicates mean average.

For a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, we let $\partial^*\Omega$ denote the `reduced boundary` of Ω, that is,

$$\partial^*\Omega \text{ consists of all points } x \in \partial\Omega \text{ satisfying the following three properties: } 0 < \sigma_*(B(x,r)) < +\infty \text{ for each } r \in (0,\infty), \text{ formula } \lim_{r\to 0^+} \fint_{B(x,r)} \nu\,\mathrm{d}\sigma_* = \nu(x) \text{ is valid, and } |\nu(x)| = 1. \tag{5.6.13}$$

For example, consider the open unit square $\Omega := (0,1)\times(0,1) \subseteq \mathbb{R}^2$. Then Ω is a set of locally finite perimeter (this may be checked using the original definition or, more directly, employing the criterion given later in (5.6.37)) for which $\partial_*\Omega = \partial\Omega$, and such that $\partial^*\Omega$ differs from $\partial\Omega$ by the vertices of the square, i.e.,

$$\partial^*\Omega = \partial\Omega \setminus \big\{(0,0),(0,1),(1,0),(1,1)\big\}. \tag{5.6.14}$$

Indeed, anticipating results we shall discuss just a little later in (5.6.22) and (5.6.18), for the set $\Omega := (0,1)\times(0,1) \subseteq \mathbb{R}^2$ we may explicitly identify the limit (5.6.12) at the vertex $x := (0,0)$ as being

$$\lim_{r\to 0^+} \fint_{B((0,0),r)\cap\partial\Omega} \nu\,\mathrm{d}\mathcal{H}^1 = (-1,-1) \tag{5.6.15}$$

and since $|(-1,-1)| = \sqrt{2} \neq 1$, it follows from (5.6.13) that $(0,0) \notin \partial^*\Omega$. Similar considerations apply to all other remaining vertices and this ultimately justifies (5.6.14).

Returning to the general setting we wish to note that since $\mathbf{1}_{\mathbb{R}^n\setminus\Omega} = 1 - \mathbf{1}_\Omega$, from (5.6.3) we see that

$$\text{if } \Omega \subseteq \mathbb{R}^n \text{ is a set of locally finite perimeter with geometric measure theoretic outward unit normal } \nu, \text{ then } \mathbb{R}^n\setminus\Omega \text{ is a set of locally finite perimeter, its geometric measure theoretic outward unit normal is } -\nu, \text{ and } \partial^*(\mathbb{R}^n\setminus\Omega) = \partial^*\Omega. \tag{5.6.16}$$

If $\Omega \subseteq \mathbb{R}^n$ is a set of locally finite perimeter, from the Structure Theorem for sets of locally finite perimeter (cf. [80, Theorem 2, p. 205]) it follows that

$$\partial^*\Omega \quad \text{is countably rectifiable (of dimension } n-1). \tag{5.6.17}$$

See also [262, Theorem 5.7.3, p. 244] in this regard. Moreover, on each of the compact pieces of $(n-1)$-dimensional $\mathscr{C}^1$ surfaces covering $\partial^*\Omega$ up to a $\mathcal{H}^{n-1}$-nullset as in (5.3.19), the geometric measure outward unit normal ν to Ω is orthogonal to the classical tangent hyperplane to said $\mathscr{C}^1$ surface. (5.6.18)

It turns out that the measure σ_* is concentrated on $\partial^*\Omega$. In fact, a much more nuanced result is true. Specifically, first (5.6.17) implies that

$$\text{the set } \partial^*\Omega \text{ is } \mathcal{H}^{n-1}\text{-measurable,} \tag{5.6.19}$$

then [80, Theorem 2, p. 205] gives (keeping the convention made in (3.1.22) in mind) that

$$\sigma_* = \mathcal{H}^{n-1}\lfloor\partial^*\Omega. \tag{5.6.20}$$

Fix a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter. It is then well known (see e.g., [80, Lemma 1, p. 208]) that

$$\partial^*\Omega \subseteq \partial_*\Omega \subseteq \partial\Omega \ \text{ and } \ \mathcal{H}^{n-1}\big(\partial_*\Omega \setminus \partial^*\Omega\big) = 0. \tag{5.6.21}$$

Hence, σ_* is also concentrated on $\partial_*\Omega$ (cf. (3.8.3)). As such, we may also identify σ_* with $\mathcal{H}^{n-1}\lfloor\partial_*\Omega$, i.e., write

$$\sigma_* = \mathcal{H}^{n-1}\lfloor\partial_*\Omega. \tag{5.6.22}$$

In such a scenario, the geometric measure theoretic outward unit normal ν to Ω is defined σ_*-a.e. on $\partial_*\Omega$, and (5.6.2) gives

$$\begin{gathered}\nu \in \big[L^\infty(\partial_*\Omega, \sigma_*)\big]^n \text{ is an } \mathbb{R}^n\text{-valued function} \\ \text{satisfying } |\nu(x)| = 1 \text{ at } \sigma_*\text{-a.e. point } x \in \partial_*\Omega.\end{gathered} \tag{5.6.23}$$

Also, (5.6.3) and (5.6.22) imply (again, in light of the convention made in (3.1.22))

$$\begin{gathered}\text{if } \Omega \subseteq \mathbb{R}^n \text{ is a set of locally finite perimeter then} \\ \nabla\mathbf{1}_\Omega = -\nu\,\sigma_* = -\nu\,\mathcal{H}^{n-1}\lfloor\partial_*\Omega \text{ in } \big[\mathcal{D}'(\mathbb{R}^n)\big]^n.\end{gathered} \tag{5.6.24}$$

Here is an extension of (5.6.3) to more general homogeneous first-order systems.

Proposition 5.6.3 *Let $\Omega \subseteq \mathbb{R}^n$ be a set of locally finite perimeter and denote by ν its geometric measure theoretic outward unit normal. Also, let*

$$D = \Big(\sum_{j=1}^n a_j^{\alpha\beta}\partial_j\Big)_{\substack{1\le\alpha\le N \\ 1\le\beta\le N'}} \tag{5.6.25}$$

be a homogeneous $N \times N'$ first-order system (where $N, N' \in \mathbb{N}$ are arbitrary) with constant complex coefficients and recall the definition of its principal symbol given in (1.7.16). Make the convention that the action of D on an ordinary distribution u in $\mathbb{R}^n$ is the matrix of distributions

$$Du := \Big(\sum_{j=1}^{n} a_j^{\alpha\beta}\partial_j u\Big)_{\substack{1\leq\alpha\leq N\\ 1\leq\beta\leq N'}}. \tag{5.6.26}$$

Then one has

$$D\big(\mathbf{1}_\Omega\big) = \mathrm{iSym}(D;\,\nu)\sigma_* = \mathrm{iSym}(D;\,\nu)\mathcal{H}^{n-1}\lfloor\partial^*\Omega \tag{5.6.27}$$

$$= \mathrm{iSym}(D;\,\nu)\mathcal{H}^{n-1}\lfloor\partial_*\Omega \tag{5.6.28}$$

in the sense of distributions in $\mathbb{R}^n$.

Proof This is clear from (1.7.12), (5.6.10), (1.7.16), (5.6.20), and (5.6.22). □

We also wish to note that in the two-dimensional setting we may define the geometric measure theoretic (positively oriented) unit tangent vector τ to a given set of locally finite perimeter $\Omega \subseteq \mathbb{R}^2$ as the unique S^1-valued function $\tau \in \big[L^\infty(\partial_*\Omega, \sigma_*)\big]^2$ satisfying (cf. (3.1.22))

$$\nabla^{\mathrm{R}}\mathbf{1}_\Omega = \tau\,\sigma_* = \tau\,\mathcal{H}^1\lfloor\partial_*\Omega \ \text{ in } \ \big[\mathcal{D}'(\mathbb{R}^2)\big]^2, \tag{5.6.29}$$

where $\nabla^{\mathrm{R}} := (\partial_y, -\partial_x)$ is the (clockwise, 90°) "rotated" gradient in the xy-plane. In particular, with $\nu = (\nu_1, \nu_2)$ denoting the geometric measure theoretic outward unit normal to Ω, we have

$$\tau = (-\nu_2, \nu_1) \ \text{ at } \sigma_*\text{-a.e. point on } \ \partial_*\Omega. \tag{5.6.30}$$

Equivalently, under the identification $\mathbb{R}^2 \equiv \mathbb{C}$,

$$\tau = \mathrm{i}\nu \ \text{ at } \sigma_*\text{-a.e. point on } \ \partial_*\Omega. \tag{5.6.31}$$

As expected, alterations by negligible sets do not affect the geometric measure theoretic boundary, nor do they change the geometric measure theoretic outward unit normal. For further use, we formally state this below.

Proposition 5.6.4 *Let $\Omega, \widetilde{\Omega} \subseteq \mathbb{R}^n$ be two sets of locally finite perimeter with the property that $\mathcal{L}^n(\Omega\triangle\widetilde{\Omega}) = 0$. Then $\partial_*\Omega = \partial_*\widetilde{\Omega}$ and, if $\nu, \widetilde{\nu}$ denote the geometric measure theoretic outward unit normals to Ω and $\widetilde{\Omega}$, respectively, it follows that $\nu = \widetilde{\nu}$ at $\mathcal{H}^{n-1}$-a.e. point on $\partial_*\Omega = \partial_*\widetilde{\Omega}$.*

Proof Our assumptions imply $\mathbf{1}_\Omega = \mathbf{1}_{\widetilde{\Omega}}$ as functions in $L^1_{\mathrm{loc}}(\mathbb{R}^n, \mathcal{L}^n)$, so all desired conclusions are consequences of (5.2.5), (5.6.3), (5.6.20), and (5.6.21). □

In the class of sets of locally finite perimeter, the set $\partial\Omega \setminus \partial_*\Omega$ can be quite large (as may be seen by taking Ω to be a slit disk in the plane). This being said, given a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter,

if $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$ then $\sigma_* = \mathcal{H}^{n-1}\lfloor\partial\Omega$, the outward unit normal ν is defined $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, and the set $\partial\Omega$ is countably rectifiable (of dimension $n-1$). (5.6.32)

Moreover, from (5.2.6), (5.6.17), (5.6.21), and (3.0.3) we deduce that if $\Omega \subseteq \mathbb{R}^n$ has locally finite perimeter then

$\partial_*\Omega$ is a Borel set (in particular, $\mathcal{H}^{n-1}$-measurable), which happens to be countably rectifiable (of dimension $n-1$). (5.6.33)

As a byproduct, we note that

$$\sigma_* \ll \sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega \quad\text{and}\quad \frac{d\sigma_*}{d\sigma} = \mathbf{1}_{\partial_*\Omega}. \tag{5.6.34}$$

Let us also note here that (cf. [80, Theorem 1, p. 222])

given $\Omega \subseteq \mathbb{R}^n$ which is $\mathcal{L}^n$-measurable, the set Ω is of locally finite perimeter if and only if $\mathcal{H}^{n-1}(K \cap \partial_*\Omega) < \infty$ for each compact set $K \subset \mathbb{R}^n$. (5.6.35)

In particular Lemma 3.6.4 (in concert with (5.6.22), (5.6.33), and (5.6.35)) shows that

if $\Omega \subseteq \mathbb{R}^n$ is a set of locally finite perimeter then σ_* is a complete, locally finite, Borel-regular measure on $\partial_*\Omega$. (5.6.36)

From (5.6.35) and (5.2.3) we also see that

if $\Omega \subseteq \mathbb{R}^n$ is an $\mathcal{L}^n$-measurable set with the property that $\mathcal{H}^{n-1}(K \cap \partial\Omega) < \infty$ for each compact $K \subset \mathbb{R}^n$, then Ω is of locally finite perimeter. (5.6.37)

As a consequence of (5.6.37) and the monotonicity of the Hausdorff measure, we see that

if $\Omega \subseteq \mathbb{R}^n$ is an $\mathcal{L}^n$-measurable set with $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ it follows that Ω is a set of locally finite perimeter. (5.6.38)

Lemma 5.6.5 *The class of sets of locally finite perimeter in $\mathbb{R}^n$ is an algebra of subsets of $\mathbb{R}^n$, i.e., it is closed under finite unions, finite intersections, as well as complements.*

Proof Suppose $E, F \subseteq \mathbb{R}^n$ are two sets of locally finite perimeter, and consider an arbitrary compact set $K \subseteq \mathbb{R}^n$. Then E, F are $\mathcal{L}^n$-measurable, and Lemma 5.2.2 gives that

$$\begin{aligned} K \cap \partial_*(E \cup F) &\subseteq (K \cap \partial_* E) \cup (K \cap \partial_* F), \\ K \cap \partial_*(E \cap F) &\subseteq (K \cap \partial_* E) \cup (K \cap \partial_* F). \end{aligned} \tag{5.6.39}$$

As such,

$$\begin{aligned} \mathcal{H}^{n-1}\big(K \cap \partial_*(E \cup F)\big) &\leq \mathcal{H}^{n-1}(K \cap \partial_* E) + \mathcal{H}^{n-1}(K \cap \partial_* F) < \infty, \\ \mathcal{H}^{n-1}\big(K \cap \partial_*(E \cap F)\big) &\leq \mathcal{H}^{n-1}(K \cap \partial_* E) + \mathcal{H}^{n-1}(K \cap \partial_* F) < \infty, \end{aligned} \tag{5.6.40}$$

which in concert with (5.6.35) implies that $E \cup F$ and $E \cap F$ are sets of locally finite perimeter. Let us also note that, thanks to (5.2.3), we have

$$\mathcal{H}^{n-1}\big(K \cap \partial_*(\mathbb{R}^n \setminus E)\big) = \mathcal{H}^{n-1}(K \cap \partial_* E) < \infty. \tag{5.6.41}$$

With the help of (5.6.35) we conclude from this that the set $\mathbb{R}^n \setminus E$ has locally finite perimeter as well. With these in hand, the claim in the statement of the lemma readily follows. $\square$

We note the following result, comparing the geometric measure theoretic outward unit normals of two sets of locally finite perimeter (on the intersection of their reduced boundaries).

Proposition 5.6.6 *Let E, F be two sets of locally finite perimeter in $\mathbb{R}^n$. If ν_E and ν_F denote the geometric measure theoretic outward unit normal vectors to E and F, respectively, then at $\mathcal{H}^{n-1}$-a.e. point $x \in \partial^* E \cap \partial^* F$ one has either $\nu_E(x) = \nu_F(x)$, or $\nu_E(x) = -\nu_F(x)$.*

Proof This is a consequence of [170, Proposition 10.5, p. 101] according to which

$$\text{any two locally } \mathcal{H}^{n-1}\text{-rectifiable sets } M_1, M_2 \subseteq \mathbb{R}^n \text{ have identical approximate tangent planes at } \mathcal{H}^{n-1}\text{-a.e. point in } M_1 \cap M_2, \tag{5.6.42}$$

and [238, Theorem 14.3, (1), pp. 72–73] where it has been shown that

$$\text{given any set of locally finite perimeter } \Omega \subseteq \mathbb{R}^n\text{, its approximate tangent plane exists at each point } x \in \partial^*\Omega \text{ and is equal to } \langle \nu(x) \rangle^{\perp} \text{ (where } \nu \text{ is the geometric measure theoretic outward unit normal vector to } \Omega). \tag{5.6.43}$$

Indeed, (5.6.35) and (5.6.21) tell us that $\partial^* E, \partial^* F$ are locally $\mathcal{H}^{n-1}$-rectifiable sets (cf. [170, p. 96]), so (5.6.42) (used with $M_1 := \partial^* E$ and $M_2 := \partial^* F$) together with (5.6.43) imply that $\langle \nu_E(x) \rangle^{\perp} = \langle \nu_F(x) \rangle^{\perp}$ at $\mathcal{H}^{n-1}$-a.e. point $x \in \partial^* E \cap \partial^* F$, from which the desired conclusion follows (bearing in mind (5.6.13)). $\square$

It turns out that large classes of principal-value limits exist a.e. when considered on the geometric measure theoretic boundary of a given set of locally finite perimeter.

Proposition 5.6.7 *Suppose $k \in \mathscr{C}^2(\mathbb{R}^n \setminus \{0\})$ is an odd complex-valued function with the property that there exists a constant $C \in (0, \infty)$ such that for each $x \in \mathbb{R}^n \setminus \{0\}$ one has*

$$|k(x)| \leq C|x|^{1-n}, \quad |(\nabla k)(x)| \leq C|x|^{-n}, \quad |(\nabla^2 k)(x)| \leq C|x|^{-1-n}. \tag{5.6.44}$$

Also, assume $\Omega \subseteq \mathbb{R}^n$ is a set of locally finite perimeter and set $\sigma_* := \mathcal{H}^{n-1}\lfloor\partial_*\Omega$. Then for each function $f \in L^1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big)$ the limit

$$\lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial_*\Omega \\ |x-y|>\varepsilon}} k(x-y) f(y)\, \mathrm{d}\sigma_*(y) \ \ \textit{exists for } \sigma_*\textit{-a.e. } x \in \partial_*\Omega. \tag{5.6.45}$$

Also, if for σ_*-a.e. $x \in \partial_*\Omega$ one denotes by $F(x)$ the value of the limit in (5.3.68), then the function F, thus defined at σ_*-a.e. point on $\partial_*\Omega$, is actually σ_*-measurable.

Moreover, if one additionally assumes that $\partial\Omega$ is countably rectifiable (of dimension $n-1$) and has locally finite $\mathcal{H}^{n-1}$ measure, then for each function $f \in L^1\big(\partial_*\Omega, \frac{\sigma_*(x)}{1+|x|^{n-1}}\big)$ the limit in (5.6.45) actually exists for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$ and gives rise to a $\mathcal{H}^{n-1}$-measurable function on $\partial\Omega$.

Proof From (5.6.33) and (5.6.35) we know that the set $\partial_*\Omega \subseteq \mathbb{R}^n$ is Borel, countably rectifiable (of dimension $n-1$), and satisfies $\mathcal{H}^{n-1}(K \cap \partial_*\Omega) < +\infty$ for each compact set $K \subseteq \mathbb{R}^n$. Granted these, we may invoke Corollary 5.3.6 with $\Sigma := \partial_*\Omega$ to conclude that both (5.6.45) and the subsequent measurability claim are true. Finally, the very last part of the statement is a consequence of Corollary 5.3.6 applied with $\Sigma := \partial\Omega$ and with

$$\widetilde{f} := \begin{cases} f & \text{on } \partial_*\Omega, \\ 0 & \text{on } \partial\Omega \setminus \partial_*\Omega, \end{cases} \tag{5.6.46}$$

in place of f. □

Sets of locally finite perimeter enjoy nice infinitesimal density properties at each point on their reduced boundary. Specifically, the following result appears in [80, Lemma 2, p. 196] and [80, Corollary 1(ii), p. 203].

Lemma 5.6.8 *Let $E \subseteq \mathbb{R}^n$ be a set of locally finite perimeter. Then there exist some purely dimensional constants $A_1, \dots, A_5 \in (0, \infty)$ with the property that for each $x \in \partial^* E$ one has*

$$\liminf_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\cap E\big)}{r^n} > A_1, \tag{5.6.47}$$

$$\liminf_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\setminus E\big)}{r^n} > A_2, \tag{5.6.48}$$

$$\liminf_{r\to 0^+} \frac{\mathcal{H}^{n-1}\big(B(x,r)\cap \partial^* E\big)}{r^{n-1}} > A_3, \tag{5.6.49}$$

$$\limsup_{r\to 0^+} \frac{\mathcal{H}^{n-1}\big(B(x,r)\cap \partial^* E\big)}{r^{n-1}} \leq A_4, \tag{5.6.50}$$

$$\limsup_{r\to 0^+} \frac{\mathcal{H}^{n-1}\big(\partial^*(B(x,r)\cap E)\big)}{r^{n-1}} \leq A_5, \tag{5.6.51}$$

and, in fact,

$$\lim_{r\to 0^+} \frac{\mathcal{H}^{n-1}\big(B(x,r)\cap \partial^* E\big)}{\alpha_{n-1} r^{n-1}} = 1, \quad \textit{where} \ \ \alpha_{n-1} := \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)}. \tag{5.6.52}$$

It is apparent from the discussion so far that sets of locally finite perimeter which differ by a set of Lebesgue measure zero have identical unit normals and surface measures, as well as reduced and geometric measure theoretic boundaries. Given a set $E \subseteq \mathbb{R}^n$ of locally finite perimeter it is therefore natural to single out a natural representative among all such zero-measure alterations of E. Typically, one chooses the complement of the measure theoretic exterior of E (cf. Definition 2.8.3), i.e.,

$$E^{\#} := \mathbb{R}^n \setminus \mathrm{ext}_*(E) = \Big\{x \in \mathbb{R}^n : \limsup_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\cap E\big)}{\mathcal{L}^n\big(B(x,r)\big)} > 0\Big\} \tag{5.6.53}$$

to play that role (see (2.8.22) in this regard). One of the upshots of such a choice is that (cf. [262, (2.10) on p. 273])

$$\begin{array}{l}\text{under the identification } E \equiv E^{\#} \text{ it}\\ \text{follows that } \partial^* E \text{ is dense in } \partial E.\end{array} \tag{5.6.54}$$

The following result appears as [263, Theorem 5.6.5, p. 241].

Theorem 5.6.9 *Let $E \subseteq \mathbb{R}^n$ be a set of locally finite perimeter, and denote by ν its geometric measure theoretic outward unit normal. Pick some $x \in \partial^* E$ and consider the half-spaces*

$$H_x^{\pm} := \{y \in \mathbb{R}^n : \pm\nu(x)\cdot(y-x) > 0\}, \tag{5.6.55}$$

whose common boundary is the $(n-1)$-plane

$$H_x^{0} := \{y \in \mathbb{R}^n : \nu(x)\cdot(y-x) = 0\}. \tag{5.6.56}$$

Then

$$\mathcal{L}^n\Big(\big(B(x,r)\cap E\big)\cap H_x^+\Big)=o(r^n)\quad as\quad r\to 0^+, \tag{5.6.57}$$

$$\mathcal{L}^n\Big(\big(B(x,r)\setminus E\big)\cap H_x^-\Big)=o(r^n)\quad as\quad r\to 0^+, \tag{5.6.58}$$

and for every $s\in(0,1)$ *one has*

$$\mathcal{H}^{n-1}\big(\{y\in\partial^*E\cap B(x,r):\ \mathrm{dist}(y,H_x^0)>s|y-x|\}\big)=o(r^{n-1})\quad as\quad r\to 0^+. \tag{5.6.59}$$

In the limit, the normal is orthogonal to the chord. A version of this property, which is characteristics to smooth domains, continues to be valid in the category of sets of locally finite perimeter which are two-sided thick.

Proposition 5.6.10 *Suppose* $E\subseteq\mathbb{R}^n$ *is a set of locally finite perimeter which is two-sided* n*-thick, and denote by* ν *its geometric measure theoretic outward unit normal. Then for each* $x\in\partial^*E$ *one has*

$$\lim_{\substack{\partial E\ni y\to x\\ y\neq x}}\Big\langle\nu(x),\frac{y-x}{|y-x|}\Big\rangle=0. \tag{5.6.60}$$

Proof Reason by contradiction and assume that there exist a number $\varepsilon\in(0,1)$ and a sequence $\{y_j\}_{j\in\mathbb{N}}\subseteq\partial E\setminus\{x\}$ which converges to x and has the property that $\big|\langle\nu(x),y_j-x\rangle\big|>\varepsilon|y_j-x|$ for each $j\in\mathbb{N}$. Passing to a subsequence and working with $E^c:=\mathbb{R}^n\setminus E$ in place of E if necessary, there is no loss of generality in assuming that actually $\langle\nu(x),y_j-x\rangle>\varepsilon|y_j-x|$ for each $j\in\mathbb{N}$. In turn, this amounts to saying that each y_j belongs to the infinite circular (one-component) cone with vertex at x, symmetry axis along $\nu(x)$, and aperture angle $2\arccos\varepsilon$. Such a membership further implies that

$$B\big(y_j,\varepsilon|y_j-x|\big)\subseteq H_x^+\cap B\big(x,2|y_j-x|\big),\qquad\forall j\in\mathbb{N}, \tag{5.6.61}$$

where H_x^+ is as in (5.6.55). Consequently,

$$B\big(y_j,\varepsilon|y_j-x|\big)\setminus E\subseteq\Big(B\big(x,2|y_j-x|\big)\setminus E\Big)\cap H_x^+,\qquad\forall j\in\mathbb{N}, \tag{5.6.62}$$

hence $\mathcal{L}^n\left(B\big(y_j,\varepsilon|y_j-x|\big)\setminus E\right)\leq\mathcal{L}^n\left(\big(B\big(x,2|y_j-x|\big)\setminus E\big)\cap H_x^+\right)$ for each $j\in\mathbb{N}$ which, thanks to (5.6.57), forces

$$\mathcal{L}^n\Big(B\big(y_j,\varepsilon|y_j-x|\big)\setminus E\Big)=o\big(|y_j-x|^n\big)\quad\text{as}\quad j\to\infty. \tag{5.6.63}$$

On the other hand, the fact that E is two-sided n-thick entails the existence of a constant c such that

$$\mathcal{L}^n\Big(B\big(y_j, \varepsilon|y_j - x|\big) \setminus E\Big) \geq c|y_j - x|^n \quad \text{for each} \quad j \in \mathbb{N}. \tag{5.6.64}$$

This contradicts (5.6.63) and ultimately establishes (5.6.60). □

Recall the notion of approximate tangent $(n-1)$-plane from (5.4.1)–(5.4.2). Its relation with the geometric measure theoretic outward unit normal is made transparent in the proposition below.

Proposition 5.6.11 *Given a set $E \subseteq \mathbb{R}^n$ of locally finite perimeter, define*

$$\partial_T E := \big\{x \in \partial_* E : \textit{there exists an approximate tangent } (n-1)\textit{-plane to } \partial_* E \textit{ at } x\big\}. \tag{5.6.65}$$

Then

$$\partial^* E \subseteq \partial_T E \subseteq \partial_* E \tag{5.6.66}$$

and

$$\mathcal{H}^{n-1}\big(\partial_T E \setminus \partial^* E\big) = 0, \qquad \mathcal{H}^{n-1}\big(\partial_* E \setminus \partial_T E\big) = 0. \tag{5.6.67}$$

Also,

if ν denotes the geometric measure theoretic outward unit normal to E, then for each $x \in \partial^* E$ it follows that $\nu(x)$ is orthogonal to the approximate tangent $(n-1)$-plane to $\partial_* E$ at x. (5.6.68)

Proof The inclusions in (5.6.66) along with the orthogonality property described in (5.6.68) are seen from (5.6.65), (5.4.1)–(5.4.2), (5.6.21), (5.6.49), and (5.6.59) (keeping in mind the fact that the approximate tangent $(n-1)$-plane is unique; cf. (5.4.3)). Finally, the properties recorded in (5.6.67) are consequences of (5.6.66) and (5.6.21) (alternatively, property $\mathcal{H}^{n-1}\big(\partial_* E \setminus \partial_T E\big) = 0$ is implied by (5.6.33) and Proposition 5.4.1). □

Yet another point of view on the matter of outward unit normal is as follows. Given some $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$, define

$$\begin{aligned}\partial^N E := \Big\{x \in \mathbb{R}^n : \ &\text{there exists } N(x) \in S^{n-1} \text{ with} \\ &\lim_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r) \cap \{y \in E : (y-x)\cdot N(x) > 0\}\big)}{\mathcal{L}^n\big(B(x,r)\big)} = 0 \text{ and} \\ &\lim_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r) \cap \{y \in \mathbb{R}^n \setminus E : (y-x)\cdot N(x) < 0\}\big)}{\mathcal{L}^n\big(B(x,r)\big)} = 0\Big\}.\end{aligned} \tag{5.6.69}$$

As our next lemma shows, in the class of sets of locally finite perimeter, this construction yields, up to $\mathcal{H}^{n-1}$-nullsets, the familiar reduced boundary and geometric measure theoretic outward unit normal defined earlier (cf. (5.6.13) and (5.6.2)–(5.6.3)).

Lemma 5.6.12 *The following statements are true.*

(i) If $E \subseteq \mathbb{R}^n$ is a $\mathcal{L}^n$-measurable set, then for each $x \in \partial^N E$ the vector $N(x)$ doing the job described in (5.6.69) is unique. Henceforth, the notation $N_E(x)$ is used for this vector in order to stress its dependence on the underlying set E.

(ii) If $E \subseteq \mathbb{R}^n$ is a set of locally finite perimeter, then

$$\partial^* E \subseteq \partial^N E \;\; \textit{and} \;\; \mathcal{H}^{n-1}\big(\partial^N E \setminus \partial^* E\big) = 0. \tag{5.6.70}$$

Moreover, if ν denotes the geometric measure theoretic outward unit normal to E, then actually

$$N_E(x) = \nu(x) \;\; \textit{for each point} \;\; x \in \partial^* E. \tag{5.6.71}$$

Proof Consider a $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$. To facilitate the proof of the claim in item *(i)*, for each point $x \in \mathbb{R}^n$ and vector $N \in S^{n-1}$ define the half-spaces

$$H^{\pm}(x, N) := \big\{y \in \mathbb{R}^n : \pm(y - x) \cdot N > 0\big\} \tag{5.6.72}$$

then, for each $r > 0$, consider the half-balls

$$B_N^{\pm}(x, r) := B(x, r) \cap H^{\pm}(x, N). \tag{5.6.73}$$

With this piece of notation we have that $x \in \partial^N E$ if and only if there exists $N \in S^{n-1}$ such that

$$\mathcal{L}^n\big(B_N^-(x, r)\Delta\big[B(x, r) \cap E\big]\big) = o(r^n) \;\; \text{as} \;\; r \to 0^+. \tag{5.6.74}$$

Fix now $x \in \mathbb{R}^n$ and suppose there are two vectors $N_1, N_2 \in S^{n-1}$ doing the job in (5.6.74). Since, generally speaking, for any sets A, B, C we have

$$A\Delta B \subseteq (A\Delta C) \cup (B\Delta C), \tag{5.6.75}$$

we conclude that

$$\mathcal{L}^n\big(B_{N_1}^-(x, r)\Delta B_{N_2}^-(x, r)\big) = o(r^n) \;\; \text{as} \;\; r \to 0^+. \tag{5.6.76}$$

However, since

$$\mathcal{L}^n\big(B_{N_1}^-(x, r)\Delta B_{N_2}^-(x, r)\big) = c(N_1, N_2, n) \cdot r^n \;\; \text{for each} \;\; r > 0 \tag{5.6.77}$$

where $c(N_1, N_2, n)$ is a strictly positive constant if $N_1 \neq N_2$, we see that (5.6.76) forces $N_1 = N_2$. This proves the claim in item *(i)* of the statement.

Suppose next $E \subseteq \mathbb{R}^n$ is a set of locally finite perimeter. Then from (5.6.57)–(5.6.58), (5.6.69) and the uniqueness result established in item *(i)*, we conclude that

$$\partial^* E \subseteq \partial^N E \quad \text{and} \quad N_E(x) = \nu(x) \quad \text{for each} \quad x \in \partial^* E. \tag{5.6.78}$$

In addition, [263, Corollary 5.6.8/(5.6.14), p. 242] gives that $\mathcal{H}^{n-1}\big(\partial^N E \setminus \partial^* E\big) = 0$, finishing the proof of (5.6.70)–(5.6.71). □

Later on, it is going to be useful to know that the Cartesian product of a set of locally finite perimeter with a Euclidean space enjoys the properties described in the lemma below.

Lemma 5.6.13 *If $\Omega \subseteq \mathbb{R}^n$ is a set of locally finite perimeter with geometric measure theoretic outward unit normal ν, then $\widetilde{\Omega} := \mathbb{R}^m \times \Omega$ is also a set of locally finite perimeter with geometric measure theoretic outward unit normal $\widetilde{\nu} = (0, \nu)$.*

Proof From definitions, $\widetilde{\Omega}$ is Lebesgue measurable and $\partial_* \widetilde{\Omega} = \mathbb{R}^m \times \partial_* \Omega$. In particular, from this, the fact that Ω is a set of locally finite perimeter, and (5.6.35) we conclude that $\widetilde{\Omega}$ is a set of locally finite perimeter. Lemma 5.6.12 implies

$$\begin{aligned} &\partial^* \Omega \subseteq \partial^N \Omega, \quad \mathcal{H}^{n-1}\big(\partial^N \Omega \setminus \partial^* \Omega\big) = 0, \\ &\text{and} \quad N_\Omega(x) = \nu(x) \quad \text{for each} \quad x \in \partial^* \Omega. \end{aligned} \tag{5.6.79}$$

Note that if, for each $x \in \mathbb{R}^n$ and $r > 0$, one replaces the ball $B(x, r)$ in (5.6.69) with the cube $Q_n(x, r)$ (the subscript n indicates that this is a cube in $\mathbb{R}^n$) centered at x and of side-length r, then the set $\partial^N E$ does not change. Hence, for each $x \in \partial^N \Omega$ we have

$$\lim_{r \to 0^+} \frac{\mathcal{L}^n\big(Q_n(x, r) \cap \{y \in \Omega : (y - x) \cdot N_\Omega(x) > 0\}\big)}{r^n} = 0. \tag{5.6.80}$$

Let us denote by X, Y points in $\mathbb{R}^m$, by $Q_m(X, r)$ a cube in $\mathbb{R}^m$ centered at X and of side-length r, and by $Q_{m+n}\big((X, x), r\big)$ the cube in $\mathbb{R}^{m+n}$ centered at (X, x) and of side-length r. Observe that, for each $X \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $r > 0$, and $N \in S^{n-1}$ we have

$$\begin{aligned} &Q_{m+n}\big((X, x), r\big) \cap \Big\{(Y, y) \in \widetilde{\Omega} : \big((Y, y) - (X, x)\big) \cdot (0, N) > 0\Big\} \\ &\qquad = Q_m(X, r) \times \Big[Q_n(x, r) \cap \Big\{y \in \Omega : (Y - x) \cdot N > 0\Big\}\Big] \end{aligned} \tag{5.6.81}$$

and

$$Q_{m+n}\big((X,x),r\big)\cap\Big\{(Y,y)\in\mathbb{R}^{m+n}\setminus\widetilde{\Omega}:\ \big((Y,y)-(X,x)\big)\cdot(0,N)>0\Big\}$$
$$=Q_m(X,r)\times\Big[Q_n(x,r)\cap\Big\{y\in\mathbb{R}^n\setminus\Omega:\ (Y-x)\cdot N>0\Big\}\Big]. \tag{5.6.82}$$

This implies that

$$\begin{aligned}\partial^N\widetilde{\Omega}=\mathbb{R}^m\times\partial^N\Omega\ \text{ and}\\ N_{\widetilde{\Omega}}(X,x)=\big(0,N_\Omega(x)\big)\ \text{ for each }\ X\in\mathbb{R}^m,\ x\in\partial^N\Omega.\end{aligned} \tag{5.6.83}$$

Hence, from (5.6.83) and (5.6.80) we obtain that

$$N_{\widetilde{\Omega}}(X,x)=\big(0,\nu(x)\big)\ \text{ for each }\ X\in\mathbb{R}^m\ \text{ and } \mathcal{H}^{n-1}\text{-a.e. }\ x\in\partial^*\Omega. \tag{5.6.84}$$

Invoking again Lemma 5.6.12 for the set $\widetilde{\Omega}$ we conclude that $\widetilde{\nu}$, the geometric measure theoretic outward unit normal to $\widetilde{\Omega}$, satisfies

$$\widetilde{\nu}(X,x)=\big(0,\nu(x)\big)\ \text{ for each }\ X\in\mathbb{R}^m\ \text{ and } \mathcal{H}^{n-1}\text{-a.e. }\ x\in\partial^*\Omega, \tag{5.6.85}$$

and the desired conclusion follows. □

The next proposition elaborates on the structure of the reduced boundary for sets of locally finite perimeter (since the reduced boundary is a countably rectifiable set, this should be compared with Proposition 5.3.3). It is a slight version of the so-called Structure Theorem for sets of locally finite perimeter (cf. [80, Theorem 2, p. 205]) with the added bonus that the compact pieces in which the reduced boundary is decomposed are mutually disjoint. To state it, the reader is made aware that by a $\mathscr{C}^1$-hypersurfaces in $\mathbb{R}^n$ we shall understand a set $\mathcal{S}$ of the form $f^{-1}(\{0\})$ where $f\in\mathscr{C}^1(\mathbb{R}^n)$ is a real-valued function with the property that

$$\inf\{|(\nabla f)(x)|:\ x\in\mathbb{R}^n,\ \ f(x)=0\}>0. \tag{5.6.86}$$

Proposition 5.6.14 *Let $E\subseteq\mathbb{R}^n$ be a set of locally finite perimeter, and denote by ν its geometric measure theoretic outward unit normal. Then there exist a sequence of $\mathscr{C}^1$-hypersurfaces $\{\mathcal{S}_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}^n$, along with a sequence $\{K_j\}_{j\in\mathbb{N}}$ of mutually disjoint subsets of ∂^*E, such that each K_j is a compact subset of $\mathcal{S}_j$,*

$$\mathcal{H}^{n-1}\Big(\partial^*E\setminus\bigcup_{j\in\mathbb{N}}K_j\Big)=0, \tag{5.6.87}$$

and $\nu\big|_{K_j}$ is normal to $\mathcal{S}_j$ for each $j\in\mathbb{N}$.

Proof From (5.6.57)–(5.6.58) we know that for each $x\in\partial^*E$ we have

$$\lim_{r\to 0^+} \frac{\mathcal{L}^n\Big(\big(B(x,r)\cap E\big)\cap H_x^+\Big)+\mathcal{L}^n\Big(\big(B(x,r)\setminus E\big)\cap H_x^-\Big)}{r^n}=0. \tag{5.6.88}$$

For each $k\in\mathbb{N}$ consider $A_k := \partial^* E\cap\big(B(0,k+1)\setminus B(0,k)\big)$. The fact that E is of locally finite perimeter implies that $\mathcal{H}^{n-1}(A_k)<+\infty$ for each $k\in\mathbb{N}$ (cf. (5.6.35) and (5.6.21)). We claim that, for each fixed $k\in\mathbb{N}$,

$$\begin{array}{c}\text{there is a sequence } \{B_{i,k}\}_{i\in\mathbb{N}} \text{ of mutually disjoint subsets of } A_k\\ \text{with the property that } \mathcal{H}^{n-1}\Big(A_k\setminus\bigcup_{i\in\mathbb{N}}B_{i,k}\Big)=0 \text{ and such that}\\ \text{for each } i\in\mathbb{N} \text{ the convergence in (5.6.88) is uniform for } x\in B_{i,k}.\end{array} \tag{5.6.89}$$

The existence of such a sequence is justified inductively, via repeated applications of Egoroff's Theorem (see, e.g., [80, Theorem 3, p. 16]). Specifically, a first application of Egoroff's Theorem ensures the existence of an $\mathcal{H}^{n-1}$-measurable subset $B_{1,k}$ of A_k satisfying $\mathcal{H}^{n-1}(A_k\setminus B_{1,k})<1$ and such that the convergence in (5.6.88) is uniform for all points $x\in B_{1,k}$, then a second application of Egoroff's Theorem gives an $\mathcal{H}^{n-1}$-measurable subset $B_{2,k}$ of $A_k\setminus B_{1,k}$ with $\mathcal{H}^{n-1}\big((A_k\setminus B_{1,k})\setminus B_{2,k}\big)<1/2$ and such that the convergence in (5.6.88) is uniform for $x\in B_{2,k}$, and so on.

Relabeling $\{B_{i,k}\}_{i,k\in\mathbb{N}}$ simply as $\{F_i\}_{i\in\mathbb{N}}$ then proves that

$$\begin{array}{c}\text{there exists some sequence } \{F_i\}_{i\in\mathbb{N}} \text{ of mutually disjoint subsets of } \partial^*E\\ \text{with } \mathcal{H}^{n-1}\Big(\partial^*E\setminus\bigcup_{i\in\mathbb{N}}F_i\Big)=0 \text{ and such that for each } i\in\mathbb{N} \text{ we have}\\ \mathcal{H}^{n-1}(F_i)<+\infty \text{ and the convergence in (5.6.88) is uniform for } x\in F_i.\end{array} \tag{5.6.90}$$

Next, we claim that for each fixed $i\in\mathbb{N}$

$$\begin{array}{c}\text{there exists some sequence } \{E_i^j\}_{j\in\mathbb{N}} \text{ of mutually disjoint}\\ \text{compact subsets of } F_i \text{ such that } \mathcal{H}^{n-1}\Big(F_i\setminus\bigcup_{j\in\mathbb{N}}E_i^j\Big)=0\\ \text{and with the property that } \nu\big|_{E_i^j} \text{ is continuous for each } i\in\mathbb{N}.\end{array} \tag{5.6.91}$$

Such a sequence of sets may be constructed inductively, via repeated applications of Lusin's Theorem (see, e.g., [80, Theorem 2, p. 15]). Concretely, a first application of Lusin's Theorem gives an $\mathcal{H}^{n-1}$-measurable compact subset E_i^1 of F_i satisfying $\mathcal{H}^{n-1}(F_i\setminus E_i^1)<1$ and such that $\nu\big|_{E_i^1}$ is continuous, then a second application of Lusin's Theorem provides an $\mathcal{H}^{n-1}$-measurable compact subset E_i^2 of $F_i\setminus E_i^1$ with $\mathcal{H}^{n-1}\big((F_i\setminus E_i^1)\setminus E_i^2\big)<1/2$ and such $\nu\big|_{E_i^2}$ is continuous, and so on.

If we re-index the sets $\{E_i^j\}_{i,j\in\mathbb{N}}$ and call them $\{K_i\}_{i\in\mathbb{N}}$, this proves that

$$\begin{array}{c}\text{there is a sequence } \{K_i\}_{i\in\mathbb{N}} \text{ of mutually disjoint compact}\\ \text{subsets of } \partial^*E \text{ satisfying } \mathcal{H}^{n-1}\Big(\partial^*E\setminus\bigcup_{i\in\mathbb{N}}K_i\Big)=0 \text{ and}\\ \text{with the property that } \nu\big|_{K_i} \text{ is continuous for each } i\in\mathbb{N}.\end{array} \tag{5.6.92}$$

Having identified the sequence $\{K_i\}_{i\in\mathbb{N}}$ as in (5.6.92), the remainder of the proof now proceeds as in [80, items 2–5 on pp. 206–207]. □

We next discuss a result in the spirit of [124, Proposition 2.9], establishing a link between the direction of the geometric measure theoretic outward unit normal and the cone property. To facilitate the statement of such a result, we first make a definition. By a (open, convex, truncated, one-component, circular) cone in $\mathbb{R}^n$ we understand any set of the form

$$\mathrm{C}_{\theta,b}(x,h) := \{y \in \mathbb{R}^n : \cos(\theta/2)\,|y-x| < (y-x)\cdot h < b\}, \tag{5.6.93}$$

where the point $x \in \mathbb{R}^n$ is the vertex of the cone, the vector $h \in S^{n-1}$ is the direction of its symmetry axis, the angle $\theta \in (0,\pi)$ is the (full) aperture of the cone, and the number $b \in (0,\infty)$ is the height of the cone.

Lemma 5.6.15 *Let E be a subset of $\mathbb{R}^n$ of locally finite perimeter. Fix a point x belonging to $\partial^* E$ with the property that there exist $b > 0$, $\theta \in (0,\pi)$, and $h \in S^{n-1}$ such that*

$$\mathrm{C}_{\theta,b}(x,h) \subseteq E. \tag{5.6.94}$$

Then, if $\nu(x)$ denotes the geometric measure theoretic outward unit normal to E at x, there holds

$$\nu(x) \in \overline{\mathrm{C}_{\pi-\theta,1}(0,-h)}. \tag{5.6.95}$$

Proof Since $x \in \partial^* E$, from Theorem 5.6.9 we know that (5.6.57)–(5.6.58) hold. In particular, (5.6.57) and (5.6.94) imply that, on the one hand,

$$\lim_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\cap\{y \in \mathrm{C}_{\theta,b}(x,h) : (y-x)\cdot\nu(x) > 0\}\big)}{\mathcal{L}^n\big(B(x,r)\big)} = 0. \tag{5.6.96}$$

On the other hand, for each $r \in (0,b)$ we have

$$\mathcal{L}^n\big(B(x,r)\cap\{y \in \mathrm{C}_{\theta,b}(x,h) : (y-x)\cdot\nu(x) > 0\}\big) = c(\theta,h,\nu(x))\cdot r^n \tag{5.6.97}$$

where $c(\theta,h,\nu(x))$ is a strictly positive constant whenever

$$\mathrm{C}_{\theta,b}(x,h) \cap \{y \in \mathbb{R}^n : (y-x)\cdot\nu(x) > 0\} \neq \varnothing. \tag{5.6.98}$$

Collectively, (5.6.96) and (5.6.97) then force

$$\mathrm{C}_{\theta,b}(x,h) \cap \{y \in \mathbb{R}^n : (y-x)\cdot\nu(x) > 0\} = \varnothing \tag{5.6.99}$$

which further implies that the angle between $-h$ and $\nu(x)$ is at most $(\pi-\theta)/2$. Given that $|\nu(x)| = 1$, the latter property places $\nu(x)$ in the closure of the cone $\mathrm{C}_{\pi-\theta,1}(0,-h)$, proving (5.6.95). □

In the proposition below we recall a useful differentiability criterion, of a purely geometrical nature, from [9, Proposition 3.15, p. 315].

Proposition 5.6.16 *Fix $n \in \mathbb{N}$ with $n \geq 2$. Assume that $U \subseteq \mathbb{R}^{n-1}$ is an arbitrary set, and pick a point $x_0' \in \mathring{U}$. Given a function $f : U \to \mathbb{R}$, denote by G_f the graph of f, i.e.,*

$$G_f := \big\{(x', f(x')) : x' \in U\big\} \subseteq \mathbb{R}^n. \tag{5.6.100}$$

Then f is differentiable at the point x_0' if and only if f is continuous at x_0' and there exists a non-horizontal vector $h \in S^{n-1}$ (i.e., satisfying $h \cdot \mathbf{e}_n \neq 0$) with the following significance. For every angle $\theta \in (0, \pi)$ there exists $b > 0$ with the property that $G_f \cap B\big((x_0', f(x_0')), b\big)$ lies in between the cones $\mathrm{C}_{\theta,b}\big((x_0', f(x_0')), h\big)$ and $\mathrm{C}_{\theta,b}\big((x_0', f(x_0')), -h\big)$, i.e.,

$$\begin{aligned} &G_f \cap B\big((x_0', f(x_0')), b\big) \\ &\quad \subseteq \mathbb{R}^n \setminus \Big[\mathrm{C}_{\theta,b}\big((x_0', f(x_0')), h\big) \cup \mathrm{C}_{\theta,b}\big((x_0', f(x_0')), -h\big)\Big]. \end{aligned} \tag{5.6.101}$$

If this happens, then necessarily

$$h = \frac{\big((\nabla' f)(x_0'), -1\big)}{\sqrt{1 + |(\nabla' f)(x_0')|^2}}, \tag{5.6.102}$$

where ∇' is the gradient operator in $\mathbb{R}^{n-1}$.

The result in the proposition below shows that the geometric measure theoretic notion of outward unit normal for a set of locally finite perimeter is naturally compatible with the standard, analytical concept of outward unit normal in the case the set in question may be locally described as the upper-graph of a real-valued differentiable function.

Proposition 5.6.17 *Assume $E \subseteq \mathbb{R}^n$, where $n \geq 2$, is a set of locally finite perimeter and suppose $x = (x', x_n) \in \partial^* E$ has the property that there exist a continuous function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ and some $r > 0$ satisfying*

$$\begin{gathered} f(x') = x_n, \quad f \text{ is differentiable at } x', \quad \text{and} \\ B(x, r) \cap \{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : y_n > f(y')\} \subseteq E. \end{gathered} \tag{5.6.103}$$

Then the geometric measure theoretic outward unit normal to E at x is given by

$$\nu(x) = \frac{\big((\nabla' f)(x'), -1\big)}{\sqrt{1 + |(\nabla' f)(x')|^2}}. \tag{5.6.104}$$

Proof Consider the vector

$$h = (h', h_n) := \frac{\big(-(\nabla' f)(x'), 1\big)}{\sqrt{1 + |(\nabla' f)(x')|^2}} \in S^{n-1}. \tag{5.6.105}$$

The fact that the function f is differentiable at x' entails

$$\begin{aligned} f(x'+th') &= f(x') + t(\nabla' f)(x')\cdot h' + o(t) \\ &= x_n - \frac{t|(\nabla' f)(x')|^2}{\sqrt{1+|(\nabla' f)(x')|^2}} + o(t) \\ &= x_n - t\big|(\nabla' f)(x')\big|^2 h_n + o(t) \\ &= (x_n + th_n) - t\big(1+|(\nabla' f)(x')|^2\big)h_n + o(t) \\ &= (x_n + th_n) - t\sqrt{1+|(\nabla' f)(x')|^2} + o(t) \quad \text{as } t\to 0^+. \end{aligned} \tag{5.6.106}$$

This proves that $f(x'+th') < x_n + th_n$ for all $t > 0$ sufficiently small, which ultimately goes to show that

$$\begin{aligned} &\text{the point } x+th \text{ belongs to the upper-graph of} \\ &\text{the function } f \text{ for each } t>0 \text{ sufficiently small.} \end{aligned} \tag{5.6.107}$$

Next, Proposition 5.6.16 implies that for each angle $\theta \in (0,\pi)$ there exists some $b > 0$ with the property that G_f, the graph of the function f, is disjoint from $\mathrm{C}_{\theta,b}(x,h)$. Since f is continuous, $U_f^{\pm} := \{y = (y', y_n) \in \mathbb{R}^n : \pm y_n > f(y')\}$ are open and, based on what we have just mentioned, cover the cone $\mathrm{C}_{\theta,b}(x,h)$. Given that the latter is a connected set, (5.6.107) implies that

$$\begin{aligned} &\text{for every angle } \theta\in(0,\pi) \text{ there exists } b>0 \text{ with the property that} \\ &\mathrm{C}_{\theta,b}(x,h) \text{ is contained in } U_f^+, \text{ the upper-graph of the function } f. \end{aligned} \tag{5.6.108}$$

In light of the last condition in (5.6.103), this further gives that

$$\begin{aligned} &\text{for every angle } \theta\in(0,\pi) \text{ there exists a small} \\ &\text{height } b\in(0,r) \text{ such that } \mathrm{C}_{\theta,b}(x,h)\subseteq E. \end{aligned} \tag{5.6.109}$$

Having established this, we may invoke Lemma 5.6.15 to conclude that

$$\nu(x) \in \bigcap_{\theta\in(0,\pi)} \overline{\mathrm{C}_{\pi-\theta,1}(0,-h)} = \{-h\} \tag{5.6.110}$$

which, in light of (5.6.105), proves (5.6.104). □

The above result should be compared and contrasted with the proposition below, which also serves as a rich source of examples of sets of locally finite perimeter.

Proposition 5.6.18 *Fix $n \in \mathbb{N}$ with $n \geq 2$ and pick a real-valued function*

$$\phi \in \mathscr{C}^0(\mathbb{R}^{n-1}) \quad \textit{with} \quad \nabla'\phi \in \big[L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1}. \tag{5.6.111}$$

Then the upper-graph of the function ϕ, *i.e.*,

$$\Omega := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \phi(x')\}, \tag{5.6.112}$$

is a set of locally finite perimeter, with the property that

$$\mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial_*\Omega\big) = 0. \tag{5.6.113}$$

Moreover, the geometric measure theoretic outward unit normal of Ω *is given by*

$$\nu\big(x', \phi(x')\big) = \frac{\big(\nabla'\phi(x'), -1\big)}{\sqrt{1 + |(\nabla'\phi)(x')|^2}} \quad \textit{for } \mathcal{L}^{n-1}\textit{-a.e. } x' \in \mathbb{R}^{n-1}, \tag{5.6.114}$$

and for each $\mathcal{L}^{n-1}$*-measurable set* $\mathcal{O}' \subseteq \mathbb{R}^{n-1}$ *one has*

$$\mathcal{H}^{n-1}\big(\{(x', \phi(x')) : x' \in \mathcal{O}'\}\big) = \int_{\mathcal{O}'} \sqrt{1 + |(\nabla'\phi)(x')|^2}\, \mathrm{d}x'. \tag{5.6.115}$$

All the aforementioned properties are seen from [125, Proposition 2.3, p. 2578], [125, Proposition 2.4, p. 2581], [125, Proposition 2.5, p. 2582], and [125, p. 2583] The proof of the fact that Ω is a set of locally finite perimeter is an adaptation of an argument in [263, Remark 5.8.3, p. 248] showing that upper-graphs of Lipschitz functions are sets of locally finite perimeter.

If in place of (5.6.111) we assume that

$$\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}) \ \text{ has } \ \nabla'\phi \in \big[L^{n-1,1}_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1} \tag{5.6.116}$$

then [241] guarantees that (up to being redefined on a $\mathcal{L}^{n-1}$-nullset) the function ϕ is continuous at each point in $\mathbb{R}^{n-1}$ and also differentiable at $\mathcal{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$. In view of Lemma 6.2.4, this shows that (5.6.116) implies (5.6.111). In addition, the claim in (5.6.114) may now be seen directly from Proposition 5.6.17.

We continue by presenting a criterion for identifying "pieces of Lipschitz graphs" within arbitrary sets. Recall from (5.6.93) that $\mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n)$ denotes the open, convex, one-component, circular cone in $\mathbb{R}^n$, with vertex at $x \in \mathbb{R}^n$, symmetry axis along the unit vector $\mathbf{e}_n \in S^{n-1}$, and (full) aperture the angle $\theta \in (0, \pi)$, in other words, the set $\mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) = \{y \in \mathbb{R}^n : \cos(\theta/2)\,|y - x| < (y - x) \cdot \mathbf{e}_n\}$.

Proposition 5.6.19 *Let* E *be an arbitrary set in* $\mathbb{R}^n$, *fix* $\theta \in (0, \pi)$, *and define*

$$\Sigma := \big\{x \in E : \mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \cap E = \varnothing\big\}. \tag{5.6.117}$$

Then either $\Sigma = \varnothing$ *or there exists a Lipschitz function* $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ *with Lipschitz constant less than or equal to* $\cot(\theta/2)$ *and such that*

$$\Sigma = E \cap \operatorname{Graph}\varphi \quad \textit{and} \quad E \cap \operatorname{Sepigraph}\varphi = \varnothing, \tag{5.6.118}$$

where $\mathrm{Sepigraph}\,\varphi := \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \varphi(x')\}$ *is the strict epigraph of* φ.

Proof Suppose that $\Sigma \neq \varnothing$. For each point $x = (x', x_n) \in \Sigma$, consider the function $\psi_x : \mathbb{R}^{n-1} \to \mathbb{R}$ whose graph is $\partial \mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n)$. Specifically, a direct calculation based on (5.6.93) yields the formula

$$\psi_x(z') = x_n + |z' - x'| \cot(\theta/2) \quad \text{for each} \quad z' \in \mathbb{R}^{n-1}. \tag{5.6.119}$$

Consequently, ψ_x is a Lipschitz function with Lipschitz constant equal to $\cot(\theta/2)$. Next, we claim that

$$\inf_{x \in \Sigma} \psi_x(z') > -\infty \quad \text{for each} \quad z' \in \mathbb{R}^{n-1}. \tag{5.6.120}$$

To see why (5.6.120) is true, first observe that if $x, y \in \Sigma$ are two arbitrary points then necessarily $x \in E$ and $\mathrm{C}_{\theta,\infty}(y, \mathbf{e}_n) \cap E = \varnothing$, hence $x \notin \mathrm{C}_{\theta,\infty}(y, \mathbf{e}_n)$. Consequently, $(x - y) \cdot \mathbf{e}_n \leq |x - y| \cos(\theta/2)$. Since the same formula holds by interchanging x and y, we obtain $|(x - y) \cdot \mathbf{e}_n| \leq |x - y| \cos(\theta/2)$ which, after some algebra, implies

$$|x_n - y_n| \leq |x' - y'| \cot(\theta/2), \ \ \forall x = (x', x_n) \in \Sigma \ \text{ and } \forall y = (y', y_n) \in \Sigma. \tag{5.6.121}$$

Now fix $y = (y', y_n) \in \Sigma$. Starting with (5.6.121), then using the triangle inequality and (5.6.119), for each $x = (x', x_n) \in \Sigma$ we may write

$$\begin{aligned} |x_n - y_n| &\leq |x' - y'| \cot(\theta/2) \leq |x' - z'| \cot(\theta/2) + |y' - z'| \cot(\theta/2) \\ &= \psi_x(z') + \psi_y(z') - x_n - y_n \quad \text{if} \quad z' \in \mathbb{R}^{n-1}. \end{aligned} \tag{5.6.122}$$

From (5.6.122) we have $\psi_x(z') \geq x_n + |x_n - y_n| + y_n - \psi_y(z') \geq 2y_n - \psi_y(z')$ for each $x = (x', x_n) \in \Sigma$ and each $z' \in \mathbb{R}^{n-1}$. Thus $\inf_{x \in \Sigma} \psi_x(z') \geq 2y_n - \psi_y(z')$ for each $z' \in \mathbb{R}^{n-1}$. This proves (5.6.120). In turn, (5.6.120) allows us to define the function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ by setting

$$\varphi(z') := \inf_{x \in \Sigma} \psi_x(z') \quad \text{for every} \quad z' \in \mathbb{R}^{n-1}. \tag{5.6.123}$$

From [188, Lemma 4.4, pp. 159–160] it follows that φ is a Lipschitz function with Lipschitz constant bounded by $\cot(\theta/2)$. To complete the proof of the proposition we are left with showing that the conditions in (5.6.118) hold for this choice of φ. With this goal in mind, we first claim that

$$\mathrm{Sepigraph}\,\varphi = \bigcup_{x \in \Sigma} \mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n). \tag{5.6.124}$$

Indeed, a point $z = (z', z_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ belongs to $\mathrm{Sepigraph}\,\varphi$ if and only if $z_n > \varphi(z') = \inf_{x \in \Sigma} \psi_x(z')$, which is equivalent with the existence of some $x \in \Sigma$

such that $z_n > \psi_x(z')$. The latter is equivalent with the existence of some $x \in \Sigma$ such that z belongs to the strict epigraph of ψ_x, i.e., $z \in C_{\theta,\infty}(x, \mathbf{e}_n)$, completing the proof of (5.6.124). Now the second property in (5.6.118) follows from (5.6.124) and the fact that $C_{\theta,\infty}(x, \mathbf{e}_n) \cap E = \varnothing$ for each $x \in \Sigma$. In turn, the second property in (5.6.118) and the inclusion $\Sigma \subseteq E$ imply

$$\Sigma \cap \text{Sepigraph}\,\varphi = \varnothing. \tag{5.6.125}$$

Also, from (5.6.124) and the fact that any vertex $x \in \Sigma$ may be approximated from within the cone $C_{\theta,\infty}(x, \mathbf{e}_n)$ we conclude that

$$\Sigma \subseteq \overline{\text{Sepigraph}\,\varphi} = \big\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n \geq \varphi(x')\big\}. \tag{5.6.126}$$

Together, (5.6.125)–(5.6.126) imply $\Sigma \subseteq \overline{\text{Sepigraph}\,\varphi} \setminus \text{Sepigraph}\,\varphi = \text{Graph}\,\varphi$, hence

$$\Sigma \subseteq E \cap \text{Graph}\,\varphi. \tag{5.6.127}$$

The proof of the first property in (5.6.118) will be finished once we establish the opposite inclusion, i.e.,

$$E \cap \text{Graph}\,\varphi \subseteq \Sigma. \tag{5.6.128}$$

With this goal in mind, let us fix an arbitrary point $x \in E \cap \text{Graph}\,\varphi$. Given that we have $\text{Graph}\,\varphi = \partial\big(\text{Sepigraph}\,\varphi\big)$, the identity in (5.6.124) allows us to invoke Lemma 5.6.20 (formulated a little later below) to conclude that

$$C_{\theta,\infty}(x, \mathbf{e}_n) \subseteq \text{Sepigraph}\,\varphi. \tag{5.6.129}$$

Upon recalling the second property in (5.6.118), from (5.6.129) we see that $C_{\theta,\infty}(x, \mathbf{e}_n) \cap E = \varnothing$. In light of the fact that $x \in E$, this forces $x \in \Sigma$, proving (5.6.128). The proof of the first property in (5.6.118) is finished, and this completes the proof of the proposition, modulo that of Lemma 5.6.20. □

Here is the lemma invoked in the proof of Proposition 5.6.19.

Lemma 5.6.20 *Let $\{\Gamma_i\}_{i\in I}$ be a family of open, one-component, infinite, circular cones, with fixed (full) aperture $\theta \in (0, \pi)$, and symmetry axes along the vector $\mathbf{e}_n$. Then the set $\Omega := \bigcup_{i\in I} \Gamma_i$ has the property that*

$$C_{\theta,\infty}(x, \mathbf{e}_n) \subseteq \Omega \;\; \textit{for every} \;\; x \in \overline{\Omega}. \tag{5.6.130}$$

Proof The inclusion $C_{\theta,\infty}(x, \mathbf{e}_n) \subseteq \Omega$ for every $x \in \Omega$ follows from the definition of Ω and the general fact that for each $z \in \mathbb{R}^n$ we have $C_{\theta,\infty}(y, \mathbf{e}_n) \subseteq C_{\theta,\infty}(z, \mathbf{e}_n)$ for every $y \in C_{\theta,\infty}(z, \mathbf{e}_n)$. There remains to show that the inclusion in (5.6.130) also holds when $x \in \partial\Omega$. Pick $x \in \partial\Omega$ and consider a sequence $\{x_j\}_{j\in\mathbb{N}} \subseteq \Omega$ convergent to x. Then based on what we have already proved, $C_{\theta,\infty}(x_j, \mathbf{e}_n) \subseteq \Omega$ for every integer

$j \in \mathbb{N}$. Fix a point $y \in \mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n)$. Then necessarily $(y - x) \cdot \mathbf{e}_n > |y - x| \cos(\theta/2)$ which, in concert with the fact that $\lim_{j\to\infty} x_j = x$, implies that there exists some $j \in \mathbb{N}$ with the property that $(y - x_j) \cdot \mathbf{e}_n > |y - x_j| \cos(\theta/2)$. This proves that $y \in \mathrm{C}_{\theta,\infty}(x_j, \mathbf{e}_n)$ thus, ultimately, $y \in \Omega$ as wanted. $\square$

In turn, Proposition 5.6.19 is the main ingredient in the proof of the following geometric characterization of strict epigraphs of Lipschitz functions.

Proposition 5.6.21 *Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$. Then the following statements are equivalent:*

(1) The set Ω is a graph Lipschitz domain, i.e., the strict epigraph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$;
(2) There exists an angle $\theta \in (0, \pi)$ such that $\mathrm{C}_{\theta,\infty}(x, -\mathbf{e}_n) \cap \Omega = \varnothing$ for every $x \in \partial\Omega$;
(3) One has $\partial\Omega = \partial(\overline{\Omega})$ and there exists $\theta \in (0, \pi)$ such that $\mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \subseteq \Omega$ for every $x \in \partial\Omega$;
(4) There exists an angle $\theta \in (0, \pi)$ such that $\mathrm{C}_{\theta,\infty}(x, -\mathbf{e}_n) \cap \overline{\Omega} = \varnothing$ for every $x \in \partial\Omega$.

Proof We start by observing that whenever $E \subseteq \mathbb{R}^n$ is an arbitrary set we have

$$\big\{x \in E : \mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \cap E = \varnothing\big\} = \big\{x \in E \cap \partial E : \mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \cap E = \varnothing\big\}. \tag{5.6.131}$$

Indeed, this is a consequence of $E \setminus E^\circ = E \cap \partial E$ and the fact that for every $x \in \mathbb{R}^n$, every $\theta \in (0, \pi)$, and every $r \in (0, \infty)$ we have $\mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \cap B(x, r) \neq \varnothing$. In what follows, for each arbitrary set $E \subseteq \mathbb{R}^n$ we denote $-E := \{-x : x \in E\}$.

Clearly *(4)* $\Rightarrow$ *(2)*. To prove the opposite implication, reason by contradiction, i.e., suppose *(2)* holds and for some $x \in \partial\Omega$ there exists $y \in \mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \cap \overline{\Omega}$. Then $\mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n)$ is an open neighborhood for y, hence $\mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \cap \Omega \neq \varnothing$ contradicting *(2)*. This proves that *(2)* $\Rightarrow$ *(4)*.

Next, we show that *(4)* $\Rightarrow$ *(1)*. Suppose the property formulated in *(4)* is true and introduce

$$\Sigma := \big\{x \in -\overline{\Omega} : \mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \cap (-\overline{\Omega}) = \varnothing\big\} \tag{5.6.132}$$

Since Ω is an open set, we have $\partial(\overline{\Omega}) = \overline{\Omega} \setminus (\overline{\Omega})^\circ \subseteq \overline{\Omega} \setminus \Omega^\circ = \overline{\Omega} \setminus \Omega = \partial\Omega$, which further implies $\partial(-\overline{\Omega}) \subseteq -\partial\Omega$. This, (5.6.132), and (5.6.131) (used with $E := -\overline{\Omega}$) then yield $\Sigma \subseteq -\partial\Omega$. We claim that the opposite inclusion is also true. Indeed, if $x \in -\partial\Omega$, then $-x \in \partial\Omega$ and, by assumption, it follows that $\mathrm{C}_{\theta,\infty}(-x, -\mathbf{e}_n) \cap \overline{\Omega} = \varnothing$. Given that $\mathrm{C}_{\theta,\infty}(-x, -\mathbf{e}_n) = -\mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n)$ (a fact easily seen from the definition of cones given in (5.6.93)), the latter property becomes $\mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \cap (-\overline{\Omega}) = \varnothing$. Together with (5.6.132) this gives $x \in \Sigma$, proving $-\partial\Omega \subseteq \Sigma$, as wanted. In summary, we have $\Sigma = -\partial\Omega$, hence

$$\partial\Omega = -\Sigma. \tag{5.6.133}$$

From Proposition 5.6.19 applied with $E := -\overline{\Omega}$ it follows that there exists a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Sigma = \text{Graph}\,\varphi \cap (-\overline{\Omega}) \quad \text{and} \quad (-\overline{\Omega}) \cap \text{Sepigraph}\,\varphi = \varnothing. \tag{5.6.134}$$

Define the function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ by $\psi(x') := -\varphi(-x')$ for every $x' \in \mathbb{R}^{n-1}$. Then (5.6.133) and the first identity of sets in (5.6.134) imply

$$\begin{aligned} \partial\Omega &= \Big\{\big(-x', -\varphi(x')\big) : x' \in \mathbb{R}^{n-1} \text{ is such that } \big(-x', -\varphi(x')\big) \in \overline{\Omega}\Big\} \\ &= \Big\{\big(y', \psi(y')\big) : y' \in \mathbb{R}^{n-1} \text{ is such that } \big(y', \psi(y')\big) \in \overline{\Omega}\Big\} = \text{Graph}\,\psi \cap \overline{\Omega}. \end{aligned} \tag{5.6.135}$$

Thus,

$$\partial\Omega \subseteq \text{Graph}\,\psi. \tag{5.6.136}$$

In addition, the second identity of sets in (5.6.134) implies

$$\begin{aligned} &\overline{\Omega} \cap \big\{(-x', -x_n) : x' \in \mathbb{R}^{n-1},\ -x_n < -\varphi(x')\big\} = \varnothing \\ &\Longleftrightarrow \overline{\Omega} \cap \big\{(y', y_n) : y' \in \mathbb{R}^{n-1},\ y_n < \psi(y')\big\} = \varnothing \\ &\Longleftrightarrow \overline{\Omega} \subseteq \big\{(y', y_n) : y' \in \mathbb{R}^{n-1},\ y_n \geq \psi(y')\big\} \\ &\Rightarrow \Omega \subseteq (\overline{\Omega})^\circ \subseteq \big\{(y', y_n) : y' \in \mathbb{R}^{n-1},\ y_n > \psi(y')\big\} = \text{Sepigraph}\,\psi. \end{aligned} \tag{5.6.137}$$

Since Sepigraph ψ is a connected set and $\partial(\text{Sepigraph}\,\psi) = \text{Graph}\,\psi$, in light of (5.6.136) and (5.6.137), we may invoke Lemma 5.6.22 (formulated right after the current proof) for the current set Ω and for $O := \text{Sepigraph}\,\psi$ to conclude we have that $\Omega = \text{Sepigraph}\,\psi$. This finishes the proof of *(4)* $\Rightarrow$ *(1)*.

Next we will show that *(3)* $\Rightarrow$ *(1)*. Consider the set $E := \mathbb{R}^n \setminus \Omega$. Then $\partial E = \partial\Omega$ and since E is closed we also have $E \cap \partial E = \partial E = \partial\Omega$. In particular, if we define the set

$$\Sigma := \big\{x \in E \cap \partial E : \mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \cap E = \varnothing\big\}, \tag{5.6.138}$$

then using the last assumption in *(3)* we obtain

$$\Sigma = \big\{x \in \partial\Omega : \mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \cap E = \varnothing\big\} = \partial\Omega. \tag{5.6.139}$$

From (5.6.132), Proposition 5.6.19 applied for the current E, and (5.6.139) it follows that there exists a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\partial\Omega = (\text{Graph}\,\varphi) \cap (\mathbb{R}^n \setminus \Omega) \quad \text{and} \quad (\mathbb{R}^n \setminus \Omega) \cap \text{Sepigraph}\,\varphi = \varnothing. \tag{5.6.140}$$

From the second property in (5.6.140) it follows that

$$\mathbb{R}^n \setminus \Omega \subseteq \big\{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n \leq \varphi(x')\big\} \tag{5.6.141}$$

which, after taking interiors, further implies

$$\mathbb{R}^n \setminus \overline{\Omega} = (\mathbb{R}^n \setminus \Omega)^\circ \subseteq \big\{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n < \varphi(x')\big\}. \tag{5.6.142}$$

Upon recalling the assumption $\partial\Omega = \partial(\overline{\Omega})$, from the first property in (5.6.140) we also deduce that

$$\partial\big(\mathbb{R}^n \setminus \overline{\Omega}\big) = \partial(\overline{\Omega}) = \partial\Omega \subseteq \big\{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n = \varphi(x')\big\}. \tag{5.6.143}$$

Granted (5.6.142)–(5.6.143) and since $\partial\Omega \subseteq \operatorname{Graph}\varphi$ (recall (5.6.140)), we may invoke Lemma 5.6.22 with $\big\{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n < \varphi(x')\big\}$ in place of $\mathcal{O}$ and $\mathbb{R}^n \setminus \overline{\Omega}$ in place of Ω to obtain

$$\mathbb{R}^n \setminus \overline{\Omega} = \big\{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n < \varphi(x')\big\}. \tag{5.6.144}$$

After taking first complements and then interiors in (5.6.144), we arrive at

$$\Omega \subseteq (\overline{\Omega})^\circ \subseteq \Big(\big\{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n \geq \varphi(x')\big\}\Big)^\circ = \operatorname{Sepigraph}\varphi. \tag{5.6.145}$$

One more application of Lemma 5.6.22 for the current set Ω and $\mathcal{O} := \operatorname{Sepigraph}\varphi$ (bearing in mind that $\partial\Omega \subseteq \operatorname{Graph}\varphi$; cf. (5.6.140)) allows us to conclude that $\Omega = \operatorname{Sepigraph}\varphi$. Hence, *(1)* holds so the proof of the implication *(3)* $\Rightarrow$ *(1)* is finished.

Finally, suppose now that *(1)* is true, and let $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ be the Lipschitz function such that $\Omega = \operatorname{Sepigraph}\varphi$. In particular $\partial\Omega = \operatorname{Graph}\varphi$ and the condition $\partial\Omega = \partial(\overline{\Omega})$ is satisfied. If M denotes the Lipschitz constant of φ, choose $\theta \in (0, \pi)$ such that $\cot(\theta/2) > M$. Fix $x = \big(x', \varphi(x')\big) \in \partial\Omega$ arbitrary. Then *(3)* and *(4)* will be true if we prove that

$$\mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \subseteq \Omega \ \text{ and } \ \mathrm{C}_{\theta,\infty}(x, -\mathbf{e}_n) \subseteq \mathbb{R}^n \setminus \overline{\Omega}. \tag{5.6.146}$$

To prove the first inclusion in (5.6.146), pick $y = (y', y_n) \in \mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n)$ with $y' \in \mathbb{R}^{n-1}$ and $y_n \in \mathbb{R}$. Then $y_n - \varphi(x') = (y - x) \cdot \mathbf{e}_n > |y - x| \cos(\theta/2)$. Simple algebra shows that the last inequality also implies $y_n - x_n > |y' - x'| \cot(\theta/2)$. Hence, $y_n - \varphi(x') > |y' - x'| \cot(\theta/2)$. In concert with the choice of θ and the definition of M, this permits us to estimate

$$y_n - \varphi(x') > |y' - x'| \cot(\theta/2) > M|y' - x'| \geq \varphi(y') - \varphi(x'). \tag{5.6.147}$$

Consequently, $y_n > \varphi(y')$ which implies $y \in \Omega$. The first inclusion in (5.6.146) therefore follows. This establishes *(3)*. Similarly, if $y = (y', y_n) \in \mathrm{C}_{\theta,\infty}(x, -\mathbf{e}_n)$ with $y' \in \mathbb{R}^{n-1}$ and $y_n \in \mathbb{R}$, then $\varphi(x') - y_n = (y - x) \cdot (-\mathbf{e}_n) > |y - x| \cos(\theta/2)$. Much as before, this implies

$$\varphi(x') - y_n > M|y' - x'| \geq \varphi(x') - \varphi(y'), \tag{5.6.148}$$

which forces $y_n < \varphi(y')$, hence $y \in \mathbb{R}^n \setminus \overline{\Omega}$. As such *(4)* also holds. Thus, *(1)* $\Rightarrow$ *(3)* and *(1)* $\Rightarrow$ *(4)*, completing the proof of the proposition. □

The auxiliary topological result used in the proof of Proposition 5.6.21 is discussed next.

Lemma 5.6.22 *Let Ω and $\mathcal{O}$ be nonempty open sets in $\mathbb{R}^n$ such that $\Omega \subseteq \mathcal{O}$, the set $\mathcal{O}$ is connected, and $\partial\Omega \subseteq \partial\mathcal{O}$. Then necessarily $\Omega = \mathcal{O}$.*

Proof Suppose there exists $x_* \in \mathcal{O}$ such that $x_* \notin \Omega$. The latter condition self-improves to $x_* \notin \overline{\Omega}$ (otherwise we would have $x \in \partial\Omega \subseteq \partial\mathcal{O}$, in contradiction with $x_* \in \mathcal{O}$, given that $\mathcal{O}$ is open). Next, since $\Omega \neq \varnothing$ there exists $x_0 \in \Omega$. Then $x_0 \in \mathcal{O}$. Having $\mathcal{O}$ open connected Euclidean set implies that this is path-connected. Hence, there exists a continuous map $\gamma : [0, 1] \to \mathbb{R}^n$ with $\gamma([0, 1]) \subseteq \mathcal{O}$ and such that $\gamma(0) = x_0$ and $\gamma(1) = x_*$. We necessarily have $\gamma([0, 1]) \cap \partial\Omega \neq \varnothing$ (else Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ constitute a nontrivial open cover of the set $\gamma([0, 1])$, violating its connectivity). This further implies $\gamma([0, 1]) \cap \partial\mathcal{O} \neq \varnothing$. Since $\mathcal{O}$ is open, this contradicts the fact that $\gamma([0, 1]) \subseteq \mathcal{O}$. Hence our assumption that $\mathcal{O} \setminus \Omega \neq \varnothing$ is false, forcing $\Omega = \mathcal{O}$. □

Let us define the angle of a given a unitary $n \times n$ matrix U as

$$\theta_U := \arccos\Big(\inf_{x \in S^{n-1}} \langle x, Ux\rangle\Big) \in [0, \pi]. \tag{5.6.149}$$

Observe that

$$\text{if } \theta \in (0, \pi) \text{ and } U \text{ is a unitary } n \times n \text{ matrix of angle } \theta_U < \theta/2, \text{ then } \mathbf{e}_n \in U\big(\mathrm{C}_{\theta,\infty}(0, \mathbf{e}_n)\big). \tag{5.6.150}$$

To justify this, write

$$\langle U^{-1}\mathbf{e}_n, \mathbf{e}_n\rangle = \langle U^{\top}\mathbf{e}_n, \mathbf{e}_n\rangle = \langle \mathbf{e}_n, U\mathbf{e}_n\rangle \geq \cos\theta_U > \cos(\theta/2). \tag{5.6.151}$$

Thus, $U^{-1}\mathbf{e}_n \in \mathrm{C}_{\theta,\infty}(0, \mathbf{e}_n)$ which, in turn, implies the claim in (5.6.150). As a consequence of this we have that

$$\text{if } \theta \in (0, \pi) \text{ and } U \text{ is a unitary } n \times n \text{ matrix of angle } \theta_U < \theta/2, \text{ then there exists a small angle } \alpha \in (0, \theta/2) \text{ with the property that } \mathrm{C}_{\alpha,\infty}(0, \mathbf{e}_n) \subseteq U\big(\mathrm{C}_{\theta,\infty}(0, \mathbf{e}_n)\big). \tag{5.6.152}$$

Corollary 5.6.23 *Let $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a given Lipschitz function with Lipschitz constant $M \in [0, \infty)$ and denote by $\Sigma := \big\{(x', \varphi(x')) : x' \in \mathbb{R}^{n-1}\big\}$ its graph.*

Consider $\theta \in (0, \pi)$ such that $\cot(\theta/2) > M$. Then the image of Σ under a rotation in $\mathbb{R}^n$ of angle less than $\theta/2$ is still the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$.

Proof From Proposition 5.6.21 and (5.6.152) we conclude that if $\Omega := \text{Sepigraph}\,\varphi$ then the image of Ω under a rotation of angle $< \theta/2$ is still the strict epigraph of a Lipschitz function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$. From this the desired conclusion readily follows. □

The result proved in Proposition 5.6.24 below generalizes the fact that

$$\text{any bounded open convex subset of } \mathbb{R}^n \text{ is a Lipschitz domain.} \tag{5.6.153}$$

To set the stage for dealing with this more general version of (5.6.153), recall that a set given $\Omega \subseteq \mathbb{R}^n$ is called `star-like with respect to` some subset $B \subseteq \Omega$ provided $\mathcal{I}(x, x_0) \subseteq \Omega$ for all $x \in \Omega$ and all $x_0 \in B$ where, generally speaking,

$$\mathcal{I}(x, y) \text{ denotes the open segment with endpoints } x \text{ and } y. \tag{5.6.154}$$

Here is a result elaborating on the structure and properties of Euclidean open sets which are star-like with respect to a ball.

Proposition 5.6.24 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with the property that $0 \in \Omega$. Then Ω is star-like with respect to some ball centered at the origin in $\mathbb{R}^n$ if and only if*

$$\begin{gathered} \Omega = \big\{ r\omega : \ \omega \in S^{n-1} \ \textit{and} \ 0 \leq r < \varphi(\omega) \big\} \\ \textit{for some Lipschitz function} \ \ \varphi : S^{n-1} \to (0, \infty). \end{gathered} \tag{5.6.155}$$

Moreover, whenever (5.6.155) holds, it follows that Ω is a Lipschitz domain (in the sense of Definition 2.8.12), with topological boundary

$$\partial\Omega = \big\{ \varphi(\omega)\omega : \ \omega \in S^{n-1} \big\}, \tag{5.6.156}$$

with outward unit normal

$$\nu\big(\varphi(\omega)\omega\big) = \frac{\varphi(\omega)\omega - (\nabla_{\tan}\varphi)(\omega)}{\sqrt{|(\nabla_{\tan}\varphi)(\omega)|^2 + |\varphi(\omega)|^2}} \ \ \textit{for } \mathcal{H}^{n-1}\textit{-a.e.} \ \ \omega \in S^{n-1}, \tag{5.6.157}$$

(where $\nabla_{\tan}\varphi$ denotes the tangential gradient of φ on S^{n-1}), and with surface measure σ satisfying

$$\int_{\partial\Omega} f \, \mathrm{d}\sigma = \int_{S^{n-1}} f\big(\varphi(\omega)\omega\big)\big[\varphi(\omega)\big]^{n-2}\sqrt{|(\nabla_{\tan}\varphi)(\omega)|^2 + |\varphi(\omega)|^2} \, \mathrm{d}\mathcal{H}^{n-1}(\omega), \tag{5.6.158}$$

for every non-negative, σ-measurable function f on $\partial\Omega$.

Proof Assume Ω is star-like with respect to the ball $B := B(0, \rho)$, for some radius $\rho \in \big(0, \operatorname{dist}(0, \partial\Omega)\big)$. The goal is to prove (5.6.155), and we shall do so in a number of steps. Throughout, we shall employ the piece of notation introduced in (5.6.154).

`Step I.` *We have $\mathcal{I}(x, x_0) \subseteq \Omega$ for all $x \in \overline{\Omega}$, and $x_0 \in B$.*

Proof of Step I. Since $x \in \overline{\Omega}$, there exists a sequence $\{x_j\}_{j\in\mathbb{N}} \subseteq \Omega$ such that $x_j \to x$ as $j \to \infty$. Fix $y \in \mathcal{I}(x, x_o)$. Then there exists $t \in (0, 1)$ such that $y = x_0 + t(x - x_0)$. For each $j \in \mathbb{N}$ consider $y_j := x_0 + t(x_j - x_0) \in \mathcal{I}(x, x_0) \subseteq \Omega$. Then

$$\lim_{j\to\infty} y_j = \lim_{j\to\infty} [x_0 + t(x_j - x_0)] = x_0 + t(x - x_0) = y. \tag{5.6.159}$$

Since $y_j \in \Omega$ for every $j \in \mathbb{N}$, this entails $y \in \overline{\Omega}$. Given that y is an arbitrary point in $\mathcal{I}(x, x_0)$, this implies that

$$\mathcal{I}(x, x_0) \subseteq \overline{\Omega}. \tag{5.6.160}$$

The argument so far proves that

$$\mathcal{I}(x, x_0) \subseteq \overline{\Omega} \quad \text{for all} \quad x \in \overline{\Omega} \quad \text{and} \quad x_0 \in B. \tag{5.6.161}$$

We next claim that

$$\mathcal{I}(x, x_0) \cap \partial\Omega = \varnothing \quad \text{for all} \quad x \in \overline{\Omega} \quad \text{and} \quad x_0 \in B. \tag{5.6.162}$$

Reasoning by contradiction, assume that there exist $x \in \overline{\Omega}$, $x_0 \in B$, and $z \in \mathcal{I}(x, x_0)$ such that $z \in \partial\Omega$. Define the cone-like region

$$C := \bigcup_{y\in B} \mathcal{I}(z, y). \tag{5.6.163}$$

Then C is open and $C \subseteq \overline{\Omega}$ (recall (5.6.161)). Also, $\mathcal{I}(x, z) \subseteq \mathcal{I}(x, x_0) \subseteq \overline{\Omega}$ and we claim that $\mathcal{I}(x, z) \cap \Omega = \varnothing$. Indeed, the existence of some $w \in \mathcal{I}(x, z) \cap \Omega$ would force $z \in \mathcal{I}(w, x_0) \subseteq \Omega$, in contradiction with the fact that $z \in \partial\Omega$. Hence,

$$\mathcal{I}(x, z) \subseteq \overline{\Omega} \setminus \Omega = \partial\Omega. \tag{5.6.164}$$

To proceed, fix an arbitrary point $z_0 \in \mathcal{I}(x, z)$. Then there exists a ball B_0 centered at z_0 such that $B_0 \subseteq C \subseteq \overline{\Omega}$. We may then pick three collinear points a, b, c (mutually different) with $a \in B_0$, $b \in \mathcal{I}(x, z)$, and $c \in B$ (Fig. 5.1).

This means that $a \notin \Omega$, otherwise $\mathcal{I}(a, c) \subseteq \Omega$, which implies that $b \in \Omega$. However, $b \in \mathcal{I}(x, z) \subseteq \partial\Omega$. Since, nonetheless, $a \in B_0 \subseteq C \subseteq \overline{\Omega}$, it follows that we necessarily have $a \in \partial\Omega$. Perturbing a, we see that there exists some small radius $r > 0$ such that $B(z_0, r) \subseteq \partial\Omega$. However, having $z_0 \in \partial\Omega$ implies $B(z_0, r) \cap \Omega \neq \varnothing$, in contradiction with

$$B(z_0, r) \cap \Omega \subseteq \partial\Omega \cap \Omega = \varnothing. \tag{5.6.165}$$

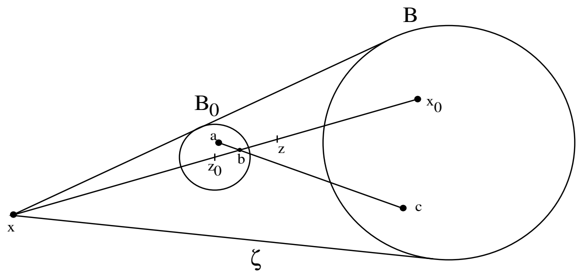

Fig. 5.1 The points a, b, c

This contradiction proves (5.6.162). From (5.6.161) and (5.6.162), the conclusion in Step I follows.

`Step II.` *For all $\omega \in S^{n-1}$, the open half line*

$$L_\omega := \{r\omega : r > 0\} \tag{5.6.166}$$

intersects $\partial\Omega$.

Proof of Step II. Note that $L_\omega \subseteq \mathbb{R}^n = \Omega \cup \partial\Omega \cup (\overline{\Omega})^c$, disjoint union. Thus, having $L_\omega \cap \partial\Omega = \varnothing$ would imply that $\{\Omega, (\overline{\Omega})^c\}$ is a covering of the connected set L_ω by disjoint open sets. Consequently, either $L_\omega \cap \Omega = \varnothing$, or $L_\omega \cap (\overline{\Omega})^c = \varnothing$. Since Ω is open and $0 \in \Omega$ the first eventuality cannot occur. On the other hand, the second scenario implies that $L_\omega \subseteq \overline{\Omega}$ which is impossible since Ω is bounded. Thus this contradiction proves Step II.

`Step III.` *For each $\omega \in S^{n-1}$the set $L_\omega \cap \partial\Omega$ contains precisely one point.*

Proof of Step III. From Step II we know that $L_\omega \cap \partial\Omega \neq \varnothing$. Suppose $x_1, x_2 \in L_\omega \cap \partial\Omega$ are two disjoint points. Then either $x_2 \in \mathcal{I}(x_1, 0)$, or $x_1 \in \mathcal{I}(x_2, 0)$. By Step I, in the first eventuality we have $\partial\Omega \ni x_2 \in \mathcal{I}(x_1, 0) \subseteq \Omega$ which is a contradiction. In the second eventuality, Step I gives $\partial\Omega \ni x_1 \in \mathcal{I}(x_2, 0) \subseteq \Omega$, which once again is a contradiction. This finishes the proof of Step III.

`Step IV.` *If for all $\omega \in S^{n-1}$ we set $\varphi(\omega) :=$ the distance from the point in $L_\omega \cap \partial\Omega$ to the origin then*

$$\varphi : S^{n-1} \longrightarrow (0, \infty) \tag{5.6.167}$$

is a well-defined function with the property that

$$\begin{aligned} \Omega &= \{r\omega : \omega \in S^{n-1} \text{ and } 0 \le r < \varphi(\omega)\} \\ \text{and } \partial\Omega &= \{\varphi(\omega)\omega : \omega \in S^{n-1}\}. \end{aligned} \tag{5.6.168}$$

Proof of Step IV. This follows directly from the fact that $0 \in \Omega$ and Step III.

`Step V.` *There exists a constant* $C \in (0, \infty)$ *such that for all*

$$\omega_1, \omega_2 \in S^{n-1} \quad \text{with} \quad |\omega_1 - \omega_2| < 1 \tag{5.6.169}$$

we have

$$|\varphi(\omega_1) - \varphi(\omega_2)| \le C\,|\omega_1 - \omega_2|\,. \tag{5.6.170}$$

Proof of Step V. Define $x_j := \varphi(\omega_j)\omega_j \in \partial\Omega$ for $j \in \{1, 2\}$. By Step IV, $x_1, x_2 \in \partial\Omega$. The triangle inequality in the triangle with vertices at $0, x_1, x_2$ then implies

$$|\varphi(\omega_1) - \varphi(\omega_2)| \le |x_1 - x_2|\,. \tag{5.6.171}$$

Let us also note that if $\theta := \sphericalangle(\omega_1, \omega_2)$ then $\theta \in [0, \pi]$ and

$$|\omega_1 - \omega_2|^2 = 2 - 2\omega_1 \cdot \omega_2 = 2 - 2\cos\theta = 4\sin^2\Big(\frac{\theta}{2}\Big). \tag{5.6.172}$$

In concert with (5.6.169), this permits us to estimate

$$1 > |\omega_1 - \omega_2| = 2\sin\Big(\frac{\theta}{2}\Big), \tag{5.6.173}$$

hence

$$\theta \in \Big(0, \frac{\pi}{3}\Big). \tag{5.6.174}$$

Recall that $B = B(0, \rho)$.

Claim 1. The line L_{x_1,x_2} passing through x_1, x_2 does not intersect $B(0, \frac{\rho}{2})$.

Proof of Claim 1. Seeking a contradiction, assume there exists $z \in B(0, \frac{\rho}{2}) \cap L_{x_1,x_2}$. Since $z \in B = B(0, \rho)$ and $x_1, x_2 \in \partial\Omega$, Step I implies that $\mathcal{I}(x_j, z) \subseteq \Omega$ for $j \in \{1, 2\}$. The fact that $x_1, x_2 \in \partial\Omega$ further gives that (Fig. 5.2)

$$z \in \mathcal{I}(x_1, x_2). \tag{5.6.175}$$

Indeed, having, say, $x_2 \in \mathcal{I}(z, x_1)$ would place x_2 in Ω, a contradiction. Similarly, we cannot have $x_1 \in \mathcal{I}(z, x_2)$, so ultimately we are left with (5.6.175) as the only viable option.
Now $x_1, x_2 \in \partial\Omega$ and $B(0, \rho) \subseteq \Omega$ implies that $|x_1|\,, |x_2| \ge \rho$, hence

$$\big|x_j - z\big| \ge \big|x_j\big| - |z| \ge \rho - \frac{\rho}{2} = \frac{\rho}{2}, \qquad j \in \{1, 2\}. \tag{5.6.176}$$

Thus, $|z|$ is the shortest side in the triangle with vertices at $0, z, x_j$. Consequently, if for $j \in \{1, 2\}$ we denote by α_j the angle made by L_{x_1,x_2} with L_{ω_j} within the triangle with vertices at $0, x_1, x_2$, then α_j is the smallest angle in the triangle with vertices at

$0, z, x_j$. This further forces

$$\alpha_j \leq \frac{\pi}{3} \quad \text{for} \quad j \in \{1, 2\}. \tag{5.6.177}$$

However, since from (5.6.174) we know that $\theta < \frac{\pi}{3}$, this makes it impossible to have $\alpha_1 + \alpha_2 + \theta = \pi$. This contradiction proves Claim 1 made above.

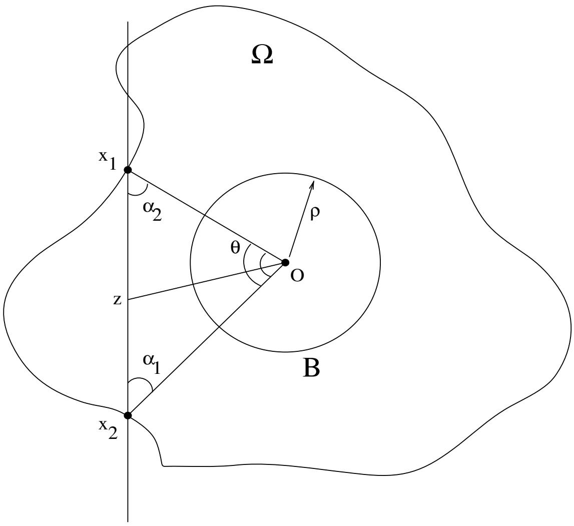

Fig. 5.2 A star-like set with respect to a ball centered at the origin

Claim 2. We have

$$\text{dist}\big(0, L_{x_1,x_2}\big) = \frac{|x_1|\ |x_2|\ \sin\theta}{|x_1 - x_2|}. \tag{5.6.178}$$

To justify this, denote by Δ the triangle with vertices at $0, x_1, x_2$.

With c denoting the length of the side joining x_1, x_2, with a, b denoting the lengths of the sides emerging from 0, and with h denoting the height from the vertex 0, we may then write

$$\frac{h \cdot c}{s} = \text{area}\,\Delta = \frac{b \cdot a \cdot \sin\theta}{2}. \tag{5.6.179}$$

For us, in the scenario just described,

$$h = \text{dist}\big(0, L_{x_1,x_2}\big), \quad a = |x_2|\,, \quad b = |x_1|\,, \quad c = |x_1 - x_2|\,. \tag{5.6.180}$$

From (5.6.179), the desired conclusion readily follows.

Combining Claims 1 and 2 then yields

$$\frac{|x_1|\ |x_2|\ \sin\theta}{|x_1 - x_2|} \geq \frac{\rho}{2} \tag{5.6.181}$$

which, bearing the equality in (5.6.173) in mind, further implies

$$\begin{aligned} |x_1 - x_2| &\leq 2\rho^{-1}\,|x_1|\ |x_2|\ \sin\theta \\ &= 4\rho^{-1}\,|x_1|\ |x_2|\ \sin\Big(\frac{\theta}{2}\Big)\ \cos\Big(\frac{\theta}{2}\Big) \\ &\leq 2\rho^{-1}\big(\mathrm{diam}(\Omega)\big)^2\ |\omega_1 - \omega_2|\,. \end{aligned} \tag{5.6.182}$$

All in all,

$$|\varphi(\omega_1) - \varphi(\omega_2)| \leq |x_1 - x_2| \leq C\,|\omega_1 - \omega_2|\,, \tag{5.6.183}$$

with $C := 2\rho^{-1}\big(\mathrm{diam}(\Omega)\big)^2 \in (0, \infty)$, proving Step V.

`Step VI.` *The function*

$$\varphi : S^{n-1} \to (0, \infty) \;\text{ is Lipschitz.} \tag{5.6.184}$$

Proof of Step VI. From Step V we know that φ is locally Lipschitz. Since S^{n-1} is compact, we may conclude that φ is Lipschitz in the context of (5.6.184).

At this stage, from Steps IV and VI we conclude that (5.6.155) holds.

In the opposite direction, suppose next that the set $\Omega \subseteq \mathbb{R}^n$ is as in (5.6.155). The goal is now to show that Ω is star-like with respect to some ball centered at the origin. To get started, given any $\omega \in S^{n-1}$ and $\theta \in (0, \pi)$, recall from (5.6.93) that

$$\mathrm{C}_{\theta,\omega} := \big\{ y \in \mathbb{R}^n :\ \cos(\theta/2)\,|y - \varphi(\omega)\omega| < \varphi(\omega) - y\cdot\omega < \varphi(\omega) \big\} \tag{5.6.185}$$

is the open, convex, truncated, one-component, circular cone in $\mathbb{R}^n$ with symmetry axis along $-\omega \in S^{n-1}$, vertex at the point $\varphi(\omega)\omega \in \partial\Omega$, height $\varphi(\omega) \in (0, \infty)$, and (full) aperture θ. In relation to this we claim that

$$\begin{aligned} &\text{there exists } \theta \in (0, \pi) \text{ sufficiently small (depending only on } \varphi) \text{ with} \\ &\text{the property that we have } \mathrm{C}_{\theta,\omega} \subseteq \Omega \text{ for each } \omega \in S^{n-1}. \end{aligned} \tag{5.6.186}$$

Using polar coordinates, it becomes apparent from (5.6.185) and (5.6.155) that we need to prove the existence of some $\theta = \theta(\varphi) \in (0, \pi)$ such that

$$\begin{aligned} &\text{whenever } r \in [0, \infty) \text{ and } \omega, \omega_o \in S^{n-1} \text{ are such that} \\ &\cos(\theta/2)\,|r\omega - \varphi(\omega_o)\omega_o| < \varphi(\omega_o) - r\omega\cdot\omega_o < \varphi(\omega_o) \\ &\text{then we necessarily have } r < \varphi(\omega). \end{aligned} \tag{5.6.187}$$

To justify (5.6.187), work with $\theta \in (0, \pi)$ to be determined later (in terms of φ) and fix $r \in [0, \infty)$ along $\omega, \omega_o \in S^{n-1}$ such that the estimate in the middle line of

(5.6.187) is satisfied. The conclusion in (5.6.187) is trivially satisfied if $r = 0$, so we shall assume that $r \in (0, \infty)$ in the remainder of the proof. In particular, the last inequality in the middle line of (5.6.187) entails $\omega \cdot \omega_o > 0$. If we now abbreviate

$$\begin{gathered}\rho := \frac{r}{\varphi(\omega_o)} \in (0, \infty), \qquad \alpha := \frac{\theta}{2} \in (0, \pi/2),\\ \text{and } \ \beta := \arccos(\omega \cdot \omega_o) \in (0, \pi/2),\end{gathered} \tag{5.6.188}$$

then

$$|\omega - \omega_o| = \sqrt{2 - 2\omega \cdot \omega_o} = \sqrt{2(1 - \cos\beta)} = 2\sin(\beta/2), \tag{5.6.189}$$

and first inequality in the middle line of (5.6.187) implies

$$(\cos\alpha)\,|\rho\omega - \omega_o| < 1 - \rho\cos\beta. \tag{5.6.190}$$

In turn, (5.6.190) is equivalent to having

$$\rho < \frac{1}{\cos\beta} \ \text{ and } \ \rho^2\big(\cos^2\alpha - \cos^2\beta\big) + 2\rho\cos\beta\sin^2\alpha - \sin^2\alpha < 0. \tag{5.6.191}$$

For further use it is convenient to observe that

$$\cos^2\alpha - \cos^2\beta = \sin(\beta - \alpha)\sin(\alpha + \beta). \tag{5.6.192}$$

Also, recall from the last line in (5.6.187) that the goal is to show that

$$\rho < \frac{\varphi(\omega)}{\varphi(\omega_o)}. \tag{5.6.193}$$

We proceed by distinguishing several cases.

`Case 1.` *Assume* $\alpha = \beta$. In this scenario, the second inequality in (5.6.191) becomes

$$2\rho\cos\alpha\sin^2\alpha - \sin^2\alpha < 0, \quad \text{or, equivalently,} \quad \rho < \frac{1}{2\cos\alpha}. \tag{5.6.194}$$

In view of the goal set in (5.6.193), it suffices to check that matters may be arranged, by tanking $\alpha \in (0, \pi/2)$ small (in terms of φ alone), so that

$$\frac{1}{2\cos\alpha} < \frac{\varphi(\omega)}{\varphi(\omega_o)}. \tag{5.6.195}$$

Note that (5.6.195) is further equivalent to

$$\frac{\varphi(\omega_o)}{\varphi(\omega)} - 1 < 2\cos\alpha - 1, \tag{5.6.196}$$

and that

$$\frac{\varphi(\omega_o)}{\varphi(\omega)} - 1 = \frac{\varphi(\omega_o) - \varphi(\omega)}{\varphi(\omega)} \leq M\Big(\inf_{S^{n-1}} \varphi\Big)^{-1} |\omega - \omega_o|, \tag{5.6.197}$$

where $M \in (0, \infty)$ is the Lipschitz constant of the function φ. Since (5.6.189) presently gives

$$|\omega - \omega_o| = 2\sin(\alpha/2), \tag{5.6.198}$$

from (5.6.197) and (5.6.198) we see that (5.6.196) is satisfied provided

$$M\Big(\inf_{S^{n-1}} \varphi\Big)^{-1} < \frac{2\cos\alpha - 1}{2\sin(\alpha/2)}. \tag{5.6.199}$$

Given that

$$\lim_{\alpha\to 0^+} \frac{2\cos\alpha - 1}{2\sin(\alpha/2)} = +\infty, \tag{5.6.200}$$

it follows that (5.6.199) can be made true by taking $\alpha \in (0, \pi/2)$ small (solely in terms of φ).

`Case 2.` *Assume* $\alpha < \beta$. Hence, $0 < \alpha < \beta < \pi/2$. Also, from (5.6.191)–(5.6.192) we know that $f(\rho) < 0$ where, for each $t \in \mathbb{R}$, we have set

$$f(t) := t^2 \sin(\beta - \alpha)\sin(\alpha + \beta) + 2t\cos\beta\sin^2\alpha - \sin^2\alpha. \tag{5.6.201}$$

It turns out that the quadratic equation $f(t) = 0$ has two distinct real roots, namely

$$f(t) = 0 \iff \text{either } t = \frac{\sin\alpha}{\sin(\alpha+\beta)} \text{ or } t = \frac{\sin\alpha}{\sin(\alpha-\beta)}. \tag{5.6.202}$$

Given that in the current case the leading coefficient in (5.6.201) is positive, having $f(\rho) < 0$ then forces

$$\rho < \frac{\sin\alpha}{\sin(\alpha+\beta)}. \tag{5.6.203}$$

Granted this, and given the goal set in (5.6.193), it is then enough to verify that matters may be arranged, by choosing $\alpha \in (0, \pi/2)$ sufficiently small (exclusively in terms of φ alone), so that

$$\frac{\sin\alpha}{\sin(\alpha+\beta)} < \frac{\varphi(\omega)}{\varphi(\omega_o)}. \tag{5.6.204}$$

Observe that (5.6.204) is further equivalent to

$$\frac{\varphi(\omega_o)}{\varphi(\omega)} - 1 < \frac{\sin(\alpha+\beta)}{\sin\alpha} - 1 = \frac{\sin(\alpha+\beta) - \sin\alpha}{\sin\alpha}. \tag{5.6.205}$$

In view of (5.6.197) and (5.6.189), the inequality in (5.6.205) is satisfied provided

$$2M\Big(\inf_{S^{n-1}} \varphi\Big)^{-1} < \frac{\sin(\alpha+\beta) - \sin\alpha}{\sin(\beta/2)\sin\alpha}. \tag{5.6.206}$$

We momentarily digress to discuss some useful elementary inequalities. Specifically, since the function $(0, \pi/2) \ni y \mapsto (\sin y)/y \in \mathbb{R}$ is strictly decreasing (given that it has a negative derivative), and since for each fixed $y \in (0, \pi/2)$ the function

$$(0, \pi/2) \ni x \longmapsto \frac{\sin(x+y) - \sin x}{y} \in \mathbb{R} \tag{5.6.207}$$

is strictly increasing (as it has a positive derivative), we conclude (after also invoking the Mean Value Theorem) that

$$1 > \frac{\sin(x+y) - \sin x}{y} > \frac{\sin y}{y} > \frac{2}{\pi} \quad \text{for each } \ x, y \in (0, \pi/2). \tag{5.6.208}$$

From (5.6.208) we then see that

$$\frac{4}{\pi \sin\alpha} < \frac{\sin(\alpha+\beta) - \sin\alpha}{\sin(\beta/2)\sin\alpha}, \tag{5.6.209}$$

so (5.6.206) holds provided

$$2M\Big(\inf_{S^{n-1}} \varphi\Big)^{-1} < \frac{4}{\pi \sin\alpha}. \tag{5.6.210}$$

Given that

$$\lim_{\alpha \to 0^+} \frac{4}{\pi \sin\alpha} = +\infty, \tag{5.6.211}$$

we see that (5.6.210) becomes valid if $\alpha \in (0, \pi/2)$ is sufficiently small (exclusively in terms of φ).

`Case 3.` *Assume* $\beta < \alpha$. Thus $0 < \beta < \alpha < \pi/2$, and from (5.6.191)–(5.6.192) we know that $f(\rho) < 0$ with the quadratic function f as in (5.6.201). Presently, the leading coefficient in (5.6.201) is negative, so having $f(\rho) < 0$ forces (in view of (5.6.202))

$$\rho \in \mathbb{R} \setminus \Big[\frac{\sin\alpha}{\sin(\alpha+\beta)}, \frac{\sin\alpha}{\sin(\alpha-\beta)}\Big]. \tag{5.6.212}$$

From (5.6.201) and (5.6.192), let us also observe that

$$f\Big(\frac{1}{\cos\beta}\Big) = \frac{\cos^2\alpha\sin^2\beta}{\cos^2\beta} > 0 \tag{5.6.213}$$

so we presently have

$$\frac{1}{\cos\beta} \in \Big(\frac{\sin\alpha}{\sin(\alpha+\beta)}, \frac{\sin\alpha}{\sin(\alpha-\beta)}\Big). \tag{5.6.214}$$

Since from (5.6.191) we also know that $\rho < 1/(\cos\beta)$, by combining (5.6.212) with (5.6.214) we ultimately conclude that in the current case we have

$$\rho < \frac{\sin\alpha}{\sin(\alpha+\beta)}. \tag{5.6.215}$$

Granted this, in view of the goal set in (5.6.193) it is enough to make sure that matters may be arranged, by tanking $\alpha \in (0, \pi/2)$ small (purely in terms of φ), so that

$$\frac{\sin\alpha}{\sin(\alpha+\beta)} < \frac{\varphi(\omega)}{\varphi(\omega_o)}. \tag{5.6.216}$$

This, however, may be ensured by once again reasoning as in (5.6.204)–(5.6.211).

The argument so far shows that there exists $\theta = \theta(\varphi) \in (0, \pi)$ for which (5.6.187) holds. This finishes the proof of (5.6.186). Henceforth assume $\theta = \theta(\varphi) \in (0, \pi)$ is as in (5.6.186) and pick some small radius r, say

$$0 < r < \min\Big\{\Big(\inf_{S^{n-1}}\varphi\Big)\tan(\theta/2), \operatorname{dist}(0, \partial\Omega)\Big\}. \tag{5.6.217}$$

In particular, the ball $B := B(0, r)$ is contained in Ω. Then from (5.6.186) and elementary geometry we conclude that for any $\omega \in S^{n-1}$, any point x belonging to the axis of symmetry for the cone $\mathrm{C}_{\theta,\omega}$, and any point $x_0 \in B$ we have

$$\mathcal{I}(x, x_0) \subseteq \mathrm{C}_{\theta,\omega} \cup B \subseteq \Omega. \tag{5.6.218}$$

Suppose now that $x \in \Omega$ and $x_0 \in B$ are two arbitrary points. If $x \in B$ then $\mathcal{I}(x, x_0) \subseteq B \subseteq \Omega$. If $x \notin B$ then $|x| \neq 0$, so $\omega := x/|x|$ is a well-defined vector in S^{n-1} with the property that x belongs to the axis of symmetry for the cone $\mathrm{C}_{\theta,\omega}$. Granted this, we may invoke (5.6.218) to conclude that $\mathcal{I}(x, x_0) \subseteq \Omega$. In view of the arbitrariness of the points x, x_0, this ultimately proves that Ω is star-like with respect to B.

Finally, the fact that any set Ω as in (5.6.155) is a Lipschitz domain, as well as the claims pertaining to (5.6.157) and (5.6.158), have been justified in [124, Corollary 4.13]. This finishes the proof of Proposition 5.6.24. □

Here is an application of Proposition 5.6.24, in the two-dimensional setting.

Proposition 5.6.25 *Start with a function*

$$F : \mathbb{R} \to \mathbb{R} \ \textit{ Lipschitz and periodic, with period } 2\pi, \tag{5.6.219}$$

and consider the curve

$$\Sigma := \Big\{ \mathrm{e}^{F(\theta)+\mathrm{i}\theta} : \theta \in [0, 2\pi) \Big\} \subseteq \mathbb{C}. \tag{5.6.220}$$

Then Σ is the boundary of a Lipschitz domain in $\mathbb{R}^2 \equiv \mathbb{C}$, which is star-like with respect to some ball centered at the origin in $\mathbb{R}^2$.

Proof The idea is to eventually invoke Proposition 5.6.24 (with $n = 2$). To set the stage, consider the function $\varphi : S^1 \to (0, \infty)$ given for each $\omega \in S^1$ by $\varphi(\omega) := \mathrm{e}^{F(\theta)}$ whenever $\theta \in \mathbb{R}$ is such that $\omega = \mathrm{e}^{\mathrm{i}\theta}$. The fact that F is periodic, with period 2π, ensures that φ is well defined. We claim that φ is actually a Lipschitz function.

To justify this claim, pick $\omega_0, \omega_1 \in S^1$ arbitrary. Observe that there exist $\theta_0, \theta_1 \in \mathbb{R}$ such that

$$|\theta_0 - \theta_1| \leq \pi \text{ and } \omega_0 = \mathrm{e}^{\mathrm{i}\theta_0}, \omega_1 = \mathrm{e}^{\mathrm{i}\theta_1}. \tag{5.6.221}$$

Indeed, start with $\theta_0, \theta_1 \in [0, 2\pi)$ such that $\omega_0 = \mathrm{e}^{\mathrm{i}\theta_0}$ and $\omega_1 = \mathrm{e}^{\mathrm{i}\theta_1}$. If $|\theta_0 - \theta_1| \leq \pi$ we are done. If $|\theta_0 - \theta_1| > \pi$, replace the smaller angle by its value increased by 2π (while retaining the same symbol), and then all properties in (5.6.221) are now satisfied.

To proceed, we remark that, in general,

$$|\mathrm{e}^{\mathrm{i}x} - 1| = 2|\sin(x/2)| \quad \text{for each} \quad x \in \mathbb{R}. \tag{5.6.222}$$

In concert with (5.6.208) this gives

$$|\mathrm{e}^{\mathrm{i}x} - 1| \geq 2x/\pi \quad \text{for each} \quad x \in [0, \pi]. \tag{5.6.223}$$

Replacing x by $-x$ in (5.6.223) we then arrive at

$$|\mathrm{e}^{\mathrm{i}x} - 1| \geq 2|x|/\pi \quad \text{for each} \quad x \in \mathbb{R} \quad \text{with} \quad |x| \leq \pi. \tag{5.6.224}$$

Hence, if $M \in (0, \infty)$ is the Lipschitz constant of F, we may estimate

$$\begin{aligned} |\omega_1 - \omega_0| &= |\mathrm{e}^{\mathrm{i}\theta_1} - \mathrm{e}^{\mathrm{i}\theta_0}| = \big|\mathrm{e}^{\mathrm{i}\theta_0}\big(\mathrm{e}^{\mathrm{i}(\theta_1-\theta_0)} - 1\big)\big| \\ &= \big|\mathrm{e}^{\mathrm{i}(\theta_1-\theta_0)} - 1\big| \geq \frac{2}{\pi}|\theta_1 - \theta_0| \geq \frac{2}{\pi M}|F(\theta_1) - F(\theta_0)|, \end{aligned} \tag{5.6.225}$$

thanks to (5.6.221) and (5.6.224). Let us momentarily digress to observe that, since $\mathrm{e}^x \geq 1 + x$ for each $x \in \mathbb{R}$, it follows (after replacing x by $-x$ and some simple algebra) that

$$\mathrm{e}^x(x - 1) + 1 \geq 0 \quad \text{for each} \quad x \in \mathbb{R}. \tag{5.6.226}$$

Consider the function $f(x) := (e^x - 1)/x$ for each $x \in \mathbb{R} \setminus \{0\}$ and $f(0) := 1$. Then f is differentiable on $\mathbb{R}$ and $f'(x) = [e^x(x-1)+1]/x^2 \geq 0$ $x \in \mathbb{R} \setminus \{0\}$ and $f'(0) = 1/2$. In view of (5.6.226) we therefore have $f'(x) \geq 0$ for each $x \in \mathbb{R}$ which goes to show that f is non-decreasing. Consequently, given any $a \in (0, \infty)$ we have

$$|e^x - 1| \leq \Big(\frac{e^a - 1}{a}\Big)|x| \quad \text{for each} \quad x \in \mathbb{R} \quad \text{with} \quad |x| \leq a. \tag{5.6.227}$$

Returning to the main topic of conversation, since F is Lipschitz and periodic it follows that F is bounded. Hence, there exists $C \in (0, \infty)$ such that $|F(\theta)| \leq C$ for all $\theta \in \mathbb{R}$. In particular, $e^{F(\theta_0)} \leq e^C$ and $|F(\theta_1) - F(\theta_0)| \leq 2C$ which, in concert with (5.6.227), permit us to estimate

$$\begin{aligned} |\varphi(\omega_1) - \varphi(\omega_0)| &= |e^{F(\theta_1)} - e^{F(\theta_0)}| = e^{F(\theta_0)}\big|e^{F(\theta_1)-F(\theta_0)} - 1\big| \\ &\leq e^C\Big(\frac{e^{2C} - 1}{2C}\Big)|F(\theta_1) - F(\theta_0)|. \end{aligned} \tag{5.6.228}$$

Finally, from (5.6.225) and (5.6.228) we conclude that

$$|\varphi(\omega_1) - \varphi(\omega_0)| \leq \frac{\pi M e^C}{2}\Big(\frac{e^{2C} - 1}{2C}\Big)|\omega_1 - \omega_0|. \tag{5.6.229}$$

Ultimately, this proves that $\varphi : S^1 \to (0, \infty)$ is indeed a Lipschitz function.

Granted this, Proposition 5.6.24 applies and gives that

$$\Omega := \big\{r\omega : \omega \in S^1 \text{ and } 0 \leq r < \varphi(\omega)\big\} \subseteq \mathbb{R}^2 \tag{5.6.230}$$

is a Lipschitz domain which is star-like with respect to some ball centered at the origin in $\mathbb{R}^2$, and whose boundary may be described as

$$\partial\Omega = \big\{\varphi(\omega)\omega : \omega \in S^1\big\}. \tag{5.6.231}$$

From this, (5.6.220), and the definition of the function φ we then see that $\Sigma = \partial\Omega$, so the desired conclusion then follows. □

The next lemma elaborates on the manner in which Lipschitz upper-graphs are locally star-like with respect to balls.

Lemma 5.6.26 *Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function and consider*

$$\Omega := \big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > \phi(x')\big\}. \tag{5.6.232}$$

Then for each $t > 0$ there exists $\varepsilon > 0$ such that if for each $s > 0$ and $x' \in \mathbb{R}^{n-1}$ one defines[8]

[8] Where $B_{n-1}(x', \varepsilon)$ denotes the ball in $\mathbb{R}^{n-1}$ centered at x' and of radius ε.

$$Q_{t,s}(x') := B_{n-1}(x', \varepsilon) \times \big(-\infty, \phi(x') + t + s \big) \tag{5.6.233}$$

then $\Omega \cap Q_{t,s}(x')$ *is a star-like set with respect to some ball centered at* $\big(x', \phi(x') + t\big)$.

Proof Recall that $\mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) = \{y \in \mathbb{R}^n : \cos(\theta/2)\, |y - x| < (y - x) \cdot \mathbf{e}_n\}$ is the open, convex, one-component, infinite, circular cone in $\mathbb{R}^n$, with vertex at $x \in \mathbb{R}^n$, symmetry axis along the vector $\mathbf{e}_n \in S^{n-1}$, and (full) aperture $\theta \in (0, \pi)$ (cf. (5.6.93)). From Proposition 5.6.21, we know that there exists $\theta \in (0, \pi)$ such that

$$\mathrm{C}_{\theta,\infty}(x, \mathbf{e}_n) \subseteq \Omega \quad \text{for each} \quad x \in \partial\Omega. \tag{5.6.234}$$

Given $t > 0$, we may then perturb this to find $r > 0$ and $\varepsilon > 0$ such that

$$\begin{aligned} & B\big((x', \phi(x') + t), r\big) \subseteq \mathrm{C}_{\theta,\infty}\big((y', \phi(y')), \mathbf{e}_n\big) \\ & \text{for each} \quad x' \in \mathbb{R}^{n-1} \quad \text{and} \quad y' \in B_{n-1}(x', \varepsilon). \end{aligned} \tag{5.6.235}$$

Fix now $x' \in \mathbb{R}^{n-1}$ and $s > 0$ arbitrary. If $z \in \Omega \cap Q_{t,s}(x')$ then $z = (z', z_n)$ with $z' \in B_{n-1}(x', \varepsilon)$ and $\phi(z') < z_n < \phi(x') + t + s$. In particular, z belongs to $\mathrm{C}_{\theta,\infty}\big((z', \phi(z')), \mathbf{e}_n\big)$. Since the latter set is convex, and so is $Q_{t,s}(x')$, it follows from (5.6.235) and (5.6.234) that any line segment joining z with a point in the ball $B\big((x', \phi(x') + t), r\big)$ is contained in $\Omega \cap Q_{t,s}(x')$. This proves that $\Omega \cap Q_{t,s}(x')$ is indeed star-like with respect to the ball $B\big((x', \phi(x') + t), r\big)$. □

There are simple examples of bounded open sets which are star-like with respect to a point and yet they fail to be Lipschitz domains (e.g., they may have inner or outer cusps). This being said, it turns out that local star-likeness with respect to balls fully characterizes the class of bounded Lipschitz domains.

Proposition 5.6.27 *Let* Ω *be a nonempty, bounded, open subset of* $\mathbb{R}^n$. *Then* Ω *is a Lipschitz domain if and only if for each* $x \in \partial\Omega$ *there exists an open set* $O_x \subseteq \mathbb{R}^n$ *with the property that* $\Omega \cap O_x$ *is a star-like set with respect to a ball centered at* x.

Proof The direct implication is a consequence of Lemmas 2.8.13 and 5.6.26. The converse implication follows from Proposition 5.6.24 and Definition 2.8.12. □

5.7 Sets of Finite Perimeter

Turning to the topic of sets of finite perimeter, we first define the relative perimeter function. Specifically, let $E \subseteq \mathbb{R}^n$ be an $\mathcal{L}^n$-measurable set, and pick an arbitrary open set $\Omega \subseteq \mathbb{R}^n$. In this context, define the (relative) `perimeter of` E `in` Ω to be

$$
\begin{aligned}
P_\Omega(E) &:= \mathbf{V}\big(\mathbf{1}_{E\cap\Omega};\,\Omega\big) \qquad (5.7.1)\\
&= \sup\Big\{\Big|\int_E \operatorname{div}\vec{\varphi}\,\mathrm{d}\mathcal{L}^n\Big| : \vec{\varphi}\in\big[\mathscr{C}_c^\infty(\Omega)\big]^n \text{ and } \sup_\Omega|\vec{\varphi}|\le 1\Big\}\in[0,+\infty].
\end{aligned}
$$

In the case when $\Omega=\mathbb{R}^n$ we agree to abbreviate $P(E):=P_{\mathbb{R}^n}(E)=\mathbf{V}\big(\mathbf{1}_E;\mathbb{R}^n\big)$. The perimeter of E may also be defined relative to any given closed subset G of $\mathbb{R}^n$ according to $P_G(E):=\inf\big\{P_\Omega(E):\ G\subseteq\Omega\subseteq\mathbb{R}^n,\ \Omega \text{ open}\big\}$.

A classical result due to De Giorgi in [70] (cf. also [106, Theorem 4.4, p. 54] and [88, Sect. 4.5]) asserts that

$$
\begin{aligned}
&P_\Omega(E)=|\nabla\mathbf{1}_{E\cap\Omega}|(\Omega)=\mathcal{H}^{n-1}(\Omega\cap\partial_*E)=\mathcal{H}^{n-1}(\Omega\cap\partial^*E)\\
&\text{if } E\subseteq\mathbb{R}^n \text{ has locally finite perimeter, and } \Omega\subseteq\mathbb{R}^n \text{ is open.}
\end{aligned}
\qquad (5.7.2)
$$

In particular, corresponding to $\Omega=\mathbb{R}^n$ we have

$$
\begin{aligned}
&P(E)=\mathbf{V}\big(\mathbf{1}_E;\mathbb{R}^n\big)=\mathcal{H}^{n-1}(\partial_*E)\\
&\text{for each } \mathcal{L}^n\text{-measurable set } E\subseteq\mathbb{R}^n.
\end{aligned}
\qquad (5.7.3)
$$

Given a Lebesgue measurable set $E\subseteq\mathbb{R}^n$ along with $\Omega\subseteq\mathbb{R}^n$ open, we shall say that E has `finite perimeter in` Ω provided $P_\Omega(E)<+\infty$. This is equivalent with demanding that the locally integrable function $\mathbf{1}_{E\cap\Omega}$ has finite variation in Ω (i.e., $\mathbf{V}\big(\mathbf{1}_{E\cap\Omega};\,\Omega\big)<+\infty$). In particular, the distributional partial derivatives $\partial_j\mathbf{1}_{E\cap\Omega}$, $1\le j\le n$, are Borel-regular measures in Ω of finite total variation. In such a scenario, the perimeter of E in Ω may be expressed as

$$
P_\Omega(E)=|\nabla\mathbf{1}_{E\cap\Omega}|(\Omega)=\|\nabla\mathbf{1}_{E\cap\Omega}\|_{\mathrm{TV}(\Omega)}, \qquad (5.7.4)
$$

the total variation norm of the vector measure $\nabla\mathbf{1}_{E\cap\Omega}=(\partial_1\mathbf{1}_{E\cap\Omega},\dots,\partial_n\mathbf{1}_{E\cap\Omega})$ in Ω.

Simply call $E\subseteq\mathbb{R}^n$ a `set of finite perimeter` (or just say that E `has finite perimeter`) if E has finite perimeter in $\Omega:=\mathbb{R}^n$. In other words,

a given set $E\subseteq\mathbb{R}^n$ has finite perimeter if E is $\mathcal{L}^n$-measurable and $P(E)<+\infty$. The latter finiteness condition is further equivalent with $\mathbf{V}\big(\mathbf{1}_E;\mathbb{R}^n\big)<+\infty$, or $\mathcal{H}^{n-1}(\partial_*E)<+\infty$ (compare with (5.6.35)). (5.7.5)

From (5.7.1) and (5.5.4) one may conclude without difficulty that the (relative) perimeter function P_Ω is lower-semicontinuous, in the sense that

if $E\subseteq\mathbb{R}^n$ is $\mathcal{L}^n$-measurable, then for any sequence $\{E_j\}_{j\in\mathbb{N}}$ of Lebesgue measurable sets in $\mathbb{R}^n$ such that $\mathbf{1}_{E_j}\to\mathbf{1}_E$ as $j\to\infty$ at $\mathcal{L}^n$-a.e. point in Ω we have $P_\Omega(E)\le\liminf\limits_{j\to\infty}P_\Omega(E_j)$. (5.7.6)

In general, the (relative) perimeter function lacks genuine continuity, i.e., we may have strict inequality in (5.7.6). A concrete example is offered by a sequence of sets shaped like “castle towers” with smaller and smaller embrasures. Concretely, for each odd integer $j \in \mathbb{N}$ consider the family of rectangles

$$R_k^{(j)} := \begin{cases} \left[\frac{k}{j}, \frac{k+1}{j}\right] \times [0, 1] & \text{if } k \text{ is even,} \\ \left[\frac{k}{j}, \frac{k+1}{j}\right] \times \left[0, 1 - \frac{1}{j}\right] & \text{if } k \text{ is odd,} \end{cases} \tag{5.7.7}$$

indexed by $k \in \{0, 1, \ldots, j-1\}$, then define

$$E_j := \bigcup_{k=0}^{j-1} R_k^{(j)} \quad \text{and} \quad E := [0, 1] \times [0, 1]. \tag{5.7.8}$$

Then $P(E) = \mathcal{H}^1(\partial_* E) = \mathcal{H}^1(\partial E) = 4$ and $\mathbf{1}_{E_j} \to \mathbf{1}_E$ as $j \to \infty$ at $\mathcal{L}^n$-a.e. point in $\mathbb{R}^n$, yet $P(E_j) = \mathcal{H}^1(\partial_* E_j) = \mathcal{H}^1(\partial E_j) = 5 - 1/j$ for each odd number $j \in \mathbb{N}$, hence $\liminf\limits_{j \to \infty} P(E_j) = 5 > 4$.

It is worth pointing out that

whenever $E \subseteq \mathbb{R}^n$ is an $\mathcal{L}^n$-measurable set with compact boundary, it follows that E has finite perimeter if and only if E is a set of locally finite perimeter. (5.7.9)

Also, if $E \subseteq \mathbb{R}^n$ has finite perimeter then E has locally finite perimeter as well. In particular, (5.6.38) may be strengthened to

whenever $E \subseteq \mathbb{R}^n$ is an $\mathcal{L}^n$-measurable set such that $\mathcal{H}^{n-1}(\partial E) < \infty$ it follows that E has finite perimeter (in particular, E also has locally finite perimeter) and one has $P(E) = \mathcal{H}^{n-1}(\partial_* E) \leq \mathcal{H}^{n-1}(\partial E)$. (5.7.10)

Regarding the above inequality, the gap between $P(E)$ and $\mathcal{H}^{n-1}(\partial E)$ could be quite large in general. Indeed,

there exists $E \subseteq \mathbb{R}^n$ compact set, with empty interior, of finite perimeter, and with $\mathcal{L}^n(\partial E) > 0$ hence, in particular, with $\mathcal{H}^{n-1}(\partial E) = +\infty$. (5.7.11)

For example, consider a countable dense subset $\{x_j\}_{j \in \mathbb{N}}$ of $B(0, 1)$ and take

$$E := \overline{B(0, 1)} \setminus \Big(\bigcup_{j=1}^{\infty} B(x_j, r_j) \Big) \tag{5.7.12}$$

where $\{r_j\}_{j\in\mathbb{N}} \subseteq (0,1)$ is a numerical sequence satisfying $\sum_{j=1}^{\infty} r_j^{n-1} < 1$. Then, clearly, E is a compact set with $\mathring{E} = \varnothing$. In particular, $\partial E = \overline{E} \setminus \mathring{E} = E$, and since ω_{n-1}/n is the volume of the n-dimensional unit ball in $\mathbb{R}^n$ we have

$$
\begin{aligned}
\mathcal{L}^n(\partial E) = \mathcal{L}^n(E) &\geq \mathcal{L}^n\big(\overline{B(0,1)}\big) - \sum_{j=1}^{\infty} \mathcal{L}^n\big(B(x_j, r_j)\big) \\
&= \omega_{n-1}/n - \sum_{j=1}^{\infty} (\omega_{n-1}/n) r_j^n \geq (\omega_{n-1}/n)\Big(1 - \sum_{j=1}^{\infty} r_j^{n-1}\Big) > 0. \quad (5.7.13)
\end{aligned}
$$

If for every $N \in \mathbb{N}$ we now consider $E_N := \overline{B(0,1)} \setminus \Big(\bigcup_{j=1}^{N} B(x_j, r_j)\Big)$, then

$$
\partial E_N \subseteq S^{n-1} \cup \Big(\bigcup_{j=1}^{N} \partial B(x_j, r_j)\Big) \quad (5.7.14)
$$

which, together with (5.7.10), permits us to conclude that each E_N is a set of finite perimeter and

$$
\begin{aligned}
P(E_N) \leq \mathcal{H}^{n-1}(\partial E_N) &\leq \mathcal{H}^{n-1}(S^{n-1}) + \sum_{j=1}^{N} \mathcal{H}^{n-1}\big(\partial B(x_j, r_j)\big) \\
&= \omega_{n-1} + \sum_{j=1}^{N} \omega_{n-1} r_j^{n-1} \leq \omega_{n-1}\Big(1 + \sum_{j=1}^{\infty} r_j^{n-1}\Big) < 2\omega_{n-1}. \quad (5.7.15)
\end{aligned}
$$

Moreover, since $E_N \searrow E$ as $N \to \infty$, from this and (5.7.6) we see that

$$
P(E) \leq \liminf_{N\to\infty} P(E_N) \leq 2\omega_{n-1}, \quad (5.7.16)
$$

proving that E has finite perimeter. The set E constructed in (5.7.12) thus satisfies all properties listed in (5.7.11) (this is often referred to as a "Swiss cheese" type set).

The isoperimetric inequality in the category of sets of finite perimeter takes the following form ([71]; see also [80, p. 190] and [263, p. 81]): if the set $E \subseteq \mathbb{R}^n$ has finite perimeter, then either $\mathcal{L}^n(E) < +\infty$, or $\mathcal{L}^n(\mathbb{R}^n \setminus E) < +\infty$ and

$$
\begin{aligned}
\min\Big\{\big[\mathcal{L}^n(E)\big]^{(n-1)/n}, \big[\mathcal{L}^n(\mathbb{R}^n \setminus E)\big]^{(n-1)/n}\Big\} \\
\leq \frac{P(E)}{n(\mathrm{vol}_n)^{1/n}} = \frac{\mathcal{H}^{n-1}(\partial_* E)}{n^{1-1/n}(\omega_{n-1})^{1/n}}, \quad (5.7.17)
\end{aligned}
$$

where, with Γ denoting here the classical gamma-function,

$$\mathrm{vol}_n := \mathcal{L}^n\big(B(0,1)\big) = \frac{\pi^{n/2}}{\Gamma\big(\frac{n}{2}+1\big)} = \frac{\omega_{n-1}}{n} \tag{5.7.18}$$

is the volume of the unit ball in $\mathbb{R}^n$. The version of the isoperimetric inequality proved by H. Federer (cf. [88, 3.2.43–3.2.44, p. 278]) reads

$$E \subseteq \mathbb{R}^n \text{ with } \mathcal{L}^n(\overline{E}) < \infty \Rightarrow \big[\mathcal{L}^n(\overline{E})\big]^{(n-1)/n} \leq \frac{1}{n(\mathrm{vol}_n)^{1/n}}\,\mathcal{H}^{n-1}(\partial E). \tag{5.7.19}$$

As seen from (5.7.2), for every set $E \subseteq \mathbb{R}^n$ of finite perimeter we have

$$P(E) = \mathcal{H}^{n-1}(\partial_* E) = \mathcal{H}^{n-1}(\partial^* E) < +\infty. \tag{5.7.20}$$

In concert with (5.2.12)–(5.2.13) in Lemma 5.2.2, this implies (cf. [12, Proposition 1] and the discussion in [264, p. 197]) that whenever $E, F \subseteq \mathbb{R}^n$ are sets of finite perimeter

$$P(E \cup F) + P(E \cap F) \leq P(E) + P(F). \tag{5.7.21}$$

Consequently, (compare with Lemma 5.6.5)

> the class of sets of finite perimeter in $\mathbb{R}^n$ is an algebra of subsets of $\mathbb{R}^n$, i.e., it is closed under finite unions, finite intersections, and complements. (5.7.22)

Lemma 5.7.1 *Assume $E \subseteq \mathbb{R}^n$ is a set of locally finite perimeter, and $F \subseteq \mathbb{R}^n$ is a bounded set of finite perimeter. Then $E \cap F$ is a set of finite perimeter.*

Proof The assumptions on F imply that $\overline{F}$ is a compact set and $\mathcal{H}^{n-1}(\partial_* F) < \infty$. Based on (5.2.10) in Lemma 5.2.2 and (5.6.35) we may then estimate

$$\mathcal{H}^{n-1}\big(\partial_*(E \cap F)\big) \leq \mathcal{H}^{n-1}\big((\partial_* E) \cap \overline{F}\big) + \mathcal{H}^{n-1}(\partial_* F) < +\infty, \tag{5.7.23}$$

which ultimately proves the lemma. □

The local version of the Divergence Theorem recorded in the lemma below refines similar results in the literature (cf., e.g., [80, Lemma 1, p. 195], [162, Lemma 5.5.2, p. 143], or [263, Lemma 5.5.2, p. 234]) by allowing the exceptional nullset involved in the statement to be independent of the chosen vector field. This improvement is going to come into play shortly.

Lemma 5.7.2 *Let $E \subseteq \mathbb{R}^n$ be a set of locally finite perimeter. Set $\sigma_* := \mathcal{H}^{n-1}\lfloor\partial_* E$ and denote by ν the geometric measure theoretic outward unit normal to E. Also, fix some point $x \in \mathbb{R}^n$. Then*

> $E \cap B(x,r),\ E \setminus B(x,r),\ E \cap \overline{B(x,r)},\ E \setminus \overline{B(x,r)}$ *are all sets of locally finite perimeter.* (5.7.24)

In addition, there exists some

$$\mathcal{L}^1\text{-measurable set } N_x \subset (0,\infty)\text{, such that } \mathcal{L}^1(N_x) = 0, \tag{5.7.25}$$

with the property that for each $r \in (0,\infty) \setminus N_x$ *and each* $\vec{F} \in \big[\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)\big]^n$ *one has*

$$\int_{E\cap B(x,r)} \mathrm{div}\vec{F}\,\mathrm{d}\mathcal{L}^n = \int_{(\partial_* E)\cap B(x,r)} \nu\cdot\vec{F}\,\mathrm{d}\sigma_* + \int_{E\cap\partial B(x,r)} \Big(\frac{y-x}{r}\Big)\cdot\vec{F}(y)\,\mathrm{d}\mathcal{H}^{n-1}(y) \tag{5.7.26}$$

and, if $\overline{E}\cap\operatorname{supp}\vec{F}$ *is bounded,*

$$\int_{E\setminus B(x,r)} \mathrm{div}\vec{F}\,\mathrm{d}\mathcal{L}^n = \int_{(\partial_* E)\setminus B(x,r)} \nu\cdot\vec{F}\,\mathrm{d}\sigma_* - \int_{E\cap\partial B(x,r)} \Big(\frac{y-x}{r}\Big)\cdot\vec{F}(y)\,\mathrm{d}\mathcal{H}^{n-1}(y). \tag{5.7.27}$$

Moreover, similar integral formulas hold with $B(x,r)$ *replaced by* $\overline{B(x,r)}$ *in all occurrences.*

Proof The claim in (5.7.24) is a direct consequence of Lemma 5.6.5. To proceed, observe that

$$\begin{array}{l}\text{for each } \vec{F} \in \big[\mathscr{C}^\infty(\mathbb{R}^n)\big]^n \text{ there exists some } \mathcal{L}^1\text{-measurable set}\\ N_{x,\vec{F}} \subset (0,\infty)\text{, such that } \mathcal{L}^1\big(N_{x,\vec{F}}\big) = 0\text{, with the property that}\\ \text{formula (5.7.26) is valid whenever } r \in (0,\infty)\setminus N_{x,\vec{F}}.\end{array} \tag{5.7.28}$$

Indeed, in the case when the vector field in question is also compactly supported this follows from [263, Lemma 5.5.2, p. 234]. To take advantage of this, pick a function $\theta \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ with the property that $\theta \equiv 1$ near the origin and define $\theta_j(x) := \theta(x/j)$ for each $j \in \mathbb{N}$ and each $x \in \mathbb{R}^n$. Then, given any $\vec{F} \in \big[\mathscr{C}^\infty(\mathbb{R}^n)\big]^n$, we may invoke [263, Lemma 5.5.2, p. 234] which applies to the vector field $\vec{F}_j := \theta_j\vec{F} \in \big[\mathscr{C}^\infty_c(\mathbb{R}^n)\big]^n$ and guarantees the existence of some $\mathcal{L}^1$-measurable set $N_{x,j} \subset (0,\infty)$ such that $\mathcal{L}^1\big(N_{x,j}\big) = 0$ with the property that formula (5.7.26) is valid with $\vec{F}_j$ in place of $\vec{F}$ whenever $r \in (0,\infty)\setminus N_{x,j}$. If we now introduce $N_{x,\vec{F}} := \cup_{j\in\mathbb{N}} N_{x,j}$ it follows that $N_{x,\vec{F}}$ is an $\mathcal{L}^1$-measurable subset of $(0,\infty)$ such that $\mathcal{L}^1\big(N_{x,\vec{F}}\big) = 0$ and with the property that whenever $r \in (0,\infty)\setminus N_{x,\vec{F}}$ formula (5.7.26) is valid with $\vec{F}_j$ in place of $\vec{F}$ for each $j \in \mathbb{N}$. Passing to limit $j \to \infty$ then proves (5.7.28).

Consider next the collection of all vector fields in $\mathbb{R}^n$ whose components are polynomials with rational coefficients. Being a countable set, this may be arranged in the form of a sequence, say $\{\vec{P}_j\}_{j\in\mathbb{N}}$. Since each $\vec{P}_j$ belongs to $\big[\mathscr{C}^\infty(\mathbb{R}^n)\big]^n$, from (5.7.28) we know that for each $j \in \mathbb{N}$ there exists an $\mathcal{L}^1$-measurable set $\widetilde{N}_{x,j} \subset (0,\infty)$ such that $\mathcal{L}^1\big(\widetilde{N}_{x,j}\big) = 0$ and so that formula (5.7.26) is valid with $\vec{P}_j$ in place of $\vec{F}$ whenever $r \in (0,\infty)\setminus\widetilde{N}_{x,j}$. Define $N_x := \cup_{j\in\mathbb{N}}\widetilde{N}_{x,j}$, so that N_x is an $\mathcal{L}^1$-measurable subset of $(0,\infty)$ such that $\mathcal{L}^1\big(N_x\big) = 0$. Pick an arbitrary $r\in(0,\infty)\setminus N_x$ along with an arbitrary vector field $\vec{F} \in \big[\mathscr{C}^\infty(\mathbb{R}^n)\big]^n$. We may then rely on [181, Lemma 2.83, p. 52] to conclude that there exists a subsequence call it $\{\vec{P}_{j_k}\}_{k\in\mathbb{N}}$, of

$\{\vec{P}_j\}_{j\in\mathbb{N}}$ such that

$$\begin{aligned}&\sup_{y\in\overline{B(x,r)}}\big|\vec{P}_{j_k}(y)-\vec{F}(y)\big|\longrightarrow 0 \text{ as } k\to\infty, \quad\text{and}\\ &\sup_{y\in\overline{B(x,r)}}\big|(\operatorname{div}\vec{P}_{j_k})(y)-(\operatorname{div}\vec{F})(y)\big|\longrightarrow 0 \text{ as } k\to\infty.\end{aligned}\tag{5.7.29}$$

As a consequence, the second line in (5.7.29) implies

$$\begin{aligned}&\Big|\int_{E\cap B(x,r)}\operatorname{div}\vec{P}_{j_k}\,d\mathcal{L}^n-\int_{E\cap B(x,r)}\operatorname{div}\vec{F}\,d\mathcal{L}^n\Big|\\ &\quad\leq\mathcal{L}^n\big(B(x,r)\big)\cdot\sup_{y\in\overline{B(x,r)}}\big|(\operatorname{div}\vec{P}_{j_k})(y)-(\operatorname{div}\vec{F})(y)\big|\to 0 \text{ as } k\to\infty,\end{aligned}\tag{5.7.30}$$

while the first line in (5.7.29) and the fact that $\mathcal{H}^{n-1}\big(\overline{B(x,r)}\cap\partial_*E\big)<\infty$ given that E is a set of locally finite perimeter (cf. (5.6.35)) imply

$$\begin{aligned}&\Big|\int_{(\partial_*E)\cap B(x,r)}\nu\cdot\vec{P}_{j_k}\,d\sigma_*-\int_{(\partial_*E)\cap B(x,r)}\nu\cdot\vec{F}\,d\sigma_*\Big|\\ &\quad\leq\mathcal{H}^{n-1}\big(\overline{B(x,r)}\cap\partial_*\Omega\big)\cdot\sup_{y\in\overline{B(x,r)}}\big|\vec{P}_{j_k}(y)-\vec{F}(y)\big|\to 0 \text{ as } k\to\infty,\end{aligned}\tag{5.7.31}$$

and, finally, the first line in (5.7.29) implies

$$\begin{aligned}&\Big|\int_{E\cap\partial B(x,r)}\Big(\frac{y-x}{r}\Big)\cdot\vec{P}_{j_k}(y)\,d\mathcal{H}^{n-1}(y)-\int_{E\cap\partial B(x,r)}\Big(\frac{y-x}{r}\Big)\cdot\vec{F}(y)\,d\mathcal{H}^{n-1}(y)\Big|\\ &\quad\leq\mathcal{H}^{n-1}\big(\partial B(x,r)\big)\cdot\sup_{y\in\overline{B(x,r)}}\big|\vec{P}_{j_k}(y)-\vec{F}(y)\big|\to 0 \text{ as } k\to\infty.\end{aligned}\tag{5.7.32}$$

Recall that our choice of the $\mathcal{L}^1$-nullset N_x ensures that for each $r\in(0,\infty)\setminus N_x$ we have

$$\begin{aligned}\int_{E\cap B(x,r)}\operatorname{div}\vec{P}_{j_k}\,d\mathcal{L}^n&=\int_{(\partial_*E)\cap B(x,r)}\nu\cdot\vec{P}_{j_k}\,d\sigma_*\\ &\quad+\int_{E\cap\partial B(x,r)}\Big(\frac{y-x}{r}\Big)\cdot\vec{P}_{j_k}(y)\,d\mathcal{H}^{n-1}(y)\end{aligned}\tag{5.7.33}$$

for all $k\in\mathbb{N}$. On account of (5.7.30)–(5.7.32), after passing to the limit $k\to\infty$ in (5.7.33) we arrive at the conclusion that

there exists some $\mathcal{L}^1$-measurable set $N_x\subset(0,\infty)$ with $\mathcal{L}^1\big(N_x\big)=0$ and such that formula (5.7.26) holds whenever $r\in(0,\infty)\setminus N_x$, for each $\vec{F}\in\big[\mathscr{C}^\infty(\mathbb{R}^n)\big]^n$. (5.7.34)

In the case when $\vec{F}$ belongs to the larger class $\big[\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)\big]^n$ and some radius $r \in (0, \infty) \setminus N_x$ has been specified, use a standard mollifier argument to produce a sequence $(\vec{F}_\varepsilon)_{\varepsilon\in(0,1)} \subset \big[\mathscr{C}^\infty(\mathbb{R}^n)\big]^n$ with the property that

$$\begin{aligned}
&\sup\nolimits_{y\in\overline{B(x,r)}} \big|\vec{F}_\varepsilon(y) - \vec{F}(y)\big| \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0^+, \\
&\sup\nolimits_{\varepsilon\in(0,1)} \sup\nolimits_{y\in\overline{B(x,r)}} \big|(\mathrm{div}\vec{F}_\varepsilon)(y)\big| < +\infty, \quad \text{and such that} \\
&(\mathrm{div}\vec{F}_\varepsilon)(y) \to (\mathrm{div}\vec{F})(y) \quad \text{as} \quad \varepsilon \to 0^+ \quad \text{for } \mathcal{L}^n\text{-a.e. } y \in \mathbb{R}^n.
\end{aligned} \tag{5.7.35}$$

From (5.7.34) we know that for each $\varepsilon \in (0, 1)$ we have

$$\begin{aligned}
\int_{E\cap B(x,r)} \mathrm{div}\vec{F}_\varepsilon \, \mathrm{d}\mathcal{L}^n = &\int_{(\partial_* E)\cap B(x,r)} \nu \cdot \vec{F}_\varepsilon \, \mathrm{d}\sigma_* \\
&+ \int_{E\cap\partial B(x,r)} \Big(\frac{y-x}{r}\Big) \cdot \vec{F}_\varepsilon(y) \, \mathrm{d}\mathcal{H}^{n-1}(y).
\end{aligned} \tag{5.7.36}$$

After passing to limit $\varepsilon \to 0^+$ in (5.7.36), on account of (5.7.35) and several applications of Lebesgue's Dominated Convergence Theorem we reach the conclusion that, as claimed in the statement of the lemma, formula (5.7.26) holds whenever $r \in (0, \infty) \setminus N_x$ and $\vec{F} \in \big[\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)\big]^n$. In fact, the version of (5.7.26) in which $B(x, r)$ has been replaced in all occurrences by $\overline{B(x, r)}$ is also valid. Indeed, this is established in a similar fashion, relying now on [80, Lemma 1, p. 195] (and the Structure Theorem for sets of locally finite perimeter from [80, Theorem 2, p. 205]) in place of [263, Lemma 5.5.2, p. 234] (which has been used to justify (5.7.28)).

With an eye towards proving formula claimed in (5.7.27), assume now that some vector field $\vec{F} \in \big[\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)\big]^n$ with the property that $\overline{E} \cap \mathrm{supp}\,\vec{F}$ is bounded has been given. Pick a scalar function $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ such that $\psi \equiv 1$ near $\overline{E} \cap \mathrm{supp}\,\vec{F}$, and define $\vec{G} := \psi\vec{F} \in \big[\mathrm{Lip}_c(\mathbb{R}^n)\big]^n$. On the one hand, we may then write

$$\int_E \mathrm{div}\vec{F}\,\mathrm{d}\mathcal{L}^n = \int_E \mathrm{div}\vec{G}\,\mathrm{d}\mathcal{L}^n = \int_{\partial_* E} \nu \cdot \vec{G}\,\mathrm{d}\sigma_* = \int_{\partial_* E} \nu \cdot \vec{F}\,\mathrm{d}\sigma_*, \tag{5.7.37}$$

where the middle equality is implied by the version of the Gauss–Green Theorem recorded in (2.8.1). On the other hand, from what we have proved up to this point we know that formula (5.7.26) (both as stated, and with $B(x, r)$ replaced by $\overline{B(x, r)}$ throughout) holds for the vector field $\vec{F} \in \big[\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)\big]^n$ and for any $r \in (0, \infty) \setminus N_x$. Subtracting the latter from (5.7.37) then yields (5.7.27) (both as formulated, and with $B(x, r)$ replaced by $\overline{B(x, r)}$ in all occurrences). □

Localizing sets of locally finite perimeter yields sets of finite perimeter in the precise sense described below. To facilitate its statement, we make the following convention:

for any two $\mathcal{H}^{n-1}$-measurable sets $A, B \subseteq \mathbb{R}^n$ we agree to write $A \equiv B$ modulo $\mathcal{H}^{n-1}$ if $\mathcal{H}^{n-1}(A \Delta B) = 0$, where $A \Delta B$ denotes the symmetric difference between A and B. (5.7.38)

Lemma 5.7.3 *Given a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter along with an arbitrary point $x \in \mathbb{R}^n$, the following assertions hold.*

(i) For each arbitrary number $r \in (0, \infty)$ the set $\Omega \cap B(x, r)$ has finite perimeter.
(ii) For $\mathcal{L}^1$-a.e. $r \in (0, \infty)$ one has, in the sense of distributions in $\mathbb{R}^n$,

$$-\nabla \mathbf{1}_{\Omega \cap B(x,r)} = \nu\, \mathcal{H}^{n-1} \lfloor \big(\partial_* \Omega \cap B(x, r)\big) + N_{x,r}\, \mathcal{H}^{n-1} \lfloor \big(\Omega \cap \partial B(x, r)\big), \tag{5.7.39}$$

where $N_{x,r}$ denotes the ordinary outward unit normal to $\partial B(x, r)$, and ν stands for the geometric measure theoretic outward unit normal to Ω.
(iii) For $\mathcal{L}^1$-a.e. $r \in (0, \infty)$ one has

$$\partial_*\big(\Omega \cap B(x, r)\big) \equiv \big(\Omega \cap \partial B(x, r)\big) \cup \big(\partial_* \Omega \cap B(x, r)\big) \ \text{ modulo } \ \mathcal{H}^{n-1}. \tag{5.7.40}$$

(iv) For $\mathcal{L}^1$-a.e. $r \in (0, \infty)$ the geometric measure theoretic outward unit normal $\nu_{x,r}$ to $\Omega \cap B(x, r)$ is given at $\mathcal{H}^{n-1}$-a.e. point on $\partial_*\big(\Omega \cap B(x, r)\big)$ by

$$\nu_{x,r} = \begin{cases} N_{x,r} & \text{at } \mathcal{H}^{n-1}\text{-a.e. point on } \ \Omega \cap \partial B(x, r), \\ \nu & \text{at } \mathcal{H}^{n-1}\text{-a.e. point on } \ \partial_* \Omega \cap B(x, r). \end{cases} \tag{5.7.41}$$

Based on (5.7.39), it has been shown in [124, Lemma 2.5] that

if $\Omega \subseteq \mathbb{R}^n$ is open, of locally finite perimeter, with $\partial\Omega = \partial(\overline{\Omega})$, then $\mathcal{H}^{n-1}\big(\partial_* \Omega \cap B(x, r)\big) > 0$ for every point $x \in \partial\Omega$ and every $r > 0$. (5.7.42)

It has been also pointed out in [124] that

for a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, the condition in the last line of (5.7.42) is actually equivalent to having $\partial_*\Omega$ dense in $\partial\Omega$. (5.7.43)

Proof of Lemma 5.7.3 The claim in item *(i)* is a direct consequence of Lemma 5.7.1. To justify (5.7.39), let $N_x \subset (0, \infty)$ be the $\mathcal{L}^1$-nullset associated with $E := \Omega$ and the point x as in (5.7.25). For a fixed $r \in (0, \infty) \setminus N_x$, write (5.7.26) for arbitrary $\vec{F} \in \big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n$ and observe that the resulting integral formula may be interpreted as

$$-\big\langle \nabla \mathbf{1}_{\Omega \cap B(x,r)}, \vec{F} \big\rangle = \big\langle \nu\, \mathcal{H}^{n-1} \lfloor \big(\partial_* \Omega \cap B(x, r)\big), \vec{F} \big\rangle + \big\langle N_{x,r}\, \mathcal{H}^{n-1} \lfloor \big(\Omega \cap \partial B(x, r)\big), \vec{F} \big\rangle, \tag{5.7.44}$$

with the brackets indicating duality between vector distributions and vector-valued test functions. From this, (5.7.39) follows in view of the arbitrariness of the vector field $\vec{F} \in \big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n$. This takes care of item *(ii)*. Turning attention to the claim in item *(iii)*, from (5.6.3), (5.6.22), and the current item *(i)* we conclude that for each $r \in (0, \infty)$ we have

$$-\nabla \mathbf{1}_{\Omega\cap B(x,r)} = \nu_{x,r}\, \mathcal{H}^{n-1}\lfloor\partial_*\big(\Omega \cap B(x,r)\big). \tag{5.7.45}$$

Above, $\nu_{x,r}$ is the geometric measure theoretic outward unit normal to the set of locally finite perimeter $\Omega \cap B(x,r)$, hence

$$|\nu_{x,r}| = 1 \text{ at } \mathcal{H}^{n-1}\text{-a.e. point on } \partial_*\big(\Omega \cap B(x,r)\big) \tag{5.7.46}$$

(cf. (5.6.13), (5.6.20), and (5.6.21)). Comparing the set on which the measure $\nabla \mathbf{1}_{\Omega\cap B(x,r)}$ is concentrated, as seen from (5.7.39) and (5.7.45), then yields (5.7.40). Finally, the claim in item *(iv)* is seen by comparing (5.7.39) with (5.7.45), bearing in mind (5.7.40). □

Given an $\mathcal{L}^n$-measurable set $\Omega \subseteq \mathbb{R}^n$, define

$$\begin{aligned}\partial_{\text{lfp}}\Omega := \big\{x \in \partial\Omega :\ &\text{there exists } r_x > 0 \text{ such that}\\ &\Omega \cap B(x, r_x) \text{ has locally finite perimeter}\big\}.\end{aligned} \tag{5.7.47}$$

From Lemma 5.7.1 it follows that for each $\mathcal{L}^n$-measurable set $\Omega \subseteq \mathbb{R}^n$ we have

$$\begin{aligned}\partial_{\text{lfp}}\Omega = \big\{x \in \partial\Omega :\ &\text{there exists } r_x > 0 \text{ such that}\\ &\Omega \cap B(x, r_x) \text{ has finite perimeter}\big\}\end{aligned} \tag{5.7.48}$$

and

$$\partial_{\text{lfp}}\Omega \text{ is a relatively open subset of } \partial\Omega, \tag{5.7.49}$$

In fact,

$$\partial_{\text{lfp}}\Omega = \bigcup_{x\in\partial_{\text{lfp}}\Omega} B(x, r_x) \cap \partial\Omega \tag{5.7.50}$$

where each $r_x > 0$ is associated with $x \in \partial_{\text{lfp}}\Omega$ as in (5.7.47). Let us also note here that while we always have $\partial_{\text{lfp}}\Omega \subseteq \partial\Omega$, Lemma 5.7.1 implies that

$$\partial_{\text{lfp}}\Omega = \partial\Omega \quad \text{whenever } \Omega \text{ is a set of locally finite perimeter.} \tag{5.7.51}$$

Before stating our next result we make a convention pertaining notation. Specifically, for any two $\mathcal{H}^{n-1}$-measurable sets $A, B \subseteq \mathbb{R}^n$ we shall write $A \subseteq B$ modulo $\mathcal{H}^{n-1}$ provided $\mathcal{H}^{n-1}(A \setminus B) = 0$, i.e., if B contains A up to an $\mathcal{H}^{n-1}$-nullset.

Proposition 5.7.4 *Let $\Omega \subseteq \mathbb{R}^n$ be an $\mathcal{L}^n$-measurable set. Then the following statements are true.*

(i) If $\mathcal{O} \subseteq \mathbb{R}^n$ is an open set with the property that $\mathcal{O} \cap \Omega$ is a set of locally finite perimeter, then

$$\mathcal{O} \cap \partial^*(\mathcal{O} \cap \Omega) \subseteq \partial^N \Omega. \tag{5.7.52}$$

Also, if ν_0 denotes the geometric measure theoretic outward unit normal to $\mathcal{O} \cap \Omega$ while N_Ω denotes the unit vector field associated with the set Ω as in item (i) of Lemma 5.6.12, then

$$\nu_0(x) = N_\Omega(x) \ \textit{for each} \ \ x \in \mathcal{O} \cap \partial^*(\mathcal{O} \cap \Omega), \tag{5.7.53}$$

and

$$\nu_0(x) = N_\Omega(x) \ \textit{for } \mathcal{H}^{n-1}\textit{-a.e.} \ \ x \in \mathcal{O} \cap \partial_* \Omega. \tag{5.7.54}$$

(ii) One has

$$\partial_* \Omega \cap \partial_{\text{lfp}} \Omega \subseteq \partial^N \Omega \ \ \textit{modulo} \ \ \mathcal{H}^{n-1}. \tag{5.7.55}$$

Proof To justify the claims in item *(i)*, pick $x \in \mathcal{O} \cap \partial^*(\mathcal{O} \cap \Omega)$ arbitrary. On the one hand, thanks to Lemma 5.6.12, the membership of x to $\partial^*(\mathcal{O} \cap \Omega)$ entails

$$\lim_{r \to 0^+} \frac{\mathcal{L}^n\big(B(x,r) \cap \{y \in \mathcal{O} \cap \Omega : \ (y-x) \cdot \nu_0(x) > 0\}\big)}{\mathcal{L}^n\big(B(x,r)\big)} = 0 \ \text{ and} \tag{5.7.56}$$

$$\lim_{r \to 0^+} \frac{\mathcal{L}^n\big(B(x,r) \cap \{y \in \mathbb{R}^n \setminus (\mathcal{O} \cap \Omega) : \ (y-x) \cdot \nu_0(x) < 0\}\big)}{\mathcal{L}^n\big(B(x,r)\big)} = 0. \tag{5.7.57}$$

On the other hand, given that x belongs to the open set $\mathcal{O}$, for each $r > 0$ sufficiently small we have (with the superscript "c" denoting the complement relative to $\mathbb{R}^n$)

$$\begin{aligned} B(x,r) \cap (\mathcal{O} \cap \Omega) &= B(x,r) \cap \Omega, \quad \text{as well as} \\ B(x,r) \cap (\mathcal{O} \cap \Omega)^c &= B(x,r) \cap (\mathcal{O}^c \cup \Omega^c) = B(x,r) \cap \Omega^c, \end{aligned} \tag{5.7.58}$$

we may simply re-write (5.7.56)–(5.7.57) as

$$\lim_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\cap\{y\in\Omega:\ (y-x)\cdot\nu_0(x)>0\}\big)}{\mathcal{L}^n\big(B(x,r)\big)} = 0 \quad\text{and} \tag{5.7.59}$$

$$\lim_{r\to 0^+} \frac{\mathcal{L}^n\big(B(x,r)\cap\{y\in\mathbb{R}^n\setminus\Omega:\ (y-x)\cdot\nu_0(x)<0\}\big)}{\mathcal{L}^n\big(B(x,r)\big)} = 0. \tag{5.7.60}$$

In turn, from (5.7.59)–(5.7.60), (5.6.69), and (5.6.71) we conclude that

$$x\in\partial^N\Omega \quad\text{and}\quad \nu_0(x)=N_\Omega(x). \tag{5.7.61}$$

The arbitrariness of x then finishes the proof of (5.7.52)–(5.7.53). Lastly, since by (5.6.21) we have

$$\mathcal{O}\cap\partial^*(\mathcal{O}\cap\Omega)\equiv\mathcal{O}\cap\partial_*(\mathcal{O}\cap\Omega) \quad\text{modulo}\quad \mathcal{H}^{n-1}, \tag{5.7.62}$$

and since Proposition 5.2.3 gives

$$\mathcal{O}\cap\partial_*(\mathcal{O}\cap\Omega)=\mathcal{O}\cap\partial_*\Omega, \tag{5.7.63}$$

the claim in (5.7.54) follows from (5.7.53), (5.7.62), and (5.7.63).

Turning our attention to (5.7.55), start by observing that, as a second-countable space, $\mathbb{R}^n$ is a strongly Lindelöf space. Hence, we may refine (5.7.50) to a countable cover of the following sort: there exist $\{x_j\}_{j\in\mathbb{N}}\subseteq\partial_{\text{lfp}}\Omega$ and $\{r_j\}_{j\in\mathbb{N}}\subseteq(0,\infty)$ such that each $\Omega\cap B(x_j,r_j)$ is a set of locally finite perimeter and

$$\partial_{\text{lfp}}\Omega=\bigcup_{j\in\mathbb{N}}B(x_j,r_j)\cap\partial\Omega. \tag{5.7.64}$$

We may then write

$$\begin{aligned}
\partial_*\Omega\cap\partial_{\text{lfp}}\Omega &= \bigcup_{j\in\mathbb{N}}B(x_j,r_j)\cap\partial_*\Omega\\
&= \bigcup_{j\in\mathbb{N}}B(x_j,r_j)\cap\partial_*\big(B(x_j,r_j)\cap\Omega\big)\\
&\equiv \bigcup_{j\in\mathbb{N}}B(x_j,r_j)\cap\partial^*\big(B(x_j,r_j)\cap\Omega\big) \quad\text{modulo}\quad \mathcal{H}^{n-1}.
\end{aligned} \tag{5.7.65}$$

Above, the first equality is implied by (5.7.64) and (5.2.3), the second equality is a consequence of Proposition 5.2.3, and the last equivalence is seen from (5.6.21). Then (5.7.55) follows by combining (5.7.65) with (5.7.52). □

5.8 Planar Curves

Recall that a `simply connected domain` is a connected open subset of $\mathbb{R}^2 \equiv \mathbb{C}$ in which all closed curves are homotopic to zero. Also, denote by $\widehat{\mathbb{C}}$ the Riemann sphere (i.e., the one-point compactification $\mathbb{C} \cup \{\infty\}$ of the complex plane).

Proposition 5.8.1 *For a connected open subset Ω of $\mathbb{R}^2 \equiv \mathbb{C}$ the following conditions are equivalent:*

(1) *Ω is simply connected;*
(2) *Ω is homeomorphic to the unit disk $\mathbb{D} := B(0, 1)$;*
(3) *the winding number of any closed curve $\gamma \subset \Omega$ with respect to any point $z \in \widehat{\mathbb{C}} \setminus \Omega$ is zero;*
(4) *$\widehat{\mathbb{C}} \setminus \Omega$ is connected;*
(5) *every holomorphic function f in Ω may be approximated by polynomials in the variable z, uniformly on compact subsets of Ω;*
(6) *$\int_\gamma f(z)\,\mathrm{d}z = 0$ for every holomorphic function f in Ω and for every rectifiable closed curve $\gamma \subset \Omega$;*
(7) *for every holomorphic function f in Ω there exists a holomorphic function F in Ω with the property that $F' = f$ in Ω;*
(8) *for every holomorphic function f in Ω which does not vanish in Ω there exists a holomorphic function g in Ω such that $f = \mathrm{e}^g$ in Ω;*
(9) *for every holomorphic function f in Ω which does not vanish in Ω and for every $n \in \mathbb{N}$ there exists a holomorphic function h in Ω such that $f = h^n$ in Ω;*
(10) *for every real-valued harmonic function u in Ω there exists a real-valued harmonic function w in Ω with the property that $u + iw$ is holomorphic in Ω.*

Proof The equivalence of conditions *(1)–(9)* may be found in [231, Theorem 13.11, p. 274]. As far as condition *(10)* is concerned, assume that Ω is simply connected and consider an arbitrary real-valued harmonic function u in Ω. Fix a point $z_o \in \Omega$ arbitrary and define $w : \Omega \to \mathbb{R}$ by setting

$$w(z) := \int_{\gamma_z} (-\partial_y u)\,\mathrm{d}x + \partial_x u\,\mathrm{d}y, \qquad \forall z \in \Omega, \tag{5.8.1}$$

where the path-integral is taken over a rectifiable curve γ_z joining z_o with z in Ω (the existence of such a curve is a consequence of the fact that connected open sets in the Euclidean setting are path-connected). Green's Formula and the harmonicity of u may be then employed to check that the above definition is unambiguous. In turn, the freedom of choosing the rectifiable curve γ_z joining z_o with z in Ω may be exploited in order to verify that u and w satisfy the Cauchy–Riemann equations in Ω. Thus, w is a real-valued harmonic function in Ω such that $u + iw$ is holomorphic in Ω. This proves that *(1)* ⇒ *(10)*.

To prove that *(10)* ⇒ *(1)*, reason by contradiction. Assume that there exist a point $z_o \in \mathbb{C} \setminus \Omega$ and a simple closed curve $\gamma \subset \Omega$ with the property that z_o belongs to the inner domain of γ. Via a translation, there is no loss of generality in assuming that z_o is actually the origin $0 \in \mathbb{C}$. In this scenario, consider

$$u(x, y) := \ln\sqrt{x^2 + y^2} \quad \text{for all } \ x + \mathrm{i}y \in \Omega, \tag{5.8.2}$$

and note that u is a well-defined, real-valued, harmonic function in Ω. Suppose that there exists a real-valued harmonic function w in Ω with the property that $F := u + \mathrm{i}w$ is holomorphic in Ω. Since

$$G(z) := \log z, \qquad \forall z \in \mathbb{C} \setminus (-\infty, 0], \tag{5.8.3}$$

is a holomorphic function whose real part coincides with u in $\Omega \setminus (-\infty, 0]$, it follows that $F - G$ is a purely imaginary constant in $\Omega \setminus (-\infty, 0]$. By taking imaginary parts this further shows that there exists $c \in \mathbb{R}$ such that $\arg z = v(z) + \mathrm{i}c$ in $\Omega \setminus (-\infty, 0]$. Since $w + \mathrm{i}c$ is harmonic, hence continuous in Ω, this shows that the function $\gamma \setminus (-\infty, 0] \ni z \mapsto \arg z \in (-\pi, \pi)$ has a continuous extension to the entire loop γ. This contradiction then finishes the proof. $\square$

Remark 5.8.2 *Assume $\Omega \subseteq \mathbb{C}$ is a nonempty, open, connected, simply connected set. If f is a holomorphic function in Ω which does not vanish in Ω, item (8) of Proposition 5.8.1 guarantees the existence of a holomorphic function g in Ω with the property that $f = \mathrm{e}^g$ in Ω. Such a function g is not unique, but if g_1, g_2 are two holomorphic functions in Ω satisfying $f = \mathrm{e}^{g_j}$ in Ω for $j = 1, 2$, then $1 = \mathrm{e}^{g_1 - g_2}$ in Ω, and after applying $\mathrm{d}/\mathrm{d}z$ we ultimately see that there exists some $m \in \mathbb{Z}$ such that $g_1 - g_2 = 2\pi\mathrm{i}m$ in Ω.*

When specializing these considerations to the case when $0 \notin \Omega$ and for the function $f(z) := z$ for each $z \in \Omega$, we see that there exists a holomorphic function in Ω, call it $\log_\Omega$ (or simply $\log$ if the set Ω is understood from context), with the property that

$$z = \mathrm{e}^{\log_\Omega z} \ \textit{ for each } \ z \in \Omega. \tag{5.8.4}$$

Moreover,

$$\textit{any holomorphic function } g \textit{ in } \Omega \textit{ satisfying } z = \mathrm{e}^{g(z)} \textit{ for each } z \in \Omega \textit{ is of the form } g = 2\pi\mathrm{i}m + \log_\Omega \textit{ for some } m \in \mathbb{Z}. \tag{5.8.5}$$

Each function of the form $2\pi\mathrm{i}m + \log_\Omega$ corresponding to a choice of $m \in \mathbb{Z}$ will be referred to as a `branch of the complex logarithm` *in Ω. For example, under the additional assumption that the real semi-axis $(0, \infty)$ is contained in Ω, the same type of argument that has led to (5.8.5) shows that there exists a unique branch of the complex logarithm in Ω (called the principal logarithmic branch) agreeing with the ordinary natural logarithmic function $\ln$ on the interval $(0, \infty)$.*

Continue to assume that $\Omega \subseteq \mathbb{C} \setminus \{0\}$ is a nonempty, open, connected, simply connected set. Then, as seen from (5.8.4), for each $(x + \mathrm{i}y) \in \Omega$, any branch $\log_\Omega$ of the complex logarithm in Ω satisfies

$$\partial_x \log_\Omega(x+\mathrm{i}y) = \frac{1}{x+\mathrm{i}y} \quad \text{and} \quad \partial_y \log_\Omega(x+\mathrm{i}y) = \frac{\mathrm{i}}{x+\mathrm{i}y}. \tag{5.8.6}$$

Formula (5.8.4) *also implies that for each* $z, w \in \Omega$ *there exists* $m(z,w) \in \mathbb{Z}$ *satisfying* $\log_\Omega(zw) = \log_\Omega z + \log_\Omega w + 2\pi \mathrm{i} m(z,w)$. *Since* $m(\cdot,\cdot) : \Omega \times \Omega \to \mathbb{Z}$ *is continuous and integer-valued, it is actually constant. In the case when the real semi-axis* $(0,\infty)$ *is contained in* Ω, *it follows that the principal logarithmic branch* log *satisfies the usual identity* $\log(zw) = \log z + \log w$ *for each* $z, w \in \Omega$.

Finally, given a nonempty, open, connected, simply connected set $\Omega \subseteq \mathbb{C} \setminus \{0\}$, *for each branch* $\log_\Omega$ *of the complex logarithm in* Ω *we can define complex powers, with base a complex number in* Ω *and a given exponent* $w \in \mathbb{C}$, *via the formula*

$$z^w := \mathrm{e}^{w \log_\Omega z} \quad \text{for each} \quad z \in \Omega. \tag{5.8.7}$$

In particular, corresponding to a fixed exponent $w \in \mathbb{C}$, *there are infinitely many complex power functions* $\Omega \ni z \mapsto z^w \in \mathbb{C}$.

A (compact) `curve` in the plane $\mathbb{R}^2$ (canonically identified with the field of complex numbers $\mathbb{C}$) is a set of the form $\Sigma = \gamma([a,b])$, where $a, b \in \mathbb{R}$, $a < b$, and $\gamma : [a,b] \to \mathbb{R}^2$ is a continuous function, called parametrization of Σ. Call the curve Σ `closed`[9] if it has a parametrization $\gamma : [a,b] \to \mathbb{R}^2$ satisfying $\gamma(a) = \gamma(b)$. Also, call $\Sigma \subset \mathbb{C}$ a `Jordan curve`, or a `simple closed curve`, provided[10] Σ is a closed curve admitting a parametrization $\gamma : [a,b] \to \mathbb{R}^2$ whose restriction to $[a,b)$ is injective.[11] Thus, a curve is Jordan if and only if it is the homeomorphic image of the unit circle S^1. Recall that the classical Jordan Curve Theorem asserts that

> the complement of $\Sigma \subset \mathbb{C}$ consists of two connected components, one bounded and simply connected Ω_+, and one unbounded Ω_-, which satisfy $\partial\Omega_\pm = \Sigma$. (5.8.8)

Recall that a simply connected domain is a connected open set in which all closed curves are homotopic to zero,[12] or, equivalently, a connected open set whose fundamental group is trivial (see also Proposition 5.8.1 for other alternative characterizations). Given a bounded planar connected domain Ω, the fact that Ω is a simply connected domain is equivalent to $\partial\Omega$ being a connected set. Also, all planar simply connected domains are homeomorphic to an open disk (see Proposition 5.8.1). A bounded open set $\Omega \subset \mathbb{C}$ is called a `Jordan domain` if $\partial\Omega$ is a Jordan curve. Then

$$\text{any Jordan domain is simply connected.} \tag{5.8.9}$$

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[9] Or, a continuous loop.

[10] Alternatively, Σ is the image of an injective continuous map of the circle S^1 into $\mathbb{R}^2$.

[11] A condition stipulating that Σ has no self-intersection points.

[12] Or null-homotopic.

The `length` $L \in [0, +\infty]$ of the (not necessarily closed) curve $\Sigma = \gamma([a, b])$ is defined as

$$L := \sup \sum_{j=1}^{N} |\gamma(t_j) - \gamma(t_{j-1})|, \tag{5.8.10}$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ of the interval $[a, b]$. As is well known (cf., e.g., [161, Theorem 4.38, p. 135]), the length L of any simple[13] curve Σ may be expressed in terms of the Hausdorff measure by

$$L = \mathcal{H}^1(\Sigma), \tag{5.8.11}$$

and

$$|z_1 - z_2| \leq \mathcal{H}^1(\Sigma) \quad \text{for each curve } \Sigma \text{ with endpoints } z_1, z_2. \tag{5.8.12}$$

Call the curve Σ `rectifiable` provided $L < +\infty$. Clearly, if $\Sigma = \gamma([a, b])$ is a rectifiable curve then $\gamma(I)$ is also a rectifiable curve, for any closed subinterval I of $[a, b]$. Also,

$$\begin{array}{l}\text{a simple curve is rectifiable if and only if}\\ \text{it has a Lipschitz parametrization}\end{array} \tag{5.8.13}$$

(see [161, Theorem 4.22, p. 129 and Theorem 4.31, p. 132]).

Remark 5.8.3 *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain satisfying $\mathcal{H}^1(\partial\Omega) < +\infty$. While in this case $\partial\Omega$ is a rectifiable curve (as seen by invoking* [69, Theorem 1.8, p. 6]*), it is not necessarily true that $\partial\Omega$ is a Jordan curve* (*with the slit disk* $B(0, 1) \setminus [0, 1]$ *serving as a counterexample*)*. On the other hand, rectifiable curves may be characterized as in* (5.3.72).

If $\Sigma = \gamma([a, b])$ is a rectifiable Jordan curve of length L, define the function $z : [0, L] \to \mathbb{C}$ by asking that for each $s \in [0, L]$ the complex number $z(s)$ is the unique point $\gamma(t) \in \Sigma$ with the property that the length of the curve $\gamma([a, t])$ is s. The map $[0, L] \ni s \mapsto z(s) \in \Sigma$ is called the `arc-length parametrization` of Σ (every rectifiable curve admits an arc-length parametrization; see, e.g., [31, 161, 255]). Then

$$z(0) = z(L) \quad \text{and} \quad |z(s_1) - z(s_2)| \leq |s_1 - s_2| \quad \text{for all} \quad s_1, s_2 \in [0, L], \tag{5.8.14}$$

by (5.8.11)–(5.8.12). As a consequence of the fact that the curve Σ is simple we also have

$$z(\cdot) : [0, L) \longrightarrow \Sigma \quad \text{bijectively}. \tag{5.8.15}$$

Furthermore, as is well known (see, e.g., [161, Definition 4.21 and Theorem 4.22, pp. 128–129]),

[13] i.e., non self-intersecting.

$$z(s) \text{ is differentiable for a.e. } s \in [0, L] \text{ and } |z'(s)| = 1 \text{ for a.e. } s \in [0, L]. \tag{5.8.16}$$

Let us also note here that since the Jacobian of the map $[0, L] \ni s \mapsto z(s) \in \mathbb{C} \equiv \mathbb{R}^2$ is equal to $|z'(s)| = 1$ for $\mathcal{L}^1$-a.e. $s \in [0, L]$, the area formula (cf. Proposition 5.3.2) yields

$$\mathcal{H}^1\big(z(A)\big) = \mathcal{L}^1(A) \tag{5.8.17}$$

for every $\mathcal{L}^1$-measurable set $A \subseteq [0, L]$. Thanks to (5.8.15) we may alternatively express this as

$$\mathcal{H}^1(E) = \mathcal{L}^1\big(z^{-1}(E)\big) \text{ for each } \mathcal{H}^1\text{-measurable set } E \subseteq \Sigma. \tag{5.8.18}$$

Formula (5.8.18) may be interpreted in terms of the push-forward of a measure. Recall that, in general, if $\mathcal{X}$ and $\widetilde{\mathcal{X}}$ are two locally compact Hausdorff topological spaces, $F : \mathcal{X} \to \widetilde{\mathcal{X}}$ is a continuous proper map, and if μ is a Borel measure on $\mathcal{X}$, the `push-forward` of μ via F is the Borel measure $F_*\mu$ on $\widetilde{\mathcal{X}}$ given by

$$(F_*\mu)(E) := \mu\big(F^{-1}(E)\big) \text{ for each Borel set } E \subseteq \widetilde{\mathcal{X}}. \tag{5.8.19}$$

In this regard, let us also note that an equivalent characterization of the measure $F_*\mu$ is via

$$\int_{\widetilde{\mathcal{X}}} f \, \mathrm{d}F_*\mu = \int_{\mathcal{X}} f \circ F \, \mathrm{d}\mu, \quad \forall f \text{ continuous, compactly supported in } \widetilde{\mathcal{X}}. \tag{5.8.20}$$

In our setting, formula (5.8.18) may then be re-phrased as

$$\mathcal{H}^1 \lfloor \Sigma = z_*\big(\mathscr{L}^1 \lfloor [0, L]\big). \tag{5.8.21}$$

Our next proposition contains useful properties of the arc-length parametrization of a rectifiable Jordan curve in the plane.

Proposition 5.8.4 *Assume that* $\Sigma \subset \mathbb{C}$ *is a rectifiable Jordan curve of length* L *and arc-length parametrization* $[0, L) \ni s \mapsto z(s) \in \Sigma$*, and define*

$$\rho(s_1, s_2) := \min\big\{|s_1 - s_2|, L - |s_1 - s_2|\big\}, \qquad \forall s_1, s_2 \in [0, L). \tag{5.8.22}$$

Then ρ *is a metric on* $[0, L)$ *and, if* τ_ρ *is the topology induced by it on the set* $[0, L)$*,*

$$\big([0, L), \tau_\rho\big) \text{ is a compact topological space}, \tag{5.8.23}$$

and

$$\begin{aligned}&\textit{if } f : [0, L] \to \mathbb{C} \textit{ is continuous in the ordinary topology then}\\ &f\big|_{[0,L)} : \big([0, L), \tau_\rho\big) \longrightarrow \mathbb{C} \textit{ is continuous} \iff f(0) = f(L).\end{aligned} \tag{5.8.24}$$

Furthermore,

$$z(\cdot) : \big([0, L), \rho\big) \longrightarrow \Sigma \textit{ is a bijective Lipschitz map with constant } \leq 1, \tag{5.8.25}$$

and if $z^{-1}(\cdot) : \Sigma \to [0, L)$ *denotes its inverse then*

$$z^{-1}(\cdot) : \Sigma \longrightarrow \big([0, L), \tau_\rho\big) \textit{ continuously.} \tag{5.8.26}$$

Finally, if for each $z_1, z_2 \in \Sigma$ *one denotes by* $\ell(z_1, z_2)$ *the length of the shortest sub-arc of* Σ *joining* z_1 *with* z_2*, then*

$$\ell(z_1, z_2) = \rho\big(z^{-1}(z_1), z^{-1}(z_2)\big) \textit{ for all } z_1, z_2 \in \Sigma, \tag{5.8.27}$$

and

$$\begin{aligned}&\ell(\cdot, \cdot) : \Sigma \times \Sigma \longrightarrow [0, L/2] \textit{ has the property that}\\ &\textit{for each } \varepsilon > 0 \textit{ there exists } \delta > 0 \textit{ such that } \ell(z_1, z_2) < \varepsilon\\ &\textit{whenever the points } z_1, z_2 \in \Sigma \textit{ are such that } |z_1 - z_2| < \delta.\end{aligned} \tag{5.8.28}$$

Let us momentarily digress for the purpose of recording the following general elementary fact, which is going to be useful shortly.

Lemma 5.8.5 *Suppose that* X *is a compact topological space,* Y *is a Hausdorff topological space, and* $f : X \to Y$ *is a given continuous bijection. Then its inverse* $f^{-1} : Y \to X$ *is also continuous.*

Proof To conclude that f^{-1} is continuous, it suffices to prove that if $A \subseteq X$ is an arbitrary closed set then $(f^{-1})^{-1}(A) = f(A)$ is closed in Y. Since X is assumed to be compact, and $A \subseteq X$ is closed, it follows that A is compact. Hence, $f(A)$ is a compact subset of Y, given that f is continuous. Finally, Y being Hausdorff entails that $f(A)$ is closed in Y, as wanted. □

Having dealt with Lemma 5.8.5, we now turn our attention to the

Proof of Proposition 5.8.4 From definitions, one may easily check that ρ is a metric on $[0, L)$. Assume next that $\{x_j\}_{j\in\mathbb{N}}$ is a sequence of numbers in $[0, L)$, and consider

$$N_1 := \big\{j \in \mathbb{N} : x_j \in [0, L/2)\big\}, \qquad N_2 := \big\{j \in \mathbb{N} : x_j \in [L/2, L)\big\}. \tag{5.8.29}$$

If the set N_1 is infinite, then by Weierstrass' theorem N_1 has an infinite subset M_1 such that the sequence $\{x_j\}_{j\in M_1}$ converges in the ordinary topology of the real line to some $x_* \in [0, L/2]$. Granted this, one can readily verify that $\{x_j\}_{j\in M_1}$ also converges to x_* in τ_ρ.

On the other hand, if N_1 is finite then necessarily N_2 is infinite. Hence, N_2 has an infinite subset M_2 with the property that the sequence $\{x_j\}_{j\in M_2}$ converges in the ordinary topology of the real line to some $x_* \in [L/2, L]$. If, in fact, $x_* \in [L/2, L)$ then one can easily verify that $\{x_j\}_{j\in M_2}$ also converges to x_* in τ_ρ. Finally, if $x_* = L$, then one can show that $\{x_j\}_{j\in M_2}$ converges to 0 in τ_ρ. Thus, in all cases, any sequence in $[0, L)$ has a subsequence that converges in τ_ρ to a number in $[0, L)$ and, given that τ_ρ is metrizable, (5.8.23) follows.

Let now $f : [0, L] \to \mathbb{C}$ be a continuous function in the ordinary topology, with the property that $f(0) = f(L)$. Pick a sequence $\{x_j\}_{j\in\mathbb{N}} \subseteq [0, L)$ which converges in τ_ρ to some $x_* \in [0, L)$ and introduce

$$N_1 := \big\{j \in \mathbb{N} : |x_* - x_j| < L/2\big\}, \qquad N_2 := \big\{j \in \mathbb{N} : |x_* - x_j| \geq L/2\big\}. \tag{5.8.30}$$

Then since

$$\rho(x_*, x_j) = \begin{cases} |x_* - x_j| & \text{if } j \in N_1, \\ L - |x_* - x_j| & \text{if } j \in N_2, \end{cases} \tag{5.8.31}$$

the fact that $\{x_j\}_{j\in\mathbb{N}}$ converges to x_* in τ_ρ has the following consequences:

$$N_1 \ \text{ infinite} \Rightarrow \{x_j\}_{j\in N_1} \text{ converges to } x_* \text{ in the ordinary topology,} \tag{5.8.32}$$

and

$$N_2 \text{ infinite} \Rightarrow x_* = 0 \text{ and } \{x_j\}_{j\in N_2} \text{ converges to } L \text{ in the ordinary topology.} \tag{5.8.33}$$

Given that $f : [0, L] \to \mathbb{C}$ is continuous in the ordinary topology, these imply

$$N_1 \ \text{ infinite} \Rightarrow \{f(x_j)\}_{j\in N_1} \text{ converges to } f(x_*) \text{ in } \mathbb{C}, \tag{5.8.34}$$

and

$$N_2 \ \text{ infinite} \Rightarrow x_* = 0 \text{ and } \{f(x_j)\}_{j\in N_2} \text{ converges to } f(L) \text{ in } \mathbb{C}. \tag{5.8.35}$$

Keeping in mind that we are currently assuming that $f(0) = f(L)$, one may readily conclude from (5.8.34)–(5.8.35) that the sequence $\{f(x_j)\}_{j\in\mathbb{N}}$ converges to $f(x_*)$ in $\mathbb{C}$. This proves that the map $f\big|_{[0,L)} : \big([0, L), \tau_\rho\big) \to \mathbb{C}$ is continuous at x_*, thus establishing the right-to-left implication in (5.8.24). In the converse direction, assume that $f : [0, L] \to \mathbb{C}$ is a continuous function in the ordinary topology with the property that the map $f\big|_{[0,L)} : \big([0, L), \tau_\rho\big) \to \mathbb{C}$ is also continuous. For each $j \in \mathbb{N}$ consider $x_j := L - L/(2j) \in [0, L)$. Then since the sequence $\{x_j\}_{j\in\mathbb{N}}$ converges to 0 in τ_ρ and converges to L in the ordinary topology on the real line, it follows that we necessarily have $f(0) = f(L)$. This finishes the proof of (5.8.24).

As regards (5.8.25), note that for any $s_1, s_2 \in [0, L)$, the points $z(s_1), z(s_2)$ divide Σ into two arcs, call them Σ_1 and Σ_2, such that

$$\begin{array}{c}\text{both arcs, } \Sigma_1 \text{ and } \Sigma_2, \text{ have } z(s_1), z(s_2) \text{ as endpoints, and}\\ \text{the lengths of } \Sigma_1, \Sigma_2 \text{ are } |s_1 - s_2| \text{ and } L - |s_1 - s_2|, \text{ respectively.}\end{array} \tag{5.8.36}$$

From this and (5.8.43) and (5.8.12) we may then conclude that $|z(s_1) - z(s_2)|$ is no larger than both $|s_1 - s_2|$ and $L - |s_1 - s_2|$. Hence,

$$|z(s_1) - z(s_2)| \leq \min\big\{|s_1 - s_2|, L - |s_1 - s_2|\big\} = \rho(s_1, s_2), \tag{5.8.37}$$

proving that $z(\cdot) : \big([0, L), \tau_\rho\big) \to \Sigma$ is Lipschitz, with constant ≤ 1.

Going further, (5.8.26) follows from (5.8.15), (5.8.23), (5.8.24) (or (5.8.25)), and Lemma 5.8.5. Next, (5.8.27) is a consequence of definitions.

Finally, we shall prove (5.8.28) reasoning by contradiction. To this end, assume there exist $\varepsilon > 0$ along with $\{z_1^{(j)}\}_{j\in\mathbb{N}} \subseteq \Sigma$ and $\{z_2^{(j)}\}_{j\in\mathbb{N}} \subseteq \Sigma$ such that for each $j \in \mathbb{N}$ we have $|z_1^{(j)} - z_2^{(j)}| < 1/j$ and $\ell\big(z_1^{(j)}, z_2^{(j)}\big) \geq \varepsilon$. Since Σ is compact, we my invoke Bolzano–Weierstrass to conclude that, after eventually passing to subsequences, both $\{z_1^{(j)}\}_{j\in\mathbb{N}}$ and $\{z_2^{(j)}\}_{j\in\mathbb{N}}$ converge to some point $z_* \in \Sigma$. Then, on the one hand, (5.8.27) gives

$$\varepsilon \leq \ell\big(z_1^{(j)}, z_2^{(j)}\big) = \rho\big(z^{-1}(z_1^{(j)}), z^{-1}(z_2^{(j)})\big) \text{ for each } j \in \mathbb{N}. \tag{5.8.38}$$

On the other hand, (5.8.26) implies

$$\rho\big(z^{-1}(z_1^{(j)}), z^{-1}(z_*)\big) \to 0 \text{ and } \rho\big(z^{-1}(z_2^{(j)}), z^{-1}(z_*)\big) \to 0 \text{ as } j \to \infty. \tag{5.8.39}$$

Given that ρ is a metric, for each $j \in \mathbb{N}$ we may write

$$\rho\big(z^{-1}(z_1^{(j)}), z^{-1}(z_2^{(j)})\big) \leq \rho\big(z^{-1}(z_1^{(j)}), z^{-1}(z_*)\big) + \rho\big(z^{-1}(z_2^{(j)}), z^{-1}(z_*)\big) \tag{5.8.40}$$

which, in view of (5.8.39), entails

$$\lim_{j\to\infty} \rho\big(z^{-1}(z_1^{(j)}), z^{-1}(z_2^{(j)})\big) = 0. \tag{5.8.41}$$

The fact that this stands in contradiction with (5.8.38) proves (5.8.28). □

Further topologic and geometric measure theoretic properties of planar domains encompassed by rectifiable Jordan curves are discussed in the proposition below.

Proposition 5.8.6 *Let $\Sigma \subset \mathbb{C}$ be a rectifiable Jordan curve and denote by Ω the bounded connected component of $\mathbb{C} \setminus \Sigma$.*

(i) Then Ω is an open set of finite perimeter which satisfies

$$\partial\Omega = \partial\big(\overline{\Omega}\big) = \Sigma \quad \textit{and} \quad \mathcal{H}^1(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{5.8.42}$$

(ii) Let L be the length of the curve Σ and let $[0, L] \ni s \mapsto z(s) \in \Sigma$ be its arc-length parametrization. Then

$$L = \mathcal{H}^1(\Sigma). \tag{5.8.43}$$

Also, if ν denotes the geometric measure theoretic outward unit normal to Ω,

$$\nu\big(z(s)\big) = -\mathrm{i}z'(s) \ \ \textit{for } \mathcal{L}^1\textit{-a.e. } s \in [0, L] \tag{5.8.44}$$

and

$$\begin{aligned} &\textit{for } \mathcal{L}^1\textit{-a.e. } s \in [0, L] \textit{ the line } \{z(s) + t\, z'(s) : t \in \mathbb{R}\} \textit{ is} \\ &\textit{an approximate tangent line to } \Sigma \textit{ at the point } z(s). \end{aligned} \tag{5.8.45}$$

Proof For (5.8.43) see (5.8.11). As a consequence of the fact that the curve Σ is simple we have (5.8.15). Furthermore, the properties recorded in (5.8.16) are presently valid. In addition, (5.8.12) implies

$$|z(s_1) - z(s_2)| \leq |s_1 - s_2|, \quad \forall s_1, s_2 \in [0, L]. \tag{5.8.46}$$

Moving on, Jordan Curve Theorem (recalled earlier in (5.8.8)) implies that we presently have $\partial\Omega = \Sigma$ and $\overline{\Omega} = \Omega \cup \partial\Omega = \Omega_+ \cup \Sigma = \mathbb{C} \setminus \Omega_-$, hence

$$\partial\big(\overline{\Omega}\big) = \partial\big(\mathbb{C} \setminus \Omega_-\big) = \partial\Omega_- = \Sigma. \tag{5.8.47}$$

Also, from $\mathcal{H}^1(\partial\Omega) = \mathcal{H}^1(\Sigma) = L < +\infty$ (with the last equality provided by [161, Theorem 4.38, p. 135]) and (5.6.38) we conclude that Ω is a domain of finite perimeter.

Next, if $s_o \in (0, L)$ is a point of differentiability for the complex-valued function $z(\cdot)$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} &z(s_o + s) \in B\big(z(s_o) + s\, z'(s_o), \varepsilon|s|\big), \\ &\forall s \in (-\delta, \delta) \ \text{ with } \ s_o + s \in (0, L). \end{aligned} \tag{5.8.48}$$

In turn, from this geometric property and (5.6.93) we deduce (keeping in mind that the vectors $z'(s_o)$ and $\mathrm{i}z'(s_o)$ are perpendicular) that for each angle $\theta \in (0, \pi)$ there exists a height $h = h(\theta) > 0$ such that if $\Gamma^{\pm}_{\theta,h}$ denote the open truncated plane sectors with common vertex at $z(s_o)$, common aperture θ, common height h, and symmetry axis along the vectors $\pm\mathrm{i}z'(s_o)$, then

$$\Gamma^{+}_{\theta,h} \subseteq \Omega \ \text{ and } \ \Gamma^{-}_{\theta,h} \subseteq \mathbb{C} \setminus \overline{\Omega}. \tag{5.8.49}$$

Together, (5.8.49) and (5.2.1) then imply that

$$\mathcal{A} := \{z(s_o) : s_o \in A\} \subseteq \partial_*\Omega, \quad \text{where we have set} \\ A := \{s_o \in (0, L) : s_o \text{ differentiability point for } z(\cdot)\}. \tag{5.8.50}$$

From (5.8.50) and (5.8.17) we conclude that

$$\mathcal{H}^1(\partial\Omega \backslash \partial_*\Omega) \leq \mathcal{H}^1(\Sigma \setminus \mathcal{A}) = \mathcal{H}^1\big(z([0, L] \setminus A)\big) = \mathcal{L}^1\big([0, L] \setminus A\big) = 0, \tag{5.8.51}$$

which completes the proof of (5.8.42). As a consequence of (5.8.51) and (5.6.21) let us also observe that $\mathcal{A} \cap \partial^*\Omega$ has full $\mathcal{H}^1$-measure in $\partial\Omega$. On account of (5.8.18) this entails

$$\mathcal{L}^1\big([0, L] \setminus z^{-1}(\mathcal{A} \cap \partial^*\Omega)\big) = 0. \tag{5.8.52}$$

Next, pick an arbitrary point $z_o \in A$ and recall that (5.8.49) holds. From this and Lemma 5.6.15 it follows that if $\Gamma_{\pi-\theta}$ is the infinite open plane sector with vertex at 0, aperture $\pi - \theta$, and symmetry axis along the vector $-\mathrm{i}z'(s_o)$, then the geometric measure theoretic outward unit normal to Ω satisfies

$$\nu(z(s_o)) \in \overline{\Gamma_{\pi-\theta}} \tag{5.8.53}$$

provided $\nu(z(s_o))$ exists, i.e., if $z(s_o) \in \partial^*\Omega$. The fact that $\theta \in (0, \pi)$ may be chosen arbitrarily close to π then forces $\nu(z(s_o)) = -\mathrm{i}z'(s_o)$ whenever $z(s_o) \in \partial^*\Omega$, i.e., for $s_o \in z^{-1}(\mathcal{A} \cap \partial^*\Omega)$. Given that by (5.8.52) the latter set has full one-dimensional Lebesgue measure in $[0, L]$, the claim in (5.8.44) is established. Lastly, the claim in (5.8.45) is a consequence of (5.6.68), and (5.8.44). □

We continue to assume that $\Sigma \subset \mathbb{C}$ is a rectifiable Jordan curve. As in the past, denote by L the length of Σ and let $[0, L] \ni s \mapsto z(s) \in \Sigma$ an arc-length parametrization of Σ. Since the Jacobian of the map $[0, L] \ni s \mapsto z(s) \in \mathbb{C} \equiv \mathbb{R}^2$ is equal to $|z'(s)| = 1$ for $\mathcal{L}^1$-a.e. $s \in [0, L]$, the change of variable formula (cf. [80, Theorem 2, p. 99]) gives that

$$\int_\Sigma g(z)\,\mathrm{d}\sigma(z) = \int_0^L g(z(s))\,\mathrm{d}s, \qquad \forall g \in L^1(\Sigma, \sigma), \tag{5.8.54}$$

where σ, the `arc-length measure`, is defined as

$$\sigma := \mathcal{H}^1 \lfloor \Sigma. \tag{5.8.55}$$

Note that (5.8.54) also follows from (5.8.20) since, in light of (5.8.55), formula (5.8.21) entails

$$\sigma = z_*\big(\mathcal{L}^1 \lfloor [0, L]\big), \tag{5.8.56}$$

i.e., the arc-length measure is the push-forward of the one-dimensional Lebesgue measure (restricted to the interval $[0, L]$) via the arc-length parametrization.

In the same context as above, we agree to define the complex measure $\mathrm{d}z$ on Σ by setting

$$\mathrm{d}z := \mathrm{i}\nu\,\mathrm{d}\sigma \tag{5.8.57}$$

where the measure σ is as in (5.8.55), and ν is the geometric measure theoretic outward unit normal to the bounded connected component of $\mathbb{C}\setminus\Sigma$ (that ν is meaningfully defined is a consequence of Lemma 5.8.6).

Lemma 5.8.7 *Suppose $\Sigma\subset\mathbb{C}$ is a rectifiable Jordan curve, of length L and arc-length parametrization $[0,L]\ni s\mapsto z(s)\in\Sigma$. Then*

$$\int_\Sigma f(z)\,\mathrm{d}z = \int_0^L f(z(s))z'(s)\,\mathrm{d}s, \qquad \forall f\in L^1(\Sigma,\sigma). \tag{5.8.58}$$

Also, with Ω denoting the inner domain of Σ,

$$\begin{aligned}\int_\Omega \overline{\partial}F\,\mathrm{d}\mathcal{L}^2 &= \frac{1}{2\mathrm{i}}\int_{\partial\Omega} F(z)\,\mathrm{d}z\\ &= \frac{1}{2\mathrm{i}}\int_0^L F(z(s))z'(s)\,\mathrm{d}s, \qquad \forall F\in \mathrm{Lip}(\overline{\Omega},\mathbb{C}),\end{aligned} \tag{5.8.59}$$

where $\overline{\partial}=\frac{1}{2}\big(\partial_x+i\,\partial_y\big)$ is the Cauchy–Riemann operator.

Proof Given any function $f\in L^1(\Sigma,\sigma)$, we may write

$$\begin{aligned}\int_\Sigma f(z)\,\mathrm{d}z &= \mathrm{i}\int_\Sigma f(z)\nu(z)\,\mathrm{d}\sigma(z) = \mathrm{i}\int_0^L f(z(s))\nu(z(s))\,\mathrm{d}s\\ &= \int_0^L f(z(s))z'(s)\,\mathrm{d}s,\end{aligned} \tag{5.8.60}$$

by (5.8.57), (5.8.54), and (5.8.44). This proves (5.8.58). As regards (5.8.59), for any function $F\in\mathrm{Lip}(\overline{\Omega},\mathbb{C})$ we have

$$\begin{aligned}\int_\Omega \overline{\partial}F\,\mathrm{d}\mathcal{L}^2 &= \frac{1}{2}\int_\Omega(\partial_x+\mathrm{i}\partial_y)F\,\mathrm{d}\mathcal{L}^2 = \frac{1}{2}\int_{\partial_*\Omega}(\nu_1+\mathrm{i}\nu_2)F\,\mathrm{d}\mathcal{H}^1\\ &= \frac{1}{2}\int_{\partial\Omega}F\nu\,\mathrm{d}\sigma = \frac{1}{2\mathrm{i}}\int_{\partial\Omega}F(z)\,\mathrm{d}z.\end{aligned} \tag{5.8.61}$$

Above, the first equality is simply the definition of the Cauchy–Riemann operator. The second equality follows from the classical De Giorgi–Federer version of the Divergence Theorem, recalled in Theorem 1.1.1 (here we use the fact that Ω has finite perimeter; cf. Lemma 5.8.6). The third equality is implied by (5.8.42) and (5.8.55), while the fourth equality is a consequence of (5.8.57). This establishes the first equality in (5.8.59), and the second equality in (5.8.59) follows from this and (5.8.58). □

Remark 5.8.8 *It is clear that the first equality in* (5.8.59) *continues to hold in the case when* Ω *is an open set in the plane whose boundary consists of finitely many rectifiable Jordan curves. A further refinement of the first equality in* (5.8.59) *is contained in* [59].

5.9 Ahlfors Regular Sets

An important class of sets of locally finite perimeter consists of open subsets of $\mathbb{R}^n$ with boundaries satisfying an upper Ahlfors regularity condition. Together with other related notions, this is defined next.

Definition 5.9.1 *Let* $\Sigma \subseteq \mathbb{R}^n$ *be an arbitrary set.*

(i) *Call* Σ `lower Ahlfors regular` *provided there exists a constant* $c \in (0, \infty)$ *such that*

$$c\, r^{n-1} \leq \mathcal{H}_*^{n-1}\big(B(x,r)\cap\Sigma\big) \text{ for each } x \in \Sigma \text{ and } r \in \big(0, 2\,\mathrm{diam}(\Sigma)\big). \tag{5.9.1}$$

(ii) *Call* Σ `upper Ahlfors regular` *if there exists* $C \in (0, \infty)$ *with the property that*

$$\mathcal{H}_*^{n-1}\big(B(x,r)\cap\Sigma\big) \leq C r^{n-1} \text{ for each } x \in \Sigma \text{ and } r > 0. \tag{5.9.2}$$

(iii) *Finally, call* Σ *simply* `Ahlfors regular`[14] *if it is both lower and upper Ahlfors regular.*

The constants intervening above will be referred to as the (*lower/upper*) ADR *constants* (*or character*) *of the set* Σ.

The Ahlfors regularity condition described in item *(iii)* of the above definition has been first introduced by L. Ahlfors for planar curves, then subsequently considered by G. David for subsets of Euclidean spaces of arbitrary dimension. Roughly speaking, Ahlfors regular sets are environments which behave much like $(n-1)$-dimensional Euclidean spaces in terms of size and mass distribution, even though they can be quite different in other aspects of their respective geometries (e.g., they may lack nontrivial rectifiable curves, like the highly disconnected four-corner planar Cantor set, described a little later below). It should be pointed out that Ahlfors regularity is not a regularity property per se, but rather a scale-invariant way of expressing the fact that the set in question is $(n-1)$-dimensional in a uniform, scale-invariant fashion, involving the Hausdorff outer measure.

In this regard, let us note that Ahlfors regularity allows the set in question to develop singularities even when the constants c, C appearing in (5.9.1)–(5.9.2) coincide. For example, it has been observed by Kowalski and Preiss in [156] that, if υ_{n-1}

[14] Or Ahlfors–David regular, or ADR for short.

denotes the volume of the unit ball in $\mathbb{R}^{n-1}$, then

$$\Sigma_n := \big\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_4^2 = x_1^2 + x_2^2 + x_3^2\big\} \text{ with } n \geq 4 \tag{5.9.3}$$

has the property that[15]

$$\mathcal{H}^{n-1}\big(\Sigma_n \cap B(x,r)\big) = v_{n-1} r^{n-1} \text{ for all } x \in \Sigma_n \text{ and all } r \in (0,\infty). \tag{5.9.4}$$

In spite of (5.9.4), the Kowalski–Preiss four-dimensional (double) cone

$$\Sigma_4 := \big\{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4^2 = x_1^2 + x_2^2 + x_3^2\big\} \tag{5.9.5}$$

has a singularity at the origin.

Moving on, it is easy to see that

for a subset Σ of $\mathbb{R}^n$, being upper Ahlfors regular is equivalent to the demand that the inequality in (5.9.2) actually holds for every point $x \in \mathbb{R}^n$ (albeit with a possibly different constant). (5.9.6)

For future reference we wish to remark that

if $\Sigma \subseteq \mathbb{R}^n$ is upper Ahlfors regular then Σ is Lebesgue measurable, has an empty interior, and $\mathcal{L}^n(\Sigma) = 0$. (5.9.7)

Indeed, this is a consequence of (5.9.2), keeping in mind that any null-set for the Hausdorff outer-measure is Hausdorff measurable (cf. [80, Remarks, p. 2]). Trivially,

any subset of an upper Ahlfors regular set is itself an upper Ahlfors regular set, (5.9.8)

and, by induction,

finite unions of lower Ahlfors regular sets are themselves lower Ahlfors regular. (5.9.9)

Also, from (5.9.6) it is clear that

a finite union of upper Ahlfors regular sets is itself an upper Ahlfors regular set. (5.9.10)

[15] In fact, [156, Main Theorem (Measure-theoretic version), p. 116] asserts that Σ as in (5.9.3) along with $(n-1)$-dimensional planes in $\mathbb{R}^n$ (in all dimensions) are, up to a rigid transformation of $\mathbb{R}^n$, the only Ahlfors regular sets satisfying (5.9.1)–(5.9.2) with the same constant (i.e., the "Global Besicovitch Property" in the terminology of [156, p. 115]). Parenthetically, we wish to remark that "tight" Ahlfors regularity estimates like (5.9.4) are true for non-flat smooth surfaces such as spheres in the three-dimensional Euclidean space. Specifically, Archimedes' formula for the area of a spherical cap in the tree-dimensional setting gives that $\mathcal{H}^2\big(S^2 \cap B(x,r)\big) = \pi r^2$ for all $x \in S^2$ and all $r \in (0,1)$.

In particular, from (5.9.9) and (5.9.10) we conclude that

$$\text{finite unions of Ahlfors regular sets are themselves Ahlfors regular sets.} \tag{5.9.11}$$

Since for each two integers $n, m \in \mathbb{N}$, each set $E \subseteq \mathbb{R}^n$, each point $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, and each number $r \in (0, \infty)$ we have $\partial\big(E \times \mathbb{R}^m\big) = (\partial E) \times \mathbb{R}^m$ and, using a self-explanatory piece of notation,

$$\begin{aligned}\big(B_n\big(x, r/\sqrt{2}\big) \cap E\big) \times B_m\big(y, r/\sqrt{2}\big) &\subseteq B_{n+m}\big((x, y), r\big) \cap \big(E \times \mathbb{R}^m\big) \\ &\subseteq \big(B_n(x, r) \cap E\big) \times B_m(y, r),\end{aligned} \tag{5.9.12}$$

we conclude that

$$\text{if } E \text{ is an Ahlfors regular set in } \mathbb{R}^n \text{ then } E \times \mathbb{R}^m \text{ happens to be an Ahlfors regular set in } \mathbb{R}^{n+m}. \tag{5.9.13}$$

Let us also note that

$$\text{assuming } \Sigma \subseteq \mathbb{R}^n \text{ is } \mathcal{H}^{n-1}\text{-measurable and Ahlfors regular, then } \mathcal{H}^{n-1}\lfloor\Sigma \text{ is a complete Radon, doubling measure on } \Sigma. \tag{5.9.14}$$

In addition, from (5.6.35) it is apparent that for each given Lebesgue measurable subset Ω of $\mathbb{R}^n$ the following implications hold:

$$\begin{aligned}\partial\Omega \text{ is upper Ahlfors regular} \Longrightarrow &\text{ the measure } \sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega \text{ is locally finite} \\ &\Longrightarrow \Omega \text{ is a set of locally finite perimeter,}\end{aligned} \tag{5.9.15}$$

and

$$\begin{aligned}\partial_*\Omega \text{ is upper Ahlfors regular} \Longrightarrow &\text{ the measure } \sigma_* := \mathcal{H}^{n-1}\lfloor\partial_*\Omega \text{ is locally finite} \\ &\Longrightarrow \Omega \text{ is a set of locally finite perimeter.}\end{aligned} \tag{5.9.16}$$

It is also clear from (5.9.15), (5.6.35), and (5.7.5) that

$$\text{any Lebesgue measurable set } E \subseteq \mathbb{R}^n \text{ with a compact upper Ahlfors regular boundary has finite perimeter.} \tag{5.9.17}$$

Also,

$$\text{whenever } \Sigma \subseteq \mathbb{R}^n \text{ is lower Ahlfors regular and } A \subseteq \Sigma \text{ is a set satisfying } \mathcal{H}^{n-1}_*(\Sigma \setminus A) = 0 \text{ then } A \text{ is dense in } \Sigma \text{ (equipped with the topology inherited from } \mathbb{R}^n). \tag{5.9.18}$$

Lastly, we wish to note that, as we shall show later on, in Proposition 8.6.12, for open subsets of $\mathbb{R}^n$ with Ahlfors regular boundaries, being n-thick is actually equivalent to the interior corkscrew property. (5.9.19)

The following criterion for upper Ahlfors regularity has been proved in [183].

Proposition 5.9.2 *Assume that $E \subseteq \mathbb{R}^n$ is a set satisfying $\overline{(E^\circ)} = \overline{E}$ as well as a uniform exterior ball condition, in the sense that there exists some $r \in (0, \infty)$ with the property that*

$$\forall x \in \partial E \quad \exists\, \omega \in S^{n-1} \quad \textit{such that} \quad B(x + r\omega, r) \subseteq \mathbb{R}^n \setminus E. \tag{5.9.20}$$

Then ∂E is upper Ahlfors regular. In particular, if E is also $\mathcal{L}^n$-measurable, then E has locally finite perimeter.

Graphs of BMO_1 functions defined in $\mathbb{R}^{n-1}$ are Ahlfors regular sets in $\mathbb{R}^n$. Specifically, if $\phi \in \mathrm{BMO}_1(\mathbb{R}^{n-1})$, i.e.,

$$\begin{aligned} &\phi : \mathbb{R}^{n-1} \to \mathbb{R} \text{ is locally integrable, } \nabla'\phi \in \big[L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1})\big]^{n-1}, \text{ and} \\ &\|\nabla'\phi\|_* := \sup_{B \text{ ball in } \mathbb{R}^{n-1}} ⨍_B \Big|(\nabla'\phi)(x') - \Big(⨍_B (\nabla'\phi)(y')\,\mathrm{d}y'\Big)\Big|\,\mathrm{d}x' < +\infty, \end{aligned} \tag{5.9.21}$$

then ϕ is continuous (thanks to John–Nirenberg inequality and Sobolev embeddings) and its graph,

$$\begin{aligned} &\Sigma := \big\{\big(x', \phi(x')\big) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}\big\} \\ &\text{is a closed Ahlfors regular set in } \mathbb{R}^n, \end{aligned} \tag{5.9.22}$$

with constants depending only on n and $\|\nabla'\phi\|_*$. See [125, Corollary 2.26, p. 2622].

Examples of Ahlfors regular sets also include certain types of fractals, such as the four-corner planar Cantor set. To describe the latter, let $E_0 := [0, 1]^2$ be the unit square in $\mathbb{R}^2$, and consider the four (closed) squares $\{Q_1^j\}_{1\leq j\leq 4}$, of side-length 4^{-1} located in the corners of E_0. Let $E_1 := \bigcup_{j=1}^4 Q_1^j$. Iteratively, for each $m \in \mathbb{N}$, consider the m-th generation of squares defined as the collection of 4^m squares $\{Q_m^j\}_{1\leq j\leq 4^m}$, of side-length $\ell(Q_m^j) = 4^{-m}$, which are located in the corners of E_{m-1} (i.e., each Q_m^j, with $j \in \{1, \dots, 4^m\}$, is located in one of the corners of a square Q_{m-1}^k, for some $k \in \{1, \dots, 4^{m-1}\}$) and define $E_m := \bigcup_{j=1}^{4^m} Q_m^j$. The `four-corner Cantor set` in $\mathbb{R}^2$ is then given by (Fig. 5.3)

$$E := \bigcap_{m=0}^{\infty} E_m. \tag{5.9.23}$$

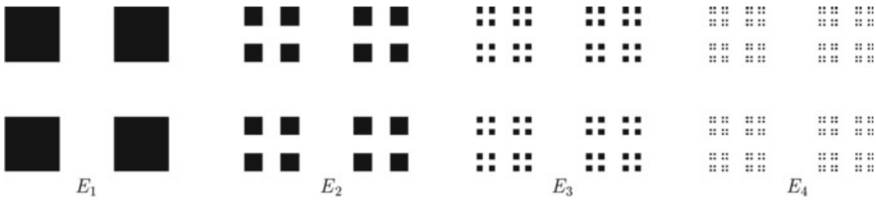

Fig. 5.3 The first four iterations in the construction of the four-corner planar Cantor set

It is then known (cf. [188, Proposition 4.79, p. 238] and [214, p. 7]) that

the four-corner planar Cantor set $E \subseteq \mathbb{R}^2$ from (5.9.23) is a compact Ahlfors regular set, satisfying $\mathcal{H}^1(E) = \sqrt{2}$, and has the property that the Euclidean distance restricted to E is equivalent (in the sense of quasi-distances) to an ultrametric. (5.9.24)

Given an Ahlfors regular set $\Sigma \subseteq \mathbb{R}^2 \equiv \mathbb{C}$, for each $\varepsilon > 0$ define the truncated "altered" Cauchy operator[16] acting on any function

$$f \in L^1\Big(\Sigma, \frac{\mathcal{H}^1(\zeta)}{1+|\zeta|}\Big) \tag{5.9.25}$$

according to

$$C_\varepsilon^{\rm alt} f(z) := \int\limits_{\substack{\zeta\in\Sigma \\ |z-\zeta|>\varepsilon}} \frac{f(\zeta)}{\zeta - z}\, d\mathcal{H}^1(\zeta) \quad \text{for all } \ z \in \Sigma, \tag{5.9.26}$$

and define the maximal "altered" Cauchy operator acting on any function f as in (5.9.25) by

$$C_{\max}^{\rm alt} f(z) := \sup_{\varepsilon>0} \big|C_\varepsilon^{\rm alt} f(z)\big| \quad \text{for all } \ z \in \Sigma. \tag{5.9.27}$$

From [98] (cf. also [68, p. 8], [65, 99, 137, 175]) we know that

if E denotes the four-corner Cantor set in $\mathbb{R}^2 \equiv \mathbb{C}$ (cf. (5.9.23)) then the truncated "altered" Cauchy operator defined as in (5.9.25)-(5.9.26) with $\Sigma := E$ fails to be bounded on $L^2\big(E, \mathcal{H}^1\lfloor E\big)$ with operator norm controlled uniformly in $\varepsilon \in (0, \infty)$. (5.9.28)

This should be contrasted with a celebrated result of G. David (originally proved in [64]; cf. also [209, Theorem 3.2, p. 285]) according to which

[16] Ordinarily, the Cauchy integral operator is considered with respect to the complex arclength $d\zeta$. Defining the latter requires making certain assumptions on the underlying set. For example, if the Cauchy operator is considered on the boundary of a set of locally finite perimeter $\Omega \subset \mathbb{R}^2 \equiv \mathbb{C}$, then we may define $d\zeta$ as $-2i\bar{\partial}\mathbf{1}_\Omega$, or $i\nu(\zeta)\,d\sigma(\zeta)$. If the underlying set is merely upper Ahlfors regular, a convenient substitute for $d\zeta$ is $\mathcal{H}^1$. In such a scenario, since we no longer deal with the classical Cauchy operator, we shall call the resulting object the "altered" Cauchy operator.

given a rectifiable curve $\Sigma \subseteq \mathbb{R}^2 \equiv \mathbb{C}$, the maximal "altered" Cauchy operator $C^{\text{alt}}_{\max}$ (cf. (5.9.27)) is bounded on $L^2\big(\Sigma, \mathcal{H}^1 \lfloor \Sigma\big)$ if and only if Σ is upper Ahlfors regular. (5.9.29)

In particular, (5.9.29) points to the prominence of connectivity in relation to the boundedness of the singular Cauchy integral operator. Let us also note that, as shown in [176],

given an Ahlfors regular set $\Sigma \subseteq \mathbb{R}^2 \equiv \mathbb{C}$, the maximal "altered" Cauchy operator $C^{\text{alt}}_{\max}$ (cf. (5.9.27)) is bounded on $L^2\big(\Sigma, \mathcal{H}^1 \lfloor \Sigma\big)$ if and only if Σ is contained in a rectifiable upper Ahlfors regular curve. (5.9.30)

Moving on, we observe from (5.8.11) and item *(ii)* in Definition 5.9.1 that

any compact upper Ahlfors regular curve $\Sigma \subset \mathbb{C}$ is rectifiable. (5.9.31)

In this vein, it is also of interest to establish the following result.

Lemma 5.9.3 *Any rectifiable curve $\Sigma \subset \mathbb{C}$ which does not reduce to a point is lower Ahlfors regular. More precisely, for any rectifiable curve $\Sigma \subset \mathbb{C}$ one has*

$$r \leq \mathcal{H}^1\big(B(z,r) \cap \Sigma\big) \;\; \textit{for all} \;\; z \in \Sigma \;\; \textit{and} \;\; r \in \big(0, \operatorname{diam} \Sigma\big). \tag{5.9.32}$$

In particular, as a consequence of this and (5.9.31),

any compact upper Ahlfors regular curve in $\mathbb{C}$ is rectifiable, hence also lower Ahlfors regular if it does not reduce to a point. (5.9.33)

Proof Pick some location $z \in \Sigma$ and some scale $r \in \big(0, \operatorname{diam} \Sigma\big)$. Then since Σ is connected it follows that $\partial B(z,r) \cap \Sigma \neq \varnothing$. Hence, there exists $z_0 \in \partial B(z,r) \cap \Sigma$. Taking z_0 to be the first exit point of the curve Σ out of the ball $B(z,r)$, there is no loss of generality in assuming that Σ_{z,z_0}, the sub-arc of Σ emerging from z and ending in z_0, is contained in $B(z,r)$. Granted this, we may then rely on (5.8.12) to estimate

$$\mathcal{H}^1\big(B(z,r) \cap \Sigma\big) \geq \mathcal{H}^1\big(\Sigma_{z,z_0}\big) \geq |z - z_0| = r, \tag{5.9.34}$$

as claimed in (5.9.32). □

Combining (5.9.32) with (5.3.72) leads to the conclusion that

any compact connected set $E \subseteq \mathbb{R}^2$ which is not a singleton and satisfies $\mathcal{H}^1(E) < \infty$ is necessarily lower Ahlfors regular. (5.9.35)

Below we give an example of a compact simple curve which is rectifiable and does not reduce to a point, hence is lower Ahlfors regular, but which is not upper Ahlfors regular.

Proposition 5.9.4 *Fix* $\alpha \in (1, 2)$ *and consider the function* $f : [0, 1] \to \mathbb{R}$ *given at each* $x \in [0, 1]$ *by*

$$f(x) := \begin{cases} x^\alpha \cdot \cos(1/x) & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases} \tag{5.9.36}$$

Define $\Sigma \subseteq \mathbb{R}^2$ *to be the graph of* f*, i.e.,*

$$\Sigma := \big\{(x, f(x)) : x \in [0, 1]\big\}. \tag{5.9.37}$$

Then Σ *is a compact simple curve which is rectifiable, lower Ahlfors regular, and has* $\mathcal{H}^1(\Sigma) < \infty$*, but is not upper Ahlfors regular.*

Proof An inspection of the definition of Σ reveals that this is indeed a compact simple curve which does not reduce to a point. Clearly, $0 \in \Sigma$. To estimate the length of Σ, first observe that

$$f'(x) = \alpha \cdot x^{\alpha-1} \cdot \cos(1/x) + x^{\alpha-2} \cdot \sin(1/x) \quad \text{for each } x \in (0, 1). \tag{5.9.38}$$

In particular,

$$|f'(x)| \leq (\alpha + 1)x^{\alpha-2} \quad \text{for each } x \in (0, 1), \tag{5.9.39}$$

so

$$\mathcal{H}^1(\Sigma) = \int_0^1 \sqrt{1 + |f'(x)|^2}\, dx \leq 1 + (\alpha + 1) \int_0^1 x^{\alpha-2}\, dx = \frac{2\alpha}{\alpha - 1}. \tag{5.9.40}$$

Thus, Σ has finite length, hence Σ is rectifiable. Granted this, Lemma 5.9.3 ensures that Σ is a lower Ahlfors regular set.

To prove that Σ fails to be upper Ahlfors regular it suffices to show that

$$\lim_{r \to 0^+} \frac{\mathcal{H}^1\big(\Sigma \cap B(0, r)\big)}{r} = +\infty. \tag{5.9.41}$$

With this goal in mind, fix an arbitrary $r \in (0, 1)$ and note that for each $x \in \big(0, r/\sqrt{2}\big)$ we have

$$\big|f(x)\big| \leq x^\alpha \leq x < r/\sqrt{2}, \quad \text{hence} \quad \sqrt{x^2 + \big(f(x)\big)^2} < r. \tag{5.9.42}$$

This shows that

$$\big\{(x, f(x)) : 0 < x < r/\sqrt{2}\big\} \subseteq \Sigma \cap B(0, r) \quad \text{for each } r \in (0, 1). \tag{5.9.43}$$

Consequently, for each $r \in (0, 1)$ we have

$$\frac{\mathcal{H}^1\big(\Sigma \cap B(0, r)\big)}{r} \geq \frac{\mathcal{H}^1\Big(\big\{(x, f(x)) : 0 < x < r/\sqrt{2}\big\}\Big)}{r}$$
$$= \frac{1}{r}\int_0^{r/\sqrt{2}} \sqrt{1 + |f'(x)|^2}\, \mathrm{d}x \geq \frac{1}{r}\int_0^r |f'(x)|\, \mathrm{d}x, \tag{5.9.44}$$

so the claim made in (5.9.41) follows as soon as we show that

$$\lim_{r \to 0^+} \frac{1}{r}\int_0^r |f'(x)|\, \mathrm{d}x = +\infty. \tag{5.9.45}$$

To justify this, start by estimating

$$|f'(x)| \geq x^{\alpha-2} \cdot |\sin(1/x)| - \alpha \cdot x^{\alpha-1} \cdot |\cos(1/x)|$$
$$\geq x^{\alpha-2} \cdot |\sin(1/x)| - \alpha \cdot x^{\alpha-1} \quad \text{for each} \quad x \in (0, 1), \tag{5.9.46}$$

and note that since $\alpha > 1$ we have

$$\lim_{r \to 0^+} \frac{1}{r}\int_0^r \alpha \cdot x^{\alpha-1}\, \mathrm{d}x = \lim_{r \to 0^+} r^{\alpha-1} = 0. \tag{5.9.47}$$

In view of the current goal, it therefore suffices to show that

$$\lim_{r \to 0^+} \frac{1}{r}\int_0^r x^{\alpha-2} \cdot |\sin(1/x)|\, \mathrm{d}x = +\infty. \tag{5.9.48}$$

To this end, use $R := 1/r$ and make the change of variable $y := 1/x$ to write

$$\lim_{r \to 0^+} \frac{1}{r}\int_0^r x^{\alpha-2} \cdot |\sin(1/x)|\, \mathrm{d}x = \lim_{R \to +\infty} \Big(R\int_R^\infty \frac{|\sin y|}{y^\alpha}\, \mathrm{d}y\Big). \tag{5.9.49}$$

For each $R \in (10, \infty)$, consider the unique $n \in \mathbb{N}$ such that $\pi n \leq R < \pi(n + 1)$. This guarantees that

$$n - 1 > (R/\pi) - 2 \quad \text{and} \quad I_R := \big[\pi(n + 1), \pi(2n)\big] \subseteq [R, 2R]. \tag{5.9.50}$$

If we then define

$$J_R := \bigcup_{j=1}^{n-1} \big[\pi(n + j) + \pi/6, \pi(n + j + 1) - \pi/6\big] \tag{5.9.51}$$

it follows that $J_R \subseteq I_R \subseteq [R, 2R]$ and the length of J_R is $(2/3)(n-1)\pi$. In view of (5.9.50), the latter implies that the length of J_R is $\geq (2/3)(R - 2\pi)$. Since we also have $|\sin y| \geq 1/2$ and $1/y^\alpha \geq 1/(2R)^\alpha$ for each $y \in J_R \subseteq [R, 2R]$, we may estimate

$$\begin{aligned}\lim_{R\to+\infty}\Big(R\int_R^\infty \frac{|\sin y|}{y^\alpha}\,dy\Big) &\geq \lim_{R\to+\infty}\Big(R\int_{J_R} \frac{|\sin y|}{y^\alpha}\,dy\Big)\\ &\geq \lim_{R\to+\infty}\Big(R\cdot\frac{1}{2}\cdot\frac{1}{(2R)^\alpha}\cdot\frac{2(R-2\pi)}{3}\Big)\\ &= +\infty, \end{aligned} \tag{5.9.52}$$

given that $\alpha < 2$. Together with (5.9.49), this finally shows that (5.9.48) holds, thus finishing the proof of Proposition 5.9.4. □

To give examples of Ahlfors regular sets of a different nature, we make one more definition. Specifically, make the following definition.

Definition 5.9.5 *Call a set $\Sigma \subseteq \mathbb{R}^2$ a* `chord-arc curve` *provided Σ is a simple locally rectifiable closed curve*[17] *satisfying*

$$\sup_{\substack{z_1\neq z_2\\ z_1,z_2\in\Sigma}} \frac{\ell(z_1,z_2)}{|z_1-z_2|} < +\infty, \tag{5.9.53}$$

where $\ell(z_1, z_2)$ is the length of the shorter sub-arc of Σ joining z_1 with z_2.

For example, the boundary of an infinite sector in the plane of (full) aperture $\theta \in (0, \pi)$ is a chord-arc curve with the corresponding supremum in (5.9.53) equal to $\big(\sin(\theta/2)\big)^{-1}$. Curves satisfying (5.9.53) have been introduced in 1936 by M.A. Lavrentiev in [160], and are sometimes called Lavrentiev curves. It is well known (cf., e.g., [218, Proposition 7.7, p. 163]) that

any simple locally rectifiable closed curve $\Sigma \subseteq \mathbb{R}^2$ satisfying (5.9.53) (hence any chord-arc curve) is an Ahlfors regular set. (5.9.54)

The converse of the statement in (5.9.54) is not true, as may be seen by considering curves with a cusp, such as a rectifiable closed curve $\Sigma \subset \mathbb{R}^2$ which passes through the origin and such that $\Sigma \cap (-1, 1)^2 = \big\{(x, \sqrt{|x|}) : x \in (-1, 1)\big\}$.

Any simple locally rectifiable closed curve in $\mathbb{R}^2$ which contains Σ defined in (5.9.37) as a sub-arc fails to be a chord-arc curve. Indeed, since Σ is not upper Ahlfors regular (cf. Proposition 5.9.4), this is implied by (5.9.54). A direct proof is as follows: Abbreviate $z_0 := (0, 0) \in \Sigma$, and for each $r \in (0, 1)$ consider the point $z_r := \big(r, f(r)\big) \in \Sigma$. The failure to be chord-arc becomes evident once we show that

[17] Possibly passing through infinity.

$$\lim_{r\to 0^+} \frac{\ell(z_0, z_r)}{|z_0 - z_r|} = +\infty, \tag{5.9.55}$$

where $\ell(z_0, z_r)$ is the length of the shorter subarc of Σ joining z_0 with z_r. To prove (5.9.55), fix an arbitrary $r \in (0, 1)$ and note that

$$\big|f(r)\big| \le r^\alpha \le r \ \text{ hence } \ |z_0 - z_r| = \sqrt{r^2 + \big(f(r)\big)^2} \le \sqrt{2}r. \tag{5.9.56}$$

Since we also have

$$\ell(z_0, z_r) = \int_0^r \sqrt{1 + |f'(x)|^2}\,\mathrm{d}x \ge \int_0^r |f'(x)|\,\mathrm{d}x, \tag{5.9.57}$$

the claim made in (5.9.55) follows from (5.9.45).

Concrete examples of chord-arc curves may be constructed from graphs of real-valued BMO_1 functions defined on the real line. More specifically, we have the following result:

Proposition 5.9.6 *Assume $f : \mathbb{R} \to \mathbb{R}$ is a locally integrable function with the property that f', its distributional derivative on the real line, belongs to the space $\mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$. Denote the graph of f by Σ, i.e., $\Sigma := \big\{(x, f(x)) : x \in \mathbb{R}\big\} \subseteq \mathbb{R}^2$. Then Σ is a locally rectifiable curve and, for any $x \in \mathbb{R}$ and $r > 0$, the length of the sub-arc of Σ with endpoints $\big(x \pm r, f(x \pm r)\big)$ is bounded by*

$$\left\{1 + \sup_{\substack{\Delta\subset\mathbb{R}\\ |\Delta|\le 2r}} ⨍_\Delta |f(y) - f_\Delta|\,\mathrm{d}y\right\} \Big|\big(x - r, f(x - r)\big) - \big(x + r, f(x + r)\big)\Big|, \tag{5.9.58}$$

where the supremum is taken over all sub-intervals Δ of $\mathbb{R}$ with length $|\Delta| \le 2r$, and where f_Δ abbreviates $⨍_\Delta f(y)\,\mathrm{d}y$. As a consequence, for any two points $z_1, z_2 \in \Sigma$,

$$\begin{aligned} &\textit{the sub-arc of } \Sigma \textit{ with endpoints } z_1 \textit{ and } z_2 \\ &\textit{has length } \le \big(1 + \|f'\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)}\big)|z_1 - z_2|. \end{aligned} \tag{5.9.59}$$

Proof Recall from [125, (2.2.34) on p. 2580] that every function $f \in \mathscr{C}^0(\mathbb{R})$ with $f' \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{L}^1)$ has the property that

$$\mathcal{H}^1\big(\{(x, f(x)) : x \in O\}\big) = \int_O \sqrt{1 + |f'(x)|^2}\,\mathrm{d}x, \qquad \forall O \subseteq \mathbb{R} \ \text{ Borel set.} \tag{5.9.60}$$

Granted this, all claims follow as soon as we show that for each $x \in \mathbb{R}$ and $r > 0$ we have

$$\int_{x-r}^{x+r} \sqrt{1 + |f'(y)|^2}\,\mathrm{d}y \le \big\{1 + M(f; r)\big\}\Big|\big(x - r, f(x - r)\big) - \big(x + r, f(x + r)\big)\Big|, \tag{5.9.61}$$

where we have set

$$M(f;r) := \sup_{\substack{\Delta\subset\mathbb{R}\\ |\Delta|\leq 2r}} \fint_\Delta |f(y) - f_\Delta|\,\mathrm{d}y. \tag{5.9.62}$$

With this goal in mind, fix $x \in \mathbb{R}$ along with $r > 0$, and introduce

$$m(x,r) := \fint_{x-r}^{x+r} f'(y)\,\mathrm{d}y. \tag{5.9.63}$$

Given that we are assuming $f' \in \mathrm{BMO}(\mathbb{R}, \mathcal{L}^1) \subset L^1_{\mathrm{loc}}(\mathbb{R}, \mathcal{L}^1)$, it follows that the function f belongs to the local Sobolev space $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$. In turn, this implies that[18] the function f is locally absolutely continuous (cf., e.g., [161, Corollary 7.14, p. 223]). As a consequence, the fundamental theorem of calculus holds[19] and we may re-express the integral average $m(x,r)$ as

$$m(x,r) = \frac{1}{2r}\int_{x-r}^{x+r} f'(y)\,\mathrm{d}y = \frac{1}{2r}\big(f(x+r) - f(x-r)\big). \tag{5.9.64}$$

Then, since the function $F(t) := \sqrt{1+t^2}$, $t \in \mathbb{R}$, is non-negative and Lipschitz with constant ≤ 1 (given that $|F'(t)| = |t|/\sqrt{1+t^2} \leq 1$ for each t), we may estimate

$$\begin{aligned}
\int_{x-r}^{x+r} \sqrt{1+|f'(y)|^2}\,\mathrm{d}y &= \int_{x-r}^{x+r} F(f'(y))\,\mathrm{d}y \qquad (5.9.65)\\
&\leq \int_{x-r}^{x+r} \big|F(f'(y)) - F(m(x,r))\big|\,\mathrm{d}y + \int_{x-r}^{x+r} F(m(x,r))\,\mathrm{d}y\\
&\leq \int_{x-r}^{x+r} \big|f'(y) - m(x,r)\big|\,\mathrm{d}y + 2r\,F(m(x,r))\\
&\leq 2r\,M(f;r) + 2r\sqrt{1 + \Big(\frac{f(x+r)-f(x-r)}{2r}\Big)^2}\\
&\leq \big\{1 + M(f;r)\big\}\sqrt{(2r)^2 + \big(f(x+r) - f(x-r)\big)^2},
\end{aligned}$$

from which (5.9.61) follows. Since (5.9.59) is an obvious consequence of (5.9.59), the proof is complete. □

Heuristically, chord-arc curves are not much worse than Lipschitz curves. Indeed, by [218, Theorem 7.9, p. 165], we have[20]

[18] After being redefined on a set of measure zero.

[19] Cf., e.g., [161, Theorem 3.30, p. 85].

[20] A version for infinite chord-arc curves, described as bi-Lipschitzian images of straight lines, may be found in [132, Proposition 1.13, p. 227]; see also [47, (ii), p. 104].

$$\left.\begin{array}{r}\text{a compact curve } \Sigma \subset \mathbb{C} \text{ is} \\ \text{a chord-arc curve}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\text{there is a bi-Lipschitz map } F \\ \text{of } \mathbb{C} \text{ onto } \mathbb{C} \text{ such that } \Sigma = F(S^1).\end{array}\right. \tag{5.9.66}$$

Hence, a compact curve $\Sigma \subseteq \mathbb{R}^2$ is chord-arc if and only if Σ is the image of the unit circle $\mathbb{T}$ under a bi-Lipschitz homeomorphism F of $\mathbb{C}$ onto $\mathbb{C}$.

Another useful characterization of the class of compact chord-arc curves is as follows.

Lemma 5.9.7 *Assume that $\Sigma \subset \mathbb{C}$ is a compact rectifiable Jordan curve. Let L be its length and denote by $[0, L] \ni s \mapsto z(s) \in \Sigma$ its arc-length parametrization. Then Σ is a chord-arc curve if and only if the mapping (5.8.25) is bi-Lipschitz, i.e., if as a mapping between metric spaces, its inverse*

$$\eta = z^{-1}(\cdot) : \big(\Sigma, |\cdot - \cdot|\big) \longrightarrow \big([0, L), d\big) \text{ is Lipschitz (hence bi-Lipschitz)}, \tag{5.9.67}$$

where $|\cdot - \cdot|$ is the standard Euclidean distance, and the metric d is as in (5.8.22).

Proof Any two distinct points $z_1, z_2 \in \Sigma$ divide Σ into two arcs, having z_1, z_2 as endpoints, and whose lengths are $\ell(z_1, z_2)$ and $L - \ell(z_1, z_2)$. A moment's reflection then shows that Σ satisfies the chord-arc condition (5.9.53) if and only if there exists $C \in (0, \infty)$ with the property that

$$\min\big\{\ell(z_1, z_2), L - \ell(z_1, z_2)\big\} \leq C|z_1 - z_2|, \qquad \forall z_1, z_2 \in \Sigma. \tag{5.9.68}$$

In turn, the validity of condition (5.9.68) for some $C \in (0, \infty)$ is readily seen to be equivalent to the demand that $\eta : \big(\Sigma, |\cdot - \cdot|\big) \to \big([0, L), d\big)$ is a Lipschitz map. □

By definition, any compact chord-arc curve $\Sigma \subseteq \mathbb{R}^2$ is a Jordan curve. In particular, Jordan's theorem implies that Σ partitions the plane into two connected components, namely a bounded simply connected open set, called the `inner domain` of Σ, and an unbounded connected open set, called the `outer domain` of Σ.

Definition 5.9.8 *Call an open set $\Omega \subseteq \mathbb{R}^2$ with compact boundary a* `chord-arc domain` *provided Ω is either the inner domain or the outer domain of a compact chord-arc curve (cf. Definition 5.9.5).*

It is well known (cf. [132, p. 92]) that

$$\begin{array}{l}\text{any bounded chord-arc domain in } \mathbb{R}^2 \text{ is a quasi-disk} \\ \text{(hence also an NTA domain; cf. Definition 5.11.1).}\end{array} \tag{5.9.69}$$

In fact, it turns out that bounded chord-arc domains are precisely bi-Lipschitz distortions of the unit disk in the plane. Specifically, we have the following result.

Proposition 5.9.9 *A bounded open set $\Omega \subseteq \mathbb{C}$ is a chord-arc domain if and only if there exists a bi-Lipschitz map F of $\mathbb{C}$ onto $\mathbb{C}$ such that $\Omega = F\big(B(0, 1)\big)$.*

Proof In one direction, if F is a bi-Lipschitz map of $\mathbb{C}$ onto $\mathbb{C}$ and $\Omega := F\big(B(0,1)\big)$, then Ω is a bounded open set whose boundary is $F(S^1)$. Since by (5.9.66) the latter is a chord-arc curve, we deduce that Ω is a chord-arc domain.

In the converse direction, suppose that $\Omega \subseteq \mathbb{C}$ is a bounded chord-arc domain. In particular, $\Sigma := \partial\Omega$ is a chord-arc curve, and this makes Ω simply connected. Let $[0, L] \ni s \mapsto z(s) \in \Sigma$ be an arc-length parametrization of Σ, where L is its length. To proceed, let

$$[0, 2\pi] \ni \theta \mapsto e^{i\theta} \in S^1 \tag{5.9.70}$$

be the standard arc-length parametrization of the unit circle. Then, as a particular case of Lemma 5.9.7, its inverse is a bijective bi-Lipschitz map in the context of metric spaces

$$\big(S^1, |\cdot - \cdot|\big) \ni e^{i\theta} \mapsto \theta \in \Big([0, 2\pi), \delta(\cdot,\cdot)\Big), \tag{5.9.71}$$

where $|\cdot - \cdot|$ is the standard Euclidean distance, and the distance $\delta(\cdot,\cdot)$ on the set $[0, 2\pi)$ is defined by

$$\delta(\theta_1, \theta_2) := \min\big\{|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|\big\}, \qquad \forall\theta_1, \theta_2 \in [0, 2\pi). \tag{5.9.72}$$

Recall the distance d on $[0, L)$ defined in (5.8.22) and define the dilation map

$$\Big([0, 2\pi), \delta(\cdot,\cdot)\Big) \ni \theta \mapsto \theta \cdot \frac{L}{2\pi} \in \big([0, L), \mathrm{d}\big), \tag{5.9.73}$$

which is a bijection with the property that $d\big(\theta_1 \cdot \frac{L}{2\pi}, \theta_2 \cdot \frac{L}{2\pi}\big) = \frac{L}{2\pi} \cdot \delta(\theta_1, \theta_2)$, for each $\theta_1, \theta_2 \in [0, 2\pi)$. Given that we are currently assuming that Σ is a chord-arc curve, Lemma 5.9.7 also ensures that

$$\big([0, L), \mathrm{d}\big) \ni s \mapsto z(s) \in \big(\Sigma, |\cdot - \cdot|\big) \tag{5.9.74}$$

is a bijective bi-Lipschitz map. Consider now the map defined as the compositions of the bijective bi-Lipschitz maps from (5.9.71), (5.9.73), (5.9.74), as well as the isometric embedding of $\big(\Sigma, |\cdot - \cdot|\big)$ into $\big(\mathbb{C}, |\cdot - \cdot|\big)$. Thus, the map in question operates as

$$\big(S^1, |\cdot - \cdot|\big) \ni e^{i\theta} \mapsto z\big(\theta L/2\pi\big) \in \big(\mathbb{C}, |\cdot - \cdot|\big) \tag{5.9.75}$$

and is bi-Lipschitz. By [218, Theorem 7.10, p. 166], every bi-Lipschitz map of S^1 into $\mathbb{C}$ may be extended to a bi-Lipschitz map of $\mathbb{C}$ onto $\mathbb{C}$. Consequently, there exists a bijective bi-Lipschitz map $F : \mathbb{C} \to \mathbb{C}$ which extends (5.9.75). In particular, $F(S^1) = \Sigma$ which further implies that F maps the bounded component of $\mathbb{C} \setminus S^1$ onto the bounded component of $\mathbb{C} \setminus \Sigma$. Hence, $F\big(B(0,1)\big) = \Omega$ as wanted. □

In view of the transformational properties under bi-Lipschitz maps established in [124], the characterization from Proposition 5.9.9 implies that

$$\text{if } \Omega \subset \mathbb{C} \text{ is a chord-arc domain then } \Omega \text{ is a two-sided NTA domain (Definition 5.11.1), } \partial\Omega \text{ is Ahlfors regular, and } \mathcal{H}^1(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{5.9.76}$$

We note that, by design, any bounded chord-arc domain is a Jordan domain, hence simply connected. Bounded simply connected Lipschitz domains in the plane are chord-arc domains, but chord-arc domains need not be locally given by graphs of functions. For example, this is seen by noting that logarithmic spiral domains are chord-arc. We elaborate on this in (5.10.45) (and the subsequent comment).

Let us also note that, as may be seen with the help of Proposition 5.9.6,

$$\text{all bounded simply connected BMO}_1\text{-domains in the plane } (\mathbb{R}^2 \equiv \mathbb{C}) \text{ are chord-arc domains.} \tag{5.9.77}$$

Moving on, we record the following version of Definition 5.9.8.

Definition 5.9.10 *Call a bounded open set $\Omega \subseteq \mathbb{C}$ a* `chord-arc domain with vanishing constant` *if Ω is a chord-arc domain and, in addition,*

$$\lim_{\partial\Omega \ni z_2 \to z_1} \frac{\ell(z_1, z_2)}{|z_1 - z_2|} = 1, \quad \textit{uniformly for } z_1 \in \partial\Omega, \tag{5.9.78}$$

where, as before, $\ell(z_1, z_2)$ denotes the length of the shorter arc of Σ joining z_1 and z_2.

It is clear that bounded, simply connected, $\mathscr{C}^1$ domains in the plane are chord-arc domains with vanishing constant. This being said, the class of chord-arc domains with vanishing constant contains non-Lipschitz domains. To make this transparent, call a bounded open set Ω of the Euclidean space a VMO_1-`domain` if locally, in a suitable system of coordinates, Ω coincides with the upper-graph of a locally integrable function f with distributional derivatives belonging to Sarason space VMO. Then from the first part of Proposition 5.9.6 we see that

$$\text{any bounded simply connected VMO}_1\text{-domain in } \mathbb{R}^2 \text{ is a chord-arc domain with vanishing constant.} \tag{5.9.79}$$

A closely related result is as follows:

Lemma 5.9.11 *Assume $u : \mathbb{R} \to \mathbb{C}$ is a function in $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ such that $u'(t) = \mathrm{e}^{ib(t)}$ for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, where $b \in \mathrm{BMO}(\mathbb{R}, \mathcal{L}^1))$ is a real-valued function satisfying*

$$\mathrm{dist}\big(b, \mathrm{VMO}(\mathbb{R}, \mathcal{L}^1)\big) < 1, \tag{5.9.80}$$

with the distance measured in $\mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)$.

Then for every number

$$\varkappa > \frac{\operatorname{dist}\big(b, \mathrm{VMO}(\mathbb{R}, \mathcal{L}^1)\big)}{1 - \operatorname{dist}\big(b, \mathrm{VMO}(\mathbb{R}, \mathcal{L}^1)\big)} \tag{5.9.81}$$

there exists some $r > 0$ *with the property that*

$$|t_1 - t_2| \leq (1+\varkappa)|u(t_1) - u(t_2)| \;\; \text{for all} \;\; t_1, t_2 \in \mathbb{R} \;\; \text{with} \;\; |t_1 - t_2| < r. \tag{5.9.82}$$

Proof Assume that some $\varkappa > 0$ as in (5.9.81) has been fixed, and note that this entails

$$\operatorname{dist}\big(b, \mathrm{VMO}(\mathbb{R}, \mathcal{L}^1)\big) < \frac{\varkappa}{1+\varkappa}. \tag{5.9.83}$$

Since b is real-valued, this estimate ensures that there exists a real-valued function $w \in \mathrm{VMO}(\mathbb{R}, \mathcal{L}^1)$ such that

$$\|b - w\|_{\mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)} < \frac{\varkappa}{1+\varkappa}. \tag{5.9.84}$$

Pick $\delta > 0$ such that

$$\delta + \|b - w\|_{\mathrm{BMO}(\mathbb{R}, \mathcal{L}^1)} < \frac{\varkappa}{1+\varkappa}. \tag{5.9.85}$$

Since $w \in \mathrm{VMO}(\mathbb{R}, \mathcal{L}^1)$, there exists $r > 0$ with the property that

$$\sup_{\substack{I \subset \mathbb{R} \\ |I| \leq r}} \min_I \big|w(t) - w_I\big| \, \mathrm{d}t < \delta, \tag{5.9.86}$$

where the supremum is taken over all intervals I of $\mathbb{R}$ with length $|I| \leq r$, and where we have set $w_I := \min_I w(t)\, \mathrm{d}t$.

To proceed, select two arbitrary distinct numbers $t_1, t_2 \in \mathbb{R}$ satisfying $|t_1 - t_2| < r$ and denote by I the closed sub-interval of $\mathbb{R}$ with endpoints t_1, t_2. To fix ideas, assume $t_1 < t_2$, hence $I = [t_1, t_2]$. Also, introduce

$$b_I := ⨍_I b(t)\, \mathrm{d}t, \qquad m_I := \mathrm{e}^{\mathrm{i} b_I}, \tag{5.9.87}$$

and note that the fact that b is real-valued implies $|m_I| = 1$. Upon observing that $m_I^{-1} = \mathrm{e}^{-\mathrm{i} b_I}$, this permits us to estimate

$$\begin{aligned}
|u(t_1) - u(t_2) - m_I(t_1 - t_2)| &= \left| \int_{t_1}^{t_2} (u'(t) - m_I)\, \mathrm{d}t \right| = \left| \int_{t_1}^{t_2} \big(u'(t) m_I^{-1} - 1\big)\, \mathrm{d}t \right| \\
&= \left| \int_{t_1}^{t_2} \big(\mathrm{e}^{\mathrm{i}(b(t) - b_I)} - 1\big)\, \mathrm{d}t \right| \\
&\leq \int_{t_1}^{t_2} \big|\mathrm{e}^{\mathrm{i}(b(t) - b_I)} - 1\big|\, \mathrm{d}t \leq \int_{t_1}^{t_2} |b(t) - b_I|\, \mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{t_1}^{t_2} |(b-w)(t) - (b-w)_I|\,\mathrm{d}t + \int_{t_1}^{t_2} |w(t) - w_I|\,\mathrm{d}t \\
&\leq |t_1 - t_2| \|b - w\|_{\mathrm{BMO}(\mathbb{R},\mathcal{L}^1)} + \delta |t_1 - t_2| \\
&< \Big(\frac{\varkappa}{1+\varkappa}\Big)|t_1 - t_2|, \qquad (5.9.88)
\end{aligned}$$

where, in addition to (5.9.84)–(5.9.86) and (5.9.85), we have used the fact that functions in $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ are locally absolutely continuous (hence, the Fundamental Theorem of Calculus applies), as well as the elementary inequality $|e^{i\theta} - 1| \leq |\theta|$ for each $\theta \in \mathbb{R}$. From (5.9.88), we obtain

$$\begin{aligned}
|t_1 - t_2| = |m_I(t_1 - t_2)| &\leq |u(t_1) - u(t_2)| + |u(t_1) - u(t_2) - m_I(t_1 - t_2)| \\
&\leq |u(t_1) - u(t_2)| + \Big(\frac{\varkappa}{1+\varkappa}\Big)|t_1 - t_2|, \qquad (5.9.89)
\end{aligned}$$

which then readily yields (5.9.82). □

Remark 5.9.12 *It is also natural to consider Jordan curves* `passing through infinity` *in the plane. This class consists of sets of the form $\Sigma = \gamma(\mathbb{R})$, such that $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a continuous injective function with $\lim_{t\to\pm\infty} |\gamma(t)| = \infty$. For this class of curves a version of the Jordan separation theorem is also valid, namely*

if Σ is a Jordan curve passing through infinity, then its complement in $\mathbb{C}$ consists precisely of two open connected components, called $\Omega_\pm$, which satisfy $\partial\Omega_+ = \Sigma = \partial\Omega_-$. (5.9.90)

A proof may be found in [171]*, where it has also been noted that*

in the context of (5.9.90), the sets $\Omega_\pm$ are simply connected. (5.9.91)

The above considerations are directly relevant in the context of the following definition:

Definition 5.9.13 *A nonempty, proper, open subset Ω of $\mathbb{R}^2$ is called a* `chord-arc domain with unbounded boundary` *if $\partial\Omega$ is a locally rectifiable Jordan curve passing through infinity in $\mathbb{C} \equiv \mathbb{R}^2$ with the property that*

$$C_o := \sup_{\substack{z_1, z_2 \in \partial\Omega \\ z_1 \neq z_2}} \frac{\ell(z_1, z_2)}{|z_1 - z_2|} < +\infty, \qquad (5.9.92)$$

where $\ell(z_1, z_2)$ denotes the length of the arc of $\partial\Omega$ joining z_1 and z_2.

Then

any chord-arc domain with unbounded boundary $\Omega \subseteq \mathbb{R}^2$ is a connected, simply connected, unbounded, two-sided NTA domain with a connected Ahlfors regular boundary (hence also an Ahlfors regular domain satisfying a two-sided local John condition and, in particular, a UR domain). (5.9.93)

In fact, it has been shown in [171] that, for such a set Ω, the following companion of Proposition 5.9.9 holds:

there exists some bi-Lipschitz homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $120^{-1}C_o^{-1}|z_1 - z_2| \leq |F(z_1) - F(z_2)| \leq 2000|z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$, and with the property that $\Omega = F(\mathbb{R}^2_+)$, $\mathbb{R}^2 \setminus \overline{\Omega} = F(\mathbb{R}^2_-)$, as well as $\partial\Omega = F(\mathbb{R} \times \{0\})$, (5.9.94)

where the constant $C_o \in [1, \infty)$ is as in (5.9.92). Moreover, given any chord-arc domain with unbounded boundary $\Omega \subseteq \mathbb{R}^2$, if $\mathbb{R} \ni s \mapsto z(s) \in \partial\Omega$ denotes the arc-length parametrization of $\partial\Omega$ then the following properties have also been noted in [171] (compare with Proposition 5.8.6):

(i) For each $s_1, s_2 \in \mathbb{R}$ one has

$$|z(s_1) - z(s_2)| \leq |s_1 - s_2| \leq C_o|z(s_1) - z(s_2)|, \tag{5.9.95}$$

and

$$z(\cdot) \text{ is differentiable at } \mathcal{L}^1\text{-a.e. point in } \mathbb{R}, \\ \text{with } |z'(s)| = 1 \text{ for } \mathcal{L}^1\text{-a.e. } s \in \mathbb{R}. \tag{5.9.96}$$

(ii) For each $z_o \in \partial\Omega$ and $r \in (0, \infty)$ abbreviate $\Delta(z_o, r) := B(z_o, r) \cap \partial\Omega$. Then for each $s_o \in \mathbb{R}$ and $r \in (0, \infty)$ one has

$$(s_o - r, s_o + r) \subseteq z^{-1}\big(\Delta(z(s_o), r)\big) \subseteq \big(s_o - C_o r, s_o + C_o r\big). \tag{5.9.97}$$

(iii) For every Lebesgue measurable set $A \subseteq \mathbb{R}$ one has

$$\mathcal{H}^1\big(z(A)\big) = \mathcal{L}^1(A), \tag{5.9.98}$$

and for each $\mathcal{H}^1$-measurable set $E \subseteq \partial\Omega$ one has

$$\mathcal{H}^1(E) = \mathcal{L}^1\big(z^{-1}(E)\big). \tag{5.9.99}$$

(iv) With the arc-length measure σ on $\partial\Omega$ defined as $\sigma := \mathcal{H}^1 \lfloor \partial\Omega$, it follows that for each σ-measurable set $E \subseteq \partial\Omega$ and each non-negative σ-measurable function g on E one has

$$\int_E g \, d\sigma = \int_{z^{-1}(E)} g(z(s)) \, ds. \tag{5.9.100}$$

(v) The geometric measure theoretic outward unit normal ν to Ω is given by

$$\nu(z(s)) = -\mathrm{i}z'(s) \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in \mathbb{R}. \tag{5.9.101}$$

Consequently, for $\mathcal{L}^1$-a.e. $s \in \mathbb{R}$ the line $\{z(s) + t\,z'(s) : t \in \mathbb{R}\}$ is an approximate tangent line to $\partial\Omega$ at the point $z(s)$. In particular, Ω has an approximate tangent line at $\mathcal{H}^1$-almost every point on $\partial\Omega$.

Pressing on, we note the following elementary result.

Lemma 5.9.14 *Assume that* $-\infty < a < b < +\infty$ *and that* f *is a real-valued monotonic function on* $[a, b]$ *which is locally absolutely continuous on* (a, b). *Then*

$$\operatorname{dist}\big((a, f(a)), (b, f(b))\big) \leq \textit{length graph } f \leq \sqrt{2}\operatorname{dist}\big((a, f(a)), (b, f(b))\big). \tag{5.9.102}$$

Proof Recall that the length of the graph of f is originally defined as

$$L_f := \sup_{a \leq x_0 \leq \cdots \leq x_N = b} \Big\{ \sum_{i=1}^{N} \operatorname{dist}\big((x_{i-1}, f(x_{i-1})), (x_i, f(x_i))\big) \Big\} \tag{5.9.103}$$

This readily implies the first inequality in (5.9.102). To prove the second inequality in (5.9.102), we begin by making several remarks. First, the present hypotheses imply that actually f is absolutely continuous on $[a, b]$, i.e.,

$$f \in \mathrm{AC}\big([a, b]\big). \tag{5.9.104}$$

Second, standard one-variable analysis (cf., e.g., [161, Corollary 3.9, p. 76]) implies that f is differentiable $\mathcal{L}^1$-a.e. in $[a, b]$ and f' is Lebesgue integrable on $[a, b]$. Third, it is well known (see, e.g., [161, Remark 4.10, p. 119]) that whenever f is as in (5.9.104) then

$$L_f = \int_a^b \sqrt{1 + |f'(x)|^2}\,\mathrm{d}x. \tag{5.9.105}$$

Fourth, we may assume that f is non-decreasing (otherwise work with $-f$ in place of f). Granted this, it follows (from the Fundamental Theorem of Calculus and Lebesgue's Differentiation Theorem) that

$$f' \geq 0 \text{ at } \mathcal{L}^1\text{-a.e. point in } [a, b]. \tag{5.9.106}$$

At this stage, based on (5.9.104)–(5.9.106) and the Fundamental Theorem of Calculus for absolutely continuous functions we may estimate

$$\begin{aligned} L_f &\leq \int_a^b \big(1 + f'(x)\big)\,\mathrm{d}x = (b - a) + \big(f(b) - f(a)\big) \\ &\leq \sqrt{2}\sqrt{(b-a)^2 + \big(f(b) - f(a)\big)^2} = \sqrt{2}\operatorname{dist}\big((a, f(a)), (b, f(b))\big), \end{aligned} \tag{5.9.107}$$

as desired. □

As a consequence of Lemma 5.9.14, we have that half-graphs of power functions

$$\Sigma_\alpha := \big\{(x, x^\alpha) : x \geq 0\big\}, \quad \text{with } \alpha \in (0, \infty), \quad \text{are simple locally rectifiable curves satisfying (5.9.53).} \tag{5.9.108}$$

Without the monotonicity assumption in Lemma 5.9.14 the conclusion in (5.9.102) may fail. For example, (5.9.108) may be used to show that

$$\text{the parabola } \mathcal{P} := \big\{(x, x^2) : x \in \mathbb{R}\big\} \text{ is a simple, locally rectifiable curve, which is Ahlfors regular, yet failing to satisfy (5.9.53).} \tag{5.9.109}$$

Indeed, $\mathcal{P}$ is clearly a simple, locally rectifiable curve in $\mathbb{R}^2$. The fact that $\mathcal{P}$ is a lower Ahlfors regular set is directly implied by Lemma 5.9.3, while the upper Ahlfors regularity of $\mathcal{P}$ is established by considering two cases. First, assume that $z \in \mathcal{P}$ and $r > 0$ are such that $0 \notin \overline{B(z, r)}$. Then $B(z, r) \cap \mathcal{P}$ consists of at most two parabola arcs, and (5.9.108) implies their individual arc-lengths are comparable with the lengths of their respective subtended chords. Since such chords have length $\leq 2r$, the desired conclusion follows in this case. Second, if $0 \in \overline{B(z, r)}$ then $\partial B(z, r) \cap \mathcal{P}$ consists of at most two points and the arc-length of the piece of $\mathcal{P}$ contained in $B(z, r)$ may be controlled in terms of the distance of these points to the origin (by again relying on (5.9.108)). Given that these distances are at most $2r$, this once more yields an upper bound of the right order. This reasoning proves that $\mathcal{P}$ is an Ahlfors regular set. Finally, the fact that (5.9.53) fails for the present example is seen by taking $z_1 := (-a, a^2)$ and $z_2 := (a, a^2)$ with $a \to +\infty$.

To further elaborate on (5.9.54), fix a function

$$\theta \in \mathscr{C}^1\big((0, \infty)\big), \quad \text{real-valued, such that } M := \sup_{0<t<\infty} |t\theta'(t)| < +\infty, \tag{5.9.110}$$

then consider the closed subset of $\mathbb{R}^2$ defined as

$$\Sigma_\theta := \{0\} \cup \big\{z(t) = t e^{i\theta(t)} \in \mathbb{C} : 0 < t < \infty\big\}. \tag{5.9.111}$$

Then, as is apparent from the above definition, Σ_θ is a simple, locally rectifiable, planar curve. We also claim that

$$\Sigma_\theta \text{ in (5.9.111) is a chord-arc curve whenever } \theta \text{ is as in (5.9.110).} \tag{5.9.112}$$

To justify this claim, fix $0 < t_1 < t_2 < \infty$ arbitrary. Then, on the one hand we have

$$\begin{aligned}\ell\big(z(t_1),z(t_2)\big) &= \int_{t_1}^{t_2} |z'(t)|\,dt \le \int_{t_1}^{t_2} \big|e^{i\theta(t)}\big(1+it\theta'(t)\big)\big|\,dt \\ &= \int_{t_1}^{t_2} \sqrt{1+|t\theta'(t)|^2}\,dt \le \sqrt{1+M^2}(t_2-t_1),\end{aligned} \tag{5.9.113}$$

while on the other hand

$$\big|z(t_2)-z(t_1)\big| = \big|t_2 e^{i(\theta(t_2)-\theta(t_1))} - t_1\big| \ge \inf_{\alpha\in\mathbb{R}} \big|t_2 e^{i\alpha} - t_1\big| = t_2 - t_1. \tag{5.9.114}$$

Together, (5.9.113) and (5.9.114) establish (5.9.53), thus finishing the proof of (5.9.112).

Examples of curves Σ_θ of the sort described in (5.9.111), corresponding to functions θ as in (5.9.110), include all `logarithmic spirals` (wriggling out of the origin as in Fig. 5.4), obtained by taking

$$\theta(t) := c\ln t \text{ for } t\in(0,\infty), \text{ where } c\in\mathbb{R} \text{ is an arbitrary constant.} \tag{5.9.115}$$

In particular, the above discussion shows that

$$\begin{array}{l}\text{all logarithmic spirals in the plane are chord-arc}\\ \text{curves, hence Ahlfors regular sets in } \mathbb{R}^2.\end{array} \tag{5.9.116}$$

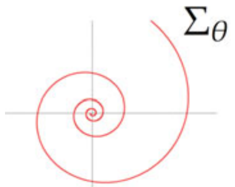

Fig. 5.4 Σ_θ as in (5.9.111), with θ as in (5.9.115)

Let us also note here that if Σ_θ is as in (5.9.111), then $\Sigma_\theta \cap \Sigma_{\theta+\alpha} = \{0\}$ for each $\alpha\in(0,2\pi)$. Consequently, if θ is as in (5.9.110), then for each $\alpha\in(0,2\pi)$ the region in the plane strictly contained in between the curves Σ_θ and $\Sigma_{\theta+\alpha}$, i.e.,

$$\Omega_{\theta,\alpha} := \big\{te^{i\beta} : 0<t<\infty \text{ and } \theta(t)<\beta<\theta(t)+\alpha\big\} \subseteq \mathbb{C}, \tag{5.9.117}$$

is an open set whose boundary $\partial\Omega_{\theta,\alpha}$ is an Ahlfors regular set and satisfies $\mathcal{H}^1\big(\partial\Omega_{\theta,\alpha}\setminus\partial_*\Omega_{\theta,\alpha}\big)=0$.

Subsets of the Euclidean space enjoying the aforementioned properties make up a distinguished class of domains. Following [189, Sect. 2.2], we make the following definition.

Definition 5.9.15 *Call a nonempty open subset* Ω *of* $\mathbb{R}^n$ *an* `Ahlfors regular domain` *if* $\partial\Omega$ *is an Ahlfors regular set* (*cf. Definition 5.9.1*) *and the geometric measure theoretic boundary of* Ω *has full* $\mathcal{H}^{n-1}$*-measure into the topological boundary of* Ω*, i.e.,*

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{5.9.118}$$

Hence, a nonempty open set $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain provided $\partial\Omega$ is an Ahlfors regular set (which, in particular, renders Ω a set of locally finite perimeter; cf. (5.9.15)) and the geometric measure theoretic outward unit normal ν to Ω is well defined at $\mathcal{H}^{n-1}$-a.e. point on $\partial\Omega$. Also, from (5.9.118), the definition of countable rectifiability (of dimension $n-1$) given in (5.3.16), and (5.6.33) we see that

if $\Omega \subseteq \mathbb{R}^n$ is an Ahlfors regular domain then $\partial\Omega$ is a closed set (in particular, $\mathcal{H}^{n-1}$-measurable), which is countably rectifiable (of dimension $n-1$). (5.9.119)

The condition imposed in (5.9.118) precludes $\partial\Omega$ from developing "too many" cusps and also prevents Ω from having "significant" cracks. For example, if

$$\begin{aligned} &\phi : \overline{B_{n-1}(0', 1/2)} \longrightarrow (-1/2, 1/2) \ \text{ is a Lipschitz function, and} \\ &\Sigma := \big\{(x', \phi(x')) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1},\ |x'| \leq 1/2\big\} \subset B(0,1) \end{aligned} \tag{5.9.120}$$

then the "crack domain" (Fig. 5.5)

$$\Omega := B(0,1) \setminus \Sigma \subset \mathbb{R}^n \tag{5.9.121}$$

is an open set with an Ahlfors regular boundary which, nonetheless, fails to be an Ahlfors regular domain in the sense of Definition 5.9.15.

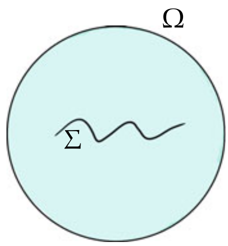

Fig. 5.5 Ω as in (5.9.121) is an open set with an Ahlfors regular boundary, but is not an Ahlfors regular domain

In the terminology introduced in Definition 5.9.15,

all logarithmic spiral domains (of the brand described in (5.9.117), (5.9.110)) are planar Ahlfors regular domains. (5.9.122)

To give higher-dimensional examples of Ahlfors regular domains, we shall say that an open subset of $\mathbb{R}^n$ is a BMO_1-`domain` provided this may be locally described, in a uniform fashion, as the upper-graph of a function as in (5.9.21). Then

$$\text{all } \mathrm{BMO}_1\text{-domains in } \mathbb{R}^n \text{ with } n \geq 2 \text{ are Ahlfors regular domains.} \tag{5.9.123}$$

To elaborate on this phenomenon, first recall that a function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ belongs to Zygmund's class $\Lambda_*(\mathbb{R}^{n-1}) = B_1^{\infty,\infty}(\mathbb{R}^{n-1})$ provided

$$\|\varphi\|_{\Lambda_*(\mathbb{R}^{n-1})} := \sup_{\substack{x\in\mathbb{R}^{n-1} \text{ and} \\ h\in\mathbb{R}^{n-1}\setminus\{0\}}} \frac{|\varphi(x+h)+\varphi(x-h)-2\varphi(x)|}{|h|} < +\infty. \tag{5.9.124}$$

Corresponding to $n = 2$, a typical example of a function in $\Lambda_*(\mathbb{R}^{n-1})$ is Weierstrass' continuous yet nowhere differentiable function

$$\varphi(x) := \sum_{j=0}^{\infty} \frac{\sin(\pi 2^j x)}{2^j}, \qquad \forall x \in \mathbb{R}. \tag{5.9.125}$$

Call an open set $\Omega \subseteq \mathbb{R}^n$ a `Zygmund domain` provided it may be locally described as upper-graphs of functions in Zygmund's class $\Lambda_*(\mathbb{R}^{n-1})$, in a uniform fashion.[21] In [132] it has been shown that

$$\begin{aligned}&\text{any Zygmund domain satisfies a two-sided corkscrew condition}\\&\text{(in fact, is a two-sided NTA domain; cf. Definition 5.11.1).}\end{aligned} \tag{5.9.126}$$

Since we also have the inclusion

$$\mathrm{BMO}_1(\mathbb{R}^{n-1}) \hookrightarrow \Lambda_*(\mathbb{R}^{n-1}) \tag{5.9.127}$$

(cf. [125, Proposition 3.15, p. 2637] for a proof), the claim in (5.9.123) follows from (5.9.22), (5.9.126)–(5.9.127), and (5.2.4).

Next, we shall establish that measurable sets which are two-sided thick have lower Ahlfors regular boundaries.

Proposition 5.9.16 *Let $E \subseteq \mathbb{R}^n$ be a $\mathcal{L}^n$-measurable set which is two-sided n-thick. Then $\partial_* E = \partial E$, and ∂E is lower Ahlfors regular.*

As a corollary, any $\mathcal{L}^n$-measurable set $E \subseteq \mathbb{R}^n$ satisfying a two-sided corkscrew condition has the property that $\partial_* E = \partial E$ and ∂E is lower Ahlfors regular.

Under the additional assumption that the set E is of locally finite perimeter, the main claim in Proposition 5.9.16 is implied more or less directly by the Relative Isoperimetric Inequality (cf., e.g., [80, Theorem 2, p. 190]) and the Structure Theorem

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[21] See also the discussion in [261, p. 64] in this regard.

for sets of locally finite perimeter (see [80, Theorem 2, p. 205]). Indeed, these classical results guarantee the existence of a finite purely dimensional constant $C_n > 0$ with the property that for each $x \in \mathbb{R}^n$ and each $r \in (0, \infty)$ we have

$$\min\Big\{\mathcal{L}^n\big(B(x,r)\cap E\big), \mathcal{L}^n\big(B(x,r)\setminus E\big)\Big\}^{(n-1)/n} \leq C_n \mathcal{H}^{n-1}\big(\partial_* E \cap \overline{B(x,r)}\big). \tag{5.9.128}$$

Given that $\partial_* E = \partial E$ (as seen from (5.2.4)) and that E is two-sided n-thick, we conclude from (5.9.128) that ∂E is indeed lower Ahlfors regular. That open subsets of $\mathbb{R}^n$ satisfying a two-sided corkscrew condition have lower Ahlfors regular topological boundaries has also been proved in [15, Lemma 2.3].

Remarkably, Proposition 5.9.16 is true without having to demand that the set in question is of locally finite perimeter. The proof we shall give is more analytic in nature than the argument just described in the latter scenario, and it makes use of the quantitative solvability of the divergence equation for Sobolev vector fields in a ball. Lemma 5.9.17 below is a consequence of much more general results of this type discussed in [191]. The reader is reminded that, given an open set $B \subseteq \mathbb{R}^n$ along with an integrability exponent $p \in (1, \infty)$, by $\mathring{W}^{1,p}(B)$ we denote the completion of $\mathscr{C}_c^\infty(B)$ in the norm $f \mapsto \|f\|_{L^p(B,\mathcal{L}^n)} + \|\nabla f\|_{[L^p(B,\mathcal{L}^n)]^n}$.

Lemma 5.9.17 *If $p \in (1, \infty)$, then there exists a constant $C_p \in (0, \infty)$ with the property that for every $x \in \mathbb{R}^n$, every $r \in (0, \infty)$ and every $f \in L^p\big(B(x,r), \mathcal{L}^n\big)$ such that $\int_{B(x,r)} f \, d\mathcal{L}^n = 0$, there exists some vector field $\vec{G}$ with components in $\mathring{W}^{1,p}\big(B(x,r)\big)$ satisfying $\operatorname{div}\vec{G} = f$ in $B(x,r)$ as well as the estimate*

$$\|\nabla \vec{G}\|_{[L^p(B(x,r),\mathcal{L}^n)]^{n^2}} + r^{-1}\|\vec{G}\|_{[L^p(B(x,r),\mathcal{L}^n)]^n} \leq C_p \|f\|_{L^p(B(x,r),\mathcal{L}^n)}. \tag{5.9.129}$$

Proof of Proposition 5.9.16 Fix a two-sided n-thick set $E \subseteq \mathbb{R}^n$. Then there exists $c > 0$ such that for each $x \in \partial E$ we have

$$\min\Big\{\mathcal{L}^n\big(B(x,r)\cap E\big), \mathcal{L}^n\big(B(x,r)\setminus E\big)\Big\} \geq c r^n, \qquad \forall r \in (0, 2\operatorname{diam} E). \tag{5.9.130}$$

From (5.2.4) we also know that $\partial_* E = \partial E$.

Recall next the point set topological inclusion

$$\partial(A \cap B) \subseteq (\partial A \cap B) \cup (\overline{A} \cap \partial B), \qquad \forall A, B \subseteq \mathbb{R}^n. \tag{5.9.131}$$

Given that for each set $A \subseteq \mathbb{R}^n$ we also have $\overline{A} = A \cup \partial A$, from (5.9.131) we deduce that

$$\partial(A \cap B) \subseteq \partial A \cup (A \cap \partial B), \qquad \forall A, B \subseteq \mathbb{R}^n. \tag{5.9.132}$$

If we now consider an arbitrary point $x \in \partial E$, the inclusion in (5.9.132) used with $A := B(x, r)$ with $r > 0$ arbitrary and $B := E$ gives

$$\partial\big(B(x,r) \cap E\big) \subseteq \partial B(x,r) \cup \big(B(x,r) \cap \partial E\big) \quad \text{for each} \quad r \in (0, \infty). \tag{5.9.133}$$

Fix now $r \in (0, 2\,\text{diam}\, E)$ and suppose first that $\mathcal{H}^{n-1}\big(B(x,r) \cap \partial E\big) < +\infty$. This and (5.9.133) then imply

$$\mathcal{H}^{n-1}\big(\partial\big(B(x,r) \cap E\big)\big) \leq \mathcal{H}^{n-1}\big(\partial B(x,r)\big) + \mathcal{H}^{n-1}\big(B(x,r) \cap \partial E\big) < +\infty. \tag{5.9.134}$$

In light of (5.6.38) and the fact that E is $\mathcal{L}^n$-measurable, the finiteness property (5.9.134) guarantees that the set

$$B(x,r) \cap E \text{ is of locally finite perimeter.} \tag{5.9.135}$$

Consider the function $f : B(x,r) \to \mathbb{R}$ defined by

$$f := \mathbf{1}_{B(x,r) \cap E} - \lambda_{x,r} \mathbf{1}_{B(x,r) \setminus E} \tag{5.9.136}$$

where the parameter $\lambda_{x,r} \in \mathbb{R}$ is given by

$$\lambda_{x,r} := \frac{\mathcal{L}^n\big(B(x,r) \cap E\big)}{\mathcal{L}^n\big(B(x,r) \setminus E\big)}. \tag{5.9.137}$$

As a consequence of (5.9.130), the function f is well defined and $\lambda_{x,r} \in (C_1, C_2)$ for some finite, positive constants C_1, C_2 independent of x and r, and depending only on n and the constant c from (5.9.130). As a consequence, $f \in L^p\big(B(x,r), \mathcal{L}^n\big)$ and

$$\|f\|_{L^p(B(x,r),\mathcal{L}^n)} \leq C r^{n/p} \tag{5.9.138}$$

for some $C \in (0, \infty)$ independent of x and r. Let us also observe that, as is apparent from (5.9.136)–(5.9.137), the function f satisfies the vanishing moment condition $\int_{B(x,r)} f \, d\mathcal{L}^n = 0$.

Fix $p \in (n, \infty)$ and note that $f \in L^p\big(B(x,r), \mathcal{L}^n\big)$. As such, Lemma 5.9.17 applies and ensures the existence of a vector field $\vec{G}$ with components in $\mathring{W}^{1,p}\big(B(x,r)\big)$, satisfying $\text{div}\vec{G} = f$ in $B(x,r)$, as well as estimate (5.9.129). Consider now a sequence of vector fields $\{\vec{F}_j\}_{j\in\mathbb{N}} \subseteq \big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n$ such that

$$\begin{aligned} &\text{supp}\,\vec{F}_j \subseteq B(x,r) \quad \text{for each } j \in \mathbb{N}, \text{ and} \\ &\vec{F}_j \to \vec{G} \quad \text{in } \big[L^p\big(B(x,r), \mathcal{L}^n\big)\big]^n \text{ as } j \to \infty, \\ &\nabla\vec{F}_j \to \nabla\vec{G} \quad \text{in } \big[L^p\big(B(x,r), \mathcal{L}^n\big)\big]^{n^2} \text{ as } j \to \infty. \end{aligned} \tag{5.9.139}$$

(That such a sequence exists is implied by the very definition of $\mathring{W}^{1,p}\big(B(x,r)\big)$, given earlier.) In particular, it follows that $\operatorname{div}\vec{F}_j \to \operatorname{div}\vec{G}$ in $L^p\big(B(x,r),\mathcal{L}^n\big)$, thus in $L^1\big(B(x,r),\mathcal{L}^n\big)$, as $j\to\infty$. On account of (5.9.130) and the De Giorgi–Federer Divergence Theorem (cf. Theorem 1.1.1, whose current applicability is ensured by (5.9.135)) we then obtain

$$\begin{aligned} cr^n &\leq \mathcal{L}^n\big(B(x,r)\cap E\big) = \int_{B(x,r)\cap E} f\,\mathrm{d}\mathcal{L}^n = \int_{B(x,r)\cap E} \operatorname{div}\vec{G}\,\mathrm{d}\mathcal{L}^n \\ &= \lim_{j\to\infty}\int_{B(x,r)\cap E}\operatorname{div}\vec{F}_j\,\mathrm{d}\mathcal{L}^n = \lim_{j\to\infty}\int_{\partial_*(B(x,r)\cap E)} \nu_{E,r}\cdot\vec{F}_j\,\mathrm{d}\mathcal{H}^{n-1} \end{aligned} \tag{5.9.140}$$

where $\nu_{E,r}$ is the geometric measure theoretic outward unit normal to $B(x,r)\cap E$. Next, based on (5.2.3) and (5.9.133) we may write

$$\begin{aligned} B(x,r)\cap\partial_*\big(B(x,r)\cap E\big) &\subseteq B(x,r)\cap\partial\big(B(x,r)\cap E\big) \\ &\subseteq B(x,r)\cap\big[\partial B(x,r)\cup\big(B(x,r)\cap\partial E\big)\big] \\ &= B(x,r)\cap\partial E. \end{aligned} \tag{5.9.141}$$

Combining (5.9.140)–(5.9.141) we may therefore estimate

$$\begin{aligned} cr^n &\leq \limsup_{j\to\infty}\int_{B(x,r)\cap\partial E}|\vec{F}_j|\,\mathrm{d}\mathcal{H}^{n-1} \\ &\leq \limsup_{j\to\infty}\Big\{\sup_{B(x,r)}|\vec{F}_j|\Big\}\mathcal{H}^{n-1}\big(B(x,r)\cap\partial E\big). \end{aligned} \tag{5.9.142}$$

Since we selected $p>n$, the embedding $W^{1,p}\big(B(0,1)\big)\hookrightarrow L^\infty\big(B(0,1),\mathcal{L}^n\big)$ is valid. Based on this, Poincaré's inequality and a natural rescaling, we then obtain

$$\sup_{B(x,r)}|\vec{F}_j| \leq Cr^{1-\frac{n}{p}}\|\nabla\vec{F}_j\|_{[L^p(B(x,r),\mathcal{L}^n)]^{n^2}} \quad\text{for each}\quad j\in\mathbb{N}, \tag{5.9.143}$$

for some finite constant $C>0$ independent of j, x, r, f. Hence, combining (5.9.143), (5.9.139), (5.9.129), and (5.9.138) further gives

$$\begin{aligned} \limsup_{j\to\infty}\Big\{\sup_{B(x,r)}|\vec{F}_j|\Big\} &\leq Cr^{1-\frac{n}{p}}\limsup_{j\to\infty}\|\nabla\vec{F}_j\|_{[L^p(B(x,r),\mathcal{L}^n)]^{n^2}} \\ &= Cr^{1-\frac{n}{p}}\|\nabla\vec{G}\|_{[L^p(B(x,r),\mathcal{L}^n)]^{n^2}} \\ &\leq Cr^{1-\frac{n}{p}}\|f\|_{L^p(B(x,r),\mathcal{L}^n)} = Cr, \end{aligned} \tag{5.9.144}$$

for some constant $C\in(0,\infty)$ independent of x and r. Together, estimates (5.9.142) and (5.9.144) imply that, whenever the radius $r\in(0,2\operatorname{diam}E)$ is such that

$\mathcal{H}^{n-1}\big(B(x,r)\cap\partial E\big) < +\infty$, we have

$$cr^{n-1} \leq \mathcal{H}^{n-1}\big(B(x,r)\cap\partial E\big), \tag{5.9.145}$$

where the constant $c > 0$ in (5.9.145) is independent of x and r. The fact that (5.9.145) also holds whenever $\mathcal{H}^{n-1}\big(B(x,r)\cap\partial E\big) = \infty$ is trivial. All together, estimate (5.9.145) is valid for every $r \in (0, 2\,\mathrm{diam}\,E)$. Together with the fact that $\mathrm{diam}\,E = \mathrm{diam}(\overline{E}) \geq \mathrm{diam}(\partial E)$, this ultimately proves that the closed set $\partial E \subseteq \mathbb{R}^n$ is indeed lower Ahlfors regular (cf. (5.9.1)). □

5.10 Uniformly Rectifiable Sets

We begin by formally introducing the notion of uniform rectifiability of G. David and S. Semmes. The following is a slight variant of the original definition in [68].

Definition 5.10.1 *Call* $\Sigma \subset \mathbb{R}^n$ *a* `uniformly rectifiable` (UR) `set` *provided* Σ *is closed, upper Ahlfors regular, and has Big Pieces of Lipschitz Images* (BPLI). *The latter property signifies the existence of* $\varepsilon > 0$ *and* $M \in (0,\infty)$ *such that, for each location* $x \in \Sigma$ *and each scale* $r \in (0, 2\,\mathrm{diam}\,\Sigma)$*, one can find a Lipschitz map* $\Phi : B_{n-1}(0',r) \to \mathbb{R}^n$ *(where* $B_{n-1}(0',r)$ *is the* $(n-1)$*-dimensional ball of radius* r *centered at the origin* $0'$ *in* $\mathbb{R}^{n-1}$*), having Lipschitz constant* $\leq M$*, and with the property that*

$$\mathcal{H}^{n-1}\Big(\Sigma\cap B(x,r)\cap\Phi\big(B_{n-1}(0',r)\big)\Big) \geq \varepsilon r^{n-1}. \tag{5.10.1}$$

All constitutive constants involved are collectively referred to as the UR *constants of* Σ.

Since, in the class of closed sets, having BPLI implies lower Ahlfors regularity, it follows that

$$\text{any UR set is Ahlfors regular.} \tag{5.10.2}$$

Also, since the property of having BPLI is preserved under arbitrary unions, it follows from (5.9.11) that

$$\text{finite unions of UR sets are themselves UR sets.} \tag{5.10.3}$$

In general, UR sets can be quite wild, e.g., may have infinitely many spirals, holes, or handles, though not without certain restrictions.

Observe that

$$\begin{array}{l}\text{any given compact, upper Ahlfors regular,}\\ \text{curve } \Sigma \subset \mathbb{C} \text{ is a uniformly rectifiable set.}\end{array} \tag{5.10.4}$$

Indeed, (5.9.33) ensures that Σ is an Ahlfors regular set, while (5.8.13) guarantees that the entire Σ is a Lipschitz image. According to Definition 5.10.1, this renders Σ a uniformly rectifiable set.

It is also known (cf. [68], [214, Theorem 25, p. 25], and Lemma 5.9.3) that an Ahlfors regular set $\Sigma \subseteq \mathbb{R}^2 \equiv \mathbb{C}$ may be contained in a rectifiable upper Ahlfors regular curve if and only if Σ is a UR set. In concert with (5.9.30) this shows that

given an Ahlfors regular set $\Sigma \subseteq \mathbb{R}^2 \equiv \mathbb{C}$, the maximal "altered" Cauchy operator $C^{\text{alt}}_{\max}$ (cf. (5.9.27)) is bounded on $L^2\big(\Sigma, \mathcal{H}^1 \lfloor \Sigma\big)$ if and only if Σ is a UR set. (5.10.5)

As far as the higher-dimensional case is concerned, according to a deep result of G. David and S. Semmes (cf. [68, Theorem, pp. 10–14]), given a closed set $\Sigma \subseteq \mathbb{R}^n$ which is Ahlfors regular, we have that

Σ is a UR set if and only if the truncated singular integral operator $T_{k,\varepsilon} f(x) := \int_{y\in\Sigma\setminus\overline{B(x,\varepsilon)}} k(x-y) f(y)\, \mathcal{H}^{n-1}(y)$, for $x \in \Sigma$, is bounded on $L^2(\Sigma, \mathcal{H}^{n-1})$ with norm majorized by a constant independent of $\varepsilon > 0$, whenever the integral kernel $k \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\})$ is odd and satisfies $\sup_{x\in\mathbb{R}^n\setminus\{0\}} |x|^{n-1+\ell} |(\nabla^\ell k)(x)| < +\infty$ for each $\ell \in \mathbb{N}_0$. (5.10.6)

When Σ is a $(n-1)$-dimensional plane or smooth submanifold of $\mathbb{R}^n$ the boundedness of the singular integral operators described in (5.10.6) is a classical result (going back to the work of Calderón and Zygmund), but the corresponding boundedness result for non-smooth sets is much deeper. In relation to (5.10.6), F. Nazarov, X. Tolsa, and A. Volberg have proved in [208] that

under the background assumption of Ahlfors regularity on the closed set $\Sigma \subseteq \mathbb{R}^n$, operator norm bounds on $L^2\big(\Sigma, \mathcal{H}^{n-1} \lfloor \Sigma\big)$ for the truncated Riesz transforms on Σ (corresponding to the kernels $k_j(x) := x_j/|x|^n$ for $1 \leq j \leq n$) which are in fact uniform with respect to the truncation parameter imply that Σ is actually a UR set. (5.10.7)

The story that emerges is that of a strong, two-way link, between singular integral operators on Lebesgue spaces and uniform rectifiability. We summarize some of these results in the next two theorems.

Theorem 5.10.2 *Given a closed set $\Sigma \subseteq \mathbb{R}^n$ which is Ahlfors regular, the following conditions are equivalent:*

(i) *Σ is a* UR *set;*

(ii) *The maximal singular integral operator acting on any given $f \in L^1\Big(\Sigma, \frac{\mathcal{H}^{n-1}(x)}{1+|x|^{n-1}}\Big)$ according to*

$$\big(T_{k,\max} f\big)(x) := \sup_{\varepsilon>0} \Big| \int_{y\in\Sigma\setminus\overline{B(x,\varepsilon)}} k(x-y) f(y)\,\mathcal{H}^{n-1}(y) \Big| \ \textit{for}\ \ x \in \Sigma, \tag{5.10.8}$$

is bounded on $L^2\big(\Sigma, \mathcal{H}^{n-1}\lfloor\Sigma\big)$ *for each integral kernel*

$$\begin{gathered} k \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\}) \ \ \textit{which is odd and satisfies} \\ \sup_{x\in\mathbb{R}^n\setminus\{0\}} |x|^{n-1+\ell} |(\nabla^\ell k)(x)| < +\infty \ \ \textit{for all} \ \ \ell \in \mathbb{N}_0. \end{gathered} \tag{5.10.9}$$

(iii) *The maximal singular integral operator* $T_{k,\max}$ *from* (5.10.8) *is bounded on* $L^2\big(\Sigma, \mathcal{H}^{n-1}\lfloor\Sigma\big)$ *for each integral kernel*

$$\begin{gathered} k \in \mathscr{C}^\infty(\mathbb{R}^n \setminus \{0\}) \ \ \textit{which is odd and} \\ \textit{positive homogeneous of degree}\ 1-n. \end{gathered} \tag{5.10.10}$$

(iv) *The maximal singular integral operator* $T_{k,\max}$ *from* (5.10.8) *is bounded on* $L^2\big(\Sigma, \mathcal{H}^{n-1}\lfloor\Sigma\big)$ *for each integral kernel of the form* $k(x) := x_j/|x|^n$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$, *where* $j \in \{1, \dots, n\}$.

Proof That the implication *(i)* ⇒ *(ii)* is true follows from (5.10.6) and the fact that

the boundedness of the family of truncated operators, in a uniform fashion with respect to the truncation parameter, ensures the boundedness of the corresponding maximal operator (cf. [186, Sect. 2.2]). (5.10.11)

Next, the implications *(ii)* ⇒ *(iii)* ⇒ *(iv)* are obvious, while the implication *(iv)* ⇒ *(i)* is a consequence of (5.10.7). □

The theorem below further elaborates on the role of uniform rectifiability and higher-dimensional Cauchy operators (within the framework of Clifford algebras, properly discussed in Sect. 6.4), as well as variety of brands of Riesz transforms.

Theorem 5.10.3 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an Ahlfors regular domain. Set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and denote by* ν *the geometric measure theoretic outward unit normal to* Ω. *Then the following conditions are equivalent:*

(a) *The boundary* $\partial\Omega$ *is a* UR *set* (*which makes* Ω *a* UR *domain in the sense of Definition 5.10.6*);
(b) *The maximal Cauchy–Clifford operator acting on any Clifford algebra-valued function* $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \otimes \mathcal{C}\!\ell_n$ *according to*

$$\big(\mathfrak{C}_{\max} f\big)(x) := \sup_{\varepsilon>0} \left| \frac{1}{\omega_{n-1}} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\,\mathrm{d}\sigma(y) \right|, \ \forall x \in \partial\Omega, \tag{5.10.12}$$

is bounded on $L^2(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$.

(c) *The truncated Cauchy–Clifford singular integral operators* $\mathfrak{C}_\varepsilon$ *defined for each* $\varepsilon > 0$ *and each function* $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \otimes \mathcal{C}\ell_n$ *as*

$$\big(\mathfrak{C}_\varepsilon f\big)(x) := \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, \mathrm{d}\sigma(y), \ \ \forall x \in \partial\Omega, \tag{5.10.13}$$

are continuous on $L^2(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$ *with operator norms bounded uniformly with respect to the truncation parameter* $\varepsilon \in (0, \infty)$.

(d) *The boundary-to-boundary* (*or principal-value*) *Cauchy–Clifford singular integral operator* $\mathfrak{C}$ *acting on each* $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \otimes \mathcal{C}\ell_n$ *according to*

$$\big(\mathfrak{C} f\big)(x) := \lim_{\varepsilon\to 0^+} \frac{1}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x-y}{|x-y|^n} \odot \nu(y) \odot f(y)\, \mathrm{d}\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.10.14}$$

is well defined, linear, and bounded on $L^2(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$.

(e) *For each* $j \in \{1, \dots, n\}$, *the maximal Riesz transform* $R_{j,\max}$ *defined as the mapping acting on each* $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ *according to*

$$(R_{j,\max} f)(x) := \sup_{\varepsilon>0} \frac{2}{\omega_{n-1}} \left| \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x_j - y_j}{|x-y|^n} f(y)\, \mathrm{d}\sigma(y) \right|, \ \ \forall x \in \partial\Omega, \tag{5.10.15}$$

is bounded on $L^2(\partial\Omega, \sigma)$.

(f) *For each* $j \in \{1, \dots, n\}$, *the* j*-th truncated Riesz transform defined for each* $\varepsilon > 0$ *and each function* $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$ *as*

$$\big(R_{j,\varepsilon} f\big)(x) := \frac{2}{\omega_{n-1}} \int\limits_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x_j - y_j}{|x-y|^n} f(y)\, \mathrm{d}\sigma(y), \ \ \forall x \in \partial\Omega, \tag{5.10.16}$$

is continuous on $L^2(\partial\Omega, \sigma)$ *with operator norm bounded uniformly with respect to the truncation parameter* $\varepsilon \in (0, \infty)$.

(g) *For each* $j \in \{1, \dots, n\}$, *the boundary-to-boundary* (*or principal-value*) *Riesz transform* R_j *defined as the mapping acting on any given* $f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\Big)$

according to

$$(R_j f)(x) := \lim_{\varepsilon\to 0^+} \frac{2}{\omega_{n-1}} \int_{\substack{y\in\partial\Omega \\ |x-y|>\varepsilon}} \frac{x_j - y_j}{|x-y|^n} f(y)\,d\sigma(y) \text{ for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{5.10.17}$$

is well defined, linear, and bounded on $L^2(\partial\Omega, \sigma)$.

(h) For each $j \in \{1, \dots, n\}$*, the distributional Riesz transform* R_j^{weak} *defined as the mapping*

$$R_j^{\text{weak}} : \mathrm{Lip}_c(\partial\Omega) \longrightarrow \mathrm{Lip}_c(\partial\Omega)' \tag{5.10.18}$$

acting for all $f, g \in \mathrm{Lip}_c(\partial\Omega)$ *according to*

$$\begin{aligned} &{}_{\mathrm{Lip}_c(\partial\Omega)'}\Big\langle R_j^{\text{weak}} f, g \Big\rangle_{\mathrm{Lip}_c(\partial\Omega)} \\ &:= \frac{2}{\omega_{n-1}} \int_{\partial\Omega}\int_{\partial\Omega} \frac{x_j - y_j}{|x-y|^n}\big[f(y)g(x) - f(x)g(y)\big]\,d\sigma(y)\,d\sigma(x) \end{aligned} \tag{5.10.19}$$

induces a linear and bounded operator on $L^2(\partial\Omega, \sigma)$.

Proof The fact that *(a)* $\Leftrightarrow$ *(b)* follows from the equivalence *(i)* $\Leftrightarrow$ *(ii)* in Theorem 5.10.2, bearing in mind Definitions 5.9.15, 5.10.6 and that (see (6.4.59))

$$\begin{gathered} \text{Clifford algebra multiplication by } \nu \\ \text{is an isomorphism of } L^2(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n. \end{gathered} \tag{5.10.20}$$

Next, *(b)* $\Rightarrow$ *(c)* is justified upon noting that $\mathfrak{C}_{\max} f = \sup_{\varepsilon>0} |\mathfrak{C}_\varepsilon f|$ pointwise on $\partial\Omega$ for each function $f \in L^2(\partial\Omega, \sigma) \otimes \mathcal{C}\ell_n$, while the converse implication follows from (5.10.11), again keeping in mind (5.10.20). Hence, *(b)* $\Leftrightarrow$ *(c)*.

To proceed, from Definition 5.9.15, (5.9.119), and Corollary 5.3.6 (applied with $\Sigma := \partial\Omega$) we conclude that

$$\text{the limit } \lim_{\varepsilon\to 0^+} \int_{\substack{y\in\partial\Omega \\ 1>|x-y|>\varepsilon}} \frac{x-y}{|x-y|^n}\,d\sigma(y) \text{ exists for } \sigma\text{-a.e. } x \in \partial\Omega. \tag{5.10.21}$$

Let us also recall a general principle in the theory of singular integral operators to the effect that

given a standard kernel and a complete Borel-semiregular measure for which the corresponding principal-value singular integral operator exists, then this is bounded (on Lebesgue spaces) if and only if the corresponding maximal operator associated with said kernel and measure, is bounded (on Lebesgue spaces); see the discussion in [186, Sect. 2.2]. (5.10.22)

Granted (5.10.21), from (5.10.22) we conclude (once again on account of (5.10.20)) that *(b)* $\Leftrightarrow$ *(d)*.

Going further, from the definitions of the operators involved and (5.10.20) we see that *(b)* $\Leftrightarrow$ *(e)*. Also, since for each $j \in \{1, \dots, n\}$ we have $R_{j,\max} f = \sup_{\varepsilon>0} |R_{j,\varepsilon} f|$ pointwise on $\partial\Omega$ for each function $f \in L^2(\partial\Omega, \sigma)$, with the help of (5.10.11) we deduce that *(e)* $\Leftrightarrow$ *(f)*. Next, (5.10.21) together with (5.10.22) show that *(e)* $\Leftrightarrow$ *(g)*. Finally, from (5.10.22) (whose applicability in the present setting is ensured by (5.10.21)) we see that *(g)* $\Leftrightarrow$ *(h)*. $\square$

Moving on, on account of (5.10.6) and the fact that boundedness on L^2 is hereditary, it readily follows that

$$\text{any lower Ahlfors regular closed subset of a UR set happens to be itself a UR set.} \tag{5.10.23}$$

G. David and D. Jerison have identified in [66] the following purely geometric sufficient condition guaranteeing uniform rectifiability.

Proposition 5.10.4 *Let $\Sigma \subset \mathbb{R}^n$ be a closed Ahlfors regular set which satisfies the following "two disk" condition: there exists $C \in (0, \infty)$ such that for each $x \in \Sigma$ and each $r \in (0, 2\,\mathrm{diam}\,\Sigma)$ one can find two $(n-1)$-dimensional disks of radius r/C with centers at distance $\leq r$ from x and which are contained in two different connected components of $\mathbb{R}^n \setminus \Sigma$. Then Σ is a uniformly rectifiable set.*

The somewhat more restrictive case of Proposition 5.10.4 where the disks are replaced by balls has been established earlier by S. Semmes in [234]. As pointed out on [66, p. 844], the same conclusion holds if the two disks can be replaced by bi-Lipschitz images of disks. What David and Jerison actually prove is that any set Σ as in the statement of Proposition 5.10.4 contains "big pieces of Lipschitz graphs" (cf. [66, Theorem 1, p. 840]).

A significant consequence of Propositions 5.10.4 and 5.9.16 states that

$$\begin{array}{l}\text{any open set } \Omega \subseteq \mathbb{R}^n \text{ with an upper Ahlfors regular boundary and} \\ \text{satisfying a two-sided corkscrew condition has the property that } \partial\Omega \\ \text{is a UR set; in particular, if } \Omega \subseteq \mathbb{R}^n \text{ is an NTA domain (in the sense} \\ \text{of Definition 5.11.1, given a little later) with an upper Ahlfors regular} \\ \text{boundary then its topological boundary, } \partial\Omega\text{, is a UR set.}\end{array} \tag{5.10.24}$$

As seen from Proposition 5.10.5 below, being uniformly rectifiable is stronger than being countably rectifiable and, in fact, uniform rectifiability may be regarded as a quantitative version of the latter property.

Proposition 5.10.5 *Any uniformly rectifiable set in $\mathbb{R}^n$ is countably rectifiable* (*of dimension $n-1$*).

Proof To see this, suppose $\Sigma \subset \mathbb{R}^n$ is a uniformly rectifiable set. Let $(x_j)_{j\in\mathbb{N}}$ be a countable, dense subset of Σ, and consider $(r_k)_{k\in\mathbb{N}}$ an enumeration of $\mathbb{Q}_+$ (or $(0, 1) \cap$

$\mathbb{Q}_+$ if Σ happens to be a compact set). For each $j, k \in \mathbb{N}$ let $\Delta_{jk} := \Sigma \cap B(x_j, r_k)$ and $L_{jk} := \Phi_{jk}(B'_{jk})$, where B'_{jk} is the $(n-1)$-dimensional ball of radius r_k centered at the origin $0'$ in $\mathbb{R}^{n-1}$, and $\Phi_{jk} : B'_{jk} \to \mathbb{R}^n$ is a Lipschitz function with Lipschitz constant $\leq M$ such that

$$\mathcal{H}^{n-1}\Big(\Sigma \cap B(x_j, r_k) \cap \Phi_{jk}(B'_{jk})\Big) \geq \varepsilon r_k^{n-1}, \tag{5.10.25}$$

where $\varepsilon > 0$ and $M \in (0, \infty)$ are two constants (cf. Definition 5.10.1), independent of j, k. Introduce

$$E := \bigcup_{j,k\in\mathbb{N}} (L_{jk} \cap \Sigma) \quad \text{and} \quad N := \Sigma \setminus E, \tag{5.10.26}$$

hence

$$\Sigma = \Big(\bigcup_{j,k\in\mathbb{N}} (L_{jk} \cap \Sigma)\Big) \cup N. \tag{5.10.27}$$

Then, using the fact that Σ is upper-Ahlfors regular and (5.10.25), we may estimate

$$\begin{aligned} \fint_{\Delta_{jk}} \mathbf{1}_E \, d\mathcal{H}^{n-1} &\geq \fint_{\Delta_{jk}} \mathbf{1}_{L_{jk}\cap\Sigma} \, d\mathcal{H}^{n-1} \\ &= \frac{\mathcal{H}^{n-1}(L_{jk} \cap \Delta_{jk})}{\mathcal{H}^{n-1}(\Delta_{jk})} \geq \frac{\mathcal{H}^{n-1}(L_{jk} \cap \Delta_{jk})}{C r_k^{n-1}} \geq \varepsilon/C, \end{aligned} \tag{5.10.28}$$

for every $j, k \in \mathbb{N}$. By density (eventually also making use of the Ahlfors regularity of Σ), this further entails the existence of a constant $c > 0$ with the property that

$$\fint_{B(x,r)\cap\Sigma} \mathbf{1}_E \, d\mathcal{H}^{n-1} \geq c, \qquad \forall x \in \Sigma, \quad \forall r > 0. \tag{5.10.29}$$

Granted (5.10.29), Lebesgue–Besicovitch Differentiation Theorem (cf., e.g., [80, Theorem 1, p. 43]) gives that $\mathbf{1}_E(x) > 0$ at $\mathcal{H}^{n-1}$-a.e. point $x \in \Sigma$ which proves that $\mathcal{H}^{n-1}(N) = 0$. In turn, this and (5.10.27) show that (5.3.18) holds, thus Σ is countably rectifiable. □

Following [125, Definition 3.7, p. 2631], let us now introduce the class of uniformly rectifiable domains (UR domains, for short).

Definition 5.10.6 *Call a nonempty open subset Ω of $\mathbb{R}^n$ a* UR domain *provided $\partial\Omega$ is a* UR set *(cf. Definition 5.10.1) and $\partial_*\Omega$ has full measure (relative to the $(n-1)$-dimensional Hausdorff measure) in the topological boundary $\partial\Omega$, i.e.,*

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0. \tag{5.10.30}$$

We emphasize that, by definition, the topological boundary of any UR domain is an Ahlfors regular set. As such, any UR domain is an Ahlfors regular domain.

As in the case of Ahlfors regular domains, condition (5.10.30) prevents $\partial\Omega$ from developing "too many" cusps and also precludes Ω from having "significant" cracks. For example, the "crack domain" Ω defined in (5.9.120)–(5.9.121) is an open set whose boundary is a UR set, and yet it fails to be a UR domain in the sense of Definition 5.10.6 (Fig. 5.6).

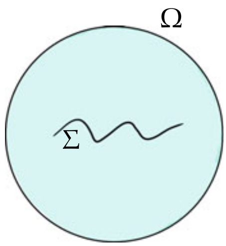

Fig. 5.6 An open set with a UR boundary, which fails to be a UR domain

Let us note that, as seen from (5.9.21)–(5.9.22), (5.9.126)–(5.9.127), Proposition 5.10.4, and (5.2.4),

$$\text{any } \mathrm{BMO}_1\text{-domain is a UR domain.} \tag{5.10.31}$$

In particular, any Lipschitz domain is a UR domain. In the plane, (5.9.76) and (5.10.24) imply that

$$\text{any bounded chord-arc domain in } \mathbb{C} \equiv \mathbb{R}^2 \text{ is a UR domain.} \tag{5.10.32}$$

From (5.10.24), (5.2.4), and Definition 5.10.6 we also see that

any open set $\Omega \subseteq \mathbb{R}^n$ with an upper Ahlfors regular boundary and satisfying a two-sided corkscrew condition is, in fact, a UR domain (in a quantitative fashion). (5.10.33)

Further examples of two-dimensional UR domains are offered by the following result.

Proposition 5.10.7 *Assume that $\Omega \subseteq \mathbb{C}$ is a connected, bounded, open set, whose boundary is a finite union of mutually disjoint, upper Ahlfors regular, Jordan curves, each of which is the boundary of a connected component of $\mathbb{C} \setminus \Omega$. Then Ω is a* UR *domain and $\partial\Omega = \partial(\overline{\Omega})$.*

Proof It suffices to treat the case when $\partial\Omega$ is an upper Ahlfors regular Jordan curve. From Lemma 5.9.3 and assumptions, it follows that $\partial\Omega$ is a rectifiable Ahlfors regular curve. In concert with the second formula in (5.8.42), this implies that Ω is an Ahlfors regular domain (cf. Definition 5.9.15). Since from (5.10.4) we also know that $\partial\Omega$ is a uniformly rectifiable set, we may conclude at this stage that Ω is a UR domain. Finally, $\partial\Omega = \partial\big(\overline{\Omega}\big)$ thanks to the first formula in (5.8.42). □

It is also useful to record here the following equivalence.

Corollary 5.10.8 *For a bounded connected open set $\Omega \subseteq \mathbb{C}$ the following are equivalent:*

(1) $\partial\Omega$ is a finite union of mutually disjoint, upper Ahlfors regular, Jordan curves, each of which is the boundary of a connected component of $\mathbb{C} \setminus \Omega$;
(2) Ω is a UR *domain satisfying $\partial\Omega = \partial(\overline{\Omega})$ and such that $\partial\Omega$ is a finite union of mutually disjoint rectifiable Jordan curves, each of which is the boundary of a connected component of $\mathbb{C} \setminus \Omega$.*

Proof The implication (1) $\Rightarrow$ (2) is a direct consequence of Proposition 5.10.7, while the implication (2) $\Rightarrow$ (1) is clear from Definition 5.10.1. □

In the context of Proposition 5.10.7, in the absence of any type of (local) connectivity for its boundary, Ω may fail to be a UR domain even if $\partial\Omega$ is compact and Ahlfors regular. To give an example, let G denote the Cantor-type set obtained from the usual procedure starting with $[0, 1]$, except that one now excludes the "middle-centered half" (i.e., the second and third quarter) of the interval at each stage. Then G is compact and has an empty interior. As such, the Cartesian product $E := G \times G$ is a compact subset of $[0, 1] \times [0, 1]$ with empty interior (an alternative description is provided in (5.9.23)). This implies that if $\Omega := B(0, 10) \setminus E$, then Ω is a connected bounded open set in $\mathbb{C}$ with the property that $\partial\Omega = \partial B(0, 10) \cup E$. In particular, (5.9.24) implies that $\partial\Omega$ is Ahlfors regular. Yet, in light of (5.9.28) and (5.10.6) we conclude that Ω is not a UR domain.

Proposition 5.10.7 already hints to the fact that UR domains can be topologically intricate and, indeed, in [195, Sect. A.4, p. 755] a class of UR domains in $\mathbb{R}^n$ (with $n \geq 2$ arbitrary) of infinite topological type was constructed.

To give other concrete examples of UR domains, of a different nature, fix two real numbers $a, b > 0$ satisfying $\mathrm{e}^{-2\pi} b < a < b$ and define the following logarithmic spiral domain in the complex plane:

$$\Omega_{a,b} := \big\{r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} : \theta \in \mathbb{R} \text{ and } r \in \big(a\mathrm{e}^{-\theta}, b\mathrm{e}^{-\theta}\big)\big\} \subset \mathbb{R}^2. \tag{5.10.34}$$

To study its nature, consider the family of functions $\Psi_t : \mathbb{R}^2 \to \mathbb{R}^2$, indexed by $t \in \mathbb{R}$, defined as follows:

$$\Psi_t(x, y) := \begin{cases} \Big(x\cos(t\ln r) - y\sin(t\ln r), x\sin(t\ln r) + y\cos(t\ln r)\Big) \\ \qquad \text{if } (x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}, \text{ where } r := \sqrt{x^2 + y^2}, \\ (0, 0) \qquad \text{if } (x, y) = (0, 0). \end{cases} \tag{5.10.35}$$

In terms of $z = x + \mathrm{i}y \in \mathbb{C}$ we may re-express Ψ_t as

$$\Psi_t(z) = \begin{cases} z\mathrm{e}^{\mathrm{i}t\ln|z|} & \text{if } z \in \mathbb{C}\setminus\{0\}, \\ 0 & \text{if } z = 0, \end{cases} \qquad \text{for each } t \in \mathbb{R}, \tag{5.10.36}$$

which readily implies that

$$\begin{gathered} \Psi_{t_1+t_2} = \Psi_{t_1} \circ \Psi_{t_2} \quad \text{for every} \quad t_1, t_2 \in \mathbb{R}, \\ \text{and} \quad \Psi_0(z) = z \quad \text{for each} \quad z \in \mathbb{C}. \end{gathered} \tag{5.10.37}$$

Moreover,

$$\Psi_t \in \mathscr{C}^0(\mathbb{R}^2) \cap \mathscr{C}^\infty(\mathbb{R}^2 \setminus \{(0,0)\}), \tag{5.10.38}$$

and the Jacobian matrix $D\,\Psi_t(x, y)$ is given at each $(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ by

$$\begin{pmatrix} (1 - t\frac{xy}{r^2})\cos(t\ln r) - t\frac{x^2}{r^2}\sin(t\ln r) & (-1 - t\frac{xy}{r^2})\sin(t\ln r) - t\frac{y^2}{r^2}\cos(t\ln r) \\ (1 - t\frac{xy}{r^2})\sin(t\ln r) + t\frac{x^2}{r^2}\cos(t\ln r) & (1 + t\frac{xy}{r^2})\cos(t\ln r) - t\frac{y^2}{r^2}\sin(t\ln r) \end{pmatrix}. \tag{5.10.39}$$

In particular, $|D\,\Psi_t| \le C_t < \infty$ in $\mathbb{R}^2 \setminus \{(0,0)\}$ which goes to show that Ψ_t is Lipschitz in $\mathbb{R}^2$. Upon recalling (5.10.37), we may therefore conclude that

$$\text{each } \Psi_t : \mathbb{R}^2 \to \mathbb{R}^2 \text{ a bi-Lipschitz map, with inverse } \Psi_{-t}. \tag{5.10.40}$$

If we now define the sector in $\mathbb{R}^2$ by setting

$$S_{a,b} := \big\{\rho\,\mathrm{e}^{\mathrm{i}\omega} \in \mathbb{C} :\ \rho > 0 \ \text{ and } \ \omega \in \big(\ln a, \ln b\big)\big\}, \tag{5.10.41}$$

a direct calculation shows that

$$\Omega_{a,b} = \Psi_{-1}\big(S_{a,b}\big). \tag{5.10.42}$$

Note that $S_{a,b}$ is a Lipschitz domain in $\mathbb{R}^2$. In concert with the transformational properties studied in [124], this implies that

$$\begin{gathered}\text{the planar logarithmic spiral domain } \Omega_{a,b} \text{ defined in (5.10.34) is a} \\ \text{UR domain satisfying a two-sided corkscrew condition (hence, in} \\ \text{particular, } \partial_*\Omega_{a,b} = \partial\Omega_{a,b}\text{); in addition, } \Omega_{a,b} \text{ is a two-sided NTA} \\ \text{domain, in the sense of Definition 5.11.1.}\end{gathered} \tag{5.10.43}$$

Moreover, if $z, w \in \partial\Omega_{a,b}$ and $\ell(z, w)$ denotes the length of the arc (z, w), with endpoints z, w, then borrowing notation and results from [124] we may write

$$\begin{aligned}\ell(z,w) &= \mathcal{H}^1\big((z,w)\big) = \mathcal{H}^1\Big(\Psi_{-1}\big((\Psi_1(z),\Psi_1(w))\big)\Big)\\ &= \int_{(\Psi_1(z),\Psi_1(w))} J_{\Psi_{-1}}\,\mathrm{d}\mathcal{H}^1\\ &\leq C\mathcal{H}^1\big((\Psi_1(z),\Psi_1(w))\big) = C\ell\big(\Psi_1(z),\Psi_1(w)\big)\\ &\leq C|\Psi_1(z)-\Psi_1(w)| \leq C|z-w|. \end{aligned} \tag{5.10.44}$$

In particular, the above reasoning proves that

the planar logarithmic spiral domain $\Omega_{a,b}$ defined in (5.10.34) is a chord-arc domain with unbounded boundary (cf. Remark 5.9.12 and subsequent discussion). (5.10.45)

One may further fashion a bounded chord-arc domain out of $\Omega_{a,b}$ by considering a suitable truncated version near the spiral point.

In the next lemma we study how the quality of being an Ahlfors regular domain, or UR domain, is preserved under passing to complements (of the closure).

Lemma 5.10.9 *Suppose $\Omega \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable. Abbreviate $\sigma_* := \mathcal{H}^{n-1}\lfloor\partial_*\Omega$ as well as $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and also define*

$$\Omega_+ := \Omega \;\; \textit{and} \;\; \Omega_- := \mathbb{R}^n \setminus \overline{\Omega}. \tag{5.10.46}$$

Then the following statements are true.

(1) The set $\Omega_- \subseteq \mathbb{R}^n$ is open, $\partial(\Omega_-) \subseteq \partial\Omega$, and

$$\partial\Omega \setminus \partial(\Omega_-) \; \textit{is a relatively open subset of} \; \partial\Omega. \tag{5.10.47}$$

Moreover, if $\partial\Omega$ is upper Ahlfors regular, then $\partial(\Omega_-)$ is upper Ahlfors regular.

(2) One has

$$\mathcal{L}^n(\partial\Omega\setminus\Omega)=0 \iff \mathcal{L}^n(\overline{\Omega}\setminus\Omega)=0 \Rightarrow \begin{cases}\partial_*(\Omega_-)=\partial_*\Omega=\partial_*(\Omega_+),\\ \textit{and}\;\; \partial\Omega\setminus\partial(\Omega_-)\subseteq\partial\Omega\setminus\partial_*\Omega.\end{cases} \tag{5.10.48}$$

(3) Make the additional assumptions that

$$\mathcal{L}^n(\partial\Omega\setminus\Omega)=0 \;\; \textit{and} \;\; \Omega \;\; \textit{has locally finite perimeter}. \tag{5.10.49}$$

In such a scenario, denote by ν the geometric measure theoretic outward unit normal to Ω.
Then Ω_- is a set of locally finite perimeter with the property that

the geometric measure theoretic outward unit normal to Ω_- is $-\nu$ at σ_*-a.e. point on $\partial_*\Omega$. (5.10.50)

(4) One has

$$\left.\begin{aligned} \mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0 \\ \textit{and } \mathcal{H}^{n-1}(\partial\Omega \cap B(x,r)) > 0 \\ \textit{for all } x \in \partial\Omega \textit{ and all } r > 0 \end{aligned}\right\} \Rightarrow \partial(\Omega_-) = \partial\Omega. \tag{5.10.51}$$

In particular,

$$\textit{whenever } \Omega \textit{ is an Ahlfors regular domain (cf. Definition 5.9.15) one has } \partial(\overline{\Omega}) = \partial\Omega. \tag{5.10.52}$$

Also,

$$\left.\begin{aligned} \textit{assuming } \partial\Omega \textit{ is a UR set, and} \\ \partial(\overline{\Omega}) \textit{ is lower Ahlfors regular} \end{aligned}\right\} \Rightarrow \partial(\Omega_-) \;\; \textit{is a UR set.} \tag{5.10.53}$$

(5) Assume $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$, as well as $\mathcal{H}^{n-1}(\partial\Omega \cap B(x,r)) > 0$ for each $x \in \partial\Omega$ and $r > 0$. Then $\Omega_- = \varnothing$ if and only if $\Omega = \mathbb{R}^n$. As a corollary, an Ahlfors regular domain (in the sense of Definition 5.9.15) is dense in $\mathbb{R}^n$ if and only if it coincides with $\mathbb{R}^n$.

(6) If $\Omega \subsetneq \mathbb{R}^n$ is an Ahlfors regular domain (in the sense of Definition 5.9.15) then Ω_- is also an Ahlfors regular domain, whose topological boundary coincides with that of Ω, and whose geometric measure theoretic boundary agrees with that of Ω, i.e.,

$$\partial(\Omega_-) = \partial\Omega \;\; \textit{and} \;\; \partial_*(\Omega_-) = \partial_*\Omega. \tag{5.10.54}$$

Moreover, the geometric measure theoretic outward unit normal to Ω_- is $-\nu$ at σ-a.e. point on $\partial\Omega$.

(7) If $\Omega \subsetneq \mathbb{R}^n$ is a UR *domain (in the sense of Definition 5.10.6) then Ω_- is also a* UR *domain, whose topological boundary coincides with that of Ω, and whose geometric measure theoretic boundary agrees with that of Ω (i.e., $\partial(\Omega_-) = \partial\Omega$ and $\partial_*(\Omega_-) = \partial_*\Omega$). In addition, the geometric measure theoretic outward unit normal to Ω_- is $-\nu$ at σ-a.e. point on $\partial\Omega$.*

Proof Since $\partial(\Omega_-)$ is a closed subset of $\mathbb{R}^n$, it follows that $\partial\Omega \setminus \partial(\Omega_-)$ is a relatively open subset of $\partial\Omega$. Also,

$$\partial(\Omega_-) = \partial(\mathbb{R}^n \setminus \overline{\Omega}) = \partial(\overline{\Omega}) = \overline{\overline{\Omega}} \setminus \mathring{\overline{\Omega}} \subseteq \overline{\Omega} \setminus \mathring{\Omega} = \partial\Omega. \tag{5.10.55}$$

In addition, if $\partial\Omega$ is upper Ahlfors regular, then it is clear from (5.9.2) that $\partial(\Omega_-)$ is also upper Ahlfors regular. This proves *(1)*.

As regards the claims in *(2)*, observe that

$$\overline{\Omega} \setminus \Omega = \Big((\overline{\Omega} \setminus \mathring{\Omega}) \cup \mathring{\Omega} \Big) \setminus \Omega = (\partial\Omega \setminus \mathring{\Omega}) \cup (\mathring{\Omega} \setminus \Omega) = (\partial\Omega \setminus \Omega) \cup \varnothing = \partial\Omega \setminus \Omega. \tag{5.10.56}$$

This readily gives the equivalence in (5.10.48). To prove the subsequent implication, make the assumption that $\mathcal{L}^n(\overline{\Omega} \setminus \Omega) = 0$. Then the geometric measure theoretic boundary of Ω_- may be expressed as

$$\partial_*(\Omega_-) = \partial_*(\mathbb{R}^n \setminus \overline{\Omega}) = \partial_*(\overline{\Omega}) = \partial_*\Big(\Omega \sqcup (\overline{\Omega} \setminus \Omega)\Big) = \partial_*\Omega, \tag{5.10.57}$$

where we have also used (5.2.3) and (5.2.5). Hence $\partial_*\Omega = \partial_*(\Omega_-)$. Since the latter set is contained in $\partial(\Omega_-)$ (cf. (5.2.3)), it follows that $\mathbb{R}^n \setminus \partial(\Omega_-) \subseteq \mathbb{R}^n \setminus \partial_*\Omega$. Intersecting both sides of this inclusion with $\partial\Omega$ then gives $\partial\Omega \setminus \partial(\Omega_-) \subseteq \partial\Omega \setminus \partial_*\Omega$, finishing the proof of (5.10.48).

Turning to the claims in *(3)*, work under the additional assumptions stipulated in (5.10.49). In particular, the fact that $\mathcal{L}^n(\overline{\Omega} \setminus \Omega) = 0$ entails

$$\mathbf{1}_{\Omega_-} = 1 - \mathbf{1}_{\Omega} \text{ at } \mathcal{L}^n\text{-a.e. point in } \mathbb{R}^n \tag{5.10.58}$$

which, after taking gradients, yields

$$\nabla \mathbf{1}_{\Omega_-} = -\nabla \mathbf{1}_{\Omega} \text{ as (vector) distributions in } \mathbb{R}^n. \tag{5.10.59}$$

Hence, Ω_- is a set of locally finite perimeter and, in concert with (5.6.3), this also readily implies (5.10.50).

As regards the first claim in item *(4)*, since $\partial(\Omega_-) \subseteq \partial\Omega$ (cf. item *(1)*), there remains to prove that $\partial\Omega \subseteq \partial(\Omega_-)$. With this goal in mind, reason by contradiction and assume that there exists a point $x \in \partial\Omega \setminus \partial(\Omega_-)$. In concert with (5.10.47) this implies that we may select some $r \in (0, \infty)$ such that $B(x, r) \cap \partial\Omega \subseteq \partial\Omega \setminus \partial(\Omega_-)$. In particular,

$$\mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial(\Omega_-)\big) \geq \mathcal{H}^{n-1}\big(B(x, r) \cap \partial\Omega\big) > 0, \tag{5.10.60}$$

thanks to the second assumption made in (5.10.51). On the other hand, the first assumption made in (5.10.51) implies that $\mathcal{L}^n(\partial\Omega \setminus \partial_*\Omega) = 0$. In view of Lemma 5.2.1, this forces $\mathcal{L}^n(\partial\Omega) = 0$. Granted this, we may invoke (5.10.48) which, together with the first assumption made in (5.10.51), permits us to conclude that $\mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial(\Omega_-)\big) = 0$. However, this contradicts (5.10.60). This finishes the proof of (5.10.51).

Concerning the last claim in item *(4)*, assume $\partial\Omega$ is a UR set, and $\partial(\overline{\Omega})$ is lower Ahlfors regular. Then from (5.10.55) and (5.10.23) we conclude that $\partial(\Omega_-)$ is a UR set.

Let us now turn our attention to item *(5)*. Together with (5.10.51), the present assumptions imply $\partial\Omega = \partial(\Omega_-)$. As such, having $\Omega_- = \varnothing$ forces $\partial\Omega = \partial(\varnothing) = \varnothing$ which ultimately implies $\Omega = \mathbb{R}^n$. The converse implication is obvious.

Consider next the claims in item *(6)*. If Ω is an Ahlfors regular domain, then $\mathcal{L}^n(\partial\Omega) = 0$ and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$. Granted these, (5.10.51) implies $\partial(\Omega_-) = \partial\Omega$ while item *(1)* guarantees that $\partial_*(\Omega_-) = \partial_*\Omega$. In particular, we have $\mathcal{H}^{n-1}\big(\partial(\Omega_-) \setminus \partial_*(\Omega_-)\big) = \mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial_*\Omega\big) = 0$. Since, by design, Ω_- is open, and since having $\Omega \neq \mathbb{R}^n$ entails $\Omega_- \neq \varnothing$ (on account of item *(5)*), we ultimately conclude that Ω_- is itself an Ahlfors regular domain. Lastly, that the geometric measure theoretic outward unit normal to Ω_- is $-\nu$ at σ-a.e. point on $\partial\Omega$ is a consequence of (5.10.50).

Finally, the claims in item *(7)* are clear from *(6)* and (5.10.53). □

We augment Lemma 5.10.9 with the following purely topological result which is going to be relevant later.

Lemma 5.10.10 *Given $n \in \mathbb{N}$ along with an arbitrary subset Ω of $\mathbb{R}^n$, consider the statements:*

(i) *$\mathbb{R}^n \setminus \Omega$ is bounded;*
(ii) *Ω is unbounded and $\partial\Omega$ is bounded.*

Then (i)$\Rightarrow$(ii) and, if $n \geq 2$, then (i)$\Longleftrightarrow$(ii).

Proof If $\mathbb{R}^n \setminus \Omega$ is bounded, the fact that $\mathbb{R}^n = (\mathbb{R}^n \setminus \Omega) \cup \Omega$ forces Ω to be unbounded. Also, $\partial\Omega$ is bounded since it matches the boundary of the bounded set $\mathbb{R}^n \setminus \Omega$. This establishes *(i)$\Rightarrow$(ii)*.

Assume next that $n \geq 2$ and that Ω is an unbounded subset of $\mathbb{R}^n$ with the property that $\partial\Omega$ is bounded. The goal is to prove *(i)*. To see that this is the case, if $r > 0$ is such that

$$\partial\Omega \subseteq B(0, r), \tag{5.10.61}$$

we claim that $\mathbb{R}^n \setminus \Omega \subset B(0, r)$, from which the desired conclusion follows. In turn, the claim is easily justified when $\overline{\Omega} = \mathbb{R}^n$, since in this case we have

$$\mathbb{R}^n \setminus \Omega = \overline{\Omega} \setminus \Omega \subseteq \overline{\Omega} \setminus \mathring{\Omega} = \partial\Omega \subseteq B(0, r). \tag{5.10.62}$$

The remaining case, when $\overline{\Omega} \neq \mathbb{R}^n$, is a consequence of a connectivity argument. Specifically, if the aforementioned inclusion fails, then there exists $x \in \mathbb{R}^n$ such that $x \notin B(0, r)$ and $x \notin \Omega$. In concert with (5.10.61), these imply that $x \notin \Omega \cup \partial\Omega = \overline{\Omega}$. Given that Ω is unbounded, we may find $y \in \Omega \setminus B(0, r)$. In view of (5.10.61), we therefore have $y \in \Omega \setminus \partial\Omega = \mathring{\Omega}$. Since $\mathbb{R}^n \setminus B(0, r)$ is pathwise connected (as $n \geq 2$), we may join x and y with a continuous curve L which is contained in $\mathbb{R}^n \setminus B(0, r)$. Writing $\mathbb{R}^n$ as the disjoint union $\mathring{\Omega} \cup \partial\Omega \cup (\mathbb{R}^n \setminus \overline{\Omega})$ and recalling that $\partial\Omega$ is contained in $B(0, r)$, it follows the connected set L is covered by the disjoint nonempty open sets $\mathring{\Omega}$ and $\mathbb{R}^n \setminus \overline{\Omega}$. Hence, either $L \subseteq \mathring{\Omega}$, or $L \subseteq \mathbb{R}^n \setminus \overline{\Omega}$, both of which cannot materialize since $x, y \in L$ with $x \in \mathbb{R}^n \setminus \overline{\Omega}$ and $y \in \mathring{\Omega}$. This contradiction finishes the proof of the lemma. □

We shall employ the following characterization of the category of locally Lipschitz domains (originally introduced in Definition 2.8.12), itself proved in [124], in the proof of Proposition 5.10.12 below.

Theorem 5.10.11 *Let Ω be a nonempty, proper open subset of $\mathbb{R}^n$ which has locally finite perimeter. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by ν the geometric measure theoretic outward unit normal to Ω. Then Ω is a locally Lipschitz domain if and only if*

$$\partial\Omega = \partial(\overline{\Omega}) \tag{5.10.63}$$

and Ω has continuous locally transverse vector fields, in the sense that for each point $x \in \partial\Omega$ there exist $r > 0$, $c > 0$, and some continuous vector field $\vec{h}$ defined on $B(x, r) \cap \partial\Omega$ satisfying

$$\nu \cdot \vec{h} \geq c \quad \textit{at } \sigma\textit{-a.e. point on} \quad B(x, r) \cap \partial_*\Omega. \tag{5.10.64}$$

Here is a useful geometric measure theoretic characterization of half-spaces (compare with [170, Proposition 15.15, p. 174]).

Proposition 5.10.12 *Let $\Omega \subset \mathbb{R}^n$ be an Ahlfors regular domain. Then Ω is a half-space if and only if its geometric measure theoretic outward unit normal is a constant vector.*

Proof Of course, the geometric measure theoretic outward unit normal of any half-space is a constant vector. The crux of the matter is establishing the converse implication, namely that if $\Omega \subset \mathbb{R}^n$ is an Ahlfors regular domain whose geometric measure theoretic outward unit normal is a constant vector then Ω is a half-space.

To this end, in a first stage we may invoke Theorem 5.10.11 (keeping in mind (5.10.63) is presently satisfied thanks to (5.10.52), and that (5.10.64) holds with $\vec{h} := \nu$, the constant outward unit normal to Ω) to conclude that

$$\Omega \text{ is a locally Lipschitz domain.} \tag{5.10.65}$$

Fix an arbitrary point $x_0 \in \partial\Omega$. From (5.10.65) and Lemma 2.8.13, we know that near x_0 the topological boundary $\partial\Omega$ agrees (up to a rotation and a translation) with the graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$. This, Proposition 5.6.17, and the fact that ν is assumed to be constant then imply that

$$\frac{\big(\nabla'\phi(x'), -1\big)}{\sqrt{1 + |(\nabla'\phi)(x')|^2}} \tag{5.10.66}$$

is a constant vector for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$ such that $\big(x', \phi(x')\big)$ is (up to the earlier rotation and translation) near x_0. Thus ϕ is actually an affine function when restricted to the aforementioned patch. As a consequence, there exists an $(n-1)$-dimensional plane $\pi_{x_0} \subseteq \mathbb{R}^n$ with the property that $\partial\Omega$ and π_{x_0} coincide near x_0, say

$$\mathcal{O} \cap \partial\Omega = \mathcal{O} \cap \pi_{x_0} \quad \text{for some open neighborhood } \mathcal{O} \text{ of } x_0. \tag{5.10.67}$$

Consider $\Sigma := \partial\Omega \cap \pi_{x_0}$ which is therefore a closed subset of π_{x_0}, and denote by Σ° the interior of Σ relative to π_{x_0} (viewed as a topological space, with the structure inherited from $\mathbb{R}^n$). From (5.10.67) we see that

$$\mathcal{O} \cap \pi_{x_0} = \mathcal{O} \cap \partial\Omega \cap \pi_{x_0} = \mathcal{O} \cap \Sigma \subseteq \Sigma \tag{5.10.68}$$

which, in particular, shows that $x_0 \in \Sigma^\circ$. The claim we make at this stage is that $\partial(\Sigma^\circ)$, considered in π_{x_0}, is empty. Indeed, if $x \in \partial(\Sigma^\circ)$ then $x \in \Sigma$ and $x \notin \Sigma^\circ$. As such, $x \in \partial\Omega$ and we can run the same argument as above (this time, for $x \in \partial\Omega$ in place of $x_0 \in \partial\Omega$) to conclude that there exist an open neighborhood U of x in $\mathbb{R}^n$ along with an $(n-1)$-dimensional plane $\pi \subseteq \mathbb{R}^n$, such that $U \cap \partial\Omega = U \cap \pi$. Note that $x \in U \cap \pi_{x_0}$ since $x \in U$ and $x \in \Sigma \subseteq \pi_{x_0}$, and that $U \cap \pi_{x_0}$ is a relatively open subset of π_{x_0}. Bearing in mind that $x \in \partial(\Sigma^\circ)$, we then see that $W := (U \cap \pi_{x_0}) \cap \Sigma^\circ$ is a nonempty subset of π_{x_0}, which is relative open in the topology of π_{x_0}, with the property that $W \subseteq U \cap \partial\Omega = U \cap \pi \subseteq \pi$. As a consequence, the hyperplanes π_{x_0} and π must coincide. In turn, this implies

$$U \cap \pi_{x_0} = U \cap \pi \cap \pi_{x_0} = U \cap \partial\Omega \cap \pi_{x_0} = U \cap \Sigma \subseteq \Sigma \tag{5.10.69}$$

which shows that $U \cap \pi_{x_0}$ is a relatively open subset of π_{x_0} contained in Σ. Hence, $U \cap \pi_{x_0} \subseteq \Sigma^\circ$. However, since $x \in U \cap \pi_{x_0}$ we further conclude that $x \in \Sigma^\circ$, which is a contradiction. Thus, Σ° is an open, nonempty, boundaryless subset of the connected topological space π_{x_0}. We can therefore conclude that $\Sigma^\circ = \pi_{x_0}$. As such $\Sigma \subseteq \pi_{x_0} = \Sigma^\circ \subseteq \Sigma$, which goes to show that $\Sigma = \pi_{x_0}$, i.e., $\pi_{x_0} \subseteq \partial\Omega$.

To summarize the analysis so far, with the arbitrary point $x_0 \in \partial\Omega$ we have associated an $(n-1)$-dimensional plane $\pi_{x_0} \subseteq \mathbb{R}^n$ which is orthogonal to the constant vector ν, with the property that $x_0 \in \pi_{x_0} \subseteq \partial\Omega$, and such that $\partial\Omega$ coincides with π_{x_0} near x_0. In concert with (5.10.65), this also implies that Ω coincides with the half-space $H^+_{x_0} := \{x \in \mathbb{R}^n : \langle x - x_0, \nu\rangle < 0\}$ near x_0, say

$$H^+_{x_0} \cap B(x_0, R) = \Omega \cap B(x_0, R) \;\text{ for some }\; R > 0. \tag{5.10.70}$$

In the case when $\partial\Omega \cap H^+_{x_0} \neq \varnothing$, consider $\{\pi_z : z \in \partial\Omega \cap H^+_{x_0}\}$, i.e., the family of $(n-1)$-dimensional planes associated as above with points in $\partial\Omega \cap H^+_{x_0}$. By design, these are all parallel (since they are all orthogonal to the constant vector ν) and contained in $\partial\Omega$. In view of (5.10.70), these properties further imply that the distance from each π_z with $z \in \partial\Omega \cap H^+_{x_0}$ to π_{x_0} is at least R (otherwise the corresponding plane, which is contained in $\partial\Omega$, would intersect $H^+_{x_0} \cap B(x_0, R)$, which is contained in Ω, an impossibility given that Ω is open). Consequently, there exists an $(n-1)$-dimensional plane, call it π_{z_0} for some $z_0 \in \partial\Omega \cap H^+_{x_0}$, which is closest to π_{x_0} among all planes in the family $\{\pi_z : z \in \partial\Omega \cap H^+_{x_0}\}$. From (5.10.65) and Lemma 2.8.13 we know that $\mathbb{R}^n \setminus \overline{\Omega}$ coincides with the half-space $H^-_{z_0} := \{x \in \mathbb{R}^n : \langle x - z_0, \nu\rangle > 0\}$ near z_0, i.e.,

$$H^-_{z_0} \cap B(z_0, r) = \big(\mathbb{R}^n \setminus \overline{\Omega}\big) \cap B(z_0, r) \quad \text{for some} \quad r \in (0, R). \tag{5.10.71}$$

Let us denote by $L := [y_0, y_1]$ the line segment joining some point $y_0 \in \Omega \cap B(x_0, R)$ with some point $y_1 \in \big(\mathbb{R}^n \setminus \overline{\Omega}\big) \cap B(z_0, r)$. Then from (5.10.70)–(5.10.71) we see that L is contained in $H^+_{x_0}$ and a connectivity argument shows that L must intersect $\partial\Omega$. Consequently, L contains a point $z_* \in \partial\Omega \cap H^+_{x_0}$. As such, the $(n-1)$-dimensional plane π_{z_*} is strictly closer to π_{x_0} than π_{z_0}, in contradiction with the choice of z_0. This contradiction proves that actually $\partial\Omega \cap H^+_{x_0} = \varnothing$.

Having established this property, we may now deduce that any point $x_* \in \partial\Omega$ cannot belong to the half-space $H^-_{x_0} := \{x \in \mathbb{R}^n : \langle x - x_0, \nu\rangle > 0\big\}$, since otherwise what we have just proved (applied to x_* in place of x_0) would imply $\partial\Omega \cap H^+_{x_*} = \varnothing$, in contradiction with the fact that π_{x_0} is contained both in $\partial\Omega$ and in $H^+_{x_*}$. Ultimately, this shows that $\partial\Omega$ is precisely the $(n-1)$-dimensional plane π_{x_0} and, hence, Ω coincides with the half-space $H^+_{x_0}$. □

5.11 Nontangentially Accessible Domains

The class of nontangentially accessible domains has been introduced by D. Jerison and C. Kenig in [132] in their quest to address the question raised by E. Stein of extending classical results in Harmonic Analysis pertaining to the nature of harmonic functions near the boundary to the most general domains "for which non-tangential behavior is meaningful."

Definition 5.11.1 *Let Ω be a nonempty open proper subset of $\mathbb{R}^n$.*

(i) Say that the set Ω satisfies the `Harnack chain condition` *provided there exist two parameters $M \in (1, \infty)$ and $R \in \big(0, \operatorname{diam} \partial\Omega\big]$ (the latter required to be ∞ if $\partial\Omega$ is unbounded) with the following significance. First, given $x, y \in \Omega$, a Harnack chain from x to y in Ω is a sequence of balls $B_1, \dots, B_N \subset \Omega$ such that $x \in B_1$, $y \in B_N$, $B_j \cap B_{j+1} \neq \varnothing$ for $1 \le j \le N-1$, and such that each B_j has a radius r_j satisfying $M^{-1} r_j < \operatorname{dist}(B_j, \partial\Omega) < M r_j$. The length of the chain is N. With this piece of terminology, one then demands that if*

$$\begin{gathered} \varepsilon \in (0, \infty),\, k \in \mathbb{N},\, z \in \partial\Omega, \text{ and } x, y \in \Omega \text{ with} \\ \max\big\{|x - z|, |y - z|\big\} < R/4 \ \text{ are such that} \\ \operatorname{dist}(x, \partial\Omega) > \varepsilon,\ \operatorname{dist}(y, \partial\Omega) > \varepsilon, \text{ and } \operatorname{dist}(x, y) < 2^k \varepsilon, \end{gathered} \tag{5.11.1}$$

then there exists a Harnack chain $B_1, \dots, B_N$ from x to y, whose length N satisfies $N \le Mk$, which further has the property that the diameter of each ball B_j is $\ge M^{-1} \min\big(\operatorname{dist}(x, \partial\Omega), \operatorname{dist}(y, \partial\Omega)\big)$.

(ii) Call Ω a `one-sided NTA domain` *(or an* `interior NTA domain`*) provided Ω satisfies an interior corkscrew condition as well as a Harnack chain condition. More precisely, it is assumed that there exist $M \in (1, \infty)$ and*

$R \in \big(0, \operatorname{diam}\partial\Omega\big]$ (*the latter required to be* ∞ *if* $\partial\Omega$ *is unbounded*) *with the property that* Ω *satisfies an interior corkscrew condition up to scale* R*, as in item (iv) of Definition 5.1.3 with constant* $\theta := M^{-1} \in (0, 1)$*, and* Ω *satisfies a Harnack chain condition as in item (i) above with constants* M, R.

(iii) *Call* Ω *an* NTA `domain` *provided* Ω *satisfies a two-sided corkscrew condition, as well as a Harnack chain condition* (*in the same precise quantitative sense as before*).

(iv) *Call* Ω *a* `two-sided` NTA `domain` *provided both* Ω *and* $\mathbb{R}^n \setminus \overline{\Omega}$ *are* NTA *domains* (*again, in the same quantitative sense*).

The Harnack chain condition in item *(i)* of Definition 5.11.1 roughly asserts that, locally, points $x, y \in \Omega$ may be "joined" by a pairwise overlapping family of Whitney balls, whose cardinality is cardinality does not exceed a multiple of

$$\log_2\left(\frac{|x-y|}{\min\big\{\operatorname{dist}(x,\partial\Omega),\operatorname{dist}(y,\partial\Omega)\big\}}\right). \tag{5.11.2}$$

In particular, more such Whitney balls are allowed if x, y are further apart, or if either of these points gets closer to the boundary.

In the context of Definition 5.11.1, whenever necessary to emphasize the role of the constants R, M, in item *(ii)* we shall say that Ω is a one-sided NTA domain with constants (R, M), with similar conventions for items *(iii)* and *(iv)*. In this regard, let us note that[22]

if $\lambda \in (0, \infty)$ and $\Omega \subseteq \mathbb{R}^n$ is a one-sided NTA domain with constants (R, M) then $\lambda\Omega$ is a one-sided NTA domain with constants $(\lambda R, M)$, plus similar results, for "ordinary" NTA domains and two-sided NTA domains. (5.11.3)

Remark 5.11.2 *If* $\Omega \subseteq \mathbb{R}^n$ *has compact boundary and is a one-sided* NTA *domain with constants* (R, M) *as in item (ii) of Definition 5.11.1, then* Ω *satisfies a* (*global*) *interior corkscrew condition as formulated in item (i) of Definition 5.1.3 with the constant* $\theta := R/(2M \cdot \operatorname{diam}\partial\Omega)$.

Indeed, up to scale R the existence of a corkscrew point with constant $\theta := M^{-1}$ *is guaranteed by definition, while for scales in the interval* $(R, 2\operatorname{diam}\partial\Omega)$ *the existence of a corkscrew point follows from this by adjusting the constant as indicated.*

As noted in (5.10.43), planar logarithmic spiral domains (defined in (5.10.34)) are two-sided NTA domains. Next we note that Proposition 5.9.16 implies that

if $\Omega \subseteq \mathbb{R}^n$ is an NTA domain then $\partial\Omega$ is a lower Ahlfors regular set which coincides with $\partial_*\Omega$. (5.11.4)

From (5.10.24), (5.2.4), and Definition 5.10.6 we also see that

[22] See, e.g., [211, Lemma 2.2, p. 341].

if $\Omega \subseteq \mathbb{R}^n$ is an NTA domain, in the sense of Definition 5.11.1, with an upper Ahlfors regular boundary then Ω is also a UR domain. (5.11.5)

Let us also mention here that

if $\Omega \subseteq \mathbb{R}^n$ is an one-sided NTA domain with an upper Ahlfors regular boundary, then $\partial\Omega$ is a UR set if and only if Ω satisfies an exterior corkscrew condition (thus rendering Ω both an NTA domain, and a UR domain). (5.11.6)

Indeed, the left-pointing implication is seen from Propositions 5.9.16 and 5.10.4, while the right-pointing implication has been established in [14].

The Harnack chain condition should be thought of as a (local) quantitative connectivity condition. At the qualitative level, any locally path-connected open set[23] with a compact boundary has finitely many connected components, which are separated (i.e., have mutually disjoint closures). In particular, we have the following result.

Lemma 5.11.3 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \in \mathbb{N}$ with $n \geq 2$) be an open set satisfying a Harnack chain condition (in particular, any one-sided* NTA *domain will do). Then Ω has finitely many connected components. Moreover, if $\partial\Omega$ is bounded then the distance between any two connected components of Ω is strictly positive, while if $\partial\Omega$ is unbounded then actually Ω is connected.*

Proof If $\partial\Omega$ is unbounded, then there is no restriction on the scale up to which the Harnack chain condition is valid, so item *(i)* in Definition 5.11.1 implies that the open set Ω is pathwise connected, hence connected. For the remainder of the proof assume that $\partial\Omega$ is bounded. Seeking a contradiction, assume Ω has infinitely many distinct connected components, say $\{O_j\}_{j\in\mathbb{N}}$. Then $\partial O_j \subseteq \partial\Omega$ for each $j \in \mathbb{N}$. Picking a point $x_j \in \partial O_j$ for each $j \in \mathbb{N}$ then yields a sequence of distinct points $\{x_j\}_{j\in\mathbb{N}} \subseteq \partial\Omega$, which is therefore bounded. By Bolzano–Weierstrass' theorem, this has an accumulation point, call it z, which necessarily belongs to $\partial\Omega$ since this set is closed. From item *(i)* in Definition 5.11.1 we then know that there exists some $\rho \in (0,\infty)$ with the property that $\Omega \cap B(z,\rho)$ is pathwise connected (since if $\rho > 0$ is sufficiently small any two points in this set may be joined by a Harnack chain, relative to some small $\varepsilon \in (0,\infty)$ and some large $k \in \mathbb{N}$). Since z is an accumulation point for $\{x_j\}_{j\in\mathbb{N}}$, there exist $j_1, j_2 \in \mathbb{N}$ such that $x_{j_1}, x_{j_2} \in B(z,\rho)$. Given that $x_{j_1} \in \partial O_{j_1}$ and $x_{j_2} \in \partial O_{j_2}$, we have $B(z,\rho) \cap O_{j_1} \neq \varnothing$ and $B(z,\rho) \cap O_{j_2} \neq \varnothing$. As such, we may select two points $y_1 \in B(z,\rho) \cap O_{j_1}$ and $y_2 \in B(z,\rho) \cap O_{j_2}$. Upon recalling that $O_{j_1}, O_{j_2} \subseteq \Omega$, it follows that y_1, y_2 belong to the pathwise connected set $\Omega \cap B(z,\rho)$. Hence, there exists a continuous curve $\gamma \subseteq \Omega \cap B(z,\rho)$ joining the point y_1 with the point y_2. Since $y_2 \in O_{j_2}$ which is an open set disjoint from O_{j_1}, it follows that $y_2 \in \mathbb{R}^n \setminus \overline{O_{j_1}}$. The fact that $y_1 \in O_{j_1}$ then forces (via a standard connectivity argument) γ to intersect ∂O_{j_1}. Upon recalling that ∂O_{j_1} is contained in

[23] i.e., an open set $\Omega \subseteq \mathbb{R}^n$ with the property that for each $x \in \Omega$ there exists $r > 0$ such that any two points in $B(x,r) \cap \Omega$ may be joined by a continuous path in Ω.

$\partial\Omega$, we see that γ intersects $\partial\Omega$ contradicting the fact that γ is contained in the open set Ω. This proves that Ω has finitely many connected components.

If the distance distance between two connected components of Ω, call them O_1 and O_2 is zero, then there exists a common point $x \in \partial O_1 \cap \partial O_2 \subseteq \partial\Omega$ (recall that we are assuming that $\partial\Omega$ is bounded). For some $r > 0$ sufficiently small, the Harnack chain condition then guarantees that $\Omega \cap B(x, r)$ is connected. Now, for $j = 1, 2$, the fact that $x \in \partial O_j$ implies that $B(x, r) \cap O_j \neq \varnothing$, hence there exists some $x_j \in B(x, r) \cap O_j \subseteq B(x, r) \cap \Omega$. Since the latter set is open and connected, there exists a closed curve C joining x_1 with x_2 in $B(x, r) \cap \Omega$. Observe that $C \cup O_1$ is a connected subset of Ω, since both C and O_1 are connected subsets of Ω and $C \cap O_1 \neq \varnothing$. By the maximality of O_1, this forces $C \subseteq O_1$, hence $x_2 \in C \subseteq O_1$. Having $O_1 \cap O_2 \neq \varnothing$ is a contradiction, and the desired conclusion follows. $\square$

The Harnack chain condition allows us to connect interior corkscrew points. Specifically, as is apparent from part *(i)* of Definition 5.11.1,

if Ω is a one-sided NTA domain with constants (R, M), as in item *(ii)* of Definition 5.11.1, then for each given $\theta \in (0, 1)$ there exist $N_\theta \in \mathbb{N}$ and $C_M \in (1, \infty)$ with the property that for each $x \in \partial\Omega$ and $r \in (0, R)$ one may connect any two interior corkscrew points relative to x and scales θr and r, respectively, via a Harnack chain of length $\leq N_\theta$ and whose balls have radii belonging to $(r/C_M, C_M r)$. (5.11.7)

(See also the comment in [132, p. 93] when $\theta = 1/2$.) In fact, it turns out that one-sided NTA domains are literally nontangentially accessible, in the precise sense described in the lemma below (which is a slight improvement upon [125, Lemma 3.13, p. 2634]). To state it, for any given open set $\Omega \subseteq \mathbb{R}^n$, parameter $\kappa > 0$, and point $x \in \partial\Omega$ define the nontangential approach region

$$\Gamma_\kappa(x) := \big\{ y \in \Omega : \, |x - y| < (1 + \kappa)\,\mathrm{dist}(y, \partial\Omega) \big\}. \tag{5.11.8}$$

Lemma 5.11.4 *Let $\Omega \subseteq \mathbb{R}^n$ be a one-sided* NTA *domain with constants (R, M). Suppose $x \in \Omega$, $y \in \partial\Omega$ and $r \in (0, R)$, $C > 1$ satisfy $B(x, r) \subset B(y, Cr) \cap \Omega$. Then there exists $C_o > 1$ which depends only on C and M along with a rectifiable simple curve $\gamma_{x,y}$ joining x with y, of length $\leq C_o r$, such that $\gamma_{x,y} \setminus \{y\} \subseteq \Omega$, and with the property that*

$$\mathrm{dist}(z, \partial\Omega) \geq |z - y|/C_o \quad \textit{for each point} \quad z \in \gamma_{x,y}. \tag{5.11.9}$$

In particular, with the piece of notation introduced in (5.11.8), there exists $\kappa > 0$ which depends only on C, M, R, such that

$$\gamma_{x,y} \setminus \{y\} \subseteq \Gamma_\kappa(y). \tag{5.11.10}$$

Proof Given x, y as in the statement, set $x_0 := x$ and, for each $j \in \mathbb{N}$, let x_j denote a corkscrew point relative to y at scale $\approx 2^{-j}r$. From (5.11.7) we know that there exist a number $N_o \in \mathbb{N}$ and a constant $C_1 > 1$ (both depending only on the NTA constants of Ω and the constant C in the statement of the lemma) with the property that, for each $j \in \mathbb{N}$, one can find a family of balls $\{B_k\}_{1 \leq k \leq N}$, with $N \leq N_o$, of radii $\approx 2^{-j}r$ such that $C_1 B_k \subset \Omega$, $x_{j-1} \in B_1$, $x_j \in B_N$, and $B_k \cap B_{k+1} \neq \varnothing$ for $k \in \{1, \dots, N-1\}$. In particular, there exists a polygonal path γ_j joining x_{j-1} with x_j which stays roughly at distance $2^{-j}r$ from $\partial\Omega$, and has length $\leq C_2 2^{-j}r$, for some $C_2 = C_2(M, C) > 1$. If we now take $\gamma_{x,y}$ to be the union of the paths γ_j for all $j \in \mathbb{N}$ (and discard all loops), it follows that $\gamma_{x,y}$ is a rectifiable simple curve, of length $\leq C_3 \sum_{j=1}^{\infty} 2^{-j}r = C_3 r$, for some constant $C_3 = C_3(M, C) > 1$. Furthermore, if $z \in \gamma_{x,y}$, say $z \in \gamma_j$ for some $j \in \mathbb{N}$, then, on the one hand, $\operatorname{dist}(z, \partial\Omega) \geq C_4 2^{-j}r$, while on the other hand

$$|z - y| \leq |z - x_j| + |x_j - y| \leq \operatorname{length}(\gamma_j) + C_5 \operatorname{dist}(x_j, \partial\Omega) \leq C_6 2^{-j}r. \tag{5.11.11}$$

Altogether, $|z - y| \leq C_o \operatorname{dist}(z, \partial\Omega)$ for some finite $C_o = C_o(M, C) > 1$, finishing the proof of the lemma. □

The same type of argument used in the proof of Lemma 5.11.4 yields the result recorded in the remark below (see also Lemma 5.11.6 in this regard).

Remark 5.11.5 *If $\Omega \subseteq \mathbb{R}^n$ is a one-sided* NTA *domain, then there exist $C \in (0, \infty)$ and $R \in (0, \infty]$ (with $R = \infty$ allowed if $\partial\Omega$ is unbounded) having the following significance: Whenever $y \in \partial\Omega$ and $0 < r_1 \leq r_2 < R$, any two corkscrew points x_1, x_2 relative to the location y and scales r_1, r_2, respectively, may be join by a rectifiable simple curve γ in Ω satisfying*

$$\begin{aligned} &\operatorname{length}(\gamma) \leq C|x_1 - x_2| \quad \textit{and} \\ &\operatorname{length}(\gamma_{x_1,z}) \leq C \cdot \operatorname{dist}(z, \partial\Omega) \quad \textit{for each} \quad z \in \gamma, \end{aligned} \tag{5.11.12}$$

where $\gamma_{x_1,z}$ denotes the arc of the curve γ joining x_1 with z.

A result related to Lemma 5.11.4 (which also implies the result in Remark 5.11.5) is proved below.

Lemma 5.11.6 *Let $\Omega \subseteq \mathbb{R}^n$ be an interior* NTA *and fix some $\kappa \in (0, \infty)$. Then there exist parameters $R \in (0, \infty)$, $\widetilde{\kappa} \in (0, \infty)$, and $C \in (0, \infty)$, with the property that whenever*

$$r \in (0, R), \quad x \in \partial\Omega, \quad \textit{and} \quad y_0, y_1 \in \Gamma_\kappa(x) \cap B(x, r), \tag{5.11.13}$$

one may find

$$\begin{aligned} &\textit{a rectifiable path } \gamma \subseteq \Gamma_{\widetilde{\kappa}}(x) \textit{ joining } y_0 \textit{ with } y_1 \textit{ and} \\ &\textit{whose length is} \leq C \cdot \max\big\{|x - y_0|, |x - y_1|\big\}. \end{aligned} \tag{5.11.14}$$

Proof Let $R \in \big(0, \operatorname{diam} \partial\Omega\big]$ be as in part *(i)* of Definition 5.11.1, and fix an arbitrary $r \in (0, R)$. Also, let $j_0, j_1 \in \mathbb{N}$ be such that

$$\begin{aligned} 2^{-j_0} r \leq |x - y_0| < 2^{-(j_0-1)} r \quad \text{and} \\ 2^{-j_1} r \leq |x - y_1| < 2^{-(j_1-1)} r. \end{aligned} \tag{5.11.15}$$

To fix ideas, assume $j_0 \leq j_1$. In turn, this entails

$$2^{-j_0} r \leq \max\big\{|x - y_0|, |x - y_1|\big\}. \tag{5.11.16}$$

For each $j \in \{j_0, \dots, j_1\}$, let z_j be a corkscrew point relative to x at scale $2^{-j} r$. That is, for some constant $C \in (1, \infty)$ which depends only on Ω, we have

$$B(z_j, 2^{-j} r/C) \subset B(x, 2^{-j} r) \cap \Omega. \tag{5.11.17}$$

Relabel z_{j_0} to be y_0 and z_{j_1} to be y_1. As observed at the bottom of p. 93 in [132], the Harnack chain condition implies that we may find $\theta \in (0, 1)$ with the property that, for each j, there exists a polygonal path γ_j joining z_{j-1} with z_j in Ω, such that

$$\inf_{z \in \gamma_j} \operatorname{dist}(z, \partial\Omega) > \theta 2^{-j} r \quad \text{and} \quad \operatorname{length}(\gamma_j) \leq \theta^{-1} 2^{-j} r. \tag{5.11.18}$$

Note that for each $z \in \gamma_j$ we have

$$\begin{aligned} |z - x| &\leq \operatorname{length}(\gamma_j) + |z_j - x| \leq \theta^{-1} 2^{-j} r + 2^{-j} r = (1 + \theta^{-1}) 2^{-j} r \\ &< (1 + \theta^{-1}) \theta^{-1} \operatorname{dist}(z, \partial\Omega), \end{aligned} \tag{5.11.19}$$

thanks to (5.11.18) and (5.11.17). Hence, if we set

$$\widetilde{\kappa} := (1 + \theta^{-1}) \theta^{-1} - 1 \in (0, \infty), \tag{5.11.20}$$

it follows from (5.11.19) and (5.11.8) that

$$\gamma_j \subseteq \Gamma_{\widetilde{\kappa}}(x) \quad \text{for each} \quad j. \tag{5.11.21}$$

If we now define γ to be the union of the γ_j's, then γ is a rectifiable path joining y_0 with y_1 in Ω, with

$$\begin{aligned} \operatorname{length}(\gamma) &\leq \sum_{j_0 \leq j \leq j_1} \operatorname{length}(\gamma_j) \leq \sum_{j=j_0}^{\infty} \theta^{-1} 2^{-j} r = 2\theta^{-1} 2^{-j_0} r \\ &\leq 2\theta^{-1} \max\big\{|x - y_0|, |x - y_1|\big\}, \end{aligned} \tag{5.11.22}$$

by (5.11.18) and (5.11.16). Also, (5.11.21) implies $\gamma \subset \Gamma_{\widetilde{\kappa}}(x)$, finishing the proof of (5.11.14). □

Going further, we recall the following definition from [125].

Definition 5.11.7 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. This is said to satisfy a* `local John condition` *if there exist $\theta \in (0,1)$, $M_o \in (1,\infty)$, and $R \in \big(0, \operatorname{diam}\partial\Omega\big]$ (the latter required to be ∞ if $\partial\Omega$ is unbounded), called the* `John constants` *of Ω, with the following significance. For every location $x \in \partial\Omega$ and every scale $r \in (0, R_*)$ one can find a point*

$$x_r \in B(x,r) \cap \Omega, \tag{5.11.23}$$

called `John center` *relative to the surface ball $\Delta(x,r) := B(x,r) \cap \partial\Omega$, such that*

$$B(x_r, \theta r) \subseteq \Omega \tag{5.11.24}$$

and for each $y \in \Delta(x,r)$ one can find

> *a rectifiable path $\gamma_y : [0,1] \to \overline{\Omega}$ whose length is $\leq M_o \cdot r$, which satisfies $\gamma_y(0) = y$, $\gamma_y(1) = x_r$, and such that for every $t \in (0,1]$ one has $\operatorname{dist}\big(\gamma_y(t), \partial\Omega\big) > \theta \cdot |\gamma_y(t) - y|$.* (5.11.25)

Finally, Ω is said to satisfy a `two-sided local John condition` *if both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ satisfy a local John condition.*

Note that, by design,

> any open set satisfying a local John condition automatically satisfies an interior corkscrew condition, and any open set which satisfies a two-sided local John condition also satisfies a two-sided corkscrew condition. (5.11.26)

In concert with (5.10.24) this further implies that

> any open set $\Omega \subseteq \mathbb{R}^n$ satisfying a two-sided local John condition and with an upper Ahlfors regular boundary has the property that its topological boundary, $\partial\Omega$, is a UR set in a quantitative fashion; in particular, Ω is a UR domain (in the sense of Definition 5.10.6) in a quantitative fashion. (5.11.27)

Let also remark here that, as a corollary of Definition 5.11.7 and Lemma 5.11.4,

> any one-sided NTA domain with constants (R, M) satisfies an interior local John condition with constants $\theta = \theta(M) \in (0,1)$, $M_o = M_o(M) \in (1,\infty)$, $R \in (0,\infty)$, and any two-sided NTA domain satisfies a two-sided local John condition (once again, in the same quantitative fashion). (5.11.28)

We next consider the class of domains introduced by P. Jones in [135].

Definition 5.11.8 *Assume that* $\varepsilon \in (0,1]$ *and* $\delta \in (0,\infty]$. *A nonempty, open, proper subset* Ω *of* $\mathbb{R}^n$ *is called an* (ε,δ)-`domain` *if for any* $x, y \in \Omega$ *with* $|x-y| < \delta$ *there exists a rectifiable curve* $\gamma : [0,1] \to \Omega$ *such that* $\gamma(0) = x$, $\gamma(1) = y$, *and*

$$\begin{gathered} \mathrm{length}(\gamma) \leq \tfrac{1}{\varepsilon}|x-y| \ \textit{and}\ \tfrac{|z-x|\,|z-y|}{|x-y|} \leq \tfrac{1}{\varepsilon}\mathrm{dist}(z,\partial\Omega) \\ \textit{for each}\ \ z \in \gamma([0,1]). \end{gathered} \tag{5.11.29}$$

Informally, the first condition in (5.11.29) says that Ω is locally path-connected in some quantitative sense (such domains are sometimes called quasi-convex), while the second condition in (5.11.29) says that there exists some type of "croissant-like" region C, with $\gamma([0,1]) \subseteq C \subseteq \Omega$ and the width of C at any point z on the curve is at least a fixed fraction of $\min\{|z-x|, |z-y|\}$.

Examples of (ε,δ)-domains include Lipschitz domains, bi-Lipschitz images of Lipschitz domains, open sets whose boundaries are given locally as graphs of functions in the Zygmund class Λ_1, or of functions with gradients in the John–Nirenberg space BMO, as well as the classical von Koch snowflake domain of conformal mapping theory. We wish to note that the boundary of an (ε,δ)-domain Ω can be highly non-rectifiable and, in general, no regularity condition on $\partial\Omega$ can be inferred from the (ε,δ) property described in Definition 5.11.8. The fact that (ε,δ)-domains are, generally speaking, not even sets of finite perimeter can be seen from the fact that the classical von Koch snowflake domain fails to have this property. In fact, for each $d \in [n-1,n)$ there exists an open set $\Omega \subseteq \mathbb{R}^n$ such that Ω is an (ε,∞)-domain for some $\varepsilon = \varepsilon(d) \in (0,\infty)$ and $\partial\Omega$ has Hausdorff dimension d.

A remarkable result due to P. Jones (cf. [135, Theorem 1, p. 73]) states that

$$\begin{gathered} \text{if } \Omega \subseteq \mathbb{R}^n \text{ is an } (\varepsilon,\delta)\text{-domain and } k \in \mathbb{N}, \text{ then there exists a linear} \\ \text{and bounded extension operator } \Lambda_k : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n), \text{ for} \\ \text{all integrability exponents } p \in [1,\infty). \end{gathered} \tag{5.11.30}$$

Going further, given an arbitrary, nonempty, open set $\Omega \subseteq \mathbb{R}^n$ define

$$\begin{gathered} \mathrm{rad}(\Omega) := \inf_j \inf_{x \in \Omega_j} \sup_{y \in \Omega_j} |x-y|, \quad \text{where} \\ \{\Omega_j\}_j \ \text{ are the connected components of } \ \Omega. \end{gathered} \tag{5.11.31}$$

Unraveling definitions then yields

$$\begin{gathered} \mathrm{rad}(\Omega) = \inf\big\{r \in (0,\infty] : \text{there exists } \ x \in \Omega \text{ such that } \Omega \subseteq B(x,r)\big\}, \\ \text{for any nonempty connected open set } \Omega \subseteq \mathbb{R}^n, \end{gathered} \tag{5.11.32}$$

hence, for any nonempty open set $\Omega \subseteq \mathbb{R}^n$ with connected components $\{\Omega_j\}_j$, we have

$$\operatorname{rad}(\Omega) = \inf_j \operatorname{rad}(\Omega_j). \tag{5.11.33}$$

In particular, $\frac{1}{2}\operatorname{diam}(\Omega) \leq \operatorname{rad}(\Omega) \leq \operatorname{diam}(\Omega)$ in the class of nonempty, connected, open subsets Ω of $\mathbb{R}^n$. It is also clear that $\operatorname{rad}(\Omega) > 0$ for every nonempty, finitely connected, open set Ω in $\mathbb{R}^n$, and that for every nonempty open set $\Omega \subseteq \mathbb{R}^n$,

$$\begin{array}{c}\operatorname{rad}(\Omega) > 0 \text{ if and only if there exists } \varkappa > 0 \text{ such that}\\ \operatorname{diam}(\Omega') > \varkappa \text{ for every connected component } \Omega' \text{ of } \Omega.\end{array} \tag{5.11.34}$$

Elementary topological considerations show that

$$\begin{array}{l}\text{any } (\varepsilon, \delta)\text{-domain } \Omega \subseteq \mathbb{R}^n \text{ with compact boundary}\\ \text{is finitely connected, hence satisfies } \operatorname{rad}(\Omega) > 0.\end{array} \tag{5.11.35}$$

Lemma 5.11.9 *Any* (ε, δ)*-domain* $\Omega \subseteq \mathbb{R}^n$ *with* $\operatorname{rad}(\Omega) > 0$ *satisfies the interior corkscrew condition* (*with constants depending on* ε, δ, *and* $\operatorname{rad}(\Omega)$). *In particular, any* (ε, δ)*-domain* $\Omega \subseteq \mathbb{R}^n$ *with* $\operatorname{rad}(\Omega) > 0$ *is* n*-thick* (*cf.* (5.1.6)).

Proof Assume that $x \in \partial\Omega$ and $0 < \rho < \min\{4\delta/5, 4\operatorname{rad}(\Omega)/5\}$ have been given. The fact that $x \in \partial\Omega$ guarantees the existence of some $x_0 \in \Omega \cap B(x, \rho/4)$. Let Ω' be the connected component of Ω containing x_0. From (5.11.32)–(5.11.33) and the fact that $5\rho/4 < \operatorname{rad}(\Omega) \leq \operatorname{rad}(\Omega')$ it follows that Ω' is not contained in $B(x_0, 5\rho/4)$ and, as such, there exists $x_* \in \Omega' \setminus B(x_0, 5\rho/4)$. Note that this forces $x_* \notin B(x, \rho)$. Since Ω' is open and connected, it is path-wise connected. Let $\Gamma \subseteq \Omega'$ be a continuous path joining x_0 and x_*. Given that $x_0 \in B(x, \rho)$ and $x_* \notin B(x, \rho)$, a simple connectivity argument shows that there exists $x_1 \in \Gamma \cap \partial B(x, \rho)$. In particular, we have $x_1 \in \Omega$ and $|x - x_1| = \rho$. This implies that $|x_0 - x_1| \leq \rho + \rho/4 = 5\rho/4 < \delta$. Based on this and the fact that Ω is an (ε, δ)-domain we deduce that there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$, and

$$\begin{array}{c}\operatorname{length}(\gamma) \leq \frac{1}{\varepsilon}|x_0 - x_1| \text{ and } \frac{|z-x_0|\,|z-x_1|}{|x_0-x_1|} \leq \frac{1}{\varepsilon}\operatorname{dist}(z, \partial\Omega)\\ \text{for all } \ z \in \gamma([0, 1]).\end{array} \tag{5.11.36}$$

Elementary connectivity considerations imply that the set $\gamma([0, 1]) \cap \partial B(x, \rho/2)$ is nonempty. Pick a point $P_\rho(x)$ belonging to this set. Then $P_\rho(x) \in \gamma([0, 1]) \subseteq \Omega$ and the fact that $|x_0 - x_1| \leq 5\rho/4$, $|x_0 - P_\rho(x)| \geq \rho/4$, $|x_1 - P_\rho(x)| \geq \rho/2$ yields (by the second inequality in (5.11.36))

$$\operatorname{dist}\big(P_\rho(x), \partial\Omega\big) \geq \frac{\varepsilon|P_\rho(x) - x_0|\,|P_\rho(x) - x_1|}{|x_0 - x_1|} \geq \big(\tfrac{\varepsilon}{10}\big)\rho. \tag{5.11.37}$$

Keeping in mind that $|P_\rho(x) - x| = \rho/2 < \rho$, the desired conclusion follows. $\square$

We continue by recording the definition of the class of uniform domains introduced by O. Martio and J. Sarvas in [173].

Definition 5.11.10 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set.*

(i) Call Ω a c-`uniform domain` *for some $c \in (0, \infty)$ (or, simply, c-*`uniform`*) if*

> *for any $x, y \in \Omega$ there exists a rectifiable simple curve γ joining x with y in Ω such that $\text{length}(\gamma) \leq c|x-y|$ and with the property that*
> $$\min\left\{\text{length}(\gamma_{x,z}), \text{length}(\gamma_{z,y})\right\} \leq c \cdot \text{dist}(z, \partial\Omega)$$
> *for each $z \in \gamma$, where $\gamma_{x,z}$ and $\gamma_{z,y}$ stand for the two connected components of the curve γ, joining x with z, and z with y, respectively.* (5.11.38)

(ii) Call Ω a `uniform domain` *(or, simply,* `uniform`*) if there exists $c \in (0, \infty)$ such that Ω is a c-uniform domain.*

Condition (5.11.38) asserts that the length of γ is comparable to the distance between its endpoints and that, away from its endpoints, the curve γ stays correspondingly far from $\partial\Omega$. Hence, heuristically, condition (5.11.38) implies that points in Ω can be joined in Ω by a curvilinear (or twisted) double cone which is neither too crocked nor too thin. The class of uniform domains is bi-Lipschitz invariant. Examples of uniform domains include bounded Lipschitz domains and their bi-Lipschitz images, as well as bounded logarithmic spiral domains (as seen from (5.10.40)–(5.10.42)). The classical von Koch snowflake is a uniform domain, and so is any quasidisk (the image of a disk or half-plane under a quasiconformal mapping of $\mathbb{C}$ onto itself). At the expense of increasing c, an equivalent reformulation of condition (5.11.38) is as follows (for a proof see [256, Sect. 2.4, p. 102]):

> for any two points $x, y \in \Omega$ there exists a rectifiable simple curve γ joining x with y in Ω, such that $\text{length}(\gamma) \leq c|x-y|$ and $\frac{|z-x|\,|z-y|}{|x-y|} \leq c \cdot \text{dist}(z, \partial\Omega)$ for all $z \in \gamma$. (5.11.39)

There is yet also another competing definition of uniformity which involves a weaker, more rudimentary version of the inequality in (5.11.38); cf. (5.11.40) below. Given that many papers in the literature use the latter notion of uniformity, it is of interest to establish the equivalence of these two definitions.

Proposition 5.11.11 *Let Ω be an open set in $\mathbb{R}^n$. Then Ω is a uniform domain (in the sense of Definition 5.11.10) if and only if there exists a constant $c \in [1, \infty)$ such that*

> *for each pair of given points $x, y \in \Omega$ one may find some rectifiable simple curve Υ joining x with y in Ω, with $\text{length}(\Upsilon) \leq c|x-y|$ and such that $\min\left\{|x-z|, |z-y|\right\} \leq c \cdot \text{dist}(z, \partial\Omega)$ for each point $z \in \Upsilon$.* (5.11.40)

Proof This follows by combining results from [172, 256]. To elaborate, we note that there are several types of competing definitions describing uniformity in the literature. For one thing, there is the notion of uniformity recalled in Definition 5.11.10. For another thing, there is the notion of uniformity proposed in (5.11.40). In [256, Theorem 2.10, p. 104], it is shown that the latter is equivalent to the so-called Möbius uniformity (cf. [256, Sect. 2.7, p. 103]). Möbius uniformity has been introduced earlier by O. Martio in [172, Definition 5.1, p. 201], who has shown that this is equivalent to the notion of uniformity from Definition 5.11.10 (this is seen by combining [172, Theorem 5.4, p. 201] with [172, Theorem 3.4, p. 199]). Together, these show that an open set $\Omega \subseteq \mathbb{R}^n$ is a uniform domain in the sense of Definition 5.11.10 if and only if there exists a constant $c \in [1, \infty)$ such that (5.11.40) holds.

Below we present an alternative, direct proof,[24] which is also going to be useful for establishing a local version of Proposition 5.11.11 (cf. Proposition 5.11.13, stated a little later).

If Ω is a uniform domain in the sense of Definition 5.11.10 then for any points $x, y \in \Omega$ the conditions in (5.11.40) are satisfied by taking Υ to be the curve γ joining x with y as in (5.11.38).

For the remainder of the proof assume $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying (5.11.40). Fix two arbitrary distinct points $x, y \in \Omega$ and let $\Upsilon : [0, 1] \to \Omega$ be a rectifiable simple curve joining x with y as in (5.11.40). Abbreviate $r := |x - y| > 0$ and define

$$\begin{aligned} t_* &:= \inf\big\{t \in [0, 1] : \Upsilon(t) \in \overline{B(y, r/4)}\big\}, \\ s_* &:= \sup\big\{s \in [0, t_*] : \Upsilon(s) \in \overline{B(x, r/4)}\big\}. \end{aligned} \tag{5.11.41}$$

It is then apparent from these definitions that $y_* := \Upsilon(t_*) \in \partial B(y, r/4)$ is the first point from the curve Υ on $\partial B(y, r/4)$, while $x_* := \Upsilon(s_*) \in \partial B(x, r/4)$ is the last point on the curve Υ belonging to $\partial B(x, r/4)$, before said curve hits $\partial B(y, r/4)$. Let $\widetilde{\Upsilon} := \Upsilon\big|_{[s_*, t_*]}$. This is a rectifiable simple curve joining the point x_* with the point y_* in the set $\Omega \setminus \big(B(x, r/4) \cup B(y, r/4)\big)$, satisfying

$$\operatorname{length}(\widetilde{\Upsilon}) \leq \operatorname{length}(\Upsilon) \leq c|x - y| = cr, \tag{5.11.42}$$

and with the property that

$$r \leq 4 \cdot \min\big\{|x - z|, |z - y|\big\} \leq 4c \cdot \operatorname{dist}(z, \partial\Omega) \quad \text{for each } z \in \widetilde{\Upsilon}. \tag{5.11.43}$$

We are going to alter $\Upsilon : [0, 1] \to \Omega$ by replacing the initial arc Υ_{x,x_*} with a rectifiable simple curve $\Upsilon^{(0)}$ joining x with x_* in Ω, and replacing the final arc $\Upsilon_{y_*,y}$ with a rectifiable simple curve $\Upsilon^{(1)}$ joining y_* with y in Ω. In addition, we shall arrange matters so that $\Upsilon^{(0)}, \Upsilon^{(1)}$ are disjoint from $\widetilde{\Upsilon}$, and we shall ensure the existence of a constant $C \in (0, \infty)$ for which

[24] We are grateful to Chema Martell for useful discussions on this topic.

$$\max\left\{\operatorname{length}(\Upsilon^{(0)}),\operatorname{length}(\Upsilon^{(1)})\right\}\leq Cr \tag{5.11.44}$$

and

$$\begin{aligned}\operatorname{length}(\Upsilon^{(0)}_{x,z})&\leq C\cdot\operatorname{dist}(z,\partial\Omega)\quad\text{for every}\quad z\in\Upsilon^{(0)},\\ \operatorname{length}(\Upsilon^{(1)}_{z,y})&\leq C\cdot\operatorname{dist}(z,\partial\Omega)\quad\text{for every}\quad z\in\Upsilon^{(1)}.\end{aligned} \tag{5.11.45}$$

Assume for now that the aforementioned alterations are possible, and have been implemented. We claim that the resulting concatenated curve, i.e.,

$$\gamma:=\Upsilon^{(0)}\cup\widetilde{\Upsilon}\cup\Upsilon^{(1)}, \tag{5.11.46}$$

is as in (5.11.38) (for a possibly different constant c). For starters, observe that, by design, γ is a rectifiable simple curve joining x with y in Ω. Also, from (5.11.46), (5.11.44), and (5.11.42) we see that $\operatorname{length}(\gamma)\leq(c+2C)|x-y|$. Finally, on account of (5.11.42)–(5.11.43), (5.11.45), and (5.11.46) we conclude that

$$\min\left\{\operatorname{length}(\gamma_{x,z}),\operatorname{length}(\gamma_{z,y})\right\}\leq 2c(c+2C)\cdot\operatorname{dist}(z,\partial\Omega) \tag{5.11.47}$$

for each $z\in\gamma$, finishing the justification of the claim.

At this stage, we turn to the construction of a rectifiable curve $\Upsilon^{(0)}$ joining x with x_* in Ω, along with a rectifiable curve $\Upsilon^{(1)}$ joining y_* with y in Ω, which satisfy the properties listed in (5.11.44)–(5.11.45). We shall only carry out the construction of $\Upsilon^{(0)}$, since $\Upsilon^{(1)}$ may be dealt with in a completely similar fashion. To this end, set $x_0:=x_*$ and for each $k\in\mathbb{N}$ pick $x_k\in\Upsilon\cap\partial B(x,2^{-k-2}r)$. Using (5.11.40), for each $k\in\mathbb{N}$ we may find a rectifiable curve Υ_k such that

$$\begin{aligned}&\Upsilon_k\ \text{joins}\ x_{k-1}\ \text{with}\ x_k\ \text{in}\ \Omega,\ \text{has}\ \operatorname{length}(\Upsilon_k)<c|x_{k-1}-x_k|,\ \text{and}\\ &\min\left\{|x_{k-1}-z|,|z-x_k|\right\}<c\cdot\operatorname{dist}(z,\partial\Omega)\ \text{for each point}\ z\in\Upsilon_k.\end{aligned} \tag{5.11.48}$$

Next, define

$$\Upsilon^{(0)}:=\bigcup_{k\in\mathbb{N}}\Upsilon_k. \tag{5.11.49}$$

Then $\Upsilon^{(0)}$ is a rectifiable curve joining x with x_* in Ω and satisfies

$$\operatorname{length}(\Upsilon^{(0)})=\sum_{k=1}^{\infty}\operatorname{length}(\Upsilon_k)\leq c\sum_{k=1}^{\infty}|x_{k-1}-x_k| \tag{5.11.50}$$

$$\leq c\sum_{k=1}^{\infty}2^{-k-1}r=(3c/4)r. \tag{5.11.51}$$

To proceed, fix an arbitrary $k\in\mathbb{N}\cup\{0\}$. On the one hand, the fact that $x_k\in\Upsilon$ implies, in light of (5.11.40), that

$$\min\left\{|x-x_k|,\,|x_k-y|\right\}\le c\cdot\operatorname{dist}(x_k,\partial\Omega). \tag{5.11.52}$$

On the other hand, since $x_k\in\partial B(x,2^{-k-2}r)$ we have $|x-x_k|=2^{-k-2}r\le r/4$ which permits us to estimate $r=|x-y|\le|x-x_k|+|x_k-y|\le r/4+|x_k-y|$. Hence $|x_k-y|\ge 3r/4$ which, together with (5.11.52), further entails

$$2^{-k-2}r=|x-x_k|=\min\left\{|x-x_k|,\,|x_k-y|\right\}\le c\cdot\operatorname{dist}(x_k,\partial\Omega). \tag{5.11.53}$$

Consider now $k\in\mathbb{N}$ fixed, arbitrary. We make the claim that

$$\operatorname{dist}(z,\partial\Omega)\ge c^{-2}\cdot 2^{-k-3}r\quad\text{for each}\quad z\in\Upsilon_k. \tag{5.11.54}$$

To justify this claim, consider first the case when $z\in\Upsilon_k$ is such that

$$|x_{k-1}-z|\ge c^{-1}\cdot 2^{-k-3}r\quad\text{and}\quad|x_k-z|\ge c^{-1}\cdot 2^{-k-3}r. \tag{5.11.55}$$

In such a scenario, we may rely on the second line in (5.11.48) to write

$$\operatorname{dist}(z,\partial\Omega)\ge c^{-1}\cdot\min\left\{|x_{k-1}-z|,\,|z-x_k|\right\}\ge c^{-2}\cdot 2^{-k-3}r, \tag{5.11.56}$$

as wanted. Next, assume that $z\in\Upsilon_k$ satisfies

$$|x_{k-1}-z|<c^{-1}\cdot 2^{-k-3}r. \tag{5.11.57}$$

Then based on the triangle inequality, (5.11.53) (written with $k-1$ in place of k), and (5.11.57) we may estimate

$$\begin{aligned}\operatorname{dist}(z,\partial\Omega)&\ge\operatorname{dist}(x_{k-1},\partial\Omega)-|x_{k-1}-z|\\&\ge c^{-1}\cdot 2^{-k-1}r-c^{-1}\cdot 2^{-k-3}r=3c^{-1}\cdot 2^{-k-3}r,\end{aligned} \tag{5.11.58}$$

which suits our purposes (recall that $c\ge 1$). Finally, the case when in place of (5.11.57) we have $|x_k-z|<c^{-1}\cdot 2^{-k-3}r$ is handled similarly, and this finishes the proof of (5.11.54).

We are now ready to check the first inequality in (5.11.45). Specifically, pick an arbitrary point $z\in\Upsilon^{(0)}$ and recall from (5.11.49) that there exists some $k_o\in\mathbb{N}$ such that $z\in\Upsilon_{k_o}$. Then, on the one hand, from the first line in (5.11.48) and (5.11.49) we see that

$$\begin{aligned}\operatorname{length}(\Upsilon^{(0)}_{x,z})&\le\sum_{k=k_o}^{\infty}\operatorname{length}(\Upsilon_k)\le c\sum_{k=k_o}^{\infty}|x_{k-1}-x_k|\\&\le c\sum_{k=k_o}^{\infty}3\cdot 2^{-k-2}r=3c2^{-k_o-1}r.\end{aligned} \tag{5.11.59}$$

On the other hand, from (5.11.54) we know that $\mathrm{dist}(z, \partial\Omega) \geq c^{-2} \cdot 2^{-k_o-3} r$ which, in concert with (5.11.59), ultimately implies

$$\mathrm{length}(\Upsilon^{(0)}_{x,z}) \leq 12c^3 \cdot \mathrm{dist}(z, \partial\Omega). \tag{5.11.60}$$

This establishes the first inequality in (5.11.45), with $C := 12c^3$. To finish the proof of the proposition there remains to observe that we may prune γ by discarding loops as to make the resulting curve simple, without affecting the validity of (5.11.38). □

We shall also need a local version of the uniformity property, as described in the following definition.

Definition 5.11.12 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set.*

(i) *Call Ω a (c, r)-*`locally uniform domain`*, for some $c, r \in (0, \infty)$, (or, simply, (c, r)-*`locally uniform`*) if (5.11.38) holds for all points $x, y \in \Omega$ satisfying $|x - y| < r$.*
(ii) *Call Ω a* `locally uniform domain` *(or, simply,* `locally uniform`*) if there exist $c, r \in (0, \infty)$ such that Ω is a (c, r)-locally uniform domain.*

Examples of domains which are locally uniform but not uniform are an infinite cylinder and the complement of a semi-infinite slab. Obviously, if Ω is a uniform domain then Ω is a locally uniform domain. In the converse direction, [256, Theorem 4.1, p. 114] implies that

$$\begin{aligned}&\text{if } \Omega \text{ is a locally uniform domain and } \partial\Omega \text{ is bounded, then}\\ &\text{any connected component of } \Omega \text{ is a uniform domain.}\end{aligned} \tag{5.11.61}$$

It turns out that a local version of the equivalence established earlier in Proposition 5.11.11 is also true. Specifically, we have the following result.

Proposition 5.11.13 *Let Ω be an open set in $\mathbb{R}^n$. Then Ω is a locally uniform domain (in the sense of Definition 5.11.12) if and only if there exist two constant $c \in [1, \infty)$ and $R \in (0, \infty)$ such that*

$$\begin{aligned}&\textit{for each } x, y \in \Omega \textit{ with } |x - y| < R \textit{ one may find some rectifiable sim-}\\ &\textit{ple curve } \Upsilon \textit{ joining } x \textit{ with } y \textit{ in } \Omega, \textit{ with } \mathrm{length}(\Upsilon) \leq c|x - y| \textit{ and such}\\ &\textit{that } \min\big\{|x - z|, |z - y|\big\} \leq c \cdot \mathrm{dist}(z, \partial\Omega) \textit{ for each point } z \in \Upsilon.\end{aligned} \tag{5.11.62}$$

Proof This is seen from a cursory inspection of the second proof of Proposition 5.11.11. □

In view of the fact that both the Harnack chain condition and (5.11.39) may be thought of as quantitative connectivity properties, it should not be too surprising that the class of uniform domains interfaces tightly both with the Jerison–Kenig class of NTA domains (cf. Definition 5.11.1) and with Jones' class of (ε, δ)-domains (cf. Definition 5.11.8). For the benefit of the reader, below we summarize some of the most basic interconnections among the various classes of domains discussed above, which are scattered in the literature.

Proposition 5.11.14 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then the following statements are true.*

(1) The set Ω is an (ε, δ)-domain for some $\varepsilon > 0$ and $\delta > 0$ if and only if Ω is a locally uniform domain.
(2) The set Ω is an (ε, ∞)-domain for some $\varepsilon > 0$ if and only if Ω is a uniform domain.[25]
(3) If Ω is a uniform domain, or Ω is a locally uniform domain with compact boundary, then Ω satisfies an interior corkscrew condition as well as a Harnack chain condition (i.e., Ω is one-sided NTA*, or interior* NTA*).*
(4) If Ω satisfies an interior corkscrew condition and a Harnack chain condition (i.e., if Ω is one-sided NTA*, or interior* NTA*) then any connected component of Ω is a uniform domain.*

Proof As a preliminary matter, we note that if $\gamma \subseteq \Omega$ is a rectifiable curve with end-points $x, y \in \Omega$ satisfying $\text{length}(\gamma) \le c|x - y|$ for some number $c \in [1, \infty)$, then for each point $z \in \gamma$ we have

$$\frac{1}{2} \cdot \min\big\{|x - z|, |z - y|\big\} \le \frac{|z - x|\,|z - y|}{|x - y|} \le c \cdot \min\big\{|x - z|, |z - y|\big\}. \tag{5.11.63}$$

Indeed, if $z \in \gamma$ has $|x - z| \le |z - y|$ then $|x - y| \le |x - z| + |z - y| \le 2|z - y|$ which readily gives the first inequality in (5.11.63) in this case. The same type of reasoning works when $z \in \gamma$ is such that $|y - z| \le |x - y|$, and this finishes the proof of the first inequality in (5.11.63). As regards the remaining inequality, for each point $z \in \gamma$ write

$$\begin{aligned} \frac{|z - x|\,|z - y|}{|x - y|} &\le \frac{|z - x| \cdot \text{length}(\gamma)}{|x - y|} \le c|x - z|, \\ \frac{|z - x|\,|z - y|}{|x - y|} &\le \frac{|z - y| \cdot \text{length}(\gamma)}{|x - y|} \le c|y - z|, \end{aligned} \tag{5.11.64}$$

from which the second inequality in (5.11.63) follows.

The claim in item *(1)* is seen from by definitions, (5.11.63), and Proposition 5.11.13. The claim in item *(2)* is a consequence of definitions, (5.11.63), and Proposition 5.11.11. Also, the statement in item *(3)* is implied by [113, Proposition A.2, p. 481] and its proof.

[25] It is known that there are unbounded (ε, δ)-domains which are not uniform; this is seen from the example in [136, p. 71].

Consider next the version of *(4)* in the class of sets with compact boundaries. Specifically, assume Ω is an open set with compact boundary and which satisfies both an interior corkscrew condition as well as a Harnack chain condition, as described in Definition 5.11.1. In view of Lemma 5.11.3, these properties are inherited by any connected component of Ω. Hence, if we fix such a connected component Ω' of Ω, it follows that Ω' satisfies both an interior corkscrew condition as well as a Harnack chain condition with constants (R, M). Then, on the one hand, from the proof of [113, Proposition A.3, p. 481] we see that any two points $x, y \in \Omega'$ with the property that there exists $w \in \partial\Omega$ such that $x, y \in \Omega' \cap B(w, R/100)$ may be joined with a rectifiable curve γ in Ω' satisfying (5.11.38) for some $c \in (0, \infty)$ depending only on R, M. On the other hand, it is known that uniformity of an open connected set with compact boundary is a local property of the boundary. Indeed, this follows from [256, Theorem 4.1, p. 114] (applied to the set in question intersected with a large ball centered at the origin) In concert, these properties allow us to conclude that Ω' is a uniform domain.

Finally, the version of *(4)* for an one-sided NTA domain Ω for which $\partial\Omega$ is unbounded is proved in [171, Lemma 2.19]. See also [14, Theorem 2.15], [23, Lemmas 4.2–4.3], [102, 256] for more in this regard (here we also wish to note that the fact that a uniform domain satisfies a Harnack chain condition may also be found in [39, Proposition 4.2]). □

A consequence of item *(3)* in Proposition 5.11.14 is that if the complement of a uniform domain $\Omega \subseteq \mathbb{R}^n$ satisfies an interior corkscrew condition, then Ω is an NTA domain. We also wish to observe that, as seen from items *(3)–(4)* in Proposition 5.11.14,

the class of uniform domains coincides, in a quantitative fashion, with the class of connected one-sided NTA domains (or interior NTA domains), i.e., the class of open connected sets satisfying an interior corkscrew condition as well as a Harnack chain condition, (5.11.65)

while items *(1)–(4)* in Proposition 5.11.14 together with Lemma 5.11.3 ensure that, in a quantitative fashion,

the class of (ε, δ)-domains with compact boundaries coincides with that of one-sided NTA (or interior NTA domains; i.e., open sets satisfying an interior corkscrew condition and a Harnack chain condition) with compact boundaries. (5.11.66)

Another useful result is that being an NTA domain is a property locally inherited near boundary points. More specifically, the following version of the main result in [136] holds; see also [132, Theorem 3.11, p. 96] (a direct proof in the smaller class of Zygmund domains is given in [132, Appendix, p. 140]).

Proposition 5.11.15 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an* NTA *domain* (*in the sense of Definition 5.11.1*). *Then there exist a constant* $C \in (1, \infty)$ *and a threshold* $R \in (0, \infty]$, *depending only on the* NTA *constants of* Ω (*and such that* $R = \infty$ *if* $\partial\Omega$ *is unbounded*), *with the property that for each* $x \in \partial\Omega$ *and* $r \in (0, R)$ *one may find a connected* NTA *domain* $\Omega_{x,r}$ (*with constants depending only on those of* Ω) *such that*

$$\Omega_{x,r} \subseteq \Omega \ \textit{ and } \ \Omega \cap B(x, r) \subseteq \Omega_{x,r} \subseteq \Omega \cap B(x, Cr). \tag{5.11.67}$$

Proof Suppose first that $\partial\Omega$ is unbounded. Then the Harnack chain condition from item *(i)* in Definition 5.11.1 holds without any limitations on the size of the scale. In turn, this implies that any two points in Ω may be joined via a polygonal path, hence Ω is connected. Granted this, items *(4)* and *(2)* in Proposition 5.11.14 imply that there exists some $\varepsilon > 0$ such that Ω is an (ε, δ)-domain for each $\delta > 0$.

To summarize, in the present case we conclude that there exists some $\varepsilon > 0$ with the property that Ω is a connected open subset of $\mathbb{R}^n$ which satisfies an exterior corkscrew condition and is an (ε, δ)-domain for each $\delta > 0$. This is what P. Jones calls in [136, p. 71] "an (ε, δ) nontangentially accessible domain" (or "(ε, δ) $N.T.A.$" for short). For this class of domains Ω, [136, Theorem, p. 71] guarantees that for each $x \in \partial\Omega$ and each $r \in (0, \infty)$ there exists a connected NTA domain $\Omega_{x,r}$ (with constants depending only on Ω) enjoying the properties listed in (5.11.67).

Consider next the case when $\partial\Omega$ is compact. From Lemma 5.11.3 we know that Ω has finitely many connected components, call them $\Omega_1, \dots, \Omega_N$ (where $N \in \mathbb{N}$), with the distance between any two of them strictly positive. In particular,

$$R := \inf_{1 \le j \ne k \le N} \operatorname{dist}\big(\Omega_j, \Omega_k\big) > 0. \tag{5.11.68}$$

To proceed, fix an arbitrary $j \in \{1, \dots, N\}$. Then Ω_j is itself a connected NTA domain with compact boundary. Once again, we may invoke items *(4)* and *(2)* in Proposition 5.11.14 to conclude that there exists some number $\varepsilon_j > 0$ such that Ω_j is an (ε_j, δ)-domain for each $\delta > 0$. Hence, in the terminology employed by P. Jones in [136, p. 71], the set Ω_j is an (ε_j, δ) $N.T.A.$ for each given $\delta > 0$. As such, [136, Theorem, p. 71] applies and guarantees the existence of a positive constant A_j, depending only on ε_j and n, with the property that for each $x \in \partial\Omega_j$ and each $r \in (0, \infty)$ we can find a connected NTA domain $\Omega_{x,r}$ (with constants depending only on Ω_j) such that

$$\Omega_{x,r} \subseteq \Omega_j \ \text{ and } \ \Omega_j \cap B(x, r) \subseteq \Omega_{x,r} \subseteq \Omega_j \cap B(x, A_j r). \tag{5.11.69}$$

Introduce $C := \max_{1 \le j \le N} A_j$. Restricting r to the interval $(0, R)$ then ultimately implies

$$\begin{gathered}\Omega_{x,r} \subseteq \Omega \ \text{ and } \ \Omega \cap B(x, r) \subseteq \Omega_{x,r} \subseteq \Omega \cap B(x, Cr) \\ \text{for each } x \in \partial\Omega \text{ and } r \in (0, R).\end{gathered} \tag{5.11.70}$$

The proof is therefore complete. □

Moving on, we discuss the following result, refining work in [193, Lemma 2.10, p. 972].

Lemma 5.11.16 *Let $\Omega \subset \mathbb{R}^n$ be a locally uniform domain. Recall from Definition 5.11.12 that this means that Ω is an open subset of $\mathbb{R}^n$ and there exist $c \in (0, \infty)$ and $R \in (0, \infty]$ such that (5.11.38) holds for all points $x, y \in \Omega$ satisfying $|x - y| < R$. As before, let $\delta_{\partial\Omega}(x)$ abbreviate* $\operatorname{dist}(x, \partial\Omega)$ *for each $x \in \mathbb{R}^n$.*

Next, fix a non-increasing function $\phi : (0, \infty) \to [0, \infty)$ and define

$$\widetilde{\phi} : [0, \infty) \to [0, \infty), \quad \widetilde{\phi}(r) := \int_0^{r/2} \phi(t)\, \mathrm{d}t \ \textit{ for each } \ r \geq 0. \tag{5.11.71}$$

Then there exists a constant $C \in (0, \infty)$, depending only on the uniform character of Ω, such that for every function $u \in \mathscr{C}^1(\Omega)$ with the property that

$$|\nabla u(x)| \leq \phi\big(\delta_{\partial\Omega}(x)\big) \ \textit{ for each } \ x \in \Omega \tag{5.11.72}$$

one has

$$|u(x) - u(y)| \leq C\widetilde{\phi}\big(|x - y|\big) \textit{ for each } \ x, y \in \Omega \ \textit{ with } \ |x - y| < R. \tag{5.11.73}$$

As a corollary, let us note that if $\Omega \subseteq \mathbb{R}^n$ is a (c, R)-locally uniform domain then for each $\alpha \in (0, 1)$ there exists a finite constant $C_{\alpha,c} > 0$ such that for each function $u \in \mathscr{C}^1(\Omega)$ we have

$$\sup_{\substack{x,y\in\Omega \\ 0<|x-y|<R}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C_{\alpha,c} \cdot \sup_{x\in\Omega} \Big\{ |(\nabla u)(x)| \cdot \operatorname{dist}(x, \partial\Omega)^{1-\alpha} \Big\}. \tag{5.11.74}$$

To justify this, fix some $u \in \mathscr{C}^1(\Omega)$ and denote by M the supremum in the right-hand side of (5.11.74). We can assume that $M < +\infty$, since otherwise there is nothing to prove. Granted this, (5.11.74) follows from (5.11.73) and (5.11.71) used with $\phi(t) := M \cdot t^{\alpha-1}$ for each $t > 0$.

In particular, (5.11.74) implies that if $\Omega \subseteq \mathbb{R}^n$ is a locally uniform domain then for each $\alpha \in (0, 1)$ there exists a finite constant $C = C(\Omega, \alpha) > 0$ such that

$$\|u\|_{\mathscr{C}^\alpha(\overline{\Omega})} \leq C \cdot \sup_{x\in\Omega} \Big\{ |(\nabla u)(x)| \cdot \operatorname{dist}(x, \partial\Omega)^{1-\alpha} + |u(x)| \Big\}, \ \ \forall u \in \mathscr{C}^1(\Omega). \tag{5.11.75}$$

A related version of the result recorded in (5.11.75) reads as follows: if the set $\Omega \subseteq \mathbb{R}^n$ is a locally uniform domain possessing a compact boundary, then there exists some closed set $D \subseteq \Omega$ satisfying $\operatorname{dist}(D, \partial\Omega) > 0$ and with the property that for each given exponent $\alpha \in (0, 1)$ there exists a finite constant $C = C(\Omega, \alpha) > 0$ such that for each function $u \in \mathscr{C}^1(\Omega)$ we have

$$\|u\|_{\mathscr{C}^\alpha(\overline{\Omega})} \leq C \cdot \sup_{x\in\Omega} \Big\{|(\nabla u)(x)| \cdot \operatorname{dist}(x, \partial\Omega)^{1-\alpha}\Big\} + C \cdot \sup_{x\in D} |u(x)|. \tag{5.11.76}$$

To justify this, assume Ω is a (c, R)-locally uniform domain. Then there exists some small threshold $r > 0$ such that $D := \big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq r\big\}$ has the property that for each point $x \in \Omega$ we can find some $x' \in D$ with $|x - x'| < R$. Since according to (5.11.74) this further entails that for each $x \in \Omega$ we have

$$\begin{aligned} |u(x)| &\leq |u(x')| + |u(x) - u(x')| \\ &\leq \sup_D |u| + R^\alpha \cdot \sup_\Omega \Big\{|\nabla u| \cdot \operatorname{dist}(\cdot, \partial\Omega)^{1-\alpha}\Big\}, \end{aligned} \tag{5.11.77}$$

we may now invoke (5.11.75) to conclude that (5.11.76) holds.

For further use it is also of interest to note that if $\Omega \subseteq \mathbb{R}^n$ is actually a uniform domain then we may take $R = \infty$ in (5.11.74) and conclude that for each $\alpha \in (0, 1)$ there exists a finite constant $C_{\Omega,\alpha} > 0$ such that

$$\begin{aligned} \|u\|_{\dot{\mathscr{C}}^\alpha(\overline{\Omega})} &= \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \\ &\leq C_{\Omega,\alpha} \cdot \sup_{x\in\Omega} \Big\{|(\nabla u)(x)| \cdot \operatorname{dist}(x, \partial\Omega)^{1-\alpha}\Big\}, \qquad \forall u \in \mathscr{C}^1(\Omega). \end{aligned} \tag{5.11.78}$$

Another useful observation is that if $\widetilde{\phi}$ is not identically $+\infty$, i.e., if

$$\int_0^\varepsilon \phi(t)\,\mathrm{d}t < +\infty \quad \text{for some} \quad \varepsilon > 0, \tag{5.11.79}$$

then Lebesgue's Dominated Convergence Theorem gives $\lim\limits_{r\to 0^+} \widetilde{\phi}(r) = 0$. In concert with (5.11.73), this shows that u is a uniformly continuous function on Ω. Consequently,

> if $\Omega \subseteq \mathbb{R}^n$ is a locally uniform domain and $u \in \mathscr{C}^1(\Omega)$ satisfies (5.11.72) for a non-increasing function $\phi : (0, \infty) \to [0, \infty)$ such that (5.11.79) holds, it follows that u has a unique extension to a function in $\mathscr{C}^0(\overline{\Omega})$. (5.11.80)

There is also a local version of Lemma 5.11.16 in which, having fixed some number $r_o \in (0, \infty)$, we now ask that

$$\begin{gathered} \phi : (0, r_o) \to [0, \infty) \text{ is non-increasing, and } \widetilde{\phi} : [0, 2r_o) \to [0, \infty) \\ \text{is given by } \widetilde{\phi}(r) := \textstyle\int_0^{r/2} \phi(t)\,\mathrm{d}t \text{ for each } r \in [0, 2r_o). \end{gathered} \tag{5.11.81}$$

For each $\rho > 0$ define $\Omega_\rho := \{x \in \Omega : \delta_{\partial\Omega}(x) < \rho\}$. Then, if $u \in \mathscr{C}^1(\Omega)$ is a function with the property that

$$|\nabla u(x)| \le \phi\big(\delta_{\partial\Omega}(x)\big) \quad \text{for each} \quad x \in \Omega_{r_o}, \tag{5.11.82}$$

the same type of proof given below shows that there exist $C_0, C_1, C_2 \in (1, \infty)$, depending only on the uniform character of Ω, such that

$$|u(x) - u(y)| \le C_0\widetilde{\phi}\big(|x - y|\big) \text{ for all } x, y \in \Omega_{r_o/C_1} \text{ with } |x - y| < C_2. \tag{5.11.83}$$

Granted this, much as in (5.11.80) we see that

$$\begin{array}{l}\text{if } \Omega \subseteq \mathbb{R}^n \text{ is a locally uniform domain and } u \in \mathscr{C}^1(\Omega) \text{ satisfies} \\ \text{(5.11.82) for a non-increasing function } \phi : (0, r_o) \to [0, \infty) \text{ such} \\ \text{that (5.11.79) holds, it follows that } u \text{ has a unique extension to a} \\ \text{function in } \mathscr{C}^0(\overline{\Omega}).\end{array} \tag{5.11.84}$$

For example, if $\Omega \subseteq \mathbb{R}^n$ is a locally uniform domain and $u \in \mathscr{C}^1(\Omega)$ satisfies

$$|\nabla u(x)| \le C\big|\ln\big(\delta_{\partial\Omega}(x)\big)\big| \text{ for each } x \in \Omega \text{ with } \delta_{\partial\Omega}(x) < 1/2, \tag{5.11.85}$$

then u extends uniquely to a function in $\mathscr{C}^0(\overline{\Omega})$, since $\int_0^{1/2} |\ln t|\, \mathrm{d}t < +\infty$. In fact, whenever (5.11.85) holds we conclude from (5.11.83) that there exist $C_0 \in (0, \infty)$ and some $\rho > 0$ such that

$$\begin{aligned} &|u(x) - u(y)| \le C_0\, |x - y|\big|\ln |x - y|\big| \\ &\text{for all } x, y \in \Omega_\rho \text{ with } |x - y| < \rho. \end{aligned} \tag{5.11.86}$$

We now present the proof of Lemma 5.11.16.

Proof of Lemma 5.11.16 Consider $c > 0$ such that condition (5.11.38) is satisfied. Let then $x, y \in \Omega$ be two arbitrary points with $|x - y| < R$ and assume γ is as in Definition 5.11.10. To proceed, denote by $L \in (0, \infty)$ and $s \in [0, L]$, respectively, the length of the curve $\gamma^* := \gamma([0, 1])$ and the arc-length parameter on γ^*. Also, let $[0, L] \ni s \mapsto \gamma(s) \in \gamma^*$ be the canonical arc-length parametrization of γ^*. In particular, $s \mapsto \gamma(s)$ is absolutely continuous and $\left|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\right| = 1$ for almost every point $s \in [0, L]$. In addition, as seen from (5.11.38), we have $L \le c|x - y|$ and

$$\min\big\{s, L - s\big\} \le c \cdot \delta_{\partial\Omega}\big(\gamma(s)\big) \quad \text{for each} \quad s \in [0, L], \tag{5.11.87}$$

hence

$$\begin{aligned} \delta_{\partial\Omega}\big(\gamma(s)\big) &\ge s/c \quad \text{for} \quad s \in (0, L/2) \quad \text{and} \\ \delta_{\partial\Omega}\big(\gamma(s)\big) &\ge (L - s)/c \quad \text{for} \quad s \in (L/2, L). \end{aligned} \tag{5.11.88}$$

We may therefore write

$$
\begin{aligned}
\int_0^L \phi\big(\delta_{\partial\Omega}(\gamma(s))\big)\,\mathrm{d}s &= \int_0^{L/2} \phi\big(\delta_{\partial\Omega}(\gamma(s))\big)\,\mathrm{d}s + \int_{L/2}^{L} \phi\big(\delta_{\partial\Omega}(\gamma(s))\big)\,\mathrm{d}s \\
&\leq \int_0^{L/2} \phi(s/c)\,\mathrm{d}s + \int_{L/2}^{L} \phi((L-s)/c)\,\mathrm{d}s \\
&= 2\int_0^{L/2} \phi(s/c)\,\mathrm{d}s = 2c\int_0^{L/(2c)} \phi(s)\,\mathrm{d}s \\
&\leq 2c\int_0^{|x-y|/2} \phi(s)\,\mathrm{d}s = 2c\widetilde{\phi}(|x-y|), \qquad (5.11.89)
\end{aligned}
$$

bearing in mind (5.11.88), that ϕ is non-increasing, the definition in (5.11.71), and the fact that $L \leq c|x-y|$ (cf. Definition 5.11.10). Then, since $\left|\frac{\mathrm{d}\gamma}{\mathrm{d}s}\right| = 1$ for almost every $s \in [0, L]$, for each function $u \in \mathscr{C}^1(\Omega)$ satisfying (5.11.72) we may estimate

$$
\begin{aligned}
|u(x) - u(y)| &= \left|\int_0^L \frac{\mathrm{d}}{\mathrm{d}s}\big[u(\gamma(s))\big]\,\mathrm{d}s\right| \\
&\leq \int_0^L \big|(\nabla u)(\gamma(s))\big|\,\mathrm{d}s \leq \int_0^L \phi\big(\delta_{\partial\Omega}(\gamma(s))\big)\,\mathrm{d}s \\
&\leq 2c\widetilde{\phi}(|x-y|), \qquad (5.11.90)
\end{aligned}
$$

finishing the proof of (5.11.73). □

The following diagram charts some of the known relations among various classes of domains in $\mathbb{R}^n$ that are relevant in the present work:

$$
\begin{aligned}
\big\{\mathscr{C}^1 \text{ domains}\big\} &\subsetneqq \big\{\text{domains locally given as upper-graphs} \\
&\qquad \text{of functions with gradients in } \mathrm{VMO} \cap L^\infty\big\} \\
&= \big\{\text{Lipschitz domains with VMO normals}\big\} \\
&\subsetneqq \big\{\text{Lipschitz domains}\big\} \subsetneqq \big\{\mathrm{BMO}_1\text{-domains}\big\} \\
&\subsetneqq \big\{\text{Zygmund domains}\big\} \cap \big\{\text{Ahlfors regular domains}\big\} \\
&\subsetneqq \big\{\text{two-sided NTA domains}\big\} \cap \big\{\text{Ahlfors regular domains}\big\} \\
&= \big\{\text{two-sided NTA domains with Ahlfors regular boundaries}\big\} \\
&\subsetneqq \big\{\text{UR domains}\big\} \subsetneqq \big\{\text{two-sided NTA domains}\big\} \\
&\subsetneqq \big\{\text{open sets satisfying a two-sided corkscrew condition}\big\} \\
&\subsetneqq \big\{\text{open sets which are two-sided } n\text{-thick}\big\} \\
&\subsetneqq \big\{\text{open sets with lower Ahlfors regular boundaries}\big\}. \qquad (5.11.91)
\end{aligned}
$$

Also,

$$\{\mathscr{C}^1 \text{ domains}\} \subsetneq \{\text{regular SKT domains}\} \subsetneq \{\text{UR domains}\}. \tag{5.11.92}$$

The label[26] "regular SKT domain" was proposed in [125], where it has been shown that this class of domains can be characterized as the family of sets

$\Omega \subset \mathbb{R}^n$ open with a compact Ahlfors regular boundary, satisfying a two-sided local John condition, and whose geometric measure theoretic unit normal ν has components belonging to the Sarason space $\mathrm{VMO}(\partial\Omega, \sigma)$, where $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. (5.11.93)

Moreover, it has been shown in [125] that this class of sets coincides with what was previously called "chord-arc domains with vanishing constant", and the above characterization eliminates certain redundancies in the original definition of the class of chord-arc domains with vanishing constant.

The following definition also originates in [125].

Definition 5.11.17 *Given $\delta > 0$, call $\Omega \subseteq \mathbb{R}^n$ a δ-*`regular` SKT `domain` *provided Ω is an open set satisfying a two-sided local John condition (cf. Definition 5.11.7), with a compact Ahlfors regular boundary, and whose geometric measure theoretic outward unit normal ν satisfies*

$$\mathrm{dist}\Big(\nu, \big[\mathrm{VMO}(\partial\Omega, \sigma)\big]^n\Big) := \inf_{\phi\in[\mathrm{VMO}(\partial\Omega,\sigma)]^n} \|\nu - \phi\|_{[\mathrm{BMO}(\partial\Omega,\sigma)]^n} < \delta, \tag{5.11.94}$$

where $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$.

The above class of sets interfaces tightly with the category of Reifenberg flat domains, hence also with the family of NTA domains.

Theorem 5.11.18 *Let $\Omega \subseteq \mathbb{R}^n$ be a δ-regular* SKT *domain. If $\delta > 0$ is sufficiently small relative to the dimension n, the John constants of Ω, and Ahlfors regularity constants of $\partial\Omega$, then there exists a geometric constant $C \in (1, \infty)$, which depends only on the dimension n, the John constants of Ω, and the Ahlfors regularity constants of $\partial\Omega$, with the property that Ω is a $(C\delta)$-Reifenberg flat domain, hence also a two-sided* NTA *domain.*

Proof This theorem is a consequence of [125, Theorem 4.19, p. 2710] and Definition 5.11.17. □

In the last portion of this section we review the notion of flatness, in the sense of Reifenberg. To facilitate the subsequent discussion, the reader is reminded that the Pompeiu–Hausdorff distance between two arbitrary nonempty sets $A, B \subset \mathbb{R}^n$ is defined as (cf. (2.8.131))

[26] With SKT acronym for Semmes–Kenig–Toro.

$$\mathrm{Dist}[A, B] := \max\Big\{\sup\{\mathrm{dist}(a, B) : a \in A\}, \sup\{\mathrm{dist}(b, A) : b \in B\}\Big\}. \tag{5.11.95}$$

We start by recalling the following definitions (see [149]).

Definition 5.11.19 *Fix $R \in (0, \infty]$ along with $\delta \in (0, \infty)$, and let $\Sigma \subset \mathbb{R}^n$ be a closed set. Then Σ is called a (R, δ)-*`Reifenberg flat set` *if for each for each $x \in \Sigma$ and each $r \in (0, R)$ there exists an $(n-1)$-dimensional plane $\pi(x, r)$ in $\mathbb{R}^n$ which contains x and satisfies*

$$\mathrm{Dist}\big[\Sigma \cap B(x, r), \pi(x, r) \cap B(x, r)\big] \leq \delta r. \tag{5.11.96}$$

For example, given $\delta > 0$, the graph of a real-valued Lipschitz function defined in $\mathbb{R}^{n-1}$ whose Lipschitz constant is $\leq \delta$ happens to be a δ-Reifenberg flat set (since (5.11.96) is then satisfied for the choice $\pi(x, r) := x + (\mathbb{R}^{n-1} \times \{0\})$).

Definition 5.11.20 *Fix $R \in (0, \infty]$ along with $\delta \in (0, \infty)$. A nonempty, proper subset Ω of $\mathbb{R}^n$ is said to satisfy the (R, δ)-*`separation property` *if for each $x \in \partial\Omega$ and $r \in (0, R)$ there exist an $(n-1)$-dimensional plane $\widetilde{\pi}(x, r)$ in $\mathbb{R}^n$ passing through x and a choice of unit normal vector $\vec{n}_{x,r}$ to $\widetilde{\pi}(x, r)$ such that*

$$\begin{aligned} &\big\{y + t\,\vec{n}_{x,r} \in B(x, r) : \ y \in \widetilde{\pi}(x, r), \ \ t > 2\delta r\big\} \subset \Omega \ \ \textit{and} \\ &\big\{y + t\,\vec{n}_{x,r} \in B(x, r) : \ y \in \widetilde{\pi}(x, r), \ \ t < -2\delta r\big\} \subset \mathbb{R}^n \setminus \Omega. \end{aligned} \tag{5.11.97}$$

Definition 5.11.21 *Fix $R \in \big(0, \mathrm{diam}\,\partial\Omega\big]$ along with $\delta \in (0, \infty)$. A nonempty, proper subset Ω of $\mathbb{R}^n$ is called an (R, δ)-*`Reifenberg flat domain` *(or simply a* `Reifenberg flat domain` *if the particular values of R, δ are not important) provided Ω satisfies the (R, δ)-separation property and $\partial\Omega$ is an (R, δ)-Reifenberg flat set.*

As is apparent from definitions,

the class of (R, δ)-Reifenberg flat domains in $\mathbb{R}^n$ becomes larger (in a set theoretic sense) as $\delta \in (0, \infty)$ increases. (5.11.98)

It also turns out that sufficiently flat Reifenberg domains are NTA domains. More specifically, from [149, Theorem 3.1, p. 524] and its proof we see that

there exists a purely dimensional constant $\delta_n \in (0, 1)$ with the property that any (R, δ)-Reifenberg flat domain $\Omega \subseteq \mathbb{R}^n$ is an NTA domain in the sense of Definition 5.11.1 with constants (R, M), where M is controlled in terms of δ and R. (5.11.99)

Chapter 6
Tools from Harmonic Analysis

In this chapter we collect and develop tools from Harmonic Analysis which are central to our subsequent work. Concretely, in Sect. 6.1 we discuss the regularized distance function and Whitney's Extension Theorem, while Sect. 6.2 amounts to a brief survey of Lorentz spaces in generic measure spaces. Also, in Sect. 6.3 we introduce and study the fractional Hardy–Littlewood maximal operator in a very general, non-metric setting. Next, in Sect. 6.4 we review the setting of Clifford algebras. These are higher-dimensional versions of the field of complex numbers that happen to be highly non-commutative, in which a brand of complex analysis may be developed (for example, there is a natural version in this setting of the classical Cauchy integral operator from the complex plane). We shall put the Clifford algebra machinery to good use later, in Volumes III–IV, to build a Calderón–Zygmund theory for singular integral operators on uniformly rectifiable sets. The discussion in Sect. 6.5 pertains to subaveraging functions, reverse Hölder estimates, and interior estimates. Finally, in Sect. 6.6 we introduce and study the solid maximal function introduced and maximal Lebesgue spaces.

6.1 The Regularized Distance Function and Whitney's Extension Theorem

Given a nonempty closed subset F of $\mathbb{R}^n$, define

$$\delta_F(x) := \operatorname{dist}(x, F) = \inf\Big\{|x-y| : y \in F\Big\}, \qquad \forall x \in \mathbb{R}^n. \tag{6.1.1}$$

In general, the function δ_F is merely Lipschitz, and a more regular version of it is described below.

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Proposition 6.1.1 *Let F be a nonempty closed subset of $\mathbb{R}^n$. Then there exist two finite purely dimensional constants $C_0, C_1 > 0$ along with a function $\delta_F^{\rm reg} : \mathbb{R}^n \to [0, \infty)$, the regularized distance to the set F, such that*

(1) for all $x \in \mathbb{R}^n$,

$$C_0 \cdot \delta_F(x) \leq \delta_F^{\rm reg}(x) \leq C_1 \cdot \delta_F(x), \tag{6.1.2}$$

(2) $\delta_F^{\rm reg} \in \mathscr{C}^\infty(\mathbb{R}^n \setminus F)$,

(3) for every multi-index $\alpha \in \mathbb{N}_0^n$ there exists a finite constant $C_\alpha > 0$, depending only on α and n, such that

$$\big|(\partial^\alpha \delta_F^{\rm reg})(x)\big| \leq C_\alpha \cdot \delta_F(x)^{1-|\alpha|}, \qquad \forall x \in \mathbb{R}^n \setminus F. \tag{6.1.3}$$

For a proof, see [240, Theorem 2, p. 171]. We shall next use Proposition 6.1.1 in the construction of a family of smooth functions, which vanish near the boundary, suitably approximating the characteristic function of an arbitrary open set.

Lemma 6.1.2 *Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$, and for each $\varepsilon > 0$ introduce the (open, one-sided) collar neighborhood $\mathcal{O}_\varepsilon$ of $\partial\Omega$ by setting*

$$\mathcal{O}_\varepsilon := \big\{x \in \Omega : \delta_{\partial\Omega}(x) < \varepsilon\big\}, \tag{6.1.4}$$

where $\delta_{\partial\Omega}(x)$ denotes the distance from the point x to the boundary $\partial\Omega$.

Then there exist a number $N > 1$ and a family of functions $\{\Phi_\varepsilon\}_{\varepsilon>0}$ satisfying the following properties for each $\varepsilon > 0$:

$$\Phi_\varepsilon \in \mathscr{C}^\infty(\Omega),\ \operatorname{supp}\Phi_\varepsilon \subseteq \Omega \setminus \mathcal{O}_{\varepsilon/N},\ 0 \leq \Phi_\varepsilon \leq 1,\ \Phi_\varepsilon \equiv 1 \text{ on } \Omega \setminus \mathcal{O}_\varepsilon, \tag{6.1.5}$$

$$\begin{array}{c}\text{and for each } \alpha \in \mathbb{N}_0^n \text{ there is } C_\alpha \in (0,\infty) \text{ so that} \\ \sup_{x\in\Omega}\big|(\partial^\alpha \Phi_\varepsilon)(x)\big| \leq C_\alpha \varepsilon^{-|\alpha|}.\end{array} \tag{6.1.6}$$

Proof To justify the existence of a family of functions $\{\Phi_\varepsilon\}_{\varepsilon>0}$ satisfying (6.1.5)–(6.1.6), let $\delta_{\partial\Omega}^{\rm reg}$ be the regularized distance to $\partial\Omega$ and for $\phi \in \mathscr{C}^\infty(\mathbb{R})$, $0 \leq \phi \leq 1$, and $\phi \equiv 0$ on $(-\infty, C_1/N)$ and $\phi \equiv 1$ on (C_0, ∞), where $N > C_1/C_0$ is fixed and $0 < C_0 \leq C_1 < +\infty$ are such that (6.1.2) holds with $F := \partial\Omega$. For each $\varepsilon > 0$ consider the function $\Phi_\varepsilon(x) := \phi\Big(\frac{\delta_{\partial\Omega}^{\rm reg}(x)}{\varepsilon}\Big)$ for all $x \in \Omega$. For each point $x \in \mathcal{O}_{\varepsilon/N}$ we have $\frac{\delta_{\partial\Omega}^{\rm reg}(x)}{\varepsilon} \leq C_1 \frac{\delta_{\partial\Omega}(x)}{\varepsilon} < \frac{C_1}{N}$, thus $\Phi_\varepsilon(x) = 0$. Moreover, if $x \in \Omega \setminus \mathcal{O}_\varepsilon$, then $x \in \Omega$ and $\delta_{\partial\Omega}(x) \geq \varepsilon$ hence $\delta_{\partial\Omega}^{\rm reg}(x) \geq C_0\delta_{\partial\Omega}(x) \geq C_0\varepsilon$ which forces $\Phi_\varepsilon(x) = 1$. Thus, all properties listed in (6.1.5) hold. Finally, the estimate in (6.1.6) is implied by (6.1.3). □

Next, we recall Whitney's extension operator. The reader is advised to recall the homogeneous Hölder space and its accompanying semi-norm from (7.3.1)–(7.3.2).

Theorem 6.1.3 *Let F be a nonempty, proper, closed subset of $\mathbb{R}^n$. Then there exists a linear operator $\mathcal{E}$, mapping the vector space of real-valued functions defined on F into the vector space of real-valued functions defined on $\mathbb{R}^n$, enjoying the following properties:*

(1) $\mathcal{E}$ is an extension operator, in the sense that $(\mathcal{E}f)\big|_F = f$ on F for every function $f : F \to \mathbb{R}$.
(2) $\mathcal{E}$ preserves constants (i.e., maps constant functions on F into constant functions on $\mathbb{R}^n$).
(3) $\mathcal{E}$ maps bounded functions on F into bounded functions on $\mathbb{R}^n$, in the precise quantitative sense that

$$\sup_{x\in\mathbb{R}^n} |(\mathcal{E}f)(x)| \leq \sup_{x\in F} |f(x)|, \qquad \forall f : F \to \mathbb{R}. \tag{6.1.7}$$

(5) $\mathcal{E}$ maps continuous real-valued functions defined on F into continuous real-valued functions defined on $\mathbb{R}^n$.
(6) For each $\gamma \in (0, 1]$, the operator $\mathcal{E}$ maps functions from the homogeneous Hölder space $\dot{\mathscr{C}}^\gamma(F)$ (cf. (7.3.1)–(7.3.2)) into functions belonging to the space $\dot{\mathscr{C}}^\gamma(\mathbb{R}^n) \cap \mathscr{C}^\infty(\mathbb{R}^n \setminus F)$ in the quantitative sense that there exists a constant $C_{n,\gamma} \in (0, \infty)$ with the property that, for every $f \in \dot{\mathscr{C}}^\gamma(F)$, we have

$$\|\mathcal{E}f\|_{\dot{\mathscr{C}}^\gamma(\mathbb{R}^n)} + \sup_{x\in\mathbb{R}^n\setminus F} \Big\{\delta_F^{1-\gamma}(x)\big|\nabla(\mathcal{E}f)(x)\big|\Big\} \leq C_{n,\gamma}\|f\|_{\dot{\mathscr{C}}^\gamma(F)}. \tag{6.1.8}$$

This result is a version of [240, Proposition on p. 172, Theorem 3 on p. 174], with some extra features noted in [10, Theorem 7.1] where the more general setting of geometrically doubling quasi-metric ambient has been considered. See also [188, Theorem 4.11, p. 164] in this regard. Classically, the extension operator $\mathcal{E}$ is constructed as

$$\mathcal{E}f(x) := \begin{cases} f(x) & \text{if } x \in F, \\ \sum\limits_{j\in J} f(x_j^*)\varphi_j(x) & \text{if } x \in \mathbb{R}^n \setminus F, \end{cases} \tag{6.1.9}$$

where, starting with a Whitney decomposition $\{Q_j\}_{j\in J}$ of $\mathbb{R}^n \setminus F$, the points x_j^* are arbitrarily chosen in Q_j for each $j \in J$, and $\{\varphi_j\}_{j\in J}$ is a suitable partition of unity associated with this Whitney decomposition. Specifically, fix $\varepsilon > 0$ sufficiently small (depending on the chosen Whitney decomposition of $\mathbb{R}^n \setminus F$), and pick a function

$$\begin{gathered} \psi^\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^n) \ \text{ with } \ 0 \leq \psi^\varepsilon \leq 1, \\ \psi^\varepsilon \equiv 1 \text{ on } Q_0, \ \text{ and } \ \operatorname{supp}\psi^\varepsilon \subseteq (1+\varepsilon)Q_0, \end{gathered} \tag{6.1.10}$$

where $Q_0 := \big[-\frac{1}{2}, \frac{1}{2}\big]^n \subseteq \mathbb{R}^n$. If for each cube $Q \subseteq \mathbb{R}^n$, of side-length $\ell(Q)$ and center x_Q, we now define

$$\psi_Q^\varepsilon(x) := \psi^\varepsilon\big(\tfrac{x-x_Q}{\ell(Q)}\big), \qquad \forall x \in \mathbb{R}^n, \tag{6.1.11}$$

then the family of functions

$$\varphi_j := \frac{\psi^{\varepsilon}_{Q_j}}{\sum_{i\in J}\psi^{\varepsilon}_{Q_i}}, \qquad \forall j \in J, \tag{6.1.12}$$

is the aforementioned Whitney partition of unity in $\mathbb{R}^n \setminus F$.

6.2 Short Foray into Lorentz Spaces

The scale of Lorentz spaces constitutes an environment in which a number of basic operators in Harmonic Analysis, including the Hardy–Littlewood maximal function and large classes of singular integral operators, act in a natural fashion. In this section we record some basic information about this scale.

To set the stage, let (X, μ) be an arbitrary measure space and fix a μ-measurable set $E \subseteq X$. For a given μ-measurable function $f : E \to \mathbb{R}$, introduce

$$m_E(\lambda, f) := \mu\big(\{x \in E : |f(x)| > \lambda\}\big), \qquad \forall \lambda \geq 0. \tag{6.2.1}$$

The non-increasing rearrangement of the function f is then defined as

$$\begin{aligned} f^*_E(t) &:= \inf\,\big\{\lambda \geq 0 : m_E(\lambda, f) \leq t\big\} \\ &= \sup\,\big\{\lambda \geq 0 : m_E(\lambda, f) > t\big\}, \qquad \forall t \in [0, \infty). \end{aligned} \tag{6.2.2}$$

The following proposition summarizes a number of properties of the non-increasing rearrangement of a function.

Proposition 6.2.1 *Let (X, μ) be a measure space and fix some μ-measurable set $E \subseteq X$. Suppose $f, g : E \to \mathbb{R}$ are two arbitrary μ-measurable functions. Then the following properties are true.*

(1) The function $f^*_E : [0, \infty) \to [0, \infty]$ is non-increasing, continuous to the right, and satisfies $f^*_E(0) = \|f\|_{L^\infty(E,\mu)}$. Furthermore, for every $\lambda > 0$ one has $m_E(\lambda, f) = m_{\mathbb{R}_+}(\lambda, f^*_E)$. Moreover, $(cf)^*_E = |c| f^*_E$ for every scalar c, and $(|f|^p)^*_E = (f^*_E)^p$ for every $p \in (0, \infty)$.

(2) Assuming $|f| \leq |g|$ at μ-a.e. point in E it follows that $g^*_E(t) \leq f^*_E(t)$ for each $t \in [0, \infty)$. In addition, $(f + g)^*_E(t_1 + t_2) \leq f^*_E(t_1) + g^*_E(t_2)$ for each $t_1, t_2 \in [0, \infty)$.

(3) For any sequence $\{f_j\}_{j\in\mathbb{N}}$ of μ-measurable functions on E, if $|f| \leq \liminf\limits_{j\to\infty} |f_j|$ at μ-a.e. point in E then $f^*_E \leq \liminf\limits_{j\to\infty} (f_j)^*_E$ on $[0, \infty)$. In particular, if $|f_j| \nearrow |f|$ as $j \to \infty$ at μ-a.e. point in E then $(f_j)^*_E \nearrow f^*_E$ as $j \to \infty$ at each point in the interval $[0, \infty)$.

(4) If $t \in [0, \infty)$ is such that $f^*_E(t) < +\infty$ then

$$\mu\big(\{x \in E : |f(x)| > f_E^*(t)\}\big) \le t \le \mu\big(\{x \in E : |f(x)| \ge f_E^*(t)\}\big). \quad (6.2.3)$$

(5) For every $p \in (0, \infty]$ *one has* (*with the usual convention when* $p = \infty$)

$$\Big(\int_E |f|^p \,\mathrm{d}\mu\Big)^{1/p} = \Big(\int_0^\infty \big(f_E^*(s)\big)^p \,\mathrm{d}s\Big)^{1/p}. \quad (6.2.4)$$

(6) Fix $p \in (0, \infty)$ *and suppose* $t \in [0, \infty)$ *is such that* $f_E^*(t) < +\infty$. *Then*

$$\int_{\{x\in E:\,|f(x)|>f_E^*(t)\}} |f(x)|^p \,\mathrm{d}\mu(x) \le \int_0^t \big(f_E^*(s)\big)^p \,\mathrm{d}s. \quad (6.2.5)$$

Proof For *(1)–(5)* we refer the reader to, e.g., [30, Propositions 1.9.2–1.9.6, pp. 49–51], and [107, Proposition 1.4.5, p. 47]. To prove (6.2.5), for each $M \in [0, \infty)$ introduce

$$[f]^M(x) := \begin{cases} f(x) & \text{if } |f(x)| > M, \\ 0 & \text{if } |f(x)| \le M, \end{cases} \qquad \forall x \in E. \quad (6.2.6)$$

Then it is obvious that $|[f]^M| \le |f|$ on E, so by *(2)*, we have

$$\big([f]^M\big)_E^* \le f_E^* \;\text{ on } E, \text{ for all }\; M \in [0, \infty). \quad (6.2.7)$$

From definitions we also have that

$$\big([f]^M\big)_E^*(s) = \inf\big\{\tau \ge 0 : \mu\big(\{x \in E : [f]^M(x) > \tau\}\big) \le s\big\} \quad (6.2.8)$$

for each $s \in [0, \infty)$ and

$$\mu\big(\{x \in E : [f]^M(x) > \tau\}\big) = \mu\big(\{x \in E : |f(x)| > M\}\big) \quad (6.2.9)$$

for all $\tau \in [0, M]$. Hence, given any cutoff parameter $M \in [0, \infty)$, we have

$$s > \mu\big(\{x \in E : |f(x)| > M\}\big) \Longrightarrow \big([f]^M\big)_E^*(s) = 0. \quad (6.2.10)$$

Now set $M := f_E^*(t) \in [0, \infty)$. Combining *(4)* with (6.2.10) yields

$$s > t \Longrightarrow s > \mu\big(\{x \in E : |f(x)| > f_E^*(t)\}\big) \Longrightarrow \big([f]^M\big)_E^*(s) = 0. \quad (6.2.11)$$

Therefore, for each given $p \in (0, \infty)$ we may write

$$\int_{\{x\in E:\,|f(x)|>f_E^*(t)\}} |f(x)|^p \,\mathrm{d}\mu(x) = \int_E \big([f]^M(x)\big)^p \,\mathrm{d}\mu(x) = \int_0^\infty \Big(\big([f]^M\big)_E^*(s)\Big)^p \,\mathrm{d}s$$
$$= \int_0^t \Big(\big([f]^M\big)_E^*(s)\Big)^p \,\mathrm{d}s \le \int_0^t \big(f_E^*(s)\big)^p \,\mathrm{d}s, \tag{6.2.12}$$

where for the first equality in (6.2.12) we used (6.2.6) (given the current choice of M), for the second equality we used *(5)*, the implication in (6.2.11) was used in the third equality, while the last inequality is based on (6.2.7). This completes the proof of (6.2.5). □

Continuing to assume that (X, μ) is an arbitrary measure space, we next define the scale of `Lorentz spaces`, $L^{p,q}(X, \mu)$ with $p, q \in (0, \infty]$. Specifically, we set

$$L^{p,q}(X,\mu) := \Big\{ f : X \to \overline{\mathbb{R}},\ \ \mu\text{-measurable} :\ \|f\|_{L^{p,q}(X,\mu)} < +\infty \Big\} \tag{6.2.13}$$

where, for each μ-measurable function f on X, the quasi-norm $\|f\|_{L^{p,q}(X,\mu)}$ is definedas[1]

$$\|f\|_{L^{p,q}(X,\mu)} := \begin{cases} \left(\displaystyle\int_0^\infty \big[t^{1/p} f_X^*(t)\big|^q \,\frac{\mathrm{d}t}{t} \right)^{1/q} & \text{if } \ 0 < p, q < \infty, \\ \sup_{t>0} \big[t^{1/p} f_X^*(t)\big] & \text{if } \ 0 < p \le \infty,\ \ q = \infty, \\ \|f\|_{L^\infty(X,\mu)} & \text{if } \ p = \infty,\ \ 0 < q \le \infty. \end{cases} \tag{6.2.14}$$

Since for each μ-measurable function f we have $\sup_{t>0}\big[f_X^*(t)\big] = \|f\|_{L^\infty(X,\mu)}$, the overlap in the last two lines of (6.2.14) causes no problems. Also, it is clear that

$$L^{\infty,q}(X,\mu) = L^\infty(X,\mu) \ \text{ for all } \ q \in (0,\infty]. \tag{6.2.15}$$

It is also apparent from the above definitions and Proposition 6.2.1 that

$$\begin{aligned} &\text{if } f, g : X \to \mathbb{R} \text{ are two } \mu\text{-measurable functions such that} \\ &|g| \le |f| \text{ at } \mu\text{-a.e. point on } X \text{ and } f \in L^{p,q}(X,\mu), \text{ then} \\ &g \in L^{p,q}(X,\mu) \text{ and one has } \|g\|_{L^{p,q}(X,\mu)} \le \|f\|_{L^{p,q}(X,\mu)}. \end{aligned} \tag{6.2.16}$$

In addition, it is useful to observe that, much as for ordinary Lebesgue spaces, for each μ-measurable set $E \subseteq X$ we have (with tilde denoting extension by zero from E to X)

$$\begin{aligned} L^{p,q}(E,\mu) \ni f &\longmapsto \widetilde{f} \in L^{p,q}(X,\mu) \ \text{ continuously}, \\ L^{p,q}(X,\mu) \ni f &\longmapsto f\big|_E \in L^{p,q}(E,\mu) \ \text{ continuously}. \end{aligned} \tag{6.2.17}$$

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[1] The choice of the quasi-norm in the case when $p = \infty$ is natural; indeed, since f_X^* is non-increasing, having $\int_0^1 \big[f_X^*(t)\big|^q \frac{\mathrm{d}t}{t} < +\infty$ for some $q \in (0, \infty)$ would always force f to vanish μ-a.e. on X.

Based on (6.2.16) one may justify that the sum of any two Lorentz spaces is a lattice. Specifically, given any $p_i, q_i \in (0, \infty]$ with $i \in \{0, 1\}$,

$$\begin{array}{l}\text{if } f, g : X \to \mathbb{R} \text{ are } \mu\text{-measurable functions with } |g| \leq |f|, \\ \mu\text{-a.e. on } X, \text{ and the function } f \in L^{p_0,q_0}(X,\mu) + L^{p_1,q_1}(X,\mu), \\ \text{then the function } g \in L^{p_0,q_0}(X,\mu) + L^{p_1,q_1}(X,\mu) \text{ and there holds} \\ \|g\|_{L^{p_0,q_0}(X,\mu)+L^{p_1,q_1}(X,\mu)} \leq \|f\|_{L^{p_0,q_0}(X,\mu)+L^{p_1,q_1}(X,\mu)}.\end{array} \tag{6.2.18}$$

Indeed, if $f = f_0 + f_1$ with $f_i \in L^{p_i,q_i}(X,\mu)$ for $i \in \{0, 1\}$, then we may decompose $g = g_0 + g_1$, with $g_0 := g \cdot \mathbf{1}_{\{|f_0| \geq |f_1|\}}$ belonging to the space to $L^{p_0,q_0}(X,\mu)$, and $g_1 := g \cdot \mathbf{1}_{\{|f_1| > |f_0|\}}$ belonging to $L^{p_1,q_1}(X,\mu)$.

We also wish to note that for each μ-measurable function f on X and each $p, q \in (0, \infty]$ simple power-dilation considerations show that

$$\big\| \, |f|^{\alpha} \big\|_{L^{p,q}(X,\mu)} = \|f\|^{\alpha}_{L^{\alpha p,\alpha q}(X,\mu)} \quad \text{for each } \alpha \in (0, \infty). \tag{6.2.19}$$

Recall next (cf., e.g., [233, p. 332]) that whenever $0 < p, q < \infty$ we have

$$\|f\|_{L^{p,q}(X,\mu)} \approx \Big(\int_0^{\infty} \big[\lambda^p \cdot m_X(\lambda, f)\big]^{q/p} \, \frac{d\lambda}{\lambda} \Big)^{1/q}, \tag{6.2.20}$$

uniformly in the class of μ-measurable functions f on X. Also, corresponding to the case when $q = \infty$ and $0 < p < \infty$,

$$\|f\|_{L^{p,\infty}(X,\mu)} \approx |||f|||_{L^{p,\infty}(X,\mu)} := \sup \big\{\lambda \cdot m_X(\lambda, f)^{1/p} : \lambda > 0\big\}, \tag{6.2.21}$$

uniformly in the class of μ-measurable functions f on X.

Example 6.2.2 *Given an upper Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \Sigma$ and fix an arbitrary number $d \in (0, \infty)$. Then there exists a purely geometric constant $C \in (0, \infty)$ with the property that for each point $x_o \in \mathbb{R}^n$ the function*

$$\begin{array}{c} f_{x_o} : \Sigma \to \mathbb{R}, \quad f_{x_o}(x) := |x - x_o|^{-d}, \quad \forall x \in \Sigma, \\ \textit{satisfies } \; f_{x_o} \in L^{(n-1)/d,\infty}(\Sigma, \sigma) \; \textit{ and } \; \|f_{x_o}\|_{L^{(n-1)/d,\infty}(\Sigma,\sigma)} \leq C. \end{array} \tag{6.2.22}$$

Indeed, having picked some point $x_o \in \mathbb{R}^n$ together with an arbitrary threshold $\lambda > 0$, for each $x \in \Sigma$ we have

$$|f_{x_o}(x)| > \lambda \Longleftrightarrow |x - x_o| < \lambda^{-1/d} \Longleftrightarrow x \in \Sigma \cap B(x_o, \lambda^{-1/d}) \tag{6.2.23}$$

hence, in view of the upper Ahlfors regularity of Σ and (5.9.6),

$$\begin{aligned} m_{\Sigma}(\lambda, f) &= \sigma\big(\{x \in \Sigma : |f_{x_o}(x)| > \lambda\}\big) = \sigma\big(\Sigma \cap B(x_o, \lambda^{-1/d})\big) \\ &\leq C \cdot \lambda^{-(n-1)/d}. \end{aligned} \tag{6.2.24}$$

From this and (6.2.21) (used with $p := (n-1)/d$) we conclude that (6.2.22) holds.

Moving on, we note that the scale of Lorentz spaces contains the ordinary Lebesgue spaces,

$$L^{p,p}(X,\mu) = L^p(X,\mu), \quad 0 < p \leq \infty, \tag{6.2.25}$$

and is (quantitatively) increasing in the second exponent, in the sense that the following inclusions are well-defined and continuous

$$L^{p,q_1}(X,\mu) \hookrightarrow L^{p,q_2}(X,\mu), \quad 0 < p \leq \infty, \quad 0 < q_1 \leq q_2 \leq \infty. \tag{6.2.26}$$

In particular,

$$\begin{gathered} L^{p,q}(X,\mu) \hookrightarrow L^p(X,\mu) \hookrightarrow L^{p,\infty}(X,\mu) \ \text{ continuously,} \\ \text{for each } \ p \in (0,\infty] \ \text{ and each } \ q \in (0,p]. \end{gathered} \tag{6.2.27}$$

Lemma 6.2.3 *Let (X,μ) be an arbitrary measure space. Then the following properties are valid:*

$$\begin{gathered} \textit{if } 0 < r < p < \infty \textit{ then for each } \mu\textit{-measurable function } f \textit{ on } X \textit{ one has} \\ \sup_{\substack{E \subseteq X \ \mu\text{-measurable} \\ 0 < \mu(E) < \infty}} \mu(E)^{\frac{1}{p}-\frac{1}{r}} \Big(\int_E |f|^r \, d\mu \Big)^{1/r} \leq \Big(\frac{p}{p-r} \Big)^{\frac{1}{r}} \|f\|_{L^{p,\infty}(X,\mu)}, \end{gathered} \tag{6.2.28}$$

and

$$\begin{gathered} \textit{if } p, r \in (0,\infty) \textit{ then for each function } f \in L^{p,\infty}(X,\mu) \textit{ one has} \\ \|f\|_{L^{p,\infty}(X,\mu)} \leq \sup_{\substack{E \subseteq X \ \mu\text{-measurable} \\ 0 < \mu(E) < \infty}} \mu(E)^{\frac{1}{p}-\frac{1}{r}} \Big(\int_E |f|^r \, d\mu \Big)^{1/r}. \end{gathered} \tag{6.2.29}$$

Hence, as a consequence of (6.2.28) and (6.2.29),

$$\begin{gathered} \|f\|_{L^{p,\infty}(X,\mu)} \approx \sup_{\substack{E \subseteq X \ \mu\text{-measurable} \\ 0 < \mu(E) < \infty}} \mu(E)^{\frac{1}{p}-\frac{1}{r}} \Big(\int_E |f|^r \, d\mu \Big)^{1/r}, \\ \textit{for each fixed } r \in (0,p), \textit{ uniformly in } f \in L^{p,\infty}(X,\mu). \end{gathered} \tag{6.2.30}$$

Finally, there is the following variant of (6.2.29), valid for μ-measurable functions:

$$\begin{gathered} \textit{if } (X,\mu) \textit{ is sigma-finite, then for each } p, r \in (0,\infty) \\ \textit{and each } \mu\textit{-measurable function } f \textit{ on } X \textit{ one has} \\ \|f\|_{L^{p,\infty}(X,\mu)} \leq \sup_{\substack{E \subseteq X \ \mu\text{-measurable} \\ 0 < \mu(E) < \infty}} \mu(E)^{\frac{1}{p}-\frac{1}{r}} \Big(\int_E |f|^r \, d\mu \Big)^{1/r}. \end{gathered} \tag{6.2.31}$$

Proof To justify (6.2.28), fix an arbitrary μ-measurable function f on X, along with some $r \in (0,p)$. In this setting, note that if $E \subseteq X$ is a μ-measurable set with the property $\mu(E) \in (0, \infty)$ then

$$\begin{aligned}\int_E |f|^r \,\mathrm{d}\mu &= r\int_0^\infty \lambda^{r-1}\mu\big(\{x \in E : |f(x)| > \lambda\}\big)\,\mathrm{d}\lambda \\ &\leq r\int_0^\infty \lambda^{r-1}\cdot\min\left\{\mu(E), \left(\frac{\|f\|_{L^{p,\infty}(X,\mu)}}{\lambda}\right)^p\right\}\mathrm{d}\lambda \\ &= r\mu(E)\int_0^{\|f\|_{L^{p,\infty}(X,\mu)}/\mu(E)^{1/p}} \lambda^{r-1}\,\mathrm{d}\lambda \\ &\quad + r\|f\|^p_{L^{p,\infty}(X,\mu)}\int_{\|f\|_{L^{p,\infty}(X,\mu)}/\mu(E)^{1/p}}^\infty \lambda^{r-p-1}\,\mathrm{d}\lambda \\ &= \mu(E)^{1-r/p}\|f\|^r_{L^{p,\infty}(X,\mu)} + \Big(\frac{r}{p-r}\Big)\mu(E)^{1-r/p}\|f\|^r_{L^{p,\infty}(X,\mu)} \\ &= \Big(\frac{p}{p-r}\Big)\mu(E)^{1-r/p}\|f\|^r_{L^{p,\infty}(X,\mu)}. \end{aligned} \tag{6.2.32}$$

From this, (6.2.28) readily follows.

To justify (6.2.29), fix $f \in L^{p,\infty}(X,\mu)$. Hence, if for each $\lambda \in (0,\infty)$ we define

$$E_\lambda := \{x \in X : |f(x)| > \lambda\} \tag{6.2.33}$$

then each E_λ is a μ-measurable subset of X satisfying $\mu(E_\lambda) < \infty$. In the process of proving the estimate in (6.2.29) it suffices to treat the case when $\|f\|_{L^{p,\infty}(X,\mu)} > 0$, since otherwise this is trivially satisfied. In such a scenario, we have that there exists some $\lambda \in (0,\infty)$ with the property that $\mu(E_\lambda) > 0$. In light of these comments we may then write

$$\begin{aligned}\sup_{\substack{E\subseteq X\ \mu\text{-measurable}\\ 0<\mu(E)<\infty}} \mu(E)^{\frac{1}{p}-\frac{1}{r}}\Big(\int_E |f|^r\,\mathrm{d}\mu\Big)^{1/r} &\geq \sup_{\substack{\lambda\in(0,\infty)\\ \mu(E_\lambda)>0}} \mu(E_\lambda)^{\frac{1}{p}-\frac{1}{r}}\Big(\int_{E_\lambda} |f|^r\,\mathrm{d}\mu\Big)^{1/r} \\ &\geq \sup_{\substack{\lambda\in(0,\infty)\\ \mu(E_\lambda)>0}} \mu(E_\lambda)^{\frac{1}{p}-\frac{1}{r}}\cdot\mu(E_\lambda)^{\frac{1}{r}}\cdot\lambda \\ &= \sup_{\lambda\in(0,\infty)} \lambda\cdot\mu\big(\{x \in X : |f(x)| > \lambda\}\big)^{\frac{1}{p}} \\ &= \|f\|_{L^{p,\infty}(X,\mu)}, \end{aligned} \tag{6.2.34}$$

establishing (6.2.29). As regards (6.2.31), assume that $X = \cup_{i\in\mathbb{N}} X_i$ with each X_i a μ-measurable set satisfying $\mu(X_i) < \infty$. There is no loss of generality in assuming that the family $\{X_i\}_{i\in\mathbb{N}}$ is nested and exhausts X. In this context, (6.2.31) is proved in a similar manner to (6.2.29), this time working with $X_i \cap E_\lambda$ in place of E_λ, and eventually passing to limit as $i \to \infty$. □

Going further, given an arbitrary measure space (X,μ), for each $r \in (0,\infty)$ we define

$$L^r_{\text{fin}}(X,\mu) := \Big\{ f\,\mu\text{-measurable on } X : \int_E |f|^r \, d\mu < \infty \text{ for each} \\ \mu\text{-measurable set } E \subseteq X \text{ with } \mu(E) < \infty \Big\}. \tag{6.2.35}$$

Lemma 6.2.4 *If (X,μ) is an arbitrary measure space then*

$$L^{p,q}(X,\mu) \subseteq L^r_{\text{fin}}(X,\mu) \text{ whenever } 0 < r < p \leq \infty \text{ and } 0 < q \leq \infty, \tag{6.2.36}$$

in a quantitative fashion. In fact, if $0 < r < p < \infty$ and $f \in L^{p,\infty}(X,\mu)$, then for each μ-measurable subset E of X one has

$$\int_E |f|^r \, d\mu \leq \Big(\frac{p}{p-r}\Big) \mu(E)^{1-\frac{r}{p}} \|f\|^r_{L^{p,\infty}(X,\mu)}. \tag{6.2.37}$$

As a corollary of this and (6.2.17),

$$L^{p,q}(E,\mu) \hookrightarrow L^{p^*}(E,\mu) \text{ continuously whenever } E \subseteq X \text{ is} \\ \text{a } \mu\text{-measurable set with } \mu(E) < +\infty \text{ and } p, q, p^* \text{ are such} \\ \text{that } 0 < p^* < p \leq \infty \text{ and } 0 < q \leq \infty. \tag{6.2.38}$$

Proof Having fixed $0 < r < p < \infty$ along with $f \in L^{p,\infty}(X,\mu)$, for each given μ-measurable subset E of X the estimate in (6.2.37) follows from (6.2.32) in the case when $0 < \mu(E) < \infty$ and is otherwise trivially true. In turn, (6.2.37) readily implies (6.2.36) on account of (6.2.26). □

Additional basic properties of functions belonging to Lorentz spaces are contained in the lemma below.

Lemma 6.2.5 *Assume (X,μ) is an arbitrary measure space.*

(a) If $f \in L^{p,q}(X,\mu)$ with $p, q \in (0,\infty]$ then

$$|f(x)| < +\infty \text{ for } \mu\text{-a.e. point } x \in X \text{ and} \\ \mu\big(\{x \in X : |f(x)| > \lambda\}\big) < +\infty \text{ for each } \lambda > 0. \tag{6.2.39}$$

(b) For each $p \in (0,\infty)$ and $q \in (0,\infty]$ one has

$$\big\|\mathbf{1}_A\big\|_{L^{p,q}(X,\mu)} \approx \mu(A)^{1/p}, \text{ uniformly in} \\ \text{the class of } \mu\text{-measurable subsets } A \text{ of } X. \tag{6.2.40}$$

(c) For each μ-measurable function f on X and each $p \in (0,\infty)$ one has

$$f^*_X(t) \leq C t^{-1/p} \|f\|_{L^{p,\infty}(X,\mu)} \text{ for each } t \in (0,\infty), \tag{6.2.41}$$

where $C \in (0,\infty)$ is independent of f.

Proof To prove *(a)*, fix some $f \in L^{p,q}(X,\mu)$ with $p, q \in (0,\infty]$. Consider the claim in the first line of (6.2.39). If $p = \infty$ this is clear from (6.2.15), so assume $p \in (0,\infty)$. Define $A := \{x \in X : |f(x)| = +\infty\}$ and note that A is μ-measurable and satisfies $\mu(A) \leq m_X(\lambda, f)$ for each $\lambda \in (0,\infty)$. Consequently, having $\mu(A) > 0$ would prevent the integral in (6.2.20) to converge at the end-point $+\infty$ when $q < \infty$ and, corresponding to $q = \infty$, having $\mu(A) > 0$ would force the supremum in (6.2.21) to be $+\infty$. These eventualities contradict the fact that $\|f\|_{L^{p,q}(X,\mu)} < +\infty$, thus finishing the proof of the claim in the first line of (6.2.39).

Turning to the claim in the second line of (6.2.39), observe that if $q = \infty$ then the result is clear from the nature of the norm in $L^{p,\infty}(X,\mu)$. Assume next that $q < \infty$. Then

$$\Big(\int_0^\infty \big[t^{\frac{1}{p}} f_X^*(t)\big]^q \,\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{q}} = \|f\|_{L^{p,q}(X,\mu)} < +\infty, \tag{6.2.42}$$

which implies

$$f_X^*(t) < +\infty \quad \text{for } \mathcal{L}^1\text{-a.e. } t > 0. \tag{6.2.43}$$

Since f_X^* is non-increasing, (6.2.43) further implies

$$\inf\big\{s > 0 : m_X(s, f) \leq t\big\} = f_X^*(t) < +\infty \quad \text{for all} \quad t > 0. \tag{6.2.44}$$

Therefore, for each $t > 0$ there exists $s_t > 0$ such that $m_X(s_t, f) \leq t$. Fix now $\lambda > 0$, and assume that $m_X(\lambda, f) = +\infty$. Then necessarily, $f_X^*(t) \geq \lambda$ for all $t > 0$. But then, (6.2.42) gives

$$+\infty > \Big(\int_0^\infty \big[t^{\frac{1}{p}} f_X^*(t)\big]^q \,\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{q}} \geq \lambda\Big(\int_0^\infty t^{\frac{q}{p}-1}\,\mathrm{d}t\Big)^{\frac{1}{q}} = +\infty, \tag{6.2.45}$$

a contradiction which finishes the proof of (6.2.39).

Next, the claim in (6.2.40) is clear from (6.2.20) if $q < \infty$ and from (6.2.21) if $q = \infty$, bearing in mind that $m_X(\lambda, \mathbf{1}_A) = \mu(A)\mathbf{1}_{(0,1)}(\lambda)$ for each $\lambda \in (0,\infty)$. This takes care of part *(b)*.

As regards part *(c)*, since (6.2.21) implies $\lambda \cdot m_X(\lambda, f)^{1/p} \leq C\|f\|_{L^{p,\infty}(X,\mu)}$ for each $\lambda > 0$, it follows that

$$m_X(\lambda, f) \leq t \quad \text{whenever} \quad t \in (0,\infty) \quad \text{and} \quad \lambda \geq C t^{-1/p}\|f\|_{L^{p,\infty}(X,\mu)}. \tag{6.2.46}$$

Then (6.2.41) becomes a consequence of (6.2.46) and (6.2.2). □

Using the real interpolation method between Lebesgue spaces over a measure space (X,μ) yields that if $0 < p_0, p_1, q_0, q_1 \leq \infty$ then

$$\begin{aligned}&\big(L^{p_0,q_0}(X,\mu), L^{p_1,q_1}(X,\mu)\big)_{\theta,q} = L^{p,q}(X,\mu),\\ &\text{if } 0 < q \leq \infty,\ \ 0 < \theta < 1,\ \text{ if } p_0 = p_1 = p \text{ and } \tfrac{1}{q} = \tfrac{1-\theta}{q_0} + \tfrac{\theta}{q_1},\\ &\text{or if } \ \tfrac{1}{p} = \tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1} \ \text{ when } \ p_0 \neq p_1.\end{aligned} \tag{6.2.47}$$

See [19, Theorem 5.3.1, p. 113]. In particular,

$$\begin{aligned} &\big(L^{p_0}(X,\mu), L^{p_1}(X,\mu)\big)_{\theta,q} = L^{p,q}(X,\mu) \\ &\text{if } \ 0 < p_0, p_1 \leq \infty \ \text{ with } \ p_0 \neq p_1, \ \text{ and} \\ &0 < q \leq \infty, \ 0 < \theta < 1, \ \tfrac{1}{p} = \tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1}. \end{aligned} \tag{6.2.48}$$

There are also natural log-convex estimates accompanying (6.2.47)–(6.2.48). For example,

$$\begin{aligned} &\text{if } 0 < p_0 < p_1 \leq \infty,\ q \in (0,\infty], \text{ and } f \in L^{p_0}(X,\mu) \cap L^{p_1}(X,\mu), \\ &\text{it follows that } f \in L^{p,q}(X,\mu) \text{ for all } p \in [p_0,p_1], \text{ and for each} \\ &\theta \in [0,1] \text{ there exists a constant } C = C(p_0,p_1,q,\theta) \in (0,\infty) \text{ with} \\ &\text{the property that } \|f\|_{L^{p_\theta,q}(X,\mu)} \leq C\|f\|^{1-\theta}_{L^{p_0}(X,\mu)}\|f\|^{\theta}_{L^{p_1}(X,\mu)} \text{ where} \\ &p_\theta := \big(\tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1}\big)^{-1}. \end{aligned} \tag{6.2.49}$$

Recall that for any pair of compatible quasi-Banach spaces X_0, X_1 one has (cf. [185, Sect. 1.3])

$$\begin{aligned} &X_0 \cap X_1 \hookrightarrow (X_0, X_1)_{\theta,q} \hookrightarrow X_0 + X_1 \ \text{ continuously,} \\ &\qquad \text{for each } \theta \in (0,1) \text{ and } q \in (0,\infty]. \end{aligned} \tag{6.2.50}$$

Then, if $0 < p_0 < p < p_1 \leq \infty$ and $0 < q \leq \infty$ we may invoke (6.2.48) and we conclude that

$$\begin{aligned} &L^{p_0}(X,\mu) \cap L^{p_1}(X,\mu) \hookrightarrow L^{p,q}(X,\mu) \ \text{ continuously,} \\ &\qquad \text{and also densely if } q < \infty, \end{aligned} \tag{6.2.51}$$

while (6.2.50), (6.2.48), (3.1.11), and (6.2.40) imply

$$\begin{aligned} &L^{p,q}(X,\mu) \hookrightarrow L^{p_0}(X,\mu) + L^{p_1}(X,\mu) \ \text{ continuously,} \\ &\qquad \text{and also densely if } p_1 < \infty. \end{aligned} \tag{6.2.52}$$

It turns out that restricting functions from Lorentz spaces to subsets of finite measure yields functions in Lebesgue spaces, albeit for a smaller integrability exponent. A precise statement is as follows.

Lemma 6.2.6 *Let (X,μ) be a measure space and let $E \subseteq X$ be a μ-measurable set with $\mu(E) < \infty$. Then, for $0 < p^* < p < \infty$ and $0 < q_1, q_2 \leq \infty$,*

$$L^{p,q_1}(X,\mu) \ni f \longmapsto f\big|_E \in L^{p^*,q_2}(E,\mu) \tag{6.2.53}$$

is a well-defined, linear, and bounded operator.

Proof This follows from Lemma 6.2.4 and interpolation. A direct proof is as follows. If $\mu(E) = 0$ the statement is trivial, so assume $\mu(E) \in (0,\infty)$. According to (6.2.26) it suffices to establish (6.2.53) in the case when $q_1 = \infty$ and $q_2 \in (0,\infty)$. Assuming this is the case, fix an arbitrary function $f \in L^{p,\infty}(X, \mu)$ and, for ease of notation, abbreviate $c := \|f\|_{L^{p,\infty}(X,\mu)}$. Then

$$m_E(\lambda, f) \leq m_X(\lambda, f) \leq \frac{c^p}{\lambda^p} \text{ and also } m_E(\lambda, f) \leq \mu(E) \text{ for all } \lambda > 0, \tag{6.2.54}$$

so we may estimate

$$\begin{aligned} \big\| f|_E \big\|_{L^{p^*,q_2}(E,\mu)} &\leq C\Big(\int_0^\infty \big[\lambda^{p^*} \cdot m_E(\lambda, f)\big]^{q_2/p^*} \frac{d\lambda}{\lambda}\Big)^{1/q_2} \\ &\leq C\Big(\int_0^\infty \big[\lambda^{p^*} \cdot \min\big\{\mu(E), c^p\lambda^{-p}\big\}\big]^{q_2/p^*} \frac{d\lambda}{\lambda}\Big)^{1/q_2} \\ &\leq C\mu(E)^{1/p^*}\Big(\int_0^{\frac{c}{\mu(E)^{1/p}}} \lambda^{q_2} \frac{d\lambda}{\lambda}\Big)^{1/q_2} \\ &\quad + C \cdot c^{p/p^*}\Big(\int_{\frac{c}{\mu(E)^{1/p}}}^\infty \big[\lambda^{p^*-p}\big]^{q_2/p^*} \frac{d\lambda}{\lambda}\Big)^{1/q_2} \\ &\leq C\mu(E)^{\frac{1}{p^*}-\frac{1}{p}} \|f\|_{L^{p,\infty}(X,\mu)}, \end{aligned} \tag{6.2.55}$$

from which the desired conclusion follows. □

We also have the following useful result, pertaining to the completeness, separability, and pointwise convergence of sequences of functions in Lorentz spaces.

Proposition 6.2.7 *Let (X, μ) be a measure space which is sigma-finite and complete. Denote by $L^0(X, \mu)$ the vector space of scalar-valued, μ-measurable functions which are μ-a.e. finite on X, and equip this space with the topology induced by convergence in measure on sets of finite measure.*

Then for each $p \in (0, \infty)$ and $q \in (0, \infty]$ the Lorentz space $L^{p,q}(X, \mu)$ is a complete quasi-metric space, containing all simple functions on X, and with the property that any of its convergent sequences has a subsequence which converges (to its limit in $L^{p,q}(X, \mu)$) in a pointwise μ-a.e. fashion. In fact, whenever $q < \infty$, the space of simple functions is dense in $L^{p,q}(X, \mu)$.

In addition, whenever the measure μ is sigma-finite, the space $L^{p,q}(X, \mu)$ embeds continuously into $L^0(X, \mu)$. Also, if $p, q \in (0, \infty)$ and the measure μ is separable and sigma-finite, then the Lorentz space $L^{p,q}(X, \mu)$ is separable.

Finally, if $q < \infty$ then $\|\cdot\|_{L^{p,q}(X,\mu)}$ is absolutely continuous, in the sense that for any given $f \in L^{p,q}(X, \mu)$ the following property holds:

$$\begin{aligned} &\textit{if } A_j \subseteq X \textit{ is } \mu\textit{-measurable for each } j \in \mathbb{N} \textit{ and } \mathbf{1}_{A_j} \to 0 \textit{ as } j \to \infty \\ &\textit{at } \mu\textit{-a.e. point on } X\textit{, then one has } \lim_{j\to\infty} \big\| |f| \cdot \mathbf{1}_{A_j} \big\|_{L^{p,q}(X,\mu)} = 0. \end{aligned} \tag{6.2.56}$$

Proof All claims follow from (6.2.40), the discussion in [190, Example 6, pp. 4776–4777], and [30, Theorem 1.9.9, p. 55]. □

There is a companion result of the absolute continuity property recorded in (6.2.56) in which the a.e. pointwise convergence to zero for the characteristic functions $\mathbf{1}_{A_j}$ is replaced by the condition that the measures of the A_j's go to zero.

Lemma 6.2.8 *Given an arbitrary measure space (X, μ) along with $p, q \in (0, \infty)$, then for each fixed function $f \in L^{p,q}(X, \mu)$ one has*

$$\begin{aligned} &\lim_{j\to\infty} \left\| f \cdot \mathbf{1}_{A_j} \right\|_{L^{p,q}(X,\mu)} = 0 \textit{ provided each } A_j \subseteq X \textit{ is} \\ &\textit{a } \mu\textit{-measurable set and } \mu(A_j) \to 0 \textit{ as } j \to \infty. \end{aligned} \tag{6.2.57}$$

Proof If for each $j \in \mathbb{N}$ we define

$$F_j(\lambda) := m_X\big(\lambda, f \cdot \mathbf{1}_{A_j}\big) = \mu\big(\{x \in A_j : |f(x)| > \lambda\}\big), \qquad \forall \lambda > 0, \tag{6.2.58}$$

then (6.2.20) gives

$$\left\| f \cdot \mathbf{1}_{A_j} \right\|_{L^{p,q}(X,\mu)} \approx \left(\int_0^\infty \left[\lambda^p \cdot F_j(\lambda)\right]^{q/p} \frac{d\lambda}{\lambda} \right)^{1/q}, \text{ uniformly in } j \in \mathbb{N}. \tag{6.2.59}$$

Since for each $\lambda > 0$ fixed we have $0 \le F_j(\lambda) \le m_X\big(\lambda, f\big)$ for all $j \in \mathbb{N}$ and $0 \le F_j(\lambda) \le \mu(A_j) \to 0$ as $j \to \infty$, Lebesgue's Dominated Convergence Theorem applies and gives that

$$\lim_{j\to\infty} \left(\int_0^\infty \left[\lambda^p \cdot F_j(\lambda)\right]^{q/p} \frac{d\lambda}{\lambda} \right)^{1/q} = 0. \tag{6.2.60}$$

Thus, (6.2.57) follows from (6.2.59) and (6.2.60). □

A natural version of Hölder's inequality (due to O'Neil) is valid on the scale of Lorentz spaces. Specifically, if $f \in L^{p_1,q_1}(X, \mu)$ and $g \in L^{p_2,q_2}(X, \mu)$ with $0 < p_1, p_2, q_1, q_2 \le \infty$ then $fg \in L^{p_3,q_3}(X, \mu)$ provided $1/p_3 = 1/p_1 + 1/p_2$ and $1/q_3 = 1/q_1 + 1/q_2$, and the naturally accompanying estimate

$$\|fg\|_{L^{p_3,q_3}(X,\mu)} \le C \|f\|_{L^{p_1,q_1}(X,\mu)} \|g\|_{L^{p_2,q_2}(X,\mu)} \tag{6.2.61}$$

holds for some finite constant $C = C(p_1, p_2, q_1, q_2) > 0$ independent of f, g.

Let us also note here that when X is sigma-finite and non-atomic then (cf., e.g., [19, p. 126] and [107, Theorem 1.4.17, p. 52]),

$$\big(L^{p,q}(X, \mu)\big)^* = \begin{cases} \{0\} \text{ if } 0 < p < 1 \text{ and } 0 < q \le \infty, \text{ or } p = 1 \text{ and } 1 < q < \infty, \\ L^\infty(X, \mu) \text{ when } p = 1 \text{ and } 0 < q \le 1, \\ L^{p',\infty}(X, \mu) \text{ when } 1 < p < \infty \text{ and } 0 < q \le 1, \\ L^{p',q'}(X, \mu) \text{ whenever } 1 < p, q < \infty, \end{cases} \tag{6.2.62}$$

where, as usual, p' and q' are defined by $1/p + 1/p' = 1$, $1/q + 1/q' = 1$. In all cases, the duality pairing is realized via integration, i.e.,

$$
\begin{gathered}
{}_{(L^{p,q}(X,\mu))^*}\big\langle f, g\big\rangle_{L^{p,q}(X,\mu)} = \int_X fg \,\mathrm{d}\mu, \\
\text{for all } \; f \in \big(L^{p,q}(X,\mu)\big)^* \; \text{ and } \; g \in L^{p,q}(X,\mu).
\end{gathered}
\tag{6.2.63}
$$

In addition, it is known (cf., e.g., [43, p. 112]) that if $1 < p < \infty$ and $1 \leq q \leq \infty$, or $p = q = 1$, or $p = q = \infty$, then with $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ the assignment

$$
L^{p,q}(X,\mu) \ni f \longmapsto \sup\left\{\left|\int_X fg \,\mathrm{d}\mu\right| : \|g\|_{L^{p',q'}(X,\mu)} \leq 1\right\}
\tag{6.2.64}
$$

is a genuine norm on $L^{p,q}(X,\mu)$ which is equivalent to the original quasi-norm $\|\cdot\|_{L^{p,q}(X,\mu)}$. Moreover, when equipped with this norm, $L^{p,q}(X,\mu)$ becomes a Banach space and, when $q < \infty$, its dual Banach space is $L^{p',q'}(X,\mu)$ under the natural integral pairing.

Lemma 6.2.9 *For each Lebesgue measurable set $\Omega \subseteq \mathbb{R}^n$ and each $m \in (0,n)$ one has the continuous embedding*

$$
L^{n/(n-m),1}(\Omega, \mathcal{L}^n) \hookrightarrow L^1\Big(\Omega, \frac{\mathrm{d}y}{1+|y|^m}\Big).
\tag{6.2.65}
$$

Proof From (6.2.21) we know that

$$
\sup_{x\in\mathbb{R}^n} \big\| |x-\cdot|^{-m} \big\|_{L^{n/m,\infty}(\Omega,\mathcal{L}^n)} < +\infty,
\tag{6.2.66}
$$

while from (6.2.62), (6.2.63) we see that

$$
L^{n/m,\infty}(\Omega, \mathcal{L}^n) = \Big(L^{n/(n-m),1}(\Omega,\mathcal{L}^n)\Big)^*,
\tag{6.2.67}
$$

with the duality pairing realized via integration on Ω. Granted these, Lemma 3.5.7 applies and yields the desired conclusion. □

We shall also need the following result concerning the convergence of series in weak Lebesgue spaces.

Proposition 6.2.10 *Let (X,μ) be a measure space, fix $p \in (0,\infty)$ arbitrary, and define*

$$
\alpha_p := \begin{cases} \frac{p}{p+1} & \textit{if } \; 0 < p \leq 1, \\ 2^{-1} & \textit{if } \; 1 \leq p < \infty. \end{cases}
\tag{6.2.68}
$$

Also, recall the quasi-norm $|||\cdot|||_{L^{p,\infty}(X,\mu)}$ from (6.2.21). Consider an at most countable family $\{f_i\}_{i\in I}$ of μ-measurable real-valued functions defined on X with the property that

$$\sum_{i\in I} |||f_i|||^{\alpha_p}_{L^{p,\infty}(X,\mu)} < \infty. \tag{6.2.69}$$

Then the series $\sum_{i\in I} f_i$ *converges in the quasi-Banach space* $L^{p,\infty}(X,\mu)$*, and*

$$\Big|\Big|\Big| \sum_{i\in I} f_i \Big|\Big|\Big|_{L^{p,\infty}(X,\mu)} \leq 2^{2/\alpha_p}\Big(\sum_{i\in I} |||f_i|||^{\alpha_p}_{L^{p,\infty}(X,\mu)}\Big)^{1/\alpha_p}. \tag{6.2.70}$$

Proof This is a consequence of the sort of quantitative Aoki–Rolewicz result proved in [188, Theorem 3.39, pp. 130–131]. □

We conclude this section by discussing an abstract embodiment of the principle that, for a family of operators mapping vectors from a common quasi-normed space into functions defined on a certain measurable space, pointwise almost everywhere convergence for a dense subset of the quasi-normed space, along with the boundedness of the maximal operator naturally associated with such a family into a Lorentz space, ultimately imply pointwise almost everywhere convergence for the family of functions obtained by applying the given operators to *any* vector from the given quasi-normed space.

Proposition 6.2.11 *Let* $(\mathscr{X},\tau)$ *be a topological space. Consider an arbitrary set* $X \subseteq \mathscr{X}$ *and suppose* μ *is a complete measure on* X*. Assume that for each* $x \in X$ *a set* $\Gamma(x) \subseteq \mathscr{X}\setminus X$ *has been assigned, with the property that*

$$x \in \overline{\Gamma(x)} \ \text{ for } \mu\text{-a.e. } \ x \in X. \tag{6.2.71}$$

Next, let $(Y, \|\cdot\|_Y)$ *be a quasi-normed space and suppose* $\mathcal{T}$ *is an operator mapping vectors from* Y *into real-valued functions defined on* $\mathscr{X}\setminus X$ *with the property that*

$$|\mathcal{T}f - \mathcal{T}g| \leq |\mathcal{T}(f-g)| \ \text{ on } \ \mathscr{X}\setminus X, \quad \text{for all } \ f,g \in Y. \tag{6.2.72}$$

Furthermore, assume that the associated maximal operator, acting on each $f \in Y$ *according to*

$$T_\star f(x) := \sup_{y\in\Gamma(x)} |(\mathcal{T}f)(y)| \ \text{ at } \mu\text{-a.e. } \ x \in X, \tag{6.2.73}$$

has the property that there exists an integrability exponent $p \in (0,\infty)$ *along with a constant* $C \in (0,\infty)$ *such that*

$$\begin{array}{l}\text{for each } f \in Y \text{ the function } T_\star f \text{ belongs to the Lorentz space}\\ L^{p,\infty}(X,\mu) \text{ and one has } \|T_\star f\|_{L^{p,\infty}(X,\mu)} \leq C\|f\|_Y.\end{array} \tag{6.2.74}$$

Lastly, suppose that there exists a set $\mathcal{V} \subseteq Y$*, which is dense in* Y *with respect to the topology induced by the quasi-norm* $\|\cdot\|_Y$*, and such that for every* $f \in \mathcal{V}$ *the limit*

$$Tf(x) := \lim_{\Gamma(x)\ni y\to x} (\mathcal{T}f)(y) \ \text{ exists for } \mu\text{-a.e. } \ x \in X. \tag{6.2.75}$$

Then (6.2.75) *actually holds for every* $f \in Y$*, and* $|Tf| \leq T_\star f$ *at* μ*-a.e. point in* X.

Proof To justify the existence of the limit in (6.2.75), let $f \in Y$ be arbitrary. To avoid measurability issues, we find it useful to consider the outer measure μ^* canonically associated with the given μ as in (3.3.13). In view of Lemma 3.3.1 and (3.3.15), it is then enough to show

$$\mu^*\Bigg(\Big\{x \in X : \limsup_{\Gamma(x)\ni y \to x} (\mathcal{T} f)(y) \neq \liminf_{\Gamma(x)\ni y \to x} (\mathcal{T} f)(y)\Big\}\Bigg) = 0. \tag{6.2.76}$$

For each threshold $\theta > 0$, consider the disagreement set

$$S_\theta := \left\{x \in X : \left|\limsup_{\Gamma(x)\ni y \to x} (\mathcal{T} f)(y) - \liminf_{\Gamma(x)\ni y \to x} (\mathcal{T} f)(y)\right| > \theta\right\}. \tag{6.2.77}$$

Since

$$\Big\{x \in X : \limsup_{\Gamma(x)\ni y \to x} (\mathcal{T} f)(y) \neq \liminf_{\Gamma(x)\ni y \to x} (\mathcal{T} f)(y)\Big\} = \bigcup_{j=1}^{\infty} S_{1/j}, \tag{6.2.78}$$

it suffices (cf. (3.3.1)) to prove that $\mu^*(S_\theta) = 0$ for each fixed $\theta \in (0, \infty)$. To this end, fix $\theta, \delta > 0$ arbitrary and select

$$g \in \mathcal{V} \ \text{ such that } \ \|f - g\|_Y < \delta. \tag{6.2.79}$$

Observe that

$$S_\theta \subseteq A_1 \cup A_2 \tag{6.2.80}$$

where

$$\begin{aligned} A_1 &:= \left\{x \in X : \left|\limsup_{\Gamma(x)\ni y \to x} (\mathcal{T} f)(y) - \lim_{\Gamma(x)\ni y \to x} (\mathcal{T} g)(y)\right| > \theta/2\right\} \\ &= \left\{x \in X : \left|\limsup_{\Gamma(x)\ni y \to x} (\mathcal{T} f - \mathcal{T} g)(y)\right| > \theta/2\right\} \\ &\subseteq \Big\{x \in X : T_\star(f - g)(x) > \theta/2\Big\}, \end{aligned} \tag{6.2.81}$$

and

$$
\begin{aligned}
A_2 &:= \Big\{x \in X : \Big|\liminf_{\Gamma(x)\ni y\to x} (\mathcal{T} f)(y) - \lim_{\Gamma(x)\ni y\to x} (\mathcal{T} g)(y)\Big| > \theta/2\Big\} \\
&= \Big\{x \in X : \Big|\liminf_{\Gamma(x)\ni y\to x} (\mathcal{T} f - \mathcal{T} g)(y)\Big| > \theta/2\Big\} \\
&\subseteq \big\{x \in X : T_\star(f-g)(x) > \theta/2\big\}. \qquad (6.2.82)
\end{aligned}
$$

It follows from (6.2.80), (6.2.81), (6.2.82), (6.2.74), (6.2.21), and (6.2.79) that there exists a constant $C \in (0,\infty)$ independent of f, g, θ, δ with the property that

$$
\begin{aligned}
\mu^*(S_\theta) &\le \mu^*(A_1) + \mu^*(A_2) \le 2\,\mu^*\big(\{x \in X : T_\star(f-g)(x) > \theta/2\}\big) \\
&= 2\,\mu\big(\{x \in X : T_\star(f-g)(x) > \theta/2\}\big) \\
&\le \frac{C}{\theta^p}\|f-g\|_Y^p \le \frac{C}{\theta^p}\delta^p. \qquad (6.2.83)
\end{aligned}
$$

The equality in (6.2.83) is a consequence of the fact that $T_\star(f-g)$ belongs to $L^{p,\infty}(X,\mu)$ (cf. the membership condition in (6.2.74)), and (3.3.15) (recall that the measure μ is assumed to be complete). Sending $\delta \to 0^+$ in (6.2.83) then proves that $\mu^*(S_\theta) = 0$. Hence (6.2.76) holds. Finally, that for every $f \in Y$ we have $|Tf| \le T_\star f$ at μ-a.e. point in X is clear from definitions. □

6.3 The Fractional Hardy–Littlewood Maximal Operator in a Non-Metric Setting

The main result in this section is Theorem 6.3.3 describing the mapping properties in Lorentz spaces of a maximal operator constructed using the same blueprint as for the fractional Hardy–Littlewood maximal operator in the classical setting of $\mathbb{R}^n$ but substituting the balls used in this scheme with a more general family of sets satisfying some basic axioms which are formulated independently of the notion of distance. As a preamble, we first establish an appropriate version of Vitali's Covering Lemma in this abstract setting.

Lemma 6.3.1 *Let X be a nonempty set and assume $\mathcal{A} := \{A(x,r)\}_{x\in X, r>0}$ is a family of subsets of X indexed by pairs $(x,r) \in X \times (0,\infty)$ satisfying the following two properties:*

(a) (*Monotonicity*) *If $x \in X$ and $0 < r_1 \le r_2$ then $A(x,r_1) \subseteq A(x,r_2)$.*
(b) (*Enveloping*) *There exists $C > 0$ with the property that whenever $x_1, x_2 \in X$ and $r_1, r_2 > 0$ are such that $r_1 \ge r_2$ and $A(x_1,r_1) \cap A(x_2,r_2) \neq \varnothing$ it follows that $A(x_2,r_2) \subseteq A(x_1,Cr_1)$.*

Then for any fixed number $\lambda > 1$ and any set $I \subseteq X \times (0,\infty)$ with the property that

$$
M := \sup\{r > 0 : \exists\, x \in X \ \text{ such that } \ (x,r) \in I\} < +\infty, \qquad (6.3.1)
$$

there exists $J \subseteq I$ satisfying

$$A(x_1, r_1) \cap A(x_2, r_2) = \varnothing \ \textit{ for all distinct } \ (x_1, r_1),\ (x_2, r_2) \in J, \tag{6.3.2}$$
$$\textit{and } \forall (x, r) \in I \ \exists\, (x_0, r_0) \in J \ \textit{ such that } \ A(x, r) \subseteq A(x_0, \lambda C r_0). \tag{6.3.3}$$

If, in addition, the set I has the property that

$$\begin{array}{c}\textit{for every sequence } \{(x_j, r_j)\}_{j\in\mathbb{N}} \subseteq I \textit{ such that } \{A(x_j, r_j)\}_{j\in\mathbb{N}} \\ \textit{consists of mutually disjoint sets one necessarily has } \lim_{j\to\infty} r_j = 0,\end{array} \tag{6.3.4}$$

then the set J in (6.3.2)–(6.3.3) can be taken to be at most countable.

Proof Let $\lambda > 1$ be fixed and, for each $k \in \mathbb{N}$, consider

$$I_k := \{(x, r) \in I : \lambda^{-k} M < r \leq \lambda^{-k+1} M\}. \tag{6.3.5}$$

In particular, I is the disjoint union of the I_k's. When equipped with the inclusion, the set

$$\{J_1 \subseteq I_1 : A(x_1, r_1) \cap A(x_2, r_2) = \varnothing \text{ for all distinct } (x_1, r_1), (x_2, r_2) \in J_1\} \tag{6.3.6}$$

becomes partially ordered and any of its totally ordered subsets has an upper bound. Therefore, by Zorn's lemma, the set (6.3.6) contains a maximal element which we will denote by J_1^*. Inductively, we then construct a family $\{J_k^*\}_{k\in\mathbb{N}}$ as follows. Assume that $k \in \mathbb{N}$ is such that $J_1^*, J_2^*, \dots, J_k^*$ have been already introduced. We then consider

$$\begin{aligned}\Big\{ J_{k+1} \subseteq I_{k+1} : \ & A(x_1, r_1) \cap A(x_2, r_2) = \varnothing \text{ for all } (x_1, r_1) \in J_{k+1} \\ & \text{and all } \ (x_2, r_2) \in J_{k+1} \cup \Big(\cup_{j=1}^{k} J_j^* \Big) \text{ such that } (x_2, r_2) \neq (x_1, r_1) \Big\}\end{aligned} \tag{6.3.7}$$

and note that, again, this is partially ordered by the inclusion of sets and any of its totally ordered subsets has an upper bound. Zorn's lemma then ensures that the set (6.3.7) has a maximal element which we define to be J_{k+1}^*.

Having constructed the family $\{J_k^*\}_{k\in\mathbb{N}}$, we then claim that

$$J := \bigcup_{k=1}^{\infty} J_k^* \tag{6.3.8}$$

does the job advertised in (6.3.2)–(6.3.3). That (6.3.2) holds is clear from the fact that J_{k+1}^* belongs to the set described in (6.3.7), for each $k \in \mathbb{N}$. As far as (6.3.3) is concerned, if $(x, r) \in I$ pick $k \in \mathbb{N}$ such that $(x, r) \in I_k$. Then, by the maximality of J_k^*, there exist $h \in \mathbb{N}$ and $(x_0, r_0) \in J_h^*$ such that $h \leq k$ and $A(x, r) \cap A(x_0, r_0) \neq \varnothing$. Then, since $J_h^* \subseteq I_h$, it follows that $r_0 > \lambda^{-h} M \geq \lambda^{-k} M \geq \lambda^{-1} r$. Hence, we have $\lambda r_0 > r$ and also $A(x_0, \lambda r_0) \cap A(x, r) \neq \varnothing$ (given that $A(x_0, r_0) \subseteq A(x_0, \lambda r_0)$ by monotonicity). As such, the enveloping property ensures that $A(x, r) \subseteq A(x_0, \lambda C r_0)$. This justifies (6.3.3).

To complete the proof of the lemma, there remains to show that J can be taken to be at most countable in the case when (6.3.4) holds. In this scenario, however, each of the sets I_k, $k \in \mathbb{N}$, is finite which, in turn, implies that each of the sets J_k^*, $k \in \mathbb{N}$, is finite. Hence, J defined in (6.3.8) is indeed at most countable. □

Remark 6.3.2 *In the context of Lemma 6.3.1, if X is a separable topological space and the sets in the family $\mathcal{A}$ have nonempty interiors it follows, a posteriori, that any set of indices J satisfying (6.3.2) is countable.*

The stage is now set for discussing the following extension of the analysis pertaining to the fractional Hardy–Littlewood maximal operator in a measure theoretic, non-metric setting.

Theorem 6.3.3 *Let $(X, \mathfrak{M}, \mu)$ be a given measure space, and consider a family of μ-measurable subsets of X, say $\mathcal{A} = \{A(x, r)\}_{x \in X, r>0} \subseteq \mathfrak{M}$, satisfying the following conditions:*

(i) $x \in A(x, r)$ *and* $0 < \mu(A(x, r)) < \infty$ *for each* $x \in X$ *and* $r > 0$*;*
(ii) $A(x, r_1) \subseteq A(x, r_2)$ *whenever* $x \in X$ *and* $0 < r_1 \leq r_2$*, and* $A(x, r) \nearrow X$ *as* $r \nearrow \infty$ *for each* $x \in X$*;*
(iii) for each $x \in X$*, each* $r > 0$*, and each sequence* $\{r_j\}_{j \in \mathbb{N}}$ *such that* $r_j \nearrow r$ *as* $j \to \infty$*, one has* $A(x, r_j) \nearrow A(x, r)$ *as* $j \to \infty$*;*
(iv) there exists a constant $C \in (0, \infty)$ *with the property that whenever* $x_1, x_2 \in X$ *and* $r_1, r_2 > 0$ *are such that* $r_1 \geq r_2$ *and* $A(x_1, r_1) \cap A(x_2, r_2) \neq \varnothing$ *it follows that* $A(x_2, r_2) \subseteq A(x_1, C r_1)$*;*
(v) the measure μ *is doubling relative to the family* $\mathcal{A}$*, i.e., there exists a finite constant* $c > 0$ *such that* $\mu(A(x, 2r)) \leq c \cdot \mu(A(x, r))$ *for each* $x \in X$ *and each* $r > 0$*;*
(vi) for each $r > 0$ *and* $E \in \mathfrak{M}$*, the function* $X \ni x \mapsto \mu(A(x, r) \cap E) \in [0, \infty)$ *is* μ*-measurable.*

Having fixed $s \in (0, \infty)$ and $\alpha \in [0, 1/s)$, define the action of the L^s-based fractional maximal operator of order α associated with the family $\mathcal{A}$ on each μ-measurable function f on X as

$$\mathcal{M}_{\mathcal{A},s,\alpha} f(x) := \sup_{r>0} \left[\mu\big(A(x, r)\big)^{\alpha} \Big(⨍_{A(x,r)} |f|^s \, d\mu \Big)^{\frac{1}{s}} \right], \qquad \forall x \in X. \tag{6.3.9}$$

Then

$$\begin{gathered} \mathcal{M}_{\mathcal{A},s,\alpha} : L^{p,q}(X, \mu) \longrightarrow L^{p^*,q}(X, \mu) \text{ is well defined,} \\ \text{sub-linear and bounded, provided } \ 0 < s < p < \tfrac{1}{\alpha} \leq \infty, \\ p^* = \big(\tfrac{1}{p} - \alpha\big)^{-1}, \ \text{ and } \ 0 < q \leq \infty. \end{gathered} \tag{6.3.10}$$

Moreover, corresponding to the limiting case $p = s$, the operator

$$\begin{aligned}&\mathcal{M}_{\mathcal{A},s,\alpha} : L^s(X,\mu) \longrightarrow L^{\frac{s}{1-\alpha s},\infty}(X,\mu) \ \textit{ is well defined,}\\ &\textit{sub-linear and bounded, for each } s \in (0,\infty) \textit{ and } 0 \le \alpha < 1/s,\end{aligned} \tag{6.3.11}$$

and, corresponding to the limiting case $p = 1/\alpha$, *the operator*

$$\begin{aligned}&\mathcal{M}_{\mathcal{A},s,\alpha} : L^{1/\alpha}(X,\mu) \longrightarrow L^{\infty}(X,\mu) \ \textit{ is well defined,}\\ &\textit{sub-linear and bounded, for each } s \in (0,\infty) \textit{ and } 0 \le \alpha < 1/s.\end{aligned} \tag{6.3.12}$$

A direct consequence of Theorem 6.3.3 is the following corollary, of independent interest, pertaining to the mapping properties of a Hardy–Littlewood maximal type operator (defined as in (6.3.9) with $\alpha = 0$) in a non-metric setting.

Corollary 6.3.4 *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space and consider a family of sets* $\mathcal{A} = \{A(x,r)\}_{x\in X, r>0} \subseteq \mathfrak{M}$ *satisfying conditions (i)–(vi) from the statement of Theorem 6.3.3. In this context, having fixed some* $s \in (0,\infty)$, *for each* μ*-measurable function* f *on* X *define*

$$\mathcal{M}_{\mathcal{A},s} f(x) := \sup_{r>0} \Big(⨍_{A(x,r)} |f|^s \,\mathrm{d}\mu \Big)^{\frac{1}{s}}, \qquad \forall x \in X. \tag{6.3.13}$$

Then

$$\begin{aligned}&\mathcal{M}_{\mathcal{A},s} : L^{p,q}(X,\mu) \longrightarrow L^{p,q}(X,\mu)\\ &\textit{is well defined, sub-linear and bounded}\\ &\textit{provided } \ 0 < s < p \le \infty \ \textit{ and } \ 0 < q \le \infty,\end{aligned} \tag{6.3.14}$$

with the convention that $q = \infty$ *if* $p = \infty$. *In particular, for every* $s \in (0,\infty)$ *the operator* $\mathcal{M}_{\mathcal{A},s}$ *is well-defined, sub-linear and bounded in each of the following settings:*

$$\mathcal{M}_{\mathcal{A},s} : L^p(X,\mu) \longrightarrow L^p(X,\mu) \ \textit{ with } \ p \in (s,\infty], \tag{6.3.15}$$

$$\mathcal{M}_{\mathcal{A},s} : L^s(X,\mu) \longrightarrow L^{s,\infty}(X,\mu). \tag{6.3.16}$$

We are now ready to present the proof of Theorem 6.3.3.

Proof of Theorem 6.3.3 As a preamble, we shall first establish that

$$\begin{aligned}&\text{given a sequence } \{A(x_j,r_j)\}_{j\in\mathbb{N}} \subseteq \mathcal{A} \text{ with the property that there exist}\\ &x_0 \in X \text{ and } r_0 \in (0,\infty) \text{ such that } A(x_j,r_j) \subseteq A(x_0,r_0) \text{ for each } j \in \mathbb{N}\\ &\text{and } \lim_{j\to\infty} \mu(A(x_j,r_j)) = 0, \text{ we necessarily have } \lim_{j\to\infty} r_j = 0.\end{aligned} \tag{6.3.17}$$

Seeking a contradiction, after eventually passing to a subsequence, we may assume that there exists $r_* \in (0,\infty)$ such that $r_j \ge r_*$ for each $j \in \mathbb{N}$. By property *(ii)*, we then have $A(x_j,r_*) \subseteq A(x_j,r_j)$, hence $0 \le \mu\big(A(x_j,r_*)\big) \le \mu\big(A(x_j,r_j)\big)$ for each $j \in \mathbb{N}$. Since we are currently assuming $\lim_{j\to\infty} \mu\big(A(x_j,r_j)\big) = 0$, this forces

$$\lim_{j\to\infty} \mu\big(A(x_j, r_*)\big) = 0. \tag{6.3.18}$$

To proceed, fix an arbitrary index $j \in \mathbb{N}$. Then property *(i)* implies that $x_j \in A(x_j, r_0)$ and also that $x_j \in A(x_j, r_j) \subseteq A(x_0, r_0)$, with the last inclusion being part of the assumptions made in (6.3.17). Thus, $x_j \in A(x_j, r_0) \cap A(x_0, r_0)$ which goes to show that $A(x_0, r_0) \cap A(x_j, r_0) \neq \varnothing$. Granted this, property *(iv)* guarantees that $A(x_0, r_0) \subseteq A(x_j, Cr_0)$. Hence, if we pick $N \in \mathbb{N}$ such that $Cr_0 < 2^N r_*$, then property *(ii)* further implies $A(x_0, r_0) \subseteq A(x_j, 2^N r_*)$. On account of this and property *(v)*, we may then iteratively estimate

$$0 < \mu\big(A(x_0, r_0)\big) \leq \mu\big(A(x_j, 2^N r_*)\big) \leq c^N \mu\big(A(x_j, r_*)\big). \tag{6.3.19}$$

Upon letting $j \to \infty$ and invoking (6.3.18), this leads to a contradiction. Thus, (6.3.17) is established.

Our next goal is to prove (6.3.11). To this end, fix $s \in (0, \infty)$ and $0 \leq \alpha < 1/s$ and consider a truncated version of (6.3.9). Namely, for each fixed $R \in (0, \infty)$ define

$$\mathcal{M}^R_{\mathcal{A},s,\alpha} f(x) := \sup_{0<r<R} \left[\mu\big(A(x,r)\big)^{\alpha} \Big(⨍_{A(x,r)} |f|^s \, d\mu \Big)^{\frac{1}{s}} \right], \quad \forall x \in X, \tag{6.3.20}$$

for each μ-measurable function f on X. The first order of business is to show that, for each $R > 0$,

$$\mathcal{M}^R_{\mathcal{A},s,\alpha} f \text{ is } \mu\text{-measurable, for every } \mu\text{-measurable function } f. \tag{6.3.21}$$

To prove this, fix a μ-measurable function f on X. The first observation is that

$$\mathcal{M}^R_{\mathcal{A},s,\alpha} f(x) = \sup_{\substack{0<r<R \\ r \text{ rational}}} \left[\mu\big(A(x,r)\big)^{\alpha} \Big(⨍_{A(x,r)} |f|^s \, d\mu \Big)^{\frac{1}{s}} \right], \quad \forall x \in X. \tag{6.3.22}$$

Indeed, this is a consequence of the fact that if $x \in X$ is arbitrary and fixed, then for each $r > 0$ and each sequence $\{r_j\}_{j\in\mathbb{N}}$ such that $r_j \nearrow r$ as $j \to \infty$ one has, by virtue of *(iii)* and Lebesgue's Monotone Convergence Theorem,

$$\mu\big(A(x,r_j)\big)^{\alpha} \Big(⨍_{A(x,r_j)} |f|^s \, d\mu \Big)^{\frac{1}{s}} \longrightarrow \mu\big(A(x,r)\big)^{\alpha} \Big(⨍_{A(x,r)} |f|^s \, d\mu \Big)^{\frac{1}{s}} \text{ as } j \to \infty. \tag{6.3.23}$$

Granted (6.3.22) and since the supremum of a countable family of μ-measurable functions is itself a μ-measurable function, it suffices to show that, for each fixed μ-measurable function f and each fixed $r > 0$, the assignment

$$X \ni x \longmapsto \Phi_{f,r}(x) := \mu\big(A(x,r)\big)^{\alpha} \Big(⨍_{A(x,r)} |f|^s \, d\mu \Big)^{\frac{1}{s}} \in [0, \infty] \tag{6.3.24}$$

is a μ-measurable function. With this goal in mind, recall from (3.1.12) that given any μ-measurable function f on X one can find a sequence $\{h_j\}_{j\in\mathbb{N}}$ of simple functions on X with the property that $0 \leq h_j(x) \nearrow |f(x)|$ as $j \to \infty$ for every $x \in X$. Since

$$\Phi_{h_j,r}(x) \nearrow \Phi_{f,r}(x) \quad \text{as } j \to \infty, \text{ for every } x \in X, \tag{6.3.25}$$

it therefore suffices to prove that for each fixed $r > 0$ and each fixed $j \in \mathbb{N}$ the function $\Phi_{h_j,r}$ is μ-measurable. In turn, given the structure of simple functions it suffices to prove that, if $r > 0$ and $E \in \mathfrak{M}$ are fixed, the function

$$\begin{gathered} X \ni x \longmapsto \mu\big(A(x,r)\big)^{\alpha s-1}\mu\big(A(x,r)\cap E\big) \in [0,\infty] \\ \text{is } \mu\text{-measurable.} \end{gathered} \tag{6.3.26}$$

This, however, is a consequence of assumption *(vi)* (used twice: first as stated, and second for the choice $E := X$). This finishes the proof of the claim made in (6.3.21).

To proceed, fix an arbitrary function $f \in L^s(X,\mu)$. Then, having fixed $\lambda > 0$ along with $x_o \in X$ and $0 < R < R_o < \infty$, consider

$$E_{R,\lambda}(x_o,R_o) := \big\{x \in A(x_o,R_o) : (\mathcal{M}^R_{\mathcal{A},s,\alpha} f)(x) > \lambda\big\} \subseteq A(x_o,R_o). \tag{6.3.27}$$

By (6.3.21), we have $E_{R,\lambda}(x_o,R_o) \in \mathfrak{M}$. Furthermore, for each $x \in E_{R,\lambda}(x_o,R_o)$ there exists a number $r_x \in (0,R)$ such that

$$\mu\big(A(x,r_x)\big)^{\alpha}\Big(⨍_{A(x,r_x)} |f|^s \, d\mu\Big)^{\frac{1}{s}} > \lambda, \tag{6.3.28}$$

i.e.,

$$\mu\big(A(x,r_x)\big)^{1-\alpha s} < \lambda^{-s}\int_{A(x,r_x)} |f|^s \, d\mu. \tag{6.3.29}$$

Consider $\big\{A(x,r_x)\big\}_{x\in E_{R,\lambda}(x_o,R_o)}$ which, by condition *(i)*, covers $E_{R,\lambda}(x_o,R_o)$. Also, by design,

$$\sup_{x\in E_{R,\lambda}(x_o,R_o)} r_x \leq R < \infty. \tag{6.3.30}$$

Moreover, for every point x belonging to $E_{R,\lambda}(x_o,R_o)$ the first part in condition *(i)* ensures that we have $x \in A(x,r_x)\cap A(x_o,R_o)$, hence $A(x,r_x)\cap A(x_o,R_o) \neq \varnothing$. Based on this observation, the fact that $0 < r_x < R < R_o$ for every $x \in E_{R,\lambda}(x_o,R_o)$, and condition *(iv)*, we may then conclude that

$$\bigcup_{x\in E_{R,\lambda}(x_o,R_o)} A(x,r_x) \subseteq A(x_o,CR_o). \tag{6.3.31}$$

Assume now that $\big\{A(x_j,r_{x_j})\big\}_{j\in\mathbb{N}}$ is a sequence of mutually disjoint sets selected from the family $\big\{A(x,r_x)\big\}_{x\in E_{R,\lambda}(x_o,R_o)}$, associated with an arbitrary sequence of points $\{x_j\}_{j\in\mathbb{N}} \subseteq E_{R,\lambda}(x_o,R_o)$. Then it follows from (6.3.29) that

$$\sum_{j=1}^{\infty} \mu\big(A(x_j, r_{x_j})\big)^{1-\alpha s} \leq \lambda^{-s} \sum_{j=1}^{\infty} \int_{A(x_j, r_{x_j})} |f|^s \, d\mu \qquad (6.3.32)$$
$$= \lambda^{-s} \int_{\cup_{j\in\mathbb{N}} A(x_j, r_{x_j})} |f|^s \, d\mu \leq \lambda^{-s} \int_X |f|^s \, d\mu < \infty.$$

In turn, given that we are assuming $\alpha s < 1$, this forces

$$\lim_{j\to\infty} \mu\big(A(x_j, r_{x_j})\big) = 0. \qquad (6.3.33)$$

Consequently, from (6.3.31), (6.3.33), and (6.3.17) we deduce that, in the scenario we are currently considering, one necessarily has

$$\lim_{j\to\infty} r_{x_j} = 0. \qquad (6.3.34)$$

This analysis shows that condition (6.3.4) in Lemma 6.3.1 is satisfied in the current setting. Furthermore, by virtue of the first part in condition *(ii)*, condition *(iv)*, and (6.3.30), we also have that conditions *(a)*, *(b)*, and (6.3.1) in Lemma 6.3.1 are satisfied. Thus, Lemma 6.3.1 applies and yields an at most countable family $\{A(x, r_x)\}_{x\in J}$, with $J \subseteq E_{R,\lambda}(x_o, R_o)$, of pairwise disjoint sets with the property that for some finite positive constant, which without loss of generality can be assumed to be of the form 2^N for some $N \in \mathbb{N}$, one has

$$E_{R,\lambda}(x_o, R_o) \subseteq \bigcup_{x\in J} A(x, 2^N r_x). \qquad (6.3.35)$$

By availing ourselves of this condition and keeping in mind the doubling property of μ relative to the family $\mathcal{A}$ (cf. condition *(v)* where the constant c used below first appears) we may write

$$\mu\big(E_{R,\lambda}(x_o, R_o)\big) \leq \sum_{x\in J} \mu\big(A(x, 2^N r_x)\big) \leq c^N \sum_{x\in J} \mu\big(A(x, r_x)\big)$$
$$\leq c^N \Big(\sum_{x\in J} \mu\big(A(x, r_x)\big)^{1-\alpha s}\Big)^{\frac{1}{1-\alpha s}}$$
$$\leq c^N \Big(\sum_{x\in J} \lambda^{-s} \int_{A(x, r_x)} |f|^s \, d\mu\Big)^{\frac{1}{1-\alpha s}}$$
$$\leq c^N \Big(\lambda^{-s} \int_X |f|^s \, d\mu\Big)^{\frac{1}{1-\alpha s}} = c^N \left(\frac{\|f\|_{L^s(X,\mu)}}{\lambda}\right)^{\frac{s}{1-\alpha s}}, \qquad (6.3.36)$$

where we have also made use of the fact that $0 \leq \alpha < 1/s$, and (6.3.29). Thus, there exists a finite positive constant C, independent of f, λ, R, and R_o, with the property that

$$\sup_{\lambda>0}\Big(\lambda\cdot\mu\big(E_{R,\lambda}(x_o,R_o)\big)^{\frac{1-\alpha s}{s}}\Big)\leq C\|f\|_{L^s(X,\mu)}. \tag{6.3.37}$$

Going further, for each $\lambda > 0$ and $R > 0$ introduce the set

$$E_{R,\lambda} := \big\{x\in X:\ (\mathcal{M}^R_{\mathcal{A},s,\alpha}f)(x)>\lambda\big\}\subseteq X. \tag{6.3.38}$$

Based on (6.3.21) we see that $E_{R,\lambda}\in\mathfrak{M}$ and, by design, for each fixed $\lambda>0$, $R>0$, and any $R_o>R$, we also have $E_{R,\lambda}\cap A(x_o,R_o)=E_{R,\lambda}(x_o,R_o)$. From this and the second part of condition *(ii)*, we may therefore conclude by sending $R_o\nearrow\infty$ (via a countable sequence of values) that

$$\sup_{\lambda>0}\Big(\lambda\cdot\mu(E_{R,\lambda})^{\frac{1-\alpha s}{s}}\Big)\leq C\|f\|_{L^s(X,\mu)}, \tag{6.3.39}$$

for some finite constant $C>0$ which is independent of f, λ, and R.

We next make the observation that since $(\mathcal{M}^R_{\mathcal{A},s,\alpha}f)(x)\nearrow(\mathcal{M}_{\mathcal{A},s,\alpha}f)(x)$ as $R\nearrow\infty$ for each $x\in X$, we may conclude that $\mathcal{M}_{\mathcal{A},s,\alpha}f$ is a μ-measurable function on X. Furthermore, if for each $\lambda>0$ we introduce

$$E_\lambda := \big\{x\in X:\ (\mathcal{M}_{\mathcal{A},s,\alpha}f)(x)>\lambda\big\}\subseteq X, \tag{6.3.40}$$

it follows that for each fixed $\lambda>0$ we have $E_\lambda\in\mathfrak{M}$ and $E_{R,\lambda}\nearrow E_\lambda$ as $R\nearrow\infty$. Consequently, $\mu(E_{R,\lambda})\nearrow\mu(E_\lambda)$ as $R\nearrow\infty$, for each fixed $\lambda>0$, hence passing to the limit $R\nearrow\infty$ in (6.3.39) yields

$$\sup_{\lambda>0}\Big(\lambda\cdot\mu(E_\lambda)^{\frac{1-\alpha s}{s}}\Big)\leq C\|f\|_{L^s(X,\mu)},\qquad\forall\lambda>0, \tag{6.3.41}$$

for some finite constant $C>0$ independent of the function $f\in L^s(X,\mu)$. This proves (6.3.11).

Moving on, consider the case when

$$1\leq p\leq\infty,\quad 0<s<\infty\ \text{ and }\ \alpha=\tfrac{1}{sp}, \tag{6.3.42}$$

and assume that $f\in L^{\frac{1}{\alpha}}(X,\mu)$ is an arbitrary function. If p is finite, then for each $x\in X$ and each $r>0$ Hölder's inequality gives

$$\begin{aligned}\mu\big(A(x,r)\big)^{\alpha}\Big(\fint_{A(x,r)}|f|^s\,\mathrm{d}\mu\Big)^{\frac{1}{s}}&\leq\mu\big(A(x,r)\big)^{\alpha}\Big(\fint_{A(x,r)}|f|^{sp}\,\mathrm{d}\mu\Big)^{\frac{1}{sp}}\\&=\mu\big(A(x,r)\big)^{\alpha-\frac{1}{sp}}\Big(\int_{A(x,r)}|f|^{sp}\,\mathrm{d}\mu\Big)^{\frac{1}{sp}}\\&\leq\|f\|_{L^{sp}(X,\mu)},\end{aligned} \tag{6.3.43}$$

i.e.,

$$\mu\big(A(x,r)\big)^{\alpha}\Big(\fint_{A(x,r)}|f|^s\,\mathrm{d}\mu\Big)^{\frac{1}{s}}\leq\|f\|_{L^{\frac{1}{\alpha}}(X,\mu)},\qquad\forall x\in X,\ \forall r>0.\tag{6.3.44}$$

Moreover, (6.3.44) is trivially true in the case when $\alpha=0$ (corresponding to $p=\infty$). The bottom line is that (6.3.44) holds whenever $0<s<\infty$ and $0\leq\alpha<1/s$. This shows that, as claimed in (6.3.12), we also have

$$\begin{gathered}\mathcal{M}_{\mathcal{A},s,\alpha}:L^{\frac{1}{\alpha}}(X,\mu)\longrightarrow L^{\infty}(X,\mu)\ \text{ is well defined, sub-linear and bounded}\\ \text{granted } 0<s<\infty \text{ and } 0\leq\alpha<1/s.\end{gathered}\tag{6.3.45}$$

The next step is to interpolate between the boundedness results established in (6.3.11) and (6.3.45). Given that the operator $\mathcal{M}_{\mathcal{A},s,\alpha}$ is quasi-additive, the real interpolation theorem for sub-linear operators (cf. [185, Sect. 1.3]) applies and gives that if $0<s<\infty$ and $0\leq\alpha<1/s$ then for each $\theta\in(0,1)$ and each $q\in(0,\infty]$,

$$\begin{gathered}\mathcal{M}_{\mathcal{A},s,\alpha}:\big(L^s(X,\mu),L^{\frac{1}{\alpha}}(X,\mu)\big)_{\theta,q}\longrightarrow\big(L^{\frac{s}{1-\alpha s},\infty}(X,\mu),L^{\infty}(X,\mu)\big)_{\theta,q}\\ \text{is a well-defined, sub-linear and bounded operator.}\end{gathered}\tag{6.3.46}$$

It remains to identify the intermediate spaces appearing in (6.3.46) for a suitable choice of θ, which we do using (6.2.48). Specifically, given $p\in(s,1/\alpha)$, taking

$$\theta:=\frac{\frac{1}{s}-\frac{1}{p}}{\frac{1}{s}-\alpha}\in(0,1)\tag{6.3.47}$$

yields

$$\big(L^s(X,\mu),L^{\frac{1}{\alpha}}(X,\mu)\big)_{\theta,q}=L^{p,q}(X,\mu)\tag{6.3.48}$$

and, if $p^*:=\big(\frac{1}{p}-\alpha\big)^{-1}$,

$$\big(L^{\frac{s}{1-\alpha s},\infty}(X,\mu),L^{\infty}(X,\mu)\big)_{\theta,q}=L^{p^*,q}(X,\mu).\tag{6.3.49}$$

This finishes the proof Theorem 6.3.3. □

6.4 Clifford Algebra Fundamentals

Taking "square roots" of quadratic forms is one of the primary reasons for working in the Clifford algebra context. In particular, in such an algebraic setting one can consider the square root of the Laplacian and still be within the class of differential operators[2] (see (6.4.56) further below). Originating in the pioneering

[2] As opposed to, say, pseudodifferential calculus.

work of Grigore Moisil, Nicolae Teodorescu[3] [202–204, 252], and Rud Fueter [93] among others, the study of the resulting elliptic first-order differential operator in higher-dimensional settings, much in the spirit of the Cauchy–Riemann $\bar\partial$ operator in the plane, has become by now a well-established area of mathematics (see, for instance, the monographs [26, 105, 120, 196], and the references therein). While we shall delve into some of these aspects later, for now the goal is to elaborate on the algebraic framework.

The `Clifford algebra` with n imaginary units is the minimal enlargement of $\mathbb{R}^n$ to a unitary real algebra $(\mathcal{C}\ell_n, +, \odot)$, which is not generated (as an algebra) by any proper subspace of $\mathbb{R}^n$, and such that

$$x \odot x = -|x|^2 \quad \text{for each} \quad x \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n. \tag{6.4.1}$$

This identity is equivalent to the demand that, if $\{\mathbf{e}_j\}_{1\le j\le n}$ is the standard orthonormal basis in $\mathbb{R}^n$, then

$$\mathbf{e}_j \odot \mathbf{e}_j = -1 \quad \text{and} \quad \mathbf{e}_j \odot \mathbf{e}_k = -\mathbf{e}_k \odot \mathbf{e}_j \quad \text{whenever} \quad 1 \le j \ne k \le n. \tag{6.4.2}$$

In particular, identifying the canonical basis $\{\mathbf{e}_j\}_{1\le j\le n}$ from $\mathbb{R}^n$ with the n imaginary units generating $\mathcal{C}\ell_n$ yields the embedding[4]

$$\mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n, \qquad \mathbb{R}^n \ni x = (x_1, \dots, x_n) \equiv \sum_{j=1}^{n} x_j\,\mathbf{e}_j \in \mathcal{C}\ell_n. \tag{6.4.3}$$

Note that if

$$\begin{aligned} \mathbb{R}^n \ni x = (x_1, \dots, x_n) \equiv \textstyle\sum_{j=1}^n x_j\,\mathbf{e}_j \in \mathcal{C}\ell_n, \\ \mathbb{R}^n \ni y = (y_1, \dots, y_n) \equiv \textstyle\sum_{k=1}^n y_k\,\mathbf{e}_k \in \mathcal{C}\ell_n, \end{aligned} \tag{6.4.4}$$

then (6.4.2) implies

$$x \odot y = -\langle x, y\rangle + \frac{1}{2}\sum_{j,k=1}^{n} (x_j y_k - x_k y_j)\mathbf{e}_j \odot \mathbf{e}_k. \tag{6.4.5}$$

[3] Théodoresco, in the French spelling of the early 1990s.

[4] As the alerted reader might have noted, for $n = 2$ the identification in (6.4.3) amounts to embedding $\mathbb{R}^2$ into quaternions, i.e., $\mathbb{R}^2 \hookrightarrow \mathbb{H} := \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} : x_0, x_1, x_2, x_3 \in \mathbb{R}\}$ via $\mathbb{R}^2 \ni (x_1, x_2) \equiv x_1\mathbf{i} + x_2\mathbf{j} \in \mathbb{H}$. The reader is reassured that this is simply a matter of convenience, and we might as well have arranged things so that the embedding (6.4.3) comes down, when $n = 2$, to perhaps the more familiar identification $\mathbb{R}^2 \equiv \mathbb{C}$, by taking $\mathbb{R}^n \ni x = (x_0, x_1, \dots, x_{n-1}) \equiv x_0 + x_1\mathbf{e}_1 + \dots x_{n-1}\mathbf{e}_{n-1} \in \mathcal{C}\ell_{n-1}$. The latter choice leads to a parallel theory to the one presented here, entailing only minor natural alterations. Indeed, if in the latter scenario we factor out $\mathbf{e}_1$ and re-denote $\widetilde{\mathbf{e}}_1 := \mathbf{e}_1 \odot \mathbf{e}_2, \dots, \widetilde{\mathbf{e}}_{n-1} := \mathbf{e}_1 \odot \mathbf{e}_n$, then $\{\widetilde{\mathbf{e}}_j\}_{1\le j\le n-1}$ become anti-commuting imaginary units and the identification (6.4.3) becomes, modulo multiplication by $\mathbf{e}_1$, equivalent to $\mathbb{R}^n \ni x = (x_0, x_1, \dots, x_{n-1}) \equiv x_0 + x_1\widetilde{\mathbf{e}}_1 + \dots x_{n-1}\widetilde{\mathbf{e}}_{n-1} \in \mathcal{C}\ell_{n-1}$.

Hence, the scalar component of the Clifford algebra product $x \odot y$ is precisely (-1) times the inner product of the vectors $x, y \in \mathbb{R}^n$, whereas the bi-vector component of $x \odot y$ contains as much information as the vector product of x and y. This points to the fact that the Clifford algebra has strong ties to the geometry of the Euclidean space (indeed, $\mathcal{C}\ell_n$ is occasionally referred to as geometric algebra). For further reference let us also note here that (6.4.5) implies the following generalization of (6.4.1):

$$x \odot y + y \odot x = -2\langle x, y\rangle \quad \text{for all} \quad x, y \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n. \tag{6.4.6}$$

The fact that $\mathcal{C}\ell_n$ is not generated (as an algebra) by any proper subspace of $\mathbb{R}^n$ ensures that any element $u \in \mathcal{C}\ell_n$ may be uniquely represented in the form

$$u = \sum_I u_I e_I := \sum_{\ell=0}^{n} {\sum_{|I|=\ell}}' u_I\, e_I, \quad u_I \in \mathbb{C}. \tag{6.4.7}$$

Here $\sum'$ indicates that the sum is performed only over strictly increasing multi-indices I with ℓ components, i.e., ordered ℓ-tuples of the form $I = (i_1, i_2, \dots, i_\ell)$ with $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$. Also, we abbreviate

$$e_I := \mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_\ell} \quad \text{if} \quad I = (i_1, i_2, \dots, i_\ell), \tag{6.4.8}$$

and denote the multiplicative unit in $\mathcal{C}\ell_n$ by

$$e_0 := e_\varnothing := 1. \tag{6.4.9}$$

Let us momentarily digress for the purpose of establishing the existence of such a Clifford algebra. An example may be produced as a suitable sub-algebra of the matrix algebra $\mathbb{R}^{2^n \times 2^n}$. Specifically, for each $M \in \mathbb{N}$ denote by I_M the $M \times M$ identity matrix, and consider the double-indexed family of matrices $\{E_j^m\}_{\substack{1 \leq m \leq n \\ 1 \leq j \leq m}}$ defined inductively by

$$E_1^1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{2 \times 2} \tag{6.4.10}$$

and, in general, for $m \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, m\}$,

$$\begin{aligned} E_j^{m+1} &:= \begin{pmatrix} E_j^m & 0 \\ 0 & -E_j^m \end{pmatrix} \in \mathbb{R}^{2^{m+1} \times 2^{m+1}} \\ \text{and} \quad E_{m+1}^{m+1} &:= \begin{pmatrix} 0 & -I_{2^m} \\ I_{2^m} & 0 \end{pmatrix} \in \mathbb{R}^{2^{m+1} \times 2^{m+1}}. \end{aligned} \tag{6.4.11}$$

Observe that if $m \in \{2, \dots, n\}$, $\ell \in \{1, \dots, m\}$, and $(j_1, j_2, \dots, j_\ell) \in \{1, \dots, m\}^\ell$ are such that $1 \leq j_1 < j_2 < \cdots < j_\ell \leq m$, we have

$$E^m_{j_1} E^m_{j_2} \cdots E^m_{j_\ell} = \begin{pmatrix} E^{m-1}_{j_1} E^m_{j_2} \cdots E^{m-1}_{j_\ell} & 0 \\ 0 & E^{m-1}_{j_1} E^m_{j_2} \cdots E^{m-1}_{j_\ell} \end{pmatrix} \text{ if } j_\ell \leq m-1, \quad (6.4.12)$$

and

$$E^m_{j_1} E^m_{j_2} \cdots E^m_{j_\ell} = \begin{pmatrix} 0 & -E^{m-1}_{j_1} E^m_{j_2} \cdots E^{m-1}_{j_{\ell-1}} \\ -E^{m-1}_{j_1} E^m_{j_2} \cdots E^{m-1}_{j_{\ell-1}} & 0 \end{pmatrix} \text{ if } j_\ell = m. \quad (6.4.13)$$

In particular, an induction on $m \in \{1, \ldots, n\}$ shows that

$$\begin{gathered} \big(E^m_j\big)^2 = -I_{2^m} \text{ for each } m \in \{1, \ldots, n\} \text{ and } j \in \{1, \ldots, m\}, \\ \text{and that } E^m_{j_1} E^m_{j_2} = -E^m_{j_2} E^m_{j_1} \text{ for all } m \in \{1, \ldots, n\} \\ \text{and } j_1, j_2 \in \{1, \ldots, m\} \text{ with } j_1 \neq j_2. \end{gathered} \quad (6.4.14)$$

Hence,

the conditions in (6.4.2) are satisfied if we identify $\mathbf{e}_j := E^n_j$ for each $j \in \{1, \ldots, n\}$ and take $\odot$ to be the ordinary multiplication of matrices. (6.4.15)

In addition, from (6.4.10)–(6.4.11) and induction we see that

$$\begin{gathered} (E^m_j)^\top = -E^m_j \text{ for each } m \in \{1, \ldots, n\} \text{ and } j \in \{1, \ldots, m\}; \\ \text{in particular, } (E^n_j)^\top = -E^n_j \text{ for each } j \in \{1, \ldots, n\}. \end{gathered} \quad (6.4.16)$$

With the identification from (6.4.15) and the conventions from (6.4.8)–(6.4.9), we also claim that

the matrices $\{e_I\}_I$ are linearly independent over $\mathbb{R}$. (6.4.17)

To justify this, note that (6.4.12)–(6.4.13) and induction on $m \in \{1, \ldots, n\}$ imply

$$\begin{gathered} \mathrm{Trace}\big(E^m_{j_1} E^m_{j_2} \cdots E^m_{j_\ell}\big) = 0 \text{ if } m \in \{1, \ldots, n\},\ \ell \in \{1, \ldots, m\}, \text{ and} \\ (j_1, j_2, \ldots, j_\ell) \in \{1, \ldots, m\}^\ell \text{ with } 1 \leq j_1 < j_2 < \cdots < j_\ell \leq m. \end{gathered} \quad (6.4.18)$$

As such,

$$\mathrm{Trace}(e_I) = 0 \text{ whenever } I \neq \varnothing, \text{ and } \mathrm{Trace}(e_\varnothing) = n. \quad (6.4.19)$$

Consequently, if $\sum_I u_I e_I = 0$ for some coefficients $u_I \in \mathbb{R}$, then for each given ℓ-tuple of integers $J = (i_1, i_2, \ldots, i_\ell)$ with $1 \leq i_1 < i_2 < \cdots < i_\ell \leq n$ we have $\sum_I u_I e_J e_I = 0$, hence

$$\sum_I u_I \mathrm{Trace}\big(e_J e_I\big) = 0. \quad (6.4.20)$$

However, since each $e_J e_I$ is of the form $\pm e_{J\Delta I}$ where $J\Delta I$ is the symmetric difference between J and I, on account of (6.4.19) and (6.4.20) we conclude that $u_J = 0$. Given that J has been arbitrarily selected, the claim in (6.4.17) follows. The bottom line is that

a concrete model for the Clifford algebra $\mathcal{C}\ell_n$ is the sub-algebra of the matrix algebra $\mathbb{R}^{2^n\times 2^n}$ consisting of all matrices of the form $u = \sum_I u_I e_I$ with $u_I \in \mathbb{R}$ (with the identification from (6.4.15) and the conventions from (6.4.8)–(6.4.9)). (6.4.21)

This concludes the digression aimed at clarifying the existence of a Clifford algebra.

Returning to the mainstream discussion, we endow $\mathcal{C}\ell_n$ with the natural Euclidean metric

$$|u| := \Big\{\sum_I |u_I|^2\Big\}^{1/2} \quad \text{for each} \quad u = \sum_I u_I e_I \in \mathcal{C}\ell_n. \tag{6.4.22}$$

The Clifford conjugation on $\mathcal{C}\ell_n$, denoted by "bar," is defined as the unique real-linear involution on $\mathcal{C}\ell_n$ for which $\overline{e_I} \odot e_I = e_I \odot \overline{e_I} = 1$ for any multi-index I. More specifically, given $u = \sum_I u_I e_I \in \mathcal{C}\ell_n$ we set $\overline{u} := \sum_I u_I \overline{e_I}$ where, for each $I = (i_1, i_2, \dots, i_\ell)$ with $1 \le i_1 < i_2 < \cdots < i_\ell \le n$,

$$\overline{e_I} = (-1)^\ell \mathbf{e}_{i_\ell} \odot \mathbf{e}_{i_{\ell-1}} \odot \cdots \odot \mathbf{e}_{i_1}. \tag{6.4.23}$$

In particular,

$$\overline{\mathbf{e}_j} = -\mathbf{e}_j \quad \text{for each} \quad j \in \{1, \dots, n\}. \tag{6.4.24}$$

For each $u = \sum_I u_I e_I \in \mathcal{C}\ell_n$ define

$$u_{\text{vect}} := \sum_{j=1}^n u_j \mathbf{e}_j \in \mathbb{R}^n \quad \text{the vector part of} \quad u, \tag{6.4.25}$$

and denote by

$$u_{\text{scal}} := u_\varnothing \mathbf{e}_\varnothing = u_\varnothing \in \mathbb{R}, \quad \text{the scalar part of} \quad u. \tag{6.4.26}$$

We shall endow $\mathcal{C}\ell_n$ with the natural Hilbert space structure

$$\langle u, w\rangle := \sum_I u_I w_I, \quad \text{if} \quad u = \sum_I u_I e_I, \quad w = \sum_I w_I e_I \in \mathcal{C}\ell_n. \tag{6.4.27}$$

If we define a complex conjugation on $\mathcal{C}\ell_n$ by setting $u^c := \sum_I u_I^c e_I$ for each $u = \sum_I u_I e_I$, where z^c denotes the usual complex conjugation of $z \in \mathbb{C}$, then the aforementioned Hilbert space structure and Euclidean metric are related via

$$|u| = \sqrt{\langle u, u^c \rangle} = \Big\{ \sum_I |u_I|^2 \Big\}^{1/2} \quad \text{for each} \quad u = \sum_I u_I e_I \in \mathcal{C}\ell_n. \tag{6.4.28}$$

It also follows directly from definitions that

$$\overline{x} = -x \quad \text{for each} \quad x \in \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n. \tag{6.4.29}$$

Moreover, for any $u, w \in \mathcal{C}\ell_n$ one has

$$|u|^2 = (u \odot \overline{u})_0 = (\overline{u} \odot u)_0, \tag{6.4.30}$$

$$\langle u, w \rangle = (u \odot \overline{w})_0 = (\overline{u} \odot w)_0, \tag{6.4.31}$$

$$\overline{u \odot w} = \overline{w} \odot \overline{u}, \tag{6.4.32}$$

$$|\overline{u}| = |u|, \quad \overline{\overline{u}} = u, \tag{6.4.33}$$

$$|u \odot w| \leq 2^{n/2} |u||w|, \tag{6.4.34}$$

and

$$|u \odot w| = |u||w| \quad \text{if either } u \text{ or } w \text{ belongs to } \mathbb{R}^n \hookrightarrow \mathcal{C}\ell_n. \tag{6.4.35}$$

Also,

$$u + \overline{u} = 2\, u_0 \quad \text{for each} \quad u \in \mathbb{R}^n \odot \mathbb{R}^n. \tag{6.4.36}$$

More detailed accounts of these and related matters can be found in [26] and [196].

Lemma 6.4.1 *Let $a \in \mathcal{C}\ell_n$ be arbitrary. Then*

$$\langle a \odot u, w \rangle = \langle u, \overline{a} \odot w \rangle, \qquad \forall u, w \in \mathcal{C}\ell_n. \tag{6.4.37}$$

In particular, as a consequence of (6.4.37) *and* (6.4.24)*, for each $j \in \{1, \dots, n\}$ one has*

$$\langle \boldsymbol{e}_j \odot u, w \rangle = -\langle u, \boldsymbol{e}_j \odot w \rangle, \qquad \forall u, w \in \mathcal{C}\ell_n. \tag{6.4.38}$$

Proof By linearity, it suffices to show that for any two multi-indices I, K and any index $j \in \{1, \dots, n\}$, there holds

$$\langle \mathbf{e}_j \odot e_I, e_K \rangle = -\langle e_I, \mathbf{e}_j \odot e_K \rangle. \tag{6.4.39}$$

In turn, this identity may be justified by analyzing three cases. First, when $j \notin I$ and $j \notin K$, both sides in (6.4.39) vanish. Second, consider the case when $j \notin I$ and $j \in K$, say $K = K_1 \cup \{j\} \cup K_2$ with $K_1 = \{k \in K : k < j\}$ and $K_2 = \{k \in K : k > j\}$. On the one hand, if $I \neq K_1 \cup K_2$ then once again both sides of (6.4.39) vanish. If, on the other hand, $I = K_1 \cup K_2$ then both sides in (6.4.39) become $(-1)^{|K_1|}$. The third (and final) case, when $j \in I$ and $j \notin K$ is handled in a similar fashion. □

Consider an open nonempty set $\Omega \subseteq \mathbb{R}^n$. We will work with $\mathcal{C}\ell_n$-valued distributions in Ω, i.e., with $\mathcal{D}'(\Omega) \otimes \mathcal{C}\ell_n$. More specifically,

$$u \in \mathcal{D}'(\Omega) \otimes \mathcal{C}\!\ell_n \overset{def}{\Longleftrightarrow} u = \sum_{\ell=0}^{n} {\sum_{|I|=\ell}}' u_I \, e_I \quad \text{with each} \quad u_I \in \mathcal{D}'(\Omega). \tag{6.4.40}$$

We agree to pair such a $\mathcal{C}\!\ell_n$-valued distribution u in Ω with a scalar-valued test function $\varphi \in \mathscr{C}_c^\infty(\Omega)$ according to

$$_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\!\ell_n}\big\langle u, \varphi \big\rangle_{\mathcal{D}(\Omega)} := \sum_{\ell=0}^{n} {\sum_{|I|=\ell}}' {}_{\mathcal{D}'(\Omega)}\langle u_I, \varphi\rangle_{\mathcal{D}(\Omega)} \, e_I \in \mathcal{C}\!\ell_n. \tag{6.4.41}$$

It is also natural to consider the subspace $\mathscr{E}'(\Omega) \otimes \mathcal{C}\!\ell_n$ of $\mathcal{D}'(\Omega) \otimes \mathcal{C}\!\ell_n$, consisting of $\mathcal{C}\!\ell_n$-valued distributions with compact support in Ω. The pairing of $u \in \mathscr{E}'(\Omega) \otimes \mathcal{C}\!\ell_n$ with a scalar-valued smooth function $\varphi \in \mathscr{C}^\infty(\Omega)$ is defined in a similar fashion to (6.4.41).

Formula (6.4.41) is a particular case of a more general way of pairing a $\mathcal{C}\!\ell_n$-valued distribution u as in (6.4.40) with a $\mathcal{C}\!\ell_n$-valued test function $\phi \in \mathscr{C}_c^\infty(\Omega) \otimes \mathcal{C}\!\ell_n$, say

$$\phi = \sum_{\ell=0}^{n} {\sum_{|J|=\ell}}' \phi_J \, e_J \quad \text{with each} \quad \phi_J \in \mathscr{C}_c^\infty(\Omega), \tag{6.4.42}$$

of the following sort:

$$_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\!\ell_n}\big\langle u, \phi \big\rangle_{\mathcal{D}(\Omega)\otimes\mathcal{C}\!\ell_n} := {\sum_{I,J}}' {}_{\mathcal{D}'(\Omega)}\langle u_I, \phi_J\rangle_{\mathcal{D}(\Omega)} \, e_I \odot e_J \in \mathcal{C}\!\ell_n. \tag{6.4.43}$$

This should be contrasted with the *scalar* pairing

$$\begin{aligned} &_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\!\ell_n}\big(u, \phi \big)_{\mathcal{D}(\Omega)\otimes\mathcal{C}\!\ell_n} := \sum_{\ell=0}^{n} {\sum_{|I|=\ell}}' {}_{\mathcal{D}'(\Omega)}\langle u_I, \phi_I\rangle_{\mathcal{D}(\Omega)} \in \mathbb{C} \\ &\text{for every} \quad u = \sum_{\ell=0}^{n} {\sum_{|I|=\ell}}' u_I \, e_I \quad \text{with each} \quad u_I \in \mathcal{D}'(\Omega), \\ &\text{and every} \quad \phi = \sum_{\ell=0}^{n} {\sum_{|J|=\ell}}' \phi_J \, e_J \quad \text{with each} \quad \phi_J \in \mathscr{C}_c^\infty(\Omega). \end{aligned} \tag{6.4.44}$$

Given a set $\Sigma \subseteq \mathbb{R}^n$, in analogy with (6.4.43) we agree to define the Clifford algebra pairing (from the left) of a $\mathcal{C}\!\ell_n$-valued distribution u on Σ (in the sense of (4.1.34)) with a $\mathcal{C}\!\ell_n$-valued test function ψ on Σ (in the sense of (4.1.4)) as

$$_{\mathrm{Lip}_c(\Sigma)\otimes\mathcal{C}\!\ell_n}\big\langle \psi, u \big\rangle_{\mathrm{Lip}_c(\Sigma)'\otimes\mathcal{C}\!\ell_n} := {\sum_{I,J}}' {}_{\mathrm{Lip}_c(\Sigma)}\langle \psi_I, u_J\rangle_{\mathrm{Lip}_c(\Sigma)'} \, e_I \odot e_J \in \mathcal{C}\!\ell_n \tag{6.4.45}$$

whenever

$$\begin{aligned}\psi &= \textstyle\sum_{\ell=0}^{n} \sum'_{|I|=\ell} \psi_I\, e_I \quad \text{with each} \quad \psi_I \in \mathrm{Lip}_c(\Sigma) \quad \text{and} \\ u &= \textstyle\sum_{\ell=0}^{n} \sum'_{|J|=\ell} u_J\, e_J \quad \text{with each} \quad u_J \in \mathrm{Lip}_c(\Sigma)'.\end{aligned} \tag{6.4.46}$$

Objects from $\mathcal{D}'(\Omega) \otimes \mathcal{C}\ell_n$ and $\mathcal{E}'(\Omega) \otimes \mathcal{C}\ell_n$ obey natural rules, much as ordinary distributions, with the added bonus that the Clifford algebra formalism is in full effect. For example, while ordinary distributions could be multiplied by smooth scalar-valued functions, Clifford algebra-valued distributions can be multiplied, via $\odot$, by smooth Clifford algebra-valued functions. We may also consider the action of differential operators with Clifford algebra coefficients. A prominent example is the classical (homogeneous) `Dirac operator`, given by

$$D := \sum_{j=1}^{n} \mathbf{e}_j \odot \partial_j. \tag{6.4.47}$$

This acts on each $u \in \mathcal{D}'(\Omega) \otimes \mathcal{C}\ell_n$ from the left and from the right according to

$$D_L u := \sum_{j=1}^{n} \mathbf{e}_j \odot (\partial_j u), \tag{6.4.48}$$

and, respectively,

$$D_R u := \sum_{j=1}^{n} (\partial_j u) \odot \mathbf{e}_j, \tag{6.4.49}$$

where the right-hand sides in (6.4.48)–(6.4.49) are regarded as $\mathcal{C}\ell_n$-valued distribution in Ω. When no subscript "L" or "R" is employed, it is understood that D acts from the left.

For example, with these conventions in mind it may be easily verified from definitions that for each function $u \in \mathscr{C}^1(\Omega) \otimes \mathcal{C}\ell_n$ we have

$$D_R u = -\overline{D_L(\overline{u})} \quad \text{in} \quad \Omega. \tag{6.4.50}$$

Let us also observe that, for each distribution $u \in \mathcal{D}'(\Omega) \otimes \mathcal{C}\ell_n$ and each test function $\phi \in \mathscr{C}_c^\infty(\Omega) \otimes \mathcal{C}\ell_n$ we have

$${}_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\ell_n}\big\langle D_L u, \phi\big\rangle_{\mathcal{D}(\Omega)\otimes\mathcal{C}\ell_n} = -\,{}_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\ell_n}\big\langle u, D_R\phi\big\rangle_{\mathcal{D}(\Omega)\otimes\mathcal{C}\ell_n}, \tag{6.4.51}$$

whereas (6.4.38) entails

$${}_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\ell_n}\big(Du, \phi\big)_{\mathcal{D}(\Omega)\otimes\mathcal{C}\ell_n} = {}_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\ell_n}\big(u, D\,\phi\big)_{\mathcal{D}(\Omega)\otimes\mathcal{C}\ell_n}. \tag{6.4.52}$$

Let us also note that if $\Omega \subseteq \mathbb{R}^n$ is a set of locally finite perimeter, and ν denotes its geometric measure theoretic outward unit normal, then in the sense of distributions in $\mathbb{R}^n$ we have

$$D(\mathbf{1}_\Omega) = -\nu \odot \sigma_* = -\nu \odot \mathcal{H}^{n-1} \lfloor \partial^* \Omega = -\nu \odot \mathcal{H}^{n-1} \lfloor \partial_* \Omega, \qquad (6.4.53)$$

where the vector $\nu = (\nu_1, \dots, \nu_n)$ is canonically identified with the $\mathcal{C}\ell_n$-valued function $\nu = \nu_1 \mathbf{e}_1 + \cdots + \nu_n \mathbf{e}_n$. In view of the matrix formalism associated with the Clifford algebras (see (6.4.21)), this is a special case of the more general result described in Proposition 5.6.3.

As just mentioned, the Dirac operator (6.4.47) may be naturally regarded as a homogeneous, constant coefficient, first-order $2^n \times 2^n$ system in $\mathbb{R}^n$, by adopting the model for the Clifford algebra $\mathcal{C}\ell_n$ described in (6.4.21). Specifically, with the identification from (6.4.15) we have

$$D = \sum_{j=1}^{n} E_j^n \partial_j. \qquad (6.4.54)$$

To illustrate this point of view, consider the case $n = 2$, in which scenario the aforementioned model identifies the Dirac operator (6.4.47) with the $2^2 \times 2^2$ system

$$D = \begin{pmatrix} 0 & -\partial_1 & -\partial_2 & 0 \\ \partial_1 & 0 & 0 & -\partial_2 \\ \partial_2 & 0 & 0 & \partial_1 \\ 0 & \partial_2 & -\partial_1 & 0 \end{pmatrix}. \qquad (6.4.55)$$

Following Moisil and Teodorescu [203, 204], we shall call u `monogenic` in Ω if $Du = 0$ in Ω. One of the most fundamental properties of Dirac operator introduced above is that this may be thought of as the square-root of the (minus) Laplacian. More precisely, D satisfies

$$D^2 = -\Delta \qquad (6.4.56)$$

where $\Delta := \sum_{j=1}^{n} \partial_j^2$ is the usual Laplace operator in $\mathbb{R}^n$. In particular, monogenic functions are null-solutions of the Laplacian (i.e., harmonic functions).

In general, if $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ is a Banach space then by $\mathscr{X} \otimes \mathcal{C}\ell_n$ we shall denote the Banach space consisting of elements of the form

$$u = \sum_{\ell=0}^{n} {\sum_{|I|=\ell}}' u_I \, e_I, \quad u_I \in \mathscr{X}, \qquad (6.4.57)$$

equipped with the natural norm

$$\|u\|_{\mathscr{X} \otimes \mathcal{C}\ell_n} := \sum_{\ell=0}^{n} {\sum_{|I|=\ell}}' \|u_I\|_{\mathscr{X}}. \qquad (6.4.58)$$

A simple but useful observation in this context is that, as is apparent from (6.4.1) and (6.4.35), if $\Omega \subset \mathbb{R}^n$ is a set of locally finite perimeter, ν denotes its geometric measure theoretic outward unit normal, and $\sigma_* := \mathcal{H}^{n-1}\lfloor\partial_*\Omega$, then $\nu\odot : L^p(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n \to L^p(\partial_*\Omega, \sigma_*) \otimes \mathcal{C}\ell_n$ happens to be an isometric isomorphism, with inverse $-\nu\odot$, for each $p \in (0, \infty]$. (6.4.59)

To state our next result we need some notation. Specifically, for any two elements A, B of a given ring $\mathscr{R}$, denote by

$$[A; B] := AB - BA \tag{6.4.60}$$

the commutator of A and B, and by

$$\{A; B\} := AB + BA \tag{6.4.61}$$

the anti-commutator of A and B.

Lemma 6.4.2 *Let $\mathscr{R}$ be a unitary ring and, given some $n \in \mathbb{N}$, let $\{B_{jk}\}_{1\le j,k,\le n}$ be a family of elements of $\mathscr{R}$ with the property that*

$$B_{kj} = -B_{jk} \ \text{ for every } \ j, k \in \{1, \dots, n\}. \tag{6.4.62}$$

Also, fix $A \in \mathscr{R}$ arbitrary. Then one has the following quadratic expansion formula in $\mathcal{C}\ell_n \otimes \mathscr{R}$:

$$\begin{aligned} 6\Big(A + 2\cdot\sum_{j,k=1}^{n}(\boldsymbol{e}_j \odot \boldsymbol{e}_k)\otimes B_{jk}\Big)^2 &= 6A^2 - 48\cdot\sum_{j,k=1}^{n}(B_{jk})^2 \\ &- 12\cdot\sum_{j,k=1}^{n}(\boldsymbol{e}_j \odot \boldsymbol{e}_k)\otimes\sum_{\ell=1}^{n}\Big(4[B_{j\ell}; B_{\ell k}] - \{A; B_{jk}\}\Big) \\ &+ 4\cdot\sum_{i,j,k,\ell=1}^{n}(\boldsymbol{e}_i \odot \boldsymbol{e}_j \odot \boldsymbol{e}_k \odot \boldsymbol{e}_\ell)\otimes\Big(\{B_{ij}; B_{k\ell}\} + \{B_{i\ell}; B_{jk}\} - \{B_{ik}; B_{j\ell}\}\Big). \end{aligned} \tag{6.4.63}$$

Proof For starters, the fact that $2B_{jj} = 0$ for every $j \in \{1, \dots, n\}$ implies

$$2\cdot\sum_{j,k=1}^{n}(\mathbf{e}_j \odot \mathbf{e}_k)\otimes B_{jk} = 2\cdot\sum_{1\le j\ne k\le n}(\mathbf{e}_j \odot \mathbf{e}_k)\otimes B_{jk}. \tag{6.4.64}$$

Also,

$$\begin{aligned}
A\Big(2\cdot \sum_{1\leq j\neq k\leq n} (\mathbf{e}_j \odot \mathbf{e}_k) \otimes B_{jk}\Big) &+ \Big(2\cdot \sum_{1\leq j\neq k\leq n} (\mathbf{e}_j \odot \mathbf{e}_k) \otimes B_{jk}\Big)A \\
&= 2\cdot \sum_{1\leq j\neq k\leq n} (\mathbf{e}_j \odot \mathbf{e}_k) \otimes (B_{jk}A + AB_{jk}) \\
&= 2\cdot \sum_{1\leq j\neq k\leq n} (\mathbf{e}_j \odot \mathbf{e}_k) \otimes \{A;\, B_{jk}\} \\
&= 2\cdot \sum_{j,k=1}^{n} (\mathbf{e}_j \odot \mathbf{e}_k) \otimes \{A;\, B_{jk}\}.
\end{aligned} \tag{6.4.65}$$

Consider next,

$$\begin{aligned}
\Big(2\cdot \sum_{1\leq j\neq k\leq n} (\mathbf{e}_j \odot \mathbf{e}_k) \otimes B_{jk}\Big)^2 & \qquad\qquad (6.4.66)\\
&= \Big(2\cdot \sum_{1\leq a\neq b\leq n} (\mathbf{e}_a \odot \mathbf{e}_b) \otimes B_{ab}\Big)\Big(2\cdot \sum_{1\leq c\neq d\leq n} (\mathbf{e}_c \odot \mathbf{e}_d) \otimes B_{cd}\Big) \\
&= \sum_{\substack{1\leq a\neq b\leq n\\ 1\leq c\neq d\leq n}} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd}).
\end{aligned}$$

Note that the sets

$$J_{\pm} := \Big\{(a,b,c,d) \in \{1,\dots,n\}^4 :\ a \neq b,\ c \neq c,\ \ \mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d = \pm 1\Big\} \tag{6.4.67}$$

may be explicitly described as

$$\begin{aligned}
J_+ &= \Big\{(a,b,c,d) \in \{1,\dots,n\}^4 :\ a \neq b,\ a = d,\ b = c\Big\}, \\
J_- &= \Big\{(a,b,c,d) \in \{1,\dots,n\}^4 :\ a \neq b,\ a = c,\ b = d\Big\}.
\end{aligned} \tag{6.4.68}$$

Moreover,

$$\begin{aligned}
\sum_{(a,b,c,d)\in J_+} & 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd}) \\
&= 4\cdot \sum_{1\leq a\neq b\leq n} B_{ab}B_{ba} = -4\cdot \sum_{j,k=1}^{n} (B_{jk})^2,
\end{aligned} \tag{6.4.69}$$

and

$$\sum_{(a,b,c,d)\in J_-} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd})$$
$$= -4 \cdot \sum_{1\le a\neq b\le n} B_{ab}B_{ab} = -4 \cdot \sum_{j,k=1}^{n} (B_{jk})^2. \tag{6.4.70}$$

From these identities we then conclude that

$$\text{the scalar component of the left side of (6.4.63) is } 6A^2 - 48 \cdot \textstyle\sum_{j,k=1}^{n} (B_{jk})^2. \tag{6.4.71}$$

To identify the bi-vector component of the left-hand side of (6.4.63), fix two distinct indices $j, k \in \{1, \dots, n\}$ which are otherwise arbitrary. Then, on the one hand, the set

$$Q_{jk} := \Big\{(a, b, c, d) \in \{1, \dots, n\}^4 : a \neq b,\ c \neq c,\ \mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d = \pm\mathbf{e}_j \odot \mathbf{e}_k\Big\} \tag{6.4.72}$$

may be written as the disjoint union of the following eight sets:

$$\begin{aligned}
I_{jk} &:= \Big\{(a, b, c, d) \in \{1, \dots, n\}^4 : a = j,\ b = c \notin \{j, k\},\ \mathrm{d} = k\Big\},\\
II_{jk} &:= \Big\{(a, b, c, d) \in \{1, \dots, n\}^4 : a = k,\ b = c \notin \{j, k\},\ \mathrm{d} = j\Big\},\\
III_{jk} &:= \Big\{(a, b, c, d) \in \{1, \dots, n\}^4 : a = j,\ b = d \notin \{j, k\},\ c = k\Big\},\\
IV_{jk} &:= \Big\{(a, b, c, d) \in \{1, \dots, n\}^4 : a = k,\ b = d \notin \{j, k\},\ c = j\Big\},\\
V_{jk} &:= \Big\{(a, b, c, d) \in \{1, \dots, n\}^4 : a = c \notin \{j, k\},\ b = j,\ \mathrm{d} = k\Big\},\\
VI_{jk} &:= \Big\{(a, b, c, d) \in \{1, \dots, n\}^4 : a = c \notin \{j, k\},\ b = k,\ \mathrm{d} = j\Big\},\\
VII_{jk} &:= \Big\{(a, b, c, d) \in \{1, \dots, n\}^4 : a = d \notin \{j, k\},\ b = j,\ c = k\Big\},\\
VIII_{jk} &:= \Big\{(a, b, c, d) \in \{1, \dots, n\}^4 : a = d \notin \{j, k\},\ b = k,\ c = j\Big\}.
\end{aligned} \tag{6.4.73}$$

On the other hand,

$$\sum_{(a,b,c,d)\in I_{jk}} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd}) \tag{6.4.74}$$
$$= -4(\mathbf{e}_j \odot \mathbf{e}_k) \otimes \Big(\sum_{\substack{b\in\{1,\dots,n\}\\ b\neq j,\, b\neq k}} B_{jb}B_{bk}\Big) = -4(\mathbf{e}_j \odot \mathbf{e}_k) \otimes \Big(\sum_{\ell=1}^{n} B_{j\ell}B_{\ell k}\Big),$$

and

$$\sum_{(a,b,c,d)\in II_{jk}} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd})$$
$$= 4(\mathbf{e}_j \odot \mathbf{e}_k) \otimes \Big(\sum_{\substack{b\in\{1,\dots,n\}\\ b\neq j,\, b\neq k}} B_{kb}B_{bj} \Big) = 4(\mathbf{e}_j \odot \mathbf{e}_k) \otimes \Big(\sum_{\ell=1}^{n} B_{k\ell}B_{\ell j} \Big)$$
$$= 4(\mathbf{e}_j \odot \mathbf{e}_k) \otimes \Big(\sum_{\ell=1}^{n} B_{\ell k}B_{j\ell} \Big), \tag{6.4.75}$$

hence

$$\sum_{(a,b,c,d)\in I_{jk}\cup II_{jk}} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd})$$
$$= -4(\mathbf{e}_j \odot \mathbf{e}_k) \otimes \Big(\sum_{\ell=1}^{n} B_{j\ell}B_{\ell k} \Big) + 4(\mathbf{e}_j \odot \mathbf{e}_k) \otimes \Big(\sum_{\ell=1}^{n} B_{\ell k}B_{j\ell} \Big)$$
$$= -4(\mathbf{e}_j \odot \mathbf{e}_k) \otimes \sum_{\ell=1}^{n} [B_{j\ell};\, B_{\ell k}]. \tag{6.4.76}$$

In a similar fashion,

$$\sum_{(a,b,c,d)\in III_{jk}\cup IV_{jk}} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd})$$
$$= -4(\mathbf{e}_j \odot \mathbf{e}_k) \otimes \sum_{\ell=1}^{n} [B_{j\ell};\, B_{\ell k}], \tag{6.4.77}$$

$$\sum_{(a,b,c,d)\in V_{jk}\cup VI_{jk}} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd})$$
$$= -4(\mathbf{e}_j \odot \mathbf{e}_k) \otimes \sum_{\ell=1}^{n} [B_{j\ell};\, B_{\ell k}], \tag{6.4.78}$$

and

$$\sum_{(a,b,c,d)\in VII_{jk}\cup VIII_{jk}} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd})$$
$$= -4(\mathbf{e}_j \odot \mathbf{e}_k) \otimes \sum_{\ell=1}^{n} [B_{j\ell};\, B_{\ell k}]. \tag{6.4.79}$$

Consequently, the bi-vector component of the left-hand side of (6.4.63) is given by

$$
\begin{aligned}
6 \cdot \sum_{1 \le j < k \le n} \sum_{(a,b,c,d) \in Q_{jk}} & 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab} B_{cd}) \\
&= -96 \cdot \sum_{1 \le j < k \le n} (\mathbf{e}_j \odot \mathbf{e}_k) \otimes \sum_{\ell=1}^{n} [B_{j\ell};\, B_{\ell k}] \\
&= -48 \cdot \sum_{j,k=1}^{n} (\mathbf{e}_j \odot \mathbf{e}_k) \otimes \sum_{\ell=1}^{n} [B_{j\ell};\, B_{\ell k}]. \qquad (6.4.80)
\end{aligned}
$$

We are left with identifying the quartet-vector component of the left-hand side of (6.4.63). To this end, fix $i, j, k, \ell \in \{1, \dots, n\}$ mutually distinct, and consider the set

$$
\begin{aligned}
S_{ijk\ell} := \Big\{ & (a, b, c, d) \in \{1, \dots, n\}^4 : a \ne b,\ c \ne c, \quad \text{and} \\
& \mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d = \pm \mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_\ell \Big\}. \qquad (6.4.81)
\end{aligned}
$$

The above set may be written as the disjoint union of the following six sets:

$$
\begin{aligned}
S_1 &:= \big\{(i, j, k, \ell),\ (j, i, \ell, k),\ (i, j, \ell, k),\ (j, i, k, \ell)\big\}, \\
S_2 &:= \big\{(i, k, j, \ell),\ (k, i, \ell, j),\ (i, k, \ell, j),\ (k, i, j, \ell)\big\}, \\
S_3 &:= \big\{(i, \ell, j, k),\ (\ell, i, j, k),\ (i, \ell, k, j),\ (\ell, i, k, j)\big\}, \\
S_4 &:= \big\{(j, k, i, \ell),\ (k, j, i, \ell),\ (j, k, \ell, i),\ (k, j, \ell, i)\big\}, \\
S_5 &:= \big\{(j, \ell, i, k),\ (\ell, j, i, k),\ (j, \ell, k, i),\ (\ell, j, k, i)\big\}, \\
S_6 &:= \big\{(k, \ell, i, j),\ (\ell, k, i, j),\ (k, \ell, j, i),\ (\ell, k, j, 1)\big\}. \qquad (6.4.82)
\end{aligned}
$$

Then since

$$
\begin{aligned}
\sum_{(a,b,c,d) \in S_1} & 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab} B_{cd}) \\
&= 16(\mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_\ell) \otimes (B_{ij} B_{k\ell}), \\
\sum_{(a,b,c,d) \in S_2} & 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab} B_{cd}) \\
&= -16(\mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_\ell) \otimes (B_{ik} B_{j\ell}), \\
\sum_{(a,b,c,d) \in S_3} & 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab} B_{cd}) \\
&= 16(\mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_\ell) \otimes (B_{i\ell} B_{jk}), \\
\sum_{(a,b,c,d) \in S_4} & 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab} B_{cd}) \\
&= 16(\mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_\ell) \otimes (B_{jk} B_{i\ell}),
\end{aligned}
$$

$$\sum_{(a,b,c,d)\in S_5} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd})$$
$$= -16(\mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_\ell) \otimes (B_{j\ell}B_{ik}),$$
$$\sum_{(a,b,c,d)\in S_6} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd})$$
$$= 16(\mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_\ell) \otimes (B_{k\ell}B_{ij}), \tag{6.4.83}$$

it follows that

$$\sum_{(a,b,c,d)\in S_{ijk\ell}} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd}) \tag{6.4.84}$$
$$= 16(\mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_\ell) \otimes \Big(\{B_{ij};\, B_{k\ell}\} + \{B_{i\ell};\, B_{jk}\} - \{B_{ik},\, B_{j\ell}\}\Big).$$

From this we then conclude that the quartet-vector component of the left-hand side of (6.4.63) is given by

$$6 \cdot \sum_{1\le i<j<k<\ell\le n} \sum_{(a,b,c,d)\in S_{ijk\ell}} 4(\mathbf{e}_a \odot \mathbf{e}_b \odot \mathbf{e}_c \odot \mathbf{e}_d) \otimes (B_{ab}B_{cd}) \tag{6.4.85}$$
$$= 4! \cdot \sum_{1\le i<j<k<\ell\le n} 4(\mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_\ell)$$
$$\otimes \Big(\{B_{ij};\, B_{k\ell}\} + \{B_{i\ell};\, B_{jk}\} - \{B_{ik};\, B_{j\ell}\}\Big)$$
$$= 4 \cdot \sum_{i,j,k,\ell=1}^{n} (\mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k \odot \mathbf{e}_\ell) \otimes \Big(\{B_{ij};\, B_{k\ell}\} + \{B_{i\ell};\, B_{jk}\} - \{B_{ik};\, B_{j\ell}\}\Big).$$

Formula (6.4.63) now follows from (6.4.71), (6.4.80), and (6.4.85). □

The next lemma elaborates on the structure of a double Clifford product of three arbitrary vectors from $\mathbb{R}^n$.

Lemma 6.4.3 *Consider the vectors* $a = (a_i)_{1\le i\le n} \in \mathbb{R}^n$, $b = (b_j)_{1\le j\le n} \in \mathbb{R}^n$, $c = (c_k)_{1\le k\le n} \in \mathbb{R}^n$, *canonically identified with the Clifford algebra elements*

$$a = \sum_{i=1}^{n} a_i \mathbf{e}_i \in \mathcal{C}\ell_n, \quad b = \sum_{j=1}^{n} b_j \mathbf{e}_j \in \mathcal{C}\ell_n, \quad c = \sum_{i=1}^{n} c_k \mathbf{e}_k \in \mathcal{C}\ell_n. \tag{6.4.86}$$

Then

$$a \odot b \odot c = -\langle a, b\rangle c + \langle a, c\rangle b - \langle b, c\rangle a + \sum_{\substack{1\le i,j,k\le n \\ \text{pairwise distinct}}} a_i b_j c_k \mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k. \tag{6.4.87}$$

Proof For starters, write

$$a \odot b \odot c = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_i b_j c_k \mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k. \tag{6.4.88}$$

Let us focus on the piece of the above sum corresponding to $k = i$, i.e.,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j c_i \mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_i. \tag{6.4.89}$$

This further breaks up into the piece corresponding to $j = i$, i.e.,

$$-\sum_{i=1}^{n} a_i b_i c_i \mathbf{e}_i, \tag{6.4.90}$$

and the piece corresponding to $j \neq i$, that is,

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j \neq i} a_i b_j c_i \mathbf{e}_j &= \sum_{i=1}^{n} a_i c_i \Big(\sum_{j \neq i} b_j \mathbf{e}_j \Big) = \sum_{i=1}^{n} a_i c_i \Big(\sum_{j=1}^{n} b_j \mathbf{e}_j - b_i \mathbf{e}_i \Big) \\ &= \langle a, c \rangle b - \sum_{i=1}^{n} a_i b_i c_i \mathbf{e}_i. \end{aligned} \tag{6.4.91}$$

Next, let us look at the piece of the sum in (6.4.88) corresponding to $k \neq i$, i.e.,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k \neq i} a_i b_j c_k \mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k. \tag{6.4.92}$$

The portion of this sum corresponding to $j = i$ is

$$\begin{aligned} -\sum_{i=1}^{n} \sum_{k \neq i} a_i b_i c_k \mathbf{e}_k &= -\sum_{i=1}^{n} a_i b_i \Big(\sum_{k \neq i} c_k \mathbf{e}_k \Big) = -\sum_{i=1}^{n} a_i b_i \Big(\sum_{k=1}^{n} c_k \mathbf{e}_k - c_i \mathbf{e}_i \Big) \\ &= -\langle a, b \rangle c + \sum_{i=1}^{n} a_i b_i c_i \mathbf{e}_i. \end{aligned} \tag{6.4.93}$$

The portion of the sum in (6.4.92) corresponding to $j \neq i$ breaks up into the piece where we also have $k \neq j$, i.e.,

$$\sum_{\substack{1 \leq i,j,k \leq n \\ \text{pairwise distinct}}} a_i b_j c_k \mathbf{e}_i \odot \mathbf{e}_j \odot \mathbf{e}_k, \tag{6.4.94}$$

and the piece where we have $k = j$, i.e.,

$$-\sum_{i=1}^{n}\sum_{j\neq i} a_i b_j c_j \mathbf{e}_i = -\sum_{i=1}^{n} a_i \mathbf{e}_i \Big(\sum_{j\neq i} b_j c_j\Big) = -\sum_{i=1}^{n} a_i \mathbf{e}_i \Big(\sum_{j=1}^{n} b_j c_j - b_i c_i\Big)$$
$$= -\langle b, c\rangle a + \sum_{i=1}^{n} a_i b_i c_i \mathbf{e}_i. \tag{6.4.95}$$

Gathering everything and canceling like-terms, the identity claimed in (6.4.87) follows. □

Given an arbitrary open subset Ω of $\mathbb{R}^n$, recall that $\mathrm{CBM}(\Omega) \otimes \mathcal{C}\ell_n$ consists of elements of the form $\mu = \sum_I \mu_I e_I$ where each μ_I is a complex Borel measure in Ω. In such a case, define $|\mu| := \sum_I |\mu_I|$. For each Clifford algebra-valued measure $\mu = \sum_I \mu_I e_I \in \mathrm{CBM}(\Omega) \otimes \mathcal{C}\ell_n$ and each Borel function $f = \sum_J f_J e_J : \Omega \to \mathcal{C}\ell_n$ we define $\int_\Omega f \odot \mathrm{d}\mu$ in a natural fashion, i.e., as $\sum_{I,J} \Big(\int_\Omega f_J \,\mathrm{d}\mu_I\Big) e_J \odot e_I$, assuming all scalar integrals are absolutely convergent.

Continue to assume that Ω is an open subset of $\mathbb{R}^n$. The `Cauchy–Clifford transform` of a Clifford algebra-valued measure $\mu \in \mathrm{CBM}(\Omega) \otimes \mathcal{C}\ell_n$ is defined as

$$\widehat{\mu}(x) := \frac{1}{\omega_{n-1}} \int_\Omega \frac{x-y}{|x-y|^n} \odot \mathrm{d}\mu(y) \tag{6.4.96}$$

at each point $x \in \Omega$ with the property that

$$\int_\Omega \frac{1}{|x-y|^{n-1}}\,\mathrm{d}|\mu|(y) < +\infty. \tag{6.4.97}$$

Lemma 6.4.4 *Let Ω be an open subset of $\mathbb{R}^n$, and pick an arbitrary Clifford algebra-valued measure $\mu \in \mathrm{CBM}(\Omega) \otimes \mathcal{C}\ell_n$. Then its Cauchy–Clifford transform, the function $\widehat{\mu}$ introduced in (6.4.96), is well-defined at $\mathcal{L}^n$-a.e. point in Ω and, in fact, $\widehat{\mu}$ belongs to $L^p_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\ell_n$ whenever $1 \le p < n/(n-1)$. Moreover,*

$$D\widehat{\mu} = \mu \quad \textit{in} \quad \mathcal{D}'(\Omega) \otimes \mathcal{C}\ell_n. \tag{6.4.98}$$

Proof Lemma 3.5.6 shows that, indeed, $\widehat{\mu}$ is well-defined as a function in $L^p_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\ell_n$ whenever $1 \le p < n/(n-1)$. As regards (6.4.98), given any scalar test function $\varphi \in \mathscr{C}^\infty_c(\Omega)$ we may compute

$$_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\ell_n}\langle D\widehat{\mu}, \varphi\rangle_{\mathcal{D}(\Omega)} = -\sum_{j=1}^{n} {}_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\ell_n}\langle \mathbf{e}_j \odot \widehat{\mu}, \partial_j\varphi\rangle_{\mathcal{D}(\Omega)}$$
$$= -\sum_{j=1}^{n} \int_\Omega (\partial_j\varphi)\mathbf{e}_j \odot \widehat{\mu}\,\mathrm{d}\mathcal{L}^n$$

$$
\begin{aligned}
&= -\frac{1}{\omega_{n-1}} \int_{\Omega} (D\varphi)(x) \odot \Big(\int_{\Omega} \frac{x-y}{|x-y|^n} \odot \mathrm{d}\mu(y) \Big) \,\mathrm{d}x \\
&= \int_{\Omega} \Big(-\frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} (D\varphi)(x) \odot \frac{x-y}{|x-y|^n} \,\mathrm{d}x \Big) \odot \mathrm{d}\mu(y) \\
&= \int_{\Omega} \varphi(y) \,\mathrm{d}\mu(y) = {}_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\ell_n}\langle \mu, \varphi \rangle_{\mathcal{D}(\Omega)} \qquad (6.4.99)
\end{aligned}
$$

where we have used the fact that $\widehat{\mu} \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\ell_n$ in the second equality, Fubini's theorem in the fourth equality, and took advantage of the fact that the inner integral in the fifth equality involves the fundamental solution (with pole at y) for the Dirac operator D in $\mathbb{R}^n$. □

In the two-dimensional setting considered in [99, Problem 4.2, p. 55], J. Garnett asks to

$$\textit{describe the sets whose characteristic functions are Cauchy transforms almost everywhere.} \qquad (6.4.100)$$

A solution in the class of compact subsets of $\mathbb{R}^n$ with $n \geq 2$ arbitrary is offered in our next proposition.

Proposition 6.4.5 *Let $E \subseteq \mathbb{R}^n$ be a compact set. Then E has finite perimeter if and only if there exists a Clifford algebra-valued measure $\mu \in \mathrm{CBM}(\mathbb{R}^n) \otimes \mathcal{C}\ell_n$ such that*

$$\widehat{\mu} = \mathbf{1}_E \ \text{ at } \mathcal{L}^n\text{-a.e. point in } \ \mathbb{R}^n. \qquad (6.4.101)$$

Proof Suppose first that $E \subseteq \mathbb{R}^n$ is a compact set with the property that there exists a measure $\mu \in \mathrm{CBM}(\mathbb{R}^n) \otimes \mathcal{C}\ell_n$ such that (6.4.101) holds. Then (6.4.98) (used with $\Omega := \mathbb{R}^n$) gives

$$D\mathbf{1}_E = D\widehat{\mu} = \mu \ \text{ in } \ \mathcal{D}'(\mathbb{R}^n) \otimes \mathcal{C}\ell_n, \qquad (6.4.102)$$

hence

$$\partial_j \mathbf{1}_E \in \mathrm{CBM}(\mathbb{R}^n) \ \text{ for each } \ j \in \{1, \dots, n\}. \qquad (6.4.103)$$

Ultimately, this shows that E has finite perimeter (cf. (5.6.4) in this regard).

In the converse direction, suppose $E \subseteq \mathbb{R}^n$ is a compact set of finite perimeter. Then the Clifford algebra-measure $\mu := D\mathbf{1}_E$ belongs to $\mathrm{CBM}(\mathbb{R}^n) \otimes \mathcal{C}\ell_n$ and satisfies $\operatorname{supp}\mu \subseteq \partial E$, hence μ is compactly supported. Fix an arbitrary scalar test function $\varphi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ and, for each $j \in \{1, \dots, n\}$, define

$$\psi_j(y) := \frac{1}{\omega_{n-1}} \int_{\mathbb{R}^n} \frac{z_j}{|z|^n} \varphi(z+y) \,\mathrm{d}z, \qquad \forall y \in \mathbb{R}^n. \qquad (6.4.104)$$

Then $\psi_j \in \mathscr{C}^\infty(\mathbb{R}^n)$ for each $j \in \{1, \dots, n\}$ and

$$D\Big(\sum_{j=1}^{n}\psi_j\mathbf{e}_j\Big)=-\varphi \ \text{ in } \ \mathbb{R}^n. \tag{6.4.105}$$

It is useful to further refashion the latter property as

$$\sum_{j,k=1}^{n}\partial_k\psi_j\mathbf{e}_k\odot\mathbf{e}_j=-\varphi \ \text{ in } \ \mathbb{R}^n. \tag{6.4.106}$$

Bearing in mind that both φ and the ψ_j's are scalar-valued, this entails

$$\sum_{j=1}^{n}\partial_j\psi_j=\varphi, \tag{6.4.107}$$

and

$$\sum_{1\leq j\neq k\leq n}\partial_k\psi_j\mathbf{e}_k\odot\mathbf{e}_j=0, \tag{6.4.108}$$

hence (given that $\mathbf{e}_k\odot\mathbf{e}_j=-\mathbf{e}_j\odot\mathbf{e}_k$ for $j\neq k$) also

$$\sum_{1\leq j\neq k\leq n}\partial_k\psi_j\mathbf{e}_j\odot\mathbf{e}_k=0. \tag{6.4.109}$$

Re-combining (6.4.107) and (6.4.109) then yields

$$\sum_{j,k=1}^{n}\partial_k\psi_j\mathbf{e}_j\odot\mathbf{e}_k=-\varphi \ \text{ in } \ \mathbb{R}^n. \tag{6.4.110}$$

Since μ is compactly supported, on account of (6.4.96), (6.4.104), and (6.4.110) we may then compute (keeping in mind that $\widehat{\mu}$ belongs to $L^1_{\rm loc}(\mathbb{R}^n,\mathcal{L}^n)\otimes\mathcal{C}\!\ell_n$; cf. Lemma 6.4.4)

$$\begin{aligned}
{}_{\mathcal{D}'(\mathbb{R}^n)\otimes\mathcal{C}\!\ell_n}\langle\widehat{\mu},\varphi\rangle_{\mathcal{D}(\mathbb{R}^n)}&=\int_{\mathbb{R}^n}\Big(\frac{1}{\omega_{n-1}}\int_{\mathbb{R}^n}\frac{x-y}{|x-y|^n}\varphi(x)\,\mathrm{d}x\Big)\odot\mathrm{d}\mu(y)\\
&=\sum_{j=1}^{n}\int_{\mathbb{R}^n}\psi_j\mathbf{e}_j\odot\mathrm{d}\mu=\sum_{j=1}^{n}{}_{\mathscr{E}'(\mathbb{R}^n)\otimes\mathcal{C}\!\ell_n}\langle\mathbf{e}_j\odot\mu,\psi_j\rangle_{\mathscr{E}(\mathbb{R}^n)}\\
&=\sum_{j=1}^{n}{}_{\mathscr{E}'(\mathbb{R}^n)\otimes\mathcal{C}\!\ell_n}\langle\mathbf{e}_j\odot D\mathbf{1}_E,\psi_j\rangle_{\mathscr{E}(\mathbb{R}^n)}
\end{aligned}$$

$$
\begin{aligned}
&= - \sum_{j,k=1}^{n} {}_{\mathscr{E}'(\mathbb{R}^n)\otimes \mathcal{C}\ell_n}\langle \mathbf{e}_j \odot \mathbf{e}_k \mathbf{1}_E, \partial_k \psi_j \rangle_{\mathscr{E}(\mathbb{R}^n)} \\
&= {}_{\mathcal{D}'(\mathbb{R}^n)\otimes \mathcal{C}\ell_n}\langle \mathbf{1}_E, \varphi \rangle_{\mathcal{D}(\mathbb{R}^n)},
\end{aligned}
\tag{6.4.111}
$$

which ultimately proves (6.4.101). □

For more flexibility, it is useful to work with a Clifford algebra $\mathcal{C}\ell_m$ in which $m \in \mathbb{N}$ is independent of the dimension n of the ambient Euclidean space. We shall do so in what follows. For each $\ell \in \{0, 1, \dots, m\}$ consider the projection map Π_m^ℓ onto the ℓ-homogeneous part of u, i.e.,

$$
\Pi_m^\ell u := \sum_{|I|=\ell}{}' u_I e_I \quad \text{if} \quad u = \sum_{\ell=0}^{m} \sum_{|I|=\ell}{}' u_I \, e_I \in \mathcal{C}\ell_m, \tag{6.4.112}
$$

and denote by Λ_m^ℓ the range of $\Pi_m^\ell : \mathcal{C}\ell_m \to \mathcal{C}\ell_m$. It follows that

$$
\mathcal{C}\ell_m = \Lambda_m^0 \oplus \Lambda_m^1 \oplus \cdots \oplus \Lambda_m^m. \tag{6.4.113}
$$

Elements in Λ_m^0, Λ_m^1, and Λ_m^2 are called `scalars`, `vectors`, and `bi-vectors`, respectively. As is apparent from definitions, multiplication of a homogeneous Clifford element by a vector consists of two parts, in which the degree of homogeneity is either increased or decreased by one unit, so if $a \in \Lambda_m^1$ and $u \in \Lambda_m^j$ for some $j \in \{0, 1, 2, \dots, m\}$, then

$$
a \odot u \in \Lambda_m^{j-1} \oplus \Lambda_m^{j+1}, \tag{6.4.114}
$$

with the convention that $\Lambda_m^{-1} = \varnothing$ and $\Lambda_m^{m+1} = \varnothing$.

Next, fix arbitrary $j \in \{1, \dots, m\}$ and $\ell \in \{0, 1, \dots, m\}$, along with an array $I = (i_1, i_2, \dots, i_\ell)$ with $1 \le i_1 < i_2 < \cdots < i_\ell \le m$. Define the `exterior product` $\wedge$ and the `interior product` $\vee$ of the imaginary unit $\mathbf{e}_j$ with e_I by

$$
\mathbf{e}_j \wedge e_I := \sum_{|J|=\ell+1}{}' \varepsilon_J^{jI} \, e_J \quad \text{and} \quad \mathbf{e}_j \vee e_I := \sum_{|J|=\ell-1}{}' \varepsilon_I^{jJ} \, e_J \tag{6.4.115}
$$

where, for any two arrays A, B the generalized Kronecker symbol ε_B^A is[5]

$$
\varepsilon_B^A := \begin{cases} \det\big((\delta_{ab})_{a\in A, b\in B}\big) & \text{if } |A| = |B|, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4.116}
$$

Several useful properties of the generalized Kronecker symbol are contained in the lemma below (see [189, Lemma 2.1, p. 51] for a proof).

[5] Recall that $\delta_{ab} := 1$ if $a = b$ and $\delta_{ab} := 0$ if $a \neq b$.

Lemma 6.4.6 *The following identities hold:*

(i) $\varepsilon^J_K = \varepsilon^K_J$ *for all arrays* J, K;
(ii) $\sum'_J \varepsilon^I_J \varepsilon^J_K = \varepsilon^I_K$ *for all arrays* I, K;
(iii) $\varepsilon^{IK}_{JK} = \varepsilon^{KI}_{KJ} = \varepsilon^I_J$ *if the arrays* I, J, K *satisfy* $K \cap (I \cup J) = \varnothing$ *as sets, where* IK *is the array obtained by concatenating* I *with* K *(in this order), etc.;*
(iv) $\varepsilon^{IJ}_K = (-1)^{|I||J|}\varepsilon^{JI}_K$ *for all arrays* I, J, K;
(v) $\varepsilon^{IJ}_{JI} = (-1)^{|I||J|}$ *if the arrays* I, J *satisfy* $I \cap J = \varnothing$ *as sets, while* $\varepsilon^{IJ}_K = 0$ *if* $I \cap J \neq \varnothing$ *as sets;*
(vi) $\sum'_A \varepsilon^{iA}_I \varepsilon^{AJ}_L + (-1)^{|I|} \sum'_B \varepsilon^{iB}_J \varepsilon^{IB}_L = \varepsilon^{IJ}_{iL}$ *for all arrays* I, J, L *and every index* $i \in \{1, \dots, n\}$.

Going further, note that

$$\mathbf{e}_j \wedge e_I = 0 \text{ if } j \in I, \quad \text{while } \mathbf{e}_j \vee e_I = 0 \text{ if } j \notin I. \tag{6.4.117}$$

Also, as a consequence of (6.4.115), we have that

$$\mathbf{e}_j \odot e_I = \mathbf{e}_j \wedge e_I - \mathbf{e}_j \vee e_I. \tag{6.4.118}$$

To see why (6.4.118) holds, let I_j^- and I_j^+ be the increasingly ordered arrays (one possibly empty) with the property that $I_j^- j I_j^+$ is an ordered array and either $I = I_j^- j I_j^+$ if $j \in I$ or $I = I_j^- I_j^+$ if $j \notin I$. Then

$$\begin{aligned} j \in I \implies \mathbf{e}_j \odot e_I &= (-1)^{|I_j^-|}\mathbf{e}_j \odot \mathbf{e}_j \odot e_{I_j^-} \odot e_{I_j^+} = -(-1)^{|I_j^-|} e_{I_j^-} \odot e_{I_j^+} \\ &= -\varepsilon_I^{jI_j^- I_j^+} = -\sum_{|J|=\ell-1}{}' \varepsilon^{jJ}_I\, e_J = -\mathbf{e}_j \vee e_I, \end{aligned} \tag{6.4.119}$$

while

$$\begin{aligned} j \notin I \implies \mathbf{e}_j \odot e_I &= (-1)^{|I_j^-|} e_{I_j^-} \odot \mathbf{e}_j \odot e_{I_j^+} = \varepsilon^{jI_j^- I_j^+}_{I_j^- j I_j^+} \\ &= \sum_{|J|=\ell+1}{}' \varepsilon_J^{jI_j^- I_j^+}\, e_J = \mathbf{e}_j \wedge e_I. \end{aligned} \tag{6.4.120}$$

As a byproduct of the above proof, for every $j \in \{1, \dots, m\}$ and every array I we have the following useful identities:

$$\mathbf{e}_j \wedge e_I = \begin{cases} \mathbf{e}_j \odot e_I & \text{if } j \notin I, \\ 0 & \text{if } j \in I, \end{cases} \quad \text{and} \quad \mathbf{e}_j \vee e_I = \begin{cases} -\mathbf{e}_j \odot e_I & \text{if } j \in I, \\ 0 & \text{if } j \notin I. \end{cases} \tag{6.4.121}$$

As a consequence, an inductive argument shows that given any $I = (i_1, i_2, \dots, i_\ell)$, with components $1 \leq i_1 < i_2 < \cdots < i_\ell \leq m$, we have

$$\mathbf{e}_{i_1} \odot \mathbf{e}_{i_2} \odot \cdots \odot \mathbf{e}_{i_\ell} = e_I = \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_\ell}. \tag{6.4.122}$$

In general, if $a \in \mathbb{C}^m$ and $u = \sum_I u_I\, e_I \in \mathcal{C}\ell_m$ then $a \wedge u$ and $a \vee u$ are meaningfully defined by extending (6.4.115) by linearity. In particular, (6.4.118) yields via linearity the identity

$$a \odot u = a \wedge u - a \vee u, \tag{6.4.123}$$

as well as

$$\begin{aligned} a \wedge u &= \sum_{\ell=0}^{m} \Pi_m^{\ell+1}\Big(a \odot {\sum_{|I|=\ell}}' u_I\, e_I\Big) \quad \text{and} \\ a \vee u &= -\textstyle\sum_{\ell=0}^{m} \Pi_m^{\ell-1}\Big(a \odot \sum_{|I|=\ell}{}' u_I\, e_I\Big). \end{aligned} \tag{6.4.124}$$

From (6.4.115) it is immediate that $\mathbf{e}_j \wedge \mathbf{e}_k = -\mathbf{e}_k \wedge \mathbf{e}_j$ for every $j, k \in \{1, \dots, m\}$. Thus, by linearity we have

$$a \wedge b = -b \wedge a \quad \text{for all} \quad a, b \in \Lambda^1. \tag{6.4.125}$$

Let us also point out here the (simple but useful) fact that

$$\begin{aligned} &\text{if } a \text{ is a vector and } u \text{ is a scalar then } a \wedge u = ua \text{ and } a \vee u = 0, \\ &\text{whereas if both } a \text{ and } u \text{ are vectors then } a \vee u = \langle a, u\rangle. \end{aligned} \tag{6.4.126}$$

Other useful algebraic properties are collected in the next lemma.

Lemma 6.4.7 *Suppose $a, b \in \Lambda_m^1$ and $u, w \in \mathcal{C}\ell_m$. Then the following identities hold.*

(1) $a \wedge (a \wedge u) = 0$ *and* $a \vee (a \vee u) = 0$.
(2) $a \wedge (b \vee u) + b \vee (a \wedge u) = \langle a, b\rangle u$.
(3) $\langle a \wedge u, w\rangle = \langle u, a \vee w\rangle$ *and* $\langle a \vee u, w\rangle = \langle u, a \wedge w\rangle$.
(4) $\langle a \wedge u, a \vee w\rangle = 0$.
(5) $\langle a \wedge u, b \wedge w\rangle + \langle b \vee u, a \vee w\rangle = \langle a, b\rangle\langle u, w\rangle$.
(6) $a \wedge (b \wedge u) = -b \wedge (a \wedge u)$.
(7) $a \vee (b \vee u) = -b \vee (a \vee u)$.

Proof By the linearity of $\wedge$ and $\vee$, in order to prove *(1)* and *(2)* it suffices to consider the case $u = e_I \in \Lambda_m^\ell$. Then the identities in *(1)* follow directly from (6.4.115). To prove *(2)*, starting with the second formula in (6.4.115) and (6.4.119), then using the first formula in (6.4.115) and (6.4.120), we write

$$\begin{aligned} a \wedge (b \vee e_I) &= \sum_{i \in I} a \wedge (b_i \mathbf{e}_i \vee e_I) = -\sum_{i \in I} a \wedge (b_i \mathbf{e}_i \odot e_I) \\ &= -\sum_{i \in I} \sum_{j \notin I \setminus \{i\}} a_j b_i \mathbf{e}_j \wedge (\mathbf{e}_i \odot e_I) = -\sum_{i \in I} \sum_{j \notin I \setminus \{i\}} a_j b_i \mathbf{e}_j \odot \mathbf{e}_i \odot e_I \\ &= \Big(\sum_{i \in I} a_i b_i\Big) e_I - \sum_{i \in I} \sum_{j \notin I} a_j b_i \mathbf{e}_j \odot \mathbf{e}_i \odot e_I. \end{aligned} \tag{6.4.127}$$

A similar computation gives

$$b \vee (a \wedge e_I) = \Big(\sum_{i \notin I} a_i b_i\Big) e_I - \sum_{i \notin I} \sum_{j \in I} a_i b_j \mathbf{e}_j \odot \mathbf{e}_i \odot e_I. \tag{6.4.128}$$

Now *(2)* follows by summing (6.4.127)–(6.4.128) and using the anti-commutativity formulas in (6.4.2).

Moving on, note that by (6.4.31), (6.4.32), and (6.4.29), for $u, w \in \mathcal{C}\ell_m$ we have

$$\langle a \odot u, w\rangle = (\overline{a \odot u} \odot w)_{\rm scal} = -(\overline{u} \odot a \odot w)_0 = \langle u, -a \odot w\rangle. \tag{6.4.129}$$

Making use of (6.4.123) in the left- and right-most sides of (6.4.129) yields

$$\langle a \wedge u, w\rangle - \langle a \vee u, w\rangle = \langle u, a \vee w\rangle - \langle u, a \wedge w\rangle. \tag{6.4.130}$$

Assume for the moment that $u \in \Lambda^\ell_m$ and $w \in \Lambda^s_m$, for some $\ell, s \in \{1, \dots, m\}$. Then by degree considerations at least one term on each side of (6.4.130) should be zero. If $\langle a \wedge u, w\rangle \neq 0$ then we should have $\ell + 1 = s$ which implies $\langle a \wedge u, w\rangle = \langle u, a \vee w\rangle$. Similarly, if $\langle a \vee u, w\rangle \neq 0$ we obtain $\langle a \vee u, w\rangle = \langle u, a \wedge w\rangle$. This proves *(3)* in the case $u \in \Lambda^\ell_m$ and $w \in \Lambda^s_m$. The general statement in *(3)* now follows by linearity and symmetry of the inner product. The identity in *(4)* is an immediate consequence of *(3)* and *(1)*. Identity *(5)* is implied by *(2)* and *(3)* since

$$\begin{aligned}\langle a, b\rangle\langle u, w\rangle &= \big\langle \langle a, b\rangle u, w\big\rangle = \big\langle a \wedge (b \vee u), w\big\rangle + \big\langle b \vee (a \wedge u), w\big\rangle \\ &= \langle b \vee u, a \vee w\rangle + \langle a \wedge u, b \wedge w\rangle.\end{aligned} \tag{6.4.131}$$

Formula *(6)* is directly seen from (6.4.125) and the associativity of the exterior product. Finally, as a consequence of the second identity in *(3)* and *(6)* for each $w \in \mathcal{C}\ell_m$ we may write

$$\langle a \vee (b \vee u), w\rangle = \langle u, b \wedge a \wedge w\rangle = -\langle u, a \wedge b \wedge w\rangle = -\langle b \vee (a \vee u), w\rangle. \tag{6.4.132}$$

Granted this, formula in item *(7)* follows given the arbitrariness of w. □

We continue by establishing certain orthogonal decompositions of Clifford algebra elements, and an accompanying Pythagorean Theorem. Recall that the superscript "c" denotes complex conjugation.

Corollary 6.4.8 *For each $u \in \mathcal{C}\ell_m$, and each $a \in \Lambda^1_m$ with $|a| = 1$ and $a = a^c$, one has*

$$u = a \wedge (a \vee u) + a \vee (a \wedge u) \quad \text{and} \quad |u|^2 = |a \wedge u|^2 + |a \vee u|^2. \tag{6.4.133}$$

Moreover,

$$a \vee \big(a \wedge (a \vee u)\big) = a \vee u \quad and \quad a \wedge \big(a \vee (a \wedge u)\big) = a \wedge u. \tag{6.4.134}$$

Proof The first formula in (6.4.133) is a direct consequence of item *(2)* in Lemma 6.4.7 and the assumptions on $a \in \Lambda^1_m$. To justify the second formula in (6.4.133), write

$$\begin{aligned} |a \wedge u|^2 + |a \vee u|^2 &= \langle a \wedge u, (a \wedge u)^c \rangle + \langle a \vee u, (a \vee u)^c \rangle \\ &= \langle a \wedge u, a \wedge (u^c) \rangle + \langle a \vee u, a \vee (u^c) \rangle \\ &= \langle u, a \vee (a \wedge (u^c)) \rangle + \langle u, a \wedge (a \vee (u^c)) \rangle \\ &= \big\langle u, a \vee (a \wedge (u^c)) + a \wedge (a \vee (u^c)) \big\rangle \\ &= \langle u, u^c \rangle = |u|^2, \end{aligned} \tag{6.4.135}$$

using (6.4.28), the fact that $a = a^c$, item *(3)* in Lemma 6.4.7, and the identity established in the first part of this proof, with u^c in place of u. Finally, the formulas in (6.4.134) are seen by applying $a\vee$ and $a\wedge$ to the first identity in (6.4.133) (while bearing in mind item *(1)* in Lemma 6.4.7). □

Given an open nonempty set $\Omega \subseteq \mathbb{R}^n$, recall that $\mathcal{D}'(\Omega)$ stands for the space of ordinary distributions in Ω. We shall denote by $\mathcal{D}'(\Omega, \mathcal{C}\ell_m)$ the space of $\mathcal{C}\ell_m$-valued distributions in Ω. More specifically,

$$u \in \mathcal{D}'(\Omega, \mathcal{C}\ell_m) \overset{def}{\Longleftrightarrow} u = \sum_{\ell=0}^{m} {\sum_{|I|=\ell}}' u_I \, e_I \quad \text{with each} \quad u_I \in \mathcal{D}'(\Omega). \tag{6.4.136}$$

Similarly, for each $\ell \in \{0, 1, \dots, m\}$, denote by $\mathcal{D}'(\Omega, \Lambda^\ell_m)$ the subspace of $\mathcal{D}'(\Omega, \mathcal{C}\ell_m)$ consisting of those Clifford algebra-valued distributions representable as $u = \sum'_{|I|=\ell} u_I \, e_I$ with $u_I \in \mathcal{D}'(\Omega)$. We shall refer to these as `differential forms of degree` ℓ (or, simply, ℓ-forms). Since any $u \in \mathcal{D}'(\Omega, \mathcal{C}\ell_m)$ may be regarded as linear combination of differential forms of various degrees, it is convenient to also think of such an object u as being a differential form (of mixed degrees).

These objects obey natural rules, much as ordinary distributions, with the added bonus that the Clifford algebra formalism is in full effect. For example, while ordinary distributions could be multiplied by smooth scalar-valued functions, Clifford algebra-valued distributions can be multiplied with Clifford algebra-valued functions via $\odot$, $\wedge$, and $\vee$. In particular, if $w \in \mathcal{D}'(\Omega, \Lambda^1_m)$ and $u \in \mathscr{C}^\infty(\Omega, \mathcal{C}\ell_m)$, then

$$w \odot u = w \wedge u - w \vee u. \tag{6.4.137}$$

Moving on, we make the following convention, valid throughout:

when simultaneously dealing with two Clifford algebras, say $\mathcal{C}\ell_{m_1}$ and $\mathcal{C}\ell_{m_2}$, we canonically view them as the sub-algebras of $\mathcal{C}\ell_m$ where $m := \max\{m_1, m_2\}$ freely generated by the first m_1 and, respectively, first m_2 imaginary units of $\mathcal{C}\ell_m$. (6.4.138)

Here is a concrete case of interest where this convention is called for. Let Ω be an open set in $\mathbb{R}^n$. Recall the classical (homogeneous) Dirac operator in $\mathbb{R}^n$ from (6.4.47), i.e., $D := \sum_{j=1}^n \mathbf{e}_j \odot \partial_j$. This acts on some $u \in \mathcal{D}'(\Omega, \mathcal{C}\ell_m)$ where $m \in \mathbb{N}_0$ according to

$$Du = \sum_{j=1}^{n} \mathbf{e}_j \odot (\partial_j u) \tag{6.4.139}$$

with the right-hand side regarded as a $\mathcal{C}\ell_N$-valued distribution, for $N := \max\{n, m\}$.

One convenient way to introduce the exterior and interior derivative operators d, δ is to set, for each $u \in \mathcal{D}'(\Omega, \mathcal{C}\ell_m)$,

$$\mathrm{d}u := \sum_{j=1}^{n} \mathbf{e}_j \wedge (\partial_j u) \quad \text{and} \quad \delta u := -\sum_{j=1}^{n} \mathbf{e}_j \vee (\partial_j u). \tag{6.4.140}$$

In particular, if $u = \sum_{\ell=0}^m \sum'_{|I|=\ell} u_I\, e_I$ has components $u_I \in \mathcal{D}'(\Omega)$, then (6.4.140) and (6.4.115) yield

$$\mathrm{d}u = \sum_{\ell=0}^{m} \sum_{|I|=\ell}{}' \sum_{j=1}^{n} (\partial_j u_I)\, \mathbf{e}_j \wedge e_I = \sum_{\ell=0}^{m} \sum_{|J|=\ell+1}{}' \sum_{|I|=\ell}{}' \sum_{j=1}^{n} \varepsilon_J^{jI} (\partial_j u_I)\, e_J \tag{6.4.141}$$

and

$$\delta u = -\sum_{\ell=0}^{m} \sum_{|J|=\ell-1}{}' \sum_{|I|=\ell}{}' \sum_{j=1}^{n} \varepsilon_I^{jJ} (\partial_j u_I)\, e_J. \tag{6.4.142}$$

Note that based on (6.4.139) and (6.4.118) we may write

$$Du = \sum_{j=1}^{n} \mathbf{e}_j \wedge (\partial_j u) - \sum_{j=1}^{n} \mathbf{e}_j \vee (\partial_j u) = \mathrm{d}u + \delta u, \quad \forall u \in \mathcal{D}'(\Omega, \mathcal{C}\ell_m). \tag{6.4.143}$$

As a consequence of (6.4.143) it follows that

$$\mathrm{d}u = \sum_{\ell=0}^{m} \Pi_m^{\ell+1}(Du) \text{ and } \delta u = -\sum_{\ell=0}^{m} \Pi_m^{\ell-1}(Du), \quad \forall u \in \mathcal{D}'(\Omega, \mathcal{C}\ell_m). \tag{6.4.144}$$

A simple but useful observation is that for any scalar distribution u one has $\mathrm{d}u = \nabla u$ and $\delta u = 0$. In addition, d maps any Λ_m^m-valued distributions into 0. From definitions, it is also straightforward to check that, in the context of $\mathcal{C}\ell_m$-valued distributions in an open subset of $\mathbb{R}^n$, the following operator identities are valid:

$$\mathrm{d}^2 = 0, \quad \delta^2 = 0, \quad \Delta = -\mathrm{d}\delta - \delta\mathrm{d} = -(\mathrm{d} + \delta)^2. \tag{6.4.145}$$

6.5 Subaveraging Functions, Reverse Hölder Estimates, and Interior Estimates

We begin by making the following definition.

Definition 6.5.1 *Let Ω be an arbitrary open subset of $\mathbb{R}^n$, and fix some $p \in (0, \infty)$. A (complex-valued) function u defined in Ω is said to be p-*`subaveraging` *if $u \in L^p_{\rm loc}(\Omega, \mathcal{L}^n)$ and there exists a finite constant $C > 0$ with the property that*

$$|u(x)| \leq C \left(⨍_{B(x,r)} |u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \tag{6.5.1}$$
$$\text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega \text{ and every } r \in \big(0, \operatorname{dist}(x, \partial\Omega)\big).$$

It is clear from definitions that being p-subaveraging is hereditary (with respect to the domain). More specifically, if $O \subseteq \Omega \subseteq \mathbb{R}^n$ are open sets and $p \in (0, \infty)$, then

$$\text{for any function } u \text{ which is } p\text{-subaveraging in } \Omega, \text{ the restriction } u\big|_O \text{ is } p\text{-subaveraging in the set } O. \tag{6.5.2}$$

The class of p-subaveraging functions also exhibits a number of self-improving properties discussed in the next three lemmas (refining work in [181, Sect. 11.5, p. 411]).

Lemma 6.5.2 *Assume that u is a Lebesgue measurable (complex-valued) function defined in an open subset Ω of $\mathbb{R}^n$, and fix $p \in (0, \infty)$.*

Consider a p-subaveraging function u in Ω and denote by $C \in (0, \infty)$ the constant associated with u as in (6.5.1). Then for every $x \in \Omega$, every $r \in \big(0, \operatorname{dist}(x, \partial\Omega)\big)$, and every $\lambda \in (0, 1)$ one has

$$\|u\|_{L^\infty(B(x,\lambda r),\mathcal{L}^n)} \leq C(1-\lambda)^{-n/p} \left(⨍_{B(x,r)} |u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}}. \tag{6.5.3}$$

In particular, any p-subaveraging function u in Ω belongs to $L^\infty_{\rm loc}(\Omega, \mathcal{L}^n)$.

In the converse direction, any function $u \in L^p_{\rm loc}(\Omega, \mathcal{L}^n)$ with the property that there exist two constants, $C \in (0, \infty)$ and $\lambda \in (0, 1)$, such that for every $x \in \Omega$ and every $r \in \big(0, \operatorname{dist}(x, \partial\Omega)\big)$ one has

$$\|u\|_{L^\infty(B(x,\lambda r),\mathcal{L}^n)} \leq C \left(⨍_{B(x,r)} |u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \tag{6.5.4}$$

is necessarily p-subaveraging.

Proof The fact that u is p-subaveraging ensures the existence of a $\mathcal{L}^n$-nullset $A \subseteq \Omega$ with the property that the estimate in (6.5.1) holds at each point in $\Omega \setminus A$.

Fix now an arbitrary point $x \in \Omega$ along with $r \in \big(0, \mathrm{dist}(x, \partial\Omega)\big)$, $\lambda \in (0, 1)$, and pick some $z \in B(x, \lambda r) \setminus A$. Then, if $R := (1-\lambda)r$, it follows that $z \in \Omega \setminus A$ and $0 < R < \mathrm{dist}(z, \partial\Omega)$. Furthermore, $B(z, R) \subseteq B(x, r)$. Consequently, with C as in (6.5.1),

$$|u(z)| \leq C \left(\min_{B(z,R)} |u|^p \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{p}} = C \left(\frac{(1-\lambda)^{-n}}{\mathcal{L}^n\big(B(z,r)\big)} \int_{B(z,R)} |u|^p \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{p}}$$
$$\leq C(1-\lambda)^{-n/p} \left(\fint_{B(x,r)} |u|^p \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{p}}, \tag{6.5.5}$$

which readily implies (6.5.3) by taking the supremum over $z \in B(x, \lambda r) \setminus A$.

Conversely, assume $u \in L^p_{\mathrm{loc}}(\Omega, \mathcal{L}^n)$ has the property that there exist a constant $C \in (0, \infty)$ and $\lambda \in (0, 1)$ such that (6.5.4) holds for every $x \in \Omega$ and every $r \in \big(0, \mathrm{dist}(x, \partial\Omega)\big)$. The goal is to show that u is p-subaveraging, i.e., that (6.5.1) holds. To this end, fix an arbitrary $x \in \Omega$. Then (6.5.4) implies that for each $r \in \big(0, \mathrm{dist}(x, \partial\Omega)\big)$ there exists a $\mathcal{L}^n$-nullset $A_{x,r} \subseteq B(x, \lambda r)$ such that

$$|u(z)| \leq C \left(\fint_{B(x,r)} |u|^p \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{p}} \quad \text{for each } \; z \in B(x, \lambda r) \setminus A_{x,r}. \tag{6.5.6}$$

Arrange $\big(0, \mathrm{dist}(x, \partial\Omega)\big) \cap \mathbb{Q}$ as a sequence, say $\big\{r^x_j\big\}_{j\in\mathbb{N}}$ and set

$$A_x := \bigcup_{j\in\mathbb{N}} A_{x, r^x_j}. \tag{6.5.7}$$

Then $A_x \subseteq \Omega$ is Lebesgue measurable and $\mathcal{L}^n(A_x) = 0$. Pick $r \in \big(0, \mathrm{dist}(x, \partial\Omega)\big)$ and some point $z \in B(x, \lambda r) \setminus A_x$. Then there exists a subsequence $\big\{r^x_{j_k}\big\}_{k\in\mathbb{N}}$ of $\big\{r^x_j\big\}_{j\in\mathbb{N}}$ with the property that $r^x_{j_k} \searrow r$ as $k \to \infty$. Consequently, if k is large then $z \in B\big(x, \lambda r^x_{j_k}\big) \setminus A_{x, r^x_{j_k}}$ and $r^x_{j_k} \in \big(0, \mathrm{dist}(x, \partial\Omega)\big)$ which permits us to invoke (6.5.6) to write

$$|u(z)| \leq C \left(\fint_{B(x, r^x_{j_k})} |u|^p \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{p}} \quad \text{whenever } k \in \mathbb{N} \text{ is large.} \tag{6.5.8}$$

By passing to the limit $k \to \infty$ and relying on Lebesgue's Dominated Convergence Theorem, we then arrive at the conclusion that for each $x \in \Omega$ there exists some Lebesgue measurable set $A_x \subseteq \Omega$ satisfying $\mathcal{L}^n(A_x) = 0$ and with the property that

$$|u(z)| \leq C \left(\fint_{B(x,r)} |u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \tag{6.5.9}$$

for each $r \in \big(0, \operatorname{dist}(x, \partial\Omega)\big)$ and each $z \in B(x, \lambda r) \setminus A_x$.

To proceed, consider now a sequence $\{x_j\}_{j\in\mathbb{N}}$ which is dense in Ω and observe that

$$A := \bigcup_{j\in\mathbb{N}} A_{x_j} \implies A \subseteq \Omega \text{ is Lebesgue measurable and } \mathcal{L}^n(A_x) = 0. \tag{6.5.10}$$

Pick an arbitrary point $x \in \Omega$ along with some $r \in \big(0, \operatorname{dist}(x, \partial\Omega)\big)$. Also, consider an arbitrary point $z \in B(x, \lambda r) \setminus A$. We may then find a subsequence $\big\{x_{j_k}\big\}_{k\in\mathbb{N}}$ of $\{x_j\}_{j\in\mathbb{N}}$ such that $x_{j_k} \to x$ as $k \to \infty$. In particular, if k is large then $z \in B\big(x_{j_k}, \lambda r\big) \setminus A_{x_{j_k}}$ and $r \in \big(0, \operatorname{dist}(x_{j_k}, \partial\Omega)\big)$. Granted these, (6.5.9) applies and gives that

$$|u(z)| \leq C \left(\fint_{B(x_{j_k},r)} |u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \quad \text{whenever } k \in \mathbb{N} \text{ is large.} \tag{6.5.11}$$

After passing to the limit $k \to \infty$ and invoking Lebesgue's Dominated Convergence Theorem we therefore obtain the following improvement over (6.5.9):

there exists some Lebesgue-nullset $A \subseteq \Omega$ with the property that

$$|u(z)| \leq C \left(\fint_{B(x,r)} |u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \tag{6.5.12}$$

for each $x \in \Omega$, each $r \in \big(0, \operatorname{dist}(x, \partial\Omega)\big)$, and each $z \in B(x, \lambda r) \setminus A$.

In particular, if $x \in \Omega \setminus A$ and $r \in \big(0, \operatorname{dist}(x, \partial\Omega)\big)$, then trivially $x \in B(x, \lambda r) \setminus A$, so the estimate in (6.5.12) is valid for $z := x$. This ultimately allows us to conclude that (6.5.1) holds, hence u is indeed p-subaveraging. □

The second self-improvement within the class of p-subaveraging functions is the fact that the value of the integrability exponent p is immaterial.

Lemma 6.5.3 *Let u be a (complex-valued) function defined in some open subset Ω of $\mathbb{R}^n$. If there exists $p_0 \in (0, \infty)$ such that u is a p_0-subaveraging function, then u is in fact a p-subaveraging function for each $p \in (0, \infty)$.*

In light of Lemma 6.5.3 we may (unequivocally) refer to a complex-valued function u simply as being `subaveraging` if u is p-subaveraging for some $p \in (0, \infty)$; the optimal constant which can be used in (6.5.1) is referred to as the p-`subaveraging constant` of u. (6.5.13)

Proof of Lemma 6.5.3 The proof is based on ideas used in the work of G. Hardy and J. Littlewood [112] (cf. also [90, Lemma 2, pp. 172–173]). The case when $p > p_0$ can be handled directly utilizing Hölder's inequality with $q = \frac{p}{p_0} > 1$. Henceforth we shall focus on the case when $p < p_0$. Replacing u by a suitable power of $|u|$, there is no loss of generality in assuming that, in fact, $p_0 = 1$ and $p \in (0, 1)$.

Fix some $x \in \Omega$ along with $r \in \big(0, \operatorname{dist}(x, \partial\Omega)\big)$. In view of the second part in Lemma 6.5.2, the goal is to show that there exist $C \in (0, \infty)$ and $\lambda \in (0, 1)$, independent of u, x, r, such that (6.5.4) holds. We may then assume (by rescaling and making a translation) that $\overline{B(0, 1)} \subseteq \Omega$, that $x := 0$, and that the function u is normalized so that $\int_{B(0,1)} |u|^p \, d\mathcal{L}^n = 1$. Working with $u_r := r^{-n/p} u(\cdot/r)$ in place of u, it suffices to prove the version of (6.5.4) when $r = 1$. If we therefore abbreviate

$$m_\infty(r) := \|u\|_{L^\infty(B(0,r),\mathcal{L}^n)} \quad \text{for each} \quad r \in (0, 1], \tag{6.5.14}$$

(recall from Lemma 6.5.2 that u is locally bounded) then the ultimate goal is to show that

there exist some $\lambda \in (0, 1)$ and some $C \in (0, \infty)$, both independent of u, such that $m_\infty(\lambda) \leq C$. (6.5.15)

With this aim in mind, fix some parameter

$$a \in (1, \infty) \quad \text{such that} \quad \tfrac{1}{a} - 1 + p > 0. \tag{6.5.16}$$

If $m_\infty(2^{-a}) \leq 1$ we are done, so it suffices to henceforth consider the case when

$$m_\infty(2^{-a}) > 1. \tag{6.5.17}$$

Suppose now that $0 < \rho < r < 1$. Then (6.5.3) written for $p := 1$, $x := 0$, and with $\lambda := \rho/r \in (0, 1)$ (recall that we are currently assuming that u is 1-subaveraging and $\int_{B(0,1)} |u|^p \, d\mathcal{L}^n = 1$) gives that

$$\begin{aligned} m_\infty(\rho) &= \|u\|_{L^\infty(B(0,\rho),\mathcal{L}^n)} = \|u\|_{L^\infty(B(0,\lambda r),\mathcal{L}^n)} \\ &\leq C(1-\lambda)^{-n}\left(⨍_{B(0,r)} |u| \, d\mathcal{L}^n\right) = C(r-\rho)^{-n}\left(\int_{B(0,r)} |u|^p |u|^{1-p} \, d\mathcal{L}^n\right) \\ &\leq \frac{C}{(r-\rho)^n} \|u\|^{1-p}_{L^\infty(B(0,r),\mathcal{L}^n)} = \frac{C}{(r-\rho)^n} m_\infty(r)^{1-p}. \end{aligned} \tag{6.5.18}$$

Hence,

$$m_\infty(\rho) \le \frac{C}{(r-\rho)^n} m_\infty(r)^{1-p} \quad \text{whenever} \quad 0 < \rho < r < 1. \tag{6.5.19}$$

To continue, specialize (6.5.19) to the case when $r \in (0, 1)$ and $\rho := r^a$ with a as in (6.5.16), then apply the natural logarithm and, finally, integrate with respect to the Haar measure $\mathrm{d}r/r$ to obtain

$$\begin{aligned}\int_{1/2}^{1} \ln\, m_\infty(r^a)\, \frac{\mathrm{d}r}{r} \le C + n \int_{1/2}^{1} \ln\, \frac{1}{(r-r^a)}\, \frac{\mathrm{d}r}{r} \\ + (1-p) \int_{1/2}^{1} \ln\, m_\infty(r)\, \frac{\mathrm{d}r}{r}.\end{aligned} \tag{6.5.20}$$

For the first integral above, the change of variables $t := r^a$ gives

$$\int_{1/2}^{1} \ln\, m_\infty(r^a)\, \frac{\mathrm{d}r}{r} = \frac{1}{a} \int_{(1/2)^a}^{1} \ln\, m_\infty(t)\, \frac{\mathrm{d}t}{t}. \tag{6.5.21}$$

Since our assumption in (6.5.17) implies $m_\infty(t) \ge 1$, the right-hand side of (6.5.21) is bounded from below by

$$\frac{1}{a} \int_{1/2}^{1} \ln\, m_\infty(r)\, \frac{\mathrm{d}r}{r}. \tag{6.5.22}$$

Therefore, (6.5.20)–(6.5.22) imply

$$\Big(\frac{1}{a} - 1 + p\Big) \int_{1/2}^{1} \ln\, m_\infty(r)\, \frac{\mathrm{d}r}{r} \le C + C \int_{1/2}^{1} \ln\, \frac{1}{(r-r^a)}\, \frac{\mathrm{d}r}{r} \le C < \infty. \tag{6.5.23}$$

Then (6.5.23) and the choice of a in (6.5.16) force

$$\int_{1/2}^{1} \ln\, m_\infty(r)\, \mathrm{d}r \le C, \tag{6.5.24}$$

hence $\ln\, m_\infty(1/2) \le C$ for some finite constant $C > 0$ independent of initial function u. Thus, we have $m_\infty(1/2) \le \mathrm{e}^C$. This establishes (6.5.15) and finishes the proof of the lemma. □

Jensen's inequality implies that the composition between a non-decreasing convex function and a subaveraging function is itself subaveraging. For example, we have the following result.

Lemma 6.5.4 *Let Ω be an open subset of $\mathbb{R}^n$ and let u be a subaveraging function in Ω. Then for each $p \in (0, \infty)$ the function $|u|^p$ is also subaveraging in Ω.*

Proof The fact that u is subaveraging implies (cf. (6.5.13) and Lemma 6.5.3) that u is p-subaveraging. As such, (6.5.1) holds, and this entails the existence of a constant $C \in (0, \infty)$ with the property that

$$|u(x)|^p \leq C \fint_{B(x,r)} |u|^p \,\mathrm{d}\mathcal{L}^n \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega \text{ and every } r \in \big(0, \operatorname{dist}(x, \partial\Omega)\big). \tag{6.5.25}$$

Hence, $|u|^p$ is 1-subaveraging in Ω which, in view of the convention in (6.5.13), amounts to saying that $|u|^p$ is subaveraging in Ω. □

There are certain connections between the subaveraging property and reverse Hölder estimates, brought to light by the next two results.

Lemma 6.5.5 *Let u be a subaveraging function in an open set $\Omega \subseteq \mathbb{R}^n$. Then for every $p, q \in (0, \infty)$ and $\lambda \in (0, 1)$ the following reverse Hölder estimate holds:*

$$\Big(\fint_{B(x,\lambda r)} |u|^q \,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{q}} \leq C\Big(\fint_{B(x,r)} |u|^p \,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}}, \tag{6.5.26}$$

for $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$, where $C \in (0, \infty)$ depends only on p, q, λ, n, and the p-subaveraging constant of u.

Proof Given $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$, we have

$$\Big(\fint_{B(x,\lambda r)} |u|^q \,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{q}} \leq \|u\|_{L^\infty(B(x,\lambda r),\mathcal{L}^n)} \leq C\Big(\fint_{B(x,r)} |u|^p \,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}}, \tag{6.5.27}$$

thanks to Lemma 6.5.2. □

Our next lemma contains weighted reverse Hölder estimates for subaveraging functions, both global and local near the boundary.

Lemma 6.5.6 *Assume that Ω is an arbitrary open, nonempty, proper subset of $\mathbb{R}^n$. Having fixed a Lebesgue measurable set $E \subseteq \Omega$ along with some number $\lambda \in (0, 1)$, define*

$$E_{\Omega,\lambda} := \bigcup_{x \in E} B\big(x, \lambda\delta_{\partial\Omega}(x)\big) \tag{6.5.28}$$

(*where $\delta_{\partial\Omega}$ is as in* (6.1.1) *with $F := \partial\Omega$*)*. Also, suppose $0 < q \leq p \leq \infty$ and $s \in \mathbb{R}$. Then for each subaveraging function u in Ω there exists a constant $C \in (0, \infty)$ which depends exclusively on p, q, s, λ, n, and the subaveraging character of u with the property that*

$$\Big(\int_E \big[\delta_{\partial\Omega}(x)^{s+n(\frac{1}{q}-\frac{1}{p})}|u(x)|\big]^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq C\Big(\int_{E_{\Omega,\lambda}} \big[\delta_{\partial\Omega}(x)^s|u(x)|\big]^q \,\mathrm{d}x\Big)^{\frac{1}{q}}. \tag{6.5.29}$$

Before presenting the proof of this result, we mention a few special cases of interest. For example, given an arbitrary $z \in \partial\Omega$ together with some $R > 0$, consider $E := \Omega \cap B(z, R)$ and note that, in this case, $E_{\Omega,1/2} \subseteq \Omega \cap B(z, 2R)$. As a consequence, there exists $C \in (0, \infty)$ which depends only on p, q, s, n, and the subaveraging character of u such that

$$\Big(\int_{\Omega \cap B(z,R)} \big[\delta_{\partial\Omega}(x)^{s+n(\frac{1}{q}-\frac{1}{p})} |u(x)| \big]^p \, dx \Big)^{\frac{1}{p}} \leq C \Big(\int_{\Omega \cap B(z,2R)} \big[\delta_{\partial\Omega}(x)^s |u(x)| \big]^q \, dx \Big)^{\frac{1}{q}}. \tag{6.5.30}$$

Also, taking $E := \Omega$ in (6.5.29) (or, upon letting $R \to \infty$ in (6.5.30)) yields

$$\Big(\int_{\Omega} \big[\delta_{\partial\Omega}(x)^{s+n(\frac{1}{q}-\frac{1}{p})} |u(x)| \big]^p \, dx \Big)^{\frac{1}{p}} \leq C \Big(\int_{\Omega} \big[\delta_{\partial\Omega}(x)^s |u(x)| \big]^q \, dx \Big)^{\frac{1}{q}}. \tag{6.5.31}$$

Since, as is apparent from definitions,

$$E_{\Omega,\lambda} \subseteq \widetilde{E}_{\Omega,\lambda} := \Big\{ x \in \Omega : \operatorname{dist}(x, E) < \lambda \sup_{z \in E} \delta_{\partial\Omega}(z) \Big\}, \tag{6.5.32}$$

we may further conclude from (6.5.29) that

$$\Big(\int_{E} \big[\delta_{\partial\Omega}(x)^{s+n(\frac{1}{q}-\frac{1}{p})} |u(x)| \big]^p \, dx \Big)^{\frac{1}{p}} \leq C \Big(\int_{\widetilde{E}_{\Omega,\lambda}} \big[\delta_{\partial\Omega}(x)^s |u(x)| \big]^q \, dx \Big)^{\frac{1}{q}}. \tag{6.5.33}$$

Finally, we wish to note that with the piece of notation introduced in (1.5.5), if $E := \mathcal{O}_\varepsilon$ for some $\varepsilon > 0$ then $\widetilde{E}_{\Omega,\lambda} \subseteq \mathcal{O}_{(1+\lambda)\varepsilon}$. In particular, (6.5.33) implies

$$\Big(\int_{\mathcal{O}_\varepsilon} \big[\delta_{\partial\Omega}(x)^{s+n(\frac{1}{q}-\frac{1}{p})} |u(x)| \big]^p \, dx \Big)^{\frac{1}{p}} \leq C \Big(\int_{\mathcal{O}_{2\varepsilon}} \big[\delta_{\partial\Omega}(x)^s |u(x)| \big]^q \, dx \Big)^{\frac{1}{q}}. \tag{6.5.34}$$

We now turn to the proof of Lemma 6.5.6.

Proof of Lemma 6.5.6 Observe that for each $x \in E$ we have $B\big(x, \lambda\delta_{\partial\Omega}(x)\big) \subseteq E_{\Omega,\lambda}$. Thus, for each fixed $x \in E$ we may estimate

$$\begin{aligned} |u(x)|^q &\leq C\delta_{\partial\Omega}(x)^{-n-qs} \int_{B(x,\lambda\delta_{\partial\Omega}(x))} \big(\delta_{\partial\Omega}(y)^s |u(y)|\big)^q \, dy \\ &\leq C\delta_{\partial\Omega}(x)^{-n-qs} \int_{E_{\Omega,\lambda}} \big(\delta_{\partial\Omega}(y)^s |u(y)|\big)^q \, dy, \end{aligned} \tag{6.5.35}$$

owing to the subaveraging property of the function u and the fact that we have $\delta_{\partial\Omega}(x) \approx \delta_{\partial\Omega}(y)$ uniformly for $y \in B\big(x, \lambda\delta_{\partial\Omega}(x)\big)$. When $p = \infty$, the estimate claimed in (6.5.30) follows straight from (6.5.35). Suppose next that $p < \infty$ and, having chosen some $\theta \in (0, 1]$, use (6.5.35) to write

$$
\begin{aligned}
|u(x)| &= |u(x)|^{\theta}|u(x)|^{1-\theta} \\
&\leq C|u(x)|^{\theta}\delta_{\partial\Omega}(x)^{-(\frac{n}{q}+s)(1-\theta)}\Big(\int_{E_{\Omega,\lambda}}\big(\delta_{\partial\Omega}(y)^{s}|u(y)|\big)^{q}\,\mathrm{d}y\Big)^{\frac{1-\theta}{q}}. \qquad (6.5.36)
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\delta_{\partial\Omega}(x)^{p\left[s+n\left(\frac{1}{q}-\frac{1}{p}\right)\right]}|u(x)|^{p} \qquad (6.5.37)\\
&\quad\leq C|u(x)|^{\theta p}\delta_{\partial\Omega}(x)^{-n+\theta p(s+\frac{n}{q})}\Big(\int_{E_{\Omega,\lambda}}\big(\delta_{\partial\Omega}(y)^{s}|u(y)|\big)^{q}\,\mathrm{d}y\Big)^{(1-\theta)\frac{p}{q}}.
\end{aligned}
$$

At this stage, set $\theta := \frac{q}{p}$ (so that $0 < \theta \leq 1$ given that $q \leq p$) and integrate both sides of the above inequality in $x \in E$. In view of the fact that $E \subseteq E_{\Omega,\lambda}$, this readily leads to (6.5.29). □

We conclude by recalling [181, Theorem 11.12, p. 415], which combines interior estimates and reverse Hölder estimates. The reader is reminded that the symbol $\mathcal{D}'(\Omega)$ denotes the space of distributions in a nonempty open set $\Omega \subseteq \mathbb{R}^n$. Also, we shall work with higher-order homogeneous constant (complex) coefficient elliptic systems in $\mathbb{R}^n$. Specifically, for $m, M \in \mathbb{N}$ we shall consider an $M \times M$ system L in $\mathbb{R}^n$ of order $2m$ of the form

$$
L = \sum_{|\alpha|=|\beta|=m} \partial^{\alpha} A_{\alpha\beta}\partial^{\beta} \tag{6.5.38}
$$

where each $A_{\alpha\beta} \in \mathbb{C}^{M\times M}$. The characteristic matrix of L is defined as the $M \times M$ matrix-valued function[6] given by

$$
L(\xi) := (-1)^m \sum_{|\alpha|=|\beta|=m} \xi^{\alpha+\beta} A_{\alpha\beta}, \qquad \forall \xi \in \mathbb{R}^n. \tag{6.5.39}
$$

Theorem 6.5.7 *Let $n, m, M \in \mathbb{N}$ and suppose L is a constant (complex) coefficient homogeneous $M \times M$ system of order $2m$ in the Euclidean space $\mathbb{R}^n$, with the property that* $\det[L(\xi)] \neq 0$ *for each $\xi \in \mathbb{R}^n \setminus \{0\}$. Assume also that $\Omega \subseteq \mathbb{R}^n$ is a nonempty open set and $u \in \big[\mathcal{D}'(\Omega)\big]^M$ is such that $Lu = 0$ in $\big[\mathcal{D}'(\Omega)\big]^M$.*

Then

$$
\begin{aligned}
&u \text{ belongs to } \big[\mathscr{C}^{\infty}(\Omega)\big]^M, \text{ is subaveraging} \\
&(\text{and so are all its partial derivatives}),
\end{aligned} \tag{6.5.40}
$$

and there exists some constant $C = C(L, n) \in (0, \infty)$ such that given any $p \in (0, \infty)$ one may find some $c = c(L, n, p) \in (0, \infty)$ with the property that

[6] The reader is alerted to the fact that our present definition of $L(\xi)$ differs by a factor of $(-1)^m$ from the one in [181, (11.3.2), p. 391].

$$\max_{y\in\overline{B(x,\lambda r)}} |\partial^\alpha u(y)| \leq c(1-\lambda)^{-|\alpha|-n/p} \cdot \frac{C^{|\alpha|}|\alpha|!}{r^{|\alpha|}} \Big(\fint_{B(x,r)} |u|^p \, d\mathcal{L}^n \Big)^{1/p}, \tag{6.5.41}$$

whenever $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)$, $\lambda \in (0, 1)$, *and* $\alpha \in \mathbb{N}_0^n$.

As a corollary of (6.5.40)–(6.5.41) *and* [181, Lemma 6.24, p. 229], *a stronger version of the membership in* (6.5.40) *holds, namely*

$$\text{the function } u \text{ is real-analytic in } \Omega. \tag{6.5.42}$$

In particular (*cf.* [181, Theorem 6.25, pp. 229-230]),

u vanishes identically in any connected component of Ω *containing a point* x_0 *such that* $(\partial^\alpha u)(x_0) = 0$ *for each multi-index* $\alpha \in \mathbb{N}_0^n$ *(which is the case if, e.g., u happens to be identically zero in a neighborhood of the point* x_0*).* (6.5.43)

6.6 The Solid Maximal Function and Maximal Lebesgue Spaces

Suppose Ω is an open, nonempty, proper subset of $\mathbb{R}^n$, and abbreviate

$$\delta_{\partial\Omega}(x) := \operatorname{dist}(x, \partial\Omega) = \inf\big\{|x-y| : y \in \partial\Omega\big\} \text{ for each } x \in \mathbb{R}^n. \tag{6.6.1}$$

Also, fix some parameter $\theta \in (0, 1)$. Then, if $u : \Omega \to \mathbb{C}$ is Lebesgue measurable, we define its `solid maximal function` as

$$u_{\star,\theta}(x) := \|u\|_{L^\infty(B(x,\theta\delta_{\partial\Omega}(x)),\mathcal{L}^n)} \in [0, \infty], \qquad \forall x \in \Omega. \tag{6.6.2}$$

In particular,

$$\text{whenever } u \in \mathscr{C}^0(\Omega) \text{ one has } u_{\star,\theta}(x) = \sup_{\substack{|x-y|<\delta_{\partial\Omega}(x)\\ y\in\Omega}} |u(y)| \text{ for all } x \in \Omega. \tag{6.6.3}$$

As is apparent from (6.6.2), for each Lebesgue measurable function u in Ω and each scalar λ we have

$$|u|_{\star,\theta} = u_{\star,\theta} \text{ and } (\lambda u)_{\star,\theta} = |\lambda| u_{\star,\theta} \text{ in } \Omega. \tag{6.6.4}$$

Other properties of interest are discussed in the lemma below.

Lemma 6.6.1 *Let* Ω *be an arbitrary open, nonempty, proper subset of* $\mathbb{R}^n$, *and fix* $\theta \in (0, 1)$. *Assume* $u : \Omega \to \mathbb{C}$ *is a Lebesgue measurable function. Then its associated solid maximal function*

$$u_{\star,\theta} : \Omega \to [0, \infty] \ \textit{is lower-semicontinuous}, \tag{6.6.5}$$

hence $u_{\star,\theta}$ is a Borel-measurable function in Ω. Moreover,

$$|u| \leq u_{\star,\theta} \ \ \textit{at } \mathcal{L}^n\textit{-a.e. point in} \ \ \Omega, \tag{6.6.6}$$

$$\big\| u_{\star,\theta} \big\|_{L^\infty(\Omega,\mathcal{L}^n)} = \|u\|_{L^\infty(\Omega,\mathcal{L}^n)}, \tag{6.6.7}$$

and for each $\gamma \in \mathbb{R}$ one has

$$\big(u \cdot \delta^{\gamma}_{\partial\Omega}\big)_{\star,\theta} \approx u_{\star,\theta} \cdot \delta^{\gamma}_{\partial\Omega} \ \ \textit{in} \ \ \Omega. \tag{6.6.8}$$

Also, given any other Lebesgue measurable function $w : \Omega \to \mathbb{R}$ one has

$$(u + w)_{\star,\theta} \leq u_{\star,\theta} + w_{\star,\theta} \ \ \textit{at } \mathcal{L}^n\textit{-a.e. point in} \ \ \Omega, \tag{6.6.9}$$

$$\big| u_{\star,\theta} - w_{\star,\theta} \big| \leq (u - w)_{\star,\theta} \ \ \textit{at } \mathcal{L}^n\textit{-a.e. point in} \ \ \Omega. \tag{6.6.10}$$

Proof To prove the claim made in (6.6.5), pick some $\lambda \in \mathbb{R}$ and consider a point $x \in \Omega$ such that $u_{\star,\theta}(x) > \lambda$. This implies that there exists some number $\varepsilon > 0$ along with some set $A \subseteq B\big(x, \theta\delta_{\partial\Omega}(x)\big)$ which is Lebesgue measurable, with $\mathcal{L}^n(A) > 0$, and has the property that $|u| > \lambda + \varepsilon$ at $\mathcal{L}^n$-a.e. every point in A. If for each $j \in \mathbb{N}$ we define

$$r_j := \left(\frac{j}{j+1}\right) \theta\delta_{\partial\Omega}(x), \tag{6.6.11}$$

it follows that $B(x, r_j) \cap A \nearrow A$ as $j \nearrow \infty$. In particular, there exists $j_o \in \mathbb{N}$ with the property that

$$\mathcal{L}^n\big(B(x, r_{j_o}) \cap A\big) > 0. \tag{6.6.12}$$

Selecting

$$0 < \eta < \left(\frac{\theta}{\theta+1}\right)\left(\frac{1}{j_o+1}\right)\delta_{\partial\Omega}(x) \tag{6.6.13}$$

then ensures that $B(x, \eta) \subseteq \Omega$ and

$$B(x, r_{j_o}) \subseteq B\big(z, \theta\delta_{\partial\Omega}(z)\big) \ \ \text{for every} \ \ z \in B(x, \eta). \tag{6.6.14}$$

Indeed, since $\delta_{\partial\Omega}$ is a Lipschitz function with Lipschitz constant ≤ 1, we have

$$\delta_{\partial\Omega}(x) - \eta \leq \delta_{\partial\Omega}(z) \ \ \text{for every} \ \ z \in B(x, \eta). \tag{6.6.15}$$

Consequently, having fixed an arbitrary point $z \in B(x, \eta)$, for every $y \in B(x, r_{j_o})$ we may estimate

$$\begin{aligned}|z-y| \le |z-x| + |x-y| &< \eta + \Big(\frac{j_o}{j_o+1}\Big)\theta\delta_{\partial\Omega}(x)\\ &\le \theta\delta_{\partial\Omega}(x) - \theta\eta,\end{aligned} \tag{6.6.16}$$

where the last inequality is a consequence of the choice of η in (6.6.13). Then (6.6.14) follows from (6.6.15) and (6.6.16).

Next, since the set A is Lebesgue measurable, from (6.6.14) and (6.6.12) we deduce that for every $z \in B(x,\eta)$ the set $A \cap B\big(z, \theta\delta_{\partial\Omega}(z)\big)$ is Lebesgue measurable and satisfies

$$\mathcal{L}^n\Big(A \cap B\big(z, \theta\delta_{\partial\Omega}(z)\big)\Big) \ge \mathcal{L}^n\Big(B(x, r_{j_o}) \cap A\Big) > 0. \tag{6.6.17}$$

Since we also know that $|u| > \lambda + \varepsilon$ at $\mathcal{L}^n$-a.e. every point in $A \cap B\big(z, \theta\delta_{\partial\Omega}(z)\big)$, it ultimately follows that

$$u_{\star,\theta}(z) = \|u\|_{L^\infty(B(z,\theta\delta_{\partial\Omega}(z)),\mathcal{L}^n)} > \lambda \quad \text{for every} \quad z \in B(x,\eta). \tag{6.6.18}$$

This finishes the proof of the claim made in (6.6.5).

Turning our attention to (6.6.6), introduce

$$\Omega_\infty := \big\{x \in \Omega : u_{\star,\theta}(x) = +\infty\big\}. \tag{6.6.19}$$

In this regard, we make the claim that

$$\text{for every } x \in \Omega \setminus \Omega_\infty \text{ we have } |u| \le u_{\star,\theta} \text{ at } \mathcal{L}^n\text{-a.e. point in } B\big(x, \theta\delta_{\partial\Omega}(x)\big). \tag{6.6.20}$$

Indeed, if $x \in \Omega \setminus \Omega_\infty$ then $u \in L^\infty\big(B\big(x, \theta\delta_{\partial\Omega}(x)\big), \mathcal{L}^n\big) \subset L^1\big(B\big(x, \theta\delta_{\partial\Omega}(x)\big), \mathcal{L}^n\big)$, hence Lebesgue's Differentiation Theorem gives that there exists an $\mathcal{L}^n$-measurable set $N_x \subseteq B\big(x, \theta\delta_{\partial\Omega}(x)\big)$ with $\mathcal{L}^n(N_x) = 0$ such that

$$|u(y)| = \lim_{r\to 0^+} \fint_{B(y,r)} |u(z)|\, dz \quad \text{for every} \quad y \in B\big(x, \theta\delta_{\partial\Omega}(x)\big) \setminus N_x. \tag{6.6.21}$$

Note that for each $y \in B\big(x, \theta\delta_{\partial\Omega}(x)\big) \setminus N_x$ and each $0 < r < \theta\delta_{\partial\Omega}(y)$ we have $B(y,r) \subseteq B\big(y, \theta\delta_{\partial\Omega}(y)\big)$, hence $|u(z)| \le \|u\|_{L^\infty(B(y,\theta\delta_{\partial\Omega}(y)),\mathcal{L}^n)} = u_{\star,\theta}(y)$ for $\mathcal{L}^n$-a.e. point $z \in B(y,r)$. In concert with (6.6.21), this implies that

$$|u(y)| \le u_{\star,\theta}(y) \quad \text{for every} \quad y \in B\big(x, \theta\delta_{\partial\Omega}(x)\big) \setminus N_x. \tag{6.6.22}$$

Since $\big\{B\big(x, \theta\delta_{\partial\Omega}(x)\big)\big\}_{x\in\Omega\setminus\Omega_\infty}$ is a cover of $\Omega \setminus \Omega_\infty$, by Lindelöf's theorem there exists a sequence $\{x_j\}_{j\in\mathbb{N}} \subseteq \Omega \setminus \Omega_\infty$ with the property that

$$\Omega \setminus \Omega_\infty \subseteq \bigcup_{j\in\mathbb{N}} B\big(x_j, \theta\delta_{\partial\Omega}(x_j)\big). \tag{6.6.23}$$

Introducing $N := \bigcup_{j\in\mathbb{N}} N_{x_j}$, it follows that N is a Lebesgue measurable subset of Ω with $\mathcal{L}^n(N) = 0$ and, in view of (6.6.22), with the property that

$$|u(y)| \leq u_{\star,\theta}(y) \quad \text{for every} \quad y \in \bigcup_{j\in\mathbb{N}} B\big(x_j, \theta\delta_{\partial\Omega}(x_j)\big) \setminus N. \tag{6.6.24}$$

Combining (6.6.23) and (6.6.24), we therefore arrive at the conclusion that

$$|u| \leq u_{\star,\theta} \quad \text{at } \mathcal{L}^n\text{-a.e. point in} \quad \Omega \setminus \Omega_\infty. \tag{6.6.25}$$

At this stage, (6.6.6) follows from (6.6.25) since, obviously, $|u| \leq u_{\star,\theta}$ at every point in Ω_∞. In turn, (6.6.7) is implied by (6.6.6) and (6.6.2). Going further, the claim in (6.6.8) is a consequence of (6.6.2) and the observation that

$$\begin{aligned} &(1-\theta)\delta_{\partial\Omega}(x) \leq \delta_{\partial\Omega}(y) \leq (1+\theta)\delta_{\partial\Omega}(x) \\ &\text{for each} \quad x \in \Omega \quad \text{and} \quad y \in B\big(x, \theta\delta_{\partial\Omega}(x)\big). \end{aligned} \tag{6.6.26}$$

Finally, given any other Lebesgue measurable function $w : \Omega \to \mathbb{R}$, the estimate claimed in (6.6.9) is a direct consequence of (6.6.2), and (6.6.10) follows by using (6.6.9) with u replaced by $u - w$ (bearing (6.6.4) in mind). □

It turns out that the choice of the parameter $\theta \in (0, 1)$ is largely immaterial, as far as the size of the solid maximal function $u_{\star,\theta}$ on Lorentz spaces is concerned. This is made precise in the lemma below.

Lemma 6.6.2 *Let Ω be an arbitrary open, nonempty, proper subset of $\mathbb{R}^n$, and select two integrability exponents, $p \in (0,\infty)$ and $q \in (0,\infty]$. Also, fix two parameters $\theta_1, \theta_2 \in (0, 1)$ along with some power $\gamma \in \mathbb{R}$. Then*

$$\|u_{\star,\theta_1}\|_{L^{p,q}(\Omega,\delta^\gamma_{\partial\Omega}\mathcal{L}^n)} \approx \|u_{\star,\theta_2}\|_{L^{p,q}(\Omega,\delta^\gamma_{\partial\Omega}\mathcal{L}^n)}, \tag{6.6.27}$$

uniformly in the class of Lebesgue measurable functions $u : \Omega \to \mathbb{C}$.

As a consequence of (6.6.27) *and* (6.2.25)*, corresponding to the case when $p = q$ one has*

$$\|u_{\star,\theta_1}\|_{L^{p}(\Omega,\delta^\gamma_{\partial\Omega}\mathcal{L}^n)} \approx \|u_{\star,\theta_2}\|_{L^{p}(\Omega,\delta^\gamma_{\partial\Omega}\mathcal{L}^n)}, \tag{6.6.28}$$

uniformly in the class of Lebesgue measurable functions $u : \Omega \to \mathbb{C}$.

Proof We begin by noting that, thanks to (6.6.8), it suffices to prove (6.6.27) in the particular case when $\gamma = 0$. To this end, choose two numbers (whose role is going to become clear later), namely $\theta_3 \in (\theta_1, 1)$ and

$$\varepsilon \in \Big(0, \min\big\{\theta_3 - \theta_1, \tfrac{\theta_2(1-\theta_1)}{2+\theta_2}\big\}\Big). \tag{6.6.29}$$

Then there exists a number $N = N(\theta_1, \varepsilon) \in \mathbb{N}$ with the following significance. Having fixed an arbitrary point $x \in \Omega$, and with $r := \delta_{\partial\Omega}(x)$, there exists a family of points $\{x_j\}_{1\leq j\leq N}$ in $B(x,\theta_1 r)$ with the property that

$$B(x,\theta_1 r)\subseteq\bigcup_{j=1}^{N}B(x_j,\varepsilon r). \tag{6.6.30}$$

In a straightforward manner, it can be verified that the choice of ε in (6.6.29) ensures

$$B(x_j,\varepsilon r)\subseteq B\big(z,\theta_2\delta_{\partial\Omega}(z)\big),\qquad \forall j\in\{1,\dots,N\},\ \forall z\in B(x_j,\varepsilon r). \tag{6.6.31}$$

If we now let $u:\Omega\to\mathbb{C}$ be an arbitrary Lebesgue measurable function, we then obtain from (6.6.31) that

$$u_{\star,\theta_2}(z)\geq\|u\|_{L^\infty(B(x_j,\varepsilon r),\mathcal{L}^n)}\qquad\forall j\in\{1,\dots,N\},\quad\forall z\in B(x_j,\varepsilon r), \tag{6.6.32}$$

thus

$$\inf_{B(x_j,\varepsilon r)}u_{\star,\theta_2}(z)\geq\|u\|_{L^\infty(B(x_j,\varepsilon r),\mathcal{L}^n)}\qquad\forall j\in\{1,\dots,N\}, \tag{6.6.33}$$

and furthermore, for each $s\in(0,\infty)$, that

$$⨍_{B(x_j,\varepsilon r)}\big(u_{\star,\theta_2}\big)^s\,\mathrm{d}\mathcal{L}^n\geq\|u\|^s_{L^\infty(B(x_j,\varepsilon r),\mathcal{L}^n)},\qquad\forall j\in\{1,\dots,N\}. \tag{6.6.34}$$

Making use of (6.6.30), (6.6.34), and the fact that $\varepsilon<\theta_3-\theta_1$, we can write

$$\begin{aligned}\big(u_{\star,\theta_1}(x)\big)^s&=\|u\|^s_{L^\infty(B(x,\theta_1 r),\mathcal{L}^n)}\leq\sum_{j=1}^{N}\|u\|^s_{L^\infty(B(x_j,\varepsilon r),\mathcal{L}^n)}\\&\leq\sum_{j=1}^{N}⨍_{B(x_j,\varepsilon r)}\big(u_{\star,\theta_2}\big)^s\,\mathrm{d}\mathcal{L}^n\leq C⨍_{B(x,(\theta_1+\varepsilon)r)}\big(u_{\star,\theta_2}\big)^s\,\mathrm{d}\mathcal{L}^n\\&\leq C⨍_{B(x,\theta_3 r)}\big(u_{\star,\theta_2}\big)^s\,\mathrm{d}\mathcal{L}^n.\end{aligned} \tag{6.6.35}$$

Combining (6.6.35) with the fact that $\delta_{\partial\Omega}(z)\approx\delta_{\partial\Omega}(x)$ uniformly for $z\in B(x,\theta_3 r)$, we obtain

$$u_{\star,\theta_1}(x)\leq C\Big(⨍_{B(x,\theta_3\delta_{\partial\Omega}(x))}\big(u_{\star,\theta_2}\big)^s\,\mathrm{d}\mathcal{L}^n\Big)^{1/s},\qquad\forall x\in\Omega. \tag{6.6.36}$$

This is the key estimate in the proof of (6.6.27).

Here is the end-game in the proof of the lemma. Consider the non-negative, Borel-measurable functions

$$f_j := \begin{cases} u_{\star,\theta_j} & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \qquad j \in \{1, 2\}. \tag{6.6.37}$$

Fix $s \in (0, p)$ and recall the L^s-based Hardy–Littlewood maximal operator $\mathcal{M}_s$ in the Euclidean space. This acts on any $\mathcal{L}^n$-measurable function f defined in $\mathbb{R}^n$ according to (cf. (7.6.7))

$$\big(\mathcal{M}_s f\big)(x) := \sup_{r>0} \Big(⨍_{B(x,r)} |f|^s \, d\mathcal{L}^n \Big)^{\frac{1}{s}}, \qquad \forall x \in \mathbb{R}^n. \tag{6.6.38}$$

Then estimate (6.6.36) translates into

$$f_1(x) \leq C\big(\mathcal{M}_s f_2\big)(x), \qquad \forall x \in \mathbb{R}^n. \tag{6.6.39}$$

Thus, by invoking the fact that $\mathcal{M}_s$ is bounded on $L^{p,q}(\mathbb{R}^n, \mathcal{L}^n)$ given that $s < p$ (cf. Corollary 7.6.2), it follows from (6.6.39) that

$$\begin{aligned} \|u_{\star,\theta_1}\|_{L^{p,q}(\Omega,\mathcal{L}^n)} &\approx \|f_1\|_{L^{p,q}(\mathbb{R}^n,\mathcal{L}^n)} \leq C\|f_2\|_{L^{p,q}(\mathbb{R}^n,\mathcal{L}^n)} \\ &\approx \|u_{\star,\theta_2}\|_{L^{p,q}(\Omega,\mathcal{L}^n)}. \end{aligned} \tag{6.6.40}$$

At this stage, the equivalence in (6.6.27) in the case when $\gamma = 0$ readily follows from (6.6.40). □

Let Ω be an arbitrary open, nonempty, proper subset of $\mathbb{R}^n$ and consider a Borel measure μ in Ω. For each fixed integrability exponents $p, q \in (0, \infty)$ and parameter $\theta \in (0, 1)$, introduce what we shall call the `maximal Lorentz space`

$$L^{p,q}_\star(\Omega, \mu) := \big\{u : \Omega \to \mathbb{C} : u \text{ is } \mathcal{L}^n\text{-measurable and } u_{\star,\theta} \in L^{p,q}(\Omega, \mu)\big\}, \tag{6.6.41}$$

equipped with the quasi-norm $\|u\|_{L^{p,q}_\star(\Omega,\mu)} := \|u_{\star,\theta}\|_{L^{p,q}(\Omega,\mu)}$ for each u in $L^{p,q}_\star(\Omega, \mu)$. The obvious connection between the ordinary Lorentz scale in Ω and the scale of maximal Lorentz scale just defined is that we have a continuous embedding

$$L^{p,q}_\star(\Omega, \mu) \hookrightarrow L^{p,q}(\Omega, \mu). \tag{6.6.42}$$

Indeed, this is a consequence of the pointwise inequality (6.6.6), plus the fact that Lorentz spaces are function latices (cf. (6.2.16)). The diagonal $p = q$ in the scale of maximal Lorentz spaces corresponds to what we shall refer to as `maximal Lebesgue spaces`. Specifically, with Ω, μ as above, for each $p \in (0, \infty)$ and $\theta \in (0, 1)$ we define

$$\begin{aligned} L^p_\star(\Omega, \mu) &:= L^{p,p}_\star(\Omega, \mu) \\ &= \big\{u : \Omega \to \mathbb{C} : \ u \ \text{ is } \mathcal{L}^n\text{-measurable and } \ u_{\star,\theta} \in L^p(\Omega, \mu)\big\}, \end{aligned} \tag{6.6.43}$$

and equip this space with the quasi-norm $\|u\|_{L^p_\star(\Omega,\mu)} := \|u_{\star,\theta}\|_{L^p(\Omega,\mu)}$ for each function $u \in L^p_\star(\Omega, \mu)$. Then (6.6.42) shows that we have a continuous embedding

$$L^p_\star(\Omega, \mu) \hookrightarrow L^p(\Omega, \mu), \tag{6.6.44}$$

while from definitions and (6.6.5) it follows that

$$\begin{array}{l}\text{if the Borel measure } \mu \text{ is such that } \mu\big(B(x,r)\big) > 0 \text{ whenever } x \in \Omega \\ \text{and } 0 < r < \delta_{\partial\Omega}(x) \text{ then } L^p_\star(\Omega,\mu) \subseteq L^\infty_{\rm loc}(\Omega, \mathcal{L}^n) \text{ for all } p \in (0,\infty).\end{array} \tag{6.6.45}$$

We are primarily interested in the case when $\mu = \delta^\gamma_{\partial\Omega}\mathcal{L}^n$ for some $\gamma \in \mathbb{R}$. In such a scenario, Lemma 6.6.2 together with (6.6.8) ensures that the choice of the parameter $\theta \in (0, 1)$ does not affect the maximal Lorentz space $L^{p,q}_\star\big(\Omega, \delta^\gamma_{\partial\Omega}\mathcal{L}^n\big)$, or the maximal Lebesgue space $L^p_\star\big(\Omega, \delta^\gamma_{\partial\Omega}\mathcal{L}^n\big)$. In this case, (6.6.45) yields a continuous embedding

$$L^p_\star\big(\Omega, \delta^\gamma_{\partial\Omega}\mathcal{L}^n\big) \hookrightarrow L^\infty_{\rm loc}(\Omega, \mathcal{L}^n) \ \text{ for each } \ p \in (0,\infty) \ \text{ and } \ \gamma \in \mathbb{R}. \tag{6.6.46}$$

Recall Definition 6.5.1 and the convention made in (6.5.13).

Proposition 6.6.3 *Let Ω be an arbitrary open, nonempty, proper subset of $\mathbb{R}^n$, and fix a parameter $\theta \in (0, 1)$. Also, consider an arbitrary function $u \in L^\infty_{\rm loc}(\Omega, \mathcal{L}^n)$. Then the solid maximal function $u_{\star,\theta}$ is subaveraging in Ω.*

Proof From definitions and (6.6.5) it follows that $u_{\star,\theta} \in L^\infty_{\rm loc}(\Omega, \mathcal{L}^n)$. Select an arbitrary integrability exponent $p \in (0, \infty)$ and fix an arbitrary point $x \in \Omega$. Our goal is to show that there exists a constant $C \in (0, \infty)$ which is independent of u, p, x with the property that for each $r \in \big(0, \delta_{\partial\Omega}(x)\big)$ we have

$$u_{\star,\theta}(x) \leq C\Big(⨍_{B(x,r)} \big|u_{\star,\theta}\big|^p \, d\mathcal{L}^n\Big)^{\frac{1}{p}}. \tag{6.6.47}$$

To this end, abbreviate $R := \theta\delta_{\partial\Omega}(x) \in (0, \infty)$. In addition, select some small scaling factor $\mu \in \big(0, (\theta + 1)^{-1}\big)$ and suppose first that $r \in (0, \mu R)$. Let us also pick a second scaling factor $\lambda \in \big(0, 1 - \mu(\theta + 1)\big)$. Given any $\zeta \in B(x, r)$ we may then write

$$\theta^{-1}R = \delta_{\partial\Omega}(x) \leq \delta_{\partial\Omega}(\zeta) + |x - \zeta| \leq \delta_{\partial\Omega}(\zeta) + r, \tag{6.6.48}$$

from which we then conclude that

$$\theta\delta_{\partial\Omega}(\zeta) \geq R - \theta r. \tag{6.6.49}$$

Since for any point $q \in B(x, \lambda R)$ we have

$$\begin{aligned}|q-\zeta| \leq |q-x|+|x-\zeta| &< \lambda R + r < (1-\mu\theta-\mu)R + \mu R \\ &= R - \theta(\mu R) < R - \theta r \leq \theta\delta_{\partial\Omega}(\zeta),\end{aligned} \tag{6.6.50}$$

it follows that

$$B(x,\lambda R) \subseteq B\big(\zeta, \theta\delta_{\partial\Omega}(\zeta)\big). \tag{6.6.51}$$

Having established this, we conclude that

$$\begin{aligned}\|u\|_{L^\infty(B(x,\lambda R),\mathcal{L}^n)} &\leq \|u\|_{L^\infty(B(\zeta,\theta\delta_{\partial\Omega}(\zeta)),\mathcal{L}^n)} \\ &= u_{\star,\theta}(\zeta) \quad \text{for each} \quad \zeta \in B(x,r).\end{aligned} \tag{6.6.52}$$

From (6.6.52) we readily obtain

$$\|u\|_{L^\infty(B(x,\lambda R),\mathcal{L}^n)} \leq \Big(\fint_{B(x,r)} \big|u_{\star,\theta}\big|^p \, d\mathcal{L}^n\Big)^{\frac{1}{p}}. \tag{6.6.53}$$

Next we wish to estimate the essential norm of u near the boundary of $B(x,R)$ in a similar fashion. Pick $y \in \partial B(x,R)$ and consider $z := x + (r/R)(y-x)$. Then the points x, y, z are collinear and since $|z-x| = (r/R)|y-x| = r$, we conclude that $z \in \partial B(x,r)$. Assume another small scaling factor $\tau \in (0,1)$ has been chosen. To estimate the distance from z to $B(y,\tau R) \cap B(x,R)$ observe that it is realized as $|z-\xi|$ where $\xi \in \partial B(y,\tau R) \cap B(x,R)$. Denote by α the angle between the line joining x with y and the line joining x with ξ. Working in the two-dimensional plane spanned by these two lines, we may compute

$$\begin{aligned}\operatorname{dist}\big(z, B(y,\tau R) \cap B(x,R)\big) = |z-\xi| &= \big|(R\cos\alpha, R\sin\alpha) - (r,0)\big| \\ &= \sqrt{R^2 + r^2 - 2rR\cos\alpha}.\end{aligned} \tag{6.6.54}$$

Let us also note that since $\sin(\alpha/2) = (\tau R/2)/R = \tau/2$, we have

$$\alpha = \arcsin(\tau/2). \tag{6.6.55}$$

To proceed, bring in yet another small scaling factor $\varepsilon \in (0,1)$ and pick some arbitrary point $w \in B(z,\varepsilon r)$. Then

$$\begin{aligned}\theta^{-1}R = \delta_{\partial\Omega}(x) &\leq \delta_{\partial\Omega}(w) + |x-w| \leq \delta_{\partial\Omega}(w) + |x-z| + |z-w| \\ &\leq \delta_{\partial\Omega}(w) + r + \varepsilon r,\end{aligned} \tag{6.6.56}$$

which further entails

$$\theta\delta_{\partial\Omega}(w) \geq R - \theta(1+\varepsilon)r. \tag{6.6.57}$$

From (6.6.54) and (6.6.57) we then see that

$$\operatorname{dist}\big(w, B(y, \tau R) \cap B(x, R)\big) \leq \operatorname{dist}\big(z, B(y, \tau R) \cap B(x, R)\big) + |z - w| < \sqrt{R^2 + r^2 - 2rR\cos\alpha} + \varepsilon r. \tag{6.6.58}$$

We claim that matters maybe arranged so that

$$\sqrt{R^2 + r^2 - 2rR\cos\alpha} + \varepsilon r < R - \theta(1+\varepsilon)r. \tag{6.6.59}$$

For starters, choose $\varepsilon \in \big(0, \frac{1-\theta}{1+\theta}\big)$ which ensures that

$$a := \theta(1+\varepsilon) + \varepsilon \ \text{ belongs to } \ (0, 1). \tag{6.6.60}$$

Next, by choosing $\tau \in (0, 1)$ small enough (6.6.55) guarantees that α is sufficiently close to zero so that $\cos\alpha > a$. Some elementary algebra shows that (6.6.59) may be recast as

$$r < 2\Big[\frac{\cos\alpha - a}{1 - a^2}\Big]R \tag{6.6.61}$$

and, given the original choice of r, this is satisfied if to begin with we pick

$$\mu \in \big(0, \tfrac{2(\cos\alpha - a)}{1-a^2}\big). \tag{6.6.62}$$

In summary, taking the scaling factors ε, τ, μ as above ensures that (6.6.59) holds.

In turn, from (6.6.58), (6.6.59), and (6.6.57) we conclude that

$$\operatorname{dist}\big(w, B(y, \tau R) \cap B(x, R)\big) < \theta\delta_{\partial\Omega}(w), \tag{6.6.63}$$

hence

$$B(y, \tau R) \cap B(x, R) \subseteq B\big(w, \theta\delta_{\partial\Omega}(w)\big). \tag{6.6.64}$$

From this we deduce that

$$\|u\|_{L^\infty(B(y,\tau R)\cap B(x,R),\mathcal{L}^n)} \leq \|u\|_{L^\infty(B(w,\theta\delta_{\partial\Omega}(w)),\mathcal{L}^n)} = u_{\star,\theta}(w) \ \text{ for each } \ w \in B(z, \varepsilon r). \tag{6.6.65}$$

Raising the most extreme sides of (6.6.65) to the p-th power and taking the integral average over the set $E := B(z, \varepsilon r) \cap B(x, r)$ leads to the conclusion that

$$\|u\|_{L^\infty(B(y,\tau R)\cap B(x,R),\mathcal{L}^n)} \leq \Big(\fint_E |u_{\star,\theta}|^p \, d\mathcal{L}^n\Big)^{\frac{1}{p}} \leq C\Big(\fint_{B(x,r)} |u_{\star,\theta}|^p \, d\mathcal{L}^n\Big)^{\frac{1}{p}} \tag{6.6.66}$$

for some $C \in (0, \infty)$ independent of x, y, r, with the last inequality above a consequence of the fact that

$$E \subseteq B(x,r) \ \text{ and } \ \mathcal{L}^n(E) \approx \mathcal{L}^n\big(B(x,r)\big). \tag{6.6.67}$$

For estimate (6.6.66) to be useful we need $\big\{B(y,\tau R)\big\}_{y\in\partial B(x,R)}$ to be a cover of $\overline{B(x,R)} \setminus B(x,\lambda R)$. This is indeed the case if $\tau R > R - \lambda R$, i.e., if $\lambda > 1-\tau$. Since we have originally assumed that $\lambda < 1-\mu(\theta+1)$, in order to be able to select λ satisfying these requirements we shall demand that $1-\tau < 1-\mu(\theta+1)$ or, equivalently, $\mu < \tau/(\theta+1)$. In view of (6.6.62) we ultimately should choose

$$\mu \in \Big(0, \min\big\{\tfrac{2(\cos\alpha - a)}{1-a^2}, \tfrac{\tau}{\theta+1}\big\}\Big). \tag{6.6.68}$$

Such a choice guarantees that we may find a finite family $\{y_j\}_{1\le j\le N} \subseteq \partial B(x,R)$, where $N \in \mathbb{N}$ depends exclusively on the dimension n and the various scaling factors considered earlier, with the property that

$$\big\{B(y_j,\tau R)\big\}_{1\le j\le N} \text{ covers } \overline{B(x,R)} \setminus B(x,\lambda R). \tag{6.6.69}$$

Collectively, (6.6.69) and (6.6.66) then imply

$$\|u\|_{L^\infty(B(x,R)\setminus B(x,\lambda R),\mathcal{L}^n)} \le C\Big(\fint_{B(x,r)} \big|u_{\star,\theta}\big|^p \, d\mathcal{L}^n\Big)^{\frac{1}{p}}. \tag{6.6.70}$$

By combining this with (6.6.53) we arrive at the conclusion that there exists some $C \in (0,\infty)$ with the property that for each $x \in \Omega$ and each $r \in \big(0, \mu\delta_{\partial\Omega}(x)\big)$ we have

$$\begin{aligned} u_{\star,\theta}(x) &= \|u\|_{L^\infty(B(x,\theta\delta_{\partial\Omega}(x)),\mathcal{L}^n)} = \|u\|_{L^\infty(B(x,R),\mathcal{L}^n)} \\ &\le C\Big(\fint_{B(x,r)} \big|u_{\star,\theta}\big|^p \, d\mathcal{L}^n\Big)^{\frac{1}{p}}. \end{aligned} \tag{6.6.71}$$

Finally, in the case when $x \in \Omega$ and we have $r \in \big[\mu\delta_{\partial\Omega}(x), \delta_{\partial\Omega}(x)\big)$, we may employ (6.6.71) with $r := 2^{-1}\mu\delta_{\partial\Omega}(x)$ to write

$$\begin{aligned} u_{\star,\theta}(x) &\le C\Big(\fint_{B(x,2^{-1}\mu\delta_{\partial\Omega}(x))} \big|u_{\star,\theta}\big|^p \, d\mathcal{L}^n\Big)^{\frac{1}{p}} \\ &\le C\Big(\fint_{B(x,r)} \big|u_{\star,\theta}\big|^p \, d\mathcal{L}^n\Big)^{\frac{1}{p}}. \end{aligned} \tag{6.6.72}$$

Collectively, (6.6.71), (6.6.72), and (6.5.13) allow us to conclude that (6.6.47) holds. Hence, the function $u_{\star,\theta}$ is subaveraging in Ω. □

Since, as we have discussed earlier, subaveraging functions satisfy global weighted reverse Hölder estimates, Proposition 6.6.3 permits us to establish useful embedding results for weighted maximal Lebesgue spaces of the sort presented below.

Proposition 6.6.4 *Let Ω be an arbitrary open, nonempty, proper subset of $\mathbb{R}^n$. Also, fix $\alpha, \gamma \in \mathbb{R}$ with $\alpha \geq 0$, and pick an integrability exponent $p \in (0, \infty)$ such that $1/p > \alpha$. Then*

$$L^p_\star\big(\Omega, \delta_{\partial\Omega}^{\gamma p}\mathcal{L}^n\big) \hookrightarrow L^{p_\alpha}_\star\big(\Omega, \delta_{\partial\Omega}^{(\gamma+n\alpha)p_\alpha}\mathcal{L}^n\big) \ \textit{where} \ \ p_\alpha := \big(\tfrac{1}{p} - \alpha\big)^{-1}. \tag{6.6.73}$$

In particular, whenever $\alpha \in (0, \infty)$ and $0 < p < \frac{1}{\alpha}$, one has the continuous embedding

$$L^p_\star\big(\Omega, \delta_{\partial\Omega}^{-n\alpha p}\mathcal{L}^n\big) \hookrightarrow L^{\frac{p}{1-\alpha p}}(\Omega, \mathcal{L}^n). \tag{6.6.74}$$

Proof Combining Proposition 6.6.3 with (6.5.31) yields (6.6.73). In turn, (6.6.73) and (6.6.42) imply (6.6.74) by choosing $\gamma := -n\alpha$.

An alternative proof of the embedding (6.6.73) is as follows. Fix $\theta \in (0, 1)$ and pick $\theta_3 \in (\theta, 1)$. Also, select some $s \in (0, \infty)$. Making use of (6.6.8) and (6.6.36) (with $\theta_1 = \theta_2 = \theta$) we see that there exists a constant $C \in (0, \infty)$ independent of u with the property that

$$\begin{aligned} u_{\star,\theta}(x)\delta_{\partial\Omega}(x)^{\gamma+n\alpha} &\leq C\mathcal{L}^n\big(B(x, \theta_3\delta_{\partial\Omega}(x)\big)^\alpha \times \\ &\quad\times \Big(⨍_{B(x,\theta_3\delta_{\partial\Omega}(x))} \big(u_{\star,\theta}\delta_{\partial\Omega}^\gamma\big)^s \, \mathrm{d}\mathcal{L}^n\Big)^{1/s}, \end{aligned} \tag{6.6.75}$$

for each $x \in \Omega$. In terms of the non-negative Borel-measurable function

$$f := \begin{cases} u_{\star,\theta} \cdot \delta_{\partial\Omega}^\gamma & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{6.6.76}$$

and the fractional Hardy–Littlewood maximal operator defined in (7.6.1) (presently used with $X := \mathbb{R}^n$ and $\mu := \mathcal{L}^n$), inequality (6.6.75) gives that

$$f(x)\delta_{\partial\Omega}(x)^{n\alpha} \leq C\big(\mathcal{M}_{s,\alpha} f\big)(x), \qquad \forall x \in \mathbb{R}^n. \tag{6.6.77}$$

Based on Theorem 7.6.1 and the fact that we have the freedom of selecting $s \in (0, \infty)$, we may therefore conclude that if $0 < p < \frac{1}{\alpha}$ and $p_\alpha := \big(\frac{1}{p} - \alpha\big)^{-1}$ then

$$\begin{aligned} \|u\|_{L^{p_\alpha}_\star(\Omega,\delta_{\partial\Omega}^{(\gamma+n\alpha)p_\alpha}\mathcal{L}^n)} &\approx \big\|u_{\star,\theta}\delta_{\partial\Omega}^{\gamma+n\alpha}\big\|_{L^{p_\alpha}(\Omega,\mathcal{L}^n)} = \big\|f\delta_{\partial\Omega}^{n\alpha}\big\|_{L^{p_\alpha}(\mathbb{R}^n,\mathcal{L}^n)} \\ &\leq C\|f\|_{L^p(\mathbb{R}^n,\mathcal{L}^n)} = C\big\|u_{\star,\theta}\delta_{\partial\Omega}^\gamma\big\|_{L^p(\Omega,\mathcal{L}^n)} \\ &\approx \|u\|_{L^p_\star(\Omega,\delta_{\partial\Omega}^{\gamma p}\mathcal{L}^n)}, \end{aligned} \tag{6.6.78}$$

finishing the alternative proof of (6.6.73). □

We can go a step further and consider a local version of the solid maximal function of the following sort. Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$, and fix a parameter $\theta \in (0, 1)$. Also, let E be an arbitrary Lebesgue measurable subset of Ω. Given $u : E \to \mathbb{C}$ Lebesgue-measurable, define its `local solid maximal function` on E as

$$u^E_{\star,\theta}(x) := \|u\|_{L^\infty(E\cap B(x,\theta\delta_{\partial\Omega}(x)),\mathcal{L}^n)} \in [0, +\infty]. \qquad \forall x \in \Omega. \tag{6.6.79}$$

Directly from definitions we see that, if $\widetilde{u}$ denotes the extension of u by zero outside E to the entire Ω, then

$$u^E_{\star,\theta} = \big(\widetilde{u}\big)_{\star,\theta} \ \text{ pointwise in } \ \Omega. \tag{6.6.80}$$

Granted this, Lemma 6.6.1 applies and gives that

$$u^E_{\star,\theta} : \Omega \to [0, +\infty] \ \text{ is lower-semicontinuous,} \tag{6.6.81}$$

and

$$|u| \leq u^E_{\star,\theta} \ \text{ at } \mathcal{L}^n\text{-a.e. point in } \ E. \tag{6.6.82}$$

Moreover, Lemma 6.6.2 ensures that, having fixed $p \in (0, \infty)$ and $q \in (0, \infty]$ along with any two parameters $\theta_1, \theta_2 \in (0, 1)$, we have

$$\|u^E_{\star,\theta_1}\|_{L^{p,q}(\Omega,\mathcal{L}^n)} \approx \|u^E_{\star,\theta_2}\|_{L^{p,q}(\Omega,\mathcal{L}^n)}, \tag{6.6.83}$$

uniformly in the class of Lebesgue measurable functions $u : \Omega \to \mathbb{C}$.

Moving on, recall from Theorem 6.5.7 that null-solutions of elliptic systems are subaveraging functions, so it should be possible to prove weighted reverse Hölder estimates (both of a local and global nature) for such functions in a general geometric setting. We do this in Proposition 6.6.6. As a preamble, we first discuss some pointwise inequalities in the lemma below.

Lemma 6.6.5 *Let $n, m, M \in \mathbb{N}$ and suppose L is a constant (complex) coefficient homogeneous $M \times M$ system of order $2m$ in $\mathbb{R}^n$, with the property that* $\det[L(\xi)] \neq 0$ *for each $\xi \in \mathbb{R}^n \setminus \{0\}$ (with the characteristic matrix $L(\xi)$ defined as in (6.5.39)). Assume also that $\Omega \subseteq \mathbb{R}^n$ is an open set and $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ is such that $Lu = 0$ in Ω. Finally, suppose $0 < \theta_1 < \theta_2 < 1$, $k \in \mathbb{N}_0$, $s \in \mathbb{R}$, and $p \in (0, \infty)$. Then there exists $C \in (0, \infty)$ which depends only on $L, n, \theta_1, \theta_2, k, p$ such that for each $x \in \Omega$ one has*

$$\begin{aligned} \delta_{\partial\Omega}(x)^{k+s}|(\nabla^k u)(x)| &\leq \delta^{k+s}_{\partial\Omega}(x)|\nabla^k u|_{\star,\theta_1}(x) \leq C\big(\delta^{k+s}_{\partial\Omega}|\nabla^k u|\big)_{\star,\theta_1}(x) \\ &\leq C\Big(\fint_{B(x,\theta_2\delta_{\partial\Omega}(x))} \big[\delta^s_{\partial\Omega}|u|\big]^p \, d\mathcal{L}^n\Big)^{\frac{1}{p}} \leq C \cdot \delta^s_{\partial\Omega}(x) u_{\star,\theta_2}(x), \end{aligned} \tag{6.6.84}$$

where $\nabla^k u$ is the vector consisting of all partial derivatives of order k of u.

Proof This is a consequence of Theorem 6.5.7, (6.6.6), (6.6.8), and (6.6.2). □

Here are the weighted reverse Hölder estimates for null-solutions of elliptic systems in arbitrary open sets, referred to earlier.

Proposition 6.6.6 *Let L be a constant (complex) coefficient homogeneous $M \times M$ system of order $2m$ in $\mathbb{R}^n$, where $M, m \in \mathbb{N}$, with the property that* $\det[L(\xi)] \neq 0$ *for each $\xi \in \mathbb{R}^n \setminus \{0\}$ (with the characteristic matrix $L(\xi)$ defined as in (6.5.39)). Also, assume that Ω is an arbitrary open (nonempty, proper) subset of $\mathbb{R}^n$ and that the function $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ is such that $Lu = 0$ in Ω. Finally, suppose $0 < q \leq p \leq \infty$ and fix $s \in \mathbb{R}$, $\theta \in (0, 1)$, and $k \in \mathbb{N}_0$.*

Then there exists a finite constant $C = C(L, p, q, s, \theta, k) > 0$, independent of u, with the property that

$$\Big(\int_{\Omega\cap B(z,R)} \big(\delta_{\partial\Omega}(x)^{s+k+n(\frac{1}{q}-\frac{1}{p})}|\nabla^k u|_{\star,\theta}(x)\big)^p\,\mathrm{d}x\Big)^{\frac{1}{p}} \leq C\Big(\int_{\Omega\cap B(z,CR)} \big(\delta_{\partial\Omega}(x)^s|u(x)|\big)^q\,\mathrm{d}x\Big)^{\frac{1}{q}} \tag{6.6.85}$$

for each $z \in \partial\Omega$ and each $R > 0$. Also, whenever $0 < p \leq \infty$, $s \in \mathbb{R}$, $k \in \mathbb{N}_0$, and $\theta \in (0, 1)$ there exists some constant $C = C(L, p, s, k, \theta) \in (0, \infty)$ with the property that

$$\begin{aligned}\Big(\int_\Omega \big(\delta_{\partial\Omega}^{s+k}|\nabla^k u|\big)^p\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}} &\leq \Big(\int_\Omega \big(\delta_{\partial\Omega}^{s+k}|\nabla^k u|_{\star,\theta}\big)^p\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}} \\ &\leq C\Big(\int_\Omega \big(\delta_{\partial\Omega}^{s}|u|\big)^p\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}} \\ &\leq C\Big(\int_\Omega \big(\delta_{\partial\Omega}^{s}|u|_{\star,\theta}\big)^p\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{p}}\end{aligned} \tag{6.6.86}$$

for each $u \in \big[\mathscr{C}^\infty(\Omega)\big]^M$ satisfying $Lu = 0$ in Ω.

Proof Choose $z \in \partial\Omega$ along with $R > 0$, and fix some $\theta_2 \in (\theta, 1)$. Observe that for each

$$B\big(x, \theta_2\delta_{\partial\Omega}(x)\big) \subseteq \Omega \cap B\big(z, (1+\theta_2)R\big) \ \text{ for each } \ x \in \Omega \cap B(z, R). \tag{6.6.87}$$

Bearing this in mind, in the case when $p < \infty$ we raise the penultimate inequality in (6.6.84) (written with $\theta_1 := \theta$) to the p-th power, integrate over $\Omega \cap B(z, R)$, and then use Fubini's theorem to write

$$
\begin{aligned}
&\int_{\Omega\cap B(z,R)} \big[\big(\delta_{\partial\Omega}^{k+s}|\nabla^k u|\big)_{\star,\theta}(x)\big]^p\,\mathrm{d}x \\
&\qquad \le C\int_{\Omega\cap B(z,R)}\Big(\fint_{B(x,\theta_2\delta_{\partial\Omega}(x))}\big[\delta_{\partial\Omega}(y)^s|u(y)|\big]^p\,\mathrm{d}y\Big)\,\mathrm{d}x \\
&\qquad \le C\int_{\Omega\cap B(z,(1+\theta_2)R)}\big[\delta_{\partial\Omega}(y)^s|u(y)|\big]^p\times \\
&\qquad\qquad\qquad \times\Big(\int_{\Omega\cap B(z,R)}\delta_{\partial\Omega}(x)^{-n}\mathbf{1}_{B(x,\theta_2\delta_{\partial\Omega}(x))}(y)\,\mathrm{d}x\Big)\,\mathrm{d}y \\
&\qquad \le C\int_{\Omega\cap B(z,(1+\theta_2)R)}\big[\delta_{\partial\Omega}(y)^s|u(y)|\big]^p\,\mathrm{d}y,
\end{aligned}
\tag{6.6.88}
$$

where the last inequality is based on the fact that for each fixed $x\in\Omega$ we have $\delta_{\partial\Omega}(y)\approx\delta_{\partial\Omega}(x)$, uniformly for $y\in B\big(x,\theta_2\delta_{\partial\Omega}(x)\big)$. The version of (6.6.88) corresponding to $p=\infty$ is a direct consequence of (6.6.84) and (6.6.87).

Having proved (6.6.88) when $0<p\le\infty$, the estimate claimed in (6.6.85) then follows by re-denoting s by $s+n(\frac{1}{q}-\frac{1}{p})$ and making use of (6.5.30) (while also bearing in mind (6.5.40)). Also, specializing (6.6.85) to the case when $q:=p$ and letting $R\to\infty$ yields the second estimate in (6.6.86). Finally, the remaining estimates in (6.6.86) are direct consequences of (6.6.6). □

In the last result in this section, we record a refinement of the estimates in Proposition 6.6.6 corresponding to $k=0$ (by only assuming that the function in question is subaveraging rather than a null-solution of an elliptic system). This lemma also strengthens the estimate in (6.5.30) through the consideration of the solid maximal function.

Lemma 6.6.7 *Let u be a subaveraging function defined in an arbitrary open (nonempty, proper) subset Ω of $\mathbb{R}^n$, and suppose $0<q\le p\le\infty$, $s\in\mathbb{R}$, and $\theta\in(0,1)$. Then there exists some $C\in(0,\infty)$ which depends only on p,q,s,θ, and the subaveraging character of u, with the property that*

$$
\begin{aligned}
&\Big(\int_{\Omega\cap B(z,R)}\big(\delta_{\partial\Omega}(x)^{s+n(\frac{1}{q}-\frac{1}{p})}u_{\star,\theta}(x)\big)^p\,\mathrm{d}x\Big)^{\frac{1}{p}} \\
&\qquad\qquad \le C\Big(\int_{\Omega\cap B(z,CR)}\big(\delta_{\partial\Omega}(x)^s|u(x)|\big)^q\,\mathrm{d}x\Big)^{\frac{1}{q}}
\end{aligned}
\tag{6.6.89}
$$

for each $z\in\partial\Omega$ and each $R>0$. In particular, upon letting $R\to\infty$ and specializing $q:=p$, it follows that if u is a subaveraging function defined in Ω, $0<p\le\infty$, and $s\in\mathbb{R}$, then there exists $C\in(0,\infty)$ depending only on p,s,θ, and the subaveraging character of u such that

$$\Big(\int_\Omega \big(\delta_{\partial\Omega}(x)^s u_{\star,\theta}(x)\big)^p\,dx\Big)^{\frac{1}{p}} \\ \leq C\Big(\int_\Omega \big(\delta_{\partial\Omega}(x)^s |u(x)|\big)^p\,dx\Big)^{\frac{1}{p}}. \tag{6.6.90}$$

As a corollary, for each $p \in (0,\infty]$ and $\gamma \in \mathbb{R}$ one has the inclusions

$$\Big\{u \in L^p\big(\Omega, \delta^\gamma_{\partial\Omega}\mathcal{L}^n\big) : u \text{ subaveraging in } \Omega\Big\} \subseteq L^p_\star\big(\Omega, \delta^\gamma_{\partial\Omega}\mathcal{L}^n\big) \\ \subseteq L^p(\Omega, \delta^\gamma_{\partial\Omega}\mathcal{L}^n), \tag{6.6.91}$$

in a quantitative fashion.

Proof This is justified by largely reasoning as in the proof of Proposition 6.6.6 with $k := 0$, with one noticeable difference. Specifically, in place of the penultimate inequality in (6.6.84), we now use the fact that for each fixed $\theta_2 \in (\theta, 1)$ and $p \in (0,\infty)$ there exists $C \in (0,\infty)$ such that

$$u_{\star,\theta}(x) \leq C\left(\fint_{B(x,\theta_2\delta_{\partial\Omega}(x))} |u|^p\,d\mathcal{L}^n\right)^{\frac{1}{p}} \quad \text{for each } x \in \Omega. \tag{6.6.92}$$

In turn, (6.6.92) is clear from Lemma 6.5.2 and (6.6.2). Parenthetically we wish to note that a direct proof of (6.6.90) may be given by performing a Whitney decomposition of Ω and relying on (6.6.92). □

Chapter 7
Quasi-Metric Spaces and Spaces of Homogeneous Type

Ever since the late 1970s it has been fully recognized that a significant portion of real analysis may be developed using very little of the structural richness of the Euclidean ambient. Maximal operators (à la Hardy–Littlewood), differentiation theorems (à la Lebesgue), function spaces (à la John–Nirenberg, Sarason, Hölder, Lipschitz, etc.), various classes of weights (à la Muckenhoupt), covering theorems (à la Vitali, Whitney), and even a full-fledged Calderón–Zygmund theory, among many other topics, make perfect sense and retain their potency in fairly scarce environments, such as spaces of homogeneous type. These are quasi-metric spaces equipped with a doubling Borel measure, and the goal here is to review and further develop those aspects of analysis on spaces of homogeneous type that are deemed relevant to our present work.

7.1 Quasi-Metric Spaces and a Sharp Metrization Result

To get started, fix a nonempty set X, which is going to play the role of the ambient space. Call two functions $\rho, \rho' : X \times X \to [0, \infty)$ `equivalent`, and write $\rho \approx \rho'$, if there exists $c \in (0, 1)$ with the property that

$$c\,\rho \leq \rho' \leq c^{-1}\rho \quad \text{on} \quad X \times X. \tag{7.1.1}$$

Next, call a function $\rho : X \times X \to [0, \infty)$ a `quasi-distance`, or `quasi-metric`, on X provided there exist two finite constants $C_0, C_1 \geq 1$ with the property that for any given points $x, y, z \in X$ we have

$$\begin{gathered}\rho(x, y) = 0 \Longleftrightarrow x = y, \quad \rho(y, x) \leq C_0 \cdot \rho(x, y), \\ \text{and} \quad \rho(x, y) \leq C_1 \cdot \max\{\rho(x, z), \rho(z, y)\}.\end{gathered} \tag{7.1.2}$$

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It is then clear that if ρ is a quasi-distance on X and $\rho' : X \times X \to [0, \infty)$ is such that $\rho' \approx \rho$ then ρ' is a quasi-distance on X as well.

For each quasi-distance ρ on X we define C_ρ to be the smallest constant which can play the role of C_1 in the last inequality in (7.1.2), i.e.,

$$C_\rho := \sup_{\substack{x,y,z\in X\\ \text{not all equal}}} \frac{\rho(x,y)}{\max\{\rho(x,z),\rho(z,y)\}} \in [1,\infty), \tag{7.1.3}$$

and define $\widetilde{C}_\rho$ to be the smallest constant which can play the role of C_0 in the first inequality in (7.1.2), i.e.,

$$\widetilde{C}_\rho := \sup_{\substack{x,y\in X\\ x\neq y}} \frac{\rho(y,x)}{\rho(x,y)} \in [1,\infty), \tag{7.1.4}$$

with the convention that $C_\rho := \widetilde{C}_\rho := 1$ when X is a mere singleton.

By a `quasi-metric space` we shall understand a pair (X, ρ) where X is a set and ρ is a quasi-distance on X. Given a quasi-metric space (X, ρ), define the ρ-`ball` centered at $x \in X$ with radius $r > 0$ to be

$$B_\rho(x,r) := \{y \in X : \rho(x,y) < r\}. \tag{7.1.5}$$

Also, for every $A, B \subseteq X$ and $x \in X$ define

$$\begin{aligned} \operatorname{diam}_\rho(A) &:= \sup\{\rho(x,y) : \ x, y \in A\},\\ \operatorname{dist}_\rho(x,A) &:= \inf\{\rho(x,y) : \ y \in A\},\\ \text{and}\ \ \operatorname{dist}_\rho(A,B) &:= \inf\{\rho(x,y) : \ x \in A,\ \ y \in B\}. \end{aligned} \tag{7.1.6}$$

Turning to topological considerations, we note that any quasi-metric space (X, ρ) has a canonical topology, naturally induced by the quasi-distance ρ which we will denote by τ_ρ. The latter is defined as the largest topology on X with the property that for each point $x \in X$ the family $\{B_\rho(x,r)\}_{r>0}$ is a fundamental system of neighborhoods of x. In concrete terms, for each set $\mathcal{O} \subseteq X$ we have

$$\mathcal{O} \text{ is open in } \tau_\rho \overset{\text{def}}{\Longleftrightarrow} \text{for each } x \in \mathcal{O} \text{ there exists } r > 0 \text{ such that } B_\rho(x,r) \subseteq \mathcal{O}. \tag{7.1.7}$$

It is then clear from definitions that if (X, ρ) is a quasi-metric space

$$\tau_\rho = \tau_{\rho'} \text{for any quasi-distance } \rho' \text{ on } X \text{ which is equivalent to } \rho. \tag{7.1.8}$$

Also, if $X_o \subseteq X$ is arbitrary, then

$$\rho_o := \rho\big|_{X_o\times X_o} \text{ is a quasi-metric on } X_o \text{ and } \tau_{\rho_o} = \tau_\rho\big|_{X_o}. \tag{7.1.9}$$

Lemma 7.1.1 *Suppose that (X, ρ) is a quasi-metric space with the property that (X, τ_ρ) is a separable topological space. Then any subset of X is separable.*

Proof Pick a dense subset $\{x_j\}_{j\in\mathbb{N}}$ of X and let $Y \subseteq X$ be an arbitrary nonempty set. For each $j \in \mathbb{N}$ consider $y_j \in Y$ such that $\rho(x_j, y_j) < \mathrm{dist}_\rho(x_j, Y) + 1/j$. We claim that $\{y_j\}_{j\in\mathbb{N}}$ is dense in Y. Indeed, given any $y \in Y \subseteq X$ we may find a subsequence $\{x_{j_k}\}_{k\in\mathbb{N}}$ convergent in τ_ρ to y. Given that

$$\rho(x_{j_k}, y_{j_k}) < \mathrm{dist}_\rho(x_{j_k}, Y) + 1/j_k \leq \rho(x_{j_k}, y) + 1/j_k \to 0 \;\text{ as }\; k \to \infty, \tag{7.1.10}$$

one readily concludes that the sequence $\{y_{j_k}\}_{k\in\mathbb{N}}$ is convergent in τ_ρ to y. Hence, the desired conclusion follows. □

As is well known, the topology induced by the given quasi-distance on a quasi-metric space is metrizable. Below we shall review a result proved in [188] which is a sharp quantitative version of this fact. To facilitate the subsequent discussion we first make a couple of definitions. Assume that X is an arbitrary, nonempty set. Given an arbitrary function $\rho : X \times X \to [0, \infty]$ and an arbitrary exponent $\alpha \in (0, \infty]$ define the function

$$\rho_\alpha : X \times X \longrightarrow [0, \infty] \tag{7.1.11}$$

by setting for each $x, y \in X$

$$\begin{aligned}\rho_\alpha(x, y) := \inf\Big\{\Big(\sum_{i=1}^{N} \rho(\xi_i, \xi_{i+1})^\alpha\Big)^{\frac{1}{\alpha}} : N \in \mathbb{N} \text{ and } \xi_1, \dots, \xi_{N+1} \in X,\\ \text{(not necessarily distinct) such that} \xi_1 = x \text{ and } \xi_{N+1} = y\Big\},\end{aligned} \tag{7.1.12}$$

whenever $\alpha < \infty$, and its natural counterpart corresponding to the case when $\alpha = \infty$, i.e.,

$$\begin{aligned}\rho_\infty(x, y) := \inf\Big\{\max_{1\leq i\leq N} \rho(\xi_i, \xi_{i+1}) : N \in \mathbb{N} \text{ and } \xi_1, \dots, \xi_{N+1} \in X,\\ \text{(not necessarily distinct) such that} \xi_1 = x \text{ and } \xi_{N+1} = y\Big\}.\end{aligned} \tag{7.1.13}$$

It is then clear from definitions that

$$\begin{aligned}&\text{for every quasi-distance } \rho \text{ on } X \text{ and every } \alpha \in (0, \infty], \text{ the function}\\ &\rho_\alpha \text{ is also a quasi-distance on } X, \text{ which satisfies } \rho_\alpha \leq \rho \text{ on } X \times X.\end{aligned} \tag{7.1.14}$$

Going further, if $\rho : X \times X \to [0, \infty]$ is an arbitrary function, consider its symmetrization ρ_{sym} defined by

$$\begin{aligned}&\rho_{sym} : X \times X \longrightarrow [0, \infty],\\ &\rho_{sym}(x, y) := \max\{\rho(x, y), \rho(y, x)\}, \;\; \forall x, y \in X.\end{aligned} \tag{7.1.15}$$

Then ρ_{sym} is symmetric, i.e., $\rho_{sym}(x,y)=\rho_{sym}(y,x)$ for every $x,y\in X$, and $\rho_{sym}\geq\rho$ on $X\times X$. In fact, ρ_{sym} is the smallest $[0,\infty]$-valued function defined on $X\times X$ which is symmetric and $\geq\rho$ pointwise on $X\times X$. Furthermore, if ρ is as in (7.1.2) then

$$\begin{aligned}&\rho_{sym} \text{ is a quasi-distance on } X \text{ satisfying } C_{\rho_{sym}}\leq C_\rho \text{ and}\\ &\widetilde{C}_{\rho_{sym}}=1 \text{ as well as } \rho\leq\rho_{sym}\leq\widetilde{C}_\rho\cdot\rho \text{ pointwise on } X\times X.\end{aligned}\tag{7.1.16}$$

Here is the quantitative metrization theorem from [188] alluded to above.

Theorem 7.1.2 *Let (X,ρ) be a quasi-metric space and assume that C_ρ, $\widetilde{C}_\rho\in[1,\infty)$ are as in (7.1.3)–(7.1.4). In this context, define (cf. (7.1.12)–(7.1.13))*

$$\rho_\# := (\rho_{sym})_\alpha \quad \textit{for} \quad \alpha := (\log_2 C_\rho)^{-1}\in(0,\infty].\tag{7.1.17}$$

Then

$$\rho_\# : X\times X\longrightarrow[0,\infty) \quad \textit{is a continuous function},\tag{7.1.18}$$

when $X\times X$ is equipped with the natural product topology $\tau_\rho\times\tau_\rho$. Furthermore, for each finite exponent $\beta\in(0,\alpha]$, the function

$$d_{\rho,\beta}:X\times X\to[0,\infty),\quad d_{\rho,\beta}(x,y):=\big[\rho_\#(x,y)\big]^\beta,\ \ \forall x,y\in X,\tag{7.1.19}$$

is a genuine distance on X, i.e., for every $x,y,z\in X$ one has

$$\begin{aligned}d_{\rho,\beta}(x,y)=0 \iff x=y,\quad d_{\rho,\beta}(x,y)=d_{\rho,\beta}(y,x),\\ \textit{and}\ \ d_{\rho,\beta}(x,y)\leq d_{\rho,\beta}(x,z)+d_{\rho,\beta}(z,y).\end{aligned}\tag{7.1.20}$$

In addition, $\big[d_{\rho,\beta}\big]^{1/\beta}\approx\rho$. More specifically,

$$(C_\rho)^{-2}\rho(x,y)\leq\big[d_{\rho,\beta}(x,y)\big]^{1/\beta}=\rho_\#(x,y)\leq\widetilde{C}_\rho\cdot\rho(x,y),\quad\forall x,y\in X.\tag{7.1.21}$$

In particular, the topology induced by the distance $d_{\rho,\beta}=(\rho_\#)^\beta$ on X is precisely τ_ρ, thus the topology induced by any quasi-metric is metrizable.

Moreover, for each finite exponent $\beta\in\big(0,(\log_2 C_\rho)^{-1}\big]$, the function $\rho_\#$ satisfies the following local Hölder-type regularity condition of order β in both variables simultaneously (which is a stronger property than (7.1.18)):

$$\begin{aligned}&\big|\rho_\#(x,y)-\rho_\#(w,z)\big|\\ &\qquad\leq\beta^{-1}\cdot\max\big\{\rho_\#(x,y)^{1-\beta},\ \rho_\#(w,z)^{1-\beta}\big\}\big(\rho_\#(x,w)^\beta+\rho_\#(y,z)^\beta\big),\end{aligned}\tag{7.1.22}$$

for all $x,y,w,z\in X$ such that $\min\{\rho(x,y),\rho(w,z)\}<\infty$ and, if $\beta\geq1$, also assuming $x\neq y$, $w\neq z$. In particular, in the case $x=w$, formula (7.1.22) becomes

$$\big|\,\rho_{\#}(x,y)-\rho_{\#}(x,z)\big| \le \beta^{-1}\cdot\max\big\{\rho_{\#}(x,y)^{1-\beta},\ \rho_{\#}(x,z)^{1-\beta}\big\}\big[\rho_{\#}(y,z)\big]^{\beta}, \tag{7.1.23}$$

for all $x, y, z \in X$ *such that* $\min\{\rho(x,y), \rho(x,z)\} < \infty$ *and, if* $\beta \ge 1$*, also assuming* $x \notin \{y, z\}$.

Finally, the Hölder-type results from (7.1.22)–(7.1.23) *are sharp in the sense that they may fail if* $\beta > (\log_2 C_\rho)^{-1}$.

The key feature of the result discussed in Theorem 7.1.2 is the fact that if (X, ρ) is any quasi-metric space then ρ^β is equivalent to a distance on X for any finite number $\beta \in (0, (\log_2 C_\rho)^{-1}]$. This result is sharp and improves upon an earlier version due to R.A. Macías and C. Segovia [169], in which these authors have identified a non-optimal upper bound for the exponent β.

7.2 Estimating Integrals Involving the Quasi-Distance

Often, we are in a position to estimate integrals involving a power of the quasi-distance in a quasi-metric space equipped with a measure whose action on balls obeys a growth condition with respect to the radii. A general result of this flavor is discussed in the lemma below.

Lemma 7.2.1 *Let* (X, ρ) *be a given quasi-metric space endowed with a quasi-distance* $\rho : X \times X \to [0, \infty)$ *which is continuous*[1] *in the product topology* $\tau_\rho \times \tau_\rho$. *Assume* μ *is a Borel measure on* (X, τ_ρ) *with the property that there exist* $d \in (0, \infty)$ *and* $c \in (0, \infty)$ *such that*

$$\mu\big(B_\rho(x, R)\big) \le cR^d \qquad \forall x \in X, \quad \forall R \in \big(0, 2\,\mathrm{diam}_\rho(X)\big). \tag{7.2.1}$$

Also, suppose $\Phi : (0, \infty) \to [0, \infty)$ *has the property that there exists some* $m \in \mathbb{R}$ *such that the function*

$$(0, \infty) \ni t \longmapsto \frac{\Phi(t)}{t^m} \in [0, \infty) \ \text{ is monotone}. \tag{7.2.2}$$

Then there exists a finite constant $C = C(d, c, m) > 0$ *such that for each radius* $r \in \big(0, 2\,\mathrm{diam}_\rho(X)\big)$ *and* $x \in X$ *one has*

$$\int_{X\setminus B_\rho(x,r)} \Phi\big(\rho(x,y)\big)\,\mathrm{d}\mu(y) \le C\int_{r/2}^{16\,\mathrm{diam}_\rho(X)} t^{d-1}\Phi(t)\,\mathrm{d}t \tag{7.2.3}$$

and

$$\int_{B_\rho(x,r)} \Phi\big(\rho(x,y)\big)\,\mathrm{d}\mu(y) \le C\int_0^{2r} t^{d-1}\Phi(t)\,\mathrm{d}t. \tag{7.2.4}$$

[1] From Theorem 7.1.2 we know that any quasi-metric space has an equivalent quasi-distance which satisfies this property.

Particular cases of (7.2.3) and (7.2.4), frequently used in the sequel, are as follows: given any $\delta \in (0, \infty)$ there exists a finite constant $C = C(d, c, \delta) > 0$ such that, for each $r > 0$ and $x \in X$,

$$\int_{B_\rho(x,r)} \frac{\mathrm{d}\mu(y)}{\rho(x,y)^{d-\delta}} \le Cr^{\delta} \quad \text{and} \quad \int_{X\setminus B_\rho(x,r)} \frac{\mathrm{d}\mu(y)}{\rho(x,y)^{d+\delta}} \le Cr^{-\delta}. \tag{7.2.5}$$

***Proof of Lemma* 7.2.1** To begin with, observe that Φ is $\mathscr{L}^1$-measurable (since we have $\Phi(t) = t^m \cdot \big(\Phi(t)/t^m\big)$ for each $t \in (0, \infty)$, and the function $t \mapsto \Phi(t)/t^m$ is assumed to be monotone), hence the last integrals in (7.2.3) and (7.2.4) are meaningful. Granted the hypotheses on ρ and μ, this also ensures that for each fixed $x \in X$ the function $X \ni y \mapsto \Phi\big(\rho(x, y)\big) \in [0, \infty)$ is μ-measurable. Thus, the first integral in (7.2.3) is also meaningful. Finally, since for each fixed $x \in X$ we have $B_\rho(x, R) \searrow \{x\}$ as $R \searrow 0$, we conclude from (7.2.1) that $\mu(\{x\}) = 0$ hence, even though Φ is not defined at zero, the first integral in (7.2.4) is, nonetheless, meaningful as well.

To deal with (7.2.3), fix $x \in X$ along with $r \in \big(0, 2\,\mathrm{diam}_\rho(X)\big)$, then set

$$j_r := \Big[\log_2\Big(\frac{\mathrm{diam}_\rho(X)}{r}\Big)\Big] + 1 \in \mathbb{N} \cup \{\infty\}. \tag{7.2.6}$$

Throughout, for each number $a \in \mathbb{R}$ we abbreviate $(a)_+ := \max\{a, 0\}$. Assume first that

$$\begin{gathered}\text{there exists } m \in \mathbb{R} \text{ with the property that}\\ (0,\infty) \ni t \mapsto \Phi(t)/t^m \in [0,\infty) \ \text{ is decreasing.}\end{gathered} \tag{7.2.7}$$

In such a scenario, we combine (7.2.7) with (7.2.1) in order to estimate

$$\begin{aligned}
&\int_{X\setminus B_\rho(x,r)} \Phi\big(\rho(x,y)\big)\,\mathrm{d}\mu(y) \qquad\qquad (7.2.8)\\
&\quad = \sum_{j=0}^{j_r} \int_{\substack{y\in X\\ 2^j r\le\rho(x,y)<2^{j+1}r}} \rho(x,y)^m \Big(\frac{\Phi\big(\rho(x,y)\big)}{\rho(x,y)^m}\Big)\,\mathrm{d}\mu(y)\\
&\quad \le 2^{(m)_+} \sum_{j=0}^{j_r} \int_{\substack{y\in X\\ 2^j r\le\rho(x,y)<2^{j+1}r}} \big(2^j r\big)^m \Big(\frac{\Phi\big(2^j r\big)}{(2^j r)^m}\Big)\,\mathrm{d}\mu(y)\\
&\quad \le 2^{(m)_+} \sum_{j=0}^{j_r} \big(2^j r\big)^m \Big(\frac{\Phi\big(2^j r\big)}{(2^j r)^m}\Big) \mu\big(B_\rho(x, 2^{j+1} r)\big)\\
&\quad \le c2^{(m)_+} \sum_{j=0}^{j_r} \big(2^{j+1} r\big)^d \big(2^j r\big)^m \Big(\frac{\Phi\big(2^j r\big)}{(2^j r)^m}\Big)
\end{aligned}$$

$$\le c2^d \cdot 2^{(m)_+} \cdot 2^{(m+d)_+} \sum_{j=0}^{j_r} \Big(\inf_{2^{j-1}r<t\le 2^j r} t^{d+m} \Big) \cdot \Big(\inf_{2^{j-1}r<t\le 2^j r} \frac{\Phi(t)}{t^m} \Big)$$

$$\le c \frac{2^d \cdot 2^{(m)_+} \cdot 2^{(m+d)_+}}{\ln 2} \sum_{j=0}^{j_r} \int_{2^{j-1}r}^{2^j r} t^{d+m} \frac{\Phi(t)}{t^m} \frac{\mathrm{d}t}{t}$$

$$\le c \frac{2^d \cdot 2^{(m)_+} \cdot 2^{(m+d)_+}}{\ln 2} \int_{r/2}^{4\,\mathrm{diam}_\rho(X)} t^{d-1} \Phi(t)\,\mathrm{d}t,$$

proving (7.2.3) in this case. Suppose next that

$$\begin{gathered}\text{there exists } m \in \mathbb{R} \text{ with the property that} \\ (0,\infty) \ni t \mapsto \Phi(t)/t^m \in [0,\infty) \ \text{ is increasing.}\end{gathered} \tag{7.2.9}$$

In such a scenario, based on (7.2.9) and (7.2.1) we may write

$$\int_{X\setminus B_\rho(x,r)} \Phi\big(\rho(x,y)\big)\,\mathrm{d}\mu(y) \tag{7.2.10}$$

$$= \sum_{j=0}^{j_r} \int\limits_{\substack{y\in X \\ 2^j r\le \rho(x,y)<2^{j+1}r}} \rho(x,y)^m \left(\frac{\Phi\big(\rho(x,y)\big)}{\rho(x,y)^m} \right) \mathrm{d}\mu(y)$$

$$\le 2^{(-m)_+} \sum_{j=0}^{j_r} \int\limits_{\substack{y\in X \\ 2^j r\le \rho(x,y)<2^{j+1}r}} \big(2^{j+1}r\big)^m \left(\frac{\Phi\big(2^{j+1}r\big)}{(2^{j+1}r)^m} \right) \mathrm{d}\mu(y)$$

$$\le 2^{(-m)_+} \sum_{j=0}^{j_r} \big(2^{j+1}r\big)^m \left(\frac{\Phi\big(2^{j+1}r\big)}{(2^{j+1}r)^m} \right) \mu\big(B_\rho(x,2^{j+1}r)\big)$$

$$\le c2^{(-m)_+} \sum_{j=0}^{j_r} \big(2^{j+1}r\big)^d \big(2^{j+1}r\big)^m \left(\frac{\Phi\big(2^{j+1}r\big)}{(2^{j+1}r)^m} \right)$$

$$\le c2^{(-m)_+} \cdot 2^{(m+d)_+} \sum_{j=0}^{j_r} \Big(\inf_{2^{j+1}r\le t<2^{j+2}r} t^{d+m} \Big) \cdot \Big(\inf_{2^{j+1}r\le t<2^{j+2}r} \frac{\Phi(t)}{t^m} \Big)$$

$$\le c \frac{2^{(-m)_+} \cdot 2^{(m+d)_+}}{\ln 2} \sum_{j=0}^{j_r} \int_{2^{j+1}r}^{2^{j+2}r} t^{d+m} \frac{\Phi(t)}{t^m} \frac{\mathrm{d}t}{t}$$

$$\le c \frac{2^{(-m)_+} \cdot 2^{(m+d)_+}}{\ln 2} \int_{2r}^{16\,\mathrm{diam}_\rho(X)} t^{d-1} \Phi(t)\,\mathrm{d}t,$$

which implies (7.2.3) in this case. This finishes the proof of (7.2.3).

As regards (7.2.4), fix $x \in X$ and $r \in \big(0, 2\,\mathrm{diam}_\rho(X)\big)$. In the case when (7.2.7) holds, we write

$$\begin{aligned}
&\int_{B_\rho(x,r)} \Phi\big(\rho(x,y)\big)\,d\mu(y) \qquad (7.2.11)\\
&\quad = \sum_{j=0}^{\infty} \int_{\substack{y\in X\\ 2^{-j-1}r\le\rho(x,y)<2^{-j}r}} \rho(x,y)^m \left(\frac{\Phi\big(\rho(x,y)\big)}{\rho(x,y)^m}\right) d\mu(y)\\
&\quad \le 2^{(m)_+}\sum_{j=0}^{\infty} \int_{\substack{y\in X\\ 2^{-j-1}r\le\rho(x,y)<2^{-j}r}} \big(2^{-j-1}r\big)^m \left(\frac{\Phi\big(2^{-j-1}r\big)}{(2^{-j-1}r)^m}\right) d\mu(y)\\
&\quad \le 2^{(m)_+}\sum_{j=0}^{\infty} \big(2^{-j-1}r\big)^m \left(\frac{\Phi\big(2^{-j-1}r\big)}{(2^{-j-1}r)^m}\right) \mu\big(B_\rho(x,2^{-j}r)\big)\\
&\quad \le c2^d\cdot 2^{(m)_+}\sum_{j=0}^{\infty} \big(2^{-j-1}r\big)^d \big(2^{-j-1}r\big)^m \left(\frac{\Phi\big(2^{-j-1}r\big)}{(2^{-j-1}r)^m}\right)\\
&\quad \le c2^d\cdot 2^{(m)_+}\cdot 2^{(-m-d)_+}\sum_{j=0}^{\infty} \Big(\inf_{2^{-j-1}r\le t<2^{-j}r} t^{d+m}\Big)\cdot\Big(\inf_{2^{-j-1}r\le t<2^{-j}r} \frac{\Phi(t)}{t^m}\Big)\\
&\quad \le c\frac{2^d\cdot 2^{(m)_+}\cdot 2^{(-m-d)_+}}{\ln 2}\sum_{j=0}^{\infty}\int_{2^{-j-1}r}^{2^{-j}r} t^{d+m}\frac{\Phi(t)}{t^m}\frac{dt}{t}\\
&\quad = c\frac{2^d\cdot 2^{(m)_+}\cdot 2^{(-m-d)_+}}{\ln 2}\int_0^r t^{d-1}\Phi(t)\,dt,
\end{aligned}$$

which implies (7.2.4) in this case. Finally, when (7.2.9) holds we write

$$\begin{aligned}
&\int_{B_\rho(x,r)} \Phi\big(\rho(x,y)\big)\,d\mu(y) \qquad (7.2.12)\\
&\quad = \sum_{j=0}^{\infty} \int_{\substack{y\in X\\ 2^{-j-1}r\le\rho(x,y)<2^{-j}r}} \rho(x,y)^m \left(\frac{\Phi\big(\rho(x,y)\big)}{\rho(x,y)^m}\right) d\mu(y)\\
&\quad \le 2^{(m)_+}\sum_{j=0}^{\infty} \int_{\substack{y\in X\\ 2^{-j-1}r\le\rho(x,y)<2^{-j}r}} \big(2^{-j-1}r\big)^m \left(\frac{\Phi\big(2^{-j}r\big)}{(2^{-j}r)^m}\right) d\mu(y)\\
&\quad \le 2^{(m)_+}\sum_{j=0}^{\infty} \big(2^{-j-1}r\big)^m \left(\frac{\Phi\big(2^{-j}r\big)}{(2^{-j}r)^m}\right) \mu\big(B_\rho(x,2^{-j}r)\big)\\
&\quad \le c2^d\cdot 2^{(m)_+}\sum_{j=0}^{\infty} \big(2^{-j-1}r\big)^d \big(2^{-j-1}r\big)^m \left(\frac{\Phi\big(2^{-j}r\big)}{(2^{-j}r)^m}\right)\\
&\quad \le c2^d\cdot 2^{(m)_+}\cdot 2^{(-m-d)_+}\sum_{j=0}^{\infty} \Big(\inf_{2^{-j}r\le t<2^{-j+1}r} t^{d+m}\Big)\cdot\Big(\inf_{2^{-j}r\le t<2^{-j+1}r} \frac{\Phi(t)}{t^m}\Big)
\end{aligned}$$

$$\leq c\frac{2^d \cdot 2^{(m)_+} \cdot 2^{(-m-d)_+}}{\ln 2}\sum_{j=0}^{\infty}\int_{2^{-j}r}^{2^{-j+1}r} t^{d+m}\frac{\Phi(t)}{t^m}\frac{\mathrm{d}t}{t}$$
$$= c\frac{2^d \cdot 2^{(m)_+} \cdot 2^{(-m-d)_+}}{\ln 2}\int_0^{2r} t^{d-1}\Phi(t)\,\mathrm{d}t,$$

finishing the proof of (7.2.4). □

Here is another useful estimate of a similar nature but where the integral in question exhibits multiple point singularities.

Lemma 7.2.2 *Let (X, ρ) be a given quasi-metric space endowed with a quasi-distance $\rho : X \times X \to [0, \infty)$ which is continuous in the product topology $\tau_\rho \times \tau_\rho$. Assume μ is a Borel measure on (X, τ_ρ) with the property that there exist $d \in (0, \infty)$ and $c \in (0, \infty)$ such that $\mu\big(B_\rho(x, R)\big) \leq cR^d$ for all $x \in X$ and $R \in \big(0, 2\,\mathrm{diam}_\rho(X)\big)$. Then for each $M, N \in [0, d)$ satisfying $M + N > d$ one can fine a constant $C \in (0, \infty)$ for which*

$$\int_X \frac{d\mu(x)}{\rho(x,y)^M\rho(x,z)^N} \leq C\rho(y,z)^{d-M-N} \quad \textit{for all} \quad y, z \in X \quad \textit{with} \quad y \neq z. \tag{7.2.13}$$

Proof In view of Theorem 7.1.2 there is no loss of generality in assuming that ρ is actually a genuine distance on X. Assume this is the case, fix a pair or arbitrary distinct points $y, z \in X$, and abbreviate $r := \rho(y, z) \in \big(0, \mathrm{diam}_\rho(X)\big]$. Note that, thanks to (7.2.5),

$$\begin{aligned}\int_{B_\rho(y,r/2)} \frac{\mathrm{d}\mu(x)}{\rho(x,y)^M\rho(x,z)^N} &\leq (r/2)^{-N}\int_{B_\rho(y,r/2)}\frac{\mathrm{d}\mu(x)}{\rho(x,y)^M}\\ &\leq C(r/2)^{-N}(r/2)^{d-M} = C\rho(y,z)^{d-M-N},\end{aligned} \tag{7.2.14}$$

since $\rho(x, z) \geq \rho(y, z) - \rho(x, y) \geq r/2$ for each $x \in B_\rho(y, r/2)$. Likewise,

$$\int_{B_\rho(z,r/2)} \frac{\mathrm{d}\mu(x)}{\rho(x,y)^M\rho(x,z)^N} \leq C\rho(y,z)^{d-M-N}. \tag{7.2.15}$$

In addition, if $S := X \setminus \big(B_\rho(y, r/2) \cup B_\rho(z, r/2)\big)$ then

$$\begin{aligned}\int_{S\cap B_\rho(z,2r)} \frac{\mathrm{d}\mu(x)}{\rho(x,y)^M\rho(x,z)^N} &\leq (r/2)^{-M}(r/2)^{-N}\int_{B_\rho(z,2r)}\mathrm{d}\mu(x)\\ &\leq Cr^{d-M-N} = C\rho(y,z)^{d-M-N},\end{aligned} \tag{7.2.16}$$

since the growth assumption on the measure μ implies that $\mu\big(B_\rho(z, 2r)\big) \leq Cr^d$. Finally, whenever $x \in S \setminus B_\rho(z, 2r)$ we may estimate

$$\rho(x,y) \geq \rho(x,z) - \rho(y,z) \geq \rho(x,z) - \tfrac{1}{2}\rho(x,z) = \tfrac{1}{2}\rho(x,z) \tag{7.2.17}$$

as well as $\rho(x,y) \le \rho(x,z) + \rho(y,z) \le \frac{3}{2}\rho(x,z)$, hence

$$\int_{S\setminus B_\rho(z,2r)} \frac{\mathrm{d}\mu(x)}{\rho(x,y)^M \rho(x,z)^N} \le C \int_{X\setminus B_\rho(z,2r)} \frac{\mathrm{d}\mu(x)}{\rho(x,z)^{M+M}}$$
$$\le C r^{d-M-N} = C\rho(y,z)^{d-M-N}, \tag{7.2.18}$$

once again by (7.2.5). At this stage, (7.2.13) follows from (7.2.14)–(7.2.18). □

7.3 Hölder Spaces on Quasi-Metric Spaces

Given a quasi-metric space (X,ρ), for $U \subseteq X$ arbitrary set and $\alpha \in (0,\infty)$ arbitrary exponent, define the `homogeneous Hölder space of order` α on U as

$$\dot{\mathscr{C}}^\alpha(U,\rho) := \Big\{ f : U \to \mathbb{R} :\ \|f\|_{\dot{\mathscr{C}}^\alpha(U,\rho)} < +\infty \Big\}, \tag{7.3.1}$$

where $\|\cdot\|_{\dot{\mathscr{C}}^\alpha(U,\rho)}$ stands for the semi-norm defined as[2]

$$\|f\|_{\dot{\mathscr{C}}^\alpha(U,\rho)} := \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|f(x)-f(y)|}{\rho(x,y)^\alpha}. \tag{7.3.2}$$

Clearly,

$$\|f\|_{\dot{\mathscr{C}}^\alpha(U,\rho)} = 0 \iff f \text{ is constant on } U. \tag{7.3.3}$$

Given this, we find it occasionally useful to "mod out" the null-space of the semi-norm $\|\cdot\|_{\dot{\mathscr{C}}^\alpha(U,\rho)}$ in order to render the resulting quotient space genuinely normed. Specifically, for any two functions f, g defined on U write $f \sim g$ provided the difference $f - g$ is constant on U. This is an equivalence relation and we denote the equivalence class of any given function f defined on U by

$$[f] := \big\{ g : U \to \mathbb{R} :\ f \sim g \big\}. \tag{7.3.4}$$

If for any function f defined on U we now set

$$\big\| [f] \big\|_{\dot{\mathscr{C}}^\alpha(U,\rho)/\sim} := \|f\|_{\dot{\mathscr{C}}^\alpha(U,\rho)}, \tag{7.3.5}$$

then $\big\| [\cdot] \big\|_{\dot{\mathscr{C}}^\alpha(U,\rho)/\sim}$ becomes a genuine norm on the quotient space

$$\dot{\mathscr{C}}^\alpha(U,\rho)/\sim := \big\{ [f] :\ f \in \dot{\mathscr{C}}^\alpha(U,\rho) \big\}. \tag{7.3.6}$$

[2] With the convention that $\|f\|_{\dot{\mathscr{C}}^\alpha(U,\rho)} := 0$ when the cardinality of U is at most one.

In fact, when equipped with the norm (7.3.5), the space (7.3.6) is complete (hence Banach).

Occasionally, we shall work with the local version of the homogeneous Hölder space introduced in (7.3.1), namely

$$\dot{\mathscr{C}}^{\alpha}_{\rm loc}(U,\rho) := \Big\{ f : U \to \mathbb{C} : f\big|_{B_\rho(x,r)\cap U} \in \dot{\mathscr{C}}^{\alpha}\big(B_\rho(x,r)\cap U,\rho\big) \\ \text{for each } x \in U \text{ and } r \in (0,\infty)\Big\}. \tag{7.3.7}$$

The lower triangle inequality readily implies that for any function $f : U \to \mathbb{R}$ we have

$$\big\| \, |f| \, \big\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)} \leq \|f\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)}, \tag{7.3.8}$$

hence,

$$f \in \dot{\mathscr{C}}^{\alpha}(U,\rho) \Rightarrow |f| \in \dot{\mathscr{C}}^{\alpha}(U,\rho). \tag{7.3.9}$$

Given that $\max\{f,g\} = \frac{1}{2}(f+g+|f-g|)$ and $\min\{f,g\} = \frac{1}{2}(f+g-|f-g|)$, it follows from (7.3.9) that $\dot{\mathscr{C}}^{\alpha}(U,\rho)$ is a lattice and, for every $f, g \in \dot{\mathscr{C}}^{\alpha}(U,\rho)$,

$$\begin{aligned} \big\| \max\{f,g\}\big\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)} &\leq \|f\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)} + \|g\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)}, \\ \big\| \min\{f,g\}\big\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)} &\leq \|f\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)} + \|g\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)}. \end{aligned} \tag{7.3.10}$$

In particular, given any real-valued function f on U, if for each $N \in \mathbb{N}$ we define

$$f_N := \min\Big\{ \max\{f,-N\}, N\Big\} = \max\Big\{ \min\{f,N\}, -N\Big\}, \tag{7.3.11}$$

i.e.,

$$f_N(x) := \begin{cases} N & \text{if } f(x) > N \\ f(x) & \text{if } -N \leq f(x) \leq N, \\ -N & \text{if } f(x) < -N, \end{cases} \quad \forall x \in U, \tag{7.3.12}$$

then

$$\begin{aligned} &|f_N(x)| \leq \min\{|f(x)|, N\} \ \text{ for each } \ x \in U, \\ \text{and } &\lim_{N\to\infty} f_N(x) = f(x) \ \text{ for each point } x \in U. \end{aligned} \tag{7.3.13}$$

Moreover, from (7.3.11) and (7.3.10) we see that

$$\|f_N\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)} \leq \|f\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)}, \qquad \forall N \in \mathbb{N}. \tag{7.3.14}$$

In the same setting as above there is a log-convexity law for the homogeneous Hölder scale, to the effect that if $0 < \alpha_0 < \alpha_1 < \infty$ and $f \in \dot{\mathscr{C}}^{\alpha_0}(U,\rho) \cap \dot{\mathscr{C}}^{\alpha_1}(U,\rho)$ then $f \in \dot{\mathscr{C}}^{\eta}(U,\rho)$ for each $\eta \in [\alpha_0,\alpha_1]$ and

$$\|f\|_{\dot{\mathscr{C}}^{\eta}(U,\rho)} \le \|f\|^{1-\theta}_{\dot{\mathscr{C}}^{\alpha_0}(U,\rho)} \cdot \|f\|^{\theta}_{\dot{\mathscr{C}}^{\alpha_1}(U,\rho)} \quad \text{if } \theta \in [0,1] \text{ and } \eta \in [\alpha_0, \alpha_1] \text{ are such that } \eta = (1-\theta)\alpha_0 + \theta\alpha_1. \tag{7.3.15}$$

This is easily seen, by multiplying two inequalities, obtained by raising

$$|f(x) - f(y)| \le \rho(x,y)^{\alpha_j} \|f\|_{\dot{\mathscr{C}}^{\alpha_j}(U,\rho)} \quad \text{for all } \ x, y \in U, \tag{7.3.16}$$

to the power $1-\theta$ if $j=0$ and, respectively, the power θ if $j=1$. In fact, the same type of proof also allows the inclusion of the end-point $\alpha_0 = 0$ in the following format

$$\|f\|_{\dot{\mathscr{C}}^{\eta}(U,\rho)} \le 2^{1-\eta/\alpha}\big(\sup_U |f|\big)^{1-\eta/\alpha} \cdot \|f\|^{\eta/\alpha}_{\dot{\mathscr{C}}^{\alpha}(U,\rho)} \quad \text{if } \ 0 < \eta \le \alpha. \tag{7.3.17}$$

Indeed, this is seen by multiplying the two inequalities obtained by raising

$$\begin{aligned} |f(x) - f(y)| \le 2 \cdot \sup\nolimits_U |f| \ \text{ and} \\ |f(x) - f(y)| \le \rho(x,y)^{\alpha} \|f\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)} \ \text{ for all } \ x, y \in U, \end{aligned} \tag{7.3.18}$$

respectively, to the power $1 - \eta/\alpha$ and η/α.

Moving on, the `inhomogeneous Hölder space of order` α on U is defined as

$$\mathscr{C}^{\alpha}(U,\rho) := \big\{ f \in \dot{\mathscr{C}}^{\alpha}(U,\rho) : \ f \ \text{ is bounded in} U \big\}, \tag{7.3.19}$$

and is equipped with the norm

$$\|f\|_{\mathscr{C}^{\alpha}(U,\rho)} := \sup_U |f| + \|f\|_{\dot{\mathscr{C}}^{\alpha}(U,\rho)}, \qquad \forall f \in \mathscr{C}^{\alpha}(U,\rho). \tag{7.3.20}$$

When the quasi-distance ρ is naturally understood from the context we omit to drop it in the above notation (this is often the case when working with subsets of $\mathbb{R}^n$, in which case the Euclidean distance plays the role of ρ). Observe that

$$\mathscr{C}^{\alpha}(U,\rho) \subseteq \dot{\mathscr{C}}^{\alpha}(U,\rho), \ \text{with equality if the set } U \text{ happens to be bounded,} \tag{7.3.21}$$

and

$$\mathscr{C}^{\alpha}(U,\rho) \ \text{ is an algebra.} \tag{7.3.22}$$

In addition,

if $f, g \in \dot{\mathscr{C}}^{\alpha}(U,\rho)$ and g vanishes identically outside of a bounded subset of U then the product $f \cdot g$ belongs to $\mathscr{C}^{\alpha}(U,\rho)$ and a naturally accompanying estimate holds. (7.3.23)

For further reference we also wish to note that since functions in $\dot{\mathscr{C}}^\alpha(U,\rho)$ are uniformly continuous, it follows that

$$\begin{aligned}&\dot{\mathscr{C}}^\alpha(U,\rho)\text{ may be canonically identified with }\dot{\mathscr{C}}^\alpha(\overline{U},\rho),\\&\text{and }\mathscr{C}^\alpha(U,\rho)\text{ may be canonically identified with }\mathscr{C}^\alpha(\overline{U},\rho),\end{aligned}\tag{7.3.24}$$

where $\overline{U}$ denotes the closure of U in the topology τ_ρ.

It turns out that the inhomogeneous Hölder scale is nested, in a quantitative sense. Specifically, if $0<\alpha_0<\alpha_1<\infty$ and $f\in\mathscr{C}^{\alpha_1}(U,\rho)$ then $f\in\mathscr{C}^{\alpha_0}(U,\rho)$ and, with $\theta:=\alpha_0/\alpha_1\in(0,1)$,

$$\|f\|_{\mathscr{C}^{\alpha_0}(U,\rho)}\le\sup_U|f|+2^{1-\theta}\big(\sup_U|f|\big)^{1-\theta}\|f\|^\theta_{\dot{\mathscr{C}}^{\alpha_1}(U,\rho)}\le 3\|f\|_{\mathscr{C}^{\alpha_1}(U,\rho)}.\tag{7.3.25}$$

The first estimate in (7.3.25) is seen from (7.3.17), and the second estimate in (7.3.25) then follows on account of this and (7.3.20).

In the same setting as above, let us also agree to define

$$\mathscr{C}^\alpha_c(U,\rho):=\big\{f\in\dot{\mathscr{C}}^\alpha(U,\rho):f\text{ vanishes outside of a }\rho\text{-bounded subset of}U\big\}.\tag{7.3.26}$$

It is then clear from definitions that

$$\mathscr{C}^\alpha_c(U,\rho)=\big\{f\in\mathscr{C}^\alpha(U,\rho):f\text{ vanishes outside of a }\rho\text{-bounded subset of }U\big\}.\tag{7.3.27}$$

Lemma 7.3.1 *Let (X,ρ) be a metric space and let μ be a Borel measure on X with the property that there exist $c,d\in(0,\infty)$ such that*[3]

$$c\,r^d\le\mu\big(B_\rho(x,r)\big)\ \textit{ for all }\ x\in X\ \textit{ and all }\ r\in\big(0,\infty).\tag{7.3.28}$$

Then any function $f\in\dot{\mathscr{C}}^\alpha(X,\rho)\cap L^p(X,\mu)$ with $\alpha\in(0,\infty)$ and $p\in(0,\infty)$ is bounded and

$$\begin{aligned}&\sup\nolimits_{x\in X}|f(x)|\le 2c^{-\theta/p}\|f\|^{1-\theta}_{\dot{\mathscr{C}}^\alpha(X,\rho)}\cdot\|f\|^\theta_{L^p(X,\mu)}\\&\qquad\textit{where }\ \theta:=\big(1+d/(p\alpha)\big)^{-1}.\end{aligned}\tag{7.3.29}$$

As a consequence of (7.3.29), for any given $\alpha\in(0,\infty)$ and $p\in(0,\infty)$ there exists a constant $C=C(d,p,\alpha,c)\in(0,\infty)$ with the property that

$$\begin{aligned}&\sup\nolimits_{x\in X}|f(x)|\le C\big(\|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)}+\|f\|_{L^p(X,\mu)}\big)\\&\qquad\textit{for all }\ f\in\dot{\mathscr{C}}^\alpha(X,\rho)\cap L^p(X,\mu).\end{aligned}\tag{7.3.30}$$

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[3] Note that (7.3.28) entails $\mu(X)=+\infty$.

Proof Assume first that $\|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} = 0$. Then f is a constant on X, and since (7.3.28) entails $\mu(X) = +\infty$, the membership of f to $L^p(X, \mu)$ forces $f \equiv 0$ on X. In such a case, (7.3.29) is trivially true. Assume next that $\|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} > 0$. Fix an arbitrary point $x_o \in X$, with the goal of proving that

$$|f(x_o)| \leq 2c^{-\theta/p}\|f\|^{1-\theta}_{\dot{\mathscr{C}}^\alpha(X,\rho)} \cdot \|f\|^{\theta}_{L^p(X,\mu)}. \tag{7.3.31}$$

This obviously holds if $|f(x_o)| = 0$, so assume in what follows that $|f(x_o)| > 0$. Abbreviate

$$r := \Big(\frac{|f(x_o)|}{2\|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)}}\Big)^{\frac{1}{\alpha}} \in (0, \infty), \tag{7.3.32}$$

and note that for each $x \in B_\rho(x_o, r)$ we have

$$|f(x_o) - f(x)| \leq \|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \cdot \rho(x_o, x)^\alpha \leq \|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \cdot r^\alpha = |f(x_o)|/2. \tag{7.3.33}$$

Thus $|f(x_o)| \leq |f(x_o) - f(x)| + |f(x)| \leq |f(x_o)|/2 + |f(x)|$ hence, further,

$$|f(x_o)|/2 \leq |f(x)| \quad \text{for each} \quad x \in B_\rho(x_o, r). \tag{7.3.34}$$

In concert with (7.3.28), this permits us to estimate

$$\begin{aligned} \frac{|f(x_o)|}{2} \cdot r^{d/p} &\leq \frac{|f(x_o)|}{2c^{1/p}} \cdot \mu\big(B_\rho(x_o, r)\big)^{1/p} \\ &\leq c^{-1/p}\Big(\int_{B_\rho(x_o,r)} |f(x)|^p \,\mathrm{d}\mu(x)\Big)^{1/p} \leq c^{-1/p}\|f\|_{L^p(X,\mu)}. \end{aligned} \tag{7.3.35}$$

In turn, from (7.3.32) and (7.3.35) we readily conclude that (7.3.31) holds. Now, (7.3.29) follows from (7.3.31) after taking the supremum over $x_o \in X$.

Finally, the estimate claimed in (7.3.30) is a direct consequence of (7.3.29) and the sequence of inequalities $a^{1-\theta} \cdot b^\theta \leq \max\{a, b\} \leq a + b$ where $a, b \in [0, \infty)$ and $\theta \in (0, 1)$. $\square$

7.4 Functions of Bounded Mean Oscillations on Spaces of Homogeneous Type

To set the stage, recall the definition of a space of homogeneous type, introduced by R. Coifman and G. Weiss in [57]. A `space of homogeneous type` is a triplet (X, ρ, μ), where the ambient X is an arbitrary set, ρ is a quasi-distance on X, and μ is a `doubling measure` on X, i.e.,

μ is a positive measure on a sigma-algebra containing all ρ-balls in X with the property that there exists some constant $C \in [1, \infty)$ so that $0 < \mu\big(B_\rho(x, 2r)\big) \leq C\mu\big(B_\rho(x, r)\big) < \infty$ for all $x \in X$, $r \in (0, \infty)$. (7.4.1)

We emphasize that

a doubling measure μ on a quasi-metric space (X, ρ) is Borel (relative to the topological space (X, τ_ρ)), locally finite, sigma-finite, as well as both strictly positive and finite on ρ-balls (7.4.2)

(cf. the discussion pertaining to [11, (3.10), p. 74]). Whenever the measure μ is doubling, the number

$$D_\mu := \log_2 \left[\sup_{x \in X,\, r > 0} \frac{\mu(B_\rho(x, 2r))}{\mu(B_\rho(x, r))} \right] \in [0, \infty) \tag{7.4.3}$$

is referred to as the `doubling order` of μ. Iterating (7.4.1) then shows that there exists some constant $C_\mu \in (0, \infty)$ with the property that

$$1 \leq \frac{\mu(B_1)}{\mu(B_2)} \leq C_\mu \Big(\frac{\text{radius of } B_1}{\text{radius of } B_2} \Big)^{D_\mu}, \quad \text{for all } \rho\text{-balls } \; B_2 \subseteq B_1 \subseteq X. \tag{7.4.4}$$

Example 7.4.1 *Given an arbitrary closed set $\Sigma \subseteq \mathbb{R}^n$ which is Ahlfors regular, denote by $C, c \in (0, \infty)$ its upper and lower Ahlfors regularity constants (cf. Definition 5.9.1), and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \Sigma$. Then for each dilation factor $\lambda \in [1, \infty)$, each point $x \in \Sigma$, and each radius $r \in (0, \infty)$ one has*

$$0 < \sigma\big(B(x, \lambda r) \cap \Sigma\big) \leq \Big(\frac{C}{c}\Big)\lambda^{n-1} \cdot \sigma\big(B(x, r) \cap \Sigma\big) < +\infty. \tag{7.4.5}$$

Indeed, in the regime $r \in \big(0, 2\,\mathrm{diam}(\Sigma)\big)$ this follows from (5.9.1)–(5.9.2), while in the case when Σ is bounded and $r > \mathrm{diam}(\Sigma)$ the claim in (7.4.5) is obviously true (since $C \geq c$ and $\lambda \geq 1$). In particular, (7.4.5) shows that $\big(\Sigma, |\cdot - \cdot|, \sigma\big)$ is a space of homogeneous type.

It is also of interest to note that (cf., e.g., [188])

for any space of homogeneous type (X, ρ, μ), the topological space (X, τ_ρ) is separable. (7.4.6)

We shall use this fact in the lemma below to show that, given any space of homogeneous type, the underlying measure space is non-atomic if and only if the corresponding measure does not charge singletons.

Lemma 7.4.2 *If (X, ρ, μ) is a space of homogeneous type with the property that μ does not charge singletons.*[4] *Then the measure space (X, μ) is non-atomic.*[5]

Proof As noted in (7.4.6), the topological space (X, τ_ρ) is separable, so there exists a dense set $\{x_j\}_{j\in\mathbb{N}} \subseteq X$. Arrange the countable family of ρ-balls with centers $\{x_j\}_{j\in\mathbb{N}}$ and radii in $\mathbb{Q}_+$ as a sequence $\{B_j\}_{j\in\mathbb{N}} \subseteq \mathfrak{M}$. Seeking a contradiction, assume $A \in \mathfrak{M}$ is an atom. Fix $x_* \in X$ and note that $\mu\big(A \cap B_\rho(x_*, j)\big) \nearrow \mu(A) > 0$ as $j \nearrow \infty$. Thus, there exists $j_* \in \mathbb{N}$ so that $\mu\big(A \cap B_\rho(x_*, j)\big) > 0$. Since $A \cap B_\rho(x_*, j_*) \in \mathfrak{M}$ is a subset of A with $0 < \mu\big(A \cap B_\rho(x_*, j_*)\big) \leq \mu\big(B_\rho(x_*, j)\big) < \infty$, the fact that A is an atom forces $\mu(A) = \mu\big(A \cap B_\rho(x_*, j_*)\big) \leq \mu\big(B_\rho(x_*, j)\big) < \infty$. This proves that $0 < \mu(A) < \infty$.

Next, for each $j \in \mathbb{N}$, the fact that A is an atom ensures that either $\mu(A \cap B_j) = 0$, or $\mu(A \cap B_j) = \mu(A)$. In the latter case we have $\mu(A \setminus B_j) = 0$ since $\mu(A) < \infty$. Hence, if for each $j \in \mathbb{N}$ we set $E_j := X \setminus B_j$ when $\mu(A \cap B_j) = 0$, and $E_j := B_j$ when $\mu(A \cap B_j) = \mu(A)$, it follows that $E_j \in \mathfrak{M}$ satisfies $\mu(A \setminus E_j) = 0$ for each $j \in \mathbb{N}$. Define

$$E := A \cap \Big(\bigcap_{j\in\mathbb{N}} E_j\Big). \tag{7.4.7}$$

Then $E \in \mathfrak{M}$ is a subset of A and

$$\begin{aligned} A \setminus E &= A \setminus \Big(\bigcap_{j\in\mathbb{N}} E_j\Big) = A \cap \Big(\bigcap_{j\in\mathbb{N}} E_j\Big)^c = A \cap \Big(\bigcup_{j\in\mathbb{N}} E_j^c\Big) \\ &= \bigcup_{j\in\mathbb{N}} (A \cap E_j^c) = \bigcup_{j\in\mathbb{N}} (A \setminus E_j), \end{aligned} \tag{7.4.8}$$

so $\mu(A \setminus E) \leq \sum_{j\in\mathbb{N}} \mu(A \setminus E_j) = 0$. In view of the fact that $\mu(A) < \infty$, this forces $\mu(E) = \mu(A) \in (0, \infty)$ and, since A is an atom, we ultimately conclude that E is an atom. Suppose E is not a singleton. Then there exist $x, y \in E$ with $x \neq y$. Upon recalling the nature of $\{B_j\}_{j\in\mathbb{N}}$, we see that there exists $j_o \in \mathbb{N}$ such that $x \in B_{j_o}$ but $y \notin B_{j_o}$. However, since $x, y \in E_{j_o}$, the former implies $E_{j_o} = B_{j_o}$, while the latter implies $E_{j_o} = X \setminus B_{j_o}$, an impossibility. Thus E is necessarily a singleton. Given that, by assumption, μ does not charge singletons, we then have $\mu(E) = 0$. This stands in contradiction with the fact that, as noted earlier, $\mu(E) \in (0, \infty)$. □

For future references let us also note in the lemma below that suitably weighted non-atomic sigma-finite measure spaces continue to be non-atomic sigma-finite.

Lemma 7.4.3 *Let $(X, \mathfrak{M}, \mu)$ be a non-atomic measure space with the property that there exists a family $(X_j)_{j\in\mathbb{N}} \subseteq \mathfrak{M}$ with $\mu(X_j) < \infty$ and $X = \bigcup_{j\in\mathbb{N}} X_j$ (hence*

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[4] I.e., $\mu(\{x\}) = 0$ for each $x \in X$.

[5] That is, there are no sets $A \in \mathfrak{M}$ with $\mu(A) > 0$ such that either $\mu(E) = 0$ or $\mu(E) = \mu(A)$ for any $E \in \mathfrak{M}$ with $E \subseteq A$.

$(X, \mathfrak{M}, \mu)$ is also sigma-finite). Suppose $w : X \to [0, \infty]$ is a μ-measurable function satisfying $w > 0$ at σ-a.e. point in X and $\int_{X_j} w \, \mathrm{d}\mu < +\infty$ for each $j \in \mathbb{N}$. Then $(X, \mathfrak{M}, w\mu)$ is a non-atomic sigma-finite measure space as well.

Proof From assumptions, $0 < w < \infty$ at σ-a.e. point in X, so w is a weight function. Let us identify this weight with the measure $w\mu$, i.e., set $dw := w \, \mathrm{d}\mu$. In view of the fact that $w(X_j) = \int_{X_j} w \, \mathrm{d}\mu < +\infty$ for each $j \in \mathbb{N}$, we conclude that the measure space $(X, \mathfrak{M}, w)$ is sigma-finite. There remains to show that this measure space is also non-atomic. To this end, assume $A \in \mathfrak{M}$ is a set with $w(A) > 0$. The latter forces $\mu(A) > 0$ and, since $(X, \mathfrak{M}, \mu)$ is non-atomic, we conclude that there exists $E \in \mathfrak{M}$ with $E \subseteq A$ and $0 < \mu(E) < \mu(A)$. Hence, $w(E) > 0$. If for each $N \in \mathbb{N}$ we define $E_N := E \cap \big(\bigcup_{j=1}^N X_j\big)$, then $E_N \in \mathfrak{M}$ and $E_N \subseteq E_{N+1} \subseteq E \subseteq A$ for each $N \in \mathbb{N}$, and $w(E_N) \nearrow w(E) > 0$ as $N \to \infty$. The latter property guarantees that $w(E_N) > 0$ for all N's large. If there exists such a number N for which we also have $w(E_N) < w(A)$, then A cannot be an atom, and we are done. Seeking a contradiction, assume $w(E_N) = w(A)$ for all N large. From this and $w(E_N) < \infty$ we thus obtain $w(A \setminus E_N) = 0$ for all N large, hence also $\mu(A \setminus E_N) = 0$ for all N large. In turn, this permits us to write $\mu(A) = \mu(A \setminus E_N) + \mu(E_N) = \mu(E_N)$ for all N large. Passing now to limit $N \to \infty$ then yields $\mu(A) = \mu(E)$, which stands in contradiction with the fact that $\mu(E) < \mu(A)$. □

Moving on, in the context of a generic space of homogeneous type (X, ρ, μ), given any function $f \in L^1_{\mathrm{loc}}(X, \mu)$ we agree to abbreviate

$$f_{B_\rho(x,r)} := \fint_{B_\rho(x,r)} f \, \mathrm{d}\mu := \frac{1}{\mu\big(B_\rho(x,r)\big)} \int_{B_\rho(x,r)} f(y) \, \mathrm{d}\mu(y), \tag{7.4.9}$$

for each $x \in X$ and $r > 0$. In the next proposition we record a sharp version of Lebesgue's Differentiation Theorem.

Proposition 7.4.4 *Let (X, ρ, μ) be a space of homogeneous type. Recall (7.3.26) and denote by $\mathscr{C}^0_c(X, \tau_\rho)$ the space of functions $f : X \to \mathbb{R}$ which are continuous with respect to the topology τ_ρ and vanish outside of a ρ-ball in X. Then the following conditions are equivalent:*

(1) the measure μ is Borel-semiregular on (X, τ_ρ);
(2) for every function $f \in L^1_{\mathrm{loc}}(X, \mu)$ one has

$$\lim_{r \to 0^+} \fint_{B_\rho(x,r)} |f(y) - f(x)| \, \mathrm{d}\mu(y) = 0 \quad \textit{for } \mu\textit{-almost every } x \in X; \tag{7.4.10}$$

(3) for every function $f \in L^1_{\mathrm{loc}}(X, \mu)$ there holds

$$\lim_{r \to 0^+} \fint_{B_\rho(x,r)} f \, \mathrm{d}\mu = f(x) \quad \textit{for } \mu\textit{-almost every } x \in X; \tag{7.4.11}$$

(4) for some (or all) $p \in (0, \infty)$ one has

$$\mathscr{C}_c^0(X, \tau_\rho) \hookrightarrow L^p(X, \mu) \ \ \textit{densely;} \tag{7.4.12}$$

(5) for any finite $\beta \in \big(0, (\log_2 C_\rho)^{-1}\big]$ (where the number $C_\rho \in [1, \infty)$ is associated with ρ as in (7.1.4)) and some (or all) $p \in (0, \infty)$ one has

$$\mathscr{C}_c^\beta(X, \rho) \hookrightarrow L^p(X, \mu) \ \ \textit{densely.} \tag{7.4.13}$$

Proof Bring in the regularized version $\rho_\#$ of the original quasi-distance ρ, constructed in Theorem 7.1.2. From (7.1.21) we see that

$$\begin{gathered} B_{\rho_\#}(x, C_\rho^{-2} r) \subseteq B_\rho(x, r) \subseteq B_{\rho_\#}(x, \widetilde{C}_\rho r), \\ \text{for every point } x \in X \text{ and every radius } r > 0. \end{gathered} \tag{7.4.14}$$

Based on this and the fact that the measure μ is doubling, we then conclude that there exist two constants $c, C \in (0, \infty)$ with the property that for every function $f \in L^1_{\rm loc}(X, \mu)$, every point $x \in X$, and every radius $r > 0$ we have

$$c \fint_{B_{\rho_\#}(x, C_\rho^{-2} r)} |f| \, d\mu \leq \fint_{B_\rho(x,r)} |f| \, d\mu \leq C \fint_{B_{\rho_\#}(x, \widetilde{C}_\rho r)} |f| \, d\mu. \tag{7.4.15}$$

In particular, for each function $f \in L^1_{\rm loc}(X, \mu)$ and each fixed point $x \in X$,

$$\lim_{r \to 0^+} \fint_{B_\rho(x,r)} |f(y) - f(x)| \, d\mu(y) = 0 \iff \lim_{r \to 0^+} \fint_{B_{\rho_\#}(x,r)} |f(y) - f(x)| \, d\mu(y) = 0 \tag{7.4.16}$$

and, as a consequence,

$$\lim_{r \to 0^+} \fint_{B_\rho(x,r)} f \, d\mu = f(x) \iff \lim_{r \to 0^+} \fint_{B_{\rho_\#}(x,r)} f \, d\mu = f(x). \tag{7.4.17}$$

Having established these equivalences, all desired conclusions are now provided by [11, Theorem 3.14, p. 93]. □

Proposition 7.4.4 is frequently employed in the future. For now, we use it to prove the following density result.

Lemma 7.4.5 *Suppose (X, ρ, μ) is a space of homogeneous type with the property that the measure μ is Borel-semiregular on (X, τ_ρ). Fix a finite number $\beta \in \big(0, (\log_2 C_\rho)^{-1}\big]$ along with an integrability exponent $p \in (0, \infty)$. Then for each μ-measurable set $E \subseteq X$ it follows that*

$$\mathscr{C}_c^\beta(E, \rho) \hookrightarrow L^p(E, \mu) \ \ \textit{densely.} \tag{7.4.18}$$

Proof Given some μ-measurable set $E \subseteq X$, for each $f \in L^p(E, \mu)$ apply Proposition 7.4.4 to the function

$$\widetilde{f} := \begin{cases} f & \text{in } E, \\ 0 & \text{in } X \setminus E, \end{cases} \tag{7.4.19}$$

to conclude that there exists a sequence $\{\phi_j\}_{j\in\mathbb{N}} \subseteq \mathscr{C}_c^\beta(X, \rho)$ which converges to $\widetilde{f}$ in $L^p(X, \mu)$. Then $\{\phi_j|_E\}_{j\in\mathbb{N}} \subseteq \mathscr{C}_c^\beta(E, \rho)$ is a sequence convergent to f in $L^p(E, \mu)$, so (7.4.18) is established. □

In turn, Lemma 7.4.5 plays a role in the proof of the density result reminiscent of the fact that test functions (i.e., smooth compactly supported functions) are dense in Lebesgue spaces in open subsets of the Euclidean ambient.

Lemma 7.4.6 *Assume (X, ρ, μ) is a space of homogeneous type with the property that the measure μ is Borel-semiregular on (X, τ_ρ). Fix a finite number $\beta \in \big(0, (\log_2 C_\rho)^{-1}\big]$ along with an open set O in (X, τ_ρ). Denote by $\mathscr{C}_0^\beta(O, \rho)$ the collection of functions $f \in \mathscr{C}^\beta(O, \rho)$ with the property that there exists a ρ-bounded subset K of O such that* $\mathrm{dist}_\rho(K, X \setminus O) > 0$ *and $f \equiv 0$ on $O \setminus K$. Then for each integrability exponent $p \in (0, \infty)$ it follows that*

$$\mathscr{C}_0^\beta(O, \rho) \hookrightarrow L^p(O, \mu) \ \textit{ densely}. \tag{7.4.20}$$

Proof By eventually replacing ρ by $\rho_\#$ (cf. Theorem 7.1.2), there is no loss of generality in assuming that the quasi-distance $\rho : X \times X \to [0, \infty)$ is continuous in the product topology $\tau_\rho \times \tau_\rho$. Fix an arbitrary function $f \in L^p(O, \mu)$ together with a reference point $x_* \in X$. Since $f\mathbf{1}_{O\cap B_\rho(x_*, j)} \to f$ in $L^p(O, \mu)$ as $j \to \infty$ by Lebesgue's Dominated Convergence Theorem, it suffices to prove (7.4.20) with O replaced by $O \cap B_\rho(x_*, j)$ for each fixed $j \in \mathbb{N}$. In other words, there is no loss of generality in assuming that the open set O is also ρ-bounded.

Pick an arbitrary threshold $\delta > 0$. Then Lemma 7.4.5 guarantees that there exists $g \in \mathscr{C}_c^\beta(O, \rho)$ such that $\|f - g\|_{L^p(O,\mu)} < \delta/2$. To proceed, for each $\varepsilon > 0$ define

$$O_\varepsilon := \big\{x \in O : \ \mathrm{dist}_\rho(x, X \setminus O) > \varepsilon\big\}. \tag{7.4.21}$$

Then according to the version of Urysohn's lemma proved in [188, Theorem 4.12, p. 165] for each $\varepsilon > 0$ there exists $\psi_\varepsilon \in \dot{\mathscr{C}}^\beta(X, \rho)$ satisfying

$$0 \le \psi_\varepsilon \le 1 \ \text{ on } \ X, \quad \psi_\varepsilon \equiv 1 \ \text{ on } \ O_\varepsilon, \quad \psi_\varepsilon \equiv 0 \ \text{ on } \ X \setminus O_{\varepsilon/2}. \tag{7.4.22}$$

In particular, $\psi_\varepsilon \in \mathscr{C}^\beta(X, \rho)$ given that we are presently assuming that O is ρ-bounded (cf. (7.3.21)). Then $g\psi_\varepsilon$ belongs to $\mathscr{C}_0^\beta(O, \rho)$ for each $\varepsilon > 0$, and $g\psi_\varepsilon \to g$ in $L^p(O, \mu)$ as $\varepsilon \to 0^+$ by Lebesgue's Dominated Convergence Theorem. Hence, $\|f - g\psi_\varepsilon\|_{L^p(O,\mu)} < \delta$ if $\varepsilon > 0$ is small enough, and the desired conclusion follows. □

We next concern ourselves with the following characterization of Hölder continuity in terms of integral mean oscillations. Results similar in spirit may be found in [169, Theorem 4, p. 259], [96, Theorem 3.3, p. 292].

Proposition 7.4.7 *Let (X, ρ, μ) be a space of homogeneous type with the property that the measure μ is Borel-semiregular on (X, τ_ρ). Fix an integrability exponent $p \in [1, \infty)$ along with some number $\alpha \in (0, \infty)$. Then there exists some constant $C \in (0, \infty)$ which depends only on the ambient with the following significance: for each $f \in L^1_{\rm loc}(X, \mu)$ satisfying*

$$M_{f,p,\alpha} := \sup_{r>0} \sup_{\substack{B \subset X \\ \rho\text{-ball of radius } r}} \left\{ r^{-\alpha} \Big(⨍_B \Big| f - ⨍_B f \, d\mu \Big|^p d\mu \Big)^{1/p} \right\} < +\infty \tag{7.4.23}$$

one can find a function $\widetilde{f} \in \dot{\mathscr{C}}^\alpha(X, \rho)$ which coincides with f at μ-a.e. point in X and has the property that

$$\|\widetilde{f}\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \leq C M_{f,p,\alpha}. \tag{7.4.24}$$

In the opposite direction, there exists a constant $C \in (0, \infty)$ which depends only on ρ with the property that for any function $f \in \dot{\mathscr{C}}^\alpha(X, \rho)$ and each ρ-ball $B \subseteq X$ of radius r one has

$$r^{-\alpha} \Big(⨍_B \Big| f - ⨍_B f \, d\mu \Big|^p d\mu \Big)^{1/p} \leq C \|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)}. \tag{7.4.25}$$

Proof To set the stage, we introduce a piece of notation and make an observation. Specifically, given any function $f \in L^1_{\rm loc}(X, \mu)$ we agree to abbreviate

$$f_{B_\rho(x,r)} := ⨍_{B_\rho(x,r)} f \, d\mu \quad \text{for each } x \in X \text{ and } r > 0. \tag{7.4.26}$$

In this regard, we observe that if a function $f \in L^1_{\rm loc}(X, \mu)$ satisfies (7.4.23) then for each $x \in X$, $r \in (0, \infty)$, and $R \in (r, 2r)$ we may employ (7.4.23) to write

$$\begin{aligned} \big| f_{B_\rho(x,r)} - f_{B_\rho(x,R)} \big| &= \Big| ⨍_{B_\rho(x,r)} \big(f - f_{B_\rho(x,R)} \big) \, d\mu \Big| \leq ⨍_{B_\rho(x,r)} \big| f - f_{B_\rho(x,R)} \big| \, d\mu \\ &= \frac{1}{\mu\big(B_\rho(x,r)\big)} \int_{B_\rho(x,r)} \big| f - f_{B_\rho(x,R)} \big| \, d\mu \\ &\leq \frac{\mu\big(B_\rho(x,2r)\big)}{\mu\big(B_\rho(x,r)\big)} ⨍_{B_\rho(x,R)} \big| f - f_{B_\rho(x,R)} \big| \, d\mu \\ &\leq C R^\alpha \cdot M_{f,p,\alpha}, \end{aligned} \tag{7.4.27}$$

where $C \in [1, \infty)$ is the doubling constant of μ (cf. (7.4.1)). To proceed, denote by L_f the set of all Lebesgue points of f, i.e.,

$$L_f := \Big\{x \in X : \lim_{r \to 0^+} \fint_{B_\rho(x,r)} f(y)\, \mathrm{d}\mu(y) = f(x)\Big\}. \tag{7.4.28}$$

From Proposition 7.4.4 we know that $N_f := X \setminus L_f$ is a nullset for the measure μ. In particular,

$$L_f \text{ is a dense subset of } X, \text{ in the topology } \tau_\rho. \tag{7.4.29}$$

Next, pick two distinct points $x_1, x_2 \in X \setminus N_f = L_f$ and set $R := \rho(x_1, x_2) > 0$. Fix some $r \in (0, R)$ and define $j_* := \min\big\{j \in \mathbb{N} : 2^j r \geq R\big\}$. Note that the later choice entails

$$2^{j_*} r \geq R > 2^{j_*-1} r, \quad \text{hence also} \;\; 2R > 2^{j_*} r. \tag{7.4.30}$$

Based on this and (7.4.27) we may then estimate

$$\begin{aligned}
\big|f_{B_\rho(x_1,r)} - f_{B_\rho(x_1,2R)}\big| &\leq \sum_{j=1}^{j_*-1} \big|f_{B_\rho(x_1,2^j r)} - f_{B_\rho(x_1,2^{j+1} r)}\big| + \big|f_{B_\rho(x_1,2^{j_*} r)} - f_{B_\rho(x_1,2R)}\big| \\
&\leq C\Big(\sum_{j=1}^{j_*} (2^j r)^\alpha\Big) \cdot M_{f,p,\alpha} \leq C(2^{j_*} r)^\alpha \cdot M_{f,p,\alpha} \\
&\leq C R^\alpha \cdot M_{f,p,\alpha} = C\rho(x_1, x_2)^\alpha \cdot M_{f,p,\alpha},
\end{aligned} \tag{7.4.31}$$

given the significance of R. After sending $r \to 0^+$ we arrive at

$$\big|f(x_1) - f_{B_\rho(x_1,2R)}\big| \leq C\rho(x_1, x_2)^\alpha \cdot M_{f,p,\alpha}, \tag{7.4.32}$$

on account of Proposition 7.4.4 (here is where we use the fact that μ is Borel-semiregular). Likewise, we obtain

$$\big|f(x_2) - f_{B_\rho(x_2,2R)}\big| \leq C\rho(x_1, x_2)^\alpha \cdot M_{f,p,\alpha}. \tag{7.4.33}$$

Going further, since for each $y \in B_\rho(x_1, 2R)$ we have

$$\rho(y, x_2) \leq C_\rho \cdot \max\big\{\rho(y, x_1), \rho(x_1, x_2)\big\} = 2C_\rho R \tag{7.4.34}$$

it follows that

$$B_\rho(x_1, 2R) \subseteq B_\rho(x_2, 2C_\rho R) \tag{7.4.35}$$

and, similarly,

$$B_\rho(x_2, 2C_\rho R) \subseteq B_\rho(x_1, \lambda R) \;\; \text{where} \;\; \lambda := C_\rho^2 \cdot \max\{2, \widetilde{C}_\rho\}. \tag{7.4.36}$$

Bearing these inclusions in mind, we may now estimate

$$\big|f_{B_\rho(x_1,2R)} - f_{B_\rho(x_2,2R)}\big| \le \mathrm{I} + \mathrm{II}, \tag{7.4.37}$$

where, reasoning much as in (7.4.27),

$$\mathrm{I} := \big|f_{B_\rho(x_2,2R)} - f_{B_\rho(x_2,2C_\rho R)}\big| \le C\rho(x_1,x_2)^\alpha \cdot M_{f,p,\alpha}, \tag{7.4.38}$$

and

$$\begin{aligned}
\mathrm{II} &:= \big|f_{B_\rho(x_2,2C_\rho R)} - f_{B_\rho(x_1,2R)}\big| \\
&= \Big|⨍_{B_\rho(x_1,2R)} \big(f - f_{B_\rho(x_2,2C_\rho R)}\big)\,\mathrm{d}\mu\Big| \le ⨍_{B_\rho(x_1,2R)} \big|f - f_{B_\rho(x_2,2C_\rho R)}\big|\,\mathrm{d}\mu \\
&= \frac{1}{\mu\big(B_\rho(x_1,2R)\big)} \int_{B_\rho(x_1,2R)} \big|f - f_{B_\rho(x_2,2C_\rho R)}\big|\,\mathrm{d}\mu \\
&\le \frac{\mu\big(B_\rho(x_2,2C_\rho R)\big)}{\mu\big(B_\rho(x_1,2R)\big)} ⨍_{B_\rho(x_2,2C_\rho R)} \big|f - f_{B_\rho(x_1,2C_\rho R)}\big|\,\mathrm{d}\mu \\
&\le \frac{\mu\big(B_\rho(x_1,\lambda R)\big)}{\mu\big(B_\rho(x_1,2R)\big)} ⨍_{B_\rho(x_2,2C_\rho R)} \big|f - f_{B_\rho(x_1,2C_\rho R)}\big|\,\mathrm{d}\mu \\
&\le CR^\alpha \cdot M_{f,p,\alpha}.
\end{aligned} \tag{7.4.39}$$

Gathering (7.4.32), (7.4.33), and (7.4.37)–(7.4.39) leads to the conclusion that

$$\big|f(x_1) - f(x_2)\big| \le C\rho(x_1,x_2)^\alpha \cdot M_{f,p,\alpha}. \tag{7.4.40}$$

In view of the arbitrariness of $x_1, x_2 \in L_f$, the reasoning so far shows that

$$f \in \dot{\mathscr{C}}^\alpha(L_f,\rho) \quad\text{and}\quad \|f\|_{\dot{\mathscr{C}}^\alpha(L_f,\rho)} \le CM_{f,p,\alpha}. \tag{7.4.41}$$

Since L_f is a dense subset of X in the topology τ_ρ (cf. (7.4.29)), we conclude from (7.4.41) and (7.3.24) that there exists a unique function

$$\widetilde{f} \in \dot{\mathscr{C}}^\alpha(L_f,\rho) \quad\text{with}\quad \|\widetilde{f}\|_{\dot{\mathscr{C}}^\alpha(L_f,\rho)} \le CM_{f,p,\alpha} \quad\text{and}\quad \widetilde{f}\big|_{L_f} = f. \tag{7.4.42}$$

In particular, $\widetilde{f} = f$ at μ-a.e. point in X. This completes the proof of the first claim in the statement.

As regards the second claim, assume the measure μ does not charge singletons and fix a function $f \in \dot{\mathscr{C}}^\alpha(X,\rho)$. In particular, f is continuous with respect to the topology τ_ρ, hence f is μ-measurable, since μ is a Borel measure (cf. (7.4.2)). As such, for each given ρ-ball $B \subseteq X$ of radius r it is meaningful to write

$$\Big(\fint_B \Big|f - \fint_B f\,\mathrm{d}\mu\Big|^p\,\mathrm{d}\mu\Big)^{1/p} \leq \Big(\mu(B)^{-2}\int_B\int_B |f(x)-f(y)|^p\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\Big)^{1/p}$$
$$\leq \|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)}\cdot\big(C_\rho\tilde{C}_\rho\cdot r\big)^\alpha, \tag{7.4.43}$$

thanks to Hölder's inequality and the fact that $\rho(x,y) < C_\rho\tilde{C}_\rho\cdot r$ for each pair of points $x, y \in B$. Now (7.4.25) readily follows from (7.4.43). □

The next proposition contains a Hölder regularity criterion which prefigures the duality result between Hardy spaces and Hölder spaces (cf. [185, Sect. 4.6]).

Proposition 7.4.8 *Let (X,ρ,μ) be a space of homogeneous type with the property that the measure μ is Borel-semiregular on (X,τ_ρ) and there exist $C, d \in (0,\infty)$ such that*

$$\mu\big(B_\rho(x,r)\big) \leq Cr^d \ \text{ for all } x \in X \text{ and } r \in (0,\infty). \tag{7.4.44}$$

Fix an integrability exponent $p \in (0,1)$ and define $\alpha := d\big(\frac{1}{p}-1\big) \in (0,\infty)$. Then there exists a constant $C \in (1,\infty)$ which depends only on the ambient with the following significance. Suppose $f \in L^1_{\mathrm{loc}}(X,\mu)$ is such that

$$M_f := \sup_g \Big|\int_X fg\,\mathrm{d}\mu\Big| < +\infty, \tag{7.4.45}$$

where the supremum is taken over all μ-measurable functions $g : X \to \mathbb{R}$ for which there exists a ρ-ball B in X such that

$$\operatorname{supp} g \subseteq B, \quad \|g\|_{L^\infty(X,\mu)} \leq \mu(B)^{-1/p}, \quad \int_X g\,\mathrm{d}\mu = 0. \tag{7.4.46}$$

Then there exists a function $\widetilde{f} \in \dot{\mathscr{C}}^\alpha(X,\rho)$ which coincides with f at μ-a.e. point in X and has the property that

$$\|\widetilde{f}\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \leq C\cdot M_f. \tag{7.4.47}$$

Finally, if in place of (7.4.44) one now assumes that there exist $c, d \in (0,\infty)$ such that

$$c\,r^d \leq \mu\big(B_\rho(x,r)\big) \ \text{ for all } x \in X \text{ and } r \in \big(0, \operatorname{diam}_\rho(X)\big), \tag{7.4.48}$$

then for each function $f \in \dot{\mathscr{C}}^\alpha(X,\rho)$ one has

$$\Big|\int_X fg\,\mathrm{d}\mu\Big| \leq C\|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \tag{7.4.49}$$

for each $g \in L^\infty(X,\mu)$ enjoying the properties listed in (7.4.46) for some ρ-ball B in X.

Proof Fix an arbitrary ρ-ball $B \subset X$. Bearing in mind that we have $0 < \mu(B) < \infty$ (cf. (7.4.2)) as well as $\int_B \big(f - ⨍_B f \,\mathrm{d}\mu\big)\,\mathrm{d}\mu = 0$, from Lemma 7.4.11 and (7.4.45)–(7.4.46) we conclude that

$$\begin{aligned} &⨍_B \Big| f - ⨍_B f \,\mathrm{d}\mu \Big| \,\mathrm{d}\mu \\ &\quad \leq 2\mu(B)^{-1} \sup\Big\{ \Big| \int_B fg \,\mathrm{d}\mu \Big| : g \in L^\infty(B,\mu),\ \|g\|_{L^\infty(B,\mu)} \leq 1,\ \int_B g \,\mathrm{d}\mu = 0 \Big\} \\ &\quad \leq 2M_f \cdot \mu(B)^{1/p-1} \leq CM_f \cdot r^\alpha, \end{aligned} \tag{7.4.50}$$

thanks to (7.4.48) plus the fact that $\frac{1}{p} - 1 = \frac{\alpha}{d}$. In turn, from this and (7.4.23) we see that

$$M_{f,1,\alpha} = \sup_{r>0} \sup_{\substack{B \subset X \\ \rho\text{-ball of radius } r}} \Big\{ r^{-\alpha} ⨍_B \Big| f - ⨍_B f \,\mathrm{d}\mu \Big| \,\mathrm{d}\mu \Big\} \leq CM_f < +\infty. \tag{7.4.51}$$

Granted this, Proposition 7.4.7 applies and implies the existence of $\widetilde{f} \in \dot{\mathscr{C}}^\alpha(X,\rho)$ which coincides with f at μ-a.e. point in X and such that (7.4.47) holds.

Finally, work under the assumption (7.4.48) in place of (7.4.44). We now consider a function $f \in \dot{\mathscr{C}}^\alpha(X,\rho)$ along with some $g \in L^\infty(X,\mu)$ satisfying the properties listed in (7.4.46) for some ρ-ball $B \subseteq X$ of radius r. Then (7.4.48) permits us to estimate

$$\begin{aligned} \Big| \int_X fg \,\mathrm{d}\mu \Big| &= \Big| \int_X \Big(f - ⨍_B f \,\mathrm{d}\mu \Big) g \,\mathrm{d}\mu \Big| = \Big| \int_B \Big(f - ⨍_B f \,\mathrm{d}\mu \Big) g \,\mathrm{d}\mu \Big| \\ &\leq \|g\|_{L^\infty(X,\mu)} \mu(B)^{-1} \int_B \int_B |f(x) - f(y)| \,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\ &\leq \|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \mu(B)^{1-1/p} \sup_{x,y \in B} \rho(x,y)^\alpha \\ &\leq C\|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)} \mu(B)^{1-1/p} r^\alpha \\ &\leq C\|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)}, \end{aligned} \tag{7.4.52}$$

proving (7.4.49). □

The following local version of Proposition 7.4.7 extends the characterization of the class of Hölder functions in the Euclidean space given by N. Meyers in [180] to the setting of spaces of homogeneous type.

Proposition 7.4.9 *Let (X,ρ,μ) be a space of homogeneous type with the property that the measure μ is Borel-semiregular on (X,τ_ρ). Fix an integrability exponent $p \in [1,\infty)$ along with some $\alpha \in (0,\infty)$. Then there exist two constants $C \in (0,\infty)$ and $\lambda \in (2,\infty)$ which depend only on the ambient with the following significance. Suppose B is a ρ-ball in X and $f \in L^1(\lambda B,\mu)$ satisfies*

$$M_{f,p,\alpha,B} := \sup_{B' \subseteq \lambda B} \left\{ r^{-\alpha} \Big(⨍_{B'} |f - f_{B'}|^p \, d\mu \Big)^{1/p} \right\} < +\infty, \tag{7.4.53}$$

where the supremum is taken over all ρ-balls B' contained in B, the radius of B' is denoted by r, and $f_{B'} := \mu(B')^{-1} \int_{B'} f \, d\mu$ for each such ρ-ball B'. Then there exists a function $\widetilde{f} \in \dot{\mathscr{C}}^{\alpha}(B, \rho)$ which coincides with f at μ-a.e. point in B and has the property that

$$\|\widetilde{f}\|_{\dot{\mathscr{C}}^{\alpha}(B,\rho)} \leq C M_{f,p,\alpha,B}. \tag{7.4.54}$$

Proof The global version of this result (formally corresponding to $B := X$) is contained in Proposition 7.4.7, and the local version described here may be established in the very same manner. The only novel aspect is the observation that for each pair of distinct points $x_1, x_2 \in B$ and any constant $C \in (0, \infty)$ the ρ-ball $B' := B_\rho\big(x_1, C\rho(x_1, x_2)\big) \subseteq \lambda B$ for some sufficiently large $\lambda \in (0, \infty)$, depending only on the ambient and C, which makes (7.4.53) applicable. □

Let $\big(X, \rho, \mu\big)$ be a `measure metric space`. That is, (X, ρ) is a metric space and μ is a doubling measure on X (in particular, the triplet $\big(X, \rho, \mu\big)$ is a space of homogeneous type). Recall the piece of notation introduced in (7.4.9). In relation to this, we note that if Δ and Δ' are two ρ-balls in X with the property that $\Delta' \subseteq \Delta$, then for any $f \in L^1_{\rm loc}(X, \mu)$ and any $p \in [1, \infty)$ we have

$$\Big(⨍_{\Delta'} |f - f_{\Delta'}|^p \, d\mu \Big)^{\frac{1}{p}} \leq 2 \Big(\frac{\mu(\Delta)}{\mu(\Delta')} \Big)^{\frac{1}{p}} \Big(⨍_{\Delta} |f - f_{\Delta}|^p \, d\mu \Big)^{\frac{1}{p}}, \tag{7.4.55}$$

$$\Big(⨍_{\Delta} |f - f_{\Delta'}|^p \, d\mu \Big)^{\frac{1}{p}} \leq \Big[1 + \Big(\frac{\mu(\Delta)}{\mu(\Delta')} \Big)^{\frac{1}{p}} \Big] \Big(⨍_{\Delta} |f - f_{\Delta}|^p \, d\mu \Big)^{\frac{1}{p}}, \tag{7.4.56}$$

$$\frac{1}{2} \Big(⨍_{\Delta} |f - f_{\Delta}|^p \, d\mu \Big)^{\frac{1}{p}} \leq \inf_{c \in \mathbb{C}} \Big(⨍_{\Delta} |f - c|^p \, d\mu \Big)^{\frac{1}{p}} \leq \Big(⨍_{\Delta} |f - f_{\Delta}|^p \, d\mu \Big)^{\frac{1}{p}}, \tag{7.4.57}$$

$$\Big(⨍_{\Delta} |f - f_{\Delta}|^p \, d\mu \Big)^{\frac{1}{p}} \leq 2 \Big(⨍_{\Delta} |f|^p \, d\mu \Big)^{\frac{1}{p}}, \tag{7.4.58}$$

as well as

$$\begin{aligned} \Big(⨍_{\Delta} |f(x) - f_{\Delta}|^p \mu(x) \Big)^{\frac{1}{p}} &\leq \Big(⨍_{\Delta} ⨍_{\Delta} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x) \Big)^{\frac{1}{p}} \\ &\leq 2 \Big(⨍_{\Delta} |f(x) - f_{\Delta}|^p \mu(x) \Big)^{\frac{1}{p}}. \end{aligned} \tag{7.4.59}$$

Next, given an arbitrary ρ-ball $\Delta \subseteq X$ along with some function $f \in L^1(\Delta, \mu)$, let us agree to abbreviate

$$\|f\|_*(\Delta) := \sup_{\Delta' \subseteq \Delta} \fint_{\Delta'} |f - f_{\Delta'}| \, \mathrm{d}\mu \tag{7.4.60}$$

with the supremum taken over all ρ-balls $\Delta' \subseteq X$ contained in Δ. In relation to this, we make several simple but useful observations. First we note that there is no loss of generality in assuming that

$$\begin{array}{l}\text{the supremum in (7.4.60) is taken over all } \rho\text{-balls } \Delta' \subseteq X \\ \text{contained in } \Delta \text{ with radii less than or equal to } \operatorname{diam}_\rho(X).\end{array} \tag{7.4.61}$$

Second, (7.4.56) implies that there exists some constant $C \in (0, \infty)$ which depends only on n and the doubling constant of μ with the property that for each function $f \in L^1_{\mathrm{loc}}(X, \mu)$ and each ρ-ball $\Delta \subseteq X$ we have

$$\big|f_{2\Delta} - f_\Delta\big| \leq C\|f\|_*(2\Delta). \tag{7.4.62}$$

In turn, (7.4.62) may be used to estimate

$$\big|f_{2^j\Delta} - f_\Delta\big| \leq \sum_{k=1}^{j} \big|f_{2^k\Delta} - f_{2^{k-1}\Delta}\big| \leq Cj\|f\|_*(2^j\Delta), \tag{7.4.63}$$

for each function $f \in L^1_{\mathrm{loc}}(X, \mu)$, each ρ-ball $\Delta \subseteq X$, and each integer $j \in \mathbb{N}$. For future use, let us also note here that there exists some $C \in (0, \infty)$ which depends only on the doubling constant of μ with the property that for each function $f \in L^1_{\mathrm{loc}}(X, \mu)$ we have

$$\begin{array}{c}\big|f_{\Delta(x,R)} - f_{\Delta(y,R)}\big| \leq C\|f\|_*(\Delta(x, 2R)) \\ \text{for each pair of points } x, y \in X \text{ and each radius } R > \rho(x, y).\end{array} \tag{7.4.64}$$

Finally, we note that the same arguments that have produced (7.4.62)–(7.4.64) may be used to show that there exists $C \in (0, \infty)$ depending only on the doubling constant of μ with the property that for each $f \in L^1_{\mathrm{loc}}(X, \mu)$ we have

$$\begin{array}{c}\big|f_\Delta - f_{\Delta'}\big| \leq C\left[1 + \log_2\left|\dfrac{\mu(\Delta)}{\mu(\Delta')}\right|\right] \cdot \max\Big\{\|f\|_*(9\Delta), \|f\|_*(9\Delta')\Big\} \\ \text{for every pair of } \rho\text{-balls } \Delta, \Delta' \text{ in } X \text{ with } \Delta \cap \Delta' \neq \varnothing.\end{array} \tag{7.4.65}$$

Let us now recall the John–Nirenberg inequality in the present setting which asserts that (see, e.g., [1, Theorem 5.2], [22, Theorem 3.15], [78, Theorem 3.1, p. 1397], as well as [153, Lemma 2.4, p. 409] and the references therein) there exist two constants $C_1, C_2 \in (0, \infty)$, depending only on the doubling constant of the measure μ, with the property that for any ρ-ball $\Delta \subseteq X$ and any function $f \in L^1(5\Delta, \mu)$ with $0 < \|f\|_*(5\Delta) < +\infty$ there holds

$$\mu\big(\{x \in \Delta : |f(x) - f_\Delta| > \lambda\}\big) \leq C_1 \exp\Big\{-\Big(\frac{C_2}{\|f\|_*(5\Delta)}\Big)\lambda\Big\}\,\mu(\Delta), \quad \forall\lambda > 0. \tag{7.4.66}$$

As a corollary of this John–Nirenberg inequality, we obtain the following basic result.

Lemma 7.4.10 *Assume (X, ρ, μ) is some given measure metric space. Then for each ρ-ball $\Delta \subseteq X$, each function $f \in L^1(5\Delta, \mu)$, and each exponent $p \in [1, \infty)$, one has*

$$\begin{aligned}\Big(⨍_\Delta |f - f_\Delta|^p \, d\mu\Big)^{\frac{1}{p}} &\leq \frac{\sqrt[p]{C_1 \cdot \Gamma(p+1)}}{C_2}\|f\|_*(5\Delta) \\ &= C_{\mu,p} \cdot \sup_{\Delta' \subseteq 5\Delta} ⨍_{\Delta'} |f - f_{\Delta'}| \, d\mu \end{aligned} \tag{7.4.67}$$

where $C_1, C_2 \in (0, \infty)$ are the constants intervening in the local John–Nirenberg level set estimate recorded in (7.4.66), and

$$\Gamma(t) := \int_0^\infty \lambda^{t-1} e^{-\lambda} \, d\lambda \ \text{ for all } \ t \in (0, \infty) \tag{7.4.68}$$

is the classical Gamma function.

Proof Since (7.4.67) is clear when $\|f\|_*(5\Delta)$ is 0 or $+\infty$, we may assume that $0 < \|f\|_*(5\Delta) < +\infty$. Granted this, the local John–Nirenberg level set estimate with exponential bound from (7.4.66) plus a natural change of variables permit us to write

$$\begin{aligned} ⨍_\Delta |f - f_\Delta|^p \, d\mu &= \frac{p}{\mu(\Delta)} \int_0^\infty \lambda^{p-1} \mu\Big(\big\{x \in \Delta : |f(x) - f_\Delta| > \lambda\big\}\Big) \, d\lambda \\ &\leq C_1 p \int_0^\infty \lambda^{p-1} \exp\Big(\frac{-C_2\lambda}{\|f\|_*(5\Delta)}\Big) \, d\lambda \\ &= C_1 p \Big(C_2^{-1}\|f\|_*(5\Delta)\Big)^p \int_0^\infty t^{p-1} e^{-t} \, dt \\ &= C_1 p \, \Gamma(p) \Big(C_2^{-1}\|f\|_*(5\Delta)\Big)^p. \end{aligned} \tag{7.4.69}$$

Since $p\,\Gamma(p) = \Gamma(p+1)$, this readily justifies (7.4.67). □

Moving on, with each $f \in L^1_{\mathrm{loc}}(X, \mu)$ associate the quantity (which may happen to be $+\infty$)

$$\|f\|_{\dot{\mathrm{BMO}}(X,\mu)} := \sup_{x \in X,\, r > 0} ⨍_{B_\rho(x,r)} \big|f - f_{B_\rho(x,r)}\big| \, d\mu. \tag{7.4.70}$$

For further use, let us observe here that the John–Nirenberg inequality (cf. (7.4.67)) implies that for each integrability exponent $p \in [1, \infty)$ we have (again, with Δ denoting arbitrary ρ-balls in the ambient X)

$$\|f\|_{\mathrm{B\dot{M}O}(X,\mu)} \approx \sup_{\Delta\subseteq X}\Big(⨍_\Delta |f-f_\Delta|^p\,\mathrm{d}\mu\Big)^{\frac{1}{p}} \approx \sup_{\Delta\subseteq X}\inf_{c\in\mathbb{C}}\Big(⨍_\Delta |f-c|^p\,\mathrm{d}\mu\Big)^{\frac{1}{p}}, \tag{7.4.71}$$

uniformly for $f\in L^1_{\mathrm{loc}}(X,\mu)$.

Proposition 7.4.12 deals with the end-point case $p=1$ in Proposition 7.4.8. Before stating it, we recall a basic duality result and prove a useful variant. Specifically, if (X,μ) is an arbitrary measure space, and $p\in[1,\infty)$ together with $p'\in(1,\infty]$ are Hölder conjugate exponents, then it is well known (cf., e.g., [91, Proposition 6.13, p. 188]) that

$$\|f\|_{L^p(X,\mu)} = \sup\Big\{\Big|\int_X fg\,\mathrm{d}\mu\Big| : g\in L^{p'}(X,\mu) \ \text{ with } \ \|g\|_{L^{p'}(X,\mu)}=1\Big\} \tag{7.4.72}$$

for each function $f\in L^p(X,\mu)$. We are interested in a version of this formula for functions with a vanishing moment, as described in the lemma below.

Lemma 7.4.11 *Assume (X,μ) is a measure space satisfying $\mu(X)<\infty$. In addition, suppose $p\in[1,\infty)$ and $p'\in(1,\infty]$ are two Hölder conjugate exponents. Then*

$$\|f\|_{L^p(X,\mu)} \leq 2\sup\Big\{\Big|\textstyle\int_X fg\,\mathrm{d}\mu\Big| : g\in L^{p'}(X,\mu),\ \|g\|_{L^{p'}(X,\mu)}=1,\ \int_X g\,\mathrm{d}\mu=0\Big\}$$
$$\text{for each function } \ f\in L^p(X,\mu) \ \text{ with } \ \textstyle\int_X f\,\mathrm{d}\mu=0. \tag{7.4.73}$$

Proof If μ identically zero, there is nothing to prove, so assume $\mu(X)>0$ in what follows. Consider an arbitrary function $g\in L^{p'}(X,\mu)$ with $\|g\|_{L^{p'}(X,\mu)}=1$. Since we are presently assuming $\mu(X)<\infty$, it follows that $h:=g-\mu(X)^{-1}\int_X g\,\mathrm{d}\mu$ belongs to $L^{p'}(X,\mu)\subseteq L^1(X,\mu)$ and satisfies $\|h\|_{L^{p'}(X,\mu)}\leq 2$ as well as $\int_X h\,\mathrm{d}\mu=0$. Fix now an arbitrary function $f\in L^p(X,\mu)$ with $\int_X f\,\mathrm{d}\mu=0$. In particular, the latter property ensures that $\int_X fh\,\mathrm{d}\mu=\int_X fg\,\mathrm{d}\mu$ which, in turn, implies

$$\Big|\int_X fg\,\mathrm{d}\mu\Big| \leq 2\sup\Big\{\Big|\int_X f\phi\,\mathrm{d}\mu\Big| : \phi\in L^{p'}(X,\mu),\ \|\phi\|_{L^{p'}(X,\mu)}\leq 1,\ \int_X \phi\,\mathrm{d}\mu=0\Big\}. \tag{7.4.74}$$

Taking the supremum over all functions $g\in L^{p'}(X,\mu)$ with $\|g\|_{L^{p'}(X,\mu)}=1$ and relying on (7.4.72) then yield

$$\|f\|_{L^p(X,\mu)} \leq 2\sup\Big\{\Big|\int_X fg\,\mathrm{d}\mu\Big| : g\in L^{p'}(X,\mu),\ \|g\|_{L^{p'}(X,\mu)}\leq 1,\ \int_X g\,\mathrm{d}\mu=0\Big\}. \tag{7.4.75}$$

Let us also observe that for each $g\in L^{p'}(X,\mu)$ which is not identically zero and satisfies $\|g\|_{L^{p'}(X,\mu)}\leq 1$ as well as $\int_X g\,\mathrm{d}\mu=0$, the function $\widetilde{g}:=g\big/\|g\|_{L^{p'}(X,\mu)}$

enjoys the following properties:

$$\widetilde{g} \in L^{p'}(X,\mu), \ \ \|\widetilde{g}\|_{L^{p'}(X,\mu)} = 1, \ \int_X \widetilde{g}\,\mathrm{d}\mu = 0, \\ \text{and} \ \left|\int_X fg\,\mathrm{d}\mu\right| \leq \left|\int_X f\widetilde{g}\,\mathrm{d}\mu\right|. \tag{7.4.76}$$

Now (7.4.73) is seen from this observation and (7.4.75). □

Here is the proposition advertised earlier, prefiguring[6] a basic duality result between Hardy spaces and Hölder spaces (cf. [185, Sect. 4.6]).

Proposition 7.4.12 *Let* (X,ρ,μ) *be a measure metric space. Pick an integrability exponent* $p \in [1,\infty)$ *and denote by* $p' \in (1,\infty]$ *its Hölder conjugate exponent. Then there exist constants* $c, C \in (0,\infty)$ *with the property that for each function* $f \in L^p_{\mathrm{loc}}(X,\mu)$ *one has*

$$c\|f\|_{\dot{\mathrm{BMO}}(X,\mu)} \leq \sup_g \left|\int_X fg\,\mathrm{d}\mu\right| \leq C\|f\|_{\dot{\mathrm{BMO}}(X,\mu)}, \tag{7.4.77}$$

where the supremum is taken over all μ*-measurable functions* $g: X \to \mathbb{R}$ *for which there exists a* ρ*-ball* B *in* X *such that*

$$\operatorname{supp} g \subseteq B, \quad \|g\|_{L^{p'}(X,\mu)} \leq \mu(B)^{-1/p}, \quad \int_X g\,\mathrm{d}\mu = 0. \tag{7.4.78}$$

Proof Pick an arbitrary ρ-ball $B \subseteq X$. In view of the fact that we have $\mu(B) < \infty$ (cf. (7.4.2)) and $\int_B \big(f - ⨍_B f\,\mathrm{d}\mu\big)\,\mathrm{d}\mu = 0$, Lemma 7.4.11 applies and, with $p' \in (1,\infty]$ denoting the Hölder conjugate exponent of p, gives

$$\Big(\int_B \Big|f - ⨍_B f\,\mathrm{d}\mu\Big|^p\,\mathrm{d}\mu\Big)^{1/p} \\ \leq 2\sup\Big\{\Big|\int_B fg\,\mathrm{d}\mu\Big| : g \in L^{p'}(B,\mu), \ \|g\|_{L^{p'}(B,\mu)} \leq 1, \ \int_B g\,\mathrm{d}\mu = 0\Big\}. \tag{7.4.79}$$

Keeping in mind (7.4.71), this readily yields the first inequality in (7.4.77). To justify the second inequality in (7.4.77), assume $g \in L^{p'}(X,\mu)$ is a function satisfying the properties listed in (7.4.78) for some ρ-ball $B \subseteq X$. Then

$$\left|\int_X fg\,\mathrm{d}\mu\right| = \left|\int_X \Big(f - ⨍_B f\Big)g\,\mathrm{d}\mu\right| = \left|\int_B \Big(f - ⨍_B f\Big)g\,\mathrm{d}\mu\right| \\ \leq \Big(⨍_B \Big|f - ⨍_B f\,\mathrm{d}\mu\Big|^p\,\mathrm{d}\mu\Big)^{1/p} \leq C\|f\|_{\dot{\mathrm{BMO}}(X,\mu)}, \tag{7.4.80}$$

[6] See the discussion in [144, pp. 327–328].

by Hölder's inequality and (7.4.71). Granted this, the second inequality in (7.4.77) is obtained after taking the supremum over all such functions g. □

Going further, for each $f \in L^1_{\rm loc}(X,\mu)$ let us set

$$\|f\|_{\mathrm{BMO}(X,\mu)} := \begin{cases} \|f\|_{\dot{\mathrm{BMO}}(X,\mu)} & \text{if } X \text{ is unbounded,} \\ \|f\|_{\dot{\mathrm{BMO}}(X,\mu)} + \left|\int_X f\,\mathrm{d}\mu\right| & \text{if } X \text{ is bounded.} \end{cases} \tag{7.4.81}$$

Note that $\|\cdot\|_{\mathrm{BMO}(X,\mu)}$ is non-degenerate if X is bounded, but vanishes on constants if X is unbounded. Regarding the format of $\|\cdot\|_{\mathrm{BMO}(X,\mu)}$ when X is bounded, observe that for each $f \in L^1(X,\mu)$ and each $p \in [1,\infty)$ we have

$$\begin{aligned} &\left|\int_X f\,\mathrm{d}\mu\right| \leq \mu(X)^{1-1/p}\|f\|_{L^p(X,\mu)} \quad \text{and} \\ &\|f\|_{L^p(X,\mu)} \leq \mu(X)^{1/p}\|f\|_{\dot{\mathrm{BMO}}(X,\mu)} + \mu(X)^{1/p-1}\left|\int_X f\,\mathrm{d}\mu\right|, \end{aligned} \tag{7.4.82}$$

where the inequality in the second line uses (7.4.71). Hence,

$$\text{if } X \text{ is bounded, } \|f\|_{\mathrm{BMO}(X,\mu)} \approx \|f\|_{L^p(X,\mu)} + \|f\|_{\dot{\mathrm{BMO}}(X,\mu)} \text{ uniformly for functions } f \in L^p(X,\mu) \text{ with } 1 \leq p < \infty. \tag{7.4.83}$$

In the Euclidean setting, it is well known that the logarithm is a prototype of a BMO function. The point of our next lemma is that this continues to be the case in a much more general geometric environment.

Lemma 7.4.13 *Let (X,ρ) be a metric space and consider a nonempty closed*[7] *set $\Sigma \subseteq X$. Also, let μ be a doubling measure on Σ with the property that there exist $C, d \in (0,\infty)$ such that*

$$\frac{\mu\big(B_\rho(x,r_1)\cap\Sigma\big)}{\mu\big(B_\rho(x,r_2)\cap\Sigma\big)} \leq C\Big(\frac{r_1}{r_2}\Big)^d \quad \textit{for all} \quad x \in \Sigma \quad \textit{and} \quad 0 < r_1 < r_2 < \infty. \tag{7.4.84}$$

In particular, this is the case if there exists $c \in (0,1)$ such that

$$cr^d \leq \mu\big(B_\rho(x,r)\cap\Sigma\big) \leq c^{-1}r^d \quad \textit{for all} \quad x \in \Sigma \quad \textit{and} \quad r \in \big(0, 2\,\mathrm{diam}_\rho\Sigma\big). \tag{7.4.85}$$

Then μ does not charge singletons[8] *and*

[7] In the topology induced by ρ on X.

[8] I.e., $\mu(\{x\}) = 0$ for each $x \in \Sigma$.

$$\sup_{x\in X}\Big\| \ln(\rho(x,\cdot))\big|_\Sigma \Big\|_{\dot{\mathrm{BMO}}(\Sigma,\mu)} < +\infty. \tag{7.4.86}$$

Proof That $\mu(\{x\}) = 0$ for each $x \in \Sigma$ is seen by sending r_1 to zero in (7.4.84):

$$\mu(\{x\}) = \lim_{n\to\infty} \mu\big(B_\rho(x, 1/n) \cap \Sigma\big) \leq C \lim_{n\to\infty} \Big(\frac{1}{n}\Big)^d \mu\big(B_\rho(x, 1) \cap \Sigma\big) = 0. \tag{7.4.87}$$

To proceed, fix $x \in X$, $x_o \in \Sigma$, and $r \in (0,\infty)$. Set $f(y) := \ln(\rho(x, y))$ for each $y \in \Sigma \setminus \{x\}$, hence for μ-a.e. $y \in \Sigma$. Then f is continuous on $\Sigma \setminus \{x\}$. With Δ abbreviating $B_\rho(x_o, r) \cap \Sigma$, the goal is to estimate $⨍_\Delta |f - c|\,\mathrm{d}\mu$ for a suitable choice of a constant $c \in \mathbb{R}$. To describe the actual value of c, we distinguish two cases.

`Case I`: *Assume* $\rho(x, x_o) > 3r$. In this scenario, for each point $y \in \Delta$ we may estimate

$$\begin{gathered}\rho(x, x_o) < \rho(y, x) + \rho(y, x_o) < \rho(y, x) + r < \rho(y, x) + \rho(x, x_o)/3,\\ \rho(y, x) < \rho(y, x_o) + \rho(x, x_o) < r + \rho(x, x_o) < 4\rho(x, x_o)/3.\end{gathered} \tag{7.4.88}$$

Thus $2/3 < \rho(x, y)\big/\rho(x, x_o) < 4/3$, so if we take $c := \ln(\rho(x, x_o))$ then

$$⨍_\Delta |f(y) - c|\,\mathrm{d}\mu(y) = ⨍_\Delta \Big|\ln\Big(\frac{\rho(x, y)}{\rho(x, x_o)}\Big)\Big|\,\mathrm{d}\mu(y) \leq \ln(3/2). \tag{7.4.89}$$

`Case II`: *Assume* $\rho(x, x_o) \leq 3r$. First, we claim that there exists some point x_* belonging to Σ such that $\rho(x, x_*) \leq 2\,\mathrm{dist}_\rho(x, \Sigma)$. Indeed, if $\mathrm{dist}_\rho(x, \Sigma) > 0$ then this is a direct consequence of definitions. On the other hand, if $\mathrm{dist}_\rho(x, \Sigma) = 0$ then for each $n \in \mathbb{N}$ there exists $x_n \in \Sigma$ such that $\rho(x, x_n) < 1/n$. In particular, $\lim_{n\to\infty} x_n = x$ in the topology induced by ρ on X, so $x \in \Sigma$ given that Σ is closed. Hence, $x_* := x$ will do in this scenario.

Moving on, observe that $\rho(x, x_*) \leq 2\,\mathrm{dist}_\rho(x, \Sigma) \leq 2\rho(x, x_o) \leq 6r$. Hence, for each point $y \in \Delta$ we have $\rho(y, x_*) < \rho(y, x_o) + \rho(x_o, x) + \rho(x, x_*) < 10r$. Also,

$$\begin{gathered}\rho(y, x_*) \leq \rho(y, x) + \rho(x, x_*) \leq \rho(y, x) + 2\,\mathrm{dist}_\rho(x, \Sigma) \leq 3\rho(y, x),\\ \text{and}\ \ \rho(x, y) \leq \rho(x, x_*) + \rho(y, x_*) < 16r.\end{gathered} \tag{7.4.90}$$

Consequently, $0 \leq \rho(y, x_*)/(48r) < \rho(x, y)/(16r) < 1$. If we now take $c := \ln(16r)$ we may use (7.4.4) and (7.4.84) to write (with $D_\mu \in [0,\infty)$ denoting the doubling order of μ; cf. (7.4.3))

$$
\begin{aligned}
&\fint_{\Delta} |f(y)-c|\,\mathrm{d}\mu(y) \qquad (7.4.91)\\
&\quad\le \frac{1}{\mu(\Delta)}\int_{B_\rho(x_*,10r)\cap\Sigma}\Big|\ln\Big(\frac{\rho(x,y)}{16r}\Big)\Big|\,\mathrm{d}\mu(y)\\
&\quad\le \frac{1}{\mu(\Delta)}\int_{B_\rho(x_*,10r)\cap\Sigma}\Big|\ln\Big(\frac{\rho(y,x_*)}{48r}\Big)\Big|\,\mathrm{d}\mu(y)\\
&\quad= \frac{1}{\mu(\Delta)}\sum_{j=0}^{\infty}\int_{[B_\rho(x_*,10r2^{-j})\setminus B_\rho(x_*,10r2^{-j-1})]\cap\Sigma}\Big|\ln\Big(\frac{\rho(y,x_*)}{48r}\Big)\Big|\,\mathrm{d}\mu(y)\\
&\quad\le \sum_{j=0}^{\infty}\Big|\ln\Big(\frac{10\cdot 2^{-j-1}}{48}\Big)\Big|\frac{\mu\big(B_\rho(x_*,10r\cdot 2^{-j})\cap\Sigma\big)}{\mu(\Delta)}\\
&\quad\le C_\mu\sum_{j=0}^{3}\Big|\ln\Big(\frac{10\cdot 2^{-j-1}}{48}\Big)\Big|(10\cdot 2^{-j})^{D_\mu}\\
&\qquad + C\sum_{j\ge 4}\Big|\ln\Big(\frac{10\cdot 2^{-j-1}}{48}\Big)\Big|(10\cdot 2^{-j})^{d} < +\infty.
\end{aligned}
$$

From the analysis carried out in Cases I–II we conclude that there exists some $C=C(\mu)\in(0,\infty)$ with the property that if for each $x_o\in\Sigma$ and $r\in(0,\infty)$ we set $\Delta:=B_\rho(x_o,r)\cap\Sigma$ then we may find $c_\Delta\in\mathbb{R}$ such that $\fint_\Delta |f-c_\Delta|\,\mathrm{d}\mu\le C$. Then the conclusion in (7.4.86) follows in view of (7.4.71). □

The John–Nirenberg space, denoted by[9] $\mathrm{BMO}\big(X,\mu\big)$, consists of functions of bounded mean oscillations on X. Specifically,

$$
\mathrm{BMO}\big(X,\mu\big) := \Big\{f\in L^1_{\mathrm{loc}}(X,\mu):\ \|f\|_{\mathrm{BMO}(X,\mu)}<+\infty\Big\}. \qquad (7.4.92)
$$

Note that while $\mathrm{BMO}(X,\mu)$ is a Banach space when X is bounded, $\|\cdot\|_{\mathrm{BMO}(X,\mu)}$ is only a semi-norm on $\mathrm{BMO}(X,\mu)$ when X is unbounded, since

$$
\begin{aligned}
&\text{if } X \text{ is unbounded then, for each } f\in L^1_{\mathrm{loc}}(X,\mu),\\
&\|f\|_{\mathrm{BMO}(X,\mu)}=0 \iff f \text{ is constant } \mu\text{-a.e. on } X.
\end{aligned} \qquad (7.4.93)
$$

Given this, we find it occasionally useful to mod out its null-space, in order to render the resulting quotient space Banach. Specifically, for two μ-measurable functions f,g defined on X we write $f\sim g$ provided the difference $f-g$ is constant μ-a.e. on X. This is an equivalence relation and we let

$$
[f] := \big\{g: X\to\mathbb{C}:\ g \text{ is } \mu\text{-measurable and } f\sim g\big\} \qquad (7.4.94)
$$

[9] A more precise notation, which also indicates the dependence on the background quasi-distance ρ, would be $\mathrm{BMO}\big(X,\rho,\mu\big)$.

denote the equivalence class of any given μ-measurable function f defined on X. In the case when X is unbounded, if for each function $f \in \mathrm{BMO}(X,\mu)$ we now set

$$\big\| [f] \big\|_{\widetilde{\mathrm{BMO}}(X,\mu)} := \|f\|_{\mathrm{BMO}(X,\mu)}, \tag{7.4.95}$$

then $\big\| [\cdot] \big\|_{\widetilde{\mathrm{BMO}}(X,\mu)}$ becomes a genuine norm on the quotient space

$$\widetilde{\mathrm{BMO}}(X,\mu) := \mathrm{BMO}(X,\mu)\big/ \sim = \big\{[f] :\ f \in \mathrm{BMO}(X,\mu)\big\}. \tag{7.4.96}$$

In fact, when equipped with the norm (7.4.95), the space (7.4.96) is complete (hence Banach).

We wish to note that, given any $f \in L^1_{\mathrm{loc}}(X,\mu)$, from (7.4.59) (used with $p = 1$), the lower triangle inequality (which implies $\big||f(x)| - |f(y)|\big| \leq |f(x) - f(y)|$), and (7.4.81)–(7.4.92), we obtain that

$$\big\| \, |f| \, \big\|_{\overset{\bullet}{\mathrm{BMO}}(X,\mu)} \leq 2\|f\|_{\overset{\bullet}{\mathrm{BMO}}(X,\mu)}. \tag{7.4.97}$$

In turn, with $C_X := 2$ if X is unbounded and $C_X := 2 + \mu(X)$ if X is bounded, this readily implies that

$$\begin{aligned} &\text{if } f \in \mathrm{BMO}(X,\mu) \text{ then } |f| \in \mathrm{BMO}(X,\mu) \\ &\text{and } \big\| \, |f| \, \big\|_{\mathrm{BMO}(X,\mu)} \leq C_X \|f\|_{\mathrm{BMO}(X,\mu)}. \end{aligned} \tag{7.4.98}$$

Since $\max\{f,g\} = \frac{1}{2}(f + g + |f - g|)$ and $\min\{f,g\} = \frac{1}{2}(f + g - |f - g|)$, it follows from (7.4.98) that $\mathrm{BMO}(X,\mu)$ is a lattice, and (7.4.97) implies that for every two functions $f, g \in \mathrm{BMO}(X,\mu)$ we have

$$\begin{aligned} \big\| \max\{f,g\}\big\|_{\overset{\bullet}{\mathrm{BMO}}(X,\mu)} &\leq \tfrac{3}{2}\Big\{\|f\|_{\overset{\bullet}{\mathrm{BMO}}(X,\mu)} + \|g\|_{\overset{\bullet}{\mathrm{BMO}}(X,\mu)}\Big\}, \\ \big\| \min\{f,g\}\big\|_{\overset{\bullet}{\mathrm{BMO}}(X,\mu)} &\leq \tfrac{3}{2}\Big\{\|f\|_{\overset{\bullet}{\mathrm{BMO}}(X,\mu)} + \|g\|_{\overset{\bullet}{\mathrm{BMO}}(X,\mu)}\Big\}. \end{aligned} \tag{7.4.99}$$

In particular, given any $f \in \mathrm{BMO}(X,\mu)$, if for each $N \in \mathbb{N}$ we define

$$f_N := \min\Big\{\max\{f, -N\}, N\Big\} = \max\Big\{\min\{f, N\}, -N\Big\}, \tag{7.4.100}$$

i.e.,

$$f_N(x) := \begin{cases} N & \text{if } f(x) > N \\ f(x) & \text{if } -N \leq f(x) \leq N, \\ -N & \text{if } f(x) < -N, \end{cases} \qquad \forall\, x \in X, \tag{7.4.101}$$

then

$$\begin{aligned} & f_N \in L^\infty(X,\mu), \quad \text{hence} \quad f_N \in \mathrm{BMO}(X,\mu), \\ & |f_N(x)| \leq \min\{|f(x)|, N\} \ \text{ for all } \ x \in X, \\ & \lim_{N\to\infty} f_N(x) = f(x) \ \text{ for each point } \ x \in X. \end{aligned} \tag{7.4.102}$$

Moreover, from (7.4.100) and (7.4.99) we see that

$$\|f_N\|_{\dot{\mathrm{BMO}}(X,\mu)} \leq \tfrac{9}{4}\|f\|_{\dot{\mathrm{BMO}}(X,\mu)}, \qquad \forall N \in \mathbb{N}, \tag{7.4.103}$$

which, in concert with (7.4.81)–(7.4.82) and (7.4.102), further implies that, for each $N \in \mathbb{N}$,

$$\|f_N\|_{\mathrm{BMO}(X,\mu)} \leq \begin{cases} \frac{9}{4}\|f\|_{\mathrm{BMO}(X,\mu)} & \text{if } X \text{ is unbounded,} \\ \big(\frac{9}{4} + \mu(X)\big)\|f\|_{\mathrm{BMO}(X,\mu)} & \text{if } X \text{ is bounded.} \end{cases} \tag{7.4.104}$$

We next note that (7.4.67) and (7.4.81) imply the following continuous embeddings:

$$L^\infty(X,\mu) \hookrightarrow \mathrm{BMO}(X,\mu) \hookrightarrow \bigcap_{0<p<\infty} L^p_{\mathrm{loc}}(X,\mu). \tag{7.4.105}$$

In particular,

$$\text{if } X \text{ is bounded, then the space } \mathrm{BMO}(X,\mu) \text{ embeds continuously into } L^p(X,\mu) \text{ for each given } p \in [1,\infty). \tag{7.4.106}$$

In what follows, for each $p \in [1,\infty)$ and $r \in (0,\infty)$ we define the L^p-based mean oscillations of a function $f \in L^1_{\mathrm{loc}}(X,\mu)$ at a given scale $R \in (0,\infty)$ as

$$\mathrm{osc}_p(f;R) := \sup_{x\in X,\, r\in(0,R)} \Big(⨍_{B_\rho(x,r)} \big|f(y) - f_{B_\rho(x,r)}\big|^p \,\mathrm{d}\mu(y)\Big)^{\frac{1}{p}} \in [0,+\infty]. \tag{7.4.107}$$

Then, (7.4.67) and (7.4.81) imply that for each fixed $p \in [1,\infty)$ we have

$$\|f\|_{\mathrm{BMO}(X,\mu)} \approx \begin{cases} \sup\limits_{0<R<\infty} \mathrm{osc}_p(f;R) & \text{if } X \text{ is unbounded,} \\ \Big|\displaystyle\int_X f\,\mathrm{d}\mu\Big| + \sup\limits_{0<R<\infty} \mathrm{osc}_p(f;R) & \text{if } X \text{ is bounded,} \end{cases} \tag{7.4.108}$$

uniformly for $f \in L^1_{\mathrm{loc}}(X,\mu)$. As a consequence, for each $p \in [1,\infty)$ we have

$$\mathrm{BMO}(X,\mu) = \Big\{ f \in L^1_{\mathrm{loc}}(X,\mu) : \sup_{0<R<\infty} \mathrm{osc}_p(f;R) < +\infty \Big\}. \tag{7.4.109}$$

An alternative point of view to the John–Nirenberg space involves the Fefferman–Stein sharp maximal operator. Specifically, having fixed $p \in [1,\infty)$, for each function $f \in L^1_{\mathrm{loc}}(X,\mu)$ define

$$f^\#_p(x) := \sup_{r>0} \Big(⨍_{B_\rho(x,r)} \big|f(y) - f_{B_\rho(x,r)}\big|^p \,\mathrm{d}\mu(y)\Big)^{1/p}, \qquad \forall x \in X. \tag{7.4.110}$$

This turns out to be a μ-measurable function,[10] and for each fixed $p \in [1, \infty)$ the first equivalence in (7.4.71) implies

$$\|f\|_{\dot{\mathrm{BMO}}(X,\mu)} \approx \|f_p^{\#}\|_{L^\infty(X,\mu)}, \quad \text{uniformly for } f \in L^1_{\mathrm{loc}}(X,\mu). \tag{7.4.111}$$

In particular, for each fixed $p \in [1, \infty)$ we have

$$\|f\|_{\mathrm{BMO}(X,\mu)} \approx \begin{cases} \|f_p^{\#}\|_{L^\infty(X,\mu)} & \text{if } X \text{ is unbounded,} \\ \Big|\displaystyle\int_X f\,\mathrm{d}\mu\Big| + \|f_p^{\#}\|_{L^\infty(X,\mu)} & \text{if } X \text{ is bounded,} \end{cases} \tag{7.4.112}$$

uniformly for $f \in L^1_{\mathrm{loc}}(X,\mu)$. In particular, for each $p \in [1, \infty)$ we have

$$\mathrm{BMO}(X,\mu) = \Big\{f \in L^1_{\mathrm{loc}}(X,\mu) : \ f_p^{\#} \in L^\infty(X,\mu)\Big\}. \tag{7.4.113}$$

In the lemma below we discuss the manner in which global integrability properties of a given function are related to the behavior at infinity of its mean oscillation function.

Lemma 7.4.14 *Assume (X, ρ, μ) is a measure metric space with the property that there exist two numbers, $d \in (0,\infty)$ and $c \in (0,\infty)$, such that*

$$\mu\big(B_\rho(x,R)\big) \le cR^d \qquad \forall x \in X, \quad \forall R \in (0,\infty). \tag{7.4.114}$$

Then whenever $1 \le p \le q < \infty$ and $\varepsilon > 0$ there exists a constant $C \in (0,\infty)$, depending only on d, ρ, ε, p, the constant c from (7.4.114), and the doubling constant of μ, such that for each function $f \in L^1_{\mathrm{loc}}(X,\mu)$, each point $x_0 \in X$, and each radius $r \in (0,\infty)$, there holds

$$\begin{aligned} &\bigg(\int_X \frac{\big|f(x) - f_{B_\rho(x_0,r)}\big|^p}{\big[r + \rho(x,x_0)\big]^{d+\varepsilon}}\,\mathrm{d}\mu(x)\bigg)^{1/p} \\ &\quad \le \frac{C}{r^{\varepsilon/p}} \int_1^\infty \bigg(⨍_{B_\rho(x_0,\lambda r)} \big|f(x) - f_{B_\rho(x_0,\lambda r)}\big|^q\,\mathrm{d}\mu(x)\bigg)^{1/q} \frac{\mathrm{d}\lambda}{\lambda^{1+\varepsilon/p}} \\ &\quad \le \frac{C}{r^{\varepsilon/p}} \min\Big\{f_q^{\#}(x_0), \int_1^\infty \mathrm{osc}_q\big(f;\lambda r\big)\,\frac{\mathrm{d}\lambda}{\lambda^{1+\varepsilon/p}}\Big\}. \end{aligned} \tag{7.4.115}$$

As a consequence, whenever $1 \le p \le q < \infty$ and $\varepsilon > 0$ there exists a constant $C \in (0,\infty)$ with the property that for each function $f \in L^1_{\mathrm{loc}}(X,\mu)$ and each point $x_0 \in X$ one has

[10] The same argument used in the proofs of Theorems 6.3.3 and 7.6.1 applies.

$$\Big(\int_X \frac{|f(x)|^p}{1+\rho(x,x_0)^{d+\varepsilon}}\,\mathrm{d}\mu(x)\Big)^{1/p}$$
$$\leq C\int_1^\infty \mathrm{osc}_q(f;\lambda)\,\frac{\mathrm{d}\lambda}{\lambda^{1+\varepsilon/p}} + C\Big|\fint_{B_\rho(x_0,1)} f\,\mathrm{d}\mu\Big|. \tag{7.4.116}$$

This further implies that if $1 \leq p \leq q < \infty$ *then for each function* $f \in L^1_{\mathrm{loc}}(X,\mu)$, *each number* $\varepsilon > 0$, *and each point* $x_0 \in X$ *one has*

$$\int_1^\infty \mathrm{osc}_q(f;\lambda)\,\frac{\mathrm{d}\lambda}{\lambda^{1+\varepsilon/p}} < +\infty \Rightarrow f \in L^p\Big(X, \frac{\mu}{1+\rho(\cdot,x_0)^{d+\varepsilon}}\Big). \tag{7.4.117}$$

In particular, for each fixed point $x_0 \in X$,

$$\mathrm{BMO}\big(X,\mu\big) \subset \bigcap_{1\leq p<\infty} L^p\Big(X, \frac{\mu}{1+\rho(\cdot,x_0)^{d+\varepsilon}}\Big), \qquad \forall\varepsilon > 0, \tag{7.4.118}$$

and for each $x_0 \in X$, $\varepsilon > 0$, *and* $p \in [1,\infty)$,

$$\dot{\mathscr{C}}^\alpha(X,\rho) \subset L^p\Big(X, \frac{\mu}{1+\rho(\cdot,x_0)^{d+\varepsilon}}\Big), \qquad \forall\alpha \in (0,\varepsilon/p). \tag{7.4.119}$$

Proof Suppose $1 \leq p \leq q < \infty$. Given $f \in L^1_{\mathrm{loc}}(X,\mu)$, along with $x_0 \in X$ and $r \in (0,\infty)$, breaking up the domain of integration in a dyadic fashion and using Minkowski's inequality allow us to estimate

$$\Big(\int_X \frac{\big|f(x)-f_{B_\rho(x_0,r)}\big|^p}{\big[r+\rho(x,x_0)\big]^{d+\varepsilon}}\,\mathrm{d}\mu(x)\Big)^{1/p} \tag{7.4.120}$$
$$\leq r^{-d/p-\varepsilon/p}\Big(\int_{B_\rho(x_0,r)} \big|f(x)-f_{B_\rho(x_0,r)}\big|^p\,\mathrm{d}\mu(x)\Big)^{1/p}$$
$$+\sum_{j=0}^\infty\Big(\int_{2^j r\leq\rho(x,x_0)<2^{j+1}r} \frac{\big|f(x)-f_{B_\rho(x_0,r)}\big|^p}{\rho(x,x_0)^{d+\varepsilon}}\,\mathrm{d}\mu(x)\Big)^{1/p}$$
$$\leq Cr^{-\varepsilon/p}\Big(\fint_{B_\rho(x_0,r)} \big|f(x)-f_{B_\rho(x_0,r)}\big|^p\,\mathrm{d}\mu(x)\Big)^{1/p}$$
$$+Cr^{-\varepsilon/p}\sum_{j=0}^\infty 2^{-j\varepsilon/p}\Big(\fint_{B_\rho(x_0,2^{j+1}r)} \big|f(x)-f_{B_\rho(x_0,r)}\big|^p\,\mathrm{d}\mu(x)\Big)^{1/p}$$
$$\leq Cr^{-\varepsilon/p}\Big(\fint_{B_\rho(x_0,r)} \big|f(x)-f_{B_\rho(x_0,r)}\big|^q\,\mathrm{d}\mu(x)\Big)^{1/q}$$
$$+Cr^{-\varepsilon/p}\sum_{j=0}^\infty 2^{-j\varepsilon/p}\Big(\fint_{B_\rho(x_0,2^{j+1}r)} \big|f(x)-f_{B_\rho(x_0,r)}\big|^q\,\mathrm{d}\mu(x)\Big)^{1/q},$$

where the last step is based on Hölder's inequality (recall that $q \geq p$). Next, for each $j \in \mathbb{N}_0$ we have

$$\begin{aligned}
&\Big(\fint_{B_\rho(x_0,2^{j+1}r)} \big|f(x) - f_{B_\rho(x_0,r)}\big|^q \,\mathrm{d}\mu(x)\Big)^{1/q} \qquad (7.4.121)\\
&\quad \leq \Big(\fint_{B_\rho(x_0,2^{j+1}r)} \big|f(x) - f_{B_\rho(x_0,2^{j+1}r)}\big|^q \,\mathrm{d}\mu(x)\Big)^{1/q}\\
&\qquad + \sum_{\ell=0}^{j} \big|f_{B_\rho(x_0,2^\ell r)} - f_{B_\rho(x_0,2^{\ell+1}r)}\big|\\
&\quad \leq \Big(\fint_{B_\rho(x_0,2^{j+1}r)} \big|f(x) - f_{B_\rho(x_0,2^{j+1}r)}\big|^q \,\mathrm{d}\mu(x)\Big)^{1/q}\\
&\qquad + C\sum_{\ell=0}^{j} \fint_{B_\rho(x_0,2^{\ell+1}r)} \big|f(x) - f_{B_\rho(x_0,2^{\ell+1}r)}\big| \,\mathrm{d}\mu(x)\\
&\quad \leq \Big(\fint_{B_\rho(x_0,2^{j+1}r)} \big|f(x) - f_{B_\rho(x_0,2^{j+1}r)}\big|^q \,\mathrm{d}\mu(x)\Big)^{1/q}\\
&\qquad + C\sum_{\ell=0}^{j} \Big(\fint_{B_\rho(x_0,2^{\ell+1}r)} \big|f(x) - f_{B_\rho(x_0,2^{\ell+1}r)}\big|^q \,\mathrm{d}\mu(x)\Big)^{1/q}\\
&\quad \leq C\sum_{\ell=0}^{j} \Big(\fint_{B_\rho(x_0,2^{\ell+1}r)} \big|f(x) - f_{B_\rho(x_0,2^{\ell+1}r)}\big|^q \,\mathrm{d}\mu(x)\Big)^{1/q}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{-j\varepsilon/p}\Big(\fint_{B_\rho(x_0,2^{j+1}r)} \big|f(x) - f_{B_\rho(x_0,r)}\big|^q \,\mathrm{d}\mu(x)\Big)^{1/q} \qquad (7.4.122)\\
&\quad \leq C\sum_{j=0}^{\infty} 2^{-j\varepsilon/p}\left\{\sum_{\ell=0}^{j} \Big(\fint_{B_\rho(x_0,2^{\ell+1}r)} \big|f(x) - f_{B_\rho(x_0,2^{\ell+1}r)}\big|^q \,\mathrm{d}\mu(x)\Big)^{1/q}\right\}\\
&\quad = C\sum_{\ell=0}^{\infty} 2^{-\ell\varepsilon/p}\Big(\fint_{B_\rho(x_0,2^{\ell+1}r)} \big|f(x) - f_{B_\rho(x_0,2^{\ell+1}r)}\big|^q \,\mathrm{d}\mu(x)\Big)^{1/q},
\end{aligned}$$

where the equality is the result of interchanging the sums in j and ℓ (here we make use of the fact that $\varepsilon > 0$). Collectively, (7.4.120) and (7.4.122) permit us to conclude that

$$\Big(\int_X \frac{\big|f(x) - f_{B_\rho(x_0,r)}\big|^p}{\big[r + \rho(x, x_0)\big]^{d+\varepsilon}}\,\mathrm{d}\mu(x)\Big)^{1/p} \tag{7.4.123}$$
$$\leq Cr^{-\varepsilon/p}\sum_{j=0}^{\infty} 2^{-j\varepsilon/p}\Big(⨍_{B_\rho(x_0,2^j r)} \big|f(x) - f_{B_\rho(x_0,2^j r)}\big|^q\,\mathrm{d}\mu(x)\Big)^{1/q}.$$

To proceed, from (7.4.55) we deduce that there exists a constant $C \in (0, \infty)$ with the property that

$$\Big(⨍_{B_\rho(x_0,2^j r)} \big|f(x) - f_{B_\rho(x_0,2^j r)}\big|^q\,\mathrm{d}\mu(x)\Big)^{1/q}$$
$$\leq C\Big(⨍_{B_\rho(x_0,\lambda r)} \big|f(x) - f_{B_\rho(x_0,\lambda r)}\big|^q\,\mathrm{d}\mu(x)\Big)^{1/q}, \tag{7.4.124}$$
$$\text{for each } j \in \mathbb{N}_0 \text{ and each } \lambda \in [2^j, 2^{j+1}].$$

In turn, via integration, this implies that for each $j \in \mathbb{N}_0$ we have

$$2^{-j\varepsilon/p}\Big(⨍_{B_\rho(x_0,2^j r)} \big|f(x) - f_{B_\rho(x_0,2^j r)}\big|^q\,\mathrm{d}\mu(x)\Big)^{1/q} \tag{7.4.125}$$
$$\leq C\int_{2^j}^{2^{j+1}} \Big(⨍_{B_\rho(x_0,\lambda r)} \big|f(x) - f_{B_\rho(x_0,\lambda r)}\big|^q\,\mathrm{d}\mu(x)\Big)^{1/q}\,\frac{\mathrm{d}\lambda}{\lambda^{1+\varepsilon/p}}.$$

Availing ourselves of this estimate back into (7.4.123) then establishes the first inequality in (7.4.115). The second inequality in (7.4.115) is a direct consequence of (7.4.107) and (7.4.110). Going further, (7.4.116) follows from the second inequality in (7.4.115) and Lemma 7.2.1. In turn, (7.4.116) readily implies (7.4.117), as well as (7.4.118) (keeping in mind (7.4.109)).

As regards the inclusion in (7.4.119), we first note that for each $p \in [1, \infty)$ and each $\alpha > 0$ there exists $C \in (0, \infty)$ such that for every function $f \in \dot{\mathscr{C}}^\alpha(X, \rho)$ we have

$$\mathrm{osc}_p(f; r) \leq Cr^\alpha \|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)}, \qquad \forall r \in (0, \infty). \tag{7.4.126}$$

Indeed, given an arbitrary ρ-ball $\Delta \subseteq X$ of radius $\leq r$, based on (7.4.59) we may estimate

$$\Big(⨍_\Delta |f(x) - f_\Delta|^p \mu(x)\Big)^{\frac{1}{p}} \leq \Big(⨍_\Delta ⨍_\Delta |f(x) - f(y)|^p\,\mathrm{d}\mu(y)\,\mu(x)\Big)^{\frac{1}{p}}$$
$$\leq Cr^\alpha \|f\|_{\dot{\mathscr{C}}^\alpha(X,\rho)}. \tag{7.4.127}$$

On account of (7.4.127) and (7.4.107), the inequality in (7.4.126) now readily follows. Having proved this, (7.4.119) is now implied by (7.4.117) and (7.4.126). □

For later purposes, we find it useful to supplement the results established in Lemma 7.4.14 with two other related lemmas. In the first such lemma we estimate for the difference of two integral averages over concentric balls.

Lemma 7.4.15 *Suppose* (X, ρ, μ) *is a measure metric space and fix some exponent* $p \in [1, \infty)$. *Then there exists a constant* $C \in (0, \infty)$ *with the property that for each function* $f \in L^1_{loc}(X, \mu)$ *and each point* $x_0 \in X$ *one has* (*recall* (7.4.9))

$$\begin{aligned} \big|f_{B_\rho(x_0,r)} - f_{B_\rho(x_0,R)}\big| &\le C \int_r^{2R} \Big(⨍_{B_\rho(x_0,\,t)} \big|f - f_{B_\rho(x_0,\,t)}\big|^p \,\mathrm{d}\mu\Big)^{1/p} \frac{\mathrm{d}t}{t} \qquad (7.4.128)\\ &\le C \int_r^{2R} \mathrm{osc}_p(f;t)\, \frac{\mathrm{d}t}{t} \quad \textit{whenever} \ \ 0 < r < R < \infty. \end{aligned}$$

Proof Assume $r, R \in (0, \infty)$ with $r < R$ have been given, and denote by $N \in \mathbb{N}_0$ the unique integer such that $2^N r < R \le 2^{N+1} r$. Then based on Hölder's inequality and reasoning as in (7.4.121) permits us to estimate

$$\begin{aligned} \big|f_{B_\rho(x_0,r)} - f_{B_\rho(x_0,2^N r)}\big| &\le ⨍_{B_\rho(x_0,2^N r)} \big|f - f_{B_\rho(x_0,r)}\big| \,\mathrm{d}\mu \\ &\le \Big(⨍_{B_\rho(x_0,2^N r)} \big|f - f_{B_\rho(x_0,r)}\big|^p \,\mathrm{d}\mu\Big)^{1/p} \\ &\le C \sum_{\ell=1}^{N} \Big(⨍_{B_\rho(x_0,2^\ell r)} \big|f - f_{B_\rho(x_0,2^\ell r)}\big|^p \,\mathrm{d}\mu\Big)^{1/p}. \qquad (7.4.129) \end{aligned}$$

On the other hand, from (7.4.55) and (7.4.107) it follows that there exists a constant $C \in (0, \infty)$ with the property that for each integer $\ell \in \mathbb{N}_0$ we have

$$\begin{aligned} &\Big(⨍_{B_\rho(x_0,2^\ell r)} \big|f - f_{B_\rho(x_0,2^\ell r)}\big|^p \,\mathrm{d}\mu\Big)^{1/p} \\ &\qquad \le C\Big(⨍_{B_\rho(x_0,\,tr)} \big|f - f_{B_\rho(x_0,\,tr)}\big|^p \,\mathrm{d}\mu\Big)^{1/p} \\ &\qquad \le C \cdot \mathrm{osc}_p(f; tr) \quad \text{for each} \ \ t \in [2^\ell, 2^{\ell+1}]. \qquad (7.4.130) \end{aligned}$$

After integrating over the interval $[2^\ell, 2^{\ell+1}]$ with respect to the Haar measure $\mathrm{d}t/t$ this further shows that for each $\ell \in \mathbb{N}_0$ we have

$$\begin{aligned} &\Big(⨍_{B_\rho(x_0,2^\ell r)} \big|f - f_{B_\rho(x_0,2^\ell r)}\big|^p \,\mathrm{d}\mu\Big)^{1/p} \\ &\qquad \le C \int_{2^\ell}^{2^{\ell+1}} \Big(⨍_{B_\rho(x_0,\,tr)} \big|f - f_{B_\rho(x_0,\,tr)}\big|^p \,\mathrm{d}\mu\Big)^{1/p} \frac{\mathrm{d}t}{t} \\ &\qquad \le C \int_{2^\ell}^{2^{\ell+1}} \mathrm{osc}_p(f; tr)\, \frac{\mathrm{d}t}{t}. \qquad (7.4.131) \end{aligned}$$

Combining (7.4.129) with (7.4.131) and keeping in mind that $2^{N+1}r = 2\cdot 2^N r \leq 2R$ then yield (after making the change of variables $\lambda := rt$)

$$\begin{aligned}\big|f_{B_\rho(x_0,r)} - f_{B_\rho(x_0,2^N r)}\big| &\leq C\int_{2r}^{2R}\Big(\fint_{B_\rho(x_0,\lambda)}\big|f - f_{B_\rho(x_0,\lambda)}\big|^p\,\mathrm{d}\mu\Big)^{1/p}\frac{\mathrm{d}\lambda}{\lambda}\\ &\leq C\int_{2r}^{2R}\mathrm{osc}_p(f;\lambda)\,\frac{\mathrm{d}\lambda}{\lambda}. \end{aligned} \tag{7.4.132}$$

In addition,

$$\begin{aligned}\big|f_{B_\rho(x_0,R)} - f_{B_\rho(x_0,2^N r)}\big| &\leq \fint_{B_\rho(x_0,2^N r)}\big|f - f_{B_\rho(x_0,R)}\big|\,\mathrm{d}\mu\\ &\leq C\Big(\fint_{B_\rho(x_0,R)}\big|f - f_{B_\rho(x_0,R)}\big|^p\,\mathrm{d}\mu\Big)^{1/p}\\ &\leq C\Big(\fint_{B_\rho(x_0,tR)}\big|f - f_{B_\rho(x_0,tR)}\big|^p\,\mathrm{d}\mu\Big)^{1/p} \end{aligned} \tag{7.4.133}$$

for all $t \in [1, 2]$, thanks to (7.4.55). Integrating over $[1, 2]$ with respect to the Haar measure $\mathrm{d}t/t$ this gives

$$\begin{aligned}\big|f_{B_\rho(x_0,R)} - f_{B_\rho(x_0,2^N r)}\big| &\leq C\int_{1}^{2}\Big(\fint_{B_\rho(x_0,tR)}\big|f - f_{B_\rho(x_0,tR)}\big|^p\,\mathrm{d}\mu\Big)^{1/p}\frac{\mathrm{d}t}{t}\\ &= C\int_{R}^{2R}\Big(\fint_{B_\rho(x_0,\lambda)}\big|f - f_{B_\rho(x_0,\lambda)}\big|^p\,\mathrm{d}\mu\Big)^{1/p}\frac{\mathrm{d}\lambda}{\lambda}\\ &\leq C\int_{r}^{2R}\Big(\fint_{B_\rho(x_0,\lambda)}\big|f - f_{B_\rho(x_0,\lambda)}\big|^p\,\mathrm{d}\mu\Big)^{1/p}\frac{\mathrm{d}\lambda}{\lambda}\\ &\leq C\int_{r}^{2R}\mathrm{osc}_p(f;\lambda)\,\frac{\mathrm{d}\lambda}{\lambda}. \end{aligned} \tag{7.4.134}$$

At this point, (7.4.128) is seen from (7.4.132) and (7.4.134). □

Here is the second companion result to Lemma 7.4.14, referred to above.

Lemma 7.4.16 *Let (X, ρ, μ) be a measure metric space and fix some $p \in [1, \infty)$. Also, assume*

$$\phi : \big(1/4, \infty\big) \to [0, \infty) \ \ \textit{is a non-increasing function} \tag{7.4.135}$$

and define

$$\widetilde{\phi} : (1, \infty) \to [0, \infty), \quad \widetilde{\phi}(\lambda) := \frac{1}{\lambda}\int_{\lambda/4}^{\infty}\phi(t)\frac{\mathrm{d}t}{t} \ \ \textit{for each} \ \ \lambda > 1. \tag{7.4.136}$$

Then there exists a constant $C \in (0, \infty)$*, independent of* ϕ*, with the property that for each function* $f \in L^1_{\rm loc}(X, \mu)$*, each point* $x_0 \in X$*, and each radius* $r > 0$ *one has* (*recall* (7.4.9))

$$\sum_{j=0}^{\infty} \phi(2^j)\Big(⨍_{B_\rho(x_0,2^j r)} \big|f - f_{B_\rho(x_0,r)}\big|^p \, d\mu\Big)^{1/p} \leq C \int_1^{\infty} \Big(⨍_{B_\rho(x_0,\lambda r)} \big|f - f_{B_\rho(x_0,\lambda r)}\big|^p \, d\mu\Big)^{1/p} \widetilde{\phi}(\lambda)\, d\lambda. \tag{7.4.137}$$

Proof Much as in (7.4.121), for each $j \in \mathbb{N}_0$ we have

$$\Big(⨍_{B_\rho(x_0,2^{j+1} r)} \big|f - f_{B_\rho(x_0,r)}\big|^p \, d\mu\Big)^{1/p} \leq C \sum_{\ell=0}^{j} \Big(⨍_{B_\rho(x_0,2^{\ell+1} r)} \big|f - f_{B_\rho(x_0,2^{\ell+1} r)}\big|^p \, d\mu\Big)^{1/p}. \tag{7.4.138}$$

Multiply by $\phi(2^j)$, sum up in $j \in \mathbb{N}_0$, and interchange the sums in j and ℓ to obtain

$$\sum_{j=0}^{\infty} \phi(2^j)\Big(⨍_{B_\rho(x_0,2^{j+1} r)} \big|f - f_{B_\rho(x_0,r)}\big|^p \, d\mu\Big)^{1/p} \leq C \sum_{\ell=0}^{\infty} \Big(\sum_{j=\ell}^{\infty} \phi(2^j)\Big)\Big(⨍_{B_\rho(x_0,2^{\ell+1} r)} \big|f - f_{B_\rho(x_0,2^{\ell+1} r)}\big|^p \, d\mu\Big)^{1/p}. \tag{7.4.139}$$

Next, observe that since ϕ is non-increasing we have

$$\phi(2^j) \leq (\ln 2) \int_{2^{j-1}}^{2^j} \phi(t)\, \frac{dt}{t} \quad \text{for each} \quad j \in \mathbb{N}_0, \tag{7.4.140}$$

hence for each $\ell \in \mathbb{N}_0$ we may write

$$\sum_{j=\ell}^{\infty} \phi(2^j) \leq (\ln 2) \int_{2^{\ell-1}}^{\infty} \phi(t)\, \frac{dt}{t} \leq C \cdot \inf_{\lambda \in [2^\ell, 2^{\ell+1}]} \big(\lambda \widetilde{\phi}(\lambda)\big), \tag{7.4.141}$$

in view of the definition made in (7.4.136). Also, after integrating in $\lambda \in [2^\ell, 2^{\ell+1}]$ with respect to the Haar measure $d\lambda/\lambda$, from (7.4.124) (written with ℓ in place of j) we see that for each $\ell \in \mathbb{N}_0$ we have

$$\Big(\fint_{B_\rho(x_0,2^\ell r)} \big|f - f_{B_\rho(x_0,2^\ell r)}\big|^p \, d\mu\Big)^{1/p}$$
$$\leq C \int_{2^\ell}^{2^{\ell+1}} \Big(\fint_{B_\rho(x_0,\lambda r)} \big|f - f_{B_\rho(x_0,\lambda r)}\big|^p \, d\mu\Big)^{1/p} \frac{d\lambda}{\lambda}. \tag{7.4.142}$$

Thanks to (7.4.141), this further implies

$$\Big(\sum_{j=\ell}^{\infty}\phi(2^j)\Big)\Big(\fint_{B_\rho(x_0,2^\ell r)} \big|f - f_{B_\rho(x_0,2^\ell r)}\big|^p \, d\mu\Big)^{1/p}$$
$$\leq C\Big(\inf_{\lambda\in[2^\ell,2^{\ell+1}]}\big(\lambda\widetilde{\phi}(\lambda)\big)\Big) \int_{2^\ell}^{2^{\ell+1}} \Big(\fint_{B_\rho(x_0,\lambda r)} \big|f - f_{B_\rho(x_0,\lambda r)}\big|^p \, d\mu\Big)^{1/p} \frac{d\lambda}{\lambda}$$
$$\leq C \int_{2^\ell}^{2^{\ell+1}} \Big(\fint_{B_\rho(x_0,\lambda r)} \big|f - f_{B_\rho(x_0,\lambda r)}\big|^p \, d\mu\Big)^{1/p} \widetilde{\phi}(\lambda)\, d\lambda, \tag{7.4.143}$$

for each $\ell \in \mathbb{N}_0$. Gathering (7.4.139) and (7.4.143) then leads to (7.4.137). □

7.5 Whitney Decompositions on Geometrically Doubling Quasi-Metric Spaces

We begin by defining the quality of being geometrically doubling in the category of quasi-metric spaces (cf. [56, p. 67]).

Definition 7.5.1 *A quasi-metric space (X, ρ) is called* `geometrically doubling` *if there exists a number $N \in \mathbb{N}$, called the geometrically doubling constant of (X, ρ), with the property that any ρ-ball of radius r in X may be covered by a family of cardinality at most N, consisting of ρ-balls in X of radii $r/2$.*

Via iterations it follows that if (X, ρ) is a geometrically doubling quasi-metric space then

for any $\theta \in (0, 1)$ there exists $N \in \mathbb{N}$ so that any ρ-ball of radius $r > 0$ in X may be covered by at most N ρ-balls in X of radii θr. (7.5.1)

It is also useful to note that (cf., e.g., [188])

any space of homogeneous type (X, ρ, μ) is geometrically doubling, in the sense described in Definition 7.5.1. (7.5.2)

The property of being geometrically doubling is hereditary in the following natural sense.

Lemma 7.5.2 *Given a geometrically doubling quasi-metric space* (X, ρ), *for any* $Y \subseteq X$ *the pair* $\big(Y, \rho\big|_{Y\times Y}\big)$ *is also a geometrically doubling quasi-metric space* (*with control of the geometrically doubling constant*).

Proof Any $\big(\rho\big|_{Y\times Y}\big)$-ball in Y is of the form $B_\rho(y, r) \cap Y$ for some $y \in Y$ and $r > 0$. Fix such a ball and, relying on (7.5.1), pick $x_1, \dots, x_N \in X$ with the property that

$$B_\rho(y, r) \subseteq \bigcup_{j=1}^{N} B_\rho\big(x_j, r/2C_\rho^2\big). \tag{7.5.3}$$

Let J be the collection of all $j \in \{1, \dots, N\}$ such that $B_\rho\big(x_j, r/2C_\rho^2\big)$ intersects Y. If for each $j \in J$ we select $y_j \in B_\rho\big(x_j, r/2C_\rho^2\big) \cap Y$, we see from (7.1.2) that $B_\rho\big(x_j, r/2C_\rho^2\big) \subseteq B_\rho(y_j, r/2)$. This goes to show that

$$B_\rho(y, r) \cap Y \subseteq \bigcup_{j\in J} B_\rho(y_j, r/2) \cap Y, \tag{7.5.4}$$

and the desired conclusion follows. □

The version of Whitney's decomposition theorem recorded below is a particular case of [188, Theorem 4.21, p. 184].

Proposition 7.5.3 *Assume* (X, ρ) *is a geometrically doubling quasi-metric space. Then for each* $\lambda \in (1, \infty)$ *there exist constants* $\Lambda \in (\lambda, \infty)$ *and* $M \in \mathbb{N}$, *both depending only on* λ, C_ρ, *and the geometrically doubling constant of* (X, ρ), *and which have the following significance.*

For each open, nonempty, proper subset $\mathcal{O}$ *of the topological space* (X, τ_ρ) *there exists a sequence of* ρ*-balls,* $\big\{B_\rho(x_j, r_j)\big\}_{j\in\mathbb{N}}$, *with centers* $\{x_j\}_{j\in\mathbb{N}} \subset \mathcal{O}$ *and radii* $\{r_j\}_{j\in\mathbb{N}} \subset (0, \infty)$, *for which the following properties are valid:*

(1) $\mathcal{O} = \bigcup_{j\in\mathbb{N}} B_\rho(x_j, r_j)$ *and* $\sum_{j\in\mathbb{N}} \mathbf{1}_{B_\rho(x_j, \lambda r_j)} \leq M$ *on* $\mathcal{O}$. *In fact, there exists* $\varepsilon \in (0, 1)$, *which depends only on* C_ρ, λ, *and the geometrically doubling constant of* (X, ρ), *with the property that for any given point* $x \in \mathcal{O}$ *one has*

$$\#\Big\{ j \in \mathbb{N} : \ B_\rho\big(x, \varepsilon \cdot \mathrm{dist}_\rho(x, X \setminus \mathcal{O})\big) \cap B_\rho(x_j, \lambda r_j) \neq \varnothing \Big\} \leq M. \tag{7.5.5}$$

(2) $B_\rho(x_j, \lambda r_j) \subseteq \mathcal{O}$ *and* $B_\rho(x_j, \Lambda r_j) \cap \big[X \setminus \mathcal{O}\big] \neq \varnothing$ *for every* $j \in \mathbb{N}$.
(3) $r_i \approx r_j$ *uniformly for* $i, j \in \mathbb{N}$ *such that* $B_\rho(x_i, \lambda r_i) \cap B_\rho(x_j, \lambda r_j) \neq \varnothing$.

In the context of Proposition 7.5.3 we shall refer to the collection $\big\{B_\rho(x_j, r_j)\big\}_{j\in\mathbb{N}}$ as being a family of `Whitney balls` (or, alternatively, amounting to a `Whitney decomposition`) for the set $\mathcal{O}$.

In certain situations it is particularly useful to have a Whitney decomposition of a set into mutually disjoint pieces, and a result of this flavor is presented in Proposition 7.5.6 below. As a preamble, we first discuss the existence of a `dyadic grid structure` on geometrically doubling quasi-metric spaces which plays a key role in the proof of Proposition 7.5.6. The following result is essentially due to M. Christ [48] (see also [67, Theorem 3.2, p. 143]), with two refinements, worked out in [123, Proposition 2.11, p. 19]. First, Christ's dyadic grid result is established in the presence of a background doubling, Borel-regular measure, which is more restrictive than merely assuming that the ambient quasi-metric space is geometrically doubling. Second, Christ's dyadic grid result involves a scale $\delta \in (0, 1)$ and it was shown in [123] that we may always take $\delta = \frac{1}{2}$, as in the Euclidean setting.

Proposition 7.5.4 *Let (X, ρ) be a geometrically doubling quasi-metric space and fix an integer $\kappa_X \in \mathbb{Z} \cup \{-\infty\}$ with the property that*

$$2^{-\kappa_X - 1} \leq \operatorname{diam}_\rho(X) \leq 2^{-\kappa_X}. \tag{7.5.6}$$

Then there are finite constants $a_1 \geq a_0 > 0$ such that, for each $k \in \mathbb{Z}$ with $k \geq \kappa_X$, there exists a collection

$$\mathbb{D}_k(X) := \{Q_\alpha^k\}_{\alpha \in I_k} \tag{7.5.7}$$

of subsets of X indexed by a nonempty, at most countable set of indices I_k, as well as a family $\{x_\alpha^k\}_{\alpha \in I_k}$ of points in X, for which the collection of all `dyadic cubes` *in X, i.e.,*

$$\mathbb{D}(X) := \bigcup_{k \in \mathbb{Z},\, k \geq \kappa_X} \mathbb{D}_k(X), \tag{7.5.8}$$

has the following properties:

(1) [All dyadic cubes are open]
For each $k \in \mathbb{Z}$ with $k \geq \kappa_X$ and each $\alpha \in I_k$, the set Q_α^k is open in τ_ρ.

(2) [Dyadic cubes are mutually disjoint within the same generation]
For each $k \in \mathbb{Z}$ with $k \geq \kappa_X$ and each $\alpha, \beta \in I_k$ with $\alpha \neq \beta$ there holds $Q_\alpha^k \cap Q_\beta^k = \varnothing$.

(3) [No partial overlap across generations]
For each $k, \ell \in \mathbb{Z}$ with $\ell > k \geq \kappa_X$, and each $\alpha \in I_k$, $\beta \in I_\ell$, either $Q_\beta^\ell \subseteq Q_\alpha^k$ or $Q_\alpha^k \cap Q_\beta^\ell = \varnothing$.

(4) [Any dyadic cube has a unique ancestor in any earlier generation]
For each $k, \ell \in \mathbb{Z}$ with $k > \ell \geq \kappa_X$ and each $\alpha \in I_k$ there is a unique $\beta \in I_\ell$ such that $Q_\alpha^k \subseteq Q_\beta^\ell$.

(5) [The size is dyadically related to the generation]
For each $k \in \mathbb{Z}$ with $k \geq \kappa_X$ and each $\alpha \in I_k$ one has

$$B_\rho(x_\alpha^k, a_0 2^{-k}) \subseteq Q_\alpha^k \subseteq B_\rho(x_\alpha^k, a_1 2^{-k}). \tag{7.5.9}$$

In particular, given a measure μ *on* X *for which* (X, ρ, μ) *is a space of homogeneous type, there exists some constant* $c \in (0, \infty)$ *such that* $\mu(Q_\beta^{k+1}) \geq c\mu(Q_\alpha^k)$ *whenever* $\alpha \in I_k$ *and* $\beta \in I_{k+1}$ *are such that* $Q_\beta^{k+1} \subseteq Q_\alpha^k$.

(6) [Control of the number of children]
There exists an integer $N \in \mathbb{N}$ *with the property that for each* $k \in \mathbb{Z}$ *with* $k \geq \kappa_X$ *one has*

$$\#\{\beta \in I_{k+1} : Q_\beta^{k+1} \subseteq Q_\alpha^k\} \leq N, \quad \textit{for every} \;\; \alpha \in I_k. \tag{7.5.10}$$

Furthermore, this integer may be chosen such that, for each $k \in \mathbb{Z}$ *with* $k \geq \kappa_X$, *each* $x \in X$, *and* $r \in (0, 2^{-k})$, *the number of* Q*'s in* $\mathbb{D}_k(X)$ *that intersect* $B_\rho(x, r)$ *is at most* N.

(7) [Any generation covers a dense subset of the entire space]
For each $k \in \mathbb{Z}$ *with* $k \geq \kappa_X$, *the set* $\bigcup_{\alpha \in I_k} Q_\alpha^k$ *is dense in* (X, τ_ρ). *In particular, for each* $k \in \mathbb{Z}$ *with* $k \geq \kappa_X$ *one has*

$$X = \bigcup_{\alpha \in I_k} \{x \in X : \operatorname{dist}_\rho(x, Q_\alpha^k) \leq \varepsilon 2^{-k}\}, \qquad \forall \varepsilon > 0, \tag{7.5.11}$$

and there exist $b_0, b_1 \in (0, \infty)$ *depending only on the geometrically doubling character of* X *with the property that*

$$\begin{gathered} \textit{for each } x_o \in X \textit{ and each } r \in \big(0, 2\operatorname{diam}_\rho(X)\big) \textit{ there} \\ \textit{exist } k \in \mathbb{Z} \textit{ with } k \geq \kappa_X \textit{ and } \alpha \in I_k \textit{ such that} \\ Q_\alpha^k \subseteq B_\rho(x_o, r) \;\textit{ and }\; b_0 r \leq 2^{-k} \leq b_1 r. \end{gathered} \tag{7.5.12}$$

Moreover, for each $k \in \mathbb{Z}$ *with* $k \geq \kappa_X$ *and each* $\alpha \in I_k$

$$\bigcup_{\beta \in I_{k+1},\, Q_\beta^{k+1} \subseteq Q_\alpha^k} Q_\beta^{k+1} \;\; \textit{is dense in} \;\; Q_\alpha^k, \tag{7.5.13}$$

and, for each $\varepsilon > 0$, *we have*

$$Q_\alpha^k \subseteq \bigcup_{\beta \in I_{k+1},\, Q_\beta^{k+1} \subseteq Q_\alpha^k} \{x \in X : \operatorname{dist}_\rho(x, Q_\beta^{k+1}) \leq \varepsilon 2^{-k-1}\}. \tag{7.5.14}$$

(8) [Dyadic cubes have thin boundaries with respect to a background doubling measure]
Given a measure μ *on* X *for which* (X, ρ, μ) *is a space of homogeneous type, a collection* $\mathbb{D}(X)$ *may be constructed as in* (7.5.8) *such that properties (1)–(7) above hold and, in addition, there exist constants* $\vartheta \in (0, 1)$ *and* $c \in (0, \infty)$ *such that for each* $k \in \mathbb{Z}$ *with* $k \geq \kappa_X$ *and each* $\alpha \in I_k$ *one has*

$$\mu\left(\{x \in Q_\alpha^k : \operatorname{dist}_{\rho_\#}(x, X \setminus Q_\alpha^k) \leq t \cdot 2^{-k}\}\right) \leq c \cdot t^\vartheta \mu(Q_\alpha^k), \tag{7.5.15}$$

for all $t > 0$. *Moreover, in such a context matters may be arranged so that, for each* $k \in \mathbb{Z}$ *with* $k \geq \kappa_X$ *and each* $\alpha \in I_k$,

$$\big(Q_\alpha^k, \rho|_{Q_\alpha^k}, \mu\lfloor Q_\alpha^k\big) \text{ is a space of homogeneous type,} \tag{7.5.16}$$

and the doubling constant of the measure $\mu\lfloor Q_\alpha^k$ *is independent of* k, α (*i.e.,* the quality of being a space of homogeneous type is hereditary at the level of dyadic cubes, in a uniform fashion).

(9) [Each generation covers the space a.e. with respect to a background doubling measure]
If μ *is a measure on* X *for which* (X, ρ, μ) *is a space of homogeneous type, then a collection* $\mathbb{D}(X)$ *as in* (7.5.8) *may be associated with the doubling measure* μ *such that properties (1)–(8) above hold and, in addition,*

$$\text{if } N_k := X \setminus \bigcup_{\alpha \in I_k} Q_\alpha^k \text{ then } \mu(N_k) = 0, \text{ for each } k \in \mathbb{Z},\ k \geq \kappa_X. \tag{7.5.17}$$

In particular, in such a setting, for each $k \in \mathbb{Z}$ *with* $k \geq \kappa_X$ *one has*

$$\mu\Big(Q_\alpha^k \setminus \bigcup_{\beta \in I_{k+1},\ Q_\beta^{k+1} \subseteq Q_\alpha^k} Q_\beta^{k+1}\Big) = 0, \text{ for every } \alpha \in I_k. \tag{7.5.18}$$

(10) [The boundaries of dyadic cubes are nullsets of any given background doubling measure]
If μ *is a measure on* X *for which* (X, ρ, μ) *is a space of homogeneous type, then a collection* $\mathbb{D}(X)$ *as in* (7.5.8) *may be associated with the doubling measure* μ *such that properties (1)–(9) above hold and, in addition,*

$$\begin{array}{c} \partial Q_\alpha^k \subseteq N_k \text{ for each } k \in \mathbb{Z} \text{ with } k \geq \kappa_X, \\ \text{where the set } N_k \text{ is defined as in (7.5.17).} \end{array} \tag{7.5.19}$$

In particular, in such a setting, for each $k \in \mathbb{Z}$ *with* $k \geq \kappa_X$ *one has*

$$\mu(\partial Q_\alpha^k) = 0 \text{ for every } \alpha \in I_k. \tag{7.5.20}$$

(11) [Dyadic tiling of open sets with respect to a given background doubling measure]
If μ *is a measure on* X *for which* (X, ρ, μ) *is a space of homogeneous type, then a collection* $\mathbb{D}(X)$ *as in* (7.5.8) *may be associated with the doubling measure* μ *such that properties (1)–(10) above hold and such that, given any open set* $\mathcal{O} \subseteq X$ *with* $\mu(\mathcal{O}) < \infty$, *if*

$$\begin{array}{l} \mathbb{D}_{\mathcal{O}} \text{ denotes the collection of all “maximal dyadic cubes” contained} \\ \text{in } \mathcal{O} \text{ (i.e., if } \mathbb{D}_{\mathcal{O}} \text{ is the family of sets } Q \in \mathbb{D}(X) \text{ with } Q \subseteq \mathcal{O} \text{ and for} \\ \text{which one cannot find any } \widetilde{Q} \in \mathbb{D}(X) \text{ such that } Q \subseteq \widetilde{Q} \subseteq \mathcal{O} \text{ and} \\ Q \neq \widetilde{Q}) \end{array} \tag{7.5.21}$$

it follows that any two sets in $\mathbb{D}_{\mathcal{O}}$ are disjoint and

$$\mu\Big(\mathcal{O}\setminus\bigcup_{Q\in\mathbb{D}_{\mathcal{O}}}Q\Big)=0. \tag{7.5.22}$$

Proof The proofs of the claims in items *(1)–(9)* may be found in [123, Proposition 2.11, p. 19]. As regards item *(10)*, having fixed $k\in\mathbb{Z}$ with $k\geq\kappa_X$ we claim that

$$\overline{Q^k_\alpha}\cap Q^k_\beta=\varnothing \text{ for every } \alpha,\beta\in I_k \text{ with } \alpha\neq\beta. \tag{7.5.23}$$

Indeed, if $\overline{Q^k_\alpha}\cap Q^k_\beta\neq\varnothing$ then, since Q^k_β is open, we would have $Q^k_\alpha\cap Q^k_\beta\neq\varnothing$ which cannot happen for any two distinct indices $\alpha,\beta\in I_k$. In turn, from (7.5.23) and (7.5.17) we conclude (bearing in mind the property from item *(2)*) that for each $\alpha\in I_k$ we have

$$\begin{aligned}\partial Q^k_\alpha\setminus N_k&=\big(\overline{Q^k_\alpha}\setminus Q^k_\alpha\big)\setminus N_k=\big(\overline{Q^k_\alpha}\cap(X\setminus N_k)\big)\cap\big((X\setminus Q^k_\alpha)\cap(X\setminus N_k)\big)\\&=\Big(\overline{Q^k_\alpha}\cap\Big(\bigcup_{\beta\in I_k}Q^k_\beta\Big)\Big)\cap\Big(\Big(\bigcup_{\beta\in I_k}Q^k_\beta\Big)\setminus Q^k_\alpha\Big)\\&=Q^k_\alpha\cap\Big(\bigcup_{\beta\in I_k\setminus\{\alpha\}}Q^k_\beta\Big)=\varnothing.\end{aligned} \tag{7.5.24}$$

This proves (7.5.19). Then (7.5.20) follows on account of (7.5.17).

To justify (7.5.22), fix an open set $\mathcal{O}\subseteq X$ with $\mu(\mathcal{O})<+\infty$. With the set N_k defined as in (7.5.17) for each $k\in\mathbb{Z}$ with $k\geq\kappa_X$, introduce

$$N:=\bigcup_{k\in\mathbb{Z},\,k\geq\kappa_X}N_k. \tag{7.5.25}$$

Then (7.5.17) ensures that

$$N \text{ is a } \mu\text{-measurable subset of } X \text{ with } \mu(N)=0. \tag{7.5.26}$$

In relation to this, we claim that

$$\begin{aligned}&\text{for each } x\in\mathcal{O}\setminus N \text{ there exists a unique}\\&Q\in\mathbb{D}_{\mathcal{O}} \text{ with the property that } x\in Q.\end{aligned} \tag{7.5.27}$$

Indeed, given $x\in\mathcal{O}\setminus N$ it follows that there exists $r\in\big(0,2\,\mathrm{diam}_\rho(X)\big)$ such that $B_\rho(x,r)\subseteq\mathcal{O}$. Pick $k\in\mathbb{Z}$ with $k\geq\kappa_X$ large enough so that

$$C_\rho\widetilde{C}_\rho a_1 2^{-k}<r \tag{7.5.28}$$

where $C_\rho, \widetilde{C}_\rho \in [1,\infty)$ are as in (7.1.3) and (7.1.4). Next, the fact that $x \in X \setminus N_k$ implies (cf. (7.5.17)) that there exists $\alpha \in I_k$ such that $x \in Q_\alpha^k$. In concert with (7.5.9), this gives

$$x \in Q_\alpha^k \subseteq B_\rho(x_\alpha^k, a_1 2^{-k}). \tag{7.5.29}$$

Hence, further, $\rho(x, x_\alpha^k) < a_1 2^{-k}$. Consequently, for each $y \in B_\rho(x_\alpha^k, a_1 2^{-k})$ we may estimate

$$\begin{aligned}\rho(x,y) &\leq C_\rho \max\{\rho(x,x_\alpha^k), \rho(x_\alpha^k, y)\} \leq C_\rho \max\Big\{\rho(x,x_\alpha^k), \widetilde{C}_\rho \rho(y, x_\alpha^k)\Big\} \\ &\leq C_\rho \max\big\{a_1 2^{-k}, \widetilde{C}_\rho a_1 2^{-k}\big\} = C_\rho \widetilde{C}_\rho a_1 2^{-k} < r. \end{aligned}\tag{7.5.30}$$

Thus the point y belongs to $B_\rho(x,r)$ and since $y \in B_\rho(x_\alpha^k, a_1 2^{-k})$ was arbitrary, this proves that $B_\rho(x_\alpha^k, a_1 2^{-k}) \subseteq B_\rho(x,r) \subseteq \mathcal{O}$. Based on this and (7.5.29) we ultimately conclude that

$$x \in Q_\alpha^k \subseteq \mathcal{O}. \tag{7.5.31}$$

To proceed, denote by J_x the set of all $\ell \in \mathbb{Z}$ with $\ell \geq \kappa_X$ and such that there is $\beta \in I_\ell$ such that $x \in Q_\beta^\ell \subseteq \mathcal{O}$. From (7.5.31) we know that $k \in J_x$, so J_x is a nonempty subset of $\mathbb{Z}$. We claim that J_x is bounded from below. This is clear if $\kappa_X > -\infty$. Suppose next that $\kappa_X = -\infty$, a scenario in which X is unbounded (cf. (7.5.6)). As is well known (cf., e.g., [11]), this is further equivalent to having $\mu(X) = +\infty$. Seeking a contradiction, assume that J_x is not actually bounded from below. Then we would be able to find a sequence $\{\ell_j\}_{j\in\mathbb{N}} \subseteq \mathbb{Z}$ convergent to $-\infty$ such that for each $j \in \mathbb{N}$ there exists $\alpha_j \in I_{\ell_j}$ for which $x \in Q_{\alpha_j}^{\ell_j} \subseteq \mathcal{O}$. From this, (7.5.9), and the fact that μ is doubling we may then conclude that there exists a constant $C \in (0,\infty)$ such that

$$\mu\big(B_{\rho_\#}(x, 2^{-\ell_j})\big) \leq C\mu(Q_{\alpha_j}^{\ell_j}) \ \text{ for each } \ j \in \mathbb{N}. \tag{7.5.32}$$

However, since we presently have $\mu\big(B_{\rho_\#}(x, 2^{-\ell_j})\big) \nearrow \mu(X) = +\infty$ as $j \to \infty$ while at the same time $\mu(Q_{\alpha_j}^{\ell_j}) \leq \mu(\mathcal{O}) < +\infty$ for each $j \in \mathbb{N}$ leads to a contradiction. This ultimately shows that J_x is indeed bounded from below. As such $k_0 := \inf J_x$ is a well-defined integer, belonging to J_x. Consequently, there exists $\alpha_0 \in I_{k_0}$ such that $x \in Q_{\alpha_0}^{k_0} \subseteq \mathcal{O}$. We claim that $Q_{\alpha_0}^{k_0} \in \mathbb{D}_{\mathcal{O}}$. To justify this, assume we can find some $\widetilde{Q} \in \mathbb{D}(X)$ such that $Q_{\alpha_0}^{k_0} \subseteq \widetilde{Q} \subseteq \mathcal{O}$ and $Q_{\alpha_0}^{k_0} \neq \widetilde{Q}$, with the goal of finding a contradiction. Then there exist $\ell \in \mathbb{Z}$ with $\ell \geq \kappa_X$ along with $\beta \in I_\ell$ such that $\widetilde{Q} = Q_\beta^\ell$. This places the number ℓ in the set J_x, so we necessarily have $\ell \geq k_0$. Since $x \in Q_{\alpha_0}^{k_0} \cap Q_\beta^\ell$, items *(2)–(3)* imply that $Q_\beta^\ell \subseteq Q_{\alpha_0}^{k_0}$. Via double inclusion we therefore have $Q_\beta^\ell = Q_{\alpha_0}^{k_0}$, contradicting the fact that $Q_{\alpha_0}^{k_0} \neq \widetilde{Q} = Q_\beta^\ell$. This shows that, as claimed, $Q_{\alpha_0}^{k_0} \in \mathbb{D}_{\mathcal{O}}$.

To complete the proof of (7.5.27) there remains to prove uniqueness. In this regard, observe that if $Q, Q' \in \mathbb{D}_{\mathcal{O}}$ are such that $x \in Q$ and $x \in Q'$ then $Q, Q' \in \mathbb{D}(X)$ satisfy $Q \cap Q' \neq \varnothing$. From items *(2)–(3)* we then see that either $Q \subseteq Q' \subseteq \mathcal{O}$ or $Q' \subseteq Q \subseteq \mathcal{O}$. In any eventuality we may then conclude that $Q = Q'$ by maximality.

At this point, (7.5.27) is established. In turn, from (7.5.27) we deduce that

$$\mathcal{O} \setminus N \subseteq \bigcup_{Q \in \mathbb{D}_{\mathcal{O}}} Q \subseteq \mathcal{O}, \tag{7.5.33}$$

so

$$\mathcal{O} \setminus \bigcup_{Q \in \mathbb{D}_{\mathcal{O}}} Q \subseteq N \tag{7.5.34}$$

from which (7.5.22) follows in view of (7.5.26). □

A few words clarifying terminology inspired by Proposition 7.5.4 are in order.

Remark 7.5.5 *As already mentioned in the statement, sets Q belonging to $\mathbb{D}(X)$ will be referred to as* `dyadic cubes` *(on X). Moreover, for each $k \in \mathbb{Z}$ with $k \geq \kappa_X$, we shall call $\mathbb{D}_k(X)$ the* `dyadic cubes of generation` *k and, for each $Q \in \mathbb{D}_k(X)$, define the* `side-length` *of Q to be $\ell(Q) := 2^{-k}$, and the* `center` *of Q to be the point $x_Q := x^k_\alpha \in X$ if $Q = Q^k_\alpha$.*

Finally, we make the convention that saying that $\mathbb{D}(X)$ is a `dyadic cube structure` *(or* `dyadic grid`*)* `on` *X will always indicate that the collection $\mathbb{D}(X)$ is associated with X as in Proposition 7.5.4. This presupposes that X is the ambient set for a geometrically doubling quasi-metric space, in which case $\mathbb{D}(X)$ satisfies properties (1)–(7) above and that, in the presence of a background Borel doubling measure μ, properties (8) and (9) also hold.*

Here is the Whitney decomposition result advertised earlier which, in contrast to the one presented in Proposition 7.5.3, allows breaking up a set into mutually disjoint pieces; on occasions, this is quite a desirable feature.

Proposition 7.5.6 *Let (X, ρ, μ) be a space of homogeneous type. In the case when X is unbounded, make the additional assumption that for each sequence of ρ-balls $\{B_j\}_{j\in\mathbb{N}}$ in X there holds*

$$\lim_{j\to\infty} \mu(B_j) = +\infty \quad \textit{if} \quad \text{radius}(B_j) \to +\infty \quad \textit{as} \quad j \to \infty. \tag{7.5.35}$$

Also, consider a dyadic grid $\mathbb{D}(X)$ on X, in the sense of Proposition 7.5.4; in particular recall the constant a_1 from (7.5.9).

Then for each $\lambda \in (a_1, \infty)$ there exists a constant $\Lambda \in (0, \infty)$, depending only on λ and the ambient, such that for each open, nonempty, proper subset $\mathcal{O}$ of the topological space (X, τ_ρ) with the property that $\mu(\mathcal{O}) < +\infty$ there exists a sequence of dyadic cubes $\{Q_j\}_{j\in\mathbb{N}} \subseteq \mathbb{D}(X)$ for which

(1) the cubes $\{Q_j\}_{j\in\mathbb{N}}$ are mutually disjoint, are contained in $\mathcal{O}$, and also satisfy $\mu\Big(\mathcal{O} \setminus \bigcup_{j\in\mathbb{N}} Q_j\Big) = 0$;

(2) $B_\rho\big(x_j, \lambda\,\ell(Q_j)\big) \subseteq \mathcal{O}$ and $\text{dist}_\rho\big(Q_j, X \setminus \mathcal{O}\big) \leq \Lambda \cdot \ell(Q_j)$ for each $j \in \mathbb{N}$, where x_j and $\ell(Q_j)$ denote, respectively, the center and side-length of Q_j.

Proof With N_k as in (7.5.17), observe that if

$$N := \bigcup_{k\in\mathbb{Z},\ k\geq\kappa_X} N_k \quad \text{then} \quad \mu(N) = 0. \tag{7.5.36}$$

From items *(2)* and *(9)* in Proposition 7.5.4 we see that for each $x \in X \setminus N$ and each $k \in \mathbb{Z}$ with $k \geq \kappa_X$ there exists a unique dyadic cube $Q_k^{(x)} \in \mathbb{D}_k(X)$ containing x. In addition, item *(3)* in Proposition 7.5.4 implies

$$x \in Q_{k+1}^{(x)} \subseteq Q_k^{(x)} \quad \text{for each} \quad k \in \mathbb{Z} \quad \text{with} \quad k \geq \kappa_X. \tag{7.5.37}$$

Henceforth, assume some open, nonempty, proper subset $\mathcal{O}$ of X has been given, and fix a point $x \in \mathcal{O} \setminus N$. For each $k \in \mathbb{Z}$ with $k \geq \kappa_X$ introduce

$$c_k := 2^k \cdot \operatorname{dist}_\rho\big(Q_k^{(x)}, X \setminus \mathcal{O}\big) \geq 0. \tag{7.5.38}$$

In relation to these numbers we make five useful observations. First, the fact that $\mathcal{O}$ is open implies that c_k becomes strictly positive if k is large enough. Second, since each cube $Q_k^{(x)}$ contains the point x, from (7.5.38) we see that

$$c_k \leq 2^k \cdot \operatorname{dist}_\rho(x, X \setminus \mathcal{O}) \quad \text{for each} \quad k \in \mathbb{Z} \quad \text{with} \quad k \geq \kappa_X. \tag{7.5.39}$$

Third, from (7.5.39) and (7.5.6) we obtain

$$c_{\kappa_X} \leq 2^{\kappa_X} \cdot \operatorname{diam}_\rho(X) \leq 1 \quad \text{in the case when } X \text{ is bounded.} \tag{7.5.40}$$

Fourth, observe that (7.5.37) entails

$$2c_k \leq c_{k+1} \quad \text{for each} \quad k \in \mathbb{Z} \quad \text{with} \quad k \geq \kappa_X. \tag{7.5.41}$$

Fifth, since $\operatorname{dist}_\rho\big(Q_{k+1}^{(x)}, X \setminus \mathcal{O}\big) \leq C \cdot \operatorname{dist}_\rho\big(Q_k^{(x)}, X \setminus \mathcal{O}\big) + C \cdot 2^{-k}$, we conclude that there exists some purely geometric number $\theta \in (0, 1)$, independent of k and x, such that

$$\theta \cdot c_{k+1} - 1 \leq c_k \quad \text{for each} \quad k \in \mathbb{Z} \quad \text{with} \quad k \geq \kappa_X. \tag{7.5.42}$$

Pressing on, bring in an arbitrary $\lambda \in (a_1, \infty)$. Make the general observation, of geometric nature, that for each given constant $A > 0$ there exists some small number $\varepsilon = \varepsilon(\lambda, A, \rho) > 0$, independent of x, such that

$$\begin{aligned} B_\rho(z, \lambda r) \subseteq \mathcal{O} \ \text{ if } \ 0 < r < \varepsilon \operatorname{dist}_\rho(x, X \setminus \mathcal{O}) \\ \text{and } \ z \in X \ \text{ with } \ \rho(x, z) < A \cdot r. \end{aligned} \tag{7.5.43}$$

In turn, such a number ε corresponding to the choice $A := a_1$ with a_1 as in (7.5.9) guarantees that, with $z_k^{(x)}$ and $\ell(Q_k^{(x)}) = 2^{-k}$ denoting the center and, respectively, the side-length of the dyadic cube $Q_k^{(x)}$, we have

$$B_\rho\big(z_k^{(x)}, \lambda\,\ell(Q_k^{(x)})\big) \subseteq \mathcal{O} \text{ provided } \varepsilon^{-1} \leq 2^k \cdot \operatorname{dist}_\rho(x, X \setminus \mathcal{O}). \tag{7.5.44}$$

Suppose now that some number $M > \max\{1, \varepsilon^{-1}\}$ has been fixed, and let $k_x \in \mathbb{Z}$ with $k_x \geq \kappa_X$ be the smallest integer $k \in \mathbb{Z}$ with $k \geq \kappa_X$ for which $c_{k+1} > M$. Then the number k_x is well defined and, by design, $c_{k_x+1} > M \geq c_{k_x}$. Since we also know from (7.5.42) that $c_{k_x} \geq \theta \cdot c_{k_x+1} - 1$, we eventually conclude that

$$M < c_{k_x+1} \leq \theta^{-1}(M+1). \tag{7.5.45}$$

In light of (7.5.38), the second inequality in (7.5.45) ultimately entails

$$\operatorname{dist}_\rho\big(Q_{k_x+1}^{(x)}, X \setminus \mathcal{O}\big) \leq \Lambda \cdot \ell\big(Q_{k_x+1}^{(x)}\big) \text{ where } \Lambda := \theta^{-1}(M+1). \tag{7.5.46}$$

Bearing in mind that $M > \varepsilon^{-1}$, from (7.5.44), (7.5.38), and the first inequality in (7.5.45) we also conclude that

$$B\big(z_{k_x+1}^{(x)}, \lambda\,\ell\big(Q_{k_x+1}^{(x)}\big)\big) \subseteq \mathcal{O}. \tag{7.5.47}$$

In particular, since $\lambda > a_1$ to begin with, from (7.5.47) and (7.5.9) we conclude that

$$Q_{k_x+1}^{(x)} \subseteq \mathcal{O}. \tag{7.5.48}$$

Consider next the family of all dyadic cubes of the form $Q_{k_x+1}^{(x)}$ corresponding to $x \in \mathcal{O} \setminus N$. Assume all redundant duplications have been eliminated, and denote the resulting set of distinct dyadic cubes by $\mathcal{F}$. Then we have

$$Q \subseteq \mathcal{O} \text{ for each } Q \in \mathcal{F} \text{ and } \mu\Big(\mathcal{O} \setminus \bigcup_{Q \in \mathcal{F}} Q\Big) = 0, \tag{7.5.49}$$

as well as

$$B_\rho\big(x_Q, \lambda\,\ell(Q)\big) \subseteq \mathcal{O} \text{ and } \operatorname{dist}_\rho\big(Q, X \setminus \mathcal{O}\big) \leq \Lambda \cdot \ell(Q) \text{ for each } Q \in \mathcal{F}, \text{ where } x_Q \text{ and } \ell(Q) \text{ denote, respectively, the center and side-length of } Q. \tag{7.5.50}$$

Since the two inequalities in (7.5.50) imply the existence of a finite constant $C > 0$ with the property that $\operatorname{dist}_\rho\big(x_Q, X \setminus \mathcal{O}\big) \leq C \cdot \ell(Q)$ for each $Q \in \mathcal{F}$, it follows that we cannot have an infinite nested sequence $Q_1 \supset Q_2 \supset \cdots \supset Q_j \supset Q_{j+1} \supset \cdots$ of cubes in $\mathcal{F}$ for which $\ell(Q_j) \to 0$ as $j \to \infty$. Indeed, this would imply that $\{x_{Q_j}\}_{j\in\mathbb{N}} \subseteq Q_1$ with $\lim_{j\to\infty} \operatorname{dist}_\rho\big(x_{Q_j}, X \setminus \mathcal{O}\big) = 0$, contradicting the fact that $Q_1 \subseteq \mathcal{O}$ is quantitatively separated from $X \setminus \mathcal{O}$ (cf. (7.5.50)). Thus, any nested subfamily of in $\mathcal{F}$ necessarily has a smallest cube.

At the other end of the spectrum, if X is bounded, $\ell(Q) \leq 2^{-\kappa_X} < +\infty$ for each $Q \in \mathbb{D}(X)$ by design. Hence, in particular, all cubes in $\mathcal{F}$ obey a uniform upper bound on their side-lengths. If X is unbounded, then the additional property (7.5.35) is in effect. As such, the fact that $B_\rho\big(x_Q, \lambda\,\ell(Q)\big) \subseteq \mathcal{O}$ for each $Q \in \mathcal{F}$ together with the knowledge that $\mathcal{O}$ has finite measure imply that $\sup_{Q\in\mathcal{F}} \ell(Q) < +\infty$.

Let us record our progress. The above reasoning shows that any nested subfamily of $\mathcal{F}$ necessarily is finite. As such, if we now consider the subset of $\mathcal{F}$ consisting of all maximal cubes (with respect to inclusion), then this may be relabeled as $\{Q_j\}_{j\in\mathbb{N}}$ yielding a sequence of mutually disjoint dyadic cubes $\{Q_j\}_{j\in\mathbb{N}} \subseteq \mathbb{D}(X)$ which, thanks to (7.5.49)–(7.5.50), satisfy properties *(1)–(2)* in the statement. □

We conclude by discussing a brand of Vitali's Covering Lemma in generic quasi-metric spaces, and one of its useful corollaries. These should be compared with [57, Theorem 3.1, p. 623], [35, Lemma 3, p. 299] in the less general setting of spaces of homogeneous type, and with [56, Theorem 1.2, p. 69] for a version which requires the underlying set to be bounded. See also [118, Theorem 1.2, p. 2] for a version in metric spaces.

Lemma 7.5.7 *Let (X, ρ) be a quasi-metric space and fix $C_o \in \big(0, C_\rho^2 \cdot \widetilde{C}_\rho\big)$, where $C_\rho, \widetilde{C}_\rho$ are as in (7.1.3) and (7.1.4). Consider a family of ρ-balls*

$$\mathcal{F} = \big\{B_\rho(x_\alpha, r_\alpha)\big\}_{\alpha\in I}, \quad x_\alpha \in X, \; r_\alpha > 0 \;\; \textit{for every} \;\; \alpha \in I, \tag{7.5.51}$$

such that

$$\sup_{\alpha\in I} r_\alpha < +\infty. \tag{7.5.52}$$

In addition, suppose that either

$$(X, \tau_\rho) \;\textit{is separable}, \tag{7.5.53}$$

(recall from [188, (4.49), p. 164] that this condition is always satisfied if the quasi-metric space (X, ρ) is geometrically doubling in the sense of Definition 7.5.1), or

$$\begin{array}{l}\textit{for every sequence } \{B_\rho(x_j, r_j)\}_{j\in\mathbb{N}} \subseteq \mathcal{F} \textit{ consisting} \\ \textit{of mutually disjoint of } \rho\textit{-balls one has } \lim\limits_{j\to\infty} r_j = 0.\end{array} \tag{7.5.54}$$

Then there exists an at most countable set $J \subseteq I$ with the property that

$$B_\rho(x_j, r_j) \cap B_\rho(x_{j'}, r_{j'}) = \varnothing \;\; \textit{for each} \;\; j, j' \in J \;\; \textit{with} \;\; j \neq j', \tag{7.5.55}$$

and each ρ-ball from $\mathcal{F}$ is contained in a dilated ρ-ball of the form $B_\rho(x_j, C_o r_j)$ for some $j \in J$. In particular,

$$\bigcup_{\alpha\in I} B_\rho(x_\alpha, r_\alpha) \subseteq \bigcup_{j\in J} B_\rho(x_j, C_o r_j). \tag{7.5.56}$$

Proof This is a consequence of Lemma 6.3.1, used here with $A(x, r) := B_\rho(x, r)$ for each $x \in X$ and $r > 0$. The version when (7.5.53) holds is based on Remark 6.3.2 and the readily verified fact that

$$B_\rho(x, r) \subseteq \big(B_\rho(x, \theta^{-1} r)\big)^\circ, \quad \forall x \in X, \ \forall r > 0, \ \forall \theta \in (0, C_\rho^{-1}), \tag{7.5.57}$$

(where S° denotes the interior of a set $S \subseteq X$ with respect to the topology τ_ρ), whereas the version when (7.5.54) holds is modeled upon (6.3.4). In either case, a direct computation shows that the enveloping property stated in part *(b)* of Lemma 6.3.1 holds for the choice $C := C_\rho^2 \cdot \widetilde{C}_\rho$. Consequently, if we choose $\lambda > 1$ such that $\lambda C < C_o$ to being with, it follows from (6.3.3) that the condition described just above (7.5.56) holds as well. □

Here is a rather versatile corollary of Lemma 7.5.7, more in line with the original Euclidean result.

Lemma 7.5.8 *Assume (X, ρ) is a geometrically doubling quasi-metric space. Then there exists a constant $c = c(X, \rho) \in (1, \infty)$ with the following property. Suppose $E \subseteq X$ and $r : E \to (0, \infty)$ are such that either the set E is ρ-bounded, or the function r is bounded.*

Then there exists an at most countable sequence of points $\{x_j\}_{j \in J}$ in E such that

$$B_\rho(x_j, r(x_j)) \cap B_\rho(x_{j'}, r(x_{j'})) = \varnothing \ \text{ for all } \ j, j' \in J, \ \ j \neq j' \tag{7.5.58}$$

and

$$E \subseteq \bigcup_{j \in J} B_\rho(x_j, c r(x_j)). \tag{7.5.59}$$

Proof The case when $r : E \to (0, \infty)$ is bounded is a direct consequence of Lemma 7.5.7. If the function $r : E \to (0, \infty)$ is unbounded but the set E is ρ-bounded, then there exists some $x_* \in E$ such that $E \subseteq B_\rho(x_*, r(x_*)\big)$, so the conclusion is trivially true (by taking family $\{x_j\}_{j \in J}$ to be the singleton $\{x_*\}$). □

7.6 The Hardy–Littlewood Maximal Operator on Spaces of Homogeneous Type

Let (X, ρ, μ) be a space of homogeneous type with the property that the quasi-distance $\rho : X \times X \to [0, \infty)$ is continuous[11] in the product topology $\tau_\rho \times \tau_\rho$. Given $s \in (0, \infty)$ and $0 \leq \alpha < \frac{1}{s}$, define the L^s-based fractional Hardy–Littlewood maximal operator of order α as

[11] Recall from Theorem 7.1.2 that any quasi-metric space has an equivalent quasi-distance which satisfies this property.

$$\mathcal{M}_{X,s,\alpha} f(x) := \sup_{r>0} \left[\mu(B_\rho(x,r))^\alpha \Big(⨍_{B_\rho(x,r)} |f|^s \, d\mu \Big)^{\frac{1}{s}} \right], \quad \forall x \in X, \tag{7.6.1}$$

for each μ-measurable function f on X.

Theorem 7.6.1 *Let (X, ρ, μ) be a space of homogeneous type having the property that the quasi-distance $\rho : X \times X \to [0, \infty)$ is continuous in the product topology $\tau_\rho \times \tau_\rho$. Having fixed an integrability exponent $s \in (0, \infty)$ along with some power $\alpha \in [0, 1/s)$ consider $\mathcal{M}_{X,s,\alpha}$, the L^s-based fractional Hardy–Littlewood operator of order α defined in this setting as in (7.6.1). Then*

$$\mathcal{M}_{X,s,\alpha} f \text{ is } \mu\text{-measurable, for each } \mu\text{-measurable function } f \text{ on } X. \tag{7.6.2}$$

Moreover,

$$\begin{gathered} \mathcal{M}_{X,s,\alpha} : L^{p,q}(X, \mu) \longrightarrow L^{p^*,q}(X, \mu) \\ \text{is well defined, sub-linear, and bounded whenever} \\ s < p < \tfrac{1}{\alpha}, \ \ p^* := \big(\tfrac{1}{p} - \alpha\big)^{-1}, \text{ and } 0 < q \leq \infty. \end{gathered} \tag{7.6.3}$$

In particular,

$$\begin{gathered} \mathcal{M}_{X,s,\alpha} : L^{p}(X, \mu) \longrightarrow L^{p^*}(X, \mu) \\ \text{is well defined, sub-linear, and bounded} \\ \text{whenever } s < p < \tfrac{1}{\alpha} \text{ and } p^* := \big(\tfrac{1}{p} - \alpha\big)^{-1}. \end{gathered} \tag{7.6.4}$$

Furthermore, corresponding to the end-point case $p = s$,

$$\begin{gathered} \mathcal{M}_{X,s,\alpha} : L^{s}(X, \mu) \longrightarrow L^{\frac{s}{1-\alpha s},\infty}(X, \mu) \\ \text{is well defined, sub-linear, and bounded,} \end{gathered} \tag{7.6.5}$$

and, corresponding to the end-point case $p = 1/\alpha$,

$$\begin{gathered} \mathcal{M}_{X,s,\alpha} : L^{1/\alpha}(X, \mu) \longrightarrow L^{\infty}(X, \mu) \\ \text{is well defined, sub-linear, and bounded.} \end{gathered} \tag{7.6.6}$$

In the sequel, for each $s \in (0, \infty)$ it is convenient to abbreviate $\mathcal{M}_{X,s} := \mathcal{M}_{X,s,0}$, i.e., define the action of the L^s-based Hardy–Littlewood maximal operator $\mathcal{M}_{X,s}$ on each μ-measurable function f on X as

$$\mathcal{M}_{X,s} f(x) := \big[\mathcal{M}_X(|f|^s)\big]^{1/s}(x) = \sup_{r>0} \Big(⨍_{B_\rho(x,r)} |f|^s \, d\mu \Big)^{\frac{1}{s}}, \ \forall x \in X. \tag{7.6.7}$$

As a direct consequence of Theorem 7.6.1, we obtain the following result:

Corollary 7.6.2 *Suppose (X, ρ, μ) is a space of homogeneous type with the property that the quasi-distance $\rho : X \times X \to [0, \infty)$ is continuous in the product topology $\tau_\rho \times \tau_\rho$, and fix some $s \in (0, \infty)$. Then*

$$\mathcal{M}_{X,s} f \text{ is } \mu\text{-measurable, for each } \mu\text{-measurable function } f \text{ on } X. \tag{7.6.8}$$

Furthermore,

$$\begin{gathered}\mathcal{M}_{X,s} : L^{p,q}(X,\mu) \longrightarrow L^{p,q}(X,\mu) \text{ is well defined, sub-linear,} \\ \text{and bounded if } s < p \leq \infty \text{ and } 0 < q \leq \infty, \\ \text{with the convention that } q = \infty \text{ if } p = \infty,\end{gathered} \tag{7.6.9}$$

and, corresponding to the case $p = s$,

$$\begin{gathered}\mathcal{M}_{X,s} : L^{s}(X,\mu) \longrightarrow L^{s,\infty}(X,\mu) \\ \text{is well defined, sub-linear, and bounded.}\end{gathered} \tag{7.6.10}$$

In particular,

$$\begin{gathered}\mathcal{M}_{X,s} : L^{p}(X,\mu) \longrightarrow L^{p}(X,\mu) \text{ is well defined,} \\ \text{sub-linear, and bounded if } p \in (s,\infty].\end{gathered} \tag{7.6.11}$$

We shall occasionally need to consider the truncated (or local) version of the above L^s-based Hardy–Littlewood maximal operator. Specifically, suppose (X, ρ, μ) is a space of homogeneous type in which the quasi-distance $\rho : X \times X \to [0, \infty)$ is continuous in the product topology $\tau_\rho \times \tau_\rho$. Also, pick an exponent $s \in (0, \infty)$ and a scale $R \in (0, \infty)$. In this context, for each μ-measurable function f defined on X set

$$\mathcal{M}^R_{X,s} f(x) := \sup_{0<r\leq R} \Big(⨍_{B_\rho(x,r)} |f|^s \, d\mu \Big)^{\frac{1}{s}}, \qquad \forall x \in X. \tag{7.6.12}$$

In relation to this, we mention that

$$\begin{gathered}\mathcal{M}^R_{X,s} : L^p_{\mathrm{loc}}(X,\mu) \longrightarrow L^p_{\mathrm{loc}}(X,\mu) \text{ is well defined ,} \\ \text{sub-linear, and bounded if } p \in (s,\infty].\end{gathered} \tag{7.6.13}$$

Explicitly, the boundedness in question amounts to saying that for every ρ-ball $B \subseteq X$ there exist a constant $C \in (0, \infty)$ and some ρ-ball $\widetilde{B} \subseteq X$ such that

$$\big\| \mathcal{M}^R_{X,s} f \big\|_{L^p(B,\mu)} \leq C \|f\|_{L^p(\widetilde{B},\mu)} \text{ for each } f \in L^p_{\mathrm{loc}}(X,\mu). \tag{7.6.14}$$

To see this, it suffices to observe that for each $f \in L^p_{\mathrm{loc}}(X, \mu)$ and each ρ-ball $B_\rho(x_o, R_o)$ in X we have

$$\big(\mathcal{M}^R_{X,s} f\big)(x) \leq \big(\mathcal{M}_{X,s}(f \cdot \mathbf{1}_{B_\rho(x_o, C(R+R_o))})\big)(x) \tag{7.6.15}$$

for each $x \in B_\rho(x_o, R_o)$, where $C \in (0, \infty)$ depends only on the quasi-distance ρ. Based on this and (7.6.11), the claim made in (7.6.14) follows.

Moving on, it is worth dealing separately with the standard Hardy–Littlewood maximal operator, defined as in (7.6.7) with $s := 1$. Specifically, by further specializing Corollary 7.6.2 to this choice of s yields the following result.[12]

Corollary 7.6.3 *Let (X, ρ, μ) be a space of homogeneous type with the property that the quasi-distance $\rho : X \times X \to [0, \infty)$ is continuous[13] in the product topology $\tau_\rho \times \tau_\rho$. In this context, define the action of the standard the Hardy–Littlewood maximal operator $\mathcal{M}_X$ on each μ-measurable function $f : X \to \mathbb{C}$ as*

$$\mathcal{M}_X f(x) := \sup_{r\in(0,\infty)} \frac{1}{\mu\big(B_\rho(x, r)\big)} \int_{B_\rho(x,r)} |f|\, d\mu, \qquad \forall x \in X. \tag{7.6.16}$$

Then

$$\begin{aligned} &\mathcal{M}_X f : X \longrightarrow [0, \infty] \ \text{ is a well-defined } \mu\text{-measurable function} \\ &\qquad \text{for each } \mu\text{-measurable function } \ f : X \to \mathbb{C}, \end{aligned} \tag{7.6.17}$$

and the mapping

$$\begin{aligned} &\mathcal{M}_X : L^p(X, \mu) \longrightarrow L^p(X, \mu) \ \text{ is well defined,} \\ &\text{sub-linear, and bounded for every } \ p \in (1, \infty], \\ \text{with } \ &\|\mathcal{M}_X\|_{L^p(X,\mu)\to L^p(X,\mu)} = O\big((p-1)^{-1}\big) \ \text{ as } \ p \to 1^+. \end{aligned} \tag{7.6.18}$$

Furthermore, corresponding to the case $p = 1$, the mapping

$$\begin{aligned} &\mathcal{M}_X : L^1(X, \mu) \longrightarrow L^{1,\infty}(X, \mu) \\ &\text{is well defined, sub-linear, and bounded.} \end{aligned} \tag{7.6.19}$$

Finally, the standard Hardy–Littlewood maximal operator defined as in (7.6.16) induces a mapping

$$\begin{aligned} &\mathcal{M}_X : L^{p,q}(X, \mu) \longrightarrow L^{p,q}(X, \mu) \ \text{ which is well defined,} \\ &\text{sub-linear, and bounded for each } \ p \in (1, \infty] \ \text{ and } \ q \in (0, \infty]. \end{aligned} \tag{7.6.20}$$

We now take up the task of providing the proof of Theorem 7.6.1.

Proof of Theorem 7.6.1 The idea is to specialize the result proved in Theorem 6.3.3 to the case when the family $\mathcal{A}$ of subsets of X is given by

$$\mathcal{A} := \{B_\rho(x, r)\}_{x\in X, r>0}. \tag{7.6.21}$$

Accordingly, the bulk of the proof consists of a verification that conditions *(i)*-*(vi)* from the statement of Theorem 6.3.3 are satisfied. To get started, recall that the

[12] A direct proof of Corollary 7.6.3 may be found in [11, Theorem 3.7, p. 82].

[13] Theorem 7.1.2 guarantees that any quasi-metric space has an equivalent quasi-distance which satisfies this property.

measure μ is defined on a sigma-algebra $\mathfrak{M}$ of subsets of X with the property that $Borel_{\tau_\rho}(X) \subseteq \mathfrak{M}$.

Consider first condition *(vi)* in the statement of Theorem 6.3.3. In the current setting, this amounts to requiring that for each $r > 0$ and each $E \in \mathfrak{M}$ fixed, the function

$$X \ni x \longmapsto \mu(B_\rho(x, r) \cap E) \in [0, \infty) \tag{7.6.22}$$

is μ-measurable. To see that this is the case, observe that it suffices to show that

$$\begin{gathered} f : X \to [0, \infty), \;\; f(x) := \mu\big(B_\rho(x, r) \cap E\big), \;\; \forall x \in X, \\ \text{is lower-semicontinuous,} \end{gathered} \tag{7.6.23}$$

since any lower-semicontinuous function is $Borel_{\tau_\rho}$-measurable. To this end, fix $x_o \in X$ arbitrary. The assumed continuity of ρ ensures that if $\{x_j\}_{j\in\mathbb{N}}$ is a sequence of points in X with the property that $x_j \to x_o$ as $j \to \infty$, with convergence understood in the (metrizable) topology τ_ρ, then

$$\liminf_{j\to\infty} \mathbf{1}_{B_\rho(x_j,r)}(y) \geq \mathbf{1}_{B_\rho(x_o,r)}(y), \qquad \forall y \in X. \tag{7.6.24}$$

Indeed, this is easily verified by analyzing separately the cases $y \in B_\rho(x_o, r)$ and $y \in X \setminus B_\rho(x_o, r)$. In turn, based on this and Fatou's lemma we may then estimate

$$\begin{aligned} f(x_o) &= \mu\big(B_\rho(x_o, r) \cap E\big) = \int_E \mathbf{1}_{B_\rho(x_o,r)}(y)\,\mathrm{d}\mu(y) \\ &\leq \int_E \liminf_{j\to\infty} \mathbf{1}_{B_\rho(x_j,r)}(y)\,\mathrm{d}\mu(y) \leq \liminf_{j\to\infty} \int_E \mathbf{1}_{B_\rho(x_j,r)}(y)\,\mathrm{d}\mu(y) \\ &= \liminf_{j\to\infty} \mu\big(B_\rho(x_j, r) \cap E\big) = \liminf_{j\to\infty} f(x_j), \end{aligned} \tag{7.6.25}$$

as desired. This concludes the verification of the version of condition (vi) from the statement of Theorem 6.3.3 corresponding to the present setting.

All the remaining conditions are basically direct consequences of definitions and assumptions. In summary, all background hypotheses made in Theorem 6.3.3 are valid for the family (7.6.21), and the conclusions in Theorem 6.3.3 prove all the claims we have set to justify here save for the very last line in (7.6.18). However, this may be justified directly, much as in the Euclidean case (cf., e.g., [107, Theorem 2.1.6, p. 80], [240, p. 7]), or based on (7.6.19), L^∞-bounds, and the Marcinkiewicz Interpolation Theorem (cf., e.g., [107, Theorem 1.3.2, pp. 32–33]). □

We continue by making four comments pertaining to the above considerations. First, given any functions $f, g \in L^0(X, \mu)$, we have

$$\mathcal{M}_X(f + g) \leq \mathcal{M}_X f + \mathcal{M}_X g \;\; \text{pointwise on} \;\; X, \tag{7.6.26}$$

from which we conclude that

$$\begin{aligned}&\text{if } x\in X \text{ is such that } \big(\mathcal{M}_X f\big)(x)<+\infty \text{ and } \big(\mathcal{M}_X g\big)(x)<+\infty\\ &\text{it follows that } \Big|\big(\mathcal{M}_X f\big)(x)-\big(\mathcal{M}_X g\big)(x)\Big|\leq \big(\mathcal{M}_X(f-g)\big)(x).\end{aligned}\tag{7.6.27}$$

In turn, from (7.6.27), (7.6.18), and (7.6.19) see that

$$\begin{aligned}&\text{for any two given functions } f,g\in L^p(X,\mu) \text{ with } p\in[1,\infty] \text{ we have}\\ &\Big|\big(\mathcal{M}_X f\big)(x)-\big(\mathcal{M}_X g\big)(x)\Big|\leq \big(\mathcal{M}_X(f-g)\big)(x)\ \text{ at } \mu\text{-a.e. point } x\in X.\end{aligned}\tag{7.6.28}$$

As a corollary of (7.6.28) and (7.6.18)–(7.6.19), we conclude that

$$\begin{aligned}&\text{the Hardy-Littlewood maximal operator } \mathcal{M}_X \text{ is a continuous}\\ &\text{mapping both in the context of (7.6.18) and of (7.6.19).}\end{aligned}\tag{7.6.29}$$

To set the stage for our second comment, assume (X,ρ,μ) is a space of homogeneous type in which the quasi-distance $\rho : X\times X\to[0,\infty)$ is continuous in the product topology $\tau_\rho\times\tau_\rho$, and suppose ω is a Borel measure on X (viewed as a topological space, equipped with the topology τ_ρ). For $\gamma\in[0,1]$ fixed, consider the maximal function

$$\phi_\gamma(x):=\sup_{\substack{B\subseteq X,\,\rho\text{-ball}\\ x\in B}}\left(\frac{\omega(B)}{\big[\mu(B)\big]^\gamma}\right),\qquad \forall x\in X.\tag{7.6.30}$$

Then the same type of argument as in the proof of Theorem 6.3.3 (based on a Vitali-type covering lemma and Hölder's inequality) yields the following estimates:

$$\|\phi_\gamma\|_{L^{1/\gamma,\infty}(X,\mu)}\leq C\|\phi_0\|_{L^\infty(X,\mu)},\quad \text{if } 0<\gamma\leq 1,\tag{7.6.31}$$

$$\|\phi_\gamma\|_{L^\infty(X,\mu)}\leq C\|\phi_1\|_{L^{1/(1-\gamma)}(X,\mu)},\quad \text{if } 0\leq\gamma<1,\tag{7.6.32}$$

for some finite $C>0$ which depends only of the quasi-distance constant of ρ and the doubling constant of μ.

Note that (7.6.31)–(7.6.32) can be viewed as a generalization of two key estimates implicit in Theorem 7.6.1. Indeed, given any μ-measurable function f on X, for $0<s<1/\alpha$ we have the pointwise equivalence

$$\begin{aligned}&(\mathcal{M}_{X,s,\alpha}f)(x)\approx\big(\phi_{1-s\alpha}\big)(x)^{1/s}\ \text{ uniformly in } x\in X,\ \text{provided}\\ &\phi_{1-s\alpha}\text{is defined as in (7.6.30) with } \gamma:=1-s\alpha \text{ and } \omega:=|f|^s\mu.\end{aligned}\tag{7.6.33}$$

This permits us to deduce (7.6.5) directly from (7.6.31) and also to conclude from (7.6.32) that

$$\begin{aligned}&\big\|\mathcal{M}_{X,s,\alpha}f\big\|_{L^\infty(X,\mu)}\leq C\big\|\mathcal{M}_X(|f|^s)\big\|^{1/s}_{L^p(X,\mu)}\\ &\text{whenever } 1<p<\infty \text{ and } \alpha=\tfrac{1}{sp}.\end{aligned}\tag{7.6.34}$$

In view of the boundedness of the standard Hardy–Littlewood maximal operator $\mathcal{M}_X$ on $L^p(X,\mu)$ with $1 < p < \infty$, the latter estimate then shows that

$$\begin{aligned} \mathcal{M}_{X,s,\alpha} : L^{sp}(X,\mu) \longrightarrow L^{\infty}(X,\mu) \ \text{ is bounded} \\ \text{whenever } 1 < p < \infty \text{ and } \alpha = \tfrac{1}{sp}. \end{aligned} \tag{7.6.35}$$

Our third comment concerns a weighted version of Theorem 7.6.1, of the sort recorded below.

Proposition 7.6.4 *Let (X,ρ,μ) be a space of homogeneous type with the property that the quasi-distance $\rho : X \times X \to [0,\infty)$ is continuous in the product topology $\tau_\rho \times \tau_\rho$. Fix $s \in (0,\infty)$ and $\alpha \in [0,1/s)$, along with p,q such that $s < p \leq q < \infty$. Also, suppose that w, v are two weights on X with the property that $v^{-s/(p-s)}$ belongs to the Muckenhoupt class $A_\infty(X,\mu)$. Then there exists a finite constant $C > 0$ with the property that*

$$\big\|\mathcal{M}_{X,s,\alpha} f\big\|_{L^q(X,w\mu)} \leq C\|f\|_{L^p(X,v\mu)} \ \textit{ for all } \ f \in L^p(X,v\mu) \tag{7.6.36}$$

if and only if

$$\sup_{B\,\rho\text{-ball in } X} \Big[\mu(B)^{-(1/s-\alpha)p}\Big(\int_B w\,\mathrm{d}\mu\Big)^{p/q}\Big(\int_B v^{-s/(p-s)}\,\mathrm{d}\mu\Big)^{p/s-1}\Big] < +\infty. \tag{7.6.37}$$

Indeed, the case corresponding to $s = 1$ is (up to re-adjusting notation) the main result in [20]. The general case $s \in (0,\infty)$ is then obtained by relying based on the equality $\mathcal{M}_{X,s,\alpha} f = \big[\mathcal{M}_{X,1,s\alpha}(|f|^s)\big]^{1/s}$. See also [206, Theorem 3, pp. 265–266] and [260, Theorem 1, p. 258] in this regard.

Our fourth (and final) comment is the following useful lower pointwise bound for the Hardy–Littlewood maximal function.

Corollary 7.6.5 *Let (X,ρ,μ) be a space of homogeneous type with the property that the quasi-distance $\rho : X \times X \to [0,\infty)$ is continuous in the product topology $\tau_\rho \times \tau_\rho$, and the measure μ is Borel-semiregular on (X,τ_ρ). Then for each $s \in (0,\infty)$ and each μ-measurable function $f : X \to \mathbb{R}$ one has*

$$|f(x)| \leq \big(\mathcal{M}_{X,s} f\big)(x) \ \textit{ at } \mu\textit{-a.e. point } \ x \in X. \tag{7.6.38}$$

Proof Fix an arbitrary μ-measurable function $f : X \to \mathbb{R}$. As far as (7.6.38) is concerned, there is no loss in generality in assuming that $s = 1$ (since matters may be reduced to this case by working with $|f|^s$ in place of f; see (7.6.7)). Suppose this is the case (and write $\mathcal{M}_X$ in place of $\mathcal{M}_{X,1}$). If $\big(\mathcal{M}_X f\big)(x) = +\infty$ for each $x \in X$ there is nothing to prove. Assume $\big(\mathcal{M}_X f\big)(x_0) < +\infty$ for some $x_0 \in X$ which, in view of (7.6.16), forces $f \in L^1_{\mathrm{loc}}(X,\mu)$. Granted this membership and the current hypotheses, Lebesgue's Differentiation Theorem (see (1) $\Leftrightarrow$ (3) in Proposition 7.4.4) applies and gives, on account of (7.6.16),

$$|f(x)| = \lim_{r\to 0^+} \fint_{B_\rho(x,r)} |f|\,\mathrm{d}\mu \leq \big(\mathcal{M}_X f\big)(x) \quad \text{for } \mu\text{-almost every } x \in X, \tag{7.6.39}$$

as wanted. □

We continue by recording the version of the classical vector-valued maximal function inequality of Fefferman and Stein in the setting of space of homogeneous type; see, e.g., [108, Theorem 1.2, p. 299].

Theorem 7.6.6 *Let (X, ρ, μ) be a space of homogeneous type and recall that $\mathcal{M}_X$ is the Hardy–Littlewood maximal operator on X (cf. (7.6.16)). Then for $p \in (1, \infty)$ and $q \in (1, \infty]$, there exists a finite positive C_X, depending only on X, such that for any sequence of μ-measurable functions $f_j : X \to \mathbb{R}$, with $j \in \mathbb{N}$, the following inequalities hold*

$$\Big\| \Big(\sum_{j\in\mathbb{N}} (\mathcal{M}_X f_j)^q \Big)^{\frac{1}{q}} \Big\|_{L^{1,\infty}(X,\mu)} \leq C_X\, C_q \Big\| \Big(\sum_{j\in\mathbb{N}} |f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^1(X,\mu)}, \tag{7.6.40}$$

and

$$\Big\| \Big(\sum_{j\in\mathbb{N}} (\mathcal{M}_X f_j)^q \Big)^{\frac{1}{q}} \Big\|_{L^{p}(X,\mu)} \leq C_X\, C_{p,q} \Big\| \Big(\sum_{j\in\mathbb{N}} |f_j|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(X,\mu)}, \tag{7.6.41}$$

where $C_q = \max\{1, \frac{1}{q-1}\}$ and $C_{p,q} = \max\{1, \frac{1}{q-1}\} \cdot \max\{p, \frac{1}{p-1}\}$ if $q \in (1, \infty)$; if $q = \infty$, C_∞ coincides with the norm of $\mathcal{M}_X : L^1(X, \mu) \to L^{1,\infty}(X, \mu)$, whereas for $p \in (1, \infty)$, $C_{p,\infty}$ coincides with the norm of $\mathcal{M}_X : L^p(X, \mu) \to L^p(X, \mu)$.

We next extend Fefferman–Stein's maximal inequality from [89, Lemma 1, p. 111] to the setting of spaces of homogeneous type. The original argument makes use of dilation and translation properties of the Euclidean space and, as such, does not fully lend itself to the present, more general, geometric setting.

Proposition 7.6.7 *Let $\big(X, \rho, \mu\big)$ be a space of homogeneous type with the property that the quasi-distance $\rho : X \times X \to [0, \infty)$ is continuous.[14] Then for each exponent $s \in (0, 1)$ there exists some constant $C = C(\mu, \rho, s) \in (0, \infty)$ such that for any two μ-measurable functions $f, g : X \to [0, \infty]$ one has*

$$\int_X \big(\mathcal{M}_{X,s} f\big) g \,\mathrm{d}\mu \leq C \int_X f \big(\mathcal{M}_X g\big)\,\mathrm{d}\mu, \tag{7.6.42}$$

[14] Theorem 7.1.2 guarantees that any quasi-metric space has an equivalent quasi-distance which satisfies this property.

where $\mathcal{M}_X$ and $\mathcal{M}_{X,s}$ are, respectively, the Hardy–Littlewood maximal operator on X (cf. (7.6.16)) and its L^s-based version (defined as in (7.6.7)).

Before presenting the proof of Proposition 7.6.7 we make two remarks. First, from (7.6.8) and (7.6.17) we know that

$$\mathcal{M}_{X,s} f \text{ and } \mathcal{M}_X g \text{ are non-negative } \mu\text{-measurable function on } X, \tag{7.6.43}$$

so the integrals in (7.6.42) are meaningful. Second, the range $s \in (0, 1)$ is sharp, in the sense that (7.6.42) fails if $s \geq 1$. To see this, it suffices to consider the case when $s = 1$. In such a scenario (7.6.42) written for $g \equiv 1$ amounts to the boundedness of $\mathcal{M}_X$ on $L^1(X, \mu)$, which generally fails (in this regard, see also (7.6.67)).

Proof of Proposition 7.6.7 For each $R \in (0, \infty)$ define the following local maximal operators acting on arbitrary μ-measurable functions $f : X \to \mathbb{R}$ according to

$$\big(\mathcal{M}_R f\big)(x) := \sup_{r \in (0,R)} ⨍_{B_\rho(x,r)} |f| \, \mathrm{d}\mu \quad \text{for all } \; x \in X. \tag{7.6.44}$$

Choose $s \in (0, 1)$ and define $p := 1/s \in (1, \infty)$. Also, pick $f : X \to [0, \infty]$ an arbitrary μ-measurable function. Fix $\lambda, R \in (0, \infty)$ arbitrary and define the set

$$\Omega_{\lambda,R} := \big\{x \in X : \big(\mathcal{M}_R f\big)(x) > \lambda\big\}. \tag{7.6.45}$$

Recalling (7.6.44), it follows that for each $x \in \Omega_{\lambda,R}$ there exists $r_x \in (0, R)$ such that

$$⨍_{B_\rho(x,r_x)} f \, \mathrm{d}\mu > \lambda. \tag{7.6.46}$$

Since, by design, $\sup_{x \in \Omega_{\lambda,R}} r_x \leq R$, we may apply Vitali's Covering Lemma in the version recorded in Lemma 7.5.8 to the family of ρ-balls $\big\{B_\rho(x, r_x)\big\}_{x \in \Omega_{\lambda,R}}$. Specifically, Lemma 7.5.8 guarantees the existence of a constant $C_0 \in (1, \infty)$ along with an at most countable family of points $\big\{x_j\big\}_{j \in J} \subseteq \Omega_{\lambda,R}$ such that

$$\begin{gathered} \text{the } \rho\text{-balls } \; \big\{B_\rho(x_j, r_{x_j})\big\}_{j \in J} \; \text{ are mutually disjoint and} \\ \Omega_{\lambda,R} \subseteq \bigcup_{x \in \Omega_{\lambda,R}} B_\rho(x, r_x) \subseteq \bigcup_{j \in J} B_\rho(x_j, C_0 r_{x_j}). \end{gathered} \tag{7.6.47}$$

Bring in another arbitrary μ-measurable function $g : X \to [0, \infty]$. Fix $j \in J$ and, starting with (7.6.46), write

$$\begin{aligned}
&\int_{B_\rho(x_j,C_0 r_{x_j})} g\,\mathrm{d}\mu \\
&\quad < \frac{1}{\lambda}\int_{B_\rho(x_j,C_0 r_{x_j})} g(x)\Big(⨍_{B_\rho(x_j,r_{x_j})} f(y)\,\mathrm{d}\mu(y)\Big)\,\mathrm{d}\mu(x) \\
&\quad \le \frac{C}{\lambda}\int_{B_\rho(x_j,r_{x_j})} f(y)\Big(⨍_{B_\rho(x_j,C_0 r_{x_j})} g(x)\,\mathrm{d}\mu(x)\Big)\,\mathrm{d}\mu(y),
\end{aligned} \tag{7.6.48}$$

where the second inequality uses the fact that μ is a doubling measure (cf. (7.4.4)). Next, observe that there exists $C_1 \in (0,\infty)$ such that $B_\rho(x_j, C_0 r_{x_j}) \subseteq B_\rho(y, C_1 r_{x_j})$ for each $y \in B_\rho(x_j, r_{x_j})$. Based on this and the doubling property of μ (see (7.4.4) once again) we have

$$\begin{aligned}
⨍_{B_\rho(x_j,C_0 r_{x_j})} g(x)\,\mathrm{d}\mu(x) &\le \frac{1}{\mu\big(B_\rho(x_j, C_0 r_{x_j})\big)}\int_{B_\rho(y,C_1 r_{x_j})} g\,\mathrm{d}\mu \\
&\le \frac{C}{\mu\big(B_\rho(y, C_1 r_{x_j})\big)}\int_{B_\rho(y,C_1 r_{x_j})} g\,\mathrm{d}\mu \\
&\le C\big(\mathcal{M}_X g\big)(y) \quad \text{for each} \quad y \in B_\rho(x_j, r_{x_j}),
\end{aligned} \tag{7.6.49}$$

where $C \in (0,\infty)$ depends only on μ and ρ. Together, (7.6.48) and (7.6.49) imply

$$\int_{B_\rho(x_j,C_0 r_{x_j})} g\,\mathrm{d}\mu \le \frac{C}{\lambda}\int_{B_\rho(x_j,r_{x_j})} f\big(\mathcal{M}_X g\big)\,\mathrm{d}\mu. \tag{7.6.50}$$

The latter when combined with (7.6.47) gives

$$\begin{aligned}
\int_{\Omega_{\lambda,R}} g\,\mathrm{d}\mu &\le \sum_{j\in J}\int_{B_\rho(x_j,C_0 r_{x_j})} g\,\mathrm{d}\mu \le \sum_{j\in J}\frac{C}{\lambda}\int_{B_\rho(x_j,r_{x_j})} f\big(\mathcal{M}_X g\big)\,\mathrm{d}\mu \\
&= \frac{C}{\lambda}\int_{\bigcup_{j\in J} B_\rho(x_j,r_{x_j})} f\big(\mathcal{M}_X g\big)\,\mathrm{d}\mu \le \frac{C}{\lambda}\int_X f\big(\mathcal{M}_X g\big)\,\mathrm{d}\mu,
\end{aligned} \tag{7.6.51}$$

where $C \in (0,\infty)$ is independent of R.

To proceed, fix an arbitrary $\lambda \in (0,\infty)$ and define

$$\Omega_\lambda := \big\{x \in X : \big(\mathcal{M}_X f\big)(x) > \lambda\big\}. \tag{7.6.52}$$

Given that for each $R \in (0,\infty)$ we have $\mathcal{M}_R f \le \mathcal{M}_X f$ pointwise in X, it follows that $\Omega_{\lambda,R} \subseteq \Omega_\lambda$. In addition, $\mathcal{M}_{R_1} f \le \mathcal{M}_{R_2} f$ whenever $R_2 \ge R_1 > 0$. Together, these imply that the sequence $\{\Omega_{\lambda,R}\}_{R>0}$ is nested. In fact, we have

$$\Omega_{\lambda,R} \nearrow \Omega_\lambda \quad \text{as} \quad R \to \infty. \tag{7.6.53}$$

Indeed, if $x \in \Omega_\lambda$, then there exists $r \in (0, \infty)$ such that $⨍_{B_\rho(x,r)} f \, d\mu > \lambda$. In turn, the latter forces $(\mathcal{M}_R f)(x) > \lambda$ whenever $R \geq r$, thus $x \in \Omega_{\lambda,R}$ whenever $R \geq r$. Having proved (7.6.53), we may pass to the limit $R \to \infty$ in the resulting inequality in (7.6.51) and apply Lebesgue's Monotone Convergence Theorem to obtain

$$\int_{\Omega_\lambda} g \, d\mu \leq \frac{C}{\lambda} \int_X f\big(\mathcal{M}_X g\big) \, d\mu. \tag{7.6.54}$$

Hence, if we regard $g\mu$ and $(\mathcal{M}_X g)\mu$ as measures on X with the same sigma-algebra of measurable sets at μ itself (something permissible, in light of (7.6.43)), we may recast (7.6.54) simply as

$$(g\mu)(\Omega_\lambda) \leq \frac{C}{\lambda} \|f\|_{L^1\left(X,(\mathcal{M}_X g)\mu\right)}. \tag{7.6.55}$$

This shows that the sub-linear operator

$$\mathcal{M}_X : L^1\big(X, (\mathcal{M}_X g)\mu\big) \longrightarrow L^{1,\infty}(X, g\mu) \text{ is bounded.} \tag{7.6.56}$$

We also claim that the sub-linear operator

$$\mathcal{M}_X : L^\infty\big(X, (\mathcal{M}_X g)\mu\big) \longrightarrow L^{\infty}(X, g\mu) \text{ is bounded.} \tag{7.6.57}$$

Note that, without loss of generality, we may assume that we do not have $g = 0$ at μ-a.e. on X, since otherwise (7.6.57) is trivially true. In turn, this implies

$$\big(\mathcal{M}_X g\big)(x) > 0 \text{ for all } x \in X. \tag{7.6.58}$$

The claim made in (7.6.57) will follow once we succeed in establishing that for every given μ-measurable function $f : X \to [0, \infty]$ there holds

$$\|\mathcal{M}_X f\|_{L^\infty(X,g\mu)} \leq \|f\|_{L^\infty\left(X,(\mathcal{M}_X g)\mu\right)}. \tag{7.6.59}$$

Given that the inequality in (7.6.59) is true whenever $\|f\|_{L^\infty\left(X,(\mathcal{M}_X g)\mu\right)} = \infty$, we shall henceforth focus on the case when $\|f\|_{L^\infty\left(X,(\mathcal{M}_X g)\mu\right)} < \infty$. Assuming this to be the case, let $\lambda \in (0, \infty)$ be such that

$$\lambda > \|f\|_{L^\infty\left(X,(\mathcal{M}_X g)\mu\right)}. \tag{7.6.60}$$

Then necessarily we have $\big[(\mathcal{M}_X g)\mu\big]\Big(\big\{x \in X : |f(x)| > \lambda\big\}\Big) = 0$, which further implies $\int_{\{x\in X: |f(x)|>\lambda\}} (\mathcal{M}_X g) \, d\mu = 0$. Since $\mathcal{M}_X g \geq 0$, we therefore obtain that $\mathcal{M}_X g = 0$ at μ-a.e. point in the set $\{x \in X : |f(x)| > \lambda\}$. In light of (7.6.58), we infer that $\mu\big(\{x \in X : |f(x)| > \lambda\}\big) = 0$ which forces $\|f\|_{L^\infty(X,\mu)} \leq \lambda$. As a result,

$\|\mathcal{M}_X f\|_{L^\infty(X,\mu)} \leq \lambda$ hence, further,

$$\mu\Big(\big\{x \in X : \big(\mathcal{M}_X f\big)(x) > \lambda\big\}\Big) = 0. \tag{7.6.61}$$

The latter implies $\int_{\{x\in X:\,(\mathcal{M}_X f)(x)>\lambda\}} g\,\mathrm{d}\mu = 0$ or, equivalently,

$$(g\mu)\Big(\big\{x \in X : \big(\mathcal{M}_X f\big)(x) > \lambda\big\}\Big) = 0. \tag{7.6.62}$$

Consequently,

$$\|\mathcal{M}_X f\|_{L^\infty(X,g\mu)} \leq \lambda. \tag{7.6.63}$$

Now (7.6.59) follows by letting $\lambda \searrow \|f\|_{L^\infty(X,(\mathcal{M}_X g)\mu)}$ in (7.6.63).

Having established (7.6.56)–(7.6.57), we may invoke the Marcinkiewicz interpolation theorem for sub-linear operators to conclude that there exists a constant $C \in (0,\infty)$ such that

$$\|\mathcal{M}_X f\|_{L^p(X,g\mu)} \leq C\|f\|_{L^p(X,(\mathcal{M}_X g)\mu)}, \tag{7.6.64}$$

for all non-negative μ-measurable functions f defined on X. Raising both sides of the estimate in (7.6.64) to the p-th power and re-denoting f^p by f, we obtain

$$\int_X \big[\mathcal{M}_X(f^{1/p})\big]^p g\,\mathrm{d}\mu \leq C^p \int_X f(\mathcal{M}g)\,\mathrm{d}\mu \tag{7.6.65}$$

for all μ-measurable functions $f \geq 0$. The inequality stated in (7.6.42) now follows from (7.6.65) by recalling that $s = 1/p$ and observing that $\big[\mathcal{M}_X(f^{1/p})\big]^p = \mathcal{M}_{X,s} f$ (as seen from (7.6.16) and (7.6.7)). □

We conclude by including a couple of useful pointwise estimates for the Hardy–Littlewood maximal operator in the context of Ahlfors regular subsets of the Euclidean ambient.

Proposition 7.6.8 *Given a closed Ahlfors regular set $\Sigma \subseteq \mathbb{R}^n$, let $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$ and denote by $\mathcal{M}_\Sigma$ the Hardy–Littlewood maximal operator associated with the space of homogeneous type $\big(\Sigma, |\cdot-\cdot|, \sigma\big)$ (cf. Example 7.4.1). Then*

$$\Big(\mathcal{M}_\Sigma\big(\mathbf{1}_{\Delta(x_0,R)}\big)\Big)(x) \approx \frac{R^{n-1}}{(R+|x-x_0|)^{n-1}}, \quad \text{uniformly for } x_0, x \in \Sigma \text{ and } R \in (0,\infty), \tag{7.6.66}$$

where $\Delta(x_0,R) := \Sigma \cap B(x_0,R)$, and the implicit proportionality constants depend only on the dimension n and the Ahlfors character of Σ. In particular, corresponding to $R = 1$,

$$\Big(\mathcal{M}_\Sigma\big(\mathbf{1}_{\Delta(x_0,1)}\big)\Big)(x) \approx \frac{1}{1+|x-x_0|^{n-1}}, \qquad \text{uniformly for } x_0, x \in \Sigma. \tag{7.6.67}$$

Furthermore, if for each $t \in [0,\infty)$ *one defines*

$$\log_+ t := \begin{cases} 0 & \text{if } t \in [0,1], \\ \ln t & \text{if } t \in [1,\infty), \end{cases} \tag{7.6.68}$$

then

$$\Big[\big(\mathcal{M}_\Sigma \circ \mathcal{M}_\Sigma\big)\big(\mathbf{1}_{\Delta(x_0,1)}\big)\Big](x) \approx \Big(\mathcal{M}_\Sigma\big(\tfrac{1}{1+|\cdot-x_0|^{n-1}}\big)\Big)(x) \approx \frac{1+\log_+|x-x_0|}{1+|x-x_0|^{n-1}}, \tag{7.6.69}$$

in a uniform fashion for $x_0, x \in \Sigma$.

Proof If c_Σ, C_Σ denote the lower and upper Ahlfors regularity constants of Σ, then $0 < c_\Sigma \leq C_\Sigma < \infty$. To deal with (7.6.66), pick $x_0, x \in \Sigma$ and $R \in (0,\infty)$. Consider first the case where $|x-x_0| \leq 2R$. In this scenario,

$$\begin{aligned} \Big(\mathcal{M}_\Sigma\big(\mathbf{1}_{\Delta(x_0,R)}\big)\Big)(x) &= \sup_{r\in(0,\infty)} \frac{1}{\sigma\big(\Delta(x,r)\big)} \int_{\Delta(x,r)} \mathbf{1}_{\Delta(x_0,R)}\, d\sigma \\ &= \sup_{r\in(0,\infty)} \frac{\sigma\big(\Delta(x,r)\cap\Delta(x_0,R)\big)}{\sigma\big(\Delta(x,r)\big)} \\ &\leq 1 = \frac{(2R)^{n-1}}{(R+R)^{n-1}} \leq \frac{(2R)^{n-1}}{(R+2^{-1}|x-x_0|)^{n-1}} \\ &\leq \frac{(4R)^{n-1}}{(R+|x-x_0|)^{n-1}}, \end{aligned} \tag{7.6.70}$$

a bound of the right order. Suppose next that $|x-x_0| > 2R$, and pick $r \in (0,\infty)$. On the one hand, if $\Delta(x,r)\cap\Delta(x_0,R) = \varnothing$ then $\sigma\big(\Delta(x,r)\cap\Delta(x_0,R)\big) = 0$. On the other hand, if $\Delta(x,r)\cap\Delta(x_0,R) \neq \varnothing$ then necessarily $r > |x-x_0| - R$. Indeed, since currently there exists some point $y \in \Delta(x,r)\cap\Delta(x_0,R)$, we may estimate

$$r > |x-y| \geq |x-x_0| - |x_0-y| > |x-x_0| - R, \tag{7.6.71}$$

as wanted. In addition, since we are presently assuming that $|x-x_0| > 2R$, it follows that we have $r > |x-x_0| - R > \frac{1}{3}(R+|x-x_0|)$. Using this analysis we may then write

$$\begin{aligned}\Big(\mathcal{M}_\Sigma\big(\mathbf{1}_{\Delta(x_0,R)}\big)\Big)(x) &= \sup_{r\in(0,2\,\mathrm{diam}\,\Sigma)} \frac{1}{\sigma\big(\Delta(x,r)\big)}\int_{\Delta(x,r)} \mathbf{1}_{\Delta(x_0,R)}\,\mathrm{d}\sigma \\ &= \sup_{r\in(|x-x_0|-R,2\,\mathrm{diam}\,\Sigma)} \frac{\sigma\big(\Delta(x,r)\cap\Delta(x_0,R)\big)}{\sigma\big(\Delta(x,r)\big)} \\ &\leq \sup_{r\in(|x-x_0|-R,2\,\mathrm{diam}\,\Sigma)} \frac{C_\Sigma R^{n-1}}{c_\Sigma r^{n-1}} \leq \frac{C_\Sigma(3R)^{n-1}}{c_\Sigma(R+|x-x_0|)^{n-1}},\end{aligned} \tag{7.6.72}$$

where the second inequality relies on the Ahlfors regularity of Σ. From (7.6.70) and (7.6.72) we then conclude that

$$\Big(\mathcal{M}_\Sigma\big(\mathbf{1}_{\Delta(x_0,R)}\big)\Big)(x) \leq \frac{C R^{n-1}}{(R+|x-x_0|)^{n-1}}, \tag{7.6.73}$$

where $C := \max\big\{4^{n-1}, 3^{n-1}(C_\Sigma/c_\Sigma)\big\}$. This is one of the desired estimates in (7.6.66).

To establish the remaining (right-pointing) inequality in (7.6.66), fix two points $x_0, x \in \Sigma$ along with some $R \in (0,\infty)$. Since $\Delta(x_0,R) \subseteq \Delta\big(x, R+|x-x_0|\big)$, we may invoke (7.4.5) to write

$$\begin{aligned}\Big(\mathcal{M}_\Sigma\big(\mathbf{1}_{\Delta(x_0,R)}\big)\Big)(x) &\geq \fint_{\Delta(x,R+|x-x_0|)} \mathbf{1}_{\Delta(x_0,R)}\,\mathrm{d}\sigma = \frac{\sigma\big(\Delta(x_0,R)\big)}{\sigma\big(\Delta(x,R+|x-z_0|)\big)} \\ &\geq \frac{c_\Sigma R^{n-1}}{C_\Sigma(R+|x-x_0|)^{n-1}},\end{aligned} \tag{7.6.74}$$

which suits our purposes. At this stage, (7.6.66) follows from (7.6.73)–(7.6.74). Also, (7.6.67) is a special case of (7.6.66).

As far as (7.6.69) is concerned, the first equivalence is a direct consequence of (7.6.67) and the definition of $\mathcal{M}_\Sigma$. There remains to deal with the second equivalence in (7.6.69). In preparation, we make three observations. First, we claim that

$$\int_{\Delta(x_0,R)} \frac{\mathrm{d}\sigma(y)}{1+|y-x_0|^{n-1}} \approx \ln(1+R^{n-1}), \quad \text{uniformly for } x_0\in\Sigma \text{ and } R\in\big(0,6\,\mathrm{diam}(\Sigma)\big). \tag{7.6.75}$$

To justify (7.6.75), choose

$$\varepsilon := \Big(\frac{c_\Sigma}{2C_\Sigma}\Big)^{1/(n-1)} \in (0,1). \tag{7.6.76}$$

Then for each $x_0 \in \Sigma$ and $r \in \big(0, 2\,\mathrm{diam}(\Sigma)\big)$ we may write

$$C_\Sigma r^{n-1} \geq \sigma\big(\Delta(x_0, r) \setminus \Delta(x_0, \varepsilon r)\big) = \sigma\big(\Delta(x_0, r)\big) - \sigma\big(\Delta(x_0, \varepsilon r)\big) \\ \geq \Big(c_\Sigma - \varepsilon^{n-1} C_\Sigma\Big) r^{n-1} = (c_\Sigma/2) r^{n-1}, \tag{7.6.77}$$

where the inequalities rely on the Ahlfors regularity of Σ, and the last equality uses (7.6.76). For each $R \in \big(0, 2\,\mathrm{diam}(\Sigma)\big)$ we are now ready to estimate

$$\begin{aligned} \int_{\Delta(x_0,R)} \frac{\mathrm{d}\sigma(y)}{1 + |y - x_0|^{n-1}} &= \sum_{j=0}^{\infty} \int_{\Delta(x_0,\varepsilon^j R)\setminus\Delta(x_0,\varepsilon^{j+1}R)} \frac{\mathrm{d}\sigma(y)}{1 + |y - x_0|^{n-1}} \\ &\approx \sum_{j=0}^{\infty} \int_{\Delta(x_0,\varepsilon^j R)\setminus\Delta(x_0,\varepsilon^{j+1}R)} \frac{\mathrm{d}\sigma(y)}{1 + (\varepsilon^j R)^{n-1}} \\ &\approx \sum_{j=0}^{\infty} \frac{(\varepsilon^j R)^{n-1}}{1 + (\varepsilon^j R)^{n-1}} \approx \sum_{j=0}^{\infty} \int_{\varepsilon^{j+1}R}^{\varepsilon^j R} \frac{t^{n-2}}{1 + t^{n-1}}\,\mathrm{d}t \\ &= \int_0^R \frac{t^{n-2}}{1 + t^{n-1}}\,\mathrm{d}t = \frac{1}{n-1} \ln(1 + t^{n-1})\Big|_{t=0}^{t=R} \\ &= \frac{1}{n-1} \ln(1 + R^{n-1}), \end{aligned} \tag{7.6.78}$$

so (7.6.75) is established in the regime $R \in \big(0, 2\,\mathrm{diam}\,\Sigma\big)$. In turn, this readily implies that (7.6.75) also holds whenever $\mathrm{diam}\,\Sigma < +\infty$ and $R \in \big(2\,\mathrm{diam}\,\Sigma, 6\,\mathrm{diam}\,\Sigma\big)$ simply by adjusting constants. Specifically, in this scenario we write

$$\int_{\Delta(x_0,R)} \frac{\mathrm{d}\sigma(y)}{1 + |y - x_0|^{n-1}} = \int_{\Delta(x_0,\mathrm{diam}\,\Sigma)} \frac{\mathrm{d}\sigma(y)}{1 + |y - x_0|^{n-1}} \\ \approx \ln\Big(1 + (\mathrm{diam}\,\Sigma)^{n-1}\Big) \approx \ln(1 + R^{n-1}) \tag{7.6.79}$$

again, with proportionality constants depending only on n and the Ahlfors regularity character of Σ.

The second preparatory observation we wish to make is that

$$\phi(t) := \Big(\frac{t^{n-1}}{1 + t^{n-1}}\Big) \cdot \frac{1 + \log_+ t}{\ln(1 + t^{n-1})} \approx 1, \quad \text{uniformly for } t \in (0, \infty). \tag{7.6.80}$$

This is a consequence of the fact that the function ϕ is continuous and does not vanish on $(0, \infty)$, and ϕ as well as $1/\phi$ have finite limits both at 0^+ and at $+\infty$ (as may be seen by once again relying on L'Hôspital's rule).

The third (and final) preliminary remark is that the function

$$\psi(t) := \frac{\ln(1 + t^{n-1})}{t^{n-1}} \quad \text{for } t \in (0, \infty) \text{ is decreasing} \tag{7.6.81}$$

and satisfies

$$\lim_{t\to 0^+} \psi(t) = 1. \tag{7.6.82}$$

In this regard, note that for each $t \in (0,\infty)$ we have

$$\psi'(t) = \frac{n-1}{t^n}\Big[\frac{t^{n-1}}{1+t^{n-1}} - \ln(1+t^{n-1})\Big]. \tag{7.6.83}$$

Since the function $[0,\infty) \ni s \mapsto \ln(1+s) - s/(1+s)$ is increasing (as its derivative is positive) and vanishes at the origin, it follows that the expression in the square brackets above is negative. Hence, $\psi'(t) < 0$ for each $t \in (0,\infty)$, which goes to show that ψ defined in (7.6.81) is indeed decreasing. Also, an application of L'Hôspital's rule gives (7.6.82).

We are now ready to check the second equivalence in (7.6.69). Let us focus on the right-pointing inequality. Fix two points $x_0, x \in \Sigma$. If $x = x_0$, then the upper Ahlfors regularity of Σ, the fact that ψ in (7.6.81) is decreasing, and (7.6.82) permit us to write

$$\begin{aligned}
\Big(\mathcal{M}_\Sigma\big(\tfrac{1}{1+|\cdot-x_0|^{n-1}}\big)\Big)(x_0) &= \sup_{r\in(0,2\,\mathrm{diam}\,\Sigma)} \frac{1}{\sigma\big(\Delta(x_0,r)\big)}\int_{\Delta(x_0,r)} \frac{d\sigma(y)}{1+|y-x_0|^{n-1}} \\
&\approx \sup_{r\in(0,2\,\mathrm{diam}\,\Sigma)} \frac{\ln(1+r^{n-1})}{r^{n-1}} = \sup_{r\in(0,2\,\mathrm{diam}\,\Sigma)} \psi(r) \\
&= \lim_{r\to 0^+}\psi(r) = 1 = \frac{1+\log_+|x-x_0|}{1+|x-x_0|^{n-1}}\Big|_{x=x_0},
\end{aligned} \tag{7.6.84}$$

in agreement with what we are trying to prove. Consider next the case when the two points $x_0, x \in \Sigma$ are distinct. In this situation, we have $|x-x_0| > 0$ and we may write

$$\begin{aligned}
\Big(\mathcal{M}_\Sigma\big(\tfrac{1}{1+|\cdot-x_0|^{n-1}}\big)\Big)(x) &= \sup_{r\in(0,\infty)} \frac{1}{\sigma\big(\Delta(x,r)\big)}\int_{\Delta(x,r)} \frac{d\sigma(y)}{1+|y-x_0|^{n-1}} \\
&\geq \frac{1}{\sigma\big(\Delta(x,2|x-x_0|)\big)}\int_{\Delta(x,2|x-x_0|)} \frac{d\sigma(y)}{1+|y-x_0|^{n-1}} \\
&\geq \frac{2^{1-n}}{C_\Sigma|x-x_0|^{n-1}}\int_{\Delta(x_0,|x-x_0|)} \frac{d\sigma(y)}{1+|y-x_0|^{n-1}},
\end{aligned} \tag{7.6.85}$$

with the last inequality implied by the fact that $\Delta(x_0,|x-x_0|) \subseteq \Delta(x,2|x-x_0|)$ and the upper Ahlfors regularity of Σ. Combining (7.6.85) with (7.6.75), written for the choice $R := |x-x_0| \in \big(0, 2\,\mathrm{diam}(\Sigma)\big)$, yields

$$\Big(\mathcal{M}_\Sigma\big(\tfrac{1}{1+|\cdot-x_0|^{n-1}}\big)\Big)(x) \geq c\,\frac{\ln(1+|x-x_0|^{n-1})}{|x-x_0|^{n-1}}, \tag{7.6.86}$$

for some constant $c \in (0, \infty)$ which depends only on n and Σ. From (7.6.86) and (7.6.80) we then conclude that

$$\Big(\mathcal{M}_\Sigma\big(\tfrac{1}{1+|\cdot - x_0|^{n-1}}\big)\Big)(x) \geq c\,\frac{1+\log_+|x-x_0|}{1+|x-x_0|^{n-1}}, \tag{7.6.87}$$

for some constant $c \in (0, \infty)$ which depends only on n and Σ. This finishes the proof of the right-pointing inequality in the final equivalence in (7.6.69).

There remains to prove the left-pointing inequality in the final equivalence claimed in (7.6.69). To set the stage, pick two arbitrary points $x_0, x \in \Sigma$ along with a scale $r \in \big(0, 2\,\mathrm{diam}(\Sigma)\big)$. The goal is to prove that

$$\fint_{\Delta(x,r)} \frac{d\sigma(y)}{1+|y-x_0|^{n-1}} \leq C\,\frac{1+\log_+|x-x_0|}{1+|x-x_0|^{n-1}}, \tag{7.6.88}$$

for a constant $C \in (0, \infty)$ which depends only on n and Σ. To this end, we analyze several cases, starting with

Case I: *Assume* $|x - x_0| \leq 1$. Then, since $1/(1 + |y - x_0|^{n-1}) \leq 1$, we have

$$\fint_{\Delta(x,r)} \frac{d\sigma(y)}{1+|y-x_0|^{n-1}} \leq 1 \leq \frac{2}{1+|x-x_0|^{n-1}} = 2\cdot\frac{1+\log_+|x-x_0|}{1+|x-x_0|^{n-1}}, \tag{7.6.89}$$

which suits our purposes.

Case II: *Assume* $|x - x_0| > 1$ *and* $r \leq |x - x_0|/2$. In this scenario, for each $y \in \Delta(x, r)$ we have

$$|x-x_0| \leq |x-y| + |y-x_0| < r + |y-x_0| \leq |x-x_0|/2 + |y-x_0|, \tag{7.6.90}$$

which forces $|x - x_0| < 2|y - x_0|$ and, further,

$$(1+|y-x_0|^{n-1})^{-1} < \big(1+(|x-x_0|/2)^{n-1}\big)^{-1} < 2^{n-1}\big(1+|x-x_0|^{n-1}\big)^{-1}. \tag{7.6.91}$$

As such, we may write

$$\fint_{\Delta(x,r)} \frac{d\sigma(y)}{1+|y-x_0|^{n-1}} < \frac{2^{n-1}}{1+|x-x_0|^{n-1}} < 2^{n-1}\frac{1+\log_+|x-x_0|}{1+|x-x_0|^{n-1}}, \tag{7.6.92}$$

which is in line with (7.6.88).

Case III: *Assume* $|x - x_0| > 1$ *and* $r > |x - x_0|/2$. Note that if $y \in \Delta(x, r)$ then

$$|y-x_0| \leq |y-x| + |x-x_0| < r + |x-x_0| < 3r, \tag{7.6.93}$$

so $\Delta(x,r) \subseteq \Delta(x_0, 3r)$. The latter allows us to write, for some $C \in (0,\infty)$ depending only on n and Σ,

$$\begin{aligned}
⨍_{\Delta(x,r)} \frac{d\sigma(y)}{1+|y-x_0|^{n-1}} &= \frac{1}{\sigma\big(\Delta(x,r)\big)} \int_{\Delta(x,r)} \frac{d\sigma(y)}{1+|y-x_0|^{n-1}} \\
&\leq \frac{1}{c_\Sigma r^{n-1}} \int_{\Delta(x_0,3r)} \frac{d\sigma(y)}{1+|y-x_0|^{n-1}} \\
&\leq C \frac{\ln(1+r^{n-1})}{r^{n-1}},
\end{aligned} \tag{7.6.94}$$

with the last inequality implied by (7.6.75). The format of the last expression above matches well with the function ψ defined in (7.6.81). Since, as noted earlier, ψ is decreasing, the current working hypotheses imply

$$\begin{aligned}
\frac{\ln(1+r^{n-1})}{r^{n-1}} &= \psi(r) < \psi\big(|x-x_0|/2\big) = \frac{\ln\big(1+(|x-x_0|/2)^{n-1}\big)}{(|x-x_0|/2)^{n-1}} \\
&< C_n \frac{\ln\big(1+|x-x_0|^{n-1}\big)}{|x-x_0|^{n-1}} < C_n \frac{1+\log_+|x-x_0|}{1+|x-x_0|^{n-1}}.
\end{aligned} \tag{7.6.95}$$

Combining (7.6.94) with (7.6.95) once again gives an estimate which is in agreement with (7.6.88).

Collectively, Cases I–III prove the inequality claimed in (7.6.88) which, after taking the supremum in $r \in \big(0, 2\,\mathrm{diam}(\Sigma)\big)$, yields

$$\Big(\mathcal{M}_\Sigma\big(\tfrac{1}{1+|\cdot-x_0|^{n-1}}\big)\Big)(x) \leq C \frac{1+\log_+|x-x_0|}{1+|x-x_0|^{n-1}}, \tag{7.6.96}$$

with $C \in (0,\infty)$ depending only on n and Σ. Hence, the claim made in (7.6.69) is now fully proved.

□

7.7 Muckenhoupt Weights on Spaces of Homogeneous Type

We shall work in the context of a space of homogeneous type (X, ρ, μ) with the property that the quasi-distance $\rho : X \times X \to [0,\infty)$ is continuous[15] in the product topology $\tau_\rho \times \tau_\rho$. In this setting, call a real-valued function w defined on X a `weight` if it is μ-measurable as well as (strictly) positive and finite μ-a.e. on X. We agree to also use the symbol w for the weighted measure $w\,\mu$, i.e., define

[15] From Theorem 7.1.2 it follows that any quasi-metric space has an equivalent quasi-distance which satisfies this property.

$$w(E) := \int_E w \, d\mu \quad \text{for each } \mu\text{-measurable set } E \subseteq X. \tag{7.7.1}$$

Then the measures w and μ have the same sigma-algebra of measurable sets and are mutually absolutely continuous with each other.

Next, given $p \in (1, \infty)$, a weight w is said to belong to the `Muckenhoupt class` $A_p(X, \rho, \mu)$ (occasionally denoted by $A_p(X, \mu)$ if the quasi-distance ρ is understood, or even just simply A_p if the background (X, ρ, μ) is clear from the context)

$$\begin{aligned}[w]_{A_p} &:= \sup_{B\ \rho\text{-ball}} \Big(\frac{1}{\mu(B)} \int_B w \, d\mu\Big)\Big(\frac{1}{\mu(B)} \int_B w^{-1/(p-1)} \, d\mu\Big)^{p-1} \\ &= \sup_{B\ \rho\text{-ball}} \Big(\frac{1}{\mu(B)} \int_B w \, d\mu\Big)\Big(\frac{1}{\mu(B)} \int_B w^{1-p'} \, d\mu\Big)^{p-1} < +\infty, \end{aligned} \tag{7.7.2}$$

where $p' \in (1, \infty)$ is the Hölder conjugate exponent of p. Corresponding to the case when $p = 1$, the class $A_1 = A_1(X, \rho, \mu) = A_1(X, \mu)$ is then defined as the collection of weights w for which

$$[w]_{A_1} := \sup_{B\ \rho\text{-ball}} \Big(\operatorname*{ess\,inf}_B w\Big)^{-1}\Big(\frac{1}{\mu(B)} \int_B w \, d\mu\Big) < +\infty. \tag{7.7.3}$$

Hence, a weight w belongs to A_1 if and only if there exists $C \in (0, \infty)$ with the property that for every ρ-ball $B \subseteq X$ we have

$$\frac{1}{\mu(B)} \int_B w \, d\mu \le C w(x) \quad \text{for } \mu\text{-a.e. } x \in B, \tag{7.7.4}$$

and the best constant in (7.7.4) is actually $[w]_{A_1}$. In all cases,

$$[\lambda w]_{A_p} = [w]_{A_p} \text{ for each } w \in A_p \text{ with } p \in [1, \infty) \text{ and } \lambda \in (0, \infty). \tag{7.7.5}$$

It is also useful to note that

$$[w]_{A_p} \ge 1 \quad \text{for each} \quad w \in A_p \quad \text{with} \quad p \in [1, \infty). \tag{7.7.6}$$

Indeed, if $p, p' \in (1, \infty)$ are such that $1/p + 1/p' = 1$ and $w \in A_p$, then for each ρ-ball $B \subseteq X$ we may use Hölder's inequality and (7.7.2) to write

$$\begin{aligned} 1 &= \frac{1}{\mu(B)} \int_B w^{1/p} \cdot w^{-1/p} \, d\mu \\ &\le \Big(\frac{1}{\mu(B)} \int_B w \, d\mu\Big)^{1/p}\Big(\frac{1}{\mu(B)} \int_B w^{-p'/p} \, d\mu\Big)^{1/p'} \le [w]_{A_p}^{1/p}, \end{aligned} \tag{7.7.7}$$

from which (7.7.6) follows when $p \in (1, \infty)$. The case $p = 1$ is seen from (7.7.3).

Next, as it is clear from definitions,

$$A_p(X,\mu) \subseteq L^1_{\rm loc}(X,\mu) \ \text{ for each } \ p \in [1,\infty). \tag{7.7.8}$$

Corresponding to the end-point $p = \infty$,

$$\begin{gathered}\text{the class } A_\infty(X,\mu) \text{ is defined as}\\ \text{the union of all } A_p(X,\mu) \text{ with } p \in [1,\infty).\end{gathered} \tag{7.7.9}$$

An equivalent characterization of the membership of a weight $w \in L^1_{\rm loc}(X,\mu)$ to the Muckenhoupt class $A_p(X,\rho,\mu)$ with $p \in [1,\infty)$ is the demand that there exists a constant $C \in (0,\infty)$ with the property that

$$\begin{gathered}\frac{1}{\mu(B)}\int_B |f|\,d\mu \leq C\Big(\frac{1}{w(B)}\int_B |f|^p w\,d\mu\Big)^{1/p}\\ \text{where } \ w(B) := \textstyle\int_B w\,d\mu, \ \text{ for each}\mu\text{-measurable}\\ \text{function } f \text{ defined on } X, \ \text{ and each } \rho\text{-ball } B \subseteq X.\end{gathered} \tag{7.7.10}$$

Indeed, if $p \in (1,\infty)$ then specializing (7.7.10) to the case when $f := w^{-1/(p-1)}$ (which ensures that we also have $|f|^p w = w^{-1/(p-1)}$) yields

$$\Big(\frac{1}{\mu(B)}\int_B w\,d\mu\Big)\Big(\frac{1}{\mu(B)}\int_B w^{-1/(p-1)}\,d\mu\Big)^{p-1} \leq C \tag{7.7.11}$$

for each ρ-ball $B \subseteq X$, which proves that $w \in A_p(X,\rho,\mu)$ (cf. (7.7.2)). In the case when $p = 1$, for each ρ-ball $B \subseteq X$ and each $\lambda > \operatorname{ess\,inf}_B w$ specialize the estimate in (7.7.10) to the case $f := \mathbf{1}_{E_\lambda}$ where $E_\lambda := \{x \in B : w(x) < \lambda\}$. This yields $\mu(E_\lambda)/\mu(B) \leq C w(E_\lambda)/w(B)$, and since Chebyshev's inequality guarantees that $w(E_\lambda) \leq \lambda\mu(E_\lambda)$ we ultimately obtain (bearing in mind that $0 < \mu(E_\lambda) < +\infty$, thanks to the choice of λ) that $w(B)/\mu(B) \leq C\lambda$. By taking the infimum over all λ's which are greater than $\operatorname{ess\,inf}_B w$ we arrive at the conclusion that (7.7.4) holds, hence $w \in A_1(X,\rho,\mu)$. In the opposite direction, assume $w \in A_p(X,\rho,\mu)$ with $p \in (1,\infty)$, pick an arbitrary μ-measurable function f defined on X along with some ρ-ball $B \subseteq X$, and denote by $p' \in (1,\infty)$ the Hölder conjugate exponent of p. Then, based on Hölder's inequality and (7.7.2), we may estimate

$$\begin{aligned}\frac{1}{\mu(B)}\int_B |f|\,d\mu &= \frac{1}{\mu(B)}\int_B \big(|f|w^{1/p}\big)w^{-1/p}\,d\mu\\ &\leq \frac{1}{\mu(B)}\Big(\int_B |f|^p w\,d\mu\Big)^{1/p}\Big(\int_B w^{-p'/p}\,d\mu\Big)^{1/p'}\\ &= \Big(\frac{1}{w(B)}\int_B |f|^p w\,d\mu\Big)^{1/p}\Big(\frac{1}{\mu(B)}\int_B w\,d\mu\Big)^{1/p}\times\\ &\qquad\times\Big(\frac{1}{\mu(B)}\int_B w^{-p'/p}\,d\mu\Big)^{(p-1)/p}\\ &\leq [w]_{A_p}^{1/p}\Big(\frac{1}{w(B)}\int_B |f|^p w\,d\mu\Big)^{1/p},\end{aligned} \tag{7.7.12}$$

proving (7.7.10) with $C := [w]_{A_p}^{1/p} \in (0, \infty)$. Finally, when $w \in A_1(X, \rho, \mu)$, we may rely on (7.7.3) to write

$$\begin{aligned}
\frac{1}{\mu(B)} \int_B |f| \, d\mu &= \frac{1}{\mu(B)} \int_B \big(|f|w\big) w^{-1} \, d\mu \\
&=\leq \frac{1}{\mu(B)} \Big(\int_B |f| w \, d\mu \Big) \text{ess} \sup_B w^{-1} \\
&= \Big(\frac{1}{w(B)} \int_B |f| w \, d\mu \Big) \Big(\text{ess} \inf_B w \Big)^{-1} \Big(\frac{1}{\mu(B)} \int_B w \, d\mu \Big) \\
&\leq [w]_{A_1} \Big(\frac{1}{w(B)} \int_B |f| w \, d\mu \Big),
\end{aligned} \tag{7.7.13}$$

proving the version of (7.7.10) when $p = 1$ with $C := [w]_{A_1} \in (0, \infty)$.

In the lemma below we summarize a number of other basic properties (see, e.g., [7, 55, 97, 103, 107, 128, 129, 141, 154, 247] for proofs and other pertinent references). The reader is alerted to the fact that some natural examples of Muckenhoupt weights are given later, in Propositions 8.7.4–8.7.5 and Examples 8.7.6–8.7.7.

Lemma 7.7.1 *Consider a space of homogeneous type (X, ρ, μ) with the property that the quasi-distance $\rho : X \times X \to [0, \infty)$ is continuous*[16] *in the product topology $\tau_\rho \times \tau_\rho$. In this context, the following properties hold:*

(1) [*Boundedness of Maximal Operator*] *Given a weight function w on X along with an integrability exponent $p \in (1, \infty)$, the Hardy–Littlewood maximal operator $\mathcal{M}_X$ on X is bounded on $L^p(X, w) := L^p(X, w\mu)$ (cf. (7.7.1)) if and only if $w \in A_p$, in which case there exists some constant $C = C(\rho, \mu, p) \in (0, \infty)$ with the property that*

$$[w]_{A_p}^{1/p} \leq \|\mathcal{M}_X\|_{L^p(X,w) \to L^p(X,w)} \leq C[w]_{A_p}^{1/(p-1)}. \tag{7.7.14}$$

Also, corresponding to $p = 1$, the operator $\mathcal{M}_X$ satisfies the weak-$(1, 1)$ inequality

$$\begin{aligned}
&\sup_{0<\lambda<\infty} \lambda \cdot w\big(\{x \in X : \mathcal{M}_X f(x) > \lambda\}\big) \leq C\|f\|_{L^1(X,w)} \\
&\textit{for all } f \in L^1(X, w), \ \textit{with } C \in (0, \infty) \textit{ independent of } f,
\end{aligned} \tag{7.7.15}$$

if and only if $w \in A_1$.

(2) [*Dual/Conjugate Weights*] *If $1 < p < \infty$ then the weight $w \in A_p$ if and only if $w^{1-p'} = w^{-p'/p} \in A_{p'}$, where p' is the Hölder conjugate exponent of p, in which case $[w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{1/(p-1)}$.*

[16] Theorem 7.1.2 guarantees that any quasi-metric space has an equivalent quasi-distance which satisfies this property.

(3) [*Products/Reverse Factorizations*] *Given any two weights* $w_0, w_1 \in A_1$ *along with an arbitrary exponent* $p \in [1, \infty)$*, it follows that* $w_0 \cdot w_1^{1-p} \in A_p$ *and one has* $[w_0 \cdot w_1^{1-p}]_{A_p} \leq [w_0]_{A_1}[w_1]_{A_1}^{p-1}$.

(4) [*Membership to* A_1] *There exists some* $C_X \in (0, \infty)$ *which depends only on the quasi-distance* ρ (*via the constants* $C_\rho, \widetilde{C}_\rho$ *appearing in* (7.1.3)–(7.1.4)) *and the doubling charter of* μ*, with the property that for any* $C \in (0, \infty)$ *and any weight function* w *on* X *satisfying* $\mathcal{M}_X(w) \leq Cw$ *on* X *one has* $w \in A_1$ *and* $[w]_{A_1} \leq C_X \cdot C$.

(5) [*Monotonicity*] *If* $w \in A_p$ *for some* $p \in [1, \infty)$ *then* $w \in A_q$ *for any exponent* $q \in [p, \infty)$ *and one has* $[w]_{A_q} \leq [w]_{A_p}$*. In particular, one has* $A_p \subseteq A_q$ *whenever* $1 \leq p \leq q \leq \infty$.

(6) [*Doubling*] *If* $w \in A_p$ *with* $p \in [1, \infty)$ *then for every* ρ*-ball* $B \subseteq X$ *and every* μ*-measurable set* $E \subseteq B$ *one has*[17]

$$\left(\frac{\mu(E)}{\mu(B)}\right)^p \leq [w]_{A_p} \cdot \frac{w(E)}{w(B)}. \tag{7.7.16}$$

In particular, $w\,\mu$ *is a doubling measure on* X (*in the sense of* (7.4.1)).

(7) [*Logarithmic Convexity*] *If* $w_0, w_1 \in A_p$ *with* $p \in [1, \infty)$ *then for each exponent* $\theta \in [0, 1]$ *one has* $w_0^\theta \cdot w_1^{1-\theta} \in A_p$ *and* $[w_0^\theta \cdot w_1^{1-\theta}]_{A_p} \leq [w_0]_{A_p}^\theta \cdot [w_1]_{A_p}^{1-\theta}$ (*in particular, choosing* $w_1 \equiv$*constant, if* $w \in A_p$ *for some* $p \in [1, \infty)$ *then* $w^\theta \in A_p$ *for each* $\theta \in [0, 1]$).

(8) [*Building* A_1 *Weights*] *There exists* $C \in (0, \infty)$ *which depends only on the ambient with the property that if* $f \in L^1_{\rm loc}(X, \mu)$ *is not identically zero and satisfies* $\mathcal{M}_X f < \infty$ *at* μ*-a.e. point on* X *then for each* $\theta \in (0, 1)$ *one has* $(\mathcal{M}_X f)^\theta \in A_1$ *and* $\big[(\mathcal{M}_X f)^\theta\big]_{A_1} \leq C(1-\theta)^{-1}$.

(9) [*Self-Improving*] *If the measure* μ *is Borel-semiregular, then for each* $w \in A_p$ *with* $p \in (1, \infty)$ *there exists some small* $\varepsilon \in (0, p-1)$*, which depends only on the ambient, p, and* $[w]_{A_p}$*, such that* $w \in A_{p-\varepsilon}$ *with* $[w]_{A_{p-\varepsilon}}$ *controlled in terms of the ambient, p, and* $[w]_{A_p}$.

(10) [*Power Stability*] *If the measure* μ *is Borel-semiregular then for each* $w \in A_p$ *with* $p \in [1, \infty)$ *there exists some small* $\delta > 0$ (*which depends only on the ambient, p, and* $[w]_{A_p}$) *such that* $w^{1+\delta} \in A_p$ *with* $\big[w^{1+\delta}\big]_{A_p}$ *controlled in terms of the ambient, p, and* $[w]_{A_p}$.

(11) [*Reverse Hölder Inequalities*] *Given* $q \in (1, \infty)$*, denote by* $RH_q(X, \rho, \mu)$ (*or simply* $RH_q(X, \mu)$*, or even just* RH_q*, if the ambient is clear from context*) *the collection of weight functions* w *on* X *with the property that*

$$[w]_{RH_q} := \sup_{B\ \rho\text{-ball}} \left(\frac{1}{\mu(B)}\int_B w^q\, d\mu\right)^{\frac{1}{q}} \left(\frac{1}{\mu(B)}\int_B w\, d\mu\right)^{-1} < +\infty. \tag{7.7.17}$$

Then

[17] With the piece of notation introduced in (7.7.1).

$$\bigcup_{1\leq p<\infty} A_p(X,\rho,\mu) \subseteq \bigcup_{1<q<\infty} RH_q(X,\rho,\mu). \tag{7.7.18}$$

More specifically, for every $w \in A_p$ *with* $p \in [1,\infty)$ *there exist some* $q \in (1,\infty)$ *and some* $C \in (0,\infty)$ (*which both depend only on* p, $[w]_{A_p}$, *and the ambient space of homogeneous type*) *such that*[18]

$$\left(⨍_B w^q \,\mathrm{d}\mu\right)^{1/q} \leq C ⨍_B w \,\mathrm{d}\mu \ \ \textit{for every } \rho\textit{-ball } B \subseteq X. \tag{7.7.19}$$

As a consequence of (7.7.19) *and Hölder's inequality, for each* μ*-measurable function* f *on* X *and each* ρ*-ball* $B \subseteq X$ *one has*

$$⨍_B |f| \,\mathrm{d}w \leq C \left(⨍_B |f|^{q'} \,\mathrm{d}\mu\right)^{1/q'} \tag{7.7.20}$$

where $q' \in (1,\infty)$ *is the Hölder conjugate exponent of* q, *and* $C \in (0,\infty)$ *is as in* (7.7.19). *In particular, there exist some power* $\tau > 0$ *and some constant* $C \in (0,\infty)$ (*in fact,* C *is the same as in* (7.7.19) *and* $\tau = 1/q'$ *where* q' *is the Hölder conjugate of the exponent* q *from* (7.7.19)) *such that*

$$\frac{w(E)}{w(B)} \leq C \left(\frac{\mu(E)}{\mu(B)}\right)^{\tau} \tag{7.7.21}$$

for every ρ*-ball* $B \subseteq X$ *and every* μ*-measurable set* $E \subseteq B$.

(12) [BMO *and Weights*] *For each weight* $w \in A_p$ *with* $p \in (1,\infty)$ *there exist* $\varepsilon > 0$ *and* $C \in (0,\infty)$, *both depending only on* X, p, $[w]_{A_p}$, *such that for each function* $b \in \mathrm{BMO}(X,\mu)$ *with* $\|b\|_{\mathrm{BMO}(X,\mu)} < \varepsilon$ *one has* $w \cdot \mathrm{e}^b \in A_p$ *and* $\big[w \cdot \mathrm{e}^b\big]_{A_p} \leq C$. *In particular, for each integrability exponent* $p \in (1,\infty)$ *the set* $\mathcal{U}_p := \big\{b \in \mathrm{BMO}(X,\mu) : \mathrm{e}^b \in A_p\big\}$ *is open in* $\mathrm{BMO}(X,\mu)$. *Also, for each weight* $w \in A_1$, *the function* $\log w$ *belongs to* $\mathrm{BMO}(X,\mu)$ *and one has* $\|\log w\|_{\mathrm{BMO}(X,\mu)} \leq C(X,[w]_{A_1})$. *Finally, as a consequence of* (7.7.9), *item (5) above,* (7.7.20), *and* (7.4.105),

$$\mathrm{BMO}(X,\mu) \subseteq L^1_{\mathrm{loc}}(X,w) \ \ \textit{continuously for each} \ \ w \in A_\infty(X,\mu). \tag{7.7.22}$$

We wish to augment the list of properties in Lemma 7.7.1 with two other related results. The first such result may be regarded as a "precise openness property" for Muckenhoupt weights (compare with item *(9)* in Lemma 7.7.1).

Lemma 7.7.2 *Let* (X,ρ,μ) *be a space of homogeneous type with the property that the quasi-distance* $\rho : X \times X \to [0,\infty)$ *is continuous in the product topology* $\tau_\rho \times \tau_\rho$. *Having fixed an exponent* $p \in (1,\infty)$, *select some* $w \in A_p$ *and denote by*

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[18] Using the convention adopted in (7.7.1).

$w' := w^{-p'/p} = w^{1-p'} \in A_{p'}$ the conjugate weight of w (where p' is the Hölder conjugate exponent of p; see item (2) in Lemma 7.7.1). Then if w' belongs to the reverse Hölder class RH_q for some $q \in (1, \infty)$ it follows that

$$w \in A_{p-\varepsilon} \ \ \textit{where} \ \ \varepsilon := \frac{p-1}{q'} \in (0, p-1), \tag{7.7.23}$$

and q' is the Hölder conjugate exponent of q. As a consequence,

$$\begin{aligned} &\textit{for each } w \in A_p \cap RH_q \textit{ with } p, q \in (1, \infty) \textit{ one} \\ &\textit{has } w' \in A_{p'-\varepsilon} \textit{ with } \varepsilon := \tfrac{p'-1}{q'} \in (0, p'-1). \end{aligned} \tag{7.7.24}$$

Proof The choice of ε in (7.7.23) entails

$$p - \varepsilon - 1 = \frac{p-1}{q}. \tag{7.7.25}$$

Bearing this in mind, for each ρ-ball $B \subseteq X$ we may then write

$$\begin{aligned} &\Big(\frac{1}{\mu(B)}\int_B w\,\mathrm{d}\mu\Big)\Big(\frac{1}{\mu(B)}\int_B w^{-1/(p-\varepsilon-1)}\,\mathrm{d}\mu\Big)^{p-\varepsilon-1} \\ &\qquad = \Big(\frac{1}{\mu(B)}\int_B w\,\mathrm{d}\mu\Big)\Big(\frac{1}{\mu(B)}\int_B w^{-q/(p-1)}\,\mathrm{d}\mu\Big)^{(p-1)/q} \\ &\qquad = \Big(\frac{1}{\mu(B)}\int_B w\,\mathrm{d}\mu\Big)\Big(\frac{1}{\mu(B)}\int_B (w')^{q}\,\mathrm{d}\mu\Big)^{(p-1)/q} \\ &\qquad \leq C\Big(\frac{1}{\mu(B)}\int_B w\,\mathrm{d}\mu\Big)\Big(\frac{1}{\mu(B)}\int_B w'\,\mathrm{d}\mu\Big)^{p-1} \\ &\qquad = C\Big(\frac{1}{\mu(B)}\int_B w\,\mathrm{d}\mu\Big)\Big(\frac{1}{\mu(B)}\int_B w^{-1/(p-1)}\,\mathrm{d}\mu\Big)^{p-1} \leq C[w]_{A_p}, \end{aligned} \tag{7.7.26}$$

thanks to the fact that $w' \in RH_q$. In view of (7.7.2), this shows that $w \in A_{p-\varepsilon}$, finishing the proof of (7.7.23).

As far as (7.7.24) is concerned, assume $w \in A_p \cap RH_q$ with $p, q \in (1, \infty)$. Then $w' \in A_{p'}$ by item *(2)* in Lemma 7.7.1 and $(w')' = w \in RH_q$, so (7.7.23) (applied for w' in place of w and p' in place of p) shows that we have $w' \in A_{p'-\varepsilon}$ for the choice $\varepsilon := \frac{p'-1}{q'} \in (0, p'-1)$. □

Here is the second result mentioned above (refining work in the Euclidean setting from [134]).

Lemma 7.7.3 *Let (X, ρ, μ) be some space of homogeneous type with the property that the quasi-distance $\rho : X \times X \to [0, \infty)$ is continuous in the product topology $\tau_\rho \times \tau_\rho$. Select two arbitrary exponents, $p \in [1, \infty)$ and $q \in (1, \infty)$. Then for each weight $w \in A_p \cap RH_q$ one has $w^q \in A_{q(p-1)+1}$.*

Proof First consider the case when $p = 1$. Assume $w \in A_1 \cap RH_q$, and let $B \subseteq X$ be an arbitrary ρ-ball. Then for some $C \in (0, \infty)$ independent of B we may estimate

$$\begin{aligned}\Big(\operatorname{essinf}_B w^q\Big)^{-1}\Big(\frac{1}{\mu(B)}\int_B w^q \,\mathrm{d}\mu\Big) &= \Big(\operatorname{essinf}_B w\Big)^{-q}\Big(\frac{1}{\mu(B)}\int_B w^q \,\mathrm{d}\mu\Big)\\ &\leq C\Big(\operatorname{essinf}_B w\Big)^{-q}\Big(\frac{1}{\mu(B)}\int_B w \,\mathrm{d}\mu\Big)^q\\ &\leq C[w]_{A_1}^q, \end{aligned}\tag{7.7.27}$$

with the first inequality above a consequence of the fact that $w \in RH_q$, while the last inequality uses the fact that $w \in A_1$. In turn, (7.7.27) readily implies that $w^q \in A_1$.

There remains to consider the case when $p \in (1, \infty)$. In such a scenario, define $s := q(p-1)+1$, and denote by p', q', and s' the Hölder conjugate exponents of p, q, and s, respectively. Pick an arbitrary ρ-ball $B \subseteq X$. Then there exists some $C \in (0, \infty)$, independent of B, such that

$$\begin{aligned}&\Big(\frac{1}{\mu(B)}\int_B w^q \,\mathrm{d}\mu\Big)\Big(\frac{1}{\mu(B)}\int_B w^{q(1-s')} \,\mathrm{d}\mu\Big)^{s-1}\\ &\qquad = \Big(\frac{1}{\mu(B)}\int_B w^q \,\mathrm{d}\mu\Big)\Big(\frac{1}{\mu(B)}\int_B w^{1-p'}\Big)^{q(p-1)}\\ &\qquad \leq C\Big(\frac{1}{\mu(B)}\int_B w \,\mathrm{d}\mu\Big)^q\Big(\frac{1}{\mu(B)}\int_B w^{1-p'} \,\mathrm{d}\mu\Big)^{q(p-1)}\\ &\qquad \leq C[w]_{A_p}^q. \end{aligned}\tag{7.7.28}$$

The equality above employs the observation that $q(1-s') = 1-p'$, which may be checked from definitions. The first inequality above is based on the membership $w \in RH_q$. The last inequality above is implied by the fact that $w \in A_p$. Finally, from (7.7.28) we see that $w^q \in A_{q(p-1)+1}$, as wanted. □

Here is the third result alluded to earlier, complementing the list of properties in Lemma 7.7.1 (compare with [107, 9.1.5, p. 683]).

Lemma 7.7.4 *Let (X, ρ, μ) be a space of homogeneous type with the property that the quasi-distance $\rho : X \times X \to [0, \infty)$ is continuous in the product topology $\tau_\rho \times \tau_\rho$. Suppose $w_0 \in A_{p_0}$ with $p_0 \in (1, \infty)$, and $w_1 \in A_{p_1}$ with $p_1 \in (1, \infty)$. Also, pick some $\theta \in [0, 1]$ and define*

$$p := \Big(\tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1}\Big)^{-1} \in (1, \infty).\tag{7.7.29}$$

Then the weight

$$w := \Big(w_0^{\frac{1-\theta}{p_0}} \cdot w_1^{\frac{\theta}{p_1}}\Big)^p\tag{7.7.30}$$

belongs to Muckenhoupt's class $A_p(X, \rho, \mu)$ and satisfies

$$[w]_{A_p} \leq C\Big([w_0]_{A_{p_0}}^{\frac{p}{p_0-1}}\Big)^{1-\theta} \cdot \Big([w_1]_{A_{p_1}}^{\frac{p}{p_1-1}}\Big)^{\theta} \tag{7.7.31}$$

for some constant $C = C(\rho, \mu, p) \in (0, \infty)$.

Proof From item *(1)* in Lemma 7.7.1 we know that the Hardy–Littlewood maximal operator $\mathcal{M}_X$ on X is bounded on $L^{p_i}(X, w_i)$ for $i = 0, 1$ and there exists some constant $C = C(\rho, \mu, p) \in (0, \infty)$ with the property that

$$\|\mathcal{M}_X\|_{L^{p_i}(X,w_i)\to L^{p_i}(X,w_i)} \leq C[w_i]_{A_{p_i}}^{1/(p_i-1)} \text{ for } i = 0, 1. \tag{7.7.32}$$

From this and the real interpolation theorem for sub-linear operators (cf. [185, Sect. 1.3]) we then conclude that

$$\mathcal{M}_X : \Big(L^{p_0}(X, w_0), L^{p_1}(X, w_1)\Big)_{\theta,p} \longrightarrow \Big(L^{p_0}(X, w_0), L^{p_1}(X, w_1)\Big)_{\theta,p} \tag{7.7.33}$$

is a bounded operator with norm dominated by

$$C\Big([w_0]_{A_{p_0}}^{1/(p_0-1)}\Big)^{1-\theta} \cdot \Big([w_1]_{A_{p_1}}^{1/(p_1-1)}\Big)^{\theta}. \tag{7.7.34}$$

In addition, we know (see [19, Theorem 5.5.1, p. 119], [92]) that

$$\Big(L^{p_0}(X, w_0), L^{p_1}(X, w_1)\Big)_{\theta,p} = L^p(X, w) \text{ where } w := \Big(w_0^{\frac{1-\theta}{p_0}} \cdot w_1^{\frac{\theta}{p_1}}\Big)^p. \tag{7.7.35}$$

In view of the converse results in item *(1)* in Lemma 7.7.1 and (7.7.14), we may then conclude that w from (7.7.30) belongs to Muckenhoupt's class $A_p(X, \rho, \mu)$ and satisfies (7.7.31). □

For each weight function w on X we have $L^\infty(X, \mu) = L^\infty(X, w)$, i.e., these vector spaces coincide and they also have identical norms. Remarkably, whenever $w \in A_\infty(X, \mu)$ it follows that the BMO spaces on X with respect to μ and w are once again identical. Here is a formal statement of this fact (compare with [207, Theorem 5, p. 236]).

Lemma 7.7.5 *Suppose* (X, ρ, μ) *is a measure metric space. Then for each weight* $w \in A_\infty(X, \mu)$ *it follows that*

$$\begin{gathered} \mathrm{BMO}(X, \mu) \textit{ and } \mathrm{BMO}(X, w) \textit{ coincide as sets, and} \\ \|f\|_{\mathrm{BMO}(X,\mu)} \approx \|f\|_{\mathrm{BMO}(X,w)} \textit{ uniformly for } f \textit{ in this set.} \end{gathered} \tag{7.7.36}$$

Proof From (7.7.9) and item *(5)* in Lemma 7.7.1 we know that $w \in A_p(X, \mu)$ for some $p \in (1, \infty)$. In particular, w is a doubling measure on X, so it makes sense to consider $\mathrm{BMO}(X, w)$. Pick a function $f \in L^1_{\mathrm{loc}}(X, \mu) \cap L^1_{\mathrm{loc}}(X, w)$. Our first goal is to show that there exists a constant $C \in [1, \infty)$ which depends only on p, $[w]_{A_p}$, and the ambient, such that

$$C^{-1}\|f\|_{\dot{\mathrm{BMO}}(X,\mu)} \leq \|f\|_{\dot{\mathrm{BMO}}(X,w)} \leq C\|f\|_{\dot{\mathrm{BMO}}(X,\mu)}. \tag{7.7.37}$$

To prove the first inequality claimed in (7.7.37), use (7.4.71) (twice) and (7.7.10) with f replaced by $f - \fint_B f\,\mathrm{d}w$, for some arbitrary ρ-ball $B \subseteq X$, to obtain

$$\begin{aligned}\|f\|_{\dot{\mathrm{BMO}}(X,\mu)} &\leq C \sup_{B\subseteq X} \inf_{c\in\mathbb{C}} \Big(\fint_B |f-c|\,\mathrm{d}\mu\Big) \leq C \sup_{B\subseteq X} \fint_B \Big|f - \fint_B f\,\mathrm{d}w\Big|\,\mathrm{d}\mu \\ &\leq C \sup_{B\subseteq X} \Big(\fint_B \Big|f - \fint_B f\,\mathrm{d}w\Big|^p \mathrm{d}w\Big)^{1/p} \\ &\leq C\|f\|_{\dot{\mathrm{BMO}}(X,w)},\end{aligned} \tag{7.7.38}$$

for some constant $C \in (0,\infty)$ as in the statement. To prove the second inequality in (7.7.37), observe first that w belongs to some Reverse Hölder class, i.e., w satisfies (7.7.19) for some $q \in (1,\infty)$. If $q' \in (1,\infty)$ denotes the Hölder conjugate exponent of q, then (7.7.20) allows to estimate

$$\begin{aligned}\inf_{c\in\mathbb{C}} \Big(\fint_B |f-c|\,\mathrm{d}w\Big) &\leq \fint_B \Big|f - \fint_B f\,\mathrm{d}\mu\Big|\,\mathrm{d}w \\ &\leq C\Big(\fint_B \Big|f - \fint_B f\,\mathrm{d}\mu\Big|^{q'} \mathrm{d}\mu\Big)^{1/q'},\end{aligned} \tag{7.7.39}$$

for some constant $C \in (0,\infty)$ of the same nature as before. Taking the supremum in (7.7.39) over all ρ-balls $B \subseteq X$ and then using (7.4.71), we ultimately obtain $\|f\|_{\dot{\mathrm{BMO}}(X,w)} \leq C\|f\|_{\dot{\mathrm{BMO}}(X,\mu)}$, as desired.

We next claim that for each μ-measurable function f on X one has the equivalence

$$f \in \mathrm{BMO}(X,\mu) \Longleftrightarrow f \in \mathrm{BMO}(X,w) \tag{7.7.40}$$

and if either of these memberships materializes then

$$\|f\|_{\mathrm{BMO}(X,\mu)} \approx \|f\|_{\mathrm{BMO}(X,w)}, \tag{7.7.41}$$

where the implicit proportionality constants depend only on p, $[w]_{A_p}$, and the ambient.

As regards the equivalence in (7.7.40), assume first that $f \in \mathrm{BMO}(X,\mu)$. Then (7.7.22) and (7.4.81) imply that $f \in L^1_{\mathrm{loc}}(X,\mu) \cap L^1_{\mathrm{loc}}(X,w)$, so (7.7.37) holds, thanks to what we have proved so far. In particular, $f \in \mathrm{BMO}(X,w)$ if X is unbounded (cf. (7.4.81) and (7.4.92)), and (7.7.37) presently yields (7.7.41) (cf. (7.4.81)). Consider now the case when X is bounded. Then (7.7.22) gives that $f \in L^1(X,w)$ and $\|f\|_{L^1(X,w)} \leq C\|f\|_{\mathrm{BMO}(X,\mu)}$. With this in hand, (7.4.83) (written for the measure w) together with (7.7.37) allow us to write

$$\begin{aligned}\|f\|_{\mathrm{BMO}(X,w)} &\approx \|f\|_{L^1(X,w)} + \|f\|_{\dot{\mathrm{BMO}}(X,w)}\\ &\leq C\|f\|_{\mathrm{BMO}(X,\mu)} + \|f\|_{\dot{\mathrm{BMO}}(X,\mu)} \leq C\|f\|_{\mathrm{BMO}(X,\mu)}. \end{aligned} \tag{7.7.42}$$

Hence, $f \in \mathrm{BMO}(X, w)$ and the right-pointing inequality in (7.7.41) holds.

Conversely, assume $f \in \mathrm{BMO}(X, w)$. In particular, $f \in L^1_{\mathrm{loc}}(X, w)$ (cf. (7.4.81)) and the John–Nirenberg inequality (for the doubling measure w) guarantees that we also have $f \in L^p_{\mathrm{loc}}(X, w)$. In concert with (7.7.10) the latter membership implies that $f \in L^1_{\mathrm{loc}}(X, \mu)$. Hence, in a quantitative fashion,

$$f \in L^1_{\mathrm{loc}}(X, \mu) \cap L^1_{\mathrm{loc}}(X, w). \tag{7.7.43}$$

Thus, once again (7.7.37) applies. If X is unbounded, this gives $f \in \mathrm{BMO}(X, \mu)$, which ultimately finishes the proof of (7.7.40) and (7.7.41) in the case when X is unbounded.

Finally, consider the situation when X is bounded and pick $f \in \mathrm{BMO}(X, w)$. Then (7.7.43) presently gives $f \in L^1(X, \mu)$ and $\|f\|_{L^1(X,\mu)} \leq C\|f\|_{\mathrm{BMO}(X,w)}$. Granted this, (7.4.83) and (7.7.37) permit us to estimate

$$\begin{aligned}\|f\|_{\mathrm{BMO}(X,\mu)} &\approx \|f\|_{L^1(X,\mu)} + \|f\|_{\dot{\mathrm{BMO}}(X,\mu)}\\ &\leq C\|f\|_{\mathrm{BMO}(X,w)} + \|f\|_{\dot{\mathrm{BMO}}(X,w)} \leq C\|f\|_{\mathrm{BMO}(X,w)}. \end{aligned} \tag{7.7.44}$$

As such, $f \in \mathrm{BMO}(X, \mu)$ and the left-pointing inequality in (7.7.41) is valid. In summary, we proved that $\mathrm{BMO}(X, \mu) = \mathrm{BMO}(X, w)$ as sets and that the equivalence $\|f\|_{\mathrm{BMO}(X,\mu)} \approx \|f\|_{\mathrm{BMO}(X,w)}$ holds uniformly for functions f in the space $\mathrm{BMO}(X, \mu) = \mathrm{BMO}(X, w)$. □

A fundamental result in the theory of Muckenhoupt weights is Rubio de Francia's extrapolation theorem. Its philosophy is perfectly summed up in Antonio Córdoba's delightful quip[19]:

$$\textit{There are no } L^p \textit{ spaces, only weighted } L^2. \tag{7.7.45}$$

We shall present a version of this result in the context of spaces of homogeneous type (refining work in [58], [125, Proposition 2.17, p. 2603]). A more general result in this spirit, extrapolating Muckenhoupt weighted Lebesgue space estimates to norm estimates on Generalized Banach Function Spaces (of the sort introduced in [185, Sect. 5.1]) is discussed later, in [185, Sect. 5.2].

Proposition 7.7.6 *Consider a space of homogeneous type (X, ρ, μ) with the property that the quasi-distance $\rho : X \times X \to [0, \infty)$ is continuous*[20] *in the product*

[19] "L^p no existe; sólo existe L^2 con peso" [94, p. 8].

[20] Theorem 7.1.2 guarantees that any quasi-metric space has an equivalent quasi-distance satisfying this property.

topology $\tau_\rho \times \tau_\rho$. Let f, g be two non-negative μ-measurable real-valued functions defined on X with the property that there exists some integrability exponent $p_0 \in [1, \infty)$ such that for every Muckenhoupt weight $w \in A_{p_0}(X, \rho, \mu)$ one has

$$\Big(\int_X f^{p_0}\, w\,\mathrm{d}\mu\Big)^{1/p_0} \leq C_w\Big(\int_X g^{p_0}\, w\,\mathrm{d}\mu\Big)^{1/p_0}, \tag{7.7.46}$$

for some constant $C_w \in (0, \infty)$ which depends only on ρ, μ, p_0, and w.

Then for each integrability exponent $p \in (1, \infty)$ there exists some $C \in (0, \infty)$, depending only on the quasi-distance ρ (via the constants $C_\rho, \widetilde{C}_\rho$ appearing in (7.1.3)–(7.1.4)*), the doubling charter of μ, as well as p and p_0, such that for each weight $\omega \in A_p(X, \rho, \mu)$ if one defines*

$$W_{\omega,p} := C[\omega]_{A_p}^{1+(p_0-1)/(p-1)} \tag{7.7.47}$$

(hence $W_{\omega,p} \in (0, \infty)$ is a constant which depends only on ρ, μ, p_0, p, and $[\omega]_{A_p}$) it follows that

$$\Big(\int_X f^p \omega\,\mathrm{d}\mu\Big)^{1/p} \leq C_{\omega,p}\Big(\int_X g^p \omega\,\mathrm{d}\mu\Big)^{1/p}, \tag{7.7.48}$$

where

$$C_{\omega,p} := 2^{2-1/p_0} \cdot \Bigg(\sup_{\substack{w \in A_{p_0}(X,\rho,\mu)\\ [w]_{A_{p_0}} \leq W_{\omega,p}}} C_w\Bigg) \in (0, \infty]. \tag{7.7.49}$$

Proof For each integrability exponent $p \in [1, \infty)$ and each weight function ω on X abbreviate $L^p(\omega) := L^p(X, \omega\,\mu)$. Throughout, denote by $\mathcal{M}$ the Hardy–Littlewood maximal operator on (X, ρ, μ). Whenever $p \in (1, \infty)$ and $\omega \in A_p(X, \mu)$, implement Rubio de Francia's iterative algorithm, i.e., define

$$\begin{gathered} T = T(p, \omega) : L^p(\omega) \longrightarrow L^p(\omega) \ \text{ by setting} \\ Th := \sum_{j=0}^{\infty} \frac{\mathcal{M}^j h}{2^j \|\mathcal{M}\|^j_{L^p(\omega)\to L^p(\omega)}} \ \text{ for each function } \ h \in L^p(\omega), \end{gathered} \tag{7.7.50}$$

where $\mathcal{M}^0 h := |h|$ for each $h \in L^p(\omega)$, and $\mathcal{M}^j := \mathcal{M} \circ \cdots \circ \mathcal{M}$ (involving j factors) for each $j \in \mathbb{N}$. Then, thanks to item *(1)* in Lemma 7.7.1, T is a well-defined sub-linear operator for which

$$\|T\|_{L^p(\omega)\to L^p(\omega)} := \sup\Big\{\|Th\|_{L^p(\omega)} : \|h\|_{L^p(\omega)} = 1\Big\} \leq 2. \tag{7.7.51}$$

In addition,

$$\begin{aligned}&\text{for each function } h \in L^p(\omega) \text{ with } h \geq 0 \text{ one has}\\ &h \leq Th \quad\text{and}\quad \mathcal{M}(Th) \leq 2\|\mathcal{M}\|_{L^p(\omega)\to L^p(\omega)}\, Th,\end{aligned} \tag{7.7.52}$$

by virtue of the first term in (7.7.50) being equal to h in this case, plus the sub-linearity and boundedness of $\mathcal{M}$ on $L^p(\omega)$. Given a non-negative function $h \in L^p(\omega)$ which is not identically zero (μ-a.e.), it follows from (7.7.52) and (7.7.50) that $0 < Th < \infty$ at μ-a.e. point on X. In particular, Th is a weight on X, hence

$$\begin{aligned}&T \text{ acting on any non-negative function in } L^p(\omega) \text{ which is not identically}\\ &\text{zero } (\mu\text{-a.e.}) \text{ is a weight on } X \text{ (i.e., a } \mu\text{-measurable function which is}\\ &\text{strictly positive and finite } \mu\text{-a.e. on } X).\end{aligned} \tag{7.7.53}$$

Next, observe that there exist $C_1 \in (0,\infty)$, which depends only on the quasi-distance ρ (via the constants $C_\rho, \widetilde{C}_\rho$ appearing in (7.1.3)–(7.1.4)), along with $C_2 \in (0,\infty)$ and the doubling charter of μ, with the property that for each non-negative function $h \in L^p(\omega)$, each ρ-ball $B = B_\rho(x_0, r) \subseteq X$, and each point $x \in B$ we may write

$$\begin{aligned}⨍_B Th\, d\mu &\leq C_2 ⨍_{B_\rho(x, C_1 r)} Th\, d\mu\\ &\leq C_2\big(\mathcal{M}(Th)\big)(x) \leq 2C_2\|\mathcal{M}\|_{L^p(\omega)\to L^p(\omega)}(Th)(x),\end{aligned} \tag{7.7.54}$$

thanks to the last property in (7.7.52). In view of (7.7.4) (plus the subsequent comment), (7.7.14), and (7.7.53), this implies that

$$\begin{aligned}&\text{if } 0 \leq h \in L^p(\omega) \text{ is not identically zero } (\mu\text{-a.e.}),\\ &\qquad\text{then } Th \in A_1(X,\mu) \quad\text{and}\\ &[Th]_{A_1} \leq 2C_2\|\mathcal{M}\|_{L^p(\omega)\to L^p(\omega)} \leq C_3[\omega]_{A_p}^{1/(p-1)}\end{aligned} \tag{7.7.55}$$

where $C_3 \in (0,\infty)$ depends only on the quasi-distance ρ (via the constants $C_\rho, \widetilde{C}_\rho$ appearing in (7.1.3)–(7.1.4)), the doubling charter of μ, and p (see also item *(4)* in Lemma 7.7.1).

With the goal of proving (7.7.48), fix some integrability exponent $p \in (1,\infty)$ along with an arbitrary Muckenhoupt weight $\omega \in A_p(X,\rho,\mu)$. If $g = 0$ at μ-a.e. point in X then (7.7.46) (with, say, $w \equiv 1$) forces $f = 0$ at μ-a.e. point in X, in which case (7.7.48) is trivially true. Since the claim in (7.7.48) is also trivially true if $\|g\|_{L^p(\omega)} = \infty$, henceforth assume that

$$f \text{ is not identically zero } (\mu\text{-a.e.}), \text{ and } 0 < \|g\|_{L^p(\omega)} < \infty. \tag{7.7.56}$$

In particular, the latter property permits us to define

$$\widetilde{g} := \frac{g}{\|g\|_{L^p(\omega)}}, \tag{7.7.57}$$

which satisfies

$$0 \leq \widetilde{g} \in L^p(\omega) \quad \text{and} \quad \|\widetilde{g}\|_{L^p(\omega)} = 1. \tag{7.7.58}$$

The remainder of the proof is divided into several steps, starting with

`Step I.` *Suppose* $p_0 \in (1, \infty)$ *and* $f \in L^p(\omega)$. Let $p' \in (1, \infty)$ denote the Hölder conjugate exponent of p, and set $\omega' := \omega^{1-p'} \in A_{p'}(X, \mu)$ (see item *(2)* in Lemma 7.7.1). Denote by $T = T(p, \omega)$ the sub-linear operator associated as in (7.7.50) with the integrability exponent p and the weight $\omega \in A_p(X, \mu)$, and by $T' = T'(p', \omega')$ the sub-linear operator associated as in (7.7.50) with the integrability exponent p' and the weight $\omega' \in A_{p'}(X, \mu)$. Then, as a consequence of (7.7.56) and the current working assumptions, if

$$h := \frac{|f|^{p-1}\omega}{\|f\|_{L^p(\omega)}^{p-1}} \tag{7.7.59}$$

then

$$\begin{aligned}&h \text{ is a well-defined, non-negative, } \mu\text{-measurable function,}\\ &\text{not identically zero } (\mu\text{-a.e.}), \text{ and belonging to } L^{p'}(\omega')\end{aligned} \tag{7.7.60}$$

since we actually have

$$\begin{aligned}\|h\|_{L^{p'}(\omega')} &= \Big(\int_X |h|^{p'}\omega'\,\mathrm{d}\mu\Big)^{1/p'} = \|f\|_{L^p(\omega)}^{1-p}\Big(\int_X |f|^{p'(p-1)}\omega^{p'}\omega^{1-p'}\,\mathrm{d}\mu\Big)^{1/p'}\\ &= \|f\|_{L^p(\omega)}^{1-p}\Big[\Big(\int_X |f|^p\omega\,\mathrm{d}\mu\Big)^{1/p}\Big]^{p/p'}\\ &= \|f\|_{L^p(\omega)}^{1-p}\|f\|_{L^p(\omega)}^{p-1} = 1,\end{aligned} \tag{7.7.61}$$

given that $p'(p-1) = p$ and $p/p' = p-1$. Granted (7.7.60) and (7.7.58), we conclude from (7.7.55) and item *(2)* in Lemma 7.7.1 that

$$\begin{aligned}&T\widetilde{g},\, T'h \in A_1(X, \mu) \;\text{ with }\; [T\widetilde{g}]_{A_1} \leq C_3[\omega]_{A_p}^{1/(p-1)} \;\text{ and}\\ &[T'h]_{A_1} \leq C_4[\omega']_{A_{p'}}^{1/(p'-1)} = C_4[\omega]_{A_p}^{1/[(p'-1)(p-1)]} = C_4[\omega]_{A_p}\end{aligned} \tag{7.7.62}$$

since $(p'-1)(p-1) = 1$, where $C_4 \in (0, \infty)$, much like C_3, depends only on the quasi-distance ρ (via the constants C_ρ, $\widetilde{C}_\rho$ appearing in (7.1.3)–(7.1.4)), the doubling charter of μ, and p. In turn, based on these properties and item *(3)* in Lemma 7.7.1 we see that if

$$w := (T\widetilde{g})^{1-p_0}(T'h) \tag{7.7.63}$$

then

$$w \in A_{p_0}(X, \mu) \;\text{ and }\; [w]_{A_{p_0}} \leq C_5[\omega]_{A_p}^{1+(p_0-1)/(p-1)} \leq W_{\omega,p} \tag{7.7.64}$$

for some $C_5 \in (0, \infty)$ depending only on the quasi-distance ρ (via the constants $C_\rho, \widetilde{C}_\rho$ appearing in (7.1.3)–(7.1.4)), the doubling charter of μ, as well as p and p_0. The last inequality in (7.7.64) is implied by (7.7.47) assuming $C \geq C_5$. With $q_0 \in (1, \infty)$ denoting the Hölder conjugate exponent of p_0, we may now write

$$
\begin{aligned}
\|f\|_{L^p(\omega)} &= \Big(\int_X |f|^p \omega \, \mathrm{d}\mu \Big)^{1/p} = \|f\|_{L^p(\omega)}^{(p-1)/p} \Big(\int_X |f| h \, \mathrm{d}\mu \Big)^{1/p} \\
&\leq \|f\|_{L^p(\omega)}^{1/p'} \Big(\int_X |f| (T'h) \, \mathrm{d}\mu \Big)^{1/p} \\
&= \|f\|_{L^p(\omega)}^{1/p'} \Big(\int_X |f| (T\widetilde{g})^{p_0-1} \, w \, \mathrm{d}\mu \Big)^{1/p} \\
&= \|f\|_{L^p(\omega)}^{1/p'} \Big(\int_X |f| (T\widetilde{g})^{p_0-1} \, \mathrm{d}w \Big)^{1/p} \\
&\leq \|f\|_{L^p(\omega)}^{1/p'} \Big(\int_X |f|^{p_0} \, \mathrm{d}w \Big)^{1/(pp_0)} \Big(\int_X (T\widetilde{g})^{(p_0-1)q_0} \, \mathrm{d}w \Big)^{1/(pq_0)} \\
&= \|f\|_{L^p(\omega)}^{1/p'} \|f\|_{L^{p_0}(w)}^{1/p} \Big(\int_X (T\widetilde{g})^{(p_0-1)q_0} (T\widetilde{g})^{1-p_0} (T'h) \, \mathrm{d}\mu \Big)^{1/(pq_0)} \\
&= \|f\|_{L^p(\omega)}^{1/p'} \|f\|_{L^{p_0}(w)}^{1/p} \Big(\int_X (T\widetilde{g})(T'h) \, \mathrm{d}\mu \Big)^{1/(pq_0)} \\
&\leq \|f\|_{L^p(\omega)}^{1/p'} \|f\|_{L^{p_0}(w)}^{1/p} \Big(\|T\widetilde{g}\|_{L^p(\omega)} \|T'h\|_{L^{p'}(\omega')} \Big)^{1/(pq_0)} \\
&\leq 4^{1/(pq_0)} \|f\|_{L^p(\omega)}^{1/p'} \|f\|_{L^{p_0}(w)}^{1/p}.
\end{aligned}
\tag{7.7.65}
$$

Above, the first equality uses the fact that $\mathrm{d}\omega = \omega \mathrm{d}\mu$, while the second equality is based on (7.7.59). The first inequality is a consequence of the first property in (7.7.52), and the subsequent equality comes from (7.7.63). To write the next equality we have used $w \, \mathrm{d}\mu = \mathrm{d}w$, and then we have employed Hölder's inequality. Next, we have again employed (7.7.63), then took advantage of the identity $(p_0 - 1)(q_0 - 1) = 1$, and once more applied Hölder's inequality. Lastly, we have invoked (7.7.51) (twice) together with the last property in (7.7.58) and (7.7.61). Having established (7.7.65), bearing in mind the current working assumptions we obtain that

$$
\|f\|_{L^p(\omega)} \leq 4^{1/q_0} \|f\|_{L^{p_0}(w)}. \tag{7.7.66}
$$

We next estimate g. To get started, recall from (7.7.53) and (7.7.58) that

$$
0 < T\widetilde{g} < \infty \quad \text{at } \mu\text{-a.e. point in } X. \tag{7.7.67}
$$

Also, use (7.7.57) and the first property in (7.7.52) to write

$$
|g| = \|g\|_{L^p(\omega)} \widetilde{g} \leq \|g\|_{L^p(\omega)} T\widetilde{g}. \tag{7.7.68}
$$

Bearing (7.7.67) in mind, this implies

$$|g|(T\widetilde{g})^{-1} \leq \|g\|_{L^p(\omega)} \quad \text{at } \mu\text{-a.e. point in } X. \tag{7.7.69}$$

After rising both sides of (7.7.69) to the power $p_0 - 1 > 0$ and multiplying by $|g|$, we arrive (again, bearing (7.7.67) in mind) at the conclusion that

$$|g|^{p_0}\big(T\widetilde{g}\big)^{1-p_0} \leq \|g\|_{L^p(\omega)}^{p_0-1}|g| \quad \text{at } \mu\text{-a.e. point in } X. \tag{7.7.70}$$

Based on the definition of the weight w given in (7.7.63), (7.7.70), Hölder's inequality, and (7.7.51) (written for T') we therefore obtain

$$\begin{aligned}
\|g\|_{L^{p_0}(w)} &= \left(\int_X |g|^{p_0}\big(T\widetilde{g}\big)^{1-p_0}(T'h)\,\mathrm{d}\mu\right)^{1/p_0} \leq \|g\|_{L^p(\omega)}^{\frac{p_0-1}{p_0}}\left(\int_X |g|(T'h)\,\mathrm{d}\mu\right)^{1/p_0} \\
&\leq \|g\|_{L^p(\omega)}^{\frac{p_0-1}{p_0}}\Big(\|g\|_{L^p(\omega)}\|T'h\|_{L^{p'}(\omega')}\Big)^{1/p_0} = \|g\|_{L^p(\omega)}\|T'h\|_{L^{p'}(\omega')}^{1/p_0} \\
&\leq 2^{1/p_0}\|g\|_{L^p(\omega)},
\end{aligned} \tag{7.7.71}$$

hence

$$\|g\|_{L^{p_0}(w)} \leq 2^{1/p_0}\|g\|_{L^p(\omega)}. \tag{7.7.72}$$

At this stage, (7.7.48) readily follows by combining (7.7.66), (7.7.46), and (7.7.72) (in this order) and upon recalling (7.7.64).

`Step II.` *Suppose* $p_0 = 1$ *and* $f \in L^p(\omega)$. In this scenario, in place of (7.7.63) we now simply define

$$w := T'h \tag{7.7.73}$$

which in view of (7.7.62) implies

$$w \in A_1(X,\mu) \quad \text{and} \quad [w]_{A_1} \leq C_4[\omega]_{A_p} \leq W_{\omega,p} \tag{7.7.74}$$

with the last inequality in (7.7.74) a consequence of (7.7.47), assuming $C \geq C_4$. Much as in (7.7.65), we then have

$$\begin{aligned}
\|f\|_{L^p(\omega)} &= \left(\int_X |f|^p\omega\,\mathrm{d}\mu\right)^{1/p} = \|f\|_{L^p(\omega)}^{(p-1)/p}\left(\int_X |f|h\,\mathrm{d}\mu\right)^{1/p} \\
&\leq \|f\|_{L^p(\omega)}^{1/p'}\left(\int_X |f|(T'h)\,\mathrm{d}\mu\right)^{1/p} = \|f\|_{L^p(\omega)}^{1/p'}\left(\int_X |f|\,\mathrm{d}w\right)^{1/p} \\
&= \|f\|_{L^p(\omega)}^{1/p'}\|f\|_{L^1(w)}^{1/p}
\end{aligned} \tag{7.7.75}$$

which then implies

$$\|f\|_{L^p(\omega)} \leq \|f\|_{L^1(w)}. \tag{7.7.76}$$

As far as the function g is concerned, (7.7.73), Hölder's inequality, (7.7.61), and the version of (7.7.51) written for T' yield

$$\|g\|_{L^1(w)} = \int_X |g|(T'h)\,\mathrm{d}\mu \leq \|g\|_{L^p(\omega)}\|T'h\|_{L^{p'}(\omega')} \leq 2\|g\|_{L^p(\omega)}. \tag{7.7.77}$$

The claim made in (7.7.48) (with $p_0 = 1$) then becomes a consequence of (7.7.75), (7.7.46), and (7.7.77) (in this order), bearing in mind (7.7.74).

`Step III.` *The end-game in the proof of* (7.7.48). Fix $p \in (1, \infty)$ and an arbitrary weight $\omega \in A_p(X, \rho, \mu)$. From Steps I–II we know that (7.7.48) is true if $f \in L^p(\omega)$. To eliminate the latter additional assumption, fix a reference point $x_0 \in X$ and define

$$f_N := \min\{f, N\} \cdot \mathbf{1}_{B_\rho(x_0,N)} \quad \text{for each} \quad N \in \mathbb{N}. \tag{7.7.78}$$

Then each f_N is a non-negative μ-measurable function on X, and (7.7.46) implies that for every Muckenhoupt weight $w \in A_{p_0}(X, \rho, \mu)$ we have

$$\Big(\int_X f_N^{p_0}\, w\,\mathrm{d}\mu\Big)^{1/p_0} \leq \Big(\int_X f^{p_0}\, w\,\mathrm{d}\mu\Big)^{1/p_0} \leq C_w\Big(\int_X g^{p_0}\, w\,\mathrm{d}\mu\Big)^{1/p_0}. \tag{7.7.79}$$

Since, as is apparent from (7.7.78), each f_N belongs to $L^p(\omega)$, we conclude from (7.7.79) and Steps I–II that

$$\Big(\int_X f_N^p\omega\,\mathrm{d}\mu\Big)^{1/p} \leq C_{\omega,p}\Big(\int_X g^p\omega\,\mathrm{d}\mu\Big)^{1/p} \quad \text{for each} \quad N \in \mathbb{N}, \tag{7.7.80}$$

where $C_{\omega,p}$ is as in (7.7.49). Passing to limit $N \to \infty$ and relying on Lebesgue's Monotone Convergence Theorem we then arrive at the conclusion that (7.7.48) holds as stated. □

As a byproduct of the proof of Rubio De Francia's extrapolation theorem included above, we obtain the following useful embedding result for Muckenhoupt weighted Lebesgue spaces.

Lemma 7.7.7 *Let* (X, ρ, μ) *be a space of homogeneous type with the property that the quasi-distance* $\rho : X \times X \to [0, \infty)$ *is continuous*[21] *in the product topology* $\tau_\rho \times \tau_\rho$. *Then*

$$\bigcup_{\substack{1\leq p<\infty\\ \omega\in A_p(X,\mu)}} L^p(X, \omega\mu) = \bigcup_{w\in A_1(X,\mu)} L^1(X, w\mu) \tag{7.7.81}$$

and for each $p_0 \in (1, \infty)$ *one has*

$$\bigcup_{\substack{1<p<\infty\\ \omega\in A_p(X,\mu)}} L^p(X, \omega\mu) = \bigcup_{w\in A_{p_0}(X,\mu)} L^{p_0}(X, w\mu). \tag{7.7.82}$$

[21] Theorem 7.1.2 guarantees that any quasi-metric space has an equivalent quasi-distance satisfying this property.

Proof We shall freely borrow notation and results from the proof of Proposition 7.7.6. To deal with (7.7.82), fix two integrability exponents $p_0, p \in (1, \infty)$ along with an arbitrary Muckenhoupt weight $\omega \in A_p(X, \mu)$. Let $p' \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$, and define $\omega' := \omega^{1-p'} \in A_{p'}(X, \mu)$. Also, fix an arbitrary non-negative function $h \in L^{p'}(\omega')$ with the property that $\|h\|_{L^{p'}(\omega')} = 1$. For this choice of h, given any nontrivial non-negative function $g \in L^p(\omega)$, run the argument (7.7.67)–(7.7.71), which has produced (7.7.72). This shows that there exists $w \in A_{p_0}(X, \mu)$ with the property that $g \in L^{p_0}(w)$, ultimately proving the left-to-right inclusion in (7.7.82). Since the opposite inclusion is trivially satisfied, the claim in (7.7.82) is established. Finally, (7.7.81) is justified in a similar manner, now relying on (7.7.77) and (7.7.74). □

Moving on, it turns out that there is a closely related version of (7.7.9), in the context of measures. Membership to this class amounts to a quantitative version of mutual absolute continuity.

Definition 7.7.8 *Suppose (X, ρ, μ) is a measure metric space. Let $\widetilde{\mu}$ be a non-negative measure defined on the same sigma-algebra of subsets of X as μ, such that $0 < \widetilde{\mu}\big(B_\rho(x, r)\big) < \infty$ for every $x \in X$ and $r > 0$. The measure $\widetilde{\mu}$ belongs to the class $A_\infty(\mu)$ if for any $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that for each $x \in X$, $r > 0$, and each μ-measurable set $E \subseteq B_\rho(x, r)$ one has*

$$\frac{\mu(E)}{\mu(B_\rho(x, r))} < \delta \Longrightarrow \frac{\widetilde{\mu}(E)}{\widetilde{\mu}(B_\rho(x, r))} < \varepsilon. \tag{7.7.83}$$

Some of the main properties of the class of measures $A_\infty(\mu)$ are summarized below, following [52, 247], [147, Theorem 1.4.13, pp. 17–18], and [154].

Proposition 7.7.9 *Let (X, ρ, μ) be measure metric space and suppose $\widetilde{\mu}$ is a non-negative measure defined on the same sigma-algebra of subsets of X as μ with the property that $0 < \widetilde{\mu}\big(B_\rho(x, r)\big) < \infty$ for every $x \in X$ and $r > 0$. Then the following are true.*

(i) If $\widetilde{\mu} \in A_\infty(\mu)$ then $\widetilde{\mu}$ is doubling and absolutely continuous with respect to μ.
(ii) If $\widetilde{\mu} \in A_\infty(\mu)$ then $\mu \in A_\infty(\widetilde{\mu})$. In particular, if $\widetilde{\mu} \in A_\infty(\mu)$ then μ and $\widetilde{\mu}$ are actually mutually absolute continuous.
(iii) One has $\widetilde{\mu} \in A_\infty(\mu)$ if and only if there exist some $\varepsilon, \delta \in (0, 1)$ such that if $x \in X$, $r > 0$, and $E \subseteq B_\rho(x, r)$ is a μ-measurable set, then

$$\frac{\mu(E)}{\mu(B_\rho(x, r))} < \delta \Longrightarrow \frac{\widetilde{\mu}(E)}{\widetilde{\mu}(B_\rho(x, r))} < \varepsilon. \tag{7.7.84}$$

(iv) One has $\widetilde{\mu} \in A_\infty(\mu)$ if and only if there exist $C \in (0, \infty)$ along with $\eta, \theta > 0$, such that for each $x \in X$, $r > 0$, and μ-measurable set $E \subseteq B_\rho(x, r)$ one has

$$\frac{\widetilde{\mu}(E)}{\widetilde{\mu}(B_\rho(x,r))} \leq C\Big(\frac{\mu(E)}{\mu(B_\rho(x,r))}\Big)^\theta \quad \textit{and} \qquad (7.7.85)$$
$$\frac{\mu(E)}{\mu(B_\rho(x,r))} \leq C\Big(\frac{\widetilde{\mu}(E)}{\widetilde{\mu}(B_\rho(x,r))}\Big)^\eta.$$

(v) One has $A_\infty(\mu) = \bigcup_{1<q<\infty} B_q(\mu)$*, where each reverse Hölder class* $B_q(\mu)$ *consists of locally finite Borel measures* λ *on* X*, which are absolutely continuous with respect to* μ*, the Radon–Nikodym derivative* $k := \frac{d\lambda}{d\mu}$ *belongs to* $L^q_{\rm loc}(X, \mu)$*, and there exists some* $C \in (0, \infty)$ *such that the following reverse Hölder condition is satisfied*[22]*:*

$$\Big(\fint_{B_\rho(x,r)} k^q \,d\mu\Big)^{\frac{1}{q}} \leq C \fint_{B_\rho(x,r)} k \,d\mu, \quad \forall x \in X, \ \forall r > 0. \qquad (7.7.86)$$

(vi) One has $B_q(\mu) \subseteq B_p(\mu)$ *whenever* $1 < p \leq q < \infty$*, and if* $\lambda \in B_q(\mu)$ *for some* $q \in (1, \infty)$ *then there exists* $\varepsilon > 0$ *such that* $\lambda \in B_{q+\varepsilon}(\mu)$*.*

(vii) Let λ *be a non-negative Borel measure on* X *satisfying* $0 < \lambda\big(B_\rho(x,r)\big) < \infty$ *for every* $x \in X$ *and* $r > 0$*. Also, fix* $q, q' \in (1, \infty)$ *with the property that* $\frac{1}{q} + \frac{1}{q'} = 1$*. Then* $\lambda \in B_q(\mu)$ *if and only if the Hardy–Littlewood maximal operator associated with* λ*, i.e.,*

$$\big(\mathcal{M}_\lambda f\big)(x) := \sup_{r>0}\Big(\fint_{B_\rho(x,r)} |f| \,d\lambda\Big), \qquad \forall x \in X, \qquad (7.7.87)$$

has the property that there exists $C \in (0, \infty)$ *such that for each* $f \in L^{q'}(X, \mu)$ *one has*

$$\|\mathcal{M}_\lambda f\|_{L^{q'}(X,\mu)} \leq C\|f\|_{L^{q'}(X,\mu)}. \qquad (7.7.88)$$

(viii) One has

$$A_\infty(\mu) = \big\{w\,\mu : \ w \in A_p(X, \mu) \ \textit{for some} \ \ p \in [1, \infty)\big\}. \qquad (7.7.89)$$

Our next goal is to prove that the property of being a compact operator extrapolates on the scales of Muckenhoupt weighted Lebesgue spaces (see Lemma 7.7.12 further below). To set the stage, we begin by recalling some relevant abstract interpolation results. First, we have the following remarkable one-sided compactness property for the real method of interpolation[23] for (compatible) Banach couples established in [49, Theorem 2.3, p. 286], [50, 60]:

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[22] Hence, $B_q(\mu)$ consists of all locally finite Borel measures λ on X satisfying $\lambda << \mu$ and $d\lambda/d\mu$ belongs to the reverse Hölder class RH_q with respect to μ; cf. (7.7.19).

[23] The corresponding result for the complex method of interpolation remains open.

Proposition 7.7.10 *Let X_j, Y_j, where $j \in \{0, 1\}$, be two compatible Banach couples and suppose that the linear operator $T : X_j \to Y_j$ is bounded for $j = 0$ and compact for $j = 1$. Then the linear operator $T : (X_0, X_1)_{\theta,q} \to (Y_0, Y_1)_{\theta,q}$ is compact for all $\theta \in (0, 1)$ and $q \in [1, \infty]$.*

Second, we have the following useful real interpolation result for weighted Lebesgue spaces (cf. [19, Theorem 5.5.1, p. 119]).

Proposition 7.7.11 *Let (X, μ) be a measure space, and consider two weights*[24] *w_0, w_1 on X. Also, let*

$$\begin{gathered}\theta \in (0,1), \quad 0 < p_0, p_1 < \infty, \quad p := \big(\tfrac{1-\theta}{p_0} + \tfrac{\theta}{p_1}\big)^{-1}, \\ w := w_0^{p(1-\theta)/p_0} \cdot w_1^{p\theta/p_1}.\end{gathered} \tag{7.7.90}$$

Then

$$\Big(L^{p_0}(X, w_0\,\mu), L^{p_1}(X, w_1\,\mu)\Big)_{\theta,p} = L^p(X, w\,\mu). \tag{7.7.91}$$

Here is the extrapolation result of compactness on the scales of Muckenhoupt weighted Lebesgue spaces, advertised earlier.

Proposition 7.7.12 *Let (X, ρ, μ) be a measure metric space and suppose T is a linear operator mapping each Muckenhoupt weighted Lebesgue space $L^p(X, w\,\mu)$ with $p \in (1, \infty)$ and $w \in A_p(X, \rho, \mu)$ boundedly into itself. Then*

$$\begin{gathered}\text{there exists } p_o \in (1,\infty) \text{ and } w_o \in A_{p_o}(X,\rho,\mu) \text{ so} \\ \text{that } T \text{ is a compact operator on } L^{p_o}(X, w_o\,\mu)\end{gathered} \tag{7.7.92}$$

if and only if

$$\begin{gathered}T \text{ is a compact operator on } L^p(X, w\,\mu) \text{ for each} \\ p \in (1,\infty) \text{ and each weight } w \in A_p(X,\rho,\mu).\end{gathered} \tag{7.7.93}$$

Proof Work under the assumption made in (7.7.92). If $p_o' \in (1, \infty)$ denotes the Hölder conjugate exponent of p_o then item *(2)* in Lemma 7.7.1 guarantees that $w_o^{1-p_o'} \in A_{p_o'}(X, \rho, \mu)$. As such, it follows that

$$T : L^{p_o'}(X, w_o^{1-p_o'}\mu) \longrightarrow L^{p_o'}(X, w_o^{1-p_o'}\mu) \quad \text{is bounded.} \tag{7.7.94}$$

From (7.7.92), (7.7.94), and Proposition 7.7.10 we then conclude that, on the one hand,

$$\begin{gathered}T \quad \text{is compact on} \quad \big(L^{p_o'}(X, w_o^{1-p_o'}\mu), L^{p_o}(X, w_o\mu)\big)_{\theta,q} \\ \text{for each } \theta \in (0,1) \text{ and } q \in [1,\infty].\end{gathered} \tag{7.7.95}$$

[24] That is, μ-measurable functions defined on X which are finite and strictly positive at μ-a.e. point in X.

On the other hand, from (7.7.91) we see that

$$\big(L^{p_o'}(X, w_o^{1-p_o'}\mu), L^{p_o}(X, w_o\mu)\big)_{\frac{1}{2},2} = L^2(X, \mu). \tag{7.7.96}$$

Together, (7.7.95) and (7.7.96) prove that

$$T : L^2(X, \mu) \longrightarrow L^2(X, \mu) \text{ is compact.} \tag{7.7.97}$$

By once again appealing to Proposition 7.7.10 and interpolation we then deduce from (7.7.97) that

$$T : L^p(X, \mu) \longrightarrow L^p(X, \mu) \text{ is compact for each } p \in (1, \infty). \tag{7.7.98}$$

Fix now some integrability exponent $p \in (1, \infty)$ along with some Muckenhoupt weight $w \in A_p(X, \rho, \mu)$. From item *(10)* in Lemma 7.7.1 we know that there exists some small $\delta > 0$ such that $w^{1+\delta} \in A_p(X, \rho, \mu)$. Then since from (7.7.91) we see that

$$\big(L^p(X, \mu), L^p(X, w^{1+\delta}\mu)\big)_{\theta,p} = L^p(X, w\,\mu) \text{ if } \theta := (1+\delta)^{-1} \in (0, 1), \tag{7.7.99}$$

and since T is bounded on $L^p(X, w^{1+\delta}\mu)$ (by hypotheses) and compact on $L^p(X, \mu)$ (by (7.7.98)), Proposition 7.7.10 ultimately gives that T is compact on $L^p(X, w\,\mu)$. This finishes the proof of the fact that (7.7.92) implies (7.7.93). Finally, it is clear that (7.7.93) implies (7.7.92). □

We shall also need more specialized properties of Muckenhoupt weights, such as

if (X, ρ) is a metric space and μ is a non-negative measure defined on a sigma-algebra of subsets of X which contains all ρ-balls and for which there exists $d \in (0, \infty)$ such that $\mu\big(B_\rho(x, r)\big) \approx r^d$ uniformly for all $x \in X$ and $r \in \big(0, \operatorname{diam}_\rho(X)\big)$, then given any $w \in A_p(X, \rho, \mu)$ with $p \in (1, \infty)$ there exists a constant $C \in (0, \infty)$, depending only on p and the ambient (X, ρ), with the property that for each $\lambda \in (1, \infty)$ we have $\int_{\lambda B} w\, d\mu \leq C[w]_{A_p}\lambda^{dp} \int_B w\, d\mu$ for each ρ-ball $B \subseteq X$ (where λB denotes the concentric dilate of the ρ-ball B by a factor of $\lambda > 0$). (7.7.100)

This is a direct consequence of the doubling property (7.7.16). See also [97, Lemma 2.2, p. 396] for a proof in the Euclidean case that readily adapts to the setting considered in (7.7.100).

In particular, (7.7.100) is relevant in establishing the embeddings in Lemma 7.7.13 below, which are going to be useful for us later on. Before stating this, we introduce a piece of notation frequently used in the sequel. Specifically, given a closed set $\Sigma \subseteq \mathbb{R}^n$ along with a Borel measure σ on Σ and an exponent $m \in [0, \infty)$, we shall denote by $L^1\big(\Sigma, \frac{\sigma(x)}{1+|x|^m}\big)$ the Lebesgue space of absolutely integrable functions on Σ with respect to the weighted measure $\mu := (1+|x|^m)^{-1}\sigma$.

Lemma 7.7.13 *Suppose $\Sigma \subseteq \mathbb{R}^n$, where $n \in \mathbb{N}$ with $n \geq 2$, is a closed set which is Ahlfors regular, and let $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Fix $p \in (1,\infty)$ along with a Muckenhoupt weight $w \in A_p(\Sigma, |\cdot - \cdot|, \sigma)$. Then there exists $\varepsilon = \varepsilon(\Sigma, n, p, [w]_{A_p}) \in (0,1)$ for which*

$$\int_\Sigma \frac{w(x)}{(1+|x|^{n-1-\varepsilon})^p}\,\mathrm{d}\sigma(x) < +\infty. \tag{7.7.101}$$

Moreover,

$$\begin{gathered}\text{there exists } \varepsilon = \varepsilon(\Sigma, n, p, [w]_{A_p}) \in (0,1) \text{ such that}\\ L^p(\Sigma, w\sigma) \hookrightarrow L^1\Big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1-\varepsilon}}\Big),\end{gathered} \tag{7.7.102}$$

and one can find an exponent $p_o = p_o(\Sigma, n, p, [w]_{A_p}) \in (1, p]$ with the property that

$$\begin{gathered}\text{for each } q \in (0, p_o) \text{ there exists } \varepsilon = \varepsilon(\Sigma, n, p, [w]_{A_p}, q) \in (0,1) \text{ such that}\\ L^p(\Sigma, w\sigma) \hookrightarrow L^q\Big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1-\varepsilon}}\Big) \hookrightarrow L^q\Big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\Big).\end{gathered} \tag{7.7.103}$$

As a consequence,

$$L^p(\Sigma, w\sigma) \hookrightarrow L^1\Big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \text{ continuously}, \tag{7.7.104}$$

and

$$L^p(\Sigma, w\sigma) \subseteq \bigcup_{1<q<p} L^q_{\mathrm{loc}}(\Sigma, \sigma). \tag{7.7.105}$$

In particular, (7.7.104) with $w \equiv 1$ entails

$$L^p(\Sigma, \sigma) \hookrightarrow L^1\Big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \text{ continuously for each } p \in [1,\infty), \tag{7.7.106}$$

while (7.7.106), (6.2.48), (6.2.50), and (6.2.36) imply that

$$\begin{gathered}L^{p,q}(\Sigma, \sigma) \hookrightarrow L^1\Big(\Sigma, \frac{\sigma(x)}{1+|x|^{n-1}}\Big) \cap \Big(\bigcap_{1<s<p} L^s_{\mathrm{loc}}(\Sigma, \sigma)\Big)\\ \text{continuously, for each } p \in (1,\infty) \text{ and each } q \in (0,\infty].\end{gathered} \tag{7.7.107}$$

Proof The self-improving property of Muckenhoupt weights from item *(9)* in Lemma 7.7.1 (whose present validity is ensured by (3.6.26)) together with the monotonicity of the Muckenhoupt classes of weights (cf. item *(5)* in Lemma 7.7.1) guarantee that

$$\text{there exists some small number } \varepsilon \in (0, p-1) \text{ with the property that } w \text{ belongs to } A_{p-\varepsilon}(\Sigma, |\cdot - \cdot|, \sigma). \tag{7.7.108}$$

Pick $x_0 \in \Sigma$ and observe that there exists some constant $C = C_{x_0} \in (0, \infty)$ with the property that $1 + |x - x_0| \le C(1 + |x|)$ for all $x \in \mathbb{R}^n$. Keeping this in mind, for each choice

$$\theta \in \big(-\infty, \tfrac{(n-1)\varepsilon}{p}\big) \tag{7.7.109}$$

we may estimate

$$\begin{aligned}
&\int_\Sigma \frac{w(x)}{(1+|x|^{n-1-\theta})^p}\,\mathrm{d}\sigma(x)\\
&\quad \le C\int_\Sigma \frac{w(x)}{(1+|x-x_0|^{n-1-\theta})^p}\,\mathrm{d}\sigma(x) \le C\int_{B(x_0,1)\cap\Sigma} w(x)\,\mathrm{d}\sigma(x)\\
&\qquad + C\sum_{j=0}^\infty \int_{[B(x_0,2^{j+1})\setminus B(x_0,2^j)]\cap\Sigma} \frac{w(x)}{|x-x_0|^{(n-1-\theta)p}}\,\mathrm{d}\sigma(x)\\
&\quad \le C\int_{B(x_0,1)\cap\Sigma} w\,\mathrm{d}\sigma + C\sum_{j=0}^\infty \frac{1}{2^{j(n-1-\theta)p}}\int_{B(x_0,2^{j+1})\cap\Sigma} w\,\mathrm{d}\sigma\\
&\quad \le C[w]_{A_{p-\varepsilon}}\int_{B(x_0,1)\cap\Sigma} w\,\mathrm{d}\sigma + C\sum_{j=0}^\infty \frac{2^{j(n-1)(p-\varepsilon)}}{2^{j(n-1-\theta)p}}\int_{B(x_0,1)\cap\Sigma} w\,\mathrm{d}\sigma\\
&\quad \le C[w]_{A_{p-\varepsilon}}\Big(1+\sum_{j=0}^\infty 2^{-j[(n-1-\theta)p-(n-1)(p-\varepsilon)]}\Big)\int_{B(x_0,1)\cap\Sigma} w\,\mathrm{d}\sigma\\
&\quad = C\big(\Sigma, x_0, p, [w]_{A_p}\big)\int_{B(x_0,1)\cap\Sigma} w\,\mathrm{d}\sigma < +\infty,
\end{aligned} \tag{7.7.110}$$

where we have made use of (7.7.108), (7.7.100), (7.7.8), (7.7.6), item *(9)* in Lemma 7.7.1, and the observation that $(n-1-\theta)p - (n-1)(p-\varepsilon) > 0$ due to the choice in (7.7.109). This establishes (7.7.101).

To proceed, by once again relying on the self-improving property of Muckenhoupt weights from item *(9)* in Lemma 7.7.1, and keeping in mind the monotonicity of the Muckenhoupt classes of weights (cf. item *(5)* in Lemma 7.7.1), we conclude that

$$\text{there exists some } r_o \in [1, p) \text{ with the property that } w \text{ belongs to } A_r(\Sigma, |\cdot - \cdot|, \sigma) \text{ for each } r \in (r_o, \infty). \tag{7.7.111}$$

Define $p_o := p/r_o \in (1, p]$, pick $q \in (0, p_o)$ arbitrary, and set $r := p/q \in (r_o, \infty)$. Denote by $r' \in (1, \infty)$ the conjugate exponent of r. Then (7.7.111) and item *(2)* in Lemma 7.7.1 guarantee that

$$\omega := w^{-r'/r} = w^{1-r'} \in A_{r'}(\Sigma, |\cdot - \cdot|, \sigma). \tag{7.7.112}$$

Having fixed an arbitrary function $f \in L^p(\Sigma, w\sigma)$, for each θ associated with the weight ω much as the choice in (7.7.109) has been associated with w we then estimate

$$\begin{aligned}
&\Big(\int_\Sigma \frac{|f(x)|^q}{1+|x|^{n-1-\theta}}\,\mathrm{d}\sigma(x)\Big)^{1/q}\\
&\quad= \bigg(\int_\Sigma \big(|f(x)|^q w(x)^{q/p}\big)\frac{w(x)^{-q/p}}{1+|x|^{n-1-\theta}}\,\mathrm{d}\sigma(x)\bigg)^{1/q}\\
&\quad\leq \Big[\int_\Sigma \big(|f|^q\, w^{p/q}\big)^r\,\mathrm{d}\sigma\Big]^{1/(qr)}\Big[\int_\Sigma \frac{w(x)^{-(q/p)r'}}{(1+|x|^{n-1-\theta})^{r'}}\,\mathrm{d}\sigma(x)\Big]^{1/(qr')}\\
&\quad= \|f\|_{L^p(\Sigma,w\sigma)}\Big[\int_\Sigma \frac{\omega(x)}{(1+|x|^{n-1-\theta})^{r'}}\,\mathrm{d}\sigma(x)\Big]^{1/(qr')}\\
&\quad\leq C\big(\Sigma, x_0, p, q, [w^{1-r'}]_{A_{r'}}\big)\Big(\int_{B(x_0,1)\cap\Sigma}\omega\,\mathrm{d}\sigma\Big)^{1/(qr')}\|f\|_{L^p(\Sigma,w\sigma)}\\
&\quad\leq C\big(\Sigma, x_0, p, q, [w]_{A_r}\big)[\omega]_{A_{r'}}^{1/(qr')}\times\\
&\qquad\quad\times\Big(\int_{B(x_0,1)\cap\Sigma}\omega^{-1/(r'-1)}\,\mathrm{d}\sigma\Big)^{(1-r')/(qr')}\|f\|_{L^p(\Sigma,w\sigma)}\\
&\quad\leq C\big(\Sigma, x_0, p, q, [w]_{A_p}\big)\Big(\int_{B(x_0,1)\cap\Sigma} w\,\mathrm{d}\sigma\Big)^{-1/p}\|f\|_{L^p(\Sigma,w\sigma)},
\end{aligned}\tag{7.7.113}$$

using Hölder's inequality, (7.7.110) (used with w replaced by ω, and with p replaced by r'), (7.7.112), (7.7.2), the Ahlfors regularity of Σ, Lemma 7.7.1, and the definition of r. At this stage, (7.7.103) becomes a consequence of (7.7.113). Finally, (7.7.102), (7.7.104), and (7.7.105) are obvious consequences of (7.7.103). □

The measure induced by a Muckenhoupt weight enjoys the properties described in the next lemma.

Lemma 7.7.14 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set, and set $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix some integrability exponent $p \in (1, \infty)$ along with a Muckenhoupt weight $w \in A_p(\Sigma, |\cdot - \cdot|, \sigma)$. Then*

the measure $w\,\sigma$ is complete, locally finite (hence also sigma-finite), separable, Borel-regular on Σ, and the Lebesgue space $L^q(\Sigma, w\sigma)$ is separable for each integrability exponent $q \in (0, \infty)$. (7.7.114)

Also, for each integrability exponent $q \in (0, \infty)$ the natural inclusion

$$\big\{\phi\big|_\Sigma : \phi \in \mathscr{C}_c^\infty(\mathbb{R}^n)\big\} \hookrightarrow L^p(\Sigma, w\sigma) \;\; \textit{has dense range,} \tag{7.7.115}$$

and for every function $f \in L^1_{\rm loc}(\Sigma, w\sigma)$ *and every open set* $\mathcal{O} \subseteq \mathbb{R}^n$ *one has*

$$f = 0 \ \text{at } \sigma\text{-a.e. point on } \mathcal{O} \cap \Sigma \iff \int_{\mathcal{O}\cap\Sigma} f\phi\, w\, d\sigma = 0 \ \text{for every } \phi \in \mathscr{C}_c^\infty(\mathcal{O}). \tag{7.7.116}$$

Proof From assumptions, (5.2.6), and Lemma 3.6.4 we see that

$$\begin{array}{l}\sigma \text{ is a complete, locally finite (hence sigma-finite as well),}\\ \text{separable, Borel-regular measure on } \Sigma \text{ (considered equipped}\\ \text{with the topology canonically inherited from } \mathbb{R}^n).\end{array} \tag{7.7.117}$$

Since the weight w is a real-valued function which is (strictly) positive and finite σ-a.e. on Σ, and which belongs to $L^1_{\rm loc}(\Sigma, \sigma)$, we conclude from (7.7.117) and definitions that the measure $w\,\sigma$ is complete, locally finite, and Borel-regular on Σ. Having established this, Proposition 3.6.3 applies (upon noting that, as a topological space, $\mathbb{R}^n$ is second-countable, and this property is hereditary) and gives that the measure $w\,\sigma$ is separable and the Lebesgue space $L^q(\Sigma, w\sigma)$ is separable for each integrability exponent $q \in (0, \infty)$. This finishes the proof of (7.7.114). Granted this, Proposition 3.7.1 then applies and guarantees that (7.7.115) holds, while Proposition 3.7.2 ensures that (7.7.116) is true. The proof is therefore complete. □

For future reference, let us also note here the following result.

Lemma 7.7.15 *Suppose* $\Sigma \subseteq \mathbb{R}^n$ *is a closed upper Ahlfors regular set, and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. *Then*

$$(1 + |x|)^{1-n} \in L^{p,q}(\Sigma, \sigma) \ \text{ for each } p \in (1, \infty) \text{ and each } \ q \in (0, \infty]. \tag{7.7.118}$$

Proof This is implied by Lemma 7.2.1, (6.2.50), and (6.2.48). □

We conclude by including an elementary but useful weighted estimate for fractional integration operators.

Lemma 7.7.16 *Let* $\Sigma \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$ *with* $n \geq 2$) *be a closed set which is Ahlfors regular and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. *Also, fix some* $\alpha \in (0, \infty)$ *together with some* $R \in (0, \infty)$ *and, given* $f \in L^1_{\rm loc}(\Sigma, \sigma)$, *define*

$$F(x) := \int\limits_{\substack{y\in\Sigma\\ |x-y|<R}} \frac{f(y)}{|x-y|^{n-1-\alpha}}\, d\sigma(y), \qquad x \in \Sigma. \tag{7.7.119}$$

Then there exists $C = C(\Sigma, n, \alpha) \in (0, \infty)$ *such that, with* $\mathcal{M}_\Sigma$ *denoting the Hardy–Littlewood maximal operator associated with* Σ (*cf.* (7.6.16)), *one has*

$$|F| \leq C R^\alpha \mathcal{M}_\Sigma f \ \text{ on } \ \Sigma. \tag{7.7.120}$$

As a consequence, having fixed an integrability exponent $p \in (1, \infty)$ *along with some Muckenhoupt weight* $w \in A_p(\Sigma, |\cdot - \cdot|, \sigma)$*, for each given function* $f \in L^p(\Sigma, w\sigma)$ *it follows that* $F \in L^p(\Sigma, w\sigma)$ *and there exists some constant* $C = C(\Sigma, n, p, \alpha, [w]_{A_p}) \in (0, \infty)$ *with the property that*

$$\|F\|_{L^p(\Sigma,w\sigma)} \leq C R^\alpha \|f\|_{L^p(\Sigma,w\sigma)}. \tag{7.7.121}$$

Proof For each $x \in \Sigma$ we may estimate

$$\begin{aligned}
&\int_{\substack{y\in\Sigma\\ |x-y|<R}} \frac{|f(y)|}{|x-y|^{n-1-\alpha}}\,\mathrm{d}\sigma(y) \\
&\quad = \sum_{j=0}^{\infty} \int_{\substack{y\in\Sigma\\ 2^{-j-1}R\leq|x-y|<2^{-j}R}} \frac{|(\mathbf{1}_{B(x,R)}f)(y)|}{|x-y|^{n-1-\alpha}}\,\mathrm{d}\sigma(y) \\
&\quad \leq C\sum_{j=0}^{\infty}(2^{-j}R)^{-(n-1-\alpha)} \int_{\substack{y\in\Sigma\\ 2^{-j-1}R\leq|x-y|<2^{-j}R}} |(\mathbf{1}_{B(x,R)}f)(y)|\,\mathrm{d}\sigma(y) \\
&\quad \leq C\sum_{j=0}^{\infty}(2^{-j}R)^{-(n-1-\alpha)} \int_{\substack{y\in\Sigma\\ |x-y|<2^{-j}R}} |(\mathbf{1}_{B(x,R)}f)(y)|\,\mathrm{d}\sigma(y) \\
&\quad \leq C\sum_{j=0}^{\infty}(2^{-j}R)^{-(n-1-\alpha)}\sigma\big(\Sigma\cap B(x,2^{-j}R)\big) ⨍_{\Sigma\cap B(x,2^{-j}R)} |\mathbf{1}_{B(x,R)}f|\,\mathrm{d}\sigma \\
&\quad \leq C R^\alpha \sum_{j=0}^{\infty}(2^{-\alpha})^j\big(\mathcal{M}_\Sigma(\mathbf{1}_{B(x,R)}f)\big)(x).
\end{aligned} \tag{7.7.122}$$

This proves that F in (7.7.119) is a well-defined function, given by an absolutely convergent integral for σ-a.e. $x \in \Sigma$ (cf. (7.7.15)), and that (7.7.120) holds.

Since F is also σ-measurable, from (7.7.120) and item *(1)* in Lemma 7.7.1 we conclude that F belongs to the space $L^p(\Sigma, w\sigma)$ and (7.7.121) holds for any the function $f \in L^p(\Sigma, w\sigma)$. □

7.8 The Fractional Integration Theorem

The goal here is to recall the Fractional Integration Theorem in a general setting, described next. Let (X, ρ) be a metric space and suppose μ is a Borel measure on (X, τ_ρ) which is doubling and has the property that there exist $C > 0$ and $d > 0$ such that

$$\mu\big(B_\rho(x,r)\big) \le Cr^d, \qquad \forall x \in X,\ \forall r \in (0,\infty). \tag{7.8.1}$$

In particular, as a measure space, (X,μ) is sigma-finite. Also, sending the radius to zero in (7.8.1) implies that the measure μ does not charge singletons, i.e., $\mu(\{x\}) = 0$ for each $x \in X$.

Fix a reference point $x_0 \in X$. Given some μ-measurable set $E \subseteq X$ together with some parameter $\alpha \in (0,d)$, consider the fractional integral operator $I_{E,\alpha}$ of order α on E,

$$I_{E,\alpha} : L^1\Big(E, \frac{\mu(x)}{1+\rho(x,x_0)^{d-\alpha}}\Big) \longrightarrow L^1_{\rm loc}(E,\mu), \tag{7.8.2}$$

acting on functions $f \in L^1\big(E, \frac{\mu(x)}{1+\rho(x,x_0)^{d-\alpha}}\big)$ according to

$$I_{E,\alpha} f(x) := \int_E \frac{f(y)}{\rho(x,y)^{d-\alpha}}\,\mathrm{d}\mu(y) \ \text{ for } \mu\text{-a.e. } \ x \in E. \tag{7.8.3}$$

Note that if $f \in L^1\big(E, \frac{\mu(x)}{1+\rho(x,x_0)^{d-\alpha}}\big)$ then for each $R>0$ we may use Fubini's Theorem (bearing in mind that, as noted earlier, (X,μ) is sigma-finite) and the first estimate in (7.2.5) to write

$$\begin{aligned}
&\int_{E\cap B_\rho(x_0,R)}\Big(\int_E \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\,\mathrm{d}\mu(y)\Big)\,\mathrm{d}\mu(x)\\
&\quad = \int_{E\cap B_\rho(x_0,R)}\Big(\int_{E\cap B_\rho(x_0,2R)} \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\,\mathrm{d}\mu(y)\Big)\,\mathrm{d}\mu(x)\\
&\qquad + \int_{E\cap B_\rho(x_0,R)}\Big(\int_{E\setminus B_\rho(x_0,2R)} \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\,\mathrm{d}\mu(y)\Big)\,\mathrm{d}\mu(x)\\
&\quad \le \int_{E\cap B_\rho(x_0,2R)} |f(y)|\Big(\int_{B_\rho(x_0,R)} \frac{\mathrm{d}\mu(x)}{\rho(x,y)^{d-\alpha}}\Big)\,\mathrm{d}\mu(y)\\
&\qquad + 2^{d-\alpha}\int_{B_\rho(x_0,R)}\Big(\int_{E\setminus B_\rho(x_0,2R)} \frac{|f(y)|}{\rho(y,x_0)^{d-\alpha}}\,\mathrm{d}\mu(y)\Big)\,\mathrm{d}\mu(x)\\
&\quad \le CR^d \int_{E\cap B_\rho(x_0,2R)} |f(y)|\,\mathrm{d}\mu(y)\\
&\qquad + 2^{d-\alpha}\mu\big(B_\rho(x_0,R)\big)\int_{E\setminus B_\rho(x_0,2R)} \frac{|f(y)|}{1+\rho(y,x_0)^{d-\alpha}}\,\mathrm{d}\mu(y)\\
&\quad \le C_R \int_E \frac{|f(y)|}{1+\rho(y,x_0)^{d-\alpha}}\,\mathrm{d}\mu(y) < +\infty,
\end{aligned} \tag{7.8.4}$$

for some finite constant $C_R > 0$. This goes to show that

$$\begin{array}{l}\text{for each function } f \in L^1\big(E, \tfrac{\mu(x)}{1+\rho(x,x_0)^{d-\alpha}}\big) \text{ it follows that } (I_{E,\alpha} f)(x)\\ \text{is well defined, via an absolutely convergent integral, at } \mu\text{-a.e. point}\\ x \in E, \text{ and } I_{E,\alpha} f \in L^1_{\rm loc}(E,\mu); \text{ in fact, the fractional integral operator}\\ I_{E,\alpha} \text{ in (7.8.2) is well defined, linear, and continuous.}\end{array} \tag{7.8.5}$$

In addition, the same type of estimate as in (7.8.4) shows that Fubini's Theorem may be applied to conclude that

$$\begin{array}{l}\displaystyle\int_E (I_{E,\alpha} f) g \,\mathrm{d}\mu = \int_E f (I_{E,\alpha} g)\,\mathrm{d}\mu \;\; \text{for any two given functions}\\ f \in L^1\big(E, \tfrac{\mu(x)}{\rho(x,x_0)^{d-\alpha}}\big), \;\; \text{and} \;\; g \in L^\infty(E,\mu) \;\; \text{with bounded support.}\end{array} \tag{7.8.6}$$

As regards mapping properties on ordinary Lebesgue spaces, in the same setting as above the following result holds:

$$\begin{array}{l}\text{if } p \in \big(1, d/\alpha\big) \text{ and } p^* := \big(\tfrac{1}{p} - \tfrac{\alpha}{d}\big)^{-1}, \text{ then there exists a finite con-}\\ \text{stant } C = C(p,\alpha) > 0 \text{ such that for each } f \in L^p(E,\mu) \text{ the expression}\\ \big(I_{E,\alpha} f\big)(x) \text{ is defined in (7.8.3) by an absolutely convergent integral at}\\ \mu\text{-a.e. point } x \in E, \text{ and we have } \big\| I_{E,\alpha} f \big\|_{L^{p^*}(E,\mu)} \leq C \|f\|_{L^p(E,\mu)}.\end{array} \tag{7.8.7}$$

Indeed, this is proved in [95, Corollary 3.3, p. 249] when $E = X$ and the present, more flexible, version follows from this on account of (6.2.17) after observing that, with tilde denoting extension by zero from E to X, we have

$$I_{E,\alpha} f = \big(I_{X,\alpha} \widetilde{f}\,\big)\Big|_E \quad \text{on} \quad E. \tag{7.8.8}$$

In a similar manner, based on [95, Theorem 3.2, p. 248], (7.8.8), and (6.2.17), we see that the following result, corresponding to the end-point case $p = 1$ of (7.8.7) and now involving a weak Lebesgue space, holds:

$$\begin{array}{l}\text{for each } f \in L^1(E,\mu) \text{ the expression } \big(I_{E,\alpha} f\big)(x) \text{ is defined in (7.8.3)}\\ \text{by an absolutely convergent integral at } \mu\text{-a.e. point } x \in E, \text{ and } I_{E,\alpha}\\ \text{maps } L^1(E,\mu) \text{ into the Lorentz space } L^{\frac{d}{d-\alpha},\infty}(E,\mu) \text{ in a linear and}\\ \text{bounded fashion.}\end{array} \tag{7.8.9}$$

In fact, since much as in Example 6.2.2,

$$\begin{array}{l}\text{there exists some constant } C \in (0,\infty) \text{ such that for each given point}\\ x_o \in X \text{ the function } f_{x_o} : E \to \mathbb{R} \text{ defined at each } x \in E \setminus \{x_o\} \text{ by}\\ f_{x_o}(x) := \rho(x,x_o)^{-(d-\alpha)} \text{ belongs to the space } L^{d/(d-\alpha),\infty}(E,\mu) \text{ and}\\ \|f_{x_o}\|_{L^{d/(d-\alpha),\infty}(E,\mu)} \leq C,\end{array} \tag{7.8.10}$$

which, together with (6.2.61), implies

$$I_{E,\alpha} : L^{d/\alpha,1}(E,\mu) \longrightarrow L^{\infty}(E,\mu) \ \text{boundedly,} \tag{7.8.11}$$

we may also obtain (7.8.7) by interpolating (7.8.9) with (7.8.11) (cf. (6.2.47)).

It is also worth pointing out that (7.8.7) self-extends, via real interpolation, to an estimate involving the more inclusive scale of Lorentz spaces. More specifically,

$$\text{if } 1 < p < \tfrac{d}{\alpha},\ 0 < q \leq \infty, \text{ and } \tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{\alpha}{d}, \text{ there exists } C \in (0,\infty) \text{ such that } \|I_{E,\alpha} f\|_{L^{p^*,q}(E,\mu)} \leq C\|f\|_{L^{p,q}(E,\mu)} \text{ for each } f \in L^{p,q}(E,\mu). \tag{7.8.12}$$

In particular, corresponding to the case $q := p$ we have

$$\text{if } 1 < p < \tfrac{d}{\alpha} \text{ and } \tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{\alpha}{d}, \text{ there exists } C \in (0,\infty) \text{ such that } \|I_{E,\alpha} f\|_{L^{p^*,p}(E,\mu)} \leq C\,\|f\|_{L^{p}(E,\mu)} \text{ for each function } f \in L^{p}(E,\mu). \tag{7.8.13}$$

In the case when E is bounded, estimates in the spirit of (7.8.7) are also valid in the range $p \in \big[d/\alpha, \infty\big)$. Concretely, Hölder's inequality and the first estimate in (7.2.5) (presently used with $r := \operatorname{diam}_\rho(E)$) imply that

$$\text{given any } p \in \big(d/\alpha, \infty\big) \text{ there exists some constant } C \in (0,\infty), \text{ depending only on } p, d, \alpha \text{ and the constant in (7.8.1), such that } \|I_{E,\alpha} f\|_{L^{\infty}(E,\mu)} \leq C\big[\operatorname{diam}_\rho(E)\big]^{\alpha-d/p}\|f\|_{L^{p}(E,\mu)} \text{ holds for each function } f \in L^{p}(E,\mu). \tag{7.8.14}$$

The critical value $p = d/\alpha$ may be then covered from (7.8.9) and (7.8.14) via interpolation, which gives that

$$\text{if } E \text{ is also bounded then for each } q \in (0,\infty) \text{ there exists some } C = C(E) \in (0,\infty) \text{ such that } \|I_{E,\alpha} f\|_{L^{q}(E,\mu)} \leq C\|f\|_{L^{d/\alpha}(E,\mu)} \text{ for every function } f \in L^{d/\alpha}(E,\mu). \tag{7.8.15}$$

There is a close relationship between fractional integral operators and the Hardy–Littlewood maximal operator, which may be employed to give conceptually natural proofs of the Fractional Integration Theorem. Specifically, we note the following result.

Proposition 7.8.1 *Assume (X,ρ) is a quasi-metric space with the property that the quasi-distance $\rho : X \times X \to [0,\infty)$ is continuous*[25] *in the product topology $\tau_\rho \times \tau_\rho$. In addition, suppose μ is a Borel measure on (X,τ_ρ) with the property that there exist $C_\mu \in (0,\infty)$ and $d \in (0,\infty)$ such that*

$$0 < \mu\big(B_\rho(x,r)\big) \leq C_\mu\, r^d \ \ \textit{for each}\ \ x \in X \ \ \textit{and}\ \ r > 0. \tag{7.8.16}$$

[25] Theorem 7.1.2 guarantees that any quasi-metric space has an equivalent quasi-distance which satisfies this property.

Then for each $\alpha \in (0, d)$ and each exponent $p \in [1, d/\alpha)$ there exists some constant $C = C(d, \alpha, p, C_\mu) \in (0, \infty)$ with the property that for each μ-measurable function f on X one has

$$\int_X \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\, \mathrm{d}\mu(y) \leq C\|f\|_{L^p(X,\mu)}^{p\alpha/d}\big[(\mathcal{M}_X f)(x)\big]^{1-p\alpha/d}, \ \ \forall x \in X, \tag{7.8.17}$$

where $\mathcal{M}_X$ is the Hardy–Littlewood maximal operator on X, defined as in (7.6.16).

Moreover, if in place of (7.8.16) one assumes the lower bound

$$\begin{aligned} c_\mu\, r^d \leq \mu\big(B_\rho(x,r)\big) < +\infty \ \ &\textit{for each} \ \ x \in X \\ \textit{and each finite} \ \ r &\in \big(0, \mathrm{diam}_\rho(X)\big], \end{aligned} \tag{7.8.18}$$

for some $c_\mu \in (0, \infty)$ and $d \in (0, \infty)$, then for each $\alpha \in (0, d)$ there exists a constant $C = C(d, \alpha, c_\mu) \in (0, \infty)$ with the property that for each μ-measurable function f on X one has

$$\mathcal{M}_{X,1,\alpha/d} f(x) \leq C \int_X \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\, \mathrm{d}\mu(y), \qquad \forall x \in X, \tag{7.8.19}$$

where $\mathcal{M}_{X,1,\alpha/d}$ is the L^1-based fractional Hardy–Littlewood maximal operator of order α/d on X, defined as in (7.6.1).

Proof Fix $\alpha \in (0, d)$ along with an arbitrary μ-measurable function f on X. Then for each $x \in X$ and each $r \in (0, \infty)$ we may write, based on (7.8.16) and (7.6.16),

$$\begin{aligned} &\int_{B_\rho(x,r)} \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\, \mathrm{d}\mu(y) \\ &\quad = \sum_{j=0}^{\infty} \int_{B_\rho(x,2^{-j}r)\setminus B_\rho(x,2^{-j-1}r)} \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\, \mathrm{d}\mu(y) \\ &\quad \leq C \sum_{j=0}^{\infty} (2^{-j}r)^{\alpha-d} \mu\big(B_\rho(x,2^{-j}r)\big) \fint_{B_\rho(x,2^{-j}r)} |f(y)|\, \mathrm{d}\mu(y) \\ &\quad \leq C\Big\{\sum_{j=0}^{\infty} (2^{-j}r)^{\alpha-d}(2^{-j}r)^d\Big\}(\mathcal{M}_X f)(x) \\ &\quad = C r^\alpha (\mathcal{M}_X f)(x), \end{aligned} \tag{7.8.20}$$

for a constant $C \in (0, \infty)$ which is independent of x, r, f. Assume $p \in \big[1, d/\alpha\big)$ has been fixed and choose $p' \in \big(d/(d-\alpha), \infty\big]$ such that $1/p + 1/p' = 1$. Observe that this choice entails $(d-\alpha)p' > d$. Granted this, for each $r > 0$ and each $x \in X$ we may use Hölder's inequality and (7.2.5) to estimate

$$\begin{aligned}
&\int_{X\setminus B_\rho(x,r)} \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\,\mathrm{d}\mu(y)\\
&\quad\leq \Big(\int_{X\setminus B_\rho(x,r)} |f(y)|^p\,\mathrm{d}\mu(y)\Big)^{1/p}\Big(\int_{X\setminus B_\rho(x,r)} \frac{\mathrm{d}\mu(y)}{\rho(x,y)^{(d-\alpha)p'}}\Big)^{1/p'}\\
&\quad\leq \|f\|_{L^p(X,\mu)}\Big(\int_{X\setminus B_\rho(x,r)} \frac{\mathrm{d}\mu(y)}{\rho(x,y)^{(d-\alpha)p'}}\Big)^{1/p'}\\
&\quad\leq Cr^{\alpha-d/p}\|f\|_{L^p(X,\mu)}. \qquad (7.8.21)
\end{aligned}$$

From (7.8.20) and (7.8.21) we then conclude that for each $x \in X$ we have

$$\begin{aligned}
\int_X \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\,\mathrm{d}\mu(y) &\leq C\inf_{r>0}\Big\{r^\alpha(\mathcal{M}_X f)(x) + r^{\alpha-d/p}\|f\|_{L^p(X,\mu)}\Big\}\\
&= C\|f\|_{L^p(X,\mu)}^{p\alpha/d}\big[(\mathcal{M}_X f)(x)\big]^{1-p\alpha/d}, \qquad (7.8.22)
\end{aligned}$$

for some constant $C \in (0,\infty)$ independent of x and f. This establishes (7.8.17).

As regards (7.8.19), making use of (7.8.18), for each point $x \in X$ and each finite $r \in \big(0, \mathrm{diam}_\rho(X)\big]$ write

$$\begin{aligned}
\int_X \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\,\mathrm{d}\mu(y) &\geq \int_{B_\rho(x,r)} \frac{|f(y)|}{\rho(x,y)^{d-\alpha}}\,\mathrm{d}\mu(y)\\
&\geq \frac{\mu\big(B_\rho(x,r)\big)}{r^{d-\alpha}} ⨍_{B_\rho(x,r)} |f(y)|\,\mathrm{d}\mu(y)\\
&\geq C\Big[\mu\big(B_\rho(x,r)\big)\Big]^{\alpha/d} ⨍_{B_\rho(x,r)} |f(y)|\,\mathrm{d}\mu(y), \qquad (7.8.23)
\end{aligned}$$

where $C \in (0,\infty)$ is independent of f, x, r. Taking the supremum over all finite $r \in \big(0, \mathrm{diam}_\rho(X)\big]$ yields (7.8.19) on account of (7.6.1). □

In relation to Proposition 7.8.1 we remark that, in concert with the mapping properties of the Hardy–Littlewood maximal operator on spaces of homogeneous type (cf. Corollary 7.6.3), estimate (7.8.17) readily implies both (7.8.7) and (7.8.9). Indeed, (7.8.17) implies that

$$I_{X,\alpha}f(x) \leq C\|f\|_{L^p(X,\mu)}^{p\alpha/d}\big[(\mathcal{M}_X f)(x)\big]^{1-p\alpha/d}, \qquad \forall x \in X. \qquad (7.8.24)$$

Hence, assuming (X,ρ,μ) is a space of homogeneous type for which (7.8.16) holds, if $p \in \big(1, d/\alpha\big)$ and $p^* := \big(\frac{1}{p} - \frac{\alpha}{d}\big)^{-1}$ we may rely on (7.8.24) and (7.6.18) to estimate

$$\begin{aligned}\|I_{X,\alpha}f\|_{L^{p^*}(X,\mu)} &\leq C\|f\|_{L^p(X,\mu)}^{p\alpha/d}\|(\mathcal{M}_X f)^{1-p\alpha/d}\|_{L^{p^*}(X,\mu)}\\ &= C\|f\|_{L^p(X,\mu)}^{p\alpha/d}\|\mathcal{M}_X f\|_{L^p(X,\mu)}^{p/p^*}\\ &\leq C\|f\|_{L^p(X,\mu)}^{p\alpha/d}\|f\|_{L^p(X,\mu)}^{p/p^*} = C\|f\|_{L^p(X,\mu)}. \end{aligned} \tag{7.8.25}$$

From this and (7.8.8), the claim in (7.8.7) follows. The claim in (7.8.9), corresponding to $p = 1$, is proved similarly, this time using the weak boundedness result from (7.6.19).

In closing, we also wish to note that under the assumption made in (7.8.18), the pointwise estimate (7.8.19) trivially implies that for each $p \in (0, \infty)$ and $q \in (0, \infty]$ we have

$$\|\mathcal{M}_{X,1,\alpha/d}f\|_{L^{p,q}(X,\mu)} \leq C\|I_{X,\alpha}|f|\|_{L^{p,q}(X,\mu)}, \tag{7.8.26}$$

for each μ-measurable function f on X, where $C \in (0, \infty)$ is independent of f. Remarkably, under suitable assumptions on the ambient (X, ρ, μ), the opposite inequality in (7.8.26) is also true. This may be seen by establishing a good-λ inequality as in the proof of [2, Theorem 3.6.1, p. 72] where this is done in the Euclidean setting (see also [206, Theorem 1, p. 262] for a weighted version). In turn, such an estimate once again yields (7.8.7) and (7.8.9), in view of the mapping properties of the fractional Hardy–Littlewood maximal operator (cf. Theorem 7.6.1).

Chapter 8
Open Sets with Locally Finite Surface Measures and Boundary Behavior

In this chapter, we develop the main technology behind our main results pertaining to the Divergence Theorem from Sects. 1.2–1.12. Among the technical tools required in this endeavor, there are certain off-diagonal Carleson measure estimates of reverse Hölder type, which we formulate and prove in Sect. 8.6. In turn, these estimates use information on the nontangential maximal operator developed in Sects. 8.1–8.4 (as well as its version from Sect. 8.10), and the solid maximal function introduced earlier in Sect. 6.6. Other key players in this regard are the notion of nontangentially accessible boundary which we define and study in Sect. 8.8, and the pointwise nontangential boundary trace operator considered at length in Sect. 8.9.

8.1 Nontangential Approach Regions in Arbitrary Open Sets

Here we introduce the nontangential approach regions in arbitrary open sets and study some of their most basic properties.

As a preamble, we make the observation that the "distance to the boundary" function satisfies a natural monotonicity property (with respect to the underlying set), as described in the lemma below.

Lemma 8.1.1 *Assume $\mathcal{O}$, Ω are two nonempty proper subsets of $\mathbb{R}^n$ with the property that $\mathcal{O} \subseteq \Omega$. Then $\operatorname{dist}\big(x, \partial\Omega\big) \geq \operatorname{dist}\big(x, \partial\mathcal{O}\big)$ for each $x \in \mathcal{O}$.*

Proof Fix an arbitrary point $x \in \mathcal{O} \subseteq \Omega$. If $x \in \partial\mathcal{O}$, the inequality we seek to establish is trivially true, so assume $x \in \mathcal{O} \setminus \partial\mathcal{O} = \mathring{\mathcal{O}}$. Then $r := \operatorname{dist}\big(x, \partial\mathcal{O}\big)$ is a well-defined number in the interval $(0, \infty)$, and $B(x, r) \subseteq \mathcal{O}$. As a consequence, $B(x, r) \subseteq \Omega$. The latter inclusion proves that $\operatorname{dist}\big(x, \partial\Omega\big) \geq r$, as desired. □

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Throughout, fix an open, nonempty, proper subset Ω of $\mathbb{R}^n$. In agreement with notation employed in the past, by $\delta_{\partial\Omega}$, we shall denote the `distance function to` $\partial\Omega$, i.e.

$$\delta_{\partial\Omega}(y) := \mathrm{dist}(y, \partial\Omega) := \inf_{x\in\partial\Omega} |x-y|, \quad \text{for all} \quad y \in \mathbb{R}^n. \tag{8.1.1}$$

Given $\kappa > 0$ arbitrary, we define the `nontangential approach regions` (to $\partial\Omega$ from within Ω) of aperture parameter κ by setting

$$\Gamma_\kappa(x) = \Gamma_{\Omega,\kappa}(x) := \big\{y \in \Omega : |x-y| < (1+\kappa)\delta_{\partial\Omega}(y)\big\}, \qquad \forall x \in \partial\Omega. \tag{8.1.2}$$

While it is clear from (8.1.2) that each nontangential approach region is an open[1] subset of Ω, the reader is alerted that it may well happen that $\Gamma_\kappa(x) = \varnothing$ for certain points $x \in \partial\Omega$ and certain aperture parameters $\kappa > 0$. To give an example, assume $n \in \mathbb{N}$ satisfies $n \geq 2$ and fix $\theta \in (0, \pi)$. If we consider the open cone of half-aperture angle $\theta/2$ with vertex at the origin in $\mathbb{R}^n$ given by

$$\Omega_\theta := \big\{x \in \mathbb{R}^n : \langle x, \mathbf{e}_n\rangle > |x|\cos(\theta/2)\big\}, \tag{8.1.3}$$

then simple geometry shows that

$$\Gamma_\kappa(0) = \varnothing \Longleftrightarrow 0 < \kappa \leq \frac{1}{\sin(\theta/2)} - 1. \tag{8.1.4}$$

On the other hand, the nontangential approach regions at a boundary point where Ω satisfies a cone condition are always nonempty if the aperture parameter is sufficiently large. To make this precise, recall the cones $\mathrm{C}_{\theta,b}(x,h)$ defined in (5.6.93). Then, if $x \in \partial\Omega$ has the property that there exist $h \in S^{n-1}$, $\theta \in (0,\pi)$, and $b \in (0,\infty)$ for which

$$\mathrm{C}_{\theta,b}(x,h) \subseteq \Omega, \tag{8.1.5}$$

simple geometric considerations show that

$$\kappa > \frac{1}{\sin(\theta/2)} - 1 \Rightarrow \Big\{x + th : 0 < t < \frac{b}{1+\sin(\theta/2)}\Big\} \subseteq \Gamma_\kappa(x). \tag{8.1.6}$$

We also wish to note that, in the two-dimensional setting, nontangential approach regions are preserved under quasi-conformal mappings in the precise sense described in [133, Proposition 1.1, p. 223].

Moving on, it is clear from definitions that

[1] Given that both the norm and the distance function are continuous.

$$\bigcup_{x\in\partial\Omega}\Gamma_\kappa(x)=\Omega \ \text{ for each fixed } \kappa\in(0,\infty), \text{ and}$$
$$\bigcup_{\kappa>0}\Gamma_\kappa(x)=\Omega \ \text{ for each fixed } \ x\in\partial\Omega. \tag{8.1.7}$$

Simple geometry also shows that for each $x, y \in \partial\Omega$, we have

$$\begin{aligned}\operatorname{dist}\big(y,\Gamma_\kappa(x)\big)&\leq|x-y| \ \text{ if } \ x\in\overline{\Gamma_\kappa(x)},\\ \operatorname{dist}\big(y,\Gamma_\kappa(x)\big)&\geq(2+\kappa)^{-1}|x-y| \ \text{ if } \ \Gamma_\kappa(x)\neq\varnothing.\end{aligned}\tag{8.1.8}$$

In addition, the following elementary topological result is going to be useful.

Lemma 8.1.2 *Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$, and fix an aperture parameter $\kappa>0$. Then*

if $x_*\in\mathbb{R}^n$, $x\in\partial\Omega$, and $r>0$ satisfy $B(x_*,r)\subseteq\Gamma_\kappa(x)$ then there exists $\varepsilon>0$ such that $B(x_*,r/2)\subseteq\Gamma_\kappa(z)$ for all $z\in\partial\Omega\cap B(x,\varepsilon)$. (8.1.9)

Proof We proceed in a series of steps, starting with:

`Step I:` *Under the assumptions that (X,τ) is a topological space, $K\subseteq X$ is a compact set, and $F,G:X\to\mathbb{R}$ are two continuous functions with the property that $F(x)<G(x)$ for each $x\in K$, it follows that there exists $U\in\tau$ such that $K\subseteq U$ and $F(x)<G(x)$ for each $x\in U$.*

Indeed, for each $x\in K$, the number $r_x:=G(x)-F(x)$ is strictly positive which, in turn, ensures that $x\in O_x:=(F-G)^{-1}\big(-\infty,-r_x/2)\big)\in\tau$. Hence, $K\subseteq\bigcup_{x\in K}O_x$ which means that there exist $x_1,\dots,x_N\in K$ such that $K\subseteq U:=\bigcup_{1\leq j\leq N}O_{x_j}$. At this stage, there remains to observe that $U\in\tau$ and that $(F-G)\big|_U<-\min_{1\leq j\leq N}r_{x_j}<0$.

`Step II:` *Let $x_*\in\mathbb{R}^n$, $x\in\partial\Omega$, $r>0$ be such that $B(x_*,r)\subseteq\Gamma_\kappa(x)$. Define $X:=\partial\Omega\times B(x_*,r)$ equipped with the natural product topology τ (with $\partial\Omega$ considered with the relative topology induced from $\mathbb{R}^n$). Define $F,G:X\to\mathbb{R}$ by setting $F(z,y):=|y-z|$ and $G(z,y):=(1+\kappa)\operatorname{dist}(y,\partial\Omega)$ for each $(z,y)\in X$. Then there exists $\varepsilon>0$ such that $F<G$ on $\big(B(x,\varepsilon)\cap\partial\Omega\big)\times\overline{B(x_*,r/2)}$.*

Applying Step I for the compact set $K:=\{x\}\times\overline{B(x_*,r/2)}$, we conclude that there exists an open subset U of X which contains K and such that $F<G$ on U. Next, for each $y\in\overline{B(x_*,r/2)}$, there exist $\varepsilon_y>0$ and $r_y>0$ such that $\big(B(x,\varepsilon_y)\cap\partial\Omega\big)\times B(y,r_y)\subseteq U$ (since $(x,y)\in K\subseteq U\in\tau$). Given that $\overline{B(x_*,r/2)}\subseteq\bigcup_{y\in\overline{B(x_*,r/2)}}B(y,r_y)$, it follows that there exist finitely many points $y_1,\dots,y_M\in\overline{B(x_*,r/2)}$ such that $\overline{B(x_*,r/2)}\subseteq\bigcup_{1\leq j\leq M}B(y_j,r_{y_j})$. If we now make the choice $\varepsilon:=\min_{1\leq j\leq M}\varepsilon_{y_j}$, then it is immediate that $\varepsilon>0$ and we have $F<G$ on $\big(B(x,\varepsilon)\cap\partial\Omega\big)\times\overline{B(x_*,r/2)}$ since the latter set is contained in U.

`Step III:` *The claim in* (8.1.9) *holds.*

From Step II, we know that $F(z,y)<G(z,y)$ whenever $z\in B(x,\varepsilon)\cap\partial\Omega$ and $y\in B(x_*,r/2)$. Thus, $|y-z|<(1+\kappa)\operatorname{dist}(y,\partial\Omega)$ which comes down to having

$y \in \Gamma_\kappa(z)$ for each $z \in B(x, \varepsilon) \cap \partial\Omega$ and each $y \in B(x_*, r/2)$. Consequently, we have $B(x_*, r/2) \subseteq \Gamma_\kappa(z)$ for each $z \in B(x, \varepsilon) \cap \partial\Omega$, as desired. □

The nontangential approach region satisfies a natural monotonicity property (with respect to the underlying domain), of the sort described in the lemma below.

Lemma 8.1.3 *Assume $\mathcal{O} \subseteq \Omega \subseteq \mathbb{R}^n$ are open sets and $x \in \partial\mathcal{O} \cap \partial\Omega$. Also, fix an arbitrary aperture parameter $\kappa > 0$. Then $\Gamma_{\mathcal{O},\kappa}(x)$, the κ-nontangential approach region in $\mathcal{O}$ with apex at $x \in \partial\mathcal{O}$, is contained in $\Gamma_{\Omega,\kappa}(x)$, the κ-nontangential approach region in Ω with apex at $x \in \partial\Omega$.*

Proof This is a direct consequence of (8.1.2) and Lemma 8.1.1. □

Going further, we continue to assume that Ω is an open, nonempty, proper subset of $\mathbb{R}^n$. Observe that for each $x \in \partial\Omega$, we have

$$\overline{\Gamma_\kappa(x)} \cap \partial\Omega \subseteq \{x\} \text{ and } \overline{\Gamma_\kappa(x)} \subseteq \Omega \cup \{x\}. \tag{8.1.10}$$

Indeed, if $z \in \overline{\Gamma_\kappa(x)} \cap \partial\Omega$, then there exists a sequence $\{y_j\}_{j\in\mathbb{N}} \subseteq \Gamma_\kappa(x)$ such that $y_j \to z$ as $j \to \infty$. In particular, we have $|x - y_j| < (1 + \kappa)\,\mathrm{dist}(y_j, \partial\Omega)$ for each $j \in \mathbb{N}$, so passing to the limit yields $|x - z| \leq (1 + \kappa)\,\mathrm{dist}(z, \partial\Omega) = 0$ since $z \in \partial\Omega$. Thus, necessarily $z = x$, proving the first property in (8.1.10). The second property in (8.1.10) is then justified based on what we have just proved by observing that since $\Gamma_\kappa(x) \subseteq \Omega$, we have $\overline{\Gamma_\kappa(x)} \subseteq \overline{\Omega} = \Omega \cup \partial\Omega$, hence

$$\overline{\Gamma_\kappa(x)} \subseteq \Omega \cup \big(\partial\Omega \cap \overline{\Gamma_\kappa(x)}\big) \subseteq \Omega \cup \{x\}. \tag{8.1.11}$$

Lemma 8.1.4 *Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$ and fix $\kappa > 0$. Then for each $x, y \in \partial\Omega$, one has*

$$|y - x| \leq (2 + \kappa)\,\mathrm{dist}\big(y, \Gamma_\kappa(x)\big) \;\; \text{if} \;\; \Gamma_\kappa(x) \neq \varnothing, \tag{8.1.12}$$

and

$$\mathrm{dist}\big(y, \Gamma_\kappa(x)\big) \leq |y - x| \;\; \text{if} \;\; x \in \overline{\Gamma_\kappa(x)}. \tag{8.1.13}$$

Proof Pick $x, y \in \partial\Omega$. Then for each $z \in \Gamma_\kappa(x)$, we may estimate

$$\begin{aligned} |y - x| &\leq |y - z| + |z - x| < |y - z| + (1 + \kappa)\,\mathrm{dist}(z, \partial\Omega) \\ &\leq |y - z| + (1 + \kappa)|y - z| = (2 + \kappa)|y - z|. \end{aligned} \tag{8.1.14}$$

Taking the infimum over all points $z \in \Gamma_\kappa(x)$ then yields (8.1.12). As regards the claim in (8.1.13), if $x \in \overline{\Gamma_\kappa(x)}$, then there exists a sequence $\{x_j\}_{j\in\mathbb{N}} \subseteq \Gamma_\kappa(x)$ which converges to x. Hence, $\mathrm{dist}\big(y, \Gamma_\kappa(x)\big) \leq |y - x_j| \to |y - x|$ as $j \to \infty$, proving (8.1.13). □

We continue to assume that Ω is a fixed, open, nonempty, proper subset of $\mathbb{R}^n$. Also, pick $\kappa > 0$ arbitrary. The "shadow" (or projection) of a given set $E \subseteq \Omega$ onto

$\partial\Omega$ (with respect to the nontangential approach regions in (8.1.2)) is defined as

$$\pi_\kappa(E) = \pi_{\Omega,\kappa}(E) := \big\{x \in \partial\Omega : \Gamma_\kappa(x) \cap E \neq \varnothing\big\}. \tag{8.1.15}$$

Clearly

$$\pi_\kappa(E) \text{ is a relatively open nonempty subset of } \partial\Omega, \text{ for each } E \subseteq \Omega. \tag{8.1.16}$$

Also, straightforward geometry shows that

$$\pi_\kappa\big(B(x,r) \cap \Omega\big) \subseteq B\big(x, (2+\kappa)r\big) \cap \partial\Omega, \qquad \forall x \in \partial\Omega, \quad \forall r > 0. \tag{8.1.17}$$

In particular

$$\begin{array}{l}\text{for any bounded subset } E \text{ of } \Omega,\\ \pi_\kappa(E) \text{ is a bounded subset of } \partial\Omega.\end{array} \tag{8.1.18}$$

Also,

$$\begin{array}{c}B\big(x_*, \kappa\delta_{\partial\Omega}(x)\big) \cap \partial\Omega \subseteq \pi_\kappa\big(\{x\}\big) \subseteq B\big(x_*, (2+\kappa)\delta_{\partial\Omega}(x)\big) \cap \partial\Omega\\ \text{if } x \in \Omega \text{ and } x_* \in \partial\Omega \text{ are such that } \delta_{\partial\Omega}(x) = |x - x_*|.\end{array} \tag{8.1.19}$$

For further use, we remark that

$$\begin{array}{l}\kappa' \geq \kappa \Longrightarrow \Gamma_\kappa(x) \subseteq \Gamma_{\kappa'}(x) \text{ for every } x \in \partial\Omega, \text{ and}\\ \qquad \pi_\kappa(E) \subseteq \pi_{\kappa'}(E) \text{ for each } \mathscr{L}^n\text{-measurable } E \subseteq \Omega.\end{array} \tag{8.1.20}$$

Our next lemma contains several variants of the Pythagorean Theorem in a general geometric setting.

Lemma 8.1.5 *Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$ and fix $\kappa > 0$. Then the following Pythagorean-like formula holds:*

$$\begin{array}{c}|x - y| \approx |x - y_*| + \delta_{\partial\Omega}(y), \quad \textit{uniformly for}\\ x \in \partial\Omega,\ y \in \Omega, \textit{ and } y_* \in \pi_{\Omega,\kappa}(\{y\}).\end{array} \tag{8.1.21}$$

As a consequence

$$\begin{array}{l}|x - y| \approx |x - y_*| + \delta_{\partial\Omega}(y) \textit{ uniformly for } x \in \partial\Omega \textit{ and } y \in \Omega,\\ \textit{assuming the point } y_* \in \partial\Omega \textit{ is such that } \delta_{\partial\Omega}(y) = |y - y_*|,\end{array} \tag{8.1.22}$$

and

$$\begin{array}{c}|z - y| \approx |z - x| + |x - y| \quad \textit{uniformly}\\ \textit{for } x \in \partial\Omega, \ y \in \partial\Omega, \ \textit{and } z \in \Gamma_\kappa(x).\end{array} \tag{8.1.23}$$

Moreover

$$\begin{array}{c}|y - z| + \max\big\{\delta_{\partial\Omega}(y), \delta_{\partial\Omega}(z)\big\} \approx |y_* - z_*| + \max\big\{\delta_{\partial\Omega}(y), \delta_{\partial\Omega}(z)\big\},\\ \textit{uniformly for } y, z \in \Omega, \textit{ and } y_* \in \pi_{\Omega,\kappa}(\{y\}),\ z_* \in \pi_{\Omega,\kappa}(\{z\}).\end{array} \tag{8.1.24}$$

Proof If $x \in \partial\Omega$, $y \in \Omega$, and $y_* \in \pi_{\Omega,\kappa}(\{y\})$, then $|y - y_*| < (1+\kappa)\delta_{\partial\Omega}(y)$ since $y \in \Gamma_{\Omega,\kappa}(y_*)$. Thus, $|x - y| \le |x - y_*| + |y - y_*| < |x - y_*| + (1+\kappa)\delta_{\partial\Omega}(y)$. This proves the left-pointing inequality in (8.1.21).

Conversely, $|x - y| \ge \delta_{\partial\Omega}(y)$ since $x \in \partial\Omega$, so if $\delta_{\partial\Omega}(y) > \frac{1}{2(1+\kappa)}|x - y_*|$, we have $|x - y| \ge \frac{1}{4(1+\kappa)}\big(|x - y_*| + \delta_{\partial\Omega}(y)\big)$. On the other hand, if we are in the situation when $\delta_{\partial\Omega}(y) \le \frac{1}{2(1+\kappa)}|x - y_*|$, then

$$\begin{aligned} |x - y| &\ge |x - y_*| - |y - y_*| > |x - y_*| - (1+\kappa)\delta_{\partial\Omega}(y) \\ &\ge |x - y_*| - \tfrac{1}{2}|x - y_*| = \tfrac{1}{2}|x - y_*|, \end{aligned} \tag{8.1.25}$$

so that $|x - y| \ge \frac{1}{4}\big(|x - y_*| + \delta_{\partial\Omega}(y)\big)$. The argument so far proves that in all instances $|x - y| \ge c\big(|x - y_*| + \delta_{\partial\Omega}(y)\big)$ for some $c = c(\kappa) \in (0, \infty)$. This establishes the right-pointing inequality in (8.1.21), finishing its proof.

In turn, (8.1.22) is a particular case of (8.1.21), since under the conditions stipulated in (8.1.22), we have $y_* \in \pi_{\Omega,\kappa}(\{y\})$ for any $\kappa > 0$. It is also clear that (8.1.21) implies (8.1.23) since for each $x \in \partial\Omega$ and $z \in \Gamma_\kappa(x)$, we have $x \in \pi_{\Omega,\kappa}(\{z\})$ and $\delta_{\partial\Omega}(z) \approx |z - x|$.

As regards (8.1.24), if $y, z \in \Omega$, and $y_* \in \pi_{\Omega,\kappa}(\{y\})$, $z_* \in \pi_{\Omega,\kappa}(\{z\})$, then

$$\begin{aligned} |y - z| &\le |y - y_*| + |y_* - z_*| + |z_* - z| \\ &< (1+\kappa)\delta_{\partial\Omega}(y) + |y_* - z_*| + (1+\kappa)\delta_{\partial\Omega}(z) \\ &\le (1+\kappa)\Big(|y_* - z_*| + \max\big\{\delta_{\partial\Omega}(y), \delta_{\partial\Omega}(z)\big\}\Big), \end{aligned} \tag{8.1.26}$$

proving the left-pointing inequality in (8.1.24). Finally, in the opposite direction

$$\begin{aligned} |y_* - z_*| &\le |y_* - y| + |y - z| + |z - z_*| \\ &\le (1+\kappa)\Big(|y - z| + \max\big\{\delta_{\partial\Omega}(y), \delta_{\partial\Omega}(z)\big\}\Big). \end{aligned} \tag{8.1.27}$$

This finishes the justification of (8.1.24) and completes the proof of the lemma. □

The specific format of nontangential approach regions may vary wildly, and we continue by presenting several concrete examples of this nature.

Example I: Corresponding to $n = 1$, whenever $-\infty < a < b < +\infty$ for any given $\kappa \in (0, \infty)$, we have

$$\Omega = (a, b) \Rightarrow \Gamma_\kappa(a) = \Big(a, \tfrac{(1+\kappa)b+a}{2+\kappa}\Big) \text{ and } \Gamma_\kappa(b) = \Big(\tfrac{(1+\kappa)a+b}{2+\kappa}, b\Big). \tag{8.1.28}$$

Note that, in agreement with (8.1.7), we have

$$\Gamma_\kappa(a) \cup \Gamma_\kappa(b) = (a, b). \tag{8.1.29}$$

Also,

$$\Gamma_\kappa(a) = \Omega \text{ if } \Omega = (a, \infty), \text{ or } \Omega = (-\infty, a), \text{ for some } a \in \mathbb{R}, \tag{8.1.30}$$

whereas if $-\infty < a < b < +\infty$, then for each $\kappa \in (0, \infty)$, we have

$$\Omega = \mathbb{R} \setminus [a, b] \Rightarrow \begin{cases} \Gamma_\kappa(a) = (-\infty, a) \cup \big(\frac{(1+\kappa)b-a}{\kappa}, +\infty\big) \\ \Gamma_\kappa(b) = \big(-\infty, \frac{(1+\kappa)a-b}{\kappa}\big) \cup (b, +\infty). \end{cases} \tag{8.1.31}$$

Example II: Working in the Euclidean space $\mathbb{R}^n$ with $n \in \mathbb{N}$, for each $\kappa \in (0, \infty)$, we have

$$\Omega = B(0, 1) \setminus \{0\} \Rightarrow \Gamma_\kappa(0) = B\big(0, \tfrac{1+\kappa}{2+\kappa}\big) \setminus \{0\}. \tag{8.1.32}$$

Also, having $\Omega = \mathbb{R}^n \setminus \{0\}$ implies $\Gamma_\kappa(0) = \Omega$ for each $\kappa > 0$.

Example III: It is clear from definitions that

if Ω is an exterior domain in $\mathbb{R}^n$, then for each $\kappa > 0$ there exists some large $R = R(\Omega, \kappa) \in (0, \infty)$ with the property that $\mathbb{R}^n \setminus B(0, R) \subseteq \Gamma_\kappa(x)$ for each $x \in \partial\Omega$. (8.1.33)

Moving on, if E is an arbitrary subset of Ω, we define the restricted nontangential approach regions Γ_κ^E (relative to the set E) as

$$\begin{aligned} \Gamma_\kappa^E(x) &:= \big\{y \in E : |x - y| < (1 + \kappa)\delta_{\partial\Omega}(y)\big\} \\ &= \Gamma_\kappa(x) \cap E, \qquad \forall x \in \partial\Omega. \end{aligned} \tag{8.1.34}$$

It follows that

$$\Gamma_\kappa^E(x) \text{ is a (relatively) open subset of } E, \text{ for each } x \in \partial\Omega, \tag{8.1.35}$$

and the first property in (8.1.7) implies

$$\bigcup_{x\in\partial\Omega} \Gamma_\kappa^E(x) = E. \tag{8.1.36}$$

It turns out that, in the two-dimensional setting, conformal mappings preserve nontangential approach regions in a double containment sense, up to adjustments in the aperture parameter. This is made precise in the lemma below.

Lemma 8.1.6 *Let $\Omega \subseteq \mathbb{R}^2$ be a chord-arc domain with unbounded boundary (cf. Definition 5.9.13), and consider a conformal mapping $\Phi : \mathbb{R}^2_+ \equiv \mathbb{C}_+ \to \Omega$. Then the function Φ extends to a homeomorphism $\Phi : \overline{\mathbb{R}^2_+} \to \overline{\Omega}$, with $\Phi(\partial\mathbb{R}^2_+) = \partial\Omega$. In addition, for each aperture parameter $\kappa \in (0, \infty)$ there exist $\kappa_1, \kappa_2 \in (0, \infty)$ with the property that*

$$\Gamma_{\mathbb{R}^2_+,\kappa_1}\big(\Phi(x)\big) \subseteq \Phi\big(\Gamma_{\Omega,\kappa}(x)\big) \subseteq \Gamma_{\mathbb{R}^2_+,\kappa_2}\big(\Phi(x)\big), \tag{8.1.37}$$

for each $x \in \mathbb{R} \equiv \partial\mathbb{R}^2_+$.

The fact that $\Phi(\partial\mathbb{R}^2_+) = \partial\Omega$ ensures that $\Phi(x) \in \partial\Omega$ for each $x \in \mathbb{R} \equiv \partial\mathbb{R}^2_+$. In particular, (8.1.37) is meaningfully formulated.

Proof of Lemma 8.1.6 Recall (5.9.93). Then, according to Carathéodory's theorem (cf. e.g., [100, Theorem 3.1, p. 13]), Φ extends to a homeomorphism $\Phi : \overline{\mathbb{R}^2_+} \to \overline{\Omega}$, mapping $\partial\mathbb{R}^2_+$ bijectively onto $\partial\Omega$. Bearing this in mind, (8.1.37) now follows from (5.9.93), the fact that (as noted in [132, (iv), p. 92]) any NTA domain in the plane is a quasicircle, and [133, Proposition 1.1, p. 223].

A result in the same spirit, for two-dimensional upper-graph Lipschitz domains, appears in [146, Lemma 1.13, p. 136]. Here is a similar result to Lemma 8.1.6, in all space dimensions, for bi-Lipschitz maps.

Lemma 8.1.7 *Let* $\Omega, \widetilde{\Omega} \subseteq \mathbb{R}^n$ *be arbitrary sets, and suppose* $F : \Omega \to \widetilde{\Omega}$ *is a bi-Lipschitz homeomorphism. Then* F *extends in a unique fashion to a bi-Lipschitz homeomorphism mapping the closure of* Ω *onto the closure of* $\widetilde{\Omega}$ (*still denoted by* F), *and* $F(\partial\Omega) = \partial\widetilde{\Omega}$.

Moreover, if $\Omega, \widetilde{\Omega}$ *are open, then for each given aperture parameter* $\kappa > 0$ *there exists* $\widetilde{\kappa} \in (0, \infty)$, *which depends only on* κ *and* F, *with the property that*

$$F\big(\Gamma_{\Omega,\kappa}(x)\big) \subseteq \Gamma_{\widetilde{\Omega},\widetilde{\kappa}}\big(F(x)\big) \ \textit{ for every point } \ x \in \partial\Omega. \tag{8.1.38}$$

Proof The fact that F is bi-Lipschitz entails the existence of $c, C \in (0, \infty)$, such that

$$c|x-y| \leq |F(x) - F(y)| \leq C|x-y| \ \text{ for all } \ x, y \in \Omega. \tag{8.1.39}$$

In particular, F is uniformly continuous which, in turn, implies that F extends uniquely as a continuous function (still denoted by F) mapping the closure of Ω into the closure of $\widetilde{\Omega}$. Passing to limit in (8.1.40) then shows that this extension satisfies

$$c|x-y| \leq |F(x) - F(y)| \leq C|x-y| \ \text{ for all } \ x, y \in \overline{\Omega}. \tag{8.1.40}$$

Pick an arbitrary point $y \in \partial\widetilde{\Omega}$. Then there exists a sequence $\{y_j\}_{j\in\mathbb{N}}$ contained in the interior of $\widetilde{\Omega}$ converging to y. Since $F : \Omega \to \widetilde{\Omega}$ is a homeomorphism, it follows that there exists a sequence $\{x_j\}_{j\in\mathbb{N}}$ contained in the interior of Ω such that $F(x_j) = y_j$ for each $j \in \mathbb{N}$. Thus, on the one hand, $\lim\limits_{j\to\infty} F(x_j) = y$. On the other hand, since $\{y_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\widetilde{\Omega}$ it follows from (8.1.39) that $\{x_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in Ω, hence convergent to some point $x_* \in \overline{\Omega}$. Combining these two facts leads to the conclusion that $F(x_*) = y$. In view of this, we see that x_* cannot belong to the interior of Ω, since otherwise the homeomorphism F would send it into a point in the interior of $\widetilde{\Omega}$, which is not the case (recall that $y \in \partial\widetilde{\Omega}$). Hence, we necessarily have $x_* \in \partial\widetilde{\Omega}$. Ultimately, this proves that $\partial\widetilde{\Omega} \subseteq F(\partial\Omega)$. Granted this, it follows that the extension F of the original homeomorphism from Ω onto $\widetilde{\Omega}$ maps the closure of Ω onto the closure of $\widetilde{\Omega}$, and satisfies (8.1.40). The latter shows that said extension is also injective, thus a bi-Lipschitz bijection of the closure of Ω onto the closure of $\widetilde{\Omega}$. This takes care of the claims in the first part of the statement.

To deal with the claim in the second part of the statement, assume Ω, $\widetilde{\Omega}$ are open and fix some $\kappa > 0$. Also, pick a point $x \in \partial\Omega$ along with some $y \in \Gamma_{\Omega,\kappa}(x)$, then select $\xi \in \partial\widetilde{\Omega}$ with the property that $|F(y) - \xi| = \operatorname{dist}\big(F(y), \partial\widetilde{\Omega}\big)$. From what we have proved above, there exists a unique $z \in \partial\Omega$ such that $F(z) = \xi$. We may then estimate

$$\begin{aligned} |F(y) - F(x)| &\leq C|y - x| < C(1+\kappa)\operatorname{dist}(y, \partial\Omega) \leq C(1+\kappa)|y - z| \\ &\leq (C/c)(1+\kappa)|F(y) - F(z)| = (C/c)(1+\kappa)|F(y) - \xi| \\ &= (C/c)(1+\kappa)\operatorname{dist}\big(F(y), \partial\widetilde{\Omega}\big), \end{aligned} \tag{8.1.41}$$

from which (8.1.38) follows whenever $\widetilde{\kappa} \geq (C/c)(1+\kappa) - 1$. □

We conclude this section by presenting two results, of a general geometric nature, relating "solid" integrals to "surface" integrals. The first such result is a manifestation of Fubini-Tonelli's Theorem.

Lemma 8.1.8 *Assume $\Sigma \subseteq \mathbb{R}^n$ is a nonempty closed set with empty interior, σ is a sigma-finite Borel measure on Σ, and μ is a sigma-finite Borel measure on $\mathbb{R}^n \setminus \Sigma$. Then for each aperture parameter $\kappa > 0$ one has*

$$\int_\Sigma \mu\big(\Gamma_\kappa(z)\big)\, d\sigma(z) = \int_{\mathbb{R}^n \setminus \Sigma} \sigma\big(\pi_\kappa(\{x\})\big)\, d\mu(x), \tag{8.1.42}$$

with the nontangential approach regions $\Gamma_\kappa(\cdot)$ and the projection operator π_κ considered relative to the open ambient $\Sigma^c := \mathbb{R}^n \setminus \Sigma$.

Proof To justify (8.1.42), write

$$\begin{aligned} \int_\Sigma \mu\big(\Gamma_\kappa(z)\big)\, d\sigma(z) &= \int_{z\in\Sigma} \Big(\int_{\Gamma_\kappa(z)} 1\, d\mu\Big)\, d\sigma(z) \\ &= \int_{z\in\Sigma} \Big(\int_{x\in\mathbb{R}^n\setminus\Sigma} \mathbf{1}_{\Gamma_\kappa(z)}(x)\, d\mu(x)\Big)\, d\sigma(z) \\ &= \int_{x\in\mathbb{R}^n\setminus\Sigma} \Big(\int_{z\in\Sigma} \mathbf{1}_{\Gamma_\kappa(z)}(x)\, d\sigma(z)\Big)\, d\mu(x) \\ &= \int_{x\in\mathbb{R}^n\setminus\Sigma} \Big(\int_{z\in\Sigma} \mathbf{1}_{\pi_\kappa(\{x\})}(z)\, d\sigma(z)\Big)\, d\mu(x) \\ &= \int_{\mathbb{R}^n\setminus\Sigma} \sigma\big(\pi_\kappa(\{x\})\big)\, d\mu(x), \end{aligned} \tag{8.1.43}$$

thanks to (8.1.7) and Fubini-Tonelli's Theorem. □

The second result alluded to earlier uses Lemma 8.1.8 to produce estimates involving "solid" and "surface" integrals in a general geometric setting.

Lemma 8.1.9 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a nonempty closed set with empty interior, and denote by δ_Σ the distance function to Σ. Also, assume σ is a sigma-finite Borel measure on Σ, and μ is a sigma-finite Borel measure on $\mathbb{R}^n \setminus \Sigma$. Finally, fix an aperture parameter $\kappa > 0$ and denote by $\Gamma_\kappa(\cdot)$ the nontangential approach regions relative to the open ambient $\Sigma^c := \mathbb{R}^n \setminus \Sigma$.*

If there exist an exponent $d \in (0, \infty)$ and a constant $c \in (0, \infty)$, such that

$$\sigma\big(B(x,r) \cap \Sigma\big) \geq cr^d \text{ for all } x \in \Sigma \text{ and } r \in \big(0, 2\,\mathrm{diam}\,\Sigma\big) \tag{8.1.44}$$

then one can find $C \in (0, \infty)$, such that

$$\int_{\mathbb{R}^n \setminus \Sigma} \delta_\Sigma(x)^d \,\mathrm{d}\mu(x) \leq C \int_\Sigma \mu\big(\Gamma_\kappa(z)\big)\,\mathrm{d}\sigma(z). \tag{8.1.45}$$

Moreover, if in place of (8.1.44) one now assumes that there exist an exponent $d \in (0, \infty)$ and a constant $c \in (0, \infty)$, such that

$$\sigma\big(B(x,r) \cap \Sigma\big) \leq cr^d \text{ for all } x \in \Sigma \text{ and } r \in \big(0, 2\,\mathrm{diam}\,\Sigma\big) \tag{8.1.46}$$

then one can find $C \in (0, \infty)$, such that

$$\int_\Sigma \mu\big(\Gamma_\kappa(z)\big)\,\mathrm{d}\sigma(z) \leq C \int_{\mathbb{R}^n \setminus \Sigma} \delta_\Sigma(x)^d \,\mathrm{d}\mu(x). \tag{8.1.47}$$

As a corollary, if Σ is Ahlfors regular then

$$\int_\Sigma \mu\big(\Gamma_\kappa(z)\big)\,\mathrm{d}\sigma(z) \approx \int_{\mathbb{R}^n \setminus \Sigma} \delta_\Sigma(x)^{n-1} \,\mathrm{d}\mu(x). \tag{8.1.48}$$

Proof All claims are clear from (8.1.42), (8.1.19), and assumptions. □

8.2 The Definition and Basic Properties of the Nontangential Maximal Operator

Let Ω be an arbitrary, open, nonempty, proper subset of $\mathbb{R}^n$, fixed throughout. Having also fixed some number $\kappa \in (0, \infty)$, if $u : \Omega \to \mathbb{R}$ is an arbitrary Lebesgue measurable function[2] define the `nontangential maximal function` of u with

[2] All considerations in this section naturally adapt to Lebesgue measurable functions taking values in the extended real line $\overline{\mathbb{R}}$, the field of complex numbers $\mathbb{C}$, as well as the finite Cartesian products of $\overline{\mathbb{R}}$ or $\mathbb{C}$.

aperture κ as

$$\mathcal{N}_\kappa u : \partial\Omega \longrightarrow [0, +\infty], \ \ (\mathcal{N}_\kappa u)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x),\mathcal{L}^n)} \text{ for all } x \in \partial\Omega. \tag{8.2.1}$$

Explicitly, for each $x \in \partial\Omega$ we set

$$(\mathcal{N}_\kappa u)(x) := \inf\big\{\lambda \geq 0 : \ \mathcal{L}^n\big(\{y \in \Gamma_\kappa(x) : \ |u(y)| > \lambda\}\big) = 0\big\}, \tag{8.2.2}$$

with the convention that $\inf \varnothing = +\infty$. In particular, (8.2.1) implies that

$$\begin{aligned} &\text{whenever } u \in \mathscr{C}^0(\Omega) \text{ one has} \\ &\big(\mathcal{N}_\kappa u\big)(x) = \sup_{y\in\Gamma_\kappa(x)} |u(y)| \ \text{ for all } \ x \in \partial\Omega. \end{aligned} \tag{8.2.3}$$

More generally, if $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function and $E \subseteq \Omega$ is a Lebesgue measurable set, we denote by $\mathcal{N}_\kappa^E u$ the nontangential maximal function of u restricted to E

$$\begin{aligned} &\mathcal{N}_\kappa^E u : \partial\Omega \longrightarrow [0, +\infty] \ \text{ defined as} \\ &(\mathcal{N}_\kappa^E u)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x)\cap E,\mathcal{L}^n)} \ \text{ for each } \ x \in \partial\Omega. \end{aligned} \tag{8.2.4}$$

Hence, $(\mathcal{N}_\kappa^E u)(x) = 0$ whenever $x \in \partial\Omega \setminus \pi_\kappa(E)$, and for each $x \in \partial\Omega$ we may refashion the last line in (8.2.4) as

$$(\mathcal{N}_\kappa^E u)(x) = \inf\big\{\lambda \geq 0 : \ \mathcal{L}^n\big(\{y \in \Gamma_\kappa(x) \cap E : \ |u(y)| > \lambda\}\big) = 0\big\} \tag{8.2.5}$$

again, with the convention that $\inf \varnothing = +\infty$. Note that, if we work (as one usually does) with equivalence classes, obtained by identifying functions which coincide $\mathcal{L}^n$-a.e., the nontangential maximal operator is independent of the specific choice of a representative in a given equivalence class. In other words,

$$\begin{aligned} &\mathcal{N}_\kappa^E u = \mathcal{N}_\kappa^E w \text{ everywhere on } \partial\Omega, \text{ whenever } u, w : \Omega \to \mathbb{R} \\ &\text{are two Lebesgue measurable functions so that } u = w \text{ for} \\ &\mathcal{L}^n\text{-a.e. point in } \Omega. \end{aligned} \tag{8.2.6}$$

Given any two Lebesgue measurable functions $u, w : \Omega \to \mathbb{R}$ along with any Lebesgue measurable set $E \subseteq \Omega$ we have

$$\mathcal{N}_\kappa^E\big(\max\{u, w\}\big) \leq \max\big\{\mathcal{N}_\kappa^E u, \mathcal{N}_\kappa^E w\big\} \ \text{ pointwise on } \ \partial\Omega. \tag{8.2.7}$$

Indeed, this is a direct consequence of (8.2.4) and the fact that in any measure space (X, μ) we have

$$\begin{aligned} &\big\|\max\{f, g\}\big\|_{L^\infty(X,\mu)} \leq \max\big\{\|f\|_{L^\infty(X,\mu)}, \|g\|_{L^\infty(X,\mu)}\big\} \\ &\text{for any } \mu-\text{measurable functions } f, g : X \to \mathbb{R}. \end{aligned} \tag{8.2.8}$$

For any two Lebesgue measurable functions $u, w : \Omega \to \mathbb{R}$, which are assumed to be finite $\mathcal{L}^n$-a.e., and any Lebesgue measurable set $E \subseteq \Omega$ we have

$$\mathcal{N}_\kappa^E(u+w) \leq \mathcal{N}_\kappa^E u + \mathcal{N}_\kappa^E w \text{ pointwise on } \partial\Omega, \tag{8.2.9}$$

and

$$\begin{gathered}\mathcal{N}_\kappa^E(uw) \leq \big(\mathcal{N}_\kappa^E u\big) \cdot \big(\mathcal{N}_\kappa^E w\big) \text{ pointwise on } \partial\Omega, \\ \text{with the convention that } 0 \cdot \infty = \infty \cdot 0 = 0 \text{ used in the right side.}\end{gathered} \tag{8.2.10}$$

Indeed, if $x \in \partial\Omega$ is a point at which, say, $\big(\mathcal{N}_\kappa^E u\big)(x) = 0$ and $\big(\mathcal{N}_\kappa^E w\big)(x) = \infty$, then (8.2.1) forces $u = 0$ at $\mathcal{L}^n$-a.e. point in $\Gamma_\kappa(x)$, hence $uw = 0$ at $\mathcal{L}^n$-a.e. point in $\Gamma_\kappa(x)$ since w is finite $\mathcal{L}^n$-a.e. by assumption. Ultimately, this permits us to conclude that $\big(\mathcal{N}_\kappa^E(uw)\big)(x) = 0$, which goes to show that the estimate in (8.2.10) is valid at the point x if we adopt the convention that $0 \cdot \infty = 0$.

As a corollary of (8.2.9),

$$\begin{gathered}\text{if } x \in \partial\Omega \text{ is such that } \big(\mathcal{N}_\kappa^E u\big)(x) < +\infty \text{ and } \big(\mathcal{N}_\kappa^E w\big)(x) < +\infty \\ \text{we have } \Big|\big(\mathcal{N}_\kappa^E u\big)(x) - \big(\mathcal{N}_\kappa^E w\big)(x)\Big| \leq \big(\mathcal{N}_\kappa^E (u-w)\big)(x).\end{gathered} \tag{8.2.11}$$

It is also clear from earlier definitions that for each Lebesgue measurable function $u : \Omega \to \mathbb{R}$ and each $p \in (0, \infty)$ we have

$$\mathcal{N}_\kappa^E\big(|u|^p\big) = \big(\mathcal{N}_\kappa^E u\big)^p \text{ pointwise on } \partial\Omega, \tag{8.2.12}$$

and

$$\begin{aligned}&\big(\mathcal{N}_\kappa^E u\big)(x) = \sup\nolimits_{y \in \Gamma_\kappa(x) \cap E} |u(y)| \text{ at each point } x \in \partial\Omega, \text{ whenever} \\ &E \subseteq \Omega \text{ is open and } u \text{ is a Lebesgue measurable function defined} \\ &\text{on } \Omega \text{ which happens to be actually continuous on the given set } E.\end{aligned} \tag{8.2.13}$$

Also, for every Lebesgue measurable function $u : \Omega \to \mathbb{R}$ and every Lebesgue measurable set $E \subseteq \Omega$, we have

$$\mathcal{N}_\kappa^E u = \mathcal{N}_\kappa(u \cdot \mathbf{1}_E) \leq \big(\mathcal{N}_\kappa u\big) \cdot \mathbf{1}_{\pi_\kappa(E)} \text{ on } \partial\Omega. \tag{8.2.14}$$

Moreover, for any Lebesgue measurable function $u : \Omega \to \mathbb{R}$ and any two Lebesgue measurable set $E_1, E_2 \subseteq \Omega$ we have

$$\mathcal{N}_\kappa^{E_1 \cup E_2} u \leq \max\big\{\mathcal{N}_\kappa^{E_1} u, \mathcal{N}_\kappa^{E_2} u\big\} \text{ on } \partial\Omega. \tag{8.2.15}$$

To see that this is the case, use (8.2.14) and (8.2.7) to write

$$\begin{aligned}\mathcal{N}_\kappa^{E_1\cup E_2}u &= \mathcal{N}_\kappa\big(u\cdot \mathbf{1}_{E_1\cup E_2}\big) = \mathcal{N}_\kappa\big(u\cdot \max\{\mathbf{1}_{E_1},\mathbf{1}_{E_2}\}\big)\\ &= \mathcal{N}_\kappa\big(\max\{u\cdot \mathbf{1}_{E_1}, u\cdot \mathbf{1}_{E_2}\}\big) = \max\big\{\mathcal{N}_\kappa(u\cdot \mathbf{1}_{E_1}), \mathcal{N}_\kappa(u\cdot \mathbf{1}_{E_2})\big\}\\ &= \max\big\{\mathcal{N}_\kappa^{E_1}u, \mathcal{N}_\kappa^{E_2}u\big\} \ \text{ on } \ \partial\Omega, \end{aligned} \tag{8.2.16}$$

proving (8.2.15). More generally, given any countable family $(u_i)_{i\in\mathbb{N}}$ of Lebesgue measurable functions $u_i : E \to \mathbb{C}$, we claim that

$$\mathcal{N}_\kappa^E\Big(\sup_{i\in\mathbb{N}} u_i\Big)(x) = \sup_{i\in\mathbb{N}}\big(\mathcal{N}_\kappa^E u_i\big)(x) \ \text{ for each } \ x\in\partial\Omega. \tag{8.2.17}$$

This is a consequence of the definition made in (8.2.4) plus a general result contained in the lemma below.

Lemma 8.2.1 *Let (X,μ) be a measure space and consider some family $(f_i)_{i\in\mathbb{N}}$ of μ-measurable functions $f_i : X \to [0,+\infty]$. Also, fix $p\in(0,\infty]$. Then the formula*

$$\big\|\sup_{i\in\mathbb{N}} f_i\big\|_{L^p(X,\mu)} = \sup_{i\in\mathbb{N}}\|f_i\|_{L^p(X,\mu)} \tag{8.2.18}$$

is valid when either $p=\infty$, or when the sequence $(f_i)_{i\in\mathbb{N}}$ is pointwise non-decreasing, i.e., when

$$\begin{array}{c} \textit{for each } \ i\in\mathbb{N} \ \textit{ one has}\\ f_i(x)\le f_{i+1}(x) \ \textit{ for } \mu\textit{-a.e. } x\in X. \end{array} \tag{8.2.19}$$

Proof In one direction, $0\le f_i\le \sup_{i\in\mathbb{N}} f_i$ for each $i\in\mathbb{N}$. Since $\sup_{i\in\mathbb{N}} f_i$ is itself a μ-measurable function on X, taking the L^p quasi-norm in (X,μ) yields $\|f_i\|_{L^p(X,\mu)}\le \big\|\sup_{i\in\mathbb{N}} f_i\big\|_{L^p(X,\mu)}$ for each $i\in\mathbb{N}$, from which the right-pointing inequality in (8.2.18) follows. To establish the inequality in the opposite direction, we distinguish two cases. First, consider the case when $p=\infty$. Then for each $i\in\mathbb{N}$ we have $f_i(x)\le\|f_i\|_{L^\infty(X,\mu)}\le\sup_{i\in\mathbb{N}}\|f_i\|_{L^\infty(X,\mu)}$ at μ-a.e. point $x\in X$, which then further implies $0\le\sup_{i\in\mathbb{N}} f_i(x)\le\sup_{i\in\mathbb{N}}\|f_i\|_{L^\infty(X,\mu)}$ at μ-a.e. point $x\in X$. Ultimately, this shows that $\big\|\sup_{i\in\mathbb{N}} f_i\big\|_{L^\infty(X,\mu)}\le\sup_{i\in\mathbb{N}}\|f_i\|_{L^\infty(X,\mu)}$, which finishes the proof of (8.2.18) in the case when $p=\infty$.

Finally, assume $0<p<\infty$ and make the additional assumption that (8.2.19) holds. The latter implies $f_i^p\nearrow\big(\sup_{i\in\mathbb{N}} f_i\big)^p$ pointwise on X as $i\nearrow\infty$, so Lebesgue's Monotone Convergence Theorem then guarantees that

$$\int_X\big(\sup_{i\in\mathbb{N}} f_i\big)^p\,\mathrm{d}\mu = \lim_{i\to\infty}\int_X f_i^p\,\mathrm{d}\mu. \tag{8.2.20}$$

Hence,

$$\big\|\sup_{i\in\mathbb{N}} f_i\big\|^p_{L^p(X,\mu)}\le\sup_{i\in\mathbb{N}}\|f_i\|^p_{L^p(X,\mu)}, \tag{8.2.21}$$

and the desired conclusion once again follows. □

Here is another useful application of Lemma 8.2.1.

Lemma 8.2.2 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and fix a Lebesgue measurable set $E \subseteq \Omega$ along with an aperture parameter $\kappa > 0$. Let $u : \Omega \to \mathbb{R}$ be an $\mathcal{L}^n$-measurable function. Finally, suppose $\{u_j\}_{j\in\mathbb{N}}$ is a sequence of real-valued $\mathcal{L}^n$-measurable functions defined in Ω with the property that $\lim_{j\to\infty} u_j(x) = u(x)$ at $\mathcal{L}^n$-a.e. point $x \in E$. Then*

$$\mathcal{N}_\kappa^E u \leq \sup_{j\in\mathbb{N}} \mathcal{N}_\kappa^E u_j \quad \text{at each point on } \partial\Omega. \tag{8.2.22}$$

Proof Since the limit of any convergent numerical sequence is less than, or equal to, its supremum, we have $|u(x)| \leq \sup_{j\in\mathbb{N}} |u_j(x)|$ for $\mathcal{L}^n$-a.e. $x \in E$. Consequently, at each point $x_0 \in \partial\Omega$ we may write

$$\begin{aligned}(\mathcal{N}_\kappa^E u)(x_0) &= \|u\|_{L^\infty(E\cap\Gamma_\kappa(x_0),\mathcal{L}^n)} \leq \Big\| \sup_{j\in\mathbb{N}} |u_j| \Big\|_{L^\infty(E\cap\Gamma_\kappa(x_0),\mathcal{L}^n)} \\ &= \sup_{j\in\mathbb{N}} \|u_j\|_{L^\infty(E\cap\Gamma_\kappa(x_0),\mathcal{L}^n)} = \sup_{j\in\mathbb{N}} (\mathcal{N}_\kappa^E u_j)(x_0),\end{aligned} \tag{8.2.23}$$

where the penultimate equality uses (8.2.18) with $p := \infty$. □

Moving on, for further reference let us also note that

$$\begin{aligned}&\mathcal{N}_\kappa^E \mathbf{1}_F \leq \mathbf{1}_{\pi_\kappa(E\cap F)}, \quad \forall F \subseteq \Omega \text{ Lebesgue measurable,} \\ &\qquad \text{with equality if } E, F \text{ are open subsets of } \Omega.\end{aligned} \tag{8.2.24}$$

In addition, (8.1.20) implies

$$\begin{aligned}&\mathcal{N}_\kappa^E u \leq \mathcal{N}_{\kappa'}^{E'} u \text{ everywhere on } \partial\Omega, \text{ if } u : E \to \mathbb{C} \text{ is} \\ &\text{Lebesgue measurable, } \kappa' \geq \kappa > 0, \text{ and } E, E' \subseteq \Omega \text{ are any} \\ &\text{two Lebesgue measurable sets such that } E \subseteq E'.\end{aligned} \tag{8.2.25}$$

Proposition 8.2.3 *Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$, and fix an aperture parameter $\kappa \in (0, \infty)$. Also, assume $u : \Omega \to \mathbb{R}$ is an arbitrary Lebesgue measurable function and select some Lebesgue measurable set $E \subseteq \Omega$. Then*

$$\begin{aligned}&\mathcal{N}_\kappa^E u \text{ is lower-semicontinuous on } \partial\Omega, \\ &\text{and vanishes identically on } \partial\Omega \setminus \pi_\kappa(E).\end{aligned} \tag{8.2.26}$$

In particular, for every number $\lambda \in \mathbb{R}$,

$$\text{the set } \big\{x \in \partial\Omega : (\mathcal{N}_\kappa^E u)(x) > \lambda\big\} \text{ is (relatively) open in } \partial\Omega, \tag{8.2.27}$$

hence

$$\mathcal{N}_\kappa^E u : \partial\Omega \longrightarrow [0, +\infty] \ \text{ is a Borel-measurable function.} \tag{8.2.28}$$

Proof In addition to (8.2.5) we find it useful to consider

$$\dot{\mathcal{N}}_\kappa^E u : \partial\Omega \longrightarrow (-\infty, +\infty] \tag{8.2.29}$$

defined at each $x \in \partial\Omega$ by (again, with the convention that $\inf \varnothing = +\infty$)

$$\big(\dot{\mathcal{N}}_\kappa^E u\big)(x) := \inf\Big\{\lambda \in \mathbb{R} : \mathcal{L}^n\big(\{y \in \Gamma_\kappa(x) \cap E : u(y) > \lambda\}\big) = 0\Big\}. \tag{8.2.30}$$

Inspecting definitions reveals that

$$\mathcal{N}_\kappa^E u = \dot{\mathcal{N}}_\kappa^E |u| \ \text{ on } \ \partial\Omega, \tag{8.2.31}$$

and

$$\big|\dot{\mathcal{N}}_\kappa^E u\big| \leq \mathcal{N}_\kappa^E u \ \text{ on } \ \partial\Omega. \tag{8.2.32}$$

In relation to this version of the nontangential maximal operator we claim that, for every Lebesgue measurable function $u : \Omega \to \mathbb{R}$ and every number $\lambda \in \mathbb{R}$,

$$\text{the set } \Big\{x \in \partial\Omega : \big(\dot{\mathcal{N}}_\kappa^E u\big)(x) > \lambda\Big\} \text{ is (relatively) open in } \partial\Omega. \tag{8.2.33}$$

To prove this, suppose $x \in \partial\Omega$ is such that $\big(\dot{\mathcal{N}}_\kappa^E u\big)(x) > \lambda$. In view of (8.2.30), this implies that there exist

a number $\varepsilon > 0$ and a Lebesgue measurable set $A \subseteq \Gamma_\kappa(x) \cap E$, with $\mathcal{L}^n(A) > 0$, such that $u > \lambda + \varepsilon$ at $\mathcal{L}^n$-a.e. point on A. (8.2.34)

By Lebesgue's Differentiation Theorem we have

$$\lim_{r \to 0^+} \frac{\mathcal{L}^n\big(A \cap B(y, r)\big)}{\mathcal{L}^n\big(B(y, r)\big)} = 1 \ \text{ for } \mathcal{L}^n\text{-a.e. point } \ y \in A. \tag{8.2.35}$$

Given that $\mathcal{L}^n(A) > 0$, it is then possible to find $y_* \in A$ and $r_* > 0$, such that

$$\mathcal{L}^n\big(A \cap B(y_*, r)\big) > \tfrac{1}{2}\mathcal{L}^n\big(B(y_*, r)\big) \ \text{ whenever } \ r \in (0, r_*). \tag{8.2.36}$$

Since y_* belongs to $\Gamma_\kappa(x)$ which is an open set (cf. (8.1.35)), it follows that there is $r_{**} > 0$ such that $B(y_*, r_{**}) \subset \Gamma_\kappa(x)$. Define $r := \min\{r_*/2, r_{**}/2\} > 0$. Then (8.1.9) implies that there exists $\varepsilon_o > 0$ with the property that

$$B(y_*, r) \subseteq \Gamma_\kappa(z) \ \text{ for every } \ z \in \partial\Omega \cap B(x, \varepsilon_o). \tag{8.2.37}$$

In concert with the containment property in the first line of (8.2.34) (which ensures that $A \subseteq E$) this implies

$$A \cap B(y_*, r) \subseteq E \cap \Gamma_\kappa(z) \quad \text{for every} \quad z \in \partial\Omega \cap B(x, \varepsilon_o). \tag{8.2.38}$$

Moreover,

$$\mathcal{L}^n\big(A \cap B(y_*, r)\big) > 0 \tag{8.2.39}$$

by (8.2.36) and our choice of r, while the last property in (8.2.34) implies

$$u > \lambda + \varepsilon \quad \text{at } \mathcal{L}^n\text{-a.e. point on } A \cap B(y_*, r). \tag{8.2.40}$$

Collectively, (8.2.38)–(8.2.40) then permit us to conclude that

$$\big(\dot{\mathcal{N}}_\kappa^E u\big)(z) > \lambda \quad \text{for every} \quad z \in \partial\Omega \cap B(x, \varepsilon_o). \tag{8.2.41}$$

At this stage, (8.2.33) readily follows from (8.2.41). Having established (8.2.33) we then conclude that

$$\dot{\mathcal{N}}_\kappa^E u \quad \text{is lower-semicontinuous on} \quad \partial\Omega, \quad \text{and vanishes on} \quad \partial\Omega \setminus \pi_\kappa(E). \tag{8.2.42}$$

In concert with (8.2.31), this proves (8.2.27) and all desired conclusions follow. □

To close, we remark that it is useful to extend the action of the nontangential maximal operator to distributions which are of function type outside of a closed subset of their domain. Specifically, given an open set $\Omega \subseteq \mathbb{R}^n$, some $\kappa \in (0, \infty)$, and a distribution

$$\begin{gathered} u \in \mathcal{D}'(\Omega) \text{ such that there exists a closed set } K \subset \Omega \text{ for which} \\ u\big|_{\Omega\setminus K} \text{ is of function type, i.e., } u\big|_{\Omega\setminus K} \in L^1_{\rm loc}(\Omega \setminus K, \mathcal{L}^n), \end{gathered} \tag{8.2.43}$$

we agree to define

$$\mathcal{N}_\kappa^{\Omega\setminus K} u := \mathcal{N}_\kappa \widetilde{u} \quad \text{where} \quad \widetilde{u} := \begin{cases} u\big|_{\Omega\setminus K} & \text{in } \Omega \setminus K, \\ 0 & \text{in } K. \end{cases} \tag{8.2.44}$$

In particular, $\mathcal{N}_\kappa^{\Omega\setminus K} u$ is a well-defined function for each closed set $K \subset \Omega$ containing regsupp u, the regular support of the given distribution $u \in \mathcal{D}'(\Omega)$ (i.e., the smallest relatively closed subset of Ω outside of which u is a locally integrable function; cf. (1.5.4)). Finally, the reader is reminded (cf. (1.5.5)) that, throughout this work,

$$\begin{gathered} \text{whenever } 0 < \varepsilon < \operatorname{dist}\big(\operatorname{regsupp} u, \partial\Omega\big) \text{ we shall abbreviate} \\ \mathcal{N}_\kappa^\varepsilon u := \mathcal{N}_\kappa\big(u \cdot \mathbf{1}_{O_\varepsilon}\big) \quad \text{where} \quad O_\varepsilon := \big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\big\}. \end{gathered} \tag{8.2.45}$$

8.3 Elementary Estimates Involving the Nontangential Maximal Operator

Our first observation is that any function whose nontangential maximal operator is almost everywhere finite is necessarily locally bounded.

Lemma 8.3.1 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ and assume μ is a Borel measure on $\partial\Omega$ satisfying*

$$\mu\big(B(x,r)\cap\partial\Omega\big)>0 \ \text{ for every } \ x\in\partial\Omega \ \text{ and } \ r>0. \tag{8.3.1}$$

Fix $\kappa>0$ and assume that $u:\Omega\to\mathbb{C}$ is a Lebesgue measurable function with the property that

$$\mathcal{N}_\kappa u<+\infty \ \text{ at } \ \mu\text{-a.e. point on } \ \partial\Omega. \tag{8.3.2}$$

Then $u\in L^\infty_{\rm loc}(\Omega,\mathcal{L}^n)$.

Proof Fix $x_*\in\Omega$ arbitrary. From (8.1.36) (with $E=\Omega$) and (8.1.9) it follows that there exist $x\in\partial\Omega$ and $r>0$ satisfying $B(x_*,r)\subseteq\Gamma_\kappa(z)$ for each $z\in B(x,r)\cap\partial\Omega$. In turn, by relying on (8.3.1) and (8.3.2), we can find a point $z_*\in B(x,r)\cap\partial\Omega$ with $\big(\mathcal{N}_\kappa u\big)(z_*)<+\infty$, and since

$$\begin{gathered}\|u\|_{L^\infty(B(x_*,r),\mathcal{L}^n)}\leq\|u\|_{L^\infty(\Gamma_\kappa(z),\mathcal{L}^n)}=\big(\mathcal{N}_\kappa u\big)(z)\\ \text{for each point } \ z\in B(x,r)\cap\partial\Omega,\end{gathered} \tag{8.3.3}$$

the desired conclusion follows. □

Our second observation is that the essential supremum of a function matches the essential supremum of its nontangential maximal operator.

Lemma 8.3.2 *Suppose Ω is an open nonempty proper subset of $\mathbb{R}^n$ and assume μ is a Borel measure on $\partial\Omega$ with the property that*

$$0<\mu\big(B(x,r)\cap\partial\Omega\big)<\infty \ \text{ for every } \ x\in\partial\Omega \ \text{ and } \ r>0. \tag{8.3.4}$$

Also, fix an $\mathcal{L}^n$-measurable set $E\subseteq\Omega$ along with some aperture parameter $\kappa>0$. Then for each $\mathcal{L}^n$-measurable function $u:E\to\mathbb{R}$ one has

$$\mathcal{N}^E_\kappa u\in L^\infty(\pi_\kappa(E),\,\mu) \ \text{ and } \ \|\mathcal{N}^E_\kappa u\|_{L^\infty(\pi_\kappa(E),\,\mu)}=\|u\|_{L^\infty(E,\mathcal{L}^n)}. \tag{8.3.5}$$

In particular, corresponding to $E:=\Omega$, for each Lebesgue measurable function $u:\Omega\to\mathbb{R}$ one has

$$\mathcal{N}_\kappa u\in L^\infty(\partial\Omega,\,\mu) \ \text{ and } \ \|\mathcal{N}_\kappa u\|_{L^\infty(\partial\Omega,\,\mu)}=\|u\|_{L^\infty(\Omega,\mathcal{L}^n)}. \tag{8.3.6}$$

Proof It suffices to show that (8.3.6) holds for each Lebesgue measurable function $u : \Omega \to \mathbb{R}$. Granted this, given some $\mathcal{L}^n$-measurable set $E \subseteq \Omega$ together with some $\mathcal{L}^n$-measurable function $u : E \to \mathbb{R}$, formula (8.3.5) follows by applying (8.3.6) to the given function u extended by zero to the entire Ω.

On to the proof of (8.3.6). Fix a Lebesgue measurable function $u : \Omega \to \mathbb{R}$. Since $\mathcal{N}_\kappa u$ is a lower-semicontinuous function on $\partial\Omega$, hence μ-measurable, and for every $x \in \partial\Omega$ we have

$$\big(\mathcal{N}_\kappa u\big)(x) = \|u\|_{L^\infty(\Gamma_\kappa(x),\mathcal{L}^n)} \leq \|u\|_{L^\infty(\Omega,\mathcal{L}^n)}, \tag{8.3.7}$$

it follows that

$$\mathcal{N}_\kappa u \in L^\infty(\partial\Omega,\ \mu) \ \text{ and } \ \|\mathcal{N}_\kappa u\|_{L^\infty(\partial\Omega,\,\mu)} \leq \|u\|_{L^\infty(\Omega,\mathcal{L}^n)}. \tag{8.3.8}$$

We emphasize that property (8.3.4) has not been used in the proof of (8.3.8).

To prove the opposite inequality, fix an arbitrary point $x_* \in \Omega$ and pick $x \in \partial\Omega$ such that $\text{dist}(x, \partial\Omega) = |x - x_*|$. This implies that x_* belongs to $\Gamma_\kappa(x)$, and since the latter set us open, there exists $r > 0$ such that $B(x, r) \subseteq \Gamma_\kappa(x)$, Granted this, we conclude from (8.1.9) that there exists $\varepsilon > 0$, such that

$$B(x_*, r/2) \subseteq \Gamma_\kappa(z) \text{ for each } z \in \partial\Omega \cap B(x, \varepsilon). \tag{8.3.9}$$

In turn, (8.3.9) implies

$$\|u\|_{L^\infty(B(x_*,r/2),\mathcal{L}^n)} \leq \|u\|_{L^\infty(\Gamma_\kappa(z),\mathcal{L}^n)} = \big(\mathcal{N}_\kappa u\big)(z) \tag{8.3.10}$$

for each $z \in \partial\Omega \cap B(x, \varepsilon)$. Integrating (8.3.10) for $z \in \partial\Omega \cap B(x, \varepsilon)$ with respect to μ then yields

$$\begin{aligned} \|u\|_{L^\infty(B(x_*,r/2),\mathcal{L}^n)} \cdot \mu\big(\partial\Omega \cap B(x, \varepsilon)\big) &\leq \int_{\partial\Omega\cap B(x,\varepsilon)} \big(\mathcal{N}_\kappa u\big)(z)\, \mu(z) \\ &\leq \|\mathcal{N}_\kappa u\|_{L^\infty(\partial\Omega,\,\mu)} \cdot \mu\big(\partial\Omega \cap B(x, \varepsilon)\big). \end{aligned} \tag{8.3.11}$$

In light of (8.3.4), after denoting $r_* := r/2$ this further gives

$$\|u\|_{L^\infty(B(x_*,r_*),\mathcal{L}^n)} \leq \|\mathcal{N}_\kappa u\|_{L^\infty(\partial\Omega,\,\mu)}. \tag{8.3.12}$$

The end-game in the proof of (8.3.6) is as follows. Start with the open cover $\big\{B(x_*, r_*)\big\}_{x_*\in\Omega}$ of Ω such that (8.3.12) holds for each of these balls, and use Lindelöf's theorem to refine this to a countable sub-cover, say, $\big\{B(x_j, r_j)\big\}_{j\in\mathbb{N}}$. For each $j \in \mathbb{N}$, we know that there exists an $\mathcal{L}^n$-nullset $N_j \subseteq B(x_j, r_j)$, such that

$$|u(x)| \leq \|\mathcal{N}_\kappa u\|_{L^\infty(\partial\Omega,\,\mu)} \text{ for each } \ x \in B(x_j, r_j) \setminus N_j. \tag{8.3.13}$$

Then $N := \cup_{j\in\mathbb{N}} N_j$ is an $\mathcal{L}^n$-nullset contained in Ω with the property that

$$|u(x)| \leq \|\mathcal{N}_\kappa u\|_{L^\infty(\partial\Omega,\,\mu)} \text{ for each } \ x \in \Omega \setminus N. \tag{8.3.14}$$

This proves

$$\|u\|_{L^\infty(\Omega,\mathcal{L}^n)} \leq \|\mathcal{N}_\kappa u\|_{L^\infty(\partial\Omega,\,\mu)} \tag{8.3.15}$$

which, in concert with (8.3.8), establishes (8.3.6). □

Another remark closely related to Lemma 8.3.2 is as follows.

Lemma 8.3.3 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ and fix an aperture parameter $\kappa > 0$. Then for each $\mathcal{L}^n$-measurable function $u : \Omega \to \mathbb{R}$ one has*

$$\sup_{x\in\partial\Omega} (\mathcal{N}_\kappa u)(x) = \|u\|_{L^\infty(\Omega,\mathcal{L}^n)}. \tag{8.3.16}$$

Proof It is clear from (8.2.1) that

$$\sup_{x\in\partial\Omega} (\mathcal{N}_\kappa u)(x) \leq \|u\|_{L^\infty(\Omega,\mathcal{L}^n)}. \tag{8.3.17}$$

To prove the opposite inequality, pick $\lambda \in \mathbb{R}$, such that $\lambda < \|u\|_{L^\infty(\Omega,\mathcal{L}^n)}$. Then, by definition, there exists a Lebesgue measurable set $A \subseteq \Omega$ with $\mathcal{L}^n(A) > 0$ and such that $|u(x)| > \lambda$ for each $x \in A$. By Lebesgue's Differentiation Theorem,

$$\lim_{r\to 0^+} \frac{\mathcal{L}^n\big(A \cap B(x,r)\big)}{\mathrm{vol}_n \cdot r^n} = \lim_{r\to 0^+} ⨍_{B(x,r)} \mathbf{1}_A \, \mathrm{d}\mathcal{L}^n = \mathbf{1}_A(x) \tag{8.3.18}$$

for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$, where vol_n denotes the volume of the unit ball in $\mathbb{R}^n$ (cf. (5.7.18)). Since $\mathcal{L}^n(A) > 0$, we may therefore find a point $x_* \in A$ with the property that

$$\lim_{r\to 0^+} \frac{\mathcal{L}^n\big(A \cap B(x_*,r)\big)}{\mathrm{vol}_n \cdot r^n} = 1. \tag{8.3.19}$$

In particular, this implies that $x_* \in \Omega$ and

$$\mathcal{L}^n\big(A \cap B(x_*,r)\big) > 0 \ \text{ for each } \ r > 0. \tag{8.3.20}$$

Let $x_o \in \partial\Omega$ be such that $\mathrm{dist}(x_*, \partial\Omega) = |x_* - x_o|$. In light of (8.1.1), this implies that $x_* \in \Gamma_\kappa(x_o)$. Given that $\Gamma_\kappa(x_o)$ is an open set, there exists $r_* > 0$, such that we actually have $B(x_*, r_*) \subseteq \Gamma_\kappa(x_o)$. Consequently,

$$\begin{aligned} \sup_{x\in\partial\Omega} (\mathcal{N}_\kappa u)(x) &\geq (\mathcal{N}_\kappa u)(x_o) = \|u\|_{L^\infty(\Gamma_\kappa(x_o),\mathcal{L}^n)} \\ &\geq \|u\|_{L^\infty(B(x_*,r_*),\mathcal{L}^n)} \geq \lambda, \end{aligned} \tag{8.3.21}$$

where the last inequality is justified by recalling that for each $x \in A \cap B(x_*, r_*)$ we have $|u(x)| > \lambda$ and by observing that $\mathcal{L}^n\big(A \cap B(x_*, r_*)\big) > 0$ (cf. (8.3.20)). Upon sending $\lambda \nearrow \|u\|_{L^\infty(\Omega,\mathcal{L}^n)}$ in (8.3.21) we arrive at the conclusion that

$$\sup_{x\in\partial\Omega} (\mathcal{N}_\kappa u)(x) \geq \|u\|_{L^\infty(\Omega,\mathcal{L}^n)}. \tag{8.3.22}$$

Together with (8.3.17) this finishes the proof of (8.3.16). □

In the one-dimensional setting, it turns out that the essential supremum of a function is dominated by the integral of its nontangential maximal operator (with respect to the counting measure).

Lemma 8.3.4 *Let Ω be a nonempty, open, proper subset of the real line $\mathbb{R}$. Abbreviate $\sigma := \mathcal{H}^0\lfloor\partial\Omega$ (thus, σ is the counting measure on $\partial\Omega$) and fix $\kappa > 0$ arbitrary. Then for each $\mathcal{L}^1$-measurable function $u : \Omega \to \mathbb{R}$ one has*

$$\|u\|_{L^\infty(\Omega,\mathcal{L}^1)} \leq \|\mathcal{N}_\kappa u\|_{L^1(\partial\Omega,\sigma)}. \tag{8.3.23}$$

Proof The hypotheses on Ω imply that there exist an at most countable set J along with a family $\{I_j\}_{j\in J}$ of mutually disjoint intervals of the form $I_j = (a_j, b_j)$ with $-\infty \leq a_j < b_j \leq +\infty$ and $\min\{|a_j|, |b_j|\} < \infty$ for each $j \in J$, such that

$$\Omega = \bigcup_{j\in J} I_j. \tag{8.3.24}$$

In fact, $\{I_j\}_{j\in J}$ is the family of connected components of Ω. As such,

$$\bigcup_{j\in\mathbb{N}} \partial I_j \subseteq \partial\Omega \tag{8.3.25}$$

which, in view of the fact that $\sigma := \mathcal{H}^0\lfloor\partial\Omega$ is the counting measure on $\partial\Omega$, implies.

$$\sum_{j\in J} \|\mathcal{N}_\kappa u\|_{L^1(\partial I_j,\sigma)} \leq \|\mathcal{N}_\kappa u\|_{L^1(\partial\Omega,\sigma)}. \tag{8.3.26}$$

In turn, for each $j \in J$ we have

$$\|\mathcal{N}_\kappa u\|_{L^1(\partial I_j,\sigma)} = \begin{cases} \big(\mathcal{N}_\kappa u\big)(a_j) + \big(\mathcal{N}_\kappa u\big)(b_j) & \text{if } -\infty < a_j < b_j < +\infty, \\ \big(\mathcal{N}_\kappa u\big)(a_j) & \text{if } -\infty < a_j < b_j = +\infty, \\ \big(\mathcal{N}_\kappa u\big)(b_j) & \text{if } -\infty = a_j < b_j < +\infty. \end{cases} \tag{8.3.27}$$

From this and (8.1.28)–(8.1.30) we may then conclude that

$$\|u\|_{L^\infty(I_j,\mathcal{L}^1)} \leq \|\mathcal{N}_\kappa u\|_{L^1(\partial I_j,\sigma)}, \qquad \forall j \in J \tag{8.3.28}$$

which, together with (8.3.24) and (8.3.26), permits us to write

$$\begin{aligned}\|u\|_{L^\infty(\Omega,\mathcal{L}^1)} &= \sup_{j\in J}\|u\|_{L^\infty(I_j,\mathcal{L}^1)} \leq \sum_{j\in J}\|u\|_{L^\infty(I_j,\mathcal{L}^1)}\\ &\leq \sum_{j\in J}\|\mathcal{N}_\kappa u\|_{L^1(\partial I_j,\sigma)} \leq \|\mathcal{N}_\kappa u\|_{L^1(\partial\Omega,\sigma)},\end{aligned} \tag{8.3.29}$$

proving (8.3.23). □

When naturally organized as a quasi-normed space, the collection of all measurable functions whose nontangential maximal operator is p-th power integrable (for some fixed $p \in (0,\infty]$) turns out to be a reasonably behaved quasi-Banach space.

Proposition 8.3.5 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ and suppose μ is a Borel measure on $\partial\Omega$ satisfying*

$$\mu\big(B(x,r)\cap\partial\Omega\big) > 0 \ \text{ for every } \ x\in\partial\Omega \ \text{ and every } \ r>0. \tag{8.3.30}$$

In this context, for each given aperture parameter $\kappa > 0$ and each integrability exponent $p \in (0,\infty]$ define the space

$$N^p_\kappa(\Omega;\mu) := \big\{u:\Omega\to\mathbb{C} : u \text{ is } \mathcal{L}^n\text{-measurable, and } \|u\|_{N^p_\kappa(\Omega;\mu)} < +\infty\big\} \tag{8.3.31}$$

where, for each $\mathcal{L}^n$-measurable function $u:\Omega\to\mathbb{C}$,

$$\|u\|_{N^p_\kappa(\Omega;\mu)} := \|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,\,\mu)}. \tag{8.3.32}$$

Then (8.3.32) is a quasi-norm, and $N^p_\kappa(\Omega;\mu)$ is a quasi-Banach space (respectively, a genuine norm, and an actual Banach space, when $1\leq p\leq\infty$) which embeds continuously into $L^0(\Omega,\mathcal{L}^n)$ (the space of measurable, $\mathcal{L}^n$-a.e. finite functions on Ω) when the latter space is endowed with the topology induced by convergence in measure on sets of finite measure.

Moreover,

$$N^p_\kappa(\Omega;\mu) \text{ embeds continuously into } L^\infty_{\rm loc}(\Omega,\mathcal{L}^n) \tag{8.3.33}$$

and

$$N^p_\kappa(\Omega;\mu)\cap\mathscr{C}^0(\Omega) \ \text{ is a closed subspace of } \ N^p_\kappa(\Omega;\mu). \tag{8.3.34}$$

Also,

$$\begin{gathered}\text{if } 0<p\leq 1 \text{ then for all } u,w\in N^p_\kappa(\Omega;\mu) \text{ one has}\\ \|u+w\|^p_{N^p_\kappa(\Omega;\mu)} \leq \|u\|^p_{N^p_\kappa(\Omega;\mu)} + \|w\|^p_{N^p_\kappa(\Omega;\mu)},\\ \text{hence } \|\cdot\|_{N^p_\kappa(\Omega;\mu)} \text{ is a } p\text{-norm whenever } p\in(0,1].\end{gathered} \tag{8.3.35}$$

Finally, $N_\kappa^p(\Omega;\mu)$ has the property that any sequence $\{u_j\}_{j\in\mathbb{N}}$ in $N_\kappa^p(\Omega;\mu)$ which is convergent to some $u \in N_\kappa^p(\Omega;\mu)$ in the topology induced by $\|\cdot\|_{N_\kappa^p(\Omega;\mu)}$ has a subsequence which converges to u pointwise $\mathcal{L}^n$-a.e. on Ω.

Remark 8.3.6 In analogy with the classical work in [54], we may regard $N_\kappa^p(\Omega;\mu)$ from (8.3.31) as the *tent space* T_∞^p associated with the set Ω and measure μ.

Proof of Proposition 8.3.5 Let us denote by $\mathscr{M}_+(\Omega,\mathcal{L}^n)$ the collection of all equivalence classes of scalar-valued, non-negative, $\mathcal{L}^n$-measurable functions u defined in Ω. Then, thanks to (8.2.26) and the fact that μ is a Borel measure on $\partial\Omega$, the mapping

$$\|\cdot\|_{N_\kappa^p(\Omega;\mu)} : \mathscr{M}_+(\Omega,\mathcal{L}^n) \longrightarrow [0,+\infty] \tag{8.3.36}$$

introduced earlier in (8.3.32) is well defined. From (8.2.9) it follows that for each $u, w \in \mathscr{M}_+(\Omega,\mathcal{L}^n)$ we have

$$\begin{aligned}\|u+w\|_{N_\kappa^p(\Omega;\mu)} &\le c_p\big(\|u\|_{N_\kappa^p(\Omega;\mu)} + \|w\|_{N_\kappa^p(\Omega;\mu)}\big)\\ &\le 2c_p \max\big\{\|u\|_{N_\kappa^p(\Omega;\mu)}, \|w\|_{N_\kappa^p(\Omega;\mu)}\big\},\end{aligned} \tag{8.3.37}$$

where

$$c_p := 2^{\max\{0,1/p-1\}} \in [1,+\infty). \tag{8.3.38}$$

Hence, $\|\cdot\|_{N_\kappa^p(\Omega;\mu)}$ satisfies a quasi-triangle inequality. The first inequality in (8.3.37) also shows (bearing (8.3.38) in mind) that $\|\cdot\|_{N_\kappa^p(\Omega;\mu)}$ satisfies the standard triangle inequality when $1 \le p \le \infty$. Let us also note that if $0 < p \le 1$ then (8.2.9) implies that for all $u, w \in N_\kappa^p(\Omega;\mu)$ we have

$$\|u+w\|_{N_\kappa^p(\Omega;\mu)}^p \le \|u\|_{N_\kappa^p(\Omega;\mu)}^p + \|w\|_{N_\kappa^p(\Omega;\mu)}^p, \tag{8.3.39}$$

proving (8.3.39). As is apparent from definitions, (8.3.32) is positive homogeneous of degree one, in the sense that

$$\|\lambda u\|_{N_\kappa^p(\Omega;\mu)} = \lambda\|u\|_{N_\kappa^p(\Omega;\mu)}, \qquad \forall u \in \mathscr{M}_+(\Omega,\mathcal{L}^n), \quad \forall \lambda \in (0,+\infty). \tag{8.3.40}$$

Recall from the proof of Lemma 8.3.1 that for each $x_* \in \Omega$, there exist $x \in \partial\Omega$ and $r > 0$, such that (8.3.3) holds. In concert with (8.3.30), this shows that the mapping (8.3.32) satisfies the non-degeneracy condition

$$\|u\|_{N_\kappa^p(\Omega;\mu)} = 0 \iff u = 0 \text{ at } \mathcal{L}^n\text{-a.e. point in } \Omega. \tag{8.3.41}$$

It is also clear that $\|\cdot\|_{N_\kappa^p(\Omega;\mu)}$ is monotone, meaning that for every two functions $u, w \in \mathscr{M}_+(\Omega,\mathcal{L}^n)$ we have

$$u \le w \text{ at } \mathcal{L}^n\text{-a.e. point in } \Omega \Longrightarrow \|u\|_{N_\kappa^p(\Omega;\mu)} \le \|w\|_{N_\kappa^p(\Omega;\mu)}. \tag{8.3.42}$$

Next, assume the functions $\{u_i\}_{i\in\mathbb{N}}$ are $\mathcal{L}^n$-measurable on Ω and for each $i \in \mathbb{N}$ satisfy $0 \leq u_i \leq u_{i+1}$ at $\mathcal{L}^n$-a.e. point in Ω. From (8.2.17) we know that for each point $x \in \partial\Omega$ we have

$$\mathcal{N}_\kappa\Big(\sup_{i\in\mathbb{N}} u_i\Big)(x) = \sup_{i\in\mathbb{N}}\big(\mathcal{N}_\kappa u_i\big)(x). \tag{8.3.43}$$

In turn, this implies

$$\begin{aligned}\Big\|\sup_{i\in\mathbb{N}} u_i\Big\|_{N^p_\kappa(\Omega;\mu)} &= \Big\|\mathcal{N}_\kappa\Big(\sup_{i\in\mathbb{N}} u_i\Big)\Big\|_{L^p(\partial\Omega,\mu)} = \Big\|\sup_{i\in\mathbb{N}}\big(\mathcal{N}_\kappa u_i\big)\Big\|_{L^p(\partial\Omega,\mu)} \\ &= \sup_{i\in\mathbb{N}}\Big\|\big(\mathcal{N}_\kappa u_i\big)\Big\|_{L^p(\partial\Omega,\mu)} = \sup_{i\in\mathbb{N}}\|u_i\|_{N^p_\kappa(\Omega;\mu)},\end{aligned} \tag{8.3.44}$$

where the third equality comes from Lemma 8.2.1 (in view of the fact that (8.2.19) is satisfied with $f_i := \mathcal{N}_\kappa u_i$). In particular, (8.3.44) proves that $\|\cdot\|_{N^p_\kappa(\Omega;\mu)}$ satisfies the weak Fatou property. Keeping in mind that the Lebesgue measure is sigma-finite, results proved in [188, 190] imply that $\big(N^p_\kappa(\Omega;\mu), \|\cdot\|_{N^p_\kappa(\Omega;\mu)}\big)$ is a quasi-Banach space (actually a genuine Banach space when $1 \leq p \leq \infty$) which embeds continuously into $L^0(\Omega, \mathcal{L}^n)$ (equipped with the topology induced by convergence in measure on sets of finite measure), and which has the property that any of its convergent sequences has a subsequence which converges (to its limit in $N^p_\kappa(\Omega;\mu)$) in a pointwise $\mathcal{L}^n$-a.e. fashion.

The latter property, together with the fact that $N^p_\kappa(\Omega;\mu)$ embeds continuously into $L^\infty_{\rm loc}(\Omega, \mathcal{L}^n)$ (as seen from the proof of Lemma 8.3.1; cf. (8.3.3)), then implies (8.3.34) (keeping in mind that continuity is preserved under uniform convergence).

There are precise two-sided estimates for the nontangential maximal operator of functions which are powers of the distance to a fixed point, of the sort described in the next lemma.

Lemma 8.3.7 *Suppose Ω is an arbitrary nonempty, open, proper subset of $\mathbb{R}^n$. Pick an aperture parameter $\kappa > 0$ and select an exponent $N \in [0, \infty)$.*

Then, having fixed a point $x_0 \in \overline{\Omega}$, there exists a constant $C = C_{N,\kappa} \in (0, \infty)$, such that for each $x \in \partial\Omega \setminus \{x_0\}$, one has

$$\begin{aligned}\Big(\mathcal{N}_\kappa^{\Omega\setminus\overline{B(x_0,d/2)}}\big(|\cdot - x_0|^{-N}\big)\Big)(x) &= \sup_{y\in\Gamma_\kappa(x)\setminus\overline{B(x_0,d/2)}} |y - x_0|^{-N} \\ &\leq C|x - x_0|^{-N}\end{aligned} \tag{8.3.45}$$

where $d := \operatorname{dist}(x_0, \partial\Omega) \geq 0$ (with the understanding that $B(x_0, d/2) = \varnothing$ if $d = 0$). Moreover, at each point $x \in \partial\Omega \setminus \{x_0\}$ with the property that $x \in \overline{\Gamma_\kappa(x)}$, one also has

$$|x-x_0|^{-N} \leq \sup_{y\in\Gamma_\kappa(x)\setminus\overline{B(x_0,d/2)}} |y-x_0|^{-N} = \Big(\mathcal{N}_\kappa^{\Omega\setminus\overline{B(x_0,d/2)}}\big(|\cdot - x_0|^{-N}\big)\Big)(x). \tag{8.3.46}$$

Finally, there exists a constant $C = C_{\Omega,N,\kappa} \in (0,\infty)$*, such that*

$$\Big(\mathcal{N}_\kappa^{\Omega}\big((1+|\cdot|)^{-N}\big)\Big)(x) \leq \frac{C}{1+|x|^N} \quad \textit{for each} \ \ x \in \partial\Omega, \tag{8.3.47}$$

and, in the opposite direction,

$$\frac{1}{1+|x|^N} \leq \Big(\mathcal{N}_\kappa^{\Omega}\big((1+|\cdot|)^{-N}\big)\Big)(x). \quad \textit{at each point}\, x \in \partial\Omega\, \textit{for which}\, x \in \overline{\Gamma_\kappa(x)}. \tag{8.3.48}$$

Proof Fix x_0 as in the hypotheses of the lemma. If $N = 0$, there is nothing to prove. Suppose $N > 0$ is fixed and let $x_* \in \partial\Omega$ be such that $|x_0 - x_*| = d$. Choose $M \in \big[2(2+\kappa),\infty\big)$ and take an arbitrary $x \in \partial\Omega \setminus B(x_*, Md)$. In particular, we have

$$|x - x_*| \geq Md. \tag{8.3.49}$$

Also, pick some $y \in \Gamma_\kappa(x)$, hence

$$|y - x| < (1+\kappa)\,\mathrm{dist}(y,\partial\Omega) \leq (1+\kappa)|y - x_*|. \tag{8.3.50}$$

We may then estimate

$$\begin{aligned} |x-x_*| &\leq |x-y| + |y-x_*| < (2+\kappa)|y-x_*| \\ &\leq (2+\kappa)|y-x_0| + (2+\kappa)|x_0-x_*| \\ &= (2+\kappa)|y-x_0| + (2+\kappa)d. \end{aligned} \tag{8.3.51}$$

Note that from (8.3.51) and (8.3.49), we obtain $|y-x_0| > \frac{Md-(2+\kappa)d}{2+\kappa} \geq d$, thus

$$\Gamma_\kappa(x) \cap \overline{B(x_0,d/2)} = \varnothing \ \text{ for all } \ x \in \partial\Omega \setminus B(x_*, Md). \tag{8.3.52}$$

Also, (8.3.51), (8.3.49), and the definition of M imply

$$\begin{aligned} (2+\kappa)|y-x_0| &\geq |x-x_*| - (2+\kappa)d \\ &\geq \tfrac{1}{2}|x-x_*| + \tfrac{Md}{2} - (2+\kappa)d \geq \tfrac{1}{2}|x-x_*|. \end{aligned} \tag{8.3.53}$$

In addition, using (8.3.49), we have

$$|x-x_0| \leq |x-x_*| + |x_*-x_0| = |x-x_*| + d \leq \big(1+\tfrac{1}{M}\big)|x-x_*|. \tag{8.3.54}$$

Together, (8.3.53) and (8.3.54) yield

$$|y - x_0| \geq \tfrac{1}{2(2+\kappa)}|x - x_*| \geq \tfrac{M}{2(2+\kappa)(M+1)}|x - x_0| \geq \tfrac{1}{M+1}|x - x_0|. \tag{8.3.55}$$

Since (8.3.55) holds for any $x \in \partial\Omega \setminus B(x_*, Md)$ and any $y \in \Gamma_\kappa(x)$, we obtain

$$\sup_{y\in\Gamma_\kappa(x)} \big[|y - x_0|^{-N}\big] \leq (M+1)^N |x - x_0|^{-N}, \qquad \forall x \in \partial\Omega \setminus B(x_*, Md). \tag{8.3.56}$$

Next, consider $x \in \partial\Omega \cap B(x_*, Md)$ and $y \in \Gamma_\kappa(x) \setminus \overline{B(x_0, d/2)}$. Then

$$|x - x_0| \leq |x - x_*| + |x_* - x_0| \leq (M+1)d, \tag{8.3.57}$$

thus

$$|y - x_0| \geq \tfrac{1}{2}d \geq \tfrac{1}{2(M+1)}|x - x_0|. \tag{8.3.58}$$

This further shows that at each point $x \in \big(\partial\Omega \setminus \{x_0\}\big) \cap B(x_*, Md)$, we have

$$\sup_{y\in\Gamma_\kappa(x)\setminus\overline{B(x_0,d/2)}} \big[|y - x_0|^{-N}\big] \leq 2^N(M+1)^N|x - x_0|^{-N}. \tag{8.3.59}$$

At this point, (8.3.45) follows from (8.3.56), (8.3.52), and (8.3.59), by choosing $C := 2^N(M+1)^N$.

Moving on, (8.3.46) is justified by observing that for every $x \in \partial\Omega \setminus \{x_0\}$ such that $x \in \overline{\Gamma_\kappa(x)}$, we may write

$$\sup_{y\in\Gamma_\kappa(x)\setminus\overline{B(x_0,d/2)}} \big[|y - x_0|^{-N}\big] \geq \lim_{\Gamma_\kappa(x)\ni y\to x} |y - x_0|^{-N} = |x - x_0|^{-N}. \tag{8.3.60}$$

Consider next the task of establishing (8.3.47). Pick an arbitrary point $x_0 \in \Omega$ and abbreviate $d := \operatorname{dist}(x_0, \partial\Omega) \in (0, \infty)$. Since $|x - x_0| \approx 1 + |x|$ uniformly for $x \in \mathbb{R}^n \setminus \overline{B(x_0, d/2)}$, we conclude from the estimate in (8.3.45) that there exists a constant $C = C_{\Omega,N,\kappa,x_0} \in (0, \infty)$ such that

$$\begin{aligned}\Big(\mathcal{N}_\kappa^{\Omega\setminus\overline{B(x_0,d/2)}}\big((1 + |\cdot|)^{-N}\big)\Big)(x) &\leq C\Big(\mathcal{N}_\kappa^{\Omega\setminus\overline{B(x_0,d/2)}}\big(|\cdot - x_0|^{-N}\big)\Big)(x) \\ &\leq C|x - x_0|^{-N} \leq \frac{C}{1 + |x|^N} \quad \text{for each } \ x \in \partial\Omega.\end{aligned} \tag{8.3.61}$$

From (8.1.18), we know that $E := \pi_\kappa\big(\overline{B(x_0, d/2)}\big)$ is a bounded subset of $\partial\Omega$, and from (8.2.26), we see that

$$\Big(\mathcal{N}_\kappa^{\overline{B(x_0,d/2)}}\big((1 + |\cdot|)^{-N}\big)\Big)(x) \leq C\mathbf{1}_E(x) \quad \text{for each } \ x \in \partial\Omega. \tag{8.3.62}$$

Since the boundedness of E entails

$$\mathbf{1}_E(x) \le \frac{C}{1+|x|^N} \quad \text{for each} \quad x \in \partial\Omega, \tag{8.3.63}$$

we ultimately conclude from (8.3.61), (8.3.62), (8.3.63), and (8.2.15) that (8.3.47) holds.

Finally, suppose some point $x \in \partial\Omega$ having the property that $x \in \overline{\Gamma_\kappa(x)}$ has been given. It is then possible to pick a sequence $\{x_j\}_{j\in\mathbb{N}} \subseteq \Gamma_\kappa(x)$ convergent to x, and we write

$$\begin{aligned}\frac{1}{1+|x|^N} &= \lim_{j\to\infty} \frac{1}{1+|x_j|^N} \le \sup_{j\in\mathbb{N}} \frac{1}{1+|x_j|^N} \\ &\le \sup_{y\in\Gamma_\kappa(x)} \frac{1}{1+|y|^N} = \Big(\mathcal{N}_\kappa^\Omega\big((1+|\cdot|)^{-N}\big)\Big)(x),\end{aligned} \tag{8.3.64}$$

where the last equality comes from (8.2.13) (presently used with $E := \Omega$). This establishes (8.3.48), completing the proof of Lemma 8.3.7. □

We continue by presenting the following useful estimate on the nontangential maximal function.

Proposition 8.3.8 *Let Ω be an arbitrary nonempty, open, proper subset of $\mathbb{R}^n$, and pick an aperture parameter $\kappa > 0$. Also, suppose $b \in \mathscr{C}^1(\mathbb{R}^n \setminus \{0\})$ is such that, for some constant $C \in (0,\infty)$,*

$$|b(x)| \le C|x|^{1-n} \quad \text{and} \quad |(\nabla b)(x)| \le C|x|^{-n} \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{8.3.65}$$

Finally, fix two distinct points $x_1, x_2 \in \partial\Omega$ and define

$$u(x) := b(x-x_1) - b(x-x_2) \quad \text{for each} \quad x \in \Omega. \tag{8.3.66}$$

Then there exists $C \in (0,\infty)$ with the property that for each $x \in \partial\Omega$ one has

$$\big(\mathcal{N}_\kappa u\big)(x) \le \begin{cases} C|x-x_j|^{1-n} & \text{if } x \text{ is near } x_j \text{ with } j=1 \text{ or } j=2, \\ C(1+|x|)^{-n} & \text{if } x \text{ is away from both } x_1 \text{ and } x_2. \end{cases} \tag{8.3.67}$$

In particular

$$\begin{gathered}\text{if } \partial\Omega \text{ is upper Ahlfors regular and } \sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega \text{ then} \\ \mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma) \quad \text{for each} \quad p \in \big(\tfrac{n-1}{n}, 1\big).\end{gathered} \tag{8.3.68}$$

Proof This is a consequence of Lemma 8.3.7, the Mean Value Theorem, and (7.2.5). □

In the lemma below, we estimate a "solid" integral, which may be regarded as the weighted Lebesgue norm of an arbitrary function (with the weight a power of the distance to the "boundary") by a "surface" integral, involving the nontangential maximal operator of the function in question and a suitably weighted "surface measure."

Lemma 8.3.9 *Suppose $\Sigma \subseteq \mathbb{R}^n$ is a nonempty closed set with empty interior, and denote by δ_Σ the distance function to Σ. Also, assume μ is a sigma-finite Borel measure on $\mathbb{R}^n \setminus \Sigma$ which is absolutely continuous with respect to the Lebesgue measure, and σ is a sigma-finite Borel measure on Σ with the property that there exist an exponent $d \in (0, \infty)$ and a constant $c \in (0, \infty)$, such that*

$$\sigma\big(B(x,r) \cap \Sigma\big) \geq cr^d \quad \textit{for all} \quad x \in \Sigma \quad \textit{and} \quad r \in \big(0, 2\,\mathrm{diam}\,\Sigma\big). \tag{8.3.69}$$

Finally, fix an aperture parameter $\kappa > 0$, an exponent $p \in (0, \infty)$, a μ-measurable set $E \subseteq \mathbb{R}^n \setminus \Sigma$, and a μ-measurable function $u : E \to \mathbb{C}$. Then there exists a constant $C \in (0, \infty)$, such that

$$\int_E \delta_\Sigma(x)^d |u(x)|^p \,\mathrm{d}\mu(x) \leq C \int_{\pi_\kappa(E)} \big(\mathcal{N}_\kappa^E u\big)(z)^p \cdot \mu\big(\Gamma_\kappa^E(z)\big)\,\mathrm{d}\sigma(z), \tag{8.3.70}$$

where the nontangential maximal operator $\mathcal{N}_\kappa^E$, the projection operator π_κ, and the nontangential approach regions $\Gamma_\kappa^E(\cdot)$ are all considered relative to the open ambient $\Sigma^c := \mathbb{R}^n \setminus \Sigma$.

Proof This is seen by applying (8.1.45) with the measure μ replaced by $\big|\widetilde{u}\,\big|^p \mu$ where $\widetilde{u}$ is the extension of u to $\mathbb{R}^n \setminus \Sigma$ by zero outside E, and using the fact that

$$\int_{\Gamma_\kappa(z)} \big|\widetilde{u}\,\big|^p \,\mathrm{d}\mu \leq \big(\mathcal{N}_\kappa^E u\big)(z)^p \cdot \mu\big(\Gamma_\kappa^E(z)\big) \quad \text{for each} \quad z \in \Sigma, \tag{8.3.71}$$

itself a consequence of (8.2.4), the absolute continuity of μ with respect to $\mathcal{L}^n$, and (8.1.34). □

8.4 Size Estimates for the Nontangential Maximal Operator Involving a Doubling Measure

Proposition 8.4.1 below shows that the choice of the parameter κ, governing the aperture of the nontangential approach regions (cf. (8.1.2)), plays a relatively minor role when measuring the size of the nontangential maximal function on the scale of Lorentz spaces with respect to a doubling measure. This refines work in [125, Proposition 2.2], which deals with an open set with an Ahlfors regular boundary and the scale of Lebesgue spaces on its topological boundary considered with respect to the canonical surface measure. Later on, in Theorem 8.4.6 and Corollaries 8.4.7–8.4.8, we shall present more versatile versions of this result, applicable to a variety of function spaces of interest.

Proposition 8.4.1 *Assume that Ω is an open nonempty proper subset of $\mathbb{R}^n$ and consider a doubling Borel measure σ on $\partial\Omega$. Also, fix a pair of exponents p, q such that either $p \in (0,\infty)$ and $q \in (0,\infty]$, or $p = \infty$ and $q = \infty$.*

Then for every pair of aperture parameters $\kappa_1, \kappa_2 > 0$ there exist two finite constants $C_0, C_1 > 0$, which depend only on κ_1, κ_2, p, q, and the doubling character of σ, such that

$$C_0 \|\mathcal{N}^E_{\kappa_1} u\|_{L^{p,q}(\partial\Omega,\sigma)} \leq \|\mathcal{N}^E_{\kappa_2} u\|_{L^{p,q}(\partial\Omega,\sigma)} \leq C_1 \|\mathcal{N}^E_{\kappa_1} u\|_{L^{p,q}(\partial\Omega,\sigma)}, \tag{8.4.1}$$

for each Lebesgue measurable set $E \subseteq \Omega$ and each Lebesgue measurable function $u : \Omega \to \mathbb{C}$.

In particular

$$\begin{aligned} C_0 \|\mathcal{N}^E_{\kappa_1} u\|_{L^p(\pi_{\kappa_1}(E),\sigma)} &\leq \|\mathcal{N}^E_{\kappa_2} u\|_{L^p(\pi_{\kappa_2}(E),\sigma)} \\ &\leq C_1 \|\mathcal{N}^E_{\kappa_1} u\|_{L^p(\pi_{\kappa_1}(E),\sigma)}. \end{aligned} \tag{8.4.2}$$

Proof The case $p = q = \infty$ (in which scenario $L^{\infty,\infty}(\partial\Omega,\sigma) = L^\infty(\partial\Omega,\sigma)$) is seen directly from Lemma 8.3.2 (applied to the function $u\mathbf{1}_E$).

There remain to treat the case when $p \in (0,\infty)$ and $q \in (0,\infty]$, a scenario in which we shall adapt a point-of-density argument of Fefferman and Stein [90] (cf. also p. 62 in [242]). Specifically, fix $\lambda > 0$ and, for each $\kappa > 0$, define

$$\mathcal{O}_{\kappa,\lambda} := \big\{x \in \partial\Omega : \big(\mathcal{N}^E_\kappa u\big)(x) > \lambda\big\}. \tag{8.4.3}$$

By (8.2.27), this is a relatively open subset of $\partial\Omega$. As a consequence, $A := \partial\Omega \setminus \mathcal{O}_{\kappa_1,\lambda}$ is relatively closed (in the topology induced by the Euclidean ambient on $\partial\Omega$). For each $\gamma \in (0,1)$, consider

$$A^*_\gamma := \big\{x \in \partial\Omega : \sigma(A \cap \Delta(x,r)) \geq \gamma\sigma(\Delta(x,r)) \ \text{ for all } \ r > 0\big\}, \tag{8.4.4}$$

where we have set

$$\Delta(x,r) := B(x,r) \cap \partial\Omega, \quad \text{for each } x \in \partial\Omega \text{ and } r > 0. \tag{8.4.5}$$

That is, A^*_γ is the collection of points of (global) γ-density for the set A, relative to the measure σ. We now claim that there exists $\gamma \in (0,1)$, such that

$$\mathcal{O}_{\kappa_2,\lambda} \subseteq \partial\Omega \setminus A^*_\gamma. \tag{8.4.6}$$

To justify this inclusion, fix an arbitrary point $x \in \mathcal{O}_{\kappa_2,\lambda}$. Then, necessarily, we have $x \in \partial\Omega$ and $\|u\|_{L^\infty(\Gamma_{\kappa_2}(x)\cap E,\mathcal{L}^n)} = \big(\mathcal{N}^E_{\kappa_2} u\big)(x) > \lambda$ which entails that there exists some small $\varepsilon > 0$ and a set $U \subseteq \Gamma_{\kappa_2}(x) \cap E$ with the property that $\mathcal{L}^n(U) > 0$ and $|u| > \lambda + \varepsilon$ on U. Since obviously

$$\Gamma_{\kappa_2}(x) \cap E \subseteq \bigcup_{y \in \Gamma_{\kappa_2}(x) \cap E} B\big(y, \varepsilon\delta_{\partial\Omega}(y)\big), \tag{8.4.7}$$

by Lindelöff's theorem, we may refine this union to a countable one. As a consequence, there exists a family of points $\{y_j\}_{j\in\mathbb{N}} \subset \Gamma_{\kappa_2}(x) \cap E$ with the property that $U \subseteq \bigcup_{j=1}^{\infty} B(y_j, \varepsilon\delta_{\partial\Omega}(y_j))$. In particular,

$$0 < \mathcal{L}^n(U) \leq \sum_{j=1}^{\infty} \mathcal{L}^n\big(U \cap B\big(y_j, \varepsilon\delta_{\partial\Omega}(y_j)\big)\big) \tag{8.4.8}$$

which forces that $\mathcal{L}^n\big(U \cap B(y_{j_o}, \varepsilon\delta_{\partial\Omega}(y_{j_o}))\big) > 0$ for some $j_o \in \mathbb{N}$. By further decreasing ε as needed, we may conclude that

> there exist $\varepsilon \in \big(0, \frac{\kappa_1(1+\kappa_1)}{2+\kappa_1}\big)$ and $y \in \Gamma_{\kappa_2}(x) \cap E$ with the property that $|u| > \lambda + \varepsilon$ on a subset of positive Lebesgue measure of $B\big(y, \varepsilon\delta_{\partial\Omega}(y)\big) \cap E$. (8.4.9)

Next, select $y^* \in \partial\Omega$ with $|y - y^*| = \delta_{\partial\Omega}(y)$. We now make two observations of a purely geometric nature. First

$$z \in \Delta\big(y^*, \kappa_1(2+\kappa_1)^{-1}\delta_{\partial\Omega}(y)\big) \Rightarrow B\big(y, \varepsilon\delta_{\partial\Omega}(y)\big) \subseteq \Gamma_{\kappa_1}(z). \tag{8.4.10}$$

Indeed, if $z \in \partial\Omega$ with $|z - y^*| < \kappa_1(2+\kappa_1)^{-1}\delta_{\partial\Omega}(y)$ and $|w - y| < \varepsilon\delta_{\partial\Omega}(y)$, then

$$\begin{aligned} |z - w| &\leq |z - y^*| + |y^* - y| + |y - w| \\ &< \kappa_1(2+\kappa_1)^{-1}\delta_{\partial\Omega}(y) + |y - y^*| + \varepsilon\delta_{\partial\Omega}(y) \\ &< (1+\kappa_1)\delta_{\partial\Omega}(y), \end{aligned} \tag{8.4.11}$$

i.e., $w \in \Gamma_{\kappa_1}(z)$, as desired.

Our second observation is that

$$\Delta\big(y^*, \kappa_1\delta_{\partial\Omega}(y)\big) \subseteq \Delta\big(x, (2+\kappa_1+\kappa_2)\delta_{\partial\Omega}(y)\big). \tag{8.4.12}$$

To see this, we note that if $z \in \partial\Omega$ and $|z - y^*| < \kappa_1\delta_{\partial\Omega}(y)$, then, since $y \in \Gamma_{\kappa_2}(x)$, we may write

$$\begin{aligned} |x - z| &\leq |x - y| + |y - y^*| + |y^* - z| \\ &< (1+\kappa_2)\delta_{\partial\Omega}(y) + \delta_{\partial\Omega}(y) + \kappa_1\delta_{\partial\Omega}(y) \\ &= (2+\kappa_1+\kappa_2)\delta_{\partial\Omega}(y), \end{aligned} \tag{8.4.13}$$

proving (8.4.12).

To proceed, note that since $\kappa_1(2+\kappa_1)^{-1} < \kappa_1$, by combining (8.4.10), (8.4.12), and (8.4.9), we obtain

$$\Delta\big(y^*, \kappa_1(2+\kappa_1)^{-1}\delta_{\partial\Omega}(y)\big) \subseteq \mathcal{O}_{\kappa_1,\lambda} \cap \Delta\big(x, (2+\kappa_1+\kappa_2)\delta_{\partial\Omega}(y)\big). \tag{8.4.14}$$

Hence, on account of (8.4.14) and (8.4.12), we may estimate

$$\begin{aligned}\frac{\sigma\Big(\mathcal{O}_{\kappa_1,\lambda} \cap \Delta\big(x, (2+\kappa_1+\kappa_2)\delta_{\partial\Omega}(y)\big)\Big)}{\sigma\Big(\Delta\big(x, (2+\kappa_1+\kappa_2)\delta_{\partial\Omega}(y)\big)\Big)} &\geq \frac{\sigma\Big(\Delta\big(y^*, \kappa_1(2+\kappa_1)^{-1}\delta_{\partial\Omega}(y)\big)\Big)}{\sigma\Big(\Delta\big(x, (2+\kappa_1+\kappa_2)\delta_{\partial\Omega}(y)\big)\Big)} \\ &\geq c\left(\frac{\kappa_1(2+\kappa_1)^{-1}}{2+\kappa_1+\kappa_2}\right)^{D_\sigma}, \end{aligned} \tag{8.4.15}$$

where D_σ is the doubling order of σ (cf. (7.4.3)) and the constant $c \in (0, \infty)$ depends only on σ. In particular, if we abbreviate

$$r := (2+\kappa_1+\kappa_2)\delta_{\partial\Omega}(y), \tag{8.4.16}$$

then

$$\frac{\sigma\big(A \cap \Delta(x, r)\big)}{\sigma\big(\Delta(x, r)\big)} \leq 1 - c\left(\frac{\kappa_1(2+\kappa_1)^{-1}}{2+\kappa_1+\kappa_2}\right)^{D_\sigma}. \tag{8.4.17}$$

Thus, if we select γ, such that

$$1 - c\left(\frac{\kappa_1(2+\kappa_1)^{-1}}{2+\kappa_1+\kappa_2}\right)^{D_\sigma} < \gamma < 1, \tag{8.4.18}$$

then (8.4.17) entails $x \notin A^*_\gamma$. This proves the claim (8.4.6).

To proceed, bring in the Hardy–Littlewood maximal operator, $\mathcal{M}_{\partial\Omega}$, associated with the space of homogeneous type $(\partial\Omega, |\cdot - \cdot|, \sigma)$, acting on functions f in the space $L^1_{\mathrm{loc}}(\partial\Omega, \sigma)$ according to

$$\big(\mathcal{M}_{\partial\Omega} f\big)(x) := \sup_{r>0} \frac{1}{\sigma(\Delta(x, r))} \int_{\Delta(x,r)} |f| \, d\sigma, \qquad \forall x \in \partial\Omega. \tag{8.4.19}$$

Cf. the discussion in Sect. 7. Then, based on (8.4.6) and Corollary 7.6.3, we may write

$$\begin{aligned}\sigma(\mathcal{O}_{\kappa_2,\lambda}) \leq \sigma(\partial\Omega \setminus A^*_\gamma) &= \sigma\Big(\{x \in \partial\Omega : \mathcal{M}_{\partial\Omega}(\mathbf{1}_{\partial\Omega\setminus A})(x) > 1-\gamma\}\Big) \\ &\leq \frac{C}{1-\gamma}\,\sigma(\partial\Omega \setminus A) = C(\partial\Omega, \sigma, \gamma)\,\sigma(\mathcal{O}_{\kappa_1,\lambda}). \end{aligned} \tag{8.4.20}$$

Hence,

$$\sigma(\mathcal{O}_{\kappa_2,\lambda}) \leq C(\partial\Omega, \sigma, \kappa_1, \kappa_2)\,\sigma(\mathcal{O}_{\kappa_1,\lambda}), \tag{8.4.21}$$

which shows that for each $\lambda > 0$, we have

$$\sigma\big(\{x \in \partial\Omega : \big(\mathcal{N}^E_{\kappa_2} u\big)(x) > \lambda\}\big) \leq C\sigma\big(\{x \in \partial\Omega : \big(\mathcal{N}^E_{\kappa_1} u\big)(x) > \lambda\}\big), \tag{8.4.22}$$

where $C = C(\partial\Omega, \sigma, \kappa_1, \kappa_2) \in (0, \infty)$. Utilizing the abstract piece of notation from (6.2.1) in relation to the measure space $(\partial\Omega, \sigma)$ then permits us to write

$$m_{\partial\Omega}\big(\lambda, \mathcal{N}^E_{\kappa_2} u\big) \leq C m_{\partial\Omega}\big(\lambda, \mathcal{N}^E_{\kappa_1} u\big), \quad \forall \lambda > 0. \tag{8.4.23}$$

Consequently, for every $t > 0$

$$\big\{\lambda > 0 : m_{\partial\Omega}\big(\lambda, \mathcal{N}^E_{\kappa_2} u\big) \leq t\big\} \subseteq \big\{\lambda > 0 : m_{\partial\Omega}\big(\lambda, \mathcal{N}^E_{\kappa_1} u\big) \leq t\big\}. \tag{8.4.24}$$

In view of (6.2.2), this implies the rearrangement function estimate

$$\big(\mathcal{N}^E_{\kappa_1} u\big)^*_{\partial\Omega}(t) \leq \big(\mathcal{N}^E_{\kappa_2} u\big)^*_{\partial\Omega}(t), \qquad \forall t > 0. \tag{8.4.25}$$

Having established this, then invoking (6.2.14) yields

$$\big\|\mathcal{N}^E_{\kappa_1} u\big\|_{L^{p,q}(\partial\Omega,\sigma)} \leq C\big\|\mathcal{N}^E_{\kappa_2} u\big\|_{L^{p,q}(\partial\Omega,\sigma)}, \tag{8.4.26}$$

for some finite constant $C > 0$, independent of u and E. Changing the roles of κ_1 and κ_2 then finishes the proof of (8.4.1). □

Before stating our next result, the reader is reminded that the truncated nontangential maximal operator $\mathcal{N}^\varepsilon_\kappa$ has been defined in (8.2.45).

Corollary 8.4.2 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ and suppose σ is a doubling Borel measure on $\partial\Omega$. Fix an integrability exponent $p \in (0, \infty]$ along with a truncation parameter $\varepsilon \in (0, \infty)$, and consider an arbitrary Lebesgue measurable function $u : \Omega \to \mathbb{C}$. Then*

$$\mathcal{N}^\varepsilon_{\kappa_1} u \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma) \;\; \textit{if and only if} \;\; \mathcal{N}^\varepsilon_{\kappa_2} u \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma). \tag{8.4.27}$$

Proof To fix ideas, suppose $\mathcal{N}^\varepsilon_{\kappa_1} u \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma)$ and consider some $x_o \in \partial\Omega$ and $r \in (0, \infty)$. Abbreviate $\mathcal{O}_\varepsilon := \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) < \varepsilon\big\}$, and $\Delta := B(x_o, r) \cap \partial\Omega$, then define

$$E := \Big(\bigcup_{x\in\Delta} \Gamma_{\kappa_2}(x)\Big) \cap \mathcal{O}_\varepsilon. \tag{8.4.28}$$

It follows that E is a bounded open subset of Ω. Also, since $\Gamma_{\kappa_2}(x) \cap \mathcal{O}_\varepsilon = \Gamma_{\kappa_2}(x) \cap E$ for every $x \in \Delta$, from (8.2.45) and the equality in (8.2.14), we conclude that

$$\big(\mathcal{N}^\varepsilon_{\kappa_2} u\big)(x) = \big(\mathcal{N}^E_{\kappa_2} u\big)(x) \;\; \text{for each} \;\; x \in \Delta. \tag{8.4.29}$$

Also, since $E \subseteq \mathcal{O}_\varepsilon$, we have

$$\big(\mathcal{N}^E_{\kappa_1} u\big)(x) \leq \big(\mathcal{N}^\varepsilon_{\kappa_1} u\big)(x) \;\; \text{for each} \;\; x \in \partial\Omega. \tag{8.4.30}$$

On account of (8.4.29)–(8.4.30), Proposition 8.4.1, and (8.2.26), we may then write

$$
\begin{aligned}
\|\mathcal{N}^{\varepsilon}_{\kappa_2}u\|_{L^p(\Delta,\sigma)} &= \|\mathcal{N}^{E}_{\kappa_2}u\|_{L^p(\Delta,\sigma)} \leq \|\mathcal{N}^{E}_{\kappa_2}u\|_{L^p(\partial\Omega,\sigma)} \\
&\leq C\|\mathcal{N}^{E}_{\kappa_1}u\|_{L^p(\partial\Omega,\sigma)} \leq C\|\mathcal{N}^{\varepsilon}_{\kappa_1}u\|_{L^p(\partial\Omega,\sigma)} \\
&= C\|\mathcal{N}^{\varepsilon}_{\kappa_1}u\|_{L^p(\pi_{\kappa_1}(E),\sigma)} < +\infty, \qquad (8.4.31)
\end{aligned}
$$

with the final inequality a consequence of (8.1.18), (8.1.16), and the assumption that we have $\mathcal{N}^{\varepsilon}_{\kappa_1}u \in L^p_{\text{loc}}(\partial\Omega,\sigma)$. This justifies one of the implications in (8.4.27), and the other one is proved in a similar fashion. □

It turns out that Proposition 8.4.1 is in the nature of best possible, in the sense that, given a Borel measure σ which is positive and finite on balls in $\mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+$, the comparability of the Lebesgue norms of nontangential maximal operators with arbitrary apertures, acting on measurable functions defined in the upper half-space $\mathbb{R}^n_+$, is equivalent to σ being doubling. Specifically, we have the following result.

Proposition 8.4.3 *Fix $n \in \mathbb{N}$ with $n \geq 2$, and suppose σ is a Borel measure on $\mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+$ with the property that*

$$
0 < \sigma\big(B_{n-1}(x',r)\big) < +\infty, \qquad \forall x' \in \mathbb{R}^{n-1}, \quad \forall r > 0, \qquad (8.4.32)
$$

where $B_{n-1}(x',r) := \big\{y' \in \mathbb{R}^{n-1} : |x'-y'| < r\big\}$ is the $(n-1)$-dimensional ball centered at x' and of radius r. Also, pick some integrability exponent $p \in (0,\infty)$.

Then the measure σ is doubling if and only if for each $\kappa_1, \kappa_2 > 0$, one has

$$
\begin{aligned}
&\|\mathcal{N}_{\kappa_1}u\|_{L^p(\mathbb{R}^{n-1},\sigma)} \approx \|\mathcal{N}_{\kappa_2}u\|_{L^p(\mathbb{R}^{n-1},\sigma)} \ \text{ uniformly for} \\
&u : \mathbb{R}^n_+ \longrightarrow \mathbb{C} \text{ Lebesgue measurable function.}
\end{aligned} \qquad (8.4.33)
$$

Proof If σ is doubling then Proposition 8.4.1 ensures that (8.4.33) holds. Conversely, suppose (8.4.33) holds and fix $x' \in \mathbb{R}^{n-1}$ and $r > 0$ arbitrary. Simple geometric arguments show that for each $\kappa > 0$, we have

$$
\mathcal{N}_\kappa \mathbf{1}_{B((x',0),r)\cap\mathbb{R}^n_+} = \mathbf{1}_{B_{n-1}(x',r/((1+\kappa))} \ \text{ on } \ \mathbb{R}^{n-1} \qquad (8.4.34)
$$

hence

$$
\begin{aligned}
\big\|\mathcal{N}_\kappa \mathbf{1}_{B((x',0),r)\cap\mathbb{R}^n_+}\big\|^p_{L^p(\mathbb{R}^{n-1},\sigma)} &= \big\|\mathbf{1}_{B_{n-1}(x',r/(1+\kappa))}\big\|^p_{L^p(\mathbb{R}^{n-1},\sigma)} \\
&= \sigma\big(B_{n-1}(x',r/(1+\kappa))\big). \qquad (8.4.35)
\end{aligned}
$$

From this and (8.4.33), we then conclude that for each $\kappa_1, \kappa_2 > 0$ fixed, we have

$$
\sigma\big(B_{n-1}(x',r/(1+\kappa_1))\big) \approx \sigma\big(B_{n-1}(x',r/(1+\kappa_2))\big), \qquad (8.4.36)
$$

uniformly in $x' \in \mathbb{R}^{n-1}$ and $r > 0$. Changing r into $2r$ and taking $\kappa_1 = 1$, $\kappa_2 = 3$ then yields

$$\sigma\big(B_{n-1}(x',r)\big) \approx \sigma\big(B_{n-1}(x',r/2)\big), \quad \text{uniformly in } x' \in \mathbb{R}^{n-1} \text{ and } r > 0 \tag{8.4.37}$$

which, in concert with (8.4.32), proves that σ is a doubling measure on $\mathbb{R}^{n-1}$ (cf. (7.4.1)). □

We continue by presenting a weighted version of Proposition 8.4.1.

Corollary 8.4.4 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ and consider a doubling Borel measure σ on $\partial\Omega$. Also, fix a Muckenhoupt weight $w \in A_p(\partial\Omega, \sigma)$ with $p \in [1, \infty)$. Then for each $\kappa_1, \kappa_2 \in (0, \infty)$ there exist two finite constants $C_0, C_1 > 0$, which depend only on κ_1, κ_2, p, q, the doubling character of σ, and $[w]_{A_p}$, such that*

$$C_0\|\mathcal{N}^E_{\kappa_1}u\|_{L^p(\partial\Omega,w\sigma)} \leq \|\mathcal{N}^E_{\kappa_2}u\|_{L^p(\partial\Omega,w\sigma)} \leq C_1\|\mathcal{N}^E_{\kappa_1}u\|_{L^p(\partial\Omega,w\sigma)} \tag{8.4.38}$$

for each Lebesgue measurable set $E \subseteq \Omega$ and each Lebesgue measurable function $u : \Omega \to \mathbb{C}$.

Proof From item *(6)* in Lemma 7.7.1, applied to the space of homogeneous type $(\partial\Omega, |\cdot - \cdot|, \sigma)$, we know that $w\,\sigma$ is a doubling measure on $\partial\Omega$. Granted this, the estimates claimed in (8.4.38) are implied by Proposition 8.4.1. □

In turn, Corollary 8.4.4 is one of the main ingredients in the proof of the following version of Proposition 8.4.1, in the context of Morrey spaces.

Corollary 8.4.5 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ (where $n \in \mathbb{N}$ satisfies $n \geq 2$) with the property that $\partial\Omega$ is Ahlfors regular. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix an integrability exponent $p \in (1, \infty)$ along with a parameter $\lambda \in (0, n-1)$.*

Then for each $\kappa_1, \kappa_2 \in (0, \infty)$, there exist two finite constants $C_0, C_1 > 0$, which depend only on $\kappa_1, \kappa_2, p, \lambda$ and the Ahlfors regularity character of $\partial\Omega$, such that

$$C_0\|\mathcal{N}^E_{\kappa_1}u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq \|\mathcal{N}^E_{\kappa_2}u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C_1\|\mathcal{N}^E_{\kappa_1}u\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \tag{8.4.39}$$

for each Lebesgue measurable set $E \subseteq \Omega$ and each Lebesgue measurable function $u : \Omega \to \mathbb{C}$.

Proof In [185, Sect. 6.2], it is shown that, for a given pair of measurable functions, estimates in Muckenhoupt weighted Lebesgue spaces imply estimates in Morrey spaces. The conclusion we currently seek follows from this result applied to the pair of functions $f := \mathcal{N}^E_{\kappa_1}u$ and $g := \mathcal{N}^E_{\kappa_2}u$, while bearing Corollary 8.4.4 in mind. □

The proof of Proposition 8.4.1 is based on a "point-of-density" argument, whose main output is the level set estimate recorded in (8.4.22). Here we develop an alternative approach to said level set estimate which has the distinct benefit of also producing *pointwise* estimates for the nontangential maximal operators corresponding to various aperture parameters. The aforementioned pointwise estimates pin the nontangential maximal operator with a larger aperture in between the nontangential maximal operator with a smaller aperture and the action of the Hardy–Littlewood maximal function on the latter (see (8.4.41) below for a precise formulation). In particular, as we shall show a little later in Corollaries 8.4.7–8.4.8, this allows us to recover all results obtained earlier in Proposition 8.4.1, Corollary 8.4.4, and Corollary 8.4.5 and, significantly, also consider new function spaces, for which these results are not applicable. Here is the actual statement of Theorem 8.4.6 which, de facto, is our main result in this section.

Theorem 8.4.6 *Fix $n \in \mathbb{N}$ satisfying $n \geq 2$, and let Ω be an arbitrary open (nonempty, proper) subset of $\mathbb{R}^n$. Also, let σ be a doubling Borel measure on $\partial\Omega$ and, having selected an exponent $s \in (0, \infty)$, bring in the L^s-based Hardy–Littlewood maximal operator associated with $\partial\Omega$ and σ as in (7.6.7), acting on each σ-measurable function $f : \partial\Omega \to \mathbb{C}$ according to*

$$\mathcal{M}_{\partial\Omega,s} f(x) := \sup_{r>0} \Big(\fint_{B(x,r)\cap\partial\Omega} |f|^s \, d\sigma \Big)^{\frac{1}{s}}, \qquad \forall x \in \partial\Omega. \tag{8.4.40}$$

Lastly, pick two aperture parameters $\widetilde{\kappa}, \kappa \in (0, \infty)$ satisfying $\widetilde{\kappa} \geq \kappa$.

Then there exists a constant $C \in (0, \infty)$ which depends only on $\widetilde{\kappa}, \kappa, s$ and the doubling character of σ with the property that

$$\mathcal{N}_\kappa^E u \leq \mathcal{N}_{\widetilde{\kappa}}^E u \leq C \cdot \mathcal{M}_{\partial\Omega,s}\big(\mathcal{N}_\kappa^E u\big) \ \text{ at each point on } \ \partial\Omega, \tag{8.4.41}$$

for each Lebesgue measurable set $E \subseteq \Omega$ and each Lebesgue measurable function $u : \Omega \to \mathbb{C}$. In particular, corresponding to $E := \Omega$,

$$\mathcal{N}_\kappa u \leq \mathcal{N}_{\widetilde{\kappa}} u \leq C \cdot \mathcal{M}_{\partial\Omega,s}\big(\mathcal{N}_\kappa u\big) \ \text{ at each point on } \ \partial\Omega, \tag{8.4.42}$$

for each Lebesgue measurable function $u : \Omega \to \mathbb{C}$. Furthermore, corresponding to $s = 1$, for each Lebesgue measurable function $u : \Omega \to \mathbb{C}$, one has

$$\mathcal{N}_\kappa u \leq \mathcal{N}_{\widetilde{\kappa}} u \leq C \cdot \mathcal{M}_{\partial\Omega}\big(\mathcal{N}_\kappa u\big) \ \text{ at each point on } \ \partial\Omega, \tag{8.4.43}$$

where $\mathcal{M}_{\partial\Omega}$ is the standard Hardy–Littlewood maximal operator associated with $\partial\Omega$ and σ as in (7.6.16), i.e., the mapping acting on each σ-measurable function $f : \partial\Omega \to \mathbb{C}$ according to

$$\mathcal{M}_{\partial\Omega} f(x) := \sup_{r>0} \fint_{B(x,r)\cap\partial\Omega} |f| \, d\sigma, \qquad \forall x \in \partial\Omega. \tag{8.4.44}$$

Also, for some constant $C \in (0, \infty)$ which depends only on $\widetilde{\kappa}, \kappa, s$ and the doubling character of σ, one has

$$\mathcal{N}_\kappa^\rho u \leq \mathcal{N}_{\widetilde{\kappa}}^\rho u \leq C \cdot \mathcal{M}_{\partial\Omega,s}^{C\rho}\big(\mathcal{N}_\kappa^\rho u\big) \ \text{ at each point on } \ \partial\Omega, \tag{8.4.45}$$

for each $\rho > 0$ and each Lebesgue measurable function $u : \Omega \to \mathbb{C}$, where the local L^s-based Hardy–Littlewood maximal operator $\mathcal{M}^R_{\partial\Omega,s}$ with $R \in (0, \infty)$ acts on a σ-measurable function $f : \partial\Omega \to \mathbb{C}$ according to

$$\mathcal{M}^R_{\partial\Omega,s} f(x) := \sup_{0<r\leq R} \Big(⨍_{B(x,r)\cap\partial\Omega} |f|^s \, d\sigma \Big)^{\frac{1}{s}}, \qquad \forall x \in \partial\Omega. \tag{8.4.46}$$

Finally, there exists a constant $C \in (0, \infty)$ which depends only on $\widetilde{\kappa}$, κ and the doubling character of σ with the following significance. Given a Lebesgue measurable set $E \subseteq \Omega$ along with a Lebesgue measurable function $u : \Omega \to \mathbb{C}$, if for each $\lambda \in (0, \infty)$, one defines

$$\begin{aligned} \mathcal{U}^E_\kappa(\lambda) &:= \big\{x \in \partial\Omega : \big(\mathcal{N}^E_\kappa u\big)(x) > \lambda\big\}, \\ \mathcal{U}^E_{\widetilde{\kappa}}(\lambda) &:= \big\{x \in \partial\Omega : \big(\mathcal{N}^E_{\widetilde{\kappa}} u\big)(x) > \lambda\big\}, \end{aligned} \tag{8.4.47}$$

then for each $\lambda \in (0, \infty)$, one has

$$\mathbf{1}_{\mathcal{U}^E_{\widetilde{\kappa}}(\lambda)} \leq C \cdot \mathcal{M}_{\partial\Omega}\big(\mathbf{1}_{\mathcal{U}^E_\kappa(\lambda)}\big) \ \textit{ at each point on } \ \partial\Omega. \tag{8.4.48}$$

As a consequence of this and the weak-$(1, 1)$ boundedness of the Hardy–Littlewood maximal operator $\mathcal{M}_{\partial\Omega}$,

$$\sigma\big(\mathcal{U}^E_{\widetilde{\kappa}}(\lambda)\big) \leq C \cdot \sigma\big(\mathcal{U}^E_\kappa(\lambda)\big) \quad \forall \lambda > 0, \tag{8.4.49}$$

for each Lebesgue measurable set $E \subseteq \Omega$ and each Lebesgue measurable function $u : \Omega \to \mathbb{C}$. In the special case when $E := \Omega$, one, therefore, has

$$\begin{gathered} \sigma\Big(\big\{x \in \partial\Omega : \big(\mathcal{N}_{\widetilde{\kappa}} u\big)(x) > \lambda\big\}\Big) \leq C \cdot \sigma\Big(\big\{x \in \partial\Omega : \big(\mathcal{N}_\kappa u\big)(x) > \lambda\big\}\Big) \\ \textit{for each Lebesgue measurable function } u : \Omega \to \mathbb{C} \textit{ and each } \lambda > 0. \end{gathered} \tag{8.4.50}$$

It is possible to be more precise about the nature of the constant $C \in (0, \infty)$ appearing in (8.4.41). Indeed, a glance at the proof below shows that we may take

$$C = C_{\sigma,\kappa,s} \cdot \widetilde{\kappa}^{D_\sigma/s} \tag{8.4.51}$$

where $C_{\sigma,\kappa,s} \in (0, \infty)$ is independent of $\widetilde{\kappa}$, and $D_\sigma \in [0, \infty)$ is the doubling order of the measure σ (defined as in (7.4.3)). In fact, as seen from the proof of Theorem 8.4.6 presented below (cf. (8.4.61), (8.4.74), and (7.4.5) used with $\lambda := (4 + \varepsilon)/\varepsilon$),

in the particular case when $\partial\Omega$ is an Ahlfors regular set and the doubling measure σ is actually $\mathcal{H}^{n-1}\lfloor\partial\Omega$, the constant $C \in (0, \infty)$ appearing in (8.4.41) may be taken to be of the form $C = C_{\partial\Omega,n,\kappa,s} \cdot \widetilde{\kappa}^{(n-1)/s}$, where $C_{\partial\Omega,n,\kappa,s} \in (0, \infty)$ is independent of $\widetilde{\kappa} \in [\kappa, \infty)$. (8.4.52)

We now turn to the proof of Theorem 8.4.6.

Proof of Theorem 8.4.6 Throughout, we agree to abbreviate

$$\Delta(x_o, R) := B(x_o, R) \cap \partial\Omega \quad \text{for each} \quad x_o \in \partial\Omega \quad \text{and} \quad R \in (0, \infty). \tag{8.4.53}$$

To start in earnest, consider a Lebesgue measurable function $u : \Omega \to \mathbb{C}$. Fix an arbitrary point $x \in \partial\Omega$, with the goal of proving that

$$\big(\mathcal{N}_{\widetilde{\kappa}} u\big)(x) \leq C \cdot \Big(\mathcal{M}_{\partial\Omega,s}\big(\mathcal{N}_{\kappa} u\big)\Big)(x) \tag{8.4.54}$$

for some constant $C \in (0, \infty)$ which depends only on $\widetilde{\kappa}$, κ and the doubling character of σ. If the left side in (8.4.54) is zero, there is nothing to prove, so assume $\big(\mathcal{N}_{\widetilde{\kappa}} u\big)(x) > 0$. Choose a number λ with

$$0 < \lambda < \big(\mathcal{N}_{\widetilde{\kappa}} u\big)(x), \tag{8.4.55}$$

otherwise arbitrary, and abbreviate

$$A_\lambda := \big\{y \in \Gamma_{\widetilde{\kappa}}(x) : \ |u(y)| > \lambda\big\}. \tag{8.4.56}$$

Note that A_λ is a Lebesgue measurable set, given that u is Lebesgue measurable and $\Gamma_{\widetilde{\kappa}}(x)$ is an open subset of Ω. Since (8.4.55) entails $\lambda < \|u\|_{L^\infty(\Gamma_{\widetilde{\kappa}}(x), \mathcal{L}^n)}$, it follows that

$$\mathcal{L}^n(A_\lambda) > 0. \tag{8.4.57}$$

In turn, this ensures that there exists a point $y_* \in A_\lambda$ with density 1, i.e., satisfying

$$\lim_{\rho \to 0^+} \frac{\mathcal{L}^n\big(A_\lambda \cap B(y_*, \rho)\big)}{\mathcal{L}^n\big(B(y_*, \rho)\big)} = 1. \tag{8.4.58}$$

Pick a point

$$z_* \in \partial\Omega \quad \text{such that} \quad \operatorname{dist}(y_*, \partial\Omega) = |y_* - z_*|, \tag{8.4.59}$$

and define

$$r := |y_* - x| > 0. \tag{8.4.60}$$

Also, choose

$$\varepsilon := \frac{\kappa}{2(1 + \widetilde{\kappa})} \in \Big(0, \frac{\kappa}{1 + \widetilde{\kappa}}\Big). \tag{8.4.61}$$

We claim that

$$y_* \in \Gamma_\kappa(z) \quad \text{for each} \quad z \in \Delta(z_*, \varepsilon r). \tag{8.4.62}$$

To justify this, select an arbitrary point $z \in \Delta(z_*, \varepsilon r)$ and note that

$$\begin{aligned} |y_* - z| &\le |y_* - z_*| + |z_* - z| < \mathrm{dist}(y_*, \partial\Omega) + \varepsilon r \\ &= \mathrm{dist}(y_*, \partial\Omega) + \varepsilon \cdot |y_* - x| < \mathrm{dist}(y_*, \partial\Omega) + \varepsilon(1 + \widetilde{\kappa})\,\mathrm{dist}(y_*, \partial\Omega) \\ &= \big(1 + \varepsilon(1 + \widetilde{\kappa})\big)\,\mathrm{dist}(y_*, \partial\Omega) < (1 + \kappa)\,\mathrm{dist}(y_*, \partial\Omega), \end{aligned} \tag{8.4.63}$$

where the second inequality uses the fact that $y_* \in \Gamma_{\widetilde{\kappa}}(x)$, and the last inequality uses (8.4.61). From (8.4.63), we conclude that $y_* \in \Gamma_\kappa(z)$, which establishes (8.4.62).

To proceed, pick an arbitrary $z \in \Delta(z_*, \varepsilon r)$. Since y_* belongs to the open set $\Gamma_\kappa(z)$, there exists $\rho_* > 0$ with the property that $B(y_*, \rho_*) \subseteq \Gamma_\kappa(z)$. This implies

$$A_\lambda \cap B(y_*, \rho) \subseteq A_\lambda \cap \Gamma_\kappa(z) \quad \text{for each} \quad \rho \in (0, \rho_*), \tag{8.4.64}$$

which, in concert with (8.4.58), permits us to write

$$\frac{\mathcal{L}^n\big(A_\lambda \cap \Gamma_\kappa(z)\big)}{\mathcal{L}^n\big(B(y_*, \rho)\big)} \ge \frac{\mathcal{L}^n\big(A_\lambda \cap B(y_*, \rho)\big)}{\mathcal{L}^n\big(B(y_*, \rho)\big)} \longrightarrow 1 \quad \text{as} \quad \rho \to 0^+. \tag{8.4.65}$$

In turn, from (8.4.65), we see that

$$\mathcal{L}^n\big(A_\lambda \cap \Gamma_\kappa(z)\big) > 0 \tag{8.4.66}$$

which, in view of the fact that $|u| > \lambda$ in A_λ (cf. (8.4.56)), further implies

$$\|u\|_{L^\infty(\Gamma_\kappa(z), \mathcal{L}^n)} \ge \lambda. \tag{8.4.67}$$

Bearing in mind (8.2.1), this ultimately proves that

$$\big(\mathcal{N}_\kappa u\big)(z) \ge \lambda \quad \text{for each} \quad z \in \Delta(z_*, \varepsilon r). \tag{8.4.68}$$

Recall from (8.2.28) that $\mathcal{N}_\kappa u$ is a non-negative Borel-measurable function on $\partial\Omega$. Via integration, (8.4.68) then entails

$$\left(\fint_{\Delta(z_*, \varepsilon r)} \big(\mathcal{N}_\kappa u\big)^s(z)\, d\sigma(z) \right)^{1/s} \ge \lambda. \tag{8.4.69}$$

Moving on, we make a few observations. First

$$\Delta(z_*, \varepsilon r) \subseteq \Delta\big(x, (2 + \varepsilon) r\big). \tag{8.4.70}$$

Indeed, for each $z \in \Delta(z_*, \varepsilon r)$, we have

$$
\begin{aligned}
|z - x| &\leq |x - y_*| + |y_* - z_*| + |z_* - z| < r + \operatorname{dist}(y_*, \partial\Omega) + \varepsilon r \\
&\leq r + |y_* - x| + \varepsilon r = (2 + \varepsilon)r,
\end{aligned}
\tag{8.4.71}
$$

justifying (8.4.70). Second,

$$
\Delta\big(x, (2 + \varepsilon)r\big) \subseteq \Delta\big(z_*, (4 + \varepsilon)r\big).
\tag{8.4.72}
$$

To see that this is the case, for each $w \in \Delta\big(x, (2 + \varepsilon)r\big)$, write

$$
\begin{aligned}
|w - z_*| &\leq |w - x| + |x - y_*| + |y_* - z_*| < (2 + \varepsilon)r + r + \operatorname{dist}(y_*, \partial\Omega) \\
&\leq (3 + \varepsilon)r + |y_* - x| = (4 + \varepsilon)r,
\end{aligned}
\tag{8.4.73}
$$

from which (8.4.72) follows. Lastly, we observe that

$$
\sigma\Big(\Delta\big(x, (2 + \varepsilon)r\big)\Big) \leq \sigma\Big(\Delta\big(z_*, (4 + \varepsilon)r\big)\Big) \leq C \cdot \sigma\big(\Delta(z_*, \varepsilon r)\big),
\tag{8.4.74}
$$

thanks to (8.4.72) and the fact that σ is a doubling Borel measure on $\partial\Omega$. Above, the constant $C \in (0, \infty)$ can be taken to depend only on $\widetilde{\kappa}$, κ and the doubling character of σ.

At this stage, we may combine (8.4.69) with (8.4.70), (8.4.74), and (8.4.40), to write

$$
\begin{aligned}
\lambda &\leq \bigg(\fint_{\Delta(z_*, \varepsilon r)} \big(\mathcal{N}_\kappa u\big)^s(z)\, d\sigma(z) \bigg)^{1/s} \leq C \bigg(\fint_{\Delta(x, (2+\varepsilon)r)} \big(\mathcal{N}_\kappa u\big)^s(z)\, d\sigma(z) \bigg)^{1/s} \\
&\leq C \cdot \Big(\mathcal{M}_{\partial\Omega, s}\big(\mathcal{N}_\kappa u\big)\Big)(x).
\end{aligned}
\tag{8.4.75}
$$

Upon letting $\lambda \nearrow \big(\mathcal{N}_{\widetilde{\kappa}} u\big)(x)$ (something permissible, in light of (8.4.55)), we then arrive at the conclusion that (8.4.54) holds. Together with (8.2.25), this finishes the proof of (8.4.42).

Next, given any Lebesgue measurable set $E \subseteq \Omega$, from (8.4.42) written for $u \cdot \mathbf{1}_E$ in place of u, we obtain (8.4.41) (see the equality in (8.2.14) in this regard). The claim in (8.4.43) also readily follows from (8.4.42), in view of (8.4.40) and (8.4.44).

To justify (8.4.45), we run the same argument that has produced (8.4.41) for the function $u \cdot \mathbf{1}_{\mathcal{O}_\rho}$ where $\mathcal{O}_\rho := \big\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \rho\big\}$. In this scenario, having $y_* \in A_\lambda$ forces $y_* \in \mathcal{O}_\rho$ which further implies (cf. (8.4.60))

$$
r = |y_* - x| < (1 + \widetilde{\kappa}) \operatorname{dist}(y_*, \partial\Omega) < (1 + \widetilde{\kappa})\rho.
\tag{8.4.76}
$$

As a consequence of this and (8.4.46), in place of (8.4.75), we may now write

$$\lambda \leq \Big(\fint_{\Delta(z_*, \varepsilon r)} \big(\mathcal{N}_\kappa^\rho u\big)^s(z)\, \mathrm{d}\sigma(z) \Big)^{1/s} \leq C \Big(\fint_{\Delta(x,(2+\varepsilon)r)} \big(\mathcal{N}_\kappa^\rho u\big)^s(z)\, \mathrm{d}\sigma(z) \Big)^{1/s}$$
$$\leq C \cdot \Big(\mathcal{M}_{\partial\Omega,s}^{\widetilde{C}\rho} \big(\mathcal{N}_\kappa^\rho u\big) \Big)(x), \tag{8.4.77}$$

with $\widetilde{C} := (1+\widetilde{\kappa})(2+\varepsilon)$. Sending $\lambda \nearrow \big(\mathcal{N}_{\widetilde{\kappa}}^\rho u\big)(x)$ yields

$$\big(\mathcal{N}_{\widetilde{\kappa}}^\rho u\big)(x) \leq C \cdot \Big(\mathcal{M}_{\partial\Omega,s}^{\widetilde{C}\rho} \big(\mathcal{N}_\kappa^\rho u\big) \Big)(x) \tag{8.4.78}$$

for some constants $C, \widetilde{C} \in (0,\infty)$ depending only on $\widetilde{\kappa}$, κ and the doubling character of σ. Then (8.4.45) follows in view of this and (8.2.25).

Pressing on, suppose some Lebesgue measurable function $u : \Omega \to \mathbb{C}$ has been given, and for each threshold $\lambda \in (0,\infty)$, define

$$\begin{aligned} \mathcal{U}_\kappa(\lambda) &:= \big\{ x \in \partial\Omega : \big(\mathcal{N}_\kappa u\big)(x) > \lambda \big\}, \\ \mathcal{U}_{\widetilde{\kappa}}(\lambda) &:= \big\{ x \in \partial\Omega : \big(\mathcal{N}_{\widetilde{\kappa}} u\big)(x) > \lambda \big\}. \end{aligned} \tag{8.4.79}$$

Assume $0 < \lambda < \lambda' < \infty$, with the goal of showing that

$$\mathbf{1}_{\mathcal{U}_{\widetilde{\kappa}}(\lambda')} \leq C \cdot \mathcal{M}_{\partial\Omega}\big(\mathbf{1}_{\mathcal{U}_\kappa(\lambda)}\big) \ \text{ at each point on } \ \partial\Omega, \tag{8.4.80}$$

for a constant $C \in (0,\infty)$ which depends only on $\widetilde{\kappa}$, κ and the doubling character of σ.

To this end, pick an arbitrary point $x \in \mathcal{U}_{\widetilde{\kappa}}(\lambda')$. Then $x \in \partial\Omega$ and $\big(\mathcal{N}_{\widetilde{\kappa}} u\big)(x) > \lambda'$. The latter implies $\|u\|_{L^\infty(\Gamma_{\widetilde{\kappa}}(x),\mathcal{L}^n)} > \lambda'$ from which we conclude that, with

$$A_{\lambda'} := \big\{ y \in \Gamma_{\widetilde{\kappa}}(x) : |u(y)| > \lambda' \big\}, \tag{8.4.81}$$

(i.e., the version of (8.4.56) with λ replaced by λ'), we have

$$\mathcal{L}^n(A_{\lambda'}) > 0. \tag{8.4.82}$$

Then the same reasoning as in (8.4.57)–(8.4.68),run with λ' in place of λ, shows that $\big(\mathcal{N}_\kappa u\big)(z) \geq \lambda'$ for each $z \in \Delta(z_*, \varepsilon r)$. In particular, since $\lambda' > \lambda$, we have

$$\big(\mathcal{N}_\kappa u\big)(z) > \lambda \ \text{ for each } \ z \in \Delta(z_*, \varepsilon r). \tag{8.4.83}$$

Bearing in mind (8.4.79), we may recast this as

$$\Delta(z_*, \varepsilon r) \subseteq \mathcal{U}_\kappa(\lambda). \tag{8.4.84}$$

Then we may rely on (8.4.44), (8.4.70), (8.4.84), (8.4.72), and the fact that σ is a doubling Borel measure on $\partial\Omega$ to write

$$\begin{aligned}\Big(\mathcal{M}_{\partial\Omega}\big(\mathbf{1}_{\mathcal{U}_\kappa(\lambda)}\big)\Big)(x) &\geq \fint_{\Delta(x,(2+\varepsilon)r)} \mathbf{1}_{\mathcal{U}_\kappa(\lambda)}\, d\sigma \\ &\geq \frac{1}{\sigma\big(\Delta(x,(2+\varepsilon)r)\big)} \int_{\Delta(z_*,\varepsilon r)} \mathbf{1}_{\mathcal{U}_\kappa(\lambda)}\, d\sigma \\ &= \frac{\sigma\big(\Delta(z_*,\varepsilon r)\big)}{\sigma\big(\Delta(x,(2+\varepsilon)r)\big)} \geq \frac{\sigma\big(\Delta(z_*,\varepsilon r)\big)}{\sigma\big(\Delta(z_*,(4+\varepsilon)r)\big)} \geq c > 0,\end{aligned} \tag{8.4.85}$$

where $c \in (0,\infty)$ depends solely on the doubling character of σ. In turn, this shows that the inequality in (8.4.80) holds at each point $x \in \mathcal{U}_{\widetilde{\kappa}}(\lambda')$, with $C := c^{-1}$. Since the inequality in (8.4.80) is trivially true at each point $x \in \partial\Omega \setminus \mathcal{U}_{\widetilde{\kappa}}(\lambda')$, the claim in (8.4.80) is fully established.

Moving on, it is clear from (8.4.79) that

$$\mathcal{U}_{\widetilde{\kappa}}(\lambda_1) \subseteq \mathcal{U}_{\widetilde{\kappa}}(\lambda_2) \ \text{ if } \ \lambda_1 \geq \lambda_2 > 0 \tag{8.4.86}$$

and for each $\lambda \in (0,\infty)$, we have

$$\bigcup_{\lambda'>\lambda} \mathcal{U}_{\widetilde{\kappa}}(\lambda') = \mathcal{U}_{\widetilde{\kappa}}(\lambda). \tag{8.4.87}$$

In turn, these readily imply that, at each point on $\partial\Omega$,

$$\sup_{\lambda'>\lambda} \mathbf{1}_{\mathcal{U}_{\widetilde{\kappa}}(\lambda')} = \mathbf{1}_{\mathcal{U}_{\widetilde{\kappa}}(\lambda)}. \tag{8.4.88}$$

As such, given any $\lambda \in (0,\infty)$, taking the supremum in (8.4.80) over all $\lambda' > \lambda$ yields

$$\mathbf{1}_{\mathcal{U}_{\widetilde{\kappa}}(\lambda)} \leq C \cdot \mathcal{M}_{\partial\Omega}\big(\mathbf{1}_{\mathcal{U}_\kappa(\lambda)}\big) \ \text{ at each point on } \ \partial\Omega. \tag{8.4.89}$$

Having proved this, then (8.4.48) follows for each Lebesgue measurable set $E \subseteq \Omega$ simply by writing (8.4.89) with u replaced by $u \cdot \mathbf{1}_E$ (in this vein, see the equality in (8.2.14)).

With (8.4.48) in hand, for each $\lambda \in (0,\infty)$, we may now write

$$\begin{aligned}\sigma\big(\mathcal{U}^E_{\widetilde{\kappa}}(\lambda)\big) &\leq C\big\|\mathbf{1}_{\mathcal{U}^E_{\widetilde{\kappa}}(\lambda)}\big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \leq C\Big\|\mathcal{M}_{\partial\Omega}\big(\mathbf{1}_{\mathcal{U}^E_{\kappa}(\lambda)}\big)\Big\|_{L^{1,\infty}(\partial\Omega,\sigma)} \\ &\leq C\big\|\mathbf{1}_{\mathcal{U}^E_{\kappa}(\lambda)}\big\|_{L^{1}(\partial\Omega,\sigma)} = C \cdot \sigma\big(\mathcal{U}^E_\kappa(\lambda)\big),\end{aligned} \tag{8.4.90}$$

where the first inequality comes from (6.2.40) (presently employed with $p := 1$ and $q := \infty$), the second inequality is a consequence of (8.4.48) and (6.2.16), the third inequality is provided by the weak-(1, 1) boundedness of the Hardy–Littlewood maximal operator $\mathcal{M}_{\partial\Omega}$ (cf. (7.6.19)), and the final equality is clear from definitions. This proves (8.4.49). Finally, (8.4.50) is a special case of (8.4.49). □

Theorem 8.4.6 naturally lends itself to a useful, versatile, general principle for establishing norm estimates involving nontangential maximal operators with different aperture parameters. This is made precise in our next corollary.

Corollary 8.4.7 *Fix $n \in \mathbb{N}$ satisfying $n \geq 2$, and consider an arbitrary open (nonempty, proper) subset Ω of $\mathbb{R}^n$. Suppose σ is a doubling Borel measure on $\partial\Omega$, and denote by $\mathscr{M}_+(\partial\Omega, \sigma)$ the space of non-negative σ-measurable functions on $\partial\Omega$. Consider a mapping*

$$\|\cdot\| : \mathscr{M}_+(\partial\Omega, \sigma) \longrightarrow [0, +\infty] \tag{8.4.91}$$

enjoying the following three properties:

(i) [*Quasi-Monotonicity*] *There exists $C_1 \in (0, \infty)$ such that $\|f\| \leq C_1\|g\|$ for any pair of functions $f, g \in \mathscr{M}_+(\partial\Omega, \sigma)$ satisfying $f \leq g$ at σ-a.e. point on $\partial\Omega$.*
(ii) [*Quasi-Homogeneity*] *There exists $C_2 \in (0, \infty)$ such that $\|\lambda f\| \leq C_2\lambda\|f\|$ for any function $f \in \mathscr{M}_+(\partial\Omega, \sigma)$ and any number $\lambda \in (0, \infty)$.*
(iii) [*Boundedness of some L^s-based Hardy-Littlewood Maximal Operator*] *There exist an integrability exponent $s \in (0, \infty)$ along with a constant $C_3 \in (0, \infty)$, such that*

$$\big\|\mathcal{M}_{\partial\Omega,s} f\big\| \leq C_3\|f\| \ \ \text{for each} \ \ f \in \mathscr{M}_+(\partial\Omega, \sigma), \tag{8.4.92}$$

where $\mathcal{M}_{\partial\Omega,s}$ is the L^s-based Hardy–Littlewood maximal operator acting on σ-measurable functions on $\partial\Omega$ as in (8.4.40).

Then for any two aperture parameters $\widetilde{\kappa}, \kappa \in (0, \infty)$, there exists some constant $C \in (0, \infty)$ which depends only on $C_1, C_2, C_3, \widetilde{\kappa}, \kappa$ and the doubling character of σ with the property that

$$\big\|\mathcal{N}^E_{\widetilde{\kappa}} u\big\| \leq C\big\|\mathcal{N}^E_{\kappa} u\big\| \tag{8.4.93}$$

for each Lebesgue measurable set $E \subseteq \Omega$ and each Lebesgue measurable function $u : \Omega \to \mathbb{C}$. In particular, corresponding to $E := \Omega$, for any two aperture parameters $\widetilde{\kappa}, \kappa \in (0, \infty)$

$$\begin{gathered}\text{there exists } C \in (0, \infty) \text{ as before such that } \big\|\mathcal{N}_{\widetilde{\kappa}} u\big\| \leq C\big\|\mathcal{N}_{\kappa} u\big\| \\ \text{for each Lebesgue measurable function } u : \Omega \to \mathbb{C}.\end{gathered} \tag{8.4.94}$$

Proof This is a direct consequence of (8.4.41) in Theorem 8.4.6, and properties *(i)-(iii)* in the statement, also bearing in mind (8.2.28). □

In particular, Corollary 8.4.7 is applicable to any Köthe function space (defined as in [185, Sect. 1.5]) on which the Hardy–Littlewood maximal operator on $\partial\Omega$ happens to be bounded.

In the corollary below, we present concrete embodiments of the general principle established in Corollary 8.4.7, corresponding to a variety of specific function spaces for which conditions *(i)-(iii)* (formulated above) are satisfied. The reader is alerted to the fact that the scale of Morrey and block spaces on Ahlfors regular sets are discussed at length in [185, Chap. 6].

Corollary 8.4.8 *Pick $n \in \mathbb{N}$ satisfying $n \geq 2$, and suppose Ω is an arbitrary open (nonempty, proper) subset of $\mathbb{R}^n$. Also, let σ be a doubling Borel measure on $\partial\Omega$, and pick two arbitrary aperture parameters $\widetilde{\kappa}, \kappa \in (0,\infty)$. Finally, consider an arbitrary Lebesgue measurable set $E \subseteq \Omega$ and an arbitrary Lebesgue measurable function $u : \Omega \to \mathbb{C}$. Then the following results are true:*

(1) [*Ordinary Lebesgue Spaces*] *For each $p \in (0,\infty]$ there exists some constant $C \in (0,\infty)$ which depends only on $\widetilde{\kappa}, \kappa, p$ and the doubling character of σ with the property that*

$$\big\|\mathcal{N}^E_{\widetilde{\kappa}} u\big\|_{L^p(\pi_{\widetilde{\kappa}}(E),\sigma)} \leq C\big\|\mathcal{N}^E_{\kappa} u\big\|_{L^p(\pi_{\kappa}(E),\sigma)}. \tag{8.4.95}$$

In particular

$$\big\|\mathcal{N}_{\widetilde{\kappa}} u\big\|_{L^p(\partial\Omega,\sigma)} \leq C\big\|\mathcal{N}_{\kappa} u\big\|_{L^p(\partial\Omega,\sigma)}. \tag{8.4.96}$$

(2) [*Lorentz Spaces*] *Fix $p \in (0,\infty]$ and $q \in (0,\infty]$, and make the convention that $q = \infty$ if $p = \infty$. Then there exists a constant $C \in (0,\infty)$ which depends only on $\widetilde{\kappa}, \kappa, p, q$ and the doubling character of σ with the property that*

$$\big\|\mathcal{N}^E_{\widetilde{\kappa}} u\big\|_{L^{p,q}(\partial\Omega,\sigma)} \leq C\big\|\mathcal{N}^E_{\kappa} u\big\|_{L^{p,q}(\partial\Omega,\sigma)}. \tag{8.4.97}$$

(3) [*Muckenhoupt Weighted Lebesgue Spaces*] *For each exponent $p \in (1,\infty)$ and each Muckenhoupt weight $w \in A_p(\partial\Omega,\sigma)$ there exists a constant $C \in (0,\infty)$ which depends only on $\widetilde{\kappa}$, κ, p, $[w]_{A_p}$, and the doubling character of σ with the property that*

$$\big\|\mathcal{N}^E_{\widetilde{\kappa}} u\big\|_{L^p(\partial\Omega,w\sigma)} \leq C\big\|\mathcal{N}^E_{\kappa} u\big\|_{L^p(\partial\Omega,w\sigma)}. \tag{8.4.98}$$

(4) [*Orlicz-type Spaces*] *Let $\Phi : [0,\infty) \to [0,\infty)$ be a non-decreasing function which is absolutely continuous on each compact sub-interval of $[0,\infty)$, and satisfying $\Phi(0) = 0$ as well as $\lim_{t\to\infty} \Phi(t) = \infty$. Extend Φ to a function defined on $[0,\infty]$ and taking values in $[0,\infty]$, by setting $\Phi(\infty) := \infty$. Then there exists a constant $C \in (0,\infty)$ which depends only on $\widetilde{\kappa}$, κ, and the doubling character of σ with the property that*

$$\int_{\partial\Omega} \Phi\Big(\big(\mathcal{N}^E_{\widetilde{\kappa}} u\big)(x)\Big)\,d\sigma(x) \leq C \int_{\partial\Omega} \Phi\Big(\big(\mathcal{N}^E_{\kappa} u\big)(x)\Big)\,d\sigma(x). \tag{8.4.99}$$

(5) [*Morrey Spaces*] *Assume $\partial\Omega$ is Ahlfors regular and suppose $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Then for each $p \in (1,\infty)$ and $\lambda \in (0, n-1)$, there exists a constant $C \in (0,\infty)$ which depends only on $\widetilde{\kappa}$, κ, p, λ, and the Ahlfors regularity constants of $\partial\Omega$*

with the property that

$$\big\|\mathcal{N}^E_{\widetilde{\kappa}}u\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq C\big\|\mathcal{N}^E_{\kappa}u\big\|_{M^{p,\lambda}(\partial\Omega,\sigma)}. \tag{8.4.100}$$

(6) [*Block Spaces*] *Assume $\partial\Omega$ is Ahlfors regular and suppose $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Then for each $q \in (1,\infty)$ and $\lambda \in (0, n-1)$ there exists a constant $C \in (0,\infty)$ which depends only on $\widetilde{\kappa}$, κ, q, λ, and the Ahlfors regularity constants of $\partial\Omega$ with the property that*

$$\big\|\mathcal{N}^E_{\widetilde{\kappa}}u\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \leq C\big\|\mathcal{N}^E_{\kappa}u\big\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}. \tag{8.4.101}$$

Proof The claims in item *(1)* are implied by Corollary 8.4.7 with the choice $\|\cdot\| := \|\cdot\|_{L^p(\partial\Omega,\sigma)}$, bearing in mind that, for this choice, condition *(iii)* in Corollary 8.4.7 follows from (7.6.11). In the case of (8.4.95), the property in the second line of (8.2.26) is also relevant. Likewise, the claim in item *(2)* follows from Corollary 8.4.7 used with $\|\cdot\| := \|\cdot\|_{L^{p,q}(\partial\Omega,\sigma)}$. For this choice, condition *(i)* in Corollary 8.4.7 has been noted in (6.2.16),while condition *(iii)* in Corollary 8.4.7 is guaranteed by (7.6.9). Next, the claim in item *(3)* is seen from Corollary 8.4.7 this time employed with $\|\cdot\| := \|\cdot\|_{L^p(\partial\Omega,w\sigma)}$, a choice for which condition *(iii)* in Corollary 8.4.7 follows from item *(1)* in Lemma 7.7.1.

Let us now deal with the claim made in item *(4)*. As a preamble, we first observe that for each σ-measurable function $f : \partial\Omega \to [0,\infty]$, we have

$$\int_{\partial\Omega} \Phi\big(f(x)\big)\,\mathrm{d}\sigma(x) = \int_0^\infty \sigma\Big(\big\{x\in\partial\Omega:\ f(x)>t\big\}\Big)\Phi'(t)\,\mathrm{d}t. \tag{8.4.102}$$

To justify this, introduce

$$f_N(x) := \min\{f(x), N\} \ \text{ for each } \ N\in\mathbb{N} \ \text{ and } \ x\in\partial\Omega. \tag{8.4.103}$$

Also, for each $N\in\mathbb{N}$, define $F_N : \partial\Omega\times(0,\infty)\to[0,\infty)$ by setting

$$F_N(x,t) := \mathbf{1}_{\{f_N>t\}}(x)\cdot\Phi'(t) \ \text{ for each } \ x\in\partial\Omega \ \text{ and } \ \mathcal{L}^1\text{-a.e. } \ t\in(0,\infty). \tag{8.4.104}$$

Then we may write

$$\begin{aligned}
\int_{\partial\Omega} \Phi\big(f(x)\big)\,\mathrm{d}\sigma(x) &= \lim_{N\to\infty}\int_{\partial\Omega}\Phi\big(f_N(x)\big)\,\mathrm{d}\sigma(x)\\
&= \lim_{N\to\infty}\int_{\partial\Omega}\Big(\int_0^{f_N(x)}\Phi'(t)\,\mathrm{d}t\Big)\,\mathrm{d}\sigma(x)\\
&= \lim_{N\to\infty}\int_{\partial\Omega}\Big(\int_0^{\infty}F_N(x,t)\,\mathrm{d}t\Big)\,\mathrm{d}\sigma(x)\\
&= \lim_{N\to\infty}\int_0^{\infty}\Big(\int_{\partial\Omega}F_N(x,t)\,\mathrm{d}\sigma(x)\Big)\,\mathrm{d}t
\end{aligned}$$

$$= \lim_{N\to\infty} \int_0^\infty \sigma\Big(\big\{x \in \partial\Omega : f_N(x) > t\big\}\Big)\Phi'(t)\,\mathrm{d}t$$
$$= \int_0^\infty \sigma\Big(\big\{x \in \partial\Omega : f(x) > t\big\}\Big)\Phi'(t)\,\mathrm{d}t. \tag{8.4.105}$$

Above, the first equality is a consequence of Lebesgue's Monotone Convergence Theorem (here is relevant to recall that Φ is non-decreasing, and the extended version of Φ is continuous on $[0, \infty]$), the second equality is provided by the Fundamental Theorem of Calculus applied to Φ' on the finite interval $[0, f_N(x)]$ (bearing in mind that $\Phi(0) = 0$), the third equality is seen directly from (8.4.104), the fourth equality is implied by the Fubini–Tonelli's Theorem for non-negative functions (keeping in mind that Φ' is a non-negative measurable function, and that σ is a sigma-finite measure), the fifth equality follows from (8.4.104), and the sixth equality is ultimately a consequence of Lebesgue's Monotone Convergence Theorem. This establishes (8.4.102). Granted this, we may then rely on (8.4.49) (bearing in mind (8.4.47)) to write

$$\begin{aligned}\int_{\partial\Omega} \Phi\Big(\big(\mathcal{N}^E_{\widetilde{\kappa}} u\big)(x)\Big)\,\mathrm{d}\sigma(x) &= \int_0^\infty \sigma\Big(\big\{x \in \partial\Omega : \big(\mathcal{N}^E_{\widetilde{\kappa}} u\big)(x) > t\big\}\Big)\Phi'(t)\,\mathrm{d}t\\ &\leq C\int_0^\infty \sigma\Big(\big\{x \in \partial\Omega : \big(\mathcal{N}^E_{\kappa} u\big)(x) > t\big\}\Big)\Phi'(t)\,\mathrm{d}t\\ &= C\int_{\partial\Omega} \Phi\Big(\big(\mathcal{N}^E_{\kappa} u\big)(x)\Big)\,\mathrm{d}\sigma(x),\end{aligned} \tag{8.4.106}$$

proving (8.4.99). Incidentally, (8.4.99) contains (8.4.96) as a particular case, corresponding to the choice $\Phi(t) := t^p$ for each $t \geq 0$.

Going further, the claim in item *(5)* is a consequence of Corollary 8.4.7, presently used with $\|\cdot\| := \|\cdot\|_{M^{p,\lambda}(\partial\Omega,\sigma)}$. For such a choice, condition *(i)* in Corollary 8.4.7 is implied by the fact that

$$\begin{array}{l}\text{if } f, g : \partial\Omega \to \mathbb{C} \text{ are two } \sigma\text{-measurable functions such that}\\ |g| \leq |f| \text{ at } \sigma\text{-a.e. point on } \partial\Omega \text{ and } f \in M^{p,\lambda}(\partial\Omega,\sigma), \text{ then}\\ g \text{ also belongs to } M^{p,\lambda}(\partial\Omega,\sigma) \text{ and one has the estimate}\\ \|g\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \leq \|f\|_{M^{p,\lambda}(\partial\Omega,\sigma)},\end{array} \tag{8.4.107}$$

while condition *(iii)* in Corollary 8.4.7 is ensured by the fact that the Hardy–Littlewood maximal operator on $\partial\Omega$ induces a well-defined, sub-linear, and bounded mapping on Morrey spaces; see [185, Sect. 6.2] for proofs. The claim in item *(6)* is treated similarly, now employing Corollary 8.4.7 with $\|\cdot\| := \|\cdot\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}$ and relying on the lattice property for block spaces together with the boundedness of the Hardy–Littlewood maximal operator on block spaces (again, see [185, Sect. 6.2] for proofs of those properties). □

Moving on, we note that in the setting of one-sided NTA domains it is possible to obtain local control of the nontangential maximal operator of a given function in terms of the nontangential maximal operator of its gradient.

Proposition 8.4.9 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a one-sided* NTA *domain and fix an arbitrary aperture parameter* $\kappa > 0$. *Then there exist* $\widetilde{\kappa} > 0$ *large enough along with some threshold* $R \in (0, \infty)$ (*which may be taken* $+\infty$ *if* $\partial\Omega$ *is unbounded*) *and some constant* $C \in (1, \infty)$, *such that for each given* $\rho \in (0, R)$ *and every point* $x_o \in \partial\Omega$, *one may find a compact subset* K_{ρ,x_o} *of* Ω, *of diameter* $\approx \rho$ *and distance to the boundary* $\approx \rho$, *with the property that for every function* $u \in \mathscr{C}^1(\Omega)$, *one has*

$$\big(\mathcal{N}_\kappa^\rho u\big)(x) \leq C\rho \cdot \mathcal{N}_{\widetilde{\kappa}}^{C\rho}(\nabla u)(x) + \sup_{K_{\rho,x_o}} |u|, \qquad \forall x \in B(x_o, \rho) \cap \partial\Omega. \tag{8.4.108}$$

As a consequence, for any given doubling Borel measure σ *on* $\partial\Omega$ *and any function* $u \in \mathscr{C}^1(\Omega)$, *it follows that*

$$\begin{array}{l} \text{if } \mathcal{N}_\kappa^\varepsilon(\nabla u) \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma) \text{ for some } p \in (0, \infty] \text{ and } \varepsilon \in (0, \infty) \text{ then} \\ \mathcal{N}_\kappa^{\varepsilon/C} u \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma) \text{ for some sufficiently large constant } C > 1. \end{array} \tag{8.4.109}$$

Proof Fix a point $x_o \in \partial\Omega$ and recall that $\mathcal{O}_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$ for each $\varepsilon > 0$. Since Ω is a one-sided NTA domain, there exist $R \in (0, \infty)$ (which may be taken $+\infty$ if $\partial\Omega$ is unbounded) and $c \in (0, 1)$ with the property that for each $\rho \in (0, R)$, we may find a corkscrew point $A_\rho \in \Omega \setminus \mathcal{O}_{c\rho}$ relative to the location x_o and scale ρ. We may also select an aperture $\widetilde{\kappa} > 0$, large enough so that, given any $x \in B(x_o, \rho) \cap \partial\Omega$, we may join each point $y \in \Gamma_\kappa(x) \cap \mathcal{O}_\rho$ with A_ρ by a polygonal path γ of length $\leq C\rho$ which is contained in $\Gamma_{\widetilde{\kappa}}(x)$. Such a path may be constructed in a fashion analogous to the procedure described in the proof of Lemma 5.11.4 which shows that there exist $\widetilde{\kappa} > 0$ along with $C_1, C_2 \in (0, \infty)$ with the property that

$$\begin{array}{l} \text{any two corkscrew points relative to a common location } x \in \partial\Omega \\ \text{and corresponding to scales which are} \leq C_1\rho \text{ may be joined by} \\ \text{a polygonal path in } \Gamma_{\widetilde{\kappa}}(x) \text{ of length} \leq C_2\rho. \end{array} \tag{8.4.110}$$

Upon observing that y and A_ρ fit these attributes, the existence of the aforementioned path γ follows. With ds and ∂_s denoting, respectively, the arc-length measure and tangential derivative along γ, for any given function $u \in \mathscr{C}^1(\Omega)$ we may then use the Fundamental Theorem of Calculus to estimate

$$|u(y)| \leq |u(A_\rho)| + \left| \int_\gamma \partial_s u \, \mathrm{d}s \right| \leq |u(A_\rho)| + C\rho \cdot \mathcal{N}_{\widetilde{\kappa}}^{C\rho}(\nabla u)(x). \tag{8.4.111}$$

After taking the supremum over all $y \in \Gamma_\kappa(x) \cap \mathcal{O}_\rho$, the estimate recorded in (8.4.108) readily follows. Finally, given a doubling Borel measure σ on $\partial\Omega$, the claim in (8.4.109) is a consequence of (8.4.108), (8.2.28), and Corollary 8.4.2. $\square$

We next present a Hardy-type estimate, in which the nontangential maximal operator of the gradient of a function controls the nontangential maximal operator of the function (suitably normalized). This is done on the Lebesgue scale, involving an arbitrary integrability exponent $p \in (0, n-1)$ and its associated embedding exponent $p^* := \big(\frac{1}{p} - \frac{1}{n-1}\big)^{-1}$, on the boundary of upper-graph Lipschitz domains. In [187, Sect. 1.8] and [187, Sect. 2.2], we shall establish similar results for more general classes of sets, when the functions in question are null-solutions of elliptic PDE's.

Proposition 8.4.10 *Suppose $n \geq 3$ and let $\Omega \subseteq \mathbb{R}^n$ be an upper-graph Lipschitz domain. Denote by σ its surface measure and pick an aperture parameter $\kappa \in (0,\infty)$. Then for each real-valued function $w \in \mathscr{C}^1(\Omega)$ and each exponent $p \in (0, n-1)$ there exist constants $c = c(w) \in \mathbb{R}$ and $C = C(\Omega, \kappa, p) \in (0,\infty)$, such that*

$$\|\mathcal{N}_\kappa(w - c)\|_{L^{p^*}(\partial\Omega,\sigma)} \leq C\|\mathcal{N}_\kappa(\nabla w)\|_{L^p(\partial\Omega,\sigma)} \tag{8.4.112}$$

where

$$p^* := \big(\tfrac{1}{p} - \tfrac{1}{n-1}\big)^{-1}. \tag{8.4.113}$$

Prior to presenting the proof of this result, we isolate one technical aspect in the following lemma.

Lemma 8.4.11 *Let $\Omega \subseteq \mathbb{R}^n$ be an upper-graph Lipschitz domain. Assume $u \in \mathscr{C}^1(\Omega)$ is a real-valued function with the property that there exist two constants, $C \in (0,\infty)$ and $\alpha \in (1,\infty)$, such that*

$$|(\nabla u)(x)| \leq C\big[\mathrm{dist}(x, \partial\Omega)\big]^{-\alpha}, \qquad \forall x \in \Omega. \tag{8.4.114}$$

Then for each $x \in \Omega$ the limit

$$c := \lim_{t\to\infty} u(x + t\mathbf{e}_n) \tag{8.4.115}$$

exists, is independent of the point x and, moreover,

$$|u(x) - c| \leq C\big[\mathrm{dist}(x, \partial\Omega)\big]^{1-\alpha}, \qquad \forall x \in \Omega. \tag{8.4.116}$$

Proof For every $x \in \Omega$ and $t \geq 0$ set

$$c(x,t) := u(x + t\mathbf{e}_n) + \int_t^\infty (\partial_n u)(x + s\mathbf{e}_n)\,\mathrm{d}s. \tag{8.4.117}$$

By (8.4.114), the integral in (8.4.117) is absolutely convergent, and the Fundamental Theorem of Calculus ensures that the expression in the right side is independent of $t \geq 0$. We may thus abbreviate $c(x) := c(x, t)$. Thanks to this and Lebesgue's Dominated Convergence Theorem, the limit

$$\begin{aligned}\lim_{t\to\infty} u(x + t\mathbf{e}_n) &= \lim_{t\to\infty} c(x) - \lim_{t\to\infty} \int_t^{\infty} (\partial_n u)(x + s\mathbf{e}_n)\, \mathrm{d}s \\ &= c(x) - 0 = c(x) \text{ exists for every } x \in \mathbb{R}^n_+. \end{aligned} \tag{8.4.118}$$

To prove that this limit is actually independent of x, suppose Ω is the upper-graph of a function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, with Lipschitz constant $M \in (0, \infty)$. Fix two arbitrary points, $x = (x', x_n) \in \Omega$ and $y = (y', y_n) \in \Omega$. Pick $t \geq M|x' - y'|$ and consider an arbitrary point $z \in [x + t\mathbf{e}_n, y + t\mathbf{e}_n]$. We claim that $z \in \Omega$ and there exists a constant $c \in (0, \infty)$, independent of x, y, z, such that

$$\operatorname{dist}(z, \partial\Omega) \geq ct. \tag{8.4.119}$$

To justify this, observe that there exists $\theta \in [0, 1]$, such that

$$\begin{aligned} z = (z', z_n) &= \theta(x + t\mathbf{e}_n) + (1 - \theta)(y + t\mathbf{e}_n) \\ &= \big(\theta x' + (1 - \theta)y', \theta x_n + (1 - \theta)y_n + t\big). \end{aligned} \tag{8.4.120}$$

Consequently, since $x_n > \phi(x')$ and $y_n > \phi(y')$, we may write

$$\begin{aligned} z_n - \phi(z') &= \theta x_n + (1 - \theta)y_n + t - \phi\big(\theta x' + (1 - \theta)y'\big) \\ &> \theta\phi(x') + (1 - \theta)\phi(y') + t - \phi\big(\theta x' + (1 - \theta)y'\big) \\ &= t + \theta\big[\phi(x') - \phi\big(\theta x' + (1 - \theta)y'\big)\big] + (1 - \theta)\big[\phi(y') - \phi\big(\theta x' + (1 - \theta)y'\big)\big] \\ &\geq t - \theta\big|\phi(x') - \phi\big(\theta x' + (1 - \theta)y'\big)\big| - (1 - \theta)\big|\phi(y') - \phi\big(\theta x' + (1 - \theta)y'\big)\big| \\ &\geq t - \theta M\big|x' - \big(\theta x' + (1 - \theta)y'\big)\big| - (1 - \theta)M\big|y' - \big(\theta x' + (1 - \theta)y'\big)\big| \\ &= t - 2\theta(1 - \theta)M|x' - y'| \geq t - \tfrac{1}{2}M|x' - y'| \\ &\geq \tfrac{1}{2}t. \end{aligned} \tag{8.4.121}$$

In particular, $z_n > \phi(z')$ which places z in Ω. Keeping this in mind, from (8.4.121) and the fact that

$$\operatorname{dist}(z, \partial\Omega) \approx z_n - \phi(z') \tag{8.4.122}$$

we then see that (8.4.119) holds. In turn, based on the fact that $[x + t\mathbf{e}_n, y + t\mathbf{e}_n]$ is contained in Ω whenever $t \geq M|x' - y'|$, the Mean Value Theorem, (8.4.114), and (8.4.119), we may estimate

$$|u(x + t\mathbf{e}_n) - u(y + t\mathbf{e}_n)| \leq C|x - y|t^{-\alpha} \to 0 \text{ as } t \to \infty, \tag{8.4.123}$$

which ultimately shows that $c(x) = c(y)$ for every $x, y \in \Omega$. As such, we may simply write $c \in \mathbb{R}$ for $c(x)$ corresponding to arbitrary $x \in \Omega$. Since for each $x = (x', x_n) \in \Omega$, we have

$$\operatorname{dist}(x + s\mathbf{e}_n, \partial\Omega) \approx x_n + s - \phi(x') \approx \operatorname{dist}(x, \partial\Omega) + s, \tag{8.4.124}$$

uniformly for $s \geq 0$, we may use (8.4.117) with $t = 0$ and (8.4.114) to estimate

$$\begin{aligned} |u(x) - c| = |u(x) - c(x)| &= \Big| \int_0^\infty (\partial_n u)(x + s\mathbf{e}_n)\, \mathrm{d}s \Big| \\ &\leq \int_0^\infty |(\partial_n u)(x + s\mathbf{e}_n)|\, \mathrm{d}s \leq C \int_0^\infty \big[\operatorname{dist}(x + s\mathbf{e}_n, \partial\Omega)\big]^{-\alpha}\, \mathrm{d}s \\ &\leq C \int_0^\infty \big[\operatorname{dist}(x, \partial\Omega) + s\big]^{-\alpha}\, \mathrm{d}s \leq C\big[\operatorname{dist}(x, \partial\Omega)\big]^{1-\alpha}, \end{aligned} \tag{8.4.125}$$

for each point $x \in \Omega$. This establishes (8.4.116). □

We are now ready to discuss the proof of Proposition 8.4.10.

Proof of Proposition 8.4.10 To fix ideas, suppose Ω is the upper-graph of a Lipschitz function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$, i.e., $\Omega = \big\{x = (x', x_n) \in \mathbb{R}^n : x_n > \phi(x')\big\}$. For each $x = (x', x_n) \in \Omega$ denote by $\widehat{x} := \big(x', \phi(x')\big) \in \partial\Omega$ its vertical projection onto the boundary, and abbreviate $\delta_{\partial\Omega}(x) := \operatorname{dist}(x, \partial\Omega)$. Also, for each $r \in (0, \infty)$ and $x \in \partial\Omega$, set $\Delta(x, r) := B(x, r) \cap \partial\Omega$. We claim that there exists an aperture parameter $\widetilde{\kappa} \in (0, \infty)$, depending only on the Lipschitz constant of ϕ, with the property that

$$\text{if } \ x \in \Omega \ \text{ and } \ y \in \Delta\big(\widehat{x}, \delta_{\partial\Omega}(x)\big) \ \text{ then } \ x \in \Gamma_{\widetilde{\kappa}}(y). \tag{8.4.126}$$

Indeed, there exists $C_\Omega \in (0, \infty)$ such that $|x - \widehat{x}| \leq C_\Omega \cdot \delta_{\partial\Omega}(x)$ for every $x \in \Omega$. Keeping this in mind, given any $x \in \Omega$ and $y \in \Delta\big(\widehat{x}, \delta_{\partial\Omega}(x)\big)$ we may estimate

$$|x - y| \leq |x - \widehat{x}| + |y - \widehat{x}| < C_\Omega \cdot \delta_{\partial\Omega}(x) + \delta_{\partial\Omega}(x) = (1 + C_\Omega)\delta_{\partial\Omega}(x), \tag{8.4.127}$$

and (8.4.126) follows by simply taking $\widetilde{\kappa} := C_\Omega \in (0, \infty)$.

Going further, assume $\mathcal{N}_\kappa(\nabla w) \in L^p(\partial\Omega, \sigma)$, since otherwise there is nothing to prove. Proposition 8.4.1 then guarantees that

$$\begin{gathered} \mathcal{N}_{\widetilde{\kappa}}(\nabla w) \in L^p(\partial\Omega, \sigma) \ \text{ and} \\ \|\mathcal{N}_{\widetilde{\kappa}}(\nabla w)\|_{L^p(\partial\Omega,\sigma)} \approx \|\mathcal{N}_\kappa(\nabla w)\|_{L^p(\partial\Omega,\sigma)}. \end{gathered} \tag{8.4.128}$$

Fix an arbitrary point $x \in \Omega$. For any exponent $\alpha \in (0, \infty)$, we may use (8.4.126) to estimate

$$|(\nabla w)(x)| \leq C \left(⨍_{\Delta\left(\widehat{x}, \delta_{\partial\Omega}(x)\right)} |\mathcal{N}_{\widetilde{\kappa}}(\nabla w)|^\alpha \, \mathrm{d}\sigma \right)^{\frac{1}{\alpha}}. \tag{8.4.129}$$

This used with $\alpha := p$ implies that there exists some $C \in (0, \infty)$ independent of x and w, such that

$$|(\nabla w)(x)| \leq C \delta_{\partial\Omega}(x)^{-\frac{n-1}{p}} \left\| \mathcal{N}_{\widetilde{\kappa}}(\nabla w) \right\|_{L^p(\partial\Omega,\sigma)}. \tag{8.4.130}$$

From (8.4.128), (8.4.130), the fact that $p \in (0, n-1)$, and Lemma 8.4.11, we see that it is possible to choose $c \in \mathbb{R}$, such that $w - c$ vanishes at infinity, in the sense that

$$|w(x) - c| \leq C \delta_{\partial\Omega}(x)^{1-\frac{n-1}{p}}, \qquad \forall x \in \Omega. \tag{8.4.131}$$

Henceforth, abbreviate $u := w - c$. Fix now $x \in \partial\Omega$ and pick some $y \in \Gamma_{\widetilde{\kappa}}(x)$. Then by virtue of (8.4.130)–(8.4.131), the definition of u, and the Fundamental Theorem of Calculus

$$\begin{aligned} |u(y)| &= \left| \int_0^\infty (\partial_n u)(y + t\mathbf{e}_n)\, \mathrm{d}t \right| \leq \int_0^\infty |(\nabla u)(y + t\mathbf{e}_n)|\, \mathrm{d}t \\ &= \int_0^\infty |(\nabla w)(y + t\mathbf{e}_n)|\, \mathrm{d}t. \end{aligned} \tag{8.4.132}$$

Given that $n \geq 3$ and $p \in (0, n-1)$, we have $\frac{p}{n-1} < \min\{1, p\}$. As such, it is possible to choose α so that

$$\frac{p}{n-1} < \alpha < \min\{1, p\}. \tag{8.4.133}$$

Observe that there exists some $C \in (0, \infty)$ with the property that

$$\begin{aligned} &\Delta\big(\widehat{y + t\mathbf{e}_n}, \delta_{\partial\Omega}(y + t\mathbf{e}_n)\big) \subseteq \Delta\big(x, C(t + \delta_{\partial\Omega}(y))\big) \text{ and} \\ &\delta_{\partial\Omega}(y + t\mathbf{e}_n) \approx t + \delta_{\partial\Omega}(y), \quad \text{uniformly for } \ t \in (0, \infty). \end{aligned} \tag{8.4.134}$$

Combining (8.4.132) with (8.4.129) and bearing in mind (8.4.134) then yields

$$\begin{aligned} |u(y)| &\leq C \int_0^\infty \left(\fint_{\Delta\left(x, C(t+\delta_{\partial\Omega}(y))\right)} |\mathcal{N}_{\widetilde{\kappa}}(\nabla w)|^\alpha \, \mathrm{d}\sigma \right)^{\frac{1}{\alpha}} \mathrm{d}t \\ &= C \int_0^\infty \left(\fint_{\Delta\left(x, C(t+\delta_{\partial\Omega}(y))\right)} |\mathcal{N}_{\widetilde{\kappa}}(\nabla w)|^\alpha \, \mathrm{d}\sigma \right)^{\frac{1}{\alpha}-1} \times \\ &\qquad \times \left(\fint_{\Delta\left(x, C(t+\delta_{\partial\Omega}(y))\right)} |\mathcal{N}_{\widetilde{\kappa}}(\nabla w)|^\alpha \, \mathrm{d}\sigma \right) \mathrm{d}t. \end{aligned} \tag{8.4.135}$$

Let $\mathcal{M}$ denote the Hardy-Littlewood maximal operator on $\partial\Omega$. Then, by definition,

$$\fint_{\Delta\left(x, C(t+\delta_{\partial\Omega}(y))\right)} |\mathcal{N}_{\widetilde{\kappa}}(\nabla w)|^\alpha \, \mathrm{d}\sigma \leq \mathcal{M}\big(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^\alpha\big)(x) \tag{8.4.136}$$

and so, from (8.4.135) and the fact that $\frac{1}{\alpha}-1>0$, we conclude that

$$|u(y)|\leq C\big[\mathcal{M}\big(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha}\big)(x)\big]^{\frac{1}{\alpha}-1}\Bigg(\int\limits_{0}^{\infty}\int\limits_{\Delta\left(x,C(t+\delta_{\partial\Omega}(y))\right)}\frac{\big[\mathcal{N}_{\widetilde{\kappa}}(\nabla w)(z)\big]^{\alpha}}{\big(t+\delta_{\partial\Omega}(y)\big)^{n-1}}\,\mathrm{d}\sigma(z)\,\mathrm{d}t\Bigg). \tag{8.4.137}$$

For each $\theta\in(0,n-1)$ denote by $\mathfrak{I}_\theta$ the fractional integration operator acting on each non-negative measurable function f defined in $\partial\Omega$ according to

$$\big(\mathfrak{I}_\theta f\big)(x):=\int_{\partial\Omega}\frac{f(z)}{|x-z|^{n-1-\theta}}\,\mathrm{d}\sigma(z),\qquad\forall\,x\in\partial\Omega. \tag{8.4.138}$$

Then by switching the order of integration in (8.4.137) and then making the change of variables $s:=t+\delta_{\partial\Omega}(y)$, we can estimate

$$\begin{aligned}|u(y)|&\leq C\big[\mathcal{M}\big(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha}\big)(x)\big]^{\frac{1}{\alpha}-1}\times\\&\qquad\times\int\limits_{\partial\Omega}\big[\mathcal{N}_{\widetilde{\kappa}}(\nabla w)(z)\big]^{\alpha}\Bigg(\int_{\frac{|x-z|}{C}<t+\delta_{\partial\Omega}(y)}^{\infty}\frac{\mathrm{d}t}{\big(t+\delta_{\partial\Omega}(y)\big)^{n-1}}\Bigg)\mathrm{d}\sigma(z)\\&=C\big[\mathcal{M}\big(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha}\big)(x)\big]^{\frac{1}{\alpha}-1}\int\limits_{\partial\Omega}\frac{\big[\mathcal{N}_{\widetilde{\kappa}}(\nabla w)(z)\big]^{\alpha}}{|x-z|^{n-2}}\,\mathrm{d}\sigma(z)\\&\leq C\big[\mathcal{M}\big(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha}\big)(x)\big]^{\frac{1}{\alpha}-1}\,\mathfrak{I}_1\big(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha}\big)(x),\end{aligned} \tag{8.4.139}$$

where the fact that the integral in the variable t is absolutely convergent is guaranteed by the assumption that $n\geq 3$. Taking the supremum over all points $y\in\Gamma_{\widetilde{\kappa}}(x)$ in (8.4.139), we obtain

$$(\mathcal{N}_{\widetilde{\kappa}}u)(x)\leq C\big[\mathcal{M}\big(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha}\big)(x)\big]^{\frac{1}{\alpha}-1}\,\mathfrak{I}_1\big(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha}\big)(x) \tag{8.4.140}$$

for each $x\in\partial\Omega$, and so

$$\int\limits_{\partial\Omega}(\mathcal{N}_{\widetilde{\kappa}}u)^{p^*}\,\mathrm{d}\sigma\leq C\int\limits_{\partial\Omega}\big[\mathcal{M}(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha})\big]^{p^*(\frac{1}{\alpha}-1)}\big[\mathfrak{I}_1(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha})\big]^{p^*}\,\mathrm{d}\sigma. \tag{8.4.141}$$

Choose $r,r'\in(1,\infty)$ such that $1/r+1/r'=1$ and

$$(1-\alpha)r=1-\frac{p}{n-1}=\frac{p}{p^*}. \tag{8.4.142}$$

That this may be arranged is guaranteed by the fact that $(1-\alpha)^{-1}\big(1-\frac{p}{n-1}\big)>1$, itself a consequence of the choice of α in (8.4.133). Next, from (8.4.141) and Hölder's inequality,

$$\int_{\partial\Omega} (\mathcal{N}_{\widetilde{\kappa}} u)^{p^*}\, d\sigma \leq C \Big(\int_{\partial\Omega} \big[\mathcal{M}(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha})\big]^{p^*(\frac{1-\alpha}{\alpha})r}\, d\sigma \Big)^{\frac{1}{r}} \times$$
$$\times \Big(\int_{\partial\Omega} \big[\mathfrak{I}_1(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha})\big]^{p^* r'}\, d\sigma \Big)^{\frac{1}{r'}}. \tag{8.4.143}$$

Let $q := \frac{p}{\alpha}$, so $1 < q < n-1$ thanks to the manner in which α has been selected in (8.4.133), and abbreviate $q^* := \big(\frac{1}{q} - \frac{1}{n-1}\big)^{-1}$. Then from our choice of r in (8.4.142), we conclude that the following identities hold:

$$p^*\Big(\frac{1-\alpha}{\alpha}\Big)r = \frac{p^*}{\alpha} \cdot \frac{p}{p^*} = \frac{p}{\alpha} = q, \tag{8.4.144}$$

$$\frac{1}{p^* r'} = \frac{1}{p^*}\Big(1 - \frac{1}{r}\Big) = \frac{1}{p^*} - \frac{1}{p^* r} = \frac{1}{p} - \frac{1}{n-1} - \frac{1-\alpha}{p}$$
$$= \frac{\alpha}{p} - \frac{1}{n-1} = \frac{1}{q^*}, \tag{8.4.145}$$

hence $q^* = p^* r'$, and

$$\frac{1}{r} + \frac{q^*}{q r'} = \frac{p^*(1-\alpha)}{p} + \frac{p^* r'}{q r'} = \frac{p^*(1-\alpha)}{p} + \frac{p^*}{q}$$
$$= \frac{p^*(1-\alpha)}{p} + \frac{\alpha p^*}{p} = \frac{p^*}{p}. \tag{8.4.146}$$

Applying identities (8.4.144)–(8.4.145) to (8.4.141) gives

$$\int_{\partial\Omega} (\mathcal{N}_{\widetilde{\kappa}} u)^{p^*}\, d\sigma \leq C \Big(\int_{\partial\Omega} \big[\mathcal{M}(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha})\big]^{q}\, d\sigma \Big)^{\frac{1}{r}} \times$$
$$\times \Big(\int_{\partial\Omega} \big[\mathfrak{I}_1(\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha})\big]^{q^*}\, d\sigma \Big)^{\frac{1}{r'}}. \tag{8.4.147}$$

It is well known that since $1 < q < n-1$, the operator $\mathcal{M}$ is bounded from $L^q(\partial\Omega, \sigma)$ into itself, while the operator $\mathfrak{I}_1$ is bounded from $L^q(\partial\Omega, \sigma)$ into $L^{q^*}(\partial\Omega, \sigma)$. Bearing these mapping properties in mind, it follows that

$$\int_{\partial\Omega} (\mathcal{N}_{\widetilde{\kappa}} u)^{p^*}\, d\sigma \leq C \Big(\int_{\partial\Omega} (\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha})^{q}\, d\sigma \Big)^{\frac{1}{r}} \Big(\int_{\partial\Omega} (\mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{\alpha})^{q}\, d\sigma \Big)^{\frac{q^*}{q r'}}$$
$$= C \Big(\int_{\partial\Omega} \mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{p}\, d\sigma \Big)^{\frac{1}{r} + \frac{q^*}{q r'}}$$
$$= C \Big(\int_{\partial\Omega} \mathcal{N}_{\widetilde{\kappa}}(\nabla w)^{p}\, d\sigma \Big)^{\frac{p^*}{p}}, \tag{8.4.148}$$

where the last equality uses identity (8.4.146). Granted this, we can finally conclude that

$$\|\mathcal{N}_{\widetilde{\kappa}}u\|_{L^{p^*}(\partial\Omega,\sigma)} \leq C\|\mathcal{N}_{\widetilde{\kappa}}(\nabla w)\|_{L^p(\partial\Omega,\sigma)}. \tag{8.4.149}$$

Upon invoking Proposition 8.4.1 and bearing in mind that $u = w - c$, this ultimately establishes (8.4.112). □

We conclude this section by presenting an estimate for the nontangential maximal operator in terms of the Hardy–Littlewood maximal operator.

Proposition 8.4.12 *Let $\Omega \subset \mathbb{R}^n$ be an open set with an upper Ahlfors regular boundary and such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is doubling. Consider a measurable function $k(\cdot,\cdot) : \Omega \times \partial\Omega \to [0,\infty)$ with the property that there exist $\alpha > 0$ and $C_0 \in (0,\infty)$, such that*

$$k(x,y) \leq C_0 \cdot \operatorname{dist}(x,\partial\Omega)^{\alpha}|x-y|^{1-n-\alpha}, \qquad \forall x \in \Omega, \quad \forall y \in \partial\Omega, \tag{8.4.150}$$

and define the integral operator acting on each σ-measurable $f : \partial\Omega \to [0,\infty)$ according to

$$\mathscr{P}f(x) := \int_{\partial\Omega} k(x,y)f(y)\,d\sigma(y) \in [0,\infty], \qquad \forall x \in \Omega. \tag{8.4.151}$$

Then for each aperture parameter $\kappa > 0$, there exists $C \in (0,\infty)$, which depends only on the constant C_0 from (8.4.150), κ, and the upper Ahlfors regularity constant of $\partial\Omega$, with the property that for each σ-measurable function $f : \partial\Omega \to [0,\infty)$, one has

$$\mathcal{N}_\kappa(\mathscr{P}f) \leq C \cdot \mathcal{M}_{\partial\Omega}f \quad \text{at each point on} \quad \partial\Omega, \tag{8.4.152}$$

where $\mathcal{M}_{\partial\Omega}$ denotes the Hardy–Littlewood maximal operator on $\partial\Omega$ (cf. (7.6.16)).

As a consequence, for each $p \in (1,\infty]$ there exists a constant $C_p \in (0,\infty)$ such that

$$\big\|\mathcal{N}_\kappa(\mathscr{P}|f|)\big\|_{L^p(\partial\Omega,\sigma)} \leq C\|f\|_{L^p(\partial\Omega,\sigma)}, \quad \forall f \in L^p(\partial\Omega,\sigma). \tag{8.4.153}$$

In fact, estimates similar to (8.4.153) are also valid in the context of Muckenhoupt weighted Lebesgue spaces $L^p(\partial\Omega, w\sigma)$ with $p \in (1,\infty)$ and $w \in A_p(\partial\Omega,\sigma)$, Lorentz spaces $L^{p,q}(\partial\Omega,\sigma)$ with $p \in (1,\infty)$ and $q \in (0,\infty]$. In the case when $\partial\Omega$ is actually Ahlfors regular, then estimates similar to (8.4.153) are also valid for the Morrey spaces $M^{p,\lambda}(\partial\Omega,\sigma)$ with $p \in (1,\infty)$ and $\lambda \in (0,n-1)$, as well as the block spaces $\mathcal{B}^{q,\lambda}(\Sigma,\sigma)$ with $q \in (1,\infty)$ and $\lambda \in (0,n-1)$.

Proof Fix an arbitrary σ-measurable function $f : \partial\Omega \to [0,\infty)$. For starters, observe that (8.4.150) and (8.4.151) imply

$$\mathscr{P}f(x) \leq C_0 \int_{\partial\Omega} \frac{\operatorname{dist}(x,\partial\Omega)^{\alpha}}{|x-y|^{n-1+\alpha}} f(y)\,d\sigma(y), \qquad \forall x \in \Omega. \tag{8.4.154}$$

With the goal of estimating the nontangential maximal function of $\mathscr{P}f$ with aperture κ, consider an arbitrary boundary point $z \in \partial\Omega$. We claim that

$$|x-y| \geq (3+\kappa)^{-1}\big(\operatorname{dist}(x,\partial\Omega)+|y-z|\big), \quad \forall x \in \Gamma_\kappa(z), \quad \forall y \in \partial\Omega. \tag{8.4.155}$$

Indeed, $\operatorname{dist}(x,\partial\Omega) \leq |x-y|$ since $y \in \partial\Omega$, and given that for each $x \in \Gamma_\kappa(x)$, we also have

$$|y-z| \leq |x-y|+|x-z| = |x-y|+(1+\kappa)\operatorname{dist}(x,\partial\Omega) \leq (2+\kappa)|x-y|, \tag{8.4.156}$$

the estimate in (8.4.155) follows. Making use of (8.4.155) in (8.4.154), we arrive at the conclusion that there exists $C = C(\kappa) \in (0,\infty)$, such that for each $z \in \partial\Omega$ and each $x \in \Gamma_\kappa(z)$, we have

$$\mathscr{P}f(x) \leq C \int_{\partial\Omega} \frac{\operatorname{dist}(x,\partial\Omega)^\alpha}{\big(\operatorname{dist}(x,\partial\Omega)+|y-z|\big)^{n-1+\alpha}} f(y)\,\mathrm{d}\sigma(y). \tag{8.4.157}$$

Let us now fix $z \in \partial\Omega$ along with $x \in \Gamma_\kappa(z)$ and abbreviate $r := \operatorname{dist}(x,\partial\Omega)$. Also, for each $R > 0$ set $\Delta(z,R) := B(z,R) \cap \partial\Omega$. Then we may decompose $\partial\Omega$ into a family of dyadic annuli $\partial\Omega = \bigcup_{j=0}^{\infty} R_j(z)$, where $R_0(z) := \Delta(z,r)$ and $R_j(z) := \Delta(z,2^{j+1}r) \setminus \Delta(z,2^j r)$ for each $j \in \mathbb{N}$. Bearing in mind that $\partial\Omega$ is an upper Ahlfors regular set, for each $j \geq 1$ we may then estimate

$$\begin{aligned}
\int_{R_j(z)} \frac{r^\alpha}{\big(r+|y-z|\big)^{n-1+\alpha}} f(y)\,\mathrm{d}\sigma(y) &\leq \int_{R_j(z)} \frac{r^\alpha}{|y-z|^{n-1+\alpha}} f(y)\,\mathrm{d}\sigma(y) \\
&\leq \frac{C}{r^{n-1}2^{j(n-1+\alpha)}} \int_{\Delta(z,2^{j+1}r)} f\,\mathrm{d}\sigma \\
&\leq \frac{C2^{-j\alpha}}{\sigma\big(\Delta(z,2^{j+1}r)\big)} \int_{\Delta(z,2^{j+1}r)} f\,\mathrm{d}\sigma \\
&\leq C2^{-j\alpha}\big(\mathcal{M}_{\partial\Omega}f\big)(z).
\end{aligned} \tag{8.4.158}$$

Also, corresponding to $j = 0$, we have

$$\begin{aligned}
\int_{R_0(z)} \frac{r^\alpha}{\big(r+|y-z|\big)^{n-1+\alpha}} f(y)\,\mathrm{d}\sigma(y) &\leq \frac{C}{r^{n-1}} \int_{\Delta(z,r)} f\,\mathrm{d}\sigma \\
&\leq \frac{C}{\sigma\big(\Delta(z,r)\big)} \int_{\Delta(z,r)} f\,\mathrm{d}\sigma \\
&\leq C\big(\mathcal{M}_{\partial\Omega}f\big)(z).
\end{aligned} \tag{8.4.159}$$

On account of (8.4.157)–(8.4.159), we, therefore, obtain the pointwise inequality

$$\big(\mathcal{N}_\kappa(\mathscr{P}f)\big)(z) \le C\big(\mathcal{M}_{\partial\Omega}f\big)(z), \qquad \forall z \in \partial\Omega, \tag{8.4.160}$$

where $C = C(\partial\Omega, C_0, \kappa) > 0$ is a finite constant, independent of f. This establishes (8.4.152).

In turn, the claims in the last part of the proposition are consequences of (8.4.152), Corollary 7.6.3, item *(1)* in Lemma 7.7.1, plus the boundedness of the Hardy–Littlewood maximal operator on Morrey and block spaces on Ahlfors regular sets (cf. [185, Sect. 6.2]). □

8.5 Maximal Operators: Tangential Versus Nontangential

In addition to the nontangential maximal operator, introduced earlier in Sect. 8.2, there is a related version in which the essential supremum is taken over the entire open set, rather than just a nontangential approach region. Because this larger set permits approaching the boundary in a tangential fashion, it is natural to call this brand on maximal operator "tangential." For each given boundary point, the essential supremum entering its definition is suitably weighted by a power of the ratio between the distance to the boundary and the distance to the point in question. Here is a precise definition.

Definition 8.5.1 *Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$, and abbreviate*

$$\delta_{\partial\Omega}(x) := \mathrm{dist}(x, \partial\Omega) \ \textit{ for each } \ x \in \mathbb{R}^n. \tag{8.5.1}$$

Also, fix a power $M \in [0, \infty)$, and suppose $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function. In this context, introduce the `tangential maximal function` *of u (with exponent M) to be*

$$\begin{gathered} u_M^{\max} : \partial\Omega \longrightarrow [0, +\infty] \ \textit{ defined by} \\ u_M^{\max}(x) := \Big\| u(y)\Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M \Big\|_{L^\infty_y(\Omega, \mathcal{L}^n)} \ \textit{ for each } \ x \in \partial\Omega, \end{gathered} \tag{8.5.2}$$

where the subscript y indicates that the essential norm is taken with respect to the variable $y \in \Omega$.

It is then clear that, in the context of the above definition, for any Lebesgue measurable function $u : \Omega \to \mathbb{R}$ we have

$$u \notin L^\infty_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \iff u_M^{\max} \equiv +\infty \ \text{ on } \ \partial\Omega, \tag{8.5.3}$$

and

$$u_{M_2}^{\max} \le u_{M_1}^{\max} \ \text{ on } \ \partial\Omega \ \text{ whenever } \ 0 \le M_1 \le M_2 < +\infty. \tag{8.5.4}$$

Much of the subsequent material in this section is devoted to comparing the brand of tangential maximal operator from Definition 8.5.1 with its nontangential counterpart, introduced earlier in Sect. 8.2. We begin by observing that the former always dominates the latter, in a pointwise fashion, up to a multiplicative constant.

Lemma 8.5.2 *Suppose Ω is an open, nonempty, proper subset of $\mathbb{R}^n$, and fix an arbitrary power $M \in [0, \infty)$ along with some aperture parameter $\kappa \in (0, \infty)$. Then for each Lebesgue measurable function $u : \Omega \to \mathbb{R}$, one has*

$$\big(\mathcal{N}_\kappa u\big)(x) \leq (1+\kappa)^M \cdot u_M^{\max}(x) \quad \textit{for each} \quad x \in \partial\Omega. \tag{8.5.5}$$

Proof This is a direct consequence of definitions, bearing in mind that for each point $x \in \partial\Omega$, we have $\frac{1}{(1+\kappa)^M} \leq \Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M$ for all $y \in \Gamma_\kappa(x)$. □

We slightly digress for the purpose of identifying a content in which the tangential maximal function happens to be measurable.

Lemma 8.5.3 *Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$, satisfying*

$$\mathcal{H}^{n-1}(\partial\Omega \cap K) < +\infty \quad \textit{for each compact} \quad K \subset \mathbb{R}^n, \tag{8.5.6}$$

and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix some power $M \in [0, \infty)$ along with some Lebesgue measurable function $u : \Omega \to \mathbb{R}$. Then

$$u_M^{\max} : \partial\Omega \longrightarrow [0, +\infty] \quad \textit{is } \sigma\textit{-measurable}. \tag{8.5.7}$$

Proof From Lemma 3.6.4, we know that

$$\begin{array}{l}\sigma \text{ is a complete, locally finite, Borel-regular measure on } \partial\Omega \\ \text{(equipped with the relative topology, induced by } \mathbb{R}^n).\end{array} \tag{8.5.8}$$

To proceed, recall from (8.5.3) that if $u \notin L^\infty_{\rm loc}(\Omega, \mathcal{L}^n)$, then $u_M^{\max} \equiv +\infty$ on $\partial\Omega$, so (8.5.7) is trivially true in this case. As such, there remains to consider the situation when

$$u \in L^\infty_{\rm loc}(\Omega, \mathcal{L}^n). \tag{8.5.9}$$

In this scenario, for each $j \in \mathbb{N}$ introduce

$$\Omega_j := \big\{x \in \Omega : \ |x| < j \ \text{ and } \ \delta_{\partial\Omega}(x) > 1/j\big\} \tag{8.5.10}$$

and define

$$\begin{gathered} u_{M,j}^{\max} : \partial\Omega \longrightarrow [0, +\infty] \ \text{ acting according to} \\ u_{M,j}^{\max}(x) := \Big\| u(y)\Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M \Big\|_{L^\infty_y(\Omega_j, \mathcal{L}^n)} \ \text{ for each } \ x \in \partial\Omega. \end{gathered} \tag{8.5.11}$$

Since $\Omega_j \nearrow \Omega$ as $j \nearrow \infty$, it follows that

$$u_M^{\max}(x) = \lim_{j\to\infty} u_{M,j}^{\max}(x) \quad \text{for each} \quad x \in \partial\Omega. \tag{8.5.12}$$

As far as (8.5.7) is concerned, from (8.5.12), Remark 3.1.2, and (8.5.8), we see that it suffices to show that

$$u_{M,j}^{\max} : \partial\Omega \longrightarrow [0, +\infty] \quad \text{is } \sigma\text{-measurable for each} \quad j \in \mathbb{N}. \tag{8.5.13}$$

With this goal in mind, fix an arbitrary index $j \in \mathbb{N}$ and observe from (8.5.9)–(8.5.10) that

$$u\big|_{\Omega_j} \in L^\infty(\Omega_j, \mathcal{L}^n). \tag{8.5.14}$$

As a result, for each fixed point $x \in \partial\Omega$, we have

$$\left\| u(y)\Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M \right\|_{L^\infty_y(\Omega_j,\mathcal{L}^n)} < +\infty. \tag{8.5.15}$$

Since also $\mathcal{L}^n(\Omega_j) < +\infty$, a well-known property of the family of Lebesgue norms (cf., e.g., [91, Exercise 7, p. 187]) gives

$$\left\| u(y)\Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M \right\|_{L^\infty_y(\Omega_j,\mathcal{L}^n)} = \lim_{p\to\infty} \left\| u(y)\Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M \right\|_{L^p_y(\Omega_j,\mathcal{L}^n)}. \tag{8.5.16}$$

Hence, if for each $p \in \mathbb{N}$, we introduce

$$\begin{gathered} u_{M,j,p}^{\max} : \partial\Omega \longrightarrow [0, +\infty] \quad \text{defined as} \\ u_{M,j,p}^{\max}(x) := \Big(\int_{\Omega_j} |u(y)|^p \Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^{Mp} \, dy \Big)^{1/p} \quad \text{for each} \quad x \in \partial\Omega, \end{gathered} \tag{8.5.17}$$

then from (8.5.11) and (8.5.16), we see that

$$u_{M,j}^{\max}(x) = \lim_{p\to\infty} u_{M,j,p}^{\max}(x) \quad \text{for each} \quad x \in \partial\Omega. \tag{8.5.18}$$

In view of this, by once again appealing to Remark 3.1.2 and (8.5.8), we see that, as far as (8.5.13) is concerned, it suffices to prove that

$$u_{M,j,p}^{\max} : \partial\Omega \longrightarrow [0, +\infty] \quad \text{is } \sigma\text{-measurable for each} \quad j, p \in \mathbb{N}. \tag{8.5.19}$$

However, it is clear from (8.5.17) and (8.5.14) that each $u_{M,j,p}^{\max}$ is actually a continuous function (as may be seen using Vitali's convergence theorem), hence Borel measurable. Given that, as noted earlier, σ is a Borel measure, the desired conclusion follows. $\square$

Returning to the task of comparing the tangential and nontangential maximal operators, here is a pointwise inequality which goes in the opposite direction to the one considered in Lemma 8.5.2.

Proposition 8.5.4 *Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$ such that $\partial\Omega$ is an Ahlfors regular set. Fix an exponent $s \in (0, \infty)$ along with an aperture parameter $\kappa \in (0, \infty)$ and pick some real number $M \geq (n-1)/s$. Then there exists a constant $C \in (0, \infty)$ with the property that for each Lebesgue measurable function $u : \Omega \to \mathbb{R}$ one has the pointwise inequality*

$$u_M^{\max}(x) \leq C \cdot \Big(\mathcal{M}_{\partial\Omega,s}\big(\mathcal{N}_\kappa u\big)\Big)(x) \ \textit{ for each } \ x \in \partial\Omega, \tag{8.5.20}$$

where $\mathcal{M}_{\partial\Omega,s}$ stands for the L^s-based Hardy–Littlewood maximal operator on $\partial\Omega$ (defined as in (7.6.7)), with respect to the measure $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$).

In view of Lemma 8.5.2, we may regard Proposition 8.5.4 as an extension of Theorem 8.4.6, where nontangential maximal functions corresponding to various aperture parameters have been compared. We also wish to note that, in the setting of Proposition 8.5.4, from (8.5.20) and (8.5.5) we deduce that for any given power $M_o \in [0, \infty)$, there exists a constant $C \in (0, \infty)$, such that

$$u_M^{\max} \leq C \cdot \mathcal{M}_{\partial\Omega,s}\big(u_{M_o}^{\max}\big) \ \text{ on } \ \partial\Omega \tag{8.5.21}$$

for each Lebesgue measurable function $u : \Omega \to \mathbb{R}$ (this should be compared with (8.5.4)).

Proof of Proposition 8.5.4 Fix an arbitrary point $x \in \partial\Omega$ and decompose

$$\Omega = \bigcup_{j=0}^{\infty} \Gamma_{2^j\kappa}(x) = \Gamma_\kappa(x) \cup \Big(\bigcup_{j=1}^{\infty} \big(\Gamma_{2^j\kappa}(x) \setminus \Gamma_{2^{j-1}\kappa}(x)\big)\Big). \tag{8.5.22}$$

Given an arbitrary Lebesgue measurable function $u : \Omega \to \mathbb{R}$, use this and (8.5.2) to write

$$u_M^{\max}(x) = \Big\| u(y)\Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M \Big\|_{L_y^\infty(\Omega,\mathcal{L}^n)} = \max\Big\{\mathrm{I}_0, \sup_{j\in\mathbb{N}} \mathrm{I}_j\Big\}, \tag{8.5.23}$$

where

$$\mathrm{I}_0 := \Big\| u(y)\Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M \Big\|_{L_y^\infty(\Gamma_\kappa(x),\mathcal{L}^n)} \tag{8.5.24}$$

and, for each $j \in \mathbb{N}$,

$$\mathrm{I}_j := \Big\| u(y)\Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M \Big\|_{L_y^\infty(\Gamma_{2^j\kappa}(x)\setminus\Gamma_{2^{j-1}\kappa}(x),\mathcal{L}^n)}. \tag{8.5.25}$$

Since $M \geq 0$ and $\frac{\delta_{\partial\Omega}(y)}{|x-y|} \leq 1$ for each $y \in \Omega$, it follows that

$$\mathrm{I}_0 \leq \|u\|_{L^\infty(\Gamma_\kappa(x),\mathcal{L}^n)} = \big(\mathcal{N}_\kappa u\big)(x) \leq C \cdot \Big(\mathcal{M}_{\partial\Omega,s}\big(\mathcal{N}_\kappa u\big)\Big)(x), \tag{8.5.26}$$

where the last inequality is a consequence of (8.4.42), presently used with $\widetilde{\kappa} := \kappa$, bearing in mind that the measure $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is doubling (thanks to the Ahlfors regularity of $\partial\Omega$). Also, given any $j \in \mathbb{N}$, for each point $y \in \Gamma_{2^j\kappa}(x) \setminus \Gamma_{2^{j-1}\kappa}(x)$, we have $|x - y| \geq (1 + 2^{j-1}\kappa)\delta_{\partial\Omega}(y)$, hence

$$\Big(\frac{\delta_{\partial\Omega}(y)}{|x-y|}\Big)^M \leq \frac{1}{(1+2^{j-1}\kappa)^M} \leq \Big(\frac{2^M}{\kappa^M}\Big)2^{-jM}. \tag{8.5.27}$$

Consequently, for each $j \in \mathbb{N}$, we may rely on (8.5.27), (8.4.41), and (8.4.52) to estimate

$$\begin{aligned} \mathrm{I}_j &\leq \Big(\frac{2^M}{\kappa^M}\Big)2^{-jM}\|u\|_{L^\infty_y(\Gamma_{2^j\kappa}(x),\mathcal{L}^n)} = \Big(\frac{2^M}{\kappa^M}\Big)2^{-jM}\big(\mathcal{N}_{2^j\kappa}u\big)(x) \\ &\leq C_{\partial\Omega,n,\kappa,s,M} \cdot 2^{-jM} \cdot (2^j\kappa)^{(n-1)/s} \cdot \Big(\mathcal{M}_{\partial\Omega,s}\big(\mathcal{N}_\kappa u\big)\Big)(x) \\ &\leq C_{\partial\Omega,n,\kappa,s,M} \cdot \Big(\mathcal{M}_{\partial\Omega,s}\big(\mathcal{N}_\kappa u\big)\Big)(x) \end{aligned} \tag{8.5.28}$$

where the last inequality uses the fact that $M \geq (n-1)/s$. All together, from (8.5.23), (8.5.26), and (8.5.28), we deduce that (8.5.20) holds. □

At this stage, we may use the results developed so far in this section to prove a general principle for comparing the size of the nontangential maximal function with the size of the tangential maximal function in various abstract norms.

Theorem 8.5.5 *Fix $n \in \mathbb{N}$ satisfying $n \geq 2$, and consider an open, nonempty, proper subset Ω of $\mathbb{R}^n$ such that $\partial\Omega$ is Ahlfors regular. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and denote by $\mathscr{M}_+(\partial\Omega,\sigma)$ the space of non-negative σ-measurable functions on $\partial\Omega$. Consider a mapping*

$$\|\cdot\| : \mathscr{M}_+(\partial\Omega,\sigma) \longrightarrow [0,+\infty] \tag{8.5.29}$$

enjoying the following three properties:

(i) [*Quasi-Monotonicity*] *There exists $C_1 \in (0,\infty)$ such that $\|f\| \leq C_1\|g\|$ for any pair of functions $f, g \in \mathscr{M}_+(\partial\Omega,\sigma)$ satisfying $f \leq g$ at σ-a.e. point on $\partial\Omega$.*

(ii) [*Quasi-Homogeneity*] *There exists $C_2 \in (0,\infty)$ such that $\|\lambda f\| \leq C_2\lambda\|f\|$ for any function $f \in \mathscr{M}_+(\partial\Omega,\sigma)$ and any number $\lambda \in (0,\infty)$.*

(iii) [*Boundedness of some L^s-based Hardy–Littlewood Maximal Operator*] *There exist an integrability exponent $s \in (0,\infty)$ along with a constant $C_3 \in (0,\infty)$, such that*

$$\big\|\mathcal{M}_{\partial\Omega,s}f\big\| \leq C_3\|f\| \ \text{ for each } \ f \in \mathscr{M}_+(\partial\Omega,\sigma), \tag{8.5.30}$$

where $\mathcal{M}_{\partial\Omega,s}$ is the L^s-based Hardy–Littlewood maximal operator acting on σ-measurable functions on $\partial\Omega$ as in (8.4.40).

Then for any power $M \in \left[\frac{n-1}{s}, \infty\right)$ *and any aperture parameter* $\kappa \in (0, \infty)$, *there exists a constant* $C \in (1, \infty)$, *which depends only on* C_1, C_2, C_3, M, κ *and the Ahlfors regularity character of* $\partial\Omega$, *with the property that*

$$C^{-1}\left\|\mathcal{N}_\kappa u\right\| \leq \|u_M^{\max}\| \leq C\left\|\mathcal{N}_\kappa u\right\| \tag{8.5.31}$$

for each Lebesgue measurable function $u : \Omega \to \mathbb{C}$.

Proof The estimates claimed in (8.5.31) are consequences of Lemma 8.5.2, Lemma 8.5.3, Proposition 8.5.4, (8.2.28), and properties *(i)*-*(iii)* in the statement. □

Among other things, Theorem 8.5.5 is applicable to any Köthe function space on $\partial\Omega$ (defined as in [185, Sect. 1.5]) on which the Hardy–Littlewood maximal operator is bounded. Our next result contains further specific embodiments of the general principle established in Theorem 8.5.5, corresponding to a wealth of concrete function spaces for which conditions *(i)*-*(iii)* formulated above hold (in this vein, we wish to remark that the scale of Morrey and block spaces on Ahlfors regular sets are methodically discussed in [185, Sect. 6.2]).

Corollary 8.5.6 *Pick* $n \in \mathbb{N}$ *satisfying* $n \geq 2$, *and suppose* Ω *is an open, nonempty, proper subset of* $\mathbb{R}^n$ *with the property that* $\partial\Omega$ *is Ahlfors regular. Set* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and select an arbitrary aperture parameter* $\kappa \in (0, \infty)$. *Then the following results are true:*

(1) [*Ordinary Lebesgue Spaces*] *For each* $p \in (0, \infty]$ *and each* $M > (n-1)/p$, *one has*
$$\left\|u_M^{\max}\right\|_{L^p(\partial\Omega,\sigma)} \approx \left\|\mathcal{N}_\kappa u\right\|_{L^p(\partial\Omega,\sigma)}, \tag{8.5.32}$$
in a uniformly fashion for Lebesgue measurable functions $u : \Omega \to \mathbb{C}$.

(2) [*Lorentz Spaces*] *Fix* $p \in (0, \infty]$ *and* $q \in (0, \infty]$, *employing the convention that* $q = \infty$ *if* $p = \infty$, *and suppose* $M > (n-1)/p$. *Then*
$$\left\|u_M^{\max}\right\|_{L^{p,q}(\partial\Omega,\sigma)} \approx \left\|\mathcal{N}_\kappa u\right\|_{L^{p,q}(\partial\Omega,\sigma)}, \tag{8.5.33}$$
in a uniformly fashion for Lebesgue measurable functions $u : \Omega \to \mathbb{C}$.

(3) [*Muckenhoupt Weighted Lebesgue Spaces*] *Fix an exponent* $p \in (1, \infty)$ *along with a Muckenhoupt weight* $w \in A_p(\partial\Omega, \sigma)$, *and assume* $M \geq n-1$. *Then*
$$\left\|u_M^{\max}\right\|_{L^p(\partial\Omega,w\sigma)} \approx \left\|\mathcal{N}_\kappa u\right\|_{L^p(\partial\Omega,w\sigma)}, \tag{8.5.34}$$
in a uniformly fashion for Lebesgue measurable functions $u : \Omega \to \mathbb{C}$.

(4) [*Morrey Spaces*] *Fix some* $p \in (1, \infty)$ *together with* $\lambda \in (0, n-1)$, *and suppose* $M \geq n-1$. *Then*

$$\|u_M^{\max}\|_{M^{p,\lambda}(\partial\Omega,\sigma)} \approx \|\mathcal{N}_\kappa u\|_{M^{p,\lambda}(\partial\Omega,\sigma)}, \tag{8.5.35}$$

in a uniformly fashion for Lebesgue measurable functions $u : \Omega \to \mathbb{C}$.

(5) [*Block Spaces*] *Pick some $p \in (1,\infty)$ along with $\lambda \in (0, n-1)$, and assume $M \geq n-1$. Then*

$$\|u_M^{\max}\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)} \approx \|\mathcal{N}_\kappa u\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}, \tag{8.5.36}$$

in a uniformly fashion for Lebesgue measurable functions $u : \Omega \to \mathbb{C}$.

Proof The claim made in item *(1)* is a consequence of Theorem 8.5.5 used with $\|\cdot\| := \|\cdot\|_{L^p(\partial\Omega,\sigma)}$, keeping in mind that, for this choice, condition *(iii)* in Theorem 8.5.5 is implied by (7.6.11). In a similar fashion, the claim in item *(2)* follows from Theorem 8.5.5 used with $\|\cdot\| := \|\cdot\|_{L^{p,q}(\partial\Omega,\sigma)}$. For such a choice, condition *(i)* in Theorem 8.5.5 is ensured by (6.2.16), while condition *(iii)* in Theorem 8.5.5 is guaranteed by (7.6.9). Going further, the claim in item *(3)* is implied by Theorem 8.5.5 used with $\|\cdot\| := \|\cdot\|_{L^p(\partial\Omega,w\sigma)}$, a choice for which condition *(iii)* in Theorem 8.5.5 seen from item *(1)* in Lemma 7.7.1.

Moving on, the claim made in item *(4)* is a consequence of Theorem 8.5.5, presently used with $\|\cdot\| := \|\cdot\|_{M^{p,\lambda}(\partial\Omega,\sigma)}$. For this choice, condition *(i)* in Theorem 8.5.5 is noted in (8.4.107), while condition *(iii)* in Theorem 8.5.5 follows from the fact that the Hardy–Littlewood maximal operator on $\partial\Omega$ induces a well-defined, sub-linear and bounded mapping on Morrey spaces (cf. [185, Sect. 6.2]). Likewise, the claim in item *(5)* is dealt with similarly, now employing Theorem 8.5.5 with $\|\cdot\| := \|\cdot\|_{\mathcal{B}^{q,\lambda}(\partial\Omega,\sigma)}$, and relying on the lattice property for block spaces together with the boundedness of the Hardy–Littlewood maximal operator on block spaces (see [185, Sect. 6.2]). □

It is also possible to produce a version of Proposition 8.5.4 in which the assumption that the topological boundary of the underlying open set is Ahlfors regular has been replaced by the weaker demand that the measure on said boundary (with respect to which the Hardy–Littlewood maximal operator is considered) is doubling and Borel-semiregular.

Proposition 8.5.7 *Let Ω be an open, nonempty, proper subset of $\mathbb{R}^n$, and suppose σ is a doubling, Borel-semiregular measure on $\partial\Omega$. Denote by D_σ the doubling order of the measure σ* (*defined as in* (7.4.3)), *and fix an exponent $s \in (0,\infty)$. In this setting, bring in $\mathcal{M}_{\partial\Omega,s}$, the L^s-based Hardy–Littlewood maximal operator on $\partial\Omega$, defined as in* (7.6.7) *with respect to the measure σ. Finally, pick an aperture parameter $\kappa \in (0,\infty)$.*

Then, whenever $M \geq D_\sigma/s$, one can find a constant $C \in (0,\infty)$, such that

$$u_M^{\max}(x) \leq C \cdot \Big(\mathcal{M}_{\partial\Omega,s}\big(\mathcal{N}_\kappa u\big)\Big)(x) \quad \textit{for all} \quad x \in \partial\Omega, \tag{8.5.37}$$

for every Lebesgue measurable function $u : \Omega \to \mathbb{R}$.

Proof The same type of argument as in the proof of Proposition 8.5.4 continues to work, now making use of (8.4.51) (in place of (8.4.52)). □

In turn, Proposition 8.5.7 yields a version of Theorem 8.5.5 in which the assumption that the topological boundary of the underlying open set is Ahlfors regular is replaced by the weaker hypothesis that the measure on the said boundary (with respect to which the Hardy–Littlewood maximal operator is defined) is doubling and Borel-semiregular. Ultimately, this permits establishing versions of items *(1)-(3)* in Corollary 8.5.6 in this more general setting.

8.6 Off-Diagonal Carleson Measure Estimates of Reverse Hölder Type

Recall that, in the classical setting of the upper half-space $\mathbb{R}^n_+$, a Borel measure μ on $\mathbb{R}^n_+$ is called a `Carleson measure` provided there exists a finite constant $C > 0$ with the property that

$$\mu\big(B(x,r)\cap\mathbb{R}^n_+\big) \leq C r^{n-1}, \qquad \forall x \in \mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+, \quad \forall r > 0. \tag{8.6.1}$$

For such a measure, the classical Carleson estimate asserts that

$$\int_{\mathbb{R}^n_+} |u|\, d\mu \leq C_{n,\mu,\kappa} \int_{\mathbb{R}^{n-1}} \mathcal{N}_\kappa u \, d\mathcal{H}^{n-1}, \tag{8.6.2}$$

for any Borel-measurable function $u : \mathbb{R}^n_+ \to \mathbb{R}$ where $\kappa > 0$ is a fixed aperture parameter and $\mathcal{N}_\kappa$ is the corresponding nontangential maximal operator in the upper half-space. Of course, replacing u by $|u|^p$ for some $p \in (0,\infty)$ in (8.6.2) yields

$$\Big(\int_{\mathbb{R}^n_+} |u|^p \, d\mu\Big)^{1/p} \leq C_{n,\mu,\kappa,p}\Big(\int_{\mathbb{R}^{n-1}} (\mathcal{N}_\kappa u)^p \, d\mathcal{H}^{n-1}\Big)^{1/p}, \tag{8.6.3}$$

for any Borel-measurable function $u : \mathbb{R}^n_+ \to \mathbb{R}$. Inequality (8.6.3) is a basic example of what might be termed as a "collapsing" inequality, in the sense that it permits us to estimate the L^p norm of a function u in the domain $\mathbb{R}^n_+$ in terms of another L^p norm, this time taken on the boundary of the domain in question. The missing (in this case, vertical) direction is, instead, accounted for in the way the nontangential maximal operator has been defined in (8.2.1). Another key feature is the fact that the definition of the Carleson measure in (8.6.1) may be refashioned as a condition relating the growths of μ and $\mathcal{H}^{n-1}$ (the two measures intervening in (8.6.3)). With impending generalizations in mind, the aforementioned condition may be expressed as

$$\begin{gathered}\mu\big(B(x,r)\cap\mathbb{R}^n_+\big) \leq C\big[\mathcal{H}^{n-1}\big(B(x,r)\cap\mathbb{R}^{n-1}\big)\big]^\alpha, \\ \forall x \in \mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+, \ \forall r > 0, \ \text{provided } \alpha = 1.\end{gathered} \tag{8.6.4}$$

The main goal in this section is to derive significantly more general versions of the Carleson estimate (8.6.3), for pairs (μ, σ) of measures satisfying a suitable variant of (8.6.4), which we shall call a *Carleson pair condition* (see Definition 8.6.1). In addition, the estimate that we have in mind is off-diagonal, i.e., of the type L^p-L^q with p not necessarily equal to q (by way of contrast, (8.6.3) is of type L^p-L^p), and incorporates a reverse Hölder inequality, in the sense that $p > q$. It is precisely this aspect where having $\alpha > 1$ as opposed to $\alpha = 1$ plays a crucial role.

One, albeit quite specialized, estimate which shares the above features is the classical isoperimetric inequality to the effect that if $\Omega \subseteq \mathbb{R}^n$ is a bounded, reasonably decent set, then we have $\mathcal{L}^n(\Omega) \leq c_n[\mathcal{H}^{n-1}(\partial\Omega)]^{n/(n-1)}$ (cf. (5.7.19)). Indeed, this can be equivalently re-written as the L^1-$L^{(n-1)/n}$ estimate

$$\int_\Omega |u| \, d\mathcal{L}^n \leq c_n \Big(\int_{\partial\Omega} (\mathcal{N}_\kappa u)^{(n-1)/n} \, d\mathcal{H}^{n-1} \Big)^{n/(n-1)} \tag{8.6.5}$$

with the convention that $u \equiv 1$ and the nontangential maximal operator $\mathcal{N}_\kappa$ is now associated with the domain Ω. Note that, in this scenario, the correlation between the two measures involved in (8.6.5) continues to be of the form (8.6.4) though, this time, $\alpha = n/(n-1) > 1$, i.e., for all $x \in \partial\Omega$ and $r > 0$ we have

$$\mathcal{L}^n\big(B(x,r) \cap \Omega\big) \leq C\big[\mathcal{H}^{n-1}\big(B(x,r) \cap \partial\Omega\big)\big]^{n/(n-1)}. \tag{8.6.6}$$

Indeed, as we shall see later, (8.6.5) remains valid for any $\mathcal{L}^n$-measurable function $u : \Omega \to \mathbb{R}$, and not just for $u \equiv 1$ which corresponds to the classical isoperimetric inequality. As such, it is reasonable to think of (8.6.5) as a weighted isoperimetric inequality, with $|u|$ being the weight of $\mathcal{L}^n$ in Ω and $(\mathcal{N}_\kappa u)^{(n-1)/n}$ being the weight of $\mathcal{H}^{n-1}$ on $\partial\Omega$.

The philosophy that emerges is that off-diagonal Carleson estimates of reverse Hölder type ultimately involve two measures, μ defined on a metric space (X, ρ) and σ defined on $\Sigma \subseteq X$, with the property that there exists $\alpha \geq 1$ such that

$$\mu\big(B_\rho(x,r) \setminus \overline{\Sigma}\big) \leq C\big[\sigma(B_\rho(x,r) \cap \Sigma)\big]^\alpha, \ \ \forall x \in \Sigma, \ \ \forall r \in (0, \operatorname{diam}(\Sigma)]. \tag{8.6.7}$$

Note that if $X = \overline{\mathbb{R}^n_+}$, μ is a Carleson measure in $\mathbb{R}^n_+$, and we take $\sigma := \mathcal{H}^{n-1}$ on $\Sigma := \partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}$, then (8.6.7) holds with $\alpha = 1$. We shall actually work with a slightly more general condition than (8.6.7), as described in the following definition.

Definition 8.6.1 *Let (X, ρ) be a quasi-metric space such that $\rho : X \times X \to [0, \infty)$ is continuous with respect to the product topology $\tau_\rho \times \tau_\rho$, and suppose Σ is a nonempty subset of X. Given some $\alpha, \beta \in \mathbb{R}$, call (μ, σ) an (α, β)-*`Carleson pair` *for the ambient (X, Σ) provided μ is a Borel measure on $X \setminus \overline{\Sigma}$, σ is a Borel measure on Σ, and there exists a finite constant $C > 0$ such that*

$$\begin{gathered} \mu\big(B_\rho(x,r) \setminus \overline{\Sigma}\big) \leq C r^\beta \big[\sigma(B_\rho(x,r) \cap \Sigma)\big]^\alpha \\ \text{for all } \ x \in \Sigma \ \text{ and all finite } \ r \in (0, \operatorname{diam}_\rho(\Sigma)]. \end{gathered} \tag{8.6.8}$$

The quantity

$$\sup_{\substack{r\in(0,\mathrm{diam}_\rho(\Sigma)] \\ \textit{finite, and } x\in\Sigma}} \left(\frac{\mu\big(B_\rho(x,r)\setminus\overline{\Sigma}\big)}{r^\beta\big[\sigma(B_\rho(x,r)\cap\Sigma)\big]^\alpha} \right) \in [0,\infty) \tag{8.6.9}$$

is referred to as the (α,β)*-Carleson constant of the pair* (μ,σ)*.*

Here is our first off-diagonal Carleson estimate of reverse Hölder type (the reader is reminded that the local solid maximal operator $u\mapsto u^E_{\star,\theta}$ has been introduced in (6.6.79)).

Theorem 8.6.2 *Assume that* Ω *is an arbitrary nonempty open proper subset of* $\mathbb{R}^n$*. Suppose* (μ,σ) *is an* (α,β)*-Carleson pair for the ambient* $(\Omega,\partial\Omega)$ *for some* $\alpha\geq 1$ *and* $\beta\geq 0$*. Next, select* $0<p<\infty$*,* $0<q\leq\infty$*,* $\kappa>0$*,* $0<\theta<1$*, along with two* μ*-measurable sets* $E,F\subseteq\Omega$*. For each power* $\eta\in[0,\infty)$ *define the weighted measure*

$$\widetilde{\sigma}_\eta := \begin{cases} \big(\mathrm{dist}(\cdot,\partial\Omega\setminus\pi_\kappa(E))\big)^\eta\cdot\sigma & \textit{if } \eta>0,\\ \sigma & \textit{if } \eta=0.\end{cases} \tag{8.6.10}$$

Then there exists a constant $C_0\in(0,\infty)$ *such that for each* μ*-measurable function* $u:E\to\mathbb{R}$ *one has the estimate*

$$\big\|u^E_{\star,\theta}\big\|_{L^{\alpha p,q}(F,\mu)} \leq C_0\big\|\mathcal{N}^E_\kappa u\big\|_{L^{p,q}(\pi_\kappa(E),\widetilde{\sigma}_{\beta/\alpha})} \tag{8.6.11}$$

in any of the circumstances (i)-(iii) described below (*the nature of* C_0 *being elucidated in each case*)*:*

(i) $\pi_\kappa(E)\neq\partial\Omega$ *and* C_0 *is a finite, positive constant, which depends only on* p*,* κ*,* α*,* θ*,* n*, and the* (α,β)*-Carleson constant of the pair* (μ,σ)*;*

(ii) $\sigma(\partial\Omega)=+\infty$*,* $\beta=0$*, and* C_0 *is a finite, positive constant, which depends only on* p*,* κ*,* α*,* θ*,* n*, and the* $(\alpha,0)$*-Carleson constant of the pair* (μ,σ)*;*

(iii) $\pi_\kappa(E)=\partial\Omega$*,* $\sigma(\partial\Omega)<+\infty$*,* $\mu(F)<+\infty$*,* $\beta=0$*, and*

$$C_0 := \max\Big\{C,\ \mu(F)\sigma(\partial\Omega)^{-\alpha}\Big\}^{1/(\alpha p)} \in (0,+\infty) \tag{8.6.12}$$

where C *in* (8.6.12) *is a finite positive constant which depends only on* p*,* κ*,* α*,* θ*,* n*, and the* $(\alpha,0)$*-Carleson constant of the pair* (μ,σ)*.*

In particular, in either of the cases (i)-(iii) above one has the estimate

$$\|u\|_{L^{\alpha p}(E,\mu)} \leq C_0\big\|\mathcal{N}^E_\kappa u\big\|_{L^{p}(\pi_\kappa(E),\widetilde{\sigma}_{\beta/\alpha})} \quad \textit{for every } \mu\textit{-measurable function } u:E\to\mathbb{R}. \tag{8.6.13}$$

Proof By extending u to the entire Ω to be zero outside E, we may assume that we are dealing with an $\mathcal{L}^n$-measurable function $u:\Omega\to\mathbb{R}$. In light of the conclusions we presently seek to establish, Lemma 6.6.2 (cf. also (6.6.83)) guarantees that there is no loss of generality in assuming that

$$\theta \in \big(0, \kappa/(\kappa+2)\big). \tag{8.6.14}$$

Suppose that this is the case and define

$$\kappa' := (1-\theta)\kappa - 2\theta > 0. \tag{8.6.15}$$

Next, consider the restricted "tent" regions

$$\begin{gathered} T_{F,\kappa'}(O) := F \setminus \Big[\bigcup_{y\in\partial\Omega\setminus O} \Gamma_{\kappa'}(y)\Big], \text{ for each} \\ O \subseteq \partial\Omega \text{ open in the relative topology on } \partial\Omega. \end{gathered} \tag{8.6.16}$$

In relation to these we claim that, for each $\lambda \in (0,\infty)$,

$$\big\{x \in F : u^E_{\star,\theta}(x) > \lambda\big\} \subseteq T_{F,\kappa'}\big(\{z \in \pi_\kappa(E) : \big(\mathcal{N}^E_\kappa u\big)(z) > \lambda\}\big). \tag{8.6.17}$$

We shall prove (8.6.17) reasoning by contradiction. Specifically, let $\lambda > 0$ be fixed and suppose that the inclusion in (8.6.17) fails, i.e., there exists $x \in F$ satisfying $u^E_{\star,\theta}(x) > \lambda$ for which we can find $y \in \partial\Omega$ with the property that $\big(\mathcal{N}^E_\kappa u\big)(y) \leq \lambda$ and $x \in \Gamma_{\kappa'}(y)$. In particular, on the one hand we have

$$\|u\|_{L^\infty(B(x,\theta\delta_{\partial\Omega}(x))\cap E,\mathcal{L}^n)} > \lambda, \tag{8.6.18}$$

while on the other hand

$$\|u\|_{L^\infty(\Gamma_\kappa(y)\cap E,\mathcal{L}^n)} \leq \lambda. \tag{8.6.19}$$

However, these two inequalities conflict with one another since

$$B\big(x, \theta\delta_{\partial\Omega}(x)\big) \subseteq \Gamma_\kappa(y). \tag{8.6.20}$$

To justify the inclusion in (8.6.20), observe that for each point $z \in B\big(x, \theta\delta_{\partial\Omega}(x)\big)$ we necessarily have $\delta_{\partial\Omega}(z) > (1-\theta)\delta_{\partial\Omega}(x)$ which, in turn, permits us to write (bearing in mind the choice of κ' in (8.6.15))

$$\begin{aligned} |z-y| &\leq |z-x| + |x-y| < \theta\delta_{\partial\Omega}(x) + (1+\kappa')\delta_{\partial\Omega}(x) \\ &= (1+\theta+\kappa')\delta_{\partial\Omega}(x) < \frac{1+\theta+\kappa'}{1-\theta}\delta_{\partial\Omega}(z) = (1+\kappa)\delta_{\partial\Omega}(z). \end{aligned} \tag{8.6.21}$$

This shows that $z \in \Gamma_\kappa(y)$, finishing the proof of (8.6.20) which, in turn, completes the proof of (8.6.17).

Our next goal is to show that there exists a constant $C \in (0, \infty)$ with the property that

$$\mu\big(T_{F,\kappa'}(\mathcal{O})\big) \leq C\Big[\int_{\mathcal{O}} \big(\operatorname{dist}(\cdot, \partial\Omega \setminus \mathcal{O})\big)^{\beta/\alpha}\, d\sigma\Big]^{\alpha} \quad \text{for any proper (relatively) open subset } \mathcal{O} \text{ of } \partial\Omega. \tag{8.6.22}$$

Note that since $F \subseteq \Omega$, from (8.1.7) we have $T_{F,\kappa'}(\varnothing) = \varnothing$, hence (8.6.22) trivially holds when $\mathcal{O} = \varnothing$. There remains to consider the case when the set $\mathcal{O}$ as in the last line of (8.6.22) is also nonempty. Assume this is the case and, making use of Proposition 7.5.3, decompose $\mathcal{O}$ it into a finite-overlap family of Whitney surface balls $\{\Delta_j\}_j$, by considering the set $\partial\Omega$ equipped with the Euclidean distance as a geometrically doubling metric space (cf. Lemma 7.5.2). Also, for each surface ball $\Delta := B(z, r) \cap \partial\Omega$, with $z \in \partial\Omega$ and $0 < r < 2\operatorname{diam}(\partial\Omega)$, define the restricted Carleson region

$$C_\tau^F(\Delta) := B(z, \tau r) \cap F, \tag{8.6.23}$$

where $\tau > 0$ is a large constant, to be specified later. We now claim that τ may be chosen so that

$$T_{F,\kappa'}(\mathcal{O}) \subseteq \bigcup_j C_\tau^F(\Delta_j). \tag{8.6.24}$$

In order to justify (8.6.24) we note that definition (8.6.16) can be rephrased as

$$T_{F,\kappa'}(\mathcal{O}) = \big\{x \in F : \operatorname{dist}(x, \mathcal{O}) \leq (1+\kappa')^{-1} \operatorname{dist}(x, \partial\Omega \setminus \mathcal{O})\big\}. \tag{8.6.25}$$

To see this, note that $x \in T_{F,\kappa'}(\mathcal{O})$ if and only if $x \in F$ and $|x-y| \geq (1+\kappa')\delta_{\partial\Omega}(x)$ for all $y \in \partial\Omega \setminus \mathcal{O}$, and that the last condition is equivalent with

$$(1+\kappa')^{-1}\operatorname{dist}(x, \partial\Omega \setminus \mathcal{O}) \geq \delta_{\partial\Omega}(x). \tag{8.6.26}$$

Since $\operatorname{dist}(x, \mathcal{O}) \geq \operatorname{dist}(x, \partial\Omega) = \delta_{\partial\Omega}(x)$, this proves that the right-to-left inclusion in (8.6.25) holds. On the other hand, given an arbitrary point x belonging to the set in the left-hand side of (8.6.25) it follows that $x \in F \subseteq \Omega$ and we select $x_0 \in \partial\Omega$ such that $|x - x_0| = \delta_{\partial\Omega}(x)$. Of course, either $x_0 \in \mathcal{O}$ or $x_0 \in \partial\Omega \setminus \mathcal{O}$, and in the latter eventuality we may write

$$\operatorname{dist}(x, \partial\Omega \setminus \mathcal{O}) \leq |x - x_0| = \delta_{\partial\Omega}(x) < (1+\kappa')\delta_{\partial\Omega}(x), \tag{8.6.27}$$

contradicting (8.6.26). Thus, necessarily, $x_0 \in \mathcal{O}$ which together with (8.6.26) then permits us to estimate

$$\operatorname{dist}(x, \mathcal{O}) \leq |x - x_0| = \delta_{\partial\Omega}(x) \leq (1+\kappa')^{-1} \operatorname{dist}(x, \partial\Omega \setminus \mathcal{O}). \tag{8.6.28}$$

This shows that the left-to-right inclusion in (8.6.25) also holds, completing the proof of (8.6.25).

Let now x be an arbitrary point in $T_{F,\kappa'}(\mathcal{O})$. This places x in F, hence in Ω, which means that x does not belong to $\overline{\mathcal{O}} \subseteq \partial\Omega$. In particular, $\mathrm{dist}(x, \mathcal{O}) > 0$. Going further, assume that some small $\varepsilon > 0$ has been fixed. The previous discussion then shows that it is possible to pick a point $x^* \in \mathcal{O}$ with the property that

$$|x - x^*| < (1+\varepsilon)\,\mathrm{dist}(x, \mathcal{O}). \tag{8.6.29}$$

Then there exists an index j for which $x^* \in \Delta_j$ and we shall show that ε and τ can be chosen so as to guarantee that

$$x \in C_\tau^F(\Delta_j). \tag{8.6.30}$$

Indeed, assume $\Delta_j = B(z_j, r_j) \cap \Sigma$ for some $z_j \in \partial\Omega$ and $r_j \in (0, 2\,\mathrm{diam}(\partial\Omega))$, and write

$$\begin{aligned} |x - x^*| &\leq (1+\varepsilon)\,\mathrm{dist}(x, \mathcal{O}) \leq \frac{1+\varepsilon}{1+\kappa'}\,\mathrm{dist}(x, \Sigma \setminus \mathcal{O}) \\ &\leq \frac{1+\varepsilon}{1+\kappa'}\Big(|x - x^*| + \mathrm{dist}(x^*, \partial\Omega \setminus \mathcal{O})\Big) \\ &\leq \frac{1+\varepsilon}{1+\kappa'}\Big(|x - x^*| + Cr_j\Big), \end{aligned} \tag{8.6.31}$$

where $C > 0$ is a constants depending only on $\partial\Omega$. The last step above uses the fact that $x^* \in \Delta_j$ and that Δ_j is a Whitney ball for $\mathcal{O}$ relative to the ambient $\partial\Omega$ (cf. item *(2)* in Proposition 7.5.3). Choosing $\varepsilon := \kappa'/2$, this now yields

$$|x - x^*| \leq C\Big(\frac{2+\kappa'}{\kappa'}\Big)r_j \tag{8.6.32}$$

hence, further,

$$|x - z_j| \leq |x - x^*| + |x^* - z_j| < Cr_j(2+\kappa')/\kappa' + r_j. \tag{8.6.33}$$

Granted this, the membership in (8.6.30) holds provided we take $\tau := 2 + C(\frac{2+\kappa'}{\kappa'})$ to begin with.

Having established (8.6.24), we may now finish the proof of (8.6.35) by estimating

$$\begin{aligned} \mu\big(T_{F,\kappa'}(\mathcal{O})\big) &\leq \sum_j \mu\big(C_t^F(\Delta_j)\big) \leq C\sum_j r_j^\beta \sigma(\Delta_j)^\alpha \\ &\leq C\sum_j \Big[\inf_{x\in\Delta_j} \big(\mathrm{dist}(x, \partial\Omega \setminus \mathcal{O})\big)^{\beta/\alpha}\sigma(\Delta_j)\Big]^\alpha \\ &\leq C\sum_j \Big[\int_{\Delta_j} \big(\mathrm{dist}(\cdot, \partial\Omega \setminus \mathcal{O})\big)^{\beta/\alpha}\,d\sigma\Big]^\alpha \end{aligned}$$

$$\leq C\Big[\sum_j \int_{\Delta_j} \big(\mathrm{dist}(\cdot, \partial\Omega \setminus O)\big)^{\beta/\alpha}\, \mathrm{d}\sigma\Big]^{\alpha}$$
$$\leq C\Big[\int_O \big(\mathrm{dist}(\cdot, \partial\Omega \setminus O)\big)^{\beta/\alpha}\, \mathrm{d}\sigma\Big]^{\alpha}. \tag{8.6.34}$$

Above, the second inequality in (8.6.34) is based on the fact that (μ, σ) is an (α, β)-Carleson pair, the third inequality makes use of the fact that for each $x \in \Delta_j$ we have $r_j \approx \mathrm{dist}(x, \partial\Omega \setminus O)$, the fifth inequality uses the fact that $\alpha \geq 1$ and, finally, the sixth inequality uses the finite-overlap property of the Δ_j's. This finishes the proof of (8.6.22), hence also of (8.6.35). Moreover, this analysis shows that the constant in the last term in (8.6.34) depends only on n, κ, and the (α, β)-Carleson constant of the pair (μ, σ). This concludes the proof of (8.6.22).

Going further, recall the weighted measure $\widetilde{\sigma}_{\beta/\alpha}$ from (8.6.10). In relation to this, we claim that there exists $C \in (0, \infty)$ with the property that

$$\begin{gathered}\text{if } \beta = 0, \text{ or } \pi_\kappa(E) \neq \partial\Omega, \text{ then } \ \mu\big(T_{F,\kappa'}(O)\big) \leq C\widetilde{\sigma}_{\beta/\alpha}(O)^{\alpha} \\ \text{for each } O \text{ proper (relatively) open subset of } \partial\Omega \text{ contained in } \pi_\kappa(E).\end{gathered} \tag{8.6.35}$$

Indeed, if $\beta = 0$ the inequality in (8.6.35) is a direct consequence of (8.6.22) and the convention in (8.6.10). On the other hand, if $\pi_\kappa(E) \neq \partial\Omega$ then for every subset O of $\pi_\kappa(E)$ we have

$$\mathrm{dist}(\cdot, \partial\Omega \setminus O) \leq \mathrm{dist}(\cdot, \partial\Omega \setminus \pi_\kappa(E)) \ \text{ at all points in } \ \partial\Omega, \tag{8.6.36}$$

so the inequality in (8.6.35) follows from (8.6.22) and (8.6.36), bearing in mind that $\beta \geq 0$.

We divide the remaining portion of the proof into three cases, corresponding to the scenarios *(i)-(iii)* considered in the statement of the theorem.

<u>`Case I`</u>: *Assume that* $\pi_\kappa(E) \neq \partial\Omega$. In such a situation, (8.2.26) ensures that for each given $\lambda > 0$ the set

$$\begin{gathered}O := \big\{z \in \partial\Omega : \big(\mathcal{N}_\kappa^E u\big)(z) > \lambda\big\} = \big\{z \in \pi_\kappa(E) : \big(\mathcal{N}_\kappa^E u\big)(z) > \lambda\big\} \\ \text{is a proper (relatively) open subset of } \ \partial\Omega, \ \text{ contained in } \ \pi_\kappa(E).\end{gathered} \tag{8.6.37}$$

Granted this, we may then invoke (8.6.17) and (8.6.35) which yield the level set estimate

$$\mu\big(\{x \in F : |u^E_{\star,\theta}(x)| > \lambda\}\big) \leq C\widetilde{\sigma}_{\beta/\alpha}\big(\{x \in \pi_\kappa(E) : \big(\mathcal{N}_\kappa^E u\big)(x) > \lambda\}\big)^{\alpha} \tag{8.6.38}$$

for each $\lambda > 0$, where $C > 0$ is a finite constant depending only on p, κ, n, and the (α, β)-Carleson constant of the pair (μ, σ). In terms of the abstract piece of notation introduced in (6.2.1), the above estimate reads

$$m_F\big(\lambda, u^E_{\star,\theta}\big) \leq C m_{\pi_\kappa(E)}\big(\lambda, \mathcal{N}_\kappa^E u\big)^{\alpha}, \quad \text{for all } \lambda > 0, \tag{8.6.39}$$

where m_F is defined relative to the measure μ, while $m_{\pi_\kappa(E)}$ is defined relative to the measure $\widetilde{\sigma}_{\beta/\alpha}$. In particular, for every $t > 0$,

$$\big\{\lambda > 0 : m_{\pi_\kappa(E)}\big(\lambda, \mathcal{N}_\kappa^E u\big) \le t\big\} \subseteq \big\{\lambda > 0 : m_F\big(\lambda, u^E_{\star,\theta}\big) \le C t^\alpha\big\}, \tag{8.6.40}$$

which, by (6.2.2), proves the rearrangement function estimate

$$\big(u^E_{\star,\theta}\big)^*_F(s) \le \big(\mathcal{N}_\kappa^E u\big)^*_{\pi_\kappa(E)}\big((s/C)^{\frac{1}{\alpha}}\big), \quad \text{for all } s > 0. \tag{8.6.41}$$

In the case in which $0 < q < \infty$, this further implies

$$\begin{aligned}\big\|u^E_{\star,\theta}\big\|_{L^{\alpha p,q}(F,\mathcal{L}^n)} &= \left(\int_0^\infty \Big[s^{\frac{1}{\alpha p}}\big(u^E_{\star,\theta}\big)^*_F(s)\Big]^q \frac{\mathrm{d}s}{s}\right)^{\frac{1}{q}} \\ &\le \left(\int_0^\infty \Big[s^{\frac{1}{\alpha p}}\big(\mathcal{N}_\kappa^E u\big)^*_{\pi_\kappa(E)}\big((s/C)^{\frac{1}{\alpha}}\big)\Big]^q \frac{\mathrm{d}s}{s}\right)^{\frac{1}{q}} \\ &\le C^{\frac{1}{\alpha p}}\left(\int_0^\infty \Big[t^{1/p}\big(\mathcal{N}_\kappa^E u\big)^*_{\pi_\kappa(E)}(t)\Big]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\ &= C^{\frac{1}{\alpha p}}\big\|\mathcal{N}_\kappa^E u\big\|_{L^{p,q}(\pi_\kappa(E),\sigma)},\end{aligned} \tag{8.6.42}$$

where in the second inequality we have made the change of variables $t = (s/C)^{\frac{1}{\alpha}}$, and the last step uses (6.2.14). Likewise, corresponding to the case $q = \infty$, we have

$$\begin{aligned}\big\|u^E_{\star,\theta}\big\|_{L^{\alpha p,\infty}(F,\mathcal{L}^n)} &= \sup_{s>0}\Big[s^{\frac{1}{\alpha p}}\big(u^E_{\star,\theta}\big)^*_F(s)\Big] \le \sup_{s>0}\Big[s^{\frac{1}{\alpha p}}\big(\mathcal{N}_\kappa^E u\big)^*_{\pi_\kappa(E)}\big((s/C)^{\frac{1}{\alpha}}\big)\Big] \\ &\le C^{\frac{1}{\alpha p}}\sup_{t>0}\Big[t^{1/p}\big(\mathcal{N}_\kappa^E u\big)^*_{\pi_\kappa(E)}(t)\Big] \\ &= C^{\frac{1}{\alpha p}}\big\|\mathcal{N}_\kappa^E u\big\|_{L^{p,\infty}(\pi_\kappa(E),\sigma)},\end{aligned} \tag{8.6.43}$$

as desired. Collectively, (8.6.42) and (8.6.43) prove (8.6.11) in case *(i)*.

Case II: *Assume that* $\sigma(\partial\Omega) = +\infty$ *and* $\beta = 0$. Note that the latter condition entails $\widetilde{\sigma}_{\beta/\alpha} = \sigma$ (cf. (8.6.10)). Keeping this in mind, and taking into account the nature of the conclusion we seek, there is no loss of generality in assuming that $\mathcal{N}_\kappa^E u \in L^{p,q}(\pi_\kappa(E), \sigma)$. In such a scenario, from (8.2.26), (8.2.27), and (6.2.20) it follows that, for each $\lambda > 0$, the set

$$\begin{aligned}&\mathcal{O} := \big\{z \in \partial\Omega : \big(\mathcal{N}_\kappa^E u\big)(z) > \lambda\big\} = \big\{z \in \pi_\kappa(E) : \big(\mathcal{N}_\kappa^E u\big)(z) > \lambda\big\} \\ &\text{is a proper (relatively) open subset of } \partial\Omega, \text{ contained in } \pi_\kappa(E).\end{aligned} \tag{8.6.44}$$

Having made this observation, from (8.6.17) and (8.6.35) we may then conclude that the level set estimate (8.6.38) with $\beta = 0$ holds in the present setting as well.

With this in hand, the same argument that has produced (8.6.42) and (8.6.43) proves (8.6.11) in case *(ii)*.

Case III: *Assume* $\pi_\kappa(E) = \partial\Omega$, $\sigma(\partial\Omega) < +\infty$, $\mu(F) < +\infty$, *and* $\beta = 0$. In particular, $\widetilde{\sigma}_{\beta/\alpha} = \sigma$. We may once again run the argument as in Case I as soon as we establish the estimate in (8.6.35) in the extreme case when $O = \partial\Omega$. Given that $T_{F,\kappa'}(\partial\Omega) = F$, the estimate in question presently takes the form

$$\mu(F) \leq C\sigma(\partial\Omega)^\alpha. \tag{8.6.45}$$

The smallest number $C \in [0, +\infty)$ which does this job is $C := \mu(F)\,\sigma(\partial\Omega)^{-\alpha}$ and, as seen from (8.6.42)–(8.6.43), the manner in which this constant intervenes in the final estimate (8.6.11) is as $C^{\frac{1}{\alpha p}} = \big[\mu(F)\big]^{\frac{1}{\alpha p}}\sigma(\partial\Omega)^{-\frac{1}{p}}$. This explains the choice of C_0 in (8.6.12) in case *(iii)*. Hence, (8.6.38) holds in this case as well and the desired conclusion follows as before.

At this point, (8.6.11) has been fully proved. Finally, (8.6.13) is a consequence of (8.6.11), the embedding (6.2.26) and identification (6.2.25) which give

$$L^{\alpha p,p}(E,\mu) \hookrightarrow L^{\alpha p,\alpha p}(E,\mu) = L^{\alpha p}(E,\mu), \tag{8.6.46}$$

as well as the pointwise estimate (6.6.82). This finishes the proof of Theorem 8.6.2. □

One particular case of Theorem 8.6.2 worth singling out corresponds to Ω being an open subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary and $\mu := \mathcal{L}^n\lfloor\Omega$. Such a setting makes the object of Proposition 8.6.3 below. In dimensions two and higher, taking $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ renders (μ,σ) an $(n/(n-1), 0)$-Carleson pair for the ambient $(\Omega, \partial\Omega)$ in such a setting, so Theorem 8.6.2 applies. In addition, thanks to the availability of isoperimetric inequalities in the current case, we may further elaborate on the nature of the constant C_0 inherited from (8.6.12).

Proposition 8.6.3 *Fix $n \in \mathbb{N}$ and let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary. Define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, choose $\kappa \in (0,\infty)$, and pick $\theta \in (0,1)$ along with $p \in (0,\infty)$ and $q \in (0,\infty]$, all arbitrary. Also, consider some $\mathcal{L}^n$-measurable set $E \subseteq \Omega$ and make the assumption that*

$$\begin{gathered} \mathcal{L}^n(E) < +\infty \quad \textit{in the scenario in which} \\ \mathcal{L}^n(\Omega) = +\infty, \;\; \sigma(\partial\Omega) < +\infty, \quad \textit{and} \quad \pi_\kappa(E) = \partial\Omega. \end{gathered} \tag{8.6.47}$$

Then for every $\mathcal{L}^n$-measurable function $u : E \to \mathbb{R}$ one has

$$\big\| u^E_{\star,\theta} \big\|_{L^{\frac{np}{n-1},q}(E,\mathcal{L}^n)} \leq C_\# \big\| \mathcal{N}^E_\kappa u \big\|_{L^{p,q}(\pi_\kappa(E),\sigma)} \tag{8.6.48}$$

(with the understanding that $L^{\frac{np}{n-1},q}$ is L^∞ when $n = 1$) where $C_\#$ is a finite positive number which depends only on p, q when $n = 1$, and in the case when $n \geq 2$ the number $C_\# \in (0,\infty)$ is defined as

$$C_\# := \begin{cases} C_{\mathrm{geo}} & \text{if either } \sigma(\partial\Omega) = +\infty, \text{ or } \pi_\kappa(E) \neq \partial\Omega, \\ C_{\mathrm{geo}} & \text{if both } \mathcal{L}^n(\Omega) < +\infty \text{ and } \sigma(\partial\Omega) < +\infty, \\ \max\left\{C_{\mathrm{geo}}, \left[\mathcal{L}^n(E)\right]^{\frac{n-1}{np}} \sigma(\partial\Omega)^{-\frac{1}{p}}\right\} & \text{in all other cases,} \end{cases} \tag{8.6.49}$$

for some geometric constant $C_{\mathrm{geo}} \in (0, \infty)$ *which depends only on* κ, θ, p, q, n, *and the lower* ADR *constant of* $\partial\Omega$.

As a corollary of (8.6.48), (6.6.82), (6.2.16)*, and* (6.2.26)*, the estimates*

$$\|u\|_{L^{\frac{np}{n-1},q}(E,\mathcal{L}^n)} \leq C_\# \left\|\mathcal{N}_\kappa^E u\right\|_{L^{p,q}(\pi_\kappa(E),\sigma)}, \tag{8.6.50}$$

and

$$\|u\|_{L^{\frac{np}{n-1}}(E,\mathcal{L}^n)} \leq C_\# \left\|\mathcal{N}_\kappa^E u\right\|_{L^p(\pi_\kappa(E),\sigma)} \tag{8.6.51}$$

(*naturally interpreting* $L^{\frac{np}{n-1}}$ *as* L^∞ *when* $n = 1$) *hold for every* $\mathcal{L}^n$*-measurable function* $u : E \to \mathbb{R}$, *with* $C_\# \in (0, \infty)$ *as above.*

Remark 8.6.4 We would like to point out that when $n \geq 2$ estimate (8.6.51) may also be regarded as a weighted isoperimetric inequality, involving the weighted measures

$$|u|^{\frac{n}{n-1}}\mathcal{L}^n \text{ in } \Omega, \text{ and } (\mathcal{N}_\kappa u)\mathcal{H}^{n-1} \text{ on } \partial\Omega. \tag{8.6.52}$$

Specifically, given an $\mathcal{L}^n$-measurable set $E \subseteq \Omega$ and an $\mathcal{L}^n$-measurable function $u : E \to \mathbb{R}$, we may recast estimate (8.6.51) as the *weighted isoperimetric inequality*

$$\left[(|u|^{\frac{n}{n-1}}\mathcal{L}^n)(E)\right]^{\frac{n-1}{n}} \leq C_\#\big((\mathcal{N}_\kappa^E u)\mathcal{H}^{n-1}\big)(\pi_\kappa(E)), \tag{8.6.53}$$

where $C_\#$ is as in (8.6.49). Specializing this to the case when $E := \Omega$ and $u \equiv 1$ yields an estimate very much reminiscent of the classical isoperimetric inequality, namely

$$\left[\mathcal{L}^n(\Omega)\right]^{\frac{n-1}{n}} \leq C_\# \mathcal{H}^{n-1}(\partial\Omega). \tag{8.6.54}$$

Yet another way to understand the nature of Proposition 8.6.3 is from the perspective of *embeddings of tent spaces*. Concretely, assuming that either $\sigma(\partial\Omega) = +\infty$ or $\mathcal{L}^n(\Omega) < +\infty$, estimate (8.6.51) implies (taking $E := \Omega$ and $u \equiv 1$), in the language suggested in Remark 8.3.6, that the tent space T^p_∞ associated with Ω and σ embeds continuously into the Lebesgue space $L^{\frac{np}{n-1}}(\Omega, \mathcal{L}^n)$. Hence, in the scenario mentioned above, using the piece of notation introduced in (8.3.31) we have the continuous embedding

$$N^p_\kappa(\Omega; \sigma) \hookrightarrow L^{\frac{np}{n-1}}(\Omega, \mathcal{L}^n). \tag{8.6.55}$$

Here is the proof of Proposition 8.6.3.

Proof of Proposition 8.6.3 If $n = 1$, then (8.6.48) follows from Lemma 8.3.2. Consider now the case when $n \geq 2$. As noted in the buildup to the statement of Proposition 8.6.3, taking $\mu := \mathcal{L}^n \lfloor \Omega$ renders (μ, σ) an $(n/(n-1), 0)$-Carleson pair for the ambient $(\Omega, \partial\Omega)$. Granted this, Theorem 8.6.2 applied with $F := E$ (as well as $\alpha := n/(n-1) > 1$ and $\beta := 0$) yields all the desired conclusions in the statement, except for the fact that $C_\#$ is a finite, purely geometric constant, in the case when both $\mathcal{L}^n(\Omega) < +\infty$ and $\sigma(\partial\Omega) < +\infty$. In such a case, the latter condition forces $\mathcal{L}^n(\partial\Omega) = 0$ which further implies $\mathcal{L}^n(\overline{\Omega}) = \mathcal{L}^n(\Omega) < +\infty$. Having clarified this, we may invoke the isoperimetric inequality (5.7.19) which presently gives

$$\mathcal{L}^n(\Omega) \leq n^{-1}(\omega_{n-1})^{-1/(n-1)}\sigma(\partial\Omega)^{\frac{n}{n-1}}. \tag{8.6.56}$$

In turn, from (8.6.56) and (8.6.16) it follows that for every Lebesgue measurable set $E \subseteq \Omega$ we have

$$\mathcal{L}^n\big(T_{E,\kappa'}(\partial\Omega)\big) = \mathcal{L}^n(E) \leq \mathcal{L}^n(\Omega) \leq n^{-1}(\omega_{n-1})^{-1/(n-1)}\,\sigma(\partial\Omega)^{\frac{n}{n-1}}. \tag{8.6.57}$$

This shows that, in the current scenario, the estimate in (8.6.35) holds in the extreme case when $\mathcal{O} = \partial\Omega$, with a purely dimensional constant. All things considered, we conclude that the estimate in (8.6.35) presently takes the stronger form

$$\mathcal{L}^n\big(T_{E,\kappa'}(\mathcal{O})\big) \leq C\sigma(\mathcal{O})^{\frac{n}{n-1}}, \quad \forall \mathcal{O} \text{ relatively open subset of } \partial\Omega, \tag{8.6.58}$$

for some $C \in (0, \infty)$ which depends only on n, κ, and the lower ADR constant of $\partial\Omega$. With this in hand, we may now run the same argument as in Case I in the proof of Theorem 8.6.2 and obtain (8.6.48) with $C_\# = C_{\rm geo} \in (0, \infty)$, a geometric constant which depends only on κ, θ, p, q, n, and the lower AR constant of $\partial\Omega$. □

Here is a versatile local estimate near the boundary, of a purely real-variable nature, derived from Proposition 8.6.3 in a very general geometric setting.

Corollary 8.6.5 *Assume Ω is an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Then for each integrability exponent $p \in (0, \infty)$ and each aperture parameter $\kappa > 0$ there exists some $C \in (0, \infty)$, which depends only on lower Ahlfors regularity constant of $\partial\Omega$ and n, p, κ, with the property that for each Lebesgue measurable function $u : \Omega \to \mathbb{R}$, each point $x \in \partial\Omega$, and each radius $\rho \in (0, \infty)$ one has*

$$\Big(\int_{B(x,\rho)\cap\Omega} |u|^p \, d\mathcal{L}^n\Big)^{1/p} \leq C\rho^{1/p}\Big(\int_{B(x,(2+\kappa)\rho)\cap\partial\Omega} (\mathcal{N}_\kappa^\rho u)^p \, d\sigma\Big)^{1/p}. \tag{8.6.59}$$

Proof Fix $p \in (0, \infty)$ and $\kappa > 0$. Given any $\mathcal{L}^n$-measurable function $u : \Omega \to \mathbb{R}$, for each $x \in \partial\Omega$ and $\rho \in (0, \infty)$ we may estimate

$$\begin{aligned}\Big(\int_{B(x,\rho)\cap\Omega} |u|^p \, d\mathcal{L}^n\Big)^{1/p} \\ \leq \Big(\int_{B(x,\rho)\cap\Omega} |u|^{np/(n-1)} \, d\mathcal{L}^n\Big)^{(n-1)/(np)}\Big(\int_{B(x,\rho)\cap\Omega} 1 \, d\mathcal{L}^n\Big)^{1/(np)}\end{aligned}$$

$$\leq C\Big(\int_{B(x,(2+\kappa)\rho)\cap\partial\Omega}(\mathcal{N}_\kappa^\rho u)^p\,\mathrm{d}\sigma\Big)^{1/p}\cdot\rho^{1/p}, \tag{8.6.60}$$

thanks to Hölder's inequality, (8.6.51) presently used with $E := B(x,\rho)\cap\Omega$, the inclusion in (8.1.17), and the definition of the truncated nontangential maximal operator from (1.5.5). □

An elementary, yet useful, version of the estimate in (8.6.48) is discussed in the lemma below.

Lemma 8.6.6 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Then for each $p\in(0,\infty)$, $\theta\in(0,1)$, and $\kappa>0$ there exists a finite constant $C>0$ depending on lower Ahlfors regularity constant of $\partial\Omega$, and θ, p, κ, with the property that for every Lebesgue measurable set $E\subseteq\Omega$ and every Lebesgue measurable function $u: E\to\mathbb{R}$ there holds*

$$\big\|\delta_{\partial\Omega}^{(n-1)/p}\cdot u^E_{\star,\theta}\big\|_{L^\infty(\Omega,\mathcal{L}^n)}\leq C\|\mathcal{N}_\kappa^E u\|_{L^p(\pi_\kappa(E),\sigma)}. \tag{8.6.61}$$

Hence, as a consequence of (8.6.61) *and* (6.6.82),

$$\big\|\delta_{\partial\Omega}^{(n-1)/p}\cdot u\big\|_{L^\infty(E,\mathcal{L}^n)}\leq C\|\mathcal{N}_\kappa^E u\|_{L^p(\pi_\kappa(E),\sigma)}. \tag{8.6.62}$$

Proof Fix some $z\in\Omega$ and pick $x\in E\cap B\big(z,\theta\delta_{\partial\Omega}(z)\big)$ arbitrary. If we now choose $x^*\in\partial\Omega$ such that $|x-x^*|=\delta_{\partial\Omega}(x)$ then $x\in\Gamma_\kappa(y)$ for every point y in the surface ball $\Delta\big(x^*,\kappa\delta_{\partial\Omega}(x)\big):=B\big(x^*,\kappa\delta_{\partial\Omega}(x)\big)\cap\partial\Omega$. Hence,

$$|u(x)|^p\leq\inf_{y\in\Delta(x^*,\kappa\delta_{\partial\Omega}(x))}\big(\mathcal{N}_\kappa^E u\big)(y)^p\leq\fint_{\Delta(x^*,\kappa\delta_{\partial\Omega}(x))}\big(\mathcal{N}_\kappa^E u\big)(y)^p\,\mathrm{d}\sigma(y). \tag{8.6.63}$$

Since $(1-\theta)\delta_{\partial\Omega}(z)<\delta_{\partial\Omega}(x)<(1+\theta)\delta_{\partial\Omega}(z)$, using (8.6.63) and the fact that $\partial\Omega$ is lower Ahlfors regular we may now estimate

$$\begin{aligned}\delta_{\partial\Omega}(z)^{(n-1)/p}|u(x)|&\leq C\cdot\delta_{\partial\Omega}(x)^{(n-1)/p}|u(x)|\leq C\cdot\sigma\big(\Delta(x^*,\kappa\delta_{\partial\Omega}(x))\big)^{1/p}|u(x)|\\&\leq C\|\mathcal{N}_\kappa^E u\|_{L^p(\pi_\kappa(E),\sigma)},\end{aligned} \tag{8.6.64}$$

where the constants in (8.6.64) depend only on p,κ,θ, and the lower Ahlfors regularity constants of $\partial\Omega$. On account of (6.6.79) this yields

$$\delta_{\partial\Omega}(z)^{(n-1)/p}\cdot u^E_{\star,\theta}(z)\leq C\|\mathcal{N}_\kappa^E u\|_{L^p(\pi_\kappa(E),\sigma)}, \tag{8.6.65}$$

and since $z\in\Omega$ was selected arbitrarily, the desired conclusion follows. □

We can go a step further and interpolate between (8.6.48) and (8.6.61) to obtain the estimate in the next proposition.

Proposition 8.6.7 *Suppose Ω is an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Select some arbitrary $\kappa \in (0,\infty)$, $\theta \in (0,1)$, $p \in (0,\infty)$, and $q \in \big[\frac{np}{n-1},\infty\big)$. In addition, consider some $\mathcal{L}^n$-measurable set $E \subseteq \Omega$ satisfying (8.6.47).*

Then for every $\mathcal{L}^n$-measurable function $u : E \to \mathbb{R}$ one has

$$\Big(\int_E \delta_{\partial\Omega}^{(n-1)q/p-n} \cdot \big|u^E_{\star,\theta}\big|^q \,\mathrm{d}\mathcal{L}^n\Big)^{1/q} \leq C\big\|\mathcal{N}^E_\kappa u\big\|_{L^p(\pi_\kappa(E),\sigma)} \tag{8.6.66}$$

where $C \in (0,\infty)$ depends only on p, q, κ, θ, and the constant $C_\#$ defined in (8.6.49).

In particular, (8.6.66) and (6.6.82) imply

$$\Big(\int_E \delta_{\partial\Omega}^{(n-1)q/p-n} \cdot |u|^q \,\mathrm{d}\mathcal{L}^n\Big)^{1/q} \leq C\big\|\mathcal{N}^E_\kappa u\big\|_{L^p(\pi_\kappa(E),\sigma)}. \tag{8.6.67}$$

Proof Upon observing that at $\mathcal{L}^n$-a.e. point in Ω we have

$$\begin{aligned}\delta_{\partial\Omega}^{(n-1)q/p-n} \cdot \big|u^E_{\star,\theta}\big|^q &= \Big[\delta_{\partial\Omega}^{(n-1)/p} \cdot u^E_{\star,\theta}\Big]^{q-np/(n-1)} \cdot \big|u^E_{\star,\theta}\big|^{np/(n-1)} \\ &\leq \big\|\delta_{\partial\Omega}^{(n-1)/p} \cdot u^E_{\star,\theta}\big\|^{q-np/(n-1)}_{L^\infty(\Omega,\mathcal{L}^n)} \cdot \big|u^E_{\star,\theta}\big|^{np/(n-1)},\end{aligned} \tag{8.6.68}$$

we may write

$$\begin{aligned}&\Big(\int_E \delta_{\partial\Omega}^{(n-1)q/p-n} \cdot \big|u^E_{\star,\theta}\big|^q \,\mathrm{d}\mathcal{L}^n\Big)^{1/q} \\ &\quad\leq \big\|\delta_{\partial\Omega}^{(n-1)/p} \cdot u^E_{\star,\theta}\big\|^{1-\frac{np}{(n-1)q}}_{L^\infty(\Omega,\mathcal{L}^n)} \Big(\int_E \big|u^E_{\star,\theta}\big|^{np/(n-1)} \,\mathrm{d}\mathcal{L}^n\Big)^{1/q} \\ &\quad\leq C\big\|\mathcal{N}^E_\kappa u\big\|^{1-\frac{np}{(n-1)q}}_{L^p(\pi_\kappa(E),\sigma)} \cdot \big\|\mathcal{N}^E_\kappa u\big\|^{\frac{np}{(n-1)q}}_{L^p(\pi_\kappa(E),\sigma)} \\ &\quad= C\big\|\mathcal{N}^E_\kappa u\big\|_{L^p(\pi_\kappa(E),\sigma)},\end{aligned} \tag{8.6.69}$$

on account of (8.6.61) and (8.6.48) (used here with $q := p$, bearing in mind (6.2.25)–(6.2.26)). □

Integrability properties corresponding to large exponents for the nontangential maximal operator of the gradient of a given function imply Hölder regularity of the function in question.

Corollary 8.6.8 *Let $\Omega \subseteq \mathbb{R}^n$ (where $n \geq 2$) be a uniform domain (in the sense of Definition 5.11.10) with a lower Ahlfors regular boundary. Set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, fix an aperture parameter $\kappa \in (0,\infty)$, and pick an integrability exponent $p \in (n-1,\infty)$. In this setting, assume $u \in \mathscr{C}^1(\Omega)$ is a function satisfying $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega,\sigma)$. Then $u \in \dot{\mathscr{C}}^\alpha(\overline{\Omega})$ where $\alpha := 1 - \frac{n-1}{p} \in (0,1)$ and, for some $C = C(\Omega,n,p,\kappa)$ in $(0,\infty)$, one has*

$$\|u\|_{\dot{\mathscr{C}}^\alpha(\overline{\Omega})} \leq C\|\mathcal{N}_\kappa(\nabla u)\|_{L^p(\partial\Omega,\sigma)}. \tag{8.6.70}$$

Proof This is a consequence of (5.11.78) and (8.6.62). □

It is also possible to produce a weighted version of the estimate in (8.6.51) from Proposition 8.6.3 of the kind described in our next result.

Proposition 8.6.9 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ (where $n \geq 2$) with a lower Ahlfors regular boundary and set $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an aperture parameter $\kappa \in (0,\infty)$, an integrability exponent $p \in (0,\infty)$, and suppose $0 \leq N \leq M$. Also, consider some $\mathcal{L}^n$-measurable set $E \subseteq \Omega$ and retain the assumption made in (8.6.47).*

Then for each $\mathcal{L}^n$-measurable function $u : E \to \mathbb{R}$ and each $y \in \partial\Omega$ one has

$$\left[\int_E \frac{\delta_{\partial\Omega}(x)^N}{|x-y|^M}|u(x)|^{\frac{np}{n-1}}\,\mathrm{d}\mathcal{L}^n(x)\right]^{\frac{n-1}{np}} \leq C_\#\left[\int_{\pi_\kappa(E)}\big(\mathcal{N}_\kappa^E u\big)(x)^p|x-y|^{(N-M)\frac{n-1}{n}}\,\mathrm{d}\sigma(x)\right]^{\frac{1}{p}} \tag{8.6.71}$$

where $C_\# \in (0,\infty)$ is defined as in (8.6.49) (with the understanding that now the constant $C_{\rm geo} \in (0,\infty)$ depends on κ, p, n, N, M, and the lower AR *constant of* $\partial\Omega$).

Proof Applying (8.6.51) to the function $E \ni x \mapsto \Big(\frac{\delta_{\partial\Omega}(x)^N}{|x-y|^M}\Big)^{\frac{n-1}{np}} \cdot u(x) \in \mathbb{R}$ yields

$$\left[\int_E \frac{\delta_{\partial\Omega}(x)^N}{|x-y|^M}|u(x)|^{\frac{np}{n-1}}\,\mathrm{d}\mathcal{L}^n(x)\right]^{\frac{n-1}{np}} \leq C_\#\left[\int_{\pi_\kappa(E)}\big(\mathcal{N}_\kappa^E u\big)(x)^p\Big(\sup_{z\in\Gamma_\kappa(x)\cap E}\frac{\delta_{\partial\Omega}(z)^N}{|z-y|^M}\Big)^{\frac{n-1}{np}}\,\mathrm{d}\sigma(x)\right]^{\frac{1}{p}}. \tag{8.6.72}$$

Since (8.1.23) and the fact that $0 \leq N \leq M$ entail that for each $x \in \pi_\kappa(E)\setminus\{y\}$ we have

$$\sup_{z\in\Gamma_\kappa(x)}\frac{\delta_{\partial\Omega}(z)^N}{|z-y|^M} \approx \sup_{z\in\Gamma_\kappa(x)}\frac{|z-x|^N}{(|z-x|+|x-y|)^M} \leq \sup_{0<t<\infty}\frac{t^N}{(t+|x-y|)^M} \approx |x-y|^{N-M}, \tag{8.6.73}$$

and since $\mathcal{H}^{n-1}(\{y\}) = 0$ (given that $n \geq 2$), the estimate claimed in (8.6.71) follows by combing (8.6.72) with (8.6.73). □

We next revisit Corollary 8.6.5, with the goal of establishing a similar estimate, now involving a full one-sided collar neighborhood of the boundary in the left side. To state it, recall that for any given open set $\Omega \subseteq \mathbb{R}^n$ and any $\varepsilon > 0$, the one-sided collar neighborhood $\mathcal{O}_\varepsilon$ of $\partial\Omega$ has been introduced in (6.1.4), and the truncated nontangential maximal operator $\mathcal{N}_\kappa^\varepsilon$ is defined as

$$\mathcal{N}_\kappa^\varepsilon u := \mathcal{N}_\kappa^{\mathcal{O}_\varepsilon} u = \mathcal{N}_\kappa\big(u\mathbf{1}_{\mathcal{O}_\varepsilon}\big), \tag{8.6.74}$$

for each Lebesgue measurable function $u : \Omega \to \mathbb{C}$.

Proposition 8.6.10 *Fix $n \in \mathbb{N}$ and suppose Ω is an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary. Consider the measure $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$.*

Then for each $p \in (0,\infty)$ and $\kappa \in (0,\infty)$ there exists $C = C(\partial\Omega, n, \kappa, p)$ in $(0,\infty)$ depending only on n, κ, p, and the lower ADR *constant of $\partial\Omega$, with the property that, if*

$$\varepsilon \in \big(0, \varepsilon_{\Omega,\kappa}\big) \ \textit{with} \ \ \varepsilon_{\Omega,\kappa} := \frac{\operatorname{diam}(\partial\Omega)}{n(2+\sqrt{n})(3+2\kappa)} \in (0, +\infty], \tag{8.6.75}$$

for each $\mathcal{L}^n$-measurable function $u : \Omega \to \mathbb{C}$ one has

$$\Big(\int_{\mathcal{O}_\varepsilon} |u|^p \, d\mathcal{L}^n\Big)^{1/p} \leq C\varepsilon^{1/p} \cdot \big\|\mathcal{N}_\kappa^\varepsilon u\big\|_{L^p(\partial\Omega,\sigma)}. \tag{8.6.76}$$

When the measure σ is also locally finite (which is the case if, for example, Ω is a set of locally finite perimeter), then Lemma 8.3.2 shows that (8.6.76), suitably interpreted, continues to be valid for the end-point value $p = \infty$.

Proof of Proposition 8.6.10 For each $\varepsilon > 0$ fixed, denote by $\mathcal{Q}_\varepsilon := \varepsilon\mathbb{Z}^n + [0,\varepsilon]^n$ the standard grid in $\mathbb{R}^n$ consisting of cubes of side-length ε. Also, let $\mathcal{J}_\varepsilon(\Omega)$ be the collection of cubes $Q \in \mathcal{Q}_\varepsilon$ such that $Q \cap \partial\Omega \neq \varnothing$. In general, for a cube Q and a positive number λ, we denote by x_Q the center of Q and by λQ the concentric cube dilated by λ. We divide the proof of (8.6.76) into four steps.

Step (i). For each $\varepsilon > 0$ one has

$$\mathcal{O}_\varepsilon \subseteq \bigcup_{Q \in \mathcal{J}_\varepsilon(\Omega)} (2+\sqrt{n})Q. \tag{8.6.77}$$

Indeed, for $x \in \mathcal{O}_\varepsilon$, there exists $x^* \in \partial\Omega$ such that $|x - x^*| = \delta_{\partial\Omega}(x) < \varepsilon$. Since $x^* \in \partial\Omega$, there exists $Q^* \in \mathcal{J}_\varepsilon(\Omega)$ such that $x^* \in Q^*$. Then

$$|x - x_{Q^*}| \leq |x - x^*| + |x^* - x_{Q^*}| \leq (1 + \sqrt{n}/2)\varepsilon, \tag{8.6.78}$$

so that x belongs to the cube centered at x_{Q^*} with side-length $(2+\sqrt{n})\varepsilon$.

Step (ii). Having fixed $\varepsilon > 0$ arbitrary, for each $\lambda \geq 1$, $\kappa > 0$, and $Q \in \mathcal{J}_\varepsilon(\Omega)$, one has

$$\pi_\kappa(\lambda Q) \subseteq \lambda\sqrt{n}(3+2\kappa)Q. \tag{8.6.79}$$

To prove (8.6.79), first observe that

$$\lambda \geq 1 \text{ and } Q \in \mathcal{J}_\varepsilon(\Omega) \Rightarrow \varnothing \neq Q \cap \partial\Omega \subseteq \lambda Q \cap \partial\Omega \Rightarrow \lambda Q \cap \partial\Omega \neq \varnothing$$
$$\Rightarrow \delta_{\partial\Omega}(y) \leq \lambda\sqrt{n}\,\varepsilon \quad \text{for each} \quad y \in \lambda Q. \tag{8.6.80}$$

Assume next that $x \in \pi_\kappa(\lambda Q)$. Then $\Gamma_\kappa(x) \cap \lambda Q \neq \varnothing$. Pick $y \in \Gamma_\kappa(x) \cap \lambda Q$. Thanks to (8.1.2) and (8.6.80) we may then estimate

$$\begin{aligned} |x - x_Q| &\leq |x-y| + |y - x_Q| < (1+\kappa)\delta_{\partial\Omega}(y) + \frac{\lambda\sqrt{n}}{2}\,\varepsilon \\ &\leq (1+\kappa)\lambda\sqrt{n}\,\varepsilon + \frac{\lambda\sqrt{n}}{2}\,\varepsilon = \frac{1}{2}\,\lambda\sqrt{n}(3+2\kappa)\varepsilon. \end{aligned} \tag{8.6.81}$$

This places x in $\lambda\sqrt{n}(3+2\kappa)Q$, as wanted.

Step (iii). For each $M \in (0,\infty)$, the collection of cubes $\{MQ\}_{Q\in\mathcal{J}_\varepsilon(\Omega)}$ has a finite overlap. More precisely,

$$\sum_{Q\in\mathcal{J}_\varepsilon(\Omega)} \mathbf{1}_{MQ} \leq (1 + M\sqrt{n})^n. \tag{8.6.82}$$

To see that this is the case, assume $Q_1, \dots, Q_m \in \mathcal{J}_\varepsilon(\Omega)$ are such that the intersection $(MQ_1) \cap \dots \cap (MQ_m)$ is nonempty and pick $x_0 \in (MQ_1) \cap \dots \cap (MQ_m)$. For such a point we have $|x_0 - x_{Q_j}| \leq \frac{M\sqrt{n}\varepsilon}{2}$ for each $j \in \{1,\dots,m\}$, which implies $\{x_{Q_1},\dots,x_{Q_m}\} \subseteq B\big(x_0, \frac{M\sqrt{n}\varepsilon}{2}\big)$. By design, $|x_{Q_i} - x_{Q_j}| \geq \varepsilon$ for every i, j in $\{1,\dots,m\}$ with $i \neq j$. In other words, the family $\{x_{Q_1},\dots,x_{Q_m}\}$ is ε-disperse. This readily implies that $m \leq (1+M\sqrt{n})^n$, as wanted.

Step (iv). The end-game in the proof of (8.6.76). Denoting $\lambda := 2 + \sqrt{n} \in [1,\infty)$, for each given $\mathcal{L}^n$-measurable function $u : \Omega \to \mathbb{C}$ we may write

$$\begin{aligned} \int_{O_\varepsilon} |u|\,\mathrm{d}\mathcal{L}^n &\leq \sum_{Q\in\mathcal{J}_\varepsilon(\Omega)} \int_{\lambda Q} |u|\,\mathrm{d}\mathcal{L}^n \leq \sum_{Q\in\mathcal{J}_\varepsilon(\Omega)} \|\mathbf{1}_{\lambda Q}\|_{L^n(\Omega,\mathcal{L}^n)} \|u\|_{L^{\frac{n}{n-1}}(\lambda Q,\mathcal{L}^n)} \\ &\leq C\varepsilon \sum_{Q\in\mathcal{J}_\varepsilon(\Omega)} \big\|\mathcal{N}_\kappa^{\lambda Q} u\big\|_{L^1(\pi_\kappa(\lambda Q),\sigma)} \\ &= C\varepsilon \sum_{Q\in\mathcal{J}_\varepsilon(\Omega)} \int_{\partial\Omega} \mathbf{1}_{\pi_\kappa(\lambda Q)} \mathcal{N}_\kappa^{\lambda Q} u \,\mathrm{d}\sigma \\ &\leq C\varepsilon \int_{\partial\Omega} \Big(\sum_{Q\in\mathcal{J}_\varepsilon(\Omega)} \mathbf{1}_{\pi_\kappa(\lambda Q)}\Big) \mathcal{N}_\kappa u \,\mathrm{d}\sigma \\ &\leq C\varepsilon \int_{\partial\Omega} \Big(\sum_{Q\in\mathcal{J}_\varepsilon(\Omega)} \mathbf{1}_{\lambda\sqrt{n}(3+2\kappa)Q}\Big) \mathcal{N}_\kappa u \,\mathrm{d}\sigma \\ &\leq C\varepsilon\big(1 + \lambda\,n(3+2\kappa)\big)^n \int_{\partial\Omega} \mathcal{N}_\kappa u \,\mathrm{d}\sigma, \end{aligned} \tag{8.6.83}$$

for some $C \in (0, \infty)$ which depends only on κ, n, the lower ADR constant of $\partial\Omega$, and the doubling character of σ. Above, the first inequality is a consequence of (8.6.77), given the present choice of λ. The second inequality is simply Hölder's inequality. The third inequality follows from the fact that $\|\mathbf{1}_{\lambda Q}\|_{L^n(\Omega,\mathcal{L}^n)} \leq \lambda\varepsilon$, together with Proposition 8.6.3. The latter is applicable once we choose ε small enough so that

$$\pi_\kappa(\lambda Q) \neq \partial\Omega \quad \text{for each} \quad Q \in \mathcal{J}_\varepsilon(\Omega). \tag{8.6.84}$$

Note that, in light of (8.6.79), condition (8.6.84) is satisfied whenever

$$\operatorname{diam}(\partial\Omega) > \operatorname{diam}\big(\lambda\sqrt{n}(3+2\kappa)Q\big) = \lambda\, n(3+2\kappa)\varepsilon, \tag{8.6.85}$$

hence whenever ε is as in (8.6.75). Next, the equality in (8.6.83) is clear from definitions, while the fourth inequality in (8.6.83) is trivial. The fifth inequality in (8.6.83) is implied by (8.6.79), while the last inequality in (8.6.83) is seen from (8.6.82) applied with $M := \lambda\sqrt{n}(3+2\kappa)$. Estimate (8.6.83) establishes

$$\int_{O_\varepsilon} |u|\, d\mathcal{L}^n \leq C\varepsilon \cdot \big\|\mathcal{N}_\kappa u\big\|_{L^1(\partial\Omega,\sigma)}. \tag{8.6.86}$$

Finally, replacing the function u by $\big|u \cdot \mathbf{1}_{O_\varepsilon}\big|^p$ in (8.6.86) readily yields (8.6.76), on account of (1.5.5) and (8.2.12). □

Having established the estimate in Proposition 8.6.10 we now derive a very useful geometric measure theoretic inequality.

Corollary 8.6.11 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary, and recall the family of one-sided collar neighborhoods O_ε of $\partial\Omega$, indexed by $\varepsilon > 0$, introduced in (6.1.4). Then for each $\lambda \in (2, \infty)$ there exist $c, C \in (0, \infty)$ with the property that*

$$\begin{gathered}\mathcal{L}^n\big(B(x,r) \cap O_\varepsilon\big) \leq C\varepsilon \cdot \sigma\big(B(x,\lambda r) \cap \partial\Omega\big) \\ \textit{for each} \quad x \in \partial\Omega, \ \ \varepsilon \in \big(0, c\cdot \operatorname{diam}(\partial\Omega)\big), \quad \textit{and} \quad r \in (0,\infty).\end{gathered} \tag{8.6.87}$$

As a consequence, if $\Sigma \subseteq \mathbb{R}^n$ is a nonempty closed Ahlfors regular set and for each threshold $\varepsilon > 0$ one defines $\mathcal{U}_\varepsilon := \big\{x \in \mathbb{R}^n : \operatorname{dist}(x,\Sigma) < \varepsilon\big\}$, then there exist two constants $c, C \in (0, \infty)$ such that

$$\begin{gathered}\mathcal{L}^n\big(B(x,r) \cap \mathcal{U}_\varepsilon\big) \leq C\varepsilon\, r^{n-1} \\ \textit{for each} \quad x \in \Sigma, \ \ \varepsilon \in \big(0, c\cdot \operatorname{diam}(\Sigma)\big), \quad \textit{and} \quad r \in (0,\infty).\end{gathered} \tag{8.6.88}$$

Proof Given $\lambda \in (2, \infty)$, consider the aperture parameter $\kappa := \lambda - 2 \in (0, \infty)$ and recall the threshold $\varepsilon_{\Omega,\kappa}$ from (8.6.75). Also, abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix $x \in \partial\Omega$ along with $r \in (0, \infty)$. Then (8.1.17) and (8.2.24) imply

$$\mathcal{N}_\kappa(\mathbf{1}_{B(x,r)\cap\Omega}) \leq \mathbf{1}_{B(x,(2+\kappa)r)\cap\partial\Omega}. \tag{8.6.89}$$

For each $\varepsilon \in (0, \varepsilon_{\Omega,\kappa})$ we may now estimate

$$\begin{aligned}\mathcal{L}^n\big(B(x,r)\cap O_\varepsilon\big) &= \int_{O_\varepsilon} \mathbf{1}_{B(x,r)\cap\Omega}\,\mathrm{d}\mathcal{L}^n \leq C\varepsilon\big\|\mathcal{N}_\kappa^\varepsilon(\mathbf{1}_{B(x,r)\cap\Omega})\big\|_{L^1(\partial\Omega,\sigma)}\\ &\leq C\varepsilon\big\|\mathcal{N}_\kappa(\mathbf{1}_{B(x,r)\cap\Omega})\big\|_{L^1(\partial\Omega,\sigma)} \leq C\varepsilon\big\|\mathbf{1}_{B(x,(2+\kappa)r)\cap\partial\Omega}\big\|_{L^1(\partial\Omega,\sigma)}\\ &= C\varepsilon\,\sigma\big(B(x,(2+\kappa)r)\cap\partial\Omega\big),\end{aligned} \tag{8.6.90}$$

based on Proposition 8.6.10 and (8.6.89). In view of the fact that $\kappa + 2 = \lambda$, this establishes (8.6.87).

In turn, (8.6.88) follows by applying (8.6.87) to $\Omega := \mathbb{R}^n \setminus \Sigma$, which is an open set whose boundary $\partial\Omega = \Sigma$ is Ahlfors regular, keeping in mind that $\mathcal{L}^n(\Sigma) = 0$. □

With Corollary 8.6.11 in hand, we now momentarily digress for the purpose of providing proof of the claim made earlier in (5.9.19).

Proposition 8.6.12 *Assume $\Omega \subseteq \mathbb{R}^n$ is an open set with an Ahlfors regular boundary. Then Ω is n-thick if and only if Ω satisfies an interior corkscrew property.*

Proof The fact that if Ω satisfies an interior corkscrew property implies that Ω is n-thick has been observed in (5.1.6). In the opposite direction, assume $\Omega \subseteq \mathbb{R}^n$ is an open n-thick set with an Ahlfors regular boundary, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an arbitrary point $x \in \partial\Omega$ along with some $r \in \big(0, \operatorname{diam}(\partial\Omega)\big)$. Also, recall the constants $c, C \in (0,\infty)$ from Corollary 8.6.11 and assume that

$$0 < \varepsilon < c \cdot \operatorname{diam}(\partial\Omega). \tag{8.6.91}$$

Finally, bring in the one-sided collar neighborhood O_ε of $\partial\Omega$ from (6.1.4). Based on (8.6.87) we may then estimate

$$\begin{aligned}\mathcal{L}^n\big((B(x,r)\cap\Omega)\setminus O_\varepsilon\big) &= \mathcal{L}^n\big(B(x,r)\cap\Omega\big) - \mathcal{L}^n\big(B(x,r)\cap O_\varepsilon\big)\\ &\geq C_1 r^n - C\varepsilon\, r^{n-1},\end{aligned} \tag{8.6.92}$$

where $C_1 \in (0,\infty)$ is the constant associated with the n-thickness condition of Ω. Hence, if we now choose

$$0 < \theta < \min\{c, C_1/C\} \tag{8.6.93}$$

and take $\varepsilon := \theta r$, then (8.6.91) is satisfied so we may conclude from (8.6.92) that

$$\mathcal{L}^n\big((B(x,r)\cap\Omega)\setminus O_{\theta r}\big) > 0. \tag{8.6.94}$$

In particular, this implies that the set $(B(x,r)\cap\Omega)\setminus O_{\theta r}$ is not empty. There remains to observe that for any point $z \in (B(x,r)\cap\Omega)\setminus O_{\theta r}$ we have

$$z \in \Omega, \quad \operatorname{dist}(z, \partial\Omega) \geq \theta r, \quad |z - x| < r, \tag{8.6.95}$$

which ultimately proves that Ω satisfies an interior corkscrew property. □

As a consequence of Proposition 8.6.12, we see that complements of closed Ahlfors regular set do satisfy an interior corkscrew property. Here is a formal statement and proof.

Corollary 8.6.13 *If $\Sigma \subseteq \mathbb{R}^n$ is a closed Ahlfors regular set, then $\mathbb{R}^n \setminus \Sigma$ satisfies an interior corkscrew property.*

Proof Note that $\Omega := \mathbb{R}^n \setminus \Sigma$ is an open set with $\partial\Omega := \Sigma$. Moreover, since Σ has zero Lebesgue measure, Ω is obviously n-thick. Granted these observations, Proposition 8.6.12 applies and gives that Ω satisfies an interior corkscrew property. □

In this vein, it is worth recalling the following definition.

Definition 8.6.14 *Call a nonempty closed set $\Sigma \subseteq \mathbb{R}^n$* `porous` *provided*

there exists $\eta \in (0, 1)$ with the property that for each x in Σ and r in $(0, \infty)$ it is possible to find $y \in \mathbb{R}^n$ such that $B(y, \eta r) \subseteq B(x, r) \setminus \Sigma$. (8.6.96)

As is apparent from Corollary 8.6.13 and Definition 8.6.14,

$$\text{any closed Ahlfors regular set } \Sigma \subseteq \mathbb{R}^n \text{ is porous.} \tag{8.6.97}$$

For more general results of this flavor see [33] (where (8.6.96) is referred to as a ball condition).

Pressing on, we discuss yet another off-diagonal Carleson measure estimate of reverse Hölder type which is going to be of basic importance for our later work.

Proposition 8.6.15 *Assume $\Sigma \subseteq \mathbb{R}^n$ is a nonempty closed set with empty interior, and σ is a doubling Borel measure on Σ with the property that there exist $\beta \in \mathbb{R}$ and $c \in (0, \infty)$ such that*

$$\sigma\big(B(x, r) \cap \Sigma\big) \geq c r^{\beta}, \qquad \forall x \in \Sigma, \quad \forall r \in \big(0, 2 \operatorname{diam} \Sigma\big). \tag{8.6.98}$$

Then for each $\alpha \in [1, \infty)$, $p \in (0, \infty)$, $\kappa \in (0, \infty)$, $\theta \in (0, 1)$, and $\lambda < n - \alpha\beta$, there exists a constant $C \in (0, \infty)$ which depends only on α, β, λ, p, κ, θ, c, and the doubling constant of σ, with the property that, for every $\mathcal{L}^n$-measurable subset E of $\mathbb{R}^n \setminus \Sigma$ and every $\mathcal{L}^n$-measurable function $u : E \to \mathbb{C}$, one has

$$\int_E \delta_\Sigma^{-\lambda} \left|u^E_{\star,\theta}\right|^p \, d\mathcal{L}^n \leq C\Big(\sup_E \delta_\Sigma\Big)^{n-\alpha\beta-\lambda} \left\|\mathcal{N}^E_\kappa u\right\|^p_{L^{p/\alpha}(\pi_\kappa(E),\sigma)} \tag{8.6.99}$$

where the intervening nontangential maximal operator $\mathcal{N}_\kappa^E$*, the projection operator* π_κ*, and local maximal function* $u^E_{\star,\theta}$ *(cf.* (6.6.79)*) are all considered relative to the open ambient* $\mathbb{R}^n \setminus \Sigma$*.*

Hence, as a corollary of (8.6.99) *and* (6.6.82)*,*

$$\int_E \delta_\Sigma^{-\lambda}\,|u|^p\,\mathrm{d}\mathcal{L}^n \le C\Big(\sup_E \delta_\Sigma\Big)^{n-\alpha\beta-\lambda}\,\big\|\mathcal{N}_\kappa^E u\big\|^p_{L^{p/\alpha}(\pi_\kappa(E),\sigma)}. \tag{8.6.100}$$

Before presenting the proof of Proposition 8.6.15 we make four comments. First, the particular version of (8.6.100) corresponding to taking $u := 1$ gives that, in the geometric context described in Proposition 8.6.15, we have

$$\int_E \delta_\Sigma^{-\lambda}\,\mathrm{d}\mathcal{L}^n \le C\Big(\sup_E \delta_\Sigma\Big)^{n-\alpha\beta-\lambda}\cdot\big[\sigma\big(\pi_\kappa(E)\big)\big]^\alpha. \tag{8.6.101}$$

Second, specializing (8.6.100) to the case when $u := 1$ and $E := B(x,r)\setminus\Sigma$ with $x \in \Sigma$ and $r > 0$ arbitrary proves that

$$\begin{array}{l}\text{if we abbreviate } \mu_\lambda := \delta_\Sigma^{-\lambda}\mathcal{L}^n, \text{ then } (\mu_\lambda,\sigma) \text{ is}\\ \text{a } (\alpha, n-\alpha\beta-\lambda)\text{-Carleson pair for the ambient}\\ (\mathbb{R}^n\setminus\Sigma,\ \Sigma), \text{ in the sense of Definition 8.6.1.}\end{array} \tag{8.6.102}$$

Third, in the case when $\Sigma := \partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is an open set with a lower Ahlfors regular boundary, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is doubling, $\alpha := 1$, $\beta := n-1$, and we take $\lambda := 0$, it follows that the estimate recorded in (8.6.100) becomes

$$\begin{array}{c}\displaystyle\int_E |u|^p\,\mathrm{d}\mathcal{L}^n \le C\Big(\sup_E \delta_{\partial\Omega}\Big)\cdot\big\|\mathcal{N}_\kappa^E u\big\|^p_{L^p(\pi_\kappa(E),\sigma)}\\ \text{whenever } E \text{ is a } \mathcal{L}^n\text{-measurable subset of } \Omega\\ \text{and } u : E \to \mathbb{C} \text{ is a } \mathcal{L}^n\text{-measurable function.}\end{array} \tag{8.6.103}$$

In particular, when $E := \mathcal{O}_\varepsilon$ defined as in (6.1.4) for an arbitrary $\varepsilon > 0$, we recover the estimate in (8.6.76) under the assumptions made in Proposition 8.6.15. This being said, observe that Proposition 8.6.10 does not require the measure σ to be doubling, as we presently do.

Fourth, in the context of Proposition 8.6.15, the values $\alpha := \frac{n}{n-1}$ and $\beta := n-1$ are permissible, and they force $\lambda < n - \alpha\beta = 0$. In the (precluded) limiting case $\lambda = 0$, estimate (8.6.99) formally corresponds to (8.6.48), after readjusting the exponent p. From this point of view we may regard Proposition 8.6.3 as a (prohibited) limiting case of Proposition 8.6.15.

Proof of Proposition 8.6.15 For starters, observe that the topological conditions imposed on Σ ensure that $\mathbb{R}^n \setminus \Sigma$ is an open set with boundary Σ. We claim that in place of (8.6.99) it suffices to show that there exists $C \in (0,\infty)$ such that

$$\int_E \delta_\Sigma^{-\lambda} |u|^p \, d\mathcal{L}^n \leq C\Big(\sup_E \delta_\Sigma\Big)^{n-\alpha\beta-\lambda} \big\|\mathcal{N}_\kappa^E u\big\|^p_{L^{p/\alpha}(\pi_\kappa(E),\sigma)} \tag{8.6.104}$$

for every $\mathcal{L}^n$-measurable subset E of $\mathbb{R}^n \setminus \Sigma$ and every $\mathcal{L}^n$-measurable function $u : E \to \mathbb{R}$. To see that (8.6.104) self-improves to (8.6.99) we reason as follows. First, it is easy to check based on definitions that

$$B\big(y, \theta\delta_\Sigma(y)\big) \subseteq \Gamma_{\kappa+\theta}(x), \qquad \forall x \in \Sigma, \quad \forall y \in \Gamma_\kappa(x). \tag{8.6.105}$$

In concert with (6.6.79), this readily implies

$$\mathcal{N}_\kappa^E\big(u^E_{\star,\theta}\big) \leq \mathcal{N}^E_{\kappa+\theta}(u) \;\text{ everywhere on }\; \Sigma. \tag{8.6.106}$$

In turn, from (8.6.106), (8.1.20), and (8.4.2) (bearing in mind that σ is doubling), we deduce that

$$\begin{aligned}\big\|\mathcal{N}_\kappa^E\big(u^E_{\star,\theta}\big)\big\|^p_{L^{p/\alpha}(\pi_\kappa(E),\sigma)} &\leq \big\|\mathcal{N}^E_{\kappa+\theta}u\big\|^p_{L^{p/\alpha}(\pi_\kappa(E),\sigma)} \\ &\leq \big\|\mathcal{N}^E_{\kappa+\theta}u\big\|^p_{L^{p/\alpha}(\pi_{\kappa+\theta}(E),\sigma)} \leq C\big\|\mathcal{N}^E_\kappa u\big\|^p_{L^{p/\alpha}(\pi_\kappa(E),\sigma)},\end{aligned} \tag{8.6.107}$$

for some finite constant $C = C(\Sigma, \kappa, p, \alpha) > 0$. Writing (8.6.104) for $u^E_{\star,\theta}$ (restricted to E) in place of u then yields (8.6.99) on account of (6.6.81) and (8.6.107).

Henceforth we focus on proving (8.6.104). To this end, fix an $\mathcal{L}^n$-measurable subset E of $\mathbb{R}^n \setminus \Sigma$ and note that, given the goals we have in mind, there is no loss of generality in assuming that $\sup\{\delta_\Sigma(x) : x \in E\} < \infty$. Suppose this is indeed the case, and also fix some $\mathcal{L}^n$-measurable function $u : E \to \mathbb{R}$. For each $\varepsilon > 0$ introduce

$$\mathcal{O}^\varepsilon_\Sigma := \big\{x \in \mathbb{R}^n : \delta_\Sigma(x) < \varepsilon\big\}, \tag{8.6.108}$$

then define the Whitney strip

$$\widetilde{\mathcal{O}}^\varepsilon_{E,\Sigma} := E \cap \Big(\mathcal{O}^\varepsilon_\Sigma \setminus \mathcal{O}^{\varepsilon/2}_\Sigma\Big). \tag{8.6.109}$$

Note that in order to prove (8.6.104) it suffices to show that

$$\int_{\widetilde{\mathcal{O}}^\varepsilon_{E,\Sigma}} \delta_\Sigma^{-\lambda} |u|^p \, d\mathcal{L}^n \leq C\varepsilon^{n-\alpha\beta-\lambda}\big\|\mathcal{N}^E_\kappa u\big\|^p_{L^{p/\alpha}(\Sigma,\sigma)}. \tag{8.6.110}$$

Indeed, since $\delta_\Sigma(x) \approx \varepsilon$ uniformly for $x \in \widetilde{\mathcal{O}}^\varepsilon_{E,\Sigma}$, writing (8.6.110) with the parameter $\varepsilon > 0$ replaced by $2^{-j}r$ where $r := \sup\{\delta_\Sigma(x) : x \in E\} \in (0, \infty)$ and summing over $j \in \mathbb{N}_0$, yields (8.6.104) on account of the fact that $\lambda < n - \alpha\beta$.

To continue, consider the collection of balls

$$C := \big\{B(y, \varepsilon/20) : y \in \widetilde{\mathcal{O}}^\varepsilon_{E,\Sigma}\big\} \tag{8.6.111}$$

covering the Whitney strip $\widetilde{O}^{\varepsilon}_{E,\Sigma}$. By Vitali's covering lemma (cf. [240, Lemma 1.6, p. 9]) there exists an at most countable sub-collection $C_0 \subseteq C$ such that

$$\forall B, B' \in C_0 \Rightarrow B \cap B' = \varnothing \quad \text{and} \quad \widetilde{O}^{\varepsilon}_{E,\Sigma} \subseteq \bigcup_{B \in C_0} 5B. \tag{8.6.112}$$

For each such $B \in C_0$, we will compare $\int_B \delta_\Sigma^{-\lambda} |u|^p \, d\mathcal{L}^n$ with the integral of $\big(\mathcal{N}^E_\kappa u\big)^{p/\alpha}$ over the Σ-ball

$$\Delta_B := B\big(y_*, \delta_\Sigma(y)\big) \cap \Sigma, \tag{8.6.113}$$

where y is the center of B and $y_* \in \Sigma$ is closest to y. In order to facilitate this comparison, we first discuss a preliminary geometrical result. Specifically, for each ball $B \in C_0$ set

$$\mathfrak{A}(B) := \big\{x \in \Sigma : \ B \subseteq \Gamma_\kappa(x)\big\}, \tag{8.6.114}$$

where $\Gamma_\kappa(x)$ is the nontangential approach region (with apex at x and aperture parameter κ) associated with the open set $\mathbb{R}^n \setminus \Sigma$. The claim we make in this regard is that

$$\Delta_B \subseteq \mathfrak{A}(5B), \qquad \forall B \in C_0. \tag{8.6.115}$$

To justify this inclusion, pick some $B \in C_0$ assume that y and y_* retain their earlier significance relative to B. Then for each point $z \in 5B = B(y, \varepsilon/4)$ we have

$$\begin{aligned} \delta_\Sigma(z) &\geq |y - y_*| - |y - z| \geq \varepsilon/2 - \varepsilon/4 = \varepsilon/4 \\ \text{and} \ \ |z - y_*| &\leq |z - y| + |y - y_*| \leq \varepsilon/4 + \delta_\Sigma(y). \end{aligned} \tag{8.6.116}$$

Given that $\delta_\Sigma(y) \leq \varepsilon$, we thus obtain $|z - y_*| \leq 5\varepsilon/4 \leq 5\delta_\Sigma(z)$. Consequently, for every point $x \in \Delta_B = B\big(y_*, \delta_\Sigma(y)\big) \cap \Sigma$ we may write

$$|x - z| \leq |x - y_*| + |y_* - z| \leq \delta_\Sigma(y) + 5\delta_\Sigma(z) \leq 9\delta_\Sigma(z). \tag{8.6.117}$$

Choosing $\kappa' > 8$ then guarantees, in light of (8.6.117), that

$$|x - z| < (1 + \kappa')\delta_\Sigma(z). \tag{8.6.118}$$

This goes to show that

$$z \in \Gamma_{\kappa'}(x) \quad \text{whenever} \quad z \in 5B \quad \text{and} \quad x \in \Delta_B, \tag{8.6.119}$$

i.e., that $5B \subseteq \Gamma_{\kappa'}(x)$ for every $x \in B\big(y_*, \delta_\Sigma(y)\big) \cap \Sigma = \Delta_B$. As a consequence, $x \in \mathfrak{A}(5B)$ for every $x \in \Delta_B$, and (8.6.115) follows.

The usefulness of Δ_B in establishing (8.6.110) is apparent from the estimate

$$\|u\|_{L^\infty(5B\cap E,\mathcal{L}^n)} \leq \big(\mathcal{N}^E_{\kappa'}u\big)(x), \qquad \forall x \in \Delta_B, \tag{8.6.120}$$

which is clear from (8.6.119). In turn (8.6.120) implies that, on the one hand,

$$\|u\|_{L^\infty(5B\cap E,\mathcal{L}^n)} \leq \inf_{\Delta_B} \mathcal{N}^E_{\kappa'}u. \tag{8.6.121}$$

On the other hand, for every ball $B \in C_0$ centered at y we have

$$5B = B(y, \varepsilon/4) \subseteq B(y_*, 5\varepsilon/4) \subseteq B\left(y_*, \tfrac{5}{2}\delta_\Sigma(y)\right), \tag{8.6.122}$$

since $|y - y_*| < \varepsilon$ and $\delta_\Sigma(y) \geq \varepsilon/2$. Thus, making use of (8.6.122), (8.6.98), (8.6.121), the fact that $\varepsilon/2 \leq \delta_\Sigma(y) < \varepsilon$, and that σ is doubling, we may now estimate

$$\begin{aligned}
\int_{5B\cap E} |u|^p \,\mathrm{d}\mathcal{L}^n &\leq \mathcal{L}^n(5B)\|u\|^p_{L^\infty(5B\cap E,\mathcal{L}^n)} \leq \mathcal{L}^n\big(B(y_*, \tfrac{5}{2}\delta_\Sigma(y))\big)\|u\|^p_{L^\infty(5B\cap E,\mathcal{L}^n)} \\
&\leq C\delta_\Sigma(y)^{n-\alpha\beta}\Big[\sigma\Big(B\big(y_*, \tfrac{5}{2}\delta_\Sigma(y)\big)\cap\Sigma\Big)\Big]^\alpha \|u\|^p_{L^\infty(5B\cap E,\mathcal{L}^n)} \\
&\leq C\varepsilon^{n-\alpha\beta}\big[\sigma(\Delta_B)\big]^\alpha \cdot \inf_{\Delta_B}\big(\mathcal{N}^E_{\kappa'}u\big)^p \\
&= C\varepsilon^{n-\alpha\beta}\Big[\sigma(\Delta_B)\,\inf_{\Delta_B}\big(\mathcal{N}^E_{\kappa'}u\big)^{p/\alpha}\Big]^\alpha \\
&\leq C\varepsilon^{n-\alpha\beta}\Big[\int_{\Delta_B}\big(\mathcal{N}^E_{\kappa'}u\big)^{p/\alpha}\,\mathrm{d}\sigma\Big]^\alpha.
\end{aligned} \tag{8.6.123}$$

In turn, given that $\delta_\Sigma \approx \varepsilon$ uniformly on $\widetilde{O}^\varepsilon_{E,\Sigma}$, estimate (8.6.123) permits us to write (bearing in mind that $\alpha \geq 1$)

$$\begin{aligned}
\int_{\widetilde{O}^\varepsilon_{E,\Sigma}} \delta_\Sigma^{-\lambda}|u|^p\,\mathrm{d}\mathcal{L}^n &\leq C\varepsilon^{-\lambda}\int_{\widetilde{O}^\varepsilon_{E,\Sigma}} |u|^p\,\mathrm{d}\mathcal{L}^n \leq C\varepsilon^{-\lambda}\sum_{B\in C_0}\int_{5B\cap E}|u|^p\,\mathrm{d}\mathcal{L}^n \\
&\leq C\varepsilon^{n-\alpha\beta-\lambda}\sum_{B\in C_0}\Big[\int_{\Delta_B}\big(\mathcal{N}^E_{\kappa'}u\big)^{p/\alpha}\,\mathrm{d}\sigma\Big]^\alpha \\
&\leq C\varepsilon^{n-\alpha\beta-\lambda}\Big[\sum_{B\in C_0}\int_{\Delta_B}\big(\mathcal{N}^E_{\kappa'}u\big)^{p/\alpha}\,\mathrm{d}\sigma\Big]^\alpha \\
&\leq C\varepsilon^{n-\alpha\beta-\lambda}\Big[\int_{\bigcup_{B\in C_0}\Delta_B}(\mathcal{N}^E_{\kappa'}u)^{p/\alpha}\,\mathrm{d}\sigma\Big]^\alpha \\
&\leq C\,\varepsilon^{n-\alpha\beta-\lambda}\Big[\int_\Sigma\Big(\sum_{B\in C_0}\mathbf{1}_{\Delta_B}\Big)\big(\mathcal{N}^E_{\kappa'}u\big)^{p/\alpha}\,\mathrm{d}\sigma\Big]^\alpha.
\end{aligned} \tag{8.6.124}$$

Next we claim that

$$\sum_{B\in C_0} \mathbf{1}_{\Delta_B} \leq 81^n. \tag{8.6.125}$$

Accept for the moment (8.6.125) and notice that, in concert with (8.6.124) and Proposition 8.4.1, this implies

$$\begin{aligned}\int_{\widetilde{\mathcal{O}}^{\varepsilon}_{E,\Sigma}} \delta_\Sigma^{-\lambda}\,|u|^p\,\mathrm{d}\mathcal{L}^n &\leq C\varepsilon^{n-\alpha\beta-\lambda}\big\|\mathcal{N}^E_{\kappa'}u\big\|^p_{L^{p/\alpha}(\Sigma,\sigma)} \approx \varepsilon^{n-\alpha\beta-\lambda}\big\|\mathcal{N}^E_{\kappa}u\big\|^p_{L^{p/\alpha}(\Sigma,\sigma)}\\ &\approx \varepsilon^{n-\alpha\beta-\lambda}\big\|\mathcal{N}^E_{\kappa}u\big\|^p_{L^{p/\alpha}(\pi_{\Sigma,\kappa}(E),\sigma)}.\end{aligned} \tag{8.6.126}$$

This completes the proof of Proposition 8.6.15, granted the validity of (8.6.125).

Turning now to the proof of (8.6.125), observe that

$$\left.\begin{array}{l} B = B(y,\varepsilon/20),\ B' = B(y',\varepsilon/20)\in C_0\\ \text{with the property that }\ \Delta_B\cap\Delta_{B'}\neq\varnothing\end{array}\right\}\Rightarrow B\subseteq B(y',81\varepsilon/20). \tag{8.6.127}$$

Let $y_*, y'_*\in\Sigma$ be closest points to y and, respectively, y', and observe that we have $\delta_\Sigma(y) = |y-y_*|\leq\varepsilon$ and $\delta_\Sigma(y') = |y'-y'_*|\leq\varepsilon$. Now, if $z\in B$ is arbitrary then

$$|y'-z|\leq|y'-y'_*|+|y'_*-y_*|+|y_*-y|+|y-z|\leq 81\varepsilon/20, \tag{8.6.128}$$

since $|y'_*-y_*|$, the distance between the centers of the Σ-balls $\Delta_{B'}$ and Δ_B, is less than or equal to $\delta_\Sigma(y')+\delta_\Sigma(y)$, the sum of their radii, which in turn is $\leq 2\varepsilon$. Thus, (8.6.127) holds.

Going further, for each $x\in\Sigma$ we have $\sum\limits_{B\in C_0}\mathbf{1}_{\Delta_B}(x) = \#\{B\in C_0 : x\in\Delta_B\}$ where, generally speaking, $\#A$ denotes the cardinality of the set A. On the one hand, if $x\notin\bigcup\limits_{B\in C_0}\Delta_B$ then $\sum\limits_{B\in C_0}\mathbf{1}_{\Delta_B}(x) = 0$. On the other hand, if there exists $B' = B(y',\varepsilon/20)\in C_0$ such that $x\in\Delta_{B'}$ then

$$\{B\in C_0 :\ x\in\Delta_B\}\subseteq\{B\in C_0 :\ \Delta_B\cap\Delta_{B'}\neq\varnothing\}, \tag{8.6.129}$$

which thanks to (8.6.127) permits us to estimate (keeping (8.6.112) in mind)

$$\begin{aligned}\sum_{B\in C_0}\mathbf{1}_{\Delta_B}(x) &= \#\{B\in C_0 : x\in\Delta_B\}\leq\#\{B\in C_0 : \Delta_B\cap\Delta_{B'}\neq\varnothing\}\\ &\leq\#\Big\{B\in C_0 : B\subseteq B\big(y',81\varepsilon/20\big)\Big\}\leq\frac{\mathcal{L}^n\big(B(y',81\varepsilon/20)\big)}{\mathcal{L}^n\big(B(y',\varepsilon/20)\big)} = 81^n.\end{aligned} \tag{8.6.130}$$

This justifies (8.6.125), and finishes the proof of (8.6.110). In turn, (8.6.110) establishes (8.6.104) as indicated in the earlier part of the proof. The proof of Proposition 8.6.15 is now complete. □

We conclude by noting that estimate (8.6.100) from Proposition 8.6.15 self-improves in the fashion described in the proposition below.

Proposition 8.6.16 *Assume $\Sigma \subseteq \mathbb{R}^n$ is a nonempty closed set with empty interior, and σ is a doubling Borel measure on Σ with the property that there exist an exponent $\beta \in \mathbb{R}$ along with some constant $c \in (0,\infty)$ such that $\sigma\big(B(x,r)\cap\Sigma\big) \geq cr^\beta$ for all $x \in \Sigma$ and $r \in \big(0, 2\,\mathrm{diam}\,\Sigma\big)$.*

Then for each $\alpha \in [1,\infty)$, $p \in (0,\infty)$, $\kappa \in (0,\infty)$, and $\lambda, N, M \in \mathbb{R}$ such that $\lambda < n - \alpha\beta$ and $0 \leq N \leq M$ there exists a constant $C \in (0,\infty)$ which depends only on α, β, λ, p, κ, c, N, M, and the doubling constant of σ, with the property that for every $\mathscr{L}^n$-measurable subset E of $\mathbb{R}^n \setminus \Sigma$, every $\mathscr{L}^n$-measurable function $u : E \to \mathbb{C}$, and every point $y \in \Sigma$ one has

$$\int_E \frac{\delta_\Sigma(x)^{N-\lambda}}{|x-y|^M}|u(x)|^p\,\mathrm{d}\mathscr{L}^n(x) \tag{8.6.131}$$
$$\leq C\Big(\sup_E \delta_\Sigma\Big)^{n-\alpha\beta-\lambda}\cdot\Big[\int_{\pi_\kappa(E)}\big(\mathcal{N}_\kappa^E u\big)(x)^{p/\alpha}|x-y|^{(N-M)/\alpha}\,\mathrm{d}\sigma(x)\Big]^\alpha,$$

where the intervening nontangential maximal operator $\mathcal{N}_\kappa^E$ and the projection operator π_κ are considered relative to the open ambient $\mathbb{R}^n \setminus \Sigma$.

Proof This follows by applying (8.6.100) to the function

$$E \ni x \mapsto \Big(\frac{\delta_\Sigma(x)^N}{|x-y|^M}\Big)^{\frac{1}{p}}\cdot u(x) \in \mathbb{C} \tag{8.6.132}$$

and making use of (8.6.73) (for $\Omega := \mathbb{R}^n \setminus \Sigma$). □

8.7 Estimates for Marcinkiewicz Type Integrals and Applications

Let $\Sigma \subseteq \mathbb{R}^n$ be a nonempty closed set and denote by $\delta_\Sigma(x)$ the (Euclidean) distance from $x \in \mathbb{R}^n$ to Σ. For some fixed $\lambda, N \in \mathbb{R}$ and $r > 0$ then define the Marcinkiewicz integral

$$I_{\lambda,N}(x;r) := \int_{|z|<r}\frac{\delta_\Sigma(x+z)^\lambda}{|z|^N}\,\mathrm{d}z = \int_{B(x,r)}\frac{\delta_\Sigma(y)^\lambda}{|x-y|^N}\,\mathrm{d}y, \qquad x \in \mathbb{R}^n. \tag{8.7.1}$$

This type of integral has along history and various versions have been considered in the work of A. Zygmund (cf. [265, 266]), A.P. Calderón and A. Zygmund ([38]), L. Carleson ([41]), C. Fefferman and E. Stein ([89]). The variant (8.7.1) is modeled upon Stein's book (cf. [240, pp. 14–16]), where it is proved that

$$\text{if } \lambda > 0 \text{ then } I_{\lambda,n+\lambda}(x;1) < +\infty \text{ for } \mathcal{L}^n\text{-a.e. } x \in \Sigma \\ \text{and } I_{\lambda,n+\lambda}(x;1) = +\infty \text{ for every } x \in \mathbb{R}^n \setminus \Sigma. \tag{8.7.2}$$

Among other things, Marcinkiewicz type integrals are useful in the study of singular integral operators. Our goal here is to obtain quantitative versions of (8.7.2), under appropriate geometric measure theoretical conditions, and derive consequences that are of relevance to the present work.

We start by considering the geometric estimates formulated below and note that Proposition 8.6.15 plays a key role in their proof. The reader is reminded that for each number $a \in \mathbb{R}$ we use the abbreviation $(a)_+ := \max\{a, 0\}$.

Proposition 8.7.1 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set and denote by δ_Σ the distance function to Σ. Then the following statements are valid.*

(i) Suppose $\alpha < 1$ and $N < n - \alpha$. Then there exists a finite constant $C > 0$ (depending only on the Ahlfors regularity constants of Σ, N, and α) such that

$$\int_{B(y,r)\setminus\Sigma} \frac{\delta_\Sigma(x)^{-\alpha}}{|x-y|^N}\,\mathrm{d}x \le C r^{n-\alpha-N}, \qquad \forall r > 0, \quad \forall y \in \Sigma. \tag{8.7.3}$$

(ii) Suppose $\alpha < 1$ and $N < n - (\alpha)_+$. Also, fix a constant $c > 0$. Then there exists a finite constant $C > 0$ depending on the Ahlfors regularity constants of Σ, N, α, and c such that

$$\int_{B(y,r)\setminus\Sigma} \frac{\delta_\Sigma(x)^{-\alpha}}{|x-y|^N}\,\mathrm{d}x \le C\, r^{n-\alpha-N}, \quad \forall r > 0, \ \forall y \in \mathbb{R}^n \ \textit{with } \delta_\Sigma(y) < cr. \tag{8.7.4}$$

(iii) Suppose $n - N < \alpha < 1$ and fix a constant $c > 0$. Then there exists $C > 0$ depending only on the Ahlfors regularity constants of Σ, N, α, and c such that

$$\int_{(\mathbb{R}^n\setminus\Sigma)\setminus B(y,r)} \frac{\delta_\Sigma(x)^{-\alpha}}{|x-y|^N}\,\mathrm{d}x \le C r^{n-\alpha-N}, \ \forall r > 0, \ \forall y \in \mathbb{R}^n \ \textit{with } \delta_\Sigma(y) < cr. \tag{8.7.5}$$

(iv) Suppose $n - N < \alpha < 1$ and assume $E \subseteq \mathbb{R}^n$ is a $\mathcal{L}^n$-measurable set. Then there exists $C > 0$ depending only on the Ahlfors regularity constants of Σ, N, and α such that

$$\int_{E\setminus\Sigma} \frac{\delta_\Sigma(x)^{-\alpha}}{|x-y|^N}\,\mathrm{d}x \le C\,\mathrm{dist}(y,E)^{n-\alpha-N} \ \textit{for all } \ y \in \Sigma \setminus \overline{E}. \tag{8.7.6}$$

(v) Suppose $\alpha < 1$, $N < n - (\alpha)_+$, and make the assumption that $E \subseteq \mathbb{R}^n$ is a bounded $\mathcal{L}^n$-measurable set. Then there exists $C > 0$ depending only on the Ahlfors regularity constants of Σ, N, and α such that

$$\int_{E\setminus\Sigma} \frac{\delta_\Sigma(x)^{-\alpha}}{|x-y|^N}\,\mathrm{d}x \le C max\big\{\delta_\Sigma(y), \mathrm{dist}(y,E)+\mathrm{diam}(E)\big\}^{n-\alpha-N}, \quad \forall y\in\mathbb{R}^n. \tag{8.7.7}$$

(vi) *Suppose* $\alpha<1$, $0\le N<n-(\alpha)_+$, *and assume* $E\subseteq\mathbb{R}^n$ *is an arbitrary bounded* $\mathcal{L}^n$*-measurable set. If* Σ *is bounded then there exists some finite constant* $C_0=C_0(N,\alpha,E,\Sigma)>0$ *such that*

$$\int_{E\setminus\Sigma} \frac{\delta_\Sigma(x)^{-\alpha}}{|x-y|^N}\,\mathrm{d}x \le C_0, \qquad \forall y\in\mathbb{R}^n. \tag{8.7.8}$$

In fact, one may take $C_0=CR^{n-\alpha-N}$ *where* C *is as in* (iv)-(v) *above and* R *is any positive number with the property that there exists* $y_0\in\Sigma$ *such that* $E\cup\Sigma\subseteq B(y_0,R)$.

Proof In a first stage, note that there exists a constant $C\in(0,\infty)$ with the property that for each point $y\in\mathbb{R}^n$ and each number $r>0$ we have $B(y,r)\subseteq B(y_*,Cr)$, where $y_*\in\Sigma$ is such that $\delta_\Sigma(y)=|y-y_*|$. Granted this inclusion, we may now apply (8.6.100) (with $\alpha:=1$, $p:=1$, $\beta:=n-1$, $u\equiv 1$, and $E:=B(y_*,Cr)\setminus\Sigma$) to obtain (also bearing (8.1.17) in mind) that

$$\alpha<1 \Rightarrow \int_{B(y,r)\setminus\Sigma} \delta_\Sigma^{-\alpha}\,\mathrm{d}\mathcal{L}^n \le Cr^{n-\alpha}, \quad \forall r>0,\ \forall y\in\mathbb{R}^n \text{ with } \delta_\Sigma(y)<cr. \tag{8.7.9}$$

With (8.7.9) in hand, for an arbitrary $y\in\Sigma$ we may now write

$$\begin{aligned}
\int_{B(y,r)\setminus\Sigma} \frac{\delta_\Sigma(x)^{-\alpha}}{|x-y|^N}\,\mathrm{d}x &\le \sum_{j=0}^\infty \int_{[B(y,2^{-j}r)\setminus B(y,2^{-j-1}r)]\setminus\Sigma} \frac{\delta_\Sigma(x)^{-\alpha}}{|x-y|^N}\,\mathrm{d}x \\
&\le C\sum_{j=0}^\infty (2^{-j}r)^{-N}\Big(\int_{B(y,2^{-j}r)\setminus\Sigma} \delta_\Sigma(x)^{-\alpha}\,\mathrm{d}x\Big) \\
&\le C\sum_{j=0}^\infty (2^{-j}r)^{-N}(2^{-j}r)^{n-\alpha} = Cr^{n-N-\alpha}\Big(\sum_{j=0}^\infty 2^{-j(n-\alpha-N)}\Big) \\
&= Cr^{n-N-\alpha},
\end{aligned} \tag{8.7.10}$$

whenever $N<n-\alpha$. This establishes (8.7.3).

Consider next (8.7.4). To get started, fix $r>0$ along with $y\in\mathbb{R}^n$ satisfying $\delta_\Sigma(y)<cr$. We proceed by analyzing separately several cases.

`Case 1:` $N\le 0$ *and* $\alpha<1$. The fact that (8.7.4) holds under these circumstances is an immediate consequence of (8.7.9) and the observation that $|x-y|^{-N}\le C\,r^{-N}$ whenever $x\in B(y,r)$.

`Case 2:` $0 \leq \alpha < 1$ *and* $0 \leq N < n - \alpha$. Pick $y_* \in \Sigma$ so that $\delta_\Sigma(y) = |y - y_*|$ and split the integral in (8.7.4) into two integrals corresponding to the following additional restrictions on x: $|x - y_*| \leq 3|x - y|$ and $3|x - y| < |x - y_*|$. Denote the resulting integrals by I and II, respectively. Then, since we are currently assuming that $N \geq 0$, and since in I we have $|x - y_*| \leq 3|x - y| < 3r$ for every $x \in B(y, r)$, we may estimate

$$I \leq C \int_{B(y_*,3r)\setminus\Sigma} \frac{\delta_\Sigma(x)^{-\alpha}}{|x - y_*|^N} \, dx \leq C r^{n-\alpha-N}, \tag{8.7.11}$$

where the last inequality follows from (8.7.3). In II we have $3|x - y| < |x - y_*|$, hence

$$\delta_\Sigma(y) = |y - y_*| \geq |x - y_*| - |x - y| > 2|x - y|. \tag{8.7.12}$$

In turn, since $\delta_\Sigma(x) + |x - y| \geq \delta_\Sigma(y)$ this further implies that $\delta_\Sigma(x) > |x - y|$ and thus we can write (recall that we are assuming $\alpha \geq 0$ and $N + \alpha < n$ here)

$$II \leq C \int_{B(y,r)} \frac{dx}{|x - y|^{N+\alpha}} = C\, r^{n-\alpha-N}. \tag{8.7.13}$$

A combination of (8.7.11) and (8.7.13) proves (8.7.4) in this case as well.

`Case 3:` $\alpha \leq 0$ *and* $N < n$. Given that we are assuming $\delta_\Sigma(y) < cr$, it follows that for every $x \in B(y, r)$ we have $\delta_\Sigma(x)^{-\alpha} < Cr^{-\alpha}$. This permits us to estimate (keeping in mind that $N < n$)

$$\int_{B(y,r)\setminus\Sigma} \frac{\delta_\Sigma(x)^{-\alpha}}{|x - y|^N} \, dx \leq C\, r^{-\alpha} \int_{B(y,r)} \frac{dx}{|x - y|^N} = C\, r^{n-\alpha-N}. \tag{8.7.14}$$

Collectively, Cases 1-3 above yield (8.7.4) whenever $\alpha < 1$ and $N < n - (\alpha)_+$.

Consider next (8.7.5). To get started, fix $r > 0$ and let $y \in \mathbb{R}^n$ be such that $\delta_\Sigma(y) < cr$. Granted that $\alpha < 1$, we may once again employ (8.7.9) to estimate

$$\begin{aligned}
\int_{(\mathbb{R}^n\setminus\Sigma)\setminus B(y,r)} \frac{\delta_\Sigma(x)^{-\alpha}}{|x - y|^N} \, dx &\leq \sum_{j=0}^{\infty} \int_{(\mathbb{R}^n\setminus\Sigma)\cap[B(y,2^{j+1}r)\setminus B(y,2^j r)]} \frac{\delta_\Sigma(x)^{-\alpha}}{|x - y|^N} \, dx \\
&\leq C \sum_{j=0}^{\infty} (2^j r)^{-N} \Big(\int_{B(y,2^{j+1}r)\setminus\Sigma} \delta_\Sigma(x)^{-\alpha} \, dx \Big) \\
&\leq C \sum_{j=0}^{\infty} (2^j r)^{n-\alpha} (2^j r)^{-N}
\end{aligned}$$

$$= Cr^{n-N-\alpha}\Big(\sum_{j=0}^{\infty} 2^{-j(-n+\alpha+N)}\Big) = Cr^{n-N-\alpha}, \quad (8.7.15)$$

since the last series in (8.7.15) converges given that $-n+\alpha+N>0$. This concludes the proof of (8.7.5).

Next, the estimate in (8.7.6) corresponding to some $y \in \Sigma \setminus \overline{E}$ follows from (8.7.5) written with $r := \mathrm{dist}(y, E)$, since we have $E \subseteq \mathbb{R}^n \setminus B\big(y, \mathrm{dist}(y, E)\big)$.

Consider the claim made in item *(v)*. Now, the set E is bounded which allows us to obtain (8.7.7) by applying (8.7.4) with $r := \max\big\{\delta_\Sigma(y), \delta_E(y) + \mathrm{diam}(E)\big\}$, since this choice guarantees $E \subseteq B(y, r)$ and $\delta_\Sigma(y) < r$.

Finally, as far as (8.7.8) is concerned, pick an arbitrary point $y_0 \in \Sigma$ and let $R > 0$ be large enough so that $E \cup \Sigma \subseteq B(y_0, R)$. Then, by (8.7.7),

$$\int_{E\setminus\Sigma} \frac{\delta_\Sigma(x)^{-\alpha}}{|x-y|^N}\,\mathrm{d}x \le C\,R^{n-\alpha-N}, \qquad \forall y \in B(y_0, 2R). \quad (8.7.16)$$

On the other hand, if $y \in \mathbb{R}^n \setminus B(y_0, 2R)$, the fact that $N \ge 0$ forces $|x-y|^{-N} \le R^{-N}$ for each $x \in E$, so

$$\begin{aligned}\int_{E\setminus\Sigma} \frac{\delta_\Sigma(x)^{-\alpha}}{|x-y|^N}\,\mathrm{d}x &\le R^{-N} \int_{E\setminus\Sigma} \delta_\Sigma(x)^{-\alpha}\,\mathrm{d}x \\ &\le R^{-N} \int_{B(y_0,R)\setminus\Sigma} \delta_\Sigma(x)^{-\alpha}\,\mathrm{d}x \le C\,R^{n-\alpha-N},\end{aligned} \quad (8.7.17)$$

by (8.7.3). In concert, (8.7.16)–(8.7.17) give (8.7.8). The proof of the proposition is therefore complete. □

Our next goal is to establish growth estimates for the Lebesgue measure weighted by a power of the distance function to an Ahlfors regular set, of the sort described in the proposition below. These are going to be useful in the proofs of Propositions 8.7.3–8.7.4 where we shall show that suitable powers of the distance function to a fixed Ahlfors regular set are Muckenhoupt weights in $\mathbb{R}^n$.

Proposition 8.7.2 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set, and fix a constant $C \in (0, \infty)$.*

(i) For each exponent $\alpha \in \mathbb{R}$ there exists some $c \in (0, \infty)$, which depends only on α, C, and the Ahlfors regularity constants of Σ, with the property that for each point $x \in \mathbb{R}^n$ one has

$$\begin{aligned}&\int_{B(x,r)} \delta_\Sigma^{-\alpha}\,\mathrm{d}\mathcal{L}^n \ge c\,r^{n-\alpha} \quad \textit{for each} \quad r \in (0, \infty) \\ &\textit{if either } \alpha \le 0, \quad \textit{or } \alpha \in \mathbb{R} \textit{ and } \delta_\Sigma(x) < Cr.\end{aligned} \quad (8.7.18)$$

(ii) If $\alpha < 1$ *then there exists some* $c \in (0, \infty)$*, which depends only on* α*,* C*, and the Ahlfors regularity constants of* Σ*, such that*

$$\int_{B(x,r)} \delta_\Sigma^{-\alpha} \, d\mathcal{L}^n \approx r^{n-\alpha}, \quad \text{uniformly for } x \in \mathbb{R}^n \text{ and } r \in (0, \infty) \text{ with the property that } \delta_\Sigma(x) < Cr. \tag{8.7.19}$$

Proof To prove the claim in part *(i)*, pick $\varepsilon \in (0, 1/3)$ such that $\varepsilon < C$. Also, select $x \in \mathbb{R}^n$ and $r \in (0, \infty)$. We divide the proof of (8.7.18) into several cases. Consider first the case when $\alpha \leq 0$ and $\delta_\Sigma(x) \geq \varepsilon r$. Since $B(x, \varepsilon r/2) \subseteq B(x, r)$ and each $y \in B(x, \varepsilon r/2)$ satisfies $\delta_\Sigma(y) \geq \delta_\Sigma(y)/2 \geq \varepsilon r/2$, we may estimate

$$\int_{B(x,r)} \delta_\Sigma^{-\alpha} \, d\mathcal{L}^n \geq \int_{B(x,\varepsilon r/2)} \delta_\Sigma^{-\alpha} \, d\mathcal{L}^n \geq c \, r^{-\alpha} \mathcal{L}^n\big(B(x, \varepsilon r/2)\big) = c \, r^{n-\alpha}, \tag{8.7.20}$$

which suits our current purposes.

Next, consider the case when $\alpha \in \mathbb{R}$ and $\delta_\Sigma(x) < \varepsilon r$. Then $B(x, \varepsilon r) \cap \Sigma \neq \varnothing$, hence we may select $x_0 \in \Sigma \cap B(x, \varepsilon r)$ such that $\delta_\Sigma(x) = |x - x_0|$. On the other hand, from Corollary 8.6.13 we know that $\mathbb{R}^n \setminus \Sigma$ satisfies an interior corkscrew property. Hence, there exists $\theta \in (0, 1)$ (independent of x and r) along with $y \in \mathbb{R}^n$ such that $B\big(y, \theta(1 - 2\varepsilon)r\big) \subseteq B\big(x_0, (1 - 2\varepsilon)r\big) \setminus \Sigma$. As a result, for each $z \in B(y, \theta \, \varepsilon r)$ we have

$$\begin{aligned} |z - x| &\leq |z - y| + |y - x_0| + |x - x_0| < \theta \, \varepsilon r + (1 - 2\varepsilon)r + \delta_\Sigma(x) \\ &< \varepsilon r + (1 - 2\varepsilon)r + \varepsilon r = r, \end{aligned} \tag{8.7.21}$$

and

$$\begin{aligned} \theta(1 - 3\varepsilon)r &= \theta(1 - 2\varepsilon)r - \theta \, \varepsilon r \leq \delta_\Sigma(z) \leq |z - x_0| \\ &\leq |z - y| + |y - x_0| < \theta \, \varepsilon r + (1 - 2\varepsilon)r < (1 - \varepsilon)r. \end{aligned} \tag{8.7.22}$$

Hence, $B(y, \theta \, \varepsilon r) \subseteq B(x, r)$ and $\delta_\Sigma(z) \approx r$ uniformly for $z \in B(y, \theta \, \varepsilon r)$. In turn, these permit us to estimate

$$\begin{aligned} \int_{B(x,r)} \delta_\Sigma^{-\alpha} \, d\mathcal{L}^n &\geq \int_{B(y,\theta \, \varepsilon r)} \delta_\Sigma^{-\alpha}(z) \, d\mathcal{L}^n(z) \\ &\approx r^{-\alpha} \mathcal{L}^n\big(B(y, \theta \, \varepsilon r)\big) = c \, r^{n-\alpha}, \end{aligned} \tag{8.7.23}$$

as wanted. Finally, consider the case when $\alpha \in \mathbb{R}$ and $\varepsilon r \leq \delta_\Sigma(x) \leq Cr$. Then $B(x, \varepsilon r/2) \subseteq B(x, r)$ and for each $z \in B(x, \varepsilon r/2)$ we have

$$\begin{aligned} \varepsilon r/2 &\leq \delta_\Sigma(x) - \varepsilon r/2 \leq \delta_\Sigma(z) \\ &\leq |z - x| + \delta_\Sigma(x) < \varepsilon r/2 + Cr = (\varepsilon/2 + C)r. \end{aligned} \tag{8.7.24}$$

Thus, $\delta_\Sigma(z) \approx r$ uniformly for $z \in B(x, \varepsilon r/2)$. Consequently,

$$\int_{B(x,r)} \delta_\Sigma^{-\alpha}\, \mathrm{d}\mathcal{L}^n \geq \int_{B(x,\varepsilon r/2)} \delta_\Sigma^{-\alpha}(z)\, \mathrm{d}\mathcal{L}^n(z) \approx r^{-\alpha}\mathcal{L}^n\big(B(x, \varepsilon r/2)\big) = c\, r^{n-\alpha}, \tag{8.7.25}$$

as desired. This completes the proof of the claim in item *(i)* of the proposition.

There remains to deal with the claim in item *(ii)*. The right-pointing inequality in (8.7.19) follows from (8.7.18) (for this no restrictions on α are necessary), whereas the left-pointing inequality in (8.7.19) is a consequence of (8.7.4) with $N := 0$ (it is here that we make use of the fact that $\alpha < 1$), as well as of (5.9.7). □

Here is the first result advertised earlier, showing that the reciprocal of sub-unital power of the distance function to a given Ahlfors regular set is a Muckenhoupt weight in $A_1(\mathbb{R}^n, \mathcal{L}^n)$.

Proposition 8.7.3 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set. Then for each number $\theta \in [0, 1)$ the function $\delta_\Sigma^{-\theta}$ belongs to Muckenhoupt's class $A_1(\mathbb{R}^n, \mathcal{L}^n)$.*

Proof According to (7.7.4), we need to show that there exists some $C \in (0, \infty)$ with the property that for each $x \in \mathbb{R}^n$ and each $r \in (0, \infty)$ we have

$$⨍_{B(x,r)} \delta_\Sigma^{-\theta}\, \mathrm{d}\mathcal{L}^n \leq C\, r^n \delta_\Sigma(y)^{-\theta} \quad \text{for } \mathcal{L}^n\text{-a.e. } \ y \in B(x, r). \tag{8.7.26}$$

To this end, fix $x \in \mathbb{R}^n$ and $r > 0$ arbitrary. On the one hand, when $\delta_\Sigma(x) < 2r$ we may invoke (8.7.19) to write

$$\int_{B(x,r)} \delta_\Sigma^{-\theta}\, \mathrm{d}\mathcal{L}^n \approx r^{n-\theta} \leq C\, r^n \delta_\Sigma(y)^{-\theta}, \quad \text{for each } \ y \in B(x, r), \tag{8.7.27}$$

since, in this case, for each $y \in B(x, r)$ we have $\delta_\Sigma(y) \leq \delta_\Sigma(x) + r < 3r$. On the other hand, if $\delta_\Sigma(x) \geq 2r$, then $\frac{1}{2}\delta_\Sigma(x) \leq \delta_\Sigma(y) \leq \frac{3}{2}\delta_\Sigma(x)$ for each $y \in B(x, r)$ which, in turn, permits us to estimate

$$\int_{B(x,r)} \delta_\Sigma^{-\theta}\, \mathrm{d}\mathcal{L}^n \approx \delta_\Sigma(x)^{-\theta} \cdot \mathcal{L}^n\big(B(x, r)\big) \leq C\, r^n \delta_\Sigma(y)^{-\theta}, \tag{8.7.28}$$

for each $y \in B(x, r)$, as wanted. □

In light of factorization of A_p weights, in a fashion involving only (powers) of A_1 weights, Proposition 8.7.3 self-improves as follows.

Proposition 8.7.4 *Let $\Sigma \subseteq \mathbb{R}^n$ be a closed Ahlfors regular set. Pick an exponent $p \in [1, \infty)$ and fix a parameter $a \in \big(-\frac{1}{p}, 1 - \frac{1}{p}\big)$. Then the function δ_Σ^{ap} belongs to Muckenhoupt's class $A_p(\mathbb{R}^n, \mathcal{L}^n)$.*

Proof Proposition 8.7.3 implies that $\delta_\Sigma^{-\theta_1}, \delta_\Sigma^{-\theta_2} \in A_1(\mathbb{R}^n, \mathcal{L}^n)$ for each θ_1, θ_2 in $(0, 1)$. Based on this and item *(3)* in Lemma 7.7.1 we then conclude that

$$\delta_\Sigma^{-\theta_1-\theta_2(1-p)} = \delta_\Sigma^{-\theta_1} \cdot \delta_\Sigma^{-\theta_2(1-p)} \in A_p(\mathbb{R}^n, \mathcal{L}^n). \tag{8.7.29}$$

There remains to observe that the expression $-\theta_1 - \theta_2(1-p)$ covers precisely the interval $\big(-\frac{1}{p}, 1-\frac{1}{p}\big)$ as θ_1, θ_2 range freely in $(0, 1)$. Hence any $a \in \big(-\frac{1}{p}, 1-\frac{1}{p}\big)$ may be written as $-\theta_1 - \theta_2(1-p)$ for some $\theta_1, \theta_2 \in (0, 1)$. This completes the proof of Proposition 8.7.4.

In the range $p \in (1, \infty)$ a direct proof of the membership $\delta_\Sigma^{ap} \in A_p(\mathbb{R}^n, \mathcal{L}^n)$ is as follows. According to (7.7.2), we need to show that there exists a finite constant $C > 0$ with the property that for each $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ we have

$$\Big(⨍_{B(x,r)} \delta_\Sigma(y)^{ap}\, d\mathcal{L}^n(y)\Big)\Big(⨍_{B(x,r)} \delta_\Sigma(y)^{-\frac{ap}{p-1}}\, d\mathcal{L}^n(y)\Big)^{p-1} \leq C. \tag{8.7.30}$$

We proceed by distinguishing two cases, starting with:

Case 1. *Assume* $2r < \delta_\Sigma(x)$. Then $\frac{1}{2}\delta_\Sigma(x) \leq \delta_\Sigma(y) \leq \frac{3}{2}\delta_\Sigma(x)$ for each $y \in B(x, r)$. Making use of this, we obtain

$$\begin{aligned}\Big(⨍_{B(x,r)} \delta_\Sigma(y)^{ap}\, d\mathcal{L}^n(y)\Big)\Big(⨍_{B(x,r)} \delta_\Sigma(y)^{-\frac{ap}{p-1}}\, d\mathcal{L}^n(y)\Big)^{p-1}& \\ \leq C\delta_\Sigma(x)^{ap}\big(\delta_\Sigma(x)^{-\frac{ap}{p-1}}\big)^{p-1} = C,&\end{aligned} \tag{8.7.31}$$

for some $C \in (0, \infty)$ independent of x, r. This shows that (8.7.30) holds in this case.

Case 2. *Assume* $\delta_\Sigma(x) \leq 2r$. In this scenario, we apply (8.7.19) with $\alpha := -ap < 1$ and $C > 2$ to obtain

$$⨍_{B(x,r)} \delta_\Sigma(y)^{ap}\, d\mathcal{L}^n(y) \approx r^{-n} \cdot r^{n+ap} = r^{ap}. \tag{8.7.32}$$

One more application of (8.7.19) with $\alpha := \frac{ap}{p-1} < 1$ and $C > 2$ also gives

$$⨍_{B(x,r)} \delta_\Sigma(y)^{-\frac{ap}{p-1}}\, d\mathcal{L}^n(y) \approx r^{-\frac{ap}{p-1}}. \tag{8.7.33}$$

Combining (8.7.32) and (8.7.33), we conclude that (8.7.30) also holds in Case 2. Having dealt with the situations described in Cases 1-2, (8.7.30) follows, completing the alternative proof of Proposition 8.7.4 in the range $p \in (1, \infty)$. □

There are other results of a flavor similar to Proposition 8.7.4. Here is one of them (see [79, 182]).

Proposition 8.7.5 *Let* $\Sigma \subseteq \mathbb{R}^n$ *be a closed set which is Ahlfors regular and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. *Fix* $d \in [0, n-1)$ *and consider a* d*-set* $E \subseteq \Sigma$*, i.e., a closed* *subset E of Σ with the property that there exists some Borel outer measure μ on E satisfying*

$$\mu\big(B(x,r)\cap E\big)\approx r^d, \quad \text{uniformly for } x\in E \text{ and } r\in\big(0,2\,\mathrm{diam}(E)\big). \tag{8.7.34}$$

Then for each $p\in(1,\infty)$ and $a\in\big(d+1-n,(p-1)(n-1-d)\big)$ the function $w:=\big[\mathrm{dist}(\cdot,E)\big]^a$ is a Muckenhoupt weight in $A_p(\Sigma,\sigma)$. Moreover, $[w]_{A_p}$ depends only on the Ahlfors regularity character of Σ, the proportionality constants in (8.7.34), d, p, and a.

It is of interest to recall the following result, appearing in [103, Proposition 1.5.9, p. 42] (compare with Proposition 8.7.5 specialized to the case when E is a singleton).

Example 8.7.6 *Let $\Sigma\subseteq\mathbb{R}^n$ be a closed set which is Ahlfors regular and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix some integrability exponent $p\in(1,\infty)$ along with an arbitrary point $x_0\in\Sigma$ and some power $a\in\mathbb{R}$. Then the function*

$$w:\Sigma\to[0,\infty],\quad w(x):=|x-x_0|^a \text{ for each } x\in\Sigma, \tag{8.7.35}$$

is a Muckenhoupt weight in $A_p(\Sigma,\sigma)$ if and only if $a\in\big(1-n,(p-1)(n-1)\big)$. Furthermore, whenever this is the case, it follows that $[w]_{A_p}$ depends only on the Ahlfors regularity character of Σ, p, and a.

Example 8.7.6 should be compared with the following result, of a similar flavor:

Example 8.7.7 *Let $\Sigma\subseteq\mathbb{R}^n$ be a closed set which is Ahlfors regular and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\Sigma$. Then from (7.6.66) in Proposition 7.6.8 and item (8) in Lemma 7.7.1 we deduce that for each power $\theta\in(0,1)$ there exists a constant $C(\Sigma,\theta)\in(0,\infty)$ with the property that the function*

$$\begin{gathered} w:\Sigma\to[0,\infty],\quad w(x):=\tfrac{1}{1+|x|^{(n-1)\theta}} \text{ for each } x\in\Sigma, \\ \text{belongs to } A_1(\Sigma,\sigma) \text{ and satisfies } [w]_{A_1}\leq C(\Sigma,\theta). \end{gathered} \tag{8.7.36}$$

As a consequence of this and item (3) in Lemma 7.7.1, we then see that for each exponent $p\in(1,\infty)$ and power $a\in\mathbb{R}$, the function

$$\begin{gathered} \omega:\Sigma\to[0,\infty],\quad \omega(x):=(1+|x|)^a \text{ for each } x\in\Sigma, \\ \text{belongs to } A_p(\Sigma,\sigma) \text{ whenever } 1-n<a<(p-1)(n-1). \end{gathered} \tag{8.7.37}$$

Other examples of Muckenhoupt weights may be obtained in a similar fashion based on (7.6.69) in Proposition 7.6.8 and items (8), (3) in Lemma 7.7.1.

Going further, Proposition 8.7.1 is a key ingredient in the proof of the estimate below.

Proposition 8.7.8 *Let $\Sigma\subseteq\mathbb{R}^n$ be a closed Ahlfors regular set and denote by δ_Σ the distance function to Σ. Also, fix a parameter $\kappa>0$ and, for each $z\in\Sigma$, denote* *by* $\Gamma_{\Sigma^c,\kappa}(z)$ *nontangential approach regions of aperture* κ *relative to the open set* $\Sigma^c := \mathbb{R}^n \setminus \Sigma$ (*whose boundary is precisely* Σ*; cf.* (8.1.2))

Then for each $\beta < n$ *and* $M > n - \beta$ *there exists a finite constant* $C > 0$ (*depending only on* κ*,* M*,* β*, and* n) *such that*

$$\int_{\Gamma_{\Sigma^c,\kappa}(z)} \frac{\delta_\Sigma(x)^{-\beta}}{|x-y|^M}\,\mathrm{d}x \leq C|y-z|^{n-\beta-M} \ \ \textit{for all} \ \ z, y \in \Sigma \ \ \textit{with} \ \ z \neq y. \tag{8.7.38}$$

Proof Fix $z, y \in \Sigma$ with $z \neq y$ and set $r := |y - z| > 0$. For each $j \in \mathbb{N}_0$ introduce

$$\Gamma_{\Sigma^c,\kappa,j}(z) := \big\{x \in \Gamma_{\Sigma^c,\kappa}(z) : \ 2^{j-1}r < |x-z| < 2^j r\big\} \tag{8.7.39}$$

and define

$$I_j := \int_{\Gamma_{\Sigma^c,\kappa,j}(z)} \frac{\delta_\Sigma(x)^{-\beta}}{|x-y|^M}\,\mathrm{d}x. \tag{8.7.40}$$

For $x \in \Gamma_{\Sigma^c,\kappa,j}(z)$ we have $|x - y| \leq |x - z| + |z - y| \leq (2^j + 1)r \leq 2^{j+1}r$. Also, since $\delta_\Sigma(x) \leq |x - z| < (1 + \kappa)\delta_\Sigma(x)$, it follows that $\delta_\Sigma(x) \approx |x - z| \approx 2^j r$. Using these, for each $\alpha \in \mathbb{R}$ we may then write

$$I_j \leq C(2^j r)^{\alpha-\beta} \int_{B(y,2^{j+1}r)} \frac{\delta_\Sigma(x)^{-\alpha}}{|x-y|^M}\,\mathrm{d}x, \qquad \forall\, j \in \mathbb{N}_0. \tag{8.7.41}$$

Now we choose $\alpha < \min\{1, n - M\}$ and apply (8.7.3) from Proposition 8.7.1 to the integral in (8.7.41) to further obtain

$$I_j \leq C(2^j r)^{\alpha-\beta}(2^j r)^{n-\alpha-M} = 2^{j(n-M-\beta)} r^{n-M-\beta}, \qquad \forall\, j \in \mathbb{N}_0. \tag{8.7.42}$$

Next, observe that our hypotheses entail $n - M - \beta < 0$, so $\sum_{j=0}^{\infty} 2^{j(n-M-\beta)} < +\infty$ which, when used in combination with (8.7.42), gives that there exists some finite $C > 0$ such that

$$\int_{\Gamma_{\Sigma^c,\kappa}(z)\setminus B(z,r/2)} \frac{\delta_\Sigma(x)^{-\beta}}{|x-y|^M}\,\mathrm{d}x \leq C|y-z|^{n-\beta-M}. \tag{8.7.43}$$

There remains to find an appropriate bound for

$$\int_{\Gamma_{\Sigma^c,\kappa}(z)\cap B(z,r/2)} \frac{\delta_\Sigma(x)^{-\beta}}{|x-y|^M}\,\mathrm{d}x. \tag{8.7.44}$$

To this end, observe that if $x \in \Gamma_{\Sigma,\kappa}(z) \cap B(z, r/2)$ then $|x - y| \approx |z - y| = r$, thus it suffices to prove that

$$\int_{\Gamma_{\Sigma^c,\kappa}(z)\cap B(z,r/2)} \delta_\Sigma(x)^{-\beta}\,\mathrm{d}x \le Cr^{n-\beta}. \tag{8.7.45}$$

For each $j \in \mathbb{N}_0$ consider

$$\Gamma^j_{\Sigma,\kappa}(z) := \big\{x \in \Gamma_{\Sigma^c,\kappa}(z) : 2^{-j-1}r \le |x-z| \le 2^{-j}r\big\}. \tag{8.7.46}$$

Then, for each $j \in \mathbb{N}_0$,

$$\begin{aligned} &\delta_\Sigma(x) \approx |x-z| \approx 2^{-j}r \ \text{ uniformly for } \ x \in \Gamma^j_{\Sigma,\kappa}(z), \\ &\text{and } \ \mathcal{L}^n\big(\Gamma^j_{\Sigma^c,\kappa}(z)\big) \le \mathcal{L}^n\big(B(z,2^{-j}r)\big) = C(2^{-j}r)^n. \end{aligned} \tag{8.7.47}$$

Thus,

$$\int_{\Gamma^j_{\Sigma^c,\kappa}(z)} \delta_\Sigma(x)^{-\beta}\,\mathrm{d}x \le C(2^{-j}r)^{-\beta}(2^{-j}r)^n, \qquad \forall j \in \mathbb{N}_0 \tag{8.7.48}$$

and, furthermore,

$$\begin{aligned} \int_{\Gamma_{\Sigma^c,\kappa}(z)\cap B(z,r/2)} \delta_\Sigma(x)^{-\beta}\,\mathrm{d}x &= \sum_{j=1}^{\infty}\int_{\Gamma^j_{\Sigma,\kappa}(z)} \delta_\Sigma(x)^{-\beta}\,\mathrm{d}x \\ &\le Cr^{n-\beta}\sum_{j=0}^{\infty} 2^{j(\beta-n)} \le Cr^{n-\beta}, \end{aligned} \tag{8.7.49}$$

where for the last inequality in (8.7.49) we have used the fact that $\beta - n < 0$. This establishes (8.7.45), hence the proof of Proposition 8.7.8 is complete. □

Combining Lemma 8.3.9 with Proposition 8.7.8 yields the estimate described in our next result.

Proposition 8.7.9 *Let Σ be a closed Ahlfors regular set in $\mathbb{R}^n$, where $n \ge 2$. Denote by δ_Σ the distance function to Σ and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. Also, fix $p \in (0,\infty)$, $\kappa \in (0,\infty)$, $\beta < n$, and $M > n-\beta$. Then there exists a constant $C \in (0,\infty)$, which depends only on β, M, p, κ, and the Ahlfors regularity character of Σ, with the property that for every $\mathcal{L}^n$-measurable subset E of $\mathbb{R}^n \setminus \Sigma$, every $\mathcal{L}^n$-measurable function $u : E \to \mathbb{C}$, and every fixed point $y \in \Sigma$ one has*

$$\int_E \frac{\delta_\Sigma(x)^{n-1-\beta}}{|x-y|^M}\,|u(x)|^p\,\mathrm{d}\mathcal{L}^n(x) \le C\int_{\pi_\kappa(E)} \big(\mathcal{N}^E_\kappa u\big)(x)^p\cdot|x-y|^{n-\beta-M}\,\mathrm{d}\sigma(x), \tag{8.7.50}$$

where the intervening nontangential maximal operator $\mathcal{N}^E_\kappa$ and the projection operator π_κ are considered relative to the open ambient $\Sigma^c := \mathbb{R}^n \setminus \Sigma$.

The above estimate is closely related to the estimate in Proposition 8.6.16 specialized to the case when $\alpha := 1$, $\beta := n-1$, and $N := n-b$ (since writing

$\delta_\Sigma(x)^{N-\lambda} = \delta_\Sigma(x)^{1-\lambda}\delta_\Sigma(x)^{n-1-b}$ and pulling $\sup_E \delta_\Sigma^{1-\lambda}$ out of the integral sign in the left-hand side of (8.6.131) yields (8.6.131), on account of (8.7.50)). This being said, the set Σ is required to be Ahlfors regular in Proposition 8.7.9 whereas in Proposition 8.6.16 it was only assumed that Σ is lower Ahlfors regular and $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$ is a doubling measure.

Proof of Proposition 8.7.9 Consider the Borel measure $\mu := \frac{\delta_\Sigma(x)^{-\beta}}{|x-y|^M}\,\mathrm{d}\mathcal{L}^n(x)$ for $x \in \mathbb{R}^n \setminus \Sigma$. Since for each $j \in \mathbb{N}$ the set $O_j := \{x \in B(0,j) \setminus \Sigma : \delta_\Sigma(x) > j^{-1}\}$ is open, satisfies $\mu(O_j) < +\infty$, and $\bigcup_{j\in\mathbb{N}} O_j = \mathbb{R}^n \setminus \Sigma$, it follows that μ is sigma-finite. Granted this, the estimate claimed in (8.7.50) follows by applying (8.3.70) for the choice $d := n-1$ and with μ as described above, then making use of Proposition 8.7.8. □

Proposition 8.7.1 also has a significant involvement in the proof of the following result, containing a useful weighted norm estimate for an integral operator for which only a size condition is imposed on its kernel.

Proposition 8.7.10 *Assume that Ω is a nonempty, open, proper subset of $\mathbb{R}^n$, whose boundary is an Ahlfors regular set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider the integral operator*

$$Tf(x) := \int_{\partial\Omega} k(x,y)f(y)\,\mathrm{d}\sigma(y), \qquad x \in \Omega, \tag{8.7.51}$$

where $k(\cdot,\cdot)$ is a measurable function defined on $\Omega \times \partial\Omega$ which satisfies the estimate

$$|k(x,y)| \leq \frac{C_o}{|x-y|^{n-\varepsilon_*}}, \qquad x \in \Omega,\ \ y \in \partial\Omega, \tag{8.7.52}$$

for some finite constant $C_o > 0$ and some real number $\varepsilon_* < 1$. Finally, fix an integrability exponent $p \in (1,\infty)$ along with a parameter $\beta > 1 - \varepsilon_* - \frac{1}{p}$.

Then there exists a finite constant $C = C(\Omega, C_o, p, \beta, \varepsilon_*) > 0$ with the property that for each $r > 0$ one has

$$\Big(\int_{\{x\in\Omega:\,\delta_{\partial\Omega}(x)<r\}} \delta_{\partial\Omega}(x)^{\beta p}|Tf(x)|^p\,\mathrm{d}x\Big)^{1/p} \leq C\,r^{\beta-1+\varepsilon_*+\frac{1}{p}}\|f\|_{L^p(\partial\Omega,\sigma)}. \tag{8.7.53}$$

In particular,

$$\begin{array}{l}\textit{if } \Omega \textit{ is also assumed to be bounded then the operator}\\ T : L^p(\partial\Omega,\sigma) \longrightarrow L^p\big(\Omega, \delta_{\partial\Omega}^{\beta p}\mathcal{L}^n\big) \ \textit{ is continuous.}\end{array} \tag{8.7.54}$$

Proof Let $\{Q_j\}_{j\in\mathcal{J}}$ be a Whitney decomposition of the set Ω. For each $r > 0$ define $\mathcal{O}_r := \{x \in \Omega : \delta_{\partial\Omega}(x) < r\}$ and consider $\mathcal{J}_r := \{j \in \mathcal{J} : Q_j \cap (\mathcal{O}_r \setminus \mathcal{O}_{r/2}) \neq \varnothing\}$. We now claim that if $f \in L^p(\partial\Omega,\sigma)$ is arbitrary, then

$$\int_{\mathcal{O}_r\setminus\mathcal{O}_{r/2}} |Tf(x)|^p\,\mathrm{d}x \le Cr^{1+(\varepsilon_*-1)p}\int_{\partial\Omega}|f|^p\,\mathrm{d}\sigma, \tag{8.7.55}$$

for some finite constant $C>0$ independent of f. As a preamble, we note that if $\theta\in(0,1)$ is arbitrary and if $p'\in(1,\infty)$ is such that $1/p+1/p'=1$, then for each $x\in\Omega$ using Hölder's inequality we obtain

$$\begin{aligned}|Tf(x)|^p &\le C\Big(\int_{\partial\Omega}\frac{|f(y)|^p}{|x-y|^{(n-\varepsilon_*)\theta p}}\,\mathrm{d}\sigma(y)\Big)\Big(\int_{\partial\Omega}\frac{\mathrm{d}\sigma(y)}{|x-y|^{(n-\varepsilon_*)(1-\theta)p'}}\Big)^{p/p'}\\ &=: C\,I(x)\cdot II(x).\end{aligned} \tag{8.7.56}$$

Fix $j\in\mathcal{J}_r$ and pick an arbitrary point $x\in Q_j$. Using (7.2.5) we may then estimate

$$II(x)\le C\,r^{-(n-\varepsilon_*)(1-\theta)p+(n-1)(p-1)}, \tag{8.7.57}$$

provided $(n-\varepsilon_*)(1-\theta)p'>n-1$ or, equivalently,

$$1-\tfrac{n-1}{n-\varepsilon_*}\big(1-\tfrac{1}{p}\big)>\theta. \tag{8.7.58}$$

Assuming this is the case, for each $j\in\mathcal{J}_r$ we may then conclude from (8.7.56) and (8.7.57) that

$$\int_{Q_j}|Tf(x)|^p\,\mathrm{d}x\le C\,r^{-(n-\varepsilon_*)(1-\theta)p+(n-1)(p-1)}\int_{Q_j}\int_{\partial\Omega}\frac{|f(y)|^p}{|x-y|^{(n-\varepsilon_*)\theta p}}\,\mathrm{d}\sigma(y)\,\mathrm{d}x. \tag{8.7.59}$$

Observe that there exist $\lambda>1$ with the property that

$$\bigcup_{j\in\mathcal{J}_r}Q_j\subseteq\mathcal{O}_{\lambda r}\setminus\mathcal{O}_{r/\lambda}. \tag{8.7.60}$$

Using this and summing up over $j\in\mathcal{J}_r$ inequalities of the type described in (8.7.59) then allows us to write

$$\begin{aligned}\int_{\mathcal{O}_r\setminus\mathcal{O}_{r/2}}|Tf(x)|^p\,\mathrm{d}x &\le C\,r^{-(n-\varepsilon_*)(1-\theta)p+(n-1)(p-1)}\times\\ &\quad\times\int_{\partial\Omega}|f(y)|^p\Big(\int_{\mathcal{O}_{\lambda r}\setminus\mathcal{O}_{r/\lambda}}\frac{\mathrm{d}x}{|x-y|^{(n-\varepsilon_*)\theta p}}\Big)\,\mathrm{d}\sigma(y).\end{aligned} \tag{8.7.61}$$

For each fixed $y\in\partial\Omega$, the inner integral in the right-hand side of (8.7.61) is handled as follows. Assuming we may choose $\alpha\in\mathbb{R}$ so that $1>\alpha>n-(n-\varepsilon_*)\theta p$ or, equivalently,

$$\theta>\tfrac{n-1}{p(n-\varepsilon_*)}, \tag{8.7.62}$$

we may invoke (8.7.5) (with $\Sigma:=\partial\Omega$ and $N:=(n-\varepsilon_*)\theta p$) to estimate

$$
\begin{aligned}
\int_{O_{\lambda r}\setminus O_{r/\lambda}} \frac{\mathrm{d}x}{|x-y|^{(n-\varepsilon_*)\theta p}} &\approx r^{\alpha} \int_{O_{\lambda r}\setminus O_{r/\lambda}} \frac{\delta_{\partial\Omega}(x)^{-\alpha}}{|x-y|^{(n-\varepsilon_*)\theta p}}\,\mathrm{d}x \\
&\leq r^{\alpha} \int_{\Omega\setminus B(y,r/\lambda)} \frac{\delta_{\partial\Omega}(x)^{-\alpha}}{|x-y|^{(n-\varepsilon_*)\theta p}}\,\mathrm{d}x \\
&\leq C\, r^{n-(n-\varepsilon_*)\theta p}.
\end{aligned}
\tag{8.7.63}
$$

Having $\varepsilon_* < 1$ ensures that $\Big(\frac{n-1}{p(n-\varepsilon_*)}, 1-\frac{n-1}{n-\varepsilon_*}\big(1-\frac{1}{p}\big)\Big)$ is a nonempty sub-interval of $(0,1)$. Choosing θ in this sub-interval thus permits us to conclude that

$$
\int_{O_r\setminus O_{r/2}} |Tf(x)|^p\,\mathrm{d}x \leq C\, r^{1+p(\varepsilon_*-1)} \int_{\partial\Omega} |f(y)|^p\,\mathrm{d}\sigma(y), \tag{8.7.64}
$$

which further implies that

$$
\int_{O_r\setminus O_{r/2}} \delta_{\partial\Omega}(x)^{\beta p}|Tf(x)|^p\,\mathrm{d}x \leq C\, r^{1+p(\varepsilon_*-1)+\beta p}\|f\|^p_{L^p(\partial\Omega,\sigma)}. \tag{8.7.65}
$$

Replacing r by $2^{-k}r$ in (8.7.65) and then summing up over $k\in\mathbb{N}_0$ yields

$$
\begin{aligned}
\int_{O_r} \delta_{\partial\Omega}(x)^{\beta p}|Tf(x)|^p\,\mathrm{d}x &= \sum_{k=0}^{\infty}\int_{O_{2^{-k}r}\setminus O_{2^{-k-1}r}} \delta_{\partial\Omega}(x)^{\beta p}|Tf(x)|^p\,\mathrm{d}x \\
&\leq C\, r^{1+p(\varepsilon_*-1)+\beta p}\|f\|^p_{L^p(\partial\Omega,\sigma)}\Big(\sum_{k=0}^{\infty}(2^{-k})^{1+p(\varepsilon_*-1)+\beta p}\Big) \\
&= C\, r^{1+p(\varepsilon_*-1)+\beta p}\|f\|^p_{L^p(\partial\Omega,\sigma)},
\end{aligned}
\tag{8.7.66}
$$

given that, from assumptions, $1+p(\varepsilon_*-1)+\beta p>0$. □

Here is a companion result to Proposition 8.7.10 which is relevant in the treatment of L^p-square function estimates.

Lemma 8.7.11 *Suppose $\Omega\subseteq\mathbb{R}^n$ is a bounded open set with an Ahlfors regular boundary and define $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Let $p,p'\in(1,\infty)$ with $1/p+1/p'=1$, and fix some $\varepsilon\in(0,\frac{1}{p})$. Assume $b(x,y)$ is a measurable function satisfying*

$$
\|b(\cdot,y)\|_{L^p(B(x,r),\mathcal{L}^n)} \leq C_b|x-y|^{1-\varepsilon-n/p'}, \tag{8.7.67}
$$

for each $x\in\Omega$ and $y\in\partial\Omega$, with $r:=|x-y|/2$ and with $C_b\in(0,\infty)$ independent of x, y.

Then the integral operator

$$
\mathscr{R}f(x) := \int_{\partial\Omega} b(x,y)f(y)\,\mathrm{d}\sigma(y), \qquad x\in\Omega, \tag{8.7.68}
$$

has the property that for each power $a > 0$ there exists a finite constant $C_* > 0$, which depends only on n, a, p, ε, and the diameter of Ω, such that

$$\int_\Omega |\mathscr{R} f(x)|^p \operatorname{dist}(x, \partial\Omega)^a \, dx \leq C_*(C_b)^p \int_{\partial\Omega} |f|^p \, d\sigma, \ \ \forall f \in L^p(\partial\Omega, \sigma). \tag{8.7.69}$$

Proof Denote $d := \operatorname{diam} \Omega$ and, having fixed an arbitrary number $\delta \in (0, d]$, set

$$A_\delta := \big\{x \in \Omega : \ \delta/2 < \operatorname{dist}(x, \partial\Omega) \leq \delta\big\}. \tag{8.7.70}$$

Next, let $\{I_k\}_k$ be a decomposition of Ω into non-overlapping Whitney cubes and introduce

$$\mathcal{J}_\delta := \big\{k : \ I_k^\delta := I_k \cap A_\delta \neq \varnothing\big\}. \tag{8.7.71}$$

It follows that the side-length $\ell(I_k)$ of each I_k with $k \in \mathcal{J}_\delta$ is comparable with δ. More specifically, there exists a dimensional constant $C_n \in (1, \infty)$ such that

$$C_n^{-1}\delta \leq \ell(I_k) \leq C_n \delta. \tag{8.7.72}$$

Pick $f \in L^p(\partial\Omega, \sigma)$, arbitrary, with the goal of estimating $\int_{A_\delta} |\mathscr{R} f(x)|^p \, dx$. To do this, decompose

$$b(x, y) = \sum_{j \geq 0} b_{j,\delta}(x, y), \tag{8.7.73}$$

with $b_{0,\delta}(x, y)$ supported in the closed ball $|x - y| \leq 2\sqrt{n}\, C_n\delta$ and $b_{j,\delta}(x, y)$ supported in the annulus $2^{j-1}(2\sqrt{n}\, C_n\delta) \leq |x - y| \leq 2^j(2\sqrt{n}\, C_n\delta)$ for $j \geq 1$. Since having $x \in \Omega$ and $y \in \partial\Omega$ entails $|x - y| \leq d$, it may be assumed that the sum in (8.7.73) actually ranges only over $\mathcal{F}_\delta := \big\{j \geq 0 : \ 2^{j-1}(2\sqrt{n}\, C_n\delta) \leq d\big\}$. The integral operator associated with the kernel $b_{j,\delta}$ will be denoted by $\mathscr{R}_{j,\delta}$. Hence, $\mathscr{R} = \sum_{j\in\mathcal{F}_\delta} \mathscr{R}_{j,\delta}$. We want to estimate $\mathscr{R}_{j,\delta} f$ on A_δ. In this regard, write

$$A_\delta = \bigcup_{k\in\mathcal{J}_\delta} I_k^\delta, \quad \text{disjoint union.} \tag{8.7.74}$$

Since $\partial\Omega$ equipped with the measure σ and the Euclidean distance is a space of homogeneous type, for each $j \geq 0$ with $2^{j-1}(2\sqrt{n}\, C_n\delta) \leq d$ there exists a decomposition of $\partial\Omega$ into a grid of dyadic boundary "cubes" Q_j^δ, of side-length comparable with $2^j\delta$ (see Proposition 7.5.4). For each $k \in \mathcal{J}_\delta$, select one such boundary dyadic cube $Q_{j,k}^\delta$ with the property that

$$\operatorname{dist}\big(I_k^\delta, \partial\Omega\big) = \operatorname{dist}\big(I_k^\delta, Q_{j,k}^\delta\big). \tag{8.7.75}$$

Matters can be arranged so that the concentric dilates of these boundary dyadic cubes have bounded overlap. That is, for every $\lambda \geq 1$ there exists a finite constant $C_\lambda > 0$

such that

$$\sum_{k\in\mathcal{J}_\delta} \mathbf{1}_{\lambda\, Q^\delta_{j,k}} \leq C_\lambda \quad \text{on} \quad \partial\Omega. \tag{8.7.76}$$

Let us now estimate $\|\mathscr{R}_{j,\delta} f\|_{L^p(I^\delta_k,\mathcal{L}^n)}$. In this regard, first note that if $x \in I^\delta_k$ and $y \in \partial\Omega$ are such that $|x-y| \leq 2^j(2\sqrt{n}\,C_n\delta)$, then (8.7.75) implies that $y \in \lambda Q^\delta_{j,k}$ where $\lambda \geq 1$ is a fixed, sufficiently large constant, independent of j,k. Based on this observation and Minkowski's inequality we may therefore write

$$\begin{aligned}\|\mathscr{R}_{j,\delta} f\|_{L^p(I^\delta_k,\mathcal{L}^n)} &= \Big\| \int_{\lambda Q^\delta_{j,k}} b_{j,\delta}(x,y) f(y)\,\mathrm{d}\sigma(y)\Big\|_{L^p_x(I^\delta_k,\mathcal{L}^n)} \\ &\leq \int_{\lambda Q^\delta_{j,k}} \big\|b_{j,\delta}(\cdot,y)\big\|_{L^p(I^\delta_k,\mathcal{L}^n)} |f(y)|\,\mathrm{d}\sigma(y). \end{aligned}\tag{8.7.77}$$

The next step is to estimate $\big\|b_{j,\delta}(\cdot,y)\big\|_{L^p(I^\delta_k,\mathcal{L}^n)}$ for a fixed point $y \in \lambda Q^\delta_{j,k}$. First consider the case when $j \geq 1$. If I^δ_k is disjoint from the annulus

$$\big\{x :\, 2^{j-1}(2\sqrt{n}\,C_n\delta) \leq |x-y| \leq 2^j(2\sqrt{n}\,C_n\delta)\big\}, \tag{8.7.78}$$

where $b_{j,\delta}(\cdot,y)$ is supported, then the aforementioned norm is zero. Thus, suppose there exists $x_o \in I^\delta_k$ with $2^{j-1}(2\sqrt{n}\,C_n\delta) \leq |x_o-y| \leq 2^j(2\sqrt{n}\,C_n\delta)$. Then, if x_k is the center of I_k, using (8.7.72) we may estimate

$$\begin{aligned}|y-x_k| &\leq |y-x_o| + |x_k-x_o| \leq 2^j(2\sqrt{n}\,C_n\delta) + \sqrt{n}\,C_n\delta \\ &= (2^{j+1}+1)\sqrt{n}\,C_n\delta,\end{aligned}\tag{8.7.79}$$

and

$$\begin{aligned}|y-x_k| &\geq |y-x_o| - |x_k-x_o| \geq 2^{j-1}(2\sqrt{n}\,C_n\delta) - \sqrt{n}\,C_n\delta \\ &= (2^j-1)\sqrt{n}\,C_n\delta.\end{aligned}\tag{8.7.80}$$

In particular, since $2^j-1 \geq 1$, it follows from (8.7.80) and (8.7.72) that the radius $r := |y-x_k|/2$ has the property that $I_k \subseteq B(x_k,r)$. In concert with (8.7.67) and (8.7.79)–(8.7.80), this inclusion implies

$$\begin{aligned}\big\|b_{j,\delta}(\cdot,y)\big\|_{L^p(I^\delta_k,\mathcal{L}^n)} &\leq \big\|b_{j,\delta}(\cdot,y)\big\|_{L^p(B(x_k,r),\mathcal{L}^n)} \leq \big\|b(\cdot,y)\big\|_{L^p(B(x_k,r),\mathcal{L}^n)} \\ &\leq C|x_k-y|^{1-\varepsilon-n/p'} \approx (2^j\delta)^{1-\varepsilon-n/p'}\end{aligned}\tag{8.7.81}$$

which is the desired estimate in the case when $j \geq 1$.

We wish to derive a similar estimate in the case when $j = 0$. Use a grid to cover I_k with a finite family of balls $\big\{B(x_\alpha,\delta/6)\big\}_\alpha$ whose cardinality is independent of δ and k. Denote by $\mathscr{A}$ the collection of all indices α in this family for which

$B(x_\alpha, \delta/6)$ intersects both the set I_k^δ and the closed ball $\{x : |x-y| \leq 2\sqrt{n}\, C_n\delta\}$, where $b_{0,\delta}(\cdot, y)$ is supported. Thus, for each $\alpha \in \mathscr{A}$ there exist $z_\alpha \in I_k^\delta \cap B(x_\alpha, \delta/6)$ and $\zeta_\alpha \in B(x_\alpha, \delta/6)$ satisfying $|\zeta_\alpha - y| \leq 2\sqrt{n}\, C_n\delta$. Since $z_\alpha \in I_k^\delta \subseteq A_\delta$, we see that $|y - z_\alpha| \geq \delta/2$. As a consequence,

$$|y - x_\alpha| \geq |y - z_\alpha| - |z_\alpha - x_\alpha| > \delta/2 - \delta/6 = \delta/3. \tag{8.7.82}$$

Also,

$$|y - x_\alpha| \leq |y - \zeta_\alpha| + |\zeta_\alpha - x_\alpha| \leq 2\sqrt{n}\, C_n\delta + \delta/6 = C\delta. \tag{8.7.83}$$

All together, for each $\alpha \in \mathscr{A}$ we have

$$|y - x_\alpha| \approx \delta. \tag{8.7.84}$$

In addition, from (8.7.82) we see that $r_\alpha := |y - x_\alpha|/2 \geq \delta/6$ for each $\alpha \in \mathscr{A}$, which further ensures that $B(x_\alpha, \delta/6) \subseteq B(x_\alpha, r_\alpha)$ whenever $\alpha \in \mathscr{A}$. Together with (8.7.67) and (8.7.84), this inclusion implies

$$\begin{aligned}\big\|b_{0,\delta}(\cdot, y)\big\|_{L^p(I_k^\delta, \mathcal{L}^n)} &\leq \sum_{\alpha\in\mathscr{A}} \big\|b_{0,\delta}(\cdot, y)\big\|_{L^p(B(x_\alpha, r_\alpha), \mathcal{L}^n)} \\ &\leq \sum_{\alpha\in\mathscr{A}} \big\|b(\cdot, y)\big\|_{L^p(B(x_\alpha, r_\alpha), \mathcal{L}^n)} \\ &\leq C \sum_{\alpha\in\mathscr{A}} |x_\alpha - y|^{1-\varepsilon-n/p'} \approx \delta^{1-\varepsilon-n/p'}\end{aligned} \tag{8.7.85}$$

which matches the format of (8.7.81) in the case when $j = 0$.

Gathering (8.7.77), (8.7.81), and (8.7.85) then yields

$$\|\mathscr{R}_{j,\delta} f\|_{L^p(I_k^\delta, \mathcal{L}^n)} \leq C(2^j\delta)^{1-\varepsilon-n/p'} \|f\|_{L^1(\lambda Q_{j,k}^\delta, \mathcal{L}^n)}. \tag{8.7.86}$$

On the other hand, by Hölder's inequality and the fact that $\partial\Omega$ is Ahlfors regular,

$$\|f\|_{L^1(\lambda Q_{j,k}^\delta, \mathcal{L}^n)} \leq C(2^j\delta)^{(n-1)/p'} \|f\|_{L^p(\lambda Q_{j,k}^\delta, \mathcal{L}^n)} \tag{8.7.87}$$

so

$$\begin{aligned}\|\mathscr{R}_{j,\delta} f\|_{L^p(A_\delta, \mathcal{L}^n)}^p &= \sum_{k\in\mathcal{J}_\delta} \|\mathscr{R}_{j,\delta} f\|_{L^p(I_k^\delta, \mathcal{L}^n)}^p \\ &\leq C(2^j\delta)^{1-p\varepsilon} \sum_{k\in\mathcal{J}_\delta} \|f\|_{L^p(\lambda Q_{j,k}^\delta, \sigma)}^p \\ &\leq C(2^j\delta)^{1-p\varepsilon} \|f\|_{L^p(\partial\Omega, \sigma)}^p,\end{aligned} \tag{8.7.88}$$

by (8.7.74) and (8.7.76). Hence,

$$
\begin{aligned}
\Big(\int_{A_\delta}|\mathscr{R}f(x)|^p\,\mathrm{d}x\Big)^{1/p} &\leq \sum_{j\in\mathcal{F}_\delta}\|\mathscr{R}_{j,\delta}f\|_{L^p(A_\delta,\mathcal{L}^n)}\\
&\leq C\Big(\sum_{\{j\geq 0:\,2^{j-1}\delta\leq d\}}(2^j\delta)^{1/p-\varepsilon}\Big)\|f\|_{L^p(\partial\Omega,\sigma)}\\
&\leq C_{d,p,\varepsilon}C_b\|f\|_{L^p(\partial\Omega,\sigma)}, \qquad (8.7.89)
\end{aligned}
$$

with the constant C_b as in (8.7.67), and with $C_{d,p,\varepsilon}$ independent of δ and finite, as long as $\varepsilon\in(0,\frac{1}{p})$, which we assume to begin with. Consequently, since

$$
\Omega=\bigcup_{\mu\in\mathbb{N}_0}A_{2^{-\mu}d},\quad\text{disjoint union,} \qquad (8.7.90)
$$

we may use the estimate obtained in (8.7.89) for $\delta:=2^{-\mu}d\in(0,d]$ with $\mu\in\mathbb{N}_0$ to write

$$
\begin{aligned}
\int_\Omega|\mathscr{R}f(x)|^p\operatorname{dist}(x,\partial\Omega)^a\,\mathrm{d}x &\leq d^a\sum_{\mu\in\mathbb{N}_0}2^{-\mu a}\|\mathscr{R}f\|^p_{L^p(A_{2^{-\mu}d},\mathcal{L}^n)}\\
&\leq C_{d,a,p,\varepsilon}C_b\|f\|^p_{L^p(\partial\Omega,\sigma)}, \qquad (8.7.91)
\end{aligned}
$$

with $C_{d,a,p,\varepsilon}\in(0,\infty)$ if $a>0$. □

Several other useful estimates of a purely geometric measure theoretic nature are established in the corollary below, based on the general results from Lemma 7.2.1 (cf. (7.2.5)).

Corollary 8.7.12 *Assume $\Omega\subseteq\mathbb{R}^n$ is an open set and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Then, if $\partial\Omega$ is upper Ahlfors regular, it follows that for each $\varepsilon>0$ there exists $C\in(0,\infty)$ with the property that*

$$
\int_{\partial\Omega}\frac{\mathrm{d}\sigma(y)}{|x-y|^{n-1+\varepsilon}}\leq C\cdot\delta_{\partial\Omega}(x)^{-\varepsilon}\ \ \textit{for each}\ \ x\in\Omega, \qquad (8.7.92)
$$

and if $\partial\Omega$ is lower Ahlfors regular then for each $\varepsilon>0$ there exists $c\in(0,\infty)$ such that

$$
c\cdot\delta_{\partial\Omega}(x)^{-\varepsilon}\leq\int_{\partial\Omega}\frac{\mathrm{d}\sigma(y)}{|x-y|^{n-1+\varepsilon}}\ \ \textit{for each}\ \ x\in\Omega. \qquad (8.7.93)
$$

As a consequence, if $\partial\Omega$ is an Ahlfors regular set then for each $\varepsilon>0$ one has

$$
\int_{\partial\Omega}\frac{\mathrm{d}\sigma(y)}{|x-y|^{n-1+\varepsilon}}\approx\delta_{\partial\Omega}(x)^{-\varepsilon}\ \ \textit{uniformly for}\ \ x\in\Omega. \qquad (8.7.94)
$$

Proof Fix an arbitrary point $x\in\Omega$, introduce $r:=\delta_{\partial\Omega}(x)$, and consider a point $x_*\in\partial\Omega$ with the property that $|x_*-x|=r$. Then, under the assumption that $\partial\Omega$ is upper Ahlfors regular set, we may estimate

$$\begin{aligned}\int_{\partial\Omega} \frac{\mathrm{d}\sigma(y)}{|x-y|^{n-1+\varepsilon}} &\approx \int_{\partial\Omega} \frac{\mathrm{d}\sigma(y)}{[r+|x_*-y|]^{n-1+\varepsilon}} \\ &\leq C \int_{B(x_*,2r)\cap\partial\Omega} \frac{\mathrm{d}\sigma(y)}{r^{n-1+\varepsilon}} + C \int_{\partial\Omega\setminus B(x_*,2r)} \frac{\mathrm{d}\sigma(y)}{|x_*-y|^{n-1+\varepsilon}} \\ &\leq Cr^{-(n-1+\varepsilon)} \cdot \sigma\big(B(x_*,2r)\cap\partial\Omega\big) + Cr^{-\varepsilon} \leq Cr^{-\varepsilon}, \end{aligned} \tag{8.7.95}$$

thanks to (8.1.22), the fact that $\partial\Omega$ is upper Ahlfors regular, and the second inequality in (7.2.5). This establishes (8.7.92). If, on the other hand, $\partial\Omega$ is a lower Ahlfors regular set, we have

$$\begin{aligned}\int_{\partial\Omega} \frac{\mathrm{d}\sigma(y)}{|x-y|^{n-1+\varepsilon}} &\approx \int_{\partial\Omega} \frac{\mathrm{d}\sigma(y)}{[r+|x_*-y|]^{n-1+\varepsilon}} \geq c \int_{B(x_*,r)\cap\partial\Omega} \frac{\mathrm{d}\sigma(y)}{r^{n-1+\varepsilon}} \\ &= cr^{-(n-1+\varepsilon)} \cdot \sigma\big(B(x_*,r)\cap\partial\Omega\big) \geq cr^{-\varepsilon}, \end{aligned} \tag{8.7.96}$$

proving (8.7.93). Finally, under the assumption that $\partial\Omega$ is an Ahlfors regular set, (8.7.94) is a consequence of (8.7.92)–(8.7.93). □

Our final result in this section is going to be relevant later on when dealing with singular integral operators.

Proposition 8.7.13 *Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set with an upper Ahlfors regular boundary and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. In addition, assume $a(\cdot,\cdot)$ is a Borel-measurable function on $\Omega\times\partial\Omega$ with the property that there exist some exponent $\alpha \in [0,\infty)$ and some constant $C_o \in (0,\infty)$ such that*

$$\begin{aligned}&\textit{for } \mathcal{L}^n\textit{-a.e. point } x\in\Omega \textit{ one has } |a(x,y)| \leq C_o\,|x-y|^{-(n-1+\alpha)} \\ &\qquad \textit{at } \sigma\textit{-a.e. point } y \textit{ belonging to } \partial\Omega.\end{aligned} \tag{8.7.97}$$

Finally, define the integral operator

$$\begin{aligned}&\mathscr{A}f(x) := \int_{\partial\Omega} a(x,y)f(y)\,\mathrm{d}\sigma(y) \quad \text{at } \mathcal{L}^n\text{-a.e. } x\in\Omega, \\ &\text{for each function } f \in L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1+\alpha}}\Big).\end{aligned} \tag{8.7.98}$$

Then the integral in (8.7.98) is absolutely convergent, one has the continuous embedding

$$L^p(\partial\Omega,\sigma) \hookrightarrow L^1\Big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1+\alpha}}\Big) \quad \text{for each } p\in[1,\infty), \tag{8.7.99}$$

and for each $p \in [1,\infty)$ there exists a constant $C \in (0,\infty)$ with the property that

$$\Big\| \delta_{\partial\Omega}^{\frac{n-1}{p}+\alpha} \cdot \mathscr{A} f \Big\|_{L^\infty(\Omega,\mathcal{L}^n)} \leq C \|f\|_{L^p(\partial\Omega,\sigma)} \text{ for all } f \in L^p(\partial\Omega,\sigma). \tag{8.7.100}$$

Finally, if $\alpha > 0$ *then* $p = \infty$ *is allowed in* (8.7.99) *and* (8.7.100).

Proof All claims are consequences of (8.7.97)–(8.7.98), Hölder's inequality, and estimate (8.7.92) in Corollary 8.7.12. □

8.8 The Nontangentially Accessible Boundary

Consider an arbitrary open nonempty proper subset Ω of $\mathbb{R}^n$ and pick an arbitrary aperture parameter $\kappa \in (0,\infty)$. From (8.1.10) we know that at each point $x \in \partial\Omega$ we have the following dichotomy:

$$\text{either } x \in \overline{\Gamma_\kappa(x)}, \quad \text{or } \overline{\Gamma_\kappa(x)} \subseteq \Omega. \tag{8.8.1}$$

In fact, in the latter scenario it turns out that $\operatorname{dist}\big(\overline{\Gamma_\kappa(x)}, \partial\Omega\big) > 0$. To study the likelihood of the first eventuality in (8.8.1) materializing, we propose to study the set

$$A_\kappa(\partial\Omega) := \big\{x \in \partial\Omega : x \in \overline{\Gamma_\kappa(x)}\big\}. \tag{8.8.2}$$

Informally, $A_\kappa(\partial\Omega)$ consists of those boundary points which are "accessible" in a nontangential fashion (specifically, from within nontangential approach regions of aperture κ). From definitions, it follows that

$$A_\kappa(\partial\Omega) \cap \mathcal{O} \subseteq \pi_\kappa(\mathcal{O}) \text{ for each open set } \mathcal{O} \subseteq \mathbb{R}^n. \tag{8.8.3}$$

To proceed, recall that, in a given topological space, a G_δ set is a countable intersection of open sets.

Proposition 8.8.1 *Let* Ω *be an open nonempty proper subset of* $\mathbb{R}^n$ *and consider* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. *Then*

$$A_{\kappa_0}(\partial\Omega) \subseteq A_\kappa(\partial\Omega) \text{ whenever } 0 < \kappa_0 \leq \kappa < \infty, \tag{8.8.4}$$

and, for each $\kappa > 0$,

$A_\kappa(\partial\Omega)$ is a G_δ set in $\partial\Omega$ (equipped with the relative topology, induced by $\mathbb{R}^n$); in particular, each set $A_\kappa(\partial\Omega)$ is Borelian, hence σ-measurable. (8.8.5)

Moreover,

$$\text{if } \sigma \text{ is a doubling measure on the set } \partial\Omega, \text{ then} \\ \sigma\big(A_\kappa(\partial\Omega) \setminus A_{\kappa_0}(\partial\Omega)\big) = 0 \text{ for all } \kappa, \kappa_0 > 0. \tag{8.8.6}$$

Remarkably, (8.8.6) states that, under the assumption that σ is a doubling measure, the sets $A_\kappa(\partial\Omega)$ indexed by $\kappa \in (0, \infty)$ are equal to one another up to σ-nullsets. Later on, in Proposition 8.8.3, we shall show that in a suitable geometric setting there exist sets in this family which are "large" (i.e., cover $\partial_*\Omega$ up to a σ-nullset).

As a prelude to the proof of Proposition 8.8.1, in the lemma below we isolate a useful elementary result.

Lemma 8.8.2 *Let X be an arbitrary set. Consider a sequence $f_j : X \to \mathbb{R}$, indexed by $j \in \mathbb{N}$, such that $f(x) := \limsup\limits_{j\to\infty} f_j(x)$ exists in $\mathbb{R}$ for each $x \in X$. Then the function $f : X \to \mathbb{R}$ has the property that for each number $\lambda \in \mathbb{R}$ and each numerical sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subseteq (-\infty, \lambda)$ with $\lambda = \lim\limits_{i\to\infty} \lambda_i$ one has*

$$f^{-1}\big([\lambda, \infty)\big) = \bigcap_{k\in\mathbb{N}} \bigcap_{i\in\mathbb{N}} \bigcup_{j\geq k} f_j^{-1}\big((\lambda_i, \infty)\big). \tag{8.8.7}$$

As a consequence, if X is a topological space and each f_j is a lower-semicontinuous function, then $f^{-1}\big([\lambda, \infty)\big)$ is a G_δ set for each $\lambda \in \mathbb{R}$.

Proof If for each $k \in \mathbb{N}$ we define $g_k := \sup_{j\geq k} f_j$, then for each $x \in X$ we have

$$f(x) = \limsup_{j\to\infty} f_j(x) = \inf_{k\in\mathbb{N}} \Big(\sup_{j\geq k} f_j(x)\Big) = \inf_{k\in\mathbb{N}} g_k(x). \tag{8.8.8}$$

Consequently, for any $\lambda \in \mathbb{R}$ and $\{\lambda_i\}_{i\in\mathbb{N}} \subseteq (-\infty, \lambda)$ with $\lambda = \lim\limits_{i\to\infty} \lambda_i$ we may write

$$\begin{aligned} f^{-1}\big([\lambda, \infty)\big) &= \bigcap_{k\in\mathbb{N}} g_k^{-1}\big([\lambda, \infty)\big) = \bigcap_{k\in\mathbb{N}} g_k^{-1}\Big(\bigcap_{i\in\mathbb{N}} (\lambda_i, \infty)\Big) \\ &= \bigcap_{k\in\mathbb{N}} \bigcap_{i\in\mathbb{N}} g_k^{-1}\big((\lambda_i, \infty)\big) = \bigcap_{k\in\mathbb{N}} \bigcap_{i\in\mathbb{N}} \bigcup_{j\geq k} f_j^{-1}\big((\lambda_i, \infty)\big), \end{aligned} \tag{8.8.9}$$

proving (8.8.7). The last claim in the statement is then an immediate corollary of this (bearing in mind that each $f_j^{-1}\big((\lambda_i, \infty)\big)$ is now an open set). □

We are now ready to proceed to the proof of Proposition 8.8.1.

Proof of Proposition 8.8.1 The inclusion claimed in (8.8.4) is clear from (8.8.2) and (8.1.20). The proof of (8.8.5) requires some preparations. Given a nonempty closed set $F \subseteq \partial\Omega$ denote

$$F^\varepsilon := \{x \in \Omega : \operatorname{dist}(x, F) < \varepsilon\}, \qquad \forall \varepsilon > 0. \tag{8.8.10}$$

Thus, each F^ε is an open subset of Ω. Moreover, for each fixed $\kappa > 0$ we have

$$\pi_\kappa(F^\varepsilon) \subseteq \{x \in \partial\Omega : \operatorname{dist}(x, F) < (2+\kappa)\varepsilon\}, \qquad \forall \varepsilon > 0. \tag{8.8.11}$$

Indeed, if $x \in \pi_\kappa(F^\varepsilon)$ then $\Gamma_\kappa(x) \cap F^\varepsilon \neq \varnothing$. If y is a point in this intersection, then $|x-y| < (1+\kappa)\operatorname{dist}(y, \partial\Omega)$ and $\operatorname{dist}(y, \partial\Omega) \le \operatorname{dist}(y, F) < \varepsilon$. Then (8.8.11) follows by observing that these inequalities imply

$$\operatorname{dist}(x, F) \le |x-y| + \operatorname{dist}(y, F) < (2+\kappa)\varepsilon. \tag{8.8.12}$$

In relation to (8.8.10) we claim that for each fixed $\kappa > 0$ we have

$$\mathcal{N}_\kappa(\mathbf{1}_{F^\varepsilon}) \longrightarrow \mathbf{1}_{\{x\in F:\, x\in\overline{\Gamma_\kappa(x)}\}} \ \text{ pointwise on } \ \partial\Omega, \quad \text{as } \ \varepsilon \to 0^+. \tag{8.8.13}$$

To justify this claim, consider first a point $z \in F$ with the property that $z \in \overline{\Gamma_\kappa(z)}$ and pick some $\varepsilon > 0$. Then the set $\mathscr{U}_\varepsilon := \{x \in \mathbb{R}^n : \operatorname{dist}(x, F) < \varepsilon\}$ is a neighborhood of z. Bearing in mind that $\Gamma_\kappa(z) \subseteq \Omega$ and $\mathscr{U}_\varepsilon \cap \Omega = F^\varepsilon$, this implies that for every $\varepsilon > 0$ we have

$$\varnothing \neq \mathscr{U}_\varepsilon \cap \Gamma_\kappa(z) = \mathscr{U}_\varepsilon \cap \Omega \cap \Gamma_\kappa(z) = F^\varepsilon \cap \Gamma_\kappa(z). \tag{8.8.14}$$

Hence, $F^\varepsilon \cap \Gamma_\kappa(z)$ is a nonempty open subset of Ω on which $\mathbf{1}_{F^\varepsilon}$ is identically one. In light of (8.2.5) this implies that $\mathcal{N}_\kappa(\mathbf{1}_{F^\varepsilon})(z) = 1$ for every $\varepsilon > 0$. This suits our purposes since, by the assumptions on z, we also have $\big(\mathbf{1}_{\{x\in F:\, x\in\overline{\Gamma_\kappa(x)}\}}\big)(z) = 1$.

Consider next the case when $z \in \partial\Omega \setminus F$. Since F is a closed subset of $\partial\Omega$, this implies the existence of some $\varepsilon_z > 0$ with the property that $B(z, \varepsilon_z) \cap F = \varnothing$. Hence, $\operatorname{dist}(z, F) \ge \varepsilon_z$. In view of (8.2.24) and (8.8.11), this implies that whenever $\varepsilon \in \big(0, \varepsilon_z/(2+\kappa)\big)$ we have

$$\mathcal{N}_\kappa(\mathbf{1}_{F^\varepsilon})(z) \le \big(\mathbf{1}_{\pi_\kappa(F^\varepsilon)}\big)(z) = 0. \tag{8.8.15}$$

This proves that $\mathcal{N}_\kappa(\mathbf{1}_{F^\varepsilon})(z) = 0$ for every $\varepsilon \in \big(0, \varepsilon_z/(2+\kappa)\big)$, which once again suits our goals since $\big(\mathbf{1}_{\{x\in F:\, x\in\overline{\Gamma_\kappa(x)}\}}\big)(z) = 0$ in this case.

Finally, consider the case when $z \in \partial\Omega$ is such that $z \notin \overline{\Gamma_\kappa(z)}$. Observe that the latter condition guarantees the existence of some $\varepsilon_z > 0$ the property that

$$F^{\varepsilon_z} \cap \Gamma_\kappa(z) = \varnothing. \tag{8.8.16}$$

Otherwise, one could find a sequence $\{x_j\}_{j\in\mathbb{N}} \subset \Gamma_\kappa(z)$ with the property that $\operatorname{dist}(x_j, F) \to 0$ as $j \to \infty$. In such a scenario,

$$|z - x_j| < (1+\kappa)\operatorname{dist}(x_j, \partial\Omega) \le (1+\kappa)\operatorname{dist}(x_j, F) \to 0 \ \text{ as } \ j \to \infty, \tag{8.8.17}$$

which would then force $z = \lim_{j\to\infty} x_j \in \overline{\Gamma_\kappa(z)}$, a contradiction. In turn, (8.8.16) implies

$$z \notin \pi_\kappa(F^\varepsilon) \text{ for every } \varepsilon \in (0, \varepsilon_z). \tag{8.8.18}$$

As such, for every $\varepsilon \in (0, \varepsilon_z)$ we have $\mathcal{N}_\kappa(\mathbf{1}_{F^\varepsilon})(z) \leq \big(\mathbf{1}_{\pi_\kappa(F^\varepsilon)}\big)(z) = 0$, proving that $\mathcal{N}_\kappa(\mathbf{1}_{F^\varepsilon})(z) = 0$ for every $\varepsilon \in (0, \varepsilon_z)$. This once more serves our current aims since $\big(\mathbf{1}_{\{x\in F:\, x\in\overline{\Gamma_\kappa(x)}\}}\big)(z) = 0$ in this case. This finishes the proof of (8.8.13).

If we now specialize (8.8.13) to the case when $F := \partial\Omega$ it follows that for each fixed $\kappa > 0$

$$\mathcal{N}_\kappa(\mathbf{1}_{(\partial\Omega)^\varepsilon}) \longrightarrow \mathbf{1}_{\{x\in\partial\Omega:\, x\in\overline{\Gamma_\kappa(x)}\}} \text{ pointwise on } \partial\Omega, \text{ as } \varepsilon \to 0^+. \tag{8.8.19}$$

In turn, from (8.8.19), (8.8.2), the lower-semicontinuity on $\partial\Omega$ of each $\mathcal{N}_\kappa(\mathbf{1}_{(\partial\Omega)^\varepsilon})$ (cf. (8.2.26)), and Lemma 8.8.2 applied with $X := \partial\Omega$, the sequence $f_j := \mathcal{N}_\kappa(\mathbf{1}_{(\partial\Omega)^{1/j}})$ indexed by $j \in \mathbb{N}$ (a choice which forces $f := \lim_{j\to\infty} f_j = \mathbf{1}_{A_\kappa(\partial\Omega)}$), and $\lambda := 1$, we conclude that $A_\kappa(\partial\Omega) = f^{-1}\big([1, \infty)\big)$ is a G_δ set in $\partial\Omega$. Hence, (8.8.5) holds.

Before going any further, we wish to note that

$$\begin{array}{c}\text{if } F \subseteq \partial\Omega \text{ is a given nonempty, closed, bounded set,}\\ \text{then for each aperture parameter } \kappa \in (0, \infty) \text{ we have}\\ \mathcal{N}_\kappa(\mathbf{1}_{F^\varepsilon}) \to \mathbf{1}_{\{x\in F:\, x\in\overline{\Gamma_\kappa(x)}\}} \text{ in } L^1(\partial\Omega, \sigma), \text{ as } \varepsilon \to 0^+.\end{array} \tag{8.8.20}$$

Indeed, this is a consequence of Lebesgue's Dominated Convergence Theorem, whose applicability is ensured by (8.8.13) together with the observation that for each $\varepsilon \in (0, 1)$ we have (thanks to (8.2.24) and (8.8.11))

$$\mathcal{N}_\kappa(\mathbf{1}_{F^\varepsilon}) \leq \mathbf{1}_{\pi_\kappa(F^\varepsilon)} \leq \mathbf{1}_{\{x\in\partial\Omega:\, \mathrm{dist}(x,F)<2+\kappa\}} \in L^1(\partial\Omega, \sigma). \tag{8.8.21}$$

Moving on, make the assumption that $\sigma = \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$, and consider $\kappa, \kappa_0 > 0$ arbitrary. Then Proposition 8.4.1 implies that for each nonempty, closed, and bounded set $F \subseteq \partial\Omega$ and each $\varepsilon > 0$ we have

$$\|\mathcal{N}_\kappa \mathbf{1}_{F^\varepsilon}\|_{L^1(\partial\Omega,\sigma)} \approx \|\mathcal{N}_{\kappa_0} \mathbf{1}_{F^\varepsilon}\|_{L^1(\partial\Omega,\sigma)} \tag{8.8.22}$$

where the implicit constants depend only on κ, κ_0, n and the doubling character of σ. Upon letting $\varepsilon \to 0^+$ and availing ourselves of (8.8.20) then yields

$$\sigma\big(\{x \in F : x \in \overline{\Gamma_\kappa(x)}\}\big) \approx \sigma\big(\{x \in F : x \in \overline{\Gamma_{\kappa_0}(x)}\}\big), \tag{8.8.23}$$

uniformly in $F \subseteq \partial\Omega$, nonempty, closed, and bounded set. In fact, working with $F_j := F \cap \overline{B(0, j)}$ in place of F and passing to the limit $j \to \infty$ (while mindful of (3.1.6)), allows us to dispense with the demand that the set F is bounded. In light of (8.8.2), this version of (8.8.23) implies

$$\sigma\big(F \cap A_\kappa(\partial\Omega)\big) \approx \sigma\big(F \cap A_{\kappa_0}(\partial\Omega)\big) \quad \text{for every closed set } F \subseteq \partial\Omega, \tag{8.8.24}$$

where the implicit constants depend only on κ, κ_0, n and the doubling character of σ. Specializing (8.8.24) to the case when F is any closed subset of $A_\kappa(\partial\Omega) \setminus A_{\kappa_0}(\partial\Omega)$ yields

$$\sigma(F) = 0 \quad \text{for every closed subset } F \text{ of } A_\kappa(\partial\Omega) \setminus A_{\kappa_0}(\partial\Omega). \tag{8.8.25}$$

On the other hand, thanks to Lemma 3.4.13, Lemma 3.6.4, and the fact that (3.4.44) always holds for a space of homogeneous type, item *(3)* in Proposition 3.4.15 applies to the measure σ. In view of (8.8.25), the inner-regularity result described in (3.4.47) then yields, in the case of the σ-measurable set $A_\kappa(\partial\Omega) \setminus A_{\kappa_0}(\partial\Omega)$, that

$$\begin{aligned}\sigma\big(A_\kappa(\partial\Omega) \setminus A_{\kappa_0}(\partial\Omega)\big) \\ = \sup\big\{\sigma(F) : F \text{ closed subset of } A_\kappa(\partial\Omega) \setminus A_{\kappa_0}(\partial\Omega)\big\} = 0.\end{aligned} \tag{8.8.26}$$

This establishes (8.8.6), and finishes the proof of the proposition. □

We continue by recording a delicate zero-density result for Hausdorff measures, essentially due to Besicovitch [21], which is going to be relevant for us momentarily. Specifically, given any $s \geq 0$, if

$$E \subseteq \mathbb{R}^n \text{ is a } \mathcal{H}^s\text{-measurable set satisfying} \quad \mathcal{H}^s(E \cap K) < \infty \text{ for every compact } K \subset \mathbb{R}^n, \tag{8.8.27}$$

then

$$\lim_{r\to 0^+} \frac{\mathcal{H}^s\big(B(x,r) \cap E\big)}{r^s} = 0 \quad \text{for } \mathcal{H}^s\text{-a.e. } x \in \mathbb{R}^n \setminus E. \tag{8.8.28}$$

For a proof of (8.8.28) in the case when $E \subseteq \mathbb{R}^n$ is $\mathcal{H}^s$-measurable and satisfies $\mathcal{H}^s(E) < \infty$ see [177, Theorem 6.2(2), p. 89], or [162, Theorem 1.3.7, p. 16]. This result then readily self-improves to the slightly more general case described in (8.8.27), where the set E is only required to be of locally finite $\mathcal{H}^s$-measure.

Proposition 8.8.3 *Suppose Ω is an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary and such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a locally finite measure on $\partial\Omega$. Then Ω has locally finite perimeter and there exists some $\kappa_\Omega \in (0, \infty)$ with the property that*

$$\sigma\big(\partial_*\Omega \setminus A_\kappa(\partial\Omega)\big) = \sigma\big(\partial^*\Omega \setminus A_\kappa(\partial\Omega)\big) = 0 \quad \textit{for each } \kappa > \kappa_\Omega. \tag{8.8.29}$$

Proof From definitions and assumptions it follows that for each compact set $K \subset \mathbb{R}^n$ we may estimate $\mathcal{H}^{n-1}\big(\partial_*\Omega \cap K\big) \leq \mathcal{H}^{n-1}\big(\partial\Omega \cap K\big) = \sigma\big(\partial\Omega \cap K\big) < \infty$. In light of (5.6.35) this proves that Ω has locally finite perimeter, hence it is meaningful to talk about its reduced boundary $\partial^*\Omega$.

Let us now turn out attention to (8.8.29). The first equality is a direct consequence of (5.6.21), so we shall focus on the second equality. As a preamble, recall (5.6.19) and observe that (8.8.27)–(8.8.28) used for $s := n-1$ and $E := \partial\Omega \setminus \partial^*\Omega$ give that, on the one hand, in the current setting we have

$$\lim_{r\to 0^+} \frac{\mathcal{H}^{n-1}\big(B(x,r)\cap(\partial\Omega\setminus\partial^*\Omega)\big)}{r^{n-1}} = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in\partial^*\Omega. \tag{8.8.30}$$

On the other hand, [80, Corollary 1(ii), p. 203] implies that if

$$\mathrm{vol}_{n-1} := \pi^{(n-1)/2}\big/\Gamma\big((n+1)/2\big) = \omega_{n-2}/(n-1) \tag{8.8.31}$$

denotes the volume of the unit ball in $\mathbb{R}^{n-1}$ then

$$\lim_{r\to 0^+} \frac{\mathcal{H}^{n-1}\big(B(x,r)\cap\partial^*\Omega\big)}{r^{n-1}} = \mathrm{vol}_{n-1} \quad \text{for each } x\in\partial^*\Omega. \tag{8.8.32}$$

If for any point $x\in\partial\Omega$ and number $r>0$ we abbreviate $\Delta(x,r) := B(x,r)\cap\partial\Omega$, then (8.8.30) and (8.8.32) allow us to conclude that

$$\lim_{r\to 0^+} \frac{\sigma\big(\Delta(x,r)\big)}{r^{n-1}} = \mathrm{vol}_{n-1} \quad \text{for } \sigma\text{-a.e. } x\in\partial^*\Omega. \tag{8.8.33}$$

Next, recall the family of one-sided collar neighborhoods $\mathcal{O}_\varepsilon$ of $\partial\Omega$, indexed by $\varepsilon>0$, introduced in (6.1.4). Also, fix some $\kappa_0>0$. Then from (8.6.87) (and its proof) used with $\lambda := 2+\kappa_0\in(2,\infty)$ we know that

$$\mathcal{L}^n\big(B(x,r)\cap\mathcal{O}_\varepsilon\big) \leq C\varepsilon\,\sigma\big(\Delta(x,(2+\kappa_0)r)\big), \tag{8.8.34}$$

for every $\varepsilon\in\big(0,\varepsilon_{\Omega,\kappa_0}\big)$, where $\varepsilon_{\Omega,\kappa_0}$ is as in (8.6.75) (with $\kappa := \kappa_0$) and the constant $C = C(\partial\Omega,\kappa_0)\in(0,\infty)$ is independent of x, r, ε. In concert with (8.8.33), this implies that there exists some constant $\widetilde{C} = \widetilde{C}(\partial\Omega,\kappa_0)\in(0,\infty)$ such that

$$\begin{array}{l}\text{for } \sigma\text{-a.e. point } x\in\partial^*\Omega \text{ there exists some number} \\ r_x^{(1)}\in(0,1) \text{ such that } \mathcal{L}^n\big(B(x,r)\cap\mathcal{O}_\varepsilon\big)\leq\widetilde{C}\,\varepsilon\,r^{n-1} \\ \text{for every } r\in\big(0,r_x^{(1)}\big) \text{ as well as every } \varepsilon\in\big(0,\varepsilon_{\Omega,\kappa_0}\big).\end{array} \tag{8.8.35}$$

This property is going to be relevant momentarily.

Carrying on, from Lemma 5.6.8 we know that there exists some purely dimensional constant $c_n\in(0,\infty)$ for which

$$\liminf_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap\Omega)}{r^n} > c_n, \qquad \forall x\in\partial^*\Omega. \tag{8.8.36}$$

Consequently, for each $x\in\partial^*\Omega$ there exists $r_x^{(2)}\in(0,1)$ such that

$$\mathcal{L}^n(B(x,r)\cap\Omega) > c_n r^n, \qquad \forall r\in\big(0,r_x^{(2)}\big). \tag{8.8.37}$$

Choose

$$\vartheta = \vartheta(\partial\Omega,\kappa_0) := \min\Big\{1/2,\, \varepsilon_{\Omega,\kappa_0},\, c_n\big/\widetilde{C}\Big\}. \tag{8.8.38}$$

Then for σ-a.e. point $x\in\partial^*\Omega$ and each scale $r\in\big(0,r_x\big)$ where

$$r_x := \min\big\{r_x^{(1)}, r_x^{(2)}\big\}\in(0,1), \tag{8.8.39}$$

a combination of (8.8.37) and (8.8.35) with ε replaced by $\varepsilon_r := \vartheta r\in\big(0,\varepsilon_{\Omega,\kappa_0}\big)$ yields

$$\begin{aligned}\mathcal{L}^n\big([B(x,r)\cap\Omega]\setminus\mathcal{O}_{\varepsilon_r}\big) &= \mathcal{L}^n(B(x,r)\cap\Omega) - \mathcal{L}^n\big(B(x,r)\cap\mathcal{O}_{\varepsilon_r}\big)\\ &> c_n r^n - \widetilde{C}\,\varepsilon_r\, r^{n-1} = r^n\big(c_n - \widetilde{C}\,\vartheta\big)\ge 0,\end{aligned} \tag{8.8.40}$$

bearing in mind the choice of ϑ.

What we have just proved implies that there exists a nullset $N\subseteq\partial\Omega$ for the measure σ with the property that

$$[B(x,r)\cap\Omega]\setminus\mathcal{O}_{\varepsilon_r}\neq\varnothing \qquad \forall x\in\partial^*\Omega\setminus N,\quad \forall r\in(0,r_x). \tag{8.8.41}$$

Fix now $x\in\partial^*\Omega\setminus N$ and, for each $r\in(0,r_x)$, choose $y_r\in[B(x,r)\cap\Omega]\setminus\mathcal{O}_{\varepsilon_r}$. Then $y_r\in\Omega$ and we have $|x-y_r|<r$ as well as $\delta_{\partial\Omega}(y_r)\ge\vartheta\, r$. Hence

$$\lim_{r\to 0^+} y_r = x \tag{8.8.42}$$

and

$$y_r\in\Gamma_\kappa(x) \;\text{ for each } r\in(0,r_x),\; \text{provided }\; \kappa > (1/\vartheta)-1. \tag{8.8.43}$$

With $\kappa_\Omega := (1/\vartheta)-1$, this proves

$$\kappa>\kappa_\Omega \Rightarrow x\in\overline{\Gamma_\kappa(x)} \;\text{ for } \sigma\text{-a.e. }\; x\in\partial^*\Omega, \tag{8.8.44}$$

from which the second equality in (8.8.29) follows. □

Circumstances under which the nontangential accessibility sets $A_\kappa(\partial\Omega)$, with arbitrary apertures $\kappa\in(0,\infty)$, compare favorably (from the point of view of Hausdorff measure) with the geometric measure theoretic boundary $\partial_*\Omega$ are described in our next proposition.

Proposition 8.8.4 *Suppose Ω is an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary and such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Then Ω has locally finite perimeter and*

$$\sigma\big(\partial_*\Omega \setminus A_\kappa(\partial\Omega)\big) = \sigma\big(\partial^*\Omega \setminus A_\kappa(\partial\Omega)\big) = 0 \ \textit{ for each } \ \kappa > 0. \tag{8.8.45}$$

Proof That Ω has locally finite perimeter follows from Proposition 8.8.3 and the fact that any doubling measure is, by definition, locally finite. To proceed, given $\kappa \in (0, \infty)$ arbitrary, select $\kappa_0 > \kappa_\Omega$ where $\kappa_\Omega \in (0, \infty)$ is as in Proposition 8.8.3. Then

$$\partial^*\Omega \setminus A_\kappa(\partial\Omega) \subseteq \big(\partial^*\Omega \setminus A_{\kappa_0}(\partial\Omega)\big) \cup \big(A_{\kappa_0}(\partial\Omega) \setminus A_\kappa(\partial\Omega)\big) \tag{8.8.46}$$

which, in concert with (8.8.29) and (8.8.6), proves (8.8.45). □

The following definition is central to the present work.

Definition 8.8.5 *Define the* `nontangentially accessible boundary` *of any given nonempty open proper subset Ω of $\mathbb{R}^n$ as*

$$\partial_{\text{nta}}\Omega := \bigcap_{\kappa>0} A_\kappa(\partial\Omega) = \big\{x \in \partial\Omega : \ x \in \overline{\Gamma_\kappa(x)} \ \textit{for each } \kappa > 0\big\}. \tag{8.8.47}$$

In particular, if $\mathbb{Q}_+$ denotes the set of positive rational numbers, based on (8.8.4) it is easy to see that

$$\begin{array}{c}\text{for any open nonempty proper subset } \Omega \\ \text{of } \mathbb{R}^n \text{ we have } \partial_{\text{nta}}\Omega = \bigcap_{\kappa\in\mathbb{Q}_+} A_\kappa(\partial\Omega).\end{array} \tag{8.8.48}$$

Note that if

$$\Omega := \big\{x \in \mathbb{R}^n : \ \cos(\theta/2)\,|x - x_0| < (x - x_0)\cdot h\big\} \tag{8.8.49}$$

is the open, convex, infinite, circular cone in $\mathbb{R}^n$ with vertex at $x_0 \in \mathbb{R}^n$, symmetry axis along the vector unit $h \in S^{n-1}$, and full aperture angle $\theta \in (0, \pi)$, then for each $\kappa \in (0, \infty)$ a simple computation shows that

$$x_0 \in A_\kappa(\partial\Omega) \Longleftrightarrow \kappa > \frac{1}{\sin(\theta/2)} - 1. \tag{8.8.50}$$

In particular, $x_0 \notin \partial_{\text{nta}}\Omega$, hence

$$\begin{array}{c}\text{if } \ \Omega \text{ is the cone defined in (8.8.49) then} \\ \partial_{\text{nta}}\Omega = \partial\Omega \setminus \{x_0\} \ \text{ whereas } \ \partial_*\Omega = \partial\Omega.\end{array} \tag{8.8.51}$$

As can be seen in the case of a slit disk in the plane, it may happen that $\partial_{\text{nta}}\Omega$ is considerably larger than $\partial_*\Omega$. In the proposition below we identify qualities of Ω guaranteeing that the nontangentially accessible boundary $\partial_{\text{nta}}\Omega$ compares favorably (as regards the Hausdorff measure) with the geometric measure theoretic boundary $\partial_*\Omega$.

Proposition 8.8.6 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ and consider $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Then the following statements are true.*

(i) The set $\partial_{\text{nta}}\Omega$ is Borelian, hence σ-measurable, and satisfies $\partial_{\text{nta}}\Omega \subseteq A_\kappa(\partial\Omega)$ for each $\kappa > 0$.
(ii) If σ is a doubling measure on $\partial\Omega$ then $\sigma\big(A_\kappa(\partial\Omega) \setminus \partial_{\text{nta}}\Omega\big) = 0$ for each $\kappa > 0$.
(iii) If the set $\partial\Omega$ is lower Ahlfors regular, and the measure σ is doubling on $\partial\Omega$, then

$$\sigma\big(\partial^*\Omega \setminus \partial_{\text{nta}}\Omega\big) = 0 \ \textit{ and } \ \sigma\big(\partial_*\Omega \setminus \partial_{\text{nta}}\Omega\big) = 0. \tag{8.8.52}$$

Regarding the nature of (8.8.52), it is remarkable that the intrinsic properties of the topological boundary $\partial\Omega$ (such as lower Ahlfors regularity and the surface measure being doubling) provide information about the "thickness" of the set Ω itself, by implying that, almost all points in the geometric measure theoretic boundary $\partial_*\Omega$ may be approached nontangentially (with any fixed aperture parameter) from within the open set Ω. Incidentally, it would be misleading to think of this as being merely a "soft" topological property, since there are quantitative estimates (going back to Proposition 8.4.1) underpinning this implication.

Proof of Proposition 8.8.6 Given any open nonempty proper subset Ω of $\mathbb{R}^n$, from (8.8.47) and the fact that the family $\big\{A_\kappa(\partial\Omega)\big\}_{\kappa>0}$ is nested we conclude that

$$\partial_{\text{nta}}\Omega = \bigcap_{j=1}^{\infty} A_{1/j}(\partial\Omega). \tag{8.8.53}$$

In light of (8.8.5), this formula implies that the set $\partial_{\text{nta}}\Omega$ is indeed Borelian, therefore σ-measurable. Moreover, for any $\kappa > 0$ we have $\partial_{\text{nta}}\Omega \subseteq A_\kappa(\partial\Omega)$ by design. Hence, the claims in part *(i)* are justified.

To treat part *(ii)*, assume that σ is a doubling measure on $\partial\Omega$. Since thanks to (8.8.53) we have

$$A_\kappa(\partial\Omega) \setminus \partial_{\text{nta}}\Omega = \bigcup_{j=1}^{\infty} \big(A_\kappa(\partial\Omega) \setminus A_{1/j}(\partial\Omega)\big), \tag{8.8.54}$$

it follows from (8.8.6) that $\sigma\big(A_\kappa(\partial\Omega) \setminus \partial_{\text{nta}}\Omega\big) = 0$, as wanted.

Finally, assume that $\partial\Omega$ and σ are as in part *(iii)*. In concert with (8.8.45), the result established in part *(ii)* then implies

$$\sigma\big(\partial^*\Omega \setminus \partial_{\text{nta}}\Omega\big) = 0. \tag{8.8.55}$$

Upon recalling (5.6.21), we see from (8.8.55) that

$$\sigma\big(\partial_*\Omega \setminus \partial_{\text{nta}}\Omega\big) = 0, \tag{8.8.56}$$

completing the proof of the proposition. □

An open set $\Omega \subset \mathbb{R}^n$ with the property that

$$\text{for each}\, \kappa > 0\, \text{one has} \quad x \in \overline{\Gamma_\kappa(x)} \quad \text{for}\, \mathcal{H}^{n-1}\text{-a.e.} \quad x \in \partial\Omega, \tag{8.8.57}$$

has been called in [125] a `weakly accessible domain`. Using the piece of notation introduced in (8.8.2), we may refashion the condition formulated in (8.8.57) as

$$\mathcal{H}^{n-1}\big(\partial\Omega \setminus A_\kappa(\partial\Omega)\big) = 0 \quad \text{for each} \quad \kappa > 0. \tag{8.8.58}$$

In [125, Proposition 2.9] it has been proved that

$$\begin{array}{l}\text{if } \Omega \text{ is an open nonempty proper subset of } \mathbb{R}^n \text{ with an Ahlfors regular} \\ \text{boundary and satisfying } \mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0, \text{ then } \Omega \text{ is a weakly} \\ \text{accessible domain (in the sense that (8.8.57) holds).}\end{array} \tag{8.8.59}$$

Bearing in mind (8.8.58), the property in (8.8.59) is now an immediate consequence of Proposition 8.8.6.

Next, we remark that (8.1.5)–(8.1.6) imply that for any given open set $\Omega \subseteq \mathbb{R}^n$ we have

$$\left.\begin{array}{r}\text{if } \mathrm{C}_{\theta,b}(x,h) \subseteq \Omega \text{ for some } x \in \partial\Omega \\ h \in S^{n-1}, \;\; \theta \in (0,\pi), \;\; \text{and } b \in (0,\infty)\end{array}\right\} \Rightarrow x \in A_\kappa(\partial\Omega) \;\text{ if }\; \kappa > \frac{1}{\sin(\theta/2)} - 1. \tag{8.8.60}$$

As a consequence,

$$\begin{array}{l}\text{if the open set } \Omega \subseteq \mathbb{R}^n \text{ satisfies an interior ball condition at a point} \\ x \in \partial\Omega, \text{ in the sense that there exist } h \in S^{n-1} \text{ and } r > 0 \text{ such that} \\ B(x+rh, r) \subseteq \Omega, \text{ then actually } x \text{ belongs to } \partial_{\text{nta}}\Omega.\end{array} \tag{8.8.61}$$

A related membership criterion to the nontangentially accessible boundary of a set is presented below.

Proposition 8.8.7 *Let $\Omega \subseteq \mathbb{R}^n$, where $n \geq 2$, be an open set and suppose a point $x = (x', x_n) \in \partial\Omega$ has the property that there exist some $r > 0$ and a continuous function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying*

$$\begin{array}{c} f(x') = x_n, \;\; f \text{ is differentiable at } x', \text{ and} \\ B(x,r) \cap \{y = (y', y_n) \in \mathbb{R}^n : \; y_n > f(y')\} \subseteq \Omega. \end{array} \tag{8.8.62}$$

Then actually x belongs to $\partial_{\text{nta}}\Omega$.

Proof The same argument that has produced (5.6.109) in the proof of Proposition 5.6.17 presently yields

$$\begin{array}{l}\text{for every angle } \theta \in (0,\pi) \text{ there exists some} \\ \text{small height } b > 0 \text{ so that } \mathrm{C}_{\theta,b}(x,h) \subseteq \Omega.\end{array} \tag{8.8.63}$$

Granted this, (8.8.60) then implies that $x \in A_\kappa(\partial\Omega)$ for each $\kappa > 0$, hence x belongs to $\partial_{\text{nta}}\Omega$. □

In particular, Proposition 8.8.7 shows that

$$\text{if } \Omega \subseteq \mathbb{R}^n, \text{ where } n \geq 2, \text{ is the strict epigraph of a continuous function } f : \mathbb{R}^{n-1} \to \mathbb{R} \text{ then } \big(x', f(x')\big) \text{ belongs to } \partial_{\text{nta}}\Omega \text{ whenever } x' \in \mathbb{R}^{n-1} \text{ is a differentiability point for } f. \tag{8.8.64}$$

Another result of similar flavor is as follows.

Proposition 8.8.8 *Assume $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying an interior corkscrew condition with constant $\theta \in (0, 1)$ (cf. (5.1.5)). Then*

$$A_\kappa(\partial\Omega) = \partial\Omega \ \ \textit{for each} \ \ \kappa > 2(\theta^{-1} - 1). \tag{8.8.65}$$

Proof Fix $\kappa > 2(\theta^{-1} - 1)$. With notation introduced in Definition 5.1.3, we shall show that

$$B\big(z_r(x), \theta r/2\big) \subseteq \Gamma_\kappa(x), \qquad \forall x \in \partial\Omega, \quad \forall r \in (0, 2\,\text{diam}\,\Omega). \tag{8.8.66}$$

Indeed, having fixed a point $x \in \partial\Omega$ along with some $r \in (0, 2\,\text{diam}\,\Omega)$, for each $y \in B\big(z_r(x), \theta r/2\big)$ we have $\text{dist}(y, \partial\Omega) > \theta r/2$ (cf. (5.1.5)), so

$$|y - x| \leq r - \theta r/2 < (1 + \kappa)\theta r/2 < (1 + \kappa)\,\text{dist}(y, \partial\Omega), \tag{8.8.67}$$

proving (8.8.66). In turn, (8.8.66) implies that $x \in \overline{\Gamma_\kappa(x)}$, given that $z_r(x) \to x$ as $r \to 0^+$. Consequently, $x \in A_\kappa(\partial\Omega)$ from which (8.8.65) follows. □

Combining the above result with Proposition 8.8.6 yields the following useful corollary.

Corollary 8.8.9 *If $\Omega \subseteq \mathbb{R}^n$ is an open set satisfying an interior corkscrew condition and with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$, then*

$$\sigma\big(\partial\Omega \setminus \partial_{\text{nta}}\Omega\big) = 0. \tag{8.8.68}$$

In particular,

$$\textit{if } \Omega \subseteq \mathbb{R}^n \textit{ is an open set which is } n\textit{-thick and has an Ahlfors regular boundary it follows that } \mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial_{\text{nta}}\Omega\big) = 0. \tag{8.8.69}$$

Proof The main assertion is a consequence of Proposition 8.8.8 and item *(ii)* in Proposition 8.8.6. In turn, (8.8.69) is implied by this and Proposition 8.6.12. □

We conclude with an elementary but useful observation.

Proposition 8.8.10 *For any open set $\Omega \subseteq \mathbb{R}$ with the property that $\partial\Omega$ is of locally finite cardinality one has $\partial_{\rm nta}\Omega = \partial\Omega$.*

Proof For starters, $\partial_{\rm nta}\Omega \subseteq \partial\Omega$ by (8.8.47). To prove the opposite inclusion, write Ω as an at most countable union of open, mutually disjoint intervals (which are actually the connected components of Ω), say

$$\Omega = \bigcup_{j\in J} I_j. \tag{8.8.70}$$

We claim that

$$\partial\Omega = \overline{\bigcup_{j\in J} \partial I_j}. \tag{8.8.71}$$

Indeed, since $\bigcup_{j\in J} \partial I_j$ is a subset of $\partial\Omega$, and the latter is a closed set, the right-to-left inclusion in (8.8.71) holds. To prove the left-to-right inclusion in (8.8.71), reason by contradiction and assume that there exists

$$x \in \partial\Omega \quad \text{such that} \quad x \notin \overline{\bigcup_{j\in J} \partial I_j}. \tag{8.8.72}$$

The last property above guarantees the existence of some $r > 0$ such that

$$(x - r, x + r) \cap \partial I_j \quad \text{for each} \quad j \in J. \tag{8.8.73}$$

In turn, this shows that for each $j \in J$ we have

$$\text{either} \quad (x - r, x + r) \subseteq I_j \quad \text{or} \quad (x - r, x + r) \cap I_j = \varnothing. \tag{8.8.74}$$

However, the first eventuality above never materializes since, if it did, the existence of some $j \in J$ for which $(x - r, x + r) \subseteq I_j$ would imply that $x \in \Omega$, in contradiction with the fact that Ω is open and we are presently assuming that $x \in \partial\Omega$. Thus, we actually have

$$(x - r, x + r) \cap I_j = \varnothing \quad \text{for each} \quad j \in J. \tag{8.8.75}$$

In turn, from (8.8.75) we conclude that

$$(x - r, x + r) \cap \Omega = (x - r, x + r) \cap \Big(\bigcup_{j\in J} I_j\Big) = \varnothing. \tag{8.8.76}$$

However, this is in contradiction with the fact that $x \in \partial\Omega$. As such, (8.8.71) is proved.

Since $\partial\Omega$ is of locally finite cardinality, it follows that $\bigcup_{j\in J} \partial I_j$ is also of locally finite cardinality. As such, $\bigcup_{j\in J} \partial I_j$ is a closed set, hence in fact

$$\partial\Omega = \bigcup_{j\in J} \partial I_j. \tag{8.8.77}$$

Having established this, the end-game in the proof of the inclusion

$$\partial\Omega \subseteq \partial_{\text{nta}}\Omega \tag{8.8.78}$$

is as follows. Having fixed an arbitrary aperture parameter $\kappa > 0$, from (8.1.28), (8.1.30) we see that for each $j \in J$ we have

$$x \in \overline{\Gamma_{I_j,\kappa}(x)} \quad \text{for each} \quad x \in \partial I_j, \tag{8.8.79}$$

hence

$$x \in \overline{\Gamma_{\Omega,\kappa}(x)} \quad \text{for each} \quad x \in \partial I_j, \tag{8.8.80}$$

on account of Lemma 8.1.3 (whose applicability uses (8.8.70)–(8.8.71)). In view of (8.8.47), this proves

$$\bigcup_{j\in J} \partial I_j \subseteq \partial_{\text{nta}}\Omega. \tag{8.8.81}$$

Together with (8.8.77), this proves (8.8.78). □

8.9 The Nontangential Boundary Trace Operator

We debut by making the following definition, which is central to our work. This introduces the nontangential boundary trace of a measurable function at a (nontangentially accessible) boundary point as the "essential limit" of said function at the given point from within the corresponding nontangential approach region.

Definition 8.9.1 *Fix a background parameter $\kappa > 0$ and let u be a real-valued Lebesgue measurable function defined $\mathcal{L}^n$-a.e. in an open set $\Omega \subset \mathbb{R}^n$. Consider a point*

$$x \in A_\kappa(\partial\Omega), \quad \textit{i.e.,} \quad x \in \partial\Omega \ \ \textit{such that} \ \ x \in \overline{\Gamma_\kappa(x)}. \tag{8.9.1}$$

Then one says that the nontangential limit[3] *of u at x from within $\Gamma_\kappa(x)$ exists, and its value is the number $a \in \mathbb{R}$, provided*

$$\begin{aligned} &\textit{for every } \varepsilon > 0 \textit{ there exists some } r > 0 \textit{ such that} \\ &|u(y) - a| < \varepsilon \textit{ for } \mathcal{L}^n\textit{-a.e. point } y \in \Gamma_\kappa(x) \cap B(x,r). \end{aligned} \tag{8.9.2}$$

Whenever the nontangential limit of u at x from within $\Gamma_\kappa(x)$ exists, its value is denoted by $\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x)$.

[3] More precisely, the κ-nontangential limit.

A few comments are in order. As a preamble, for each $\varepsilon > 0$, recall the neighborhood collar $\mathcal{O}_\varepsilon$ from (6.1.4) and that, for each Lebesgue measurable function $u : \Omega \to \mathbb{C}$, we have abbreviated

$$\mathcal{N}_\kappa^\varepsilon u := \mathcal{N}_\kappa^{\mathcal{O}_\varepsilon} u = \mathcal{N}_\kappa\big(u\mathbf{1}_{\mathcal{O}_\varepsilon}\big). \tag{8.9.3}$$

Remark 1. Definition 8.9.1 naturally extends to complex-valued and vector-valued functions by working on components. In such cases, we shall retain the same symbol to denote the nontangential boundary trace operator.

Remark 2. Sometimes we will slightly abuse terminology and simply say that $\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x)$ exists as a way of indicating that the nontangential limit of u at x exists when considered from within nontangential approach regions with aperture parameter κ.

Remark 3. If (as is usually the case) one works with equivalence classes, obtained by identifying functions which coincide $\mathcal{L}^n$-a.e., the nontangential limit is independent of the specific choice of a representative in a given equivalence class. Thus,

$$\begin{array}{l} \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) = \big(w\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) \text{ if } u, w : \Omega \longrightarrow \mathbb{R} \text{ are} \\ \text{two Lebesgue measurable functions with the prop-} \\ \text{erty that } u = w \text{ at } \mathcal{L}^n\text{-a.e. point in the set } \Omega. \end{array} \tag{8.9.4}$$

Remark 4. Since for each x as in (8.9.1) the set $\Gamma_\kappa(x) \cap B(x, r)$ is nonempty and open (hence, of positive Lebesgue measure) for every $r > 0$, it is clear that there could be at most one number $a \in \mathbb{R}$ satisfying (8.9.2). As such, whenever the nontangential limit at x exists, the number $\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x)$ is unambiguously defined.

Remark 5. The κ-nontangential limit of u at x exists (in the sense of Definition 8.9.1) if and only if

$$\begin{array}{l} \text{there exists some Lebesgue measurable set } N(x) \subset \Gamma_\kappa(x), \\ \text{with } \mathcal{L}^n(N(x)) = 0, \text{ such that } \displaystyle\lim_{(\Gamma_\kappa(x)\setminus N(x))\ni y\to x} u(y) \text{ exists.} \end{array} \tag{8.9.5}$$

Furthermore, whenever the above limit exists, it actually equals $\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x)$.

Let us re-write (8.9.5) more explicitly, and contrast it with the original definition (cf. (8.9.2)). Concretely, (8.9.5) states that

$$\begin{array}{l} \text{there exist } b \in \mathbb{C} \text{ and a Lebesgue measurable set } N(x) \subset \Gamma_\kappa(x), \text{ with} \\ \mathcal{L}^n(N(x)) = 0, \text{ such that for every } \varepsilon > 0 \text{ one can find } r_\varepsilon > 0 \text{ with the} \\ \text{property that } |u(y) - b| < \varepsilon \text{ for each } y \in \big(\Gamma_\kappa(x) \setminus N(x)\big) \cap B(x, r_\varepsilon), \end{array} \tag{8.9.6}$$

whereas the original definition (8.9.2) amounts to the demand that

$$\begin{aligned}&\text{for every } \varepsilon > 0 \text{ there exist some } a \in \mathbb{C}^M, \text{ some radius } r_\varepsilon > 0 \text{ and some} \\ &\text{set } N_\varepsilon(x) \subseteq \Gamma_\kappa(x), \text{ which is } \mathcal{L}^n\text{-measurable and has } \mathcal{L}^n(N_\varepsilon(x)){=}0, \\ &\text{such that } |u(y) - a| < \varepsilon \text{ for each } y \in \big(\Gamma_\kappa(x) \cap B(x, r_\varepsilon)\big) \setminus N_\varepsilon(x).\end{aligned} \tag{8.9.7}$$

Compared with the latter, the demand stipulated in (8.9.6) involves a nullset $N(x)$ which is independent of ε. In particular, (8.9.6) implies (8.9.7). Let us check that the converse implication is also true. Specifically, choose the number $b := a$ and define $N(x) := \bigcup_{j\in\mathbb{N}} N_{1/j}(x)$. Then $N(x)$ is $\mathcal{L}^n$-measurable and $\mathcal{L}^n(N(x)) = 0$. In concert with (8.9.7), this definition also implies that for each $j \in \mathbb{N}$ there exists some $r_j > 0$ such that $|u(y) - b| < 1/j$ for each point $y \in \big(\Gamma_\kappa(x) \setminus N(x)\big) \cap B(x, r_j)$. From this, (8.9.6) now readily follows (choosing $j \in \mathbb{N}$ such that $1/j < \varepsilon$ and defining $r_\varepsilon := r_j$). This concludes the proof of the fact that the existence of the nontangential limit of u at x (in the sense of Definition 8.9.1) is equivalent to (8.9.5).

As a consequence of (8.9.5), whenever $x \in A_\kappa(\partial\Omega)$ and the κ-nontangential limit of u at x exists, for each $\varepsilon > 0$ we have

$$\Big|\Big(u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x)\Big| \leq \big(\mathcal{N}_\kappa^\varepsilon u\big)(x) \leq \big(\mathcal{N}_\kappa u\big)(x). \tag{8.9.8}$$

Remark 6. The existence of $\big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(x)$ amounts to saying that we may redefine u on an $\mathcal{L}^n$-nullset as to make $u\big|_{\Gamma_\kappa(x)}$ have a continuous extension to $\Gamma_\kappa(x) \cup \{x\}$.

Remark 7. In the class of continuous functions the definition of the nontangential boundary limit takes a simpler form, namely if x is as in (8.9.1) then

$$\Big(u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = \lim_{\Gamma_\kappa(x)\ni y\to x} u(u), \qquad \forall u \in \mathscr{C}^0(\Omega). \tag{8.9.9}$$

Moreover,

$$\begin{aligned}&\text{if } x \in A_\kappa(\partial\Omega) \text{ then } \Big(u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = \big(u\big|_{\partial\Omega}\big)(x) \\ &\text{for every function } u \text{ belonging to } \mathscr{C}^0(\overline{\Omega}).\end{aligned} \tag{8.9.10}$$

Remark 8. Whenever $x \in A_\kappa(\partial\Omega)$ and $u, w : \Omega \to \mathbb{R}$ are two Lebesgue measurable functions such that the κ-nontangential limits of u and w at x exist, it follows that $\big((uw)\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(x)$ exists as well, and

$$\Big((uw)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) = \Big(u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) \cdot \Big(w\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x). \tag{8.9.11}$$

Remark 9. The pointwise nontangentially boundary trace introduced in Definition 8.9.1 is compatible with other natural notions of boundary traces, such as those considered on Sobolev-Besov-Triebel-Lizorkin scales of spaces in certain open sets with Ahlfors regular boundaries as in [27, Corollary 5.7, p. 4257, and Theorem 8.7(iii), p. 4413].

Remark10. As seen from (8.9.1)–(8.9.2), the definition of the nontangential trace $\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x)$ may, in principle, depend on the aperture parameter κ. However, under suitable assumptions on Ω and u, this definition turns out to be independent of κ, an issue discussed in Proposition 8.9.8. This is why we overwhelmingly prefer the simpler terminology "nontangential limit" in place of the more elaborate piece of nomenclature "κ-nontangential limit."

For now we make the simple observation that, as is apparent from definitions,

$$\begin{aligned}&\text{if } x \in A_\kappa(\partial\Omega) \text{ and } \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) \text{ exists, then for any other smaller}\\&\text{aperture parameter, say } \kappa' \in (0,\kappa], \text{ it follows that } x \in A_{\kappa'}(\partial\Omega) \text{ and}\\&\big(u\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\big)(x) = \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x).\end{aligned} \tag{8.9.12}$$

Our next lemma establishes two basic facts. First, the truncated nontangential maximal operator (cf. (8.9.3)) of a given function converges to the absolute value of the nontangential pointwise limit of the said function (whenever the latter exists), as the truncation parameter approaches zero. Second, if the truncated nontangential maximal operator (cf. (8.9.3)) of a given function converges to zero then the nontangential pointwise limit of the said function exists and is equal to zero.

Lemma 8.9.2 *Given an open nonempty proper subset Ω of $\mathbb{R}^n$, some Lebesgue measurable function u defined in Ω, and some aperture parameter $\kappa > 0$,*

$$\begin{aligned}&\textit{if } x \in \partial\Omega \textit{ is such that } x \in \overline{\Gamma_\kappa(x)} \textit{ and } \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) \textit{ exists,}\\&\textit{then necessarily } \big(\mathcal{N}_\kappa^\varepsilon u\big)(x) \to \Big|\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x)\Big| \textit{ as } \varepsilon \to 0^+.\end{aligned} \tag{8.9.13}$$

and

$$\begin{aligned}&\textit{if } x \in \partial\Omega \textit{ is such that } x \in \overline{\Gamma_\kappa(x)} \textit{ and } \big(\mathcal{N}_\kappa^\varepsilon u\big)(x) \to 0 \textit{ as}\\&\varepsilon \to 0^+ \textit{ then } \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) \textit{ exists and is equal to zero.}\end{aligned} \tag{8.9.14}$$

Proof We first deal with (8.9.13). To this end, fix $x \in \partial\Omega$ such that $x \in \overline{\Gamma_\kappa(x)}$ and $a := \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x)$ exists in $\mathbb{R}$. Also, pick some $\eta > 0$ arbitrary. Then Definition 8.9.1 ensures the existence of some $r_\eta > 0$ with the property that

$$|u(y) - a| < \eta \quad \text{for } \mathcal{L}^n\text{-a.e. } \; y \in \Gamma_\kappa(x) \cap B(x, r_\eta). \tag{8.9.15}$$

To proceed, recall the notation introduced in (6.1.4). Upon observing that for every $r > 0$ we have

$$\Gamma_\kappa(x) \cap B(x,r) \subseteq \Gamma_\kappa(x) \cap O_r \subseteq \Gamma_\kappa(x) \cap B(x,(1+\kappa)r), \tag{8.9.16}$$

it follows from (8.9.15) that

$$\begin{array}{c}\forall r \in (0, r_\eta) \Longrightarrow |a| - \eta < |u(y)| < |a| + \eta \\ \text{for } \mathcal{L}^n\text{-a.e. } y \in \Gamma_\kappa(x) \cap O_{r/(1+\kappa)}.\end{array} \tag{8.9.17}$$

Bearing in mind that $x \in \overline{\Gamma_\kappa(x)}$ forces $\Gamma_\kappa(x) \cap O_{r/(1+\kappa)}$ to be a nonempty open set, hence a Lebesgue measurable set of strictly positive measure, this further entails

$$|a| - \eta \leq \|u\|_{L^\infty(\Gamma_\kappa(x)\cap O_{r/(1+\kappa)}, \mathcal{L}^n)} \leq |a| + \eta, \quad \forall r \in (0, r_\eta). \tag{8.9.18}$$

In summary, we have shown that for every $\eta > 0$ there exists $r_\eta > 0$ such that

$$|a| - \eta \leq \big(\mathcal{N}_\kappa^{r/(1+\kappa)} u\big)(x) \leq |a| + \eta, \quad \forall r \in (0, r_\eta) \tag{8.9.19}$$

which, ultimately, establishes (8.9.13).

To justify (8.9.14), fix $x \in \partial\Omega$ such that $x \in \overline{\Gamma_\kappa(x)}$ and $\big(\mathcal{N}_\kappa^\rho u\big)(x) \to 0$ as $\rho \to 0^+$. In view of (8.9.3) and (8.2.4), the latter property implies that for each $\varepsilon > 0$ there exists some $\rho > 0$ such that

$$\|u\|_{L^\infty(\Gamma_\kappa(x)\cap O_\rho, \mathcal{L}^n)} < \varepsilon \;\text{ where }\; O_\rho := \big\{y \in \Omega : \delta_{\partial\Omega}(y) < \rho\big\}. \tag{8.9.20}$$

In particular, this shows that

$$|u(y)| < \varepsilon \;\text{ for } \mathcal{L}^n\text{-a.e. } \; y \in \Gamma_\kappa(x) \cap B(x, \rho). \tag{8.9.21}$$

Finally, from (8.9.21) and (8.9.2) (with $a := 0$) we then conclude that $\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x)$ exists and is equal to zero. □

It is also useful to note that the existence of the nontangential pointwise limit for a locally bounded function guarantees the finiteness of its truncated nontangential maximal function.

Lemma 8.9.3 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ and fix some aperture parameter $\kappa > 0$. Then*

given some point $x \in \partial\Omega$ such that $x \in \overline{\Gamma_\kappa(x)}$ together with some function $u \in L^\infty_{\text{loc}}(\Omega, \mathcal{L}^n)$ such that $\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x)$ exists, it follows that $\big(\mathcal{N}_\kappa^\varepsilon u\big)(x) < +\infty$ for each truncation parameter $\varepsilon > 0$. (8.9.22)

Proof If we set $a := \big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x)$, then Lemma 8.9.2 implies that $\big(\mathcal{N}_\kappa^\varepsilon u\big)(x) \to |a|$ as $\varepsilon \to 0^+$. As such, there exists $\varepsilon_o > 0$ with the property that $\big(\mathcal{N}_\kappa^{\varepsilon_o} u\big)(x) \leq |a| + 1$. In view of (8.9.3), this implies

$$\|u\|_{L^\infty(\Gamma_\kappa(x)\cap O_{\varepsilon_o}, \mathcal{L}^n)} < +\infty. \tag{8.9.23}$$

Next, fix $\varepsilon > 0$ arbitrary and define $K := \mathcal{O}_\varepsilon \cap \big(\Gamma_\kappa(x) \setminus \mathcal{O}_{\varepsilon_o/2}\big)$. By design, any point in K is at distance $\geq \varepsilon_o/2$ from the boundary of Ω. Also, since any point $y \in K$ has $|y - x| < (1+\kappa)\,\mathrm{dist}(y, \partial\Omega) < (1+\kappa)\varepsilon$ it follows that $K \subseteq B\big(x, (1+\kappa)\varepsilon\big)$. Thus, K is also bounded, which goes to show that K is a relatively compact subset of Ω. As a consequence, $\|u\|_{L^\infty_{\rm loc}(K,\mathcal{L}^n)} < +\infty$ which, together with (8.9.23), the definition of K, and (8.9.3), ultimately implies that $\big(\mathcal{N}^\varepsilon_\kappa u\big)(x) < +\infty$, as wanted. □

Our next lemma, which builds on the result established earlier in Lemma 8.9.2, is going to play a key role in the proof of Proposition 8.9.8, formulated a little later.

Lemma 8.9.4 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ and assume μ is a complete Borel measure on $\partial\Omega$. Fix $\kappa > 0$ and recall the set $A_\kappa(\partial\Omega)$ from* (8.8.2) *which is a Borelian* (*cf.* (8.8.5)). *Also, assume*

$$\mathcal{U} \text{ is a } \mu\text{-measurable subset of } \partial\Omega \text{ satisfying } \mu\big(\mathcal{U} \setminus A_\kappa(\partial\Omega)\big) = 0. \tag{8.9.24}$$

Finally, suppose $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function with the property that

$$\begin{array}{l}\text{the nontangential limit } \Big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) \text{ exists} \\ (\text{in } \mathbb{R}) \text{ for } \mu\text{-a.e. point } x \text{ belonging to } \mathcal{U}.\end{array} \tag{8.9.25}$$

Then the function $w : \partial\Omega \to \mathbb{R}$ defined (μ*-a.e.*) *as*

$$w := \begin{cases} u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} & \text{in } \mathcal{U}, \\ 0 & \text{in } \partial\Omega \setminus \mathcal{U}, \end{cases} \tag{8.9.26}$$

is a μ-measurable function on $\partial\Omega$. In particular, the nontangential trace of u on $\partial\Omega$

$$u\big|^{\kappa-\text{n.t.}}_{\partial\Omega} \text{ is a } \mu\text{-measurable function on } \mathcal{U}. \tag{8.9.27}$$

Furthermore,

$$\begin{array}{l}\text{if one also assumes } \displaystyle\int_{\mathcal{U}} \big|\mathcal{N}^\delta_\kappa u\big|^p \,\mathrm{d}\mu < \infty \text{ for some } \delta > 0 \text{ and } p \in (0,\infty) \\ \text{then } w \in L^p(\partial\Omega,\mu) \text{ and } \mathbf{1}_{\mathcal{U}} \cdot \mathcal{N}^\varepsilon_\kappa u \to |w| \text{ in } L^p(\partial\Omega,\mu) \text{ as } \varepsilon \to 0^+,\end{array} \tag{8.9.28}$$

while

$$\begin{array}{l}\text{if also } \displaystyle\int_{\partial\Omega\setminus(A_\kappa(\partial\Omega)\setminus\mathcal{U})} \big|\mathcal{N}^\delta_\kappa u\big|^p \,\mathrm{d}\mu < \infty \text{ for some } \delta > 0 \text{ and } p \in (0,\infty), \text{ then} \\ w \in L^p(\partial\Omega,\mu) \text{ and } \mathbf{1}_{\partial\Omega\setminus(A_\kappa(\partial\Omega)\setminus\mathcal{U})} \cdot \mathcal{N}^\varepsilon_\kappa u \to |w| \text{ in } L^p(\partial\Omega,\mu) \text{ as } \varepsilon \to 0^+.\end{array} \tag{8.9.29}$$

Proof Assume $x \in \partial\Omega$ is a point satisfying $x \notin \overline{\Gamma_\kappa(x)}$. Then there exists $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \cap \Gamma_\kappa(x) = \varnothing$. Observe that

$$\text{if } \varepsilon \in \big(0, \varepsilon_x/(1+\kappa)\big) \text{ then } \Gamma_\kappa(x) \cap \mathcal{O}_\varepsilon = \varnothing. \tag{8.9.30}$$

Indeed, the existence of some $y \in \Gamma_\kappa(x) \cap \mathcal{O}_\varepsilon$ would imply

$$|x-y| < (1+\kappa)\delta_{\partial\Omega}(y) < (1+\kappa)\varepsilon < \varepsilon_x \Rightarrow y \in B(x, \varepsilon_x) \cap \Gamma_\kappa(x), \tag{8.9.31}$$

which is a contradiction. This establishes (8.9.30) which, in turn, goes to show that

$$\big(\mathcal{N}_\kappa^\varepsilon u\big)(x) = 0 \quad \text{whenever} \quad \varepsilon \in \big(0, \varepsilon_x/(1+\kappa)\big). \tag{8.9.32}$$

We are now prepared to show that

$$\lim_{\varepsilon \to 0^+} \big(\mathbf{1}_{\mathcal{U}} \cdot \mathcal{N}_\kappa^\varepsilon u\big)(x) = |w(x)| \quad \text{for } \mu\text{-a.e. } x \in \partial\Omega. \tag{8.9.33}$$

Indeed, the formula in (8.9.33) is valid for μ-a.e. $x \in \mathcal{U} \cap A_\kappa(\partial\Omega)$ thanks to (8.9.13), (8.9.25), and (8.8.2). Given that we are assuming $\mu\big(\mathcal{U} \setminus A_\kappa(\partial\Omega)\big) = 0$ and that the formula in (8.9.33) is trivially true when $x \in \partial\Omega \setminus \mathcal{U}$, we ultimately conclude that (8.9.33) holds. Next, since from (8.2.26) we know that

$$\text{for every } \varepsilon > 0, \text{ the function } \mathcal{N}_\kappa^\varepsilon u \text{ is } \mu\text{-measurable on } \partial\Omega, \tag{8.9.34}$$

and since the set $\mathcal{U}$ is assumed to be μ-measurable and the measure μ is assumed to be complete, we may then deduce from (8.9.33) (cf. Remark 3.1.2) that

$$\text{the function } |w| \text{ is } \mu\text{-measurable on } \partial\Omega. \tag{8.9.35}$$

From this, the μ-measurability of the function w defined in (8.9.26) is then derived as follows. Bring in

$$u^\pm := \max\{\pm u, 0\} = \frac{|u| \pm u}{2} : \Omega \to [0, \infty]. \tag{8.9.36}$$

These are Lebesgue measurable functions satisfying

$$0 \leq u^\pm \leq |u| \quad \text{and} \quad u = u^+ - u^- \quad \text{in} \quad \Omega. \tag{8.9.37}$$

Also, since the absolute value function is continuous, our assumptions on u entail that the pointwise nontangential limits $\big(u^\pm\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x)$ exist for μ-a.e. $x \in \mathcal{U}$, are non-negative, and

$$\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x) = \big(u^+\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x) - \big(u^-\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x) \quad \text{for } \mu\text{-a.e. } x \in \mathcal{U}. \tag{8.9.38}$$

As a consequence, similar conclusions to (8.9.35) hold for $w^\pm$ associated with $u^\pm$ much as w has been associated with u in (8.9.26). Specifically, since we are now dealing with non-negative functions, we have that $w^{\pm}$ are μ-measurable. Also, (8.9.38) implies that $w = w^+ - w^-$ on $\partial\Omega$. From these considerations we then finally conclude that the original function w defined in (8.9.26) is indeed μ-measurable.

Moving on, make the additional assumption that $\int_{\mathcal{U}} \big|\mathcal{N}_\kappa^\delta u\big|^p \, d\mu < \infty$ for some truncation height $\delta > 0$ and some integrability exponent $p \in (0, \infty)$. Given that we already know that the function w is μ-measurable, from (8.9.26) and (8.9.8) we see that $w \in L^p(\partial\Omega, \mu)$. Furthermore, for every $\varepsilon \in (0, \delta)$ we have the following pointwise inequalities on $\partial\Omega$:

$$0 \le \mathbf{1}_{\mathcal{U}} \cdot \mathcal{N}_\kappa^\varepsilon u \le \mathbf{1}_{\mathcal{U}} \cdot \mathcal{N}_\kappa^\delta u \in L^p(\partial\Omega, \mu). \tag{8.9.39}$$

Granted these, from (8.9.33) and Lebesgue's Dominated Convergence Theorem we may then conclude that $\mathbf{1}_{\mathcal{U}} \cdot \mathcal{N}_\kappa^\varepsilon u$ converges to $|w|$ in $L^p(\partial\Omega, \mu)$ as $\varepsilon \to 0^+$. This establishes (8.9.28).

As regards the claim in (8.9.29), we first observe that since $\partial\Omega \setminus (A_\kappa(\partial\Omega) \setminus \mathcal{U})$ contains $\mathcal{U}$, the hypotheses in (8.9.29) imply the hypotheses in (8.9.28). As such, $w \in L^p(\partial\Omega, \mu)$. Also,

$$\lim_{\varepsilon \to 0^+} \big(\mathbf{1}_{\partial\Omega \setminus (A_\kappa(\partial\Omega) \setminus \mathcal{U})} \cdot \mathcal{N}_\kappa^\varepsilon u\big)(x) = |w(x)| \ \text{ for } \mu\text{-a.e. } \ x \in \partial\Omega. \tag{8.9.40}$$

Indeed, this follows by writing

$$\mathbf{1}_{\partial\Omega \setminus (A_\kappa(\partial\Omega) \setminus \mathcal{U})} = \mathbf{1}_{\mathcal{U}} + \mathbf{1}_{\partial\Omega \setminus A_\kappa(\partial\Omega)} - \mathbf{1}_{\mathcal{U} \setminus A_\kappa(\partial\Omega)} \tag{8.9.41}$$

then relying on (8.9.33), observing that $\big(\mathbf{1}_{\partial\Omega \setminus A_\kappa(\partial\Omega)} \cdot \mathcal{N}_\kappa^\varepsilon u\big)(x) \to 0$ as $\varepsilon \to 0^+$ for each $x \in \partial\Omega$ (a consequence of (8.9.32)) and, finally, recalling that $\mathbf{1}_{\mathcal{U} \setminus A_\kappa(\partial\Omega)} = 0$ at μ-a.e. point on $\partial\Omega$ (cf. (8.9.24)). With this in hand, the claim in (8.9.29) becomes a consequence of Lebesgue's Dominated Convergence Theorem. □

In applications, several natural choices of the set $\mathcal{U}$ in (8.9.24) present themselves. First, we discuss a version of Lemma 8.9.4 corresponding to the case when $\mathcal{U}$ is the nontangentially accessible boundary of the underlying domain.

Proposition 8.9.5 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with the property that $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$ is a doubling measure on $\partial\Omega$. Fix $\kappa > 0$ and suppose $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function such that*

$$\begin{array}{l} \textit{the nontangential limit } \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) \textit{ exists (in } \mathbb{R}) \\ \textit{for } \sigma\textit{-a.e. point } x \textit{ belonging to the set } \partial_{\mathrm{nta}}\Omega. \end{array} \tag{8.9.42}$$

Then the function $w : \partial\Omega \to \mathbb{R}$ defined (σ-a.e.) as

$$w := \begin{cases} u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} & \text{on } \ \partial_{\mathrm{nta}}\Omega, \\ 0 & \text{on } \ \partial\Omega \setminus \partial_{\mathrm{nta}}\Omega, \end{cases} \tag{8.9.43}$$

is a σ-measurable function on $\partial\Omega$. In particular,

$$u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ is a } \sigma\text{-measurable function on } \partial_{\text{nta}}\Omega. \tag{8.9.44}$$

Moreover,

$$\begin{aligned}&\text{if } \mathcal{N}_\kappa^\delta u \in L^p(\partial\Omega,\sigma) \text{ for some } p \in (0,\infty) \text{ and } \delta > 0, \text{ then}\\ &w \in L^p(\partial\Omega,\sigma) \text{ and } \mathcal{N}_\kappa^\varepsilon u \longrightarrow |w| \text{ in } L^p(\partial\Omega,\sigma) \text{ as } \varepsilon \to 0^+.\end{aligned} \tag{8.9.45}$$

Proof The fact that w is a σ-measurable function on $\partial\Omega$ is implied by the corresponding claim in Lemma 8.9.4 applied here with $\mu := \sigma$ (which Lemma 3.6.4 guarantees, bearing in mind (7.4.1)), to be a complete Borel measure on $\partial\Omega$), and for the set $\mathcal{U} := \partial_{\text{nta}}\Omega$ (which satisfies (8.9.24), according to item *(i)* in Proposition 8.8.6). Since $\partial_{\text{nta}}\Omega$ is a σ-measurable set, the fact that w is a σ-measurable function on $\partial\Omega$ then implies (8.9.44). To justify (8.9.45), observe that $\mathbf{1}_{\partial\Omega\setminus(A_\kappa(\partial\Omega)\setminus\partial_{\text{nta}}\Omega)}(x) = 1$ for σ-a.e. $x \in \partial\Omega$ since item *(ii)* of Proposition 8.8.6 gives $\sigma\big(A_\kappa(\partial\Omega) \setminus \partial_{\text{nta}}\Omega\big) = 0$. Granted this, (8.9.45) is implied by (8.9.29). □

We continue by presenting a version of Lemma 8.9.4 corresponding to the case when $\mathcal{U}$ is the geometric measure theoretic boundary of the given domain.

Corollary 8.9.6 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ such that $\partial\Omega$ is lower Ahlfors regular and $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Fix $\kappa > 0$ and suppose $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function with the property that*

$$\begin{aligned}&\text{the nontangential limit } \Big(u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)(x) \text{ exists (in } \mathbb{R})\\ &\text{for } \sigma\text{-a.e. point } x \text{ belonging to the set } \partial_*\Omega.\end{aligned} \tag{8.9.46}$$

Then

$$u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ is a } \sigma\text{-measurable function on } \partial_*\Omega. \tag{8.9.47}$$

Regarding the formulation of Corollary 8.9.6 we wish to remark that, purely as a matter of design, the largest subset of $\partial\Omega$ on which one may even begin to contemplate the existence of the nontangential pointwise limit $u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ is $A_\kappa(\partial\Omega)$. The present assumptions on Ω ensure the applicability of Proposition 8.8.4 which, in turn, guarantees that $A_\kappa(\partial\Omega)$ covers $\partial_*\Omega$, up to a σ-nullset. Ultimately, this makes the assumption in (8.9.46) meaningful.

Proof of Corollary 8.9.6 From Lemma 3.6.4 (and (7.4.1)) we know that σ is a complete Borel measure on $\partial\Omega$. As such, (8.9.47) follows by appealing to Lemma 8.9.4 applied with the measure $\mu := \sigma$ and the set $\mathcal{U} := \partial_*\Omega$, which satisfies (8.9.24) (according to (5.6.33) and Proposition 8.8.4, granted the current assumptions). □

Finally, here is yet another useful version of Lemma 8.9.4.

Corollary 8.9.7 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ and assume μ is a complete Borel measure on $\partial\Omega$. Fix $\kappa > 0$ and recall the Borelian set $A_\kappa(\partial\Omega) \subseteq \partial\Omega$ (cf. (8.8.2) and (8.8.5)). Also, suppose $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function with the property that*

$$\begin{array}{l}\textit{the nontangential limit } \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) \textit{ exists (in } \mathbb{R})\\ \textit{for } \mu\textit{-a.e. point } x \textit{ belonging to the set } A_\kappa(\partial\Omega).\end{array} \tag{8.9.48}$$

Then $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ is a μ-measurable function on $A_\kappa(\partial\Omega)$.

Proof This is a direct consequence of Lemma 8.9.4 used with $\mathcal{U} := A_\kappa(\partial\Omega)$. □

The next result elaborates on the degree of flexibility in the choice of the parameter κ (used to regulate the aperture of the nontangential approach regions; cf. (8.1.2)) in the process of taking nontangential pointwise limits (see Definition 8.9.1).

Proposition 8.9.8 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Fix $\kappa > 0$ along with $p \in (0, \infty]$ and assume that $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function satisfying*

$$\begin{array}{l}\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma) \textit{ and the nontangential limit}\\ \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) \textit{ exists (in } \mathbb{R}) \textit{ for } \sigma\textit{-a.e. point } x \in \partial_{\mathrm{nta}}\Omega.\end{array} \tag{8.9.49}$$

Then for every other $\kappa' > 0$ one has

$$\begin{array}{l}\mathcal{N}_{\kappa'} u \in L^p(\partial\Omega, \sigma) \textit{ and the nontangential limit}\\ \big(u\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\big)(x) \textit{ exists (in } \mathbb{R}) \textit{ for } \sigma\textit{-a.e. point } x \in \partial_{\mathrm{nta}}\Omega,\end{array} \tag{8.9.50}$$

and the two nontangential limits agree, i.e.,

$$\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) = \big(u\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\big)(x) \quad \textit{at, } \sigma\textit{-a.e. point } x \in \partial_{\mathrm{nta}}\Omega. \tag{8.9.51}$$

Before proving this result we wish to note that a version in which the truncated nontangential maximal operators $\mathcal{N}_\kappa^\varepsilon$ with $\varepsilon > 0$ a fixed arbitrary number (cf. (8.9.3)), replace the ordinary nontangential maximal operators $\mathcal{N}_\kappa$ also holds. Indeed, this is seen from Proposition 8.9.8 applied to the function $\mathbf{1}_{O_\varepsilon} \cdot u$ in place of the original function u. For other related versions see Corollary 8.9.9.

Proof of Proposition 8.9.8 We shall first consider the scenario when $p \in (0, \infty)$ and defer the treatment of the case when $p = \infty$ for later.

To get started, fix $\kappa' > 0$ arbitrary. We shall make a number of definitions in relation to a given function $w : \Omega \to \mathbb{R}$ which is Lebesgue measurable and satisfies

$$\mathcal{N}_{\kappa'} w < +\infty \quad \text{at } \sigma\text{-a.e. point on } \partial\Omega. \tag{8.9.52}$$

Specifically, for every $\varepsilon > 0$ define

$$\big(\dot{\mathcal{N}}^{\varepsilon}_{\kappa'} w\big)(x) := \inf\big\{\lambda \in \mathbb{R} : \mathcal{L}^n\big(\{y \in \Gamma_{\kappa'}(x) \cap \mathcal{O}_\varepsilon : w(y) > \lambda\}\big) = 0\big\}, \tag{8.9.53}$$

at every $x \in \partial\Omega$. Since for each $x \in \partial\Omega$ the quantity $\big(\dot{\mathcal{N}}^{\varepsilon}_{\kappa'} w\big)(x)$ decreases (possibly to $-\infty$) as the parameter $\varepsilon > 0$ decreases to zero, it follows that the functions

$$w\Big|_{\partial\Omega}^{\kappa'-\sup} : \partial\Omega \to [-\infty, +\infty], \ \Big(w\Big|_{\partial\Omega}^{\kappa'-\sup}\Big)(x) := \lim_{\varepsilon\to 0^+} \big(\dot{\mathcal{N}}^{\varepsilon}_{\kappa'} w\big)(x), \ \forall x \in \partial\Omega, \tag{8.9.54}$$

and

$$w\Big|_{\partial\Omega}^{\kappa'-\inf} : \partial\Omega \to [-\infty, +\infty], \ \Big(w\Big|_{\partial\Omega}^{\kappa'-\inf}\Big)(x) := -\lim_{\varepsilon\to 0^+} \big(\dot{\mathcal{N}}^{\varepsilon}_{\kappa'} (-w)\big)(x), \ \forall x \in \partial\Omega, \tag{8.9.55}$$

are meaningfully defined. Note that these definitions imply

$$w\Big|_{\partial\Omega}^{\kappa'-\inf} = -\bigg((-w)\Big|_{\partial\Omega}^{\kappa'-\sup}\bigg) \quad \text{on } \partial\Omega, \tag{8.9.56}$$

while (8.2.42) ensures that the functions

$$w\Big|_{\partial\Omega}^{\kappa'-\inf} \ \text{ and } \ w\Big|_{\partial\Omega}^{\kappa'-\sup} \ \text{ are } \sigma\text{-measurable.} \tag{8.9.57}$$

Moreover, based on (8.9.54), the continuity of the absolute value function, and (8.2.32), for each $x \in \partial\Omega$ we may write

$$\begin{aligned} \bigg|\Big(w\Big|_{\partial\Omega}^{\kappa'-\sup}\Big)(x)\bigg| &= \lim_{\varepsilon\to 0^+} \Big|\big(\dot{\mathcal{N}}^{\varepsilon}_{\kappa'} w\big)(x)\Big| \leq \limsup_{\varepsilon\to 0^+} \big(\mathcal{N}^{\varepsilon}_{\kappa'} w\big)(x) \\ &= \lim_{\varepsilon\to 0^+} \big(\mathcal{N}^{\varepsilon}_{\kappa'} w\big)(x), \end{aligned} \tag{8.9.58}$$

where the last equality is a consequence of the fact that $\big(\mathcal{N}^{\varepsilon}_{\kappa'} w\big)(x)$ decreases as the parameter $\varepsilon > 0$ decreases to zero. Actually, the latter property implies (in light of (8.9.58)) that

$$\bigg|w\Big|_{\partial\Omega}^{\kappa'-\sup}\bigg| \leq \mathcal{N}^{\varepsilon}_{\kappa'} w \quad \text{on } \partial\Omega, \quad \text{for each } \varepsilon > 0. \tag{8.9.59}$$

From this and (8.9.56) we also deduce that

$$\bigg|w\Big|_{\partial\Omega}^{\kappa'-\inf}\bigg| \leq \mathcal{N}^{\varepsilon}_{\kappa'} w \quad \text{on } \partial\Omega, \quad \text{for each } \varepsilon > 0. \tag{8.9.60}$$

As a consequence of (8.9.59)–(8.9.60) we therefore have

$$\left|w\Big|_{\partial\Omega}^{\kappa'-\sup}\right| \le \mathcal{N}_{\kappa'}w \quad\text{and}\quad \left|w\Big|_{\partial\Omega}^{\kappa'-\inf}\right| \le \mathcal{N}_{\kappa'}w \quad\text{on}\quad \partial\Omega, \tag{8.9.61}$$

from which we conclude, in view of the assumption (8.9.52), that

$$\left|w\Big|_{\partial\Omega}^{\kappa'-\sup}\right| < +\infty \quad\text{and}\quad \left|w\Big|_{\partial\Omega}^{\kappa'-\inf}\right| < +\infty \quad\text{at } \sigma\text{-a.e. point on}\quad \partial\Omega. \tag{8.9.62}$$

Granted this, we may then meaningfully define

$$\big(G_w\big)(x) := \left(w\Big|_{\partial\Omega}^{\kappa'-\sup}\right)(x) - \left(w\Big|_{\partial\Omega}^{\kappa'-\inf}\right)(x) \quad\text{for } \sigma\text{-a.e.}\quad x \in \partial\Omega. \tag{8.9.63}$$

Thanks to (8.9.57), this is a σ-measurable function defined on $\partial\Omega$. Also, as seen from (8.9.63) and (8.9.59)–(8.9.60), this function satisfies

$$\big|G_w\big| \le 2\,\mathcal{N}^{\varepsilon}_{\kappa'}w \quad\text{at } \sigma\text{-a.e. point on}\quad \partial\Omega, \quad\text{for every}\quad \varepsilon > 0. \tag{8.9.64}$$

We next claim that

$$\phi \in \mathscr{C}^0_c(\mathbb{R}^n) \Longrightarrow (G_w)(x) = (G_{w+\phi})(x) \quad\text{for } \sigma\text{-a.e. } x \in \partial_{\text{nta}}\Omega. \tag{8.9.65}$$

In turn, the latter claim is going to be a consequence of the fact that, given any $\phi \in \mathscr{C}^0_c(\mathbb{R}^n)$, we have

$$\left((w+\phi)\Big|_{\partial\Omega}^{\kappa'-\sup}\right)(x) = \left(w\Big|_{\partial\Omega}^{\kappa'-\sup}\right)(x) + \phi(x) \quad\text{for } \sigma\text{-a.e.}\quad x \in \partial_{\text{nta}}\Omega, \tag{8.9.66}$$

and

$$\left((w+\phi)\Big|_{\partial\Omega}^{\kappa'-\inf}\right)(x) = \left(w\Big|_{\partial\Omega}^{\kappa'-\inf}\right)(x) + \phi(x) \quad\text{for } \sigma\text{-a.e.}\quad x \in \partial_{\text{nta}}\Omega. \tag{8.9.67}$$

Let us justify (8.9.66). It is useful to rephrase the definition made in (8.9.53) as

$$\big(\dot{\mathcal{N}}^{\varepsilon}_{\kappa'}w\big)(x) = \inf\big\{\lambda \in \mathbb{R} :\ w \le \lambda\ \mathcal{L}^n\text{-a.e. in}\ \ \Gamma_{\kappa'}(x) \cap \mathcal{O}_\varepsilon\big\}, \quad \forall x \in \partial\Omega. \tag{8.9.68}$$

Fix some $\phi \in \mathscr{C}^0_c(\mathbb{R}^n)$ and select some $x \in \partial\Omega$. Observe that having $w + \phi \le \lambda$ at $\mathcal{L}^n$-a.e. point in $\Gamma_{\kappa'}(x) \cap \mathcal{O}_\varepsilon$ implies

$$w \le \lambda - \inf_{\Gamma_{\kappa'}(x)\cap\mathcal{O}_\varepsilon} \phi \quad\text{at } \mathcal{L}^n\text{-a.e. point in}\quad \Gamma_{\kappa'}(x) \cap \mathcal{O}_\varepsilon. \tag{8.9.69}$$

Bearing (8.9.68) in mind, this further yields

$$\big(\dot{\mathcal{N}}^{\varepsilon}_{\kappa'} w\big)(x) \leq \big(\dot{\mathcal{N}}^{\varepsilon}_{\kappa'}(w+\phi)\big)(x) - \inf_{\Gamma_{\kappa'}(x)\cap O_\varepsilon} \phi. \tag{8.9.70}$$

By passing to the limit as $\varepsilon \to 0^+$ in the above inequality we arrive at the conclusion that

$$\begin{gathered}\left(w\Big|^{\kappa'-\sup}_{\partial\Omega} \right)(x) \leq \left((w+\phi)\Big|^{\kappa'-\sup}_{\partial\Omega} \right)(x) - \phi(x) \\ \text{for every } x \in \partial\Omega \text{ such that } x \in \overline{\Gamma_{\kappa'}(x)}, \end{gathered} \tag{8.9.71}$$

thanks to (8.9.54) and the continuity of ϕ. In particular,

$$w\Big|^{\kappa'-\sup}_{\partial\Omega} \leq (w+\phi)\Big|^{\kappa'-\sup}_{\partial\Omega} - \phi \quad \text{at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega. \tag{8.9.72}$$

In fact, this inequality self-improves (given the relative arbitrariness of ϕ and w) to the equality recorded in (8.9.66). Specifically, writing (8.9.72) for $w+\phi$ in place of w and $-\phi$ in place of ϕ (both viable choices as far as the veracity of (8.9.72) is concerned) yields

$$\begin{aligned}(w+\phi)\Big|^{\kappa'-\sup}_{\partial\Omega} &\leq \big((w+\phi)-\phi\big)\Big|^{\kappa'-\sup}_{\partial\Omega} + \phi \\ &= w\Big|^{\kappa'-\sup}_{\partial\Omega} + \phi \quad \text{at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega, \end{aligned} \tag{8.9.73}$$

from which the desired conclusion follows. This finishes the proof of (8.9.66), and formula (8.9.67) is then readily justified with the help of (8.9.56). At this stage, the proof of (8.9.65) is complete.

Pressing on, assume $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function satisfying (8.9.49). Specifically, suppose

$$\begin{gathered}\mathcal{N}_\kappa u \in L^p(\partial\Omega,\sigma) \text{ and for } \sigma\text{-a.e. point } x \in \partial_{\mathrm{nta}}\Omega \text{ one can find a} \\ \text{Lebesgue measurable set } N(x) \subseteq \Gamma_\kappa(x) \text{ with } \mathcal{L}^n(N(x)) = 0 \text{ and} \\ \text{the property that the limit } \lim_{(\Gamma_\kappa(x)\setminus N(x))\ni y\to x} u(y) \text{ exists (in } \mathbb{R}). \end{gathered} \tag{8.9.74}$$

Our earlier work sets the stage for justifying the claims made in (8.9.50). For starters, the fact that

$$\mathcal{N}_{\kappa'} u \in L^p(\partial\Omega,\sigma) \tag{8.9.75}$$

follows from (8.9.49) and Proposition 8.4.1. To proceed, for σ-a.e. point $x \in \partial_{\mathrm{nta}}\Omega$ define

$$f(x) := \lim_{(\Gamma_\kappa(x)\setminus N(x))\ni y\to x} u(y) \tag{8.9.76}$$

where $N(x) \subseteq \Gamma_\kappa(x)$ is as in (8.9.74). From Proposition 8.9.5 and (8.8.47) it follows that f belongs to $L^p(\partial_{\mathrm{nta}}\Omega,\sigma)$. For an arbitrary, fixed $\eta > 0$, pick $\phi \in \mathscr{C}^0_c(\mathbb{R}^n)$ such that

$$\Big\| \phi\big|_{\partial_{\mathrm{nta}}\Omega} - f \Big\|_{L^p(\partial_{\mathrm{nta}}\Omega,\sigma)} < \eta. \tag{8.9.77}$$

That this is possible is ensured by the density result established in Corollary 3.7.3 and the current assumptions. Then, invoking (8.9.65) and (8.9.64) (which is permissible given that (8.9.75) entails $\mathcal{N}_{\kappa'} u < +\infty$ at σ-a.e. point in $\partial\Omega$), for every $\varepsilon > 0$ we may estimate

$$|G_u| = |G_{u-\phi}| \leq 2\mathcal{N}_{\kappa'}^{\varepsilon}(u-\phi) \text{ at } \sigma\text{-a.e. point on } \partial\Omega. \tag{8.9.78}$$

When used in concert with Proposition 8.4.1 (here we use that σ is doubling), this gives (bearing in mind that, as established earlier, G_u is a σ-measurable function) that, on the one hand,

$$\|G_u\|_{L^p(\partial\Omega,\sigma)} \leq 2\|\mathcal{N}_{\kappa'}^{\varepsilon}(u-\phi)\|_{L^p(\partial\Omega,\sigma)} \leq C\|\mathcal{N}_{\kappa}^{\varepsilon}(u-\phi)\|_{L^p(\partial\Omega,\sigma)}, \tag{8.9.79}$$

for some finite constant $C > 0$ which does not depend on ε. On the other hand, from Proposition 8.9.5 (applied to $u - \phi$) we know that

$$\mathcal{N}_{\kappa}^{\varepsilon}(u-\phi) \to \begin{cases} \big|(f-\phi)\big|_{\partial_{\text{nta}}\Omega}\big| & \text{on } \partial_{\text{nta}}\Omega, \\ 0 & \text{on } \partial\Omega \setminus \partial_{\text{nta}}\Omega, \end{cases} \text{ in } L^p(\partial\Omega,\sigma) \text{ as } \varepsilon \to 0^+. \tag{8.9.80}$$

This allows us to take the limit as $\varepsilon \to 0^+$ in (8.9.79) to obtain, based also on (8.9.77), that

$$\begin{aligned} \|G_u\|_{L^p(\partial\Omega,\sigma)} &\leq C \lim_{\varepsilon\to 0^+} \|\mathcal{N}_{\kappa}^{\varepsilon}(u-\phi)\|_{L^p(\partial\Omega,\sigma)} \\ &= C\big\|(f-\phi)\big|_{\partial_{\text{nta}}\Omega}\big\|_{L^p(\partial_{\text{nta}}\Omega,\sigma)} \leq C\eta. \end{aligned} \tag{8.9.81}$$

Since $\eta > 0$ is arbitrary, (8.9.81) forces $\|G_u\|_{L^p(\partial\Omega,\sigma)} = 0$ which ultimately yields $G_u = 0$ at σ-a.e. point on $\partial\Omega$. As a consequence,

$$a := \Big(u\Big|_{\partial\Omega}^{\kappa'-\sup}\Big)(x) = \Big(u\Big|_{\partial\Omega}^{\kappa'-\inf}\Big)(x) \tag{8.9.82}$$

is a well-defined real number for σ-a.e. point $x \in \partial\Omega$, hence also for σ-a.e. point $x \in \partial_{\text{nta}}\Omega$.

In the last part of the proof the goal is to show that for every such point x we have

$$|u(y) - a| < \eta \text{ for } \mathcal{L}^n\text{-a.e. } y \in \Gamma_{\kappa'}(x) \cap B(x,\varepsilon). \tag{8.9.83}$$

To justify this, pick an arbitrary $\eta > 0$. Then from (8.9.82) and (8.9.54)–(8.9.55) we see that there exists $\varepsilon > 0$ with the property that

$$\big(\dot{\mathcal{N}}_{\kappa'}^{\varepsilon} u\big)(x) < a + \eta \text{ and } -\big(\dot{\mathcal{N}}_{\kappa'}^{\varepsilon}(-u)\big)(x) > a - \eta. \tag{8.9.84}$$

Unraveling definitions (cf. (8.9.68)), the first inequality above implies

$$u(y) < a + \eta \quad \text{for } \mathcal{L}^n\text{-a.e. } y \in \Gamma_{\kappa'}(x) \cap \mathcal{O}_\varepsilon, \tag{8.9.85}$$

while the second inequality in (8.9.84) yields

$$a - \eta < u(y) \quad \text{for } \mathcal{L}^n\text{-a.e. } y \in \Gamma_{\kappa'}(x) \cap \mathcal{O}_\varepsilon. \tag{8.9.86}$$

Together, (8.9.85) and (8.9.86) prove (8.9.83). As remarked in (8.9.5), property (8.9.83) may be equivalently rephrased as saying that for σ-a.e. point $x \in \partial\Omega$ one can find a Lebesgue measurable set $N'(x) \subseteq \Gamma_{\kappa'}(x)$ with $\mathcal{L}^n(N'(x)) = 0$ such that the limit $\lim_{(\Gamma_{\kappa'}(x)\setminus N'(x))\ni y\to x} u(y)$ exists. The proof of (8.9.50) is therefore complete, assuming $p < \infty$. In such a scenario, (8.9.51) also follows from what we proved so far and (8.1.20).

We now turn to the case when $p = \infty$. Then (8.9.49) and Proposition 8.4.1 imply that for every other $\kappa' > 0$ we have

$$\mathcal{N}_{\kappa'} u \in L^\infty(\partial\Omega, \sigma). \tag{8.9.87}$$

To proceed, select a scalar-valued function $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ with the property that $\varphi = 1$ on $B(0, 1)$ and $\varphi = 0$ on $\mathbb{R}^n \setminus B(0, 2)$ and, for each $\varepsilon > 0$, define $\varphi_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by setting $\varphi_\varepsilon(x) := \varphi(\varepsilon x)$ for every $x \in \mathbb{R}^n$. Then from (8.1.18) and (8.2.26) we may conclude that

$$\mathcal{N}_\kappa \varphi_\varepsilon \in L^\infty_{\text{comp}}(\partial\Omega, \sigma) \quad \text{for each } \varepsilon > 0. \tag{8.9.88}$$

If for each $\varepsilon > 0$ we now consider $u_\varepsilon := \big(\varphi_\varepsilon\big|_\Omega\big) \cdot u : \Omega \to \mathbb{R}$, then the function u_ε is $\mathcal{L}^n$-measurable, and (8.9.10)–(8.9.11) together with (8.9.49) ensure that

$$\begin{gathered} \Big(u_\varepsilon\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) = \varphi_\varepsilon(x) \cdot \Big(u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) \\ \text{exists (in } \mathbb{R}) \text{ for } \sigma\text{-a.e. point } x \in \partial_{\text{nta}}\Omega. \end{gathered} \tag{8.9.89}$$

In addition, from (8.2.10) and (8.9.49) (with $p = \infty$) we have

$$0 \leq \mathcal{N}_\kappa u_\varepsilon \leq \mathcal{N}_\kappa \varphi_\varepsilon \cdot \mathcal{N}_\kappa u \in L^\infty_{\text{comp}}(\partial\Omega, \sigma) \tag{8.9.90}$$

which, in concert with (8.2.26), proves that

$$\mathcal{N}_\kappa u_\varepsilon \in \bigcup_{0<q\leq\infty} L^q(\partial\Omega, \sigma). \tag{8.9.91}$$

With (8.9.89) and (8.9.91) in hand, the first part in the current proof (dealing with the scenario when $p < \infty$) applies and gives that for every other $\kappa' > 0$ the nontangential limit $\Big(u_\varepsilon\Big|^{\kappa'-\text{n.t.}}_{\partial\Omega}\Big)(x)$ exists and matches $\Big(u_\varepsilon\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x) = \varphi_\varepsilon(x) \cdot \Big(u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)(x)$ for σ-a.e. point $x \in \partial_{\text{nta}}\Omega$. Since $\varphi_\varepsilon = 1$ in $B(0, 1/\varepsilon)$ it follows that $u\Big|^{\kappa'-\text{n.t.}}_{\partial\Omega}$ exists, and is equal to $u_\varepsilon\Big|^{\kappa'-\text{n.t.}}_{\partial\Omega}$ hence to $u\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}$, at σ-a.e. point in $B(0, 1/\varepsilon) \cap \partial_{\text{nta}}\Omega$. By letting $\varepsilon \to 0^+$

we finally conclude that $\big(u\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\big)(x)$ exists at σ-a.e. point in $\partial_{\mathrm{nta}}\Omega$ and (8.9.51) holds. This completes the proof of Proposition 8.9.8. □

For the following result, which may be regarded as a local version of Proposition 8.9.8, we shall provide two proofs: one employs a localization argument, while another uses Lemma 8.9.2, Lemma 8.9.10, and Theorem 8.4.6.

Corollary 8.9.9 *Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Fix $\kappa, \varepsilon > 0$ along with $p \in (0,\infty]$ and assume that $u : \Omega \to \mathbb{R}$ is a Lebesgue measurable function satisfying*

$$\begin{gathered}\mathcal{N}_\kappa^\varepsilon u \in L^p_{\mathrm{loc}}(\partial\Omega,\sigma) \ \textit{and the nontangential limit} \\ \big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) \textit{ exists (in } \mathbb{R}) \textit{ for } \sigma\textit{-a.e. point } x \in \partial_{\mathrm{nta}}\Omega.\end{gathered} \tag{8.9.92}$$

Then for every other $\kappa' > 0$ there holds

$$\begin{gathered}\mathcal{N}_{\kappa'}^\varepsilon u \in L^p_{\mathrm{loc}}(\partial\Omega,\sigma) \ \textit{and the nontangential limit} \\ \big(u\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\big)(x) \textit{ exists (in } \mathbb{R}) \textit{ for } \sigma\textit{-a.e. point } x \in \partial_{\mathrm{nta}}\Omega,\end{gathered} \tag{8.9.93}$$

and the two nontangential limits agree, i.e.,

$$\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) = \big(u\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\big)(x) \ \textit{ at } \sigma\textit{-a.e. point } x \in \partial_{\mathrm{nta}}\Omega. \tag{8.9.94}$$

Here is the first proof of Corollary 8.9.9, while the second proof of Corollary 8.9.9 is provided a little later below.

First proof of Corollary 8.9.9 We use the same type of localization argument as in the last portion of the proof of Proposition 8.9.8. Specifically, fix a scalar-valued function $\varphi \in \mathscr{C}^\infty(\mathbb{R}^n)$ satisfying $\varphi = 1$ on $B(0,1)$ and $\varphi = 0$ on $\mathbb{R}^n \setminus B(0,2)$ and, for each $R \in (0,\infty)$ and $x \in \mathbb{R}^n$, define $\varphi_R(x) := \varphi(x/R)$. In particular, (8.1.18) and (8.2.26) imply

$$\mathcal{N}_\kappa \varphi_R \in L^\infty_{\mathrm{comp}}(\partial\Omega,\sigma) \ \text{ for each } \ R > 0. \tag{8.9.95}$$

If for each $R > 0$ we now consider $u_R := \big(\varphi_R\big|_\Omega\big)\cdot u : \Omega \to \mathbb{R}$, then the function u_R is $\mathcal{L}^n$-measurable, and (8.9.10)–(8.9.11) together with (8.9.92) imply that

$$\begin{gathered}\text{for each } \ R > 0 \ \text{ the nontangential limit } \ \big(u_R\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x) \ \text{ exists} \\ \text{(in } \mathbb{R}) \text{ for } \sigma\text{-a.e. point } x \in \partial_{\mathrm{nta}}\Omega \text{ and equals } \ \varphi_R(x)\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)(x).\end{gathered} \tag{8.9.96}$$

In addition, from (8.2.10) and (8.9.92) we have

$$0 \le \mathcal{N}_\kappa^\varepsilon u_R \le \mathcal{N}_\kappa^\varepsilon \varphi_R \cdot \mathcal{N}_\kappa^\varepsilon u \in L^p_{\mathrm{comp}}(\partial\Omega,\sigma) \ \text{ for each } \ R > 0 \tag{8.9.97}$$

which, in concert with (8.2.26), proves that

$$\mathcal{N}_{\kappa}^{\varepsilon} u_R \in L^p(\partial\Omega, \sigma) \quad \text{for each} \quad R > 0. \tag{8.9.98}$$

Having established (8.9.96) and (8.9.98), Proposition 8.9.8 applies to the function $u \cdot \mathbf{1}_{\mathcal{O}_\varepsilon}$ and gives that for every other $\kappa' > 0$ we have

$$\mathcal{N}_{\kappa'}^{\varepsilon} u_R \in L^p(\partial\Omega, \sigma) \quad \text{for each} \quad R > 0, \tag{8.9.99}$$

and the nontangential limit $\big(u_R\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big)(x) = \varphi_R(x) \cdot \big(u\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big)(x)$ exists and matches the nontangential limit $\big(u_R\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(x) = \varphi_R(x) \cdot \big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(x)$ for σ-a.e. point $x \in \partial_{\text{nta}}\Omega$. The fact that $\varphi_R = 1$ in $B(0, R)$ then implies that $u\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}$ exists and matches $u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ at σ-a.e. point in $B(0, R) \cap \partial_{\text{nta}}\Omega$. By letting $R \to \infty$ we finally conclude that $\big(u\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big)(x)$ exists at σ-a.e. point in $\partial_{\text{nta}}\Omega$ and (8.9.94) holds.

To prove the local p-th power integrability of the nontangential maximal function with aperture parameter κ', fix some $x_o \in \partial\Omega$ and $r \in (0, \infty)$ arbitrary. Observe that for each $x \in B(x_o, r) \cap \partial\Omega$ and $y \in \Gamma_{\kappa'}(x) \cap \mathcal{O}_\varepsilon$ we have

$$|y - x_o| \le |y - x| + |x - x_o| < (1 + \kappa')\varepsilon + r. \tag{8.9.100}$$

Since $B\big(x_o, (1 + \kappa')\varepsilon + r\big)$ is contained in $B(0, R)$ for each $R > 0$ sufficiently large (depending on x_o, r, ε, and κ'), it follows that $\varphi_R(y) = 1$ when $x \in B(x_o, r) \cap \partial\Omega$ and $y \in \Gamma_{\kappa'}(x) \cap \mathcal{O}_\varepsilon$ provided R is sufficiently large. Consequently, for each point $x \in B(x_o, r) \cap \partial\Omega$ we have $u = u_R$ on $\Gamma_{\kappa'}(x) \cap \mathcal{O}_\varepsilon$ if R is sufficiently large, hence

$$\mathcal{N}_{\kappa'}^{\varepsilon} u = \mathcal{N}_{\kappa'}^{\varepsilon} u_R \quad \text{on} \quad B(x_o, r) \cap \partial\Omega \quad \text{provided } R > 0 \text{ is sufficiently large.} \tag{8.9.101}$$

Together with (8.2.26) and (8.9.98), this shows that

$$\mathcal{N}_{\kappa'}^{\varepsilon} u \in L^p\big(B(x_o, r) \cap \partial\Omega, \sigma\big) \quad \text{for each} \quad x_o \in \partial\Omega \quad \text{and each} \quad r > 0, \tag{8.9.102}$$

which ultimately proves that $\mathcal{N}_{\kappa'}^{\varepsilon} u \in L^p_{\text{loc}}(\partial\Omega, \sigma)$. □

We wish to present a second proof of Corollary 8.9.9 which makes use of the pointwise inequality established in Theorem 8.4.6 together with the boundedness of the local L^s-based Hardy-Littlewood maximal operator. In this endeavor, we shall also need the extension result contained in the following lemma.

Lemma 8.9.10 *Assume Ω is an arbitrary open (nonempty, proper) subset of $\mathbb{R}^n$ and let σ be a doubling Borel-semiregular measure on $\partial\Omega$. Also, pick some integrability exponent $p \in (0, \infty]$. Then for any given real-valued function $f \in L^p_{\text{loc}}(\partial\Omega, \sigma)$ there exists a Lebesgue measurable function $w : \Omega \to \mathbb{R}$ with the property that for each aperture parameter $\kappa \in (0, \infty)$ and each truncation parameter $\rho \in (0, \infty)$ one has*

$$\mathcal{N}_\kappa^\rho w \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma) \ \ \textit{and} \ \ w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f \ \ \textit{at } \sigma\textit{-a.e. point in } \ A_\kappa(\partial\Omega). \tag{8.9.103}$$

Before presenting the proof of Lemma 8.9.10 we make two comments. First, as a consequence of the present assumptions and items *(i)-(ii)* in Proposition 8.8.6, we may equivalently rephrase the last property above as the demand that $w\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} = f$ at σ-a.e. point in $\partial_{\mathrm{nta}}\Omega$. Second, if we impose the stronger assumption $f \in L^p(\partial\Omega, \sigma)$ then the membership in (8.9.103) improves to $\mathcal{N}_\kappa w \in L^p(\partial\Omega, \sigma)$; this is apparent from the proof given below.

Proof of Lemma 8.9.10 Consider a Whitney decomposition of Ω

$$\Omega = \bigcup_{Q \in \mathcal{W}_\Omega} Q \tag{8.9.104}$$

into a family $\mathcal{W}_\Omega$ of mutually disjoint semi-open cubes in $\mathbb{R}^n$, satisfying

$$\mathrm{dist}(Q, \partial\Omega) \approx \ell(Q) \ \ \text{uniformly for} \ \ Q \in \mathcal{W}_\Omega \tag{8.9.105}$$

(with the constants involved depending only on the dimension), where $\ell(Q)$ denotes the side-length of the cube Q. See, e.g., [240, Theorem 3, p. 16]. For each $Q \in \mathcal{W}_\Omega$ pick a point $x_Q \in \partial\Omega$ with the property that

$$\mathrm{dist}(Q, \partial\Omega) = \mathrm{dist}(Q, x_Q) \tag{8.9.106}$$

and abbreviate

$$\Delta_Q := B\big(x_Q, \ell(Q)\big) \cap \partial\Omega. \tag{8.9.107}$$

To proceed, fix an arbitrary real-valued function $f \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma)$ and introduce

$$f_\pm := \tfrac{1}{2}(|f| \pm f). \tag{8.9.108}$$

Then $f_\pm$ are non-negative σ-measurable functions, satisfying

$$0 \le f_\pm \le |f|, \quad f = f_+ - f_-, \quad \text{and} \quad f_\pm \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma). \tag{8.9.109}$$

Pick some $s \in (0, p)$ and define $w_\pm : \Omega \to [0, \infty)$ by setting

$$w_\pm(x) := \Big(⨍_{\Delta_Q} (f_\pm)^s \, d\sigma \Big)^{1/s} \ \ \text{whenever} \ \ x \in Q \in \mathcal{W}_\Omega. \tag{8.9.110}$$

Thus, by design, $w_\pm$ are constant on each $Q \in \mathcal{W}_\Omega$. The finiteness of $w_\pm$ is ensured by the fact that

$$(f_\pm)^s \in L^{p/s}_{\mathrm{loc}}(\partial\Omega, \sigma) \subseteq L^1_{\mathrm{loc}}(\partial\Omega, \sigma), \tag{8.9.111}$$

given that $s \in (0, p)$. In particular, $w_\pm$ are Lebesgue measurable functions taking finite non-negative values.

Next, fix an aperture parameter $\kappa \in (0, \infty)$ and consider a truncation parameter $\rho \in (0, \infty)$. Also, pick a point $x \in \partial\Omega$ along with some point $y \in \Gamma_\kappa(x)$. Then there exists a (unique) cube $Q \in \mathcal{W}_\Omega$ such that $y \in Q$. We claim that there exists $C = C(n, \kappa) \in (0, \infty)$ with the property that

$$\Delta_Q \subseteq \Delta\big(x, C\ell(Q)\big) \quad \text{and} \quad \Delta\big(x, \ell(Q)\big) \subseteq \Delta\big(x_Q, C\ell(Q)\big). \tag{8.9.112}$$

Indeed, for each $z \in \Delta_Q$ we may use (8.9.105), (8.9.106), and (8.9.107) to estimate

$$\begin{aligned} |z - x| &\leq |z - x_Q| + |x_Q - y| + |y - x| < \ell(Q) + |x_Q - y| + (1 + \kappa)\,\mathrm{dist}(y, \partial\Omega) \\ &\leq \ell(Q) + |x_Q - y| + (1 + \kappa)|x_Q - y| = \ell(Q) + (2 + \kappa)|x_Q - y| \\ &\leq \ell(Q) + (2 + \kappa)\big(\mathrm{dist}(Q, \partial\Omega) + \mathrm{diam}(Q)\big) \leq C\ell(Q), \end{aligned} \tag{8.9.113}$$

from which the first inclusion in (8.9.112) follows. Likewise, for each $z \in \Delta\big(x, \ell(Q)\big)$ we may rely on (8.9.105)–(8.9.107) to write

$$\begin{aligned} |z - x_Q| &\leq |z - x| + |x - y| + |y - x_Q| < \ell(Q) + (1 + \kappa)\,\mathrm{dist}(y, \partial\Omega) + |y - x_Q| \\ &\leq \ell(Q) + (1 + \kappa)|y - x_Q| + |y - x_Q| = \ell(Q) + (2 + \kappa)|y - x_Q| \\ &\leq \ell(Q) + (2 + \kappa)\big(\mathrm{dist}(Q, \partial\Omega) + \mathrm{diam}(Q)\big) \leq C\ell(Q), \end{aligned} \tag{8.9.114}$$

which ultimately establishes the second inclusion in (8.9.112). In turn, from (8.9.110), (8.9.112), and the fact that σ is a doubling measure we deduce that

$$0 \leq w_\pm(y) = \Big(\fint_{\Delta_Q} (f_\pm)^s \, d\sigma\Big)^{1/s} \leq C\Big(\fint_{\Delta(x, C\ell(Q))} (f_\pm)^s \, d\sigma\Big)^{1/s}. \tag{8.9.115}$$

Let us also note that if we additionally assume that $\mathrm{dist}(y, \partial\Omega) < \rho$ then, as seen from (8.9.105),

$$\ell(Q) \approx \mathrm{dist}(Q, \partial\Omega) \leq \mathrm{dist}(y, \partial\Omega) < \rho, \tag{8.9.116}$$

hence $\ell(Q) \leq C_n\rho$, for some purely dimensional constant $C_n \in (0, \infty)$. From this, (8.9.115), and (7.6.12) we then conclude that

$$0 \leq \big(\mathcal{N}_\kappa^\rho w_\pm\big)(x) \leq C\big(\mathcal{M}_{\partial\Omega, s}^{C\rho} f_\pm\big)(x). \tag{8.9.117}$$

In light of the fact that $\mathcal{N}_\kappa^\rho w_\pm$ are Borel-measurable functions (cf. (8.2.28)), hence also σ-measurable functions since σ is a Borel measure, from (8.9.117) and (7.6.13) we see that

$$\mathcal{N}_\kappa^\rho w_\pm \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma). \tag{8.9.118}$$

If we now define

$$w := w_+ - w_-, \tag{8.9.119}$$

it follows that $w : \Omega \to \mathbb{R}$ is a well-defined Lebesgue measurable function for which

$$0 \le \mathcal{N}_\kappa^\rho w \le \mathcal{N}_\kappa^\rho w_+ + \mathcal{N}_\kappa^\rho w_- \quad \text{on} \quad \partial\Omega. \tag{8.9.120}$$

In view of (8.2.28) and (8.9.118), we then deduce from (8.9.120) that

$$\mathcal{N}_\kappa^\rho w \in L^p_{\mathrm{loc}}(\partial\Omega, \sigma). \tag{8.9.121}$$

This takes care of the first demand in (8.9.103). The second demand in (8.9.103) follows from (8.9.119) as soon as we show that

$$w_\pm\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = f_\pm \quad \text{at } \sigma\text{-a.e. point in} \quad A_\kappa(\partial\Omega). \tag{8.9.122}$$

From (8.9.5) and the continuity of $[0, \infty) \ni t \mapsto t^a$ for each fixed $a \in (0, \infty)$, we see that this is further equivalent to showing that

$$(w_\pm)^s\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = (f_\pm)^s \quad \text{at } \sigma\text{-a.e. point in} \quad A_\kappa(\partial\Omega). \tag{8.9.123}$$

Recall the sharp version of Lebesgue's Differentiation Theorem recorded in Proposition 7.4.4. Granted (8.9.111) and bearing in mind that σ is a doubling Borel-semiregular measure, this ensures the existence of a σ-measurable set $N \subseteq \partial\Omega$ with the property that $\sigma(N) = 0$ and

$$\lim_{r\to 0^+} \fint_{\Delta(x,r)} |f_\pm(z)^s - f_\pm(x)^s|\, d\sigma(z) = 0 \quad \text{for every} \quad x \in \partial\Omega \setminus N. \tag{8.9.124}$$

Fix now a point $x \in A_\kappa(\partial\Omega) \setminus N$ and pick an arbitrary $y \in \Gamma_\kappa(x)$. Then $y \in \Omega$ and we denote by Q_y the unique Whitney cube in $\mathcal{W}_\Omega$ containing y. As a consequence of (8.9.105), we have

$$\ell(Q_y) \approx \operatorname{dist}(Q_y, \partial\Omega) \le \operatorname{dist}(y, \partial\Omega) \le |x - y|, \tag{8.9.125}$$

which shows that, on the one hand,

$$\ell(Q_y) \to 0^+ \quad \text{as} \quad \Gamma_\kappa(x) \ni y \to x. \tag{8.9.126}$$

On the other hand, from (8.9.110) we see that

$$\left|w_\pm(y)^s - f_\pm(x)^s\right| \le C \fint_{\Delta\left(x, C\ell(Q_y)\right)} \left|(f_\pm)(z)^s - (f_\pm)(x)^s\right| d\sigma(z). \tag{8.9.127}$$

Together, (8.9.127), (8.9.126), (8.9.124), and the fact that $x \in \overline{\Gamma_\kappa(x)}$ (given that $x \in A_\kappa(\partial\Omega)$) show that

$$\lim_{\Gamma_\kappa(x)\ni y\to x} \left|w_\pm(y)^s - f_\pm(x)^s\right| = 0, \tag{8.9.128}$$

from which (8.9.123) follows (on account of (8.9.5)). This completes the proof of (8.9.122), thus finishing the proof of the second demand in (8.9.103). □

Here is the second proof of Corollary 8.9.9 mentioned earlier.[4]

Second proof of Corollary 8.9.9 Consider $f : \partial\Omega \to \mathbb{R}$ defined σ-a.e. as

$$f := \begin{cases} u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} & \text{on } \partial_{\text{nta}}\Omega, \\ 0 & \text{on } \partial\Omega \setminus \partial_{\text{nta}}\Omega. \end{cases} \tag{8.9.129}$$

Proposition 8.9.5 ensures that f is a σ-measurable function and from (8.9.8) we see that

$$|f| \le \mathcal{N}_\kappa^\varepsilon u \text{ on } \partial\Omega. \tag{8.9.130}$$

As a consequence,

$$f \in L^p_{\text{loc}}(\partial\Omega, \sigma). \tag{8.9.131}$$

Granted this, the extension result proved in Lemma 8.9.10 (whose present applicability is ensured by Lemma 3.6.4) guarantees that there exists a Lebesgue measurable function $w : \Omega \to \mathbb{R}$ with the property that

$$\begin{gathered} \text{for each aperture parameter } \widetilde{\kappa} \in (0,\infty) \text{ we have} \\ \mathcal{N}_{\widetilde{\kappa}}^\varepsilon w \in L^p_{\text{loc}}(\partial\Omega, \sigma) \text{ and } w\Big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} = f \text{ at } \sigma\text{-a.e. point in } \partial_{\text{nta}}\Omega. \end{gathered} \tag{8.9.132}$$

Fix now another aperture parameter $\kappa' \in (0,\infty)$ and select some integrability exponent $s \in (0,p)$. Then Theorem 8.4.6 applied to the Lebesgue measurable function $u - w : \Omega \to \mathbb{R}$ shows that there exists some constant $C \in (0,\infty)$, which depends only on κ, κ', s and the doubling character of σ, such that for each $\rho \in (0,\varepsilon]$ we have

$$\mathcal{N}_{\kappa'}^\rho(u-w) \le C \cdot \mathcal{M}_{\partial\Omega,s}^{C\varepsilon}\big(\mathcal{N}_\kappa^\rho(u-w)\big) \text{ at each point on } \partial\Omega \tag{8.9.133}$$

(with the local L^s-based Hardy-Littlewood maximal operator as in (8.4.46)). We claim that

$$\mathcal{N}_\kappa^\rho(u-w) \to 0 \text{ in } L^p_{\text{loc}}(\partial\Omega,\sigma) \text{ as } \rho \to 0^+. \tag{8.9.134}$$

To justify this claim, first observe that

[4] A similar argument may, in fact, be used to give another proof of Proposition 8.9.8.

$$0 \le \mathcal{N}_\kappa^\rho(u-w) \le \mathcal{N}_\kappa^\varepsilon(u-w) \in L^p_{\rm loc}(\partial\Omega,\sigma) \ \text{ for each } \ \rho \in (0,\varepsilon]. \tag{8.9.135}$$

Next, recall from items *(i)-(ii)* in Proposition 8.8.6 that $\partial_{\rm nta}\Omega \subseteq A_\kappa(\partial\Omega)$ and $\sigma\big(A_\kappa(\partial\Omega)\setminus\partial_{\rm nta}\Omega\big)=0$. Bearing this in mind, at σ-a.e. point $x\in A_\kappa(\partial\Omega)$ we may then compute

$$\begin{aligned}\lim_{\rho\to 0^+}\big(\mathcal{N}_\kappa^\rho(u-w)\big)(x) &= \Big|\Big((u-w)\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)(x)\Big| = \Big|\Big(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)(x)-\Big(w\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big)(x)\Big| \\ &= |f(x)-f(x)| = 0,\end{aligned} \tag{8.9.136}$$

thanks to (8.9.13) in Lemma 8.9.2, (8.9.129), and (8.9.132). Since at each point $x\in\partial\Omega\setminus A_\kappa(\partial\Omega)$ we also have

$$\begin{aligned}\big(\mathcal{N}_\kappa^\rho(u-w)\big)(x) &= \Big(\mathcal{N}_\kappa\big((u-w)\cdot\mathbf{1}_{O_\rho}\big)\Big)(x) \\ &= \big\|(u-w)\cdot\mathbf{1}_{O_\rho}\big\|_{L^\infty(\Gamma_\kappa(x),\mathcal{L}^n)} = 0 \ \text{ if } \ \rho>0 \text{ is small enough,}\end{aligned} \tag{8.9.137}$$

we ultimately conclude from (8.9.136) and (8.9.137) that

$$\lim_{\rho\to 0^+}\big(\mathcal{N}_\kappa^\rho(u-w)\big)(x) = 0 \ \text{ for each } \ x\in\partial\Omega. \tag{8.9.138}$$

At this stage, the claim made in (8.9.134) follows from (8.9.135), (8.9.138), and Lebesgue's Dominated Convergence Theorem.

In turn, (8.9.133), (8.9.134), and (7.6.13) imply

$$\lim_{\rho\to 0^+}\mathcal{N}_{\kappa'}^\rho(u-w) = 0 \ \text{ in } \ L^p_{\rm loc}(\partial\Omega,\sigma) \ \text{ as } \ \rho\to 0^+. \tag{8.9.139}$$

As such, there exist a numerical sequence $\{\rho_j\}_{j\in\mathbb{N}}\subseteq(0,\varepsilon]$ which converges to zero, together with some σ-measurable set $N\subseteq\partial\Omega$ satisfying $\sigma(N)=0$, such that

$$\lim_{j\to\infty}\Big(\mathcal{N}_{\kappa'}^{\rho_j}(u-w)\Big)(x) = 0 \ \text{ for each } \ x\in\partial\Omega\setminus N. \tag{8.9.140}$$

Given the monotonicity of the truncated nontangential maximal operator with respect to the truncation parameter, this self-improves to

$$\lim_{\rho\to 0^+}\Big(\mathcal{N}_{\kappa'}^{\rho}(u-w)\Big)(x) = 0 \ \text{ for each } \ x\in\partial\Omega\setminus N. \tag{8.9.141}$$

In concert with (8.9.14) in Lemma 8.9.2, the above property forces

$$\Big((u-w)\big|^{\kappa'-{\rm n.t.}}_{\partial\Omega}\Big)(x) = 0 \ \text{ for } \sigma\text{-a.e. } \ x\in A_{\kappa'}(\partial\Omega). \tag{8.9.142}$$

In view of item *(i)* in Proposition 8.8.6 this further permits us to conclude that

$$\Big((u-w)\Big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\Big)(x)=0 \ \text{ for } \sigma\text{-a.e. } \ x\in\partial_{\text{nta}}\Omega. \tag{8.9.143}$$

Writing $u=(u-w)+w$ and recalling (8.9.132), we ultimately see from (8.9.143) and (8.9.129) that

$$\begin{aligned}&\text{the nontangential limit } \big(u\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big)(x) \text{ exists (in } \mathbb{R}) \text{ for } \sigma\text{-a.e. } x\in\partial_{\text{nta}}\Omega\\ &\text{and we have } \big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(x)=\big(u\big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big)(x) \text{ at } \sigma\text{-a.e. point } x\in\partial_{\text{nta}}\Omega.\end{aligned} \tag{8.9.144}$$

Finally, Theorem 8.4.6 also guarantees the existence of a constant $C\in(0,\infty)$, which depends only on κ, κ', s and the doubling character of σ, such that

$$0\leq \mathcal{N}_{\kappa'}^{\varepsilon}u\leq C\cdot \mathcal{M}_{\partial\Omega,s}^{C\varepsilon}\big(\mathcal{N}_{\kappa}^{\varepsilon}u\big) \ \text{ at each point on } \ \partial\Omega. \tag{8.9.145}$$

In view of this, the fact that $\mathcal{N}_{\kappa}^{\varepsilon}u\in L^p_{\text{loc}}(\partial\Omega,\sigma)$, (7.6.13), and (8.2.28) we then conclude that $\mathcal{N}_{\kappa'}^{\varepsilon}u$ also belongs to $L^p_{\text{loc}}(\partial\Omega,\sigma)$. Together with (8.9.144), this finishes the (second) proof of Corollary 8.9.9. □

We next prove that for null-solutions of weakly elliptic systems, in arbitrary open sets in the Euclidean space, multiplication by the distance to the boundary essentially acts like integration under the action of the nontangential maximal operator. Here we also show that null-solutions of weakly elliptic systems, again in arbitrary open sets in the Euclidean space, which possess nontangential boundary traces have the property that their partial derivatives of any order multiplied by a corresponding power of the distance to the boundary actually vanish on the boundary (in the nontangential trace sense).

Proposition 8.9.11 *Let Ω be an arbitrary open subset of $\mathbb{R}^n$. Also, assume L is a homogeneous constant (complex) coefficient weakly elliptic $M\times M$ system of order $2m$ in $\mathbb{R}^n$ (for some $m\in\mathbb{N}$) and suppose*

$$u\in\big[\mathscr{C}^\infty(\Omega)\big]^M \ \textit{ satisfies } \ Lu=0 \ \textit{ in } \ \Omega. \tag{8.9.146}$$

Finally, fix two aperture parameters $\widetilde{\kappa},\kappa\in(0,\infty)$ with $\widetilde{\kappa}>\kappa$.

Then, given any multi-index $\alpha\in\mathbb{N}_0^n$ along with any number $\lambda\in\mathbb{R}$, it follows that there exists some constant $C=C(L,\widetilde{\kappa},\kappa,\alpha,\lambda)\in(0,\infty)$ with the property that

$$\mathcal{N}_{\kappa}\big(\delta_{\partial\Omega}^{|\alpha|+\lambda}\cdot\partial^\alpha u\big)\leq C\cdot\mathcal{N}_{\widetilde{\kappa}}\big(\delta_{\partial\Omega}^{\lambda}\cdot u\big) \ \textit{ at each point on } \ \partial\Omega. \tag{8.9.147}$$

In fact, for each given threshold $\delta>0$, each multi-index $\alpha\in\mathbb{N}_0^n$, and each number $\lambda\in\mathbb{R}$ there exists some constant $C=C(L,\widetilde{\kappa},\kappa,\delta,\alpha,\lambda)\in(0,\infty)$ such that for each pair of truncation parameters $\widetilde{\rho},\rho\in(0,\infty)$ satisfying $\widetilde{\rho}>(1+\delta)\rho$ one has

$$\mathcal{N}_{\kappa}^{\rho}\big(\delta_{\partial\Omega}^{|\alpha|+\lambda}\cdot\partial^\alpha u\big)\leq C\cdot\mathcal{N}_{\widetilde{\kappa}}^{\widetilde{\rho}}\big(\delta_{\partial\Omega}^{\lambda}\cdot u\big) \ \textit{ at each point on } \ \partial\Omega. \tag{8.9.148}$$

As a corollary, given any $\alpha \in \mathbb{N}_0^n$ *along with any* $\lambda \in \mathbb{R}$, *one has*

$$\begin{aligned} &\big[\delta_{\partial\Omega}^{|\alpha|+\lambda} \cdot \partial^\alpha u\big]\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) = 0 \ \ \textit{at each point} \\ &x \in A_\kappa(\partial\Omega) \ \ \textit{where} \ \ \big[\delta_{\partial\Omega}^{\lambda} \cdot u\big]\Big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}}(x) = 0. \end{aligned} \tag{8.9.149}$$

Finally,

$$\begin{aligned} &\textit{if } x \in A_\kappa(\partial\Omega) \textit{ is so that the trace } \Big(u\Big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}}\Big)(x) \textit{ exists (in } \mathbb{C}^M\textit{) then} \\ &\big[\delta_{\partial\Omega}^{|\alpha|} \cdot \partial^\alpha u\big]\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) = 0 \ \textit{for each multi-index} \ \alpha \in \mathbb{N}_0^n \ \textit{with} \ |\alpha| > 0. \end{aligned} \tag{8.9.150}$$

In particular, (8.9.149) with $|\alpha| = 1$ and $\lambda = -1$ shows that

$$\begin{aligned} &(\nabla u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}(x) = 0 \ \text{ at each point} \\ &x \in A_\kappa(\partial\Omega) \ \text{ where } \ \big[u/\delta_{\partial\Omega}\big]\Big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}}(x) = 0. \end{aligned} \tag{8.9.151}$$

Proof of Proposition 8.9.11 To justify (8.9.147), observe that it is possible to choose some small number $a \in (0, 1)$ for which

$$\frac{1+\kappa+a}{1-a} < 1 + \widetilde{\kappa}. \tag{8.9.152}$$

Next, select some multi-index $\alpha \in \mathbb{N}_0^n$ and fix an arbitrary point $x_o \in \partial\Omega$. Also, pick an arbitrary point x belonging to the nontangential approach region $\Gamma_\kappa(x_o)$ (which, in particular, places x in Ω). Then for each point

$$y \in B\big(x, a \cdot \delta_{\partial\Omega}(x)\big) \subseteq \Omega \tag{8.9.153}$$

we have (bearing in mind that the distance function $\delta_{\partial\Omega}$ is Lipschitz with constant ≤ 1),

$$\begin{aligned} \delta_{\partial\Omega}(x) \leq \delta_{\partial\Omega}(y) + |x - y| < \delta_{\partial\Omega}(y) + a \cdot \delta_{\partial\Omega}(x) \\ \Longrightarrow \delta_{\partial\Omega}(x) < \frac{1}{1-a}\delta_{\partial\Omega}(y). \end{aligned} \tag{8.9.154}$$

In view of (8.9.152), this permits us to estimate

$$\begin{aligned} |y - x_o| &\leq |y - x| + |x - x_o| < a \cdot \delta_{\partial\Omega}(x) + (1+\kappa)\,\text{dist}(x, \partial\Omega) \\ &= (1+\kappa+a)\delta_{\partial\Omega}(x) < (1+\widetilde{\kappa})(1-a) \cdot \delta_{\partial\Omega}(x) \\ &< (1+\widetilde{\kappa})\delta_{\partial\Omega}(y). \end{aligned} \tag{8.9.155}$$

In turn, this shows that $y \in \Gamma_{\widetilde{\kappa}}(x_o)$ and, ultimately, that

$$B\big(x, a \cdot \delta_{\partial\Omega}(x)\big) \subseteq \Gamma_{\widetilde{\kappa}}(x_o). \tag{8.9.156}$$

In addition,

$$(1-a)\delta_{\partial\Omega}(x) < \delta_{\partial\Omega}(y) < (1+a)\delta_{\partial\Omega}(x) \quad \text{for each} \quad y \in B\big(x, a \cdot \delta_{\partial\Omega}(x)\big). \tag{8.9.157}$$

Going further, pick a multi-index $\alpha \in \mathbb{N}_0^n$ and a number $\lambda \in \mathbb{R}$, both arbitrary. Use interior estimates for the null-solution u of the weakly elliptic system L in the ball $B\big(x, a \cdot \delta_{\partial\Omega}(x)\big) \subseteq \Omega$ (cf. Theorem 6.5.7) to write

$$\begin{aligned} \delta_{\partial\Omega}(x)^{|\alpha|+\lambda} \cdot |(\partial^\alpha u)(x)| &\le \delta_{\partial\Omega}(x)^{|\alpha|+\lambda} \cdot \frac{C}{(a \cdot \delta_{\partial\Omega}(x))^{|\alpha|}} ⨍_{B\left(x, a\cdot\delta_{\partial\Omega}(x)\right)} |u| \, \mathrm{d}\mathcal{L}^n \\ &\le C ⨍_{B\left(x, a\cdot\delta_{\partial\Omega}(x)\right)} \delta_{\partial\Omega}^{\lambda} \cdot |u| \, \mathrm{d}\mathcal{L}^n \\ &\le C\Big(\mathcal{N}_{\widetilde{\kappa}}\big(\delta_{\partial\Omega}^{\lambda} \cdot u\big)\Big)(x_o), \end{aligned} \tag{8.9.158}$$

where the penultimate inequality is based on (8.9.157), and where the last step uses (8.9.156). After taking the supremum over all $x \in \Gamma_\kappa(x_o)$ we therefore arrive at

$$\Big(\mathcal{N}_\kappa\big(\delta_{\partial\Omega}^{|\alpha|+\lambda} \cdot \partial^\alpha u\big)\Big)(x_o) \le C \cdot \Big(\mathcal{N}_{\widetilde{\kappa}}\big(\delta_{\partial\Omega}^{\lambda} \cdot u\big)\Big)(x_o). \tag{8.9.159}$$

In view of this and the arbitrariness of $x_o \in \partial\Omega$, it follows that the estimate claimed in (8.9.147) holds.

The claim in (8.9.148) is established analogously, with a couple of alterations, indicated below. For starters, in addition to the demand placed in (8.9.152), decrease the value of $a \in (0, 1)$ so that we also have $a < \delta$. Hence, $(1 + a)\rho < \widetilde{\rho}$ for any pair of truncation parameters $\widetilde{\rho}, \rho \in (0, \infty)$ satisfying $\widetilde{\rho} > (1 + \delta)\rho$. In turn, this ensures that (compare with (8.9.156))

$$B\big(x, a \cdot \delta_{\partial\Omega}(x)\big) \subseteq \Gamma_{\widetilde{\kappa}}(x_o) \cap \mathcal{O}_{\widetilde{\rho}} \quad \text{for each} \quad x \in \Gamma_\kappa(x_o) \cap \mathcal{O}_\rho, \tag{8.9.160}$$

where the neighborhood collars $\mathcal{O}_{\widetilde{\rho}}, \mathcal{O}_\rho$ are defined as in (6.1.4). With this in hand and arguing much as in (8.9.158) we then conclude (keeping in mind (8.9.3)) that (8.9.148) holds.

Next, the claim in (8.9.149) is implied by (8.9.148) and Lemma 8.9.2. As far as the last claim in the statement is concerned, fix a point $x_o \in A_\kappa(\partial\Omega) \subseteq A_{\widetilde{\kappa}}(\partial\Omega)$ at which the nontangential limit $\Big(u\big|_{\partial\Omega}^{\widetilde{\kappa}-\mathrm{n.t.}}\Big)(x_o)$ exists, and pick a multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| > 0$. The goal is to show that

$$\big[\delta_{\partial\Omega}^{|\alpha|} \cdot \partial^\alpha u\big]\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x_o) = 0. \tag{8.9.161}$$

Choose $a \in (0, 1)$ as before (cf. (8.9.152)), and pick an arbitrary point $x \in \Gamma_\kappa(x_o)$. Then the inclusion in (8.9.156) holds. Using interior estimates (cf. Theorem 6.5.7) in the ball $B\big(x, a \cdot \delta_{\partial\Omega}(x)\big) \subseteq \Omega$ for the null-solution of the weakly elliptic system L defined as

$$\widetilde{u}(y) := u(y) - \big(u\big|_{\partial\Omega}^{\widetilde{\kappa}-\mathrm{n.t.}}\big)(x_o) \quad \text{for each} \quad y \in \Omega, \tag{8.9.162}$$

we may estimate, for some constant $C = C(L, n, a, \alpha) \in (0, \infty)$ (here we use the fact that $|\alpha| > 0$),

$$\begin{aligned}
\delta_{\partial\Omega}(x)^{|\alpha|} \cdot |(\partial^\alpha u)(x)| &= \delta_{\partial\Omega}(x)^{|\alpha|} \cdot |(\partial^\alpha \widetilde{u})(x)| \\
&\leq \delta_{\partial\Omega}(x)^{|\alpha|} \cdot \frac{C}{(a \cdot \delta_{\partial\Omega}(y))^{|\alpha|}} \times \\
&\quad \times \fint_{B\left(x, a \cdot \delta_{\partial\Omega}(x)\right)} \Big| u - \big(u\big|_{\partial\Omega}^{\widetilde{\kappa}-\mathrm{n.t.}}\big)(x_o) \Big| \, d\mathcal{L}^n \\
&\leq C \cdot \sup_{\substack{y \in \Gamma_{\widetilde{\kappa}}(x_o) \\ |y - x_o| < (1+\kappa+a)\delta_{\partial\Omega}(x)}} \Big| u(y) - \big(u\big|_{\partial\Omega}^{\widetilde{\kappa}-\mathrm{n.t.}}\big)(x_o) \Big|,
\end{aligned} \tag{8.9.163}$$

since having $y \in B\big(x, a \cdot \delta_{\partial\Omega}(x)\big)$ entails $y \in \Gamma_{\widetilde{\kappa}}(x_o)$ (cf. (8.9.156)), as well as

$$\begin{aligned}
|y - x_o| &\leq |y - x| + |x - x_o| < a \cdot \delta_{\partial\Omega}(x) + (1 + \kappa)\delta_{\partial\Omega}(x) \\
&= (1 + \kappa + a)\delta_{\partial\Omega}(x).
\end{aligned} \tag{8.9.164}$$

In turn, (8.9.163) implies (in light of Definition 8.9.1)

$$\big[\delta_{\partial\Omega}^{|\alpha|} \cdot \partial^\alpha u\big]\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x_o) = \lim_{\Gamma_\kappa(x_o) \ni x \to x_o} \big(\delta_{\partial\Omega}(x)^{|\alpha|} \cdot (\partial^\alpha u)(x)\big) = 0, \tag{8.9.165}$$

proving (8.9.161). □

Other useful versions of Proposition 8.9.11 naturally present themselves, and we isolate such a result in the remark below.

Remark 8.9.12 *Retain the context of Proposition 8.9.11. Having fixed an arbitrary point $x_0 \in \Omega$ along with some $r \in \big(0, \tfrac{1}{4}\,\mathrm{dist}(x_*, \partial\Omega)\big)$, abbreviate $K := \overline{B(x_*, r)}$ and $K' := \overline{B(x_*, 3r)}$. If in place of (8.9.146) we now only assume*

$$u \in \big[\mathscr{C}^\infty\big(\Omega \setminus K\big)\big]^M \quad \textit{satisfies} \quad Lu = 0 \quad \textit{in} \quad \Omega \setminus K, \tag{8.9.166}$$

then in lieu of (8.9.147) we can conclude that for any multi-index $\alpha \in \mathbb{N}_0^n$ and any number $\lambda \in \mathbb{R}$, it follows that there exists $C = C(L, \widetilde{\kappa}, \kappa, \alpha, \lambda, x_0, r) \in (0, \infty)$ with the property that

$$\mathcal{N}_\kappa^{\Omega \setminus K'}\big(\delta_{\partial\Omega}^{|\alpha|+\lambda} \cdot \partial^\alpha u\big) \leq C \cdot \mathcal{N}_{\widetilde{\kappa}}^{\Omega \setminus K}\big(\delta_{\partial\Omega}^{\lambda} \cdot u\big) \quad \textit{at each point on} \quad \partial\Omega. \tag{8.9.167}$$

This is established by reasoning much as in the proof of (8.9.147)*, observing that if the parameter* $a \in (0,1)$ *is sufficiently small* (*relative to* x_* *and* r) *then*

$$B\big(x, a \cdot \delta_{\partial\Omega}(x)\big) \cap \overline{B(x_*, r)} = \varnothing \ \text{ for each } \ x \in \Omega \setminus \overline{B(x_*, 3r)}. \tag{8.9.168}$$

To justify (8.9.168)*, consider the number* $d := 2 \cdot \max\big\{\delta_{\partial\Omega}(z) : \overline{B(x_*, r)}\big\}$ *and (since we only need to prove* (8.9.168) *for* $a \in (0,1)$ *sufficiently small) assume that* $0 < a < \min\big\{1/2, (2r)/d\big\}$*. Also, fix* $x \in \Omega \setminus \overline{B(x_*, 3r)}$ *and* $y \in B\big(x, a \cdot \delta_{\partial\Omega}(x)\big)$ *arbitrary. We shall consider two cases. First, corresponding to the case* $\delta_{\partial\Omega}(x) \geq d$ *we have* $\delta_{\partial\Omega}(y) > (1-a)\delta_{\partial\Omega}(x) \geq (1-a)d > d/2$ *which forces* $y \notin \overline{B(x_*, r)}$*, as wanted. Second, if* $\delta_{\partial\Omega}(x) < d$*, then* $|x - y| < a \cdot \delta_{\partial\Omega}(x) < a \cdot d < 2r$*, and since we also have* $3r < |x - x_*| \leq |x_* - y| + |x - y|$ *we may conclude that* $r < |x_* - y|$*, so once again* $y \notin \overline{B(x_*, r)}$*, as wanted.*

Next we revisit Proposition 8.9.11, with the goal of deriving (local and global) integrability results out of these earlier pointwise properties.

Corollary 8.9.13 *Let* Ω *be an open nonempty proper subset of* $\mathbb{R}^n$ *with the property that* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *is a doubling measure on* $\partial\Omega$*, and fix an aperture parameter* $\kappa \in (0,\infty)$*, a truncation parameter* $\varepsilon \in (0,\infty)$*, and an integrability exponent* $p \in (0,\infty]$*. Also, assume* L *is a homogeneous constant* (*complex*) *coefficient weakly elliptic* $M \times M$ *system of order* $2m$ *in* $\mathbb{R}^n$ (*for some* $m \in \mathbb{N}$) *and suppose*

$$\begin{aligned} u \in \big[\mathscr{C}^\infty(\Omega)\big]^M \ \ \text{satisfies} \ \ Lu = 0 \ \text{ in } \ \Omega, \\ \text{as well as} \ \ \mathcal{N}_\kappa^\varepsilon u \in L^p_{\text{loc}}(\partial\Omega, \sigma). \end{aligned} \tag{8.9.169}$$

Then for each $\rho \in (0, \varepsilon)$ *one has*

$$\mathcal{N}_\kappa^\rho\big(\delta_{\partial\Omega}^{|\alpha|} \cdot \partial^\alpha u\big) \in L^p_{\text{loc}}(\partial\Omega, \sigma) \ \text{ for each multi-index } \ \alpha \in \mathbb{N}_0^n. \tag{8.9.170}$$

Furthermore, if in addition to (8.9.169) *the function* u *is also assumed to satisfy*

$$u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \ \text{exists (in } \mathbb{C}^M\text{) at } \sigma\text{-a.e. point on } \partial_{\text{nta}}\Omega, \tag{8.9.171}$$

then

$$\begin{aligned} \big[\delta_{\partial\Omega}^{|\alpha|} \cdot \partial^\alpha u\big]\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = 0 \ \text{ at } \sigma\text{-a.e. point on } \partial_{\text{nta}}\Omega \\ \text{for each multi-index } \ \alpha \in \mathbb{N}_0^n \ \text{ with } \ |\alpha| > 0. \end{aligned} \tag{8.9.172}$$

Finally, if the last property in (8.9.169) *is strengthened to*

$$\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma) \tag{8.9.173}$$

then in place of (8.9.170) *one now concludes that, for each multi-index* $\alpha \in \mathbb{N}_0^n$,

$$\begin{aligned}&\mathcal{N}_\kappa\big(\delta_{\partial\Omega}^{|\alpha|}\cdot\partial^\alpha u\big)\ \textit{belongs to}\ L^p(\partial\Omega,\sigma)\ \textit{and}\\&\big\|\mathcal{N}_\kappa\big(\delta_{\partial\Omega}^{|\alpha|}\cdot\partial^\alpha u\big)\big\|_{L^p(\partial\Omega,\sigma)}\leq C\|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,\sigma)}\end{aligned}\tag{8.9.174}$$

for some constant $C = C(\Omega, L, \kappa, \alpha) \in (0, \infty)$ *independent of* u.

Proof Pick another aperture parameter $\widetilde{\kappa} > \kappa$. From the second line in (8.9.169) and Corollary 8.4.2 we conclude that

$$\mathcal{N}_{\widetilde{\kappa}}^{\varepsilon}u \in L^p_{\rm loc}(\partial\Omega, \sigma).\tag{8.9.175}$$

In concert with (8.2.28) and (8.9.148), this establishes the claim made in (8.9.170).

Next, assume that in addition to (8.9.169) the function u also satisfies (8.9.171). The goal is to prove (8.9.172). First, based on the second line in (8.9.169), (8.9.171), and Corollary 8.9.9 we conclude that

$$\begin{aligned}&\text{the nontangential limit } \big(u\big|^{\widetilde{\kappa}-\rm n.t.}_{\partial\Omega}\big)(x) \text{ exists}\\&(\text{in } \mathbb{C}^M) \text{ for } \sigma\text{-a.e. point } x \in \partial_{\rm nta}\Omega.\end{aligned}\tag{8.9.176}$$

On the other hand, Proposition 8.8.6 ensures that $\partial_{\rm nta}\Omega$ is a Borelian set (hence also σ-measurable), and satisfies

$$\partial_{\rm nta}\Omega \subseteq A_\kappa(\partial\Omega) \ \text{ and } \ \sigma\big(A_\kappa(\partial\Omega) \setminus \partial_{\rm nta}\Omega\big) = 0.\tag{8.9.177}$$

From (8.9.176), (8.9.177), and the last claim in the statement of Proposition 8.9.11 we then see that (8.9.172) holds.

To justify the claim made in the last part of the statement, assume the last property in (8.9.169) is strengthened to (8.9.173). Then (8.9.174) follows from (8.9.147), Proposition 8.4.1, and (8.2.28). □

Moving on to a new topic, we make the following definition.

Definition 8.9.14 *Call a set* Ω `locally pathwise nontangentially accessible` (*or* LPNA *for short*) *provided* Ω *is an open nonempty proper subset of* $\mathbb{R}^n$ *for which the following holds:*

given $\kappa > 0$ *there exist* $\widetilde{\kappa} \geq \kappa$ *along with* $c \in [1, \infty)$ *and* $d > 0$ *such that* σ*-a.e. point* $x \in \partial\Omega$ *has the property that any* $y \in \Gamma_\kappa(x)$ *with* $\operatorname{dist}(y, \partial\Omega) < d$ *may be joined by a rectifiable curve* $\gamma_{x,y}$ *satisfying* $\gamma_{x,y} \setminus \{x\} \subset \Gamma_{\widetilde{\kappa}}(x)$ *and whose length is* $\leq c|x - y|$. (8.9.178)

In addition, it is agreed to adopt the adjective "globally" in the case when one can allow $d = \infty$ *in* (8.9.178). *More precisely, a set* Ω *is said to be* `globally pathwise nontangentially accessible` (*or* GPNA *for short*) *provided* Ω *is an open nonempty proper subset of* $\mathbb{R}^n$ *such that:*

given $\kappa > 0$ *there exist* $\widetilde{\kappa} \geq \kappa$ *along with* $c \in [1, \infty)$ *such that* σ*-a.e. point* $x \in \partial\Omega$ *has the property that any* $y \in \Gamma_\kappa(x)$ *may be joined by a rectifiable curve* $\gamma_{x,y}$ *satisfying* $\gamma_{x,y} \setminus \{x\} \subset \Gamma_{\widetilde{\kappa}}(x)$ *and whose length is* $\leq c|x - y|$. (8.9.179)

Lemma 5.11.4 shows that

any one-sided NTA domain is a locally pathwise nontangentially accessible set (and, in fact, a globally pathwise nontangentially accessible set if the boundary of the said domain is unbounded). (8.9.180)

Of course, being a locally pathwise nontangentially accessible (LPNA) set is a much weaker condition than being a one-sided NTA domain. For example, a partially slit disk is a locally pathwise nontangentially accessible set, but fails to satisfy the Harnack chain condition.

Our notion of local pathwise nontangential accessibility should be compared with the concept of semi-uniformity introduced by H. Aikawa and K. Hirata in [6]. The latter is a quantitative connectivity condition, which may be regarded as a less restrictive version of the uniform condition considered in Definition 5.11.10. Specifically,

an open nonempty proper subset Ω of $\mathbb{R}^n$ is said to be `semi-uniform` if there exists some $C > 1$ such that each pair of points $x \in \partial\Omega$ and $y \in \Omega$ may be joined by a rectifiable curve $\gamma = \gamma(x, y)$ with $\gamma \setminus \{x\} \subseteq \Omega$, whose length is $\leq C|x - y|$ and which satisfies the "cigar path" condition $\min\Big\{\text{length}\,\gamma(x, z), \text{length}\,\gamma(z, y)\Big\} \leq C\,\text{dist}(z, \partial\Omega)$ for each point $z \in \gamma$, where $\gamma(x, z)$ and $\gamma(z, y)$ denote the two connected components of the curve γ, joining x with z, and z with y, respectively. (8.9.181)

The quality of being semi-uniform is analogous to the property of being uniform (see Definition 5.11.10), the sole difference being that in the former, as opposed to the latter, only one of the points is inside of the domain (the second one being on the boundary). We may employ an argument based on the Arzela–Ascoli Theorem to show that, in fact,

any uniform domain is a semi-uniform set. (8.9.182)

In concert with (5.11.65), this also shows that

and connected one-sided NTA domain (i.e., an open connected set satisfying an interior corkscrew condition as well as a Harnack chain condition) is a semi-uniform set. (8.9.183)

Proposition 8.9.15 *Any semi-uniform set is globally* (*hence also locally*) *pathwise nontangentially accessible.*

Proof With $\Omega \subseteq \mathbb{R}^n$ a given semi-uniform set, let $C \in (0, \infty)$ be as in (8.9.181), fix an arbitrary aperture parameter $\kappa > 0$, and pick

$$\varepsilon \in \big(0, 2^{-1}(1+\kappa)^{-1}C^{-1}\big). \tag{8.9.184}$$

Next, select two points, $x \in \partial\Omega$ and $y \in \Gamma_\kappa(x) \subseteq \Omega$, and let $\gamma = \gamma(x, y)$ be the curve as in (8.9.181). Finally, consider an arbitrary point $z \in \gamma$.

`Case I:` *Assume* $\operatorname{length}\gamma(z, y) < \varepsilon \cdot \operatorname{length}\gamma(x, y)$. Then on account of this, (8.9.181), (8.1.2), and (8.9.184) we may estimate

$$\begin{aligned} |y-z| &\leq \operatorname{length}\gamma(z,y) < \varepsilon \cdot \operatorname{length}\gamma(x,y) \\ &\leq \varepsilon \cdot C|x-y| < \varepsilon \cdot C(1+\kappa)\operatorname{dist}(y,\partial\Omega) < \tfrac{1}{2}\operatorname{dist}(y,\partial\Omega). \end{aligned} \tag{8.9.185}$$

In turn, this implies $\operatorname{dist}(z,\partial\Omega) > \frac{1}{2}\operatorname{dist}(y,\partial\Omega)$ which, together with part of (8.9.185), permits us to write

$$\begin{aligned} \operatorname{dist}(z,\partial\Omega) &> \tfrac{1}{2}\operatorname{dist}(y,\partial\Omega) \geq \varepsilon \cdot \operatorname{length}\gamma(x,y) \geq \varepsilon \cdot \operatorname{length}\gamma(x,z) \\ &\geq \varepsilon|x-z|. \end{aligned} \tag{8.9.186}$$

This proves that, in this case,

$$z \in \Gamma_{\widetilde{\kappa}}(x) \ \text{ if } \ \widetilde{\kappa} \geq \varepsilon^{-1} - 1. \tag{8.9.187}$$

`Case II:` *Assume that we have* $\operatorname{length}\gamma(z, y) \geq \varepsilon \cdot \operatorname{length}\gamma(x, y)$ *and that* $\operatorname{length}\gamma(z, y) < \operatorname{length}\gamma(x, z)$. Then

$$\begin{aligned} \varepsilon|x-z| &\leq \varepsilon \cdot \operatorname{length}\gamma(x,z) \leq \varepsilon \cdot \operatorname{length}\gamma(x,y) \\ &\leq \min\Big\{\operatorname{length}\gamma(x,z), \operatorname{length}\gamma(z,y)\Big\} \leq C\operatorname{dist}(z,\partial\Omega) \end{aligned} \tag{8.9.188}$$

hence, in this case,

$$z \in \Gamma_{\widetilde{\kappa}}(x) \ \text{ if } \ \widetilde{\kappa} \geq C\varepsilon^{-1} - 1. \tag{8.9.189}$$

`Case III:` *Assume that we have* $\operatorname{length}\gamma(z, y) \geq \varepsilon \cdot \operatorname{length}\gamma(x, y)$ *as well as* $\operatorname{length}\gamma(z, y) \geq \operatorname{length}\gamma(x, z)$. Then

$$\begin{aligned} |x-z| &\leq \operatorname{length}\gamma(x,z) = \min\Big\{\operatorname{length}\gamma(x,z), \operatorname{length}\gamma(z,y)\Big\} \\ &\leq C\operatorname{dist}(z,\partial\Omega). \end{aligned} \tag{8.9.190}$$

Thus, in this case,

$$z \in \{x\} \cup \Gamma_{\widetilde{\kappa}}(x) \quad \text{if } \widetilde{\kappa} > C - 1. \tag{8.9.191}$$

Collectively, (8.9.187), (8.9.189), and (8.9.191) prove that

$$\gamma \setminus \{x\} \subseteq \Gamma_{\widetilde{\kappa}}(x) \quad \text{if } \widetilde{\kappa} > C\varepsilon^{-1} - 1. \tag{8.9.192}$$

As such, condition (8.9.178) is satisfied (with $c := C$ and $d \in (0, \infty)$ arbitrary), so Ω is a globally pathwise nontangentially accessible set. □

For any locally pathwise nontangentially accessible (LPNA) set with a doubling surface measure, the nontangentially accessible boundary has full (surface) measure into the topological boundary.

Proposition 8.9.16 *Let $\Omega \subset \mathbb{R}^n$ be a locally pathwise nontangentially accessible* (LPNA) *set with the property that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Then*

$$\mathcal{H}^{n-1}\big(\partial\Omega \setminus \partial_{\text{nta}}\Omega\big) = 0. \tag{8.9.193}$$

Proof From (8.9.178) it is clear that there exists some $\kappa_0 \in (0, \infty)$ with the property that $\mathcal{H}^{n-1}\big(\partial\Omega \setminus A_{\kappa_0}(\partial\Omega)\big) = 0$. Granted this, (8.9.193) becomes a consequence of item *(ii)* in Proposition 8.8.6. □

In the class of locally pathwise nontangentially accessible sets, we may estimate the nontangential maximal operator of a function which vanishes on the boundary via the nontangential maximal operator of its gradient.

Proposition 8.9.17 *Suppose $\Omega \subset \mathbb{R}^n$ is a locally pathwise nontangentially accessible set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix $\kappa > 0$ arbitrary, then let $\widetilde{\kappa} \geq \kappa$ together with $d > 0$ and $c \in [1, \infty)$ be associated with κ as in* (8.9.178)*, and define $\theta := \big[c(1+\kappa)\big]^{-1} \in (0, 1)$. Finally, consider a function $u \in \mathscr{C}^1(\Omega)$ with the property that*

$$u\Big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} \quad \textit{exists at } \sigma\textit{-a.e. point on } \partial_{\text{nta}}\Omega. \tag{8.9.194}$$

Then for each $\varepsilon \in (0, d/\theta)$ one has

$$(\mathcal{N}_{\kappa}^{\theta\,\varepsilon}\, u)(x) \leq \Big|\Big(u\Big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}}\Big)(x)\Big| + \varepsilon\Big(\mathcal{N}_{\widetilde{\kappa}}^{\varepsilon}(\nabla u)\Big)(x) \quad \textit{at } \sigma\textit{-a.e. } x \in \partial_{\text{nta}}\Omega. \tag{8.9.195}$$

In particular, if actually

$$u\Big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} = 0 \quad \textit{at } \sigma\textit{-a.e. point on } \partial_{\text{nta}}\Omega \tag{8.9.196}$$

then for each $\varepsilon \in (0, d/\theta)$ the estimate in (8.9.195) *reduces to*

$$\mathcal{N}_{\kappa}^{\theta\,\varepsilon}\, u \leq \varepsilon \cdot \mathcal{N}_{\widetilde{\kappa}}^{\varepsilon}(\nabla u) \quad \textit{at } \sigma\textit{-a.e. point on } \partial_{\text{nta}}\Omega. \tag{8.9.197}$$

Moreover,

if the set Ω *is actually globally pathwise nontangentially accessible* (*cf.* (8.9.179)) *then* (8.9.195) (*hence also the fact that* (8.9.196) *implies* (8.9.197)) *is valid for every* $\varepsilon \in (0, \infty)$. (8.9.198)

We wish to note that, according to Corollary 8.9.9,

condition (8.9.194) automatically holds whenever there exists *some* $\kappa' > 0$ such that the nontangential limit $\big(u\big|^{\kappa'-\text{n.t.}}_{\partial\Omega}\big)(x)$ exists for σ-a.e. $x \in \partial_{\text{nta}}\Omega$, the measure σ is doubling on $\partial\Omega$, and $\mathcal{N}^{\rho}_{\kappa'} u \in L^p_{\text{loc}}(\partial\Omega, \sigma)$ for some arbitrary truncation parameter $\rho > 0$ and exponent $p > 0$. (8.9.199)

Proof of Proposition 8.9.17 Without loss of generality it may be assumed that the given function u is real-valued. Let $A_0 \subseteq \partial\Omega$ be a σ-measurable set with $\sigma(A_0) = 0$ and such that any point $x \in \partial\Omega \setminus A_0$ satisfies the pathwise connectivity property described in (8.9.178). Also, the assumption in (8.9.194) implies that

there exists a σ-measurable set $A_1 \subseteq \partial_{\text{nta}}\Omega$ with $\sigma(A_1) = 0$ and so that $\big(u\big|^{\widetilde{\kappa}-\text{n.t.}}_{\partial\Omega}\big)(x)$ exists for each point $x \in \partial_{\text{nta}}\Omega \setminus A_1$. (8.9.200)

To proceed, fix $\varepsilon \in (0, d/\theta)$ and introduce

$$N := \big\{x \in \partial_{\text{nta}}\Omega \setminus A_1 : \mathcal{N}^{\varepsilon}_{\widetilde{\kappa}}(\nabla u)(x) = +\infty\big\}. \tag{8.9.201}$$

In particular, the estimate in (8.9.195) is trivially true when $x \in N$. The proof is therefore finished as soon as we show that the estimate in (8.9.195) also holds at every point $x \in \big(\partial_{\text{nta}}\Omega \setminus (A_0 \cup A_1)\big) \setminus N$.

With this goal in mind, fix an arbitrary point $x \in \big(\partial_{\text{nta}}\Omega \setminus (A_0 \cup A_1)\big) \setminus N$ and pick some $y \in \Gamma_\kappa(x)$ satisfying $\text{dist}(y, \partial\Omega) < \theta\,\varepsilon$. In particular, $\text{dist}(y, \partial\Omega) < d$, so the fact that $x \in \partial\Omega \setminus A_0$ ensures the existence of a rectifiable curve $\gamma_{x,y}$ joining x with y in Ω, with the property that

$$\gamma_{x,y} \setminus \{x\} \subseteq \Gamma_{\widetilde{\kappa}}(x), \tag{8.9.202}$$

and whose length $L := \text{length}\,\gamma_{x,y}$ satisfies

$$L \leq c|x-y| < c(1+\kappa)\,\text{dist}(y, \partial\Omega) < c(1+\kappa)\theta\,\varepsilon = \varepsilon. \tag{8.9.203}$$

As a consequence,

$$\text{dist}(\xi, \partial\Omega) \leq |\xi - x| \leq L \leq \varepsilon, \qquad \forall\,\xi \in \gamma_{x,y}. \tag{8.9.204}$$

Going further, let $[0, L] \ni s \mapsto \gamma_{x,y}(s)$ be the arc-length parametrization of the curve $\gamma_{x,y}$ and consider the function given for each $s \in [0, L]$ by $f(s) := u\big(\gamma_{x,y}(s)\big)$. By design, this is continuous on $[0, L]$ and satisfies $f(L) = u(y)$ and $f(0) = \big(u\big|^{\widetilde{\kappa}-\mathrm{n.t.}}_{\partial\Omega}\big)(x)$ (cf. (8.9.200)). Also, f is differentiable at $\mathcal{L}^1$-a.e. $s \in (0, L)$ and, bearing in mind that $\big|\frac{\mathrm{d}}{\mathrm{d}s}[\gamma_{x,y}(s)]\big| = 1$ for $\mathcal{L}^1$-a.e. $s \in (0, L)$, the Chain Rule and Cauchy-Schwarz' inequality imply that

$$
\begin{aligned}
|f'(s)| &= \Big|(\nabla u)\big(\gamma_{x,y}(s)\big) \cdot \dot{\gamma}_{x,y}(s)\Big| \leq \Big|(\nabla u)\big(\gamma_{x,y}(s)\big)\Big| \leq \sup_{\xi \in \gamma_{x,y} \setminus \{x\}} \big|(\nabla u)(\xi)\big| \\
&\leq \mathcal{N}^{\varepsilon}_{\widetilde{\kappa}}(\nabla u)(x) \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in (0, L),
\end{aligned}
\tag{8.9.205}
$$

where the last inequality uses (8.9.202) and (8.9.204). Upon recalling that the point $x \in \partial_{\mathrm{nta}}\Omega \setminus (A_0 \cup A_1)$ but $x \notin N$, this implies that f is a Lipschitz function, with Lipschitz constant $\mathcal{N}^{\varepsilon}_{\widetilde{\kappa}}(\nabla u)(x) < +\infty$. Consequently,

$$
\begin{aligned}
|u(y)| = |f(L)| &\leq |f(L) - f(0)| + |f(0)| \leq L \cdot \mathcal{N}^{\varepsilon}_{\widetilde{\kappa}}(\nabla u)(x) + \Big|\big(u\big|^{\widetilde{\kappa}-\mathrm{n.t.}}_{\partial\Omega}\big)(x)\Big| \\
&\leq \varepsilon \cdot \mathcal{N}^{\varepsilon}_{\widetilde{\kappa}}(\nabla u)(x) + \Big|\big(u\big|^{\widetilde{\kappa}-\mathrm{n.t.}}_{\partial\Omega}\big)(x)\Big|.
\end{aligned}
\tag{8.9.206}
$$

Taking the supremum in (8.9.206) over all $y \in \Gamma_{\kappa}(x)$ satisfying $\operatorname{dist}(y, \partial\Omega) < \theta\,\varepsilon$ then ultimately yields

$$
\big(\mathcal{N}^{\theta\,\varepsilon}_{\kappa} u\big)(x) \leq \varepsilon \cdot \mathcal{N}^{\varepsilon}_{\widetilde{\kappa}}(\nabla u)(x) + \Big|\big(u\big|^{\widetilde{\kappa}-\mathrm{n.t.}}_{\partial\Omega}\big)(x)\Big|, \qquad \forall x \in \big(\partial_{\mathrm{nta}}\Omega \setminus (A_0 \cup A_1)\big) \setminus N.
\tag{8.9.207}
$$

Given that $\sigma(A_0 \cup A_1 \cup N) = 0$, this ultimately implies (8.9.195). Finally, the claim in (8.9.198) is justified in an analogous manner (formally taking $d = \infty$). □

There is more we can extract from the proof of Proposition 8.9.17, such as the fact that, for functions vanishing on the boundary, division by the distance to the boundary "acts as differentiation," as far as the size of the nontangential maximal operator is concerned. This pointwise result, which should be compared with Proposition 8.9.11, is made precise in the proposition below.

Proposition 8.9.18 *Let* $\Omega \subset \mathbb{R}^n$ *be a locally pathwise nontangentially accessible set. Abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ *and define* $\delta_{\partial\Omega} := \operatorname{dist}(\cdot, \partial\Omega)$. *Next, fix an arbitrary* $\kappa > 0$, *then let* $\widetilde{\kappa} \geq \kappa$ *together with* $d > 0$ *and* $c \in [1, \infty)$ *be associated with* κ *as in* (8.9.178), *and define* $\theta := \big[c(1+\kappa)\big]^{-1} \in (0, 1)$. *Lastly, suppose*

$$
\begin{gathered}
u \in \mathscr{C}^1(\Omega) \text{ is a function with the property that} \\
u\big|^{\widetilde{\kappa}-\mathrm{n.t.}}_{\partial\Omega} = 0 \text{ at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega.
\end{gathered}
\tag{8.9.208}
$$

Then for each $\varepsilon \in (0, d/\theta)$ *one has*

$$\Big(\mathcal{N}_\kappa^{\theta\,\varepsilon}\big(u/\delta_{\partial\Omega}\big)\Big)(x) \leq c(1+\kappa)\cdot\Big(\mathcal{N}_{\widetilde{\kappa}}^{\varepsilon}(\nabla u)\Big)(x) \ \text{ at } \sigma\text{-a.e. } x \in \partial_{\text{nta}}\Omega. \tag{8.9.209}$$

Moreover, if Ω is actually globally pathwise nontangentially accessible (a scenario in which $d=\infty$; cf. Definition 8.9.14) then one may allow $\varepsilon=\infty$ in (8.9.209), i.e., one now has

$$\Big(\mathcal{N}_\kappa\big(u/\delta_{\partial\Omega}\big)\Big)(x) \leq c(1+\kappa)\cdot\Big(\mathcal{N}_{\widetilde{\kappa}}(\nabla u)\Big)(x) \ \text{ at } \sigma\text{-a.e. } x \in \partial_{\text{nta}}\Omega. \tag{8.9.210}$$

Proof The claim in (8.9.209) is implicit in the proof of Proposition 8.9.17, from which we shall freely borrow notation and results. Specifically, from (8.9.203) we know that

$$L \leq c|x-y| < c(1+\kappa)\,\mathrm{dist}(y,\partial\Omega) = c(1+\kappa)\delta_{\partial\Omega}(y), \tag{8.9.211}$$

while the first line in (8.9.206) presently yields (thanks to (8.9.208))

$$|u(y)| \leq L\cdot\mathcal{N}_{\widetilde{\kappa}}^{\varepsilon}(\nabla u)(x). \tag{8.9.212}$$

Collectively, (8.9.211) and (8.9.212) give

$$\frac{|u(y)|}{\delta_{\partial\Omega}(y)} \leq c(1+\kappa)\cdot\mathcal{N}_{\widetilde{\kappa}}^{\varepsilon}(\nabla u)(x) \tag{8.9.213}$$

so, after taking the supremum over all points $y\in\Gamma_\kappa(x)$ satisfying $\mathrm{dist}(y,\partial\Omega)<\theta\,\varepsilon$, we arrive at (8.9.209). Finally, in the case when $d=\infty$, we obtain (8.9.210) from (8.9.209) by simply sending $\varepsilon\to\infty$. □

Moreover, a cursory inspection of the above proof reveals that the following version of Proposition 8.9.18 is also true:

Remark 8.9.19 *In the context of Proposition 8.9.18, if in place of the first line in* (8.9.208) *one now assumes that $u\in\mathscr{C}^1(\Omega\setminus K)$ for some compact set $K\subseteq\Omega$, then* (8.9.209) *continues to hold provided $\varepsilon\in(0,d/\theta)$ is sufficiently small (relative to* $\mathrm{dist}(K,\partial\Omega)$).

There is one final variant of Proposition 8.9.18 which we wish to single out in Proposition 8.9.21, stated a little further below. To facilitate its proof, we isolate a technical result in the next lemma.

Lemma 8.9.20 *Suppose $\Omega\subset\mathbb{R}^n$ is a locally pathwise nontangentially accessible set, and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Recall the family of one-sided collar neighborhoods $\{\mathcal{O}_\varepsilon\}_{\varepsilon>0}$ of $\partial\Omega$ defined in* (1.5.5)*. Fix $\kappa>0$ arbitrary, then let $\widetilde{\kappa}\geq\kappa$ together with $d>0$ and $c\in[1,\infty)$ be associated with κ as in* (8.9.178)*. Also, define $\widetilde{\kappa}_o:=4+3\widetilde{\kappa}$. Next, pick an arbitrary point $x_0\in\Omega$ together with some number $\rho\in\big(0\,,\frac14\,\mathrm{dist}(x_0,\partial\Omega)\big)$ then set $K:=\overline{B(x_0,\rho)}$ and $2K:=\overline{B(x_0,2\rho)}$. Finally, choose a threshold $\varepsilon_o\in(0,d)$ and set*

$$\widetilde{c} := c + 2\pi\rho \cdot \varepsilon_o^{-1} \in (c, \infty) \;\; \textit{and} \;\; \widetilde{\theta} := \big[\widetilde{c}(1+\kappa)\big]^{-1} \in (0,1). \tag{8.9.214}$$

Then

$$\begin{aligned} &\textit{for any given function } u \in \mathscr{C}^1(\Omega \setminus K) \textit{ having the property that} \\ &u\Big|_{\partial\Omega}^{\widetilde{\kappa}_o-\text{n.t.}} = 0 \textit{ at } \sigma\textit{-a.e. point on } \partial_{\text{nta}}\Omega, \textit{ and for any } \varepsilon \in (0, d/\widetilde{\theta}), \textit{ one} \\ &\textit{has } \mathcal{N}_\kappa^{(\mathcal{O}_{\widetilde{\theta}\varepsilon} \setminus \mathcal{O}_{\varepsilon_o})\setminus(2K)} u \leq \varepsilon \cdot \mathcal{N}_{\widetilde{\kappa}_o}^{\mathcal{O}_\varepsilon \setminus K}(\nabla u) \textit{ at } \sigma\textit{-a.e. point on } \partial_{\text{nta}}\Omega. \end{aligned} \tag{8.9.215}$$

Furthermore, if Ω is actually a globally pathwise nontangentially accessible set then (8.9.215) *holds with $d := \infty$.*

Proof Select some $x \in \partial_{\text{nta}}\Omega$ and suppose

$$\begin{aligned} &y \in \big(\Gamma_\kappa(x) \setminus \mathcal{O}_{\varepsilon_o}\big) \setminus (2K) \;\text{ is such that} \\ &\text{dist}(y, \partial\Omega) < d \;\text{ and }\; \gamma_{x,y} \cap (2K) \neq \varnothing, \end{aligned} \tag{8.9.216}$$

where $\gamma_{x,y}$ the curve associated with x, y as in (8.9.179). For y as above, denote by z_0 and z_1, respectively, the first and last contact points of the curve $\gamma_{x,y}$ with the closed ball $2K$. Define $\widetilde{\gamma}_{x,y}$ to be the curve obtained by modifying $\gamma_{x,y}$ in between z_0 and z_1 by replacing the sub-arc of $\gamma_{x,y}$ with end-points z_0, z_1 by a portion of a big circle on $\partial(2K)$ joining z_0 with z_1. Then $\widetilde{\gamma}_{x,y}$ is a rectifiable curve satisfying $\widetilde{\gamma}_{x,y} \setminus \{x\} \subset \Omega \setminus K$ and whose length is

$$\text{length}\big(\widetilde{\gamma}_{x,y}\big) \leq \text{length}\big(\gamma_{x,y}\big) + 2\pi\rho \leq c|x-y| + 2\pi\rho. \tag{8.9.217}$$

Then from (8.9.217), the fact that $|x-y| > \varepsilon_o$, the formula of $\widetilde{c}$ in (8.9.214), and the last property in (8.9.179) we have

$$\max\Big\{\text{length}\big(\gamma_{x,y}\big), \text{length}\big(\widetilde{\gamma}_{x,y}\big)\Big\} \leq \widetilde{c}|x-y|. \tag{8.9.218}$$

We also claim that

$$2K \subseteq \Gamma_{\widetilde{\kappa}_o}(x). \tag{8.9.219}$$

Since having $0 < \rho < \frac{1}{4}\,\text{dist}(x_0, \partial\Omega)$ guarantees that $2K \subseteq \Omega$, there remains to check that any given point $\xi \in 2K$ satisfies

$$|\xi - x| < (1+\widetilde{\kappa}_o)\,\text{dist}(\xi, \partial\Omega). \tag{8.9.220}$$

To justify this fix a point $\xi \in 2K$. Upon observing that

$$4\rho < \text{dist}(x_0, \partial\Omega) \leq \text{dist}(\xi, \partial\Omega) + |x_0 - \xi| < \text{dist}(\xi, \partial\Omega) + 2\rho \tag{8.9.221}$$

we conclude that

$$2\rho < \operatorname{dist}(\xi, \partial\Omega). \tag{8.9.222}$$

Next, recall that $z_0 \in 2K \cap \gamma_{x,y}$ and $\gamma_{x,y} \setminus \{x\} \subset \Gamma_{\widetilde{\kappa}}(x)$. As such, $z_0 \in \Gamma_{\widetilde{\kappa}}(x)$ hence

$$\begin{aligned} |z_0 - x| &< (1+\widetilde{\kappa})\operatorname{dist}(z_0, \partial\Omega) \leq (1+\widetilde{\kappa})\operatorname{dist}(\xi, \partial\Omega) + (1+\widetilde{\kappa})|\xi - z_0| \\ &< (1+\widetilde{\kappa})\operatorname{dist}(\xi, \partial\Omega) + (1+\widetilde{\kappa})4\rho < 3(1+\widetilde{\kappa})\operatorname{dist}(\xi, \partial\Omega), \end{aligned} \tag{8.9.223}$$

thanks to (8.9.222). Consequently,

$$\begin{aligned} |\xi - x| &\leq |z_0 - x| + |z_0 - \xi| < 3(1+\widetilde{\kappa})\operatorname{dist}(\xi, \partial\Omega) + 4\rho \\ &< (5+3\widetilde{\kappa})\operatorname{dist}(\xi, \partial\Omega) = (1+\widetilde{\kappa}_o)\operatorname{dist}(\xi, \partial\Omega). \end{aligned} \tag{8.9.224}$$

This establishes (8.9.220) and finishes the proof of (8.9.219). In particular, as seen from (8.9.219) and the design of the curve $\widetilde{\gamma}_{x,y}$,

$$\begin{aligned} &\widetilde{\gamma}_{x,y} \setminus \{x\} \subseteq \Gamma_{\widetilde{\kappa}_o}(x) \setminus K \quad \text{whenever} \\ &x \in \partial_{\text{nta}}\Omega \quad \text{and } y \text{ is as in (8.9.216).} \end{aligned} \tag{8.9.225}$$

The above analysis may now be used to justify the claim made in (8.9.215). Specifically, this is established by reasoning as in the proof of Proposition 8.9.17 with $\widetilde{\kappa}_o$ in place of $\widetilde{\kappa}$, with $\widetilde{\gamma}_{x,y}$ in place of $\gamma_{x,y}$ in the scenario described in (8.9.216) (and otherwise retaining $\gamma_{x,y}$ instead if this curve is disjoint from $2K$), with $\widetilde{c}$ in place of c, and with $\widetilde{\theta}$ in place of θ (two relevant aspects to keep in mind in this regard are (8.9.218) and (8.9.225)). □

We are now prepared to prove the following variant of Proposition 8.9.17, alluded to before.

Proposition 8.9.21 *Let $\Omega \subset \mathbb{R}^n$ be a globally pathwise nontangentially accessible set, and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix an arbitrary point $x_0 \in \Omega$ together with some $\rho \in \big(0, \frac{1}{4}\operatorname{dist}(x_0, \partial\Omega)\big)$ and set $K := \overline{B(x_0, \rho)}$ and $2K := \overline{B(x_0, 2\rho)}$. Then for each aperture parameter $\kappa > 0$ there exist $\widetilde{\kappa}_o \geq \kappa$ and a constant $C \in (0, \infty)$ with the property that for each function*

$$\begin{aligned} &u \in \mathscr{C}^1(\Omega \setminus 2K) \text{ such that} \\ &u\Big|_{\partial\Omega}^{\widetilde{\kappa}_o-\text{n.t.}} = 0 \quad \text{at } \sigma\text{-a.e. point on } \partial_{\text{nta}}\Omega \end{aligned} \tag{8.9.226}$$

one has

$$\Big(\mathcal{N}_\kappa^{\Omega\setminus 2K}\big(u/\delta_{\partial\Omega}\big)\Big)(x) \leq C \cdot \Big(\mathcal{N}_{\widetilde{\kappa}_o}^{\Omega\setminus K}(\nabla u)\Big)(x) \quad \text{at } \sigma\text{-a.e. } x \in \partial_{\text{nta}}\Omega. \tag{8.9.227}$$

Proof Let $\widetilde{\kappa} \geq \kappa$ and $c \in [1, \infty)$ be associated with κ as in (8.9.178) (with $d := \infty$). Consider $\widetilde{\kappa}_o := 4 + 3\widetilde{\kappa}$. Also, fix a threshold $\varepsilon_o \in (0, \infty)$ and define $\widetilde{\theta}$ as in (8.9.214). Finally, pick a function u as in (8.9.226). Then (8.9.215) ensures that there exists some σ-measurable set $N \subseteq \partial_{\text{nta}}\Omega$ with the property that $\sigma(N) = 0$ and such that

$$\begin{aligned}&\Big(\mathcal{N}_\kappa^{(\mathcal{O}_{\widetilde{\theta}\varepsilon}\setminus\mathcal{O}_{\varepsilon_o})\setminus(2K)}\,u\Big)(x)\le\varepsilon\cdot\Big(\mathcal{N}_{\widetilde{\kappa}_o}^{\mathcal{O}_\varepsilon\setminus K}(\nabla u)\Big)(x)\\&\text{for each }\ x\in\partial_{\text{nta}}\Omega\setminus N\ \text{ and }\ \varepsilon\in(0,\infty)\cap\mathbb{Q}.\end{aligned}\tag{8.9.228}$$

Fix some $x\in\partial_{\text{nta}}\Omega\setminus N$. Given an arbitrary point

$$y\in\big(\Gamma_\kappa(x)\setminus\mathcal{O}_{\varepsilon_o}\big)\setminus(2K),\tag{8.9.229}$$

choose $\varepsilon\in(0,\infty)\cap\mathbb{Q}$ such that $\varepsilon>\delta_{\partial\Omega}(y)/\widetilde{\theta}$. This ensures $y\in(\mathcal{O}_{\widetilde{\theta}\varepsilon}\setminus\mathcal{O}_{\varepsilon_o})\setminus(2K)$, so we may invoke (8.9.228) to estimate

$$\begin{aligned}|u(y)|&\le\Big(\mathcal{N}_\kappa^{(\mathcal{O}_{\widetilde{\theta}\varepsilon}\setminus\mathcal{O}_{\varepsilon_o})\setminus(2K)}\,u\Big)(x)\le\varepsilon\cdot\Big(\mathcal{N}_{\widetilde{\kappa}_o}^{\mathcal{O}_\varepsilon\setminus K}(\nabla u)\Big)(x)\\&\le\varepsilon\cdot\Big(\mathcal{N}_{\widetilde{\kappa}_o}^{\Omega\setminus K}(\nabla u)\Big)(x).\end{aligned}\tag{8.9.230}$$

After letting $\varepsilon\searrow\delta_{\partial\Omega}(y)/\widetilde{\theta}$ we arrive at

$$|u(y)|\le\big(\delta_{\partial\Omega}(y)/\widetilde{\theta}\big)\cdot\Big(\mathcal{N}_{\widetilde{\kappa}_o}^{\Omega\setminus K}(\nabla u)\Big)(x),\tag{8.9.231}$$

hence

$$\frac{|u(y)|}{\delta_{\partial\Omega}(y)}\le\widetilde{\theta}^{-1}\cdot\Big(\mathcal{N}_{\widetilde{\kappa}_o}^{\Omega\setminus K}(\nabla u)\Big)(x).\tag{8.9.232}$$

Taking the supremum over all points y as in (8.9.229) then leads to the conclusion that

$$\Big(\mathcal{N}_\kappa^{(\Omega\setminus\mathcal{O}_{\varepsilon_o})\setminus 2K}\big(u/\delta_{\partial\Omega}\big)\Big)(x)\le C\cdot\Big(\mathcal{N}_{\widetilde{\kappa}_o}^{\Omega\setminus K}(\nabla u)\Big)(x)\ \text{ at }\sigma\text{-a.e. }\ x\in\partial_{\text{nta}}\Omega,\tag{8.9.233}$$

for some $C=C(\Omega,x_0,\rho,\kappa,\varepsilon_o)\in(0,\infty)$. The result described in Remark 8.9.19 implies

$$\Big(\mathcal{N}_\kappa^{\mathcal{O}_{\varepsilon_o}\setminus 2K}\big(u/\delta_{\partial\Omega}\big)\Big)(x)\le C\cdot\Big(\mathcal{N}_{\widetilde{\kappa}_o}^{\Omega\setminus K}(\nabla u)\Big)(x)\ \text{ at }\sigma\text{-a.e. }\ x\in\partial_{\text{nta}}\Omega.\tag{8.9.234}$$

Together, (8.9.233) and (8.9.234) yield (8.9.227). □

Moving on, under suitable geometric assumptions on the underlying domain it becomes possible to show that control of the nontangential maximal operator of the gradient of a given function implies that the function in question has nontangential pointwise traces a.e. on the boundary. Our next proposition elaborates on this topic.

Proposition 8.9.22 *Let $\Omega\subseteq\mathbb{R}^n$ be an interior* NTA *domain with the property that $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure and the set $\partial\Omega$ is lower Ahlfors regular* (*the latter condition is automatically satisfied if Ω is an* NTA *domain; cf.* (5.11.4)). *Fix an aperture $\kappa\in(0,\infty)$, a truncation parameter $\varepsilon\in(0,\infty)$, and an integrability exponent $p\in(0,\infty]$.*

Then if $u \in \mathscr{C}^1(\Omega)$ *has* $\mathcal{N}_\kappa^\varepsilon(\nabla u) \in L^p_{\rm loc}(\partial\Omega, \sigma)$ *it follows that*

$$\begin{aligned}&\textit{the nontangential trace } \Big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\Big)(x) \textit{ exists at } \sigma\textit{-a.e. } x \in \partial_*\Omega,\\ &\textit{and the function } u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \textit{ belongs to the space } L^p_{\rm loc}(\partial_*\Omega, \sigma).\end{aligned} \tag{8.9.235}$$

In particular,

$$\begin{aligned}&\textit{under the additional assumption that } \partial\Omega \textit{ is bounded, given any}\\ &\textit{function } u \in \mathscr{C}^1(\Omega) \textit{ such that } \mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma) \textit{ it follows}\\ &\textit{that the nontangential trace } \Big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\Big)(x) \textit{ exists at } \sigma\textit{-a.e. point}\\ &x \in \partial_*\Omega, \textit{ and the function } u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \textit{ belongs to the space } L^p(\partial_*\Omega, \sigma).\end{aligned} \tag{8.9.236}$$

Proof Assume $u \in \mathscr{C}^1(\Omega)$ is a function satisfying $\mathcal{N}_\kappa^\varepsilon(\nabla u) \in L^p_{\rm loc}(\partial\Omega, \sigma)$. Let $\widetilde{\kappa}, R, C \in (0, \infty)$ be associate to the given setting as in Lemma 5.11.6. Without loss of generality assume $\varepsilon \in (0, R/2)$ and pick $\delta \in \big(0, \varepsilon/(C+1)\big)$. Consider a measurable set $A \subseteq \partial_*\Omega$, of full measure (relative to σ), with the property that

$$\mathcal{N}_{\widetilde{\kappa}}^\varepsilon(\nabla u)(x) < \infty \quad\text{and}\quad x \in \overline{\Gamma_\kappa(x)} \quad\text{for every}\quad x \in A. \tag{8.9.237}$$

That such a set exists is guaranteed by the present assumptions, Proposition 8.4.1, and Proposition 8.8.4. Pick an arbitrary point $x \in A$. Then for every pair of points $y_0, y_1 \in \Gamma_\kappa(x) \cap B(x, \delta)$ consider γ as in (5.11.14). In concert with the Fundamental Theorem of Calculus, this permits us to estimate

$$\begin{aligned}|u(y_1) - u(y_0)| &\leq \int_\gamma |\nabla u|\, {\rm d}s \leq \mathcal{N}_{\widetilde{\kappa}}^\varepsilon(\nabla u)(x) \cdot {\rm length}(\gamma)\\ &\leq C\, \mathcal{N}_{\widetilde{\kappa}}^\varepsilon(\nabla u)(x) \cdot \max\big\{|x - y_0|, |x - y_1|\big\},\end{aligned} \tag{8.9.238}$$

where ds denotes the arc-length measure on γ. In turn, this readily implies that the numerical sequence $\{u(y_j)\}_{j\in\mathbb{N}}$ is Cauchy whenever $\{y_j\}_{j\in\mathbb{N}} \subseteq \Gamma_\kappa(x)$ is a sequence of points with the property that $y_j \to x$ as $j \to \infty$. This ultimately proves that

$$\Big(u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\Big)(x) = \lim_{\Gamma_\kappa(x)\ni y \to x} u(y) \quad\text{exists for } \sigma\text{-a.e. } x \in \partial_*\Omega. \tag{8.9.239}$$

Granted this, the fact that $u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}$ is σ-measurable on $\partial_*\Omega$ is implied by Corollary 8.9.6. Then Proposition 8.4.9 and (8.9.8) ensure that said function belongs to the space $L^p_{\rm loc}(\partial_*\Omega, \sigma)$, finishing the proof of (8.9.235). Finally, (8.9.236) is a particular case of what we have proved earlier. □

8.10 The Averaged Nontangential Maximal Operator

Fix an open, nonempty, proper subset Ω of $\mathbb{R}^n$. As before, $\delta_{\partial\Omega}$ denotes the distance function to $\partial\Omega$. In what follows we also fix $\kappa > 0$, $\theta \in (0, 1)$, and $r \in (0, \infty)$. Recall the nontangential approach regions Γ_κ defined in (8.1.2) and consider a function $u \in L^r_{\rm loc}(\Omega, \mathcal{L}^n)$. Then the `averaged nontangential maximal function` of u (with aperture κ, scale θ, and integrability r) is the mapping[5]

$$\begin{gathered}\widetilde{\mathcal{N}}_{\kappa,\theta,r}u : \partial\Omega \longrightarrow [0,+\infty] \ \text{ defined at each } x \in \partial\Omega \text{ by}\\ (\widetilde{\mathcal{N}}_{\kappa,\theta,r}u)(x) := \left\| \Gamma_\kappa(x) \ni y \mapsto \Big(⨍_{B(y,\theta\delta_{\partial\Omega}(y))} |u|^r \, d\mathcal{L}^n \Big)^{1/r} \right\|_{L^\infty_y(\Gamma_\kappa(x),\mathcal{L}^n)}.\end{gathered} \tag{8.10.1}$$

A special case has been considered earlier in [148]. In what follows we analyze the properties of this averaged nontangential maximal operator. The first order of business is understanding how this relates to the "ordinary" nontangential maximal operator defined earlier in (8.2.1). As a prelude, define

$$\widetilde{u}(y) := \Big(⨍_{B(y,\theta\delta_{\partial\Omega}(y))} |u|^r \, d\mathcal{L}^n \Big)^{1/r} \quad \text{for every} \quad y \in \Omega, \tag{8.10.2}$$

and note that

$$\widetilde{u} \in \mathscr{C}^0(\Omega). \tag{8.10.3}$$

To justify (8.10.3), consider $y \in \Omega$ along with a sequence $\{y_j\}_{j\in\mathbb{N}} \subseteq \Omega$ convergent to y. Then the fact that the distance function $\delta_{\partial\Omega}$ is Lipschitz (hence continuous) implies

$$\begin{gathered}|u(x)|^r \mathbf{1}_{B(y_j,\theta\delta_{\partial\Omega}(y_j))}(x) \longrightarrow |u(x)|^r \mathbf{1}_{B(y,\theta\delta_{\partial\Omega}(y))}(x) \ \text{ as } j \to \infty\\ \text{for each fixed point } \ x \in \Omega \setminus \partial B(y, \theta\delta_{\partial\Omega}(y)).\end{gathered} \tag{8.10.4}$$

In concert, the pointwise $\mathcal{L}^n$-a.e. convergence proved in (8.10.4), the membership $u \in L^r_{\rm loc}(\Omega, \mathcal{L}^n)$, Lebesgue's Dominated Convergence Theorem, and the fact that $\lim\limits_{j\to\infty} \mathcal{L}^n\big(B(y_j, \theta\delta_{\partial\Omega}(y_j))\big) = \mathcal{L}^n\big(B(y, \theta\delta_{\partial\Omega}(y))\big)$ imply $\lim\limits_{j\to\infty} \widetilde{u}(y_j) = \widetilde{u}(y)$, proving (8.10.3).

It is then immediate from (8.10.1) and (8.10.3) that

$$\widetilde{\mathcal{N}}_{\kappa,\theta,r}u = \mathcal{N}_\kappa \widetilde{u} \ \text{ pointwise on } \partial\Omega. \tag{8.10.5}$$

As a consequence of (8.10.5), (8.10.3), and (8.2.26) we see that

$$\widetilde{\mathcal{N}}_{\kappa,\theta,r}u \ \text{ is lower-semicontinuous on } \ \partial\Omega. \tag{8.10.6}$$

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[5] Where the barred integral represents mean average, and L^∞_y indicates that the L^∞ norm is taken in the y variable.

In light of (8.10.3), we may also recast (8.10.1) as

$$(\widetilde{\mathcal{N}}_{\kappa,\theta,r}u)(x) = \sup_{y\in\Gamma_\kappa(x)} \left(⨍_{B(y,\theta\delta_{\partial\Omega}(y))} |u|^r \, \mathrm{d}\mathcal{L}^n \right)^{1/r} \quad \text{for each } x \in \partial\Omega. \tag{8.10.7}$$

In turn, (8.10.7), the assumption that $u \in L^r_{\mathrm{loc}}(\Omega, \mathcal{L}^n)$, and Hölder's inequality imply

$$\widetilde{\mathcal{N}}_{\kappa,\theta,s}u \leq \widetilde{\mathcal{N}}_{\kappa,\theta,r}u \text{ on } \partial\Omega, \text{ for each } s \in (0, r]. \tag{8.10.8}$$

It is also immediate from (8.10.7) that

$$\widetilde{\mathcal{N}}_{\kappa,\theta',r}u \leq C_{\theta,\theta'}\widetilde{\mathcal{N}}_{\kappa,\theta,r}u \quad \text{whenever } 0 < \theta' \leq \theta, \tag{8.10.9}$$

$$\text{and } \widetilde{\mathcal{N}}_{\kappa',\theta,r}u \leq \widetilde{\mathcal{N}}_{\kappa,\theta,r}u \quad \text{whenever } 0 < \kappa' \leq \kappa, \tag{8.10.10}$$

for some constant $C_{\theta,\theta'} \in (0, \infty)$, where the inequalities in (8.10.9)–(8.10.10) hold pointwise on $\partial\Omega$.

Next, we claim that

$$\widetilde{\mathcal{N}}_{\kappa,\theta,r}u \leq \mathcal{N}_{\widetilde{\kappa}}u \quad \text{pointwise on } \partial\Omega, \quad \text{where } \widetilde{\kappa} := \frac{\kappa + 2\theta}{1-\theta}. \tag{8.10.11}$$

To justify the inequality in (8.10.11), take an arbitrary $x \in \partial\Omega$ and some $y \in \Gamma_\kappa(x)$. Then for each $z \in B\big(y, \theta\delta_{\partial\Omega}(y)\big)$ we have $\delta_{\partial\Omega}(z) \geq (1-\theta)\delta_{\partial\Omega}(y)$ which, in turn, permits us to estimate

$$\begin{aligned} |z - x| &\leq |z - y| + |y - x| < \theta\delta_{\partial\Omega}(y) + (1+\kappa)\delta_{\partial\Omega}(y) \\ &= (1 + \theta + \kappa)\delta_{\partial\Omega}(y) < \frac{1+\theta+\kappa}{1-\theta}\delta_{\partial\Omega}(z) = (1+\widetilde{\kappa})\delta_{\partial\Omega}(z). \end{aligned} \tag{8.10.12}$$

Hence,

$$B\big(y, \theta\delta_{\partial\Omega}(y)\big) \subseteq \Gamma_{\widetilde{\kappa}}(x) \quad \text{for every } y \in \Gamma_\kappa(x) \text{ and } x \in \partial\Omega. \tag{8.10.13}$$

Thus, if $x \in \partial\Omega$ and $y \in \Gamma_\kappa(x)$, then for $\mathcal{L}^n$-a.e. point $z \in B\big(y, \theta\delta_{\partial\Omega}(y)\big)$ we have $|u(z)| \leq (\mathcal{N}_{\widetilde{\kappa}}u)(x)$ which, in concert with (8.10.1) and (8.2.1), establishes (8.10.11).

It is useful to observe that for subaveraging functions (cf. Definition 6.5.1 and (6.5.13)) the ordinary and averaged nontangential maximal operators are essentially equivalent. Indeed, it is apparent from (8.10.11) and Lemma 6.5.3 that

given any $\kappa \in (0, \infty)$, $\theta \in (0, 1)$, and $r \in (0, \infty)$ there exists a constant $C \in (1, \infty)$ such that for each subaveraging function u in Ω we have $C^{-1} \cdot \mathcal{N}_\kappa u \leq \widetilde{\mathcal{N}}_{\kappa,\theta,r}u \leq C \cdot \mathcal{N}_{\widetilde{\kappa}}u$ on $\partial\Omega$, where $\widetilde{\kappa} := (\kappa+2\theta)/(1-\theta)$. (8.10.14)

The next claim we make is that

for any given $\kappa \in (0, \infty)$, $\theta, \theta' \in (0, 1)$, and $r \in (0, \infty)$ there exists $C_{r,\theta,\theta'} \in (0, \infty)$ with the property that $\widetilde{\mathcal{N}}_{\kappa,\theta,r} u \le C_{r,\theta,\theta'} \cdot \widetilde{\mathcal{N}}_{\widetilde{\kappa},\theta',r} u$ at every point on $\partial\Omega$, where $\widetilde{\kappa} := (\kappa + 2\theta)/(1 - \theta)$. (8.10.15)

To prove (8.10.15), let $\kappa \in (0, \infty)$, $\theta, \theta' \in (0, 1)$ and $r \in (0, \infty)$ be arbitrary. Also, fix $x \in \partial\Omega$ and pick an arbitrary $y \in \Gamma_\kappa(x)$. Then there exists a family of points $z_j \in B\big(y, \theta\delta_{\partial\Omega}(y)\big)$, where $j \in \{1, \dots, N_{\theta,\theta'}\}$ with $N_{\theta,\theta'} \in \mathbb{N}$ depending only on θ, θ', n, such that

$$B\big(y, \theta\delta_{\partial\Omega}(y)\big) \subseteq \bigcup_{j=1}^{N_{\theta,\theta'}} B\big(z_j, (1-\theta)\theta'\delta_{\partial\Omega}(y)\big). \tag{8.10.16}$$

Since $\delta_{\partial\Omega}(z_j) \ge (1-\theta)\delta_{\partial\Omega}(y)$ for each $j \in \{1, \dots, N_{\theta,\theta'}\}$, this further implies

$$B\big(y, \theta\delta_{\partial\Omega}(y)\big) \subseteq \bigcup_{j=1}^{N_{\theta,\theta'}} B\big(z_j, \theta'\delta_{\partial\Omega}(z_j)\big). \tag{8.10.17}$$

Hence,

$$\int_{B(y,\theta\delta_{\partial\Omega}(y))} |u|^r \, d\mathcal{L}^n \le \sum_{j=1}^{N_{\theta,\theta'}} \int_{B(z_j,\theta'\delta_{\partial\Omega}(z_j))} |u|^r \, d\mathcal{L}^n. \tag{8.10.18}$$

Since for each j we also have $\delta_{\partial\Omega}(z_j) \le |z_j - y| + \delta_{\partial\Omega}(y) \le (1+\theta)\delta_{\partial\Omega}(y)$, from (8.10.18) we deduce that

$$\begin{aligned}
⨍_{B(y,\theta\delta_{\partial\Omega}(y))} |u|^r \, d\mathcal{L}^n &\le \frac{1}{\mathrm{vol}_n \cdot (\theta\delta_{\partial\Omega}(y))^n} \sum_{j=1}^{N_{\theta,\theta'}} \int_{B(z_j,\theta'\delta_{\partial\Omega}(z_j))} |u|^r \, d\mathcal{L}^n \\
&\le \frac{(\theta')^n(1+\theta)^n}{\theta^n} \sum_{j=1}^{N_{\theta,\theta'}} ⨍_{B(z_j,\theta'\delta_{\partial\Omega}(z_j))} |u|^r \, d\mathcal{L}^n \\
&\le N_{\theta,\theta'} \frac{(\theta')^n(1+\theta)^n}{\theta^n} \cdot \Big[\big(\widetilde{\mathcal{N}}_{\widetilde{\kappa},\theta',r} u\big)(x)\Big]^r,
\end{aligned} \tag{8.10.19}$$

where the last inequality in (8.10.19) is a consequence of having $z_j \in B\big(y, \theta\delta_{\partial\Omega}(y)\big)$ for each $j \in \{1, \dots, N_{\theta,\theta'}\}$, (8.10.13), and (8.10.7). Now (8.10.15) follows from (8.10.19) by taking the supremum over $y \in \Gamma_\kappa(x)$ and invoking (8.10.7).

The next proposition addresses the issue of the equivalence of L^p-norms of the averaged nontangential maximal function for different apertures and scales (compare to Proposition 8.4.1).

Proposition 8.10.1 *Assume that Ω is an open nonempty proper subset of $\mathbb{R}^n$ and consider a doubling measure σ on $\partial\Omega$. Also, fix an integrability exponent $p \in (0, \infty)$.*

Then for every apertures $\kappa_1, \kappa_2 \in (0, \infty)$*, every scales* $\theta_1, \theta_2 \in (0, 1)$*, and every integrability exponent* $r \in (0, \infty)$*, there exist two finite constants* $C_0, C_1 > 0$*, which depend only on* $\kappa_1, \kappa_2, \theta_1, \theta_2, p, r$ *and the doubling character of* σ*, such that*[6]

$$C_0 \big\| \widetilde{\mathcal{N}}_{\kappa_1,\theta_1,r} u \big\|_{L^p(\partial\Omega,\sigma)} \leq \big\| \widetilde{\mathcal{N}}_{\kappa_2,\theta_2,r} u \big\|_{L^p(\partial\Omega,\sigma)} \leq C_1 \big\| \widetilde{\mathcal{N}}_{\kappa_1,\theta_1,r} u \big\|_{L^p(\partial\Omega,\sigma)}, \tag{8.10.20}$$

for each function $u \in L^r_{\rm loc}(\Omega, \mathcal{L}^n)$*.*

Proof Apply (8.10.15) with $\kappa := \kappa_1$, $\theta := \theta_1$, and $\theta' := \theta_2$ to obtain that there exists some constant $c_1 := C_{r,\theta_1,\theta_2} \in (0, \infty)$ such that

$$\big\| \widetilde{\mathcal{N}}_{\kappa_1,\theta_1,r} u \big\|_{L^p(\partial\Omega,\sigma)} \leq c_1 \big\| \widetilde{\mathcal{N}}_{\widetilde{\kappa}_1,\theta_2,r} u \big\|_{L^p(\partial\Omega,\sigma)} \text{ where } \widetilde{\kappa}_1 := \frac{\kappa_1 + 2\theta_1}{1-\theta_1}. \tag{8.10.21}$$

Next, a combination of (8.10.5) and Proposition 8.4.1 proves that there exists some finite constant $c_2 := C(\widetilde{\kappa}_1, \kappa_2, p) > 0$ with the property that

$$\big\| \widetilde{\mathcal{N}}_{\widetilde{\kappa}_1,\theta_2,r} u \big\|_{L^p(\partial\Omega,\sigma)} \leq c_2 \big\| \widetilde{\mathcal{N}}_{\kappa_2,\theta_2,r} u \big\|_{L^p(\partial\Omega,\sigma)}. \tag{8.10.22}$$

From (8.10.21) and (8.10.22) the first inequality in (8.10.20) follows. The second inequality in (8.10.20) is immediate from what we proved so far by reversing the roles of κ_1, θ_1 and κ_2, θ_2. □

Concerning the relationship between the ordinary nontangential maximal operator and the averaged one we wish to note the following result, pointing to the fact that the former encodes more nuanced information about a given function than the latter.

Proposition 8.10.2 *Let* Ω *be an open nonempty proper subset of* $\mathbb{R}^n$ *and consider a doubling measure* σ *on* $\partial\Omega$*. Fix an aperture parameter* $\kappa \in (0, \infty)$*, a scale parameter* $\theta \in (0, 1)$*, along with two integrability exponents* $p, r \in (0, \infty)$*, and consider a Lebesgue measurable function* $u : \Omega \to \mathbb{C}$*. Then there exists a constant* $C \in (0, \infty)$ *independent of* u *such that*

$$\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma) \Rightarrow \begin{cases} \widetilde{\mathcal{N}}_{\kappa,\theta,r} u \in L^p(\partial\Omega, \sigma) \ \ \textit{and one has} \\ \|\widetilde{\mathcal{N}}_{\kappa,\theta,r} u\|_{L^p(\partial\Omega,\sigma)} \leq C \|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,\sigma)}. \end{cases} \tag{8.10.23}$$

Proof Suppose $\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma)$. Then Lemma 8.3.1 implies that $u \in L^\infty_{\rm loc}(\Omega, \mathcal{L}^n)$ (hence, in particular, $u \in L^r_{\rm loc}(\Omega, \mathcal{L}^n)$). Also, if $\widetilde{\kappa} := \frac{\kappa+2\theta}{1-\theta}$, then Proposition 8.4.1 ensures that $\mathcal{N}_{\widetilde{\kappa}} u \in L^p(\partial\Omega, \sigma)$ and $\|\mathcal{N}_{\widetilde{\kappa}} u\|_{L^p(\partial\Omega,\sigma)} \approx \|\mathcal{N}_\kappa u\|_{L^p(\partial\Omega,\sigma)}$, with proportionality constants independent of u. In turn, from this, (8.10.11), and (8.10.6), the desired conclusions follow. □

Suppose Ω is an open, nonempty, proper subset of $\mathbb{R}^n$ and let $\kappa > 0$, $\theta \in (0, 1)$, and $r \in (0, \infty)$ be arbitrary. Then for each $\varepsilon \in (0, \infty)$ the truncated averaged nontangential maximal function of $u \in L^r_{\rm loc}(\Omega, \mathcal{L}^n)$ is defined as

[6] Recall (8.10.6).

$$(\widetilde{\mathcal{N}}^{\varepsilon}_{\kappa,\theta,r}u)(x) := \sup_{y\in\Gamma_\kappa(x),\,\delta_{\partial\Omega}(y)<\varepsilon} \Big(\fint_{B(y,\theta\delta_{\partial\Omega}(y))} |u|^r \, d\mathcal{L}^n \Big)^{1/r} \tag{8.10.24}$$

for each $x \in \partial\Omega$. In particular, with the truncated version of the ordinary nontangential maximal function $\mathcal{N}^{\varepsilon}_{\kappa}$ defined as in (1.5.5), it follows that

$$\widetilde{\mathcal{N}}^{\varepsilon}_{\kappa,\theta,r}u = \mathcal{N}^{\varepsilon}_{\kappa}\widetilde{u} \ \text{ pointwise on } \ \partial\Omega. \tag{8.10.25}$$

and

$$\widetilde{\mathcal{N}}^{\varepsilon}_{\kappa,\theta,r}u \leq \widetilde{\mathcal{N}}_{\kappa,\theta,r}u \ \text{ pointwise on } \ \partial\Omega. \tag{8.10.26}$$

If $x \in \partial\Omega$, $y \in \Gamma_\kappa(x)$, and $\delta_{\partial\Omega}(y) < \varepsilon$, then for each $z \in B\big(y, \theta\delta_{\partial\Omega}(y)\big)$ we have $\delta_{\partial\Omega}(z) \leq |z-y| + \delta_{\partial\Omega}(y) < (1+\theta)\varepsilon$ which further entails $z \in \mathcal{O}_{(1+\theta)\varepsilon}$, the one-sided collar neighborhood of $\partial\Omega$ introduced in (6.1.4). Granted this, from (8.10.24) we conclude that

$$\widetilde{\mathcal{N}}^{\varepsilon}_{\kappa,\theta,r}u \leq \widetilde{\mathcal{N}}_{\kappa,\theta,r}\big(u \cdot \mathbf{1}_{\mathcal{O}_{(1+\theta)\varepsilon}}\big) \ \text{ on } \ \partial\Omega. \tag{8.10.27}$$

In addition, whenever we have $x \in \partial\Omega$, $y \in \Gamma_\kappa(x)$, and $z \in \mathcal{O}_\varepsilon \cap B\big(y, \theta\delta_{\partial\Omega}(y)\big)$, then we may estimate $\delta_{\partial\Omega}(y) \leq |y-z| + \delta_{\partial\Omega}(z) < \theta\delta_{\partial\Omega}(y) + \varepsilon$ which shows that $\delta_{\partial\Omega}(y) < \frac{\varepsilon}{1-\theta}$. Together with (8.10.24) this further implies

$$\widetilde{\mathcal{N}}_{\kappa,\theta,r}\big(u \cdot \mathbf{1}_{\mathcal{O}_\varepsilon}\big) \leq \widetilde{\mathcal{N}}^{\frac{\varepsilon}{1-\theta}}_{\kappa,\theta,r}u \ \text{ on } \ \partial\Omega. \tag{8.10.28}$$

We next prove an analogue of Proposition 8.6.10 for the averaged nontangential maximal function. In this vein, we wish to remark that, given Proposition 8.10.2, estimate (8.10.30) in the proposition below may be viewed as an improvement over (8.6.76).

Proposition 8.10.3 *Fix $n \in \mathbb{N}$ and suppose Ω is an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix some $r \in (0,\infty)$ along with $\kappa \in (0,\infty)$ and $\theta \in (0,1)$. Then there exists $C = C(\partial\Omega, n, \kappa, \theta, r) \in (0,\infty)$ which depends only on n, κ, θ, r, and the lower* ADR *constant of $\partial\Omega$ with the property that if*

$$\varepsilon \in \Big(0, \tfrac{\theta+1}{2\theta+1} \cdot \varepsilon_{\Omega,\kappa}\Big) \ \text{ with } \ \varepsilon_{\Omega,\kappa} := \frac{\operatorname{diam}(\partial\Omega)}{n(2+\sqrt{n})(3+2\kappa)} \in [0,+\infty], \tag{8.10.29}$$

then for each $u \in L^r_{\rm loc}(\Omega, \mathcal{L}^n)$ one has

$$\Big(\int_{\mathcal{O}_\varepsilon} |u|^r \, d\mathcal{L}^n \Big)^{1/r} \leq C \cdot \varepsilon^{1/r} \cdot \Big\| \widetilde{\mathcal{N}}^{(\frac{2\theta+1}{1+\theta})\varepsilon}_{\kappa,\theta,r} u \Big\|_{L^r(\partial\Omega,\sigma)}. \tag{8.10.30}$$

Proof Working with $|u|^r$ in place of u, there is no loss of generality in assuming that $r = 1$. Suppose this is the case and fix $\kappa \in (0,\infty)$ and $\theta \in (0,1)$. Also, let $\widetilde{u}$ be as in

(8.10.2) corresponding to $r := 1$. By Proposition 8.6.10, applied with $u := \widetilde{u}$, there exists some positive constant $C = C(\partial\Omega, \kappa)$ such that if $\varepsilon \in \big(0, \varepsilon_{\Omega,\kappa}\big)$ then

$$\int_{\mathcal{O}_\varepsilon} |\widetilde{u}\,| \, d\mathcal{L}^n \leq C \cdot \varepsilon \cdot \big\| \mathcal{N}_\kappa^\varepsilon \widetilde{u}\,\big\|_{L^1(\partial\Omega,\sigma)} = C \cdot \varepsilon \cdot \big\| \widetilde{\mathcal{N}}_{\kappa,\theta,1}^\varepsilon u \big\|_{L^1(\partial\Omega,\sigma)} \tag{8.10.31}$$

where the last equality is a consequence of (8.10.25). In addition, we may write

$$\begin{aligned} \int_{\mathcal{O}_\varepsilon} |\widetilde{u}\,| \, d\mathcal{L}^n &= \int_{\mathcal{O}_\varepsilon} \Big(⨍_{B(y,\theta\delta_{\partial\Omega}(y))} |u(z)| \, dz \Big) \, dy \\ &= \int_\Omega \int_\Omega \frac{1}{c_n (\theta \delta_{\partial\Omega}(y))^n} |u(z)| \, \mathbf{1}_{\mathcal{O}_\varepsilon}(y) \mathbf{1}_{B(y,\theta\delta_{\partial\Omega}(y))}(z) \, dy \, dz \\ &\geq \frac{(1-\theta)^n}{c_n \theta^n} \int_\Omega \int_\Omega \frac{1}{\delta_{\partial\Omega}(z)^n} |u(z)| \, \mathbf{1}_{\mathcal{O}_\varepsilon}(y) \mathbf{1}_{B(y,\theta\delta_{\partial\Omega}(y))}(z) \, dy \, dz \end{aligned} \tag{8.10.32}$$

where for the last inequality in (8.10.32) we used the fact that whenever $y \in \Omega$ and $z \in B\big(y, \theta\delta_{\partial\Omega}(y)\big)$ we have $(1-\theta)\delta_{\partial\Omega}(y) \leq \delta_{\partial\Omega}(z)$. Furthermore, if we now take

$$z \in \mathcal{O}_{(\frac{\theta+1}{2\theta+1})\varepsilon} \quad \text{and} \quad y \in B\big(z, \big(\tfrac{\theta}{\theta+1}\big)\delta_{\partial\Omega}(z)\big), \tag{8.10.33}$$

then $\delta_{\partial\Omega}(y) > \big(1 - \frac{\theta}{\theta+1}\big)\delta_{\partial\Omega}(z) = \frac{1}{\theta+1}\delta_{\partial\Omega}(z)$ hence

$$|y - z| < \tfrac{\theta}{\theta+1} \delta_{\partial\Omega}(z) < \theta \delta_{\partial\Omega}(y) \tag{8.10.34}$$

and also

$$\delta_{\partial\Omega}(y) \leq |y-z| + \delta_{\partial\Omega}(z) \leq \Big(\tfrac{\theta}{\theta+1} + 1\Big)\delta_{\partial\Omega}(z) = \Big(\tfrac{2\theta+1}{\theta+1}\Big)\delta_{\partial\Omega}(z) < \varepsilon. \tag{8.10.35}$$

The fact that (8.10.33) implies (8.10.34) and (8.10.35) may be rephrased as the estimate

$$\mathbf{1}_{\mathcal{O}_{(\frac{\theta+1}{2\theta+1})\varepsilon}}(z) \cdot \mathbf{1}_{B(z,(\frac{\theta}{\theta+1})\delta_{\partial\Omega}(z))}(y) \leq \mathbf{1}_{\mathcal{O}_\varepsilon}(y) \cdot \mathbf{1}_{B(y,\theta\delta_{\partial\Omega}(y))}(z) \tag{8.10.36}$$

for all $y, z \in \Omega$. Invoking (8.10.36) to bound from below the last integral in (8.10.32), we may then write

$$\begin{aligned} \int_{\mathcal{O}_\varepsilon} |\widetilde{u}\,| \, d\mathcal{L}^n &\geq \frac{(1-\theta)^n}{c_n \theta^n} \int_\Omega \int_\Omega \frac{1}{\delta_{\partial\Omega}(z)^n} |u(z)| \, \mathbf{1}_{\mathcal{O}_{(\frac{\theta+1}{2\theta+1})\varepsilon}}(z) \times \\ &\qquad \times \mathbf{1}_{B(z,(\frac{\theta}{\theta+1})\delta_{\partial\Omega}(z))}(y) \, dy \, dz \\ &= \frac{(1-\theta)^n}{(1+\theta)^n} \int_{\mathcal{O}_{(\frac{\theta+1}{2\theta+1})\varepsilon}} |u(z)| \, dz. \end{aligned} \tag{8.10.37}$$

In concert, (8.10.37) and (8.10.31) prove that for every $\varepsilon \in \big(0, \varepsilon_{\Omega,\kappa}\big)$ we have

$$\int_{\mathcal{O}_{(\frac{\theta+1}{2\theta+1})\varepsilon}} |u(z)|\,\mathrm{d}z \leq C\frac{(1+\theta)^n}{(1-\theta)^n}\cdot\varepsilon\cdot\big\|\widetilde{\mathcal{N}}^{\varepsilon}_{\kappa,\theta,1}u\big\|_{L^1(\partial\Omega,\sigma)}. \tag{8.10.38}$$

If we now set $\widetilde{\varepsilon} := \big(\frac{\theta+1}{2\theta+1}\big)\varepsilon$ and $C(\partial\Omega, n, \kappa, \theta) := C\frac{(1+\theta)^n}{(1-\theta)^n}\cdot\big(\frac{2\theta+1}{\theta+1}\big)$, then (8.10.38) becomes the version (8.10.30) corresponding to $r = 1$ and with $\widetilde{\varepsilon}$ in place of ε. $\square$

Our next proposition elaborates on the relationship between the nontangential boundary trace of a given function u and the nontangential boundary trace of the function $\widetilde{u}$, associated with u as in (8.10.2).

Proposition 8.10.4 *Let Ω be an open set in $\mathbb{R}^n$ and let $r \in (0,\infty)$ and $\kappa \in (0,\infty)$ be arbitrary. Consider some $\kappa' \in (0,\kappa)$ and suppose $u \in L^r_{\mathrm{loc}}(\Omega, \mathcal{L}^n)$ and $x \in A_{\kappa'}(\partial\Omega)$ are such that the nontangential limit $\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x)$ exists. Then, if $\theta \in (0,1)$ is small enough so that $\frac{\kappa'+2\theta}{1-\theta} < \kappa$ and $\widetilde{u}$ is defined as in (8.10.2) for this θ, then the nontangential limit $\big(\widetilde{u}\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}\big)(x)$ exists and equals $\big|\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x)\big|$.*

Proof Abbreviate $a := \big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x) \in \mathbb{C}$. Then Definition 8.9.1 ensures that for every $\varepsilon > 0$ there exists $\delta > 0$ and a Lebesgue measurable set $N \subseteq \Gamma_\kappa(x)$ with $\mathcal{L}^n(N) = 0$ such that

$$\big||u(z)|^r - |a|^r\big| < \varepsilon \;\text{ if }\; z \in \big(\Gamma_\kappa(x)\setminus N\big)\cap\mathcal{O}_\delta. \tag{8.10.39}$$

To prove that $\big(\widetilde{u}\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}\big)(x)$ exists, it suffices to show that there exists $\lambda > 0$ such that

$$\big|\widetilde{u}(y)^r - |a|^r\big| < \varepsilon \;\text{ for every }\; y \in \big(\Gamma_{\kappa'}(x)\setminus N\big)\cap\mathcal{O}_\lambda. \tag{8.10.40}$$

Note that

$$\text{if }\; y \in \mathcal{O}_{\delta/(1+\theta)} \text{ and } z \in B\big(y, \theta\delta_{\partial\Omega}(y)\big) \text{ then } z \in \mathcal{O}_\delta, \tag{8.10.41}$$

since $\delta_{\partial\Omega}(z) \leq |y-z| + \delta_{\partial\Omega}(y) < (\theta+1)\delta_{\partial\Omega}(y) < \delta$. Moreover, if $\widetilde{\kappa} := \frac{\kappa'+2\theta}{1-\theta}$, then $\widetilde{\kappa} < \kappa$, hence $\Gamma_{\widetilde{\kappa}}(x) \subseteq \Gamma_\kappa(x)$. In addition, from (8.10.13) (applied with $\kappa := \kappa'$ and the current $\widetilde{\kappa}$), we know that

$$\text{if }\; y \in \Gamma_{\kappa'}(x) \text{ then } B\big(y, \theta\delta_{\partial\Omega}(y)\big) \subseteq \Gamma_{\widetilde{\kappa}}(x) \subseteq \Gamma_\kappa(x). \tag{8.10.42}$$

From (8.10.42) and (8.10.41) it follows that

$$\begin{aligned} y \in \Gamma_{\kappa'}(x)\cap\mathcal{O}_{\delta/(1+\theta)} \;\text{ and }\; z \in B(y, \theta\delta_{\partial\Omega}(y))\setminus N \\ \Longrightarrow z \in (\Gamma_\kappa(x)\setminus N)\cap\mathcal{O}_\delta. \end{aligned} \tag{8.10.43}$$

As such, taking $\lambda := \frac{\delta}{1+\theta}$, for each $y \in \big(\Gamma_{\kappa'}(x)\setminus N\big)\cap\mathcal{O}_\lambda$ we may combine (8.10.43) and (8.10.39) to conclude that

$$|a|^r - \varepsilon \leq \fint_{B(y,\theta\delta_{\partial\Omega}(y))} |u(z)|^r \, dz \leq |a|^r + \varepsilon. \tag{8.10.44}$$

Now (8.10.40) follows from (8.10.44) by recalling (8.10.2). □

In turn, Proposition 8.10.4 is one of the main ingredients in the proof of the estimate given in the corollary below.

Corollary 8.10.5 *Let Ω be an open set in $\mathbb{R}^n$ and pick some arbitrary $\kappa \in (0, \infty)$. Also, suppose $u \in L^r_{\rm loc}(\Omega, \mathcal{L}^n)$ for some $r \in (0, \infty)$ and assume the nontangential boundary trace $\big(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(x)$ exists at some point $x \in \partial_{\rm nta}\Omega$. Then for any $\varepsilon > 0$ and any $\theta \in \big(0, \kappa/(\kappa+2)\big)$ one has*

$$\Big|\big(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(x)\Big| \leq \big(\widetilde{\mathcal{N}}^{\varepsilon}_{\kappa,\theta,r} u\big)(x) \leq \big(\widetilde{\mathcal{N}}_{\kappa,\theta,r} u\big)(x). \tag{8.10.45}$$

Proof Since $\theta < \kappa/(\kappa+2)$ it follows that $2\theta/(1-\theta) < \kappa$, hence there exists some $\kappa' \in (0, \kappa)$ such that $\frac{\kappa'+2\theta}{1-\theta} < \kappa$. Combining Proposition 8.10.4, (8.8.47), (8.9.8), (8.2.25), (8.10.25), and (8.10.26) we obtain

$$\begin{aligned}\Big|\big(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(x)\Big| &= \Big|\big(\widetilde{u}\big|^{\kappa'-{\rm n.t.}}_{\partial\Omega}\big)(x)\Big| \leq (\mathcal{N}^{\varepsilon}_{\kappa'}\widetilde{u})(x) \leq (\mathcal{N}^{\varepsilon}_{\kappa}\widetilde{u})(x) \\ &= (\widetilde{\mathcal{N}}^{\varepsilon}_{\kappa,\theta,r} u)(x) \leq \big(\widetilde{\mathcal{N}}_{\kappa,\theta,r} u\big)(x),\end{aligned} \tag{8.10.46}$$

proving (8.10.45). □

Here is a result in the spirit of Proposition 8.9.5, for the averaged nontangential maximal operator.

Proposition 8.10.6 *Let $\Omega \subseteq \mathbb{R}^n$ be an open nonempty proper subset of $\mathbb{R}^n$ such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Also, fix $r \in (0, \infty)$, $\kappa \in (0, \infty)$, and $\theta \in (0, 1)$. Suppose $u \in L^r_{\rm loc}(\Omega, \mathcal{L}^n)$ is such that*

$$\big(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\big)(x) \ \ \textit{exists for } \sigma\textit{-a.e. } \ x \in \partial_{\rm nta}\Omega \tag{8.10.47}$$

and

$$\widetilde{\mathcal{N}}_{\kappa,\theta,r} u \in L^p(\partial\Omega, \sigma) \ \ \textit{for some} \ \ p \in (0, \infty). \tag{8.10.48}$$

Then $u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ is a σ-measurable function on $\partial_{\rm nta}\Omega$ and, in fact, it belongs to $L^p(\partial_{\rm nta}\Omega, \sigma)$. In addition, for any $\theta_* \in (0, 1)$ which is small enough so that $\frac{2\theta_*}{1-\theta_*} < \kappa$, and any $\kappa_* \in (0, \infty)$, one has

$$\widetilde{\mathcal{N}}^{\varepsilon}_{\kappa_*,\theta_*,r} u \longrightarrow \begin{cases} \Big|u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\Big| & on \ \ \partial_{\rm nta}\Omega, \\ 0 & on \ \ \partial\Omega \setminus \partial_{\rm nta}\Omega \end{cases} \quad in\ L^p(\partial\Omega, \sigma)\ as\ \varepsilon \to 0^+. \tag{8.10.49}$$

Proof That $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ is a σ-measurable function on $\partial_{\mathrm{nta}}\Omega$ has been already noted in (8.9.44). Choose some $\theta' \in \big(0, \kappa/(\kappa+2)\big)$. Since Proposition 8.10.1 ensures that $\widetilde{\mathcal{N}}_{\kappa,\theta',r}u \in L^p(\partial\Omega,\sigma)$, based on the aforementioned measurability property and Corollary 8.10.5 we then conclude that $u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ belongs to the space $L^p(\partial_{\mathrm{nta}}\Omega,\sigma)$. To proceed, let $\kappa_* \in (0,\infty)$ and $\theta_* \in (0,1)$ be such that $\frac{2\theta_*}{1-\theta_*} < \kappa$, then pick

$$\kappa' \in (0,\kappa) \;\text{ with the property that }\; \frac{\kappa' + 2\theta_*}{1-\theta_*} < \kappa. \tag{8.10.50}$$

The assumptions on u combined with (8.10.50) and Proposition 8.10.4 imply that

$$\widetilde{u}^*\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}} \text{ exists and equals } \Big|u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big| \text{ at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega, \tag{8.10.51}$$

where $\widetilde{u}^*$ is defined as in (8.10.2) with θ_* in place of θ. Also, from Proposition 8.10.1 and (8.10.48) it follows that $\widetilde{\mathcal{N}}_{\kappa',\theta_*,r}u \in L^p(\partial\Omega,\sigma)$, which in concert with (8.10.5) yields

$$\mathcal{N}_{\kappa'}\widetilde{u}^* \in L^p(\partial\Omega,\sigma). \tag{8.10.52}$$

Together, (8.10.51), (8.10.52), and Proposition 8.9.8 imply that

$$\begin{aligned}&\mathcal{N}_{\kappa_*}\widetilde{u}^* \text{ belongs to } L^p(\partial\Omega,\sigma) \text{ and the nontangential limit } \widetilde{u}^*\big|_{\partial\Omega}^{\kappa_*-\mathrm{n.t.}}\\ &\text{exists and happens to be equal } \Big|u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big| \text{ at } \sigma\text{-a.e. point on } \partial_{\mathrm{nta}}\Omega.\end{aligned} \tag{8.10.53}$$

Then invoking Proposition 8.9.5 it follows that the function $w : \partial\Omega \to \mathbb{R}$ defined (σ-a.e.) as

$$w := \begin{cases} \Big|u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big| & \text{on } \partial_{\mathrm{nta}}\Omega,\\ 0 & \text{on } \partial\Omega\setminus\partial_{\mathrm{nta}}\Omega,\end{cases} \tag{8.10.54}$$

belongs to $L^p(\partial\Omega,\sigma)$ and

$$\mathcal{N}^{\varepsilon}_{\kappa_*}\widetilde{u}^* \longrightarrow |w| \;\text{ in }\; L^p(\partial\Omega,\sigma) \;\text{ as }\; \varepsilon \to 0^+. \tag{8.10.55}$$

Since $\mathcal{N}^{\varepsilon}_{\kappa_*}\widetilde{u}^* = \widetilde{\mathcal{N}}^{\varepsilon}_{\kappa_*,\theta_*,r}u$ (as noted in (8.10.5)), the claim made in (8.10.49) now follows from (8.10.54)–(8.10.55). □

In view of Proposition 8.10.2, our next result may be regarded as an improvement over the version of (8.6.51) corresponding to $E := \Omega$.

Proposition 8.10.7 *Fix $n \in \mathbb{N}$ with $n \geq 2$. Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary and define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Also assume that either $\sigma(\partial\Omega) = +\infty$, or $\mathcal{L}^n(\Omega) < +\infty$ and $\sigma(\partial\Omega) < +\infty$. Finally, fix some $\kappa \in (0,\infty)$, and pick $\theta \in (0,1)$ and $p \in (0,\infty)$ arbitrary.*

Then there exists some constant $C_0 \in (0,\infty)$ which depends only on κ, θ, p, n, and the lower ADR *constant of $\partial\Omega$ such that*[7]

[7] The fact that $n \geq 2$ ensures that the interval $\big[\frac{np}{n-1},\infty\big)$ is not degenerate.

$$\|u\|_{L^{\frac{np}{n-1}}(\Omega,\mathcal{L}^n)} \leq C_0 \big\|\widetilde{\mathcal{N}}_{\kappa,\theta,r}u\big\|_{L^p(\partial\Omega,\sigma)} \quad \textit{if} \;\; r \in \big[\tfrac{np}{n-1},\infty\big), \tag{8.10.56}$$

holds for every $u \in L^r_{\rm loc}(\Omega,\mathcal{L}^n)$.

Proof Fix $u \in L^r_{\rm loc}(\Omega,\mathcal{L}^n)$ with $r \in \big[\frac{np}{n-1},\infty\big)$. Then

$$\begin{aligned}
&\|u\|^{np/(n-1)}_{L^{np/(n-1)}(\Omega,\mathcal{L}^n)} \\
&\quad = \int_\Omega |u(z)|^{np/(n-1)}\,{\rm d}z \\
&\quad = \frac{(1+\theta)^n}{c_n\theta^n}\int_\Omega |u(z)|^{np/(n-1)}\delta_{\partial\Omega}(z)^{-n}\Big(\int_\Omega \mathbf{1}_{B\left(z,\frac{\theta}{\theta+1}\delta_{\partial\Omega}(z)\right)}(y)\,{\rm d}y\Big)\,{\rm d}z \\
&\quad \leq \frac{(1+\theta)^n}{c_n\theta^n(1-\theta)^n}\int_\Omega\int_\Omega \delta_{\partial\Omega}(y)^{-n}|u(z)|^{np/(n-1)}\mathbf{1}_{B(y,\theta\delta_{\partial\Omega}(y))}(z)\,{\rm d}z\,{\rm d}y \\
&\quad = \frac{(1+\theta)^n}{(1-\theta)^n}\int_\Omega \Big(⨍_{B(y,\theta\delta_{\partial\Omega}(y))} |u|^{np/(n-1)}\,{\rm d}\mathcal{L}^n\Big)\,{\rm d}y \\
&\quad \leq \frac{(1+\theta)^n}{(1-\theta)^n}\int_\Omega \Big(⨍_{B(y,\theta\delta_{\partial\Omega}(y))} |u|^{r}\,{\rm d}\mathcal{L}^n\Big)^{\frac{np}{r(n-1)}}\,{\rm d}y \\
&\quad = \frac{(1+\theta)^n}{(1-\theta)^n}\int_\Omega |\widetilde{u}\,|^{np/(n-1)}\,{\rm d}\mathcal{L}^n \leq \frac{(1+\theta)^n}{(1-\theta)^n}\big(C_{\rm geo}\big)^{np/(n-1)}\big\|\mathcal{N}_\kappa\widetilde{u}\big\|^{np/(n-1)}_{L^p(\partial\Omega,\sigma)} \\
&\quad = \frac{(1+\theta)^n}{(1-\theta)^n}\big(C_{\rm geo}\big)^{np/(n-1)}\big\|\widetilde{\mathcal{N}}_{\kappa,\theta,r}u\big\|^{np/(n-1)}_{L^p(\partial\Omega,\sigma)}.
\end{aligned}\tag{8.10.57}$$

Above, the first inequality uses the fact that

$$(1-\theta)\delta_{\partial\Omega}(y) \leq \delta_{\partial\Omega}(z) \text{ for each } y \in \Omega \;\text{ and }\; z \in B\big(y,\theta\delta_{\partial\Omega}(y)\big) \tag{8.10.58}$$

and that

$$\mathbf{1}_{B\left(z,\frac{\theta}{\theta+1}\delta_{\partial\Omega}(z)\right)}(y) \leq \mathbf{1}_{B(y,\theta\delta_{\partial\Omega}(y))}(z) \text{ for all } y,z \in \Omega. \tag{8.10.59}$$

The former claim is clear while the latter is justified by writing

$$\begin{aligned}
z \in \Omega \;\text{ and }\; y \in B\big(z,\tfrac{\theta}{\theta+1}\delta_{\partial\Omega}(z)\big)& \\
&\Longrightarrow \delta_{\partial\Omega}(y) \geq \Big(1-\tfrac{\theta}{\theta+1}\Big)\delta_{\partial\Omega}(z) = \tfrac{1}{\theta+1}\delta_{\partial\Omega}(z) \\
&\Longrightarrow |y-z| < \tfrac{\theta}{\theta+1}\delta_{\partial\Omega}(z) < \theta\delta_{\partial\Omega}(y) \\
&\Longrightarrow y \in \Omega \;\text{ and }\; z \in B\big(y,\theta\delta_{\partial\Omega}(y)\big).
\end{aligned}\tag{8.10.60}$$

Going further, in the second inequality in (8.10.57) we have made use of Hölder's inequality with exponent $r(n-1)/(np) \in [1, \infty)$, while the subsequent equality is based in the definition of $\widetilde{u}$ from (8.10.2). The next inequality comes from Proposition 8.6.3 (cf. (8.6.51) with $u := \widetilde{u}$ and $E := \Omega$, bearing in mind (8.10.3)). Finally, the last equality in (8.10.57) is implied by (8.10.5). This proves (8.10.57) which, in turn, establishes (8.10.56). □

We continue by presenting an off-diagonal Carleson measure estimate of reverse Hölder type for the averaged nontangential maximal operator, in the spirit of Proposition 8.6.3.

Proposition 8.10.8 *Fix $n \in \mathbb{N}$ with $n \geq 2$. Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary and define $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$. Also assume that $\mathcal{L}^n(\Omega) = +\infty$ and $\sigma(\partial\Omega) < +\infty$. Finally, fix $\kappa \in (0, \infty)$, and pick $\theta \in (0, 1)$, $\varepsilon \in (0, \infty)$, and $p \in (0, \infty)$ arbitrary.*

Then there exists some constant $C_\varepsilon \in (0, \infty)$ which depends on ε as well as κ, θ, p, n, the lower ADR *constant of $\partial\Omega$, and $\sigma(\partial\Omega)$, such that*[8]

$$\|u\|_{L^{\frac{np}{n-1}}(\mathcal{O}_\varepsilon, \mathcal{L}^n)} \leq C_\varepsilon \big\| \widetilde{\mathcal{N}}_{\kappa,\theta,r} u \big\|_{L^p(\partial\Omega,\sigma)} \ \ \textit{if} \ \ r \in \big[\tfrac{np}{n-1}, \infty\big), \tag{8.10.61}$$

holds for every function $u \in L^r_{\mathrm{loc}}(\Omega, \mathcal{L}^n)$.

Proof The proof is similar to that of Proposition 8.10.7. Specifically, having fixed $u \in L^r_{\mathrm{loc}}(\Omega, \mathcal{L}^n)$ with $r \in \big[\frac{np}{n-1}, \infty\big)$, write

$$\begin{aligned}
&\|u\|^{np/(n-1)}_{L^{np/(n-1)}(\mathcal{O}_\varepsilon,\mathcal{L}^n)} \\
&= \int_{\mathcal{O}_\varepsilon} |u(z)|^{np/(n-1)}\,\mathrm{d}z = \int_\Omega |u(z)|^{np/(n-1)} \mathbf{1}_{\mathcal{O}_\varepsilon}(z)\,\mathrm{d}z \\
&= \frac{(1+\theta)^n}{c_n\theta^n} \int_\Omega \frac{|u(z)|^{np/(n-1)}}{\delta_{\partial\Omega}(z)^n} \mathbf{1}_{\mathcal{O}_\varepsilon}(z) \Big(\int_\Omega \mathbf{1}_{B\left(z, \frac{\theta}{\theta+1}\delta_{\partial\Omega}(z)\right)}(y)\,\mathrm{d}y \Big)\,\mathrm{d}z \\
&\leq \frac{(1+\theta)^n}{c_n\theta^n(1-\theta)^n} \int_\Omega \int_\Omega \frac{|u(z)|^{np/(n-1)}}{\delta_{\partial\Omega}(y)^n} \mathbf{1}_{\mathcal{O}_{(\frac{2\theta+1}{\theta+1})\varepsilon}}(y) \mathbf{1}_{B(y,\theta\delta_{\partial\Omega}(y))}(z)\,\mathrm{d}z\,\mathrm{d}y \\
&= \frac{(1+\theta)^n}{(1-\theta)^n} \int_{\mathcal{O}_{(\frac{2\theta+1}{\theta+1})\varepsilon}} \Big(\fint_{B(y,\theta\delta_{\partial\Omega}(y))} |u|^{np/(n-1)}\,\mathrm{d}\mathcal{L}^n \Big)\,\mathrm{d}y \\
&\leq \frac{(1+\theta)^n}{(1-\theta)^n} \int_{\mathcal{O}_{(\frac{2\theta+1}{\theta+1})\varepsilon}} \Big(\fint_{B(y,\theta\delta_{\partial\Omega}(y))} |u|^r\,\mathrm{d}\mathcal{L}^n \Big)^{\frac{np}{r(n-1)}}\,\mathrm{d}y \\
&= \frac{(1+\theta)^n}{(1-\theta)^n} \int_{\mathcal{O}_{(\frac{2\theta+1}{\theta+1})\varepsilon}} |\widetilde{u}|^{np/(n-1)}\,\mathrm{d}\mathcal{L}^n
\end{aligned}$$

[8] The fact that $n \geq 2$ ensures that the interval $\big[\frac{np}{n-1}, \infty\big)$ is not degenerate.

$$\leq \frac{(1+\theta)^n}{(1-\theta)^n}\big(C_\varepsilon\big)^{np/(n-1)}\big\|\mathcal{N}_\kappa\widetilde{u}\big\|^{np/(n-1)}_{L^p(\partial\Omega,\sigma)}$$
$$= \frac{(1+\theta)^n}{(1-\theta)^n}\big(C_\varepsilon\big)^{np/(n-1)}\big\|\widetilde{\mathcal{N}}_{\kappa,\theta,r}u\big\|^{np/(n-1)}_{L^p(\partial\Omega,\sigma)}. \tag{8.10.62}$$

Above, the first inequality uses (8.10.58) and the fact that, for all $y, z \in \Omega$,

$$\mathbf{1}_{\mathcal{O}_\varepsilon}(z)\cdot\mathbf{1}_{B(z,(\frac{\theta}{\theta+1})\delta_{\partial\Omega}(z))}(y) \leq \mathbf{1}_{\mathcal{O}_{(\frac{2\theta+1}{\theta+1})\varepsilon}}(y)\cdot\mathbf{1}_{B(y,\theta\delta_{\partial\Omega}(y))}(z), \tag{8.10.63}$$

itself a consequence of (8.10.36) written with ε replaced by $(\frac{2\theta+1}{\theta+1})\varepsilon$. The second inequality in (8.10.62) uses Hölder's inequality with the exponent $r(n-1)/(np) \in [1,\infty)$, whereas the subsequent equality follows from (8.10.2). The next inequality is derived from (8.6.51), presently used with $u := \widetilde{u}$ and with $E := \mathcal{O}_{(\frac{2\theta+1}{\theta+1})\varepsilon}$ (keeping in mind (8.10.3)). For these choices, the constant $C_\# \in (0,\infty)$ in the last line of (8.6.49) becomes

$$C_\varepsilon := \max\Big\{C_{\rm geo}, \big[\mathcal{L}^n\big(\mathcal{O}_{(\frac{2\theta+1}{\theta+1})\varepsilon}\big)\big]^{\frac{n-1}{np}}\cdot\sigma(\partial\Omega)^{-\frac{1}{p}}\Big\}. \tag{8.10.64}$$

Finally, the last equality in (8.10.62) is implied by (8.10.5). This proves (8.10.62) which, in turn, concludes the justification of (8.10.61). □

The reader is reminded that the notation $L^p_{\rm bdd}(\Omega,\mathcal{L}^n)$ has been introduced in (4.2.4).

Corollary 8.10.9 *Fix $n \in \mathbb{N}$ with $n \geq 2$. Let Ω be an open nonempty proper subset of $\mathbb{R}^n$ with a lower Ahlfors regular boundary and define $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix some $\kappa \in (0,\infty)$, $\theta \in (0,1)$, $r \in (0,\infty)$ arbitrary, and pick some*[9] *$p \in \big(0, r(n-1)/n\big]$. Then*

$$u \in L^r_{\rm loc}(\Omega,\mathcal{L}^n) \text{ and } \widetilde{\mathcal{N}}_{\kappa,\theta,r}u \in L^p(\partial\Omega,\sigma) \Longrightarrow u \in L^{np/(n-1)}_{\rm bdd}(\Omega,\mathcal{L}^n). \tag{8.10.65}$$

Proof In the case when $\mathcal{L}^n(\Omega) = +\infty$ and $\sigma(\partial\Omega) < +\infty$ the implication claimed in (8.10.65) is a consequence of (8.10.61). The remaining situations are covered by Proposition 8.10.7 in which scenario (8.10.56) yields an even stronger conclusion than in (8.10.65). □

[9] The fact that $n \geq 2$ ensures that the interval $\big(0, r(n-1)/n\big]$ is not degenerate.

Chapter 9
Proofs of Main Results Pertaining to Divergence Theorem

This chapter is reserved for presenting the proofs of the main results formulated in Sects. 1.1–1.11 in relation to our new brand of Divergence Theorems and related topics.

9.1 Proofs of Theorems 1.2.1 and 1.3.1 and Corollaries 1.2.2, 1.2.4, and 1.3.2

We first present the proof of Theorem 1.3.1 and subsequently deduce Theorem 1.2.1 from the latter result. The proofs of Corollaries 1.2.2, 1.2.4, and 1.3.2 are then presented in the last part of this section.

Proof of Theorem 1.3.1 We begin with a series of preliminary remarks. First, the fact that in the current setting the set Ω has locally finite perimeter may be seen from (5.9.15). Also, that for each $\kappa' > 0$ the function $\vec{F}\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}$ exists σ-a.e. on $\partial_{\mathrm{nta}}\Omega$ and is actually independent of κ' follows from assumptions and Corollary 8.9.9. In particular, there is no ambiguity in dropping the dependence on the parameter κ' and henceforth simply denoting said function by $\vec{F}\big|_{\partial\Omega}^{\mathrm{n.t.}}$. For ease of notation, let us actually abbreviate

$$\vec{f} := \vec{F}\big|_{\partial\Omega}^{\mathrm{n.t.}} \ \text{ at } \sigma-\text{a.e. point on } \ \partial_{\mathrm{nta}}\Omega. \tag{9.1.1}$$

Thanks to the assumptions on $\vec{F}$, (8.9.44), and (8.9.8), this vector-valued function satisfies

$$\vec{f} \in \big[L^1_{\mathrm{loc}}(\partial_{\mathrm{nta}}\Omega, \sigma)\big]^n \ \text{ and } \ |\vec{f}\,| \leq \mathcal{N}_\kappa \vec{F} \ \text{ at } \sigma-\text{a.e. point on } \ \partial_{\mathrm{nta}}\Omega. \tag{9.1.2}$$

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Moreover, since Proposition 8.8.6 ensures that

$$\partial_*\Omega \subseteq \partial_{\text{nta}}\Omega \cup \big(\partial_*\Omega \setminus \partial_{\text{nta}}\Omega\big) \text{ and } \sigma\big(\partial_*\Omega \setminus \partial_{\text{nta}}\Omega\big) = 0, \tag{9.1.3}$$

it follows that

$$\begin{aligned}&\vec{f}, \text{ originally defined as in (9.1.1), may be canonically viewed as}\\ &\text{a } \mathbb{C}^n\text{-valued function defined } \sigma\text{-a.e. on } \partial_*\Omega, \text{ in which scenario}\\ &\vec{f} \text{ belongs to the space } \big[L^1_{\text{loc}}(\partial_*\Omega, \sigma)\big]^n.\end{aligned} \tag{9.1.4}$$

Let us also note that the assumptions on $\vec{F}$ and Lemma 8.3.1 imply

$$\vec{F} \in \big[L^\infty_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^n. \tag{9.1.5}$$

We continue by introducing notation which will facilitate the subsequent discussion. Specifically, upon recalling that $\text{CBM}(\Omega)$ stands for the collection of all complex Borel measures on Ω, define

$$\mathscr{F}(\Omega) := \Big\{\vec{G} \in \big[L^1_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^n : \operatorname{div}\vec{G} \in \text{CBM}(\Omega)\Big\}, \tag{9.1.6}$$

and, for each vector field $\vec{G} \in \mathscr{F}(\Omega)$, abbreviate

$$\mu_{\vec{G}} := \operatorname{div}\vec{G}. \tag{9.1.7}$$

It is then clear that

$$\begin{aligned}&\mathscr{F}(\Omega) \text{ is a complex vector space and the mapping}\\ &\mathscr{F}(\Omega) \ni \vec{G} \longmapsto \mu_{\vec{G}} \in \text{CBM}(\Omega) \text{ is (complex) linear.}\end{aligned} \tag{9.1.8}$$

In our current setting, since $\vec{F} \in \mathscr{F}(\Omega)$ we may use the Polar Decomposition Theorem to write

$$\begin{aligned}&d\mu_{\vec{F}} = h\, d|\mu_{\vec{F}}|, \text{ where } |\mu_{\vec{F}}| \text{ is a finite positive Borel measure}\\ &\text{on } \Omega \text{ (the total variation of } \mu_{\vec{F}}), \text{ and } h \text{ is a } |\mu_{\vec{F}}| - \text{measurable}\\ &\text{complex-valued function defined in } \Omega, \text{ with the property that}\\ &|h(x)| = 1 \text{ for each } x \in \Omega.\end{aligned} \tag{9.1.9}$$

We divide the remainder of the proof into two steps, starting with the following:

Step I. *Retain the initial background geometric measure theoretic assumptions on Ω made in the statement of Theorem 1.3.1. Specifically, suppose Ω is an open nonempty proper subset of $\mathbb{R}^n$ (with $n \in \mathbb{N}$ arbitrary) with a lower Ahlfors regular boundary and such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Also, pick $\vec{F} : \Omega \to \mathbb{C}^n$*

with $\mathcal{L}^n$-measurable components and fix $\kappa \in (0, \infty)$. As far as this vector field is concerned, strengthen the original assumptions in Theorem 1.3.1 by now assuming in place of (1.3.6)–(1.3.7) that

$$\begin{gathered} \vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \text{ exists (in } \mathbb{C}^n) \ \sigma-\text{a.e. on } \partial_{\mathrm{nta}}\Omega, \\ \mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma), \text{ and } \operatorname{div}\vec{F} \in \mathrm{CBM}(\Omega). \end{gathered} \tag{9.1.10}$$

In this setting, we claim that the divergence formula recorded in (1.3.8) holds under the additional assumption that

$$\textit{there exists } R \in (0, \infty) \textit{ such that } \vec{F} \textit{ vanishes in } \Omega \setminus B(0, R). \tag{9.1.11}$$

To see that this is the case, recall $\vec{f}$ from (9.1.1) and fix $\eta > 0$ arbitrary. Since we are now assuming that $\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma)$, from (9.1.2) we conclude that

$$\vec{f} \in \big[L^1(\partial_{\mathrm{nta}}\Omega, \sigma)\big]^n. \tag{9.1.12}$$

Granted this, we may invoke the density result established in Corollary 3.7.3 (used here with the choices $X := \partial_{\mathrm{nta}}\Omega$, $s := n-1 \geq 0$, and $p := 1$) to guarantee the existence of a vector field

$$\vec{G} \in \big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n \text{ such that } \big\|\vec{f} - \vec{G}\big|_{\partial_{\mathrm{nta}}\Omega}\big\|_{[L^1(\partial_{\mathrm{nta}}\Omega,\sigma)]^n} < \eta. \tag{9.1.13}$$

Also, let $\{\Phi_\varepsilon\}_{\varepsilon>0}$ be the sequence of functions associated with the set Ω as in Lemma 6.1.2. Then, using the properties of this sequence, the polar decomposition discussed in (9.1.9), the fact that (as seen from (9.1.8))

$$\begin{gathered} d\mu_{\vec{F}} = d\mu_{\vec{F}-\vec{G}} + d\mu_{\vec{G}} = d\mu_{\vec{F}-\vec{G}} + (\operatorname{div}\vec{G})\, d\mathcal{L}^n \\ \text{as complex Borel measures in } \Omega, \end{gathered} \tag{9.1.14}$$

and Lebesgue's Dominated Convergence Theorem, we may write

$$\begin{aligned} \big(\operatorname{div}\vec{F}\big)(\Omega) &= \int_\Omega 1\, d\mu_{\vec{F}} = \int_\Omega h\, d|\mu_{\vec{F}}| = \lim_{\varepsilon\to 0^+} \int_\Omega \Phi_\varepsilon\, h\, d|\mu_{\vec{F}}| = \lim_{\varepsilon\to 0^+} \int_\Omega \Phi_\varepsilon\, d\mu_{\vec{F}} \\ &= \lim_{\varepsilon\to 0^+} \int_\Omega \Phi_\varepsilon\, d\mu_{\vec{F}-\vec{G}} + \lim_{\varepsilon\to 0^+} \int_\Omega \Phi_\varepsilon \operatorname{div}\vec{G}\, d\mathcal{L}^n \\ &=: I + II. \end{aligned} \tag{9.1.15}$$

To estimate term I in (9.1.15), recall that $R \in (0, \infty)$ is as in (9.1.11) and pick some scalar-valued function

$$\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n) \text{ satisfying } \psi \equiv 1 \text{ near } \overline{B(0, R)} \cup \operatorname{supp}\vec{G}. \tag{9.1.16}$$

In particular, $\psi\,\Phi_\varepsilon \in \mathscr{C}_c^\infty(\Omega)$ for each $\varepsilon > 0$. Thus, with ${}_{\mathcal{D}'(\Omega)}\langle\cdot,\cdot\rangle_{\mathcal{D}(\Omega)}$ denoting the distributional pairing in the open set Ω, we have

$$
\begin{aligned}
I &= \lim_{\varepsilon\to 0^+}\int_\Omega \psi\,\Phi_\varepsilon\,\mathrm{d}\mu_{\vec{F}-\vec{G}}\\
&= \lim_{\varepsilon\to 0^+}{}_{\mathcal{D}'(\Omega)}\big\langle \mathrm{div}(\vec{F}-\vec{G}),\,\psi\,\Phi_\varepsilon\big\rangle_{\mathcal{D}(\Omega)}\\
&= -\lim_{\varepsilon\to 0^+}{}_{[\mathcal{D}'(\Omega)]^n}\big\langle \vec{F}-\vec{G},\,\nabla(\psi\,\Phi_\varepsilon)\big\rangle_{[\mathcal{D}(\Omega)]^n}\\
&= -\lim_{\varepsilon\to 0^+}{}_{[\mathcal{D}'(\Omega)]^n}\big\langle \vec{F}-\vec{G},\,\psi\,\nabla\Phi_\varepsilon\big\rangle_{[\mathcal{D}(\Omega)]^n} = -\lim_{\varepsilon\to 0^+}\int_\Omega \psi(\vec{F}-\vec{G})\cdot\nabla\Phi_\varepsilon\,\mathrm{d}\mathcal{L}^n\\
&= -\lim_{\varepsilon\to 0^+}\int_\Omega (\vec{F}-\vec{G})\cdot\nabla\Phi_\varepsilon\,\mathrm{d}\mathcal{L}^n. \qquad (9.1.17)
\end{aligned}
$$

The first and last equalities in (9.1.17) use the fact that $\psi \equiv 1$ on the support of $\vec{F}-\vec{G}$, the second equality holds by virtue of (1.3.12) since $\psi\,\Phi_\varepsilon$ belongs to $\mathscr{C}_c^\infty(\Omega)$, the third equality uses the fact that $\mathrm{div}\vec{F}$ is considered in the sense of distributions, the fourth is based on the fact that $\nabla\psi \equiv 0$ on the support of $\vec{F}-\vec{G}$ (as seen from (9.1.16) and (9.1.11)), while the fifth equality relies on having $\vec{F}-\vec{G} \in \big[L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n)\big]^n$ (cf. (9.1.5)). Consequently, from (9.1.17) we obtain

$$
\begin{aligned}
|I| &\le \limsup_{\varepsilon\to 0^+}\int_\Omega |\vec{F}-\vec{G}|\,|\nabla\Phi_\varepsilon|\,\mathrm{d}\mathcal{L}^n\\
&\le C\limsup_{\varepsilon\to 0^+}\Big(\varepsilon^{-1}\int_{O_\varepsilon}|\vec{F}-\vec{G}|\,\mathrm{d}\mathcal{L}^n\Big)\\
&\le C\limsup_{\varepsilon\to 0^+}\big\|\mathcal{N}_\kappa^\varepsilon(\vec{F}-\vec{G})\big\|_{L^1(\partial\Omega,\sigma)}\\
&= C\big\|(\vec{F}-\vec{G})\big|^{\text{n.t.}}_{\partial\Omega}\big\|_{[L^1(\partial_{\text{nta}}\Omega,\sigma)]^n} = C\big\|\vec{F}\big|^{\text{n.t.}}_{\partial\Omega}-\vec{G}\big|_{\partial\Omega}\big\|_{[L^1(\partial_{\text{nta}}\Omega,\sigma)]^n}\\
&= C\big\|\vec{f}-\vec{G}\big|_{\partial_{\text{nta}}\Omega}\big\|_{[L^1(\partial_{\text{nta}}\Omega,\sigma)]^n} < C\eta, \qquad (9.1.18)
\end{aligned}
$$

where $C \in (0,\infty)$ depends only on Ω and κ. Above, the second inequality follows from properties (6.1.5)–(6.1.6) of Φ_ε, while the third inequality comes from (8.6.76). Moreover, the first equality in (9.1.18) is a consequence of Proposition 8.9.5, the second is implied by (8.9.10), and the third uses (9.1.1). The last inequality in (9.1.18) originates in (9.1.13).

Consider next the task of estimating II in (9.1.15). By first invoking Lebesgue's Dominated Convergence Theorem and then using De Giorgi–Federer's version of the Gauss–Green Formula stated in Theorem 1.1.1 we obtain

$$
II = \int_\Omega \mathrm{div}\vec{G}\,\mathrm{d}\mathcal{L}^n = \int_{\partial_*\Omega}\nu\cdot\vec{G}\,\mathrm{d}\sigma. \qquad (9.1.19)
$$

Granted this and bearing in mind (9.1.3)–(9.1.4), we may therefore write

$$\begin{aligned}\Big| II - \int_{\partial_*\Omega} \nu\cdot\vec{f}\,d\sigma\Big| &\leq \int_{\partial_*\Omega} \big|\vec{f}-\vec{G}\big|\,d\sigma \leq \int_{\partial_{\rm nta}\Omega} \big|\vec{f}-\vec{G}\big|\,d\sigma \\ &= \big\|\vec{f}-\vec{G}\big|_{\partial_{\rm nta}\Omega}\big\|_{[L^1(\partial_{\rm nta}\Omega,\sigma)]^n} < \eta. \end{aligned} \tag{9.1.20}$$

All together, the above analysis (which has produced (9.1.15), (9.1.18), and (9.1.20)) proves that, for each $\eta > 0$,

$$\Big|\big(\mathrm{div}\vec{F}\big)(\Omega) - \int_{\partial_*\Omega} \nu\cdot\vec{f}\,d\sigma\Big| \leq C\eta, \tag{9.1.21}$$

with $C > 0$ a finite geometric constant depending only on Ω. Since $\eta > 0$ is arbitrary, (1.3.8) follows from this (bearing in mind (9.1.1)).

Step II.

The end-game in the proof of Theorem 1.3.1 The goal is to establish the Divergence Formula (1.3.8) under the original assumptions made in Theorem 1.3.1. To get started, combine Proposition 8.6.3, the membership $\mathcal{N}_\kappa\vec{F}\in L^1_{\rm loc}(\partial\Omega,\sigma)$, and Lemma 8.3.4 (in the case when $n = 1$) to conclude that

$$\vec{F}\big|_E \in \big[L^{\frac{n}{n-1}}(E,\mathcal{L}^n)\big]^n \text{ for every bounded } \mathcal{L}^n-\text{measurable set } E\subseteq\Omega \tag{9.1.22}$$

(naturally interpreting $L^{\frac{n}{n-1}}(E,\mathcal{L}^n)$ as $L^\infty(E,\mathcal{L}^1)$ in the case when $n = 1$). In particular, $\vec{F}$ is absolutely integrable on every bounded open subset of Ω.

Next, fix a system of auxiliary functions $\{\phi_R\}_{R>0}\subseteq\mathscr{C}^\infty_c(\mathbb{R}^n)$ (in the sense of (1.3.3)) and, for each number $R\in(0,\infty)$, define

$$\vec{F}_R := \phi_R\vec{F} \text{ in } \Omega. \tag{9.1.23}$$

Hence, by design, for each $R > 0$ there exists a number $R_*\in(0,\infty)$ such that

$$\vec{F}_R \equiv 0 \text{ outside } \Omega\cap B(0,R_*), \tag{9.1.24}$$

and

$$\mathrm{div}\vec{F}_R = \phi_R\,\mathrm{div}\vec{F} + \nabla\phi_R\cdot\vec{F} \text{ in } \mathcal{D}'(\Omega). \tag{9.1.25}$$

As a consequence of (9.1.25), (1.3.7), and (9.1.22),

$$\mathrm{div}\vec{F}_R \text{ is a complex Borel measure on } \Omega. \tag{9.1.26}$$

In addition, thanks to (9.1.5), (8.2.10), the third line of (1.3.6), (8.2.26), and (8.1.18), for each fixed $R > 0$ we have

$$\vec{F}_R \in \big[L^\infty_{\rm loc}(\Omega,\mathcal{L}^n)\big]^n \quad\text{and}\quad \mathcal{N}_\kappa\vec{F}_R \in L^1(\partial\Omega,\sigma). \tag{9.1.27}$$

Also, from the first line of (1.3.6) and (8.9.9)–(8.9.10), we see that the nontangential limit

$$\begin{aligned} &\vec{F}_R\Big|^{\rm n.t.}_{\partial\Omega} \text{ exists at } \sigma-\text{a.e. point on } \partial_{\rm nta}\Omega \text{ and, in fact,}\\ &\vec{F}_R\Big|^{\rm n.t.}_{\partial\Omega} = \big(\phi_R\big|_{\partial\Omega}\big)\Big(\vec{F}\Big|^{\rm n.t.}_{\partial\Omega}\Big) \text{ at } \sigma-\text{a.e. point on } \partial_{\rm nta}\Omega. \end{aligned} \tag{9.1.28}$$

In particular,

$$\vec{F}_R\Big|^{\rm n.t.}_{\partial\Omega} \longrightarrow \vec{F}\Big|^{\rm n.t.}_{\partial\Omega} \text{ at } \sigma-\text{a.e. point on } \partial_{\rm nta}\Omega, \text{ as } R\to\infty. \tag{9.1.29}$$

Bearing in mind that $\nu\cdot\Big(\vec{F}\Big|^{\rm n.t.}_{\partial\Omega}\Big)$ belongs to the space $L^1(\partial_*\Omega,\sigma)$ (cf. (1.3.6)), from (9.1.29), (8.8.52), and Lebesgue's Dominated Convergence Theorem, we conclude that, on the one hand,

$$\int_{\partial_*\Omega}\nu\cdot\Big(\vec{F}_R\Big|^{\rm n.t.}_{\partial\Omega}\Big)\,{\rm d}\sigma \longrightarrow \int_{\partial_*\Omega}\nu\cdot\big(\vec{F}\Big|^{\rm n.t.}_{\partial\Omega}\big)\,{\rm d}\sigma \quad\text{as}\quad R\to\infty. \tag{9.1.30}$$

On the other hand, granted (9.1.27), (9.1.28), (9.1.26), and (9.1.24), the result proved in Step I applies to the vector field $\vec{F}_R$ and gives

$$\big({\rm div}\vec{F}_R\big)(\Omega) = \int_{\partial_*\Omega}\nu\cdot\Big(\vec{F}_R\Big|^{\rm n.t.}_{\partial\Omega}\Big)\,{\rm d}\sigma, \qquad \forall R>0. \tag{9.1.31}$$

Upon recalling (9.1.25), and bearing in mind that $\mu_{\vec{F}}$ abbreviates ${\rm div}\vec{F}\in{\rm CBM}(\Omega)$, the left-hand side of (9.1.31) may be refashioned as

$$\big({\rm div}\vec{F}_R\big)(\Omega) = \int_\Omega \phi_R\,{\rm d}\mu_{\vec{F}} + \int_\Omega \nabla\phi_R\cdot\vec{F}\,{\rm d}\mathcal{L}^n. \tag{9.1.32}$$

To elucidate the behavior of this expression as $R\to\infty$, we use (9.1.9) and Lebesgue's Dominated Convergence Theorem (for ordinary positive measures) to compute

$$\begin{aligned} \lim_{R\to\infty}\int_\Omega\phi_R\,{\rm d}\mu_{\vec{F}} &= \lim_{R\to\infty}\int_\Omega\phi_R\,h\,{\rm d}|\mu_{\vec{F}}| = \int_\Omega h\,{\rm d}|\mu_{\vec{F}}|\\ &= \int_\Omega 1\,{\rm d}\mu_{\vec{F}} = \big({\rm div}\vec{F}\big)(\Omega). \end{aligned} \tag{9.1.33}$$

Granted (9.1.30)-(9.1.31), (9.1.32), and (9.1.33), it follows that the limit

$$\lim_{R\to\infty}\int_{\Omega}\nabla\phi_R\cdot\vec{F}\,\mathrm{d}\mathcal{L}^n \text{ exists, is independent of the family } \{\phi_R\}_{R>0}, \text{ and equals the complex number } \int_{\partial_*\Omega}\nu\cdot\big(\vec{F}\big|^{\text{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma - \big(\mathrm{div}\vec{F}\big)(\Omega). \tag{9.1.34}$$

In light of (1.3.2), from (9.1.34) we conclude that $[\vec{F}]_\infty$, the contribution of $\vec{F}$ at infinity, is meaningfully and unambiguously defined and that formula (1.3.8) holds.

Consider next the claims made in the last portion of the statement of Theorem 1.3.1. First, by design, $[\vec{F}]_\infty = 0$ whenever Ω is bounded. Next, we shall show that $[\vec{F}]_\infty$ also vanishes if the conditions in (1.3.10) are imposed, i.e., when $\partial\Omega$ is unbounded and $\mathcal{N}_\kappa\vec{F}\in L^1(\partial\Omega,\sigma)$.

Corresponding to the case when $n=1$, the "surface" measure $\sigma=\mathcal{H}^0\lfloor\partial\Omega$ becomes the counting measure on the discrete set $\partial\Omega$ and, by assumption, σ is locally finite (see the very last inequality in (7.4.1)). As such, $\partial\Omega$ has no (finite) accumulation points. Granted this, Proposition 4.7.4 applies and presently gives $[\vec{F}]_\infty=0$. Suppose now $n\geq 2$. The assumption that $\mathcal{N}_\kappa\vec{F}$ belongs to $L^1(\partial\Omega,\sigma)$ together with (8.6.51) (presently applied with $E:=\Omega$) ensure that

$$\vec{F}\in\Big[L^{\frac{n}{n-1}}(\Omega,\mathcal{L}^n)\Big]^n. \tag{9.1.35}$$

Given that we are also assuming that $n\geq 2$, the last part in Lemma 4.7.3 guarantees that $[\vec{F}]_\infty=0$ in this case as well.

Finally, whenever the growth condition (1.3.11) holds for some $\lambda\in(1,\infty)$, Lemma 4.7.3 implies that $[\vec{F}]_\infty=0$. The proof of Theorem 1.3.1 is therefore complete.

Let us formally record the proof of Theorem 1.2.1.

Proof of Theorem 1.2.1 In view of the identification (1.3.1), all claims are direct consequences of Theorem 1.3.1 and its proof (noting that thanks to (8.9.8), (9.1.3), and (8.9.44), we now have $\vec{F}\big|^{\text{n.t.}}_{\partial\Omega}\in\big[L^1(\partial_*\Omega,\sigma)\big]^n$).

We continue by presenting the proof of Corollary 1.2.2. □

Proof of Corollary 1.2.2 Having $\mathcal{N}_\kappa\vec{F}\in L^1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ implies (cf. Lemma 8.3.1)

$$\vec{F}\in\big[L^\infty_{\text{loc}}(\mathbb{R}^n_+,\mathcal{L}^n)\big]^n\subseteq\big[L^1_{\text{loc}}(\mathbb{R}^n_+,\mathcal{L}^n)\big]^n. \tag{9.1.36}$$

In particular, it is meaningful to consider $\mathrm{div}\vec{F}$ in the sense of distributions in $\mathbb{R}^n_+$. To proceed, observe from simple geometric considerations that there exists a large constant $K=K(n,\kappa)\in(1,\infty)$ such that

$$B\big(x+\varepsilon\mathbf{e}_n,\varepsilon/K\big)\subseteq\Gamma_\kappa(z) \text{ for each } \varepsilon>0, \text{ each } z\in\partial\mathbb{R}^n_+, \text{ and each } x\in\overline{\Gamma_\kappa(z)}. \tag{9.1.37}$$

Pick a non-negative function $\theta\in\mathscr{C}^\infty_c(\mathbb{R}^n)$ with $\mathrm{supp}\,\theta\subseteq B(0,1/K)$ and with $\int_{\mathbb{R}^n}\theta\,\mathrm{d}\mathcal{L}^n=1$. For each $\varepsilon>0$ set $\theta_\varepsilon(x):=\varepsilon^{-n}\theta(x/\varepsilon)$ for all $x\in\mathbb{R}^n$, then define $\vec{G}_\varepsilon:=\vec{F}(\cdot+\varepsilon\mathbf{e}_n)*\theta_\varepsilon$ in $\mathbb{R}^n_+$. That is,

$$\vec{G}_\varepsilon(x) = \int_{\mathbb{R}^n} \vec{F}(x - y + \varepsilon\mathbf{e}_n)\theta_\varepsilon(y)\,\mathrm{d}y = \int_{\mathbb{R}^n_+} \vec{F}(z)\theta_\varepsilon(x + \varepsilon\mathbf{e}_n - z)\,\mathrm{d}z \tag{9.1.38}$$
$$= \Big({}_{\mathcal{D}'(\mathbb{R}^n_+)}\big\langle F_j, \theta_\varepsilon(x + \varepsilon\mathbf{e}_n - \cdot)\big\rangle_{\mathcal{D}(\mathbb{R}^n_+)}\Big)_{1\leq j\leq n} \quad \text{for each } \ x \in \mathbb{R}^n_+,$$

where $(F_j)_{1\leq j\leq n}$ are the scalar components of $\vec{F}$. From (9.1.38) and (9.1.36) we see that $\vec{G}_\varepsilon$ is well defined and belongs to $\big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+})\big]^n$. Moreover, for each $x \in \mathbb{R}^n_+$ we may write

$$\begin{aligned}
(\mathrm{div}\vec{G}_\varepsilon)(x) &= \sum_{j=1}^n {}_{\mathcal{D}'(\mathbb{R}^n_+)}\big\langle F_j, (\partial_j\theta_\varepsilon)(x + \varepsilon\mathbf{e}_n - \cdot)\big\rangle_{\mathcal{D}(\mathbb{R}^n_+)}\\
&= -\sum_{j=1}^n {}_{\mathcal{D}'(\mathbb{R}^n_+)}\big\langle F_j, \partial_j\big[\theta_\varepsilon(x + \varepsilon\mathbf{e}_n - \cdot)\big]\big\rangle_{\mathcal{D}(\mathbb{R}^n_+)}\\
&= \sum_{j=1}^n {}_{\mathcal{D}'(\mathbb{R}^n_+)}\big\langle \partial_j F_j, \theta_\varepsilon(x + \varepsilon\mathbf{e}_n - \cdot)\big\rangle_{\mathcal{D}(\mathbb{R}^n_+)}\\
&= {}_{\mathcal{D}'(\mathbb{R}^n_+)}\big\langle \mathrm{div}\vec{F}, \theta_\varepsilon(x + \varepsilon\mathbf{e}_n - \cdot)\big\rangle_{\mathcal{D}(\mathbb{R}^n_+)}\\
&= \int_{\mathbb{R}^n_+} (\mathrm{div}\vec{F})(z)\theta_\varepsilon(x + \varepsilon\mathbf{e}_n - z)\,\mathrm{d}z,
\end{aligned} \tag{9.1.39}$$

where we have used (9.1.38) and the last property in (1.2.14). We next claim that

$$\mathrm{div}\vec{G}_\varepsilon \in L^1(\mathbb{R}^n_+, \mathcal{L}^n) \ \text{ and } \ \int_{\mathbb{R}^n_+} \mathrm{div}\vec{G}_\varepsilon\,\mathrm{d}\mathcal{L}^n = \int_{\mathbb{R}^n_+} \mathrm{div}\vec{F}\,\mathrm{d}\mathcal{L}^n \ \text{ for each } \ \varepsilon > 0. \tag{9.1.40}$$

To justify the membership above fix $\varepsilon > 0$ arbitrary. Then $\mathrm{div}\vec{G}_\varepsilon \in \mathscr{C}^\infty(\overline{\mathbb{R}^n_+})$ and we may estimate

$$\begin{aligned}
\int_{\mathbb{R}^n_+} \big|(\mathrm{div}\vec{G}_\varepsilon)(x)\big|\,\mathrm{d}x &\leq \int_{\mathbb{R}^n_+}\Big(\int_{\mathbb{R}^n_+} \big|(\mathrm{div}\vec{F})(z)\big|\theta_\varepsilon(x + \varepsilon\mathbf{e}_n - z)\,\mathrm{d}z\Big)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^n_+} \big|(\mathrm{div}\vec{F})(z)\big|\Big(\int_{\mathbb{R}^n_+} \theta_\varepsilon(x + \varepsilon\mathbf{e}_n - z)\,\mathrm{d}x\Big)\,\mathrm{d}z\\
&= \int_{\mathbb{R}^n_+} \big|(\mathrm{div}\vec{F})(z)\big|\Big(\int_{\mathbb{R}^n} \theta_\varepsilon(y)\,\mathrm{d}y\Big)\,\mathrm{d}z\\
&= \int_{\mathbb{R}^n_+} \big|(\mathrm{div}\vec{F})(z)\big|\,\mathrm{d}z < \infty,
\end{aligned} \tag{9.1.41}$$

using (9.1.39), a change of variables (note that $\mathrm{supp}\,\theta_\varepsilon \subseteq B(0, \varepsilon/K)$), and the fact that $\int_{\mathbb{R}^n} \theta_\varepsilon\,\mathrm{d}\mathcal{L}^n = 1$. Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}^n_+}(\mathrm{div}\vec{G}_\varepsilon)(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n_+}\Big(\int_{\mathbb{R}^n_+}(\mathrm{div}\vec{F})(z)\theta_\varepsilon(x+\varepsilon\mathbf{e}_n-z)\,\mathrm{d}z\Big)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^n_+}(\mathrm{div}\vec{F})(z)\Big(\int_{\mathbb{R}^n_+}\theta_\varepsilon(x+\varepsilon\mathbf{e}_n-z)\,\mathrm{d}x\Big)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^n_+}(\mathrm{div}\vec{F})(z)\Big(\int_{\mathbb{R}^n}\theta_\varepsilon(y)\,\mathrm{d}y\Big)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^n_+}(\mathrm{div}\vec{F})(z)\,\mathrm{d}z, \end{aligned} \tag{9.1.42}$$

finishing the proof of (9.1.40). Moving on, observe that

$$\mathcal{N}_\kappa\vec{G}_\varepsilon \leq \mathcal{N}_\kappa\vec{F} \;\text{ at each point on }\; \mathbb{R}^{n-1}\equiv\partial\mathbb{R}^n_+. \tag{9.1.43}$$

Indeed, for each $z\in\partial\mathbb{R}^n_+$, each $x\in\Gamma_\kappa(z)$, and each $y\in\operatorname{supp}\theta_\varepsilon\subseteq B(0,\varepsilon/K)$ we have $x-y+\varepsilon\mathbf{e}_n\in B\big(x+\varepsilon\mathbf{e}_n,\varepsilon/K\big)$, hence (9.1.37) implies $x-y+\varepsilon\mathbf{e}_n\in\Gamma_\kappa(z)$. Keeping this in mind, we conclude from (9.1.38) that $\mathcal{N}_\kappa\vec{G}_\varepsilon(z)\leq\mathcal{N}_\kappa\vec{F}(z)$ which, in view of the arbitrariness of $z\in\partial\mathbb{R}^n_+$, establishes (9.1.43). In concert with the first membership in (1.2.14) and (8.2.28), the estimate in (9.1.43) ultimately guarantees that

$$\mathcal{N}_\kappa\vec{G}_\varepsilon\in L^1(\mathbb{R}^{n-1},\mathcal{L}^{n-1}). \tag{9.1.44}$$

Finally, since for $\varepsilon>0$ we have

$$\begin{aligned}\Big((\vec{G}_\varepsilon)_n\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\Big)(x) &= \Big((\vec{G}_\varepsilon)_n\big|_{\partial\mathbb{R}^n_+}\Big)(x)\\ &= \int_{\mathbb{R}^n}F_n(x-y+\varepsilon\mathbf{e}_n)\theta_\varepsilon(y)\,\mathrm{d}y \;\text{ for each }\; x\in\partial\mathbb{R}^n_+, \end{aligned}\tag{9.1.45}$$

it follows that for $\mathcal{L}^{n-1}$-a.e. point $x\in\partial\mathbb{R}^n_+$ we have

$$\begin{aligned}&\Big|\Big((\vec{G}_\varepsilon)_n\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\Big)(x)-\Big(F_n\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\Big)(x)\Big|\\ &\qquad\leq\int_{\mathbb{R}^n}\Big|F_n(x-y+\varepsilon\mathbf{e}_n)-\Big(F_n\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\Big)(x)\Big|\theta_\varepsilon(y)\,\mathrm{d}y. \end{aligned}\tag{9.1.46}$$

Since for each $x\in\partial\mathbb{R}^n_+$ and each $y\in\operatorname{supp}\theta_\varepsilon\subseteq B(0,\varepsilon/K)$ we have, thanks to (9.1.37),

$$\begin{aligned}&x-y+\varepsilon\mathbf{e}_n\in B\big(x+\varepsilon\mathbf{e}_n,\varepsilon/K\big)\subseteq\Gamma_\kappa(x)\\ &\text{and }\;\big|(x-y+\varepsilon\mathbf{e}_n)-x\big|\leq\varepsilon\big(1+1/K\big),\end{aligned}\tag{9.1.47}$$

we may conclude from (9.1.46), (9.1.47), Definition 8.9.1, and the first line in (1.2.14) that

$$\lim_{\varepsilon\to0^+}(\vec{G}_\varepsilon)_n\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}=F_n\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\;\text{ at }\mathcal{L}^{n-1}-\text{a.e. point on }\;\mathbb{R}^{n-1}\equiv\partial\mathbb{R}^n_+. \tag{9.1.48}$$

In turn, as a consequence of (9.1.43), (9.1.48), and Lebesgue's Dominated Convergence Theorem we have

$$\lim_{\varepsilon\to 0^+}(\vec{G}_\varepsilon)_n\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = F_n\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} \ \text{ in } \ L^1(\mathbb{R}^{n-1},\mathcal{L}^{n-1}). \tag{9.1.49}$$

Granted these properties, we may invoke Theorem 1.2.1 (with $\Omega := \mathbb{R}^n_+$, a scenario in which $\nu(x) = -\mathbf{e}_n$ for each $x \in \partial\mathbb{R}^n_+$) to conclude that

$$\int_{\mathbb{R}^n_+} \operatorname{div}\vec{G}_\varepsilon \,d\mathcal{L}^n = -\int_{\mathbb{R}^{n-1}} \Big((\vec{G}_\varepsilon)_n\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+}\Big)\,d\mathcal{L}^{n-1} \ \text{ for each } \ \varepsilon > 0. \tag{9.1.50}$$

Passing to limit $\varepsilon \to 0^+$ in (9.1.50) then yields, on account of (9.1.40) and (9.1.49),

$$\int_{\mathbb{R}^n_+} \operatorname{div}\vec{F}\,d\mathcal{L}^n = -\int_{\mathbb{R}^{n-1}} \Big(F_n\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+}\Big)\,d\mathcal{L}^{n-1}. \tag{9.1.51}$$

Finally, the fact that for any other aperture parameter $\kappa' > 0$ the nontangential trace $F_n\Big|^{\kappa'-\text{n.t.}}_{\partial\mathbb{R}^n_+}$ exists $\mathcal{L}^{n-1}$-a.e. on $\mathbb{R}^{n-1}$ and is actually independent of κ' is a consequence of assumptions and Proposition 8.9.8. □

We now present the proof of Corollary 1.2.4. In the first part, we elaborate on the specific manner in which the surface measure and outward unit normal are defined in the statement of this result.

Proof of Corollary 1.2.4 A word of clarification as to what is meant when saying that (1.2.23)–(1.2.24) are used to define the surface measure σ on $\partial\Omega$ is in order. Specifically, for each index $j \in \{1,\dots,N\}$, denote by σ_j the canonical surface measure on the (rotated and translated) Lipschitz graph Σ_j. A set $E \subseteq \partial\Omega$ is said to be σ-measurable provided each $E \cap \Sigma_j$ is σ_j-measurable, in which case we define

$$\sigma(E) := \sum_{j=1}^{N} \sigma_j(E \cap \Sigma_j). \tag{9.1.52}$$

Since $\sigma_j = \mathcal{H}^{n-1}\lfloor\Sigma_j$ for each j, from (1.2.23)–(1.2.24) we conclude that σ is a well-defined measure on $\partial\Omega$ and, in fact, $\sigma = \mathcal{H}^{n-1}\lfloor\partial\Omega$.

Let us also elaborate on what is meant when saying that (1.2.23)–(1.2.24) are used to define the outward unit normal ν to Ω at σ-a.e. point on $\partial\Omega$. For each j, denote by $\Sigma_j^{\pm}$ the two connected components of $\mathbb{R}^n \setminus \Sigma_j$ and denote by A_j the collection of points at which the Lipschitz graph Σ_j is lacking a tangent plane. If we introduce $A_o := \cup_{j=1}^{N} A_j$ and define

$$A := A_o \cup \Big(\bigcup_{1\le j_1\neq j_2\le N} \big(\Sigma_{j_1} \cap \Sigma_{j_2}\big)\Big) \tag{9.1.53}$$

it follows that $\sigma(A) = 0$, each point $x \in \partial\Omega \setminus A$ belongs precisely to one Σ_j, and the aforementioned Σ_j has a tangent plane at x. To fix notation, denote this index j as j_x in order to emphasize its dependence on the point x. Given an arbitrary $x \in \partial\Omega \setminus A$, we claim that there exists an open neighborhood $\mathcal{U}_x$ of x in $\mathbb{R}^n$ satisfying the following properties:

$$\begin{aligned} &\mathcal{U}_x \cap \Sigma_j = \varnothing \ \text{ for each } \ j \in \{1, \dots, N\} \setminus \{j_x\}, \\ &\mathcal{U}_x \cap \Sigma^+_{j_x} \ \text{ and } \ \mathcal{U}_x \cap \Sigma^-_{j_x} \ \text{ are connected open sets,} \\ &\text{precisely one of the sets } \mathcal{U}_x \cap \Sigma^{\pm}_{j_x} \text{ is contained in } \Omega. \end{aligned} \tag{9.1.54}$$

The first property in (9.1.54) may be arranged keeping in mind that $x \notin \Sigma_j$ for each $j \neq j_x$ and that the sets Σ_j are closed. We may also accommodate the second property in (9.1.54) by virtue of the fact that $\Sigma^{\pm}_j$ are rotated and translated upper-graph Lipschitz domains, hence locally star-like (cf. Lemma 5.6.26). Let us show that the aforementioned choice of $\mathcal{U}_x$ also satisfies the third property in (9.1.54). The starting point is to observe that, since Ω is open, we have (with disjoint unions)

$$\begin{aligned} \mathcal{U}_x &= (\mathcal{U}_x \cap \Omega) \sqcup (\mathcal{U}_x \cap \partial\Omega) \sqcup (\mathcal{U}_x \setminus \overline{\Omega}) \\ &= (\mathcal{U}_x \cap \Omega) \sqcup (\mathcal{U}_x \cap \Sigma_{j_x}) \sqcup (\mathcal{U}_x \setminus \overline{\Omega}) \end{aligned} \tag{9.1.55}$$

where the final equality uses (1.2.23) plus the first property in (9.1.54). As a consequence of (9.1.55) and the fact that the sets $\Sigma^{\pm}_{j_x}$ are disjoint from Σ_{j_x}, we obtain

$$\mathcal{U}_x \cap \Sigma^{\pm}_{j_x} \subseteq (\mathcal{U}_x \cap \Omega) \sqcup (\mathcal{U}_x \setminus \overline{\Omega}). \tag{9.1.56}$$

In view of this and the second property in (9.1.54), we conclude that each of the sets $\mathcal{U}_x \cap \Sigma^{\pm}_{j_x}$ is contained in either $\mathcal{U}_x \cap \Omega$ or $\mathcal{U}_x \setminus \overline{\Omega}$. If both are contained in $\mathcal{U}_x \cap \Omega$, then so is their union, i.e., $\mathcal{U}_x \setminus \Sigma_{j_x} \subseteq \mathcal{U}_x \cap \Omega$. Upon selecting some $r \in \big(0, 2\,\mathrm{diam}(\partial\Omega)\big)$ such that $B(x,r) \subseteq \mathcal{U}_x$, this implies $B(x,r)\setminus\Sigma_{j_x} \subseteq B(x,r) \cap \Omega$ which further forces $\mathcal{L}^n\big(B(x,r)\big) = \mathcal{L}^n\big(B(x,r) \cap \Omega\big)$, since $\mathcal{L}^n(\Sigma_{j_x}) = 0$. Thus, $\mathcal{L}^n\big(B(x,r) \setminus \Omega\big) = 0$, which contradicts the hypothesis made in (1.2.22). This contradiction shows that $\mathcal{U}_x \cap \Sigma^{\pm}_{j_x}$ cannot be both contained in $\mathcal{U}_x \cap \Omega$. Likewise, $\mathcal{U}_x \cap \Sigma^{\pm}_{j_x}$ cannot be both contained in $\mathcal{U}_x \setminus \overline{\Omega}$, finishing the proof of the third property in (9.1.54). Having established (9.1.54), we now

$$\begin{aligned} &\text{define the outward unit normal } \nu(x) \text{ to the set } \Omega \text{ at the point } x \text{ as being} \\ &\text{the outward unit normal to the Lipschitz domain } \Sigma^+_{j_x} \text{ at the point } x \\ &\text{if } \mathcal{U}_x \cap \Sigma^+_{j_x} \text{ is contained in } \Omega, \text{ and as the outward unit normal to the} \\ &\text{Lipschitz domain } \Sigma^-_{j_x} \text{ at the point } x \text{ if } \mathcal{U}_x \cap \Sigma^-_{j_x} \text{ is contained in } \Omega. \end{aligned} \tag{9.1.57}$$

Having made these comments, we now begin the proof of Corollary 1.2.4 in earnest. On the one hand, from (1.2.23), (5.9.8), and (5.9.10) we conclude that $\partial\Omega$ is an upper Ahlfors regular set. On the other hand, Proposition 5.9.16 guarantees that $\partial_*\Omega = \partial\Omega$ and $\partial\Omega$ is lower Ahlfors regular. Hence,

$$\partial_*\Omega = \partial\Omega \text{ and } \partial\Omega \text{ is Ahlfors regular.} \tag{9.1.58}$$

In particular, Ω is a set of locally finite perimeter and σ is a doubling measure (cf. (5.9.15), (5.9.14)). Granted these properties, we may invoke item *(iii)* of Proposition 8.8.6 to conclude that

$$\sigma\big(\partial\Omega \setminus \partial_{\text{nta}}\Omega\big) = 0. \tag{9.1.59}$$

Going further, recall that the outward unit normal vector $\nu(x)$ to the set Ω has been defined at σ-a.e. point $x \in \partial\Omega$ as in (9.1.57). If ν_Ω denotes the geometric measure theoretic outward unit normal to Ω, regarded as a generic set of locally finite perimeter, we claim that

$$\nu_\Omega(x) = \nu(x) \;\; \text{at}\,\sigma - \text{a.e. point } \; x \in \partial\Omega. \tag{9.1.60}$$

To prove this, we shall freely use notation introduced in the buildup to the definition made in (9.1.57). In addition, we bring in further notation which is relevant in this argument. First, consider $\widetilde{A} := \partial\Omega \setminus \partial^*\Omega$ which, thanks to (9.1.58) and (5.6.21), satisfies $\sigma(\widetilde{A}) = 0$. Second, for each $j \in \{1, \dots, N\}$ denote by $\nu_{\Sigma_j^+}$ the outward unit normal to the Lipschitz domain Σ_j^+ and let B_j be the $\mathcal{H}^{n-1}$-nullset contained in $\partial^*\Omega \cap \partial^*\Sigma_j^+$ with the property that

$$\begin{gathered}\text{at each point in } \; \big(\partial^*\Omega \cap \partial^*\Sigma_j^+\big) \setminus B_j \; \text{ we have} \\ \text{either } \; \nu_\Omega = \nu_{\Sigma_j^+} \; \text{ or } \; \nu_\Omega = -\nu_{\Sigma_j^+},\end{gathered} \tag{9.1.61}$$

whose existence is guaranteed by Proposition 5.6.6 (applied with $E := \Omega$ and $F := \Sigma_j^+$).

Next, pick an arbitrary point

$$x \in \partial\Omega \setminus \Big(A \cup \widetilde{A} \cup \bigcup_{j=1}^{N} B_j\Big) \tag{9.1.62}$$

and, without loss of generality, assume (see third property in (9.1.54)) that

$$\mathcal{U}_x \cap \Sigma_{j_x}^+ \text{ is contained in } \Omega. \tag{9.1.63}$$

Then, by definition (cf. (9.1.57)),

$$\nu(x) := \nu_{\Sigma_{j_x}^+}(x), \tag{9.1.64}$$

where $\nu_{\Sigma^+_{j_x}}(x)$ is the outward unit normal to the Lipschitz domain $\Sigma^+_{j_x}$ at the point x. Since the Lipschitz domain $\Sigma^+_{j_x}$ enjoys a (uniform) cone property, there exists an open, truncated, one-component, circular cone in $\mathbb{R}^n$ with vertex at x, symmetry axis along a vector $h \in S^{n-1}$, full aperture $\theta \in (0, \pi)$, and truncated at height $b \in (0, \infty)$, i.e.,

$$\mathrm{C}_{\theta,b}(x,h) := \{y \in \mathbb{R}^n : \cos(\theta/2)\,|y-x| < (y-x)\cdot h < b\}, \tag{9.1.65}$$

with the property that

$$\mathrm{C}_{\theta,b}(x,h) \subseteq \mathcal{U}_x \cap \Sigma^+_{j_x}. \tag{9.1.66}$$

In view of this and (9.1.63), we also have

$$\mathrm{C}_{\theta,b}(x,h) \subseteq \Omega. \tag{9.1.67}$$

Granted (9.1.66)–(9.1.67), we may now invoke Lemma 5.6.15 (twice, first with $E := \Omega$, then with $E := \Sigma^+_{j_x}$) to conclude that, on the one hand,

$$\text{both } \nu_{\Sigma^+_{j_x}}(x) \text{ and } \nu_\Omega(x) \text{ belong to the cone } \overline{\mathrm{C}_{\pi-\theta,1}(0,-h)}. \tag{9.1.68}$$

On the other hand, (9.1.61) guarantees that

$$\text{either } \ \nu_\Omega(x) = \nu_{\Sigma^+_{j_x}}(x) \ \text{ or } \ \nu_\Omega(x) = -\nu_{\Sigma^+_{j_x}}(x). \tag{9.1.69}$$

From (9.1.68)–(9.1.69) we conclude that, in fact, we can only have $\nu_\Omega(x) = \nu_{\Sigma^+_{j_x}}(x)$. In view of (9.1.64), this finishes the proof of (9.1.60).

At this stage, Theorem 1.2.1 applies and yields all desired conclusions, on account of (9.1.60), (9.1.58), and (9.1.59). □

We conclude by presenting the proof of Corollary 1.3.2.

Proof of Corollary 1.3.2 Abbreviate

$$\mu := \mathrm{div}\vec{F} \in \mathrm{CBM}(\mathbb{R}^n_+) \tag{9.1.70}$$

and bring back the family of vector fields $\vec{G}_\varepsilon \in \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+})\big]^n$ defined for each $\varepsilon > 0$ as in (9.1.38). This time, in place of (9.1.39), for every $\varepsilon > 0$ we have

$$\begin{aligned}(\mathrm{div}\vec{G}_\varepsilon)(x) &= {}_{\mathcal{D}'(\mathbb{R}^n_+)}\big\langle \mathrm{div}\vec{F}, \theta_\varepsilon(x+\varepsilon\mathbf{e}_n-\cdot)\big\rangle_{\mathcal{D}(\mathbb{R}^n_+)} \\ &= \int_{\mathbb{R}^n_+} \theta_\varepsilon(x+\varepsilon\mathbf{e}_n-z)\,\mathrm{d}\mu(z) \ \text{ at each } \ x \in \mathbb{R}^n_+. \end{aligned} \tag{9.1.71}$$

Then using (9.1.71) for each $\varepsilon > 0$ we may write

$$\begin{aligned}\int_{\mathbb{R}^n_+} \big|(\mathrm{div}\vec{G}_\varepsilon)(x)\big|\,\mathrm{d}x &\leq \int_{\mathbb{R}^n_+} \Big(\int_{\mathbb{R}^n_+} \theta_\varepsilon(x+\varepsilon\mathbf{e}_n - z)\,\mathrm{d}|\mu|(z)\Big)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n_+} \Big(\int_{\mathbb{R}^n_+} \theta_\varepsilon(x+\varepsilon\mathbf{e}_n - z)\,\mathrm{d}x\Big)\,\mathrm{d}|\mu|(z) \\ &= \int_{\mathbb{R}^n_+} \Big(\int_{\mathbb{R}^n} \theta_\varepsilon(y)\,\mathrm{d}y\Big)\,\mathrm{d}|\mu|(z) \\ &= |\mu|(\mathbb{R}^n_+) < \infty, \end{aligned} \tag{9.1.72}$$

hence

$$\mathrm{div}\vec{G}_\varepsilon \in L^1(\mathbb{R}^n_+, \mathcal{L}^n) \ \text{ for each } \ \varepsilon > 0. \tag{9.1.73}$$

Let us momentarily digress in order to prove a useful auxiliary result. Specifically, we claim that if $g \in L^1(\mathbb{R}^n_+, \mathcal{L}^n)$ and if for every $\varepsilon > 0$ we define

$$\begin{aligned} g_\varepsilon(x) &:= \int_{\mathbb{R}^n} g(x-y+\varepsilon\mathbf{e}_n)\theta_\varepsilon(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n_+} g(z)\theta_\varepsilon(x+\varepsilon\mathbf{e}_n - z)\,\mathrm{d}z \ \text{ at each } \ x \in \mathbb{R}^n_+, \end{aligned} \tag{9.1.74}$$

then

$$g_\varepsilon \in L^1(\mathbb{R}^n_+, \mathcal{L}^n) \ \text{ and } \ \lim_{\varepsilon\to 0^+} g_\varepsilon = g \ \text{ in } \ L^1(\mathbb{R}^n_+, \mathcal{L}^n). \tag{9.1.75}$$

In particular,

$$\begin{aligned} &\text{there exists a sequence } \{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0,\infty) \text{ with } \lim_{j\to\infty} \varepsilon_j = 0 \\ &\text{and such that } \lim_{j\to\infty} g_{\varepsilon_j}(x) = g(x) \text{ for } \mathcal{L}^n - \text{a.e. point } x \in \mathbb{R}^n_+. \end{aligned} \tag{9.1.76}$$

To justify this claim, pick an arbitrary $\delta > 0$ and select $\phi \in \mathscr{C}^\infty_c(\mathbb{R}^n_+)$ such that

$$\|g-\phi\|_{L^1(\mathbb{R}^n_+, \mathcal{L}^n)} \leq \delta. \tag{9.1.77}$$

Then

$$\begin{aligned} &\int_{\mathbb{R}^n_+} \Big(\int_{\mathbb{R}^n_+} \Big|g(x-y+\varepsilon\mathbf{e}_n) - \phi(x-y+\varepsilon\mathbf{e}_n)\Big|\theta_\varepsilon(y)\,\mathrm{d}y\Big)\,\mathrm{d}x \\ &\quad = \int_{\mathbb{R}^n_+} \theta_\varepsilon(y)\Big(\int_{\mathbb{R}^n_+} \Big|g(x-y+\varepsilon\mathbf{e}_n) - \phi(x-y+\varepsilon\mathbf{e}_n)\Big|\,\mathrm{d}x\Big)\,\mathrm{d}y \\ &\quad = \int_{\mathbb{R}^n_+} \theta_\varepsilon(y)\Big(\int_{\mathbb{R}^n_+ + B(0,\varepsilon/K)+\varepsilon\mathbf{e}_n} \big|g(z)-\phi(z)\big|\,\mathrm{d}z\Big)\,\mathrm{d}y \end{aligned}$$

$$\leq \int_{\mathbb{R}^n_+} \theta_\varepsilon(y)\Big(\int_{\mathbb{R}^n_+} \big|g(z) - \phi(z)\big|\, dz\Big)\, dy$$
$$= \|g - \phi\|_{L^1(\mathbb{R}^n_+, \mathcal{L}^n)} \leq \delta. \tag{9.1.78}$$

Also, if $S_{\phi,\varepsilon} := (\operatorname{supp}\phi) \bigcup \big[\operatorname{supp}\phi + B(0, \varepsilon/K) + \varepsilon\mathbf{e}_n\big]$, then using the fact that ϕ is Lipschitz we may estimate, for some $C = C(n, \kappa) \in (0, \infty)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} \Big(\int_{\mathbb{R}^n_+} \big|\phi(x - y + \varepsilon\mathbf{e}_n) - \phi(x)\big|\theta_\varepsilon(y)\, dy\Big)\, dx \\
&\qquad \leq C\varepsilon\mathcal{L}^n(S_{\phi,\varepsilon}) \cdot \Big(\sup_{\mathbb{R}^n_+} |\nabla\phi|\Big) \int_{\mathbb{R}^n_+} \theta_\varepsilon(y)\, dy \\
&\qquad = C\varepsilon\mathcal{L}^n(S_{\phi,\varepsilon}) \cdot \Big(\sup_{\mathbb{R}^n_+} |\nabla\phi|\Big).
\end{aligned} \tag{9.1.79}$$

Collectively, (9.1.39), (9.1.77), and (9.1.79) permit us to write

$$\begin{aligned}
\|g_\varepsilon - g\|_{L^1(\mathbb{R}^n_+, \mathcal{L}^n)} &\leq \int_{\mathbb{R}^n_+} \Big(\int_{\mathbb{R}^n_+} \big|g(x - y + \varepsilon\mathbf{e}_n) - \phi(x - y + \varepsilon\mathbf{e}_n)\big|\theta_\varepsilon(y)\, dy\Big)\, dx \\
&\quad + \int_{\mathbb{R}^n_+} \Big(\int_{\mathbb{R}^n_+} \big|\phi(x - y + \varepsilon\mathbf{e}_n) - \phi(x)\big|\theta_\varepsilon(y)\, dy\Big)\, dx \\
&\quad + \|\phi - g\|_{L^1(\mathbb{R}^n_+, \mathcal{L}^n)} \\
&\leq 2\delta + C\varepsilon\mathcal{L}^n(S_{\phi,\varepsilon}) \cdot \Big(\sup_{\mathbb{R}^n_+} |\nabla\phi|\Big).
\end{aligned} \tag{9.1.80}$$

Thus,

$$\limsup_{\varepsilon \to 0^+} \|g_\varepsilon - g\|_{L^1(\mathbb{R}^n_+, \mathcal{L}^n)} \leq 2\delta \tag{9.1.81}$$

which, in view of the arbitrariness of $\delta > 0$, ultimately establishes (9.1.75). Then (9.1.76) follows from this and basic measure theory.

Returning to the principal topic of conversation, much as in (9.1.43), for each $\varepsilon > 0$ we continue to have

$$\mathcal{N}_\kappa \vec{G}_\varepsilon \leq \mathcal{N}_\kappa \vec{F} \quad \text{at each point on} \quad \mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+ \tag{9.1.82}$$

and, in place of (9.1.44), we now conclude (based on (9.1.82) and the second line in (1.3.31)) that

$$\mathcal{N}_\kappa \vec{G}_\varepsilon \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathcal{L}^{n-1}). \tag{9.1.83}$$

Finally, the same proof as before (cf. (9.1.48)) gives

$$\lim_{\varepsilon\to 0^+} (\vec{G}_\varepsilon)_n\big|_{\partial\mathbb{R}^n_+} = F_n\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} \text{ at } \mathcal{L}^{n-1}-\text{a.e. point on } \mathbb{R}^{n-1}\equiv\partial\mathbb{R}^n_+, \tag{9.1.84}$$

which, in light of (9.1.82), further implies

$$\lim_{\varepsilon\to 0^+} (\vec{G}_\varepsilon)_n\big|_{\partial\mathbb{R}^n_+} = F_n\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathcal{L}^{n-1}). \tag{9.1.85}$$

We next reason as in the end-game of the proof of Theorem 1.3.1. Concretely, fix a system of auxiliary functions $\{\phi_R\}_{R>0}\subseteq\mathscr{C}^\infty_c(\mathbb{R}^n)$ (in the sense of (1.3.3)), and for each $R\in(0,\infty)$ and $\varepsilon>0$ define

$$\vec{G}_{\varepsilon,R} := \phi_R\vec{G}_\varepsilon \text{ in } \overline{\mathbb{R}^n_+}. \tag{9.1.86}$$

Hence, by design, for each $R>0$ there exists a number $R_*\in(0,\infty)$ such that

$$\vec{G}_{\varepsilon,R}\in\big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+})\big]^n \text{ and } \vec{G}_{\varepsilon,R}\equiv 0 \text{ outside } \overline{\mathbb{R}^n_+}\cap B(0,R_*). \tag{9.1.87}$$

Also,

$$\mathrm{div}\vec{G}_{\varepsilon,R} = \phi_R\,\mathrm{div}\vec{G}_\varepsilon + \nabla\phi_R\cdot\vec{G}_\varepsilon\in L^1(\mathbb{R}^n_+,\mathcal{L}^n). \tag{9.1.88}$$

In addition, thanks to (9.1.87), (8.2.26), and (8.1.18), for each fixed $R>0$ we have

$$\mathcal{N}_\kappa\vec{G}_{\varepsilon,R}\in L^1(\mathbb{R}^n_+,\mathcal{L}^n). \tag{9.1.89}$$

Going further, since at each point on $\partial\mathbb{R}^n_+$ we have

$$\vec{G}_{\varepsilon,R}\big|_{\partial\mathbb{R}^n_+} = \big(\phi_R\big|_{\partial\mathbb{R}^n_+}\big)\big(\vec{G}_\varepsilon\big|_{\partial\mathbb{R}^n_+}\big), \tag{9.1.90}$$

we may invoke (9.1.85) and conclude that

$$\begin{gathered}\lim_{\varepsilon\to 0^+}(\vec{G}_{\varepsilon,R})_n\big|_{\partial\mathbb{R}^n_+} = \big(\phi_R\big|_{\partial\mathbb{R}^n_+}\big)\Big(F_n\Big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\Big)\\ \text{in } L^1(\mathbb{R}^{n-1},\mathcal{L}^{n-1}) \text{ for each fixed } R>0.\end{gathered} \tag{9.1.91}$$

Granted (9.1.87), the standard version of the Divergence Theorem applies and gives that

$$\int_{\mathbb{R}^n_+}\mathrm{div}\vec{G}_{\varepsilon,R}\,d\mathcal{L}^n = -\int_{\mathbb{R}^{n-1}}\Big((\vec{G}_{\varepsilon,R})_n\big|_{\partial\mathbb{R}^n_+}\Big)\,d\mathcal{L}^{n-1} \text{ for each } \varepsilon,R>0. \tag{9.1.92}$$

In view of this and (9.1.88), for each $\varepsilon,R>0$ we therefore have

$$\int_{\mathbb{R}^n_+} \phi_R \,\mathrm{div}\vec{G}_\varepsilon \,\mathrm{d}\mathcal{L}^n + \int_{\mathbb{R}^n_+} \nabla\phi_R \cdot \vec{G}_\varepsilon \,\mathrm{d}\mathcal{L}^n = -\int_{\mathbb{R}^{n-1}} \Big((\vec{G}_{\varepsilon,R})_n \big|_{\partial\mathbb{R}^n_+} \Big) \,\mathrm{d}\mathcal{L}^{n-1}. \tag{9.1.93}$$

The idea is to send $\varepsilon \to 0^+$ and $R \to \infty$, in this order. This requires some preparations. For every ε, $R > 0$ define

$$f_{\varepsilon,R}(z) := \int_{\mathbb{R}^n_+} \phi_R(x)\theta_\varepsilon(x + \varepsilon\mathbf{e}_n - z)\,\mathrm{d}x \quad \text{at each} \quad z \in \mathbb{R}^n_+. \tag{9.1.94}$$

Then from this and (1.3.3) we see that for every ε, $R \in (0, \infty)$ we have

$$\begin{aligned} |f_{\varepsilon,R}(z)| &\leq \Big(\sup_{R>0} \sup_{\mathbb{R}^n} |\phi_R| \Big) \int_{\mathbb{R}^n} \theta_\varepsilon(x + \varepsilon\mathbf{e}_n - z)\,\mathrm{d}x \\ &= \sup_{R>0} \sup_{\mathbb{R}^n} |\phi_R| < \infty \quad \text{for each} \quad z \in \mathbb{R}^n_+. \end{aligned} \tag{9.1.95}$$

Also, since ϕ_R has compact support in $\mathbb{R}^n$, if $\varepsilon \in (0, 1)$ it follows that for each $R \in (0, \infty)$ the function $f_{\varepsilon,R}$ vanishes identically outside of a bounded subset of $\mathbb{R}^n_+$ which depends only on R. In addition, from (9.1.76) we know that there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subseteq (0, \infty)$ with $\lim_{j\to\infty} \varepsilon_j = 0$ and such that

$$\lim_{j\to\infty} f_{\varepsilon_j,R}(x) = \phi_R(x) \quad \text{for } \mathcal{L}^n - \text{a.e. point} \quad x \in \mathbb{R}^n_+. \tag{9.1.96}$$

Next, since $\mu \in \mathrm{CBM}(\mathbb{R}^n_+)$, we may use the Polar Decomposition Theorem to write

$$\begin{aligned} &\mathrm{d}\mu = h\,\mathrm{d}|\mu|, \text{ where } |\mu| \text{ is a finite positive Borel measure on } \mathbb{R}^n_+ \text{ (the} \\ &\text{total variation of } \mu\text{), and } h \text{ is a } |\mu| - \text{measurable complex-valued function} \\ &\text{defined in } \mathbb{R}^n_+, \text{ with the property that } |h(x)| = 1 \text{ for each point } x \in \mathbb{R}^n_+. \end{aligned} \tag{9.1.97}$$

Then (9.1.71), (9.1.94), and Fubini's Theorem allow us to write

$$\begin{aligned} \int_{\mathbb{R}^n_+} \phi_R \,\mathrm{div}\vec{G}_\varepsilon \,\mathrm{d}\mathcal{L}^n &= \int_{\mathbb{R}^n_+} \phi_R(x)\Big(\int_{\mathbb{R}^n_+} \theta_\varepsilon(x + \varepsilon\mathbf{e}_n - z)\,\mathrm{d}\mu(z) \Big)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n_+} \Big(\int_{\mathbb{R}^n_+} \phi_R(x)\theta_\varepsilon(x + \varepsilon\mathbf{e}_n - z)\,\mathrm{d}x \Big)\,\mathrm{d}\mu(z) \\ &= \int_{\mathbb{R}^n_+} f_{\varepsilon,R}(z)\,\mathrm{d}\mu(z), \end{aligned} \tag{9.1.98}$$

hence

$$\lim_{j\to\infty}\int_{\mathbb{R}^n_+}\phi_R\,\mathrm{div}\vec{G}_{\varepsilon_j}\,\mathrm{d}\mathcal{L}^n=\lim_{j\to\infty}\int_{\mathbb{R}^n_+}f_{\varepsilon_j,R}\,\mathrm{d}\mu=\lim_{j\to\infty}\int_{\mathbb{R}^n_+}f_{\varepsilon_j,R}\,h\,\mathrm{d}|\mu|$$
$$=\int_{\mathbb{R}^n_+}\phi_R\,h\,\mathrm{d}|\mu|=\int_{\mathbb{R}^n_+}\phi_R\,\mathrm{d}\mu,\tag{9.1.99}$$

thanks to (9.1.98), (9.1.97), and Lebesgue's Dominated Convergence Theorem (whose applicability in the present setting is ensured by (9.1.94)–(9.1.96)).

Next, from Proposition 8.6.3 and the fact that $\mathcal{N}_\kappa\vec{F}\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ we see that

$$\vec{F}\big|_E\in\big[L^{\frac{n}{n-1}}(E,\mathcal{L}^n)\big]^n\ \text{ for every bounded}\quad \mathcal{L}^n-\text{measurable set }\ E\subseteq\mathbb{R}^n_+.\tag{9.1.100}$$

In particular, $\vec{F}$ is absolutely integrable on every bounded $\mathcal{L}^n$-measurable subset of $\mathbb{R}^n_+$. Keeping this in mind, we conclude from (9.1.38) and (9.1.74)–(9.1.75) that

$$\lim_{\varepsilon\to0^+}\vec{G}_\varepsilon\big|_E=\vec{F}\big|_E\ \text{ in }\ \big[L^1(E,\mathcal{L}^n)\big]^n\ \text{ for every}\quad \text{bounded }\mathcal{L}^n-\text{measurable set }\ E\subseteq\mathbb{R}^n_+.\tag{9.1.101}$$

Having established this, for each fixed $R>0$ we may then write

$$\lim_{\varepsilon\to0^+}\int_{\mathbb{R}^n_+}\nabla\phi_R\cdot\vec{G}_\varepsilon\,\mathrm{d}\mathcal{L}^n=\int_{\mathbb{R}^n_+}\nabla\phi_R\cdot\vec{F}\,\mathrm{d}\mathcal{L}^n.\tag{9.1.102}$$

At this stage, from (9.1.93), (9.1.99), (9.1.102), and (9.1.91) we conclude that

$$\int_{\mathbb{R}^n_+}\phi_R\,\mathrm{d}\mu+\int_{\mathbb{R}^n_+}\nabla\phi_R\cdot\vec{F}\,\mathrm{d}\mathcal{L}^n=-\int_{\mathbb{R}^{n-1}}\phi_R\Big(F_n\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\Big)\,\mathrm{d}\mathcal{L}^{n-1}\tag{9.1.103}$$

for each $R>0$. Use (9.1.97) and Lebesgue's Dominated Convergence Theorem (for ordinary positive measures) to write

$$\lim_{R\to\infty}\int_{\mathbb{R}^n_+}\phi_R\,\mathrm{d}\mu=\lim_{R\to\infty}\int_{\mathbb{R}^n_+}\phi_R\,h\,\mathrm{d}|\mu|=\int_{\mathbb{R}^n_+}h\,\mathrm{d}|\mu|$$
$$=\int_{\mathbb{R}^n_+}1\,\mathrm{d}\mu=\big(\mathrm{div}\vec{F}\big)(\mathbb{R}^n_+).\tag{9.1.104}$$

Given that $F_n\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\in L^1(\mathbb{R}^{n-1},\mathcal{L}^{n-1})$ (cf. (1.3.31)), Lebesgue's Dominated Convergence Theorem also gives

$$\lim_{R\to\infty}\int_{\mathbb{R}^{n-1}}\phi_R\Big(F_n\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\Big)\,\mathrm{d}\mathcal{L}^{n-1}=\int_{\mathbb{R}^{n-1}}\Big(F_n\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\Big)\,\mathrm{d}\mathcal{L}^{n-1}.\tag{9.1.105}$$

Granted (9.1.103)–(9.1.105), it follows that the limit

$$\lim_{R\to\infty}\int_{\mathbb{R}^n_+}\nabla\phi_R\cdot\vec{F}\,\mathrm{d}\mathcal{L}^n \text{ exists, is independent of the family } \{\phi_R\}_{R>0}, \text{ and}$$
$$\text{equals the (complex) number } \quad -\int_{\mathbb{R}^{n-1}}\Big(F_n\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\Big)\,\mathrm{d}\mathcal{L}^{n-1}-\big(\mathrm{div}\vec{F}\big)(\mathbb{R}^n_+). \tag{9.1.106}$$

Upon recalling (1.3.2), from (9.1.106) we then conclude that $[\vec{F}]_\infty$, the contribution of $\vec{F}$ at infinity, is meaningfully and unambiguously defined, and we have

$$\big(\mathrm{div}\vec{F}\big)(\mathbb{R}^n_+)=-\int_{\mathbb{R}^{n-1}}\Big(F_n\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\Big)\,\mathrm{d}\mathcal{L}^{n-1}+[\vec{F}]_\infty. \tag{9.1.107}$$

In addition, from assumptions and Proposition 8.9.8 we see that for any other aperture parameter $\kappa'>0$ the nontangential trace $F_n\big|^{\kappa'-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}$ exists $\mathcal{L}^{n-1}$-a.e. on $\mathbb{R}^{n-1}$ and is actually independent of κ'. As a consequence, formula (1.3.32) holds as stated.

Finally, consider the claims made in the last portion of the statement of Corollary 1.3.2. First, assuming the growth condition (1.3.34) holds for some $\lambda\in(1,\infty)$, Lemma 4.7.3 (whose applicability is ensured by (9.1.106)) implies $[\vec{F}]_\infty=0$. Second, if (1.3.35) is imposed, then (8.6.51) (presently used with $E:=\mathbb{R}^n_+$) ensures

$$\vec{F}\in\big[L^{\frac{n}{n-1}}(\mathbb{R}^n_+,\mathcal{L}^n)\big]^n. \tag{9.1.108}$$

Since we are also assuming $n\geq 2$, the last part in Lemma 4.7.3 guarantees the validity of (1.3.34) (see also Comment 11 just before the statement of Corollary 1.3.2). The proof of Corollary 1.3.2 is therefore complete.

9.2 Proof of Theorem 1.4.1 and Corollaries 1.4.2–1.4.4

The main idea in the proof of Theorem 1.4.1 is to localize matters as to be able to invoke Theorem 1.3.1 for the "function" part of the distribution $\vec{F}\in\big[\mathcal{D}'(\Omega)\big]^n$.

Proof of Theorem 1.4.1 By eventually enlarging the compact set K appearing in (1.4.2) there is no loss of generality (cf. (4.6.3)) in assuming that (1.4.4) takes the more precise form

$$\mathrm{div}\vec{F}=w+\mu \;\text{ in }\; \mathcal{D}'(\Omega), \quad \text{where } \; w\in\mathscr{E}'_K(\Omega) \;\text{ and }\; \mu\in\mathrm{CBM}(\Omega). \tag{9.2.1}$$

Suppose this is the case and fix a cutoff function $\varphi\in\mathscr{C}^\infty_c(\Omega)$ such that $0\leq\varphi\leq 1$ in Ω, and $\varphi\equiv 1$ near K. With tilde denoting the pointwise extension of functions by zero from $\Omega\setminus K$ to Ω, we then have

$$(1-\varphi)\Big(\widetilde{\vec{F}\big|_{\Omega\setminus K}}\Big)\in\big[L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n)\big]^n\subset\big[\mathcal{D}'(\Omega)\big]^n, \tag{9.2.2}$$

and we claim that

$$(1-\varphi)\vec{F} = (1-\varphi)\Big(\widetilde{\vec{F}\big|_{\Omega\setminus K}}\Big) \ \text{ in } \ \big[\mathcal{D}'(\Omega)\big]^n. \tag{9.2.3}$$

Indeed, for each $\psi \in \big[\mathscr{C}_c^\infty(\Omega)\big]^n$ we have $(1-\varphi)\psi \in \big[\mathscr{C}_c^\infty(\Omega \setminus K)\big]^n$ which permits us to write (bearing (4.6.1) in mind)

$$\begin{aligned}
{}_{[\mathcal{D}'(\Omega)]^n}\big\langle (1-\varphi)\vec{F}, \psi\big\rangle_{[\mathcal{D}(\Omega)]^n} &= {}_{[\mathcal{D}'(\Omega)]^n}\big\langle \vec{F}, (1-\varphi)\psi\big\rangle_{[\mathcal{D}(\Omega)]^n} \\
&= {}_{[\mathcal{D}'(\Omega\setminus K)]^n}\big\langle \vec{F}\big|_{\Omega\setminus K}, (1-\varphi)\psi\big\rangle_{[\mathcal{D}(\Omega\setminus K)]^n} \\
&= \int_{\Omega\setminus K} \Big(\vec{F}\big|_{\Omega\setminus K}\Big)\cdot\big[(1-\varphi)\psi\big]\,d\mathcal{L}^n \\
&= \int_{\Omega} (1-\varphi)\Big(\widetilde{\vec{F}\big|_{\Omega\setminus K}}\Big)\cdot\psi\,d\mathcal{L}^n,
\end{aligned} \tag{9.2.4}$$

from which (9.2.3) follows.

To proceed, define (keeping in mind (9.2.2)–(9.2.3))

$$\vec{G} := (1-\varphi)\vec{F} = (1-\varphi)\Big(\widetilde{\vec{F}\big|_{\Omega\setminus K}}\Big) \in \big[L^1_{\rm loc}(\Omega,\mathcal{L}^n)\big]^n. \tag{9.2.5}$$

As such, the function

$$\vec{G} : \Omega \longrightarrow \mathbb{C}^n \ \text{ is } \mathcal{L}^n - \text{measurable} \tag{9.2.6}$$

and

$$|\vec{G}| = (1-\varphi)\Big|\widetilde{\vec{F}\big|_{\Omega\setminus K}}\Big| \le \mathbf{1}_{\Omega\setminus K}\Big|\widetilde{\vec{F}\big|_{\Omega\setminus K}}\Big| \ \text{ in } \ \Omega, \tag{9.2.7}$$

hence

$$0 \le \mathcal{N}_\kappa \vec{G} \le \mathcal{N}_\kappa^{\Omega\setminus K}\vec{F} \ \text{ on } \ \partial\Omega. \tag{9.2.8}$$

In light of the last condition in (1.4.3), from (9.2.5)–(9.2.8) and (8.2.28) we conclude that

$$\mathcal{N}_\kappa \vec{G} \in L^1_{\rm loc}(\partial\Omega,\sigma). \tag{9.2.9}$$

Moreover, as is apparent from (9.2.5) and the choice of the cutoff function,

$$\vec{G} = \vec{F}\big|_{\Omega\setminus K} \text{ pointwise near } \partial\Omega. \tag{9.2.10}$$

In particular, from (9.2.10) and (1.4.3) we conclude that the pointwise nontangential boundary trace

$$\vec{G}\Big|^{\kappa-\rm n.t.}_{\partial\Omega} \ \text{ exists (in } \mathbb{C}^n) \ \sigma \ - \text{a.e. on } \ \partial_{\rm nta}\Omega \ \text{ and equals } \ \vec{F}\Big|^{\kappa-\rm n.t.}_{\partial\Omega}. \tag{9.2.11}$$

In concert, (9.2.11), (1.4.3), and (8.8.52) further entail

$$\nu \cdot \big(\vec{G}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) = \nu \cdot \big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big) \in L^1(\partial_*\Omega, \sigma). \tag{9.2.12}$$

With (9.2.9) and (9.2.11) in hand, from Corollary 8.9.9 we may then conclude that for any $\kappa' > 0$ the nontangential trace $\vec{G}\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega}$ exists σ-a.e. on $\partial_{\mathrm{nta}}\Omega$ and is actually independent of κ'. In view of (9.2.10), (9.1.3), (8.9.44), and (8.9.8) we therefore have that, for any $\kappa' > 0$,

$$\begin{aligned}&\text{the nontangential pointwise trace } \vec{F}\big|^{\kappa'-\mathrm{n.t.}}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial_{\mathrm{nta}}\Omega,\\ &\text{is actually independent of } \kappa', \text{ and belongs to } \big[L^1_{\mathrm{loc}}(\partial_*\Omega, \sigma)\big]^n.\end{aligned} \tag{9.2.13}$$

Next, since both $(1-\varphi) \in \mathscr{C}^\infty(\Omega) \cap L^\infty(\Omega, \mathcal{L}^n)$ and $\nabla\varphi \in [\mathscr{C}^\infty_c(\Omega)]^n$ vanish identically near K, in the sense of distributions in Ω we have

$$\begin{aligned}\mathrm{div}\vec{G} &= (1-\varphi)\mathrm{div}\vec{F} - \nabla\varphi \cdot \vec{F}\\ &= \underbrace{(1-\varphi)\mathrm{div}\vec{F}}_{\in \mathrm{CBM}(\Omega)} - \underbrace{\nabla\varphi \cdot \Big(\widetilde{\vec{F}\big|_{\Omega\setminus K}}\Big)}_{\in L^1(\Omega, \mathcal{L}^n)} \in \mathrm{CBM}(\Omega),\end{aligned} \tag{9.2.14}$$

where we have used (9.2.1), the fact that $\mathrm{CBM}(\Omega)$ is a module over $L^\infty(\Omega, \mathcal{L}^n)$, the first membership in (1.4.2), the properties of φ, and (4.6.6). On the one hand, since $\nabla\varphi \in [\mathscr{C}^\infty_c(\Omega \setminus K)]^n$, we may compute

$$\begin{aligned}-\int_\Omega \nabla\varphi \cdot \Big(\widetilde{\vec{F}\big|_{\Omega\setminus K}}\Big)\, d\mathcal{L}^n &= -\int_{\Omega\setminus K} \Big(\vec{F}\big|_{\Omega\setminus K}\Big) \cdot \nabla\varphi\, d\mathcal{L}^n\\ &= -{}_{[\mathcal{D}'(\Omega\setminus K)]^n}\big\langle \vec{F}\big|_{\Omega\setminus K}, \nabla\varphi\big\rangle_{[\mathcal{D}(\Omega\setminus K)]^n}\\ &= -{}_{[\mathcal{D}'(\Omega)]^n}\big\langle \vec{F}, \nabla\varphi\big\rangle_{[\mathcal{D}(\Omega)]^n} = {}_{\mathcal{D}'(\Omega)}\big\langle \mathrm{div}\vec{F}, \varphi\big\rangle_{\mathcal{D}(\Omega)}\\ &= {}_{\mathcal{D}'(\Omega)}\langle w, \varphi\rangle_{\mathcal{D}(\Omega)} + \int_\Omega \varphi\, d\mu\\ &= {}_{\mathscr{E}'(\Omega)}\langle w, 1\rangle_{\mathscr{E}(\Omega)} + \mu(\Omega) + \int_\Omega (\varphi - 1)\, d\mu\\ &= {}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(w + \mu, 1\big)_{\mathscr{C}^\infty_b(\Omega)} + \int_\Omega (\varphi - 1)\, d\mu\\ &= {}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\Omega)} + \int_\Omega (\varphi - 1)\, d\mu,\end{aligned} \tag{9.2.15}$$

making use of (9.2.1) and (4.6.19). On the other hand, in light of (9.2.1) the total mass of the complex Borel measure $(1-\varphi)\mathrm{div}\vec{F} \in \mathrm{CBM}(\Omega)$ is

$$\big((1-\varphi)\mathrm{div}\vec{F}\big)(\Omega) = \big((1-\varphi)\mu\big)(\Omega) = \int_\Omega (1-\varphi)\, d\mu. \tag{9.2.16}$$

Consequently, from (9.2.14)–(9.2.16) we obtain

$$\big(\mathrm{div}\vec{G}\big)(\Omega) = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)}. \tag{9.2.17}$$

Thanks to the assumptions on Ω, (9.2.6), (9.2.11), (9.2.12), (9.2.9), and (9.2.14), all hypotheses of Theorem 1.3.1 are satisfied by the vector field $\vec{G}$. Consequently, Theorem 1.3.1 guarantees that $[\vec{G}]_\infty$, the contribution of $\vec{G}$ at infinity, is meaningfully and unambiguously defined. Having proved this, since $\vec{G}$ and $\vec{F}$ coincide outside of a compact subset of Ω, the definition of the contribution at infinity given in (1.3.5)–(1.3.4) implies that the contribution of $\vec{F}$ at infinity is also meaningfully and unambiguously defined and that we have

$$[\vec{G}]_\infty = [\vec{F}]_\infty. \tag{9.2.18}$$

Also, formula (1.3.8) written for $\vec{G}$, in concert with (9.2.17) and (9.2.12)–(9.2.18), permits us to write

$$\begin{aligned} {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)} &= \big(\mathrm{div}\vec{G}\big)(\Omega) \\ &= \int_{\partial_*\Omega} \nu\cdot\big(\vec{G}\,\big|^{\text{n.t.}}_{\partial\Omega}\big)\,d\sigma + [\vec{G}]_\infty \\ &= \int_{\partial_*\Omega} \nu\cdot\big(\vec{F}\,\big|^{\text{n.t.}}_{\partial\Omega}\big)\,d\sigma + [\vec{F}]_\infty. \end{aligned} \tag{9.2.19}$$

This proves (1.4.5).

Moving on, if Ω is bounded then from the very definition of the contribution at infinity we see that $[\vec{F}]_\infty = 0$. Next, work under the conditions imposed in (1.4.7), i.e., when $\partial\Omega$ is unbounded and $\mathcal{N}_\kappa^{\Omega\setminus K}\vec{F} \in L^1(\partial\Omega,\sigma)$. From the last property, (9.2.8), and (8.2.28) we then see that $\mathcal{N}_\kappa\vec{G}$ also belongs to $L^1(\partial\Omega,\sigma)$. Granted this, and bearing in mind that $\partial\Omega$ is currently assumed to be unbounded, we may invoke the last part in Theorem 1.3.1 to conclude that $[\vec{G}]_\infty = 0$. Thus, $[\vec{F}]_\infty = 0$ thanks to (9.2.18). Finally, if (1.4.8) holds, then a similar condition holds with $\vec{F}$ replaced by $\vec{G}$ (given that $\vec{G}$ and $\vec{F}$ coincide outside of a compact subset of Ω). From Theorem 1.3.1 we know that this forces $[\vec{G}]_\infty = 0$ hence, ultimately, $[\vec{F}]_\infty = 0$ by (9.2.18).

This completes the proof of Theorem 1.4.1. □

Next, here is the proof of Corollary 1.4.2.

Proof of Corollary 1.4.2 This is deduced, via the same localization procedure, from Corollary 1.3.2 in the same manner in which Theorem 1.4.1 has been established from Theorem 1.3.1. □

Finally, we present the proof of Corollaries 1.4.3–1.4.4.

Proof of Corollaries 1.4.3–1.4.4 Each of these follows from Proposition 2.8.21 by the same localization argument that has been employed to obtain Theorem 1.4.1 from Theorem 1.3.1.

9.3 Proofs of Theorem 1.5.1 and Corollary 1.5.2

As regards Theorem 1.5.1, we shall revisit a number of techniques employed in the proof of Theorem 1.4.1 and, along the way, carefully monitor the effect of the loss of the doubling property for the measure $\sigma = \mathcal{H}^{n-1} \lfloor \partial\Omega$.

Proof of Theorem 1.5.1 Let $\kappa_\Omega \in (0, \infty)$ be as in Proposition 8.8.3 and fix an arbitrary $\kappa > \kappa_\Omega$. A combination of (8.8.5), (8.8.29), and (5.6.21) then proves the claims in (1.5.6).

Fix now a vector field $\vec{F}$ as in (1.5.7) satisfying (1.5.8)–(1.5.9). In particular, with the piece of notation introduced in (1.5.4), we have

$$\begin{aligned} &K := \text{regsupp}\,\vec{F} \text{ is a compact subset of } \Omega \\ &\text{and } \vec{F}\big|_{\Omega\setminus K} \text{ belongs to } \big[L^1_{\text{loc}}(\Omega \setminus K, \mathcal{L}^n)\big]^n. \end{aligned} \tag{9.3.1}$$

We find it convenient to temporarily strengthen the hypotheses on $\vec{F}$ by assuming that

$$\begin{aligned} &\text{there exists some } R_o \in (0, \infty) \text{ such that } \vec{F} = 0 \text{ in } \Omega \setminus \overline{B(0, R_o)}, \text{ and} \\ &\text{there exists } \varepsilon_o \in \big(0, \text{dist}(K, \partial\Omega)\big) \text{ such that } \mathcal{N}_\kappa^{\varepsilon_o}\vec{F} \in L^1(\partial\Omega, \sigma). \end{aligned} \tag{9.3.2}$$

In this scenario, abbreviate

$$\vec{f} := \vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \quad \text{on} \quad A_\kappa(\partial\Omega), \tag{9.3.3}$$

and observe that Corollary 8.9.7 (whose applicability is, in turn, ensured by Lemma 3.6.4) together with (8.9.8) currently imply

$$\begin{aligned} &\vec{f} \in \big[L^1(A_\kappa(\partial\Omega), \sigma)\big]^n \quad \text{and} \\ &\|\vec{f}\|_{[L^1(A_\kappa(\partial\Omega),\sigma)]^n} \leq \big\|\mathcal{N}_\kappa^{\varepsilon_o}\vec{F}\big\|_{L^1(\partial\Omega,\sigma)} < +\infty. \end{aligned} \tag{9.3.4}$$

In addition, (1.5.6) (which has already been established) guarantees that

$$\begin{aligned} &\vec{f}, \text{ originally defined as in (9.3.3), may be canonically} \\ &\text{viewed as a } \mathbb{C}^n - \text{valued function defined } \sigma - \text{a.e. on } \partial_*\Omega. \end{aligned} \tag{9.3.5}$$

Fix $\eta > 0$ arbitrary. Granted (8.8.5) and the fact that the measure σ is locally finite, Corollary 3.7.3 applies for $X := A_\kappa(\partial\Omega)$, $s := n - 1 \geq 0$, and $p := 1$. This ensures the existence of a vector field $\vec{G} \in \big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n$ such that

$$\big\|\vec{f} - \vec{G}\big|_{A_\kappa(\partial\Omega)}\big\|_{[L^1(A_\kappa(\partial\Omega),\sigma)]^n} < \eta. \tag{9.3.6}$$

Next, with $R \in (0, \infty)$ as in (9.3.2), select a cutoff function $\psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ satisfying $\psi \equiv 1$ near $\overline{B(0, R_o)} \cup \operatorname{supp} \vec{G}$. Also, let $\{\Phi_\varepsilon\}_{\varepsilon>0}$ be the sequence constructed in Lemma 6.1.2 for the set Ω. On account of the properties of these functions, Lemmas 4.6.1 and 4.6.2, and the fact that the functional $\operatorname{div}(\vec{F} - \vec{G}) \in \big(\mathscr{C}_b^\infty(\Omega)\big)^*$ is continuous in the sense of (1.5.1)–(1.5.2), we may write

$$
\begin{aligned}
&{}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)} \\
&\quad = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\operatorname{div}\vec{F}, \psi\big)_{\mathscr{C}_b^\infty(\Omega)} \\
&\quad = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\Big(\operatorname{div}(\vec{F} - \vec{G}), \psi\Big)_{\mathscr{C}_b^\infty(\Omega)} + {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\operatorname{div}\vec{G}, \psi\big)_{\mathscr{C}_b^\infty(\Omega)} \\
&\quad = \lim_{\varepsilon\to 0^+} {}_{(\mathscr{C}_b^\infty(\Omega))^*}\Big(\operatorname{div}(\vec{F} - \vec{G}), \psi\,\Phi_\varepsilon\Big)_{\mathscr{C}_b^\infty(\Omega)} + \int_\Omega \psi \operatorname{div}\vec{G}\,\mathrm{d}\mathcal{L}^n \\
&\quad =: I + II.
\end{aligned}
\tag{9.3.7}
$$

To further handle term I in (9.3.7), observe that $\psi\,\Phi_\varepsilon \in \mathscr{C}_c^\infty(\Omega)$ for each $\varepsilon > 0$. As such, with ${}_{\mathcal{D}'(\Omega)}\langle\cdot,\cdot\rangle_{\mathcal{D}(\Omega)}$ denoting the distributional pairing in the open set Ω, we have

$$
\begin{aligned}
I &= \lim_{\varepsilon\to 0^+} {}_{\mathcal{D}'(\Omega)}\big\langle \operatorname{div}(\vec{F} - \vec{G}), \psi\,\Phi_\varepsilon\big\rangle_{\mathcal{D}(\Omega)} \\
&= -\lim_{\varepsilon\to 0^+} {}_{[\mathcal{D}'(\Omega)]^n}\big\langle \vec{F} - \vec{G}, \nabla(\psi\,\Phi_\varepsilon)\big\rangle_{[\mathcal{D}(\Omega)]^n} \\
&= -\lim_{\varepsilon\to 0^+} {}_{[\mathcal{D}'(\Omega)]^n}\big\langle \vec{F} - \vec{G}, \psi\,\nabla\Phi_\varepsilon\big\rangle_{[\mathcal{D}(\Omega)]^n} \\
&= -\lim_{\varepsilon\to 0^+} \int_\Omega \psi(\vec{F} - \vec{G}) \cdot \nabla\Phi_\varepsilon\,\mathrm{d}\mathcal{L}^n \\
&= -\lim_{\varepsilon\to 0^+} \int_\Omega (\vec{F} - \vec{G}) \cdot \nabla\Phi_\varepsilon\,\mathrm{d}\mathcal{L}^n.
\end{aligned}
\tag{9.3.8}
$$

The first equality in (9.3.8) uses the compatibility of $\operatorname{div}(\vec{F} - \vec{G})$ viewed as a functional in $\big(\mathscr{C}_b^\infty(\Omega)\big)^*$ with $\operatorname{div}(\vec{F} - \vec{G})$ viewed as a distribution in Ω. The second equality is standard, the third equality uses the fact that $\nabla\psi \equiv 0$ on the support of $\vec{F} - \vec{G}$, and the fourth equality is justified by noting that $\operatorname{supp}(\nabla\Phi_\varepsilon) \subseteq O_\varepsilon$ and $(\vec{F} - \vec{G})\big|_{O_\varepsilon} \in \big[L^1_{\mathrm{loc}}(O_\varepsilon, \mathcal{L}^n)\big]^n$ whenever $\varepsilon \in \big(0, \operatorname{dist}(K, \partial\Omega)\big)$ (thanks to (9.3.1)). Finally, the last equality in (9.3.8) uses the fact that $\psi \equiv 1$ on the support of $\vec{F} - \vec{G}$.

Going further, (9.3.8) permits us to estimate

$$
\begin{aligned}
|I| &\le \limsup_{\varepsilon\to 0^+} \int_\Omega |\vec{F} - \vec{G}|\,|\nabla\Phi_\varepsilon|\,\mathrm{d}\mathcal{L}^n \\
&\le C \limsup_{\varepsilon\to 0^+} \Big(\varepsilon^{-1} \int_{O_\varepsilon} |\vec{F} - \vec{G}|\,\mathrm{d}\mathcal{L}^n\Big) \\
&\le C \limsup_{\varepsilon\to 0^+} \big\| \mathcal{N}_\kappa^\varepsilon(\vec{F} - \vec{G}) \big\|_{L^1(\partial\Omega,\sigma)}
\end{aligned}
$$

$$
\begin{aligned}
&= C\big\|\big(\vec{F}-\vec{G}\big)\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big\|_{[L^1(A_\kappa(\partial\Omega),\sigma)]^n} \\
&= C\big\|\vec{f}-\vec{G}\big|_{A_\kappa(\partial\Omega)}\big\|_{[L^1(A_\kappa(\partial\Omega),\sigma)]^n} \\
&< C\eta, \qquad (9.3.9)
\end{aligned}
$$

where $C \in (0, \infty)$ depends only on Ω and κ. In (9.3.9), the second inequality is implied by properties (6.1.5)–(6.1.6) of Φ_ε, while the third inequality follows by applying (8.6.76) to the (components of the) $\mathbb{C}^n$-valued function

$$
u := \begin{cases} (\vec{F}-\vec{G})\big|_{\Omega\setminus K} & \text{in } \ \Omega\setminus K, \\ 0 & \text{in } \ K. \end{cases} \tag{9.3.10}
$$

For the first equality in (9.3.9), in place of Proposition 8.9.5 which was used (9.1.18) when σ was assumed to be doubling, we now invoke Lemma 8.9.4 (with $\mu := \sigma$ and $\mathcal{U} := A_\kappa(\partial\Omega)$). The final inequality in (9.3.9) is supplied by (9.3.6).

Consider next the task of estimating II in (9.3.7). Upon recalling that $\psi \equiv 1$ on $\operatorname{supp}\vec{G}$ and then using De Giorgi-Federer's version of the Gauss–Green Formula stated in Theorem 1.1.1 we obtain

$$
II = \int_\Omega \operatorname{div}\vec{G}\,\mathrm{d}\mathcal{L}^n = \int_{\partial_*\Omega} \nu\cdot\vec{G}\,\mathrm{d}\sigma. \tag{9.3.11}
$$

Granted this and bearing in mind (1.5.6) together with (9.3.6), we may therefore write

$$
\begin{aligned}
\Big|II - \int_{\partial_*\Omega}\nu\cdot\vec{f}\,\mathrm{d}\sigma\Big| &\le \int_{\partial_*\Omega}\big|\vec{f}-\vec{G}\big|\,\mathrm{d}\sigma \le \int_{A_\kappa(\partial\Omega)}\big|\vec{f}-\vec{G}\big|\,\mathrm{d}\sigma \\
&= \big\|\vec{f}-\vec{G}\big|_{A_\kappa(\partial\Omega)}\big\|_{[L^1(A_\kappa(\partial\Omega),\sigma)]^n} < \eta. \qquad (9.3.12)
\end{aligned}
$$

Collectively, (9.3.7), (9.3.9), and (9.3.12)) prove that, for each $\eta > 0$,

$$
\Big|{}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\Omega)} - \int_{\partial_*\Omega}\nu\cdot\vec{f}\,\mathrm{d}\sigma\Big| \le C\eta, \tag{9.3.13}
$$

with $C > 0$ a finite geometric constant depending only on Ω and κ. Bearing in mind (9.3.3) and recalling that $\eta > 0$ is arbitrary, from this we arrive at the following conclusion:

$$
\begin{aligned}
&\text{under the additional hypotheses in (9.3.2) we have} \\
&{}_{(\mathscr{C}^\infty_b(\Omega))^*}\big(\operatorname{div}\vec{F}, 1\big)_{\mathscr{C}^\infty_b(\Omega)} = \int_{\partial_*\Omega}\nu\cdot\big(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma.
\end{aligned} \tag{9.3.14}
$$

Moving on, the goal is to establish the Divergence Formula (1.5.10) under the original assumptions (1.5.7)–(1.5.9) made on the vector field $\vec{F}$ in the statement of Theorem 1.5.1. For starters, observe that as a consequence of the Heine–Borel theorem, property (1.5.8) implies that

$$\begin{gathered}\text{for each compact set } S \subseteq \partial\Omega \text{ there exists } \varepsilon_S \in \big(0, \operatorname{dist}(K, \partial\Omega)\big) \\ \text{with the property that } \mathcal{N}_\kappa^{\varepsilon_S}\vec{F} \in L^1(S, \sigma).\end{gathered} \tag{9.3.15}$$

Let us also fix a compact neighborhood $\widetilde{K}$ of the set $K = \operatorname{regsupp}\vec{F}$ which is contained in Ω. Consider now a bounded subset E of $\Omega \setminus \widetilde{K}$ which is $\mathcal{L}^n$-measurable. In particular, (8.1.18) ensures that $S := \overline{\pi_\kappa(E)}$ is a compact subset of $\partial\Omega$. With ε_S associated with S as in (9.3.15) we may now combine Proposition 8.6.3 and Lemma 8.3.4 to conclude that, on the one hand,

$$\vec{F}\Big|_{E\cap O_{\varepsilon_S}} \text{ belongs to } \Big[L^{\frac{n}{n-1}}\big(E \cap O_{\varepsilon_S}, \mathcal{L}^n\big)\Big]^n \subset \Big[L^1\big(E \cap O_{\varepsilon_S}, \mathcal{L}^n\big)\Big]^n \tag{9.3.16}$$

(naturally interpreting $L^{\frac{n}{n-1}}(E \cap O_{\varepsilon_S}, \mathcal{L}^n)$ as $L^\infty(E \cap O_{\varepsilon_S}, \mathcal{L}^1)$ when $n = 1$). On the other hand, $E \setminus O_{\varepsilon_S}$ is a relatively compact subset of $\Omega \setminus K$. Keeping this in mind, from (9.3.16) and (9.3.1) we then conclude that $\vec{F}\big|_E \in \big[L^1(E, \mathcal{L}^n)\big]^n$. In summary, this argument proves that

$$\begin{gathered}\text{the vector field } \vec{F} \text{ is absolutely integrable on all} \\ \text{bounded } \mathcal{L}^n\text{-measurable subsets of } \Omega \setminus \widetilde{K}.\end{gathered} \tag{9.3.17}$$

Next, fix a system of auxiliary functions $\{\phi_R\}_{R>0}$ in the sense of (1.3.3). For the purposes we have in mind, there is no loss of generality in assuming that

$$\phi_R \equiv 1 \text{ near } \widetilde{K} \text{ for each } R \in (0, \infty). \tag{9.3.18}$$

For each number $R \in (0, \infty)$ define

$$\vec{F}_R := \phi_R \vec{F} \in \big[\mathcal{D}'(\Omega)\big]^n. \tag{9.3.19}$$

Hence, by design, for each $R > 0$ there exists a number $\rho \in (0, \infty)$ such that

$$\vec{F}_R = 0 \text{ in } \Omega \setminus \overline{B(0, \rho)}, \tag{9.3.20}$$

and

$$\operatorname{div}\vec{F}_R = \nabla\phi_R \cdot \vec{F} + \phi_R \operatorname{div}\vec{F} \text{ in } \mathcal{D}'(\Omega). \tag{9.3.21}$$

As a consequence of (9.3.17), (9.3.18), and (4.6.23),

$$\nabla\phi_R \cdot \vec{F} \in L^1(\Omega, \mathcal{L}^n) \subset \big(\mathscr{C}_b^\infty(\Omega)\big)^* \text{ for each } R > 0. \tag{9.3.22}$$

To proceed, let us denote by Λ the extension of the distribution $\operatorname{div}\vec{F}$ to a continuous functional in $\big(\mathscr{C}_b^\infty(\Omega)\big)^*$ (referred to in the last line of (1.5.9)). Also, for each fixed $R \in (0, \infty)$ define the functional $\Lambda_R \in \big(\mathscr{C}_b^\infty(\Omega)\big)^*$ by setting

$$\big(\Lambda_R, f\big)_{\mathscr{C}_b^\infty(\Omega)} = \big(\Lambda, \phi_R f\big)_{\mathscr{C}_b^\infty(\Omega)} \quad \text{for each } f \in \mathscr{C}_b^\infty(\Omega). \tag{9.3.23}$$

In relation to this observe that, for each $R > 0$,

$$\begin{aligned} &\text{the functional } \widetilde{\Lambda}_R := \nabla\phi_R \cdot \vec{F} + \Lambda_R \in \big(\mathscr{C}_b^\infty(\Omega)\big)^* \text{ is} \\ &\text{a continuous extension of the distribution } \operatorname{div}\vec{F}_R. \end{aligned} \tag{9.3.24}$$

Indeed, it is clear from definitions that $\widetilde{\Lambda}_R$ is continuous in the sense of (1.5.1)–(1.5.2) and satisfies

$$\widetilde{\Lambda}_R\big|_{\mathscr{C}_c^\infty(\Omega)} = \operatorname{div}\vec{F}_R. \tag{9.3.25}$$

Pressing on, thanks to (8.2.10), (8.2.26), (8.1.18), and the original assumptions on $\vec{F}$, for each fixed $R > 0$ we have

$$\mathcal{N}_\kappa^\varepsilon \vec{F}_R \in L^1(\partial\Omega, \sigma). \tag{9.3.26}$$

Also, from the first line of (1.5.9) and (8.9.9)–(8.9.10), we see that the nontangential limit

$$\begin{aligned} &\vec{F}_R\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \sigma-\text{ a.e. point on } A_\kappa(\partial\Omega) \text{ and, in fact,} \\ &\vec{F}_R\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \big(\phi_R\big|_{\partial\Omega}\big)\Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) \text{ at } \sigma-\text{ a.e. point on } A_\kappa(\partial\Omega). \end{aligned} \tag{9.3.27}$$

As a consequence of this and (1.5.6),

$$\vec{F}_R\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \longrightarrow \vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ at } \sigma\text{-a.e. point on } \partial_*\Omega, \text{ as } R \to \infty. \tag{9.3.28}$$

Since we are assuming that $\nu \cdot \Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big)$ belongs to the space $L^1(\partial_*\Omega, \sigma)$ (see the first line of (1.5.9)), from (9.3.28) and Lebesgue's Dominated Convergence Theorem we conclude that, on the one hand,

$$\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}_R\Big|_{\partial\Omega}^{\text{n.t.}}\Big)\, d\sigma \longrightarrow \int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\Big|_{\partial\Omega}^{\text{n.t.}}\Big)\, d\sigma \quad \text{as} \quad R \to \infty. \tag{9.3.29}$$

On the other hand, granted (9.3.26), (9.3.27), and (9.3.20), the result established in (9.3.14) applies to each $\vec{F}_R$ and, on account of (9.3.24), presently gives

$${}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\widetilde{\Lambda}_R, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = \int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}_R\Big|_{\partial\Omega}^{\text{n.t.}}\Big)\, d\sigma, \qquad \forall R > 0. \tag{9.3.30}$$

Let us take a closer look at the left-hand side above. For each $R > 0$ split

$$
{}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\widetilde{\Lambda}_R, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = I_R + II_R, \tag{9.3.31}
$$

with

$$
I_R := {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\nabla\phi_R \cdot \vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)} \tag{9.3.32}
$$

and

$$
II_R := {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\Lambda_R, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\Lambda, \phi_R\big)_{\mathscr{C}_b^\infty(\Omega)}, \tag{9.3.33}
$$

where the second equality above is implied by (9.3.23). Then (9.3.22) and (4.6.23) permit us to write

$$
I_R = \int_\Omega \nabla\phi_R \cdot \vec{F}\, d\mathcal{L}^n \quad \text{for each} \quad R > 0, \tag{9.3.34}
$$

whereas thanks to (1.3.3) and the fact that the functional $\Lambda \in \big(\mathscr{C}_b^\infty(\Omega)\big)^*$ is continuous in the sense of (1.5.1)–(1.5.2) we have

$$
\begin{aligned}
\lim_{R\to\infty} II_R &= \lim_{R\to\infty} {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\Lambda, \phi_R\big)_{\mathscr{C}_b^\infty(\Omega)} \\
&= {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\Lambda, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)},
\end{aligned} \tag{9.3.35}
$$

where the last equality is clear from the definition of Λ. Granted (9.3.29), (9.3.30), (9.3.31), (9.3.34), and (9.3.35), it follows that the limit

$$
\begin{aligned}
&\lim_{R\to\infty} \int_\Omega \nabla\phi_R \cdot \vec{F}\, d\mathcal{L}^n \;\; \text{exists in } \mathbb{C}, \text{ is independent of the family } \{\phi_R\}_{R>0}, \\
&\text{and equals the number} \;\; \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)\, d\sigma - {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)}.
\end{aligned} \tag{9.3.36}
$$

In view of (1.3.2), from (9.3.36) we conclude that $[\vec{F}]_\infty$, the contribution of $\vec{F}$ at infinity, is meaningfully and unambiguously defined and that formula (1.5.10) holds.

Let us now turn to the claims made in the last part of the statement of Theorem 1.5.1. If Ω is bounded then, obviously, $[\vec{F}]_\infty = 0$. Next, work under the assumptions made in (1.5.12). The fact that we can pick some large $R_* > 0$ such that $\mathrm{regsupp}\,\vec{F} \subseteq B(0, R_*)$ and $\mathcal{N}_\kappa^{\Omega\setminus\overline{B(0,R_*)}}\vec{F} \in L^1(\partial\Omega, \sigma)$ together with (8.6.51) (presently applied with $E := \Omega \setminus \overline{B(0, R_*)}$) ensures

$$
\vec{F}\Big|_{\Omega\setminus\overline{B(0,R_*)}} \in \Big[L^{\frac{n}{n-1}}\big(\Omega \setminus \overline{B(0, R_*)}, \mathcal{L}^n\big)\Big]^n \tag{9.3.37}
$$

(with $L^{\frac{n}{n-1}}$ interpreted as L^∞ when $n = 1$). If we are also assuming that $n \geq 2$, from (9.3.37) and the second part in Lemma 4.7.3 we then conclude that $[\vec{F}]_\infty = 0$.

Next, suppose $n = 1$, while retaining (1.5.12). Introduce $\widetilde{\Omega} := \Omega \setminus \overline{B(0, R_*)}$ and define $\widetilde{F} := \vec{F}\big|_{\widetilde{\Omega}}$. Then $\widetilde{\Omega}$ is an open subset of the real line, and the fact that $\partial\Omega$ is unbounded and without finite accumulation points (given that σ is assumed to be a locally finite measure) implies that $\partial\widetilde{\Omega}$ enjoys similar properties. Also, the choice of R_* guarantees that $\widetilde{F}$ is a Lebesgue measurable function. Observe that $\vec{F}$ and $\widetilde{F}$ agree outside of a bounded set, and recall that the contribution of $\vec{F}$ at infinity is meaningfully and unambiguously defined (cf. (9.3.36)). It follows that $[\widetilde{F}]_\infty$ is also meaningfully and unambiguously defined and, in fact,

$$[\vec{F}]_\infty = [\widetilde{F}]_\infty. \tag{9.3.38}$$

In addition, the one-dimensional version of (9.3.37) yields

$$\widetilde{F} \in L^\infty\big(\widetilde{\Omega}, \mathcal{L}^1\big). \tag{9.3.39}$$

Hence, if $\widetilde{\sigma} := \mathcal{H}^0 \lfloor \partial\widetilde{\Omega}$ and $\widetilde{\mathcal{N}}_\kappa$ denotes the nontangential maximal operator relative to $\widetilde{\Omega}$, from (9.3.39) we conclude that

$$\widetilde{\mathcal{N}}_\kappa \widetilde{F} \in L^\infty\big(\partial\widetilde{\Omega}, \widetilde{\sigma}\big). \tag{9.3.40}$$

Moreover, for each $x \in \partial\Omega \cap \partial\widetilde{\Omega}$ we have $\Gamma_{\widetilde{\Omega},\kappa}(x) \subseteq \Gamma_{\Omega,\kappa}(x) \cap \widetilde{\Omega}$ (see Lemma 8.1.3) which goes to show that

$$\begin{aligned}\big(\widetilde{\mathcal{N}}_\kappa \widetilde{F}\big)(x) &= \|\widetilde{F}\|_{L^\infty(\Gamma_{\widetilde{\Omega},\kappa}(x),\mathcal{L}^1)} \leq \big\|\vec{F}\big|_{\widetilde{\Omega}}\big\|_{L^\infty(\Gamma_{\Omega,\kappa}(x)\cap\widetilde{\Omega},\mathcal{L}^1)} \\ &\leq \Big(\mathcal{N}_\kappa^{\Omega\setminus\overline{B(0,R_*)}} \vec{F}\Big)(x). \end{aligned} \tag{9.3.41}$$

Observe that $\partial\widetilde{\Omega} \subseteq \partial\Omega \cup \{\pm R_*\}$. Bearing this in mind, from (9.3.40)–(9.3.41) and the fact that we are currently assuming $\mathcal{N}_\kappa^{\Omega\setminus\overline{B(0,R_*)}} \vec{F} \in L^1(\partial\Omega, \sigma)$ we then conclude that

$$\widetilde{\mathcal{N}}_\kappa \widetilde{F} \in L^1\big(\partial\widetilde{\Omega}, \widetilde{\sigma}\big). \tag{9.3.42}$$

Granted these properties, Proposition 4.7.4 applies and gives $[\widetilde{F}]_\infty = 0$. In concert with (9.3.38), this ultimately shows that $[\vec{F}]_\infty = 0$ when $n = 1$ and the assumptions in (1.5.12) are imposed.

Finally, the fact that $[\vec{F}]_\infty = 0$ provided (1.5.13) holds is clear from the first part of Lemma 4.7.3. The proof of Theorem 1.5.1 is therefore complete. □

The proof of Corollary 1.5.2, presented next, makes essential use of Theorem 1.5.1.

Proof of Corollary 1.5.2 Work under the more general conditions specified in (1.5.23). Fix an arbitrary aperture parameter $\kappa' > 0$. Granted the current assumptions, Corollary 8.9.9 guarantees that

$$\text{the nontangential boundary trace } \vec{F}\Big|_{\partial\Omega}^{\kappa'-\text{n.t.}} \text{ exists at } \sigma-\text{a.e. point on } \partial_{\text{nta}}\Omega, \text{ is actually independent of } \kappa', \text{ and } \mathcal{N}^{\varepsilon}_{\kappa'}\vec{F} \in L^1_{\text{loc}}(\partial\Omega, \sigma). \tag{9.3.43}$$

In view of this, item *(iii)* in Proposition 8.8.6, and the last condition in (1.5.23) we conclude that

$$\nu \cdot \big(\vec{F}\,\Big|_{\partial\Omega}^{\kappa'-\text{n.t.}}\big) = \nu \cdot \big(\vec{F}\,\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) \in L^1(\partial_*\Omega, \sigma). \tag{9.3.44}$$

Also, from (9.3.43) and item *(ii)* in Proposition 8.8.6 we see that

$$\vec{F}\Big|_{\partial\Omega}^{\kappa'-\text{n.t.}} \text{ exists at } \sigma-\text{a.e. point on } A_{\kappa'}(\partial\Omega). \tag{9.3.45}$$

Finally, having $\mathcal{N}^{\varepsilon}_{\kappa'}\vec{F} \in L^1_{\text{loc}}(\partial\Omega, \sigma)$ implies that the condition formulated in (1.5.8) is presently true.

At this stage, choosing $\kappa' > 0$ sufficiently large (specifically, $\kappa' > \kappa_\Omega$ where κ_Ω is as in Theorem 1.5.1) ensures that all hypotheses of Theorem 1.5.1 are satisfied. As such, Theorem 1.5.1 applies and yields all claims in the statement of Corollary 1.5.2. □

9.4 Proofs of Theorem 1.6.1 and Corollaries 1.6.2–1.6.6

We begin by giving the proof of Theorem 1.6.1.

Proof of Theorem 1.6.1 As noted in (4.5.10), the fact that we are assuming that K has finite upper $(n-d)$-dimensional Minkowski content forces $\mathcal{L}^n(K) = 0$. In concert with (5.2.5) and (5.2.2), this implies $\partial_*\Omega = \partial_*\widetilde{\Omega} \subseteq \partial\widetilde{\Omega}$. In particular, $\mathcal{H}^{n-1}(\partial_*\Omega) \leq \mathcal{H}^{n-1}(\partial\widetilde{\Omega}) < +\infty$ which, in light of (5.6.35), proves that $\Omega, \widetilde{\Omega}$ are sets of locally finite perimeter. Having established this, Proposition 5.6.4 applies and gives that

$$\text{the geometric measure theoretic outward unit normals to } \Omega \text{ and } \widetilde{\Omega} \text{ coincide at } \mathcal{H}^{n-1}-\text{a.e. point on } \partial_*\Omega = \partial_*\widetilde{\Omega}. \tag{9.4.1}$$

Let us also observe that, as visible from (8.1.2), the fact that K is a compact subset of $\widetilde{\Omega}$ implies that

$$\text{for each aperture parameter } \kappa > 0 \text{ the nontangential approach regions } \Gamma_{\Omega,\kappa}(x) \text{ and } \Gamma_{\widetilde{\Omega},\kappa}(x) \text{ with apex at } x \text{ (considered from within } \Omega \text{ and } \widetilde{\Omega}, \text{ respectively) coincide near each point } x \in \partial\widetilde{\Omega} \subseteq \partial\Omega. \tag{9.4.2}$$

Since $\widetilde{\sigma} := \mathcal{H}^{n-1}\lfloor\partial\widetilde{\Omega}$ is currently assumed to be a finite measure, Proposition 8.8.3 ensures that there exists some $\kappa_{\widetilde{\Omega}} \in (0, \infty)$ with the property that if $\kappa > \kappa_{\widetilde{\Omega}}$ then

$$\mathcal{H}^{n-1}\big(\partial_*\widetilde{\Omega} \setminus A_\kappa(\partial\widetilde{\Omega})\big) = 0. \tag{9.4.3}$$

From (8.8.2) and the property recorded in (9.4.2) we also see that

$$A_\kappa(\partial\widetilde{\Omega}) \subseteq A_\kappa(\partial\Omega) \setminus K \text{ for each aperture parameter } \kappa > 0. \tag{9.4.4}$$

One notable consequence of (9.4.3), the fact that $\partial_*\Omega = \partial_*\widetilde{\Omega}$ (cf. (9.4.1)), and (9.4.4) is that

$$\mathcal{H}^{n-1}\Big(\partial_*\Omega \setminus \big(A_\kappa(\partial\Omega) \setminus K\big)\Big) = 0 \ \text{ if } \ \kappa > \kappa_{\widetilde{\Omega}}. \tag{9.4.5}$$

Together with the first line in (1.6.3), this implies that

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists at } \mathcal{H}^{n-1} - \text{a.e. point on } \partial_*\Omega \text{ if } \kappa > \kappa_{\widetilde{\Omega}}. \tag{9.4.6}$$

Going further, denote by $\vec{G}$ the extension by zero of $\vec{F}$ from Ω to $\widetilde{\Omega}$. Then since K_o is a compact subset of Ω, we see from (1.6.1) that

$$\vec{G} \in \Big[L^{\frac{d}{d-1}}(\widetilde{\Omega}, \mathcal{L}^n) + \mathscr{E}'_{K_o}(\widetilde{\Omega})\Big]^n \subset \big[\mathcal{D}'(\widetilde{\Omega})\big]^n. \tag{9.4.7}$$

Also, as a consequence of (9.4.6) and (9.4.2), whenever $\kappa > \kappa_{\widetilde{\Omega}}$ it follows that

$$\begin{array}{l} \text{the nontangential boundary trace } \vec{G}\Big|_{\partial\widetilde{\Omega}}^{\kappa-\text{n.t.}}, \text{ taken from within } \widetilde{\Omega}, \\ \text{exists and coincides with } \vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ at } \mathcal{H}^{n-1} - \text{a.e. point on } \partial_*\Omega. \end{array} \tag{9.4.8}$$

Our next remark pertains to $\operatorname{div}\vec{G} \in \mathcal{D}'(\widetilde{\Omega})$. Specifically, set

$$f := \operatorname{div}\vec{F} \in L^1(\Omega, \mathcal{L}^n) + \mathscr{E}'_{K_o}(\Omega) \tag{9.4.9}$$

and denote by $\widetilde{f}$ the extension by zero of f to $\widetilde{\Omega} = \Omega \cup K$. Then

$$\widetilde{f} \in L^1(\widetilde{\Omega}, \mathcal{L}^n) + \mathscr{E}'_{K_o}(\widetilde{\Omega}) \ \text{ and } \ \operatorname{div}\big(\vec{G}\big|_\Omega\big) = \widetilde{f}\,\big|_\Omega \ \text{ in } \ \mathcal{D}'(\Omega). \tag{9.4.10}$$

Granted this, the removability criterion for singularities of distributional derivatives from Proposition 4.5.2 (currently used with $D := \operatorname{div}$, with $\widetilde{\Omega}$ in place of Ω, and with $u := \vec{G}$) gives that

$$\operatorname{div}\vec{G} = \widetilde{f} \ \text{ in } \ \mathcal{D}'(\widetilde{\Omega}). \tag{9.4.11}$$

Having established (9.4.11), we may invoke Theorem 1.5.1 (used with $\widetilde{\Omega}$ in place of Ω) to conclude that

$${}_{(\mathscr{C}_b^\infty(\widetilde{\Omega}))^*}\big(\widetilde{f}, 1\big)_{\mathscr{C}_b^\infty(\widetilde{\Omega})} = \int_{\partial_*\widetilde{\Omega}} \widetilde{\nu} \cdot \Big(\vec{G}\,\Big|_{\partial\widetilde{\Omega}}^{\kappa-\text{n.t.}}\Big)\, d\mathcal{H}^{n-1}. \tag{9.4.12}$$

Keeping in mind that $\mathcal{L}^n(\widetilde{\Omega} \setminus \Omega) = 0$ and that K_o is a compact subset of Ω, it is also apparent from (4.6.19) and (9.4.9)–(9.4.10) that

$$ {}_{(\mathscr{C}_b^\infty(\widetilde{\Omega}))^*}\big(\widetilde{f}, 1\big)_{\mathscr{C}_b^\infty(\widetilde{\Omega})} = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(f, 1\big)_{\mathscr{C}_b^\infty(\Omega)} = {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)}. \tag{9.4.13} $$

At this stage, (1.6.4) becomes a consequence of (9.4.12), (9.4.1), (9.4.8), and (9.4.13). □

Next, we present the proof of Corollary 1.6.2.

Proof of Corollary 1.6.2 Observe that since Ω is a subset of $\widetilde{\Omega}$, it follows that Ω is bounded and that $\partial\Omega$ is contained in the closure of $\widetilde{\Omega}$. We claim that, in fact,

$$ \partial\Omega \subseteq \partial\widetilde{\Omega} \cup K. \tag{9.4.14} $$

To justify this, pick some $x \in \mathbb{R}^n$ which does not belong to $\partial\widetilde{\Omega} \cup K$. Then $x \in \mathbb{R}^n \setminus \partial\widetilde{\Omega}$ so either x is in $\widetilde{\Omega}$, or x does not belong to the closure of $\widetilde{\Omega}$. On the one hand, the latter scenario implies (in light of our earlier observation) that x does not belong to $\partial\Omega$. On the other hand, if $x \in \widetilde{\Omega}$ then $x \in \widetilde{\Omega} \setminus K = \Omega$, so once again x does not belong to $\partial\Omega$ (given that Ω is open, hence disjoint from $\partial\Omega$). All together, this proves that in all cases $x \notin \partial\Omega$, finishing the proof of (9.4.14). In turn, from (9.4.14) and assumptions we deduce that

$$ \mathcal{H}^{n-1}(\partial\Omega) \leq \mathcal{H}^{n-1}(\partial\widetilde{\Omega}) + \mathcal{H}^{n-1}(K) < +\infty. \tag{9.4.15} $$

Also, from (1.6.5), Proposition 8.2.3, and (9.4.15) we conclude that for each aperture parameter $\kappa > 0$ we have (cf. (8.3.8))

$$ \mathcal{N}_\kappa\vec{F} \in L^\infty(\partial\Omega, \mathcal{H}^{n-1}) \subseteq L^1(\partial\Omega, \mathcal{H}^{n-1}). \tag{9.4.16} $$

Finally, since Ω is open and bounded, from (1.6.5) we conclude that

$$ \vec{F} \in \big[L^\infty(\Omega, \mathcal{L}^n)\big]^n \subseteq \big[L^{\frac{d}{d-1}}(\Omega, \mathcal{L}^n)\big]^n. \tag{9.4.17} $$

Granted these, Theorem 1.6.1 applies and (1.6.4) yields (1.6.7) on account of (4.6.23). □

Here is the proof of Corollary 1.6.3:

Proof of Corollary 1.6.3 This is a direct consequence of Corollary 1.6.2 upon observing that $\vec{F}_{\mathrm{b}}$ agrees with $\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ at $\mathcal{H}^{n-1}$-a.e. point on $A_\kappa(\partial\Omega) \setminus K$. □

We continue by discussing the proof of Corollary 1.6.4.

Proof of Corollary 1.6.4 Let $\{x_1, \dots, x_N\}$ be the collection of connected components of $\partial\Omega$ which are singletons. Since $\partial\Omega$ is assumed to have finitely many connected components, this is a finite set of isolated points in $\partial\Omega$. Hence, there exist $r_1, \dots, r_N > 0$ with the property that

$$B(x_j, r_j) \cap \partial\Omega = \{x_j\} \quad \text{for each} \quad j \in \{1, \dots, N\}. \tag{9.4.18}$$

Fix $j \in \{1, \dots, N\}$. Then $A_j := B(x_j, r_j) \setminus \{x_j\}$ is a subset of $\mathbb{R}^2$, itself a set covered by the (disjoint) union of Ω, $\partial\Omega$, and $\mathbb{R}^2 \setminus \overline{\Omega}$. Since, as visible from (9.4.18), the set A_j is disjoint from $\partial\Omega$, it follows that A_j is covered by Ω and $\mathbb{R}^2 \setminus \overline{\Omega}$. Given that A_j is connected, we conclude that

$$\text{either } A_j \subseteq \Omega, \text{ or } A_j \subseteq \mathbb{R}^2 \setminus \overline{\Omega}. \tag{9.4.19}$$

On the other hand, the fact that x_j belongs to $\partial\Omega$ forces $B(x_j, r_j) \cap \Omega \neq \varnothing$. Consequently, there exists a point $y_j \in B(x_j, r_j) \cap \Omega$. Note that we cannot have $y_j = x_j$ simply because $y_j \in \Omega$, $x_j \in \partial\Omega$, and $\Omega \cap \partial\Omega = \varnothing$ given that Ω is open. Hence, necessarily, we have $\big(B(x_j, r_j) \setminus \{x_j\}\big) \cap \Omega \neq \varnothing$. This proves that A_j overlaps with Ω which, in light of (9.4.19), ultimately shows that

$$B(x_j, r_j) \setminus \{x_j\} \subseteq \Omega \quad \text{for each} \quad j \in \{1, \dots, N\}. \tag{9.4.20}$$

To proceed, introduce

$$\widetilde{\Omega} := \Omega \cup \{x_1, \dots, x_N\} \tag{9.4.21}$$

and note that (9.4.20) implies

$$B(x_j, r_j) \subseteq \widetilde{\Omega} \quad \text{for each} \quad j \in \{1, \dots, N\}. \tag{9.4.22}$$

From (9.4.21)–(9.4.22) and the fact that Ω is open we conclude that

$$\widetilde{\Omega} \text{ is an open subset of } \mathbb{R}^2. \tag{9.4.23}$$

Also, since (9.4.21) ensures that

$$\Omega \subseteq \widetilde{\Omega} \subseteq \Omega \cup \partial\Omega = \overline{\Omega}, \tag{9.4.24}$$

we deduce that

$$\text{the closure of } \widetilde{\Omega} \text{ is } \overline{\Omega}. \tag{9.4.25}$$

As a consequence of (9.4.21), (9.4.23), and (9.4.25) we then have

$$\partial\widetilde{\Omega} = \overline{\Omega} \setminus \widetilde{\Omega} = \overline{\Omega} \setminus \big(\Omega \cup \{x_1, \dots, x_N\}\big) = \partial\Omega \setminus \{x_1, \dots, x_N\}. \tag{9.4.26}$$

This has two notable consequences. First, (9.4.26) implies that

$$\mathcal{H}^1(\partial\widetilde{\Omega}) = \mathcal{H}^1(\partial\Omega) < +\infty. \tag{9.4.27}$$

Second, (9.4.26) guarantees that

$$\begin{aligned}&\text{the set } \partial\widetilde{\Omega} \text{ has finitely many connected components,}\\&\text{none of which happen to actually be singletons.}\end{aligned} \tag{9.4.28}$$

In turn, from (9.4.27), (9.4.28), (5.3.72), and Lemma 5.9.3 we conclude that

$$\partial\widetilde{\Omega} \text{ is lower Ahlfors regular.} \tag{9.4.29}$$

Parenthetically, we also wish to note that (9.4.2) continues to hold in the present setting and this implies that for each aperture parameter $\kappa > 0$ we have

$$A_\kappa(\partial\widetilde{\Omega}) = A_\kappa(\partial\Omega) \setminus \{x_1, \dots, x_N\}. \tag{9.4.30}$$

In addition, $\widetilde{\Omega}$ is nonempty and bounded, $K := \{x_1, \dots, x_N\}$ is a compact subset of $\widetilde{\Omega}$ having finite 0-dimensional upper Minkowski content in $\mathbb{R}^2$, and $\widetilde{\Omega} \setminus K = \Omega$. These properties make it possible to invoke Corollary 1.6.3 (with $n := 2$ and $d := 2$) and conclude that (1.6.12) holds. □

We now turn to the proof of Corollary 1.6.5.

Proof of Corollary 1.6.5 Define the open bounded set $\widetilde{\Omega} \subseteq \mathbb{R}^2$ as in the proof of Corollary 1.6.4 (cf. (9.4.21)) and set $K := \{x_1, \dots, x_N\}$, so that $\widetilde{\Omega} \setminus K = \Omega$ and $\mathcal{H}^1(K) = 0$. Granted (9.4.27) and (9.4.29), all desired conclusions follow from Theorem 1.6.1 (with $n := 2$ and $d := 2$). □

Finally, here is the proof of Corollary 1.6.6.

Proof of Corollary 1.6.6 The Divergence Formula claimed in (1.6.19) is implied by Corollary 1.6.5 employed with $\vec{F} := (Q, -P)$.

9.5 Proofs of Theorems 1.7.1, 1.7.2, and 1.7.6

Obviously, Theorem 1.7.1 is a particular case of Theorem 1.7.2, so we move on to providing the proof of the latter result.

Proof of Theorem 1.7.2 Observing that, as a consequence of Lemma 8.3.1 and the first line in (1.7.23), we have

$$u \in \big[L^\infty_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^{N'} \quad\text{and}\quad w \in \big[L^\infty_{\text{loc}}(\Omega, \mathcal{L}^n)\big]^{N}. \tag{9.5.1}$$

In particular, it makes sense to consider partial derivatives of u and w in the sense of distributions in Ω (and this is how Du and $D^\top w$ should be interpreted in (1.7.23)).

As far as the integration by parts formula (1.7.27) is concerned, the idea is to apply Theorem 1.2.1 to a suitably constructed vector field. Specifically, we define $\vec{F} : \Omega \to \mathbb{C}^n$ by the requirement that

$$\xi \cdot \vec{F}(x) = \Big\langle (-\mathrm{i})\mathrm{Sym}(D;\xi)u(x), w(x) \Big\rangle, \quad \forall \xi \in \mathbb{R}^n, \quad \text{for } \mathcal{L}^n - \text{a.e. point } x \in \Omega. \tag{9.5.2}$$

Since the right-hand side depends linearly in the variable $\xi \in \mathbb{R}^n$, the above demand determines $\vec{F}$ uniquely (as a vector-valued function defined $\mathcal{L}^n$-a.e. on Ω) and unambiguously. Moreover, from (9.5.1) and (9.5.2) it is clear that $\vec{F} \in \big[L^\infty_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n$. To compute its distributional divergence in Ω, fix an arbitrary (scalar-valued) test function $\varphi \in \mathscr{C}^\infty_c(\Omega)$ and write

$$\begin{aligned} {}_{\mathcal{D}'(\Omega)}\big\langle \mathrm{div}\vec{F}, \varphi \big\rangle_{\mathcal{D}(\Omega)} &= -{}_{[\mathcal{D}'(\Omega)]^n}\big\langle \vec{F}, \nabla\varphi \big\rangle_{[\mathcal{D}(\Omega)]^n} = -\int_\Omega \nabla\varphi \cdot \vec{F}\,\mathrm{d}\mathcal{L}^n \\ &= -\int_\Omega \big\langle (-\mathrm{i})\mathrm{Sym}(D;\nabla\varphi)u, w \big\rangle \,\mathrm{d}\mathcal{L}^n, \end{aligned} \tag{9.5.3}$$

where the last equality comes from (9.5.2) (presently used with $\xi := \nabla\varphi$). Recall that, by assumption $Du \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^N$. In concert with (9.5.1), this implies that $D(\varphi u) \in \big[L^1_{\rm comp}(\Omega, \mathcal{L}^n)\big]^N$, and since we have the commutator identity

$$(-\mathrm{i})\mathrm{Sym}(D;\nabla\varphi)u = D(\varphi u) - \varphi Du \ \text{ in } \ \Omega, \tag{9.5.4}$$

we may further express

$${}_{\mathcal{D}'(\Omega)}\big\langle \mathrm{div}\vec{F}, \varphi \big\rangle_{\mathcal{D}(\Omega)} = -\int_\Omega \big\langle D(\varphi u), w \big\rangle \,\mathrm{d}\mathcal{L}^n + \int_\Omega \big\langle \varphi Du, w \big\rangle \,\mathrm{d}\mathcal{L}^n =: I + II. \tag{9.5.5}$$

To handle I, introduce $\Omega_\varepsilon := \big\{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \varepsilon\big\}$ for each $\varepsilon > 0$ sufficiently small. Also, pick $\theta \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ such that $\mathrm{supp}\,\theta \subseteq B(0,1)$, $\int_{\mathbb{R}^n} \theta \,\mathrm{d}\mathcal{L}^n = 1$, and, for each $\varepsilon > 0$ set $\theta_\varepsilon(x) := \varepsilon^{-n}\theta(x/\varepsilon)$ for every $x \in \mathbb{R}^n$. Next, for each sufficiently small $\varepsilon > 0$ and each $x \in \Omega_\varepsilon$ define

$$w_\varepsilon(x) := \int_{\mathbb{R}^n} w(x-y)\theta_\varepsilon(y)\,\mathrm{d}y = \Big\{ {}_{\mathcal{D}'(\Omega)}\big\langle w_\alpha, \theta_\varepsilon(x - \cdot) \big\rangle_{\mathcal{D}(\Omega)} \Big\}_{1 \le \alpha \le N}. \tag{9.5.6}$$

If $D^\top$ is as in (1.7.13), for any $\varepsilon > 0$ small, any $x \in \Omega_\varepsilon$, and any $\beta \in \{1, \dots, N'\}$, we may then use (9.5.6) and the knowledge that $D^\top w \in \big[L^1_{\rm loc}(\Omega, \mathcal{L}^n)\big]^{N'}$ to compute

$$\begin{aligned} \big(D^\top w_\varepsilon\big)_\beta(x) &= {}_{\mathcal{D}'(\Omega)}\Big\langle \sum_{\alpha=1}^N \sum_{j=1}^n a_j^{\alpha\beta} w_\alpha, \partial_j[\theta_\varepsilon(x - \cdot)] \Big\rangle_{\mathcal{D}(\Omega)} \\ &\quad + {}_{\mathcal{D}'(\Omega)}\Big\langle \sum_{\alpha=1}^N b^{\alpha\beta} w_\alpha, \theta_\varepsilon(x - \cdot) \Big\rangle_{\mathcal{D}(\Omega)} \end{aligned}$$

$$
\begin{aligned}
&= {}_{\mathcal{D}'(\Omega)}\Big\langle \Big(-\sum_{\alpha=1}^{N}\sum_{j=1}^{n} a_j^{\alpha\beta}\partial_j + \sum_{\alpha=1}^{N} b^{\alpha\beta}\Big) w_\alpha, \theta_\varepsilon(x-\cdot)\Big\rangle_{\mathcal{D}(\Omega)} \\
&= {}_{\mathcal{D}'(\Omega)}\Big\langle (D^\top w)_\beta, \theta_\varepsilon(x-\cdot)\Big\rangle_{\mathcal{D}(\Omega)} \\
&= \int_\Omega (D^\top w)_\beta(y)\theta_\varepsilon(x-y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} (D^\top w)_\beta(x-y)\theta_\varepsilon(y)\,\mathrm{d}y. \qquad (9.5.7)
\end{aligned}
$$

Ultimately, from (9.5.1), (9.5.6), (9.5.7), and the fact that $D^\top w$ has locally integrable components in Ω, we conclude that

$$
\begin{aligned}
&w_\varepsilon \in \big[\mathscr{C}^\infty(\Omega_\varepsilon)\big]^{N'}, \quad w_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} w \text{ at every Lebesgue point of } w \text{ in } \Omega, \\
&\text{for each compact } K\subset\Omega \text{ one has } D^\top w_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} D^\top w \text{ in } \big[L^1(K,\mathcal{L}^n)\big]^{N'}, \qquad (9.5.8)\\
&\text{and there exists some } \varepsilon_K>0 \text{ such that } \sup_{0<\varepsilon<\varepsilon_K}\sup_{x\in K}|w_\varepsilon(x)|<\infty.
\end{aligned}
$$

Granted these, we may return to I in (9.5.5) and, with the help of Lebesgue's Dominated Convergence Theorem and (9.5.1), compute

$$
\begin{aligned}
I &= -\int_\Omega \big\langle D(\varphi u), w\big\rangle\,\mathrm{d}\mathcal{L}^n = -\lim_{\varepsilon\to 0^+}\int_\Omega \big\langle D(\varphi u), w_\varepsilon\big\rangle\,\mathrm{d}\mathcal{L}^n \\
&= -\lim_{\varepsilon\to 0^+} {}_{[\mathcal{D}'(\Omega)]^N}\big\langle D(\varphi u), w_\varepsilon\big\rangle_{[\mathcal{D}(\Omega)]^N} \\
&= -\lim_{\varepsilon\to 0^+} {}_{[\mathcal{D}'(\Omega)]^{N'}}\big\langle \varphi u, D^\top w_\varepsilon\big\rangle_{[\mathcal{D}(\Omega)]^{N'}} \\
&= -\lim_{\varepsilon\to 0^+}\int_\Omega \big\langle \varphi u, D^\top w_\varepsilon\big\rangle\,\mathrm{d}\mathcal{L}^n \\
&= -\int_\Omega \big\langle \varphi u, D^\top w\big\rangle\,\mathrm{d}\mathcal{L}^n. \qquad (9.5.9)
\end{aligned}
$$

Returning with this in (9.5.5) allows us to write

$$
\begin{aligned}
{}_{\mathcal{D}'(\Omega)}\big\langle \operatorname{div}\vec{F}, \varphi\big\rangle_{\mathcal{D}(\Omega)} &= -\int_\Omega \big\langle \varphi u, D^\top w\big\rangle\,\mathrm{d}\mathcal{L}^n + \int_\Omega \big\langle \varphi Du, w\big\rangle\,\mathrm{d}\mathcal{L}^n \\
&= \int_\Omega \Big\{\big\langle Du, w\big\rangle - \big\langle u, D^\top w\big\rangle\Big\}\varphi\,\mathrm{d}\mathcal{L}^n, \qquad (9.5.10)
\end{aligned}
$$

which goes to show that

$$
\operatorname{div}\vec{F} = \big\langle Du, w\big\rangle - \big\langle u, D^\top w\big\rangle \text{ in } \mathcal{D}'(\Omega). \qquad (9.5.11)
$$

In particular, from (9.5.11) and the last condition in (1.7.23) we conclude that

$$
\operatorname{div}\vec{F} \in L^1(\Omega,\mathcal{L}^n). \qquad (9.5.12)
$$

With $\kappa'' := \min\{\kappa, \kappa'\} > 0$, from (9.5.2), (8.2.25), and (8.2.10) we see that

$$\mathcal{N}_{\kappa''}\vec{F} \leq C\,\mathcal{N}_{\kappa}u \cdot \mathcal{N}_{\kappa'}w \text{ pointwise on } \partial\Omega. \tag{9.5.13}$$

Thanks to (8.2.26) and the second line in (1.7.23), this pointwise estimate ultimately proves that

$$\mathcal{N}_{\kappa''}\vec{F} \in L^1(\partial\Omega, \sigma). \tag{9.5.14}$$

In addition, from (9.5.2) we conclude that the nontangential trace $\vec{F}\big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}}$ exists σ-a.e. on $\partial_{\mathrm{nta}}\Omega$ and, in fact, for σ-a.e. $x \in \partial_*\Omega$ we have

$$\begin{aligned}\nu(x)\cdot\Big(\vec{F}\big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}}\Big)(x) &= \Big\langle (-\mathrm{i})\mathrm{Sym}(D;\nu)\Big(u\big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}}\Big)(x), \Big(w\big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}}\Big)(x)\Big\rangle \\ &= \Big\langle (-\mathrm{i})\mathrm{Sym}(D;\nu)\Big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x), \Big(w\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\Big)(x)\Big\rangle. \end{aligned} \tag{9.5.15}$$

Finally, it is clear from (9.5.2) that the growth condition (1.2.3) is satisfied whenever (1.7.25) holds. At this stage, formula (1.7.27) follows readily from (1.2.2), with the help of (9.5.11) and (9.5.15). This concludes the proof of Theorem 1.7.2. □

We continue by presenting the proof of Theorem 1.7.6.

Proof of Theorem 1.7.6 Lemma 8.3.1 and the first line in (1.7.41) imply

$$u \in L^\infty_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\!\ell_n \text{ and } w \in L^\infty_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\!\ell_n. \tag{9.5.16}$$

As such, it is meaningful to consider partial derivatives of u and w in the sense of distributions in Ω (and this is how $D_R u$ and $D_L w$ should be interpreted in (1.7.41)).

Next, consider the vector field $\vec{F} = (F_j)_{1\leq j\leq n}$ with $\mathcal{C}\!\ell_n$-valued components given by

$$F_j := u \odot \mathbf{e}_j \odot w \text{ in } \Omega. \tag{9.5.17}$$

From (9.5.17) and (9.5.16) it is clear that

$$\vec{F} \in \big[L^\infty_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\!\ell_n\big]^n \subseteq \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\!\ell_n\big]^n. \tag{9.5.18}$$

For an arbitrary scalar-valued test function $\varphi \in \mathscr{C}^\infty_c(\Omega)$ we may write

$$\begin{aligned} {}_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\!\ell_n}\big\langle \mathrm{div}\vec{F}, \varphi\big\rangle_{\mathcal{D}(\Omega)} &= -\sum_{j=1}^n \int_\Omega F_j \partial_j\varphi \, d\mathcal{L}^n \\ &= -\int_\Omega u \odot (D\varphi) \odot w \, d\mathcal{L}^n. \end{aligned} \tag{9.5.19}$$

To proceed, consider $\Omega_\varepsilon := \big\{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \varepsilon\big\}$ for each $\varepsilon > 0$ sufficiently small. Pick $\theta \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ such that $\mathrm{supp}\,\theta \subseteq B(0,1)$, $\int_{\mathbb{R}^n}\theta\, d\mathcal{L}^n = 1$, and for each

$\varepsilon > 0$ set $\theta_\varepsilon(x) := \varepsilon^{-n}\theta(x/\varepsilon)$ for every $x \in \mathbb{R}^n$. For each sufficiently small $\varepsilon > 0$ and each $x \in \Omega_\varepsilon$ define

$$w_\varepsilon(x) := \int_{\mathbb{R}^n} w(x-y)\theta_\varepsilon(y)\,\mathrm{d}y = {}_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\ell_n}\big\langle w, \theta_\varepsilon(x-\cdot)\big\rangle_{\mathcal{D}(\Omega)}. \tag{9.5.20}$$

Using the fact that $D_L w \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\ell_n$ we may then compute

$$\begin{aligned}\big(D_L w_\varepsilon\big)(x) &= \sum_{j=1}^n {}_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\ell_n}\Big\langle \mathbf{e}_j \odot w, \partial_j[\theta_\varepsilon(x-\cdot)]\Big\rangle_{\mathcal{D}(\Omega)} \\ &= {}_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\ell_n}\Big\langle (D_L w), \theta_\varepsilon(x-\cdot)\Big\rangle_{\mathcal{D}(\Omega)} \\ &= \int_\Omega (D_L w)(y)\theta_\varepsilon(x-y)\,\mathrm{d}y = \int_{\mathbb{R}^n} (D_L w)(x-y)\theta_\varepsilon(y)\,\mathrm{d}y. \end{aligned} \tag{9.5.21}$$

Ultimately, from (9.5.16), (9.5.20), (9.5.21), and the fact that $D_L w$ is locally integrable in Ω, we conclude that

$$\begin{aligned}&w_\varepsilon \in \mathscr{C}^\infty(\Omega_\varepsilon) \otimes \mathcal{C}\ell_n, \quad w_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} w \text{ at every Lebesgue point of } w \text{ in } \Omega, \\ &\text{for each compact } K \subset \Omega \text{ one has } D_L w_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} D_L w \text{ in } L^1(K, \mathcal{L}^n) \otimes \mathcal{C}\ell_n, \\ &\text{and there exists some } \varepsilon_K > 0 \text{ such that } \sup\nolimits_{0<\varepsilon<\varepsilon_K} \sup\nolimits_{x\in K} |w_\varepsilon(x)| < \infty.\end{aligned} \tag{9.5.22}$$

Returning to (9.5.19), we may further express

$$\begin{aligned}{}_{\mathcal{D}'(\Omega)\otimes\mathcal{C}\ell_n}\big\langle \mathrm{div}\vec{F}, \varphi\big\rangle_{\mathcal{D}(\Omega)} &= -\int_\Omega u \odot (D\varphi) \odot w\,\mathrm{d}\mathcal{L}^n \\ &= -\lim_{\varepsilon\to 0^+}\int_\Omega u \odot (D\varphi) \odot w_\varepsilon\,\mathrm{d}\mathcal{L}^n = I + II,\end{aligned} \tag{9.5.23}$$

where

$$I := -\lim_{\varepsilon\to 0^+}\int_\Omega u \odot D_L(\varphi w_\varepsilon)\,\mathrm{d}\mathcal{L}^n, \quad \text{and} \tag{9.5.24}$$

$$II := \lim_{\varepsilon\to 0^+}\int_\Omega u \odot \varphi(D_L w_\varepsilon)\,\mathrm{d}\mathcal{L}^n = \int_\Omega u \odot \varphi(D_L w)\,\mathrm{d}\mathcal{L}^n, \tag{9.5.25}$$

thanks to (9.5.22) and (9.5.16). To handle term I from (9.5.24), with the help of Lebesgue's Dominated Convergence Theorem, (9.5.16), (6.4.43), and (6.4.51), we write (bearing in mind that $D_R u \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\ell_n$)

$$
\begin{aligned}
I &= - \lim_{\varepsilon \to 0^+} {}_{\mathcal{D}'(\Omega)\otimes \mathcal{C}\ell_n}\big\langle u, D_L(\varphi w_\varepsilon)\big\rangle_{\mathcal{D}(\Omega)\otimes \mathcal{C}\ell_n} \\
&= \lim_{\varepsilon \to 0^+} {}_{\mathcal{D}'(\Omega)\otimes \mathcal{C}\ell_n}\big\langle D_R u, \varphi w_\varepsilon\big\rangle_{\mathcal{D}(\Omega)\otimes \mathcal{C}\ell_n} \\
&= \lim_{\varepsilon \to 0^+} \int_\Omega (D_R u) \odot (\varphi w_\varepsilon)\, \mathrm{d}\mathcal{L}^n \\
&= \int_\Omega (D_R u) \odot (\varphi w)\, \mathrm{d}\mathcal{L}^n.
\end{aligned} \tag{9.5.26}
$$

Collectively, (9.5.23)–(9.5.26) prove that

$$
\operatorname{div}\vec{F} = (D_R u) \odot w + u \odot (D_L w) \ \text{ in } \ \mathcal{D}'(\Omega) \otimes \mathcal{C}\ell_n. \tag{9.5.27}
$$

In concert with the last condition in (1.7.41) this shows that

$$
\operatorname{div}\vec{F} \in L^1(\Omega, \mathcal{L}^n) \otimes \mathcal{C}\ell_n. \tag{9.5.28}
$$

Considering $\kappa'' := \min\{\kappa, \kappa'\} > 0$ allows us to conclude, on account of (9.5.17), (8.2.25), (8.2.10), and (6.4.34), that $\mathcal{N}_{\kappa''}\vec{F} \leq C\, \mathcal{N}_\kappa u \cdot \mathcal{N}_{\kappa'} w$ pointwise on $\partial\Omega$. Together with (8.2.26) and the second line in (1.7.41), this yields $\mathcal{N}_{\kappa''}\vec{F} \in L^1(\partial\Omega, \sigma)$. Furthermore, from (9.5.17) and (1.7.41) we see that the nontangential trace $\vec{F}\big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}}$ exists σ-a.e. on $\partial_{\mathrm{nta}}\Omega$ and, in fact, at σ-a.e. $x \in \partial_*\Omega$ we have

$$
\begin{aligned}
\sum_{j=1}^n \nu_j(x)\Big(\vec{F}\,\big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}}\Big)_j(x) &= \Big(u\big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}}\Big)(x) \odot \nu(x) \odot \Big(w\big|_{\partial\Omega}^{\kappa''-\mathrm{n.t.}}\Big)(x) \\
&= \Big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big)(x) \odot \nu(x) \odot \Big(w\big|_{\partial\Omega}^{\kappa'-\mathrm{n.t.}}\Big)(x).
\end{aligned} \tag{9.5.29}
$$

Moreover, as is apparent from (9.5.2), the growth condition (1.2.3) is satisfied whenever (1.7.43) holds. Granted these properties, formula (1.7.42) follows from (1.2.2), with the help of (9.5.27) and (9.5.29). This finishes the proof of Theorem 1.7.6. □

9.6 Proofs of Theorems 1.8.2, 1.8.3, and 1.8.5

Before presenting the proof of Theorem 1.8.2, we establish the following auxiliary result.

Lemma 9.6.1 *For every fixed multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| > 0$ the following identity holds*

$$
\sum_{\substack{\beta \in \mathbb{N}_0^n,\, 1 \leq i \leq n \\ \beta + e_i = \alpha}} \frac{1}{\beta!} = \frac{|\alpha|}{\alpha!}. \tag{9.6.1}
$$

Proof Given an arbitrary multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| > 0$, set $\operatorname{supp}\alpha := \big\{i \in \{1, \dots, n\} : \alpha_i \neq 0\big\}$. Note it is possible to decompose $\alpha = \beta + e_i$ for some $\beta \in \mathbb{N}_0^n$ and $i \in \{1, \dots, n\}$ if and only if $i \in \operatorname{supp}\alpha$. Therefore,

$$\sum_{\substack{\beta\in\mathbb{N}_0^n,\,1\leq i\leq n\\ \beta+e_i=\alpha}} \frac{1}{\beta!} = \sum_{i\in\operatorname{supp}\alpha} \frac{1}{(\alpha-e_i)!} = \sum_{i\in\operatorname{supp}\alpha} \frac{\alpha_i}{\alpha!} = \frac{1}{\alpha!}\sum_{i\in\operatorname{supp}\alpha} \alpha_i = \frac{|\alpha|}{\alpha!}, \tag{9.6.2}$$

as wanted. □

Here is the proof of Theorem 1.8.2.

Proof of Theorem 1.8.2 The idea of the proof is to apply the Divergence Formula as featured in Theorem 1.2.1. To this end, consider the vector field $\vec{F} = (F_j)_{1\leq j\leq n}$ whose components are given by

$$F_j := \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^n\\ \beta+\gamma+e_j=\alpha\\ |\gamma|=\ell-1,\,|\beta|=|\alpha|-\ell}} (-1)^{\ell+1}\frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\beta!\gamma!}(\partial^\beta u)(\partial^\gamma w), \tag{9.6.3}$$

for each $j \in \{1, \dots, n\}$. In relation to the sum in (9.6.3) we note that if $\beta, \gamma \in \mathbb{N}_0^n$ and $j \in \{1, \dots, n\}$ are such that $\beta + \gamma + e_j = \alpha$, then the functions $f := \partial^\beta u$ and $g := \partial^\gamma w$ satisfy the hypotheses of Proposition 4.3.1. Indeed, since u, w belong to $W^{\alpha,1}_{\mathrm{loc}}(\Omega)$ and satisfy *(1)*, we have

$$f,\ g,\ \partial_j f,\ \partial_j g \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \ \text{ and } \ f\cdot g \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n). \tag{9.6.4}$$

Also, assumption *(1)* and Lemma 8.3.1 (with $\mu := \sigma$ which ensures that (8.3.1) holds, due to the lower Ahlfors regularity assumption on $\partial\Omega$) imply the membership $f, g \in L^\infty_{\mathrm{loc}}(\Omega, \mathcal{L}^n)$. The latter combined with (9.6.4) further yields

$$f(\partial_j g) \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \ \text{ and } \ g(\partial_j f) \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n). \tag{9.6.5}$$

Consequently, Proposition 4.3.1 applies and we obtain $\partial_j(fg) = f(\partial_j g) + g(\partial_j f)$ in $\mathcal{D}'(\Omega)$ which, when written in terms of u and w, becomes

$$\begin{aligned} &\partial_j\big[(\partial^\beta u)(\partial^\gamma w)\big] = (\partial^\beta u)(\partial^{\gamma+e_j} w) + (\partial^\gamma w)(\partial^{\beta+e_j} u) \ \text{ in } \ \mathcal{D}'(\Omega)\\ &\quad \text{for each } \beta, \gamma \in \mathbb{N}_0^n \text{ and } j \in \{1, \dots, n\} \text{ with } \beta+\gamma+e_j=\alpha. \end{aligned} \tag{9.6.6}$$

In light of this, we may write

$$\operatorname{div}\vec{F} = \sum_{j=1}^n \partial_j F_j = I + II \ \text{ in } \ \mathcal{D}'(\Omega), \tag{9.6.7}$$

where

$$I := \sum_{j=1}^{n} \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^n \\ \beta+\gamma+e_j=\alpha \\ |\gamma|=\ell-1,\ |\beta|=|\alpha|-\ell}} (-1)^{\ell+1} \frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\beta!\gamma!} (\partial^{\beta+e_j} u)(\partial^{\gamma} w), \tag{9.6.8}$$

and

$$II := \sum_{j=1}^{n} \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^n \\ \beta+\gamma+e_j=\alpha \\ |\gamma|=\ell-1,\ |\beta|=|\alpha|-\ell}} (-1)^{\ell+1} \frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\beta!\gamma!} (\partial^{\beta} u)(\partial^{\gamma+e_j} w). \tag{9.6.9}$$

The next task is to re-write I and II in a form that allows for cancelations when the two are combined. We first focus on I. Changing the order of the summation and applying Lemma 9.6.1 we may recast I as

$$\begin{aligned} I &= \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\gamma\in\mathbb{N}_0^n \\ |\gamma|=\ell-1 \\ \gamma<\alpha}} \Bigg(\sum_{\substack{\beta\in\mathbb{N}_0^n, j\in\{1,\dots,n\} \\ \beta+e_j=\alpha-\gamma}} \frac{1}{\beta!} (\partial^{\beta+e_j} u) \Bigg) (-1)^{\ell+1} \frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\gamma!} (\partial^{\gamma} w) \\ &= \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\gamma\in\mathbb{N}_0^n \\ |\gamma|=\ell-1 \\ \gamma<\alpha}} \frac{|\alpha-\gamma|}{(\alpha-\gamma)!} (-1)^{\ell+1} \frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\gamma!} (\partial^{\alpha-\gamma} u)(\partial^{\gamma} w) \\ &= \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\gamma\in\mathbb{N}_0^n \\ |\gamma|=\ell-1 \\ \gamma<\alpha}} \frac{|\alpha|-\ell+1}{(\alpha-\gamma)!} (-1)^{\ell+1} \frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\gamma!} (\partial^{\alpha-\gamma} u)(\partial^{\gamma} w) \\ &= \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\gamma\in\mathbb{N}_0^n \\ |\gamma|=\ell-1 \\ \gamma<\alpha}} (-1)^{\ell+1} \frac{(|\alpha|-\ell+1)!}{(\alpha-\gamma)!} \frac{\alpha!(\ell-1)!}{|\alpha|!\gamma!} (\partial^{\alpha-\gamma} u)(\partial^{\gamma} w), \end{aligned} \tag{9.6.10}$$

where in the third equality we used the fact that if $\gamma < \alpha$ and $|\gamma| = \ell - 1$, then $|\alpha - \gamma| = |\alpha| - \ell + 1$. Re-denoting $\ell - 1$ by ℓ in the last expression in (9.6.10), then setting $\beta := \alpha - \gamma$ (so that $|\beta| = |\alpha| - \ell$), we may further write

$$I = \sum_{\ell=0}^{|\alpha|-1} \sum_{\substack{\gamma\in\mathbb{N}_0^n \\ |\gamma|=\ell \\ \gamma<\alpha}} (-1)^{\ell+2} \frac{(|\alpha|-\ell)!}{(\alpha-\gamma)!} \frac{\alpha!\,\ell!}{|\alpha|!\gamma!} (\partial^{\alpha-\gamma} u)(\partial^{\gamma} w)$$

$$= \sum_{\ell=0}^{|\alpha|-1} \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^n \\ \beta+\gamma=\alpha \\ |\gamma|=\ell, |\beta|=|\alpha|-\ell}} (-1)^{\ell} \frac{|\beta|!}{\beta!} \frac{\alpha!\,\ell!}{|\alpha|!\gamma!} (\partial^{\beta} u)(\partial^{\gamma} w). \tag{9.6.11}$$

Moving on to II, by first applying Lemma 9.6.1 we obtain

$$\begin{aligned} II &= \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\beta\in\mathbb{N}_0^n \\ |\beta|=|\alpha|-\ell \\ \beta<\alpha}} \Bigg(\sum_{\substack{\gamma\in\mathbb{N}_0^n, j\in\{1,\dots,n\} \\ \gamma+e_j=\alpha-\beta}} \frac{1}{\gamma!} (\partial^{\gamma+e_j} w) \Bigg) (-1)^{\ell+1} \frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\beta!} (\partial^{\beta} u) \\ &= \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\beta\in\mathbb{N}_0^n \\ |\beta|=|\alpha|-\ell \\ \beta<\alpha}} \frac{|\alpha-\beta|}{(\alpha-\beta)!} (-1)^{\ell+1} \frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\beta!} (\partial^{\beta} u)(\partial^{\alpha-\beta} w). \end{aligned} \tag{9.6.12}$$

Furthermore, by re-denoting $\alpha-\beta$ by γ in the last sum in (9.6.12), we see that

$$\begin{aligned} II &= \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^n \\ \beta+\gamma=\alpha \\ |\gamma|=\ell, |\beta|=|\alpha|-\ell}} (-1)^{\ell+1} \frac{|\gamma|}{\gamma!} \frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\beta!} (\partial^{\beta} u)(\partial^{\gamma} w) \\ &= \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^n \\ \beta+\gamma=\alpha \\ |\gamma|=\ell, |\beta|=|\alpha|-\ell}} (-1)^{\ell+1} \frac{1}{\gamma!} \frac{\alpha!|\beta|!\ell!}{|\alpha|!\beta!} (\partial^{\beta} u)(\partial^{\gamma} w). \end{aligned} \tag{9.6.13}$$

Now we may combine (9.6.11) and (9.6.13) to conclude that

$$\begin{aligned} \operatorname{div}\vec{F} = I + II &= \sum_{\ell=0}^{|\alpha|-1} \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^n \\ \beta+\gamma=\alpha \\ |\gamma|=\ell, |\beta|=|\alpha|-\ell}} (-1)^{\ell} \frac{|\beta|!}{\beta!} \frac{\alpha!\,\ell!}{|\alpha|!\gamma!} (\partial^{\beta} u)(\partial^{\gamma} w) \qquad (9.6.14) \\ &\quad + \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^n \\ \beta+\gamma=\alpha \\ |\gamma|=\ell, |\beta|=|\alpha|-\ell}} (-1)^{\ell+1} \frac{1}{\gamma!} \frac{\alpha!|\beta|!\ell!}{|\alpha|!\beta!} (\partial^{\beta} u)(\partial^{\gamma} w) \\ &= (\partial^{\alpha} u) w + (-1)^{|\alpha|+1} u (\partial^{\alpha} w) = (\partial^{\alpha} u) w - (-1)^{|\alpha|} u (\partial^{\alpha} w). \end{aligned}$$

In summary, by also recalling (1.8.2), we have proved

$$\operatorname{div}\vec{F} = (\partial^{\alpha} u) w - (-1)^{|\alpha|} u (\partial^{\alpha} w) \in L^1(\Omega, \mathcal{L}^n). \tag{9.6.15}$$

In addition, definition (9.6.3) and assumptions *(1)–(2)* imply

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exists } \sigma-\text{a.e. on } \partial_{\text{nta}}\Omega \text{ and } \mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega,\sigma), \tag{9.6.16}$$

while (9.6.3) and (1.8.3) ensure that, whenever Ω is unbounded and $\partial\Omega$ is bounded, there exists $\lambda \in (1,\infty)$ such that

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |\vec{F}|\, d\mathcal{L}^n = o(R) \text{ as } R\to\infty. \tag{9.6.17}$$

At this point, we have checked that $\vec{F}$ satisfies all hypotheses of Theorem 1.2.1, hence (1.2.2) holds for the current vector field $\vec{F}$ and implies (1.8.4) upon observing that

$$\nu\cdot\Big(\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big) = \sum_{\ell=1}^{|\alpha|} \sum_{\substack{\beta,\gamma\in\mathbb{N}_0^n \\ j\in\{1,\dots,n\},\,\beta+\gamma+e_j=\alpha \\ |\gamma|=\ell-1,\,|\beta|=|\alpha|-\ell}} (-1)^{\ell+1}\frac{\alpha!(|\alpha|-\ell)!(\ell-1)!}{|\alpha|!\beta!\gamma!}\times$$
$$\times\nu_j(\partial^\beta u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\cdot(\partial^\gamma w)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \tag{9.6.18}$$

for σ-a.e. point on $\partial_*\Omega$ (which, up to a σ-nullset, is contained in $\partial_{\text{nta}}\Omega$; see (8.8.52)).

Next, we present the proof of Theorem 1.8.3, pertaining to the "half" Green formula for a higher-order system.

Proof of Theorem 1.8.3 The proof is similar to that of Theorem 1.8.2, so we shall only highlight the main steps involved. Once again, the idea is to apply Theorem 1.2.1, this time to the vector field $\vec{F}=(F_j)_{1\le j\le n}$ with components

$$F_j := \sum_{\substack{|\alpha|=|\beta|=m \\ \alpha\ge e_j}} \sum_{\ell=1}^{m} \sum_{\substack{\delta,\gamma\in\mathbb{N}_0^n \\ \delta+\gamma+e_j=\alpha \\ |\gamma|=\ell-1,\,|\delta|=m-\ell}} (-1)^{\ell+1}\frac{\alpha!(m-\ell)!(\ell-1)!}{m!\delta!\gamma!}\times$$
$$\times\Big\langle A_{\alpha\beta}(\partial^{\beta+\delta}u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}},(\partial^\gamma w)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}\Big\rangle \tag{9.6.19}$$

for each $j\in\{1,\dots,n\}$. Analogous to (9.6.14),

$$\text{div}\vec{F} = \langle Lu,w\rangle - \sum_{|\alpha|=|\beta|=m}(-1)^m\langle A_{\alpha\beta}\partial^\beta u,\partial^\alpha w\rangle \tag{9.6.20}$$

which, thanks to (1.8.11) and the second line in (1.8.12), belongs to $L^1(\Omega,\mathcal{L}^n)$. Also, (1.8.12) ensures that the nontangential trace $\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists σ-a.e. on $\partial_{\text{nta}}\Omega$ and $\mathcal{N}_\kappa\vec{F}\in L^1(\partial\Omega,\sigma)$, while (1.8.13) implies that, whenever Ω is unbounded and $\partial\Omega$ is bounded, there exists $\lambda\in(1,\infty)$ such that

$$\int_{[B(0,\lambda R)\setminus B(0,R)]\cap\Omega} |\vec{F}|\,\mathrm{d}\mathcal{L}^n = o(R) \quad \text{as} \quad R \to \infty. \tag{9.6.21}$$

Having verified that $\vec{F}$ satisfies all hypotheses of Theorem 1.2.1, formula (1.8.14) is obtained by writing (1.2.2) for our current $\vec{F}$. □

Proof of Theorem 1.8.5 The reasoning is similar to the one used in proving the higher-order integration by parts formula (1.8.4). The plan is to apply the Divergence Formula stated in Theorem 1.2.1 to a suitable vector field. Specifically, consider $\vec{F} = (F_j)_{1\leq j\leq n}$ whose jth component is given by

$$\begin{aligned} F_j := &\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ \alpha\geq e_j, |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\delta,\gamma\in\mathbb{N}_0^n \\ \delta+\gamma=\alpha-e_j \\ |\gamma|=\ell-1, |\delta|=m-\ell}} (-1)^{\ell+1} \frac{\alpha!(m-\ell)!(\ell-1)!}{m!\delta!\gamma!} \times \\ &\qquad\qquad \times \big\langle A_{\alpha\beta}(\partial^{\beta+\delta} u), (\partial^{\gamma} w)\big\rangle \\ &+ \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ \beta\geq e_j, |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\eta,\omega\in\mathbb{N}_0^n \\ \eta+\omega=\beta-e_j \\ |\omega|=\ell-1, |\eta|=m-\ell}} (-1)^{m+\ell+1} \frac{\beta!(m-\ell)!(\ell-1)!}{m!\eta!\,\omega!} \times \\ &\qquad\qquad \times \big\langle (\partial^{\eta} u), A_{\alpha\beta}^{\top}(\partial^{\alpha+\omega} w)\big\rangle \end{aligned} \tag{9.6.22}$$

for each $j \in \{1, \dots, n\}$. Next, fix $j \in \{1, \dots, n\}$ and $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha| = |\beta| = m$ and either $\alpha \geq e_j$ or $\beta \geq e_j$. Also, pick any $\theta, \mu \in \mathbb{N}_0^n$ such that $\theta + \mu + e_j = \alpha + \beta$ and define functions $f := \partial^{\theta} u$ and $g := \partial^{\mu} w$. Since $u, w \in [W^{2m,1}_{\mathrm{loc}}(\Omega)]^M$ and satisfy *(2)*, we have

$$f,\ g,\ \partial_j f,\ \partial_j g \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \quad \text{and} \quad f \cdot g \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n). \tag{9.6.23}$$

Also, the finiteness assumptions in *(2)* and Lemma 8.3.1 (with $\mu := \sigma$ which ensures that (8.3.1) holds, due to the lower Ahlfors regularity assumption on $\partial\Omega$) imply $f, g \in L^{\infty}_{\mathrm{loc}}(\Omega, \mathcal{L}^n)$. The latter combined with (9.6.23) further yields

$$f(\partial_j g) \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \quad \text{and} \quad g(\partial_j f) \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n). \tag{9.6.24}$$

Together, (9.6.23) and (9.6.24) ensure that f, g satisfy all hypotheses of Proposition 4.3.1 which gives $\partial_j(fg) = f(\partial_j g) + g(\partial_j f)$ in $\mathcal{D}'(\Omega)$. Recalling the definition of f and g we obtain

$$\begin{gathered} \partial_j\big[(\partial^{\theta} u)(\partial^{\mu} w)\big] = (\partial^{\theta} u)(\partial^{\mu+e_j} w) + (\partial^{\mu} w)(\partial^{\theta+e_j} u) \ \text{ in } \ \mathcal{D}'(\Omega) \ \text{ for any} \\ j \in \{1, \dots, n\} \text{ and } \alpha, \beta \in \mathbb{N}_0^n \text{ with } |\alpha| = |\beta| = m \text{ satisfying } \alpha \geq e_j \text{ or } \beta \geq e_j, \\ \text{and for any multi-indices } \theta, \mu \in \mathbb{N}_0^n \text{ such that } \theta + \mu + e_j = \alpha + \beta. \end{gathered} \tag{9.6.25}$$

Now formula (9.6.25) implies

$$\mathrm{div}\vec{F} = \sum_{j=1}^{n} \partial_j F_j = \sum_{j=1}^{n} (I_j + II_j + III_j + IV_j) \quad \text{in } \mathcal{D}'(\Omega), \tag{9.6.26}$$

where

$$I_j := \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ \alpha\geq e_j, |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\delta,\gamma\in\mathbb{N}_0^n \\ \delta+e_j+\gamma=\alpha \\ |\gamma|=\ell-1, |\delta|=m-\ell}} (-1)^{\ell+1} \frac{\alpha!(m-\ell)!(\ell-1)!}{m!\delta!\gamma!} \times \times \left\langle A_{\alpha\beta}(\partial^{\beta+\delta+e_j}u), \partial^{\gamma} w\right\rangle, \tag{9.6.27}$$

$$II_j := \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ \alpha\geq e_j, |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\delta,\gamma\in\mathbb{N}_0^n \\ \delta+e_j+\gamma=\alpha \\ |\gamma|=\ell-1, |\delta|=m-\ell}} (-1)^{\ell+1} \frac{\alpha!(m-\ell)!(\ell-1)!}{m!\delta!\gamma!} \times \times \left\langle A_{\alpha\beta}(\partial^{\beta+\delta}u), \partial^{\gamma+e_j} w\right\rangle, \tag{9.6.28}$$

$$III_j := \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ \beta\geq e_j, |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\eta,\omega\in\mathbb{N}_0^n \\ \eta+e_j+\omega=\beta \\ |\omega|=\ell-1, |\eta|=m-\ell}} (-1)^{m+\ell+1} \frac{\beta!(m-\ell)!(\ell-1)!}{m!\eta!\,\omega!} \times \times \left\langle \partial^{\eta+e_j}u, A_{\alpha\beta}^{\top}(\partial^{\alpha+\omega}w)\right\rangle, \tag{9.6.29}$$

and

$$IV_j := \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ \beta\geq e_j, |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\eta,\omega\in\mathbb{N}_0^n \\ \eta+e_j+\omega=\beta \\ |\omega|=\ell-1, |\eta|=m-\ell}} (-1)^{m+\ell+1} \frac{\beta!(m-\ell)!(\ell-1)!}{m!\eta!\,\omega!} \times \times \left\langle \partial^{\eta}u, A_{\alpha\beta}^{\top}(\partial^{\alpha+\omega+e_j}w)\right\rangle. \tag{9.6.30}$$

Considering first the terms in (9.6.27), we may write

$$\sum_{j=1}^{n} I_j = \sum_{j=1}^{n} \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n \\ |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\delta,\gamma\in\mathbb{N}_0^n \\ \delta+e_j+\gamma=\alpha \\ |\gamma|=\ell-1, |\delta|=m-\ell}} (-1)^{\ell+1} \frac{\alpha!(m-\ell)!(\ell-1)!}{m!\delta!\gamma!} \times \times \left\langle A_{\alpha\beta}(\partial^{\beta+\delta+e_j}u), \partial^{\gamma} w\right\rangle$$

$$= \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\widetilde{\delta},\gamma\in\mathbb{N}_0^n\\ \widetilde{\delta}+\gamma=\alpha\\ |\gamma|=\ell-1,|\widetilde{\delta}|=m-\ell+1}} \Bigg(\sum_{\substack{\delta\in\mathbb{N}_0^n,\,j\in\{1,\dots,n\}\\ \delta+e_j=\widetilde{\delta}}} \frac{1}{\delta!}\Bigg)\times$$
$$\times\,(-1)^{\ell+1}\frac{\alpha!(m-\ell)!(\ell-1)!}{m!\gamma!}\Big\langle A_{\alpha\beta}(\partial^{\beta+\widetilde{\delta}}u),\,\partial^{\gamma}w\Big\rangle$$
$$= \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\tilde{\delta},\gamma\in\mathbb{N}_0^n\\ \tilde{\delta}+\gamma=\alpha\\ |\gamma|=\ell-1,|\tilde{\delta}|=m-\ell+1}} (-1)^{\ell+1}\frac{\alpha!(m-\ell+1)!(\ell-1)!}{m!\,\tilde{\delta}!\gamma!}\times$$
$$\times\Big\langle A_{\alpha\beta}(\partial^{\beta+\tilde{\delta}}u),\,\partial^{\gamma}w\Big\rangle \tag{9.6.31}$$

where for the last equality in (9.6.31) we have invoked Lemma 9.6.1 which presently gives

$$\sum_{\substack{\delta\in\mathbb{N}_0^n,\,j\in\{1,\dots,n\}\\ \delta+e_j=\widetilde{\delta}}} \frac{1}{\delta!} = \frac{m-\ell+1}{\widetilde{\delta}!}. \tag{9.6.32}$$

Similarly, by summing up all II_j's, introducing $\widetilde{\gamma}=\gamma+e_j$, and once again invoking Lemma 9.6.1 for the sum

$$\sum_{\substack{\gamma\in\mathbb{N}_0^n,\,j\in\{1,\dots,n\}\\ \gamma+e_j=\widetilde{\gamma}}} \frac{1}{\gamma!} = \frac{\ell}{\widetilde{\gamma}!}, \tag{9.6.33}$$

we may also write

$$\sum_{j=1}^{n} II_j = \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\delta,\widetilde{\gamma}\in\mathbb{N}_0^n\\ \delta+\widetilde{\gamma}=\alpha\\ |\widetilde{\gamma}|=\ell,|\delta|=m-\ell}} (-1)^{\ell+1}\frac{\alpha!(m-\ell)!\ell!}{m!\delta!\widetilde{\gamma}!}\times$$
$$\times\big\langle A_{\alpha\beta}(\partial^{\beta+\delta}u),\,(\partial^{\widetilde{\gamma}}w)\big\rangle$$
$$= \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} \sum_{\ell=2}^{m+1} \sum_{\substack{\delta,\widetilde{\gamma}\in\mathbb{N}_0^n\\ \delta+\widetilde{\gamma}=\alpha\\ |\widetilde{\gamma}|=\ell-1,|\delta|=m-\ell+1}} (-1)^{\ell}\frac{\alpha!(m-\ell+1)!(\ell-1)!}{m!\delta!\widetilde{\gamma}!}\times$$
$$\times\big\langle A_{\alpha\beta}(\partial^{\beta+\delta}u),\,(\partial^{\widetilde{\gamma}}w)\big\rangle, \tag{9.6.34}$$

where the last equality in (9.6.34) is obtained by simply re-denoting $\ell+1$ by ℓ.

The resulting sums in (9.6.31) and (9.6.34) may be combined (after dropping the "tilde" in both) to obtain

$$\begin{aligned}
\sum_{j=1}^{n}(I_j + II_j) = & \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\delta,\gamma\in\mathbb{N}_0^n\\ \delta+\gamma=\alpha\\ |\gamma|=\ell-1,|\delta|=m-\ell+1}} (-1)^{\ell+1}\frac{\alpha!(m-\ell+1)!(\ell-1)!}{m!\delta!\gamma!}\times \\
& \qquad\qquad \times \left\langle A_{\alpha\beta}(\partial^{\beta+\delta}u),\, \partial^{\gamma} w\right\rangle \\
& + \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|=|\beta|=m}} \sum_{\ell=2}^{m+1} \sum_{\substack{\delta,\gamma\in\mathbb{N}_0^n\\ \delta+\gamma=\alpha\\ |\gamma|=\ell-1,|\delta|=m-\ell+1}} (-1)^{\ell}\frac{\alpha!(m-\ell+1)!(\ell-1)!}{m!\delta!\gamma!}\times \\
& \qquad\qquad \times \left\langle A_{\alpha\beta}(\partial^{\beta+\delta}u),\, \partial^{\gamma} w\right\rangle \\
= & \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|=|\beta|=m}} \left\langle A_{\alpha\beta}(\partial^{\alpha+\beta}u),\, w\right\rangle \\
& + (-1)^{m+1}\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|=|\beta|=m}} \left\langle A_{\alpha\beta}(\partial^{\beta}u),\, \partial^{\alpha} w\right\rangle. \qquad (9.6.35)
\end{aligned}$$

Based on the same type of reasoning, we also obtain

$$\begin{aligned}
\sum_{j=1}^{n} III_j = & \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\widetilde{\eta},\omega\in\mathbb{N}_0^n\\ \widetilde{\eta}+\omega=\beta\\ |\omega|=\ell-1,|\widetilde{\eta}|=m-\ell+1}} \Bigg(\sum_{\substack{\eta\in\mathbb{N}_0^n, j\in\{1,\dots,n\}\\ \eta+e_j=\widetilde{\eta}}} \frac{1}{\eta!}\Bigg)\times \\
& \qquad \times (-1)^{m+\ell+1}\frac{\beta!(m-\ell)!(\ell-1)!}{m!\,\omega!}\left\langle \partial^{\widetilde{\eta}}u,\, A_{\alpha\beta}^{\top}(\partial^{\alpha+\omega}w)\right\rangle \\
= & \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\widetilde{\eta},\omega\in\mathbb{N}_0^n\\ \widetilde{\eta}+\omega=\beta\\ |\omega|=\ell-1,|\widetilde{\eta}|=m-\ell+1}} (-1)^{m+\ell+1}\frac{\beta!(m-\ell+1)!(\ell-1)!}{m!\widetilde{\eta}!\,\omega!}\times \\
& \qquad \times \left\langle \partial^{\widetilde{\eta}}u,\, A_{\alpha\beta}^{\top}(\partial^{\alpha+\omega}w)\right\rangle, \qquad (9.6.36)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^{n} IV_j = & \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\eta,\widetilde{\omega}\in\mathbb{N}_0^n\\ \eta+\widetilde{\omega}=\beta\\ |\widetilde{\omega}|=\ell,|\eta|=m-\ell}} \Bigg(\sum_{\substack{\omega\in\mathbb{N}_0^n, j\in\{1,\dots,n\}\\ \omega+e_j=\widetilde{\omega}}} \frac{1}{\omega!}\Bigg)\times \\
& \qquad \times (-1)^{m+\ell+1}\frac{\beta!(m-\ell)!(\ell-1)!}{m!\eta!}\left\langle \partial^{\eta}u,\, A_{\alpha\beta}^{\top}(\partial^{\alpha+\widetilde{\omega}}w)\right\rangle \\
= & \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\eta,\widetilde{\omega}\in\mathbb{N}_0^n\\ \eta+\widetilde{\omega}=\beta\\ |\widetilde{\omega}|=\ell,|\eta|=m-\ell}} (-1)^{m+\ell+1}\frac{\beta!(m-\ell)!\,\ell!}{m!\eta!\widetilde{\omega}!}\left\langle \partial^{\eta}u,\, A_{\alpha\beta}^{\top}(\partial^{\alpha+\widetilde{\omega}}w)\right\rangle
\end{aligned}$$

$$= \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} \sum_{\ell=2}^{m+1} \sum_{\substack{\eta,\widetilde{\omega}\in\mathbb{N}_0^n\\ \eta+\widetilde{\omega}=\beta\\ |\widetilde{\omega}|=\ell-1,|\eta|=m-\ell+1}} (-1)^{m+\ell}\frac{\beta!(m-\ell+1)!(\ell-1)!}{m!\eta!\widetilde{\omega}!} \big\langle \partial^\eta u,\, A_{\alpha\beta}^\top(\partial^{\alpha+\widetilde{\omega}}w)\big\rangle. \tag{9.6.37}$$

Combining (9.6.36) and (9.6.37) (with the tilde dropped in the writing for both) we arrive at

$$\begin{aligned}\sum_{j=1}^n (III_j+IV_j) &= \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} \sum_{\ell=1}^{m} \sum_{\substack{\eta,\omega\in\mathbb{N}_0^n\\ \eta+\omega=\beta\\ |\omega|=\ell-1,|\eta|=m-\ell+1}} (-1)^{m+\ell+1}\frac{\beta!(m-\ell+1)!(\ell-1)!}{m!\eta!\,\omega!}\times\\ &\qquad\times\big\langle \partial^\eta u,\, A_{\alpha\beta}^\top(\partial^{\alpha+\omega}w)\big\rangle\\ &\quad+\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} \sum_{\ell=2}^{m+1} \sum_{\substack{\eta,\omega\in\mathbb{N}_0^n\\ \eta+\omega=\beta\\ |\omega|=\ell-1,|\eta|=m-\ell+1}} (-1)^{m+\ell}\frac{\beta!(m-\ell+1)!(\ell-1)!}{m!\eta!\,\omega!}\times\\ &\qquad\times\big\langle \partial^\eta u,\, A_{\alpha\beta}^\top(\partial^{\alpha+\omega}w)\big\rangle\\ &= \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} (-1)^m\big\langle(\partial^\beta u),\, A_{\alpha\beta}^\top(\partial^\alpha w)\big\rangle - \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} \big\langle u,\, A_{\alpha\beta}^\top(\partial^{\alpha+\beta}w)\big\rangle.\end{aligned} \tag{9.6.38}$$

Finally, from (9.6.35), (9.6.38), and assumption *(1)* we see

$$\begin{aligned}\operatorname{div}\vec{F} &= \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} \big\langle A_{\alpha\beta}(\partial^{\alpha+\beta}u),\, w\big\rangle + (-1)^{m+1}\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} \big\langle A_{\alpha\beta}(\partial^\beta u),\, \partial^\alpha w\big\rangle\\ &\quad+\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} (-1)^m\big\langle(\partial^\beta u),\, A_{\alpha\beta}^\top(\partial^\alpha w)\big\rangle - \sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\|\alpha|=|\beta|=m}} \big\langle u,\, A_{\alpha\beta}^\top(\partial^{\alpha+\beta}w)\big\rangle\\ &= \big\langle Lu,\, w\big\rangle - \big\langle u,\, L^\top w\big\rangle \in L^1(\Omega,\mathcal{L}^n).\end{aligned} \tag{9.6.39}$$

Moreover, the membership in (1.8.19) and the format of the components of $\vec{F}$ (recall (9.6.22)) ensure $\mathcal{N}_\kappa\vec{F}\in L^1(\partial\Omega,\sigma)$, while assumption *(3)* guarantees that $\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial_{\mathrm{nta}}\Omega$. Corresponding to the case when Ω is unbounded and $\partial\Omega$ is bounded, assumption (1.8.20) implies that $\vec{F}$ satisfies (1.2.3) as well.

In summary, we have checked that $\vec{F}$ satisfies all hypotheses of Theorem 1.2.1, thus formula (1.2.2) holds. In order to write this formula, we also need to compute $\nu\cdot(\vec{F}\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}})$. To do so, we use (9.6.22) to compute

$$\nu\cdot(\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}) = \sum_{j=1}^{n}\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ \alpha\geq e_j,|\alpha|=|\beta|=m}}\sum_{\ell=1}^{m}\sum_{\substack{\delta,\gamma\in\mathbb{N}_0^n\\ \delta+\gamma=\alpha-e_j\\ |\gamma|=\ell-1,|\delta|=m-\ell}}(-1)^{\ell+1}\frac{\alpha!(m-\ell)!(\ell-1)!}{m!\delta!\gamma!}\times$$
$$\times\Big\langle \nu_j A_{\alpha\beta}(\partial^{\beta+\delta}u)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},(\partial^\gamma w)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle$$
$$+\sum_{j=1}^{n}\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ \beta\geq e_j,|\alpha|=|\beta|=m}}\sum_{\ell=1}^{m}\sum_{\substack{\eta,\omega\in\mathbb{N}_0^n\\ \eta+\omega=\beta-e_j\\ |\omega|=\ell-1,|\eta|=m-\ell}}(-1)^{m+\ell+1}\frac{\beta!(m-\ell)!(\ell-1)!}{m!\eta!\,\omega!}\times$$
$$\times\Big\langle (\partial^\eta u)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\nu_j A^\top_{\alpha\beta}(\partial^{\alpha+\omega}w)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle$$
$$=\sum_{\gamma\in\mathbb{N}_0^n,\,|\gamma|\leq m-1}(-1)^{|\gamma|}\frac{|\gamma|!}{m!\gamma!}\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ |\alpha|=|\beta|=m,\alpha>\gamma}}\sum_{\substack{\delta\in\mathbb{N}_0^n,j\in\{1,\dots,n\}\\ \delta+e_j=\alpha-\gamma}}\frac{\alpha!\,|\delta|!}{\delta!}\times$$
$$\times\Big\langle \nu_j A_{\alpha\beta}(\partial^{\beta+\delta}u)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},(\partial^\gamma w)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle$$
$$-\sum_{\eta\in\mathbb{N}_0^n,\,|\eta|\leq m-1}(-1)^{|\eta|}\frac{|\eta|!}{m!\eta!}\sum_{\substack{\alpha,\beta\in\mathbb{N}_0^n\\ \beta>\eta,|\alpha|=|\beta|=m}}\sum_{\substack{\omega\in\mathbb{N}_0^n,j\in\{1,\dots,n\}\\ \omega+e_j=\beta-\eta}}\frac{\beta!\,|\omega|!}{\omega!}\times$$
$$\times\Big\langle (\partial^\eta u)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega},\nu_j A^\top_{\alpha\beta}(\partial^{\alpha+\omega}w)\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\Big\rangle$$
$$=\Big\langle \partial^A_\nu u,\mathrm{Tr}^{\mathrm{n.t.}}_{\mathrm{m}-1}(w)\Big\rangle-\Big\langle \mathrm{Tr}^{\mathrm{n.t.}}_{\mathrm{m}-1}(u),\partial^{A^\top}_\nu w\Big\rangle, \tag{9.6.40}$$

where for the last equality we made use of the definition of conormal derivative from (1.8.16)–(1.8.17). Now (1.8.14) is a consequence of (1.2.2), (9.6.39), and (9.6.40). This finishes the proof of Theorem 1.8.5.

9.7 Proofs of Theorems 1.9.1–1.9.4

We begin by recording two useful results pertaining to the bullet product, proved in [185, §10.2], which are relevant for the present goals.

Theorem 9.7.1 *Let $\Omega\subseteq\mathbb{R}^n$ (where $n\in\mathbb{N}$ with $n\geq 2$) be an open set with an Ahlfors regular boundary and abbreviate $\sigma:=\mathcal{H}^{n-1}\lfloor\partial\Omega$. Also, fix some aperture parameter $\kappa\in(0,\infty)$ along with an integrability exponent $p\in\big(\frac{n-1}{n},\infty\big]$. Consider a vector field $\vec{F}:\Omega\to\mathbb{C}^n$, having $\mathcal{L}^n$-measurable components, with the property that*

$$\mathcal{N}_\kappa\vec{F}\in L^p(\partial\Omega,\sigma). \tag{9.7.1}$$

In particular, $\vec{F}\in\big[L^1_{\mathrm{loc}}(\Omega,\mathcal{L}^n)\big]^n$ (cf. Lemma 8.3.1) and, with the divergence taken in the sense of distributions in Ω, assume

$$\mathrm{div}\vec{F} \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n) \ \ \textit{and} \ \ \mathfrak{P}\big(\mathrm{div}\vec{F}\big) \in L^p(\partial\Omega, \sigma). \tag{9.7.2}$$

Then, first,

$$\vec{F} \in \big[L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n)\big]^n \ \ \textit{and} \ \ \mathrm{div}\vec{F} \in L^1_{\mathrm{bdd}}(\Omega, \mathcal{L}^n), \tag{9.7.3}$$

hence it is meaningful to consider the "bullet product" $\nu \bullet \vec{F}$ as a functional in $\big(\mathrm{Lip}_c(\partial\Omega)\big)'$ (cf. Proposition 4.2.3). Second, if $p < \infty$ it follows that

$$\nu \bullet \vec{F} \ \ \textit{belongs to the Hardy space} \ \ H^p(\partial\Omega, \sigma) \tag{9.7.4}$$

and there exists a constant $C_{\Omega,\kappa,p} \in (0, \infty)$, independent of $\vec{F}$, such that

$$\big\|\nu \bullet \vec{F}\big\|_{H^p(\partial\Omega,\sigma)} \leq C_{\Omega,\kappa,p}\Big\{\big\|\mathcal{N}_\kappa \vec{F}\big\|_{L^p(\partial\Omega,\sigma)} + \big\|\mathfrak{P}(\mathrm{div}\vec{F})\big\|_{L^p(\partial\Omega,\sigma)}\Big\}. \tag{9.7.5}$$

Third, corresponding to the limiting case $p := \infty$ and $q := \infty$, one has

$$\begin{aligned} &\nu \bullet \vec{F} \in L^\infty(\partial\Omega, \sigma) \ \ \textit{and there exists } C_{\Omega,\kappa} \in (0, \infty) \textit{ such that} \\ &\big\|\nu \bullet \vec{F}\big\|_{L^\infty(\partial\Omega,\sigma)} \leq C_{\Omega,\kappa}\Big\{\big\|\mathcal{N}_\kappa \vec{F}\big\|_{L^\infty(\partial\Omega,\sigma)} + \big\|\mathfrak{P}(\mathrm{div}\vec{F})\big\|_{L^\infty(\partial\Omega,\sigma)}\Big\} \\ &\qquad = C_{\Omega,\kappa}\Big\{\|\vec{F}\|_{L^\infty(\Omega,\mathcal{L}^n)} + \big\|\mathfrak{P}(\mathrm{div}\vec{F})\big\|_{L^\infty(\partial\Omega,\sigma)}\Big\}. \end{aligned} \tag{9.7.6}$$

More generally, we have the following extension of Theorem 9.7.1 (to arbitrary first-order systems in place of the divergence):

Theorem 9.7.2 *Fix $n \in \mathbb{N}$ with $n \geq 2$, and suppose $\Omega \subseteq \mathbb{R}^n$ is an open set with an Ahlfors regular boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and select an integrability exponent $p \in \big(\frac{n-1}{n}, \infty\big)$. Also, pick two integers $N, M \in \mathbb{N}$ and consider an arbitrary $N \times M$ homogeneous first-order system D with constant complex coefficients in $\mathbb{R}^n$, along with a vector-valued function $F : \Omega \to \mathbb{C}^M$, having $\mathcal{L}^n$-measurable components, with the property that for some $\kappa \in (0, \infty)$ one has*

$$\mathcal{N}_\kappa F \in L^p(\partial\Omega, \sigma). \tag{9.7.7}$$

In particular, $F \in \big[L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)\big]^M$ (cf. Lemma 8.3.1), and one also assumes that DF, computed in the sense of distributions in Ω, has components in $L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n)$ and satisfies

$$\mathfrak{P}(DF) \in L^p(\partial\Omega, \sigma). \tag{9.7.8}$$

Then, with the bullet product defined as in (1.9.19), one has

$$\mathrm{Sym}(D; \nu) \bullet F \in \big[H^p(\partial\Omega, \sigma)\big]^N \tag{9.7.9}$$

and there exists a constant $C = C(\Omega, D, p, \kappa) \in (0, \infty)$ such that

$$\Big\|\mathrm{Sym}(D;\nu)\bullet F\Big\|_{[H^p(\partial\Omega,\sigma)]^N}\leq C\big\|\mathcal{N}_\kappa F\big\|_{L^p(\partial\Omega,\sigma)}+C\|\mathfrak{P}(DF)\|_{L^p(\partial\Omega,\sigma)}.\tag{9.7.10}$$

Moreover, if in place of (9.7.7) *and* (9.7.8) *one now assumes*

$$\begin{gathered}\mathcal{N}_\kappa F\in L^p(\partial\Omega,w\sigma)\ \ \textit{and}\ \ \mathfrak{P}(DF)\in L^p(\partial\Omega,w\sigma)\\ \textit{for some exponent}\, p\in(1,\infty),\end{gathered}\tag{9.7.11}$$

then actually

$$\mathrm{Sym}(D;\nu)\bullet F\ \ \textit{belongs to the Lebesgue space}\ \ \big[L^p(\partial\Omega,\sigma)\big]^N\tag{9.7.12}$$

and there exists a constant $C=C(\Omega,\kappa,p)\in(0,\infty)$*, independent of* F*, such that*

$$\big\|\mathrm{Sym}(D;\nu)\bullet F\big\|_{[L^p(\partial\Omega,\sigma)]^N}\leq C\Big\{\big\|\mathcal{N}_\kappa F\big\|_{L^p(\partial\Omega,\sigma)}+\big\|\mathfrak{P}(DF)\big\|_{L^p(\partial\Omega,\sigma)}\Big\}.\tag{9.7.13}$$

Granted these results, we may now present in quick succession the proofs of Theorems 1.9.1–1.9.3.

Proof of Theorem 1.9.1 As regards the claims made in item *(i)* when $p\in[1,\infty)$, the fact that $\nu\bullet\vec{F}$, originally defined as a distribution on $\partial\Omega$ in the manner described in Proposition 4.2.3, actually belongs to $L^p(\partial\Omega,\sigma)$ is a consequence of (9.7.4), bearing in mind that the Hardy space $H^p(\partial\Omega,\sigma)$ coincides, in a quantitative sense, with the Lebesgue space $L^p(\partial\Omega,\sigma)$ whenever p belongs to $(1,\infty)$. The case when $p=\infty$ is handled based on (9.7.6).

Having established that $\nu\bullet\vec{F}\in L^p(\partial\Omega,\sigma)$, on account of (3.7.3) and Proposition 4.1.4 we may rephrase (4.2.12)–(4.2.13) as the integration by parts formula (1.9.6). Corollary 3.7.3 then ensures the uniqueness of the function $\nu\bullet\vec{F}$ with these properties. In addition, estimate (1.9.7) in item *(ii)* is implied by (9.7.5) (again, keeping in mind that $H^p(\partial\Omega,\sigma)=L^p(\partial\Omega,\sigma)$ if $1<p<\infty$). The claims in item *(iii)* may be justified using Theorem 9.7.1 in the same fashion as above. Finally, the claim in item *(iv)* is a consequence of work in [185, §10.2]. □

Proof of Theorem 1.9.2 All claims are justified based on Theorem 9.7.1, Proposition 4.2.3, and the compatibility between the distributional pairing on $\partial\Omega$ with the duality brackets for the Hardy and Hölder spaces (see the discussion in [185, §4.6]).

Proof of Theorem 1.9.3 This is a consequence of Theorem 9.7.2 and the compatibility between the distributional pairing on $\partial\Omega$ with the duality brackets for the Hardy and Hölder spaces (cf. the discussion in [185, §4.6]). □

Finally, here is the proof of Theorem 1.9.4.

Proof of Theorem 1.9.4 Select a compact set K such that

$$\begin{gathered}\vec{F}\in\big[L^1_{\rm bdd}(\Omega,\mathcal{L}^n)+\mathscr{E}'_K(\Omega)\big]^n\ \text{ and}\\ \mathrm{div}\vec{F}=f+u\ \text{ in }\ \mathcal{D}'(\Omega),\ \text{ where }\ f\in L^1(\Omega,\mathcal{L}^n)\ \text{ and }\ u\in\mathscr{E}'_K(\Omega).\end{gathered}\tag{9.7.14}$$

Also, bring in a cutoff function $\eta \in \mathscr{C}_c^\infty(\Omega)$ satisfying $\eta \equiv 1$ near K. Starting with (1.9.31) then using Definition 4.2.6 and (1.9.30), we may write (keeping in mind (1.3.3))

$$\begin{aligned}
&\lim_{R\to\infty} \Big\langle \nu \bullet \vec{F}, \phi_R\big|_{\partial\Omega}\Big\rangle \\
&\quad = \lim_{R\to\infty} \Big\langle \nu \bullet \big((1-\eta)\vec{F}\big), \phi_R\big|_{\partial\Omega}\Big\rangle \\
&\quad = \lim_{R\to\infty} \int_\Omega (1-\eta)\vec{F}\cdot\nabla\phi_R \,\mathrm{d}\mathcal{L}^n + \lim_{R\to\infty}\int_\Omega \mathrm{div}\big((1-\eta)\vec{F}\big)\phi_R \,\mathrm{d}\mathcal{L}^n \\
&\quad = \lim_{R\to\infty} \int_\Omega \vec{F}\cdot\nabla\phi_R \,\mathrm{d}\mathcal{L}^n + \lim_{R\to\infty}\int_\Omega \Big\{(1-\eta)(\mathrm{div}\vec{F}) - \nabla\eta\cdot\vec{F}\Big\}\phi_R \,\mathrm{d}\mathcal{L}^n \\
&\quad = -[\vec{F}]_{\mathscr{F}} + \lim_{R\to\infty}\int_\Omega (1-\eta)f\phi_R \,\mathrm{d}\mathcal{L}^n - \lim_{R\to\infty}\int_\Omega (\nabla\eta\cdot\vec{F})\phi_R \,\mathrm{d}\mathcal{L}^n \\
&\quad = -[\vec{F}]_{\mathscr{F}} + \int_\Omega (1-\eta)f \,\mathrm{d}\mathcal{L}^n - \lim_{R\to\infty}\int_\Omega \nabla(\eta\phi_R)\cdot\vec{F} \,\mathrm{d}\mathcal{L}^n \\
&\quad = -[\vec{F}]_{\mathscr{F}} + \int_\Omega (1-\eta)f \,\mathrm{d}\mathcal{L}^n - \int_\Omega \nabla\eta\cdot\vec{F} \,\mathrm{d}\mathcal{L}^n.
\end{aligned} \tag{9.7.15}$$

Hence, the limit in (1.9.31) exists and

$$\big(\nu \bullet \vec{F}, 1\big)_{\mathscr{F}} = -[\vec{F}]_{\mathscr{F}} + \int_\Omega (1-\eta)f \,\mathrm{d}\mathcal{L}^n - \int_\Omega \nabla\eta\cdot\vec{F} \,\mathrm{d}\mathcal{L}^n. \tag{9.7.16}$$

Let us also observe that

$$\begin{aligned}
\int_\Omega \nabla\eta\cdot\vec{F}\,\mathrm{d}\mathcal{L}^n &= {}_{[\mathcal{D}'(\Omega)]^n}\big\langle \vec{F}, \nabla\eta\big\rangle_{[\mathcal{D}(\Omega)]^n} = -{}_{\mathcal{D}'(\Omega)}\big\langle \mathrm{div}\vec{F}, \eta\big\rangle_{\mathcal{D}(\Omega)} \\
&= -{}_{\mathcal{D}'(\Omega)}\langle u, \eta\rangle_{\mathcal{D}(\Omega)} - {}_{\mathcal{D}'(\Omega)}\langle w, \eta\rangle_{\mathcal{D}(\Omega)} \\
&= -\int_\Omega \eta f \,\mathrm{d}\mathcal{L}^n - {}_{\mathscr{E}'(\Omega)}\langle w, 1\rangle_{\mathscr{E}(\Omega)}.
\end{aligned} \tag{9.7.17}$$

Combining (9.7.16) and (9.7.17) then yields

$$\begin{aligned}
\big(\nu \bullet \vec{F}, 1\big)_{\mathscr{F}} &= -[\vec{F}]_{\mathscr{F}} + \int_\Omega f \,\mathrm{d}\mathcal{L}^n + {}_{\mathscr{E}'(\Omega)}\langle w, 1\rangle_{\mathscr{E}(\Omega)} \\
&= -[\vec{F}]_{\mathscr{F}} + {}_{(\mathscr{C}_b^\infty(\Omega))^*}\big(\mathrm{div}\vec{F}, 1\big)_{\mathscr{C}_b^\infty(\Omega)},
\end{aligned} \tag{9.7.18}$$

thanks to (4.6.19). From this, (1.9.32) follows. □

9.8 Proof of Theorem 1.10.1

The proof of Theorem 1.10.1 proceeds along the line of reasoning employed in the proof of Theorem 1.3.1.

Proof of Theorem 1.10.1 That in the current setting the set Ω has locally finite perimeter is a consequence of (5.9.15). If we abbreviate

$$\vec{f} := \vec{F}\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \quad \text{on} \quad \partial_{\mathrm{nta}}\Omega, \tag{9.8.1}$$

then from (1.10.2) and Proposition 8.10.6 we know that this function is well defined and satisfies

$$\vec{f} \in \big[L^1(\partial_{\mathrm{nta}}\Omega, \sigma)\big]^n. \tag{9.8.2}$$

Thanks to (9.1.3) we therefore have

$$\begin{array}{l}\vec{f}, \text{ originally defined as in (9.8.1), may be canonically} \\ \text{viewed as a } \mathbb{C}^n-\text{valued function defined } \sigma-\text{a.e. on } \partial_*\Omega.\end{array} \tag{9.8.3}$$

First we claim that the Divergence Formula recorded in (1.10.6) holds under the additional assumption that

$$\text{there exists } R \in (0,\infty) \text{ such that } \vec{F} \text{ vanishes in } \Omega \setminus B(0,R). \tag{9.8.4}$$

To see that this is the case, recall $\vec{f}$ from (9.8.1) and fix $\eta > 0$ arbitrary. In view of (9.8.2), we may invoke the density result established in Corollary 3.7.3 (used here with the choices $X := \partial_{\mathrm{nta}}\Omega$, $s := n-1 \geq 0$, and $p := 1$) to find

$$\vec{G} \in \big[\mathscr{C}_c^\infty(\mathbb{R}^n)\big]^n \quad \text{such that} \quad \big\|\vec{f} - \vec{G}\big|_{\partial_{\mathrm{nta}}\Omega}\big\|_{[L^1(\partial_{\mathrm{nta}}\Omega,\sigma)]^n} < \eta. \tag{9.8.5}$$

Also, let $\{\Phi_\varepsilon\}_{\varepsilon>0}$ be the sequence of functions associated with the set Ω as in Lemma 6.1.2. Then, as in (9.1.15)–(9.1.17), we obtain

$$\big(\mathrm{div}\vec{F}\big)(\Omega) = -\lim_{\varepsilon\to0^+}\int_\Omega (\vec{F}-\vec{G})\cdot\nabla\Phi_\varepsilon\,\mathrm{d}\mathcal{L}^n + \lim_{\varepsilon\to0^+}\int_\Omega \Phi_\varepsilon\,\mathrm{div}\vec{G}\,\mathrm{d}\mathcal{L}^n. \tag{9.8.6}$$

As regards the first limit above, pick $\theta_* \in (0,1)$ which is small enough such that $\frac{2\theta_*}{1-\theta_*} < \kappa$. For some $C \in (0,\infty)$ which depends only on Ω and κ we may then estimate

$$\begin{aligned}\limsup_{\varepsilon\to0^+}\int_\Omega |\vec{F}-\vec{G}|\,|\nabla\Phi_\varepsilon|\,\mathrm{d}\mathcal{L}^n &\leq C\limsup_{\varepsilon\to0^+}\Big(\varepsilon^{-1}\int_{O_\varepsilon}|\vec{F}-\vec{G}|\,\mathrm{d}\mathcal{L}^n\Big) \\ &\leq C\limsup_{\varepsilon\to0^+}\Big\|\widetilde{\mathcal{N}}_{\kappa,\theta_*,r}^{(\frac{2\theta+1}{1+\theta})\varepsilon}(\vec{F}-\vec{G})\Big\|_{L^1(\partial\Omega,\sigma)}\end{aligned}$$

$$= C \big\| \big(\vec{F} - \vec{G}\big)\big|_{\partial\Omega}^{\kappa-\text{n.t.}} \big\|_{[L^1(\partial_{\text{nta}}\Omega,\sigma)]^n}$$
$$= C \big\| \vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}} - \vec{G}\big|_{\partial\Omega} \big\|_{[L^1(\partial_{\text{nta}}\Omega,\sigma)]^n}$$
$$= C \big\| \vec{f} - \vec{G}\big|_{\partial_{\text{nta}}\Omega} \big\|_{[L^1(\partial_{\text{nta}}\Omega,\sigma)]^n} < C\eta. \tag{9.8.7}$$

Above, the first inequality is implied by properties (6.1.5)–(6.1.6) of Φ_ε, while the second inequality is a consequence of Proposition 8.10.3 (used with $r := 1$ and $\theta := \theta_*$) and (8.10.8) (used with $s := 1$). Next, the first equality in (9.8.7) follows from Proposition 8.10.6 (whose applicability is ensured by our choice of θ_*), the second equality is based on (8.9.10), and the third equality uses (9.8.1). Finally, the last inequality in (9.8.7) comes from (9.8.5).

Thanks to Lebesgue's Dominated Convergence Theorem and De Giorgi–Federer's version of the Gauss–Green Formula from Theorem 1.1.1 we have

$$\lim_{\varepsilon\to 0^+} \int_\Omega \Phi_\varepsilon \, \mathrm{div}\vec{G}\, d\mathcal{L}^n = \int_\Omega \mathrm{div}\vec{G}\, d\mathcal{L}^n = \int_{\partial_*\Omega} \nu\cdot\vec{G}\, d\sigma. \tag{9.8.8}$$

Also, based on (9.1.3) and (9.8.3) we may write

$$\Big| \int_{\partial_*\Omega} \nu\cdot\vec{G}\, d\sigma - \int_{\partial_*\Omega} \nu\cdot\vec{f}\, d\sigma \Big| \leq \int_{\partial_*\Omega} \big|\vec{f} - \vec{G}\,\big|\, d\sigma \leq \int_{\partial_{\text{nta}}\Omega} \big|\vec{f} - \vec{G}\,\big|\, d\sigma$$
$$= \big\| \vec{f} - \vec{G}\big|_{\partial_{\text{nta}}\Omega} \big\|_{[L^1(\partial_{\text{nta}}\Omega,\sigma)]^n} < \eta. \tag{9.8.9}$$

Combining (9.8.6)–(9.8.7) and (9.8.8)–(9.8.9) then proves that for each $\eta > 0$ we have

$$\Big| \big(\mathrm{div}\vec{F}\big)(\Omega) - \int_{\partial_*\Omega} \nu\cdot\vec{f}\, d\sigma \Big| \leq C\eta, \tag{9.8.10}$$

with $C > 0$ a finite geometric constant independent of η. Bearing in mind (9.8.1) and that $\eta > 0$ is arbitrary, (1.10.6) follows from this, in the case when the additional assumption (9.8.4) is imposed.

The next goal is to establish the Divergence Formula (1.3.8) under the original assumptions made in Theorem 1.10.1. To this end, observe from Corollary 8.10.9 (used with $p := 1$) that

$$\vec{F}\big|_E \in \big[L^{\frac{n}{n-1}}(E, \mathcal{L}^n)\big]^n \text{ for every bounded } \mathcal{L}^n - \text{measurable set } E \subseteq \Omega. \tag{9.8.11}$$

As in the proof of Theorem 1.3.1 bring in the auxiliary functions $\{\phi_R\}_{R>0} \subseteq \mathscr{C}_c^\infty(\mathbb{R}^n)$ (in the sense of (1.3.3)) and, for each number $R \in (0, \infty)$, define

$$\vec{F}_R := \phi_R \vec{F} \text{ in } \Omega. \tag{9.8.12}$$

Then, as before, for each $R > 0$ there exists a number $R_* \in (0, \infty)$ such that

$$\vec{F}_R \equiv 0 \text{ outside } \Omega \cap B(0, R_*), \tag{9.8.13}$$

and

$$\operatorname{div}\vec{F}_R = \phi_R \operatorname{div}\vec{F} + \nabla\phi_R \cdot \vec{F} \text{ in } \mathcal{D}'(\Omega). \tag{9.8.14}$$

As a consequence of (9.8.14), (1.10.3), and (9.8.11),

$$\operatorname{div}\vec{F}_R \text{ is a complex Borel measure on } \Omega. \tag{9.8.15}$$

In addition, thanks to the original integrability assumptions on $\vec{F}$, the second line of (1.10.2), (8.2.10), and (8.10.6), for each fixed $R > 0$ we have

$$\vec{F}_R \in \big[L^r_{\rm loc}(\Omega, \mathcal{L}^n)\big]^n \text{ and } \widetilde{\mathcal{N}}_{\kappa,\theta,r}\vec{F}_R \in L^1(\partial\Omega, \sigma). \tag{9.8.16}$$

Also, from the first line of (1.10.2) and (8.9.9)–(8.9.10), we see that the nontangential limit

$$\begin{array}{c} \vec{F}_R\big|^{\kappa-\rm n.t.}_{\partial\Omega} \text{ exists at } \sigma-\text{a.e. point on } \partial_{\rm nta}\Omega \text{ and, in fact,} \\ \vec{F}_R\big|^{\kappa-\rm n.t.}_{\partial\Omega} = \big(\phi_R\big|_{\partial\Omega}\big)\big(\vec{F}\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big) \text{ at } \sigma-\text{a.e. point on } \partial_{\rm nta}\Omega. \end{array} \tag{9.8.17}$$

In particular,

$$\vec{F}_R\big|^{\kappa-\rm n.t.}_{\partial\Omega} \longrightarrow \vec{F}\big|^{\kappa-\rm n.t.}_{\partial\Omega} \text{ at } \sigma-\text{a.e. point on } \partial_{\rm nta}\Omega, \text{ as } R \to \infty. \tag{9.8.18}$$

Let us also pick

$$\theta_* \in \big(0, \tfrac{\kappa}{\kappa+2}\big). \tag{9.8.19}$$

Together with the first line in (9.8.17), this choice of the scale parameter ensures that we may invoke Corollary 8.10.5 (with $\theta := \theta_*$ and $u := \vec{F}_R$) which yields

$$\begin{array}{c} \Big|\big(\vec{F}_R\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big)(x)\Big| \leq \big(\widetilde{\mathcal{N}}_{\kappa,\theta_*,r}\vec{F}_R\big)(x) \leq C\big(\widetilde{\mathcal{N}}_{\kappa,\theta_*,r}\vec{F}\big)(x) \\ \text{for each } R > 0, \text{ at } \sigma-\text{a.e. point on } \partial_{\rm nta}\Omega. \end{array} \tag{9.8.20}$$

The last inequality above, involving a constant $C \in (0, \infty)$ independent of R, is implied by (8.2.10) and (1.3.3). In relation to (9.8.20), let us also observe that the second line of (1.10.2) and Proposition 8.10.1 ensure that

$$\widetilde{\mathcal{N}}_{\kappa,\theta_*,r}\vec{F} \in L^1(\partial\Omega, \sigma). \tag{9.8.21}$$

From (9.8.18), (9.8.21), (8.8.52), and Lebesgue's Dominated Convergence Theorem, we conclude that, on the one hand,

$$\int_{\partial_*\Omega} \nu \cdot \big(\vec{F}_R\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big)\, d\sigma \longrightarrow \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|^{\kappa-\rm n.t.}_{\partial\Omega}\big)\, d\sigma \text{ as } R \to \infty. \tag{9.8.22}$$

On the other hand, granted (9.8.16), (9.8.17), (9.8.15), and (9.8.13), the result established in the first part of the proof (under the additional assumption made in (9.8.4)) applies to each vector field $\vec{F}_R$ and gives

$$\big(\operatorname{div}\vec{F}_R\big)(\Omega) = \int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}_R\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)\,\mathrm{d}\sigma, \qquad \forall R > 0. \tag{9.8.23}$$

Granted (9.8.22)–(9.8.23) and reasoning as in (9.1.32)–(9.1.33), we conclude that the limit

$$\begin{aligned} &\lim_{R\to\infty}\int_\Omega \nabla\phi_R \cdot \vec{F}\,\mathrm{d}\mathcal{L}^n \;\text{ exists, is independent of the family } \{\phi_R\}_{R>0}, \\ &\text{and equals the complex number } \int_{\partial_*\Omega} \nu \cdot \big(\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)\,\mathrm{d}\sigma - \big(\operatorname{div}\vec{F}\big)(\Omega). \end{aligned} \tag{9.8.24}$$

In light of (1.3.2), from (9.8.24) we conclude that $[\vec{F}]_\infty$, the contribution of $\vec{F}$ at infinity, is meaningfully and unambiguously defined and that the version of the Divergence Formula recorded in (1.10.4) holds.

At this stage, there remains to deal with the claim to the effect that $[\vec{F}]_\infty = 0$ under any of the assumptions *(i)-(iii)* made in (1.10.5). That $[\vec{F}]_\infty = 0$ if Ω is bounded is clear from (1.3.4)–(1.3.5). Consider next the case when $\partial\Omega$ is unbounded and $r \geq \frac{n}{n-1}$ (recall that $n \geq 2$). Then (4.7.37) implies that there exists some $C \in (0,\infty)$ such that

$$\big|[\vec{F}]_\infty\big| \leq C \limsup_{R\to\infty}\Big\{R^{-1}\cdot\int_{[B(0,2R)\setminus B(0,R)]\cap\Omega} |\vec{F}|\,\mathrm{d}\mathcal{L}^n\Big\}. \tag{9.8.25}$$

Also, since $r \geq \frac{n}{n-1}$, Proposition 8.10.7 applies in the current setting (with $p := 1$) and gives

$$\vec{F} \in \Big[L^{\frac{n}{n-1}}(\Omega,\mathcal{L}^n)\Big]^n. \tag{9.8.26}$$

Given that we are also assuming that $n \geq 2$, for each $R \in (0,\infty)$ we may use Cauchy–Schwarz' inequality and Hölder's inequality to estimate

$$\begin{aligned} &R^{-1}\cdot\int_{[B(0,2R)\setminus B(0,R)]\cap\Omega} |\vec{F}|\,\mathrm{d}\mathcal{L}^n \\ &\qquad\leq R^{-1}\Big(\int_{[B(0,2R)\setminus B(0,R)]\cap\Omega} |\vec{F}|^{\frac{n}{n-1}}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{n}}\Big(\int_{B(0,2R)} 1\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{1}{n}} \\ &\qquad\leq C\Big(\int_{[B(0,2R)\setminus B(0,R)]\cap\Omega} |\vec{F}|^{\frac{n}{n-1}}\,\mathrm{d}\mathcal{L}^n\Big)^{\frac{n-1}{n}} \end{aligned} \tag{9.8.27}$$

for some purely dimensional constant $C \in (0, \infty)$. In view of (9.8.26), Lebesgue's Dominated Convergence Theorem applies and proves that the limit in (9.8.25) is actually zero, hence $[\vec{F}]_\infty = 0$ in this case as well. Finally, when $\vec{F}$ satisfies (1.2.3) for some $\lambda \in (1, \infty)$, Lemma 4.7.3 implies that $[\vec{F}]_\infty = 0$ once more. The proof of Theorem 1.10.1 is therefore complete. □

9.9 Proofs of Theorems 1.11.3, 1.11.6, and 1.11.8–1.11.11

Here is the proof of Theorem 1.11.3.

Proof of Theorem 1.11.3 We begin by arguing as in Step I in the proof of Theorem 1.3.1, bearing in mind that the bounded support property hypothesized in (9.1.11) now happens automatically since Ω is a relatively compact set to begin with. All but one of the ingredients used in this portion of the proof of Theorem 1.3.1 (abstract measure theoretic results, purely real-variable results valid on general spaces of homogeneous type, and distributional integrations by parts) have direct analogues in the setting of compact Riemannian manifolds presently considered (replacing the ordinary flat-space divergence and gradient operators with div_g and the metric gradient operator ∇_g). The only step in this process which requires special care is (9.1.19), which we now deal with by invoking Theorem 1.11.1 in lieu of De Giorgi–Federer's version of the Gauss–Green Formula employed earlier in the Euclidean setting. Ultimately, we arrive at a version of Theorem 1.11.3 for measurable vector fields $\vec{F} : \Omega \to TM$ satisfying

$$\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma_g) \ \text{ and } \ \mathrm{div}_g \vec{F} \in \mathrm{CBM}(\Omega). \tag{9.9.1}$$

In turn, this class may be enlarged to the category of vector fields described in Theorem 1.11.3 by arguing as in the first part of the proof of Theorem 1.4.1, up to (and including) (9.2.19) (while disregarding the contribution at infinity terms). The latter argument is a localization procedure of a purely local, real-variable nature and carries over, virtually verbatim, to the setting of manifolds. □

We continue by giving the proof of Theorem 1.11.6.

Proof of Theorem 1.11.6 Note that the current hypotheses and Lemma 8.3.2 imply that

$$u \in L^\infty_{\mathrm{loc}}(\Omega, \mathcal{L}^n_g) \otimes \mathcal{E} \ \text{ and } \ w \in L^\infty_{\mathrm{loc}}(\Omega, \mathcal{L}^n_g) \otimes \mathcal{F}. \tag{9.9.2}$$

To proceed, define the vector field $\vec{F} : \Omega \to TM$ via the requirement that at $\mathcal{L}^n_g$-a.e. point $x \in \Omega$ we have

$${}_{T^*_x M}\big(\xi, \vec{F}(x)\big)_{T_x M} = \Big\langle (-\mathrm{i})\mathrm{Sym}(D; \xi)u(x), w(x)\Big\rangle_{\mathcal{E}_x} \tag{9.9.3}$$

for each $\xi \in T_x^*M$. Since the right-hand side of (9.9.3) is linear in ξ, this definition is meaningful. Moreover, it is apparent from (9.9.2) and (9.9.3) that

$$\vec{F} \in L^\infty_{\rm loc}(\Omega, \mathcal{L}^n_g) \otimes TM \subset L^1_{\rm loc}(\Omega, \mathcal{L}^n_g) \otimes TM. \tag{9.9.4}$$

Also, if $\kappa'' := \min\{\kappa, \kappa'\} > 0$, then

$$\mathcal{N}_{\kappa''}\vec{F} \le C\,\mathcal{N}_\kappa u \cdot \mathcal{N}_{\kappa'} w \quad \text{pointwise on } \; \partial\Omega. \tag{9.9.5}$$

In particular, from (9.9.5), the second line in (1.11.26), and (8.2.26) we conclude that

$$\mathcal{N}_{\kappa''}\vec{F} \in L^1(\partial\Omega, \sigma_g). \tag{9.9.6}$$

Finally,

$$\begin{aligned} &\text{the pointwise nontangential boundary} \\ &\text{trace } \vec{F}\big|^{\kappa''-\text{n.t.}}_{\partial\Omega} \text{ exists } \sigma_g - \text{a.e. on } \partial_{\text{nta}}\Omega, \end{aligned} \tag{9.9.7}$$

and at σ_g-a.e. point $x \in \partial_*\Omega$ we have

$$\begin{aligned} &{}_{T_x^*M}\Big(\nu_g(x), \big(\vec{F}\big|^{\kappa''-\text{n.t.}}_{\partial\Omega}\big)(x)\Big)_{T_xM} \\ &\quad = \Big\langle (-\mathrm{i})\mathrm{Sym}(D;\nu(x))\big(u\big|^{\kappa''-\text{n.t.}}_{\partial\Omega}\big)(x), \big(w\big|^{\kappa''-\text{n.t.}}_{\partial\Omega}\big)(x)\Big\rangle_{\mathcal{E}_x} \\ &\quad = \Big\langle (-\mathrm{i})\mathrm{Sym}(D;\nu(x))\big(u\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\big)(x), \big(w\big|^{\kappa'-\text{n.t.}}_{\partial\Omega}\big)(x)\Big\rangle_{\mathcal{E}_x}. \end{aligned} \tag{9.9.8}$$

The next goal is to compute the divergence of $\vec{F}$ (in the sense of distributions). To this end, select an arbitrary test function $\psi \in \mathscr{C}^1_c(\Omega)$ and write

$$\begin{aligned} \big\langle \mathrm{div}_g\vec{F}, \psi\big\rangle &= -\int_\Omega {}_{T^*M}\big(d\psi, \vec{F}\big)_{TM}\, \mathrm{d}\mathcal{L}^n_g \\ &= \int_\Omega \Big\langle \mathrm{i}\mathrm{Sym}(D; \mathrm{d}\psi)u, w\Big\rangle_{\mathcal{E}}\, \mathrm{d}\mathcal{L}^n_g \end{aligned} \tag{9.9.9}$$

where the last equality is implied by (9.9.3) with $\xi := d\psi$. Working locally and using Friedrichs' mollifier, construct next a sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset \mathscr{C}^1(\Omega, \mathcal{E})$ satisfying

$$\begin{aligned} &u_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} u \;\text{ at } \mathcal{L}^n_g - \text{a.e. point in } \operatorname{supp}\psi, \\ &Du_\varepsilon \xrightarrow[\varepsilon\to 0^+]{} Du \;\text{ in }\; L^1(\operatorname{supp}\psi, \mathcal{L}^n_g) \otimes \mathcal{F}, \\ &\text{and } \;\sup_{\varepsilon>0}\sup_{x\in\operatorname{supp}\psi} |u_\varepsilon(x)|_{\mathcal{E}_x} < \infty. \end{aligned} \tag{9.9.10}$$

Thanks to this, Lebesgue's Dominated Convergence Theorem, and the commutator identity

$$\mathrm{iSym}(D;\,\mathrm{d}\psi)u = \psi\,Du - D(\psi u) \tag{9.9.11}$$

(cf. (1.11.25)) we may then compute

$$\begin{aligned}
\int_{\Omega}\Big\langle \mathrm{iSym}(D;\mathrm{d}\psi)u,\, w\Big\rangle_{\mathcal{E}}\,\mathrm{d}\mathcal{L}^n_g \\
&= \lim_{\varepsilon\to 0^+}\int_{\Omega}\Big\langle \mathrm{iSym}(D;\,\mathrm{d}\psi)u_\varepsilon,\, w\Big\rangle_{\mathcal{E}}\,\mathrm{d}\mathcal{L}^n_g \\
&= \lim_{\varepsilon\to 0^+}\int_{\Omega}\big\langle \psi\,Du_\varepsilon,\, w\big\rangle_{\mathcal{E}}\,\mathrm{d}\mathcal{L}^n_g - \lim_{\varepsilon\to 0^+}\int_{\Omega}\big\langle D(\psi u_\varepsilon),\, w\big\rangle_{\mathcal{E}}\,\mathrm{d}\mathcal{L}^n_g \\
&= \int_{\Omega}\big\langle \psi\,Du,\, w\big\rangle_{\mathcal{E}}\,\mathrm{d}\mathcal{L}^n_g - \lim_{\varepsilon\to 0^+}\int_{\Omega}\big\langle \psi u_\varepsilon,\, D^\top w\big\rangle_{\mathcal{F}}\,\mathrm{d}\mathcal{L}^n_g \\
&= \int_{\Omega}\psi\Big\{\big\langle Du,\, w\big\rangle_{\mathcal{E}} - \big\langle u,\, D^\top w\big\rangle_{\mathcal{F}}\Big\}\,\mathrm{d}\mathcal{L}^n_g.
\end{aligned} \tag{9.9.12}$$

In concert with (9.9.9), this proves that, in the sense of distributions in Ω,

$$\mathrm{div}_g\vec{F} = \big\langle Du,\, w\big\rangle_{\mathcal{E}} - \big\langle u,\, D^\top w\big\rangle_{\mathcal{F}}. \tag{9.9.13}$$

In light of the last line in (1.11.26) this implies that

$$\mathrm{div}_g\vec{F} \in L^1(\Omega,\mathcal{L}^n_g). \tag{9.9.14}$$

Having established (9.9.4), (9.9.6), (9.9.7), and (9.9.14), Corollary 1.11.5 applies, and (1.11.20) yields (1.11.27) on account of (9.9.8) and (9.9.13). □

Next, we present the proof of Theorem 1.11.8.

Proof of Theorem 1.11.8 The first two claims in the conclusion of Theorem 1.11.8 are consequences of Proposition 8.9.8 and Proposition 8.8.6, so the focus becomes the integration by parts formula (1.11.45). To justify this, with $\{\mathbf{e}_i\}_{1\le i\le n}$ denoting the standard orthonormal basis in $\mathbb{R}^n$, we start by defining the vector field

$$\vec{F} := \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a^{\alpha\beta}_{jk}\,u_\beta w_\alpha (N_j\mathbf{e}_k - N_k\mathbf{e}_j) : \Omega \to \mathbb{R}^n. \tag{9.9.15}$$

Clearly, $\vec{F} \in \big[L^1_{\mathrm{loc}}(\Omega,S)\big]^n$, and since

$$N\cdot\vec{F} = \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a^{\alpha\beta}_{jk}\,u_\beta w_\alpha (N_jN_k - N_kN_j) = 0 \;\text{ on }\; \Omega, \tag{9.9.16}$$

it follows that $\vec{F}$ is tangent to ∂D. As such, $\vec{F}$ may be viewed as a vector field relative to the manifold ∂D, with locally integrable components defined in Ω, i.e.,

$$\vec{F} \in L^1_{\mathrm{loc}}(\Omega, S) \otimes T\partial D. \tag{9.9.17}$$

When regarding it as such, the assumptions on u, w also imply that

$$\mathcal{N}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma) \;\text{ and }\; \vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega} \;\text{ exists}\,\sigma - \text{a.e. on }\; \partial_{\mathrm{nta}}\Omega. \tag{9.9.18}$$

Moreover, the above nontangential boundary trace is independent of κ and, with the dependence on κ systematically suppressed, at σ-a.e. point on $\partial_*\Omega$ we have

$$\nu \cdot \big(\vec{F}\big|^{\mathrm{n.t.}}_{\partial\Omega}\big) = \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a^{\alpha\beta}_{jk}(N_j\nu_k - N_k\nu_j)\big(u_\beta\big|^{\mathrm{n.t.}}_{\partial\Omega}\big)\big(w_\alpha\big|^{\mathrm{n.t.}}_{\partial\Omega}\big). \tag{9.9.19}$$

Let $\mathrm{div}_{\partial D}$ stand for the differential geometric divergence on the manifold ∂D, and denote by $\mathrm{grad}_{\partial D}$ the differential geometric gradient on the manifold ∂D. In order to compute $\mathrm{div}_{\partial D}\vec{F}$ in the sense of distributions in Ω (regarded as an open subset of the manifold ∂D), pick an arbitrary scalar-valued function $\varphi \in \mathscr{C}^1_c(\mathbb{R}^n)$ with the property that $\mathrm{supp}\,\varphi \cap \partial D \subseteq \Omega$ and write

$$\begin{aligned}
&{}_{\mathcal{D}'(\Omega)}\Big\langle \mathrm{div}_{\partial D}\vec{F}, \varphi\big|_{\Omega}\Big\rangle_{\mathcal{D}(\Omega)} \\
&\quad = -\,{}_{[\mathcal{D}'(\Omega)]^n}\Big\langle \vec{F}, \mathrm{grad}_{\partial D}\big(\varphi\big|_{\Omega}\big)\Big\rangle_{[\mathcal{D}(\Omega)]^n} = -\int_\Omega \vec{F}\cdot \mathrm{grad}_{\partial D}\big(\varphi\big|_{\Omega}\big)\,\mathrm{d}S \\
&\quad = -\int_\Omega \vec{F}\cdot\Big((\nabla\varphi)\big|_{\Omega} - N\cdot(\nabla\varphi)\big|_{\Omega}N\Big)\,\mathrm{d}S = -\int_\Omega \vec{F}\cdot(\nabla\varphi)\,\mathrm{d}S \\
&\quad = -\int_\Omega \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a^{\alpha\beta}_{jk} u_\beta w_\alpha\Big(N_j[\partial_k\varphi]\big|_{\Omega} - N_k[\partial_j\varphi]\big|_{\Omega}\Big)\,\mathrm{d}S \\
&\quad = -\int_\Omega \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a^{\alpha\beta}_{jk} u_\beta w_\alpha \partial_{\tau_{jk}}\varphi\,\mathrm{d}S \\
&\quad = \int_\Omega \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a^{\alpha\beta}_{jk}\partial_{\tau_{jk}}(u_\beta w_\alpha)\varphi\,\mathrm{d}S \\
&\quad = \int_\Omega \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a^{\alpha\beta}_{jk}\Big\{(\partial_{\tau_{jk}}u_\beta)w_\alpha + u_\beta(\partial_{\tau_{jk}}w_\alpha)\Big\}\big(\varphi\big|_{\Omega}\big)\,\mathrm{d}S.
\end{aligned} \tag{9.9.20}$$

Above, the first equality uses that, up to a minus sign, the differential operators $\mathrm{div}_{\partial D}$ and $\mathrm{grad}_{\partial D}$ are transposed to one another on the manifold ∂D. The second equality is a consequence of (9.9.17). The third equality is implied by [77, Theorem 5.1, p. 1012], and the fourth equality is seen from (9.9.16). The fifth equality is clear from (9.9.15), while the sixth equality follows from the definition of the tangential differential operator $\partial_{\tau_{jk}}$ on smooth functions. Finally, the last two equalities may be justified by integrating by parts on the boundary, keeping in mind that our (boundary) Sobolev spaces are modules over space of compactly supported Lipschitz functions (see the discussion in [185, §11.1] for details).

Ultimately, formula (9.9.20) proves that

$$\operatorname{div}_{\partial D}\vec{F} = \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a_{jk}^{\alpha\beta}\Big\{(\partial_{\tau_{jk}}u_\beta)w_\alpha + u_\beta(\partial_{\tau_{jk}}w_\alpha)\Big\} \text{ in } \mathcal{D}'(\Omega). \tag{9.9.21}$$

In concert with (1.11.42)–(1.11.43) this shows that

$$\operatorname{div}_{\partial D}\vec{F} \text{ belongs to } L^1(\Omega, S) \tag{9.9.22}$$

and

$$\begin{aligned}\int_\Omega \operatorname{div}_{\partial D}\vec{F}\,\mathrm{d}S &= \int_\Omega \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a_{jk}^{\alpha\beta}(\partial_{\tau_{jk}}u_\beta)w_\alpha\,\mathrm{d}S\\ &\quad+\int_\Omega \sum_{j,k=1}^{n}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M'} a_{jk}^{\alpha\beta}u_\beta(\partial_{\tau_{jk}}w_\alpha)\,\mathrm{d}S.\end{aligned} \tag{9.9.23}$$

At this stage, Corollary 1.11.5 applies to the manifold $M := \partial D$ and the vector field $\vec{F} : \Omega \to TM$. On account of (9.9.17), (9.9.18), (9.9.19), and (9.9.23), the Divergence Formula (1.11.20) then yields (1.11.45). □

The proof of Theorem 1.11.9 is presented below.

Proof of Theorem 1.11.9 Following a common convention, if in local coordinates $(x_1, \dots, x_n)$ the Riemannian metric tensor is expressed as

$$g = \sum_{1\leq j,k\leq n} g_{jk}\,\mathrm{d}x_j \otimes \mathrm{d}x_k \tag{9.9.24}$$

we let $(g^{jk})_{1\leq j,k\leq n}$ denote the inverse of the (real, symmetric) matrix $(g_{jk})_{1\leq j,k\leq n}$. In particular (see, e.g., [189, (2.1.23), p. 53])

$$\big\langle \mathrm{d}x_j, \mathrm{d}x_k\big\rangle_{T^*M} = g^{jk} \text{ for each } j, k \in \{1, \dots, n\}. \tag{9.9.25}$$

It is also customary to use the symbol $\sqrt{g}$ to denote $\sqrt{\det\big[(g_{jk})_{1\leq j,k\leq n}\big]}$. In addition, for each index $j \in \{1, \dots, n\}$ we agree to abbreviate

$$\widehat{\mathrm{d}x_j} := \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{j-1} \wedge \mathrm{d}x_{j+1} \wedge \cdots \wedge \mathrm{d}x_n. \tag{9.9.26}$$

Then (cf. [189, (2.1.14), p. 50] and [189, (2.1.27), p. 53])

$$\mathrm{d}V_g = \sqrt{g}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = *1 \tag{9.9.27}$$

which, in concert with [189, Lemma 2.2(9), p. 54] and [189, Proposition 2.4, p. 55], implies that for each $i \in \{1, \dots, n\}$ we have

$$\begin{aligned} *(\mathrm{d}x_i) &= *(\mathrm{d}x_i \wedge 1) = \mathrm{d}x_i \vee (*1) = \mathrm{d}x_i \vee \mathrm{d}V_g \\ &= \sqrt{g}\sum_{j=1}^{n}(-1)^{j-1} g^{ij}\widehat{\mathrm{d}x_j}. \end{aligned} \tag{9.9.28}$$

If we now pick $k \in \{1, \dots, n\}$ then multiply the most extreme sides of (9.9.28) by g_{ik} and sum up in $i \in \{1, \dots, n\}$ we arrive at

$$\widehat{\mathrm{d}x_k} = \frac{(-1)^{k-1}}{\sqrt{g}}\sum_{i=1}^{n} g_{ik} * (\mathrm{d}x_i). \tag{9.9.29}$$

Hence, on account of $* * \mathrm{d}x_i = (-1)^{n+1}\mathrm{d}x_i$ (see [189, Lemma 2.2(1), p. 54]), from (9.9.29) we obtain that

$$*(\widehat{\mathrm{d}x_k}) = \frac{(-1)^{k+n}}{\sqrt{g}}\sum_{i=1}^{n} g_{ik}\,\mathrm{d}x_i \quad \text{for each} \quad k \in \{1, \dots, n\}. \tag{9.9.30}$$

Given an arbitrary $(n-1)$-form $\omega \in \mathscr{C}^0(M, \Lambda^{n-1}TM)$, we may locally express

$$\omega = \sum_{k=1}^{n}(-1)^{k-1}\omega_k\widehat{\mathrm{d}x_k} \tag{9.9.31}$$

for some scalar-valued continuous functions ω_k, $1 \le k \le n$. Then, based on (9.9.31), (1.11.10), (9.9.30), and (9.9.25), we may compute (bearing in mind that the matrices $(g^{jk})_{1\le j,k\le n}$ and $(g_{jk})_{1\le j,k\le n}$ are inverse to one another)

$$\begin{aligned} (-1)^{n-1}\big\langle * \omega, \nu_g\big\rangle_{T^*M}\sigma_g &= \sqrt{g}\sum_{j,k=1}^{n}(-1)^{n+k}\omega_k\nu_j^{\mathrm{E}}\big\langle * (\widehat{\mathrm{d}x_k}), \mathrm{d}x_j\big\rangle_{T^*M}\sigma^{\mathrm{E}} \\ &= \sum_{i,j,k=1}^{n}\omega_k\nu_j^{\mathrm{E}}g_{ik}\big\langle \mathrm{d}x_i, \mathrm{d}x_j\big\rangle_{T^*M}\sigma^{\mathrm{E}} \\ &= \sum_{i,j,k=1}^{n}\omega_k\nu_j^{\mathrm{E}}g_{ki}g^{ij}\sigma^{\mathrm{E}} = \sum_{j,k=1}^{n}\omega_k\nu_j^{\mathrm{E}}\delta_{jk}\sigma^{\mathrm{E}} \\ &= \sum_{j=1}^{n}\omega_j\nu_j^{\mathrm{E}}\sigma^{\mathrm{E}} = (-1)^{n-1}\big\langle *^{\mathrm{E}}\,\omega, \nu^{\mathrm{E}}\big\rangle_{\mathbb{R}^n}\sigma^{\mathrm{E}}, \end{aligned} \tag{9.9.32}$$

where $*^{\mathrm{E}}$ is the Hodge star operator associated with the standard Euclidean metric in $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ is the standard inner product in $\mathbb{R}^n$. Since the last expression above does not depend on the Riemannian metric g, we conclude from (9.9.32) and (1.11.58) that for each given $(n-1)$-form $\omega \in \mathscr{C}^0(M, \Lambda^{n-1}TM)$ the Radon measure $\iota_\#^*\omega$ is actually independent of the Riemannian metric g (chosen to define it). This proves that, as claimed, the definition of the sharp pull-back given in (1.11.58) is in fact independent of the background Riemannian metric g.

Next, fix an arbitrary $(n-1)$-form $\omega \in \mathscr{C}^0(M, \Lambda^{n-1}TM)$. That its sharp pull-back $\iota_\#^*\omega$ (originally defined in (1.11.58)) may be equivalently written as in (1.11.59) is seen with the help of [189, Lemma 2.2,(3),(8), and (2.1.27)–(2.1.29), pp. 53–54].

Consider now the task of establishing the integral formula (1.11.60) for an arbitrary $(n-1)$-form $\omega \in \mathscr{C}_c^1(M, \Lambda^{n-1}TM)$. To set the stage, recall from (1.12.139) the musical isomorphism between cotangent and tangent vectors on M, expressed in local coordinates by

$$T^*M \ni \xi = \sum_{j=1}^{n} \xi_j \mathrm{d}x_j \longmapsto \xi^\sharp := \sum_{j,k=1}^{n} g^{jk}\xi_j\partial_k \in TM. \tag{9.9.33}$$

In turn, this metric identification of T^*M with TM permits us to define the vector field

$$\vec{F} := (*\omega)^\sharp \in \mathscr{C}_c^1(M, TM). \tag{9.9.34}$$

Thanks to [189, (9.1.16), p. 373] and [189, Lemma 2.8(2), p. 63], this satisfies

$$\operatorname{div}_g \vec{F} = -\delta(*\omega) = (-1)^{n-1} * \mathrm{d}\omega \quad \text{on} \quad M, \tag{9.9.35}$$

where div_g is the differential geometric divergence, while d and δ are, respectively, the exterior derivative operator and its transpose, on the Riemannian manifold M. Bearing in mind that $**\mathrm{d}\omega = \mathrm{d}\omega$ and $*1 = \mathrm{d}V_g$ (cf. [189, Lemma 2.2(1), p. 54] and (9.9.27)), this further implies

$$\mathrm{d}\omega = (-1)^{n-1}(\operatorname{div}_g \vec{F})(*1) = (-1)^{n-1}(\operatorname{div}_g \vec{F})\,\mathrm{d}V_g \quad \text{on} \quad M. \tag{9.9.36}$$

In addition, from (9.9.34) and [189, (9.1.3), p. 372] we see that

$$\begin{aligned} {}_{T^*M}\big(\nu_g, \vec{F}\,\big|_{\partial_*\Omega}\big)_{TM} &= {}_{T^*M}\Big(\nu_g, \big((*\omega)\,\big|_{\partial_*\Omega}\big)^\sharp\Big)_{TM} \\ &= \big\langle(*\omega)\big|_{\partial_*\Omega}, \nu_g\big\rangle_{T^*M} \quad \text{at } \sigma_g - \text{a.e. point on } \partial_*\Omega. \end{aligned} \tag{9.9.37}$$

At this stage, Theorem 1.11.1 (cf. also Remark 1.11.2) may be applied to the vector field (9.9.34), in which case the Divergence Formula (1.11.13) becomes precisely (1.11.60), on account of (9.9.36), (9.9.37), and (1.11.4). □

We now turn to the proof of Theorem 1.11.10.

Proof of Theorem 1.11.10 With the piece of notation introduced in (1.12.139), consider the vector field (compare with (9.9.34))

$$\vec{F} := (*\omega)^{\sharp} \in L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n_g) \otimes TM. \tag{9.9.38}$$

Then, in a manner analogous to (9.9.35) and (9.9.36), we presently obtain

$$\mathrm{div}_g \vec{F} = (-1)^{n-1} * \mathrm{d}\omega \in L^1(\Omega, \mathcal{L}^n_g), \tag{9.9.39}$$

as well as

$$\mathrm{d}\omega = (-1)^{n-1}(\mathrm{div}_g \vec{F})\,\mathrm{d}V_g \quad \text{on} \quad \Omega. \tag{9.9.40}$$

Also, much as in (9.9.37), we now have

$${}_{T^*M}\Big(\nu_g, \vec{F}\,\Big|^{\text{n.t.}}_{\partial\Omega}\Big)_{TM} = \Big\langle(*\omega)\Big|^{\text{n.t.}}_{\partial\Omega}, \nu_g\Big\rangle_{T^*M} \quad \text{at } \sigma_g - \text{a.e. point on } \partial_*\Omega. \tag{9.9.41}$$

Granted (9.9.39)–(9.9.41), we may invoke Corollary 1.11.5, and (1.11.20) gives (1.11.70).

Going further, the fact that the nontangential pull-back of ω to $\partial_*\Omega$ defined as in (1.11.68) is actually independent of the Riemannian metric g may be justified as in the proof of Theorem 1.11.9. Specifically, if ω is locally represented as in (9.9.31), then reasoning as in (9.9.32) presently gives

$$\begin{aligned} \iota^*_{\text{n.t.}}\,\omega &= \sum_{j=1}^{n}\Big(\omega_j\Big|^{\text{n.t.}}_{\partial\Omega}\Big)\nu^{\mathrm{E}}_j\sigma^{\mathrm{E}} \\ &= (-1)^{n-1}\Big\langle *^{\mathrm{E}}\Big(\omega\Big|^{\text{n.t.}}_{\partial\Omega}\Big), \nu^{\mathrm{E}}\Big\rangle_{\mathbb{R}^n}\sigma^{\mathrm{E}} \quad \text{on} \quad \partial_*\Omega, \end{aligned} \tag{9.9.42}$$

from which the desired conclusion is clear. Another proof of the independence of $\iota^*_{\text{n.t.}}\,\omega$ on the metric g goes as follows. First observe that for each scalar function $\varphi \in \mathscr{C}^1(M)$ the differential $(n-1)$-form $\varphi\,\omega$ belongs to $L^1_{\mathrm{loc}}(\Omega, \mathcal{L}^n_g) \otimes \Lambda^{n-1}TM$ and satisfies properties which are similar to those of ω recorded in (1.11.67). Indeed, since φ is bounded we have $\mathcal{N}_\kappa(\varphi\,\omega) \in L^1(\partial\Omega, \sigma_g)$, and since φ is continuous it follows that $(\varphi\,\omega)\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists and equals $\big(\varphi\big|_{\partial_{\text{nta}}\Omega}\big)\Big(\omega\Big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)$ at σ_g-a.e. point on $\partial_{\text{nta}}\Omega$. In particular,

$$\iota^*_{\text{n.t.}}(\varphi\,\omega) = \big(\varphi\big|_{\partial_*\Omega}\big)\iota^*_{\text{n.t.}}\,\omega \quad \text{on} \quad \partial_*\Omega. \tag{9.9.43}$$

Also, with the action of the exterior derivative operator d considered in the sense of distributions in Ω, we have

$$\mathrm{d}(\varphi\,\omega) = \varphi\,\mathrm{d}\omega + \mathrm{d}\varphi \wedge \omega \in L^1(\Omega, \mathcal{L}^n_g) \otimes \Lambda^n TM, \tag{9.9.44}$$

where the membership is implied by (1.11.67) and Proposition 8.6.10. Having noted this, (1.11.70) applies to the $(n-1)$-form $\varphi\,\omega$ and, in concert with (9.9.43), permits us to write

$$\int_{\partial_*\Omega} \varphi\, \iota^*_{\text{n.t.}}\, \omega = \int_{\partial_*\Omega} \iota^*_{\text{n.t.}}(\varphi\, \omega) = \int_{\Omega} \mathrm{d}(\varphi\, \omega). \tag{9.9.45}$$

Then the desired conclusion follows from (9.9.45), on account of the arbitrariness of φ, by observing that the last expression above is independent of the metric g.

Finally, that $\iota^*_{\text{n.t.}}\, \omega$ (originally defined in (1.11.68)) may be equivalently expressed as in (1.11.69) is a consequence of [189, Lemma 2.2,(3),(8), and (2.1.27)–(2.1.29), pp. 53–54]. □

Here is the proof of Theorem 1.11.11.

Proof of Theorem 1.11.11 The idea is to reason as in the proof of Theorem 1.11.10, now making use of the full force of Theorem 1.11.3 (in lieu of Corollary 1.11.5), for the vector field given by $\vec{F} := (*\omega)^\sharp \in \mathcal{D}'(\Omega) \otimes TM$. □

References

1. D. Aalto, L. Berkovits, O.E. Maasalo, H. Yue, John-Nirenberg lemmas for a doubling measure. Studia Math. **204**(1), 21–37 (2009)
2. D.R. Adams, L.I. Hedberg, *Function Spaces and Potential Theory* (Springer, Berlin, Heidelberg, 1996)
3. V. Adolfsson, L^2-integrability of second-order derivatives for Poisson's equation in nonsmooth domains. Math. Scand. **70**(1), 146–160 (1992)
4. H. Aikawa, Harmonic functions having no tangential limits. Proc. Amer. Math. Soc. **108**(2), 457–464 (1990)
5. H. Aikawa, Harmonic functions and Green potentials having no tangential limits. J. London Math. Soc. (2) **43**(1), 125–136 (1991)
6. H. Aikawa, K. Hirata, Doubling conditions for harmonic measure in John domains. Ann. Inst. Fourier (Grenoble) **58**(2), 429–445 (2008)
7. H. Aimar, A. Bernardis, B. Iaffei, Comparison of Hardy-Littlewood and dyadic maximal functions on spaces of homogeneous type. J. Math. Anal. Appl. **312**(1), 105–120 (2005)
8. G. Alberti, M. Ottolini, On the structure of continua with finite length and Gołąb's semicontinuity theorem. Nonlinear Anal. **153**, 35–55 (2017)
9. R. Alvarado, D. Brigham, V. Maz'ya, M. Mitrea, E. Ziadé, On the regularity of domains satisfying a uniform hour-glass condition and a sharp version of the Hopf-Oleinik Boundary Point Principle. J. Math. Sci. **176**(3), 281–360 (2011)
10. R. Alvarado, I. Mitrea, M. Mitrea, Whitney-type extensions in geometrically doubling quasi-metric spaces. Commun. Pure Appl. Anal. **12**(1), 59–88 (2013)
11. R. Alvarado, M. Mitrea, *Hardy Spaces on Ahlfors-Regular Quasi-Metric Spaces. A Sharp Theory*, Lecture Notes in Mathematics, vol. 2142 (Springer, 2015)
12. L. Ambrosio, V. Caselles, S. Masnou, J.-M. Morel, Connected components of sets of finite perimeter and applications to image processing. J. Eur. Math. Soc. **3**(1), 39–92 (2001)
13. C. Andreian Cazacu, *Theorie der Funktionen mehrerer komplexer Veränderlicher* (Birkhäuser Verlag, Basel, 1975)
14. J. Azzam, S. Hofmann, J.M. Martell, K. Nyström, T. Toro, A new characterization of chord-arc domains. J. Eur. Math. Soc. **19**(4), 967–981 (2017)
15. M. Badger, Null sets of harmonic measure on NTA domains: Lipschitz approximation revisited. Mathematische Zeitschrift **270**(1), 241–262 (2012)
16. S. Bell, *The Cauchy Transform, Potential Theory and Conformal Mapping* (CRC Press, Boca Raton, Fl., 1992)
17. C. Bennett, R. Sharpley, *Interpolation of Operators*, Pure and Applied Mathematics, vol. 129 (Academic Press, 1988)

D. Mitrea et al., *Geometric Harmonic Analysis I*, Developments in Mathematics 72,
https://doi.org/10.1007/978-3-031-05950-6

18. C. Berenstein, R. Gay, *Complex Variables: An Introduction* (Springer, New York, 1991)
19. J. Bergh, J. Löfström, *Interpolation Spaces. An Introduction* (Springer, Berlin/New York, 1976)
20. A. Bernardis, O. Salinas, Two-weight norm inequalities for the fractional maximal operator on spaces of homogeneous type. Studia Math. **108**, 201–207 (1994)
21. A.S. Besicovitch, On the fundamental geometrical properties of linearly measurable plane sets of points. Math. Ann. **98**, 422–464 (1928)
22. A. Björn, J. Björn, *Nonlinear Potential Theory on Metric Spaces*, EMS Tracts in Mathematics, vol. 17 (European Mathematical Society (EMS), Zürich, 2011)
23. J. Björn, N. Shanmmugalingam, Poincaré inequalities, uniform domains, and extension properties for Newton-Sobolev functions in metric spaces. J. Math. Anal. Appl. **332**, 190–208 (2007)
24. S. Bochner, Green-Goursat theorem. Math. Z. **63**, 230–242 (1955)
25. V.I. Bogachev, *Measure Theory*, vol. I (Springer, Berlin, Heidelberg, 2007)
26. F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Research Notes in Mathematics, vol. 76 (Pittman, Boston, MA, 1982)
27. K. Brewster, D. Mitrea, I. Mitrea, M. Mitrea, Extending Sobolev functions with partially vanishing traces from locally (ε, δ)-domains and applications to mixed boundary problems. J. Funct. Anal. **266**, 4314–4421 (2014)
28. R.M. Brown, The method of layer potentials for the heat equation in Lipschitz cylinders. Amer. J. Math. **111**(2), 339–379 (1989)
29. A. Brudnyi, Y. Brudnyi, *Methods of Geometric Analysis in Extension and Trace Problems*, vol. 1, Monographs in Mathematics, 102 (Birkhäuser, 2012)
30. Y. Brudnyĭ, N. Krugljak, *Interpolation Functors and Interpolation Spaces*, vol. I, North-Holland Mathematical Library, No. 47 (North-Holland Publishing Co., Amsterdam, 1991)
31. H. Busemann, *The Geometry of Geodesics* (Academic Press, New York, 1955)
32. R. Caccioppoli, *Sulla quadratura delle superfici piane e curve*, Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali, VI, vol. 6 (1927), pp. 142–146
33. A.M. Caetano, On fractals which are not so terrible. Fund. Math. **171**(3), 249–266 (2002)
34. A.P. Calderón, On the behavior of harmonic functions near the boundary. Trans. Amer. Math. Soc. **68**, 47–54 (1950)
35. A.P. Calderón, Inequalities for the maximal function relative to a metric. Studia Math. **57**, 297–306 (1976)
36. A.P. Calderón, *Commutators, singular integrals on Lipschitz curves and applications*, pp. 85–96 in "Proceedings of the International Congress of Mathematicians" (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980
37. A.P. Calderón, *Boundary value problems for the Laplace equation in Lipschitzian domains*, pp. 33–48 in "Recent Progress in Fourier Analysis," ed. by I. Peral, J. Rubio de Francia, Math. Stud. vol. 111 (Elsevier/North-Holland, Amsterdam, 1985)
38. A.P. Calderón, A. Zygmund, Local properties of solutions of elliptic partial differential equations. Studia Math. **20**, 171–225 (1961)
39. L. Capogna, P. Tang, Uniform domains and quasiconformal mappings on the Heisenberg group. Manuscripta Math. **86**, 267–281 (1995)
40. L. Carleson, On the existence of boundary values for harmonic functions in several variables. Ark. Mat. **4**, 393–399 (1962)
41. L. Carleson, On convergence and growth of partial sums of Fourier series. Acta Math. **116**, 135–157 (1966)
42. A. Cauchy, Sur les intégrales qui s'étendent à tous les points d'une courbe fermée. Comptes Rendus **23**, 251–255 (1846)
43. H.M. Chang, R.A. Hunt, D.S. Kurtz, The Hardy-Littlewood maximal function on $L(p, q)$ spaces with weights. Indiana Univ. Math. J. **31**, 109–120 (1982)
44. G.-Q. Chen, H. Frid, Divergence-measure fields and hyperbolic conservation laws. Arch. Rational Mech. Anal. **147**, 89–118 (1999)

45. G.-Q. Chen, M. Torres, Divergence-measure fields, sets of finite perimeter, and conservation laws. Arch. Ration. Mech. Anal. **175**(2), 245–267 (2005)
46. G.-Q. Chen, M. Torres, W. Ziemer, Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws. Comm. Pure Appl. Math. **62**(2), 242–304 (2009)
47. M. Christ, *Lectures on Singular Integral Operators*, The American Mathematical Society CBMS Regional Conference Series in Mathematics, vol. 77 (Providence, RI, 1990)
48. M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. Colloq. Math. **LX/LXI**, 601–628 (1990)
49. F. Cobos, T. Kühn, T. Schonbek, One-sided compactness results for Aronszajn-Gagliardo functors. J. Funct. Anal. **106**(2), 274–313 (1992)
50. F. Cobos, L.-E. Persson, Real interpolation of compact operators between quasi-Banach spaces. Math. Scand. **82**, 138–160 (1998)
51. P.J. Cohen, On Green's theorem. Proc. Amer. Math. Soc. **10**, 109–112 (1959)
52. R.R. Coifman, C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals. Studia Math. **51**, 241–250 (1974)
53. R. Coifman, A. McIntosh, Y. Meyer, L'intégrale de Cauchy definit un opérateur borné sur L^2 pour les courbes lipschitziennes. Ann. Math. **116**, 361–388 (1982)
54. R. Coifman, Y. Meyer, E.M. Stein, Some new function spaces and their applications to harmonic analysis. J. Funct. Anal. **62**, 304–335 (1985)
55. R. Coifman, R. Rochberg, Another characterization of BMO. Proc. Amer. Math. Soc. **79**, 249–254 (1980)
56. R.R. Coifman, G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Mathematics No. 242 (Springer, 1971)
57. R.R. Coifman, G. Weiss, Extensions of Hardy spaces and their use in analysis. Bull. Amer. Math. Soc. **83**(4), 569–645 (1977)
58. D. Cruz-Uribe, J. Martell, C. Pérez, *Extensions of Rubio de Francia's extrapolation theorem*, Collect. Math., Vol. Extra (2006), pp. 195–231
59. J. Cufí, J. Verdera, *A General Form of Green Formula and Cauchy Integral Theorem*, arXiv:1306.6832v1 (2013)
60. M. Cwikel, Real and complex interpolation and extrapolation of compact operators. Duke Math. J. **65**(2), 333–343 (1992)
61. B.E.J. Dahlberg, Estimates of harmonic measure. Arch. Rational Mech. Anal. **65**(3), 275–288 (1977)
62. B.E.J. Dahlberg, On the Poisson integral for Lipschitz and C^1 domains. Studia Math. **66**, 13–24 (1979)
63. B.E.J. Dahlberg, C.E. Kenig, Hardy spaces and the Neumann problem in L^p for Laplace's equation in a Lipschitz domain. Ann. Math. **125**, 437–465 (1987)
64. G. David, Opérateurs intégraux singulière sur certaines courbes du plan complexe. Ann. Scient. École Norm. Sup. (4) **17**, 157–189 (1984)
65. G. David, Une minoration de la norme de l'opérateur de Cauchy sur le graphes lipschitziennes. Trans. Amer. Math. Soc. **302**, 741–750 (1987)
66. G. David, D. Jerison, Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals. Indiana Univ. Math. J. **39**(3), 831–845 (1990)
67. G. David, P. Mattila, Removable sets for Lipschitz harmonic functions in the plane. Rev. Mat. Iberoamericana **16**(1), 137–215 (2000)
68. G. David, S. Semmes, *Singular Integrals and Rectifiable Sets in* $\mathbb{R}^n$: *Beyond Lipschitz Graphs*, Astérisque, No. 193 (1991)
69. G. David, S. Semmes, *Analysis of and on Uniformly Rectifiable Sets*, Mathematical Surveys and Monographs, AMS Series (1993)
70. E. De Giorgi, Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni. Ric. di Mat. **4**, 95–113 (1955)
71. E. De Giorgi, *Sulla proprietà isoperimetrica dell'ipersfera, nella classe degli insiemi aventi frontiera orientata di misura finita*, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat., Sez. I, (8), vol. 5 (1958), pp. 33–44

72. E. De Giorgi, *Complementi alla teoria della misura* $(n-1)$*-dimensionale in uno spazio n-dimensionale, Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960–61* (Pisa, Editrice Tecnico Scientifica, 1961), p. 31
73. E. De Giorgi, *Frontiere orientate di misura minima*, Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61 Editrice Tecnico Scientifica, Pisa (1961), 57 pp
74. M. Dindoš, *Hardy Spaces and Potential Theory on* C^1 *Domains in Riemannian Manifolds*, Memoirs of the American Mathematical Society, vol. 191, No. 894 (Providence, Rhode Island, 2008)
75. M. Dindoš, C. Kenig, J. Pipher, BMO solvability and the A_∞ condition for elliptic operators. J. Geom. Anal. **21**(1), 78–95 (2011)
76. O. Dovgoshey, O. Martio, V. Ryazanov, M. Vuorinen, The Cantor function. Expo. Math. **24**, 1–37 (2006)
77. R. Duduchava, D. Mitrea, M. Mitrea, Differential operators and boundary value problems on surfaces. Mathematische Nachrichten **9–10**, 996–1023 (2006)
78. X.T. Duong, L. Yan, *New function spaces of BMO type, the John-Nirenberg inequality, interpolation, and applications*, Communications on Pure and Applied Mathematics, vol. LVIII, (2005), pp. 1375–1420
79. B. Dyda, L. Ihnatsyeva, J. Lehrbäck, H. Tuominen, A.V. Vähäkangas, Muckenhoupt A_p-properties of distance functions and applications to Hardy-Sobolev-type inequalities. Potential Anal. **50**, 83–105 (2019)
80. L.C. Evans, R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics (CRC Press, Boca Raton, FL, 1992)
81. E.B. Fabes, M. Jodeit Jr., N.M. Rivière, Potential techniques for boundary value problems on C^1-domains. Acta Math. **141**(3–4), 165–186 (1978)
82. K.J. Falconer, *The Geometry of Fractal Sets, Cambridge Tracts in Mathematics*, vol. 85 (Cambridge University Press, Cambridge, 1986)
83. P. Fatou, Séries trigonométriques et séries de Taylor. Acta Math. **30**, 335–400 (1906)
84. H. Federer, The Gauss-Green theorem. Trans. Amer. Math. Soc. **58**, 44–76 (1945)
85. H. Federer, Measure and area. Bull. AMS **58**, 306–378 (1952)
86. H. Federer, A note on the Gauss-Green theorem. Proc. Amer. Math. Soc. **9**, 447–451 (1958)
87. H. Federer, Curvature measures. Trans. Amer. Math. Soc. **93**, 418–491 (1959)
88. H. Federer, *Geometric Measure Theory*, reprint of the 1969 edition (Springer, 1996)
89. C. Fefferman, E.M. Stein, Some maximal inequalities. Amer. J. Math. **93**(1), 107–115 (1971)
90. C. Fefferman, E.M. Stein, H^p spaces of several variables. Acta Math. **129**(3–4), 137–193 (1972)
91. G.B. Folland, *Real Analysis, Modern Techniques and Their Applications*, 2nd edn. (Wiley, New York, 1999)
92. D. Freitag, Real interpolation of weighted L_p-spaces. Mathematische Nachrichten **86**(1), 15–18 (1978)
93. R. Fueter, Analytische Funktionen einer Quaternionenvariablen. Comment. Math. Helv. **4**, 9–20 (1932)
94. J. García-Cuerva, José Luis Rubio de Francia (1949–1988). Collect. Math. **38**(1), 3–15 (1987)
95. J. García-Cuerva, A.E. Gatto, Boundedness properties of fractional integral operators associated to non-doubling measures. Studia Math. **162**(3), 245–261 (2004)
96. J. Garcia-Cuerva, A.E. Gatto, Lipschitz spaces and Calderón-Zygmund operators associated to non-doubling measures. Publicacions Matematiques **49**(2), 285–296 (2005)
97. J. García-Cuerva, J.L. Rubio de Francia, *Weighted Norm Inequalities*, North Holland, Mathematics Studies, vol. 116 (1985)
98. J. Garnett, Positive length but zero analytic capacity. Proc. Amer. Math. Soc. **21**, 696–699 (1970)
99. J. Garnett, *Analytic Capacity and Measure*, Lecture Notes in Mathematics, vol. 297 (Springer, Berlin, Heidelberg, New York, 1972)
100. J. Garnett, D.E. Marshall, *Harmonic Measure* (Cambridge University Press, 2005)

101. C.F. Gauss, Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo nova tractata. Commentationes societatis regiae scientiarium Gottingensis recentiores **II**, 1–24 (1813)
102. F.W. Gehring, B.G. Osgood, Uniform domains and the quasihyperbolic metric. Journal d'Analyse Mathématique **36**, 50–74 (1979)
103. I. Genebashvili, A. Gogatishvili, V. Kokilashvili, M. Krbec, *Weight Theory for Integral Transforms on Spaces of Homogeneous Type*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 92 (CRC Press, 1997)
104. D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edn. (Springer, Berlin, Heidelberg, 1983)
105. J.E. Gilbert, M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis* (Cambridge University Press, 1991)
106. E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, vol. 80 (Birkhäuser, 1984)
107. L. Grafakos, *Classical and Modern Fourier Analysis* (Pearson Education Inc., New Jersey, 2004)
108. L. Grafakos, L. Liu, D. Yang, Vector-valued singular integrals and maximal functions on spaces of homogeneous type. Math. Scand. **104**(2), 296–310 (2009)
109. G. Green, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism, Nottingham* (T. Wheelhouse, England, 1828)
110. P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Monographs and Studies in Mathematics, vol. 24 (Pitman Boston, MA, 1985)
111. M. Grüter, K.-O. Widman, The Green function for uniformly elliptic equations. Manuscripta Math. **37**(3), 303–342 (1982)
112. G. Hardy, J. Littlewood, Some properties of conjugate functions. J. Reine Angew. Math. **167**, 405–423 (1932)
113. W. Hansen, Uniform boundary Harnack principle and generalized triangle property. J. Funct. Anal. **226**(2), 452–484 (2005)
114. J. Harrison, Flux across nonsmooth boundaries and fractal Gauss/Green/Stokes' theorems. J. Phys. A: Math. Gen. **32**, 5317–5327 (1999)
115. J. Harrison, A. Norton, The Gauss-Green theorem for fractal boundaries. Duke J. Math. **67**, 575–588 (1992)
116. R. Harvey, J. Polking, Removable singularities of solutions of linear partial differential equations. Acta Math. **125**, 39–56 (1970)
117. F. Hausdorff, *Grundzuege der Mengenlehre* (Viet, Leipzig, 1914)
118. J. Heinonen, *Lectures on Analysis on Metric Spaces* (Springer, New York, 2001)
119. G.M. Henkin, J. Leiterer, *Theory of Functions on Complex Manifolds*, Monographs in Mathematics (Birhäuser, 1984)
120. D. Hestenes, G. Sobczyk, *Clifford Algebra to Geometric Calculus. A Unified Language for Mathematics and Physics* (D. Reidel Publ. Comp., Dordrecht, Boston, Lancaster, 1984)
121. K. Hoffman, *Banach Spaces of Analytic Functions* (Dover Publications Inc., New York, 1962)
122. S. Hofmann, J.L. Lewis, L^2 Solvability and representation by caloric layer potentials in time-varying domains. Ann. Math. **144**(2), 349–420 (1996)
123. S. Hofmann, D. Mitrea, M. Mitrea, A.J. Morris, *L^p-Square Function Estimates on Spaces of Homogeneous Type and on Uniformly Rectifiable Sets*, Memoirs of the American Mathematical Society, vol. 245, No. 1159 (2017)
124. S. Hofmann, M. Mitrea, M. Taylor, Geometric and transformational properties of Lipschitz domains, Semmes-Kenig-Toro domains, and other classes of finite perimeter domains. J. Geom. Anal. **17**(4), 593–647 (2007)
125. S. Hofmann, M. Mitrea, M. Taylor, Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains. Int. Math. Res. Not. IMRN (14), 2567–2865 (2010)
126. L. Hörmander, *An Introduction to Complex Analysis in Several Variables* (D. van Nostrand, Princeton N.J., 1966)

127. L. Hörmander, *The Analysis of Linear Partial Differential Operators. I. Distribution Theory and Fourier Analysis*, reprint of the second (1990) edition, Classics in Mathematics (Springer, Berlin, 2003)
128. T. Hytönen, A. Kairema, Systems of dyadic cubes in a doubling metric space. Colloquium Math. **126**(6), 1–33 (2012)
129. T. Hytönen, C. Pérez, E. Rela, Sharp Reverse Hölder property for A_∞ weights on spaces of homogeneous type. J. Funct. Anal. **263**(12), 3883–3899 (2012)
130. J. Jarnik, J. Kurzweil, A nonabsolutely convergent integral which admits transformation and can be used for integration on manifolds. Czechoslovak Math. J. **35**, 116–139 (1985)
131. D. Jerison, C. Kenig, The Neumann problem on Lipschitz domains. Bull. Amer. Math. Soc. **4**(2), 203–207 (1981)
132. D.S. Jerison, C.E. Kenig, Boundary behavior of harmonic functions in nontangentially accessible domains. Adv. Math. **46**(1), 80–147 (1982)
133. D.S. Jerison, C.E. Kenig, Hardy spaces, A_∞, and singular integrals on chord-arc domains. Math. Scand. **50**, 221–247 (1982)
134. R. Johnson, C.J. Neugebauer, Change of variable results for A_p-and reverse Hölder RH_r-classes. Trans. Amer. Math. Soc. **328**(2), 639–666 (1991)
135. P.W. Jones, Quasiconformal mappings and extendability of functions in Sobolev spaces. Acta Math. **47**, 71–88 (1981)
136. P.W. Jones, A geometric localization theorem. Adv. Math. **46**, 71–79 (1982)
137. P.W. Jones, Square functions, Cauchy integrals, analytic capacity, and harmonic measure, in *Harmonic Analysis and Partial Differential Equations*, ed. by J. Garcia-Cuerva, Lecture Notes in Mathematics, vol. 1384 (Springer, 1989)
138. A. Jonsson, H. Wallin, Function spaces on subsets of $\mathbb{R}^n$. Math. Rep. **2**(1) (1984)
139. J. Jost, *Partial Differential Equations*, Graduate Texts in Mathematics, vol. 214 (Springer, 2002)
140. J. Jost, *Riemannian Geometry and Geometric Analysis*, 5th edn. (Universitext, Springer, 2008)
141. J.-L. Journé, *Calderón-Zygmund Operators, Pseudodifferential Operators, and the Cauchy Integral of Calderón*, Lecture Notes in Mathematics, vol. 994 (Springer, 1983)
142. M. Jurchescu, M. Mitrea, Pluridimensional absolute continuity for differential forms and the Stokes formula. L'Enseignement Mathématique **41**(3–4), 217–255 (1995)
143. W.B. Jurkat, D.J.F. Nonnenmacher, The fundamental theorem for the ν_1-integral on more general sets and a corresponding divergence theorem with singularities. Czechoslovak Math. J. **45**, 69–77 (1995)
144. S. Kaijser, A note on dual Banach spaces. Math. Scand. **41**, 325–330 (1977)
145. V.J. Katz, The history of Stokes' theorem. Math. Mag. **52**(3), 146–156 (1979)
146. C.E. Kenig, Weighted H^p spaces on Lipschitz domains. Amer. J. Math. **102**, 129–163 (1980)
147. C.E. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems, CBMS Regional Conference Series in Mathematics*, vol. 83 (AMS, Providence, RI, 1994)
148. C.E. Kenig, J. Pipher, The Neumann problem for elliptic equations with nonsmooth coefficients. Invent. Math. **113**(3), 447–509 (1993)
149. C. Kenig, T. Toro, Harmonic measure on locally flat domains. Duke Math. J. **87**(3), 509–551 (1997)
150. C. Kenig, T. Toro, Poisson kernel characterization of Reifenberg flat chord arc domains. Ann. Sci. École Norm. Sup. (4) **36**, 323–401 (2003)
151. B.V. Khvedelidze, Linear discontinuous boundary value problems of function theory, singular integral equations, and some of their applications, (Russian), Trudy Tbil. Matem. Instituta Akademii Nauk GSSR. **23**, 3–158 (1956)
152. J. Kinnunen, The Hardy-Littlewood maximal function of a Sobolev function. Israel J. Math. **100**, 117–124 (1997)
153. J. Kinnunen, R. Korte, N. Marola, N. Shanmugalingam, A characterization of BMO self-maps of a metric measure space. Collect. Math. **66**, 405–421 (2015)

154. J. Kinnunen, P. Shukla, The structure of reverse Hölder classes on metric measure spaces. Nonlinear Anal. **95**, 666–675 (2014)
155. P. Koosis, *Introduction to H_p Spaces* (Cambridge University Press, 1998)
156. O. Kowalski, D. Preiss, Besicovitch-type properties of measures and submanifolds. J. Reine Angew. Math. **379**, 115–151 (1987)
157. J. Král, The divergence theorem. Math. Bohem. **121**, 349–356 (1996)
158. S.G. Krantz, *Functions of Several Complex Variables*, 2nd edn. (AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island, 2001)
159. J.-L. Lagrange, *Nouvelles recherches sur la nature et la propagation du son*, Miscellanea Taurinensia, 2 (1762), 11–172 (reprinted in: Oeuvres de Lagrange, vol. 1, pp. 151–316, Paris, Gauthier-Villars, 1867)
160. M.A. Lavrentiev, Boundary problems in the theory of univalent functions. Math. Sbornik, N.S., **1**, 815–845 (1936). Amer. Math. Soc. Translations, Ser. 2, **32**, 1–35 (1963)
161. G. Leoni, *A First Course in Sobolev Spaces*, Graduate Studies in Mathematics, vol. 105 (American Mathematical Society, 2009)
162. F. Lin, X. Yang, *Geometric Measure Theory – An Introduction*, International Press, Advanced Mathematics, vol. 1 (Boston, 2002)
163. E. Lindelöf, Sur un principe générale de l'analyse et ses applications à la theorie de la représentation conforme. Acta Soc. Sci. Fennicae **46**, 1–35 (1915)
164. J.E. Littlewood, Mathematical notes (4): on a theorem of Fatou. J. London Math. Soc. **2**, 172–176 (1927)
165. J.E. Littlewood, Mathematical notes (8): on functions subharmonic in a circle (II). Proc. London Math. Soc. **28**(2), 383–394 (1928)
166. N.N. Lusin, Sur la représentation conforme. Bull. Ivanovo-Vozn. Politech. Inst. **2**, 77–80 (1919)
167. T.J. Lyons, P.S.C. Yam, On Gauss-Green theorem and boundaries of a class of Hölder domains. Journal de Mathématiques Pures et Appliquées **85**(1), 38–53 (2006)
168. A. Macdonald, Stokes' theorem. Real Anal. Exchange **27**, 739–748 (2001/2002)
169. R.A. Macías, C. Segovia, Lipschitz functions on spaces of homogeneous type. Adv. Math. **33**(3), 257–270 (1979)
170. F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems. An introduction to Geometric Measure Theory*, Cambridge Studies in Advance Mathematics, vol. 135 (Cambridge University Press, 2012)
171. J.J. Marín, J. María Martell, D. Mitrea, I. Mitrea, M. Mitrea, *Singular Integrals, Quantitative Flatness, and Boundary Problems*, Progress in Mathematics, vol. 344 (Birkhäuser, 2022)
172. O. Martio, Definitions for uniform domains. Annales Academiae Scientiarum Fennicae, Series A. I. Mathematica **5**, 197–205 (1980)
173. O. Martio, J. Sarvas, Injectivity theorems in plane and space. Ann. Acad. Sci. Fenn. Ser. A I Math. **4**, 383–401 (1978–79)
174. A. Mas, Variation for singular integrals on Lipschitz graphs: L^p and endpoint estimates. Trans. Amer. Math. Soc. **365**(11), 5759–5781 (2011)
175. P. Matilla, A class of sets with positive length and zero analytic capacity. Ann. Acad. Sci. Fenn. **10**, 387–395 (1985)
176. P. Matilla, M.S. Melnikov, J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability. Ann. Math. **144**(1), 127–136 (1996)
177. P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability*, vol. 44 (Cambridge University Press, London, 1995)
178. V. Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, 2nd, revised and augmented edition, Grundlehren der mathematischen Wissenschaften, vol. 342 (Springer, 2011)
179. Y. Meyer, *Ondelettes et Opérateurs II* (Opérateurs de Calderón-Zygmund, Hermann, Paris, 1990)
180. N.G. Meyers, Mean oscillation over cubes and Hölder continuity. Proc. Amer. Math. Soc. **15**(5), 717–721 (1964)

181. D. Mitrea, *Distributions, Partial Differential Equations, and Harmonic Analysis*, 2nd edn. (Springer Nature Switzerland, 2018)
182. D. Mitrea, I. Mitrea, M. Mitrea, *A Treatise on the Theory of Elliptic Boundary Value Problems, Singular Integral Operators, and Smoothness Spaces in Rough Domains*, book manuscript (2013)
183. D. Mitrea, I. Mitrea, M. Mitrea, *On the Geometry of Sets Satisfying Uniform Ball Conditions*, preprint (2014)
184. D. Mitrea, I. Mitrea, M. Mitrea, A sharp divergence theorem with nontangential traces. Not. Amer. Math. Soc. **67**(9), 1295–1305 (2020)
185. D. Mitrea, I. Mitrea, M. Mitrea, *Geometric Harmonic Analysis II: Function Spaces Measuring Size and Smoothness on Rough Sets*, Developments in Mathematics (Springer, Cham, 2022)
186. D. Mitrea, I. Mitrea, M. Mitrea, *Geometric Harmonic Analysis III: Integral Representations, Calderón-Zygmund Theory, Fatou Theorems, and Applications to Scattering*, Developments in Mathematics (Springer, Cham, 2022)
187. D. Mitrea, I. Mitrea, M. Mitrea, *Geometric Harmonic Analysis IV: Boundary Layer Potentials in Uniformly Rectifiable Domains, and Applications to Complex Analysis*, Developments in Mathematics (Springer, Cham, 2022)
188. D. Mitrea, I. Mitrea, M. Mitrea, S. Monniaux, *Groupoid Metrization Theory with Applications to Analysis on Quasi-Metric Spaces and Functional Analysis* (Birkhäuser, 2013)
189. D. Mitrea, I. Mitrea, M. Mitrea, M. Taylor, *The Hodge-Laplacian: Boundary Value Problems on Riemannian Manifolds*, Studies in Mathematics, vol. 64 (De Gruyter, 2016)
190. D. Mitrea, I. Mitrea, M. Mitrea, E. Ziadé, Abstract capacitary estimates and the completeness and separability of certain classes of non-locally convex topological vector spaces. J. Funct. Anal. **262**(11), 4766–4830 (2012)
191. D. Mitrea, M. Mitrea, S. Monniaux, The Poisson problem for the exterior derivative operator with Dirichlet boundary condition on nonsmooth domains. Commun. Pure Appl. Anal. **7**(6), 1295–1333 (2008)
192. D. Mitrea, M. Mitrea, M. Taylor, *Layer Potentials, the Hodge Laplacian and Global Boundary Problems in Non-Smooth Riemannian Manifolds*, Memoirs of the American Mathematical Society, vol. 150, No. 713 (Providence RI, 2001)
193. D. Mitrea, M. Mitrea, J. Verdera, Characterizing regularity of domains via the Riesz transforms on their boundaries. Anal. PDE **9**(4), 955–1018 (2016)
194. I. Mitrea, M. Mitrea, *Multi-Layer Potentials and Boundary Problems for Higher-Order Elliptic Systems in Lipschitz Domains*, Lecture Notes in Mathematics, vol. 2063 (Springer, Berlin, 2013)
195. I. Mitrea, M. Mitrea, M. Taylor, Cauchy integrals, Calderon projectors, and Toeplitz operators on uniformly rectifiable domains. Adv. Math. **268**, 666–757 (2015)
196. M. Mitrea, *Clifford Wavelets, Singular Integrals, and Hardy Spaces*, Lecture Notes in Mathematics, vol. 1575 (Springer, Berlin, 1994)
197. M. Mitrea, The method of layer potentials in electro-magnetic scattering theory on non-smooth domains. Duke Math. J. **77**(1), 111–133 (1995)
198. M. Mitrea, M. Taylor, Boundary layer methods for Lipschitz domains in Riemannian manifolds. J. Funct. Anal. **163**(2), 181–251 (1999)
199. M. Mitrea, M. Taylor, Potential theory on Lipschitz domains in Riemannian manifolds: Hölder continuous metric tensors. Commun. Partial Differ. Equ. **25**(7-8), 148 (2000)
200. M. Mitrea, F. Şabac, Pompeiu's integral representation formula. History and mathematics, Revue Roumaine de Mathématiques Pures et Appliquées **43**(1–2), 211–226 (1998)
201. M. Mitrea, M. Wright, *Boundary Value Problems for the Stokes System in Arbitrary Lipschitz Domains*, Astérisque, Soc. Math. de France, vol. 344 (2012)
202. G.C. Moisil, Sur la généralisation des fonctions conjuguées, Rendiconti della Reale Accad. Naz. dei Lincei **14**, 401–408 (1931)
203. G.C. Moisil, *Les fonctions monogènes dans les espaces à plusieurs dimensions*, p. 129 in Comptes Rendus LXIV Congrès des Soc. Savantes Clermont-Ferrant (1931)

204. G.C. Moisil, N. Teodorescu, Fonctions holomorphes dans l'espace. Mathematica (Cluj) **5**, 142–159 (1931)
205. J.-M. Morel, S. Solimini, *Variational Methods in Image Segmentation* (Birkhäuser, Boston, 1995)
206. B. Muckenhoupt, R. Wheeden, Weighted norm inequalities for fractional integrals. Trans. Amer. Math. Soc. **192**, 261–274 (1974)
207. B. Muckenhoupt, R.L. Wheeden, Weighted bounded mean oscillations and the Hilbert transform. Studia Matematica **54**, 221–237 (1976)
208. F. Nazarov, X. Tolsa, A. Volberg, On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1. Acta Math. **213**, 237–321 (2014)
209. F. Nazarov, A. Volberg, *On analytic capacity of portions of continuum and a question of T. Murai*, pp. 279–94 in "Perspectives in Partial Differential Equations, Harmonic Analysis and Applications," ed. by D. Mitrea, M. Mitrea, Proceedings of Symposia in Pure Mathematics, vol. 79 (American Mathematical Society, 2008)
210. J. Nečas, *Direct Methods in the Theory of Elliptic Equations, corrected 2nd printing*, Springer Monographs in Mathematics (Springer, Heidelberg, Dordrecht, London, New York, 2012)
211. K. Nyström, Integrability of Green potentials in fractal domains. Ark. Mat. **34**, 335–381 (1996)
212. D.J.F. Nonnenmacher, Sets of finite perimeter and the Gauss-Green theorem with singularities. J. London Math. Soc. **52**, 335–344 (1995)
213. M.V. Ostrogradsky, *Démonstration d'un théorème du calcul intégral*, communicated to the Paris Academy on February 13 (1826)
214. H.M. Pajot, *Analytic Capacity, Rectifiability, Menger Curvature and Cauchy Integral*, Lecture Notes in Mathematics, vol. 1799 (Springer, 2002)
215. W.F. Pfeffer, The Gauss-Green theorem. Adv. Math. **87**, 93–147 (1991)
216. W.F. Pfeffer, *The Divergence Theorem and Sets of Finite Perimeter* (Chapman and Hall, CRC Press, 2012)
217. N. Phuc, M. Torres, Characterizations of the existence and removable singularities of divergence-measure vector fields. Indiana Univ. Math. J. **57**(4), 1573–1597 (2008)
218. Ch. Pommerenke, *Boundary Behaviour of Conformal Maps, Grundlehren der Mathematischen Wissenschaften*, vol. 299 (Springer, Berlin, 1992)
219. D. Pompeiu, *Sur la continuité des fonctions de variables complexes* (Thése), (Gauthier-Villars, Paris, 1905); Ann. Fac. Sci. de Toulouse **7**, 264–315 (1905)
220. D. Pompeiu, Sur les fonctions dérivées. Math. Ann. **63**, 326–332 (1906)
221. D. Pompeiu, Sur une classe d'intégrales doubles. Bull. de la Sect. Scient. de l'Académie Roumaine **1**, 128–131 (1912)
222. D. Pompeiu, Sur une classe d'intégrales doubles (deuxième note). Bull. de la Sect. Scient. de l'Académie Roumaine **1**, 265–271 (1912)
223. D. Pompeiu, Sur une classe d'intégrales doubles (troisième note). Bull. de la Sect. Scient. de l'Académie Roumaine **1**, 289–296 (1912)
224. D. Pompeiu, Sur une classe de fonctions d'une variable complexe. Rendiconti del Circolo Matematico di Palermo **33**, 108–113 (1912)
225. D. Pompeiu, Sur une classe de fonctions d'une variable complexe et sur certaines équations intégrales. Rendiconti del Circolo Matematico di Palermo **35**, 277–281 (1913)
226. A.C. Ponce, Singularities of the divergence of continuous vector fields and uniform Hausdorff estimates. Indiana Univ. Math. J. **62**(4), 1055–1074 (2013)
227. R.M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables, Graduate Texts in Mathematics*, vol. 108 (Springer, New York, 1986)
228. R. Remmert, *Theory of Complex Functions* (Springer, New York, 1991)
229. B. Riemann, *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (Göttingen, 1867)
230. H.L. Royden, *Real Analysis*, 3rd edn. (Prentice Hall, 1988)
231. W. Rudin, *Real and Complex Analysis*, 3rd edn. (McGraw-Hill, Boston, Massachusetts, 1987)

232. W. Rudin, *Functional Analysis*, 2nd edn. (McGraw-Hill Inc., New York, 1991)
233. E. Sawyer, Weighted Lebesgue and Lorentz norm inequalities for the Hardy operator. Trans. Amer. Math. Soc. **281**, 329–337 (1984)
234. S. Semmes, Analysis vs. geometry on a class of rectifiable hypersurfaces in $\mathbb{R}^n$. Indiana Univ. Math. J. **39**(4), 1005–1035 (1990)
235. V.L. Shapiro, On Green's theorem. J. London Math. Soc. **32**, 261–269 (1957)
236. V.L. Shapiro, The divergence theorem for discontinuous vector fields. Ann. Math. **68**, 604–624 (1958)
237. Z. Shen, Boundary value problems for parabolic Lamé systems and a nonstationary linearized system of Navier-Stokes equations in Lipschitz cylinders. Amer. J. Math. **113**(2), 293–373 (1991)
238. L. Simon, *Lectures on Geometric Measure Theory* (Australian National University, Centre for Mathematical Analysis, Canberra, 1983)
239. E.M. Stein, *Conjugate harmonic functions in several variables*, pp. 414–420 in International Congress of Mathematicians (Stockholm, 1962)
240. E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30 (Princeton University Press, Princeton, NJ, 1970)
241. E.M. Stein, The differentiability of functions in $\mathbb{R}^n$. Ann. Math. **113**, 383–385 (1981)
242. E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, vol. 43, Monographs in Harmonic Analysis, III (Princeton University Press, Princeton, NJ, 1993)
243. E.M. Stein, G. Weiss, On the theory of harmonic functions of several variables, I. The theory of H^p spaces. Acta Math. **103**, 25–62 (1960)
244. E.M. Stein, G. Weiss, Generalization of the Cauchy-Riemann equations and representations of the rotation group. Amer. J. Math. **90**(1), 163–196 (1968)
245. E.M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces* (Princeton University Press, Princeton, N.J., 1971)
246. C.H. Stolze, A history of the divergence theorem. Historia Mathematica **5**, 437–442 (1978)
247. J.-O. Strömberg, A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Mathematics, vol. 1391 (Springer, 1989)
248. A. Taheri, *Function Spaces and Partial Differential Equations*, vol. 1, Classical Analysis, Oxford Lecture Series in Mathematics and Its Applications (2015)
249. L. Takács, An increasing continuous singular function. Amer. Math. Mon. **85**(1), 35–37 (1978)
250. M. Taylor, *Partial Differential Equations*, vol. I (Springer, 1996) (2nd edn., 2011)
251. M.E. Taylor, *Measure Theory and Integration*, Graduate Studies in Mathematics, vol. 76 (American Mathematical Society, 2006)
252. N. Théodoresco, *La Dérivée Areolaire*, Ann. Roumanine des Math., Cahier 3, Bucharest (1936)
253. B.S. Thomson, J.B. Bruckner, A.M. Bruckner, *Elementary Real Analysis* (Prentice Hall, 2001)
254. X. Tolsa, Principal values for Riesz transforms and rectifiability. J. Funct. Anal. **254**, 1811–1863 (2008)
255. J. Väisälä, *Lectures on n-Dimensional Quasiconformal Mappings*, Lecture Notes in Mathematics, vol. 299 (Springer, 1971)
256. J. Väisälä, Uniform domains. Tohoku Math. J. **40**(2), 101–118 (1988)
257. G. Verchota, Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. J. Funct. Anal. **59**(3), 572–611 (1984)
258. T. Ważewski, *Kontinua prostowalne w związku z funkcjami i odwzorowaniami absolutnie ciągłemi* (Rectifiable continua in connection with absolutely continuous functions and mappings). Dodatek do Rocznika Polskiego Towarzystwa Matematycznego (Supplement to the Annals of the Polish Mathematical Society) (1927), pp. 9–49
259. R.O. Wells, *Differential Analysis on Complex Manifolds*, Graduate Texts in Mathematics (Springer, New York, 1980)
260. R.L. Wheeden, A characterization of some weighted norm inequalities for the fractional maximal function. Studia Math. **107**(3), 257–272 (1993)

261. R. Wittmann, A non-tangential limit theorem. Osaka J. Math. **24**, 61–76 (1987)
262. W. Ziemer, *Functions of least gradient and BV functions*, pp. 270–312 in “Nonlinear Analysis, Function Spaces and Applications”, vol. 6 (Acad. Sci. Czech Repub., Prague, 1999)
263. W. Ziemer, *Weakly Differentiable Functions* (Springer, New York, 1989)
264. W.P. Ziemer, K. Zumbrun, The obstacle problem for functions of least gradient. Math. Bohem. **124**(2–3), 193–219 (1999)
265. A. Zygmund, *Trigonometric Series* (Cambridge University Press, 1968)
266. A. Zygmund, On certain lemmas of Marcinkiewicz and Carleson. J. Approx. Theory **2**, 249–257 (1969)

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