Mathematics and Lego: the untold story

Eugenia Cheng
Newnham College, Cambridge
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University of Sheffield
http://cheng.staff.shef.ac.uk/

Abstract

This is the approximate text from an informal after dinner talk I gave at Newnham College in November 2001, aimed at students and researchers of all levels and all disciplines. This is not a formally presented document, and does not include the slides, so some of it is a bit hard to imagine.

When I tell people I'm a pure mathematician, they invariably say something like "Oh, so I wont understand what you do." Well I'm now going to let out my best kept secret: my research is actually just like playing around with lego.

Lego: the universal toy. Enjoyed by people of all ages all over the place. The idea is simple and brilliant. Start with some basic blocks that can be joined together. Add creativity, imagination and a bit of ingenuity. Build anything.

Mathematics is exactly the same. We start with some basic building blocks and ways of joining them together. And then we use creativity, and, yes, imagination and certainly ingenuity, and try to build anything.

Of course, just like with lego, things start getting complicated. You get fed up with the limited range of possibilities when youve only got basic lego bricks. You might want to build a house—so you need door pieces, maybe window pieces.

Then you can built all sorts of houses...until you start to long for something more complicated...

There are then all sorts of other pieces we can buy. There are in fact whole series of themed pieces to cater for what kind of thing you want to build. Theres castle lego, spaceship lego, star wars lego and so on. And there are lego fanatics who become particularly obsessed with one branch of lego, because they just want to build castles, or spaceships, or whatever. And there are some people who just want to use basic bricks and build huge and extraordinary sculptures from scratch.

Maths is the same. We start wanting to build more and more complicated mathematical things. So we might use more complicated building blocks, or we might come up with more and more ingenious ways of sticking the blocks together.

The idea of Category Theory

I work in Category Theory. I study mathematical structures—how they are built in general. Why do we need to do this? Well if we only want to build something simple then we probably can just follow our nose and use our instinct. But if we want to build complicated things we might need to understand properly the principles behind building. Personally, I could take some lego and build an ordinary little house. But could I build...the Eiffel Tower? The Leaning Tower of Pisa? I would need to understand the principles of building huge things in order to do that.

Now we could go round this room and see whats the most complicated thing we think we could build from lego off the top of our heads. Wed probably have wildly varying capabilities. Perhaps theres a lego fanatic in here who could build something like...a moving robot with articulated arms and legs and a swivelling head?

Well, it is possible to help other people build brilliant lego models. If I go out to the shops I can buy a lego set which will explain to me the principles of building, or how to build a firestation, or a garage or something. If I only want to build something simple, I dont need the instructions. But if I want to build something complex then I do.

Maths is the same. A lot of maths can be done without really thinking about how structures are built. Thats why my field, Category Theory, only came into existence about fifty years ago. Thats when people realised that things were getting complicated, and we needed to work out the *principles of building* in order to help other mathematicians build good mathematical structures. Thats what Category Theory is trying to do.

How mathematical structures are built

So how do we build mathematical structures? Or, put another way, how does lego *work*? Imagine explaining to a lego virgin how lego works, without having any actual lego in front of you.

We would explain it slowly, bit by bit. At the most basic level we have basic bricks. I dont know about you but the 2×4 brick is what I consider to be the *standard* lego brick. Whats important about lego bricks? The key thing is that they can be stuck together. How? Well, for a start, just directly on top. Well, our lego novice might then interrupt and say "Wait, but then we'll only get tall thin towers!" and they will start doing staggering the bricks so they're not just in a big vertical column, and they'll discover that they can build a wall. But it will take them longer to work out that they can go round corners. And then they might try and build a house, and they might ask, "But how will I get doors and windows??" Well thats ok, because actually there are other lego pieces like door pieces and window pieces, so you CAN build a house...

So you see, we can gradually introduce more and more things to our lego system to build more and more complicated things. Fundamentally, we start with building blocks, and ways of sticking them together. We can then introduce more ways of sticking them together, or more kinds of building block.

Different philosophies

This is where lego fanatics—and mathematicians—go off in different philosophical directions. Think about it. Would you be more interested in using basic lego bricks to build fantastic big sculptures? Or would you be interested in buying all the complicatedly designed little pieces to build machines or working robots or trainsets?

My work

Personally, Im a purist. I am really amazed by using just basic bricks to build the Eiffel Tower or a giraffe or something. And Im the same with maths. Im a purist, a pure mathematician. I dont use complicated gadgets and machinery. I take basic bricks—*mathematical* bricks—and study the possilibities of what can be built with them.

Studying the possibilites of a system.

A mathematical system is like a lego set. You have certain types of pieces—perhaps just basic pieces, perhaps castle pieces or spaceship pieces—that can be stuck together in certain ways.

A mathematical structure is what you build with it.

So really, mathematicians are just sitting around playing with their lego. Well, perhaps I should just speak for myself. I sit around and play with my lego.

High dimensions

So far so good. We like lego. Its fun. Its simple.

Now imagine that youve woken up in a 4-dimensional world. That is, four space dimensions, not counting time. It would be weird and exciting! But now suppose they had *no lego*. Disaster! Calamity! Right, you think—*I will bring lego to the 4-dimensional world!*

This is what my research is about. Except in the mathematical world there arent just 4 dimensions—we can have as many dimensions as we want, four hundred, four million, four zillion, four squillion. . .So there are rather a lot of things out there that we cant build with mere 3-dimensional lego.

Imagine for a moment a lego set with only one-by-one pieces! You would not be able to do anything very interesting at all! You wouldnt be able to build houses, castles or anything really. Its easy to understand this lego set, but frankly rather feeble.

So we as mathematicians are trying to sort out the higher dimensions.

But what on earth does this mean? Does my analogy still work? If you found yourself in a 4-d world, how would you go about making 4-dimensional lego?

Well, its hard, especially as youre new to this extra dimension so dont have a full understanding of how it all works. You just know that the possibilities in four dimensions are suddenly incredible. We can try a few things.

Generalise

You might sit down and think very hard about how lego works in 3-dimensions, to see if we can do the same sort of thing in 4-dims. In my work I spend a lot of time doing this—I study very carefully the structures in low dimensions that are understood, to try to extend the principles into higher dimensions.

Look for evidence

You might also try to familiarise yourself a bit more with the 4-d world youre in. Whats a 4-d house? Whats a 4-d car? Whats a 4-d bottle? If we can understand the workings of the 4-d world a bit better, well be a bit closer to being able to make 4-d lego. In my work, I do just this - there are higher-dimensional mathematical structures out there so I study those carefully to try and understand the principles at work.

Come up with a prototype and try it out

Once you think you have a reasonable understanding of these things you might be able to come up with a prototype, a first guess at what 4-d lego could be. And then you test it in various ways. First of all you might just see if it work—do the pieces fit together properly? Do they fit tightly so that the structure will be secure?

Once youre happy that the pieces fit together you could start trying to build things. You might wander round the 4-d world trying to build the things that you see. You might be able to build some things but not others. So you go back to the workshop to see if you can modify your lego so that you can build those other thing, and then you try again...

And yes, this is what I do in my work as well. We have a prototype—and really its still only the basic bricks were trying to work out. We have to see if the bricks will fit together properly—which is what we call coherence theory. And we also go back and wander round the mathematical world, looking at the higher-dimensional structures and seeing if we can build them with our basic bricks.

The state of the art

So how far have we got? Well imagine that there are rival factions all trying to make what they think is the best 4-dimensional lego. Theyre all very obsessed, and they think of course that their lego is the best. But they hardly even look at what anyone else is doing. They just glance and say either thats rubbish or else accuse them of stealing their design.

But then some young idealist comes along and thinks: oh, wouldnt it be so much more productive if we all worked together, and we can then benefit from everyone elses wisdom, and work in harmony rather than fight against one another, united we stand, divided we fall, etc etc.

Well in the world of mathematical lego, that young idealist is me.

The end—nearly.

Epilogue

Last night someone asked me about this seminar and I said: in summary, my work is just like playing around with lego. She looked at me completely incredulously and said "Yes but maths is hard, but lego is easy!" And I thought—is it? If you went and looked round Legoland, could you build things like that? Well *I* certainly couldnt. But I can do maths. In fact, I've given a few talks about this carefully kept secret:

Life is hard, but maths is easy!