3rd Edition

Trigonometry

For **dummies**®

A Wiley Brand

Understand the basics of sines, cosines, and tangents

Learn how to use trigonometry outside of the classroom

Graph functions and solve tough formulas

Mary Jane Sterling

Master Pi Maker

Trigonometry

3rd Edition

by Mary Jane Sterling

for dummies®
A Wiley Brand

Trigonometry For Dummies®, 3rd Edition

Contents at a Glance

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Introduction

M any of the exciting and, yes, practical accomplishments of early humans were performed using trigonometry. Even before it was formalized into a particular topic to study or used to solve problems, trigonometry helped people to sail across large bodies of water, build gigantic structures, plot out land, and measure heights and distances — even to the stars.

We still use trigonometry for all these reasons and more. For example, if you're going to get your pilot's license, you'll need trigonometry. Trigonometry is also the basis for many courses in mathematics — starting in grade school with geometric shapes and map reading, and moving on through calculus. Trig is all over the place.

You can get as deeply into this topic or as little into it as you want, and you'll still come out of it thinking, "Gee, I didn't realize that trigonometry was used to do this! Wasn't that just loads of fun!" Well, maybe I'm pushing it a bit — loads may be a slight exaggeration.

Whether you're pursuing trigonometry so that you can go on to calculus or prepare for architecture or drafting or do that piloting thing, or you're just curious, you'll find what you need here. You can get as technical as you want. You can skip through the stuff you don't need. Just know that you'll be on the same adventure as those early humans — you'll just have the advantage of a few more tools.

About This Book

So, what's in it for you? What's in a book on trigonometry that'll ring your bell or strike your fancy or just make you pretty happy? Where do I begin?

You can start anywhere in the book, jump around, and just go any direction you please. If you're really excited about triangles or circles and how they can be used to your advantage, check them out. Everything you'd want to know about them is here.

What if you've got another angle? Or, maybe you didn't have one to begin with but wish you did. If you're looking for angles, you've come to the right place. There are big angles and little angles, all named depending on their situation.

They're measured in degrees or radians. "What's a radian?" you ask. You can find out in this book, that's for sure.

You may be very analytical. If so, you'll find that your favorite spot is among all the identities and equation-solving procedures. Hop right to it. They're waiting for you.

And if drawing pictures is your bag, go to the chapters on graphing to see what can be done with simple trig graphs, complicated trig graphs, and everything in between. There are even explanations on what the function equations mean, why they're used in an application, and how they're related to the graphs.

You'll find many sidebars throughout this book. Sidebars are those fun, little anecdotes that don't necessarily contain a lot of math content but present interesting little tidbits — fun things to read. Neither the sidebars nor the items marked with the Technical Stuff icon are necessary for your understanding of the material. Think of them as little diversions for your reading pleasure.

Within this book, you may note that some web addresses break across two lines of text. If you're reading this book in print and want to visit one of these web pages, simply key in the web address exactly as it's noted in the book, pretending as though the line break doesn't exist. If you're reading this as an e-book, you've got it easy — just click the web address to be taken directly to the web page.

Foolish Assumptions

How foolish of me to assume that you're reading *Trigonometry For Dummies* because it looks more interesting than the latest bestseller! I'm really not that foolish! To be honest, trig wouldn't be my first choice for a fun read. I'm just going to assume that you really want to do this. While writing this book, I made a few other assumptions about you as well:

» You have a goal in mind. You want to conquer some of the topics in this book so you're prepared for a course of study.

» You have a pretty solid grasp on algebra and can solve a simple algebraic equation without falling completely apart.

» You're planning on being on *Jeopardy,* and you need to bone up on the possible trig questions.

Icons Used in This Book

Icons are easy to spot. They could be called *eye-cons*, because they catch your eye. Here are the ones I use in this book:

This icon points out handy hints and shortcuts that make your life easier when it comes to solving trig problems.

Of course, trig rules — it's fun! But taken another way, this icon is used to point out to you when particular equations or expressions or formulas are used in trigonometry that you should be paying attention to. They're important. This icon helps you find them again, if you need them.

This icon refers back to information that I think you may already know. It needs to be pointed out or repeated so that the current explanation makes sense.

What about trigonometry isn't technical? Actually, there's quite a bit, but this icon points out the rules or absolutely unchangeable stuff that you may need in order to understand the situation.

There are always things that are tricky or confusing, or problems that just ask for an error to happen. This icon is there to alert you, hoping to help you avoid a mathematical pitfall.

Beyond the Book

In addition to the material in the print or e-book you're reading right now, this product also comes with some access-anywhere goodies on the web. No matter how diligent you are about reading through this material, you'll likely come across a few questions where you don't have a clue. Check out the free online Cheat Sheet for helpful information, all provided in a concise, quick-access format. Simply go to `www.dummies.com` and type **Trigonometry For Dummies Cheat Sheet** in the search box.

Where to Go from Here

Back when I was in college, my friend, Judy Christopher, once consoled me with, "Life is like a sine curve. It has its ups and downs. If you're feeling like you're really down, then just remember that you'll be going up that same amount some-day soon." So, if you're in the dumps, maybe you want to start with the graphs of the sine curves and other trig curves. Make of them what you will.

Or maybe, like me, you're a puzzle buff. I can't wait to tackle a crossword or other fun type of puzzle in the Sunday paper or online. You have to call up bits and pieces of information and make them all fit into something logical. If that's what you're interested in today, then go to proving identities and solving equations. That's a great challenge for a rainy Sunday afternoon's pleasure.

Are you into angles and directions and plans? You may want to start with the ways that angles are measured and how they all fit together in the big picture. The basics are always a good place to start when you're investigating a topic.

No matter where you start with this book, be ready to flip the pages front to back or back to front. Think of it as an adventure that can take you many interesting places. Enjoy!

1

Getting Started with Trigonometry

IN THIS PART . . .

Become acquainted with angle measures and how they relate to trig functions.

Discover formulas that provide lengths of segments, midpoints, and slopes of lines.

Become familiar with circles and the relationships between radii, diameters, centers, and arcs.

Find a simple conversion method for changing from degrees to radians and vice versa.

Observe the properties of special right triangles and use the Pythagorean Theorem to formulate the relationships between the sides of these right triangles.

Chapter **1**

Taking On Trig Technicalities

How did Columbus find his way across the Atlantic Ocean? How did the Egyptians build the pyramids? How did early astronomers measure the distance to the moon? No, Columbus didn't follow a yellow brick road. No, the Egyptians didn't have LEGO instructions. And, no, there isn't a tape measure long enough to get to the moon. The common answer to all these questions is trigonometry.

Trigonometry is the study of angles and triangles and the wonderful things about them and that you can do with them. For centuries, humans have been able to measure distances that they can't reach because of the power of this mathematical subject.

Taking Trig for a Ride: What Trig Is

"What's your angle?" That question isn't a come-on such as, "What's your astrological sign?" In trigonometry, you can measure angles in both degrees and radians. You can position the angles into triangles and circles and make them do special things. Actually, angles drive trigonometry. Sure, you have to consider

algebra and other math to make it all work. But you can't have trigonometry without angles. Put an angle measure into a trig function, and out pops a special, unique number. What do you do with that number? Read on, because that's what trig is all about.

Sizing up the basic figures

Segments, rays, and lines are some of the basic forms found in geometry, and they're just as important in trigonometry. As I explain in the following sections, you use those segments, rays, and lines to form angles and triangles and other geometric and trig forms.

Drawing segments, rays, and lines

A *segment* is a straight figure drawn between two endpoints. You usually name it by its endpoints, which you indicate by capital letters. Sometimes, a single letter names a segment; this single letter is positioned at about the middle of the segment. For example, in a triangle, a lowercase letter may refer to a segment opposite the angle labeled with the corresponding uppercase letter.

A *ray* is another straight figure that has an endpoint on one end, and then it just keeps going forever in some specified direction. You name rays by their endpoint first and then by any other point that lies on the ray. You indicate that the other end goes on forever by using an arrow point.

A *line* is a straight figure that goes forever and ever in either direction. You only need two points to determine a particular line — and only one line can go through both of those points. You can name a line by any two points that lie on it.

Figure 1-1 shows a segment, ray, and line and the different ways you can name them using points.

A   B   segment AB; $\overline{AB}$

C   D   ray CD; $\overrightarrow{CD}$
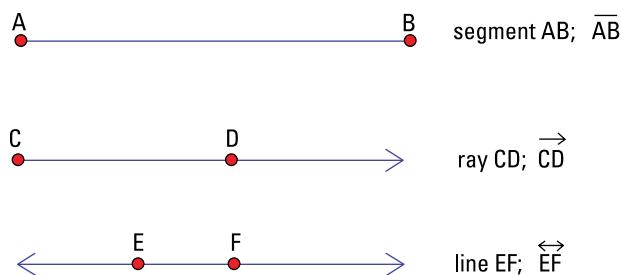
FIGURE 1-1:
Segment AB, ray CD, and line EF.

E   F   line EF; $\overleftrightarrow{EF}$

Intersecting lines

When two lines intersect — if they do intersect — they can only do so at one point. They can't double back and cross one another again. And some curious things happen when two lines intersect. The angles that form between those two lines are related to one another. Any two angles that are next to one another and share a side are called *adjacent angles.* In Figure 1-2, you see several sets of intersecting lines and marked angles. The top two figures indicate two pairs of adjacent angles. Can you spot the other two pairs? The angles that are opposite one another when two lines intersect also have a special name. Mathematicians call these angles *vertical angles.* They don't have a side in common. The two middle pairs in Figure 1-2 are vertical angles. Vertical angles are always equal in measure.
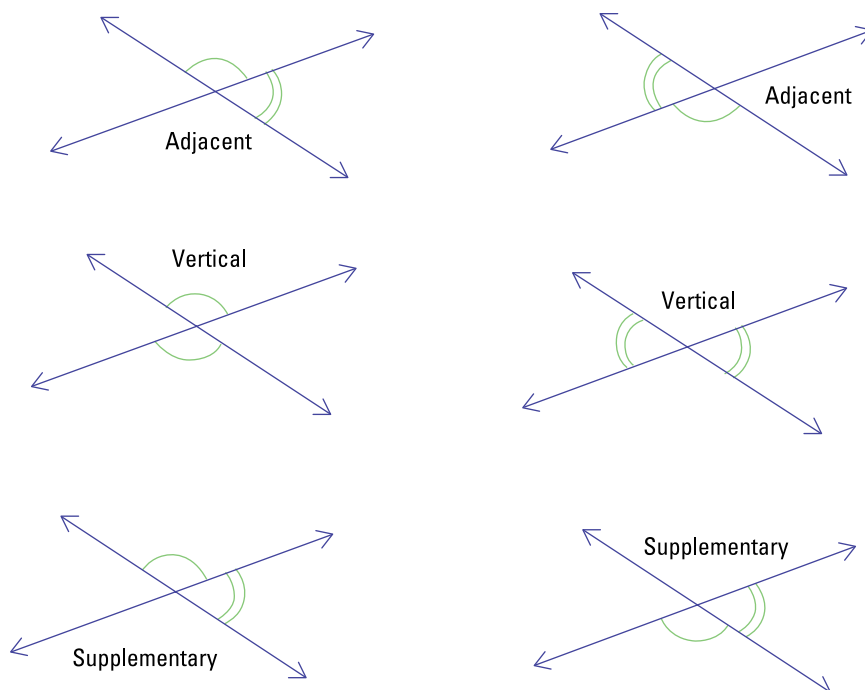
FIGURE 1-2: Intersecting lines form adjacent, vertical, and supplementary angles.

Why are these different angles so special? They're different because of how they interact with one another. The adjacent angles here are called *supplementary angles.* The sides that they don't share form a straight line, which has a measure of 180 degrees. The bottom two figures show supplementary angles. Note that these are also adjacent.

Identifying angles and their names

When two lines, segments, or rays touch or cross one another, they form an angle or angles. In the case of two intersecting lines, the result is four different angles. When two segments intersect, they can form one, two, or four angles; the same goes for two rays.

These examples are just some of the ways that you can form angles. Geometry, for example, describes an angle as being created when two rays have a common end-point. In practical terms, you can form an angle in many ways, from many figures. The business with the two rays means that you can extend the two sides of an angle out farther to help with measurements, calculations, and practical problems.

Describing the parts of an angle is pretty standard. The place where the lines, seg-ments, or rays cross is called the *vertex* of the angle. From the vertex, two sides extend.

Naming angles by size

You can name or categorize angles based on their size or measurement in degrees and radians. For more on radian measures, go to Chapter 4. Figure 1-3 shows examples of each of the following angles.

>> **Acute:** An angle with a positive measure less than 90 degrees (less than $\frac{\pi}{2}$ radians).

>> **Obtuse:** An angle measuring more than 90 degrees but less than 180 degrees (between $\frac{\pi}{2}$ and π radians).

>> **Right:** An angle measuring exactly 90 degrees (or $\frac{\pi}{2}$ radians).

>> **Straight:** An angle measuring exactly 180 degrees (a straight line or π radians).

>> **Oblique:** An angle measuring more than 180 degrees (more than π radians).

Naming angles by letters

How do you name an angle? Why does it even need a name? In most cases, you want to be able to distinguish a particular angle from all the others in a picture. When you look at a photo in a newspaper, you want to know the names of the dif-ferent people and be able to point them out. With angles, you should feel the same way.
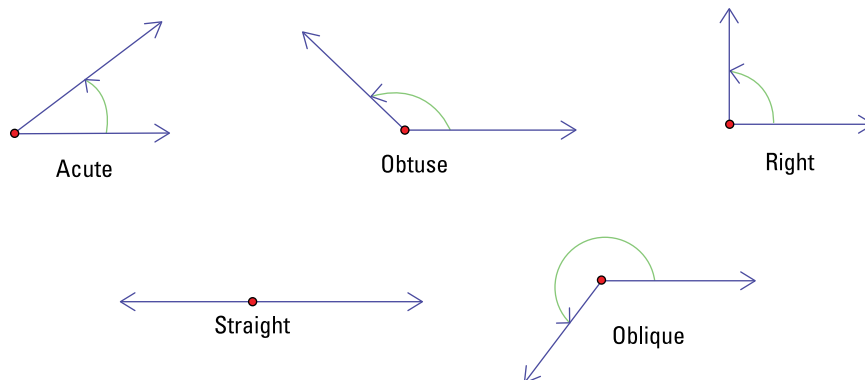
You can name an angle in one of three different ways.

» **By its vertex alone:** Often, you name an angle by its vertex alone because such a label is efficient, neat, and easy to read. In Figure 1-4, you can call the angle *A*. You only use this type of name if there aren't any angles adjacent at the vertex *A*. It has to stand alone.

» **By a point on one side, followed by the vertex, and then a point on the other side:** For example, you can call the angle in Figure 1-4 angle *BAC* or angle *CAB*. This naming method is helpful if someone may be confused as to which angle you're referring to in a picture. ***Remember:*** Make sure you always name the vertex in the middle.

» **By a letter or number written inside the angle:** Usually, that letter is Greek; in Figure 1-4, however, the angle has the letter *w*. Often, you use a number for simplicity if you're not into Greek letters or if you're going to compare different angles later.
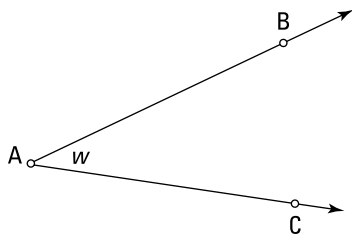
FIGURE 1-4:
Naming an angle.

Taking on triangles and their angles

All on their own, angles are certainly very exciting. But put them into a triangle, and you've got icing on the cake. Triangles are one of the most frequently studied geometric figures. When angles are part of a triangle, they have many characteristics.

Angles in triangles

A triangle always has three angles. The angles in a triangle have measures that always add up to 180 degrees — no more, no less. A triangle named ABC (often written $\triangle ABC$) has angles A, B, and C, and you can name the sides $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, depending on which two angles the side is between. The angles themselves can be acute, obtuse, or right. If the triangle has either an obtuse or right angle, then the other two angles have to be acute.

Naming triangles by their shape

Triangles can have special names based on their angles and sides. They can also have more than one name — a triangle can be both acute and isosceles, for example. Here are their descriptions, and check out Figure 1-5 for the pictures.

» **Acute triangle:** A triangle where all three angles are acute.

» **Right triangle:** A triangle with a right angle (the other two angles must be acute).

Acute and Scalene   Acute and Isoceles   Obtuse and Isosceles

Right and Scalene   Right and Isoceles   Equilateral
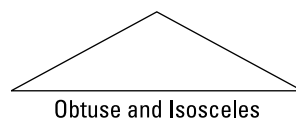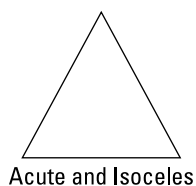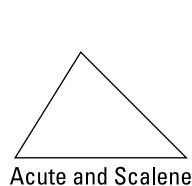
FIGURE 1-5: Triangles can have more than one name, based on their characteristics.
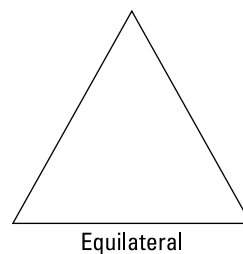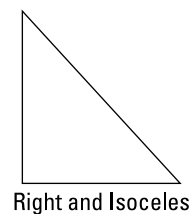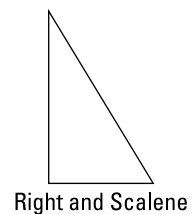
>> **Obtuse triangle:** A triangle with an obtuse angle (the other two angles must be acute).

>> **Isosceles triangle:** A triangle with two equal sides; the angles opposite those sides are equal, too. The equal angles have to be acute.

>> **Equilateral triangle:** A triangle where all three side lengths are equal, and all the angles measure 60 degrees.

>> **Scalene triangle:** A triangle with no angles or sides of the same measure.

Going outside the triangle

Another angle that comes up frequently in trigonometry is an exterior angle. When one side of a triangle is extended on both sides of its vertex, an exterior angle is formed. An angle and its exterior angle share the vertex, and the measure of the exterior angle is quickly deduced (as shown in Figure 1-6)! Here are some characteristics of an exterior angle:

>> An exterior angle to a triangle is always supplementary to the angle it's adjacent to in the triangle. This means that the sum of the angle and its exterior angle is always 180 degrees.

>> An exterior angle to a triangle has a measure that is equal to the sum of the two non-adjacent angles in the triangle.

FIGURE 1-6:
Angle *BCD* is adjacent to angle *BCA*.
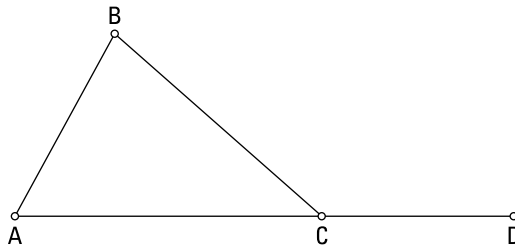
Making a circle work from every angle

A *circle* is a geometric figure that needs only two parts to identify it and classify it: its *center* (or middle) and its *radius* (the distance from the center to any point on the circle). Technically, the center isn't a part of the circle; it's just a sort of anchor or reference point. The circle consists only of all those points that are the same distance from the center.

Radius, diameter, circumference, and area

After you've chosen a point to be the center of a circle and you know how far that point is from all the points that lie on the circle, you can draw a fairly decent picture. With the measure of the radius, you can tell a lot about the circle: its *diameter* (the distance from one side to the other, passing through the center), its *circumference* (how far around it is), and its *area* (how many square inches, feet, yards, meters — what have you — fit into it). Figure 1-7 shows these features. A *chord* of a circle is a segment that joins any two points on the circle. A chord can be a *diameter.*

FIGURE 1-7:
The different features of a circle.
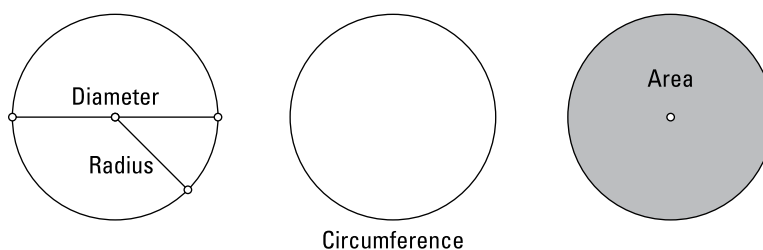
Ancient mathematicians figured out that the circumference of a circle is always a little more than three times the diameter of the circle. Since then, they narrowed that "little more than three times" to a value called *pi* (pronounced "pie"), designated by the Greek letter π. The decimal value of π isn't exact; in fact, it goes on forever and ever. However, most of the time, people refer to it as being approximately 3.14 or $\frac{22}{7}$, whichever form works best in specific computations.

The formula for figuring out the circumference of a circle is tied to π and the diameter:

Circumference of a circle $C = \pi d = 2\pi r$

The *d* represents the measure of the diameter, and *r* represents the measure of the radius. The diameter is always twice the radius, so either form of the equation works.

Similarly, the formula for the area of a circle is tied to π and the radius:

Area of a circle $A = \pi r^2$

This formula reads, "Area equals pi are squared." (And here I thought that *pies are round.*)

Example: Find the radius, circumference, and area of a circle if its diameter is equal to 10 feet in length.

DON'T GIVE ME THAT *JIVA*

The ancient Greek mathematician Ptolemy was born some time at the end of the first century. Ptolemy based his version of trigonometry on the relationships between the chords of circles and the corresponding central angles of those chords. Ptolemy came up with a theorem involving four-sided figures that you can construct with the chords. (See the section, "Finding Trig Applications in the Basics," later in this chapter.) In the meantime, mathematicians in India decided to use the measure of *half* a chord and *half* the angle to try to figure out these relationships. Drawing a radius from the center of a circle through the middle of a chord (halving it) forms a right angle, which is important in the definitions of the trig functions. These half-measures were the beginning of the sine function in trigonometry. In fact, the word *sine* actually comes from the Hindu word *jiva.*

Chord

Central angle
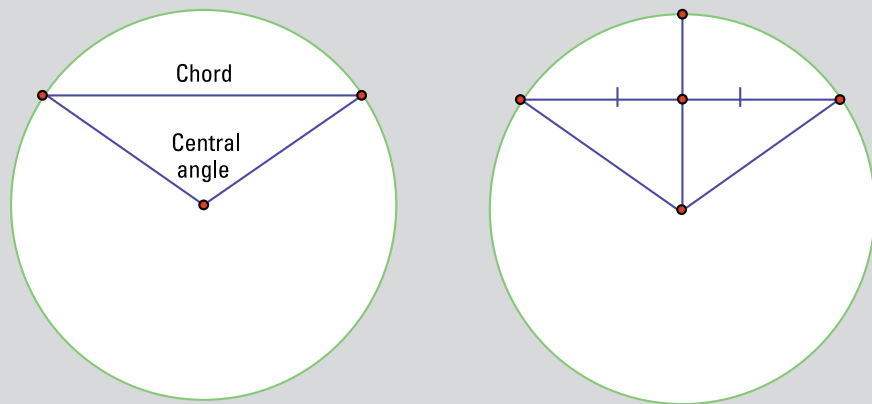
If the diameter (d) is equal to 10, you write this value as $d = 10$. The radius is half the diameter, so the radius is 5 feet, or $r = 5$. You can find the circumference by using the formula $C = \pi d = \pi \cdot 10 \approx 3.14 \cdot 10 \approx 31.4$. So, the circumference is about $31\frac{1}{2}$ feet around. You find the area by using the formula $A = \pi r^2 = \pi \cdot 5^2 = \pi \cdot 25 \approx 3.14 \cdot 25 \approx 78.5$, so the area is about $78\frac{1}{2}$ square feet.

Chord versus tangent

You show the diameter and radius of a circle by drawing segments from a point on the circle either to or through the center of the circle. But two other straight fig-ures have a place on a circle. One of these figures is called a chord, and the other is a tangent.

» A *chord* of a circle is a segment that you draw from one point on the circle to another point on the circle (see Figure 1-8). A chord always stays inside the circle. The largest chord possible is the diameter — you can't get any longer than that segment.

» A *tangent* to a circle is a line, ray, or segment that touches the outside of the circle at exactly one point, as shown in Figure 1-8. It never crosses into the circle. A tangent can't be a chord, because a chord touches a circle in two points, crossing through the inside of the circle. Any radius drawn to a tangent is perpendicular to that tangent.
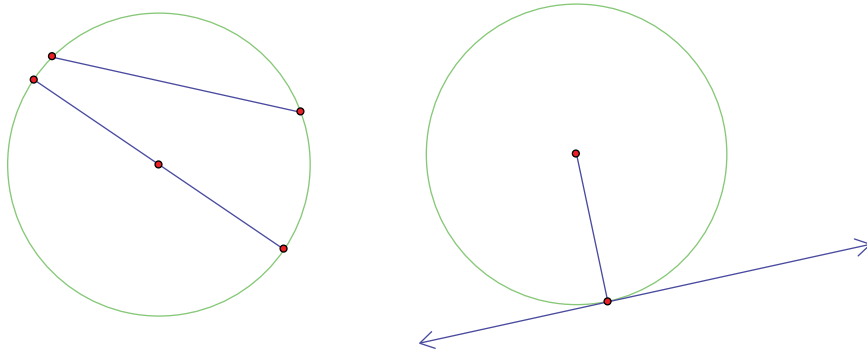
FIGURE 1-8:
Chords and a tangent of a circle.

Looking at angles in a circle

There are several ways of drawing an angle in a circle, and each has a special way of computing the size of that angle. The four different types of angles are central, inscribed, interior, and exterior. In Figure 1-9, you see examples of these different types of angles.

Central angle

A *central angle* has its vertex at the center of the circle, and the sides of the angle are two radii of the circle. The measure of the central angle is the same as the measure of the arc that the two sides cut out of the circle.

Inscribed angle

An *inscribed angle* has its vertex on the circle, and the sides of the angle are two chords of the circle. The measure of the inscribed angle is half that of the arc that the two sides cut out of the circle.

Example: Find the measure of the arc formed by the two rays of an inscribed angle if the measure of the angle is 60 degrees (see Figure 1-9).

If the measure of the inscribed angle is 60 degrees, then the arc is twice that measure, or 120 degrees.

Interior angle

An *interior angle* has its vertex at the intersection of two lines that intersect inside a circle or of two chords of the circle. The sides of the angle lie on the intersecting lines. The measure of an interior angle is the average of the measures of the two arcs that are cut out of the circle by those intersecting lines.
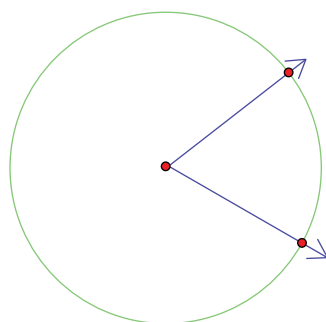
Central angle
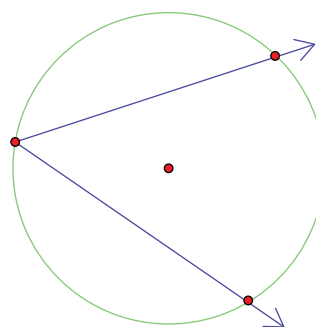
Inscribed angle
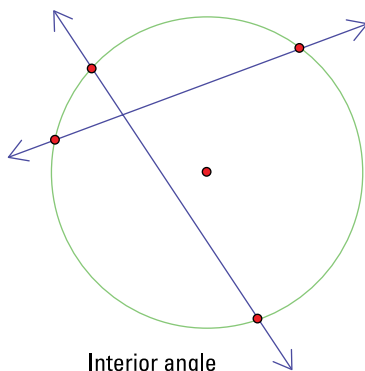
Interior angle
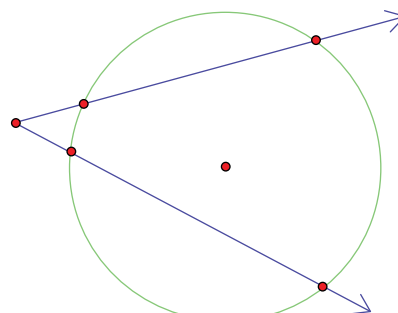
Exterior angle

FIGURE 1-9:
Measuring angles
in a circle.

Exterior angle

An *exterior angle* has its vertex where two rays share an endpoint outside a circle. The sides of the angle are those two rays. The measure of an exterior angle is found by dividing the difference between the measures of the intercepted arcs by two.

Example: Find the measure of angle *EXT*, given that the exterior angle cuts off arcs of 20 degrees and 108 degrees (see Figure 1-10).

Find the difference between the measures of the two intercepted arcs and divide by 2:

$$\frac{108 - 20}{2} = \frac{88}{2} = 44$$

The measure of angle *EXT* is 44 degrees.
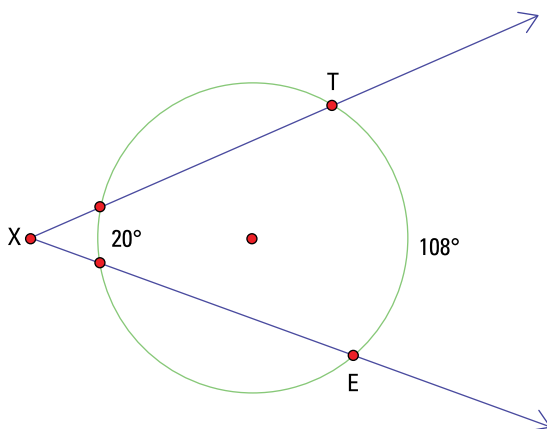
FIGURE 1-10:
Calculating the
measure of an
exterior angle.

Sectioning sectors

A *sector* of a circle is a section of the circle between two *radii* (plural for radius). You can consider this part like a piece of pie cut from a circular pie plate (see Figure 1-11).

You can find the area of a sector of a circle if you know the angle between the two radii. A circle has a total of 360 degrees all the way around the center, so if that central angle determining a sector has an angle measure of 60 degrees, then the sector takes up $\frac{60}{360}$, or $\frac{1}{6}$, of the degrees all the way around. In that case, the sector has $\frac{1}{6}$ the area of the whole circle.

FIGURE 1-11:
A sector
of a circle.
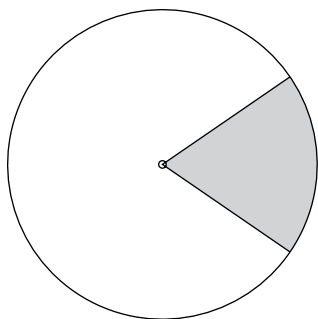
Example: Find the area of a sector of a circle if the angle between the two radii forming the sector is 80 degrees and the diameter of the circle is 9 inches.

1. **Find the area of the circle.**

   The area of the whole circle is $A = \pi r^2 = \pi \cdot (4.5)^2 \approx 3.14(20.25) \approx 63.585$, or about $63\frac{1}{2}$ square inches.

2. **Find the portion of the circle that the sector represents.**

   The sector takes up only 80 degrees of the circle. Divide 80 by 360 to get $\frac{80}{360} = \frac{2}{9} \approx 0.222$.

3. **Calculate the area of the sector.**

   Multiply the fraction or decimal from Step 2 by the total area to get the area of the sector: $0.222(63.585) \approx 14.116$. The whole circle has an area of almost 64 square inches, and the sector has an area of just over 14 square inches.

Understanding Trig Speak

Any math or science topic has its own unique vocabulary. Some very nice everyday words have new and special meanings when used in the context of that subject. Trigonometry is no exception.

Making the words fit the triangle

Every triangle has six measurable parts: three sides and three angles. If you measure the sides and then pair up those measurements (taking two at a time), you have three different pairings. Do division problems with the pairings — changing the order in each pair — and you have six different answers. These six different

answers represent the six trig functions. For example, if your triangle has sides measuring 3, 4, and 5, then the six divisions are $\frac{3}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{3}, \frac{4}{5}$, and $\frac{5}{4}$.

In Chapter 6, you find out how all these fractions work in the world of trig functions by using the different sides of a right triangle. And then, in Chapter 7, you take a whole different approach as you discover how to define the trig functions with a circle.

The six trig functions are named *sine, cosine, tangent, cotangent, secant,* and *cosecant.* Many people confuse the spoken word *sine* with *sign* — you can't really tell the difference when you hear it unless you're careful with the context. You can "go off on a tangent" in some personal dealings, but that phrase has a whole different meaning in trig. Cosigning a loan isn't what trig has in mind, either. The other three ratios are special to trig speak — you can't confuse them with anything else.

Interpreting trig abbreviations

REMEMBER

Even though the word *sine* isn't all that long, you have a three-letter abbreviation for this trig function and all the others. Mathematicians find using abbreviations easier, and those versions fit better on calculator keys. The functions and their abbreviations are

sine: sin	**cosine:** cos
tangent: tan	**cotangent:** cot
secant: sec	**cosecant:** csc

As you can see, the first three letters in the full name make up the abbreviations, except for cosecant's.

Noting notation

Angles are the main focus in trigonometry, and you can work with them even if you don't know their measure. Many angles and their angle measures have general rules that apply to them. You can name angles by one letter, three letters, or a number, but to do trig problems and computations, mathematicians commonly refer to the angle names and their measures with Greek letters.

The most commonly used letters for angle measures are α (alpha), β (beta), γ (gamma), and θ (theta). Also, many equations use the variable x to represent an angle measure.

Algebra has conventional notation involving superscripts, such as the 2 in x^2. In trigonometry, superscripts have most of the same rules and characteristics as in other mathematics. But trig superscripts often look very different. Table 1-1 presents a listing of some of the ways that trig uses superscripts.

TABLE 1-1   **How You Use Superscripts in Trig**

How to Write in Trig Notation	Alternate Notation	What the Superscript Means
$\sin^2\theta$	$(\sin\theta)^2$	Square the sine of the angle theta.
$(\sin\theta)^{-1}$	$\dfrac{1}{\sin\theta}$	Find the reciprocal of the sine of theta.
$\sin^{-1}\theta$	$\arcsin\theta$	Find the angle theta given its sine.

The first entry in Table 1-1 shows how you can save having to write parentheses every time you want to raise a trig function to a power. This notation is neat and efficient, but it can be confusing if you don't know the "code." The second entry shows you how to write the reciprocal of a trig function. It means you should take the value of the function and divide it into the number 1. The last entry in Table 1-1 shows how you write the *inverse sine* function. Using the −1 superscript between sine and the angle means that you're talking about inverse sine (or *arcsin*), not the reciprocal of the function. In Chapter 14, I cover the inverse trig functions in great detail, making this business about the notation for an inverse trig function even more clear.

Functioning with angles

The functions in algebra use many operations and symbols that are different from the common add, subtract, multiply, and divide signs in arithmetic. For example, take a look at the square-root operation, $\sqrt{25} = 5$. Putting 25 under the *radical* (square-root symbol) produces an answer of 5. Other operations in algebra, such as absolute value, factorial, and step-function, are used in trigonometry, too. But the world of trig expands the horizon, introducing even more exciting processes.

When working with trig functions, you have a whole new set of values to learn or find. For instance, putting 25 into the sine function looks like this: sin 25. The answer that pops out is either 0.423 or −0.132, depending on whether you're using degrees or radians (for more on those two important trig concepts, head on over to Chapters 3 and 4). You can't usually determine or memorize all the values that you get by putting angle measures into trig functions. Fortunately, you can use scientific calculators or computers or even cell phones to study trigonometry.

In general, when you apply a trig function to an angle measure, you get some real number (if that angle is in its domain). Some angles and trig functions have nice values, but most don't. Table 1-2 shows the trig functions for a 30-degree angle.

TABLE 1-2 **The Trig Functions for a 30-Degree Angle**

Trig Function	Exact Value	Value Rounded to Three Decimal Places
sin 30°	$\dfrac{1}{2}$	0.500
cos 30°	$\dfrac{\sqrt{3}}{2}$	0.866
tan 30°	$\dfrac{\sqrt{3}}{3}$	0.577
cot 30°	$\sqrt{3}$	1.732
sec 30°	$\dfrac{2\sqrt{3}}{3}$	1.155
csc 30°	2	2.000

Some characteristics that the entries in Table 1-2 confirm are that the sine and cosine functions always have values that are between and include –1 and 1. Also, the secant and cosecant functions always have values that are equal to or greater than 1 or equal to or less than –1. (I discuss these properties in more detail in Chapter 6.) And you'll find a complete listing of the values of the most commonly used functions in the Appendix.

How about trying this out? Do you have your cell phone handy? Or, just go to a calculator and type in **sin (45)**. Did you get 0.7071 (or maybe a few more digits)? That's the sine of 45 degrees! You'll better understand how this wonderful relationship works in later chapters.

Making triangles less radical

A *radical* is a mathematical symbol that means, "Find the number that multiplies itself by itself one or more times to give you the number under the radical." You can see why you use a symbol such as $\sqrt{}$ rather than all those words. Radicals represent values of functions that are used a lot in trigonometry. In Chapter 6, I define the trig functions by using a right triangle. To solve for the lengths of a

right triangle's sides by using the Pythagorean Theorem, you have to compute some square roots, which use radicals. Some basic answers to radical expressions are $\sqrt{16} = 4$, $\sqrt{121} = 11$, $\sqrt[3]{8} = 2$, and $\sqrt[4]{81} = 3$.

These examples are all *perfect squares, perfect cubes,* or *perfect fourth roots,* which means that the answer is a number that ends — the decimal doesn't go on forever. The following section discusses a way to simplify radicals that aren't perfect roots.

Simplifying radical forms

Simplifying a radical form means to rewrite it with a smaller number under the radical — if possible. You can simplify this form only if the number under the radical has a perfect square or perfect cube (or perfect whatever factor) that you can factor out.

Example: Simplify $\sqrt{80}$.

The number 80 isn't a perfect square, but one of its factors, 16, is. You can write the number 80 as the product of 16 and 5, write the two radicals separately, and then evaluate each radical. The resulting product is the simplified form:

$$\sqrt{80} = \sqrt{16 \cdot 5} = \sqrt{16}\sqrt{5} = 4\sqrt{5}$$

Example: Simplify $\sqrt[3]{250}$.

The number 250 isn't a perfect cube, but one of its factors, 125, is. Write 250 as the product of 125 and 2; separate, evaluate, and write the simplified product: $\sqrt[3]{250} = \sqrt[3]{125 \cdot 2} = \sqrt[3]{125}\sqrt[3]{2} = 5\sqrt[3]{2}$.

Approximating answers

As wonderful as a simplified radical is, and as useful as it is when you're doing further computations in math, sometimes you just need to know approximately how much the value's worth.

Approximating an answer means to shorten the actual value in terms of the number of decimal places. You may find approximating especially useful when the decimal value of a number goes on forever without ending or repeating. When you approximate an answer, you *round* it to a certain number of decimal places, letting the rest of the decimal values drop off. Before doing that, though, you need to consider how big a value you're dropping off. If the numbers that you're dropping off start with a 5 or greater, then bump up the last digit that you leave on by 1. If what you're dropping off begins with a 4 or less, then just leave the last remaining digit alone.

Example: Round the number 3.141592654 to four decimal places, three decimal places, and two decimal places.

>> **Four decimal places.** Look at the first five decimal places: 3.141<u>5</u>9. The 5 in the fourth place either stays as is or gets rounded up to the next digit. Because the next digit is 9, and 9 is greater than 5, you round the 5 up to a 6. Rounded to four places, 3.141592654 rounds to 3.1416.

>> **Three decimal places.** Look at the first four decimal places: 3.14<u>1</u>5. The 1 in the third place either stays as is or gets rounded up to the next digit. Because the next digit is 5, and anything 5 or greater gives a "bump up," you round the 1 to a 2. Rounded to three places, 3.141592654 rounds to 3.142.

>> **Two decimal places.** Look at the first three decimal places: 3.1<u>4</u>1. The 4 in the second place either stays as is or gets rounded up to the next digit. Because the next digit is 1, and 1 is smaller than 5, you leave the 4 as is. Rounded to two places, 3.141592654 rounds to 3.14.

Use this technique when approximating radical values. Using a calculator, the decimal value of $\sqrt{80}$ is about 8.94427191. Depending on what you're using this value for, you may want to round it to two, three, four, or more decimal places. Rounded to three decimal places, this number is 8.944.

Equating and Identifying

Trigonometry has the answers to so many questions in engineering, navigation, and medicine. The ancient astronomers, engineers, farmers, and sailors didn't have the current systems of symbolic algebra and trigonometry to solve their problems, but they did well and set the scene for later mathematical developments. People today benefit big-time by having ways to solve equations in trigonometry that are quick and efficient; trig now includes special techniques and

identities that you can play with — all thanks to the mathematicians of old who created the systems that we use today.

The methods that you use for solving equations in algebra take a completely different turn from usual solutions when you use trig *identities* (in short, equivalences that you can substitute into equations in order to simplify them). To make matters easier (or, some say, to *complicate* them), the different trig functions can be written many different ways. They almost have split personalities. When you're solving trig equations and trig identities, you're sort of like a detective working your way through to substitute, simplify, and solve. What answers should you expect when solving the equations? Why, angles, of course!

Take, for example, one trigonometric equation: $\sin\theta + \cos^2\theta = 1$.

The point of the problem is to figure out what angle or angles should replace the θ to make the equation true. In this case, if θ is 0 degrees, 90 degrees, or 180 degrees, the equation is true.

If you replace θ with 0 degrees in the equation, you get

$$\sin 0° + (\cos 0°)^2 = 1$$
$$0 + (1)^2 = 1$$
$$1 = 1$$

If you replace θ with 90 degrees in the equation, you get

$$\sin 90° + (\cos 90°)^2 = 1$$
$$1 + (0)^2 = 1$$
$$1 = 1$$

Something similar happens with 180 degrees and all the other angle measures that work in this equation (there are an infinite number of solutions). But remember that not just any angle will work here. I carefully chose the angles that are *solutions,* which are the angles that make the equation true. In order to solve trig equations like this one, you have to use inverse trig functions, trig identities, and various algebraic techniques. You can find all the details on how to use these processes in Chapters 10 through 15. And when you've got those parts figured out, dive into Chapter 16, where the equation-solving comes in.

In this particular case, you need to use an *identity* to solve the equation for all its answers. You replace the $\cos^2\theta$ with $1 - \sin^2\theta$ so that all the terms have a sine in them — or just a number. You actually have several other choices for changing the identity of $\cos^2\theta$. I chose $1 - \sin^2\theta$, but some other choices include $\dfrac{1}{\sec^2\theta}$ and $\dfrac{1+\cos 2\theta}{2}$. You can discover how to actually solve equations like this one in Chapter 16.

This example just shows you that the identity of the trig functions can change an expression significantly — but according to some very strict rules.

Finding Trig Applications in the Basics

The basics of trigonometry helped the ancients with everyday problems, and they're here now to give you some insights and help with occasional challenges.

Measuring fencing

You have a circular corral that you've divided into four equal portions so you can keep your cow, pony, pig, and rooster separate from one another, as shown in Figure 1-12.
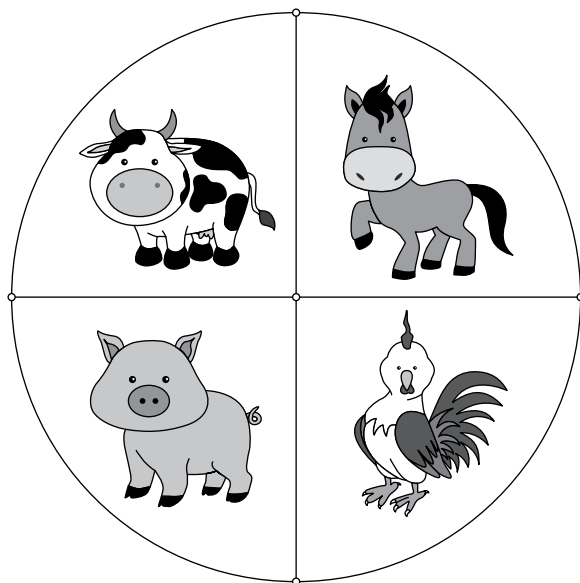
FIGURE 1-12: Dividing the circle into four equal parts.

You need some fencing to keep the animals in and apart. You know that the corral has a radius of exactly 40 feet. What is the total number of feet needed to create the corral?

Because the radius of the corral is 40 feet, and you need four of these lengths for the crisscross portion of the fencing, you need a total of 160 feet. To find the circular portion, you need the circumference of the circle. You can use either the radius or diameter in the formula $C = \pi d = 2\pi r$ to find that circumference: $C = 2\pi r = 2\pi(40) = 80\pi \approx 251.33$ feet. Add this to the crisscross portion and you have $251.33 + 160 = 411.33$ total feet of fencing.

Ptolemy's Theorem

Earlier in this chapter, I made reference to the Greek mathematician Ptolemy and his work with circles and chords. Ptolemy's Theorem says that, "When you have a quadrilateral (four-sided figure) circumscribed by a circle, the product of the measures of the two diagonals in that figure is equal to the sum of the products of the opposite sides of the figure." Take a look at Figure 1-13, where *AD* and *BC* are the diagonals, and *AB* and *CD* are one pair of opposite sides and *AC* and *BD* are the other pair.

FIGURE 1-13: The product of the diagonals equals the sum of the products of the opposite sides.
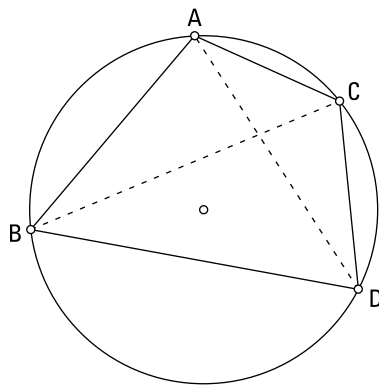
You are able to measure each segment in the figure except *BD*. Given the other measures, determine that missing measure.

Here are the measures you know: *AC* = 5, *AB* = 8, *CD* = 6, *AD* = 10, and *BC* = 10.5.

So you know the product of the diagonals: $10 \times 10.5 = 105$.

The product of the diagonals is equal to the sum of the products of the opposite sides, so: $105 = (AB)(CD) + (AD)(BC) = (8)(6) + (10)(BC) = 48 + (10)(BC)$.

Solving the equation for *BC*:

$$105 = 48 + (10)(BC)$$
$$57 = 10(BC)$$
$$\frac{57}{10} = BC$$

So *BC* is 5.7 units in length.

Dealing with radicals

One of my favorite special right triangles is the triangle with acute angles measuring 30 degrees and 60 degrees. And a great feature of this special triangle is that there's a wonderful relationship between the lengths of the sides of the triangle. In Figure 1-14, you see that the hypotenuse of the right triangle is always twice as long as the shortest side, and the middle-length side is always $\sqrt{3}$ or about 1.7 times as long as the shortest side.

30°

$x\sqrt{3}$

$2x$

60°

x
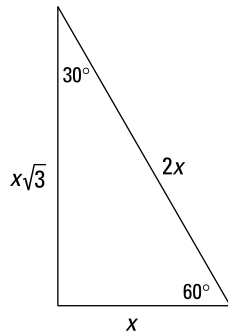
FIGURE 1-14:
The special
30-60-90 degree
right triangle.

Example: You know that the hypotenuse of one of these triangles measures 20 inches. How long are the other two sides? (Give the radical answer correct to 3 decimal places.)

The shortest side is half the hypotenuse, so it measures 10 inches.

The middle-length side (opposite the 60-degree angle) is 10 times $\sqrt{3}$. Using your calculator, you have $10\sqrt{3} \approx 10(1.732050808) = 17.32050808$. Rounded to three places, that's 17.321 inches.

Skateboarding

You're going to build a skateboard ramp that has an angle of 130 degrees as you approach and an angle of 130 degrees as you descend. This requires an isosceles triangle, as shown in Figure 1-15. What is the measure of the top angle, where you switch from going up to going down?
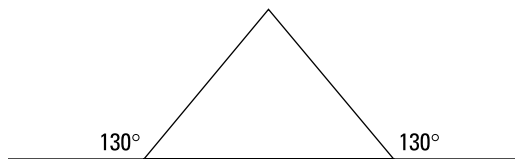
The 130-degree angle in the approach is an exterior angle of the triangle. Subtract 130 from 180, and you find that the interior angle in the triangle is 50 degrees.

The other exterior angle is also 130 degrees, so you have another angle in the triangle of 50 degrees. Add those two interior angles together, and you have 100 degrees, leaving 80 degrees for the top angle measure. Seems a bit steep to me!!!!

IN THIS CHAPTER

» **Marking points on a graph**

» **Finding the distance between two points**

» **Locating midpoints**

» **Defining the slope of a line**

» **Determining equations of circles**

Chapter **2**

Cooperating with Cartesian Coordinates

A picture is worth a thousand words. Drawing pictures or graphs of functions and equations in math helps you understand what's going on with them. In trigonometry, you often draw angles and triangles, in addition to the curves that represent the trig functions (sine, cosine, tangent, cotangent, secant, and cosecant). The standard Cartesian coordinate system, which you use when drawing graphs in algebra and other math topics, works best here. If you're looking for a refresher on this point-plotting system, you will find it in this chapter. In short, with the Cartesian coordinate system, everything reads from left to right and from bottom to top, running through the negative to the positive numbers.

Starting Out Simple: Plotting Points

Plotting points on a mathematical graph means finding the correct position for a dot that represents an ordered pair of numbers, such as (2,3), (−1,4), or (0,0). This ordered pair (*x*,*y*) is called the *Cartesian coordinates* of the point. You start with two perpendicular intersecting lines called *axes.*

Axes, axes, we all fall down

Plotting points and drawing graphs requires two axes and a defined distance or scale on them. The two intersecting, perpendicular lines that make up a graph are called the horizontal and vertical *axes* (or *coordinate axes*).

These lines extend left and right, up and down, without end. The horizontal axis is traditionally known as the *x*-axis, although in trigonometry the horizontal axis is sometimes labeled the θ axis. The vertical axis is the *y*-axis. The two axes intersect at the *origin,* labeled *O.* The part of the *x*-axis going to the right represents positive numerical values, and you use it as the starting place or initial side when drawing angles in the *standard position.* An angle in standard position has its vertex at the origin, its initial side along the positive *x*-axis, and its terminal side a ray rotated in a counterclockwise direction for positive measures.

Determining the origin of it all

The point where the two axes cross is called the *origin.* You label it with an *O* or with its ordered pair (0,0). The origin is the starting point for counting off the coordinates when graphing all other points. It's also the endpoint of the *rays* (lines that extend infinitely in one direction) that you use when drawing angles in the standard position on the coordinate axes.

Plotting x versus y

Plotting points in a coordinate system involves counting distances to the right or left and up or down from the origin. The axes serve as a starting place. The points are represented or named by the ordered pair of numbers, (*x,y*), called the *x*-coordinate and the *y*-coordinate. The designation *ordered pair* means that the order *does* matter. The *x*-coordinate always comes first, and the *y*-coordinate comes last so that this whole graphing system is universal.

PUTTING DA CART BEFORE DA HORSE

René Descartes was considered to be a mover and shaker in the 17th-century scientific community. He was responsible for many innovations in algebra and geometry. He's also credited with creating our coordinate system used for graphing mathematical objects. The *x*- and *y*-coordinates (the values that specify a location on a graph) are called *Cartesian coordinates* in honor of Descartes.

REMEMBER The *x*-coordinate is the distance to the left or right of the origin that the point lies. If the *x*-coordinate is positive, you move to the right of the origin. If it's negative, you move to the left. The second number, the *y*-coordinate, is the distance up or down from the origin. Positive numbers mean the point is up, and negatives mean you move south of the *x*-axis.

The point (2,4) is two units to the right and four units up from the origin; (−3,2) is three units to the left and two units up; (−4,−3) is four units to the left and three units down; and (5,−1) is five units to the right and one unit down. Points can lie on one of the axes, too. Those points always have a 0 for the *x*- or *y*-coordinate. The point (0,3) lies on the *y*-axis, and (1,0) lies on the *x*-axis. See how to graph all these points in Figure 2-1.
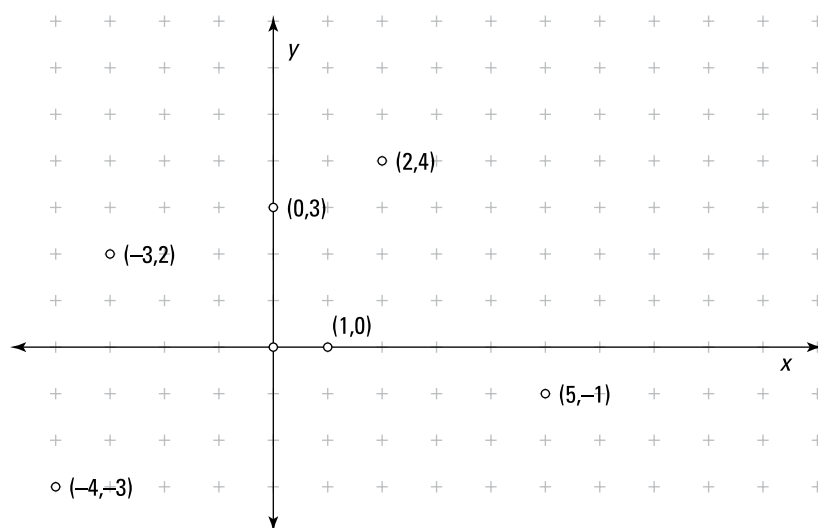
FIGURE 2-1: Six points, graphed and labeled.

Cutting the graph into four parts

The intersecting *x*- and *y*-axes divide the whole picture, or *coordinate plane,* into four separate regions called *quadrants.* The quadrants are numbered, starting in the upper-right quadrant and going counterclockwise, as shown in Figure 2-2. Traditionally, you number the quadrants with Roman numerals.

TIP These quadrant number designations are useful when referring to certain types of angles, groupings of points, and trig function properties. The points in Quadrant I all have both *x*- and *y*-coordinates that are positive. In Quadrant II, the *x*-coordinate is negative, and the *y*-coordinate is positive. The points in Quadrant III have both *x*- and *y*-coordinates that are negative. In Quadrant IV, the *x*-coordinate is positive, and the *y*-coordinate is negative.
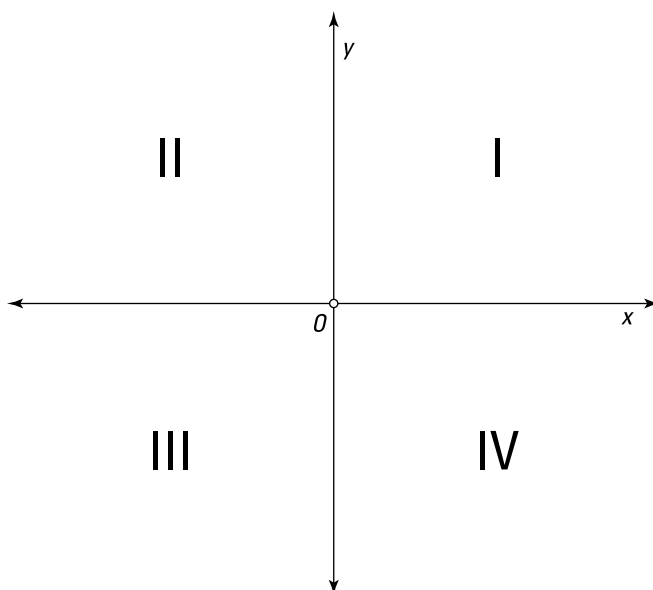
FIGURE 2-2:
The four
quadrants of the
coordinate plane.

From Here to There: Calculating Distances

The lengths of segments and distances between points play a major part in establishing the trig functions, relationships, and identities (which I cover in Chapters 6 and 13). You can compute these lengths and distances fairly easily, because the coordinate system is just so darned convenient.

Counting on vertical and horizontal distances

When the distance that you're measuring is either vertical or horizontal, the computation is a simple subtraction problem. One coordinate in each ordered pair is the same. Just find the difference between the other two coordinates.

For instance, to find the distance between the points (5,2) and (5,6), subtract 2 from 6 to get the distance of 4 units between them. This distance is vertical, because the two points have the same x-coordinate, and the second point is directly above the first. To find the distance between the two points (5,6) and (5,–3), subtract –3 from 6, which gives you $6 - (-3) = 6 + 3 = 9$ or a distance of 9 units. Always subtract the smaller number from the larger number, so that the distance you get is a positive number. (Negative distances don't make sense; after all, you can't travel –5 miles to Aunt Myrtle's house!)

TIP

Another way to deal with the different signs of the answers that occur is to use absolute value — then it doesn't matter in what order you subtract the numbers. Take, for instance, the preceding example, where I subtracted –3 from 6. The –3 is smaller, so subtracting in that order gave a positive answer. The alternative is to subtract in the opposite order and take the absolute value of the result. If you do $(-3-6)$, you get –9. The absolute value of –9, written $|-9|$, equals 9.

Horizontal distances work the same way. In Figure 2-3, you can see the horizontal distance between two points. To find the distance between the points (–8,2) and (5,2), just calculate the difference between –8 and 5, because the y-coordinates are the same. The smaller number is –8, so subtracting $5-(-8)$, the answer is 13 units. Subtracting in the other order and using absolute value, you get $-8-5=-13$, and $|-13|=13$. You can also see the vertical distance between two points, (5,6) and (5,2), in Figure 2-3. This problem uses simple arithmetic. The difference between 6 and 2 is 4.

FIGURE 2-3:
Vertical and horizontal distances between points.
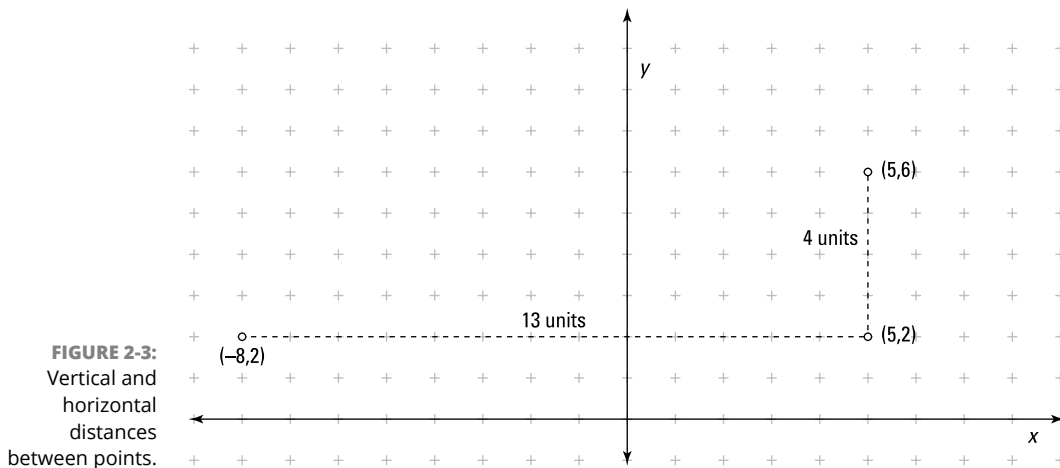
TRIG RULES

You can easily compute the distance between pairs of vertical or horizontal points, (x_1, y_1) and (x_2, y_2), using absolute value:

» Vertical distance (the x-coordinates are the same) is $|y_1 - y_2|$.

» Horizontal distance (the y-coordinates are the same) is $|x_1 - x_2|$.

Another slant: Diagonal distances

Quite often, the distances or lengths you want to determine are on a slant — they go diagonally from one point to another. The formula for determining these distances is based on the Pythagorean Theorem.

The Pythagorean Theorem

Way back when, Pythagoras discovered a relationship between the sides of any *right triangle* (where one angle is 90 degrees), as Figure 2-4 shows.
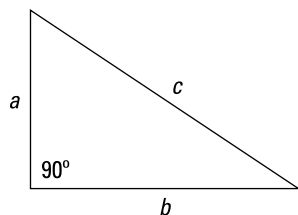
FIGURE 2-4:
A right triangle.

Pythagoras found that if a and b are the lengths of the shorter sides of the right triangle, and if c is the length of the *hypotenuse* (the side opposite the right angle), then $a^2 + b^2 = c^2$. You can use this formula to find the diagonal distances between two points on a graph, because the horizontal and vertical distances, which are the sides of the triangle, are easy to find in a coordinate system.

Determining diagonal distances

Using the Pythagorean Theorem, and solving for c, the length of the hypotenuse of a right triangle, you get

$$c = \sqrt{a^2 + b^2}$$

If length a is the horizontal distance, then you calculate that distance by subtracting the x-coordinates; if length b is the vertical distance, you get it by subtracting the y-coordinates.

To get the general distance formula, simply substitute the difference between the x- and y-values for a and b in the Pythagorean Theorem, and use the variable d (meaning distance) in place of c.

TRIG RULES

The distance, d, between two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

For example, follow these steps to find the distance between the points $(3, -4)$ and $(-2, 5)$:

1. Replace x_1 and x_2 with 3 and –2. Replace the y_1 and y_2 with –4 and 5.

$$d = \sqrt{(3 - (-2))^2 + (-4 - 5)^2}$$

2. **Subtract the coordinates.**

$$= \sqrt{5^2 + (-9)^2}$$

3. **Square the differences, add the results, and then find the square root of the sum, if possible. (It can also be simplified or estimated.)**

$$= \sqrt{25 + 81} = \sqrt{106}$$

In the preceding example, the number under the radical isn't a perfect square. You can leave the answer with the square-root symbol, simplify it, or give a decimal approximation. To three decimal places, the distance in this example is 10.296 units.

TIP When you're calculating the distance between two points, it doesn't matter in what order you subtract the points, as long as you subtract x from x and y from y. Squaring the differences always results in a positive answer, anyway.

Using exact values or estimating distances

Calculating the distance between two points often leaves you with the square root of a number that isn't a perfect square; this type of answer is called an *irrational number.* Writing the number with the square-root symbol, for example, $\sqrt{47}$, is considered to be writing the *exact value* of the distance. Using a calculator to find a decimal approximation doesn't give an exact answer, because the decimal values of irrational numbers go on forever and ever and never repeat in a pattern. Because the decimals are always estimates, mathematicians often insist on leaving the answers as exact values, complete with the square-root symbol, rather than decimals.

Although exact values are more precise, using decimal estimates of radical values is more helpful in practical situations. If you're solving for the height of a building and get $\sqrt{183}$, you get a better understanding of the height by finding a decimal estimate. A scientific calculator tells you that $\sqrt{183}$ is approximately 13.52774926. . . . Different calculators may give you fewer or more decimal values than I show you here. Usually, just two or three decimal places will do. Rounding this to two places, you get 13.53. Rounding it to three places, you get 13.528.

Getting to the Center of It All

One way to describe the middle of a triangle is to identify the *centroid.* This middle-point is the center of gravity, where you could balance the triangle and spin it around. And the middle of a line segment is its *midpoint.* When you graph a

circle, triangle, or line segment by using coordinate axes, you can name these middle points with a pair of x- and y-coordinates. All you need to find these middles are the coordinates of some other crucial points on the respective figures.

Finding the midpoint of a line segment

To find the midpoint of a line segment, you just calculate the averages of the coordinates — easy as pie.

The midpoint, M, of a segment with endpoints (x_1, y_1) and (x_2, y_2) is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

If you want to know the midpoint of the segment with endpoints $(-4, -1)$ and $(2, 5)$, then plug the numbers into the midpoint formula, and you get a midpoint of $(-1, 2)$:

$$M = \left(\frac{-4 + 2}{2}, \frac{-1 + 5}{2} \right) = \left(\frac{-2}{2}, \frac{4}{2} \right) = (-1, 2)$$

See how this segment looks in graph form in Figure 2-5.
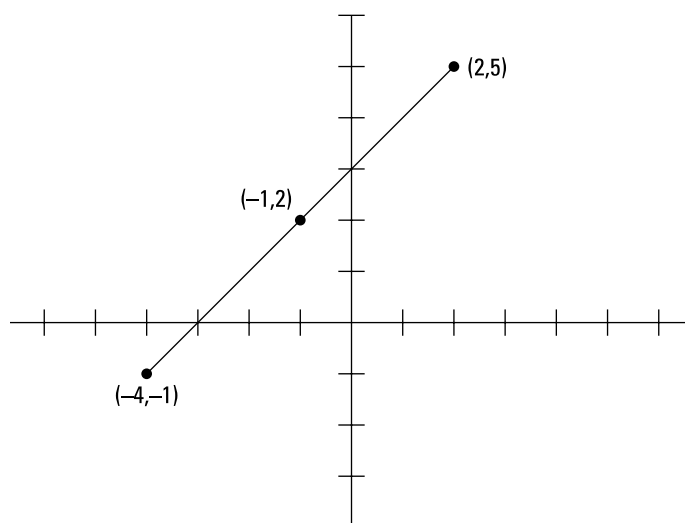
FIGURE 2-5:
The midpoint of this line segment is $(-1, 2)$.

Locating the center of a circle

If the endpoints of one diameter of a circle are (x_1, y_1) and (x_2, y_2), then the center of the circle has the coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$. You probably noticed that the center of a circle is the same as the diameter's midpoint.

The center of the circle separates the diameter into two equal segments called *radii* (plural for radius).

Figure 2-6 shows a circle with a diameter whose endpoints are (7,4) and (–1,–2). The center of the circle is at (3,1). I got the coordinates for the center by using the formula for the midpoint of a segment (see the preceding section):

$$M = \left(\frac{7+(-1)}{2}, \frac{4+(-2)}{2}\right) = \left(\frac{6}{2}, \frac{2}{2}\right) = (3,1)$$

You find the length of the diameter by using the distance formula (see the section, "Another slant: Diagonal distances," earlier in this chapter):

$$d = \sqrt{(7-(-1))^2 + (4-(-2))^2} = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10$$

For the circle shown in Figure 2-6, the diameter is 10 units long.
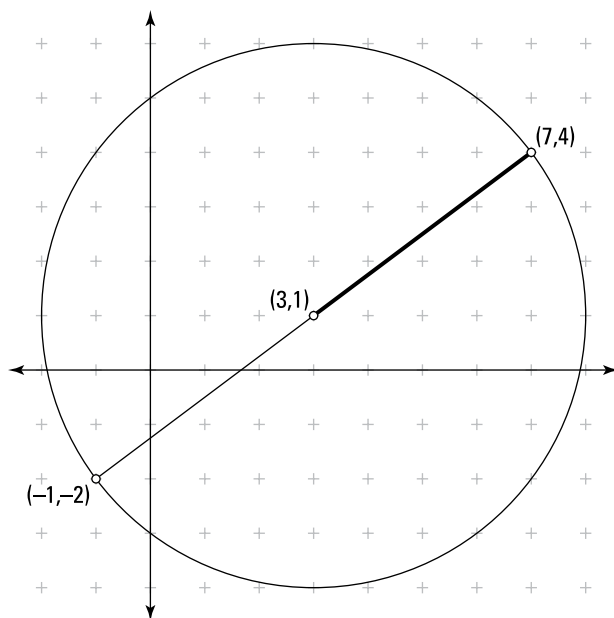
FIGURE 2-6: The center of a circle is also the midpoint of one of its diameters.

Next, I show you how to find the length of one of the radii. Either will do — they're all the same length. In this example, I figure the radius length from the center of the circle (3,1) to the endpoint of the diameter (7,4):

$$d = \sqrt{(7-3)^2 + (4-1)^2} = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$$

The radius is 5 units long. But, of course, you expected this answer, because by definition, the radius is half the length of the diameter.

Partitioning line segments further

If you can find the midpoint of a segment, you can divide it into two equal parts. Finding the middle of each of the two equal parts allows you to find the points needed to divide the entire segment into four equal parts. Finding the middle of each of these segments gives you eight equal parts, and so on.

For example, to divide the segment with endpoints $(-15,10)$ and $(9,2)$ into eight equal parts, find the various midpoints like so:

>> The midpoint of the main segment from $(-15,10)$ to $(9,2)$ is $(-3,6)$.

>> The midpoint of half of the main segment, from $(-15,10)$ to $(-3,6)$, is $(-9,8)$, and the midpoint of the other half of the main segment, from $(-3,6)$ to $(9,2)$, is $(3,4)$.

>> The midpoints of the four segments determined here are $(-12,9)$, $(-6,7)$, $(0,5)$, and $(6,3)$.

Figure 2-7 shows the coordinates of the points that divide this line segment into eight equal parts.
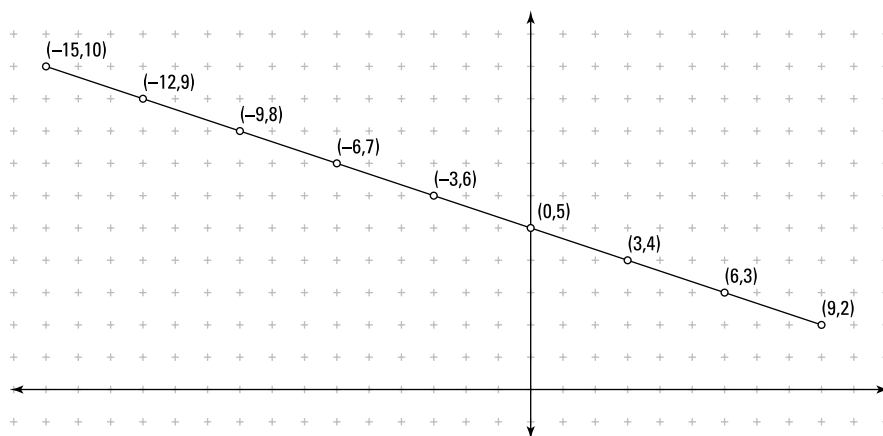
FIGURE 2-7: A line segment divided into eight equal parts using the midpoint method.

Using the midpoint method is fine, as long as you just want to divide a segment into an even number of equal segments. But your job isn't always so easy. For instance, you may need to divide a segment into three equal parts, five equal parts, or some other odd number of equal parts.

To find a point that isn't equidistant from the endpoints of a segment, just use this formula:

$$(x,y)=(x_1+k(x_2-x_1),y_1+k(y_2-y_1))$$

In this formula, (x_1,y_1) is the endpoint where you're starting, (x_2,y_2) is the other endpoint, and k is the fractional part of the segment you want.

So, to find the coordinates that divide the segment with endpoints $(-4,1)$ and $(8,7)$ into three equal parts, first find the point that's one-third of the distance from $(-4,1)$ to the other endpoint, and then find the point that's two-thirds of the distance from $(-4,1)$ to the other endpoint. The following steps show you how.

To find the point that's one-third of the distance from $(-4,1)$ to the other endpoint, $(8,7)$:

1. Replace x_1 with -4, x_2 with 8, y_1 with 1, y_2 with 7, and k with $\frac{1}{3}$.

$$(x,y)=\left(-4+\frac{1}{3}(8-(-4)),1+\frac{1}{3}(7-1)\right)$$

2. Subtract the values in the inner parentheses.

$$=\left(-4+\frac{1}{3}(12),1+\frac{1}{3}(6)\right)$$

3. Do the multiplication and then add the results to get the coordinates.

$$=(-4+4,1+2)=(0,3)$$

To find the point that's two-thirds of the distance from $(-4,1)$ to the other endpoint, $(8,7)$:

1. Replace x_1 with -4, x_2 with 8, y_1 with 1, y_2 with 7, and k with $\frac{2}{3}$.

$$(x,y)=\left(-4+\frac{2}{3}(8-(-4)),1+\frac{2}{3}(7-1)\right)$$

2. Subtract the values in the inner parentheses.

$$=\left(-4+\frac{2}{3}(12),1+\frac{2}{3}(6)\right)$$

3. Do the multiplication and then add the results to get the coordinates.

$$=(-4+8,1+4)=(4,5)$$

Figure 2-8 shows the graph of this line segment and the points that divide it into three equal parts.

FIGURE 2-8:
A line segment
divided into three
equal parts.

The points shown on the figure are (−4,1), (0,3), (4,5), and (8,7).
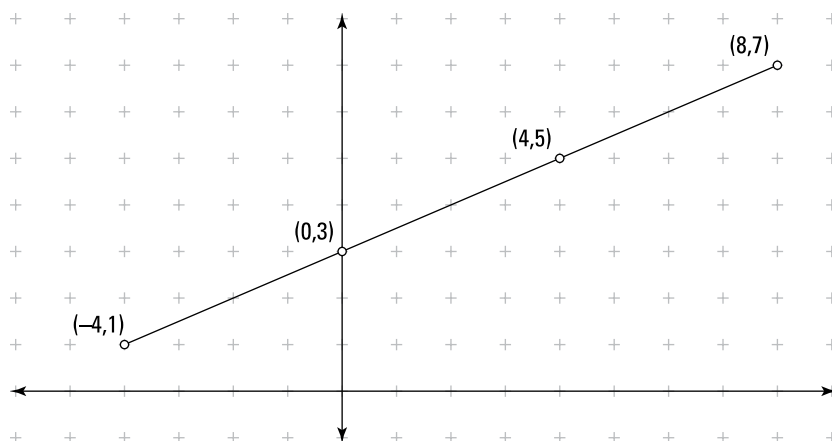
Pinpointing the center of a triangle

If you draw lines from each corner (or *vertex*) of a triangle to the midpoint of the opposite sides, then those three lines meet at a center, or *centroid,* of the triangle. The centroid is the triangle's center of gravity, where the triangle balances evenly. The coordinates of the centroid are also two-thirds of the way from each vertex along that segment. Figure 2-9 shows how the three lines drawn in the triangle all meet at the center.

FIGURE 2-9:
The lines that
intersect at the
centroid of a
triangle.
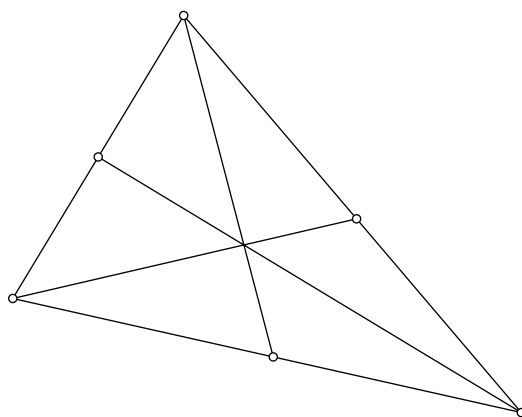
To find the centroid of a triangle, use the formula from the preceding section that locates a point two-thirds of the distance from the vertex to the midpoint of the opposite side.

CIRCUMSCRIBING A TRIANGLE

Every triangle can be *circumscribed* by a circle, meaning that one circle — and only one — goes through all three *vertices* (corners) of any triangle. In laymen's terms, any triangle can fit into some circle with all its corners touching the circle. To circumscribe a triangle, all you need to do is find the circumcenter of the circle (at the intersection of the perpendicular bisectors of the triangle's sides). You can then find the radius of the circle, because the distance from the center of the circle to one of the triangle's vertices is the radius. This exercise is a nice one to try your hand at with a compass and straightedge or with some geometry software.
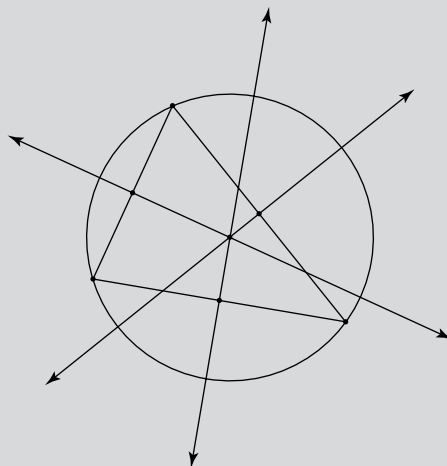
For example, to find the centroid of a triangle with vertices at (0,0), (12,0), and (3,9), first find the midpoint of one of the sides. The most convenient side is the bottom, because it lies along the *x*-axis. The coordinates of that midpoint are (6,0). Then find the point that sits two-thirds of the way from the opposite vertex, (3,9):

1. **Replace x_1, x_2, y_1, and y_2 with their respective values. Replace *k* with $\frac{2}{3}$.**

$$(x,y) = \left(3 + \frac{2}{3}(6-(3)), 9 + \frac{2}{3}(0-9)\right)$$

2. **Simplify the computation to get the point.**

$$= \left(3 + \frac{2}{3}(3), 9 + \frac{2}{3}(-9)\right) = (3+2, 9-6) = (5,3)$$

In this example, the centroid is the point (5,3), as shown in Figure 2-10.
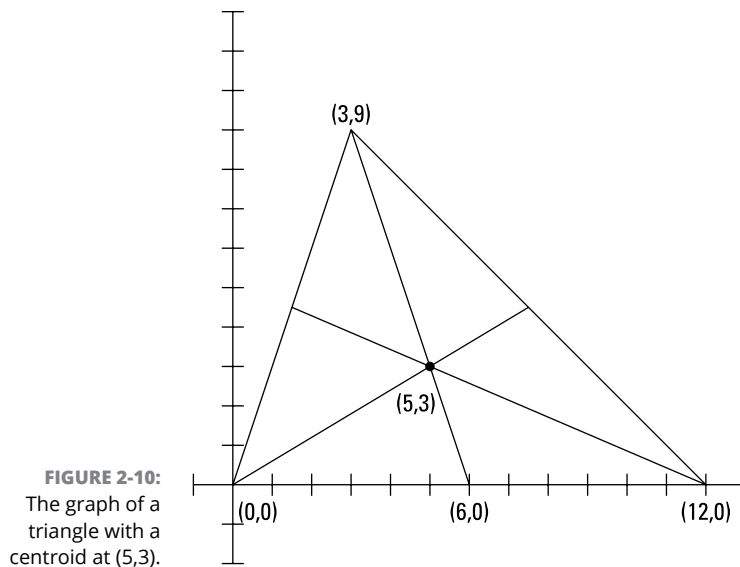
Racing Down the Slope

In mathematics, a *slope* is a particular number or value that tells you something about the nature of a line or line segment. Just by looking at the number corresponding to the slope of a line, you can tell if the line rises or falls as you read from left to right. You can also tell if the slope of the line is steep or rather flat (like the slopes in Colorado versus those in Illinois).

Slaloming slope formula

One way to find the slope of a line or segment is to choose any two points, (x_1, y_1) and (x_2, y_2), on the figure and use the formula that gives you the slope, represented by the letter m:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

For example, the slope of the line through the points $(-2, 2)$ and $(1, 8)$ is

$$m = \frac{8 - 2}{1 - (-2)} = \frac{6}{3} = 2$$

This line moves upward from left to right, which is why the slope is a positive number. Any slope greater than 1 is also considered to be *steep.*

On the other hand, the slope of the line through the points $(-5,2)$ and $(5,-1)$ is

$$m = \frac{-1-2}{5-(-5)} = \frac{-3}{10} = -\frac{3}{10}$$

This segment moves downward from left to right, so the slope is negative. The unsigned value (absolute value) of the slope is a number between 0 and 1 — so it isn't considered steep.

Figure 2-11 shows both lines, one with a slope of 2 and the other with a slope of $-\frac{3}{10}$.
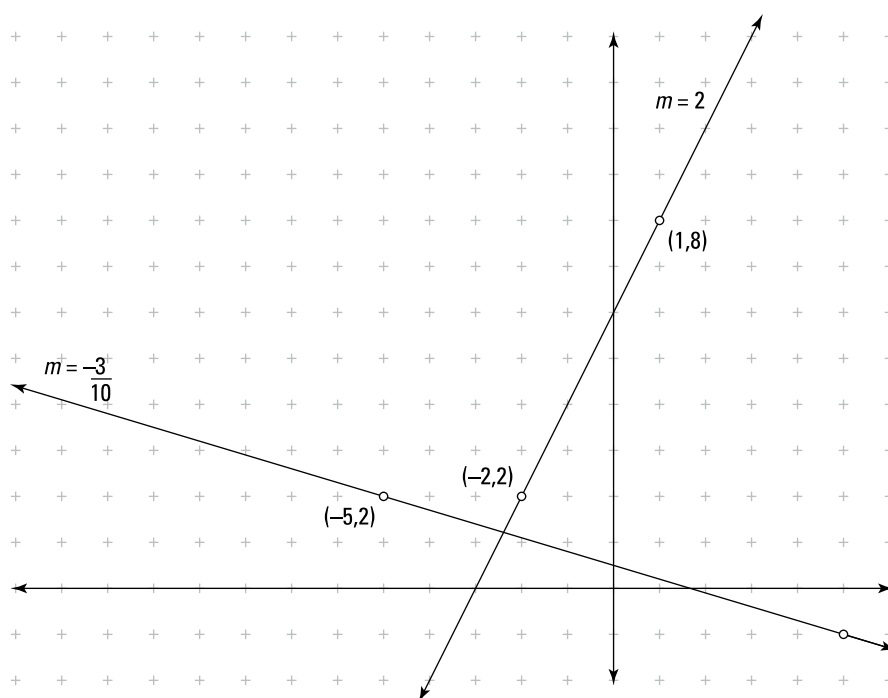
Recognizing parallel and perpendicular lines

Two lines are *parallel* if they have the same slope. Two lines are *perpendicular* if their slopes are negative reciprocals of one another. Numbers that are *negative reciprocals* have a product of –1.

Consider the following slopes of some lines or line segments:

$m_1 = \dfrac{1}{2}$	$m_2 = -2$	$m_3 = \dfrac{6}{3}$
$m_4 = \dfrac{5}{10}$	$m_5 = -\dfrac{14}{7}$	

Here are the slopes of the lines that are parallel:

» $m_1 = \dfrac{1}{2}$ and $m_4 = \dfrac{5}{10}$ have the same slope.

» $m_2 = -2$ and $m_5 = -\dfrac{14}{7}$ also have the same slope.

Here are the slopes of the lines that are perpendicular:

» $m_1 = \dfrac{1}{2}$ and $m_2 = -2$ have slopes that are negative reciprocals.

» $m_4 = \dfrac{5}{10}$ and $m_5 = -\dfrac{14}{7}$ also have slopes whose product is -1.

As a matter of fact, because the lines with slopes of $\dfrac{1}{2}$ and $\dfrac{5}{10}$ are equal to one another, they're both perpendicular to the lines with slopes of -2 and $-\dfrac{14}{7}$, which are also equal in slope. It's one big, happy family.

Defining Circles with Numbers

The circle that you use the most in trigonometry has its center at the origin and has a radius of 1 unit (called the *unit circle*). The radius of 1 in a circle makes computations so much easier when that 1 ends up in the denominator of a fraction. Fractions and circles sort of intermingle in trigonometry — in good ways, of course. But you also have many other useful circles to consider. The other circles will have different radii and different centers, but each has its place when needed. When possible, though, the *unit circle* is the circle of choice.

Centering circles at the origin

The two characteristics that define a circle are its center and its radius. The center tells where on a graph the circle is located; the radius tells how big the circle is. The location is in terms of coordinates in the coordinate plane, and those numbers end up in the equation of the circle. The *x* and *y* variables represent the coordinates

of all the points that lie on the actual circle. The standard form for the equation of a circle at the origin is $x^2 + y^2 = r^2$, where r represents the radius of the circle. So the equations for circles centered at the origin with radii of 2, 3, 4, and 5 are $x^2 + y^2 = 4$, $x^2 + y^2 = 9$, $x^2 + y^2 = 16$, and $x^2 + y^2 = 25$, respectively.

Likewise, a circle with a radius of 1 unit has the equation $x^2 + y^2 = 1$. This *unit circle* is used extensively in mathematics: The radius of 1 lends itself to the formula for changing from degrees to radians, is nice and neat when finding arc length, and makes the unit circle the easiest to use when proving properties or theorems in math.

Wandering centers

Circles don't have to have their centers at the origin. The standard form for a circle with a radius of r and its center at (h,k) is $(x - h)^2 + (y - k)^2 = r^2$, where x and y represent the coordinates of all the points on that circle. So, the equation for a circle with its center at $(3,-2)$ and with a radius of 9 is $(x - 3)^2 + (y + 2)^2 = 81$.

TECHNICAL STUFF

Notice that if you let the center of a circle be $(0,0)$ in this formula, you get $(x - 0)^2 + (y - 0)^2 = r^2$ or $x^2 + y^2 = 1$, which goes back to a circle with its center at the origin. The form works for all circles!

Circling Around with Applications

Circles are one of the most useful and versatile structures. They've been recognized since the beginning of mankind and exist in all sorts of situations nowadays. I'm beginning to see more and more traffic circles constructed to make drivers' lives easier. (But I have to admit that my first experience with a traffic circle — called a rotary or a roundabout — in Boston had me going around in circles, and not just figuratively.) Here are a couple of examples of useful circles.

Spinning a wheel

You have created a new game that involves a circle that spins around its center and a pointer that shows what prize you will win with a spin. Your triangular pointer stays in one place and has to be poised so the center of the triangle is directly over the center of the circle. The pointer is an isosceles triangle that has a base of 2 inches and a height of 6 inches. Where is the center of the triangle? Refer to Figure 2-12.
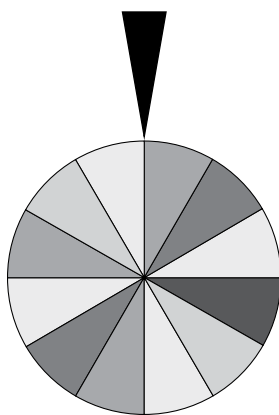
(a)
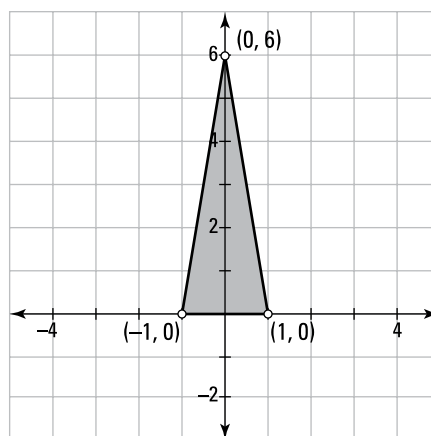
(b)

First, draw your isosceles triangle pointer on graph paper and assign coordinates to the vertices. You want to find the centroid of the triangle — which is at the intersection of the segments drawn from each vertex to the midpoint of the opposite side.

Find the midpoint of the two long sides. The midpoint of the base is sitting at the origin. Then construct segments from the two bottom corners to the midpoints of their opposites. Check this out in Figure 2-13! The intersections cross on the vertical axis at the point (0,2).
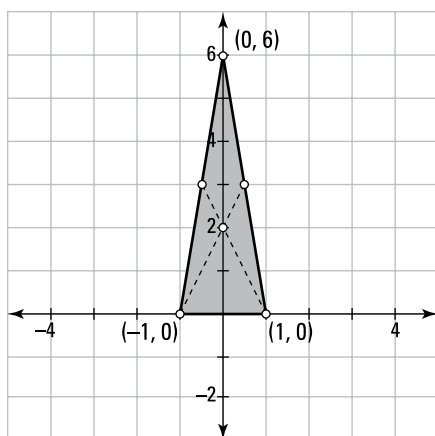
Another way to find the centroid of a triangle is to use the formula from the earlier section, "Pinpointing the center of a triangle." This formula has you locate the points two-thirds of the distance from each vertex to the midpoint of the opposite side.

Containing the cattle

A farmer has a barn in the shape of a right triangle. (I'm sure there was a good reason to construct it this way; he is probably related to Pythagoras.) The farmer wants to construct a circular fence all the way around the barn and not have the fence be any closer than 20 feet at any point. The barn is 60 feet by 80 feet on the sides forming the right angle, and the third side measures 100 feet. Figure 2-14a shows a graph of the barn (just multiply each axis number in the figure by 10). How many feet of fencing are needed to construct the smallest possible circle?

To circumscribe a triangle, you find the intersection of the perpendicular bisectors of the three sides. In Figure 2-14b, you see where the perpendicular bisectors from the left side and the bottom intersect — right on the hypotenuse. This is the cir-cumcenter of the triangle. A circle drawn with the circumcenter as center goes through all three vertices of the triangle.

FIGURE 2-14:
Finding the
center of the
fence.

(a)   (b)
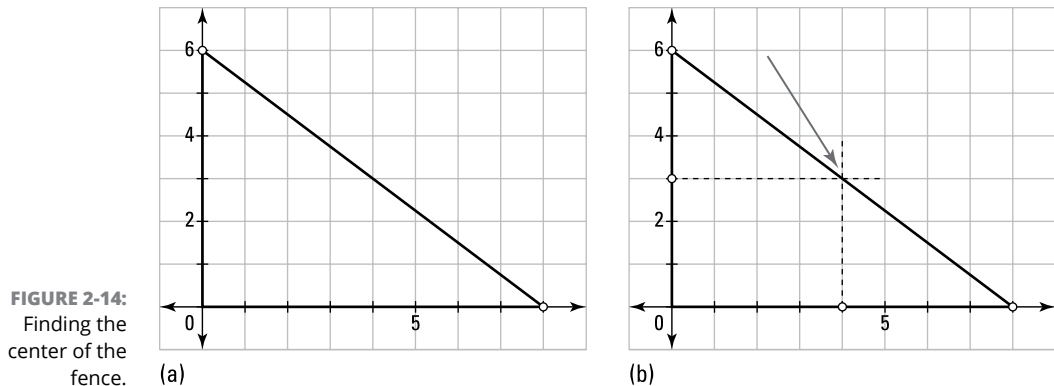
But the farmer doesn't want the fencing that close. He needs at least 20 feet of clearance around any side of the barn.

A circle drawn through the corners of the barn would have a radius of 50 feet. That's the length of the 30-40-50 triangle's hypotenuse. But the circular fence must have a radius of 70 feet (see Figure 2-15a). Using the same center and a radius of 70 feet, a circle can be drawn leaving at least 20 feet between the barn and the fencing. This closeness just occurs at the corners of the barn. Figure 2-15b shows you what the fencing looks like.

To compute the length of the fencing, you find the circumference of the circle. Using $C = \pi d = 2\pi r$, just replace the radius, r, with 70 and you have $C = 2\pi(70) = 140\pi \approx 439.6 \text{ feet}$.
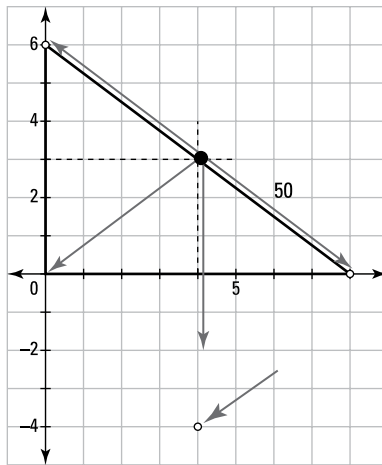
FIGURE 2-15:
Fencing in the
cattle.

(a)
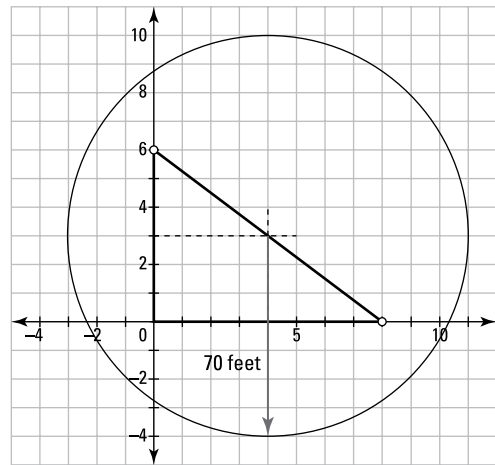
(b)

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50   PART 1 **Getting Started with Trigonometry**

Chapter **3**

Finding Degrees in Triangles and Planes

The main concept that distinguishes trigonometry from other mathematical topics is its attention to and dependence on angle measures. The trig functions (sine, cosine, tangent, cotangent, secant, and cosecant) are ratios based on the measures of an angle. What good are degrees (no, not the kind that tell you how hot or cold it is) in the real world? Navigators, carpenters, and astronauts can't do without them. How do you measure the degrees? You have many ways, dear reader, and I show you all you need to know in this chapter.

Angles, Angles Everywhere: Measuring in Degrees

What's a degree? When you graduate from college, you get your degree. The temperature outside went up a degree. When questioned, you get the third degree. All these scenarios use the word *degree,* but in trigonometry, a degree is a tiny slice of a circle. Imagine a pizza cut into 360 equal pieces (what a mess). Each little slice represents one degree. Look at Figure 3-1a to see what a degree looks like.

Slicing a coordinate plane

The first quadrant is the upper right-hand corner of the coordinate plane. (See Chapter 2 for the lowdown on quadrants and coordinate planes.) That first quadrant is $\frac{1}{4}$ of the entire plane. So, if a full circle with its center at the origin has a total of 360 degrees, then $\frac{1}{4}$ of it has 90 degrees, which is the measure of the angle that the first quadrant forms. Actually, each quadrant measures exactly 90 degrees, as you see in Figure 3-1b. You can divide each of these 90-degree measures evenly by many numbers, and you use those equal divisions frequently in trig, because they're nice, neat divisions. The most frequently used angle measures include $\frac{90}{2} = 45°$, $\frac{90}{3} = 30°$, and $\frac{90}{6} = 15°$. And then, twice the 30-degree angle is 60 degrees (another common angle in trig).

FIGURE 3-1:
One degree is $\frac{1}{360}$ of a circle.
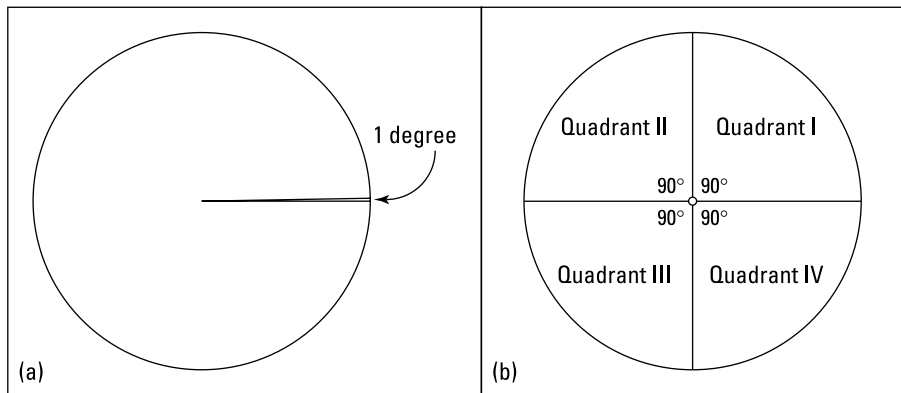
This elite group of angle measures is 0, 15, 30, 45, 60, and 90 degrees. These angles and their multiples occupy much of the discussion in trigonometry because of their convenience in computations. Figure 3-2 shows sketches of some of the angles.

FIGURE 3-2:
Some of the most commonly used angles: 90, 60, 45, 30, and 15 degrees.
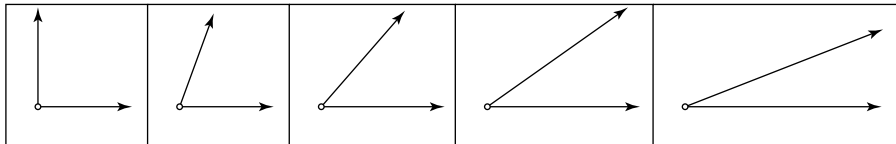
Looking elsewhere for degree measures

Your first introduction to the idea of measuring angles in degrees probably didn't come from a course in geometry or trigonometry. Most of us are exposed to this

idea through television or movies. A popular situation in such shows involves a plane flying through a storm or at night or with no one but a flight attendant at the controls. A radio transmission from the control tower comes crackling through all the static, with an air traffic controller saying, "Turn to a heading of 40 degrees." And because the flight attendant remembers their trigonometry, they save the day. Hurray for degrees!

Another type of situation that you find on television is on *This Old House,* where the stars, in all their woodworking grandeur, are able to cut boards at exact 50-degree angles so they fit perfectly in a carefully crafted wooden truss.

Navigating with degrees

In navigation and surveying, the *bearing* or *heading* is the direction that a plane, boat, or line takes. In math-speak, this bearing is the angle measured in degrees that a *ray* (a line with one endpoint that extends infinitely in the other direction) makes with a second ray that points north. The angle is measured in a clockwise direction. (Note, however, that in the *standard position* in geometry and trigonometry, you measure angles in a *counterclockwise* direction.) Figure 3-3 shows some bearings used in navigation. Notice that the direction of the arrow is always clockwise. Even though the angles in bearings are measured differently from those in trigonometry, the angle measures are still the same size — just rotated a bit. An angle of 120 degrees is still bigger than a right angle. When you're familiar with the angle sizes, translating into this bearing business is easy.

FIGURE 3-3: Bearings of 20 degrees, 100 degrees, and 250 degrees.
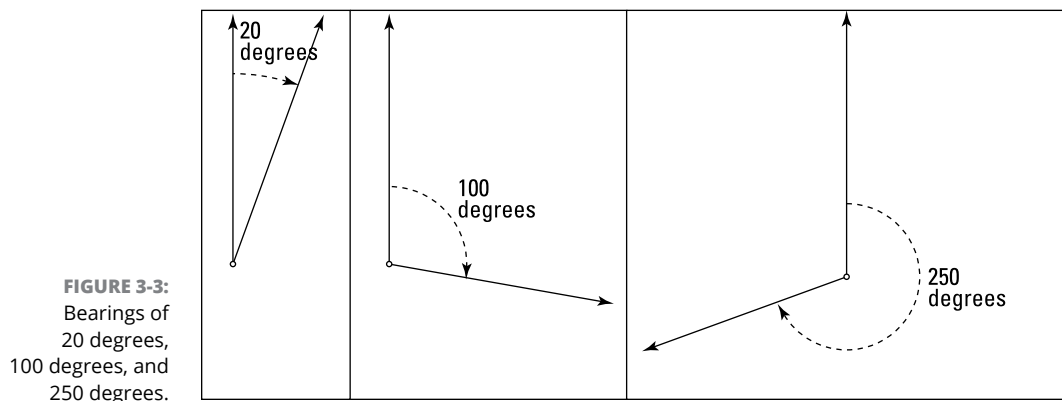
Now, take a look at Figure 3-4 to see the path of a helicopter pilot who flew for 10.5 minutes at a bearing of 36 degrees (which is northeast), then for 13.6 minutes at a bearing of 144 degrees (which is southeast), and then got back to where they started by flying for 14.4 minutes at a bearing of 280 degrees (which is west-northwest).

FIGURE 3-4: A helicopter pilot's course.

Understanding Norm's workshop

If you weren't a follower of public television's *The New Yankee Workshop* starring Norm Abram, let me fill you in: Norm is a New Englander who did woodworking projects with very expensive tools and invited his audience to do the same (with not-so-expensive tools). He set his table saw so it could cut a board straight across at a 90-degree angle, or he changed it to cut at any other angle. If Norm wanted two perpendicular pieces of wood to meet and form a right angle, he set his saw at 45 degrees. He also cut one piece at 30 degrees and the other at 60 degrees; or how about 20 and 70? Figure 3-5 shows how the two pieces of wood fit together.

If Norm wanted to create an octagonal (eight-sided) table from a single piece of wood that he cut into eight pieces, then what angles did he cut? More on that in a minute. In Figure 3-6, you see an octagonal table constructed of eight equal triangles.

FIGURE 3-5:
Two pieces of wood cut at various angles together form a 90-degree angle.

45 and 45   30 and 60   20 and 70
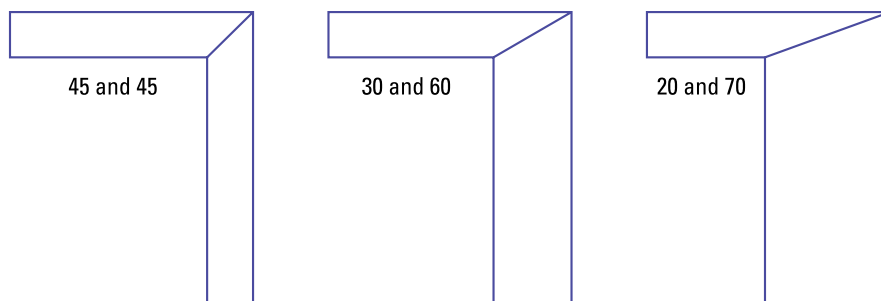
FIGURE 3-6:
An octagonal table and one of the pieces that comprises it.
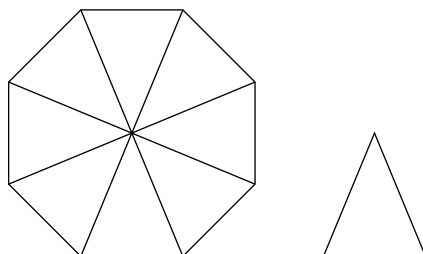
To make his octagonal table, Norm needed eight *isosceles triangles* (where the two long sides of each triangle are the same length). What are the measures of the angles he had to cut? All the way around a circle (and around the middle of the table) is 360 degrees, so each triangle has a *top angle* (the angle at the center of the table) that measures $\frac{360}{8}$, which is 45 degrees. The two *base angles* (those at the outer edge of the table) are equal in measure. The sum of the measures of the angles of a triangle is 180 degrees, so after subtracting the top angle's 45 degrees, you get 135 degrees for the other two angles together. Dividing the 135 by 2, you find that the base angles are each $67\frac{1}{2}$ degrees. Norm cut all eight triangles from a single piece of wood, because two base angles plus a top angle form a straight line. He just put the triangles together differently after cutting them out. And as you can see from Figure 3-7, he didn't have much waste.

FIGURE 3-7:
Cutting eight identical triangles out of a board.
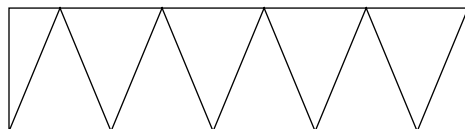
Graphing Angles in Standard Position

Navigators, surveyors, and carpenters all use the same angle measures, but the angles start out in different positions or places. In trigonometry and most other mathematical disciplines, you draw angles in a standard, universal position, so that mathematicians around the world are drawing and talking about the same thing.

Positioning initial and terminal sides

An angle in *standard position* has its vertex at the origin of the coordinate plane, as shown in Figure 3-8a. Its *initial* ray (starting side) lies along the positive *x*-axis. Its *terminal ray* (ending side) moves counterclockwise from the initial side.

If the terminal ray moves clockwise instead of counterclockwise, then the measure is a negative value. You often name angles in standard position with a Greek letter.

REMEMBER

The lengths of the rays that create the angle have nothing to do with the angle size. You can extend rays as long as you need them to be, and the angle measure doesn't change. Only the position of the terminal ray determines the angle.

FIGURE 3-8:
An angle in standard position and angles by quadrant.
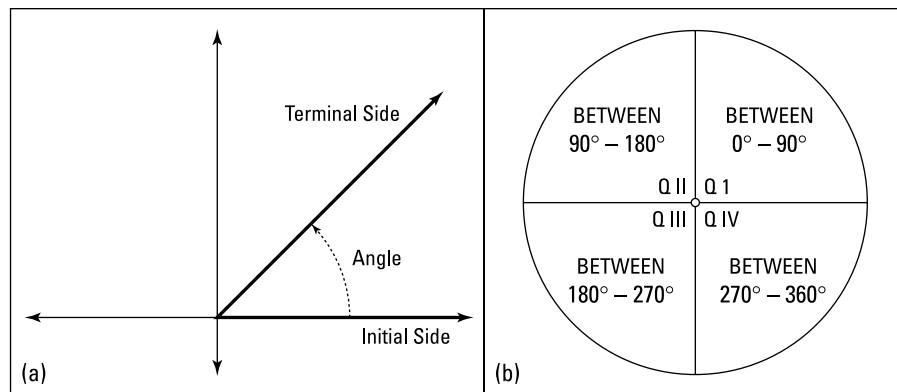
Measuring by quadrants

Angles in the standard position are used in calculus, geometry, trigonometry, and other math subjects as a basis for discussion. Being able to recognize a particular angle by the quadrant its terminal side lies in and, conversely, to know which angles have their terminal sides in a particular quadrant is helpful when working in these areas.

Check out Figure 3-8b. Angles in standard position that measure between 0 and 90 degrees have their terminal sides in Quadrant I. The angles measuring between 90 and 180 degrees have their terminal sides in Quadrant II. Angles measuring between 180 and 270 degrees have their terminal sides in Quadrant III, and those measuring between 270 and 360 degrees have their terminal sides in Quadrant IV. Angles measuring exactly 90, 180, 270, and 360 degrees do not have a terminal side that lies in a quadrant, and they're referred to as *quadrant angles.*

What's Your Angle? Labeling in Various Ways

The terminal side of an angle determines its angle measure. But more than one angle has the same terminal side — in fact, an infinite number of angles share a particular terminal side.

Using negative angle measures

If you want your angle measurement to be positive, you measure the angle in standard position in a *counterclockwise* direction. However, angles can have negative values, too, as you see in Figure 3-9. You get a negative value when you measure an angle in a *clockwise* direction. Therefore, an angle of 300 degrees has the same terminal side as an angle measuring −60 degrees. If they have the same terminal side, then why don't they have the same name/size? And which name is better? Sometimes you may want to keep the numerical part of the measure smaller. For example, picturing an angle of −30 degrees is easier than picturing one of 330 degrees. Also, pilots don't always have the choice as to which direction they can turn in, but going 10 degrees in the negative direction makes more sense than going 350 degrees — all the way around, practically — in the positive direction. One common practice is to name all angles with a number that has an absolute value less than 180 degrees. So −60 degrees is often preferable to 300 degrees.

Comingling with coterminal angles

Two angles are *coterminal* if they have the same terminal side. You have an infinite number of ways to give an angle measure for a particular terminal ray. Sometimes, using a negative angle rather than a positive angle is more convenient, or the answer to an application may involve more than one revolution (spinning around and around). Angles can have terminal sides that involve one or more full revolutions around the origin or terminal sides that go clockwise instead of counterclockwise — or both of these situations can happen.

FIGURE 3-9:
Angles with both
positive and
negative
measures.
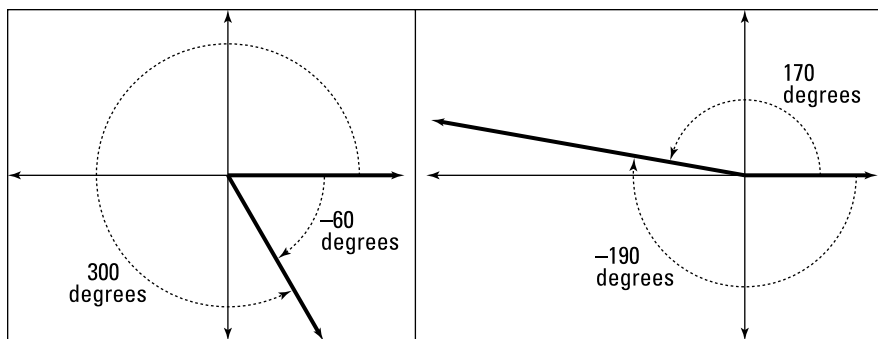
More than one revolution

An angle measuring 70 degrees is coterminal with an angle measuring 430 degrees (see Figure 3-10). The angle measuring 430 degrees is actually $360 + 70$ (one full revolution plus the original 70). These two angles are also coterminal with an angle of 790 degrees ($360 + 360 + 70 = 790$). This pattern could go on and on, with the addition of another 360 degrees each time.
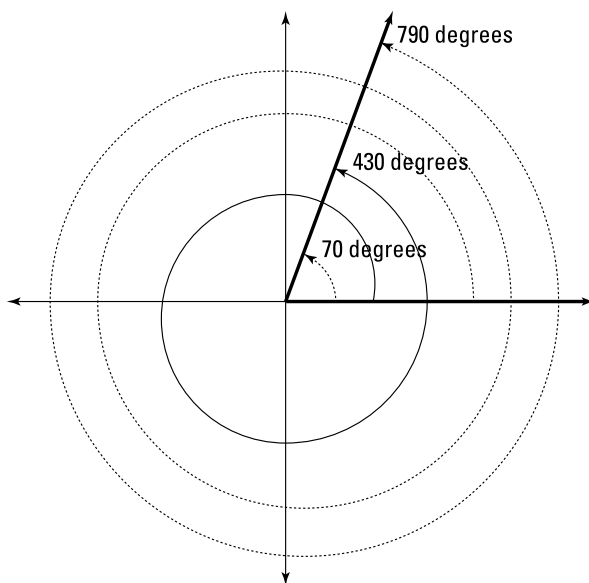
FIGURE 3-10:
Three coterminal
angles.

Negative coterminal angles

An angle of 70 degrees is coterminal with an angle of –290 degrees. Two rotations in the negative (clockwise) direction give you an angle of –650 degrees $(-290 - 360 = -650)$.

Renaming angles: So many aliases

Any angle can have many, many descriptions in terms of angle measures, because an angle is equivalent to its coterminal angles. The most frequently used positive angle measures are those that measure between 0 and 360 degrees. Rules for coterminal angles involve adding or subtracting *rotations* (or multiples of 360 degrees). The first equation that follows shows what happens when you add a full rotation over and over. The second shows what happens when you subtract a full rotation many times. The results are all coterminal angles.

$$\theta \to \theta + 360° \to \theta + 720° \to \theta + 1{,}080° \to \ldots \to \theta + 360 \cdot k°$$
$$\theta \to \theta - 360° \to \theta - 720° \to \theta - 1{,}080° \to \ldots \to \theta - 360 \cdot k°$$

So an angle measuring 100 degrees is coterminal with the following.

$$\textbf{Adding:} \ 100° \to 100° + 360° \to 100° + 720° \to 100° + 1{,}080° \to \ldots \to 100° + 360 \cdot k°$$
$$100° \to 460° \to 820° \to 1{,}180°$$
$$\textbf{Subtracting:} \ 100° \to 100° - 360° \to 100° - 720° \to 100° - 1{,}080° \to \ldots \to 100° - 360 \cdot k°$$
$$100° \to -260° \to -620° \to -980°$$

Here's an example: Suppose you want to give new measures for angles of 800 degrees and –1,040 degrees by finding an equivalent angle measure between 0 and 360 degrees.

1. **Subtract 360 degrees from 800 until the result is less than 360.**

$$800° \to 800° - 360° = 440°$$

$$440° \to 440° - 360° = 80°$$

An angle measuring 800 degrees is coterminal with an angle of 80 degrees.

2. **Add 360 degrees to –1,040 until the result is positive.**

$$-1{,}040° \to -1{,}040° + 360° = -680°$$

$$-680° \to -680° + 360° = -320°$$

$$-320° \to -320° + 360° = 40°$$

An angle measuring –1,040 degrees is coterminal with an angle of 40 degrees.

Making Degrees Work for You

Giving someone the third degree, having six degrees of separation, and freezing at 32-degrees Fahrenheit are all situations involving the same wording that also refers to angle measures. You find degree measures in many decent applications — as you find here.

Moon shining

You are out doing some International Space Station gazing one night and take two measures of your sightings. In your first measure, you see the Station to the east at a 45-degree angle from the horizon. Then, after it crosses above you and moves to the west, you measure a 65-degree angle from the western horizon. Figure 3-11 shows what happens.

FIGURE 3-11:
Tracking the International Space Station.
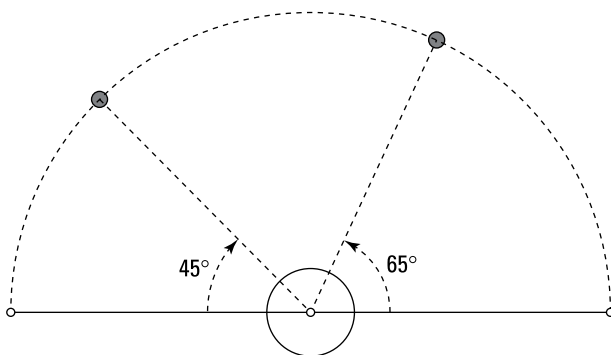
If the Space Station moves at a height of about 220 miles above the Earth and the radius of the Earth averages about 4,000 miles, then how far does the Station travel during the time you make those measurements?

You need to determine how much of a complete circle the Station makes during the time you make the measurements and then compute that proportion of the circumference that is used. The circle I'm using is the one with a radius of 4,000 miles plus the 220-mile height of the Station. Because the two angle measures use different initial sides, just add the 45 degrees and 65 degrees together and subtract from 180 degrees. You have $180 - (45 + 65) = 180 - 110 = 70$. The Station travels 70 out of the total 360 degrees of a complete circle or $\frac{7}{36}$ of the circumference — about one-sixth.

The circumference of a circle with a radius of 4,000 + 220 miles is found with: $C = 2\pi r = 2\pi(4,220) \approx 26,515$ miles.

You only want $\frac{7}{36}$ of this, and $26,515\left(\frac{7}{36}\right) \approx 5,156$ miles.

Games people play

At the latest gathering of college sorority sisters, one of the group announced that they had come up with a new game called "Twist Her." This is played by blind-folding the player and then giving her four different commands in terms of how far and in which direction to turn. If she's facing in the same direction in which she started after all the turns, then she scores ten points!

Helen was told to first turn 780 degrees clockwise. "That's silly," she thought. "780 degrees is coterminal with 60 degrees," so she turned clockwise 60 degrees.

She was then told to turn 480 degrees counterclockwise. "Golly, why don't they just say the angle, 120 degrees?" So she turned 120 degrees counterclockwise.

The next command was to turn 870 degrees clockwise. Helen was getting annoyed, but she knew that 870 degrees is coterminal with 150 degrees and turned 150 degrees clockwise.

Her final turn was to be 810 degrees counterclockwise. Happy that this was almost over, she turned 90 degrees counterclockwise.

Taking off her blindfold, Helen found herself facing in the same direction in which she started. She scored 10 points!

Had she been given all the original numbers at once, she could have saved a lot of hassle. Letting clockwise be positive angle measures and counterclockwise be negative measures and adding them together she gets $780 + (-480) + 870 + (-810) = 360$, which is coterminal with 0 degrees. Using her calculations and making the turns smaller, she had $60 + (-120) + 150 + (-90) = 0$. In both cases, she ended up in the same direction. But using her method, she wasn't quite as dizzy.

Chapter **4**

Dishing Out the Pi: Radians

A person's first introduction to angles is usually in terms of degrees. You probably have an idea of what a 30-degree angle looks like. (If not, review Chapter 3.) And even most middle-school students know that a triangle consists of 180 degrees. But most of the scientific community uses radians to measure angles and solve trig equations. Why change to radians? Why fix what ain't broke? Read on.

What's in a Radian?

A radian is much bigger than a degree. Early mathematicians decided on the size of a degree based on divisions of a full circle. A degree is the same as a slice that's $\frac{1}{360}$ of a circle. No one knows for sure how the choice of 360 degrees in a circle came to be adopted. In any case, 360 is a wonderful number, because you can divide it evenly by so many other numbers: 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, and 360. The early measures of time and distance relied on having convenient numbers to work with. A radian, on the other hand, isn't quite as nice. It isn't even a rational number. Radians probably were developed because mathematicians wanted to relate the angle measure more to

the radius or size of the circle rather than picking an arbitrary dividing number like 360. A circle has 2π radians (a little more than six radians). A radian is *almost* $\frac{1}{6}$ of a circle — it's a little more than 57 degrees. Figure 4-1 compares a degree with a radian.
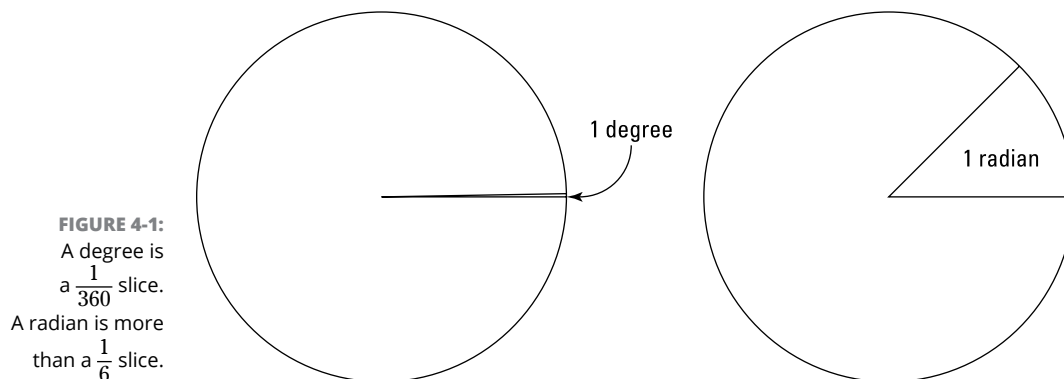
FIGURE 4-1:
A degree is
a $\frac{1}{360}$ slice.
A radian is more
than a $\frac{1}{6}$ slice.

1 degree

1 radian

Relating to a circle

The big advantage of using radians is that they're the *natural* measure for dividing up circles. Imagine taking the radius of a circle and bending it into an arc that lies along the circle. Now draw radii from the center to both ends of that arc formed by the radius. The angle formed from the radii measures one radian. Look at the right-hand drawing in Figure 4-1. There you see two radii and the arc lying on the circle starting and ending at their endpoints. You would need a little more than six of those arcs to go all the way around the circle. This fact is true of all circles. The circumference of any circle is always a little more than three times the diameter of that circle — π times the diameter, to be exact. Another way of saying this is 2π times the radius. That number may seem nicer and more civilized than the big number 360, but the disadvantage is that π doesn't have an exact decimal value. Saying 2π radians (which is equal to 360 degrees) means that each circle has *about* 6.28 radians. Even though radians are the natural measure and always relate to the radius and diameter, the decimal values get a bit messy.

Each of these measures has its own place. Measuring angles in degrees is easier, but measuring angles in radians is preferable when doing computations and applications. The radian is more exact because the radius, circumference, or area of the circle is involved. Even though π doesn't have an exact decimal value, when you use multiples of π in answers, they're exactly right. I show you an example of using π as part of an answer in the section, "Making a Clone of Arc," later in this chapter.

Converting degrees and radians

Many math problems require changing from degrees to radian measures or vice versa. You often perform mathematical computations in radians, but then convert the final answers to degrees so the answers are easier to visualize and comprehend. You can use a nifty little proportion to change from degrees to radians or radians to degrees. In this proportion, the Greek letter theta, θ, represents the name of the angle. Putting the superscripts $^\circ$ and R on θ makes the angle stand for the measure in degrees and radians, respectively.

$$\frac{\theta^\circ}{180} = \frac{\theta^R}{\pi}$$

TECHNICAL STUFF

This proportion reads: "The measure of angle θ in degrees divided by 180 is equal to the measure of angle θ in radians divided by π." (Remember that π is about 3.141592654.)

The computation required for changing degrees to radians and radians to degrees isn't difficult. This computation involves a few tricks, though, and the format is important. You don't usually write the radian measures with decimal values unless you've multiplied through by the decimal equivalent for π.

Changing degrees to radians

To change a measure in degrees to radians, start with the basic proportion for the equivalent angle measures: $\frac{\theta^\circ}{180} = \frac{\theta^R}{\pi}$.

For example, here's how you change a measure of 40 degrees to radians:

1. **Put the 40 in place of the $\theta°$ in the proportion.**

   $$\frac{40}{180} = \frac{\theta^R}{\pi}$$

2. **Reduce the fraction on the left.**

   $$\frac{\overset{2}{\cancel{40}}}{\underset{9}{\cancel{180}}} = \frac{2}{9} = \frac{\theta^R}{\pi}$$

3. **Multiply each side of the proportion by π.**

   $$\pi \cdot \frac{2}{9} = \frac{\theta^R}{\pi} \cdot \pi$$

4. **Simplify the work.**

   $$\pi \cdot \frac{2}{9} = \frac{\theta^R}{\cancel{\pi}} \cdot \cancel{\pi}$$

   $$\frac{2\pi}{9} = \theta^R$$

This example shows that 40 degrees is equivalent to $\frac{2\pi}{9}$ radians. You leave the radian measure as a fraction reduced to lowest terms.

Check out another example: Change a measure of –36 degrees to radians.

1. Put the –36 in place of the $\theta°$ in the proportion.

$$\frac{-36}{180} = \frac{\theta^R}{\pi}$$

2. Reduce the fraction on the left.

$$\frac{-\,^1\!\cancel{36}}{_5\cancel{180}} = -\frac{1}{5} = \frac{\theta^R}{\pi}$$

3. Multiply each side of the proportion by π.

$$\pi \cdot \left(-\frac{1}{5}\right) = \frac{\theta^R}{\pi} \cdot \pi$$

4. Simplify the work.

$$\pi \cdot \left(-\frac{1}{5}\right) = \frac{\theta^R}{\cancel{\pi}} \cdot \cancel{\pi}$$

$$-\frac{\pi}{5} = \theta^R$$

So you see, –36 degrees is equivalent to $-\frac{\pi}{5}$ radians. Having a negative angle is fine (see Chapter 3 for more on negative angles). You leave the expression as a fraction; don't change it to a decimal form.

Changing radians to degrees

You use the same basic proportion to change radians to degrees as you do for changing degrees to radians.

For example, to change $\frac{\pi}{12}$ radians to a degree measure:

1. Put the radian measure in place of the θ^R in the proportion.

$$\frac{\theta°}{180} = \frac{\frac{\pi}{12}}{\pi}$$

2. Simplify the complex fraction on the right by multiplying the numerator by the reciprocal of the denominator.

$$\frac{\theta°}{180} = \frac{\cancel{\pi}}{12} \cdot \frac{1}{\cancel{\pi}}$$

$$\frac{\theta°}{180} = \frac{1}{12}$$

3. **Multiply each side of the proportion by 180.**

$$180 \cdot \frac{\theta^{\circ}}{180} = \frac{1}{12} \cdot 180$$

4. **Reduce and simplify the fraction on the right.**

$$\cancel{180} \cdot \frac{\theta^{\circ}}{\cancel{180}} = \frac{1}{\cancel{12}^{\,1}} \cdot \cancel{180}^{\,15}$$

$$\theta^{\circ} = 15$$

So, $\frac{\pi}{12}$ radians is equivalent to 15 degrees.

Here's another example: Change 1.309 radians to degrees.

I changed this radian measure to a decimal by multiplying through by a decimal equivalent of π, which is approximately 3.1416. You use this same decimal equivalent to solve the problem.

1. **Put the radian measure in place of the θ^{R} in the proportion.**

$$\frac{\theta^{\circ}}{180} = \frac{1.309}{\pi}$$

2. **Change the π to a decimal approximation.** In this case, I used four decimal places.

$$\frac{\theta^{\circ}}{180} = \frac{1.309}{3.1416}$$

3. **Multiply each side of the proportion by 180.**

$$180 \cdot \frac{\theta^{\circ}}{180} = \frac{1.309}{3.1416} \cdot 180$$

JUST A MINUTE

A full circle contains 360 degrees. If you want just a part of a degree — and a degree is already pretty small — you can say you have $\frac{1}{2}$ of a degree or 0.5 of a degree, or you can use another division. You can divide one degree into 60 *minutes,* and you can divide each minute into 60 *seconds.* So, mathematically, a degree has 3,600 subdivisions — you can break it down into 3,600 seconds. The way you denote the number of minutes and seconds is with one tick mark (') for minutes and two tick marks (") for seconds. So you read the degree measure 15°45′27″ like this: "Fifteen degrees, 45 minutes, and 27 seconds." Ever since the advent of hand-held calculators, people don't use this measure much. The decimal breakdown of a degree is more universally accepted.

4. **Reduce the fractions, and simplify the value on the right.**

$$\cancel{180} \cdot \frac{\theta^\circ}{\cancel{180}} = \frac{1.309}{3.1416} \cdot 180$$

$$\theta^\circ = \frac{235.62}{3.1416} = 75$$

This result came out to be a nice number. Sometimes, however, you have a decimal answer for the degrees. Actually, you get a decimal more than sometimes — you *usually* get one.

Highlighting favorites

The *favorite* or most-used angles are those that are multiples of 15 degrees, such as 30, 45, 60, and 90 degrees. Putting these angles into the proportion for changing degrees to radians gives a nice set of angles in radian measure.

First, look at what happens when you replace θ° with 30:

$$\frac{30}{180} = \frac{\theta^R}{\pi}$$

$$\frac{1}{6} = \frac{\theta^R}{\pi}$$

$$\pi \cdot \frac{1}{6} = \frac{\theta^R}{\pi} \cdot \pi$$

$$\frac{\pi}{6} = \theta^R$$

An angle of 30 degrees is equivalent to $\frac{\pi}{6}$ radians. You get a simple fraction with a π on the top and a nice, small 6 on the bottom.

You get similar results with the other angles:

$45^\circ \rightarrow \dfrac{\pi}{4}$	$60^\circ \rightarrow \dfrac{\pi}{3}$	$90^\circ \rightarrow \dfrac{\pi}{2}$

Radian measures with denominators of 2, 3, 4, and 6 are used most frequently.

Making a Clone of Arc

The biggest advantage of using radians instead of degrees is that a radian is directly tied to a length — the length or distance around a circle, which is called its *circumference.* Using radians is very helpful when doing applications involving

the length of an arc of a circle, which is part of its circumference; measuring the sweep of a hand on a clock; and finding distance in navigation problems.

The problems in this section give you a good sampling of situations where using radians is your best bet. Of course, all these problems presume that you can make *accurate* measurements of the variables you can measure. But trigonometry does open the door to solving practical problems that aren't doable any other way.

Taking chunks out of circles

The examples in this section use features of circles. A part of a circle may be an arc, a diameter (not really a physical part, but a measure), a *sector* (a piece of the inside), or the center. The measures usually start out in degrees, and I change them to radians, when necessary, to complete the problem.

Scanning with a radar

A radar system scans a circular area that has a radius of 40 miles. In one second, it sweeps an arc of 60 degrees. What *area* does the radar cover in one second? In five seconds? Look at Figure 4-2, which shows a sweep of 60 degrees.
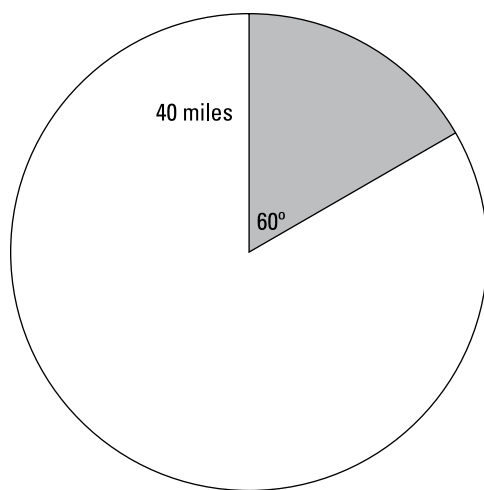
Here's how you solve this problem:

1. **Find the area of the circle.**

   Use the formula for the area of a circle, $A = \pi r^2$. Putting the 40 in for the radius, r, you get $\pi r^2 = \pi (40)^2 = 1,600\pi \approx 5,026.548$ square miles.

2. **Divide by the portion of the circle that the sweep covers.**

    The sweep of 60 degrees is only $\frac{1}{6}$ of the entire circle, so you figure the area that the sweep covers by dividing the entire area by 6. The resulting area is $\frac{5,026.548}{6} = 837.758$ square miles, which is the area the radar scans in one second. To get the area covered in five seconds, multiply that result by 5 to get 4,188.790 square miles.

The preceding problem works out nicely, because the number of degrees is a convenient value — it's a fraction of the circle. But what if the number doesn't divide evenly into 360? For example, what if the radar sweeps an angle of 76 degrees in one second?

In general, if the angle is given in degrees, then the part of a circle that the angle sweeps is $\frac{\text{angle in degrees}}{360}$. Take the fraction for that part of the circle and multiply it by the area, πr^2. A fancy name for this part of a circle is *sector.*

REMEMBER Keep the following formulas in mind when you're trying to find the area of a sector.

>> **Using degrees:** Area of sector $= \dfrac{\theta^\circ}{360} \cdot \pi r^2$

>> **Using radians:** Area of sector $= \dfrac{\theta^R \cdot r^2}{2}$

The second formula comes from the following computation. That's why there's no π in the result:

$$\frac{\theta^R}{2\pi} \cdot \pi r^2 = \frac{\theta^R}{2\cancel{\pi}} \cdot \cancel{\pi} r^2 = \frac{\theta^R \cdot r^2}{2}$$

For example, to find the area of the radar sweep in the preceding example when the radar sweeps 76 degrees per second:

1. **Put 76 in for the θ° and 40 for the radius in the formula for the area of a sector.**

    Area of sweep $= \dfrac{76}{360} \cdot \pi (40)^2$

2. **Multiply and divide to simplify the answer.**

    $= \dfrac{121,600\pi}{360} \approx \dfrac{382,017.667}{360} = 1,061.160$ square miles

To demonstrate this radar-sweep calculation if you're given measurements in radians, find the area of the radar sweep if the sweep is $\frac{\pi}{3}$ radians (which is 60 degrees).

1. **Put $\frac{\pi}{3}$ in for the θ^R and 40 in for the radius.**

$$\text{Area of sweep} = \frac{\theta^R \cdot r^2}{2} = \frac{\frac{\pi}{3}(40)^2}{2}$$

2. **Multiply and divide to simplify the answer.**

$$= \frac{\frac{\pi}{3} \cdot 1{,}600}{2} \approx \frac{1{,}675.516}{2} = 837.758 \text{ square miles}$$

Compare this result with the computation for the sweep of 60 degrees, shown earlier in this section.

Sharing pizza

Some fraternity brothers want to order pizza — and you know how hungry college men can be. The big question is, which has bigger slices of pizza: a 12-inch pizza cut into six slices, or a 15-inch pizza cut into eight slices? Figure 4-3 shows a 12-inch pizza and a 15-inch pizza, both of which are sliced. Can you tell by looking at them which slices are bigger — that is, have more area?

The 12-inch pizza is cut into six pieces. Each piece represents an angle of 60 degrees, which is $\frac{\pi}{3}$ radians, so you find the area of each sector (slice) by using the formula for the area of a sector using radians and putting the 6 in for the radius of the pizza with a 12-inch diameter. The answer is $\frac{\theta^R \cdot r^2}{2} = \frac{\frac{\pi}{3} \cdot (6)^2}{2} = \frac{\frac{\pi}{3} \cdot 36}{2} = \frac{12\pi}{2} = 6\pi$

square inches. (I leave the answer with the multiplier of π just so you can compare the sizes between the two pizzas — they'll both have a multiplier of π in them.)

The 15-inch pizza is cut into eight pieces. Each piece represents an angle of 45 degrees, which is $\frac{\pi}{4}$ radians, so, letting the radius be 7.5 this time, the area of each sector is $\frac{\theta^R \cdot r^2}{2} = \frac{\frac{\pi}{4}(7.5)^2}{2} = \frac{\frac{\pi}{4} \cdot 56.25}{2} = \frac{56.25\pi}{8} = 7.03125\pi$ square inches.

This result doesn't tell you exactly how many square inches are in each slice, but you can see that a slice of this 15-inch pizza has an area of 7.03125π square inches, and a slice of the 12-inch pizza has an area of 6π square inches. The 15-inch pizza has bigger pieces, even though you cut it into more pieces than the 12-inch pizza. And, by the way, the difference is slightly over three square inches per slice.

Sweeping hands

I discuss two scenarios in this section: the minute hand of a clock sweeping across the clock's face, and the hand of a rider on a Ferris wheel as it whooshes through the air. These examples use the formula for arc length, which is the distance around part of a circle.

REMEMBER You find the length of an arc of a circle, s, by using the following formula, where the measure of the angle is in radians, and r stands for the radius of the circle: $s = \theta^R \cdot r$.

Riding the minute hand

Suppose a ladybug settled onto the tip of a tower clock's minute hand. The minute hand is 12 feet long. How far does the ladybug travel from 3:00 until 3:20?

1. **Calculate how many degrees the minute hand swings in 20 minutes.**

   Twenty minutes is $\frac{20}{60}$ or $\frac{1}{3}$ of an hour. Translate that fraction into degrees, and you get $\frac{1}{3}$ of 360, or 120 degrees.

2. **Convert degrees to radians.**

   The formula for arc length uses angles in terms of radians, so you first need to change 120 degrees to radians. Using the proportion for changing from degrees to radians and reducing the fraction on the left, $\frac{120}{180} = \frac{\theta^R}{\pi}$ or $\frac{2}{3} = \frac{\theta^R}{\pi}$. Multiply each side of the equation by π. The final result for the angle measure is $\theta^R = \frac{2\pi}{3}$.

3. **Calculate the answer by using the formula for the length of the arc.**

Enter the angle in radians, and enter 12 (feet), which is the length of the minute hand, for the radius. Your computations should look like $s = \theta^R \cdot r = \dfrac{2\pi}{3} \cdot 12 = 8\pi$, which is the distance the ladybug traveled, about 25.13 feet.

Riding the Ferris wheel

I don't usually care for Ferris wheels and heights, but on the rare occasions when someone *does* talk me into riding one, I have to face my least favorite part of the whole event: coming down the front of the wheel (I call this part the front of the wheel because you can't see the ground below you). Take a look at Figure 4-4. Imagine that this is the London Eye, a giant Ferris wheel in London, England. The diameter of this wheel is 394 feet. If I'm in a car on the wheel and travel from the top of the wheel, halfway down to the bottom, then how far have I traveled (hyperventilating the whole way)?
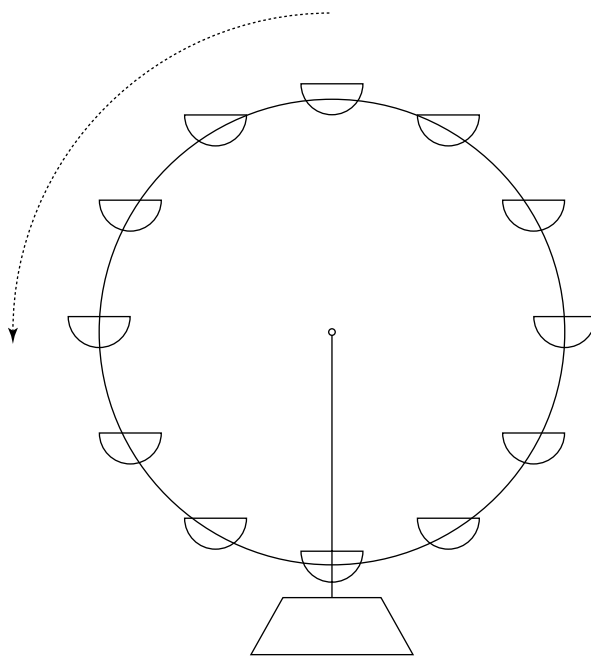
FIGURE 4-4:
Riding down the front of a Ferris wheel.

A circle with a 394-foot diameter has a 197-foot radius. From the top to the front of the wheel is $\dfrac{1}{4}$ of the circle, which is 90 degrees. In radians, 90 degrees is $\dfrac{\pi}{2}$ (see the section, "Highlighting favorites," earlier in this chapter). Using the formula

for arc length and putting in the radian measure and the radius of 197 feet, the distance is $s = \theta^R \cdot r = \frac{\pi}{2} \cdot 197 = 98.5\pi$, which is about 309.45 feet. Egad!

Going out and about

One of trigonometry's great qualities is that it lets you measure things that you can't get at or, in the case of the racetrack example in this section, things that you don't want to get close to. A circle and its angles have all sorts of applications both on earth and above.

Measuring the distance to the moon

One of the earliest applications of trigonometry was in measuring distances that you couldn't reach, such as distances to planets or the moon or to places on the other side of the world. Consider the following example.

The diameter of the moon is about 2,160 miles. When the moon is full, a person sighting the moon from the earth measures an angle of about 0.526 degrees from one side of the moon to the other (see Figure 4-5).
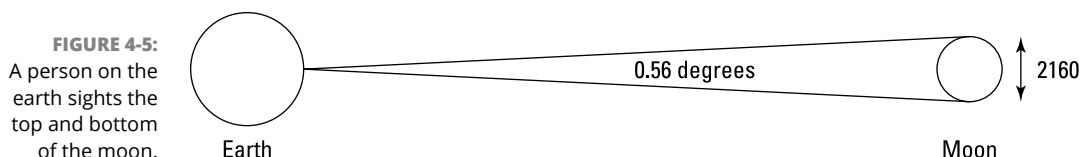
To figure out how far away the moon is from the earth, consider a circle with the earth at the center and the circumference running right through the center of the moon, along one of the moon's diameters. The moon is so far away that the straight diameter and slight curve of this big circle's circumference are essentially the same measure. The arc that runs through the moon's diameter has an angle of 0.56 degree and an arc length of 2,160 miles (the diameter). Using the arc-length formula, solve for the radius of the large circle, because the radius is the distance to the moon. To solve for the radius:

1. **First, change 0.56 degree to radians.**

    $\frac{0.56}{180} = \frac{\theta^R}{\pi} \quad \theta^R = \frac{0.56\pi}{180} \approx 0.00977$

2. **Input the numbers into the arc-length formula, $s = \theta^R \cdot r$.**

    Enter 0.00977 radian for the radian measure and 2,160 for the arc length: $2,160 = 0.00977 \cdot r$.

3. **Divide each side by 0.00977.**

The distance to the moon is $r = \dfrac{2{,}160}{0.00977} \approx 221{,}085$ miles.

Racing around the track

A race car is going around a circular track. A photographer standing at the center of the track takes a picture, turns 80 degrees, and then takes another picture 10 seconds later. If the track has a diameter of $\dfrac{1}{2}$ mile, how fast is the race car going? Figure 4-6 shows the photographer in the middle and the car in the two different positions.
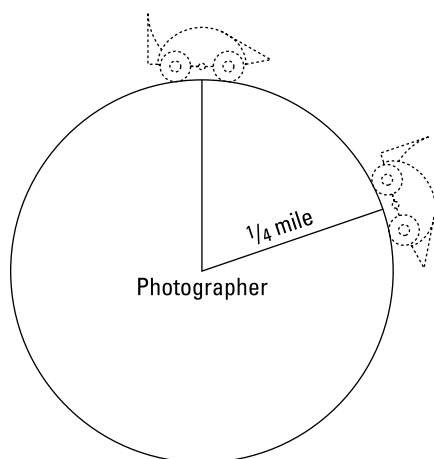
FIGURE 4-6:
A car races around the track.

How fast is the car going? Where does the problem make any mention of speed? Actually, in this situation, the car travels partway around the track in 10 seconds. By computing the arc length, you can determine how fast the car is traveling.

REMEMBER

A formula that you'll find mighty helpful is the one that says distance equals rate multiplied by time, where rate is miles per hour (or feet per second or some such measure), and time is the same measure as in the rate: $d = r \cdot t$.

1. **First, change the 80 degrees to radians.**

You end up with $\dfrac{4\pi}{9}$ radians.

2. **Input the numbers in the arc-length formula.**

Putting in the radian measure and the radius of the track, $\dfrac{1}{4}$ mile, you get arc length $= \dfrac{4\pi}{9} \cdot \dfrac{1}{4} = \dfrac{\pi}{9} \approx 0.349$ mile, which is the distance the car traveled in 10 seconds.

3. **Multiply this result by 6 (because 10 seconds is $\frac{1}{6}$ of a minute) to get miles traveled in a minute.**

This calculation gives you 2.094 miles per minute.

4. **Then multiply that number by 60 to get miles traveled in one hour.**

This calculation gives you 125.64 miles. So the car is traveling about 126 mph.

Chapter **5**

Tackling Right Triangles

Triangles are classified in many ways; one way of distinguishing one triangle from another is to use angle measurements. Because a 90-degree angle is called a *right angle,* you use the same terminology to describe a triangle with a right angle in it. This type of triangle is called a *right triangle.* And that's all right.

The measures of the sides of right triangles are used to determine the values of the trig functions. And those trig functions (along with the right triangles) are really handy when it comes to solving problems such as, "Just how high is that tree?" The special properties of a right triangle — some credited with Pythagoras — make them very useful in trigonometry and other math areas.

Sizing Up Right Triangles

If you're looking at their angles, triangles can be right, acute, or obtuse. Right triangles contain a right angle; acute triangles only have acute angles; and obtuse triangles have that one obtuse angle — makes sense! Right triangles have been of great interest to humankind for centuries. They're the basis for applications in navigation, astronomy, surveying, military engineering, and much more.

What's so right about them?

A *right triangle* has a right angle in it. But it can only have one right angle, because the total number of degrees in a triangle is 180. If it had two right angles, then those two angles would take up all 180 degrees; no degrees would be left for a third angle. So in a right triangle, the other two angles share the remaining 90 degrees.

Right triangles can come in all sorts of shapes, but they all have that corner, where the right angle sits. In Figure 5-1, you see that in all the triangles, the right angle has the two sides that are perpendicular to one another. The other two angles are *acute* angles (meaning they're less than 90 degrees).
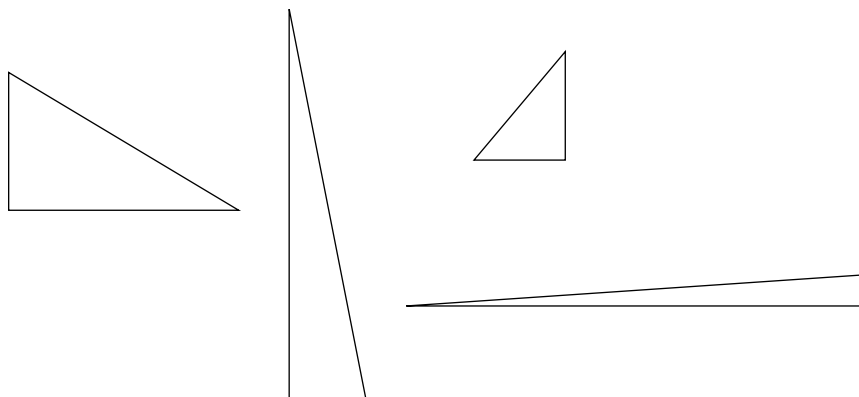
The anatomy of a right triangle

Right triangles are a familiar sight — and not just in geometry classes. Carpenters have tools for measurement that are right triangles. Architects who design by hand (rather than on a computer) draw with right-triangle templates. Even though the focus in a right triangle is the right angle, a right triangle actually has six different parts: three angles and three sides. Now, this fact is true of any triangle, but right triangles have special names for these parts. Having special names is necessary because so many properties, theorems, and applications using right triangles are out there, and the names make talking about and explaining the triangles more understandable.

REMEMBER

Figure 5-2 shows a general right triangle labeled with capital letters and lowercase letters. Since the time of Leonhard Euler, the famous Swiss mathematician, this type of labeling has been the tradition. You use capital letters to mark each *vertex* (angle), and lowercase letters to mark the sides of the triangle opposite each angle with the corresponding capital letter. This is much easier than using the geometry standard of naming sides with two letters: side *AB*, side *BC*, and side *CA*.

SQUARING THE CORNERS

Over 40 years ago, my spouse and I selected house plans that we liked and started all the processes needed to build our new home. We didn't have all that much money, so we did as much as we could by ourselves. After the lot was cleared of bushes and weeds, we went there with pegs and string to lay out the foundation. The backhoe was due the next day to dig the hole for the basement. With blueprints in hand, we had all the measurements — how long each side of the house was to be. A tape measure gave us accurate measures for lengths, but what about the right angles? A school protractor isn't big enough to make those long sides exactly perpendicular. We used a method called *squaring the corners.* We knew that some nice measures for the three sides of a right triangle are 3-4-5 or 6-8-10. At a corner in question, we would pick a point on one string that was 4 feet from the peg and a point on the other string that was 3 feet from the peg. Then we'd measure diagonally from each of those points. If the diagonal measure didn't come out to be exactly 5 feet, then we didn't have a right angle — the corner wasn't square. It took a lot of peg moving, but we got the foundation laid out accurately. The house is still standing — in fact, we still live in it — so I guess we did a good job!

The little square at the vertex C shows that the two sides meeting there are perpendicular at that vertex — that's where the right angle is. The side c, opposite the right angle, is called the *hypotenuse.* The other two sides, a and b, are called the *legs.* The hypotenuse is always the longest side, because it's opposite the largest angle.

In Figure 5-2, the angle at vertex C is the right angle, and the other two angles, A and B, are acute angles. If the measures of the angles at A and B are the same, then they're each 45 degrees, and the triangle is *isosceles.* If that's the case, then the lengths of the sides a and b are the same also.
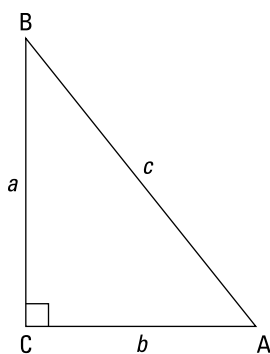
FIGURE 5-2:
Parts of a right
triangle.

If the angle at vertex *A* is bigger than that at vertex *B*, then side *a* is longer than side *b*. The measures of a triangle's angles have a direct relationship to the lengths of the sides opposite them.

Demystifying the Pythagorean Theorem

Pythagoras was a Greek philosopher and mathematician who lived around 570 B.C. Even with the relatively primitive tools at his disposal, he was able to discover and formulate a *theorem,* or rule, that became one of the most well known in all of mathematics: the Pythagorean Theorem.

The *Pythagorean Theorem* says that if *a*, *b*, and *c* are the sides of a right triangle, as shown in Figure 5-3, and if *c* is the side opposite the right angle, then their lengths have the following property: $a^2 + b^2 = c^2$.
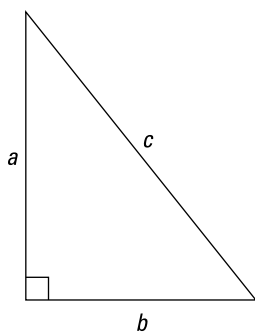
FIGURE 5-3:
Traditional labels for the sides of a right triangle.

Hitting a Pythagorean triple

A *Pythagorean triple* is a list of three numbers that works in the Pythagorean Theorem — the square of the largest number is equal to the sum of the squares of the two smaller numbers. The multiple of any Pythagorean triple (multiply each of the numbers in the triple by the same number) is also a Pythagorean triple. They seem to reinvent themselves.

Familiarizing yourself with the more frequently used Pythagorean triples is very helpful. If you recognize that you have a triple, then working with applications is much easier.

Table 5-1 shows some of the most common Pythagorean triples and some of their multiples.

TABLE 5-1

Common Pythagorean Triples

Triple	Triple × 2	Triple × 3	Triple × 4
3-4-5	6-8-10	9-12-15	12-16-20
5-12-13	10-24-26	15-36-39	20-48-52
7-24-25	14-48-50	21-72-75	28-96-100
9-40-41	18-80-82	27-120-123	36-160-164
11-60-61	22-120-122	33-180-183	44-240-244

Here's how to check out a triple and its multiple by using the Pythagorean Theorem. Try out the triple 9-40-41:

1. **Replace *a*, *b*, and *c* with 9, 40, and 41, respectively.**

$$9^2 + 40^2 = 41^2$$
$$81 + 1{,}600 = 1{,}681$$

2. **Then replace *a*, *b*, and *c* with the 9-40-41 triple multiplied by 3 (which is 27-120-123).**

$$27^2 + 120^2 = 123^2$$
$$729 + 14{,}400 = 15{,}129$$

Solving for a missing length

One of the nice qualities of right triangles is the fact that you can find the length of one side if you know the lengths of the other two sides. You don't have this luxury with just any triangle, so count your blessings now.

Practicing on triangles

The Pythagorean Theorem states that $a^2 + b^2 = c^2$ in a right triangle where *c* is the longest side. You can use this equation to figure out the length of one side if you have the lengths of the other two. Figure 5-4 shows two right triangles that are each missing one side's measure.

In the left triangle in Figure 5-4, the measure of the hypotenuse is missing. Use the Pythagorean Theorem to solve for the missing length.

1. **Replace the variables in the theorem with the values of the known sides.**

$$48^2 + 14^2 = c^2$$

FIGURE 5-4:
Solving for the
missing sides.
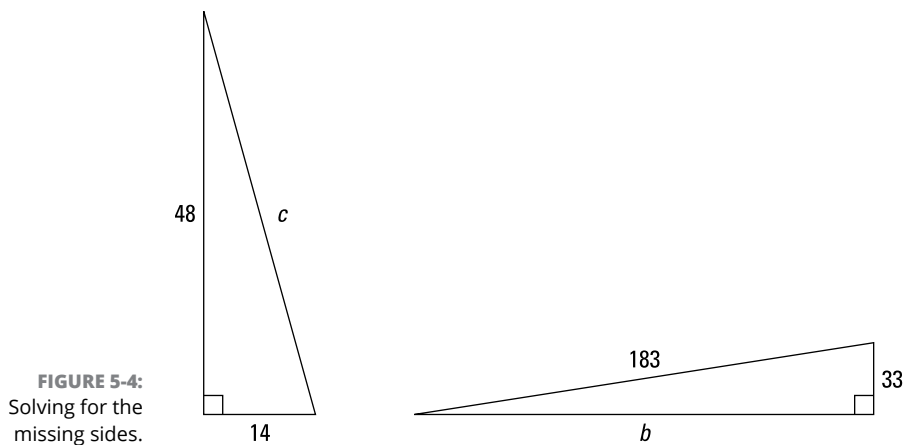
2. **Square the measures and add them together.**

$$2,304 + 196 = c^2$$
$$2,500 = c^2$$
$$\sqrt{2,500} = \sqrt{c^2}$$
$$50 = c$$

The length of the missing side, c, which is the hypotenuse, is 50.

The triangle on the right in Figure 5-4 is missing the bottom length, but you do have the length of the hypotenuse. It doesn't matter whether you call the missing length a or b.

1. **Replace the variables in the theorem with the values of the known sides.**

$$33^2 + b^2 = 183^2$$

2. **Square the measures, and subtract 1,089 from each side.**

$$1,089 + b^2 = 33,489$$
$$b^2 = 32,400$$

3. **Find the square root of each side.**

$$\sqrt{b^2} = \sqrt{32,400}$$
$$b = 180$$

The length of the missing side is 180 units. That's not much shorter than the hypotenuse, but it still shows that the hypotenuse has the longest measure.

Some of the more popular Pythagorean triples keep cropping up in discussions of the Pythagorean Theorem, and you may be asking the question: "Is that all he wrote?" The wonderful answer to that question is: "Absolutely not." The triples go on forever and ever. And a very nice way of creating your own set of triples is to follow these steps:

1. **Choose two odd numbers, *x* and *y*, where *x* is bigger than *y*.**

2. **Multiply the two numbers together to get the first number in your triple.**

3. **Square the two numbers, subtract the smaller square from the larger, then divide the difference by 2. That's your second number.**

4. **Square the two numbers, add the two squares together, then divide the sum by 2. That's your third number.**

For example, if you choose the odd numbers 9 and 5, your first number in the triple is $9 \cdot 5 = 45$. Your second number is $\frac{81-25}{2} = \frac{56}{2} = 28$. And your last number in the triple is $\frac{81+25}{2} = \frac{106}{2} = 53$. Need to check? Is $45^2 + 28^2 = 53^2$?

Last time I looked, $2{,}025 + 784 = 2{,}809$. Go ahead. Use your calculator. And, because you can choose from an infinite number of odd numbers, you can create an infinite number of "nice" triples.

Finding the distance across a pond

Trigonometry is very handy for finding distances that you can't reach to actually measure. Imagine that you want to string a cable diagonally across a pond (so you can attach a bunch of fishing line and hooks). The diagonal distance is the hypotenuse of a right triangle. You can measure the other two sides along the shore. Figure 5-5 shows the pond and the imaginary right triangle you use to figure out how long your cable needs to be.
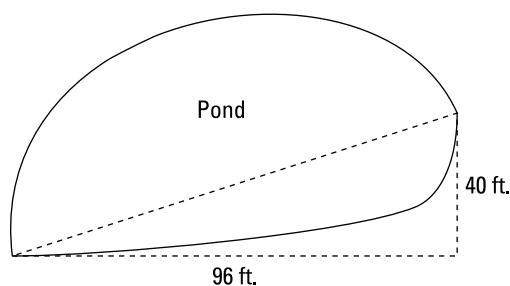
FIGURE 5-5:
Finding the distance across a pond.

Pond

40 ft.

96 ft.

The two sides of the triangle that you can measure, the length and the width of the pond, are 40 feet and 96 feet. These are the two legs of a right triangle. Use the Pythagorean Theorem to solve for the hypotenuse, which is the diagonal distance across the pond.

1. **Replace the variables in the theorem with the values of the known sides.**

   $40^2 + 96^2 = c^2$

2. **Square the measures, and add them together.**

   $1,600 + 9,216 = c^2$

   $10,816 = c^2$

3. **Find the square root of the sum.**

   $\sqrt{10,816} = \sqrt{c^2}$

   $104 = c$

The diagonal across the pond is 104 feet. String up your cable, and go fishing!

In a League of Their Own: Special Right Triangles

Right triangles are handy little structures. The relationship between the lengths of the sides helps you measure lengths that you can't reach. And just when you think that math can't get any better, along come two triangles that are the cat's meow (see Figure 5-6). One of them is an isosceles right triangle. The two legs are the same, and the hypotenuse is always a multiple of the length of a leg. The other special right triangle, called 30-60-90, has one side half as long as the other. These two triangles are very useful, because the angle measures in them are some of the most popular, and the side measures are used in trig functions.

30-60-90 right triangles

A 30-60-90 right triangle has angles measuring just what the name says. The two acute, complementary angles are 30 and 60 degrees. These triangles are great to work with because the angle measures, all being multiples of 30, have a pattern, and so do the measures of the sides. Oh, yes, the Pythagorean Theorem still holds — you have that relationship between the squares of the sides. But a, b, and c are related in another way, too. In a 30-60-90 right triangle, if a is the shortest side, then the hypotenuse, the longest side, measures twice that, or $2a$. You can

use $2a$ instead of the letter c. And the middle length is $\sqrt{3}a$, or about 1.7 times as long as the shortest side; this number replaces the letter b. The particularly nice part about this triangle is that you can write all three sides in terms of one variable, a. Look at how these lengths fit into the Pythagorean Theorem:

$$a^2 + \left(\sqrt{3}a\right)^2 = (2a)^2$$
$$a^2 + 3a^2 = 4a^2$$
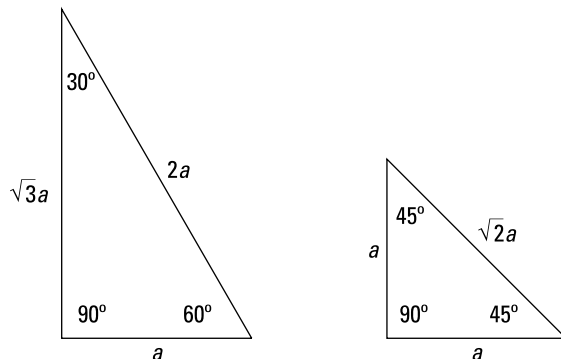
Here's a sample problem you can solve by taking advantage of the special relationships within a 30-60-90 right triangle: If the hypotenuse of a 30-60-90 right triangle is 8 units long, then how long are the other two sides?

1. **Find the length of the shorter leg.**

   The hypotenuse is twice as long as the shorter leg, a. So 8 = 2a. Divide by 2, and you get $a = 4$.

2. **Find the length of the longer leg.**

   The longer leg is $\sqrt{3}a$, so multiply $\sqrt{3}$ times 4 to get $4\sqrt{3}$, or about 6.9 units.

Isosceles right triangles

The other special right triangle is the *isosceles right triangle,* or the 45-45-90 right triangle. The two acute angles are equal, making the two legs opposite them equal, too. What's more, the lengths of those two legs have a special relationship with the hypotenuse (in addition to the one in the Pythagorean Theorem, of course). In an isosceles right triangle, if the legs are each a units in length, then the hypotenuse is $\sqrt{2}a$, or about 1.4 times as long as a leg.

Now that you know how isosceles right triangles work, try your hand at this sample problem: If an isosceles right triangle has a hypotenuse that's 16 units long, then how long are the legs?

1. **Create an equation to solve.**

The hypotenuse is $\sqrt{2}a$, where a is the length of the legs. You know that the hypotenuse is 16, so you can solve the equation $\sqrt{2}a = 16$ for the length of a.

2. **Solve for a.**

Divide each side by the radical to get

$$\frac{\sqrt{2}a}{\sqrt{2}} = \frac{16}{\sqrt{2}}$$
$$a = \frac{16}{\sqrt{2}} = \frac{16}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{16\sqrt{2}}{2} = 8\sqrt{2}$$

Each leg is about 11.3 units.

Getting the Applications Right

Using right triangles to solve everyday problems is something that you probably don't do all that frequently. But here are some examples to encourage your use of these wonderful structures.

How tall is your house?

You and a friend have a contest going. You say that your house is taller than your friend's. And, of course, there's a disagreement. The winner will be treated to an ice cream sundae!

How do you measure the height of your houses if you don't have a ladder or any measuring device that will go high enough? Here's how. Get out your handy 30-60-90 triangle. Lie on the ground far enough from your house so you can sight along the hypotenuse to the top of the house. Measure the distance from your triangle to the house and do the math! Look at Figure 5-7a.

Your 30-60-90 triangle has a measure of 6 inches, or 0.5 feet, along the side opposite the 30-degree angle. This means that the longer leg, lying along the ground, measures $0.5\sqrt{3}$ feet. You find the distance between the triangle and the base of the house directly below the peak of the roof; it's 77 feet to the house.

FIGURE 5-7:
Measuring
the height
of the house.

(a)   (b)
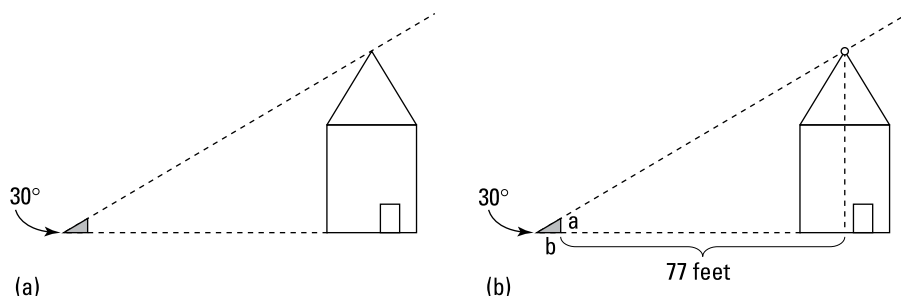
Now to compute the height of the house. Using Figure 5-7b and the proportion:

$$\frac{\text{triangle side } a}{\text{triangle side } b} = \frac{\text{height of house}}{77 \text{ feet + triangle side } b}$$

Fill in the numbers and solve for the height of the house. Multiplying each side of the equation by the denominator under the h (for height), you get a wonderful problem to stick into your calculator!

$$\frac{0.5 \text{ feet}}{0.5\sqrt{3} \text{ feet}} = \frac{\text{height of house}}{77 \text{ feet} + 0.5\sqrt{3} \text{ feet}}$$

$$\frac{0.5}{0.5\sqrt{3}} = \frac{h}{77 + 0.5\sqrt{3}}$$

$$\frac{\cancel{0.5}}{\cancel{0.5}\sqrt{3}} = \frac{h}{77 + 0.5\sqrt{3}}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{77 + 0.5\sqrt{3}}$$

$$\frac{77 + 0.5\sqrt{3}}{\sqrt{3}} = h$$

Solving for h, you get $h \approx 44.96$, or your house is about 45 feet tall. Now, on to your friend's house!

Beachfront measure

You are considering purchasing a vacation home that sits right on the shore of a lake. The cost of the property depends a lot on how much beachfront you have. The realtor isn't sure about that length yet, and you want to determine, right now, if this is going to be in your price range.

How do you measure this beachfront length if you can't get onto the beach right now? It's pretty wavy and you only have your good shoes on. But you do have tape measures and you can create a right triangle using the carpenter's square in your trunk. Take a look at Figure 5-8.

FIGURE 5-8:
Measuring the
length of the
beachfront
property.

67 feet

163 feet
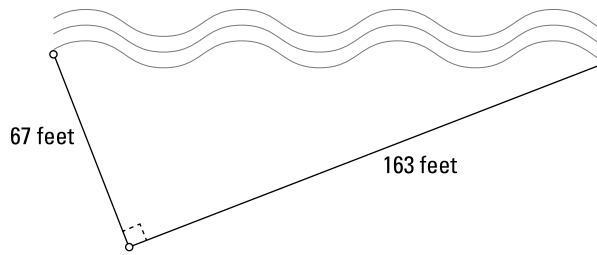
You are able to determine the two endpoints of the property along the beach (and not get soaked). Then, with your carpenter's square, you create a right triangle with its hypotenuse along the beachfront. The lengths of the two legs of the right triangle are 67 feet and 163 feet. How long is the beachfront?

Using the Pythagorean Theorem,

$$67^2 + 163^2 = c^2$$
$$4,489 + 26,569 = c^2$$
$$31,058 = c^2$$
$$\sqrt{31,058} = c$$
$$176.23 \approx c$$

176 feet of beachfront is a lot of beachfront. Have fun!

2

Trigonometric Functions

Define the basic trig functions using the lengths of the sides of a right triangle.

Determine the relationships between the trig cofunctions and their shared sides.

Extend your scope to angles greater than 90 degrees using the unit circle.

Investigate the ins and outs of the domains and ranges of the six trig functions.

Use reference angles to compute trig functions.

Apply trig functions to real-world problems.

Chapter **6**

Describing Trig Functions

B y taking the lengths of the sides of right triangles or the chords of circles and creating ratios with those numbers and variables, our ancestors initiated the birth of trigonometric functions. These functions are of infinite value, because they allow you to use the stars to navigate and to build bridges that won't fall. If you're not into navigating a boat or engineering, then you can use the trig functions at home to plan that new addition. And they're a staple for students going into calculus.

You may be asking, "What is a function? What does it have to do with trigonometry?" In mathematics, a function is a mechanism that takes a value you input into it and churns out an answer, called the output. A function is connected to rules involving mathematical operations or processes.

The six trig functions require one thing of you — inputting an *angle* measure — and then they output a number. These outputs are always real numbers, from infinitely small to infinitely large and everything in between. The results you get depend on which function you use. Although in earlier times, some of the function computations were rather tedious, today's hand-held calculators, and even phones, make everything much easier.

Discovering How Trig Functions Work

The six trig functions are named sine, cosine, tangent, cosecant, secant, and cotangent. You more frequently see the names abbreviated to save time and space: sin, cos, tan, csc, sec, cot. Each function has a different working mechanism, but they all take the measure of an angle and output an answer.

A trig function is a rule that takes in the measure of an angle and puts out a real number. These output numbers come from the relationships between the measures of the sides of triangles. Each of the basic trig functions use those measures in different ways.

The name game: A right triangle's three sides

A right triangle has two shorter sides, or *legs*, and the longest side, opposite the right angle, which is always called the *hypotenuse.* The two shorter sides have some other special designations, too, based on which acute angle of the triangle you happen to be working with at a particular time.

In reference to acute angle θ (see Figure 6-1), the leg on the other side of the triangle from θ is called the *opposite side.* That opposite side is never along one of the rays making up the angle. The other leg in the right triangle is then called the *adjacent side. Adjacent* means "next to," and in the case of right triangles, the adjacent side helps form the acute angle along with the hypotenuse because it lies along one of the angle's rays.
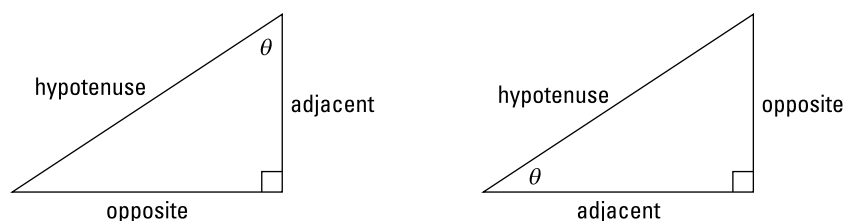
The six ratios: Relating the three sides

Each of the three sides of a right triangle — hypotenuse, opposite, and adjacent — has a respective length or measure. And those three lengths or measures form six different ratios. Each length is used four times — all getting equal credit — in the six functions. Check out Figure 6-2, which has sides of lengths 3, 4, and 5.

Hipparchus was a Greek astronomer and mathematician who lived from about 190 to 120 B.C. Also known as Hipparchus of Nicaea, he did his celestial observations from Rhodes between 146 and 127 B.C. He was the first astronomer to compile a catalog of 850 stars, well before telescopes were available, and his computations were remarkably accurate. He obtained measurements of the length of the year and the distance to the Moon. To measure the distance to the Moon, Hipparchus and a colleague each observed a solar eclipse — a total eclipse at Syene and a partial eclipse at Alexandria, where four-fifths of the Sun was blocked. Using angles and the distance between the two cities, thus creating an imaginary triangle with the lines of sight from those two cities to the Moon, he was able to make his calculations. The trig ratios he used were on a big scale, but it doesn't matter how large the triangle is, because the trig functions for the angles don't change with the triangle's size. Hipparchus is commemorated by having a Moon crater, a Mars crater, and an asteroid named after him.

FIGURE 6-2:
A right triangle
with sides
of lengths 3, 4,
and 5.
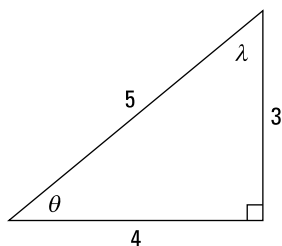
The six different ratios that you can form with the numbers 3, 4, and 5 are $\frac{3}{4}, \frac{4}{5}, \frac{3}{5}, \frac{4}{3}, \frac{5}{4},$ and $\frac{5}{3}$. These six fractions are the only ones you can make by using the three lengths of the sides. The ratios are special because they represent all the possible output values of the trig functions for the acute angles in that triangle. And even better, you can figure out the value of an unknown angle in a right triangle just by creating one of these ratios and figuring out which angle has that trig function.

The sine function: Opposite over hypotenuse

When you're using right triangles to define trig functions, the trig function *sine,* abbreviated *sin,* has input values that are angle measures and output values that you obtain from the ratio, $\frac{\text{opposite}}{\text{hypotenuse}}$. Figure 6-2 (in the preceding section)

shows two different acute angles, and each has a different value for the function sine. The two values are $\sin\theta = \frac{3}{5}$ and $\sin\lambda = \frac{4}{5}$.

The sine is always the measure of the opposite side divided by the measure of the hypotenuse. Because the hypotenuse is always the longest side, the number on the bottom of the ratio will always be larger than that on the top. For this reason, the output of the sine function will always be a proper fraction — it'll never be a number equal to or greater than 1 unless the opposite side is equal in length to the hypotenuse (which only happens when your triangle is a single segment or you're working with circles — see Chapter 7).

Even if you don't know both lengths required for the sine function, you can calculate the sine if you know any two of the three lengths of a triangle's sides. For example, to find the sine of angle α in a right triangle whose hypotenuse is 10 inches long and adjacent side is 8 inches long, follow these steps:

1. **Find the length of the side opposite α.**

Use the Pythagorean Theorem, $a^2 + b^2 = c^2$ (see Chapter 5), letting a be 8 and c be 10. When you input the numbers and solve for b, you get

$$8^2 + b^2 = 10^2$$
$$64 + b^2 = 100$$
$$b^2 = 36$$
$$b = 6$$

So, the opposite side is 6 inches long.

2. **Use the ratio for sine, opposite over hypotenuse.**

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{6}{10} = \frac{3}{5}$$

The cosine function: Adjacent over hypotenuse

The trig function *cosine*, abbreviated *cos*, works by forming this ratio: $\frac{\text{adjacent}}{\text{hypotenuse}}$. Take another look at Figure 6-2, and you see that the cosines of the two angles are $\cos\theta = \frac{4}{5}$ and $\cos\lambda = \frac{3}{5}$. The situation with the ratios is the same as with the sine function — the values are going to be less than or equal to 1 (the latter only when your triangle is a single segment or when dealing with circles), never greater than 1, because the hypotenuse is the denominator.

TIP

The two ratios for the sine and cosine of an acute angle in a right triangle are the same as those for the cosine and sine of the other acute angle in the triangle; the function values are just reversed. This property is true of the sines and cosines of *complementary angles* in any right triangle (meaning those angles that add up to 90 degrees).

TRIG RULES

If θ and λ are the two acute angles of a right triangle, then $\sin \theta = \cos \lambda$ and $\cos \theta = \sin \lambda$.

Now for an example. To find the cosine of angle β in a right triangle if the two legs are each $\sqrt{6}$ feet in length, follow these steps:

1. **Find the length of the hypotenuse.**

   Using the Pythagorean Theorem, $a^2 + b^2 = c^2$ (see Chapter 5), and replacing both a and b with the given measure, solve for c.

   $$\left(\sqrt{6}\right)^2 + \left(\sqrt{6}\right)^2 = c^2$$
   $$6 + 6 = c^2$$
   $$12 = c^2$$
   $$\sqrt{12} = c$$
   $$2\sqrt{3} = c$$

   The hypotenuse is $2\sqrt{3}$ feet long.

2. **Use the ratio for cosine, adjacent over hypotenuse, to find the answer.**

   $$\cos \beta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{6}}{2\sqrt{3}} = \frac{\sqrt{2}\sqrt{3}}{2\sqrt{3}} = \frac{\sqrt{2}}{2}$$

The tangent function: Opposite over adjacent

A third trig function, *tangent,* is abbreviated *tan.* This function uses just the measures of the two legs and doesn't use the hypotenuse at all. The tangent is described with this ratio: $\frac{\text{opposite}}{\text{adjacent}}$. No restriction or rule on the respective sizes of these sides exists — the opposite side can be larger, or the adjacent side can be larger. So, the tangent ratio produces numbers that are very large, very small, and everything in between. If you hike on back to Figure 6-2, you see that the tangents are $\tan \theta = \frac{3}{4}$ and $\tan \lambda = \frac{4}{3}$. And in case you're wondering whether the two tangents of the acute angles are always *reciprocals* (flips) of one another, the answer is yes. The trig identities in Chapter 10 explain this phenomenon.

The following example shows you how to find the values of the tangent for each of the acute angles in a right triangle where the hypotenuse is 25 inches and one leg is 7 inches.

1. **Find the measure of the missing leg.**

   Using the Pythagorean Theorem, $a^2 + b^2 = c^2$ (see Chapter 5), putting in 7 for a and 25 for c, and solving for the missing value, b, you find that the unknown length is 24 inches:

   $$7^2 + b^2 = 25^2$$
   $$b^2 = 25^2 - 7^2$$
   $$b^2 = 625 - 49$$
   $$b^2 = 576$$
   $$b = 24$$

2. **Select names for the acute angles in order to determine the *opposite* and *adjacent* designations.**

   The easiest way to do this is to draw a picture and label it — take a look at Figure 6-3.

   The two acute angles are named with the Greek letters θ and λ. The side opposite θ measures 7 inches, and the side adjacent to it measures 24 inches. For angle λ, the opposite side measures 24 inches, and the adjacent side measures 7 inches.

FIGURE 6-3: Labeling a right triangle and naming the acute angles.
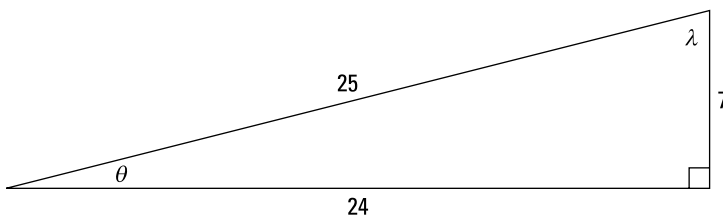
3. **Form the two tangent ratios by using the values 7, 24, and 25.**

   $$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{7}{24}$$
   $$\tan\lambda = \frac{\text{opposite}}{\text{adjacent}} = \frac{24}{7}$$

All together, now: Using one function to solve for another

Sometimes you have to solve for a trig function in terms of another function.

In the following example, the cosine of angle λ is $\frac{12}{13}$. What are the values of the sine and tangent of λ?

1. **Identify the sides given by the cosine function.**

The cosine ratio is $\frac{\text{adjacent}}{\text{hypotenuse}}$. Using the given ratio, the adjacent side measures 12 units, and the hypotenuse measures 13 units.

2. **Find the measure of the missing side.**

Using the Pythagorean Theorem, you find that the missing side (the opposite side) measures 5 units.

$$12^2 + b^2 = 13^3$$
$$b^2 = 13^2 - 12^2$$
$$b^2 = 169 - 144$$
$$b^2 = 25$$
$$b = 5$$

3. **Determine the values of the sine and tangent.**

The sine is $\frac{\text{opposite}}{\text{hypotenuse}}$ and the tangent is $\frac{\text{opposite}}{\text{adjacent}}$, so $\sin \lambda = \frac{5}{13}$ and $\tan \lambda = \frac{5}{12}$.

Similar right triangles within a right triangle

Take a right triangle and draw an altitude to the hypotenuse. What do you get? You are the proud recipient of three similar triangles — three triangles in decreasing sizes, all with the same three angle measures. Take a look at Figure 6-4a for an example of this situation. The acute angles are labeled α and β to help you find the similar features.

The angles α and β are complementary and appear in each of the three triangles. Triangle *ABC* is similar to triangle *CBG*, and both are similar to triangle *ACG.* I was careful to give these names with the first letter at the vertex with angle α, the second letter at the vertex with angle β, and the last letter at the right angle.

FIGURE 6-4:
A right triangle
with similar
right triangles.

(a)   (b)
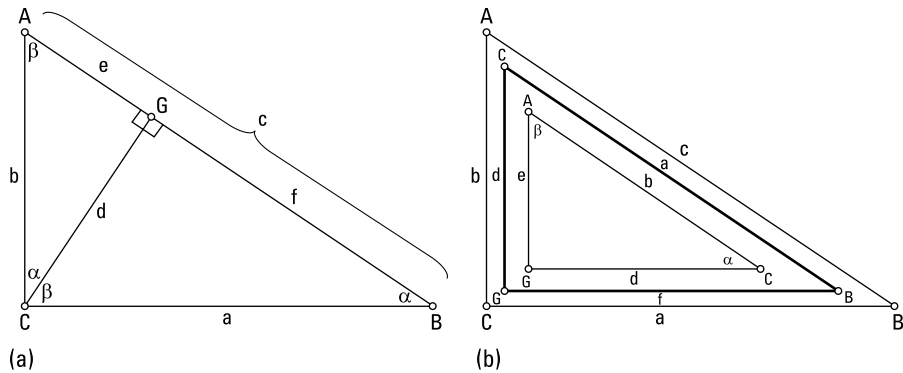
Carefully naming the triangles allows you to write the trig functions of the three different triangles in equations that compare their values. Refer to Figure 6-4b. For example, because $\triangle ABC \sim \triangle CBG \sim \triangle ACG$, you can write equations involving ratios of the corresponding sides in the triangles. If you write the ratios of the longer legs divided by the shorter legs, you have:

$$\frac{a}{b} = \frac{f}{d} = \frac{d}{e}$$

Now look at the tangents of the angle β in each triangle. In $\triangle ABC$, $\tan\beta = \frac{a}{b}$. In $\triangle CBG$, $\tan\beta = \frac{f}{d}$. In $\triangle ACG$, $\tan\beta = \frac{d}{e}$. These and other relationships allow for several equations such as the *means* and *extremes* rules.

» $d^2 = e \cdot f$: The square of the altitude is equal to the product of the two parts of the hypotenuse.

» $b^2 = e \cdot c$: The square of a leg is equal to the product of the part of the hypotenuse adjacent to the leg and the hypotenuse.

Taking It a Step Further: Reciprocal Functions

The three most basic trig functions (sine, cosine, and tangent) use the three sides of a triangle, taking two at a time and making ratios/fractions of them. But three more trig functions exist, and these are called the *reciprocal functions* because they use the *reciprocals*, or flips, of the original three functions. If these last three functions are just reciprocals of the first three, why are they even necessary? You could

probably live without them, but you'd eventually miss them when doing calculations and solving equations. Having an unknown variable in the numerator of a fraction when solving an equation is just nicer and more convenient than having one in the denominator, and these reciprocal functions make that situation possible. It's nothing more than convenience, but I'm all for such luxury.

The cosecant function: Sine flipped upside down

The *cosecant* function, abbreviated *csc,* is the reciprocal of the sine function and thus uses this ratio: $\frac{\text{hypotenuse}}{\text{opposite}}$. The hypotenuse of a right triangle is always the longest side, so the numerator of this fraction is always larger than the denominator. As a result, the cosecant function always produces values bigger than 1.

You can use the values in Figure 6-5 to determine the cosecants of the two acute angles: $\csc\theta = \frac{13}{12}$ and $\csc\lambda = \frac{13}{5}$.
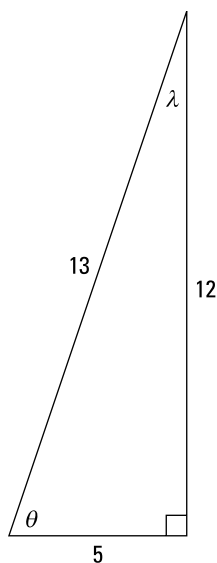
Suppose someone asks you to find the cosecant of angle α if you know that the hypotenuse is 1 unit long and that the right triangle is isosceles. Remember that an isosceles triangle has two congruent sides (flip back to Chapter 5 for a refresher).

These two sides have to be the two legs, because the hypotenuse has to have the longest side. So, to find the cosecant:

1. **Find the lengths of the two legs.**

   The Pythagorean Theorem says that $a^2 + b^2 = c^2$, but because two sides are congruent, you can take out one variable and write the equation as $a^2 + a^2 = c^2$. Put in 1 for c and solve for a.

   $$2a^2 = 1$$
   $$a^2 = \frac{1}{2}$$
   $$a = \sqrt{\frac{1}{2}}$$
   $$a = \frac{1}{\sqrt{2}}$$

   TIP

   The legs are both $\frac{1}{\sqrt{2}}$ units long. You can leave the radical in the denominator and not worry about rationalizing, because you're going to input the whole thing into the cosecant ratio, anyway, and things can change.

2. **Use the length of the opposite side in the ratio for cosecant.**

   $$\csc \alpha = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

The secant function: Cosine on its head

The *secant* function, abbreviated *sec,* is the reciprocal of the cosine. So, its ratio is $\frac{\text{hypotenuse}}{\text{adjacent}}$. Just as with the cosecant, the ratio of the sides is greater than 1. Using the triangle in Figure 6-5, the two secants are $\sec \theta = \frac{13}{5}$ and $\sec \lambda = \frac{13}{12}$.

The cotangent function: Tangent, tails side up

The last reciprocal function is the *cotangent,* abbreviated *cot.* This function is the reciprocal of the tangent (hence, the *co-*). The ratio of the sides for the cotangent is $\frac{\text{adjacent}}{\text{opposite}}$. So, if you look back at Figure 6-5, you see that the two cotangents are $\cot \theta = \frac{5}{12}$ and $\cot \lambda = \frac{12}{5}$.

TIP

The ratio for the cotangent is just that ratio, not necessarily the lengths of the sides. You could reduce the fraction made by the lengths by dividing the numerator and denominator by the same number.

Sometimes you know the value of the cotangent along with other information and have to solve for one or both of the sides. Try this example: What are the lengths of the legs of a right triangle if $\cot\alpha = \dfrac{11}{60}$ and the hypotenuse is 183 inches long?

1. **Write the adjacent and opposite sides as multiples of the same number, *m*, and put them in the Pythagorean Theorem with the hypotenuse.**

$$(11m)^2 + (60m)^2 = 183^2$$

2. **Simplify the equation and solve for *m*.**

$$(11m)^2 + (60m)^2 = 183^2$$
$$121m^2 + 3{,}600m^2 = 33{,}489$$
$$3{,}721m^2 = 33{,}489$$
$$m^2 = \frac{33{,}489}{3721}$$
$$m^2 = 9$$
$$m = 3$$

3. **Use the value of *m* to find the lengths of the two legs.**

Because you know that $m = 3$, you know that the adjacent side is $11m = 11(3) = 33$, and the opposite side is $60m = 60(3) = 180$. The three sides of the right triangle are 33, 180, and 183. You can double-check your results by plugging these three numbers into the Pythagorean Theorem and making sure the theorem holds true.

Angling In on Your Favorites

You may have a favorite television show, dessert, or color. Usually, however, a favorite *angle* isn't near the top of anyone's list. But a favorite angle isn't really out of line in the scheme of things. My favorite angle is a 30-degree angle — there's just something so acute about it.

Identifying the most popular angles

The most common or popular angles are those with measures that are multiples of 15 degrees. Topping the list are 30, 60, and 90 degrees. Another favorite is 45 degrees. The reason that these angles are favorites is because they all divide

360 degrees evenly. These exact divisions result in nicer-than-usual values for the different trig functions of the angles.

One way to capitalize even more on the four main angles — 30, 45, 60, and 90 — is to look at their multiples that go up to 360 degrees. The trig functions of the first four basic angles and the trig functions of their multiples are related (see Chapter 8). The list of all-time favorites includes multiples of 30 degrees (30, 60, 90, 120, 150, 180, 210, 240, 270, 300, and 330) and some multiples of 15 degrees that are between them: 45, 135, 225, and 315. All these multiples split the four quadrants the first time around. A 0-degree angle is also highly favored. A measure of 0 degrees is technically a multiple of any of these measures, and you need it because it's the starting point.

Determining the exact values of functions

Even though a scientific calculator gives you the values of the trig functions of any angle, not just your favorite angles, the values it shows you for most of those angles are just estimates. For example, the exact value of the sine of 60 degrees is $\frac{\sqrt{3}}{2}$. However, because radicals of numbers that aren't perfect squares are irrational and have an endless decimal value, a calculator carries that value out to a certain number of decimal places and then rounds it off. In this case, $\frac{\sqrt{3}}{2} \approx 0.8660254038$. This decimal has many more places than you usually need — normally, three or four decimal places is enough.

In trig, you frequently use the exact values of the most favorite angles because they give better results in computations and applications, so memorizing those exact values is a good idea.

The process for constructing a table of trig function values, which I explain in this section, is easy to remember, so you can create one quickly when you need to — either on paper or in your head.

A quick table for the three basic trig functions

The angles used most often in trig have trig functions with convenient exact values. Other angles don't cooperate anywhere near as nicely as these popular ones do.

A quick, easy way to memorize the exact trig-function values of the most common angles is to construct a table, starting with the sine function and working with a pattern of fractions and radicals. Create a table with the top row listing the angles, as shown in Figure 6-6. The first function, in the next row, is sine.

FIGURE 6-6:
Constructing a
table of exact
values.

$\theta°$	0°	30°	45°	60°	90°
$\sin\theta$					

The entries following $\sin\theta$ in the second row are the fractions and radicals with the following pattern:

» Each fraction has a denominator of 2.

» The numerators of the fractions are radicals with 0, 1, 2, 3, and 4 under them, in that order, as shown in Figure 6-7.

FIGURE 6-7:
Creating entries
for the second
row of the table
of exact values.

$\theta°$	0°	30°	45°	60°	90°
$\sin\theta$	$\dfrac{\sqrt{0}}{2}$	$\dfrac{\sqrt{1}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{4}}{2}$

Next, simplify the fractions that can be simplified so the table becomes what you see in Figure 6-8:

$$\frac{\sqrt{0}}{2}=0, \frac{\sqrt{1}}{2}=\frac{1}{2}, \frac{\sqrt{2}}{2}=\frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}=1$$

FIGURE 6-8:
The first row of
the table with
simplified values.

$\theta°$	0°	30°	45°	60°	90°
$\sin\theta$	0	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	1

The next row, for the cosine, is just the sine's row in reverse order, as Figure 6-9 shows. This happens because you have the angles and their complements in reverse order, too.

$\theta°$	0°	30°	45°	60°	90°
$\sin\theta$	0	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{1}{2}$	0

FIGURE 6-9:
Adding the
cosine row.

The next row is for the tangent. In a right triangle, you find the tangent of an acute angle with the ratio $\dfrac{\text{opposite}}{\text{adjacent}}$ (refer to the section, "The tangent function: Opposite over adjacent," earlier in this chapter). You get the same ratio when you divide sine by cosine. Here's how it works:

$$\frac{\text{sine}}{\text{cosine}} = \frac{\dfrac{\text{opposite}}{\text{hypotenuse}}}{\dfrac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\text{opposite}}{\text{hypotenuse}} \cdot \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{\text{opposite}}{\text{adjacent}}$$

Because you already know the values for sine and cosine, you can use this property (tangent equals sine divided by cosine) to get the tangent values for the table:

» For the tangent of 0 degrees, $\dfrac{0}{1} = 0$.

» The tangent of 30 degrees is $\dfrac{\dfrac{1}{2}}{\dfrac{\sqrt{3}}{2}} = \dfrac{1}{2} \cdot \dfrac{2}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$.

» The tangent of 45 degrees is $\dfrac{\dfrac{\sqrt{2}}{2}}{\dfrac{\sqrt{2}}{2}} = 1$.

» The tangent of 60 degrees is $\dfrac{\dfrac{\sqrt{3}}{2}}{\dfrac{1}{2}} = \dfrac{\sqrt{3}}{2} \cdot \dfrac{2}{1} = \dfrac{\sqrt{3}}{1} = \sqrt{3}$.

» The tangent of 90 degrees is $\dfrac{1}{0}$, which is undefined. So, the tangent of 90 degrees doesn't have a value — it simply doesn't exist.

See Figure 6-10 for the completed table with the tangent row.

$\theta°$	0°	30°	45°	60°	90°
$\sin\theta$	0	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{1}{2}$	0
$\tan\theta$	0	$\dfrac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

FIGURE 6-10:
The tangent row comes next.

A quick table for the three reciprocal trig functions

If you read the section, "Taking It a Step Further: Reciprocal Functions," earlier in this chapter, you know that the reciprocal functions have values that are *reciprocals,* or flips, of the values for their respective functions. The reciprocal of sine is cosecant, so each function value for cosecant is the reciprocal of sine's. The same goes for the other two reciprocal functions. The table in Figure 6-11 shows the reciprocal in each case, in their simplified forms. Whenever you see *undefined,* it's because the original function value was 0, and the reciprocal of 0 has no value.

$\theta°$	0°	30°	45°	60°	90°
csc θ	undefined	2	$\sqrt{2}$	$\dfrac{2\sqrt{3}}{3}$	1
sec θ	1	$\dfrac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undefined
cot θ	undefined	$\sqrt{3}$	1	$\dfrac{\sqrt{3}}{3}$	0

FIGURE 6-11: The reciprocal functions.

Refer to Chapter 7 for the trig values of angles measuring more than 90 degrees. These values are developed using the chart created here.

Building a Shorter Route

You live in the town of Appleton, which is at the corner of two perpendicular roads. North of you by 21 miles is Berryville, and east of you by 72 miles is Cherry Creek. And, wonder of wonders, the distance from Berryville to Cherry Creek is 75 miles, forming a right triangle! Take a look at Figure 6-12.
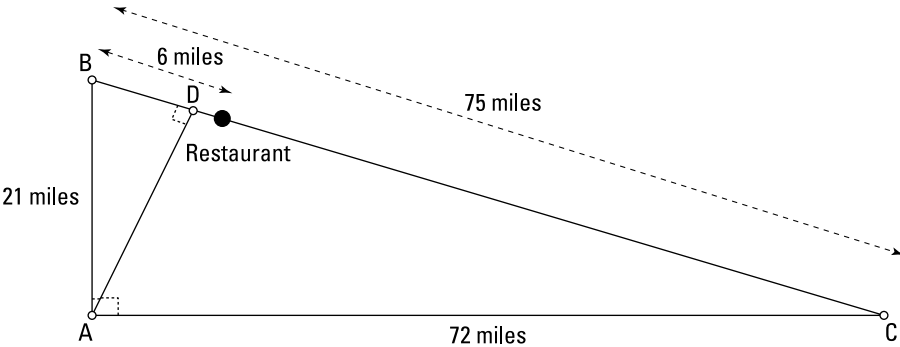
FIGURE 6-12: Driving to the restaurant.

Your favorite restaurant is located 6 miles east of Berryville, along that main road. So you drive 21 miles north plus 6 miles, equaling 27 miles, to have their wonderful food. But there are now plans to build a new road from Appleton to the road between Berryville and Cherry Creek. It will be the shortest distance possible, which, as you already know, will be a road perpendicular to that hypotenuse. Once the road is built, how many miles will you have to travel to get to that restaurant?

By drawing that perpendicular to the hypotenuse, you find that three similar triangles are created, so you can write proportions using the corresponding sides.

First, find the distance from Appleton to the hypotenuse. Use the sides of the smallest triangle and the largest triangle formed by the current roads.

$$\frac{\text{longer leg}}{\text{hypotenuse}} \rightarrow \frac{AD}{AB} = \frac{AC}{BC} \rightarrow \frac{AD}{21} = \frac{72}{75}$$

Simplifying and solving for AD:

$$\frac{AD}{21} = \frac{72}{75} \rightarrow \frac{AD}{21} = \frac{\cancel{72}^{24}}{\cancel{75}^{25}} \rightarrow \frac{AD}{21} \cdot 21 = \frac{24}{25} \cdot 21 \rightarrow \frac{AD}{\cancel{21}} \cdot \cancel{21} = \frac{24}{25} \cdot 21 \rightarrow AD = 20.16$$

The perpendicular road is 20.16 miles long. Now to find the distance from Berryville to the new intersection — the length of road you don't need to travel anymore. Using the sides of the smallest triangle and the largest triangle formed by the current roads:

$$\frac{\text{shortest side}}{\text{hypotenuse}} \rightarrow \frac{BD}{AB} = \frac{AB}{BC} \rightarrow \frac{BD}{21} = \frac{21}{75}$$

Simplifying and solving for BD:

$$\frac{BD}{21} = \frac{21}{75} \rightarrow \frac{BD}{21} = \frac{\cancel{21}^{7}}{\cancel{75}^{25}} \rightarrow \frac{BD}{21} \cdot 21 = \frac{7}{25} \cdot 21 \rightarrow \frac{BD}{\cancel{21}} \cdot \cancel{21} = \frac{7}{25} \cdot 21 \rightarrow BD = 5.88$$

Because the distance from BD is 5.88 miles, that just leaves the original 6 miles minus that distance: $6 - 5.88 = 0.12$ mile. Add that onto the length of the new road, and you now have to travel just $20.16 + 0.12 = 20.28$ miles to get to your favorite eating place. This saves you almost 7 miles!

Chapter **7**

Relating Triangles to Circular Functions

One of the ways that mathematicians first defined the trig functions was by using ratios formed from the measures of the sides of right triangles (see Chapter 6). Right triangles and the measures of their sides are convenient and easy to construct. This fact led to a sort of natural development of the trig functions, and it proved to be most useful because it allowed engineers, astronomers, and mathematicians to make accurate calculations of the heights of tall objects, areas of large expanses, and predictions of eclipses and other astronomical phenomena. But, of course, they couldn't stop there. The world of trigonometry and its applications opened up even more when they expanded the trig functions and properties to angles of *any* measure — positive and negative — not just those limited to a right triangle. This extension of the angles allowed them to calculate the areas of triangles containing obtuse angles and to conduct navigational plots. The best place to begin describing these new function values and comparing them with the old is with the most basic of all circles — the unit circle.

Getting Acquainted with the Unit Circle

The *unit circle* is a circle with its center at the origin of the coordinate plane and with a radius of 1 unit. Any circle with its center at the origin has the equation $x^2 + y^2 = r^2$, where *r* is the radius of the circle. In the case of a unit circle, the equation is $x^2 + y^2 = 1$. This equation shows that the points lying on the unit circle have to have coordinates (*x*- and *y*-values) that, when you square each of them and then add those values together, equal 1. The coordinates for the points lying on the unit circle and also on the axes are (1,0), (–1,0), (0,1), and (0,–1). These four points (called *intercepts*) are shown in Figure 7-1.
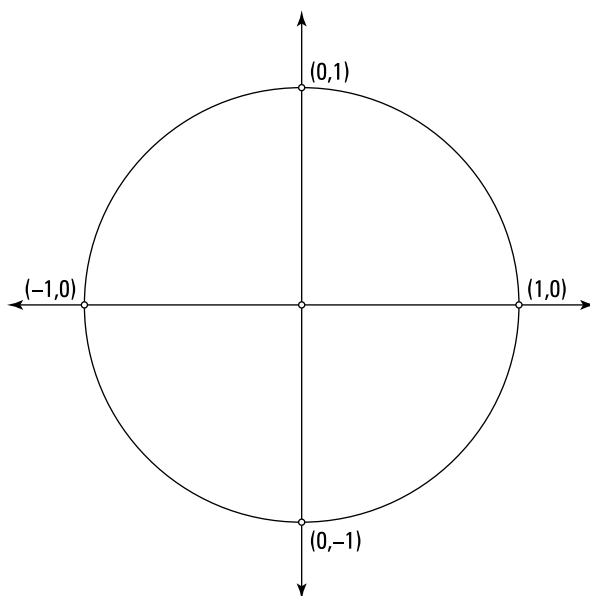
Placing points on the unit circle

The coordinates of the rest of the points on the unit circle aren't as nice and neat as those you see in Figure 7-1. The others all have fractions or radicals — or both — in them. For instance, the point $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2} \right)$ lies on the unit circle. Look at how these coordinates work in the equation of the unit circle:

$$\left(\frac{1}{2} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$

When you square each coordinate and add those values together, you get 1.

Any combination of these two coordinates, whether the coordinates are positive or negative, gives you a different point on the unit circle. They all work because whether a number is positive or negative, its square is the same positive number. Here are some combinations of those two coordinates that satisfy the unit-circle equation:

$\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$	$\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$	$\left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$
$\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$	$\left(-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$	

Another pair of coordinates that works on the unit circle is $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$, because the sum of the squares is equal to 1:

$$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + \frac{2}{4} = \frac{4}{4} = 1$$

The numbers that continually crop up as coordinates of points on the unit circle are $0, \dfrac{1}{2}, \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{3}}{2}$, and 1. If you read Chapter 6, they should look familiar — they're the sine and cosine values of the most common acute-angle measures. Figure 7-2 shows the locations of those points on the unit circle.
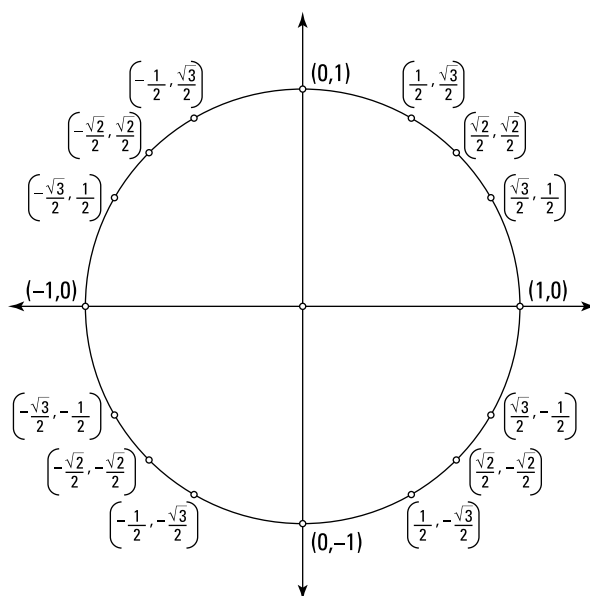
FIGURE 7-2: Points on the unit circle.

BABYLONIAN MATHEMATICS

The Babylonians were of an ancient culture that had a good deal of influence on the development of mathematics in many areas:

- **They developed a system of written symbols.** Way back in the day, the Babylonians developed a form of writing that was based on wedge-shaped symbols called *cuneiform*. Their work has been preserved in the clay tablets that they wrote on, but this way of writing was really cumbersome, so they couldn't write very fast or for very long.

- **They gave us the time of day.** The Babylonians divided the day into 24 hours, each hour into 60 minutes, and each minute into 60 seconds. This division resulted in their base-60 counting system — called the *hexasegimal* system — which means that their number system was base 60 and had different characters for 1; 10; 60; 600; 3,600; 36,000; and 216,000. Their system allowed them to easily write fractions.

- **They beat Pythagoras to the punch.** These ancients had a knowledge of trigonometry and the Pythagorean Theorem 1,000 years before Pythagoras did; they just didn't get the credit he did.

- **They were neat freaks.** Their methods for solving problems were very logical and systematic. They preferred orderly procedures based on tables and facts, which is probably why they knew about π and could approximate its value.

The Babylonians thought in terms of algebra and trigonometry, but you probably wouldn't recognize their notes if you were to pick up a tablet or two to read what they discovered. You'll find this book much more readable!

The points on the unit circle shown in Figure 7-2 are frequently used in trigonometry and other math applications, but they aren't the only points on that circle. Every circle has an infinite number of points with all sorts of interesting coordinates — even more interesting than those already shown. If you're looking for the coordinates of some other point on the unit circle, you can just pick some number between –1 and 1 to be the *x*- or the *y*-value and then solve for the other value. I describe this method for finding the other part of a coordinate in the next section. All these other coordinates come into play when you're drawing a ray that starts at the unit circle's center and want to find the trig functions of the angle formed by that ray and the positive *x*-axis.

Finding a missing coordinate

If you have the value of one of a point's coordinates on the unit circle and need to find the other, you can substitute the known value into the unit-circle equation and solve for the missing value.

You can choose any number between 1 and −1, because that's how far the unit circle extends along the x- and y-axes. For example, say $\frac{2}{5}$ is the x-coordinate of a point on the unit circle. You can find the y-coordinate like so:

1. Substitute the *x*-coordinate value into the unit-circle equation.

$$\left(\frac{2}{5}\right)^2 + y^2 = 1$$

2. Square the *x*-coordinate and subtract that value from each side.

$$\frac{4}{25} + y^2 = 1$$

$$y^2 = 1 - \frac{4}{25} = \frac{21}{25}$$

3. Take the square root of each side.

$$\sqrt{y^2} = \pm\sqrt{\frac{21}{25}}$$

$$y = \pm\frac{\sqrt{21}}{5}$$

Note that the y-coordinate can have two values, because the unit circle has two different points for every particular x-coordinate except those on the axes. Look at Figure 7-3a, and you can see how that happens. Figure 7-3b shows two points on the unit circle with the same y-coordinate. Given a particular y-coordinate, all but those special values have two x-coordinates.

Another example: Find the x-coordinate (or coordinates) if the y-coordinate is $-\frac{7}{25}$.

1. Substitute the *y*-coordinate value into the unit-circle equation.

$$x^2 + \left(-\frac{7}{25}\right)^2 = 1$$
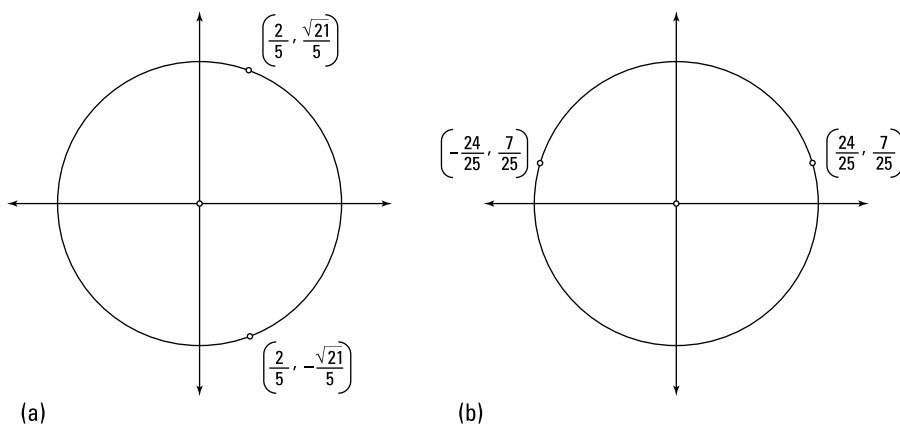
FIGURE 7-3:
Two points on the unit circle for a particular *x*-coordinate and *y*-coordinate.

(a)   (b)

2. **Square the *y*-coordinate and subtract that value from each side.**

$$x^2 + \frac{49}{625} = 1$$

$$x^2 = 1 - \frac{49}{625}$$

$$= \frac{576}{625}$$

3. **Take the square root of each side.**

$$\sqrt{x^2} = \pm\sqrt{\frac{576}{625}}$$

$$x = \pm\frac{24}{25}$$

As you can see, the *x*-coordinate here has two values, and the two points are $\left(\frac{24}{25}, -\frac{7}{25}\right)$ and $\left(-\frac{24}{25}, -\frac{7}{25}\right)$.

Sticking to rational coordinates

You may have noticed, in the last section, that one problem resulted in a coordinate with a radical in it and the other didn't. Radicals can't be avoided (as you read in the news) when doing trig problems, but sometimes you just need to keep things rational. A *rational* number is a real number that can be written as a fraction. And rational numbers have decimal values that behave — unlike the decimal values of radicals (irrational numbers). What I have for you here is a way of assuring yourself that you'll get only rational coordinates for a point on the unit circle. To do this, use the following formula, letting *m* be any rational number:

$$(x,y) = \left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$$

Want to see the formula at work? First, I show you the formula starting with a nice, civilized rational number; let $m = 4$.

1. **Replace each *m* in the formula with 4.**

$$\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right) = \left(\frac{1-4^2}{1+4^2}, \frac{2\cdot 4}{1+4^2}\right)$$

2. **Simplify.**

$$= \left(\frac{1-16}{1+16}, \frac{8}{1+16}\right) = \left(-\frac{15}{17}, \frac{8}{17}\right)$$

Skeptical? Just check the coordinates in the equation for the unit circle.

$$\left(-\frac{15}{17}\right)^2 + \left(\frac{8}{17}\right)^2 = \frac{225}{289} + \frac{64}{289} = \frac{289}{289} = 1$$

And now, to really convince even the biggest skeptics, I choose $m = -\frac{2}{3}$.

1. **Replace each *m* in the formula with $-\frac{2}{3}$.**

$$\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right) = \left(\frac{1-\left(-\frac{2}{3}\right)^2}{1+\left(-\frac{2}{3}\right)^2}, \frac{2\left(-\frac{2}{3}\right)}{1+\left(-\frac{2}{3}\right)^2}\right)$$

2. **Simplify.**

$$= \left(\frac{1-\frac{4}{9}}{1+\frac{4}{9}}, \frac{-\frac{4}{3}}{1+\frac{4}{9}}\right) = \left(\frac{\frac{5}{9}}{\frac{13}{9}}, \frac{-\frac{4}{3}}{\frac{13}{9}}\right) = \left(\frac{5}{13}, -\frac{12}{13}\right)$$

And, of course, this point checks, too.

I'm going to let you in on a little secret: This formula is based on using Pythagorean triples in the numerators and denominators of the fractions. There's a bit more to it — creating the formula using a slope of a line through an intercept of the unit circle — but you don't need all that to take advantage of the convenience of the numbers produced by the formula.

Going Full Circle with the Angles

The unit circle is a platform for describing all the possible angle measures from 0 to 360 degrees, plus all the negatives of those angles, plus all the multiples of the positive and negative angles from negative infinity to positive infinity. In Chapter 3, I discuss these co-terminal angles. Here, you're taking on their measures in terms of the unit circle. In other words, the unit circle shows you all the angles that exist. Because a right triangle can only measure angles of 90 degrees or less, the circle allows for a much broader range.

Staying positive

The positive angles on the unit circle are measured with the initial side on the positive *x*-axis and the terminal side moving counterclockwise around the origin (to figure out which side is which, see Chapter 3). Figure 7-4 shows some positive angles labeled in both degrees and radians.

In Figure 7-4, notice that the terminal sides of the pairs of angles — measuring 30 degrees and 210 degrees, measuring 60 degrees and 240 degrees, and so on — form straight lines. This is to be expected because the angle measures are 180 degrees apart, and a straight angle measures 180 degrees. You see the significance of this fact when you deal with the trig functions for these angles in Chapter 8.

Being negative or multiplying your angles

Just when you thought that angles measuring up to 360 degrees or 2π radians was enough for anyone, you're confronted with the reality that the basic angles can be written as negative values and even multiples of themselves. If you measure angles clockwise instead of counterclockwise, then the angles have negative measures: A 30-degree angle is the same as an angle measuring –330 degrees, because they have the same terminal side. Likewise, an angle of $\frac{5\pi}{3}$ is the same as an angle of $-\frac{\pi}{3}$. For rules on how to change degree measure to radian measure, refer to Chapter 1.

But wait — you have even more ways to name an angle. By doing two (or more) complete rotations and adding or subtracting 360 degrees or a multiple of it before settling on the angle's terminal side, you can get an infinite number of angle measures, both positive and negative, for the same basic angle. For example, an angle of 60 degrees has the same terminal side as that of a 420-degree angle and a –300-degree angle. Figure 7-5 shows many names for the same 60-degree angle in both degrees and radians.
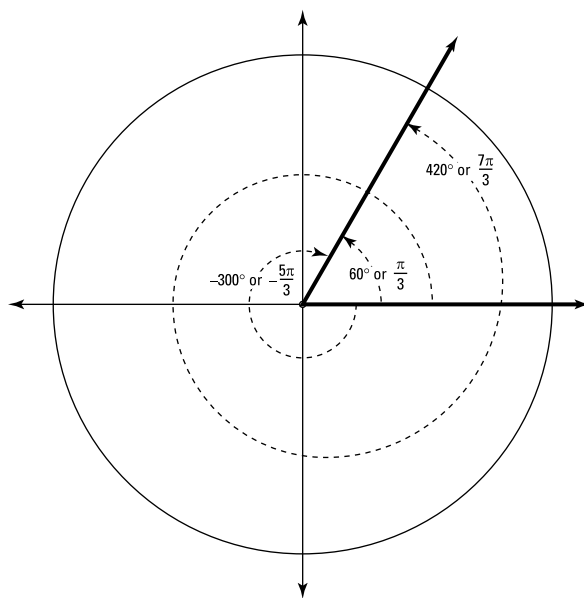
FIGURE 7-5:
You have many ways to name the same angle.

Although this name-calling of angles may seem pointless at first, there's more to it than arbitrarily using negatives or multiples of angles just to be difficult. The angles that are related to one another have trig functions that are also related, if not the same (more on that in Chapter 8).

Locating and computing reference angles

Each of the angles in a unit circle has a *reference angle,* which is always a positive acute angle (except for the angles that are already positive and acute). By identifying the reference angle, you can determine the function values for that reference angle and, ultimately, the original angle. Usually, solving for the reference angle first is much easier than trying to determine a trig function for the original angle. The trig functions have values that repeat over and over; sometimes those values are positive, and sometimes they're negative. Using a reference angle helps keep the number of different values to a minimum. You just assign the positive or negative sign after determining a numerical value for the function from the reference angle.

You determine a reference angle by looking at the terminal side of the angle you're working with and its relation with the positive or negative *x*-axis (depending on which quadrant the terminal side is in). The following guidelines tell you how to measure the reference angle when you're given the terminal side of the angle.

>> **Quadrant I (QI):** The reference angle is the same as the original angle.

>> **Quadrant II (QII):** The reference angle is the measure from the terminal side down to the negative *x*-axis.

>> **Quadrant III (QIII):** The reference angle is the measure from the negative *x*-axis down to the terminal side.

>> **Quadrant IV (QIV):** The reference angle is the measure from the terminal side up to the positive *x*-axis.

Figure 7-6a shows the positions of the reference angles in the four quadrants. And Figure 7-6b introduces you to some examples of angles and their corresponding reference angles.

As with all angles, you measure reference angles in degrees or radians. I have to admit that I sometimes prefer to work in degrees and will convert a radian measure to do these computations. Whichever method you choose is fine — it's all a matter of taste.

FIGURE 7-6:
You measure
reference angles
by using the
x-axis.
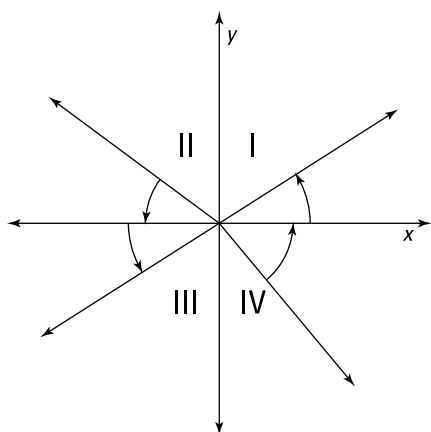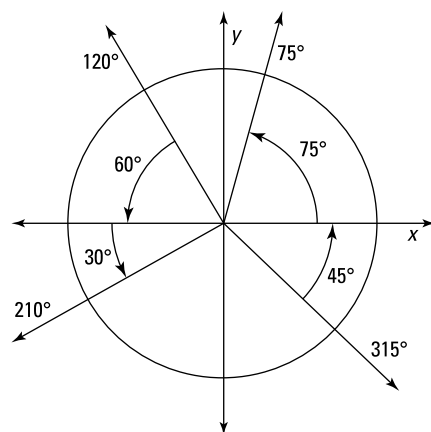
(a)   (b)

Figuring the angle measure in degrees

To compute the measure (in degrees) of the reference angle for any given angle θ, use the rules in Table 7-1.

TABLE 7-1

Finding Reference Angles in Degrees

Quadrant	Measure of Angle θ	Measure of Reference Angle
I	$0° - 90°$	θ
II	$90° - 180°$	$180° - \theta$
III	$180° - 270°$	$\theta - 180°$
IV	$270° - 360°$	$360° - \theta$

Using Table 7-1, find the reference angle for 200 degrees:

1. **Determine the quadrant in which the terminal side lies.**

A 200-degree angle is between 180 and 270 degrees, so the terminal side is in QIII.

2. **Do the operation indicated for that quadrant.**

Subtract 180 degrees from the angle, which is 200 degrees. You find that $200 - 180 = 20$, so the reference angle is 20 degrees.

Sometimes angle measures don't fit neatly in the ranges shown in Table 7-1. For example, you may need to find the reference angle for a negative angle or a multiple of an angle.

To find the reference angle for –340 degrees, follow these steps:

1. **Determine the quadrant in which the terminal side lies.**

A –340-degree angle is equivalent to a 20-degree angle. (You get the positive angle measure by adding 360, or one full revolution around the origin, to the negative measure.) A 20-degree angle has its terminal side in QI.

2. **Do the operation indicated for that quadrant.**

Angles in the first quadrant are their own reference angle, so the reference angle is 20 degrees.

Figuring the angle measure in radians

To compute the measure (in radians) of the reference angle for any given angle θ, use the rules in Table 7-2.

TABLE 7-2

Finding Reference Angles in Radians

Quadrant	Measure of Angle θ	Measure of Reference Angle
I	0 to $\dfrac{\pi}{2}$	θ
II	$\dfrac{\pi}{2}$ to π	$\pi - \theta$
III	π to $\dfrac{3\pi}{2}$	$\theta - \pi$
IV	$\dfrac{3\pi}{2}$ to 2π	$2\pi - \theta$

For example, to find the reference angle for $\dfrac{15\pi}{16}$, follow these steps:

1. **Determine the quadrant in which the terminal side lies.**

An angle measuring $\dfrac{15\pi}{16}$ has its terminal side in QII, which you know because $\dfrac{15}{16}$ is slightly less than 1, making the angle slightly less than π.

2. **Do the operation indicated for that quadrant.**

Subtract $\dfrac{15\pi}{16}$ from π. When you do so, you get $\pi - \dfrac{15\pi}{16} = \dfrac{16\pi}{16} - \dfrac{15\pi}{16} = \dfrac{\pi}{16}$, so the reference angle is $\dfrac{\pi}{16}$.

Navigating with Circular Measures

Circles are found here, there, and everywhere. Circles are one of our more useful structures. Silos and fuel storage containers are circular in shape. Traffic round-abouts are circular. Many artistic renderings contain circular figures. They have been and continue to be valuable to our daily lives.

Introducing the compass

Compasses have been around for a long time. The first compasses were probably developed about 2,000 years ago. It's also thought that Chinese scientists came up with the magnetized needles in compasses in the 11th or 12th centuries. Nowadays compasses are rather sophisticated, but the basics from those earliest users are still there: a circle, degree measures, and a needle pointing north.

Figure 7-7 shows a compass with its magnetic north-south needle pointing upward and a directional needle pointing to the left. The directional needle in this figure provides you with the *bearing* of 285 degrees. If you're going to head in that direction, though, you'll be turning to the northwest by 75 degrees.
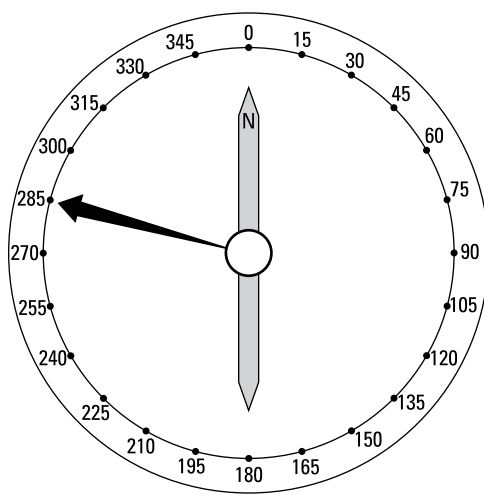
FIGURE 7-7:
Using a compass to navigate.

When using a modern compass, that ring of angle measures around the outside of the circle is called the *bezel*. It can be rotated as you face in different directions.

For example, if you're currently heading southeast with a bearing of 150 degrees and plan on turning to the left by 90 degrees, what will your new bearing be?

Just subtract 90 from 150, and your new bearing will be 60 degrees.

Cycling with a cyclic quadrilateral

A *cyclic quadrilateral* is a four-sided polygon that has each of its vertices on a single circle. Or, another way of saying it is that the quadrilateral has been *circumscribed* by a circle. Not all quadrilaterals can have circles going through all their vertices, but those that behave and cooperate mathematically do qualify. In Figure 7-8a, you see such a cooperative quadrilateral, named *ABCD*.

(a)

(b)
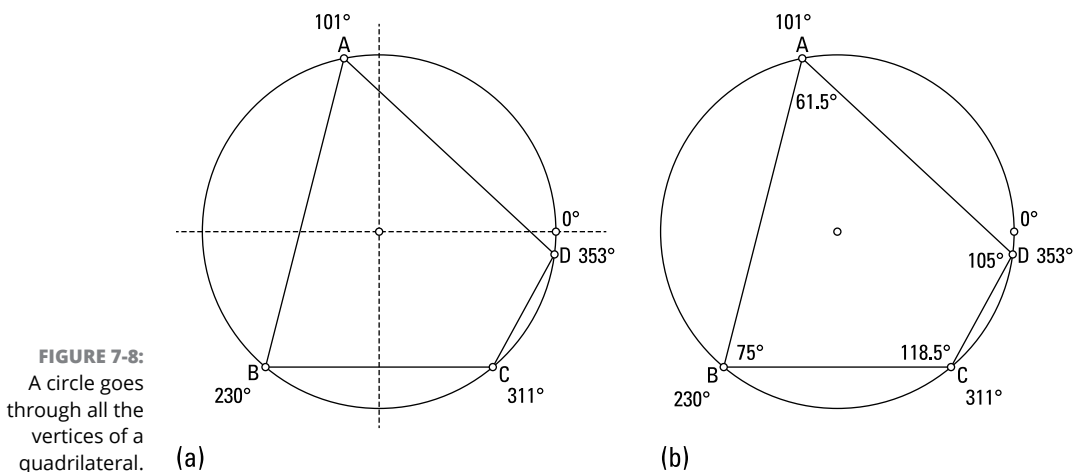
Say you want to measure the interior angles of a quadrilateral, but the figure you're working from is too large to use a protractor. Fortunately, you do have access to the degree measures around the circle, starting on the right with 0 degrees and winding counterclockwise around the circle. In Figure 7-8a, those measures are indicated.

So how does this help with the angle measures? In Chapter 1, I show you that an inscribed angle (an angle with its vertex on a circle and rays intersecting the circle) has a measure that's half the measure of the arc of the circle that it intercepts. So all you need to do is find the lengths of the arcs, divide them by 2, and you've found the angle measures!

Angle *BAD* has its rays intersecting arc *BD*. That arc goes from 230 degrees to 353 degrees. Find the difference between the two measures and divide by 2:

$$\frac{353-230}{2} = \frac{123}{2} = 61.5$$

Angle *ADC* has its rays intersecting arc *AC* (which includes the point *B*). This arc goes from 101 degrees to 311 degrees. Find the difference and divide by 2:

$$\frac{311-101}{2} = \frac{210}{2} = 105$$

Angle *DCB* has its rays intersecting arc *DB* (which includes the point *A*). This arc goes from point *D* through point *A* to 0 degrees. Then it goes from 0 to 353 degrees. It would be easier just to find the length of the arc *BD* (which I did earlier) and subtract that length (which is 123) from 360 degrees. Then divide by 2:

$$\frac{360-123}{2} = \frac{237}{2} = 118.5$$

Angle *CBA* has its rays intersecting the arc *AC* (which includes the point *D*). Again, it's easier to just find the length of the arc *AC* (which is 210) and subtract from 360 degrees. Then divide by 2:

$$\frac{360-210}{2} = \frac{150}{2} = 75$$

See Figure 7–8b for the inserted angle measures. And please note that they add up to 360 degrees minus the number of degrees in a quadrilateral!

$$61.5+105+118.5+75 = 360$$

Chapter **8**

Taking Trig Functions Global

The six basic trig functions all had humble beginnings with the right triangle and its angles. The unit circle opens up a whole new world for the input values into those functions. Because of the nature of trig functions — they repeat the same patterns over and over — the output values show up regularly. This repetition is a good thing; you recognize where in the pattern a particular input belongs and then assign the output. Life is good.

Defining Trig Functions for All Angles

So many angles are used in trigonometry and other math areas, and the majority of those angles are multiples of 30 and 45 degrees. So, having a trick up your sleeve that lets you quickly access the function values of this frequent-flier list of angles makes perfect sense. All you need to know are the values of the trig functions for 0-, 30-, 45-, 60-, and 90-degree angles in order to determine *all* the trig functions of *all* the angles, positive or negative, that are multiples of 30 or 45 degrees (these are the two basic, foundational angles). Determining the

function values for a particular angle that is a multiple of 30 or 45 degrees is a three-step process: (1) Find the measure of the angle's reference angle; (2) Assign the correct numerical value; and (3) Determine whether the function value is positive or negative.

Putting reference angles to use

The first step to finding the function value of one of the angles that's a multiple of 30 or 45 degrees is to determine which is the reference angle. You want to know which of the angles from the first quadrant — 0, 30, 45, 60, or 90 degrees — created your angle (it's a multiple of one of these angles). You can use the function value of that angle and then figure out the sign (see the next section). Use Table 7-1 or Table 7-2 (in Chapter 7) to find the reference angle.

TRIG RULES

All angles with a 30-degree reference angle have trig functions whose absolute values are the same as those of the 30-degree angle. The sines of 30, 150, 210, and 330 degrees, for example, are all either $\frac{1}{2}$ or $-\frac{1}{2}$. Likewise, using a 45-degree angle as a reference angle, the cosines of 45, 135, 225, and 315 degrees, for example, are all $\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$.

Labeling the optimists and pessimists

The function values for the sines of 30, 150, 210, and 330 degrees are $\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$, and $-\frac{1}{2}$, respectively. All these multiples of 30 degrees have an absolute value of $\frac{1}{2}$ (as I explain in the preceding section). The following rule and Figure 8-1 help you determine whether a trig-function value is positive or negative. First, note that each quadrant in the figure is labeled with a letter. The letters aren't random; they stand for trig functions.

TRIG RULES

Reading around the quadrants, starting with QI and going counterclockwise, the rule goes like this: If the terminal side of the angle is in the quadrant with letter

> » **A** — All functions are positive.

> » **S** — Sine and its reciprocal, cosecant, are positive. Tangent, cotangent, cosine, and secant are negative.

> » **T** — Tangent and its reciprocal, cotangent, are positive. Sine, cosecant, cosine, and secant are negative.

> » **C** — Cosine and its reciprocal, secant, are positive. Sine, cosecant, tangent, and cotangent are negative.

FIGURE 8-1:
Assigning positive
and negative
functions by
quadrant.

(a)   (b)
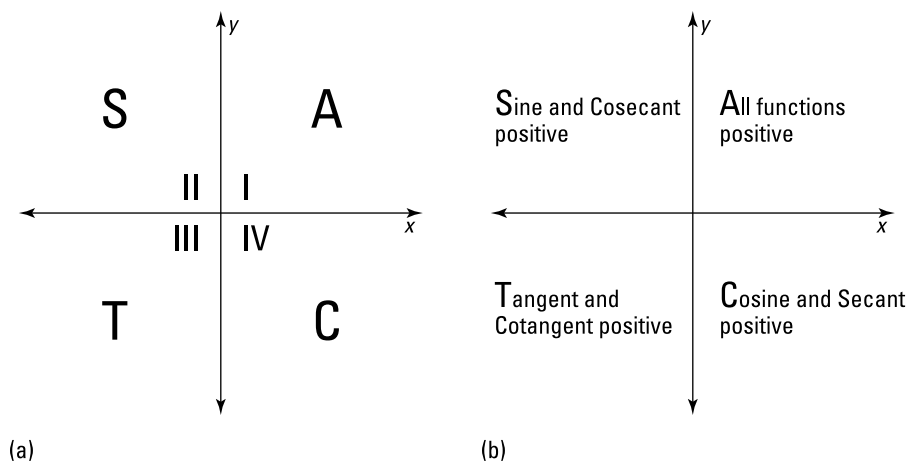
My trig teacher, Dr. Johnson, shared with the class a great way to remember the rule involving the signs of the functions using the A, S, T, C lettering: "All Students Take Calculus." (And so I did!) But if math is already giving you nightmares, maybe you'd prefer "Any Snake Teases Chickens" or "Apple Sauce Turns Colors." Make up your own! Have at it!

Combining all the rules

Using the rules for reference angles, the values of the functions of certain acute angles (see Chapter 6), and the rule for the signs of the functions, you can determine the trig functions for any angles found on the unit circle — any that are graphed in *standard position* (meaning the vertex of the angle is at the origin, and the initial side lies along the positive *x*-axis). Figure 8-2 combines information from this chapter and Chapter 7 to give you the tools you need to determine trig functions.

Now, armed with all the necessary information, find the tangent of 300 degrees.

1. **Find the reference angle**.

Using the top chart in Figure 8-2, you can see that a 300-degree angle has its terminal side in the fourth quadrant, so you find the reference angle by subtracting 300 from 360. Therefore, the measure of the reference angle is 60 degrees.

2. **Find the numerical value of the tangent**.

Using the bottom chart in Figure 8-2, you see that the numerical value of the tangent of 60 degrees is $\sqrt{3}$.

Quadrant	Measure of Angle θ	Measure of Reference Angle
I	0° to 90°	θ
II	90° to 180°	$180° - \theta$
III	180° to 270°	$\theta - 180°$
IV	270° to 360°	$360° - \theta$

$\theta°$	0°	30°	45°	60°	90°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

FIGURE 8-2: Use these tables to find function values by using reference angles.

3. **Find the sign of the tangent.**

   Because a 300-degree angle is in the fourth quadrant, and angles in that quadrant have negative tangents (refer to Figure 8-1 in the preceding section), the tangent of 300 degrees is $-\sqrt{3}$.

To try your hand at working with radians, find the cosecant of $\frac{7\pi}{6}$.

1. **Find the reference angle.**

   To use the top chart in Figure 8-2, you need to determine the degree equivalence for an angle measuring $\frac{7\pi}{6}$. In Chapter 4, you find the formula for converting from radians to degrees. Using the formula in this case, you get that $\frac{7\pi}{6}$ is equivalent to 210 degrees. This angle is in the third quadrant, so, going back to radians, you find the reference angle by subtracting π from $\frac{7\pi}{6}$, resulting in $\frac{7\pi}{6} - \pi = \frac{\pi}{6}$.

2. **Find the numerical value of the cosecant.**

   In the bottom chart of Figure 8-2, the cosecant doesn't appear. However, the reciprocal of the cosecant is sine. So find the value of the sine, and use its reciprocal. The sine of $\frac{\pi}{6}$ is $\frac{1}{2}$, which means that the cosecant of $\frac{\pi}{6}$ is 2 (the reciprocal).

3. **Find the sign of the cosecant.**

   In the third quadrant, the cosecant of an angle is negative (refer to Figure 8-1 in the preceding section), so the cosecant of $\frac{7\pi}{6}$ is –2.

Using Coordinates of Circles to Solve for Trig Functions

Another way to find the values of the trig functions for angles is to use the coordinates of points on a circle that has its center at the origin. Letting the positive x-axis be the initial side of an angle, you can use the coordinates of the point where the terminal side intersects with the circle to determine the trig functions. Figure 8-3a shows a circle with a radius of r that has an angle drawn in standard position.

TRIG RULES

The equation of a circle is $x^2 + y^2 = r^2$ (flip back to Chapter 2 for a refresher). Based on this equation and the coordinates of the point (x,y), where the terminal side of the angle intersects the circle, the six trig functions for angle θ are defined as follows:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

You can see where these definitions come from if you picture a right triangle formed by dropping a perpendicular segment from the point (x,y) to the x-axis. Figure 8-3b shows such a right triangle. Remember that the x-value is to the right (or left) of the origin, and the y-value is above (or below) the x-axis — and use those values as lengths of the triangle's sides. Therefore, the side opposite angle θ is y, the value of the y-coordinate. The adjacent side is x, the value of the x-coordinate.

TIP

Take note that for angles in the second quadrant, for example, the x-values are negative, and the y-values are positive. The radius, however, is always a positive number. With the x-values negative and the y-values positive, using the definitions for the functions listed earlier in this section, you see that the sine and cosecant are positive, but the other functions are all negative, because they all have an x in their ratios. The signs of the trig functions all fall into line when you use this coordinate system, so there's no need to worry about remembering the ASTC rule here. (For more on that rule and when to use it, see the section, "Labeling the optimists and pessimists," earlier in this chapter.)

(a)
(b)

Calculating with coordinates on the unit circle

Calculating trig functions of angles within a unit circle is easy as pie. Figure 8-4 shows a unit circle, which has the equation $x^2 + y^2 = 1$, along with some points on the circle and their coordinates.
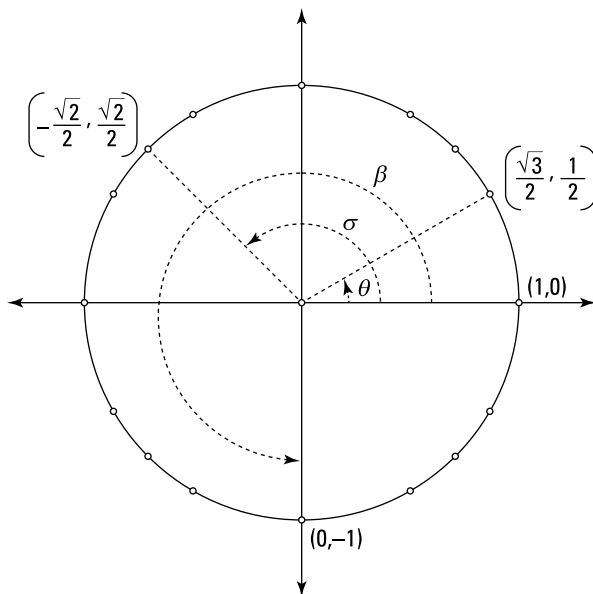
FIGURE 8-4:
Some points on the unit circle.

Using the angles in Figure 8-4, find the tangent of θ.

1. **Find the *x*- and *y*-coordinates of the point where the angle's terminal side intersects with the circle.**

   The coordinates are $x = \dfrac{\sqrt{3}}{2}$ and $y = \dfrac{1}{2}$. The radius is $r = 1$.

2. **Determine the ratio for the function and substitute in the values.**

   The ratio for the tangent is $\dfrac{y}{x}$, so you find that

   $$\frac{y}{x} = \frac{\dfrac{1}{2}}{\dfrac{\sqrt{3}}{2}} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Next, using the angles in Figure 8-4, find the cosine of σ.

1. **Find the *x*- and *y*-coordinates of the point where the terminal side of the angle intersects with the circle.**

   The coordinates are $x = -\dfrac{\sqrt{2}}{2}$ and $y = \dfrac{\sqrt{2}}{2}$; the radius is $r = 1$.

2. **Determine the ratio for the function and substitute in the values.**

   The ratio for the cosine is $\dfrac{x}{r}$, which means that you need only the *x*-coordinate,

   so $\dfrac{x}{r} = \dfrac{-\dfrac{\sqrt{2}}{2}}{1} = -\dfrac{\sqrt{2}}{2}$.

Now, using the angles in Figure 8-4, find the cosecant of β.

1. **Find the *x*- and *y*-coordinates of the point where the terminal side of the angle intersects with the circle.**

   The coordinates are $x = 0$ and $y = -1$; the radius is $r = 1$.

2. **Determine the ratio for the function and substitute in the values.**

   The ratio for cosecant is $\dfrac{r}{y}$, which means that you need only the *y*-coordinate,

   so $\dfrac{r}{y} = \dfrac{1}{-1} = -1$.

Calculating with coordinates on any circle at the origin

You don't need a unit circle to use this coordinate business when determining the function values of angles graphed in standard position on a circle. You can use a circle with any radius, as long as the center is at the origin. The standard equation for a circle centered at the origin is $x^2 + y^2 = r^2$.

Using the angles in Figure 8-5, find the sine of α.

1. **Find the *x*- and *y*-coordinates of the point where the terminal side of the angle intersects with the circle.**

   The coordinates are $x = -5$ and $y = 12$.

2. **Determine the radius of the circle.**

   The equation of the circle is $x^2 + y^2 = r^2$. Replacing the x and y in this equation with –5 and 12, respectively, you get $(-5)^2 + (12)^2 = 25 + 144 = 169 = r^2$. The square root of 169 is 13, so the radius is 13.

3. **Determine the ratio for the function, and substitute in the values.**

   The ratio for sine is $\dfrac{y}{r}$, which means that you need only the y-coordinate and radius, so $\dfrac{y}{r} = \dfrac{12}{13}$.

Next, using the angles in Figure 8-5, find the cotangent of β.

1. **Find the *x*- and *y*-coordinates of the point where the terminal side of the angle intersects with the circle.**

   The coordinates are $x = -12$ and $y = -5$.

   The cotangent function uses only the x- and y-coordinates, so you don't need to solve for the radius.

2. **Determine the ratio for the function, and substitute in the values.**

The ratio of cotangent is $\dfrac{x}{y}$, so $\dfrac{x}{y} = \dfrac{-12}{-5} = \dfrac{12}{5}$.

Now, using the angles in Figure 8-5, find the secant of γ.

1. **Find the *x*- and *y*-coordinates of the point where the terminal side of the angle intersects with the circle.**

The coordinates are $x = 0$ and $y = -13$.

2. **Determine the radius of the circle.**

Per the first example in this section, the radius is 13.

3. **Determine the ratio for the function, and substitute in the values.**

The ratio for secant is $\dfrac{r}{x}$, so you need only the *x*-coordinate; substituting in, you get $\dfrac{r}{x} = \dfrac{13}{0}$. This answer is undefined, which means that angle γ has no secant. For the reason that γ has no secant, refer to the next section.

Defining Domains and Ranges of Trig Functions

REMEMBER

The *domain* of a function consists of all the input values that a function can handle — the way the function is defined. Of course, you want to get output values (which make up the *range*) when you enter input values. But sometimes, when you input something that doesn't belong in the function, you end up with some impossible situations. In these cases, you need to limit what you put into the

function — the domain has to be restricted. For example, the cosecant is defined as the hypotenuse divided by the opposite side (see Chapter 6). If the terminal side of the angle is on the *x*-axis, then the opposite side is 0, and you're asked to divide by 0. Impossible!

Trig functions have domains that are angle measures (the inputs are all angles), in either degrees or radians. The outputs of the trig functions are real numbers. The hitch here is that the different trig functions have different domains and ranges. You can't put just any angle into some of the functions. Sine and cosine are very cooperative and have the same domain and range. The tangent function and the reciprocal functions, however, all differ. The best way to describe these different domains and ranges is visually: Refer to the coordinate plane with a circle centered at the origin and a right triangle inside it, formed by dropping a line from any point (*x*,*y*) on the circle to the *x*-axis (see Figure 8-6). Remember that *r* stands for the radius of the circle (and also the hypotenuse of the right triangle in this figure). When that hypotenuse lies along one of the axes, one of the sides of the triangle is equal to 0, which is a no-no in the denominator of a fraction.
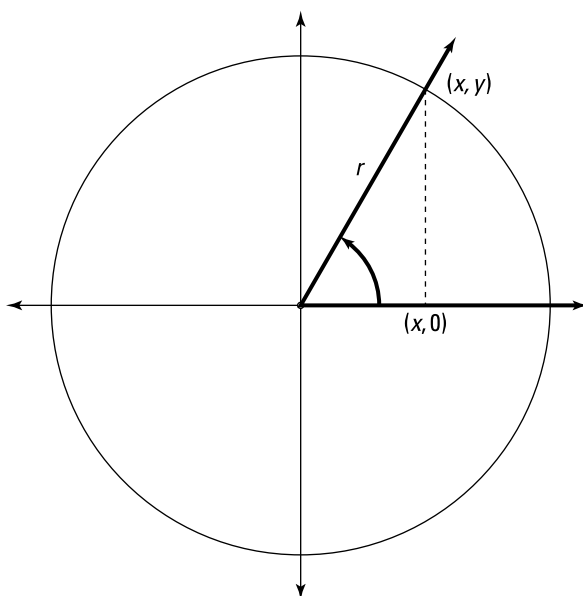
Consider the values of the variables in Figure 8-6 in relation to one another. The radius, *r*, is always positive. And the absolute values of *x* and *y* (the lengths of the segments they represent) are always smaller than *r*, unless the point (*x*,*y*) is on one of the axes — then one of the values is equal to *r* and the other is equal to 0.

Friendly functions: Sine and cosine

The sine and cosine functions are unique in the world of trig functions, because their ratios always have a value. No matter what angle you input, you get a resulting output. The value you get may be 0, but that's a number, too. In reference to the coordinate plane, sine is $\frac{y}{r}$, and cosine is $\frac{x}{r}$.

The radius, r, is always some positive number (which is why these functions always have a value, because they don't ask you to divide by 0), and r is always a number greater than (or equal to) the absolute value of x or y.

Domains of sine and cosine

TRIG RULES

The domains of sine and cosine are infinite. In trig speak, you say something like this: If θ represents all the angles in the domain of the two functions $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$, then $-\infty < \theta < \infty$, which means that θ can be any angle in degrees or radians — any real number.

Ranges of sine and cosine

TRIG RULES

The output values for sine and cosine are always between (and including) –1 and 1. In trig speak, it goes something like this: If $f(\theta)$ and $g(\theta)$ represent the output values of the functions $f(\theta) = \sin\theta$ and $g(\theta) = \cos\theta$, then $-1 \le f(\theta) \le 1$ and $-1 \le g(\theta) \le 1$.

The ratios $\frac{y}{r}$ and $\frac{x}{r}$ will never be improper fractions — the numerator can never be greater than the denominator — because the value of r, the radius, is always the biggest number. At best, if the angle θ has a terminal side on an axis (meaning that one of the sides is equal to r), then the value of those ratios is 1 or –1.

Close cousins of their reciprocals: Cosecant and secant

The cosecant and secant functions are closely tied to sine and cosine, because they're the respective reciprocals. In reference to the coordinate plane, cosecant is $\frac{r}{y}$, and secant is $\frac{r}{x}$. The value of r is the length of the hypotenuse of a right triangle — which, as you find at the beginning of this section, is always positive and always greater than x and y. The only problem that arises when computing these functions is when either x or y is 0 — when the terminal side of the angle is on an axis. A function with a 0 in the denominator creates a number or value that

doesn't exist (in math speak, the result is *undefined*), so anytime *x* or *y* is 0, you don't get any output from the cosecant or secant functions. The *x* is 0 when the terminal side is on the *y*-axis, and the *y* is 0 when the terminal side is on the *x*-axis.

Domains of cosecant and secant

The domains of cosecant and secant are restricted — you can only use the functions for angle measures with output numbers that exist.

TRIG RULES

Any time the terminal side of an angle lies along the *x*-axis (where $y = 0$), you can't perform the cosecant function on that angle. In trig speak, the rule looks like this: If $h(\theta) = \csc\theta$, then $\theta \neq 0$, 180, 360, 540,…, or any multiple of 180 degrees. In radians, $\theta \neq 0$, π, 2π, 3π,…, or any multiple of π.

TRIG RULES

Anytime the terminal side of an angle lies along the *y*-axis (where $x = 0$), you can't perform the secant function on that angle. So, in trig speak, you'd say this: If $k(\theta) = \sec\theta$, then $\theta \neq 90$, 270, 450, 630,…, or any odd multiple of 90 degrees. In radians, $\theta \neq \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}$,…, or any odd multiple of $\dfrac{\pi}{2}$.

Ranges of cosecant and secant

The ratios of the cosecant and secant functions on the coordinate plane, $\dfrac{r}{y}$ and $\dfrac{r}{x}$, have the hypotenuse, *r*, in the numerator. Because *r* is always positive and greater than or equal to *x* and *y*, these fractions are always *improper* (greater than 1) or equal to 1. The ranges of these two functions never include *proper* fractions (numbers between −1 and 1).

TRIG RULES

If $h(\theta)$ and $k(\theta)$ are the output values of the functions $h(\theta) = \csc\theta$ and $k(\theta) = \sec\theta$, then $h(\theta) \leq -1$ or $h(\theta) \geq 1$ and $k(\theta) \leq -1$ or $k(\theta) \geq 1$.

Brothers out on their own: Tangent and cotangent

The tangent and cotangent are related not only by the fact that they're reciprocals, but also by the behavior of their ranges. In reference to the coordinate plane, tangent is $\dfrac{y}{x}$, and cotangent is $\dfrac{x}{y}$. The domains of both functions are restricted, because sometimes their ratios could have zeros in the denominator, but their ranges are infinite.

Domains of tangent and cotangent

Because *x* can't equal 0 for the tangent function to work, this rule holds true: If $m(\theta) = \tan\theta$, then $\theta \neq 90, 270, 450, 630,\ldots$, or any odd multiple of 90 degrees. In radians, $\theta \neq \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2},\ldots$, or any odd multiple of $\dfrac{\pi}{2}$. Both the tangent and secant functions have ratios with *x* in the denominator, making their domains the same.

In order for the cotangent function to work, *y* can't equal 0. If $n(\theta) = \cot\theta$, then $\theta \neq 0, 180, 360, 540,\ldots$, or any multiple of 180 degrees. In radians, $\theta \neq 0,\ \pi,\ 2\pi,\ 3\pi,\ldots$, or any multiple of π.

Ranges of tangent and cotangent

The ranges of both tangent and cotangent are infinite, which, when expressed in mathematical notation, looks like this: $-\infty < m(\theta) < \infty$ and $-\infty < n(\theta) < \infty$.

The range values for these functions get very small (toward negative infinity) or very large (toward positive infinity) whenever the denominator of the respective ratio gets close to 0. When you divide some number by a very small value, such as 0.0001, the result is large. The smaller the denominator, the larger the result.

Applying the Trig Functions

When you work with triangles, you can find all sorts of applications in the real world. Circles also lend themselves to applications and discoveries. Who would have thought that the trig functions, themselves, would provide some interesting adventures?

Flying around on a Ferris wheel

The latest version of the Centennial Wheel on Navy Pier in Chicago, Illinois, has a height of about 200 feet. There's an awesome 360-degree view of the city and Lake Michigan as you ride the Ferris wheel. Refer to Figure 8-7 for a perspective.

A ride on the Centennial Wheel takes 13 minutes and involves three complete rotations. How fast are you traveling in miles per hour?
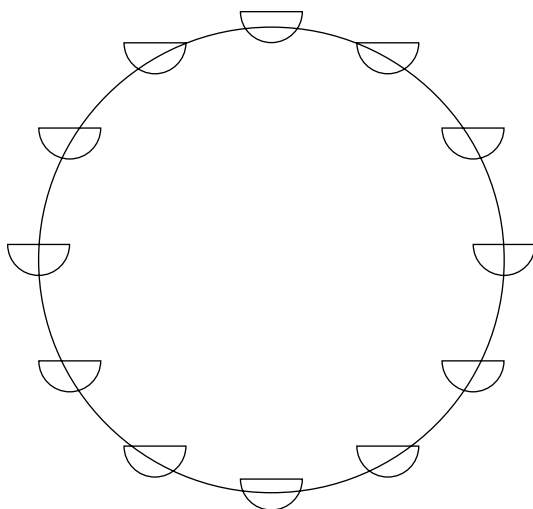
FIGURE 8-7:
Viewing the city from the Centennial Wheel.

Using the height of 200 feet for the diameter of the circular wheel, you find the circumference by multiplying by π: $C = \pi d = 3.14 \cdot 200 = 628$ feet. If you make three complete rotations (assuming they're not including the unloading of passengers), then that's $628 \cdot 3 = 1{,}884$ feet that are traveled during the ride. So you travel 1,884 feet in 13 minutes. Changing that to miles per hour, you multiply using 5,280 feet in a mile and 60 minutes in an hour:

$$\frac{1{,}884 \text{ feet}}{13 \text{ minutes}} \cdot \frac{1 \text{ mile}}{5{,}280 \text{ feet}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}}$$

$$= \frac{1{,}884 \text{ feet}}{13 \text{ minutes}} \cdot \frac{1 \text{ mile}}{5{,}280 \text{ feet}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}}$$

$$= \frac{1{,}884}{13 \text{ minutes}} \cdot \frac{1 \text{ mile}}{5{,}280} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}}$$

$$= \frac{1{,}884}{13} \cdot \frac{1 \text{ mile}}{5{,}280} \cdot \frac{60}{1 \text{ hour}} = \frac{113{,}040 \text{ miles}}{68{,}640 \text{ hours}} \approx 1.65 \text{ miles per hour}$$

Well, moving that slowly, you shouldn't be having any motion sickness!

Trying out some new trig functions

The standard, most popular, and currently used trig functions of sine, cosine, tangent, cotangent, secant, and cosecant aren't the only trig functions out there! Have you ever heard of the versine, coversine, haversine, exsecant, or excosecant? No, they aren't all that well known. The basic six functions will serve you just fine.

But, for some history: the *versine* and the others have been found in some of the earliest trig tables. The versine is sometimes called the *sagitta*, which is a Latin

word meaning "arrow." You've undoubtedly heard of the constellation Sagittarius (which is Latin for "archer"). But back to the versine.

The versine is part of a horizontal radius drawn in a circle. When an angle is created using the horizontal radius and a ray is drawn from the center to where the angle intercepts the circle, you then drop a segment perpendicular to the horizontal from the intersection. The versine is the part of the horizontal to the right of the perpendicular. Take a look at Figure 8-8. You see the versines of a 60-degree angle, a 30-degree angle, and a 45-degree angle. For convenience, they're all drawn in a unit circle; this makes the numbers nicer.

So, what is the value of versine of an angle measuring 60 degrees? Or 30 degrees? Or 45 degrees?

You've probably noticed that I've picked very special angle measures. These are from the special right triangles covered in Chapter 5. The 30-60-90 right triangle and the 45-45-90 right triangle (isosceles right triangle) have very cooperative measures of their sides.

Starting with the versine of a 60-degree angle, refer to the left-most circle in Figure 8-8. The shorter leg is always half the measure of the hypotenuse, and the longer leg is $\sqrt{3}$ times the measure of the shorter leg. This is a unit circle, so the measures of the sides are $\frac{1}{2}, \frac{\sqrt{3}}{2}$, and 1. The shorter leg, measuring $\frac{1}{2}$, shares the horizontal radius with the versine. Subtracting $\frac{1}{2}$ from 1, you're left with $\frac{1}{2}$, so you can say: versin $60° = \frac{1}{2}$.

FIGURE 8-8: The versines of angles.
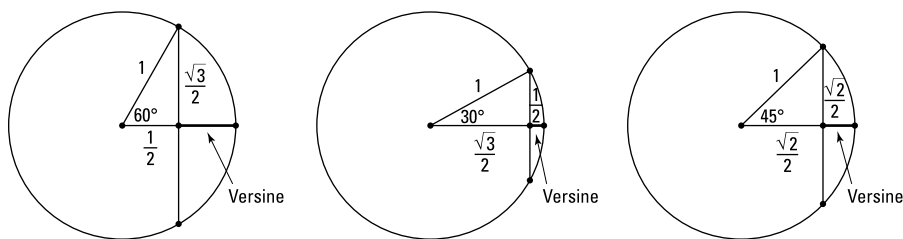
Next, onto the versine of a 30-degree angle. Refer to the middle circle in Figure 8-8. You have the same three measures of the sides as described previously. But this time, it's the longer side that shares the radius with the versine. Subtracting $\frac{\sqrt{3}}{2}$ from 1, you have $1 - \frac{\sqrt{3}}{2} = \frac{2}{2} - \frac{\sqrt{3}}{2} = \frac{2-\sqrt{3}}{2}$, so you can say: versin $30° = \frac{2-\sqrt{3}}{2}$.

And, finally, you find the versine of a 45-degree angle. Referring to the right-most circle in Figure 8-8, you see that the versine shares the radius with a triangle side measuring $\frac{\sqrt{2}}{2}$. Subtracting $\frac{\sqrt{2}}{2}$ from 1, you have $1 - \frac{\sqrt{2}}{2} = \frac{2}{2} - \frac{\sqrt{2}}{2} = \frac{2 - \sqrt{2}}{2}$, so you can say: versin $45° = \frac{2 - \sqrt{2}}{2}$.

And you can have just as much fun finding function values for the coversine, haversine, and all the others!

IN THIS CHAPTER

» **Recognizing angles of elevation and depression**

» **Determining heights of buildings**

» **Calculating the slope of a hill**

» **Measuring when objects are really high up**

» **Dealing with odd shapes and distances**

Chapter **9**

Applying Yourself to Trig Functions

Back when trig functions were first developed or recognized — centuries ago — the motivation for creating the functions wasn't so people could sit around and say, "Hey, Caesar, did you know that the sine of 45 degrees is $\frac{\sqrt{2}}{2}$?" Instead, the math gurus of the past worked out the principles of trigonometry because they needed some order or consistency to the numbers that they were applying to astronomy, agriculture, and architecture. They figured out the relationships among all these numbers and shared them with the rest of the known, civilized world.

First Things First: Elevating and Depressing

Mathematical problems that require the use of trig functions often have one of two related angles: the angle of elevation or the angle of depression. The scenarios that use these angles usually involve calculating distances that can't be physically measured. For example, these angles are used when finding the distance from an

airplane in the air down to a point on the ground or the distance up to a balloon or another object above you. Use the trig functions to solve for the missing part of the ratio or the side of the imaginary right triangle.

An *angle of elevation* is measured from the horizontal going upward. The horizontal is usually the ground, street, floor, or any other flat object. Even though the ground isn't perfectly flat or horizontal, you determine the measurements with the assumption that it is. In trig, you have to consider the optimal situation — focus on the big picture, rather than on the imperfections. Figure 9-1 shows an angle of elevation.
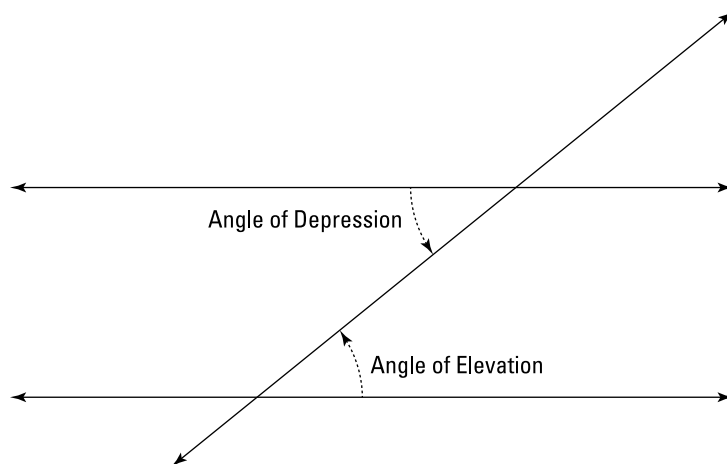
FIGURE 9-1: Angles of elevation and depression.

PTOLEMY: PART RIGHT AND PART WRONG

Ptolemy, also known as Claudius Ptolemaeus, was a Greek citizen who lived in Alexandria, Egypt, from about A.D. 87 to 150. He was an astronomer, mathematician, and geographer. Ptolemy believed that the Sun and other planets revolved around Earth. At that time, only five planets were known, and he believed that they revolved around Earth in this order: Mercury, Venus, the Sun, Mars, Jupiter, and Saturn. This theory was known as the Ptolemaic system. It predicted the positions of the planets with reasonable accuracy, considering that they were naked-eye observations, not enhanced with telescopes. But Ptolemy may not have really believed in this system of the planets — perhaps he used it only as a method of calculating their relative positions. This man did get it right, though, in determining that Earth is a sphere, not flat — this theory affected much of his important work in geography and cartography. His works, including an error that had Asia extending too far to the east, probably influenced Columbus's decision to sail west for the Indies.

An *angle of depression* is measured from the horizontal going downward. For these angles, the horizontal is, for example, an airplane's flight path or a person's line of sight while standing on a mountaintop. The angle is formed when a person on that plane or mountaintop looks at an object on the ground (or on a path parallel to the ground). Refer to Figure 9-1 for a sample angle of depression.

Measuring Tall Buildings with a Single Bound

Every day, people use trigonometry to measure things that they can't reach. How high is that building? Will this ladder reach to the top of that tree? By using the appropriate trig functions, you can find answers to such questions. Two major considerations to keep in mind when working out problems by using trig are as follows: Which trig function should you use, and what are the units or measures in the answer?

The missing values in the ratios of the trig functions represent the missing parts in the problems. You assign the known values appropriately and solve for what's left.

Rescuing a child from a burning building

Consider a day in the life of one of your city firefighters: A child is trapped in a burning building and leans out the window calling for help. The firefighter is on the ground below with a ladder. He needs to know whether it'll reach the child or whether he needs a longer ladder.

When the firefighter stands 15 feet from the base of the building under the window and looks up at the child, the angle of elevation to the window is 60 degrees. How long does the ladder have to be to reach the window? Figure 9-2 shows the situation using a right triangle.

1. **Identify the parts of the right triangle that you can use to solve the problem.**

You know that the acute angle is 60 degrees, and the adjacent side of the triangle is along the ground; the distance from the vertex of the angle (where the firefighter is standing) to the base of the building is 15 feet (the adjacent side). The hypotenuse is the length needed for the ladder — call it *x*. Figure 9-2b shows you the triangle.
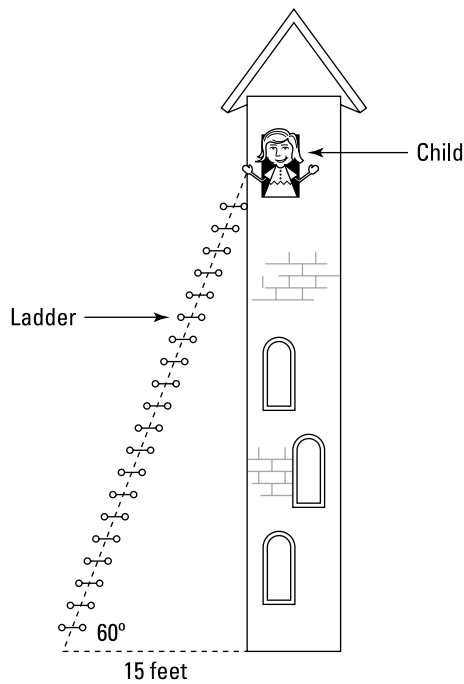
FIGURE 9-2:
The hypotenuse
of the triangle
represents the
ladder.
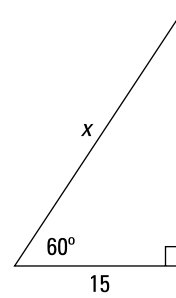
(a)   (b)

2. **Determine which trig function to use.**

The adjacent side and hypotenuse are parts of the cosine ratio. Those sides are also parts of the secant ratio, but if at all possible, you should use the three main functions, not their reciprocals.

3. **Write an equation with the trig function; then insert the values that you know.**

For a refresher on those values, look at the charts in Chapter 6. The cosine of 60 degrees is $\frac{1}{2}$, the adjacent side is 15 feet, and the hypotenuse is unknown.

$$\cos 60 = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$\frac{1}{2} = \frac{15}{x}$$

4. **Solve the equation.**

Cross-multiplying, you get

$$\frac{1}{2} = \frac{15}{x}$$
$$1 \cdot x = 15 \cdot 2$$
$$x = 30$$

The ladder needs to be 30 feet long. (That firefighter had better be pretty strong!)

Determining the height of a tree

Suppose you're flying a kite, and it gets caught at the top of a tree. You've let out all 100 feet of string for the kite, and the angle that the string makes with the ground (the angle of elevation) is 75 degrees. Instead of worrying about how to get your kite back, you wonder, "How tall is that tree?"

Figure 9-3 shows the scenario.

To find a solution to your quandary, follow these steps:

1. **Identify the parts of the right triangle that you can use to solve the problem.**

   The hypotenuse of the right triangle is the length of the string. The side opposite the 75-degree angle is what you're solving for; call it *x*.

2. **Determine which trig function to use.**

   The hypotenuse and opposite side are part of the sine ratio.

3. Write an equation with the trig function; then insert the values that you know.

The 75-degree angle isn't one of the more-common angles, so use a scientific calculator or your computer to obtain a value for the sine, correct to three decimal places. The sine of 75 degrees is about 0.966, the hypotenuse is 100 feet, and the opposite side is what is unknown.

$$\sin 75 = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$0.966 = \frac{x}{100}$$

4. Solve the equation.

Cross-multiplying, you get

$$0.966 = \frac{x}{100}$$
$$0.966 \cdot 100 = x$$
$$96.6 = x$$

The tree is over 96 feet tall. Lots of luck retrieving the kite.

Measuring the distance between buildings

Weevil Upheaval makes his living by jumping, on his motorcycle, from building to building, cliff to bluff, or any place he can get attention for doing it. His record jump is a distance of 260 feet, from one building to another. Upheaval is on to his next feat and needs to determine the distance from one building to another. His assistant, Steady Eddie, holds a 6-foot pole perpendicular to the roof he's standing on. When Upheaval, standing on top of the first building, sights straight across to a point at the base of the pole and then sights a point halfway up the pole, the angle of elevation is 1 degree. Will he be able to make the jump? See Figure 9-4 for a visual of Upheaval's calculation.

1. Identify the parts of the right triangle that you can use to solve the problem.

You know the length of the side opposite the 1-degree angle, which is half the pole length (half is 3 feet), and the adjacent side is the unknown distance. Call that distance x.

2. Determine which trig function to use.

The tangent of an angle uses opposite divided by adjacent.

FIGURE 9-4:
Upheaval's precarious situation.
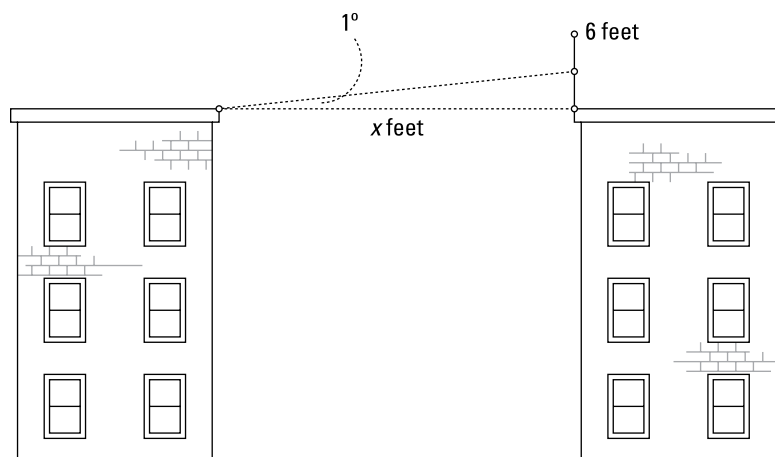
3. **Write an equation with the trig function; then insert the values that you know.**

The length of half the pole is 3 feet, so the equation looks like this:

$$\tan 1° = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{x}$$

4. **Solve for the value of x.**

Use your calculator to find the value of the tangent of 1 degree. Round the value to four decimal places. Then just one decimal place is enough for your final answer.

$$\tan 1° = \frac{3}{x}$$
$$0.0175 = \frac{3}{x}$$
$$x = \frac{3}{0.0175}$$
$$= 171.4$$

You find that the distance between the buildings is a little less than 172 feet across. Upheaval should be able to make the jump easily, because his record is 260 feet.

Measuring Slope

Have you ever noticed a surveyor along the road, peering through an instrument, looking at a fellow worker holding up a sign or a flag? Haven't you ever wondered what they're doing? Have you wanted to get out and look through the instrument,

too? With trigonometry, you can do just what those workers do — measure distances and angles. Land surveyors use trigonometry and their fancy equipment to measure things like the slope of a piece of land (how far it drops over a certain distance).

If you read the first section in this chapter, you may recognize that the slope of land downward is sort of like an angle of depression. Slopes, angles of depression, and angles of elevation are all interrelated because they use the same trig functions. It's just that in slope applications, you're solving for the angle rather than a length or distance.

To solve one of these surveying problems involving slope, you can use the trig ratios and right triangles. One side of the triangle is the distance from one worker to the other; the other side is the vertical distance from the ground to a point on a pole. You form a ratio with those measures and determine the angle — *voilà!*

Suppose that Elliott and Fred are making measurements for the road-paving crew. They need to know how much the land slopes downward along a particular stretch of road to be sure there's proper drainage. Elliott walks 80 feet from Fred and holds up a long pole, perpendicular to the ground, that has markings every inch along it. Fred looks at the pole through a sighting instrument. Looking straight across, parallel to the horizon, Fred sights a point on the pole 50 inches above the ground — call it point A. Then Fred looks through the instrument at the bottom of the pole, creating an angle of depression. See Figure 9-5 for a diagram of this situation. What is the angle of depression, or slope of the road, to where Elliott is standing?

1. **Identify the parts of the right triangle that you can use to solve the problem.**

   The values you know are for the sides adjacent to and opposite the angle of depression. Call the angle measure *x*.

2. **Determine which trig function to use.**

   The tangent of the angle with measure *x* uses opposite divided by adjacent.

3. **Write an equation with the trig function; then insert the values that you know.**

   $$\tan x = \frac{\text{opposite}}{\text{adjacent}}$$

   In this problem, you need to write the equation with a common unit of measurement — either feet or inches. Changing 80 feet to inches makes for a big number; changing 50 inches to feet involves a fraction or decimal.

Whichever unit you choose is up to you. In this example, I choose the big number, so I convert feet to inches.

$$80 \text{ feet} = 80 \cdot 12 \text{ inches} = 960 \text{ inches}$$

Substituting in the values, you get the tangent of some angle with a measure of *x* degrees:

$$\tan x = \frac{\text{opposite}}{\text{adjacent}} = \frac{50}{960} = 0.05208333$$

4. **Solve for the value of *x*.**

   Using a calculator, you plug in some angle values for the tangent (tan) function and determine that an angle of 2.9 degrees has a tangent of 0.0507, and a 3-degree angle has a tangent of 0.0524. The 3-degree angle has a tangent that's closer to 0.05208333, so you can estimate that the road slopes at a 3-degree angle between Elliott and Fred.
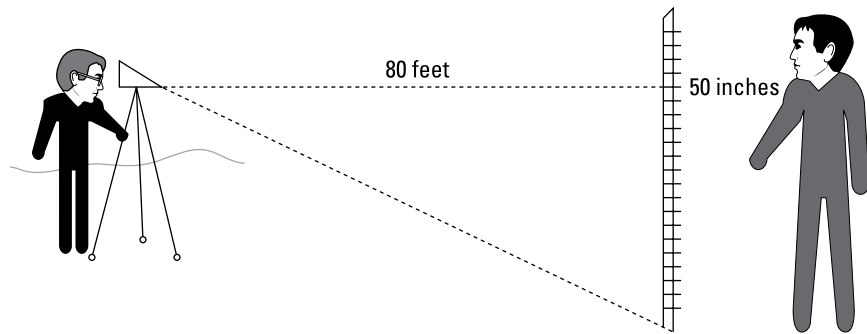
FIGURE 9-5:
Sighting along a downward slope.

Another way to solve for that angle measure is to use the inverse tangent function. I explain about inverse functions in Chapter 14, but you can jump ahead and take advantage of technology. You can find an explanation of the inverse function buttons on a calculator in the Appendix. My calculator says that the angle whose tangent is 0.05208333 is an angle of 2.98146 degrees. So, the estimate of 3 degrees is right on.

The Sky's (Not) the Limit

Early trigonometry had many earthbound applications — surveyors and engineers have used it for centuries. Over time, astronomers and navigators on journeys around the world began using trig to solve many mysteries here on Earth and

in outer space. They estimated or measured angles by sighting objects in the heavens and charting their movements. Then they used the angle between one sighting and another to solve for the distances that are unreachable.

Spotting a balloon

Cindy and Mindy, standing a mile apart, spot a hot-air balloon directly above a particular point on the ground somewhere between them. The angle of elevation from Cindy to the balloon is 60 degrees; the angle of elevation from Mindy to the balloon is 70 degrees. Figure 9-6 shows a visual representation. How high is the balloon?

If you look at Figure 9-6a, you see that two right triangles are formed. The two triangles share a side — the one opposite the measured acute angle in each. Call the length of that shared side y. The two adjacent sides add up to 1 mile, so you can keep the variables to a minimum by naming one side x and the other $1-x$. Figure 9-6b shows the triangles with the variables.
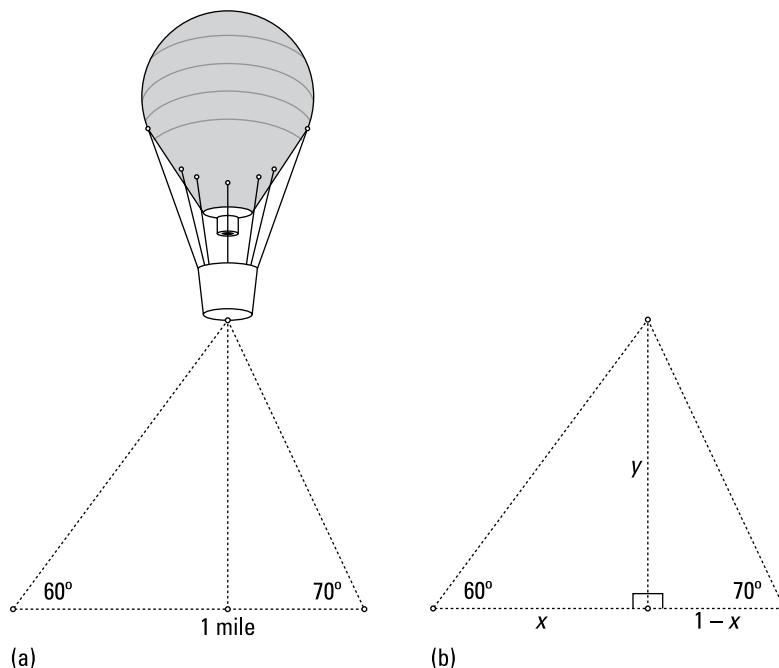
FIGURE 9-6:
Two friends spot a hot-air balloon and use a right triangle to solve the problem.

To figure out how high the balloon is, follow these steps:

1. **Identify the parts of the triangles that you can use to solve the problem.**

   In both triangles, you have variables for the adjacent and opposite sides of the acute angles of elevation.

2. **Determine which trig function to use.**

   The tangent ratio uses the opposite and adjacent sides.

3. **Write equations with the trig functions.**

   $$\tan 60° = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} \quad \text{and} \quad \tan 70° = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{1-x}$$

   $$\tan 60° \qquad\qquad = \frac{y}{x} \qquad \tan 70° \qquad\qquad = \frac{y}{1-x}$$

4. **Solve for x by setting the equations equal to one another.**

   First, solve each of the equations for y.

   $$\tan 60° = \frac{y}{x} \qquad \text{and} \quad \tan 70° = \frac{y}{1-x}$$
   $$x\tan 60° = y \qquad (1-x)\tan 70° = y$$

   Set those two equations equal to one another and solve for x.

   $$x \cdot \tan 60° = (1-x)\tan 70°$$
   $$x \cdot \tan 60° = \tan 70° - x\tan 70°$$
   $$x \cdot \tan 60° + x \cdot \tan 70° = \tan 70°$$
   $$x(\tan 60° + \tan 70°) = \tan 70°$$
   $$x = \frac{\tan 70°}{\tan 60° + \tan 70°}$$

5. **Solve for the value of x.**

   You find the value of x by finding the values of the functions with a calculator or computer. Upon doing so, you find that x is approximately 0.613 mile. Put that value into one of the equations to solve for y:

   $$x \cdot \tan 60° = y$$
   $$(0.613)\tan 60° = y$$
   $$(0.613)(1.732) = y$$
   $$1.062 = y$$

   The balloon is 1.062 miles high — sounds a tad high to me!

Tracking a rocket

In this example, a rocket is shot off and travels vertically, as a scientist watches its flight from a mile away. One second into the flight, the angle of elevation of the rocket is 30 degrees. Two seconds later, the angle of elevation is 60 degrees. How far did the rocket travel in those two seconds? Figure 9-7 shows the rocket rising vertically.

1. **Identify the parts of the triangles that you can use to solve the problem.**

In Figure 9-7, you see two right triangles. One is superimposed on the other and shares a side — the adjacent side. In both triangles, the relevant sides are those that are adjacent and opposite the angles of elevation.

2. **Determine which trig function to use.**

The ratio of the tangent uses the adjacent and opposite sides.

3. **Write equations with the trig functions.**

$$\tan 30° = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{1} = x$$

$$\tan 60° = \frac{\text{opposite}}{\text{adjacent}} = \frac{x+y}{1} = x+y$$
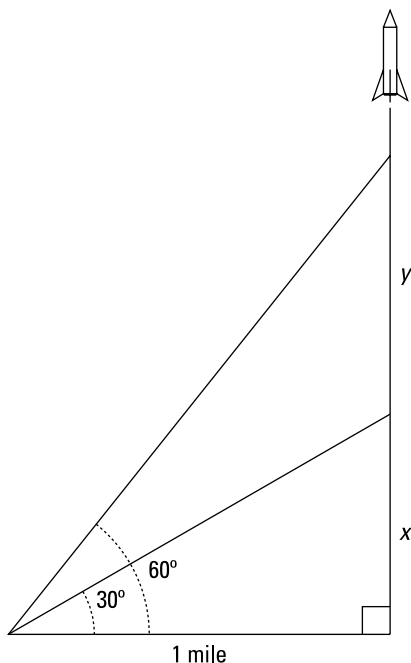
FIGURE 9-7:
A rocket, 1 mile
from a scientist,
rises vertically.

60°

30°

y

x

1 mile

SINE AND COSINE WITH ALGEBRA

You can approximate, fairly accurately, the sine and cosine of angles with an *infinite series,* which is the sum of the terms of some sequence, or list, of numbers. Take note, however, that the series for sine and cosine are accurate only for angles from about −90 degrees to 90 degrees. The series for the sine of an angle is $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \dots$ and the series for the cosine of an angle is $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \dots$. To use these formulas, you have to write the angle measure, *x,* in radians and carry out the computations to several places. The exclamation points in the formulas don't mean, "Oh, goodness! It's a 3!" The exclamation points are mathematical operations called *factorials. Factorial* means to multiply that number times every positive integer smaller than it. Going back to the series for the sine, an angle of 30 degrees is about 0.5236 radian. To find sin 0.5236, use the formula to get

$$\sin 0.5236 = 0.5236 - \frac{(0.5236)^3}{3!} + \frac{(0.5236)^5}{5!} - \frac{(0.5236)^7}{7!}$$
$$= 0.5236 - 0.0239 + 0.000328 - 0.000002$$
$$= 0.500026$$

The result is pretty close to the sine of 30 degrees, which is $\dfrac{1}{2}$. Carrying out the computations using a few more terms will make the result even closer to the actual answer. And the closer the angle measure is to 0, the more quickly the value of the sine or the cosine meets the exact value (and the fewer the terms needed for the answer).

4. **Solve for the values of *x* and *y*.**

The tangents of 30-degree and 60-degree angles are convenient values. You can refer back to Chapter 6, find the function values, and write that $\tan 30° = \dfrac{\sqrt{3}}{3} = x$ and $\tan 60° = \sqrt{3} = x + y$.

The value of *y* is the distance that the rocket traveled between the first and second sightings, so, solving for *y*, you get

$$y = x + y - x = \sqrt{3} - \frac{\sqrt{3}}{3} = \frac{3\sqrt{3}}{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3} \approx 1.155$$

The rocket rose about 1.155 miles in two seconds.

Measuring the view of satellite cameras

Consider a satellite that orbits Earth at an altitude of 750 miles. Earth has a radius of about 3,950 miles. How far in any direction can the satellite's cameras see?

Figure 9-8 shows the satellite and the length of the camera's scope due to the curvature of Earth.

1. **Identify the parts of the triangle that you can use to solve the problem.**

Because a satellite's line of sight is tangent to the curvature of Earth, and tangents to a circle form 90-degree angles with radii of the circle, you can see two right triangles in Figure 9-8. The two sides of angle θ are the radius touching the tangent to the circle and the segment extending from the center of the circle up to the satellite. These sides are the hypotenuse and adjacent side of the right triangle with acute angle θ.

2. **Determine which trig function to use.**

The adjacent side and hypotenuse are part of the ratio for the cosine of θ.
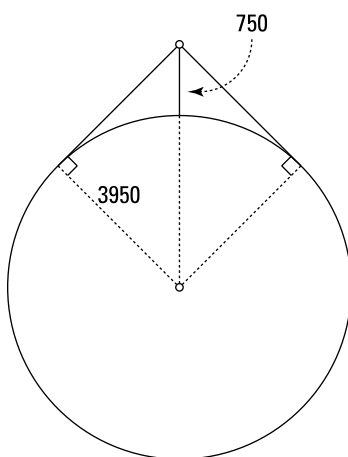
750

3950

3. **Write the equation with the trig function; then input the measures that you know and solve for cos θ.**

The adjacent side measures 3,950 miles, and the hypotenuse is the sum of the radius and height of the satellite: $3,950 + 750 = 4,700$ miles.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3,950}{4,700} \approx 0.8404$$

4. **Determine the value of θ.**

Use your calculator to find the angle whose cosine is closest to 0.8404. To the nearest degree, an angle of 33 degrees has this cosine. You can use the inverse cosine function to do this. In the Appendix, you see how to do the computation. You can find more on inverse functions in Chapter 14.

5. **Determine how much of Earth's circumference is covered in either direction from the satellite.**

The satellite's line of sight goes 33 degrees in either direction, or 66 degrees total, which is $\dfrac{66}{360}$ of the entire circumference (because all the way around would be 360 degrees). If the radius of Earth is 3,950 miles, then you can substitute that number into the equation for a circle's circumference: $C = 2\pi r = 2(3.14)(3,950) \approx 24,819$. That's Earth's circumference. The distance that the satellite scans, then, is $\dfrac{66}{360}(24,819) \approx 4,550$, or about 4,550 total miles or 2,275 miles in any direction.

Calculating Odd Shapes and Maneuvering Corners

Sometimes, finding a measure isn't so easy. You may have to deal with an irregular shape or even calculate your way around a fixed object. Whatever the case, you can use trigonometry to find the answers you've been searching for.

Finding the area of a triangular piece of land

The most commonly used formula for the area of a triangle is $A = \dfrac{1}{2}bh$, where A is the area, b is the length of the triangle's base, and h is the height of the triangle drawn perpendicular to that base. Figure 9-9a illustrates the different components of this formula.

(a) (b)
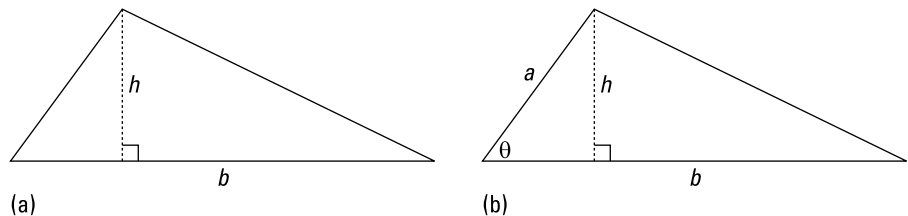
This area formula works fine if you can get the measure of the base and the height, and if you can be sure that you've measured a height that's perpendicular to the side of the triangle. But what if you have a triangular yard — a *big* triangular yard — and have no way of measuring some perpendicular segment to one of the

sides? One alternative is to use Heron's Formula, which uses the measures of all three sides. (I show you that one in the next section.) The other alternative, of course, is to use trigonometry — or, at least, a formula with an angle measure in it. To measure that angle, you can be very sophisticated and get a surveying appa-ratus, or if you've got a protractor handy, you can do a decent estimate by extend-ing the sides at an angle for a bit and eyeballing the angle size.

The trig formula for finding the area of a triangle is $A = \frac{1}{2}ab\sin\theta$, where a and b are two sides of the triangle and θ is the angle formed between those two sides. You don't need the measure of the third side at all, and you certainly don't need a perpendicular side.

Using trigonometry, I can show you where this formula comes from. Take a look at the triangle in Figure 9-9b, with sides a and b and the angle between them.

Start with the traditional formula for the area of this triangle, $A = \frac{1}{2}bh$. Then look at the smaller triangle to the left. (Because the height is drawn perpendicular to the base, the sides and height form a right triangle.) The acute angle θ has a sine equivalent to the following: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{h}{a}$. If you solve that equation for h by multiplying each side by a, you get

$$\sin\theta = \frac{h}{a}$$
$$a\sin\theta = h$$

Replace the h in the traditional formula with its equivalent from this equation, and you get

$$A = \frac{1}{2}bh = \frac{1}{2}b(a\sin\theta) = \frac{1}{2}ab\sin\theta$$

Check out how this formula works in an actual problem. The triangle in Figure 9-10 shows the measures of two of its sides and the angle between them.
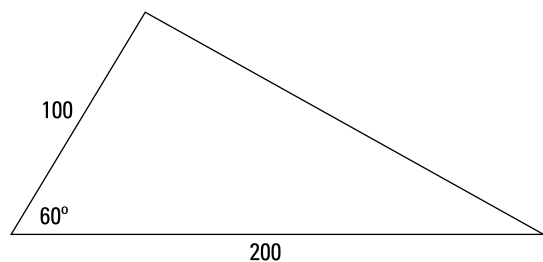
FIGURE 9-10:
Finding the area
of a triangle.

100

60°

200

To find the area of the triangle in Figure 9-10:

1. **Use the formula** $A = \dfrac{1}{2}ab\sin\theta$, **inserting the values that you know.**

$$A = \frac{1}{2}(100)(200)\sin 60°$$

2. **Solve for the value of the area.**

$$= \frac{1}{2}(100)(200)\cdot\frac{\sqrt{3}}{2}$$

$$= 10{,}000\cdot\frac{\sqrt{3}}{2} \approx 8{,}660$$

The area is about 8,660 square units.

Using Heron's Formula

As promised, in this section, I show you how to find the area of a triangle using Heron's Formula. Heron's Formula is especially helpful when you have access to the measures of the three sides of a triangle but can't draw a perpendicular height or don't have a protractor for measuring an angle.

Consider the situation where you have a large ball of string that's 100 yards long and you're told to mark off a triangular area — with the string as the marker for the border of the area. You walk 40 yards in one direction, take a turn, and walk another 25 yards; then you head back to where you started and use up that last 35 yards of string. How large an area have you created?

Heron's Formula reads: $A = \sqrt{s(s-a)(s-b)(s-c)}$ where a, b, and c are the lengths of the sides of the triangle and s is the semi-perimeter (half the perimeter).

In the case of your triangle and the string, the perimeter is $40 + 25 + 35 = 100$ yards. Half that is 50, so the formula now reads:

$$A = \sqrt{50(50-40)(50-25)(50-35)}$$

$$= \sqrt{50(10)(25)(15)} = \sqrt{187{,}500} \approx 433$$

You've marked off an area of approximately 433 square yards.

Moving an object around a corner

Here's an application of trigonometry that you may very well be able to relate to: Have you ever tried to get a large piece of furniture around a corner in a house? You twist and turn and put it up on end, but to no avail. In this example, pretend that you're trying to get a 15-foot ladder around a corner where two 4-foot-wide hallways meet at a 90-degree angle. Figure 9-11a shows a picture of the situation.

The tightest part comes when the ladder is halfway through the hallway, or when the angles where it touches the outer walls are the same. When the ladder is at the tightest point, it forms a right triangle with equal sides — half the ladder to each side of the corner. Because the sides of the right triangle are equal at this point, you've got an isosceles right triangle, which has two 45-degree angles (see Figure 9-11b, which shows the longest a ladder can be to fit around the corner). How long are the sides of the right triangle, then? When you know the dimensions of this isosceles right triangle, you can look at the hypotenuse — the ladder — and determine if it's short enough or too long to fit around the tightest part of this corner. And, of course, you don't want to scrape or punch holes in the wall!

(a) (b)
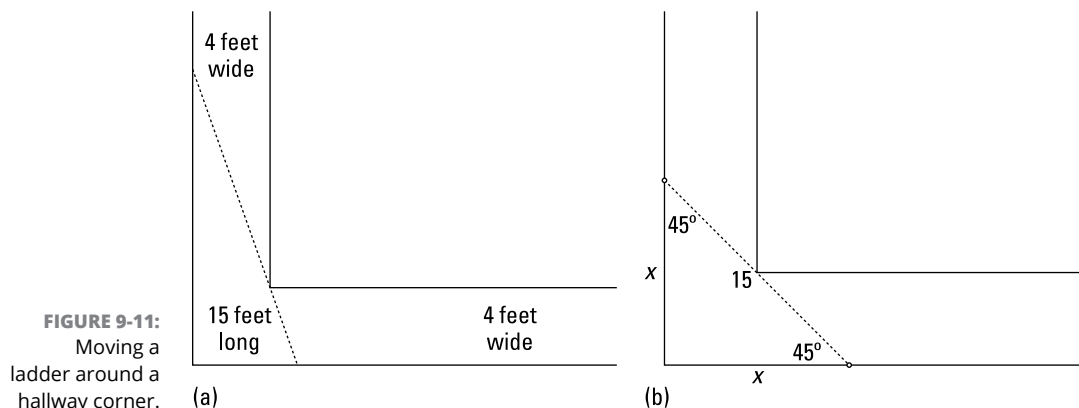
1. **Determine the trig function that you can use with the measures available.**

   The hypotenuse is the length of the ladder — 15 feet. The opposite and adjacent sides are the same in an isosceles right triangle, and in this case, those two lengths are each 8 feet. You know this measure because all the triangles are isosceles right triangles, which means they have 45-degree angles and equal leg measures (see Figure 9-12).

2. **Determine which trig function to use.**

   Both sine and cosine include the length of the hypotenuse, which is what you're solving for, so you can use either function.

3. **Write the equation with the trig function; then insert the measures that you know.**

   $$\sin 45 = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{8}{\text{hypotenuse}}$$

FIGURE 9-12:
All the triangles
have the same
angle measures.
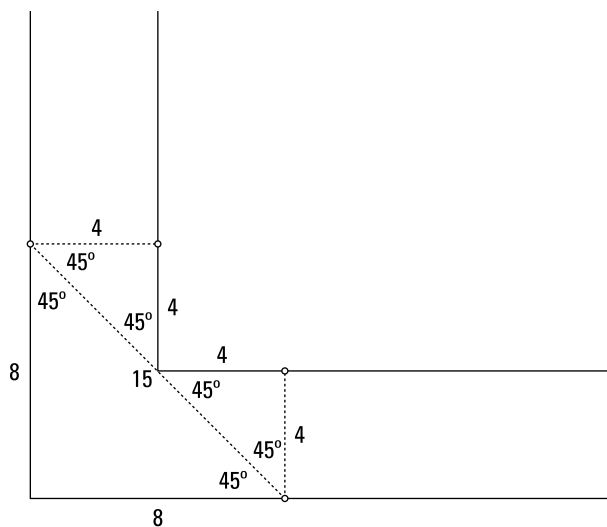
4. **Solve for the value of the hypotenuse.**

$$\frac{\sqrt{2}}{2} = \frac{8}{\text{hypotenuse}}$$

$$\sqrt{2} \cdot \text{hypotenuse} = 8 \cdot 2$$

$$\text{hypotenuse} = \frac{16}{\sqrt{2}}$$

$$\approx 11.314$$

You find that at the tightest point around the corner, the hypotenuse is only slightly more than 11 feet. That 15-foot ladder will never fit around the corner.

3

Identities

IN THIS PART . . .

Develop a working list of the relationships between the trig functions.

Use the trig identities to make statements more user-friendly.

Determine the easiest way to solve a trig identity.

Avoid going in endless loops with identities.

IN THIS CHAPTER

» **Recognizing reciprocal identities**

» **Putting functions head-over-tails by using ratio identities**

» **Performing attitude adjustments on negative angles**

» **Relating functions to Pythagoras's way of thinking**

» **Using multiple identities to simplify matters**

Chapter **10**

Introducing Basic Identities

I'm sure a thousand questions are running through your mind: What's a trig identity? Is it possible to have a mistaken trig identity? Does anyone commit trig-identity theft? Can you have an identity crisis with trig? The answer: No — probably not.

Trig identities aren't nearly as sinister as you may think. They're actually very helpful tools in simplifying trig expressions and solving equations. These identities are special to trigonometry. Basically, they're equivalences — they give you options to substitute into equations in order to simplify. For example, wouldn't you rather use the number 1 than $\frac{1,623}{1,623}$? Of course! In most cases, the number 1 is simpler. That's how trig identities work — they replace something with something simpler or more useful.

Identities are divided into different types, or categories, in order to help you remember them more easily and figure out when to use them more efficiently. In this chapter, I cover the gamut.

Flipping Functions on Their Backs: Reciprocal Identities

The simplest and most basic trig identities are those involving the reciprocals of the trig functions. To jog your memory, a *reciprocal* of a number is 1 divided by that number — for example, the reciprocal of 2 is $\frac{1}{2}$. Another way to describe reciprocals is to point out that the product of a number and its reciprocal is 1. In the case of 2 and its reciprocal, $\frac{1}{2}$, $2 \cdot \frac{1}{2} = 1$. The same principle goes for the trig reciprocals.

TRIG RULES

Here's how the reciprocal identities are defined.

>> The reciprocal of sine is cosecant: $\frac{1}{\sin \theta} = \csc \theta$

>> The reciprocal of cosine is secant: $\frac{1}{\cos \theta} = \sec \theta$

>> The reciprocal of tangent is cotangent: $\frac{1}{\tan \theta} = \cot \theta$

>> The reciprocal of cotangent is tangent: $\frac{1}{\cot \theta} = \tan \theta$

>> The reciprocal of secant is cosine: $\frac{1}{\sec \theta} = \cos \theta$

>> The reciprocal of cosecant is sine: $\frac{1}{\csc \theta} = \sin \theta$

In true fashion, when you multiply the reciprocals together, you get 1:

$$\sin \theta \cdot \csc \theta = 1$$
$$\cos \theta \cdot \sec \theta = 1$$
$$\tan \theta \cdot \cot \theta = 1$$

The reciprocal identity is a very useful one when you're solving trig equations — especially those involving fractions. If you find a way to multiply each side of an equation by a function's reciprocal, you may be able to reduce some part of the equation to 1 — and simplifying is always a good thing.

The following expression has two fractions added together — and a total of four trig identities: $\frac{\sin \theta}{\csc \theta} + \frac{\cos \theta}{\sec \theta}$. This can be simplified using the reciprocal identities.

1. **Replace the $\csc \theta$ with its identity and $\sec \theta$ with its identity.**

   $$\frac{\sin \theta}{\csc \theta} + \frac{\cos \theta}{\sec \theta} = \frac{\sin \theta}{\frac{1}{\sin \theta}} + \frac{\cos \theta}{\frac{1}{\cos \theta}}$$

2. **Multiply each numerator by the reciprocal of its denominator.**

$$\frac{\sin\theta}{\frac{1}{\sin\theta}} + \frac{\cos\theta}{\frac{1}{\cos\theta}} = \sin\theta \cdot \frac{\sin\theta}{1} + \cos\theta \cdot \frac{\cos\theta}{1} = \sin^2\theta + \cos^2\theta$$

This simplification is almost finished. You'll see what comes next after reviewing the upcoming section on Pythagorean identities.

Function to Function: Ratio Identities

Trig has two identities called *ratio identities.* This label can be confusing, because all the trig functions are defined by ratios. Somewhere along the line, however, mathematicians thought this description was perfect for these two identities, because they're basically fractions made up of two trig functions, one above the other. The ratio identities create ways to write tangent and cotangent by using the other two basic functions, sine and cosine.

TRIG RULES

The ratio identities are $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

These two identities come from the simplification of a couple of complex fractions. If you use the basic definitions for sine, cosine, and tangent — $\text{sine} = \dfrac{\text{opp}}{\text{hyp}}$,

$$\text{cosine} = \frac{\text{adj}}{\text{hyp}}, \quad \text{and} \quad \text{tangent} = \frac{\text{opp}}{\text{adj}} \quad - \quad \text{then you can use them to get}$$

$$\frac{\text{sine}}{\text{cosine}} = \frac{\dfrac{\text{opp}}{\text{hyp}}}{\dfrac{\text{adj}}{\text{hyp}}} = \frac{\text{opp}}{\text{hyp}} \cdot \frac{\text{hyp}}{\text{adj}} = \frac{\text{opp}}{\text{adj}} = \text{tangent}$$

Likewise, because cotangent is the reciprocal of tangent,

$$\text{cotangent} = \frac{1}{\text{tangent}} = \frac{1}{\dfrac{\text{sine}}{\text{cosine}}} = \frac{\text{cosine}}{\text{sine}}$$

One little trick I've used over the years, to keep track of which ratio identity has sine over cosine and which is cosine over sine, is to use the alphabet. Cotangent starts with the letter *c,* so I use this to determine that cosine over sine also starts with *c* (on the top). The ratio identity for tangent isn't quite as nice, but I know that tangent and sine are close together in the alphabet, so, since tangent starts with *t,* and *s* is pretty close to *t,* the ratio identity must start with *s.* Well, it works for me!

Now to the task of simplifying an expression involving a ratio identity: $\cot\theta\,(\sec\theta)$. Your goal is to write the product in terms of just one trig function.

1. **Start by replacing the $\cot\theta$ with its ratio identity and $\sec\theta$ with its identity.**

    $$\cot\theta\,(\sec\theta) = \frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\cos\theta}$$

2. **Reduce the fraction and multiply.**

    $$\frac{\cancel{\cos\theta}}{\sin\theta} \cdot \frac{1}{\cancel{\cos\theta}} = \frac{1}{\sin\theta}$$

3. **This may be the format you want, or you may prefer to not have a fraction. Just use the reciprocal identity.**

    $$\frac{1}{\sin\theta} = \csc\theta$$

Opposites Attract: Opposite-Angle Identities

The *opposite-angle identities* change expressions with negative angles into equivalences with positive angles. Negative angles are great for describing a situation, but they aren't really handy when it comes to sticking them in a trig function and

calculating that value. So, for example, you can rewrite the sine of −30 degrees as the sine of 30 degrees by putting a negative sign in front of the function:

$$\sin(-30°) = -\sin(30°)$$

This identity works for other angles, too. The angle measure doesn't have to be −30 degrees; any negative angle works. This negative-angle business works differently for different functions, though. First, consider the identities, and then see how they came to be.

The opposite-angle identities for the three most basic functions are

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

The rule for the sine and tangent of a negative angle almost seems intuitive. But what's with the cosine? How can the cosine of a negative angle be the same as the cosine of the corresponding positive angle? Here's how it works.

If you refer back to Chapter 7, you find that the function values of angles with their terminal sides in the different quadrants have varying signs. Sine, for example, is positive when the angle's terminal side lies in the first and second quadrants, whereas cosine is positive in the first and fourth quadrants. In addition, Chapter 3 shows you how to draw angles on a coordinate plane: Positive angles go counterclockwise from the positive *x*-axis, and negative angles go clockwise.

With those points in mind, take a look at Figure 10-1, which shows a −45-degree angle and a 45-degree angle.

FIGURE 10-1:
Angles of
−45 degrees and
45 degrees.
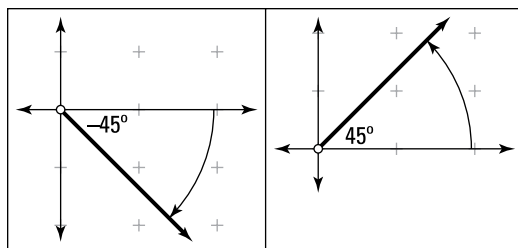
First, consider the −45-degree angle. This angle has its terminal side in the fourth quadrant, so its sine is negative. A 45-degree angle, on the other hand, has its terminal side in the first quadrant, so it has a positive sine. What about a negative angle whose terminal side ends up in the second quadrant, like −200 degrees? A +200-degree angle has a negative sine, because its terminal side is in the third

quadrant, but a −200-degree angle has a positive sine, because its terminal side is in the second quadrant. The values of the sines are opposites — further convincing you of the rule that the sine of a negative angle is the opposite value of that of the positive angle with the same measure.

Now on to the cosine function. In light of the cosine's sign with respect to the coordinate plane, you know that an angle of −45 degrees has a positive cosine. So does its counterpart, the angle of 45 degrees, which is why $\cos(-45°) = \cos(45°) = \frac{\sqrt{2}}{2}$.

So, you see, the cosine of a negative angle is the same as that of the positive angle with the same measure.

Next, try the identity on another angle, a negative angle with its terminal side in the third quadrant. Figure 10-2 shows a negative angle with the measure of −120 degrees and its corresponding positive angle, 120 degrees.

A NEW WAY TO SLICE PI(E)

Two of the best-known symbols used for two of the constants in mathematics are the Greek letter pi, π, and the lowercase letter e. The value of π, approximately 3.14159, is a decimal that goes on forever. The value of e, approximately 2.71828, goes on forever, too.

The symbol for pi was first introduced in the early 1700s by William Jones, an obscure English writer who was composing a book for math beginners. Common belief is that he chose this letter because p is the first letter in *perimetron*, meaning perimeter. He didn't realize that what he was doing would have such a long-lasting effect on the world of mathematics. A few years later, the famous mathematician Leonhard Euler used the letter p instead of the Greek letter π, but he eventually adopted the Greek notation, too, making it even more popular with the rest of the world.

The letter e represents the base for natural logarithms. Like π, this value occurs naturally — in other words, the value e occurs as a multiplier or base of the equations that represent many natural phenomena. Euler also had a hand in popularizing this symbol sometime in the early to mid-1700s. The question comes to mind as to why he chose that particular letter. Using π or p seems natural for a concept linked to the perimeter of a circle, but the letter e isn't such an obvious choice. Here are a few possible reasons why he chose this letter: First, e is the first letter in *exponential*, which is closely tied to logarithms; or perhaps he used e because it's near the beginning of the alphabet and hadn't been used as a symbol for anything else. Could it be, instead, that Euler chose the letter e because it's the first letter in his name? Whatever the reason, it appears that π and e are here to stay.

The angle of –120 degrees has its terminal side in the third quadrant, so both its sine and cosine are negative. Its counterpart, the angle measuring 120 degrees, has its terminal side in the second quadrant, where the sine is positive and the cosine is negative. So, the sine of –120 degrees is the opposite of the sine of 120 degrees, and the cosine of –120 degrees is the same as the cosine of 120 degrees. In trig notation, it looks like this:

$$\sin(-120°) = -\sin(120°) = -\frac{\sqrt{3}}{2} \text{ and } \cos(-120°) = \cos(120°) = -\frac{1}{2}$$

FIGURE 10-2: Angles of –120 degrees and 120 degrees.
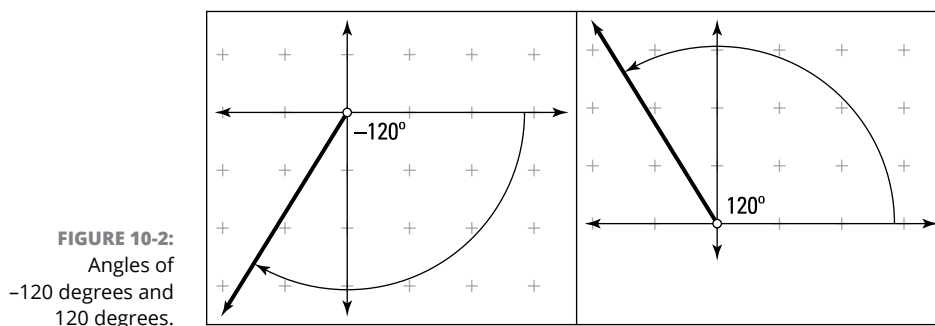
When you apply the opposite-angle identity to the tangent of a 120-degree angle (which comes out to be negative), you get that the opposite of a negative is a positive. Surprise, surprise. So, applying the identity, the opposite makes the tangent positive, which is what you get when you take the tangent of 120 degrees, where the terminal side is in the third quadrant and is, therefore, positive.

But look at the opposite-angle identity for the tangent in another way: Use the ratio identity to prove that it works.

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\frac{\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$$

Revisiting the Classic Theorem: Pythagorean Identities

Good old Pythagoras is at work everywhere — his theorem keeps cropping up in the strangest places. (Not that a chapter in this book is really a strange place, of course.) This section takes you past the basics and expands on them with the three identities called *Pythagorean identities.* (For more on the Pythagorean Theorem, refer to Chapters 2 and 5.)

The Pythagorean identities are building blocks for many of the manipulations of equations and expressions. They provide a generous number of methods for solving problems more efficiently, because they allow you to write complicated expressions in a much simpler form.

The Pythagorean identities are

TRIG RULES

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

TIP

The exponential notation used in these identities is peculiar to trigonometry. The expression $\sin^2\theta$ actually means $(\sin\theta)^2$, which says, "Find the sine of angle θ and then square that number." But mathematicians hate to waste, and having to put those big, cumbersome parentheses around "sin θ" all the time seemed wasteful. So they agreed on a condensed version: The superscript 2 right after "sin" means that you square the whole expression. The same type of notation also goes for the other trig functions ($\cos^2\theta$, $\tan^2\theta$, $\cot^2\theta$, and so on).

The mother of all Pythagorean identities

The Pythagorean identity that birthed the other two is $\sin^2\theta + \cos^2\theta = 1$. But, you may wonder, where did this identity come from, and why is it so important? Last things first: The primary Pythagorean identity is important because it sets a combination of functions equal to 1, and this simplification is very helpful for solving trig equations. As such, it's probably one of the most frequently used identities. This identity comes from putting a right triangle inside the unit circle and substituting values and equations to come up with a whole new equation (see Figure 10-3).

In Chapter 8, you discover that in a circle, $\sin\theta = \dfrac{y}{r}$ and $\cos\theta = \dfrac{x}{r}$, where (x,y) are the coordinates of the point and r is the radius of the circle. The value of x is also the length of the adjacent side of the triangle (horizontal length), and y is the length of the opposite side (vertical length). In a unit circle, the radius is equal to 1. This is shown in Figure 10-3a. When you substitute the 1 for r in the equation, you find that $\sin\theta = \dfrac{y}{r} = y$ and $\cos\theta = \dfrac{x}{1} = x$; so replace the x and y coordinates with $\cos\theta$ and $\sin\theta$, as shown in Figure 10-3b. The Pythagorean Theorem says that when you square the value of a right triangle's two legs and add the results together, you get the square of the hypotenuse. In mathematical notation, it looks like this: $a^2 + b^2 = c^2$. In the case of the right triangle on the unit circle, because the radius (which is also the hypotenuse) is 1, you can say that $x^2 + y^2 = 1^2$. Now replace the x with cos θ and the y with sin θ, switch the two terms around, and you get $\sin^2\theta + \cos^2\theta = 1$.

If all the finagling just seems like a lot of hocus-pocus to you, check out this identity in action. Suppose the angle in question is 30 degrees. Using the values for the functions of a 30-degree angle (see the Appendix), $\sin 30° = \dfrac{1}{2}$ and $\cos 30° = \dfrac{\sqrt{3}}{2}$, and putting them into the identity, you get

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\left(\frac{1}{2} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2 = \frac{1}{4} + \frac{3}{4} = 1$$
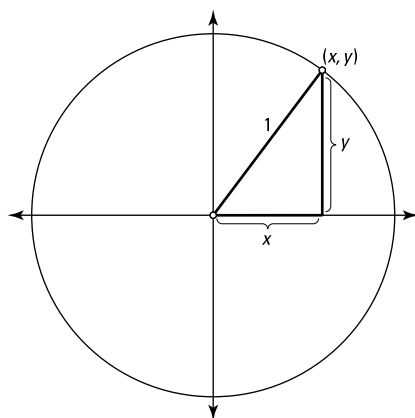
FIGURE 10-3:
Finding the Pythagorean identity on a unit circle.
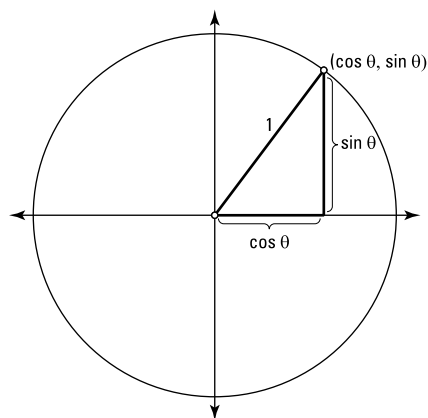
(a)   (b)

Extending to tangent and secant

The other two Pythagorean identities stem from the first one involving sine and cosine. All you do is throw in a little algebra, apply the reciprocal and ratio identities (see those sections earlier in this chapter), simplify, and — poof! — you get two new identities.

1. **Starting with the first Pythagorean identity, $\sin^2 \theta + \cos^2 \theta = 1$, divide each term by $\cos^2 \theta$.**

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

2. **Rewrite each term by using the exponential rule $\left(\dfrac{\mathbf{a}^2}{\mathbf{b}^2} \right) = \left(\dfrac{\mathbf{a}}{\mathbf{b}} \right)^2$.**

Note how the exponent 2 is pulled out of the parentheses:

$$\left(\frac{\sin \theta}{\cos \theta} \right)^2 + \left(\frac{\cos \theta}{\cos \theta} \right)^2 = \left(\frac{1}{\cos \theta} \right)^2$$

3. **Replace each of the terms with an equivalent expression.**

- Use a ratio identity, $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$, to replace the first term with tan θ.

- Replace the second term with 1: $\dfrac{\cos\theta}{\cos\theta} = 1$.

- Use a reciprocal identity, $\dfrac{1}{\cos\theta} = \sec\theta$, to replace the third term with sec θ.

Substituting these expressions into the equation and simplifying, you get

$$(\tan\theta)^2 + (1)^2 = (\sec\theta)^2$$
$$\tan^2\theta + 1 = \sec^2\theta$$

This is the second Pythagorean identity.

Finishing up with cotangent and cosecant

If you read the preceding section, you can simply do a repeat performance with cotangent and cosecant. Here's how:

1. **Starting with the first Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$, divide each term by $\sin^2\theta$.**

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

2. **Rewrite each term by using the exponential rule $\left(\dfrac{\mathbf{a}^2}{\mathbf{b}^2}\right) = \left(\dfrac{\mathbf{a}}{\mathbf{b}}\right)^2$.**

$$\left(\frac{\sin\theta}{\sin\theta}\right)^2 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1}{\sin\theta}\right)^2$$

3. **Replace each of the terms with an equivalent expression.**

- Replace the first term with 1: $\dfrac{\sin\theta}{\sin\theta} = 1$.

- Use a ratio identity, $\dfrac{\sin\theta}{\sin\theta} = 1$, to replace the second term with cot θ.

- Use a reciprocal identity, $\dfrac{1}{\sin\theta} = \csc\theta$, to replace the third term with csc θ.

Substituting these expressions into the equation and simplifying, you find that the result is

$$(1)^2 + (\cot\theta)^2 = (\csc\theta)^2$$
$$1 + \cot^2\theta = \csc^2\theta$$

This gives you the last Pythagorean identity.

Rearranging the Pythagorean identities

The preceding sections show the original definitions of the Pythagorean identities, but as you probably suspect, the forms don't end there. Familiarizing yourself with the other versions of these identities is helpful so that you can easily recognize them when solving equations or simplifying expressions.

TIP All these different versions have their places in trigonometric applications, calculus, or other math topics. You don't have to memorize them, because if you just remember the three basic Pythagorean identities, you can solve for what you need.

Changing $\sin^2\theta + \cos^2\theta = 1$

You can alter the original Pythagorean identity in myriad ways. For starters, you can isolate either $\sin^2\theta$ or $\cos^2\theta$ on one side of the equation by subtracting the other term:

$$\sin^2\theta = 1 - \cos^2\theta$$
$$\cos^2\theta = 1 - \sin^2\theta$$

Continuing on, you can factor the right side of either of these equations because that side is the difference of two perfect squares:

$$\sin^2\theta = 1 - \cos^2\theta = (1 - \cos\theta)(1 + \cos\theta)$$
$$\cos^2\theta = 1 - \sin^2\theta = (1 - \sin\theta)(1 + \sin\theta)$$

Sometimes, however, having an expression for $\sin\theta$ or $\cos\theta$, where the functions aren't squared, is helpful. Beginning with the earlier version of the basic Pythagorean identities, where one function is by itself, you can take the square root of each side to get

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta} \text{ or } \cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

Adjusting $\tan^2\theta + 1 = \sec^2\theta$

You can also adapt this second Pythagorean identity in various ways. Solving for $\tan^2\theta$ by subtracting 1 from each side of the equation, you get

$$\tan^2\theta = \sec^2\theta - 1$$

Then, factoring the difference of the squares on the right (because that side is the difference of two perfect squares), you have

$$\tan^2\theta = \sec^2\theta - 1 = (\sec\theta - 1)(\sec\theta + 1)$$

Lastly, beginning with the earlier version and taking the square root of each side, you get

$$\tan\theta = \pm\sqrt{\sec^2\theta - 1}$$

Taking another approach to this Pythagorean identity, you can subtract $\tan^2\theta$ from each side and factor the result to get

$$1 = \sec^2\theta - \tan^2\theta = (\sec\theta - \tan\theta)(\sec\theta + \tan\theta)$$

The choice of restructuring always just depends on what you want to do with the terms — what other functions they'll be interacting with in a problem.

Reconfiguring $1 + \cot^2\theta = \csc^2\theta$

You can rearrange the last Pythagorean identity, too, by subtracting 1 from each side or by subtracting $\cot^2\theta$ from each side. The two new versions are

$$\cot^2\theta = \csc^2\theta - 1$$
$$1 = \csc^2\theta - \cot^2\theta$$

Each of the preceding equations has the difference of two perfect squares, which you can factor:

$$\cot^2\theta = \csc^2\theta - 1 = (\csc\theta - 1)(\csc\theta + 1)$$
$$1 = \csc^2\theta - \cot^2\theta = (\csc\theta - \cot\theta)(\csc\theta + \cot\theta)$$

And last, the square root of each side yields an identity involving just cot θ:

$$\cot\theta = \pm\sqrt{\csc^2\theta - 1}$$

Combining the Identities

Even though each function is perfectly wonderful, being able to express each of them in terms of all the other five trig functions is frequently to your advantage. For example, you may have an equation or expression with a lot of sines, but not all the terms are sines. Having them all match — all be in terms of sine — would help you solve the equation.

Armed with the reciprocal identities, ratio identities, and Pythagorean identities, you can do just that — write any trig function in terms of the others. In this section, I show you the many variations of sine; by applying some of the same identities and following similar steps, you can form a multitude of variations of the other trig functions. These five ways of writing sine in terms of the other five functions show you how powerful, versatile, and useful all these identities are. A word of warning: Some of these equations aren't very pretty. But then, beauty is in the eye of the beholder. And you may think these are just dandy.

The many faces of sine

Here are the five ways of expressing the sine function in terms of the other functions.

» Sine in terms of cosine: $\sin\theta = \pm\sqrt{1 - \cos^2\theta}$

» Sine in terms of tangent: $\sin\theta = \dfrac{\tan\theta}{\pm\sqrt{\tan^2\theta + 1}}$

» Sine in terms of cotangent: $\sin\theta = \dfrac{1}{\pm\sqrt{1 + \cot^2\theta}}$

» Sine in terms of secant: $\sin\theta = \dfrac{\pm\sqrt{\sec^2\theta - 1}}{\sec\theta}$

» Sine in terms of cosecant: $\sin\theta = \dfrac{1}{\csc\theta}$

Working out the versions

Choosing a good starting point helps — and makes for a nicer result. You don't want any expression that's too messy or hard to remember. Take advantage of identities that have your target function isolated in a single term. This section shows you the most typical methods for changing one trig function to another.

Changing sine to cosine

You can express the sine function in terms of cosine without doing much work.

1. **Starting with the Pythagorean identity involving sin θ and cos θ, subtract cos² θ from each side**.

   $$\sin^2\theta = 1 - \cos^2\theta$$

2. **Take the square root of both sides.**

   $$\sqrt{\sin^2\theta} = \pm\sqrt{1 - \cos^2\theta}$$
   $$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

You can use either the positive root or the negative root, depending on your application.

Changing sine to tangent

To rewrite the sine function in terms of tangent:

1. **Start with the ratio identity involving sine, cosine, and tangent, and multiply each side by cosine to get the sine alone on the left.**

   $$\cancel{\cos\theta} \cdot \frac{\sin\theta}{\cancel{\cos\theta}} = \tan\theta \cdot \cos\theta$$
   $$\sin\theta = \tan\theta\cos\theta$$

2. **Replace cosine with its reciprocal function.**

   $$\sin\theta = \tan\theta\left(\frac{1}{\sec\theta}\right)$$

3. **Solve the Pythagorean identity $\tan^2\theta + 1 = \sec^2\theta$ for secant.**

   This equation gives you $\pm\sqrt{\tan^2\theta + 1} = \sec\theta$.

4. **Replace the secant in the sine equation (from Step 2).**

   You end up with $\sin\theta = \tan\theta\left(\dfrac{1}{\pm\sqrt{\tan^2\theta + 1}}\right) = \dfrac{\tan\theta}{\pm\sqrt{\tan^2\theta + 1}}$.

Not very pretty, but, if your equation has terms with tangents in them, you have a better chance of combining terms or reducing fractions.

Changing sine to cotangent

To write the sine function in terms of cotangent, begin with the equation you end up with in the preceding section,

$$\sin\theta = \tan\theta\left(\frac{1}{\pm\sqrt{\tan^2\theta+1}}\right) = \frac{\tan\theta}{\pm\sqrt{\tan^2\theta+1}}.$$

1. **Replace all the tangents with 1 over the reciprocal for tangent (which is cotangent) and simplify the expression.**

$$\sin\theta = \left(\frac{\dfrac{1}{\cot\theta}}{\pm\sqrt{\dfrac{1}{\cot^2\theta}+1}}\right)$$

The result is a *complex fraction* — it has fractions in both the numerator and denominator — so it'll look a lot better if you simplify it.

2. **Rewrite the part under the radical as a single fraction and simplify it by using the law of exponents/radicals, taking the square root of each part.**

$$\sin\theta = \frac{\dfrac{1}{\cot\theta}}{\pm\sqrt{\dfrac{1}{\cot^2\theta}+\dfrac{\cot^2\theta}{\cot^2\theta}}} = \frac{\dfrac{1}{\cot\theta}}{\pm\sqrt{\dfrac{1+\cot^2\theta}{\cot^2\theta}}}$$

$$= \frac{\dfrac{1}{\cot\theta}}{\pm\dfrac{\sqrt{1+\cot^2\theta}}{\sqrt{\cot^2\theta}}} = \frac{\dfrac{1}{\cot\theta}}{\pm\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}}$$

3. **Multiply the numerator by the reciprocal of the denominator.**

$$= \frac{\dfrac{1}{\cot\theta}}{\pm\sqrt{\dfrac{1+\cot^2\theta}{\cot\theta}}} = \frac{1}{\cot\theta}\cdot\frac{\cot\theta}{\pm\sqrt{1+\cot^2\theta}} = \frac{1}{\pm\sqrt{1+\cot^2\theta}}$$

That's it. And it even turned out simpler-looking than the sine written with tangents.

Changing sine to secant

The next function to define sine is the secant function. You see more radicals on the horizon, but radicals can be "tamed."

1. **Start with the sine in terms of the cosine (refer to the first change-up in this section).**

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

2. **Now replace the cosine with 1 over its reciprocal.**

$$\sin\theta = \pm\sqrt{1-\cos^2\theta}$$

$$= \pm\sqrt{1-\frac{1}{\sec^2\theta}}$$

The radical has a fraction in it. A better form is to simplify that fraction, so find a common denominator and split the fraction into two radicals — the bottom one of which you can further simplify:

$$\sin\theta = \pm\sqrt{\frac{\sec^2\theta}{\sec^2\theta}-\frac{1}{\sec^2\theta}} = \pm\sqrt{\frac{\sec^2\theta-1}{\sec^2\theta}} = \pm\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$$

Changing sine to cosecant

The last function to write sine in terms of is the cosecant — I saved the best (easiest) for last. The reciprocal of cosecant is sine, so this equation is just one of the basic reciprocal identities: $\sin\theta = \dfrac{1}{\csc\theta}$.

Chapter **11**

Operating on Identities

The basic building-block identities are the reciprocal, ratio, opposite angle, and Pythagorean identities, which I discuss in detail in Chapter 10. In this chapter, you take those identities a step further and develop new identities, discovering how to add, subtract, multiply, and divide the trig functions — in particular, the nice values for angles of 0, 30, 45, 60, and 90 degrees. (Those angles aren't the only ones that you can perform operations on; they're just the most convenient to use when showing how the trig identities work.) By performing such operations, you can determine the function values of even more angles than before. Whole new worlds will open up to you!

Summing It Up

The sums of angles are covered by three basic identities; these identities involve sine, cosine, and tangent. After you recognize these three identities, you can adapt them for the other three functions (cosecant, secant, and cotangent) by using the reciprocal identities (detailed in Chapter 10). All you do is start with a basic sum identity, use a reciprocal identity to change the expression to the one you want, do the necessary simplifying, and then use the new sum identity as needed. You won't have to do this very often; you can usually get by with one of the three basics.

You use the angle–sum identities to find the function values of many, many angles, but the examples in this section just show the most convenient combinations — ones with exact values that you can fill into the formulas. Suppose, for example, that you want to find the exact value of the sine of 75 degrees. To minimize fuss, you can use the sum of 30 degrees and 45 degrees and the appropriate identity.

TRIG RULES

The angle–sum identities find the function value for the sum of angle α and angle β:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Now for an example using a sum–of–angles identity.

Using the identity for the sine of a sum, find the sine of 75 degrees:

1. **Determine two angles whose sum is 75 for which you know the values for both sine and cosine.**

Choose $30 + 45$, not $50 + 25$ or $70 + 5$, because sticking to the more-common angles that have nice, exact values to use in the formula is your best bet.

2. **Input the angle measures into the identity.**

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(30 + 45) = \sin30\cos45 + \cos30\sin45$$

3. **Replace the functions of the angles with their values and simplify.**

$$\sin(30 + 45) = \sin30\cos45 + \cos30\sin45$$
$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2} + \sqrt{6}}{4} = \sin(75)$$

Sometimes, you have more than one choice for the sum. In this next example, find the cosine of 120 degrees by using the identity for the cosine of a sum.

1. **Determine two angles whose sum is 120.**

Choosing among the most convenient angles, you can use either $90 + 30$ or $60 + 60$. For this example, I use $90 + 30$, because the sine of a 90-degree angle is 1, and the cosine is equal to 0. Both of those numbers are very nice to have in a computation because they keep it simple.

2. **Input the values into the identity.**

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(90 + 30) = \cos90\cos30 - \sin90\sin30$$

3. **Replace the functions with their values and simplify.**

$$\cos(90 + 30) = \cos90\cos30 - \sin90\sin30$$
$$= 0 \cdot \frac{\sqrt{3}}{2} - 1 \cdot \frac{1}{2} = 0 - \frac{1}{2} = -\frac{1}{2} = \cos(120)$$

But what if you want to use a different set of angles to find the cosine of 120 degrees?

1. **Determine two angles whose sum is 120.**

The only other combination that comes quickly to mind is to use $60 + 60$.

2. **Input the values into the identity.**

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(60 + 60) = \cos60\cos60 - \sin60\sin60$$

3. **Replace the functions with their values and simplify.**

$$\cos(60 + 60) = \cos60\cos60 - \sin60\sin60$$
$$= \frac{1}{2} \cdot \frac{1}{2} - \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{1}{4} - \frac{3}{4} = -\frac{2}{4} = -\frac{1}{2} = \cos(120)$$

It really doesn't matter which pair you use — you get the same answer.

These identities work with radian measures, too, such as finding $\tan\dfrac{7\pi}{12}$ by using the identity for the tangent of the sum of angles.

1. **Determine two angles whose sum is $\dfrac{7\pi}{12}$.**

It may be easier to think of finding two numbers that add up to $\dfrac{7}{12}$ and leave the π off for a moment.

The two fractions that come to mind are $\dfrac{1}{3}$ and $\dfrac{1}{4}$. Because $\dfrac{1}{3} + \dfrac{1}{4} = \dfrac{4}{12} + \dfrac{3}{12} = \dfrac{7}{12}$, you have $\dfrac{7\pi}{12} = \dfrac{\pi}{3} + \dfrac{\pi}{4}$.

2. **Input the values into the identity.**

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \frac{\tan\dfrac{\pi}{3} + \tan\dfrac{\pi}{4}}{1 - \tan\dfrac{\pi}{3}\tan\dfrac{\pi}{4}}$$

3. **Replace the functions with their values and simplify.**

$$\tan\left(\frac{\pi}{3}+\frac{\pi}{4}\right)=\frac{\sqrt{3}+1}{1-\sqrt{3}\cdot 1}=\frac{\sqrt{3}+1}{1-\sqrt{3}}=\tan\frac{7\pi}{12}$$

The result in the last step doesn't leave the answer in the nicest form. The denom-inator has two terms, and one of them is a radical. One way to make the answer look a bit better and more intelligible is to use a technique called *rationalization.*

REMEMBER To rationalize the numerator or denominator of a fraction, multiply both the numerator and denominator by the *conjugate* (same terms, opposite sign) of the part that you're rationalizing. When you do so, you end up with the difference of two squares, which lets you get rid of the offending portion.

For the last example, you rationalize to get the radical out of the denominator:

$$\frac{\sqrt{3}+1}{1-\sqrt{3}}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}}=\frac{\sqrt{3}+3+1+\sqrt{3}}{1-3}=\frac{4+2\sqrt{3}}{-2}=-2-\sqrt{3}$$

The final answer is a bit nicer to understand and estimate. Because $\sqrt{3}$ is about 1.7, you can estimate that $-2-\sqrt{3}\approx-2-1.7=-3.7$.

Next, I come at these angle-sum identities from a different direction. Sometimes, you may not know what the angle measure is, but you know something about the angle's function values. For example, suppose you have two angles, α in QII and β in QI. You know that $\sin\alpha=\frac{3}{5}$ and $\cos\beta=\frac{24}{25}$. With that information, what are $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$?

1. **Find all the necessary function values for the sums.**

Both the sine and cosine angle-sum identities use the sine and cosine of each angle involved. You already know the sine of one angle and the cosine of the other angle, so you have to determine the unknown cosine and sine — you can do so by using a Pythagorean identity.

● First, use the value for sin α to solve for cos α:

$$\sin^2\alpha+\cos^2\alpha=1$$

$$\left(\frac{3}{5}\right)^2+\cos^2\alpha=1$$

$$\cos^2\alpha=1-\frac{9}{25}=\frac{16}{25}$$

$$\cos\alpha=\pm\sqrt{\frac{16}{25}}=\pm\frac{4}{5}$$

You end up with two results. Because the terminal side of angle α is in the second quadrant, the cosine of α, in this case, is negative:

$$\cos\alpha = -\frac{4}{5}$$

- Now use the value for cos β to solve for sin β:

$$\sin^2\beta + \cos^2\beta = 1$$

$$\sin^2\beta + \left(\frac{24}{25}\right)^2 = 1$$

$$\sin^2\beta = 1 - \frac{576}{625} = \frac{49}{625}$$

$$\sin\beta = \pm\sqrt{\frac{49}{625}} = \pm\frac{7}{25}$$

Again, there are two different signs to choose from for the sin of β. The terminal side of angle β is in the first quadrant, where the sine is positive: $\sin\beta = \frac{7}{25}$.

2. **Insert the function values into the identities for the sine and cosine of the sum of angles.**

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \left(\frac{3}{5}\right)\left(\frac{24}{25}\right) + \left(-\frac{4}{5}\right)\left(\frac{7}{25}\right)$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \left(-\frac{4}{5}\right)\left(\frac{24}{25}\right) - \left(\frac{3}{5}\right)\left(\frac{7}{25}\right)$$

3. **Simplify the identities and solve for the answers.**

$$\sin(\alpha+\beta) = \left(\frac{3}{5}\right)\left(\frac{24}{25}\right) + \left(-\frac{4}{5}\right)\left(\frac{7}{25}\right)$$

$$= \left(\frac{72}{125}\right) + \left(-\frac{28}{125}\right) = \frac{44}{125}$$

$$\cos(\alpha+\beta) = \left(-\frac{4}{5}\right)\left(\frac{24}{25}\right) - \left(\frac{3}{5}\right)\left(\frac{7}{25}\right)$$

$$= \left(-\frac{96}{125}\right) - \left(\frac{21}{125}\right) = -\frac{117}{125}$$

TIP By looking at the angle measures, you can predict whether the function value will be positive or negative. In the preceding example, the smaller angles, when added together, create an angle with its terminal side in the second quadrant. In Chapter 8, you find out that the sine of an angle in the second quadrant is positive. So, it's no surprise that the sine comes out to be a positive value and, likewise, that the cosine is a negative value (because cosine is negative in the second quadrant).

Overcoming the Differences

By adding angles together, you enlarge your repertoire. You have a longer list of *exact* function values — not just the basic function values, but also all the possible sums of these more-common angles. In like fashion, you have even more possibilities for finding the function values of angles when you use subtraction. For example, you can determine the sine of 15 degrees by using 45 degrees and 30 degrees and the appropriate identity.

TRIG RULES

The subtraction, or difference, identities find the function for the difference between angles α and β:

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Notice how each of the subtraction identities resembles its corresponding angle-sum identity. For the sine rule, the sign between the two products changes from $+$ to $-$, which seems to make sense. The opposite is true for cosine. The addition rule for cosine has $-$ in it, and the subtraction (or difference) rule has $+$ in it. The tangent rule has both $+$ and $-$ in it; the operation in the numerator mirrors the type of identity.

TIP

Only the original three trig functions have truly usable difference identities — the identities for the reciprocal functions are pretty darned complicated. If you want the difference of a reciprocal function, your best bet is to use the corresponding basic identity and find the reciprocal of the numerical answer after you're all finished.

To see one of the subtraction identities in action, check out the following example, which shows how you can find the sine of 15 degrees.

1. **Determine two angles with a difference of 15 degrees.**

   To keep things simple, use 45 and 30.

2. **Substitute the angles into the identity for the sine of a difference.**
   $$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
   $$\sin 15 = \sin(45 - 30) = \sin 45 \cos 30 - \cos 45 \sin 30$$

3. **Replace the terms with the function values and simplify the answer.**
   $$\sin(45 - 30) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$$
   $$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4} = \sin(15)$$

Using radians introduces even more fractions into the picture, such as finding $\tan\frac{\pi}{12}$ by using the identity for the tangent of a difference. Note that the radian measure of $\frac{\pi}{12}$ and the degree measure of $15°$ are the same angle. You're working with the same angle in these two examples.

1. **Determine which angles you need to get the difference.**

The two angles are $\frac{\pi}{3}$ and $\frac{\pi}{4}$, giving you $\frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi}{12} - \frac{3\pi}{12} = \frac{\pi}{12}$.

2. **Substitute the angles into the identity for the tangent of a difference.**

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}}$$

3. **Replace the terms with the function values and simplify the answer.**

$$\tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\sqrt{3} - 1}{1 + \sqrt{3}\cdot 1} = \frac{\sqrt{3} - 1}{1 + \sqrt{3}} = \tan\left(\frac{\pi}{12}\right)$$

The result is rather messy. You can simplify it even more by multiplying the numerator and denominator by the *conjugate* (same terms, different sign) of the denominator and simplifying the result:

$$\frac{\sqrt{3} - 1}{1 + \sqrt{3}} \cdot \frac{1 - \sqrt{3}}{1 - \sqrt{3}} = \frac{\sqrt{3} - 3 - 1 + \sqrt{3}}{1 - 3} = \frac{2\sqrt{3} - 4}{-2} = -\sqrt{3} + 2$$

In Chapter 10, I explain the opposite-angle identities. This next example uses the identity for the cosine of a difference along with the angle measuring 0 degrees to create an opposite-angle identity. You may like this explanation better than those in Chapter 10, and it just goes to show you how versatile and user-friendly trig identities are — and how they all get along so well together. In this example, find $\cos\left(-\frac{\pi}{3}\right)$ by using the identity for the difference between angles.

1. **Determine which angles you need to get the difference.**

Using 0 and $\frac{\pi}{3}$ and subtracting with the 0 first gives a negative result: $0 - \frac{\pi}{3} = -\frac{\pi}{3}$.

2. **Substitute the angles into the identity for the cosine of the difference.**

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos\left(0 - \frac{\pi}{3}\right) = \cos 0\cos\frac{\pi}{3} + \sin 0\sin\frac{\pi}{3}$$

3. **Replace the angles with the function values and simplify the answer.**

$$\cos\left(-\frac{\pi}{3}\right) = \cos 0 \cos\frac{\pi}{3} + \sin 0 \sin\frac{\pi}{3}$$

$$= 1 \cdot \frac{1}{2} + 0 \cdot \frac{\sqrt{3}}{2} = \frac{1}{2} = \cos\left(-\frac{\pi}{3}\right)$$

This answer is exactly what you get if you use the opposite-angle identity for cosine: $\cos(-\theta) = \cos\theta$ or $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

In the following example, two negatives make a positive. The angles are positive, but their function values are negative. The two angles in question are α, which is in the fourth quadrant, and β, which is in the third quadrant. The known function values are $\sin\alpha = -\frac{4}{5}$ and $\cos\beta = -\frac{5}{13}$. Find $\cos(\alpha - \beta)$.

1. **Find the necessary function values to calculate the difference.**

The cosine of the difference of two angles uses both the sine and cosine of each angle involved. You already know the sine of α and cosine of β, so you must determine the cosine of α and the sine of β. Using a Pythagorean identity, you can solve for the missing values.

First, use the value for sin α to solve for cos α:

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\left(-\frac{4}{5}\right)^2 + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\cos\alpha = \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

You have to choose between the positive and negative values. Because angle α is in the fourth quadrant, you know that the cosine of α is positive: $\cos\alpha = \frac{3}{5}$.

Now use the value for cos β to find sin β:

$$\sin^2\beta + \cos^2\beta = 1$$

$$\sin^2\beta + \left(-\frac{5}{13}\right)^2 = 1$$

$$\sin^2\beta = 1 - \frac{25}{169} = \frac{144}{169}$$

$$\sin^2\beta = 1 + \sqrt{\frac{25}{169}} = \pm\frac{12}{13}$$

Again, you choose the correct sign. Angle β is in the third quadrant, where sine is negative, so $\sin \beta = -\dfrac{12}{13}$.

2. **Insert the function values into the identity for the cosine of a difference.**

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

$$= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right)$$

3. **Simplify the identity and solve for the answer.**

$$\cos(\alpha - \beta) = \left(-\frac{15}{65}\right) + \left(\frac{48}{65}\right)$$

$$= \frac{33}{65}$$

In the preceding example, the first angle is in the fourth quadrant, so its measure is between 270 and 360 degrees. The other angle, which is in the third quadrant, is between 180 and 270 degrees. The difference between them could be anywhere between 0 and 180 degrees, meaning that the new angle is in either the first or second quadrant. The answer for the cosine of the difference came out positive. Chapter 8 tells you that the cosine is positive in the first quadrant and negative in the second quadrant, so the difference between the two angles must be somewhere between 0 and 90 degrees, which means that the new angle is in the first quadrant.

Doubling Your Money

Identities for angles that are twice as large as one of the common angles are used a lot in calculus and various math, physics, and science disciplines. These identities allow you to deal with a larger angle in terms of a smaller and more manageable one. A *double-angle* function is written, for example, as sin 2θ, cos 2α, or tan $2x$, where 2θ, 2α, and $2x$ are the angle measures and the assumption is that you mean sin (2θ), cos (2α), or tan $(2x)$. In this section, I show you how the double-angle formulas for sine and cosine came to be. I don't go off on a tangent here, but all you need to know is that because tangent is equal to the ratio of sine and cosine, its identity comes from their double-angle identities.

The double-angle identities find the function for twice the angle θ. Note that the cosine function has three different versions of its double-angle identity.

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

One plus one equals two sines

To show you where the double-angle formula for sine comes from, I start with the identity for the sine of a sum, $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. If $\alpha = \beta$, then $\alpha + \beta$ becomes $\alpha + \alpha$ or $2(\alpha)$.

You can replace β with α in the formula, giving you

$$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$$
$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

For example, you can use this double-angle identity to find the function value for the sine of 180 degrees.

1. **Determine twice which angle is 180 degrees.**

Twice 90 is 180, so the choice is 90 degrees.

2. **Substitute the measure into the double-angle identity for sine.**
$\sin 180° = \sin 2 \cdot 90° = 2\sin 90°\cos 90°$

3. **Replace the angles with the function values and simplify the answer.**
$\sin 180° = 2(1)(0) = 0$

But that angle measure is found pretty easily because the terminal side of the angle is on an axis. How about something a bit more challenging? This time, use a double-angle formula to find the sine of 150 degrees.

1. **Determine twice which angle is 150 degrees.**

Twice 75 is 150, so the choice is 75 degrees. A 75-degree angle isn't one of the basic angles, but you find the value of the sine of 75 degrees earlier in this chapter, in the section, "Summing It Up."

2. **Substitute the measure into the double-angle identity for sine.**
$\sin 150° = \sin 2 \cdot 75° = 2\sin 75°\cos 75°$

3. **Replace the angles with the function values and simplify the answer.**

You have $\sin 75° = \dfrac{\sqrt{2} + \sqrt{6}}{4}$ and also need the cosine of the angle. A Pythagorean identity comes to the rescue.

$$\sin^2 75° + \cos^2 75° = 1$$

$$\left(\frac{\sqrt{2} + \sqrt{6}}{4} \right)^2 + \cos^2 75° = 1$$

$$\cos^2 75° = 1 - \frac{8 + 2\sqrt{12}}{16}$$

$$= 1 - \frac{8 + 4\sqrt{3}}{16}$$

$$= \frac{4}{4} - \frac{2 + \sqrt{3}}{4}$$

$$= \frac{2 - \sqrt{3}}{4}$$

$$\cos 75° = \sqrt{\frac{2 - \sqrt{3}}{4}} = \sqrt{\frac{2 - \sqrt{3}}{2}}$$

Whew! Now you see why it's nice to have other options in the form of different types of trig identities.

So, you now have

$$\sin 150° = 2 \sin 75° \cos 75°$$

$$= 2 \left(\frac{\sqrt{2} + \sqrt{6}}{4} \right) \left(\frac{\sqrt{2} + \sqrt{3}}{2} \right)$$

$$= \frac{\left(\sqrt{2} + \sqrt{6} \right) \sqrt{2 - \sqrt{3}}}{4}$$

This can be simplified a bit by multiplying the numerator and denominator by the conjugate of the numerator.

$$= \frac{\left(\sqrt{2} + \sqrt{6} \right) \sqrt{2 - \sqrt{3}}}{4} \cdot \frac{\sqrt{2} - \sqrt{6}}{\sqrt{2} - \sqrt{6}}$$

$$= \frac{(2 - 6) \sqrt{2 - \sqrt{3}}}{4 \left(\sqrt{2} - \sqrt{6} \right)} = \frac{\sqrt{2} - \sqrt{3}}{\sqrt{6} - \sqrt{2}}$$

Still not a pretty sight, and it can be simplified by multiplying the numerator and denominator by a conjugate, but using your calculator to compute the value of that fraction, you get exactly $\dfrac{1}{2}$. This is *not* the most efficient way to find the sine of 150 degrees, but you see an example of applying a double-angle formula to solve for the value of a trig function.

Three's a crowd

Finding the cosine of twice an angle is easier than finding the other function values, because the cosine offers you three choices. You make your choice depending on what information is available and what looks easiest. To show you where the first of the double-angle identities for cosine comes from, I use the angle-sum identity for cosine. Because the two angles are equal, you can replace β with α, so $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ becomes

$$\cos(\alpha + \alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$$
$$\cos(2\alpha) = (\cos\alpha)^2 - (\sin\alpha)^2$$
$$= \cos^2\alpha - \sin^2\alpha$$

UNLIKELY MATHEMATICIAN

Napoleon Bonaparte is best known for his triumphs and trials in French history. But did you know that he was a closet mathematician? He even has a rule named for him. Napoleon is credited with discovering that when you construct equilateral triangles on the sides of any other triangle and then connect the centers of those triangles with segments, those segments form another equilateral triangle. Here is a figure illustrating Napoleon's theorem:
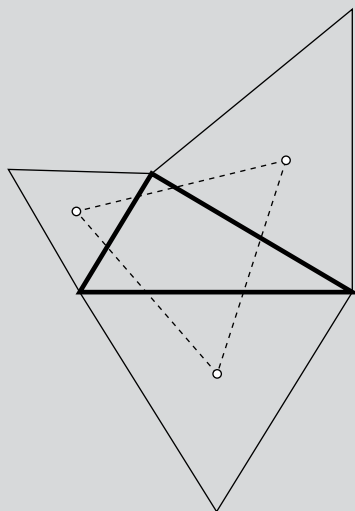
To get the second version, use the first Pythagorean identity, $\sin^2\alpha + \cos^2\alpha = 1$. Solving for $\sin^2\alpha$, you get $\sin^2\alpha = 1 - \cos^2\alpha$. Putting this result back into the double-angle identity for cosine and simplifying, you get

$$\cos(2\alpha) = \cos^2\alpha - \left(1 - \cos^2\alpha\right)$$
$$= \cos^2\alpha - 1 + \cos^2\alpha$$
$$= 2\cos^2\alpha - 1$$

To find the last version of the double-angle identity for cosine, solve the first Pythagorean identity for $\cos^2\alpha$, which gives you $\cos^2\alpha = 1 - \sin^2\alpha$. Then substitute this result into the first angle-sum identity for cosine:

$$\cos(2\alpha) = \left(1 - \sin^2\alpha\right) - \sin^2\alpha$$
$$= 1 - \sin^2\alpha - \sin^2\alpha$$
$$= 1 - 2\sin^2\alpha$$

The biggest advantage to having three different identities for the cosine of a double angle is that you can solve for the cosine with just one other function value. The sum and difference identities for sine and cosine, on the other hand, as well as the double-angle identity for sine, all involve both the sine and cosine of the angles.

Here's an example showing off that advantage. Find $\cos 2\alpha$; the angle α is in the fourth quadrant, and $\sin\alpha = -0.45$.

1. **Choose the appropriate double-angle identity.**

Because you know the value of the sine, use $\cos 2\alpha = 1 - 2\sin^2\alpha$.

2. **Insert the given value in the formula and simplify.**

$$\cos 2\alpha = 1 - 2\left(\sin\alpha\right)^2 = 1 - 2(-0.45)^2$$
$$= 1 - 2(0.2025) = 1 - 0.4050 = 0.5950$$

The resulting cosine is positive. The cosine is positive in the first and fourth quadrants, so how do you know which of those two quadrants the terminal side of this double angle lies in? Go back to the beginning of the problem — you know that the original angle is in the fourth quadrant. An angle in the fourth quadrant measures between 270 degrees and 360 degrees. If you double those numbers (because you're working with a double angle), you get 540 degrees and 720 degrees. The angles between those two values lie in the third and fourth quadrants. The cosine is positive in the fourth quadrant, so this double angle lies in the fourth quadrant.

Halving Fun Yet?

The trig identities come in sums, differences, multiples, and halves. With these identities, you can get the value of a sine for a 15-degree angle by using a formula involving half of 30 degrees. You can also get the value of the tangent of a $22\frac{1}{2}$ degree angle by using half of 45 degrees. These identities just create more and more ways to establish an exact value for many of the more commonly used trig functions.

The half-angle identities find the function value for half the measure of angle θ:

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

Notice that the tangent has two versions of its half-angle formula. "Oh, goody!" you say. Just as with the three versions of the cosine's double-angle identities, you get to choose which version is more convenient.

The half-angle identities are a result of taking the double-angle identities and scrunching them around. A more-technical term for *scrunching* is solving for the single angle in a double-angle identity. Here's how the half-angle identity for sine came to be:

1. **Write the double-angle identity for cosine that has just a sine in it.**

   $\cos 2\theta = 1 - 2\sin^2\theta$

   Using the double-angle identity for cosine works better than the double-angle identity for sine, because the sine formula has both functions on the right side of the equation, and you can't easily get rid of one or the other.

2. **Solve for sin θ. First, get the sin² θ term by itself on the left.**

   $\cos 2\theta = 1 - 2\sin^2\theta$

   $2\sin^2\theta = 1 - \cos 2\theta$

3. **Divide each side by 2, and then take the square root of each side.**

   $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

   $\sqrt{\sin^2\theta} = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$

   $\sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$

4. **Replace 2θ with α and θ with $\frac{\alpha}{2}$.**

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

By switching the letters, you can more easily see the relationship between the two angles, that one is half as big as the other.

You can find the half-angle formula for cosine by using the appropriate choice of cosine's double-angle identity and following similar steps to the preceding ones. With tangent, you use both the sine and cosine identities. But first, what's this business of + or − in the sine and cosine half-angle identities?

Explaining the ±

Trig identities are numerous. Some people say there are too many identities, and others say there just aren't enough. (Like Baby Bear, I think they're just right.) I list the most frequently used identities both in this chapter and in Chapter 10. You may be wondering, however, why these half-angle identities are in a league of their own — some of them have ± in front of them; other identities don't have that lead-in. Continue on, dear reader, for the answer to that nagging question.

What's unique about the half-angle identities for sine and cosine is the fact that the sign attached depends on what quadrant the angle that you're cutting in half is in or how big that angle is. If you want to know the sine of half a 30-degree angle, you look at what quadrant that half is in. Both 30 degrees and its half, 15 degrees, are in the first quadrant, so their sines are both positive. Not so with the sine of 300 degrees and its half, 150 degrees, though. The sine is negative in the fourth quadrant, where the 300-degree angle is located (in standard position), but the sine of 150 degrees is positive, because its terminal side is in the second quadrant. When you apply the half-angle formulas, you have to consider which quadrant each angle is in and apply the appropriate signs.

Half a tangent is double the fun

The half-angle identity for tangent has two versions. Rather than being a nuisance, having more than one option is really rather nice, because you can choose the version that works best for your situation. The half-angle formulas for the tangent involve both sine and cosine, but those functions switch places in the numerator and denominator of the fraction. Sometimes the sine of a function doesn't have a radical in its exact value when the cosine does (or vice versa). Depending on the sine and cosine values, you choose the version of the half-angle tangent identity that'll be easiest to work with after you input the values. The math is easier when you don't have to worry about those radicals in the denominator.

First, where do these half–angle tangent identities come from?

1. **Use the ratio identity for tangent and fill in the half-angle identities for sine and cosine.**

$$\tan\frac{\theta}{2} = \frac{\sin\dfrac{\theta}{2}}{\cos\dfrac{\theta}{2}} = \frac{\sqrt{\dfrac{1-\cos\theta}{2}}}{\sqrt{\dfrac{1+\cos\theta}{2}}}$$

You can leave off the ± sign because you won't have to choose which sign to use with the tangent identity.

2. **Put the numerator and denominator under the same radical and then simplify the complex fraction.**

$$= \frac{\sqrt{\dfrac{1-\cos\theta}{2}}}{\sqrt{\dfrac{1+\cos\theta}{2}}} = \sqrt{\frac{1-\cos\theta}{2}\cdot\frac{2}{1+\cos\theta}} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

3. **Multiply the numerator and denominator by the *conjugate* (same terms, different sign) of the denominator.**

$$= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}} = \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}}$$

4. **Replace the denominator by using the Pythagorean identity and then simplify by putting the radical over the numerator and denominator, individually.**

$$\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} = \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}}$$

$$= \frac{\sqrt{(1-\cos\theta)^2}}{\sqrt{\sin^2\theta}} = \frac{1-\cos\theta}{\sin\theta}$$

To find the other form of this half-angle tangent identity, change Step 3 by multiplying the numerator and denominator of the fraction by the conjugate of the *numerator* instead of the denominator.

Using half-angle identities

By adding, subtracting, or doubling angle measures, you can find lots of exact values of trig functions. This section provides some examples of the types of angles and their functions that you can find with the half–angle identities.

PI GONE WRONG

In 1853, William Shanks published his calculation of the decimal value of π to 707 decimal places, which he computed all by hand. Not until 1945 did someone discover that the last 180 digits of this computation were wrong. But an even greater error occurred in 1897. That year, the General Assembly of the State of Indiana enacted Bill Number 246, stating that π was legally equal to 3.2.

Even though you can use a difference identity to find the sine of 15 degrees, you can also use the half-angle identity.

1. **Determine which angle is double the angle you're working with.**

Half of 30 is 15, so the choice is 30 degrees. Stick to the more-common angles — the ones that have exact values (see Chapter 6) or are multiples of 30 and 45.

2. **Substitute that angle into the half-angle identity for sine.**

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\sin 15^\circ = \sin\frac{30}{2} = +\sqrt{\frac{1-\cos 30}{2}}$$

Because the sine of 15 degrees is a positive value, the sign in front of the radical becomes +.

3. **Fill in the function values and simplify the answer.**

$$\sin 15^\circ = \sqrt{\frac{1-\cos 30}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2}{2}-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

The result isn't a particularly pretty value, although beauty is in the eye of the beholder. Some would consider this answer to be wonderful, because it's the exact value and not a decimal approximation.

Now try using the half-angle identity with radians. Find $\tan\frac{\pi}{8}$.

1. **Determine which angle is double the angle you're working with.**

The angle $\frac{\pi}{4}$ is twice $\frac{\pi}{8}$.

2. **Substitute the angle measure into one of the half-angle tangent identities.**

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

$$\tan\frac{\pi}{8} = \tan\frac{\frac{\pi}{4}}{2} = \frac{1-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}$$

3. **Fill in the function values and simplify the answer.**

$$\tan\frac{\pi}{8} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\frac{2}{2}-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{2}\cdot\frac{2}{\sqrt{2}} = \frac{2-\sqrt{2}}{\sqrt{2}}$$

4. **To get the radical out of the denominator, rationalize it by multiplying both parts of the fraction by the conjugate of the denominator.**

$$= \frac{2-\sqrt{2}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}-2}{2} = \frac{2(\sqrt{2}-1)}{2} = \sqrt{2}-1$$

The other identity for the tangent of a half-angle gives you exactly the same answer. That form isn't any easier, though, because both the sine and cosine of this angle have a radical in them. If the problem involved an angle of 60 degrees, though, the story would be different. The sine of 60 degrees is $\frac{\sqrt{3}}{2}$, and the cosine is $\frac{1}{2}$, which practically begs you to use the form with the cosine in the denominator so you don't have to mess with a radical in the denominator. Both identities work — the one you use is just a matter of personal preference.

Comparing Exact Values and Estimations

In both this chapter and Chapter 10, you are given identities to find the exact values of trig functions. Some exact values are very nice, such as 0, 1, −1, $\frac{1}{2}$, and so on. But the values that are full of radicals and fractions aren't quite as pretty.

For example, when using the difference identity and the angles $\frac{\pi}{3}$ and $\frac{\pi}{4}$ to find $\tan\frac{\pi}{12}$, you end up with $\tan\frac{\pi}{12} = -\sqrt{3}+2$. What is the decimal equivalence? Well, there is no exact equivalence, because the decimal value goes on forever and ever. Using a calculator, you may get 0.2679491924, which is a rounded-off version of the exact answer. Using Excel, you get 0.267949192431123, which adds four more

decimal places. But how many decimal places do you need? It all depends on how the value is being used. If you're determining how large to make your garden, you probably need only one or two decimal places: 0.3 or 0.27. If you're sending a shuttle out into space, you want a lot more accuracy and may go to eight or nine decimal places: 0.26794919 or 0.267949192.

When using a sum identity to find $\tan\dfrac{7\pi}{12}$, you determine that $\tan\left(\dfrac{\pi}{3}+\dfrac{\pi}{4}\right)=$ $\dfrac{\sqrt{3}+1}{1-\sqrt{3}\cdot1}=\dfrac{\sqrt{3}+1}{1-\sqrt{3}}$. This can be simplified further by multiplying the numerator and denominator by the conjugate $1+\sqrt{3}$: $\dfrac{\sqrt{3}+1}{1-\sqrt{3}}\cdot\dfrac{1+\sqrt{3}}{1+\sqrt{3}}=\dfrac{4+2\sqrt{3}}{-2}=-2-\sqrt{3}$. Again, with your calculator, you can get –3.732050808. But how does that compare to your using $\dfrac{\sqrt{3}+1}{1-\sqrt{3}}$ and not bothering to simplify? You get the exact same result. You would have to carry the decimal out to many, many more places before having that radical in the denominator would make a difference.

One more thing that may have concerned you: When finding $\sin15°$ using the difference identity, you get $\sin15°=\dfrac{\sqrt{6}-\sqrt{2}}{4}$, but when using the half-angle identity, you get $\sin15°=\dfrac{\sqrt{2-\sqrt{3}}}{2}$. These really look different, so are they the same? Using a calculator, you get the exact same value: $\sin15°=0.2588190451$. It just takes a bit of algebraic simplification to show how to get from one format to the other.

Chapter **12**

Proving Identities: The Basics

One major aspect that people remember about trigonometry, if they studied it in school, is the time they spent proving identities — making one side of an equation match the other. Some people find proving identities to be the best thing ever — they can't get enough of them. Others, though, find this task of proving identities to be less than exciting — a rite of passage that is best passed by. What you find in this chapter are a game plan and suggestions so that if you aren't so fond of solving such puzzles, then you may actually begin to enjoy the process. For your reading pleasure, I divide this chapter into the methods that work best to prove the different types of identities.

Why do you need to prove identities? Don't you already know that they're correct if they're called identities? Sure you do, but proving them is still helpful down the road when you're solving complex trig problems, because the process for solving them is all in the preparation. Many of the trig expressions that you use to solve practical problems have rather complicated and nasty-looking terms. By doing substitutions and manipulations with the trig identities, you can make those expressions more usable. All this practice with proving identities prepares you and gives you a heads-up as to what's possible.

Lining Up the Players

Before starting to prove (or solve) identities, you need to look over the different equivalences that you use to solve them.

Here's a quick list for your reference:

TRIG RULES

Reciprocal Identities

$$\frac{1}{\sin\theta} = \csc\theta \qquad \frac{1}{\cos\theta} = \sec\theta \qquad \frac{1}{\tan\theta} = \cot\theta$$

$$\frac{1}{\cot\theta} = \tan\theta \qquad \frac{1}{\sec\theta} = \cos\theta \qquad \frac{1}{\csc\theta} = \sin\theta$$

Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Opposite–Angle Identities

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \text{ or } \sin^2\theta = 1 - \cos^2\theta \text{ or } \cos^2\theta = 1 - \sin^2\theta$$

$$\tan^2\theta + 1 = \sec^2\theta \text{ or } \tan^2\theta = \sec^2\theta - 1$$

$$1 + \cot^2\theta = \csc^2\theta \text{ or } \cot^2\theta = \csc^2\theta - 1$$

Sum and Difference Identities

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Double-Angle Identities

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Half-Angle Identities

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

Picking Sides

When you prove identities, you usually work on only one side of the equation or the other — not both at the same time — and for good reason. When you're working in other math areas, such as solving anti-derivatives in calculus, you need to change from one trig expression to another so you can do the problem; such a situation doesn't have any sides, so you need to work on just the one term or expression. Solving trig identities working on just one side or the other is good practice. The good news is that in these problems, you usually get to pick which side.

Here are the guidelines for choosing the side to work on in an identity. (Just as in a debate, you go for the side you can defend or work with.) You may take your pick based on any one of these options:

TRIG RULES

» **Choose the side with the greater number of terms.** Why? Combining terms into one is easier than breaking them apart.

» **Choose the side with factors that need to be multiplied together.** The reason? Trig functions have a way of merging together because of similar factors in their ratios.

» **Choose the side with terms other than sine or cosine.** Why? Because all the functions can be written in terms of sine and cosine, so this creates commonalities.

Of course, these guidelines don't include all the possibilities, but they give you a good start. Enter the puzzle part of identities. You get to look for clues in each identity to help you decide which side to work on, as in the following examples.

Prove the identity $\cot x \sin x + \tan x \cot x = \cos x + 1$.

1. **Choose the side to work on.**

The left side has more terms, the first term is a product, and the second term is the product of two reciprocal functions, so this is the best choice.

2. **Use the ratio identity to replace the first cot *x* and the reciprocal identity to replace the second cot *x*.**

$$\frac{\cos x}{\sin x} \cdot \sin x + \tan x \cdot \frac{1}{\tan x} = \cos x + 1$$

3. **Multiply the two factors together in the two terms.**

$$\frac{\cos x}{\sin x} \cdot \sin x + \tan x \cdot \frac{1}{\tan x} = \cos x + 1$$
$$\cos x + 1 = \cos x + 1$$

The identity is *proven*, because the two sides are exactly the same. This one went pretty quickly, because the ratio and reciprocal identities were chosen. Another way to approach the problem would be to factor out cot *x* from the terms on the left and then do some identity–replacing in the terms in the parentheses. This approach isn't as easy, but it can still get the job done.

The next example shows you more techniques.

Prove the identity $\sec x - \sin x \tan x = \cos x$.

1. **Choose the side to work on.**

The left side has more terms, and two of the functions aren't sine or cosine, so you use two of the guidelines in making this choice.

2. **Use the reciprocal identity to replace sec *x* and the ratio identity to replace tan *x*.** This is from the guideline suggesting you change all terms to those involving sine and cosine.

$$\frac{1}{\cos x} - \sin x \cdot \frac{\sin x}{\cos x} = \cos x$$

3. **Multiply the two factors together in the second term. Then combine the two fractions, because they have a common denominator.**

$$\frac{1}{\cos x} - \frac{\sin x}{1} \cdot \frac{\sin x}{\cos x} = \cos x$$

$$\frac{1}{\cos x} - \frac{\sin^2 x}{\cos x} = \cos x$$

$$\frac{1 - \sin^2 x}{\cos x} = \cos x$$

4. **Replace the numerator by using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, which is also written $\cos^2 x = 1 - \sin^2 x$.**

$$\frac{\cos^2 x}{\cos x} = \cos x$$

5. **Reduce the fraction on the left.**

$$\frac{\cos^{\cancel{2}} x}{\cancel{\cos x}} = \cos x$$

$$\cos x = \cos x$$

The identity is *proven*, because the two sides are exactly the same.

Next, prove the identity,

$$\frac{1 - \cos x}{\sin x} + \frac{\sin x}{1 - \cos x} = 2\csc x.$$

1. **Choose the side to work on.**

The left side has fractions that you need to add together.

2. **Find the common denominator.**

The fractions have two different denominators, so multiply each by a fraction that equals 1 — the fraction with the other term's denominator in both the numerator and denominator.

$$\frac{1 - \cos x}{\sin x} \cdot \frac{1 - \cos x}{1 - \cos x} + \frac{\sin x}{1 - \cos x} \cdot \frac{\sin x}{\sin x} = 2\csc x$$

3. **Simplify the two fractions. Then add them together, because they have the same denominators.**

$$\frac{(1 - \cos x)^2}{\sin x (1 - \cos x)} + \frac{\sin^2 x}{\sin x (1 - \cos x)} = 2\csc x$$

$$\frac{(1 - \cos x)^2 + \sin^2 x}{\sin x (1 - \cos x)} = 2\csc x$$

4. **Multiply out the squared binomial in the far-left term of the numerator.**

$$\frac{1-2\cos x + \cos^2 x + \sin^2 x}{\sin x(1-\cos x)} = 2\csc x$$

5. **Replace the last two terms in the numerator with 1, using the Pythagorean identity $\sin^2 + \cos^2 x = 1$.**

$$\frac{1-2\cos x + 1}{\sin x(1-\cos x)} = 2\csc x$$

6. **Combine the two 1s in the numerator, and then factor out the 2 from each of the terms.**

$$\frac{2-2\cos x}{\sin x(1-\cos x)} = 2\csc x$$

$$\frac{2(1-\cos x)}{\sin x(1-\cos x)} = 2\csc x$$

7. **Factor out the common multiplier in the numerator and denominator.**

$$\frac{2(1-\cos x)}{\sin x(1-\cos x)} = 2\csc x$$

$$\frac{2}{\sin x} = 2\csc x$$

8. **To finish up, use the reciprocal identity $\dfrac{1}{\sin x} = \csc x$.**

$$2\left(\frac{1}{\sin x}\right) = 2\csc x$$

$$2\csc x = 2\csc x$$

The next example uses the multiplying–out guideline and the Pythagorean identity to make for a pretty result.

Prove the identity that $\cos x(\sec x - \cos x) = \sin^2 x$.

1. **Decide which side you'll work on.**

The left side just begs to be multiplied out by distributing cos x over the two terms in the parentheses.

2. **Distribute on the left.**

$$\cos x \sec x - \cos^2 x = \sin^2 x$$

3. **Cosine and secant are reciprocals, so their product is 1. Replace the term "cos x sec x" with 1.**

$$\cos x\left(\frac{1}{\cos x}\right) - \cos^2 x = \sin^2 x$$

$$1 - \cos^2 x = \sin^2 x$$

4. **Use the Pythagorean identity to replace the terms on the left.**

$$\sin^2 x = \sin^2 x$$

Working on Both Sides

As much fun as it is to work on just one side of an identity, sometimes working on both sides at the same time is advantageous and permissible. Working on both sides of an identity is often necessary when you don't have a clear-cut way to change one side to match the other. I've even had to resort to working on both sides when it *wasn't* permissible; working backward from one side to the result on the other side can give some valuable clues on how to solve the equation.

With a trig identity, working on both sides isn't really the same as working on both sides of an algebraic equation. In algebra, you can multiply each side by the same number, square both sides, add or subtract the same thing to each side, and so on. When you solve trig identities and equations (see Chapter 16), you can use all those algebra rules *plus* you can do substitutions with the various trig identities when you need them. You can even insert a different identity on each side — the one big advantage of working on both sides of a trig identity.

This first example is rather basic, but it gets the idea across. Solve the identity $\dfrac{\sin\theta}{\csc\theta} + \dfrac{\cos\theta}{\sec\theta} = \tan\theta\cot\theta$ by working on both sides.

1. **Replace the two denominators of the fractions with their reciprocal identities. Also replace the cotangent on the right with its reciprocal.**

$$\frac{\sin\theta}{\dfrac{1}{\sin\theta}} + \frac{\cos\theta}{\dfrac{1}{\cos\theta}} = \tan\theta\left(\frac{1}{\tan\theta}\right)$$

2. **Simplify the two fractions on the left by flipping the denominators and multiplying them by their numerators. Then multiply the two factors on the right together.**

$$\sin\theta \cdot \frac{\sin\theta}{1} + \cos\theta \cdot \frac{\cos\theta}{1} = 1$$

$$\frac{\sin\theta}{1} \cdot \frac{\sin\theta}{1} + \frac{\cos\theta}{1} \cdot \frac{\cos\theta}{1} = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

3. **Replace the sum on the left using the Pythagorean identity.**

You end up with $1 = 1$.

In the next example, you change everything to sines and cosines. Prove the identity $\dfrac{\csc x}{\cos x} = \cot x + \tan x$.

1. **Change the functions to their equivalences by using the reciprocal and ratio identities.**

$$\frac{\dfrac{1}{\sin x}}{\cos x} = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$$

2. **On the left, write the denominator as a fraction and then flip it and multiply it by the numerator. On the right, multiply each fraction by a fraction equal to 1 (by using the other fraction's denominator) to get common denominators for all the fractions.**

$$\frac{\dfrac{1}{\sin x}}{\dfrac{\cos x}{1}} = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$$

$$\frac{1}{\sin x} \cdot \frac{1}{\cos x} = \frac{\cos x}{\sin x} \cdot \frac{\cos x}{\cos x} + \frac{\sin x}{\cos x} \cdot \frac{\sin x}{\sin x}$$

3. **Simplify the multiplied fractions. Add the two fractions on the right together.**

$$\frac{1}{\sin x \cos x} = \frac{\cos^2 x}{\sin x \cos x} + \frac{\sin^2 x}{\sin x \cos x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}$$

4. **Replace the numerator on the right with the value from the Pythagorean identity.**

$$\frac{1}{\sin x \cos x} = \frac{1}{\sin x \cos x}$$

When working on both sides, you're done when the two sides read the same. It's different from working on one side, where you keep something from the original equation.

This last example requires a little creativity to get the job done. But working on both sides still works best when solving $\dfrac{1 + \cot \alpha}{\cot \alpha} = \tan \alpha + \csc^2 \alpha - \cot^2 \alpha$.

1. **Split up the fraction on the left by writing each term in the numerator over the denominator.**

$$\frac{1}{\cos \alpha} + \frac{\cot \alpha}{\cot \alpha} = \tan \alpha + \csc^2 \alpha - \cot^2 \alpha$$

2. **Reduce the second fraction to 1.**

$$\frac{1}{\cot\alpha} + 1 = \tan\alpha + \csc^2\alpha - \cot^2\alpha$$

3. **Replace the csc² α on the right with its equivalent by using the Pythagorean identity.**

$$\frac{1}{\cot\alpha} + 1 = \tan\alpha + \left(1 + \cot^2\alpha\right) - \cot^2\alpha$$

4. **Simplify the terms on the right after dropping the parentheses — two of the terms are opposites of one another.**

$$\frac{1}{\cot\alpha} + 1 = \tan\alpha + 1 + \cancel{\cot^2\alpha} - \cancel{\cot^2\alpha}$$
$$= \tan\alpha + 1$$

5. **Replace the fraction on the left by using the reciprocal identity.**

$$\tan\alpha + 1 = \tan\alpha + 1$$

Going Back to Square One

With some identities, which side you should work on or what you should do with either or both sides isn't clear. And in some instances, you're faced with such a conglomeration of functions that figuring out what's going on is darn near impossible. Other times, the different terms have different powers of the same function. In such cases, simplifying matters either by changing everything to sines and cosines or by factoring out some function may be your best bet.

Changing to sines and cosines

In this first example, you can use either reciprocal or ratio identities, depending on which side you're going to work on, to change everything to sines and cosines. Here's how I'd solve the identity $\tan\theta + \cot\theta = \csc\theta\sec\theta$:

1. **Going with the guideline to work on the side with the greatest number of terms, replace the two terms on the left by using ratio identities.**

   $$\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \csc\theta\sec\theta$$

2. **To get a common denominator, multiply both terms on the left by fractions equal to 1 (by using the other term's denominator).**

   $$\frac{\sin\theta}{\cos\theta} \cdot \frac{\sin\theta}{\sin\theta} + \frac{\cos\theta}{\sin\theta} \cdot \frac{\cos\theta}{\cos\theta} = \csc\theta\sec\theta$$

3. **Simplify the fractions and then add them together, because now they have a common denominator.**

   $$\frac{\sin^2\theta}{\cos\theta\sin\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} = \csc\theta\sec\theta$$

   $$\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \csc\theta\sec\theta$$

4. **Replace the numerator on the left with its equivalent by using the Pythagorean identity.**

   $$\frac{1}{\cos\theta\sin\theta} = \csc\theta\sec\theta$$

5. **Use the reciprocal identities on the terms in the denominator and then flip each fraction and multiply.**

   $$\frac{1}{\frac{1}{\sec\theta} \cdot \frac{1}{\csc\theta}} = \csc\theta\sec\theta$$

   $$\frac{\sec\theta}{1} \cdot \frac{\csc\theta}{1} = \csc\theta\sec\theta$$

   $$\sec\theta\csc\theta = \sec\theta\csc\theta$$

In the next example, only two terms aren't already written as sines, so replacing those two with terms in sines just seems natural when solving.

$$\frac{\sin x + 8\csc x}{\sin x + 4\csc x} = \frac{\sin^2 x + 8}{\sin^2 x + 4}$$

1. **Change the two cosecants on the left by using the reciprocal identity.**

$$\frac{\sin x + 8\left(\dfrac{1}{\sin x}\right)}{\sin x + 4\left(\dfrac{1}{\sin x}\right)} = \frac{\sin^2 x + 8}{\sin^2 x + 4}$$

$$\frac{\sin x + \dfrac{8}{\sin x}}{\sin x + \dfrac{4}{\sin x}} = \frac{\sin^2 x + 8}{\sin^2 x + 4}$$

2. **Multiply each term in the numerator and denominator of the left-hand side by sin x. This action amounts to multiplying by sine over sine, or by 1.**

$$\frac{\sin x}{\sin x} \cdot \frac{\sin x + \dfrac{8}{\sin x}}{\sin x + \dfrac{4}{\sin x}} = \frac{\sin^2 x + 8}{\sin^2 x + 4}$$

$$\frac{\sin x\left(\sin x + \dfrac{8}{\sin x}\right)}{\sin x\left(\sin x + \dfrac{4}{\sin x}\right)} = \frac{\sin^2 x + 8}{\sin^2 x + 4}$$

$$\frac{\sin x \cdot \sin x + \sin x \cdot \dfrac{8}{\sin x}}{\sin x \cdot \sin x + \sin x \cdot \dfrac{4}{\sin x}} = \frac{\sin^2 x + 8}{\sin^2 x + 4}$$

3. **Simplify the numerator and denominator.**

$$\frac{\sin^2 x + \cancel{\sin x} \cdot \dfrac{8}{\cancel{\sin x}}}{\sin^2 x + \cancel{\sin x} \cdot \dfrac{4}{\cancel{\sin x}}} = \frac{\sin^2 x + 8}{\sin^2 x + 4}$$

$$\frac{\sin^2 x + 8}{\sin^2 x + 4} = \frac{\sin^2 x + 8}{\sin^2 x + 4}$$

This next example has so many different functions and terms that figuring out where to start almost seems impossible. Although you have other ways to approach it, I change the fraction on the left to all sines and cosines. If you want to see another way to solve an identity like this one, refer to Chapter 13.

Solve the identity $\dfrac{1 + \sec x}{\tan x} - \dfrac{\tan x}{\sec x} = \cot x(1 + \cos x)$.

1. **On the left side, change the secants by using the reciprocal identity and the tangents by using the ratio identity.**

$$\frac{1 + \dfrac{1}{\cos x}}{\dfrac{\sin x}{\cos x}} - \frac{\dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x}} = \cot x(1 + \cos x)$$

2. **On the left, multiply each term in the numerator and denominator by cos _x_ and simplify all the terms.**

$$\frac{\cos x}{\cos x} \cdot \frac{1+\dfrac{1}{\cos x}}{\dfrac{\sin x}{\cos x}} - \frac{\cos x}{\cos x} \cdot \frac{\dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos x}} = \cot x(1+\cos x)$$

$$\frac{\cos x\left(1+\dfrac{1}{\cos x}\right)}{\cos x\left(\dfrac{\sin x}{\cos x}\right)} - \frac{\cos x\left(\dfrac{\sin x}{\cos x}\right)}{\cos x\left(\dfrac{1}{\cos x}\right)} = \cot x(1+\cos x)$$

$$\frac{\cos x \cdot 1 + \cos x \cdot \dfrac{1}{\cos x}}{\cos x \cdot \left(\dfrac{\sin x}{\cos x}\right)} - \frac{\cos x \cdot \dfrac{\sin x}{\cos x}}{\cos x \cdot \dfrac{1}{\cos x}} = \cot x(1+\cos x)$$

$$\frac{\cos x + \cancel{\cos x} \cdot \dfrac{1}{\cancel{\cos x}}}{\cancel{\cos x} \cdot \left(\dfrac{\sin x}{\cancel{\cos x}}\right)} - \frac{\cancel{\cos x} \cdot \dfrac{\sin x}{\cancel{\cos x}}}{\cancel{\cos x} \cdot \dfrac{1}{\cancel{\cos x}}} = \cot x(1+\cos x)$$

From that mess, you get $\dfrac{\cos x+1}{\sin x} - \dfrac{\sin x}{1} = \cot x(1+\cos x)$.

3. **Find a common denominator for the two fractions on the left, add the fractions together, and simplify the result.**

$$\frac{\cos x+1}{\sin x} - \frac{\sin x}{\sin x} \cdot \frac{\sin x}{1} = \cot x(1+\cos x)$$

$$\frac{\cos x+1}{\sin x} - \frac{\sin^2 x}{\sin x} = \cot x(1+\cos x)$$

$$\frac{\cos x+1-\sin^2 x}{\sin x} = \cot x(1+\cos x)$$

4. **Replace the sin² _x_ in the numerator with its equivalent by using the Pythagorean identity, and simplify.**

$$\frac{\cos x+1-\left(1-\cos^2 x\right)}{\sin x} = \cot x(1+\cos x)$$

$$\frac{\cos x+1-1+\cos^2 x}{\sin x} = \cot x(1+\cos x)$$

$$\frac{\cos x+\cos^2 x}{\sin x} = \cot x(1+\cos x)$$

5. **Factor a cos _x_ from each term in the numerator.**

$$\frac{\cos x(1+\cos x)}{\sin x} = \cot x(1+\cos x)$$

6. Split the two factors in the numerator into two fractions that are multiplied by each other. Then replace $\dfrac{\cos x}{\sin x}$ by using the ratio identity.

$$\frac{\cos x}{\sin x} \cdot \frac{(1+\cos x)}{1} = \cot x (1+\cos x)$$

$$\cot x (1+\cos x) = \cot x (1+\cos x)$$

Factoring

The clue you'll get that suggests you should factor an identity is when powers of a particular function or repeats of that same function are in all the terms on one side of the identity.

For example, the identity $\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta = 1$ has three terms on the left that you can factor, because they're the result of squaring a binomial. The pattern you need is the algebraic equation for the square of a binomial: $a^2 + 2ab + b^2 = (a+b)^2$.

1. Factor the expression on the left as the square of a binomial.

$$\left(\sin^2\theta + \cos^2\theta\right)^2 = 1$$

2. Replace the expression in the parentheses with its equivalent by using the Pythagorean identity.

$$(1)^2 = 1$$

The preceding example was really simple — as long as you recognized the pattern of the square of a binomial. It'd be another thing altogether if you went off on some tangent (pardon the pun).

In the next example, the factoring occurs in the numerator of the fraction, where powers of sin x appear. Solve the identity $\dfrac{\sin x - \sin^3 x}{\cos x} = \tan x \cos^2 x$.

1. Factor sin *x* out of each term in the numerator.

$$\frac{\sin x \left(1 - \sin^2 x\right)}{\cos x} = \tan x \cos^2 x$$

2. Replace the expression in the parentheses with its equivalent by using the Pythagorean identity.

$$\frac{\sin x \left(\cos^2 x\right)}{\cos x} = \tan x \cos^2 x$$

3. **Split up the fraction into a product of two fractions, carefully arranging the factors in the numerator and denominator.**

$$\frac{\sin x}{\cos x} \cdot \frac{\left(\cos^2 x\right)}{1} = \tan x \cos^2 x$$

4. **Replace the first fraction with tan x by using the ratio identity.**

$$\tan x \cos^2 x = \tan x \cos^2 x$$

This next example requires factoring by using the difference between two squares. The pattern here is the algebraic equation $a^2 - b^2 = (a-b)(a+b)$ or $a^4 - b^4 = \left(a^2 - b^2\right)\left(a^2 + b^2\right)$. Solve the identity $\csc^2 \theta + \cot^2 \theta = \csc^4 \theta - \cot^4 \theta$.

1. **Factor the two terms on the right by using the difference-of-squares pattern.**

$$\csc^2 \theta + \cot^2 \theta = \left(\csc^2 \theta - \cot^2 \theta\right)\left(\csc^2 \theta + \cot^2 \theta\right)$$

2. **In the left set of parentheses only, replace csc² θ with its equivalent in the Pythagorean identity.**

You want to keep the two terms in the right parentheses as written.

$$\csc^2 \theta + \cot^2 \theta = \left(1 + \cot^2 \theta - \cot^2 \theta\right)\left(\csc^2 \theta + \cot^2 \theta\right)$$

3. **Simplify the expression, getting rid of the two opposites.**

$$\csc^2 \theta + \cot^2 \theta = \left(1 + \cancel{\cot^2 \theta} - \cancel{\cot^2 \theta}\right)\left(\csc^2 \theta + \cot^2 \theta\right)$$
$$= (1)\left(\csc^2 \theta + \cot^2 \theta\right)$$

Using a little bit of both

Just when you thought that proving identities couldn't be much more fun than what you've seen, you now find that the examples in this section involve both changing the terms to sines and cosines as well as factoring. The hardest part is deciding what to do first.

In this first example, your work goes more smoothly if you change everything to sines and cosines first. (Plus, you may not recognize right away that the expression on the left is the result of squaring a binomial.) Solve the identity:

$$\csc^2 \theta - 2\csc \theta \cot \theta + \cot^2 \theta = \frac{1 - \cos \theta}{1 + \cos \theta}$$

1. **Change the terms on the left to sines and cosines by using reciprocal and ratio identities, and then simplify the fractions.**

$$\frac{1}{\sin^2\theta} - \frac{2}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1-\cos\theta}{1+\cos\theta}$$

$$\frac{1}{\sin^2\theta} - \frac{2\cos\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1-\cos\theta}{1+\cos\theta}$$

2. **Add the three fractions on the left together, because they have the same denominator.**

$$\frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta} = \frac{1-\cos\theta}{1+\cos\theta}$$

3. **Replace the denominator of the fraction by using the Pythagorean identity.**

   You usually don't go from a simple one term to two terms, but, looking ahead, you see that you need $1+\cos\theta$ in the denominator, so this seems like a good idea.

$$\frac{1-2\cos\theta+\cos^2\theta}{1-\cos^2\theta} = \frac{1-\cos\theta}{1+\cos\theta}$$

4. **Factor the numerator as the square of a binomial; factor the denominator as the difference of two squares.**

$$\frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta}$$

5. **Factor out the common binomial in the numerator and denominator.**

$$\frac{(1-\cos\theta)^{\cancel{2}}}{\cancel{(1-\cos\theta)}(1+\cos\theta)} = \frac{1-\cos\theta}{1+\cos\theta}$$

$$\frac{1-\cos\theta}{1+\cos\theta} = \frac{1-\cos\theta}{1+\cos\theta}$$

   You find a lot of algebra in trigonometry!

In the next example, you see how to first do the factoring and then go to the basics. Solve the identity $\dfrac{\sin\theta}{\cot\theta - \cot\theta\cos^2\theta} - \sec\theta = 0$.

1. **Factor cot θ out of each term in the denominator of the fraction.**

$$\frac{\sin\theta}{\cot\theta\left(1-\cos^2\theta\right)} - \sec\theta = 0$$

2. **Replace the value in the parentheses with its equivalent by using the Pythagorean identity.**

$$\frac{\sin\theta}{\cot\theta\left(\sin^2\theta\right)} - \sec\theta = 0$$

3. **Write everything in terms of sine and cosine by using reciprocal and ratio identities.**

$$\frac{\sin\theta}{\dfrac{\cos\theta}{\sin\theta}\left(\sin^2\theta\right)} - \frac{1}{\cos\theta} = 0$$

4. **Reduce the fractions in the denominator and simplify.**

$$\frac{\sin\theta}{\dfrac{\cos\theta}{\cancel{\sin\theta}}\left(\dfrac{\sin^{\cancel{2}}\theta}{1}\right)} - \frac{1}{\cos\theta} = 0$$

$$\frac{\sin\theta}{\cos\theta\sin\theta} - \frac{1}{\cos\theta} = 0$$

5. **Divide out sin θ from the first fraction and simplify.**

$$\frac{\cancel{\sin\theta}}{\cos\theta\,\cancel{\sin\theta}} - \frac{1}{\cos\theta} = 0$$

$$\frac{1}{\cos\theta} - \frac{1}{\cos\theta} = 0$$

$$0 = 0$$

Chapter **13**

Sleuthing Out Identity Solutions

Proving a trig identity can be a simple chore, or it can be a challenge. The nice thing about an identity is that you *know* that it can be proved — it's an identity, for goodness sake. Some identities seem to just call out with the methods needed to prove them: "Look at me! Look at the three terms on the right that are begging to be combined!" Other identities just sit there — daring you to do anything about them.

In this chapter, you find more techniques and suggestions for handling identities. You always want to find the simplest way, first. . . if there is a simplest way. If the easy road fails you, then get on this super highway of trigonometric maneuvers.

Fracturing Fractions

The ratio and reciprocal identities involve fractions. The half-angle identities also use fractions. These are no big surprise since a ratio involves a fraction and a half *is* a fraction. Actually, an identity with fractions can work to your advantage. You can work toward getting rid of the fraction and, in the process, solve the problem. Some of the main techniques for working with fractions in identities are either to break

them up into separate terms or to go in the other direction and find a common denominator. You can find some examples of using a common denominator in Chapter 12 — and even more in this chapter.

Breaking up is hard to do

This section's heading is very misleading. Although breaking up with that special someone is indeed difficult, breaking up fractions really isn't all that hard to do. In fact, when you can do it, breaking up fractions is one of the most productive ways to solve identities. The trick is to break them up correctly. You can break up a fraction with several terms in the numerator and one term in the denominator — but not the other way around.

$$\text{Correct:} \frac{1+a-b}{7} = \frac{1}{7} + \frac{a}{7} - \frac{b}{7}$$

$$\text{Incorrect:} \frac{3+a-z}{2+b+c} \neq \frac{3}{2} + \frac{a}{b} - \frac{z}{c}$$

Now apply this breaking up of fractions to a trig identity. In this first example, the fraction on the left has just one term in the denominator. Solve the identity $\frac{\sin x + \cot x}{\cos x} = \tan x + \csc x$. You have the hint that breaking up fractions will work, because you see two terms on the right. You also want to have two terms on the left.

1. **Break up the fraction by writing each term in the numerator on the left over the denominator.**

$$\frac{\sin x}{\cos x} + \frac{\cot x}{\cos x} = \tan x + \csc x$$

2. **Rewrite cot *x* by using the ratio identity.**

$$\frac{\sin x}{\cos x} + \frac{\frac{\cos x}{\sin x}}{\cos x} = \tan x + \csc x$$

3. **Simplify the complex fraction by flipping the denominator and multiplying it times the numerator. Then reduce the result.**

$$\frac{\sin x}{\cos x} + \frac{\cancel{\cos x}}{\sin x} \cdot \frac{1}{\cancel{\cos x}} = \tan x + \csc x$$

$$\frac{\sin x}{\cos x} + \frac{1}{\sin x} = \tan x + \csc x$$

4. **Replace the first fraction by using the ratio identity for tangent and the second fraction by using the reciprocal identity for cosecant.**

$$\tan x + \csc x = \tan x + \csc x$$

The next example doesn't give you the hint about matching the number of terms on each side. Both sides have the same number of terms already. What catches your eye is the factoring possibilities if the left side is written as two fractions. You can break up fractions that have more than one factor (but only one term) in the denominator by carrying them both along. For example, solve the identity $\dfrac{\cot x - \cos x}{\cot x \cos x} = \dfrac{1 - \sin x}{\cos x}$.

1. **Break up the fraction on the left by writing each term in the numerator over the entire denominator.**

$$\frac{\cot x}{\cot x \cos x} - \frac{\cos x}{\cot x \cos x} = \frac{1 - \sin x}{\cos x}$$

2. **Reduce each fraction on the left side.**

$$\frac{\cancel{\cot x}}{\cancel{\cot x} \cos x} - \frac{\cancel{\cos x}}{\cot x \cancel{\cos x}} = \frac{1 - \sin x}{\cos x}$$

$$\frac{1}{\cos x} - \frac{1}{\cot x} = \frac{1 - \sin x}{\cos x}$$

3. **Rewrite cot *x* in the second denominator by using the ratio identity. Then simplify the complex fraction by flipping the denominator and multiplying.**

$$\frac{1}{\cos x} - \frac{1}{\dfrac{\cos x}{\sin x}} = \frac{1 - \sin x}{\cos x}$$

$$\frac{1}{\cos x} - 1\left(\frac{\sin x}{\cos x}\right) = \frac{1 - \sin x}{\cos x}$$

4. **The two fractions on the left now have the same denominator. Rewrite the left side as all one fraction. (What was fractured will now be rejoined.)**

$$\frac{1 - \sin x}{\cos x} = \frac{1 - \sin x}{\cos x}$$

The next example shows you a proof by breaking a fraction after involving Pythagoras. Prove the identity $\dfrac{1 + \cos x(\sin x - \cos x)}{\sin x} = \sin x + \cos x$.

1. **First, distribute cos *x* in the numerator over the two terms in the binomial.**

$$\frac{1 + \cos x \sin x - \cos^2 x}{\sin x} = \sin x + \cos x$$

2. **Regroup so that the terms 1 – cos² x appear together, suggesting a substitution using a Pythagorean triple.**

$$\frac{1 - \cos^2 x + \cos x \sin x}{\sin x} = \sin x + \cos x$$

$$\frac{\sin^2 x + \cos x \sin x}{\sin x} = \sin x + \cos x$$

3. **Break up the fraction on the left, writing each term over the denominator.**

$$\frac{\sin^2 x}{\sin x} + \frac{\cos x \sin x}{\sin x} = \sin x + \cos x$$

4. **Reduce the fractions.**

$$\frac{\sin^2 x}{\cancel{\sin x}} + \frac{\cos x \cancel{\sin x}}{\cancel{\sin x}} = \sin x + \cos x$$

$$\sin x + \cos x = \sin x + \cos x$$

You could also have gone in another direction in this last example with Step 3, factoring sin x from each term in the numerator and reducing the fraction. I probably would have gone with creating the two fractions, since I wanted two terms on the left. But, as I show you next, either method works. Starting from Step 3:

1. **Factor sin x from each term in the numerator on the left.**

$$\frac{\sin x (\sin x + \cos x)}{\sin x} = \sin x + \cos x$$

2. **Reduce the fraction on the left.**

$$\frac{\cancel{\sin x} (\sin x + \cos x)}{\cancel{\sin x}} = \sin x + \cos x$$

$$\sin x + \cos x = \sin x + \cos x$$

Finding a common denominator

Fractions are your friends. You may find this unbelievable, but the more you work with trig functions, the more you'll be swayed to my way of thinking. Finding a common denominator to combine fractions often paves the way to solving an identity.

In the identity $\dfrac{1 - \sin\theta}{\cos\theta} - \dfrac{\cos\theta}{1 + \sin\theta} = 0$, the two denominators on the left have nothing in common, so you multiply each fraction by the other's denominator — or, rather, by that denominator over itself, which equals 1.

1. **Multiply each fraction on the left by an equivalent of 1 to create a common denominator.**

$$\frac{1-\sin\theta}{\cos\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta}-\frac{\cos\theta}{1+\sin\theta}\cdot\frac{\cos\theta}{\cos\theta}=0$$

2. **Multiply the fractions together and simplify the numerators. Leave the denominator alone.**

$$\frac{(1-\sin\theta)(1+\sin\theta)}{\cos\theta(1+\sin\theta)}-\frac{\cos\theta\cos\theta}{\cos\theta(1+\sin\theta)}=0$$

$$\frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)}-\frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}=0$$

3. **Replace the first numerator with its equivalent by using the Pythagorean identity.**

   The fractions are opposites of one another.

$$\frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}-\frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}=0$$

$$0=0$$

In Chapter 12, I do a problem using that method and mention that you have another option — finding a common denominator. You have to decide which way you think is better. You may even be able to find an easier way to do this proof. Here you go: Prove the identity $\frac{1+\sec x}{\tan x}-\frac{\tan x}{\sec x}=\cot x(1+\cos x)$ by finding a common denominator.

1. **Multiply each fraction on the left by the equivalent of 1, creating a common denominator.**

$$\frac{1+\sec x}{\tan x}\cdot\frac{\sec x}{\sec x}-\frac{\tan x}{\sec x}\cdot\frac{\tan x}{\tan x}=\cot x(1+\cos x)$$

2. **Simplify the numerators by multiplying out the fractions.**

$$\frac{\sec x+\sec^2 x}{\tan x\sec x}-\frac{\tan^2 x}{\tan x\sec x}=\cot x(1+\cos x)$$

3. **Replace tan² x in the second fraction with its equivalent by using the Pythagorean identity; then combine the two numerators.**

$$\frac{\sec x+\sec^2 x}{\tan x\sec x}-\frac{\sec^2 x-1}{\tan x\sec x}=\cot x(1+\cos x)$$

$$\frac{\sec x+\sec^2 x-\sec^2 x+1}{\tan x\sec x}=\cot x(1+\cos x)$$

4. **Simplify the numerator; then rewrite the left side as the product of two fractions.**

$$\frac{\sec x + \cancel{\sec^2 x} - \cancel{\sec^2 x} + 1}{\tan x \sec x} = \cot x (1 + \cos x)$$

$$\frac{\sec x + 1}{\tan x \sec x} = \cot x (1 + \cos x)$$

$$\frac{1}{\tan x} \cdot \frac{\sec x + 1}{\sec x} = \cot x (1 + \cos x)$$

Note that this rewriting as the *product* of two fractions isn't breaking up the fraction — you still have just one term.

5. **Multiply the fraction by cos *x* divided by cos *x*, which is equivalent to 1.**

You only need to multiply one factor in the numerator and one in the denominator by cos *x*. You cleverly choose the factors with sec *x* in them.

$$\frac{1}{\tan x} \cdot \frac{\sec x + 1}{\sec x} \cdot \frac{\cos x}{\cos x} = \cot x (1 + \cos x)$$

$$\frac{1}{\tan x} \left(\frac{\sec x + 1}{\sec x} \cdot \frac{\cos x}{\cos x} \right) = \cot x (1 + \cos x)$$

6. **Multiply out the second fraction, distributing through the numerator.**

$$\frac{1}{\tan x} \left(\frac{\sec x \cos x + 1 \cdot \cos x}{\sec x \cos x} \right) = \cot x (1 + \cos x)$$

7. **Because cos *x* and sec *x* are reciprocals, their product is 1; substitute 1 for sec *x* cos *x* in both the numerator and the denominator.**

$$\frac{1}{\tan x} \left(\frac{1 + \cos x}{1} \right) = \cot x (1 + \cos x)$$

8. **Replace the reciprocal of tan *x* with cot *x*.**

$$\cot x (1 + \cos x) = \cot x (1 + \cos x)$$

ARCHIMEDES

Considered to be one of the creative geniuses of the ancient world, Archimedes was an Alexandrian mathematician who lived from about 287 to 212 B.C. One of his discoveries in which he took the most pride was a method for calculating the volume of a sphere. He found that the volume is two-thirds the volume of the smallest cylinder that the sphere can fit into. He even requested that a diagram with the cylinder and sphere be engraved on his tombstone.

Although it might be tempting to solve this identity by changing everything to sines and cosines, as it turns out, that's not a good idea. I tried it and it didn't save any steps. Also, the different necessary moves weren't very intuitive.

TIP

Using Tricks of the Trig Trade

When proving identities, sometimes the best way to handle them just leaps out at you — and sometimes the best way just stays in hiding. Usually, you can solve an identity in more than one way — the best way, the almost-as-good way, the reasonable way, and the absolutely dreadful way. The best way is the quickest and most efficient. But sometimes you have to pull something out of your hat to accomplish the task of solving a particular identity. You've already seen one little trick: multiplying a term by 1. Well, you multiply by sine over sine or some such arrangement, but it's still just multiplying by 1. Some additional little tricks amount to nothing more than multiplying a fraction by 1 in the form of a conjugate or squaring both sides of the identity.

Multiplying by a conjugate

First, what in the world is a *conjugate?* In mathematics, a conjugate consists of the same two terms as the first expression, separated by the opposite sign.

For instance, the conjugate of $x + \sqrt{y}$ is $x - \sqrt{y}$. In trig, especially, multiplying the numerator and denominator of a fraction by a conjugate can create some really nice results.

Multiplying by a conjugate is a quick, easy way of solving the identity $\dfrac{1}{\sec x - \tan x} = \tan x + \sec x$.

1. **Multiply the numerator and denominator of the fraction on the left by the conjugate of the denominator.**

$$\frac{1}{\sec x - \tan x} \cdot \frac{\sec x + \tan x}{\sec x + \tan x} = \tan x + \sec x$$

2. **The two denominators multiplied together are the difference of two squares.**

$$\frac{\tan x + \sec x}{\sec^2 x - \tan^2 x} = \tan x + \sec x$$

3. **Replace sec² *x* in the denominator with its equivalent by using the Pythagorean identity.**

$$\frac{\tan x + \sec x}{\tan^2 x + 1 - \tan^2 x} = \tan x + \sec x$$

4. **Simplify the denominator by canceling out the two opposites.**

$$\frac{\tan x + \sec x}{\cancel{\tan^2 x} + 1 - \cancel{\tan^2 x}} = \tan x + \sec x$$

$$\tan x + \sec x = \tan x + \sec x$$

In the next example, you have to decide which fraction to multiply the conjugate by. I choose the fraction on the right, because I see the conjugate of the numerator on the right in the denominator on the left. Solve the identity $\dfrac{\tan x}{1 + \cos x} = \dfrac{1 - \cos x}{\sin x \cos x}$.

1. **Multiply the numerator and denominator of the fraction on the right by the conjugate of the numerator.**

$$\frac{\tan x}{1 + \cos x} = \frac{1 - \cos x}{\sin x \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}$$

2. **Multiply the fractions together, keeping the parentheses in the denominator.**

$$\frac{\tan x}{1 + \cos x} = \frac{1 - \cos^2 x}{\sin x \cos x (1 + \cos x)}$$

3. **Substitute the equivalent from the Pythagorean identity in the numerator of the fraction on the right. Then reduce that fraction.**

$$\frac{\tan x}{1 + \cos x} = \frac{\sin^2 x}{\sin x \cos x (1 + \cos x)}$$

$$\frac{\tan x}{1 + \cos x} = \frac{\sin^{\cancel{2}} x}{\cancel{\sin x} \cos x (1 + \cos x)}$$

$$\frac{\tan x}{1 + \cos x} = \frac{\sin x}{\cos x (1 + \cos x)}$$

4. **Rewrite the fraction on the right as a product of two fractions, carefully arranging the factors.**

$$\frac{\tan x}{1 + \cos x} = \frac{\sin x}{\cos x} \cdot \frac{1}{(1 + \cos x)}$$

5. **Replace the first fraction on the right with its ratio-identity equivalent. Rewrite the expression as one fraction.**

$$\frac{\tan x}{1 + \cos x} = \tan x \cdot \frac{1}{(1 + \cos x)}$$

$$\frac{\tan x}{1 + \cos x} = \frac{\tan x}{1 + \cos x}$$

The half-angle identity for the tangent function has two different forms. Multiplying by the conjugate is a good method for showing that these two forms are equivalent. Take this example: $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$. Here, I prove that the two half-angle identities are equivalent.

1. **Multiply the numerator and denominator of the fraction on the left by the conjugate of the denominator.**

$$\frac{\sin\theta}{1+\cos\theta} \cdot \frac{1-\cos\theta}{1-\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

2. **Multiply the two denominators together, but leave the numerator in factored form.**

$$\frac{\sin\theta\,(1-\cos\theta)}{1-\cos^2\theta} = \frac{1-\cos\theta}{\sin\theta}$$

3. **Replace the denominator on the left with its equivalent by using the Pythagorean identity.**

$$\frac{\sin\theta\,(1-\cos\theta)}{\sin^2\theta} = \frac{1-\cos\theta}{\sin\theta}$$

4. **Reduce the fraction on the left.**

$$\frac{\cancel{\sin\theta}\,(1-\cos\theta)}{\sin^{\cancel{2}}\theta} = \frac{1-\cos\theta}{\sin\theta}$$

$$\frac{1-\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta}$$

Squaring both sides

One special case of working on both sides of an identity at the same time is to square both sides. Your biggest clue as to when to use this technique is usually when one side or the other has a radical. This method is also good to use when you're solving some types of trig equations. Squaring both sides has two benefits: It gets rid of radicals, and it often creates terms that can be part of one of the Pythagorean identities. The Pythagorean identities have wonderful substitutions.

This first example has only one radical, and it's on the right side. Solve the identity $\dfrac{1-\cot x}{\csc x} = \sqrt{1-2\sin x\cos x}$.

1. **Square both sides of the identity.**

   Be sure to expand the squared binomial on the left correctly.

$$\left(\frac{1-\cot x}{\csc x}\right)^2 = \left(\sqrt{1-2\sin x \cos x}\right)^2$$

$$\frac{(1-\cot x)^2}{\csc^2 x} = 1-2\sin x \cos x$$

$$\frac{1-2\cot x+\cot^2 x}{\csc^2 x} = 1-2\sin x \cos x$$

2. **Rearrange the terms in the numerator.**

$$\frac{1+\cot^2 x-2\cot x}{\csc^2 x} = 1-2\sin x \cos x$$

3. **Replace $1+\cot^2 x$ with its equivalent by using the Pythagorean identity.**

$$\frac{\csc^2 x-2\cot x}{\csc^2 x} = 1-2\sin x \cos x$$

4. **Split up the fraction by writing each term in the numerator over the denominator.**

$$\frac{\csc^2 x}{\csc^2 x} - \frac{2\cot x}{\csc^2 x} = 1-2\sin x \cos x$$

5. **Simplify the first term. Rewrite the numerator and denominator in the second term by using the ratio and reciprocal identities.**

$$1 - \frac{2\dfrac{\cos x}{\sin x}}{\dfrac{1}{\sin^2 x}} = 1-2\sin x \cos x$$

6. **Simplify the complex fraction by flipping the denominator and multiplying it by the numerator**.

$$1 - 2\frac{\cos x}{\cancel{\sin x}} \cdot \frac{\sin^2 x}{1} = 1-2\sin x \cos x$$

$$1 - 2\cos x \sin x = 1-2\sin x \cos x$$

Identifying with the Operations

Identities that have sums, differences, multiple angles, and half-angles have a suggested procedure just staring at you, because having all the functions in terms of the same angle — not twice one or the sum of the other two — is best. You just decide which angle form you want everything to be in and then apply whatever identity the terms in the equation are equal to — substitute in the equivalence of the identity — and proceed from there.

Adding it up

Sum and difference identities usually involve two different angles and then a third combined angle. To prove the identity, you need to get rid of that third angle. The first example involves a sum of two different angles.

The equation $\dfrac{\cos(x+y)}{\cos x \cos y} = 1 - \tan x \tan y$ uses the angles x and y. Get rid of the angle sum, $x + y$, by applying the appropriate identity, which contains just angle x and angle y.

1. **Replace the cosine of the sum of the two angles with its identity.**

$$\frac{\cos x \cos y - \sin x \sin y}{\cos x \cos y} = 1 - \tan x \tan y$$

2. **Break up the fraction by putting each term in the numerator over the denominator.**

$$\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y} = 1 - \tan x \tan y$$

3. **Reduce the first fraction. Rewrite the second fraction as the product of two fractions. Then replace the two fractions in that product by using the ratio identity.**

$$1 - \frac{\sin x}{\cos x} \cdot \frac{\sin y}{\cos y} = 1 - \tan x \tan y$$

$$1 - \tan x \tan y = 1 - \tan x \tan y$$

The next example shows an identity for three times an angle: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

1. **Replace the sine of 3θ with the sine of the sum of θ and 2θ to create the identity for the sum of two angles using the right side of this equation.**

$$\sin 3\theta = \sin(\theta + 2\theta)$$

$$\sin(\theta + 2\theta) = 3\sin\theta - 4\sin^3\theta$$

2. **Apply the angle-sum identity for sine on the left.**

$$\sin\theta\cos 2\theta + \cos\theta\sin 2\theta = 3\sin\theta - 4\sin^3\theta$$

3. **Now replace cos 2θ and sin 2θ by using the double-angle identities.**

You have two double-angle identities to choose from for cos 2θ. You choose the one involving the square of sine, because you see that same term on the right side of the equation.

$$\sin\theta\left(1 - 2\sin^2\theta\right) + \cos\theta\left(2\sin\theta\cos\theta\right) = 3\sin\theta - 4\sin^3\theta$$

4. **Multiply through on the left side.**

$$\sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta = 3\sin\theta - 4\sin^3\theta$$

5. **Replace cos² θ with its equivalent by using the Pythagorean identity. Then simplify the terms.**

$$\sin\theta - 2\sin^3\theta + 2\sin\theta\left(1 - \sin^2\theta\right) = 3\sin\theta - 4\sin^3\theta$$

$$\sin\theta - 2\sin^3\theta + 2\sin\theta - 2\sin^3\theta = 3\sin\theta - 4\sin^3\theta$$

$$3\sin\theta - 4\sin^3\theta = 3\sin\theta - 4\sin^3\theta$$

What difference does it make?

Using functions involving the difference between two angle measures has many of the same features as those with sums, so I added a couple of twists to the examples in this section. The first identity uses the tangent of the difference of angles, $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$.

Now, solve the identity $\cot(x - 45°) = \dfrac{1 + \tan x}{\tan x - 1}$, which uses the tangent difference identity and incorporates some function values for a 45-degree angle.

1. **Rewrite the cotangent of the difference by using the reciprocal identity, because the cotangent doesn't have a standard difference identity.**

$$\cot(x - 45°) = \frac{1}{\tan(x - 45°)}$$

2. **Replace the tangent of the difference with its identity.**

$$\frac{1}{\tan(x - 45°)} = \frac{1}{\dfrac{\tan x - \tan 45°}{1 + \tan x\tan 45°}}$$

3. **To simplify the complex fraction on the right, flip the denominator and multiply it by the numerator, which is 1.**

$$\frac{1}{\tan(x - 45°)} = \frac{1 + \tan x\tan 45°}{\tan x - \tan 45°}$$

4. **The value of the tangent of 45 degrees is 1, so replace all those terms with 1.**

$$\frac{1}{\tan(x - 45°)} = \frac{1 + \tan x(1)}{\tan x - 1}$$

$$\frac{1}{\tan(x - 45°)} = \frac{1 + \tan x}{\tan x - 1}$$

5. **Rewrite the left side in its original form.**

$$\cot(x - 45°) = \frac{1 + \tan x}{\tan x - 1}$$

The next example proves that tan *x* is equal to itself. Yes, I know that seems a bit bizarre — obviously, it must be true. Discovering the technique involved is what makes going through the steps of this identity worth the effort. The trick here is to write tan *x* as the tangent of the difference of the angles 2*x* and *x*.

1. **Write the difference identity for tangent.**

$$\tan x = \tan x$$

$$\tan x = \tan(2x - x)$$

$$= \frac{\tan 2x - \tan x}{1 + \tan 2x \tan x}$$

2. **Replace the two terms tan 2*x* with the double-angle identity.**

$$\tan x = \frac{\dfrac{2\tan x}{1 - \tan^2 x} - \tan x}{1 + \dfrac{2\tan x}{1 - \tan^2 x} \cdot \tan x}$$

3. **Get rid of the complex fraction by multiplying every term in the numerator and denominator by 1 – tan² *x*.**

$$\tan x = \frac{\left(\dfrac{2\tan x}{1 - \tan^2 x} - \tan x\right)\left(1 - \tan^2 x\right)}{\left(1 + \dfrac{2\tan x}{1 - \tan^2 x} \cdot \tan x\right)\left(1 - \tan^2 x\right)}$$

$$\tan x = \frac{\left(\dfrac{2\tan x}{1 - \tan^2 x}\right)\left(1 - \tan^2 x\right) - \tan x\left(1 - \tan^2 x\right)}{1\left(1 - \tan^2 x\right) + \left(\dfrac{2\tan x}{1 - \tan^2 x} \cdot \tan x\right)\left(1 - \tan^2 x\right)}$$

4. **Simplify the fractions.**

$$\tan x = \frac{\left(\dfrac{2\tan x}{1 - \tan^2 x}\right)\left(1 - \tan^2 x\right) - \tan x\left(1 - \tan^2 x\right)}{1\left(1 - \tan^2 x\right) + \left(\dfrac{2\tan x}{1 - \tan^2 x} \cdot \tan x\right)\left(1 - \tan^2 x\right)}$$

$$= \frac{2\tan x - \tan x\left(1 - \tan^2 x\right)}{1\left(1 - \tan^2 x\right) + \left(2\tan x \cdot \tan x\right)}$$

5. **Multiply through the parentheses.**

$$\tan x = \frac{2\tan x - \tan x + \tan^3 x}{1 - \tan^2 x + 2\tan^2 x}$$

6. **Combine the like terms in the numerator and denominator.**

$$\tan x = \frac{\tan x + \tan^3 x}{1 + \tan^2 x}$$

7. **Factor the numerator. Then reduce the fraction.**

$$\tan x = \frac{\tan x \left(1 + \cancel{\tan^2 x}\right)}{1 + \cancel{\tan^2 x}} = \tan x$$

Multiplying your fun

The only special challenge involved in dealing with identities when using the multiple-angle formulas is in deciding which version of cos 2θ to use or whether to incorporate sums of angles or double angles. Here are some examples that illustrate these situations.

Solve the identity $\dfrac{\sin 2\theta + \cos 2\theta}{\sin\theta\cos\theta} = 2 + \cot\theta - \tan\theta$. You have to make a decision as to whether to use double-angle identities or sum identities, $\sin(\theta + \theta)$ and $\cos(\theta + \theta)$.

1. **In this case, the choice is double angle: apply the double-angle identities for sin 2θ and cos 2θ.**

$$\frac{2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = 2 + \cot\theta - \tan\theta$$

Choosing the formula for the sine isn't a problem, because you have only one to choose from. The cosine, however, requires some observation. Because you have both sine and cosine in the denominator, you don't want to use the double-angle identities for cosine that have a 1 in them.

2. **Split up the fraction, writing each term in the numerator over the denominator.**

$$\frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta} = 2 + \cot\theta - \tan\theta$$

3. **Reduce the fractions.**

$$\frac{2\cancel{\sin\theta}\,\cancel{\cos\theta}}{\cancel{\sin\theta}\,\cancel{\cos\theta}} + \frac{\cos^{\cancel{2}}\theta}{\sin\theta\,\cancel{\cos\theta}} - \frac{\sin^{\cancel{2}}\theta}{\cancel{\sin\theta}\,\cos\theta} = 2 + \cot\theta - \tan\theta$$

$$2 + \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = 2 + \cot\theta - \tan\theta$$

4. **Replace the fractions with their equivalents by using the ratio identities.**

$$2 + \cot\theta - \tan\theta = 2 + \cot\theta - \tan\theta$$

Finding an identity for an angle with a multiple greater than 2 requires that you decide whether to use a sum identity or a double-angle identity. The best approach isn't always clear. Sometimes the other terms in the equation give you a hint. Often, you just flip a coin. Solve the identity $\sin 4x = 4\sin x \cos^3 x - 4\sin^3 x \cos x$.

1. **Rewrite sin 4*x* as sin (2·2*x*).**

$\sin(2 \cdot 2x) = 4\sin x \cos^3 x - 4\sin^3 x \cos x$

2. **Insert this new angle into the double-angle identity.**

$2\sin 2x \cos 2x = 4\sin x \cos^3 x - 4\sin^3 x \cos x$

3. **Replace sin 2*x* and cos 2*x* with the double-angle identities, choosing the best cosine identity for the situation.**

In this case, because you have a difference of terms involving third-degree powers of sine and cosine, the cosine identity involving second-degree powers of sine and cosine seems to be a good choice.

$2(2\sin x \cos x)(\cos^2 x - \sin^2 x) = 4\sin x \cos^3 x - 4\sin^3 x \cos x$

4. **Multiply through.**

$2(2\sin x \cos x)(\cos^2 x - \sin^2 x) = 4\sin x \cos^3 x - 4\sin^3 x \cos x$

$4\sin x \cos x(\cos^2 x - \sin^2 x) = 4\sin x \cos^3 x - 4\sin^3 x \cos x$

$4\sin x \cos^3 x - 4\sin^3 x \cos x = 4\sin x \cos^3 x - 4\sin^3 x \cos x$

Halving fun, wish you were here

The last type of identity that you can incorporate into solving an identity or doing identity problems is the half-angle identity. This identity actually comes in very handy in calculus — not by changing from a half-angle identity to angles that are larger, but by changing from larger angles to half-angle identities. The examples in this section show some of the possibilities.

Solve the identity $\tan\dfrac{\theta}{2} = \csc\theta - \cot\theta$ by using the half-angle identity for tangent. Neither side looks very promising for solving the identity until you notice that you have two different angles — one half the size of the other. You need to apply the half-angle identity to get everything in terms of the same angle.

1. **Substitute in the identity for the half-angle of tangent.**

You have two different versions to choose between: $\tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$.
The easier one to work with is the one with two terms in the numerator:
$\dfrac{1-\cos\theta}{\sin\theta} = \csc\theta - \cot\theta$.

2. **Split up the fraction on the left by putting each term in the numerator over the denominator.**

$$\frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \csc\theta - \cot\theta$$

3. **Replace the two terms on the left with their reciprocal and ratio identities.**

$$\csc\theta - \cot\theta = \csc\theta - \cot\theta$$

This last example incorporates the half–angle of the tangent, as well as the half–angle of a reciprocal function. Solve the identity $\sin\theta\sec^2\frac{\theta}{2} = 2\tan\frac{\theta}{2}$. You see here that working on both sides of the equation is necessary.

1. **Use the reciprocal of the half-angle identity for cosine to replace the half-angle of secant.**

Hold off on deciding which version of the tangent's half-angle formula to use until you see what you need.

$$\sin\theta\left(\frac{1}{\cos\frac{\theta}{2}}\right)^2 = 2\tan\frac{\theta}{2}$$

$$\sin\theta\left(\frac{1}{\sqrt{\frac{1+\cos\theta}{2}}}\right)^2 = 2\tan\frac{\theta}{2}$$

2. **Flip the fraction in the parentheses and square the radical, leaving the terms with no radical.**

$$\sin\theta\left(\frac{2}{1+\cos\theta}\right) = 2\tan\frac{\theta}{2}$$

3. **Choose the half-angle tangent identity that matches what you have on the left, and simplify.**

$$\sin\theta\left(\frac{2}{1+\cos\theta}\right) = 2\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$\frac{2\sin\theta}{1+\cos\theta} = \frac{2\sin\theta}{1+\cos\theta}$$

Applying the Magic of Trigonometry

Trigonometry has been around for quite a long time, and it has kept growing and growing in topics and capabilities and applications over those many centuries. Some procedures in trig have changed with the advent of all the technological advances, but the basics are still there. Here are some off-the-beaten-path applications.

Making some given information work

You are on a popular quiz show and are told that the sine of a particular angle is $\frac{3}{5}$ and that the tangent of that same angle is a negative number. What is the secant of the angle in question?

What identity or identities do you need to help you? There aren't any with sine and secant in them, so you have to work on at least two different identities. Since the secant is the reciprocal of cosine, and sine and cosine go together so well, you start with an identity using sine and cosine. How about: $\sin^2\theta + \cos^2\theta = 1$ or $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$ or $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$? They all have sines and cosines, but your choice is the all-time favorite, $\sin^2\theta + \cos^2\theta = 1$. Just solve for cosine and you have $\cos^2\theta = 1 - \sin^2\theta$, which becomes $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$ when you take the square root of each side. You get to choose the sign.

But first, solving for the cosine, you replace the sine with $\frac{3}{5}$ and work from there.

$$\cos\theta = \pm\sqrt{1 - \left(\frac{3}{5}\right)^2} = \pm\sqrt{1 - \frac{9}{25}} = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$$

You are told that the sine of the angle is positive and the tangent is negative, and that happens in the second quadrant — where the cosine is also negative. So you know that $\cos\theta = -\frac{4}{5}$. Now just apply the reciprocal identity: $\sec\theta = \frac{1}{\cos\theta}$, find the reciprocal, and you have $\sec\theta = -\frac{5}{4}$. You press the buzzer first and shout out the correct answer! Good work!

Off on a tangent

You need the exact value of the tangent of $285°$. You put this into your calculator and get the number –3.73205080757. But this is only rounded to the nearest ten-trillionth. Not good enough! It has to be exact.

So, rather than play around with some possible radicals to figure out what this decimal comes from, you decide to go the route of identities. What can help you with the tangent of this angle?

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \text{ or } \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta} \text{ or } \tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}?$$

The problem is that $285°$ isn't one of the multiples of 15 degrees, so you don't have an exact radical value available. But twice $285°$ is $570°$, which is coterminal with $570° - 360° = 210°$, which is the angle in the third quadrant that has a corresponding angle of $30°$. Now you're in business!

Using $\tan\dfrac{\theta}{2} = \dfrac{\sin\theta}{1+\cos\theta}$, you can say that

$$\tan\frac{570°}{2} = \frac{\sin 570°}{1+\cos 570°} = \frac{\sin 210°}{1+\cos 210°} = \frac{-\dfrac{1}{2}}{1+\left(-\dfrac{\sqrt{3}}{2}\right)}$$

$$= \frac{-\dfrac{1}{2}}{\dfrac{2-\sqrt{3}}{2}} = -\frac{1}{2} \cdot \frac{2}{2-\sqrt{3}} = \frac{-1}{2-\sqrt{3}} = \frac{1}{\sqrt{3}-2}$$

$$= \frac{1}{\sqrt{3}-2} \cdot \frac{\sqrt{3}+2}{\sqrt{3}+2} = \frac{\sqrt{3}+2}{3-4} = \frac{\sqrt{3}+2}{-1} = -\sqrt{3}-2$$

That's the exact value! Did you check your calculator? Try putting that fraction in, and you get: –3.73205080757. But you weren't surprised!

4

Equations and Applications

Become acquainted with inverse trig functions.

Identify the domains and ranges of the inverse trig functions.

Recognize the pairings of the quadrants used by each inverse function.

Solve trig equations using identities and inverse functions.

Write expressions to include infinitely many answers.

Find the areas of triangles using trig functions in the formulas.

Chapter **14**

Investigating Inverse Trig Functions

A s thrilling and fulfilling as the original six trig functions are, they just aren't complete without their inverses. An *inverse trig function* behaves like the inverse of any other type of function — it undoes what the original function did. In mathematics, functions can have inverses if they're one-to-one, meaning each output value occurs only once. This whole inverse idea is going to take some fast talking when it comes to trig functions, because they keep repeating values over and over as angles are formed with every full rotation of the circle — so you're going to wonder how these functions and inverses can be one-to-one. If you need a refresher on basic inverse functions, just refer to the section on inverses in the Appendix for the lowdown on them and how you determine one.

Writing It Right

You use inverse trig functions when you want to know what angle is involved in equations such as $\sin x = \frac{1}{2}$ or $\sec x = -2$, or $\tan 2x = 1$. In typical algebra equations, you can solve for the value of x by dividing each side of the equation by the coefficient or by adding the same thing to each side, and so on. But you can't do that with the function $\sin x = \frac{1}{2}$.

Would it make sense to divide each side by sine? "Out, out thou sine!" Here's what you'd get: $\dfrac{\sin x}{\sin} = \dfrac{\frac{1}{2}}{\sin}, \; x = \dfrac{\frac{1}{2}}{\sin}$. Goodness, no! That's silly.

Using the notation

Using inverses allows you to determine the value of x in a trig equation. To find the inverse of an equation such as $\sin x = \dfrac{1}{2}$, means to complete the following statement: "x is equal to the angle whose sine is equal to $\dfrac{1}{2}$." In trig speak, you write this statement as $x = \sin^{-1}\left(\dfrac{1}{2}\right)$. This standard notation involves putting a –1 in the superscript position immediately following the function name. Table 14-1 shows some more examples of trig equations with their corresponding inverses and the translation.

TABLE 14-1

Interpreting Trig-Speak

Function	Inverse	What It Means
$\sec x = -2$	$x = \sec^{-1}(-2)$	x is the angle whose secant is –2.
$\tan 2x = 1$	$2x = \tan^{-1}(1)$	$2x$ is the angle whose tangent is 1.
$\cos \theta = 0$	$\theta = \cos^{-1}(0)$	θ is the angle whose cosine is 0.
$\csc \alpha = -1$	$\alpha = \csc^{-1}(-1)$	α is the angle whose cosecant is –1.

You may have noticed in Table 14-1 that the same numbers and angles are showing up, but in different positions. This occurs because you're really just solving for the angle (x or Θ or . . .) by applying the inverse trig functions to both sides of the equation.

Interpreting the exponent

You've undoubtedly seen and used the exponent –1 in math expressions before now. But that exponent does a different kind of job for inverse trig functions and relations. The notation for an inverse trig relation such as $\tan^{-1}x$ means that you want an *inverse* for the expression, not the *reciprocal.* If you really want the reciprocal of tangent $\dfrac{1}{\tan x}$, then you have to use parentheses: $(\tan x)^{-1}$. Of course, the reciprocal of tangent is cotangent. The –1 exponent is where the exponential notation for trig functions makes a big exception.

When raising trig functions to a power, $\sin^2 x = (\sin x)^2$ and $\cos^4 x = (\cos x)^4$, but $\tan^{-1} x$ means the inverse function, not raising $\tan x$ to the –1 power.

REMEMBER

Alternating the notation

Inverses of trig functions have an alternate notation that avoids the confusion over what the –1 superscript means: the *arc* name. Another way of saying $\sin^{-1} x$ is arcsin *x*. The inverse cosine is $\cos^{-1} x$, or arccos *x*. The other inverse functions are arctan *x*, arccsc *x*, arcsec *x*, and arccot *x*. This notation is longer and is sometimes awkward to write out, so the original superscript notation is often preferable. You'll see inverse functions written both ways, though. Just be on the lookout.

Distinguishing between the few and the many

Technically, an inverse function is supposed to have only one answer. (Part of the definition of an inverse is that the function and inverse are one-to-one.) Each input has one output, and each output has one input. To accommodate all the practical uses of trig inverses, you have a way around this rule. You can designate whether you want one answer or many answers by using either the inverse *function* or the inverse *relation.* A relation is a bit looser than a function; it allows more than one input to have the same output. To differentiate between these two entities, I use a capital letter for the name of a function and a lowercase letter for the corresponding relation. See Table 14-2.

TABLE 14-2

Trig Functions versus Trig Relations

Trig Functions	Trig Relations
$\text{Sin}^{-1} x$ or Arcsin *x*	$\sin^{-1} x$ or arcsin *x*
$\text{Cos}^{-1} x$ or Arccos *x*	$\cos^{-1} x$ or arccos *x*
$\text{Tan}^{-1} x$ or Arctan *x*	$\tan^{-1} x$ or arctan *x*
$\text{Cot}^{-1} x$ or Arccot *x*	$\cot^{-1} x$ or arccot *x*
$\text{Sec}^{-1} x$ or Arcsec *x*	$\sec^{-1} x$ or arcsec *x*
$\text{Csc}^{-1} x$ or Arccsc *x*	$\csc^{-1} x$ or arccsc *x*

If you evaluate the function $\text{Sin}^{-1}\left(\frac{1}{2}\right)$, the result is 30 degrees (or $\frac{\pi}{6}$ radians). Just one answer exists, which is called the *principal value* of the inverse. But if

you write $\sin^{-1}\left(\dfrac{1}{2}\right)$, then the result can be 30 degrees, 150 degrees, 390 degrees, 510 degrees, and so on (or $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}, \dfrac{17\pi}{6}, \dfrac{25\pi}{6}, \dfrac{29\pi}{6}, \ldots$). It all depends on the situation — what you want at the time. Do you want just the principal value, or do you want multiple values? Or you may want a bunch of values within one full rotation — from 0 to 360 degrees.

TIP

When you want lots and lots of angles or answers, listing them all can be tedious. In fact, listing every possible solution may not even be doable. Rather than making a list, you can give a *rule*, which allows you to define an angle with all its full-rotation multiples — the angles with the same terminal side.

Let n represent any integer in the set: $\{\ldots, -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3, \ldots\}$. Using the n as a multiplier, you can write a long list of angles more efficiently. Instead of writing $x = \ldots, -135°, -90°, -45°, 0°, 45°, 90°, 135°, 180°, \ldots$, you can write $x = 45n°$.

Instead of $x = 30°,\ 150°,\ 390°,\ 510°,\ 750°,\ 870°, \ldots$, divide the list into two groups — $x = 30°,\ 390°,\ 750°,\ 1110°, \ldots$ and $x = 150°,\ 510°,\ 870°,\ 1230°, \ldots$ — and then use the two rules that follow:

$$x = 30° + 360°n \text{ or } x = 150° + 360°n$$

Also, in radians, instead of saying $x = \dfrac{\pi}{6},\ \dfrac{13\pi}{6},\ \dfrac{25\pi}{6}, \ldots$ or $x = \dfrac{5\pi}{6},\ \dfrac{17\pi}{6},\ \dfrac{29\pi}{6}, \ldots$, use these two rules: $x = \dfrac{\pi}{6} + 2\pi n$ or $x = \dfrac{5\pi}{6} + 2\pi n$.

Here's an example showing how to write all the positive angles that have a cosine equal to $\dfrac{1}{2}$. The steps involve solving the inverse relation, not just finding the principal value for the function. Solve for the values that satisfy $x = \cos^{-1}\left(\dfrac{1}{2}\right)$.

1. List several solutions in both degrees and radians.

$$\cos^{-1}\left(\dfrac{1}{2}\right) = 60°,\ 300°,\ 420°,\ 660°,\ 780°,\ 1020°, \ldots$$

$$\cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3},\ \dfrac{5\pi}{3},\ \dfrac{7\pi}{3},\ \dfrac{11\pi}{3},\ \dfrac{13\pi}{3},\ \dfrac{17\pi}{3}, \ldots$$

2. Write the answers in degrees by using just the first two angles plus multiples of 360.

$$\cos^{-1}\left(\dfrac{1}{2}\right) = 60° + 360°n \quad \text{or} \quad \cos^{-1}\left(\dfrac{1}{2}\right) = 300° + 360°n$$

3. Write the answers in radians by using the first two angles plus multiples of 2π .

$$\cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3} + 2\pi n \quad \text{or} \quad \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{5\pi}{3} + 2\pi n$$

Writing all the possible angles for inverse tangent is a bit easier than for sine or cosine. The tangent is positive in the first and third quadrants, which are at opposite corners from one another (half a full rotation). Because of this fact, the angles that have the same function values are 180 degrees apart, and you can use nice multiples of 180 degrees or π to name all the answers. This isn't the case with sine and cosine, though. The angles with the same function values are in quadrants that are adjacent to one another, so you have to use two separate rules — both with multiples of 360 degrees — to name all the answers.

Here's how to write all the angles that have a tangent equal to $-\dfrac{\sqrt{3}}{3}$. Solve for values that satisfy $x = \tan^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)$.

1. **List several answers in both degrees and radians.**

$$\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = 150°,\ 330°,\ 510°,\ 690°,\ldots$$

$$\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = \frac{5\pi}{6},\ \frac{11\pi}{6},\ \frac{17\pi}{6},\ \frac{23\pi}{6},\ldots$$

2. **Write the answers in degrees by using multiples of 180.**

$$\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = 150° + 180°n$$

3. **Write the answers in radians by using multiples of π.**

$$\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = \frac{5\pi}{6} + \pi n$$

Determining Domain and Range of Inverse Trig Functions

A **function** that has an inverse has exactly one output (belonging to the *range*) for every input (belonging to the *domain*), and vice versa. To keep inverse trig functions consistent with this definition, you have to designate ranges for them that will take care of all the possible input values and not have any duplication. The output values of the inverse trig functions are all angles — in either degrees or radians — and they're the answer to the question, "Which angle gives me this number?" In general, the output angles for the individual inverse functions are paired up as angles in Quadrants I and II or angles in Quadrants I and IV. The quadrants are selected this way for the inverse trig functions because the pairs are adjacent quadrants, allowing for both positive and negative entries. The notation for these inverse functions uses capital letters (see the preceding section).

Inverse sine function

The domain for $\text{Sin}^{-1} x$, or Arcsin x, is from –1 to 1. In mathematical notation, the domain or input values, the x's, fit into the expression $-1 \le x \le 1$, because no matter what angle measure you put into the sine function, the output of the function lies between –1 and 1, including those two numbers. The range, or output, for $\text{Sin}^{-1} x$ is all angles from –90 to 90 degrees or, in radians, $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Because the output of the inverse sine function is some angle θ, you write this range as $-90° \le \theta \le 90°$ or $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. The outputs of the inverse sine function are angles in the adjacent Quadrants I and IV, because the sine is positive in the first quadrant and negative in the second quadrant. Those angles cover all the possible input values — their function values represent all the numbers from –1 to 1.

Inverse cosine function

The domain for $\text{Cos}^{-1} x$, or Arccos x, is from –1 to 1, just like the inverse sine function. So the x (or input) values are $-1 \le x \le 1$. The range for $\text{Cos}^{-1} x$ consists of all angles from 0 to 180 degrees or, in radians, 0 to π. Because the output of the inverse cosine is some angle θ, you write these expressions for the range as $0° \le \theta \le 180°$ or $0 \le \theta \le \pi$. The outputs are angles in the adjacent Quadrants I and II, because the cosine is positive in the first quadrant and negative in the second quadrant. Those angles cover all the possible input values for the function.

Inverse tangent function

The domain for $\text{Tan}^{-1} x$, or Arctan x, is all real numbers — numbers from $-\infty$ to ∞. This is because the output of the tangent function, this function's inverse, includes all numbers. The range, or output, of $\text{Tan}^{-1} x$ is angles between –90 and 90 degrees or, in radians, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. One important note is that the range doesn't include those beginning and ending angles; the tangent function isn't defined for –90 or 90 degrees. The range of $\text{Tan}^{-1} x$ includes all the angles in the adjacent Quadrants I and IV, except for the two angles with terminal sides on the y-axis.

Inverse cotangent function

The domain of $\text{Cot}^{-1} x$, or Arccot x, is the same as that of the inverse tangent function. The domain includes all real numbers. The range, though, is different — it includes all angles between 0 and 180 degrees (between 0 and π). So any angle in Quadrants I and II is included in the range, except for those with terminal sides on the x-axis. Those two angles aren't in the domain of the cotangent function (see Chapter 7), so they aren't in the range of the inverse.

Inverse secant function

The domain of $\text{Sec}^{-1} x$, or Arcsec x, consists of all the numbers from 1 on up, plus all the numbers from −1 on down. Letting x be the input, you write this expression for the domain as $x \geq 1$ or $x \leq -1$. In other words, the domain includes all the numbers from −∞ to ∞, except for the numbers between −1 and 1. The range of $\text{Sec}^{-1} x$ is all the angles from 0 to 180 degrees (from 0 to π) — meaning all angles in Quadrants I and II, with the exception of 90 degrees, or $\frac{\pi}{2}$.

Inverse cosecant function

The domain of $\text{Csc}^{-1} x$, or Arccsc x, is the same as that for the inverse secant function, all the numbers from 1 on up plus all the numbers from −1 on down. The range is different, though — it includes all angles from −90 to 90 degrees or, in radians, from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. In short, the range is all the angles in Quadrants I and IV, with the exception of 0 degrees, or 0 radians.

Summarizing domain and range

Sometimes, looking at a chart or summary of the domains and ranges of the inverse trig functions is more informative than reading about them. Take a look at Table 14-3. You should notice some patterns — some similarities and differences. The ranges of three of the functions are in Quadrants I and II, and the other three are in Quadrants I and IV. The reciprocals sine and cosecant use the same quadrants. So do the reciprocals cosine and secant. The tangent and cotangent don't use the same quadrants, though.

TABLE 14-3

Domains and Ranges of the Inverse Trig Functions

Inverse Trig Function	Domain	Range	Quadrants in Range
$\text{Sin}^{-1} x$	$-1 \leq x \leq 1$	$-90° \leq \theta \leq 90°$	I and IV
$\text{Cos}^{-1} x$	$-1 \leq x \leq 1$	$0° \leq \theta \leq 180°$	I and II
$\text{Tan}^{-1} x$	$-\infty < x < \infty$	$-90° < \theta < 90°$	I and IV
$\text{Cot}^{-1} x$	$-\infty < x < \infty$	$0° < \theta < 180°$	I and II
$\text{Sec}^{-1} x$	$x \geq 1$ or $x \leq -1$	$0° \leq \theta \leq 180°, \theta \neq 90°$	I and II
$\text{Csc}^{-1} x$	$x \geq 1$ or $x \leq -1$	$-90° \leq \theta \leq 90°, \theta \neq 0°$	I and IV

TRIANGLE DISSECTION PARADOX

Look at the two triangles in the following figure, made up of the same four pieces. The pieces are the same size in each triangle. They're sitting on the same grid, appearing to take up the same space, but one has a hole or gap. How can that be?

This situation happens because the hypotenuse of the main outer triangle isn't really a straight line. The slopes of the sides of the two triangles (the hypotenuses) making up that larger hypotenuse aren't the same; the hypotenuse of the top triangle is slightly bent in, and the hypotenuse of the bottom triangle is slightly bent out. Of course, you're not supposed to be able to notice this bend, but you can check it by laying a ruler or straightedge along the hypotenuse. You might call this figure an optical illusion.
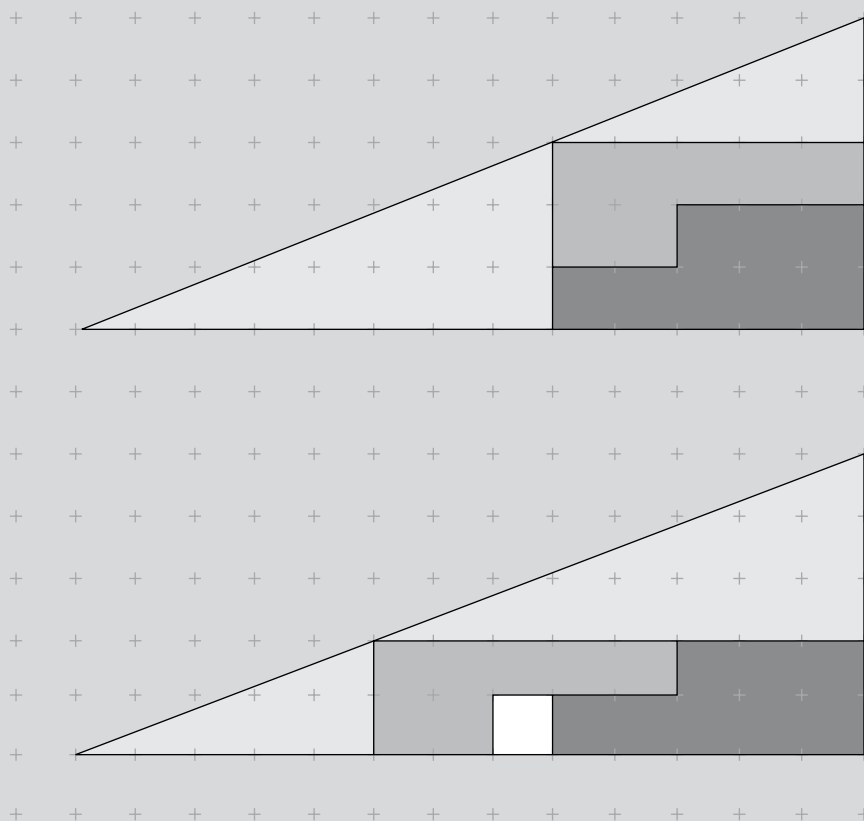
Chapter **15**

Making Inverse Trig Work for You

n Chapter 14, I introduce you to the six inverse trig functions. As with many introductions to something new, it may take a while to place the name with the face (or in this case, properties). This chapter, on the other hand, takes you deeper into the world of inverses and shows how the inverses of trig functions work. You also see why you'd even want to bother learning the names with their good or bad traits.

Working with Inverses

One of the easiest ways to work with inverse trig functions is to have a chart handy with the exact values of the most frequently used functions, which you find in Tables 15-1 and 15-2 and in the Appendix. When angles other than the most common or popular ones are involved, you can resort to your handy-dandy calculator, the computer, or even your phone.

Now, on to making good use of the functions and their values. This first example on evaluating an inverse relation uses a value from the chart.

TABLE 15-1   **Trig Values of the Most Commonly Used Functions**

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°
Radians	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	$\dfrac{2\pi}{3}$	$\dfrac{3\pi}{4}$	$\dfrac{5\pi}{6}$	π
sin x	0	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{1}{2}$	0
cos x	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{1}{2}$	0	$-\dfrac{1}{2}$	$-\dfrac{\sqrt{2}}{2}$	$-\dfrac{\sqrt{3}}{2}$	−1
tan x	0	$\dfrac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Und.	$-\sqrt{3}$	−1	$-\dfrac{\sqrt{3}}{3}$	0
cot x	Und.	$\sqrt{3}$	1	$\dfrac{\sqrt{3}}{3}$	0	$-\dfrac{\sqrt{3}}{3}$	−1	$-\sqrt{3}$	Und.
sec x	1	$\dfrac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Und.	−2	$-\sqrt{2}$	$-\dfrac{2\sqrt{3}}{3}$	−1
csc x	Und.	2	$\sqrt{2}$	$\dfrac{2\sqrt{3}}{3}$	1	$\dfrac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Und.

Note: Und. is short for "undefined."

Find $\cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$. This asks for the angles whose cosine is equal to $-\dfrac{\sqrt{2}}{2}$.

The value $-\dfrac{\sqrt{2}}{2}$ is the cosine of 135 degrees ($\dfrac{3\pi}{4}$ radians) and 225 degrees ($\dfrac{5\pi}{4}$ radians).

The angles θ for which the cosine is $-\dfrac{\sqrt{2}}{2}$ are: $\theta = 135° + 360n°$ and $\theta = 225° + 360n°$. (Refer to Chapter 7 for more on reference and coterminal angles.)

The next example involves inverse cotangent **function**. Find $\cot^{-1}\left(-\sqrt{3}\right)$. This time, you have only one angle to find. The cotangent function is defined in only two quadrants.

According to the table, the value $-\sqrt{3}$ is the cotangent of 150 degrees ($\dfrac{5\pi}{6}$ radians) or 330 degrees ($\dfrac{11\pi}{6}$ radians).

Because the cotangent function is defined in Quadrant I and Quadrant II, the answer is the angle in Quadrant II or 150 degrees.

TABLE 15-2 **Trig Values of the Most Commonly Used Functions (continued)**

Degrees	180°	210°	225°	240°	270°	300°	315°	330°	360°
Radians	π	$\dfrac{7\pi}{6}$	$\dfrac{5\pi}{4}$	$\dfrac{4\pi}{3}$	$\dfrac{3\pi}{2}$	$\dfrac{5\pi}{3}$	$\dfrac{7\pi}{4}$	$\dfrac{11\pi}{6}$	2π
sin x	0	$-\dfrac{1}{2}$	$-\dfrac{\sqrt{2}}{2}$	$-\dfrac{\sqrt{3}}{2}$	-1	$-\dfrac{\sqrt{3}}{2}$	$-\dfrac{\sqrt{2}}{2}$	$-\dfrac{1}{2}$	0
cos x	-1	$-\dfrac{\sqrt{3}}{2}$	$-\dfrac{\sqrt{2}}{2}$	$-\dfrac{1}{2}$	0	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	1
tan x	0	$\dfrac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Und.	$-\sqrt{3}$	-1	$-\dfrac{\sqrt{3}}{3}$	0
cot x	Und.	$\sqrt{3}$	1	$\dfrac{\sqrt{3}}{3}$	0	$-\dfrac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	Und.
sec x	-1	$-\dfrac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	Und.	2	$\sqrt{2}$	$\dfrac{2\sqrt{3}}{3}$	1
csc x	Und.	-2	$-\sqrt{2}$	$-\dfrac{2\sqrt{3}}{3}$	-1	$-\dfrac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	Und.

Note: Und. is short for "undefined."

And this last example asks you to find $\mathrm{Arcsin}\left(-\dfrac{\sqrt{3}}{2}\right)$. This is the inverse function for sine.

The sine function is defined in the first and fourth quadrants. So, from Table 15-2, the measure of 300° has a sine of $-\dfrac{\sqrt{3}}{2}$, meaning $\mathrm{Arcsin}\left(-\dfrac{\sqrt{3}}{2}\right) = 300°$ ($\dfrac{5\pi}{3}$ radians).

Getting Friendly with Your Calculator

Scientific calculators are wonderful tools — they make life easier and improve the quality (correctness) of the results. You can also find these calculators online or on your phone. In most instances, computing inverse trig functions with a calculator is quick and easy. You need to be aware of a couple of pitfalls, though.

Changing the mode

Scientific calculators, graphing calculators, and online calculators are very accommodating — they give you results in either degrees or radians, depending on which *mode* you set them in. This feature is great, but it trips up even the best mathematicians from time to time. Each calculator is different, but they always have either a single button or a multiple-button sequence that switches from radians to degrees and back again — sort of like a toggle switch. Some calculators even have a legend at the top or bottom of the screen that tells you whether you're in degree or radian mode, perhaps as obvious as an *R* or *D*. Problems tend to arise when you use your calculator for more than one task. Perhaps you're studying both trigonometry and physics, and one calls for degree mode while the other calls for radian mode. Just be aware, and you won't get caught.

Interpreting notation on the calculator screen

Calculator *notation*, or the mumbo jumbo on the buttons, is somewhat tricky. Even though the –1 superscript indicates an inverse trig function when written in a book or on your paper, you can't use the –1 button to find the value of an inverse trig function on a calculator. On calculator screens, the –1 or x^{-1} button means to find the **reciprocal** of a number. Look under the 2*nd functions*, which are different functions or operations written above the buttons, for the inverse trig functions. They're usually above the original sine, cosine, and tangent buttons. Some calculators have a button labeled "2nd." Others use alternative colors — usually yellow, red, or green — to denote the second use of the button.

Even when you find the inverse functions, you'll notice that they're only for the three primary trig functions. The calculator doesn't show any for cosecant, secant, or cotangent. So, where are they? First, I discuss how to use the three buttons that are available; then I tell you how to calculate the other inverses.

Using the inverse function button

To explain this button, I use an example. Here's how you find $\sin^{-1}\left(\frac{1}{2}\right)$ in degrees using a calculator.

1. **Decide whether you want your answer in radians or degrees.**

   For this example, use the mode menu or whatever method your calculator uses to change the mode to Degrees.

2. **Enter the problem as given.**

   The following are the typical keystrokes: 2nd sin .5 Enter . The result is 30, meaning 30 degrees.

Now look at what happens when you decide to find $\sin^{-1}\left(\dfrac{1}{2}\right)$ in radians using that same calculator.

1. **Specify that you want your answer in radians.**

   Use the mode menu or whatever method your calculator uses to change the mode to Radians.

2. **Enter the problem as given.**

   The following are the same keystrokes: $\boxed{\text{2nd}}\boxed{\text{sin}}\boxed{.5}\boxed{\text{Enter}}$. The result is 0.5235987756 on my calculator. This is in radians. If a radian is about 57 degrees, and this decimal is over half, then the answer is somewhere near 30 degrees — which you know from the previous problem. But what do you do if you haven't done a previous problem?

3. **Divide π by the decimal value.**

   Don't skimp on the decimal places. Use your calculator and enter: $\boxed{\pi}\boxed{\div}\boxed{.5235987756}\boxed{\text{Enter}}$. (On my calculator, I use: $\boxed{\pi}\boxed{\div}\boxed{\text{Ans}}\boxed{\text{Enter}}$, because I have the decimal sitting there from the previous computation.)

   The result is 6; the 6 is the denominator under π, which represents the radian measure equivalent to 0.5235987756. So, the radian answer in terms of π is $\dfrac{\pi}{6}$.

Calculating the inverse of a reciprocal function

To determine the inverse of a reciprocal function, such as $\mathrm{Cot}^{-1}(2)$ or $\mathrm{Sec}^{-1}(-1)$, you have to change the problem back to the function's reciprocal — one of the three basic functions — and then use the appropriate inverse button.

When changing to the function's reciprocal, you flip the input with that function, too. For example, $\mathrm{Cot}^{-1}(2)$ becomes $\tan^{-1}\left(\dfrac{1}{2}\right)$. You change $\sec^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$ to $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$, $\mathrm{Csc}^{-1}(1)$ to $\mathrm{Sin}^{-1}(1)$, and so on.

For example, find the value of $\mathrm{Sec}^{-1}(-1.1547)$. This time find the answer in degrees.

1. **Rewrite the function in terms of its reciprocal.**

   Find the reciprocal of -1.1547 using your calculator.

   $\boxed{-1.1547}\boxed{x^{-1}}\boxed{\text{Enter}}$

   The result is -0.8660258076. Just use the first four decimal places in your work.

   $\mathrm{Sec}^{-1}(-1.1547) = \mathrm{Arccos}(-0.8660)$

2. **Enter the problem.**

   2nd cos −.8660 Enter

   The calculator give you 149.9970891. This is essentially an angle of 150 degrees. All the rounding of decimals inserts a bit of an error. You can check by finding the cosine of a 150-degree angle using your calculator and then finding the reciprocal, to see if you have the correct answer. Also, the exact value of $\cos(150)$ is $-\dfrac{\sqrt{3}}{2}$, which has a decimal value of 0.8660254038. Look familiar?

Working around the inverse cotangent

The other big pitfall you encounter when using a calculator involves the inverse cotangent function. The inverse tangent function, $\text{Tan}^{-1} x$, has its range in Quadrant I and Quadrant IV, but $\text{Cot}^{-1} x$ has its range in Quadrant I and Quadrant II. If you want $\cot^{-1}\left(-\sqrt{3}\right)$, for example, and you use $\tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$ and your calculator, you get an answer in the fourth quadrant. You have to be aware that this quadrant isn't correct; you got it because you changed functions from cotangent to tangent so you could use the calculator. This is still the best way to do the problem. Just use the answer from the calculator and determine the corresponding angle in Quadrant II. Here's an example:

Find $\cot^{-1}\left(-\sqrt{3}\right)$ in degrees.

1. **Set the mode to Degrees.**

2. **Change the function and value to their reciprocals.**

   $\cot^{-1}\left(-\sqrt{3}\right)$ becomes $\tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$.

3. **Find the value of the inverse function by using a calculator.**

   Enter 2nd tan (−)1 / √3 Enter. On some calculators, parentheses automatically pop up for you to enter the tangent value inside them. If they don't, then you should insert parentheses around the fraction yourself. The result is −30 degrees. Note that this angle is in Quadrant IV, and you want Quadrant II for the inverse cotangent.

4. **Find the angle in Quadrant II that has the same reference angle.**

   The angle in Quadrant II with a 30-degree reference angle is an angle of 150 degrees. So $\cot^{-1}\left(-\sqrt{3}\right) = 150°$.

Multiplying the Input

Multiple-angle functions are those such as $\sin 2\theta$, $\cos 3x$, $\tan 6\beta$, and so on. When considering inverse relations (which give multiple answers) for these angles, the multiplier usually indicates how many more answers that problem has (it's how many more *times*) as compared to a similar problem without a multiplier. For example, the equation $\sin\theta = \dfrac{\sqrt{3}}{2}$ has two different answers if you consider all the angles between 0 and 360 degrees: θ equals 60 and 120 degrees. But if you change the equation to $\sin 2\theta = \dfrac{\sqrt{3}}{2}$, you get twice as many, or four, answers between 0 and 360 degrees: θ equals 30, 60, 210, and 240 degrees. These angles are all within one rotation, but putting them into the original equation and multiplying by 2 gives angles with the same terminal side as the angles within one rotation.

Here are some examples to show you how this multiplication works and how to find the answers. First, I show how I got the answers for $\sin 2\theta = \dfrac{\sqrt{3}}{2}$.

1. **Write the inverse equation.**

    $2\theta = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

2. **List all the angles in two rotations, $0° \le \theta \le 720°$, that have that sine, and set them equal to 2θ.**
    $2\theta = 60°,\ 120°,\ 420°,\ 480°$

    The second two angles are each 360 more than the first two.

3. **Divide each of the terms on both sides of the equation by 2 to solve for θ.**
    $\dfrac{2\theta}{2} = \dfrac{60°}{2}, \dfrac{120°}{2}, \dfrac{420°}{2}, \dfrac{480°}{2}$
    $\theta = 30°,\ 60°,\ 210°,\ 240°$

    Notice how all the solutions for θ are between 0 and 360 degrees — just as asked.

Now solve $\cos 3x = -\dfrac{\sqrt{2}}{2}$ for any x such that $0 \le x \le 2\pi$. Note that this interval indicates radian measures.

1. **Write the inverse equation**.

    $3x = \cos^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

2. **List all the angles in three rotations, $0 \le x \le 6\pi$, that have that cosine, and set them equal to 3x.**

$$3x = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{19\pi}{4}, \frac{21\pi}{4}$$

The second two angles are just 2π greater than the first two, and the last two are 2π greater than the second two. Just change 2π to $\frac{8\pi}{4}$ and add the fractions.

3. **Multiply all the terms on both sides by $\frac{1}{3}$ to solve for x.**

$$\frac{1}{3} \cdot 3x = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}\pi}{4}, \ \frac{1}{3} \cdot \frac{5\pi}{4}, \ \frac{1}{3} \cdot \frac{11\pi}{4}, \ \frac{1}{3} \cdot \frac{13\pi}{4}, \ \frac{1}{3} \cdot \frac{19\pi}{4}, \ \frac{1}{\cancel{3}} \cdot \frac{\cancel{21}^{7}\pi}{4}$$

$$x = \frac{\pi}{4}, \ \frac{5\pi}{12}, \ \frac{11\pi}{12}, \ \frac{13\pi}{12}, \ \frac{19\pi}{12}, \ \frac{7\pi}{4}$$

This result shows the big advantage of radians — the numbers don't get as big as they do with degrees. The disadvantage may be having so many fractions.

Solving Some Mixed Problems

When working with inverse trig functions, it's always more convenient when the numbers you're working with are the results of applying one of the trig functions to a common angle measure. The exact values of the functions of those more popular angles are easy to remember and work with in problems. When the angle isn't a common one, though, you need a calculator or table. Not a big deal, just not as pleasant.

By using inverse trig functions, you can solve some interesting problems, where you never even need to know what the angle measure actually is. You just need to know a function value, a quadrant, and a few trig identities.

For example, you can find $\cos\left(\sin^{-1}\left(-\frac{12}{13}\right)\right)$, which says to "find the cosine of an angle whose sine is equal to $-\frac{12}{13}$." You don't need to know the angle measure to solve this problem, but you do need to know the quadrant that the terminal side lies in, because otherwise, two different angles can be correct answers. The sine is negative in Quadrant III and Quadrant IV, so this problem could involve an angle in either of those quadrants, but the cosine is negative in Quadrant III and positive in Quadrant IV. Consider the following example, where the target quadrant is given.

Find $\cos\left(\sin^{-1}\left(-\dfrac{12}{13}\right)\right)$ if the terminal side of the angle is in Quadrant IV.

1. **Use the Pythagorean identity to find the numerical value of the cosine of the angle.**

Put the value in for $\sin\theta$, get the cosine term alone, and then take the square root of both sides:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(-\frac{12}{13}\right)^2 + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \frac{144}{169} = \frac{25}{169}$$
$$\cos\theta = \pm\sqrt{\frac{25}{169}} = \pm\frac{5}{13}$$

2. **Choose the sign of the answer.**

Because the angle's terminal side is in Quadrant IV, and the cosine is positive there, the answer is $\cos\left(\sin^{-1}\left(-\dfrac{12}{13}\right)\right) = \dfrac{5}{13}$.

The quadrant isn't a mystery in a problem that uses the inverse trig **function**. The previous example includes information on the terminal side of the angle — in which quadrant it lies. When an inverse function is involved, the quadrant is spelled out for you by the range of the function involved. You just use the assigned quadrants.

For example, to find $\tan\left(\cos^{-1}\left(-\dfrac{11}{61}\right)\right)$, you can assume that the angle involved has its terminal side in Quadrant II, because the inverse cosine function is negative in that quadrant.

1. **Use the reciprocal identity and reciprocal of the number to find the secant.**

The problem involves the angle whose cosine is $-\dfrac{11}{61}$. I call that unknown angle θ and rewrite the expression in terms of the cosine of θ with that measure. I write the expression this way in order to change from an inverse trig function to a trig function so I can use the identity.

If $\cos\theta = -\dfrac{11}{61}$, then $\sec\theta = -\dfrac{61}{11}$.

2. **Use the Pythagorean identity to solve for the tangent.**

$$\tan^2\theta + 1 = \sec^2\theta = \left(-\frac{61}{11}\right)^2 = \frac{3,721}{121}$$

$$\tan^2\theta = \frac{3,721}{121} - 1 = \frac{3,600}{121}$$

$$\tan^2\theta = \pm\sqrt{\frac{3,600}{121}} = \pm\frac{60}{11}$$

3. **Choose the sign of the answer.**

Because the terminal side is in Quadrant II and the tangent is negative in that quadrant, $\tan\left(\cos^{-1}\left(-\frac{11}{61}\right)\right) = -\frac{60}{61}$.

Finding an Unknown Angle

Trigonometry comes in handy again! You can use trig to find the distances between points when given an angle, and you can compute the measure of an angle when given distances. This is dependent on having nice, cooperative right triangles, of course, but they are usually part of the picture.

Suppose you have a 25-foot ladder and want to lean it against a building so that it reaches just under the second-floor window. Refer to Figure 15-1a.

The distance from the ground to the bottom of the window is 24 feet. What angle does that require of the ladder? And is it safe?
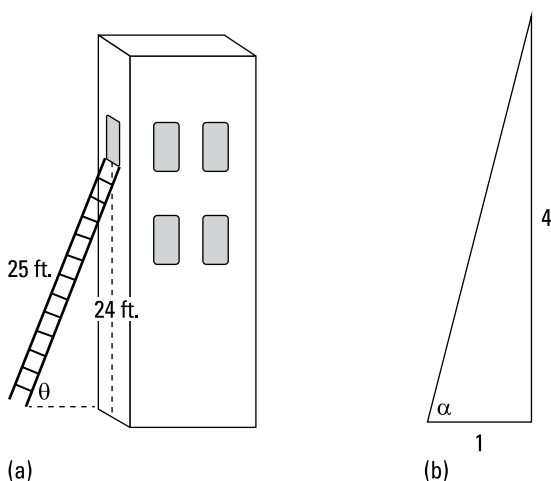
FIGURE 15-1:
Determining the angle measure.

(a)   (b)

Let the angle be represented by θ. And, with the hypotenuse and opposite side measures of a right triangle available, you can write $\sin\theta = \frac{24}{25}$. Using the inverse, you have $\sin^{-1}\left(\frac{24}{25}\right) = \theta$. Getting out your calculator, you determine that $\theta \approx 73.74°$. That's sorta steep. But does it qualify as safe? To determine that, you go to the "4-to-1 Rule."

TIP

The 4-to-1 Rule says that the base of a ladder should be positioned such that it is one foot away from the building for every four feet of height to where the ladder rests against the building.

What angle measure are we talking about here? Take a look at Figure 15-1b.

The angle α represents the safe measure for leaning the ladder against the wall. The opposite and adjacent measures are given, so $\tan\alpha = \frac{4}{1} = 4$ and $\alpha = \tan^{-1}(4) \approx 75.96°$. That's pretty close — just 2 degrees under the rule's limit — but we're probably safe here.

Chapter **16**

Solving Trig Equations

S olving equations involving trigonometric expressions takes a pinch of this, a dab of that, a gentle stirring, and just the right temperature. No, this book isn't for a cooking class, but solving these equations requires the proper preparation and some skill — just like a successful dish.

REMEMBER

Trig equations aren't identities. An identity is true for any angle in the domain of the function involved. A trig *equation* is true for some specific angles or input — if the equation has a solution at all. It's just like solving algebraic equations for the variable *x*, which can have one solution, several solutions, or no solution at all.

Some trig equations require factoring skills that you learned from algebra; some even require the quadratic formula. Successfully solving most trig equations involves incorporating trig identities at the proper time. All the equations require knowledge of the function values and how inverse trig functions work (so head on back to Chapters 8, 14, and 15 if you need a refresher). For equation-solving enthusiasts, this chapter is where all the concepts come together for maximum fun and challenge.

The methods and techniques that you see in this chapter are those that people use most frequently to solve trig equations. A few more ways exist, but they don't come up as often. Also, you usually have more than one way of solving a particular

trig equation. Your goal should always be to do it as quickly and efficiently as possible, but don't be alarmed if you seem to take the long way around. Sometimes the more circuitous route just seems to make more sense. If a particular method works for you — in other words, you get the right answer — go for it!

Generating Simple Solutions

The simplest type of trig equation is the one where you can immediately rewrite the equation using an inverse in order to determine the solutions. Some examples of these types of equations include solving for x: $\cos x = 1$, $2\sin x + 1 = 0$, and $\cot x - \sqrt{3} = 0$. Here's how to solve them.

To solve $\cos x = 1$ for all angles x between 0 and 360 degrees (including 0) and then all angles in general, follow these steps:

1. **Rewrite the equation as an inverse function equation.**

$x = \cos^{-1}(1)$

2. **List the solutions for values of x when $0 \le x < 360°$.**

$x = 0°$

The only time that the cosine is equal to 1 in this interval is when the angle, or input, is 0 degrees.

3. **List all the solutions in general.**

$x = 0° + 360°n$

Steps 2 and 3 illustrate the different ways that you can write the answers: either as a few within a certain interval, or as all that are possible, with a rule to describe them.

Now solve $2\sin x + 1 = 0$ only for values of x such that $0 \le x < 2\pi$:

1. **Rewrite the equation as an inverse function equation.**

First, subtract 1 from each side; then divide each side by 2 before applying the inverse.

$2\sin x = -1$

$\sin x = -\dfrac{1}{2}$

$x = \sin^{-1}\left(-\dfrac{1}{2}\right)$

2. **List the solutions.**

You can use the table in Chapter 15 to find the angles that work.

$$x = \frac{7\pi}{6}, \ \frac{11\pi}{6}$$

This last example involves a reciprocal function. Your best bet is to begin by using a reciprocal identity and changing the problem.

Solve the equation $\cot x - \sqrt{3} = 0$ for all the values of x, in radians, that satisfy it:

1. **Solve for the trig function by adding the radical value to each side.**

$$\cot x = \sqrt{3}$$

2. **Use the reciprocal identity and the reciprocal of the number to change to the tangent function, and then multiply both parts of the fraction by the radical in the denominator to rationalize the fraction.**

$$\cot x = \sqrt{3} \rightarrow \tan x = \frac{1}{\sqrt{3}} \rightarrow \tan x = \frac{\sqrt{3}}{3}$$

3. **Rewrite the equation as an inverse function equation.**

$$x = \tan^{-1}\left(\frac{\sqrt{3}}{3} \right)$$

4. **Write the general statements that give all the solutions. (Refer to the table in Chapter 15.)**

$$x = \frac{\pi}{6} + \pi n$$

$$x = \frac{\pi}{6}, \ \frac{7\pi}{6}, \ \frac{13\pi}{6}, \ \frac{19\pi}{6}, \dots$$

These statements mean that all the angles you find by adding or subtracting multiples of π will provide solutions for this equation. For the tangent function, you use multiples of π rather than 2π, which are used with sine and cosine. The tangent's period is only π radians, while sine and cosine have periods of 2π.

Factoring In the Solutions

The same type of factoring that algebra uses is a great help in solving trig equations. The only trick is to recognize that instead of just *x*s, *y*s, or other single-letter variables, trig variables such as sin *x* or sec *y* exist. You need the whole sin *x* or sec *y* when working with the variables; you can't factor out an *x* or a sec alone. Look for the patterns and apply the factoring techniques.

Here are a few of the most basic patterns for factoring binomials and trinomials.

>> **Greatest common factor:** $ab \pm cb = b(a \pm c)$

>> **Difference of squares:** $a^2 - b^2 = (a+b)(a-b)$

>> **Sum or difference of cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

>> **Perfect square trinomial:** $x^2 + 2ax + a^2 = (x+a)^2$, $x^2 - 2ax + a^2 = (x-a)^2$

Finding a greatest common factor

The trig equations that require finding a greatest common factor (GCF), factoring it out, and then solving the equation could look like these two equations: $2\sin x \cos x - \sin x = 0$ or $\cos x \tan x = \sqrt{3} \cos x$. I solve both of these equations in this section.

Solve $2\sin x \cos x - \sin x = 0$ for all the values of x such that $0 \le x < 360°$.

1. **Factor out sin x from each of the two terms**.
$\sin x (2\cos x - 1) = 0$

2. **Set the two different factors equal to 0.**

$\sin x = 0$ or $2\cos x - 1 = 0$

3. **Solve for the values of x that satisfy each equation.** Use the table in Chapter 15.

If $\sin x = 0$, then $x = \sin^{-1}(0) = 0°, 180°$.

If $2\cos x - 1 = 0$, $2\cos x = 1$, $\cos x = \frac{1}{2}$, then $x = \cos^{-1}\left(\frac{1}{2}\right) = 60°, 300°$.

All these values are solutions for the original equation. The complete list is $x = 0°, 60°, 180°, 300°$.

Now solve $\cos x \tan x = \sqrt{3} \cos x$ for *all* the possible values in degrees.

You don't want to divide each side by cos x, because you'll lose a solution if you do.

1. **Move the term on the right to the left by subtracting it from each side.**
$\cos x \tan x - \sqrt{3} \cos x = 0$

2. **Factor out the cos x from each term.**
$\cos x (\tan x - \sqrt{3}) = 0$

3. **Set the two different factors equal to 0.**

$\cos x = 0$ or $\tan x - \sqrt{3} = 0$

4. **Solve for the values of x that satisfy both equations.**

If $\cos x = 0$, then $x = \cos^{-1}(0) = 90°,\ 270°,\ \dots$ or $90° + 180°n$

If $\tan x - \sqrt{3} = 0,\ \tan x = \sqrt{3}$, then $x = \tan^{-1}(\sqrt{3}) = 60°,\ 240°,$ or $60° + 180°n$

So the solutions are all of the form $x = 90° + 180°n$ or $x = 60° + 180°n$

Factoring quadratics

Quadratic equations are nice to work with because, when they don't factor, you can solve them by using the quadratic formula (see the section, "Using the Quadratic Formula," later in this chapter). The types of quadratic trig equations that you can factor include $\tan^2 x = \tan x$, $4\cos^2 x - 3 = 0$, $2\sin^2 x + 5\sin x - 3 = 0$, or $\csc^2 x + \csc x - 2 = 0$. Notice that each equation has the telltale trig function raised to the second degree. I show you how to handle them in the following examples.

The first two examples have just two terms. The first has two variable terms, and the second has just one variable term. In the first example, you put both terms on the left and then factor out the variable or trig term.

Solve $\tan^2 x = \tan x$ for the values of x such that $0 \le x < 2\pi$.

1. **Move the term tan x on the right to the left by subtracting it from both sides.**

$\tan^2 x - \tan x = 0$

Don't divide through by tan x. You'll lose solutions.

2. **Factor out tan x.**
$\tan x(\tan x - 1) = 0$

3. **Set each of the two factors equal to 0.**

$\tan x = 0$ or $\tan x - 1 = 0$

4. **Solve for the values of x that satisfy both equations.**

If $\tan x = 0$, then $x = \tan^{-1}(0) = 0,\ \pi$

If $\tan x - 1 = 0,\ \tan x = 1$, then $x = \tan^{-1}(1) = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

The four solutions are $x = 0,\ \dfrac{\pi}{4},\ \pi,$ and $\dfrac{5\pi}{4}$.

In this next example, the binomial doesn't factor easily as the difference of two squares, because the 3 isn't a perfect square, and you have to use a radical in the factorization. A nice, efficient way to solve this equation is to move the 3 to the right and take the square root of each side.

Solve for all the possible solutions of $4\cos^2 x - 3 = 0$ in degrees.

1. **Move the number to the right by adding 3 to each side.**

$4\cos^2 x = 3$

2. **Take the square root of each side. Then solve for cos x by dividing each side by 2**.

$$\sqrt{4\cos^2 x} = \pm\sqrt{3}$$
$$2\cos x = \pm\sqrt{3}$$
$$\cos x = \pm\frac{\sqrt{3}}{2}$$

3. **Solve the two equations for the values of x.**

If $\cos x = \frac{\sqrt{3}}{2}$, then $x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30°, \ 330°$, with all the possible solutions being $x = 30° + 360°n$ or $330° + 360°n$.

If $\cos x = -\sqrt{\frac{3}{2}}$, then $x = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = 150°, \ 210°$, with all the possible solutions being $x = 150° + 360°n$ or $210° + 360°n$.

TIP

You may have noticed that the solutions are always 30 degrees away from 180 degrees and 360 (or 0) degrees. A simpler way to describe the solutions is to just use multiples of 180 degrees and say: $x = 30° + 180°n$ and $x = 150° + 180°n$.

The next two examples involve using un-FOIL — a technique for determining which two binomials give you a particular quadratic trinomial. When the pattern in the trinomial is obscured, you may want to first substitute some other variable for the trig function to help figure out how you factor it. I use this particular method in the next example.

Solve $2\sin^2 x + 5\sin x - 3 = 0$ for all the values of x between 0 and 360 degrees.

1. **Replace each sin x with y.**

$2y^2 + 5y - 3 = 0$

2. **Factor the trinomial as the product of two binomials.**

$(2y - 1)(y + 3) = 0$

3. **Replace each *y* with sin *x*.**

$(2\sin x - 1)(\sin x + 3) = 0$

4. **Set each factor equal to 0.**

$2\sin x - 1 = 0$ or $\sin x + 3 = 0$

5. **Solve the two equations for the values of *x* that satisfy them.**

If $2\sin x - 1 = 0$, $2\sin x = 1$, $2\sin x = \frac{1}{2}$, then $x = \sin^{-1}\left(\frac{1}{2}\right) = 30°, 150°$.

If $\sin x + 3 = 0$, $\sin x = -3$, then $x = \sin^{-1}(-3)$. This result is nonsense, because the sine function only produces values between –1 and 1 — so this factor doesn't produce any solutions. Go to Chapter 8 for information on the range of the sine function.

The only two solutions are 30 and 150 degrees.

This next example factors fairly easily, but it involves a reciprocal function. Solve $\csc^2 x + \csc x - 2 = 0$ for any angles between 0 and 2π radians.

1. **Factor the quadratic trinomial into the product of two binomials.**

$(\csc x + 2)(\csc x - 1) = 0$

2. **Set each factor equal to 0.**

$\csc x + 2 = 0$ or $\csc x - 1 = 0$

3. **Solve the two equations for the values of *x* that satisfy them.**

If $\csc x + 2 = 0$, $\csc x = -2$, then $x = \csc^{-1}(-2) = \frac{7\pi}{6}, \frac{11\pi}{6}$.

If $\csc x - 1 = 0$, $\csc x = 1$, then $x = \csc^{-1}(1) = \frac{\pi}{2}$.

An alternate way of dealing with these two binomial equations is to change them by using the reciprocal identity and writing the reciprocal of the number. For the first equation, you change from cosecant to sine: $\csc x + 2 = 0$, $\csc x = -2$, $\sin x = -\frac{1}{2}$.

Do the same for the second equation: $\csc x - 1 = 0$, $\csc x = 1$, $\sin x = 1$. You then solve the inverse equations (and get the same answers).

Increasing the degrees in factoring

Factoring quadratics is a breeze — well, I guess it gets a bit windy at times. Factoring equations with higher degrees can get a bit challenging if you don't have a nice situation such as just two terms or a quadratic-like equation. You may have the possibility of factoring by grouping, and I cover that method in the next section. In this section, the problems that I have in mind are those like $2\sin^3 x = \sin x$ or $2\cos^4 x - 9\cos^2 x + 4 = 0$.

The first equation has just two terms, so you can factor it by finding a greatest common factor.

Solve $2\sin^3 x = \sin x$ for all the possible angles in degrees.

1. **Move the term on the right to the left by subtracting it from each side.**

$2\sin^3 x - \sin x = 0$

2. **Factor out sin *x*.**

$\sin x \left(2\sin^2 x - 1\right) = 0$

3. **Set each factor equal to 0.**

$\sin x = 0$ or $2\sin^2 x - 1 = 0$

4. **Solve the two equations for the values of *x* that satisfy them.**

If $\sin x = 0$, then $x = \sin^{-1}(0) = 0°,\ 180°,\ \ldots$ or $0° + 180°n$.

If $2\sin^2 x - 1 = 0$, $2\sin^2 x = 1$, $\sin^2 x = \dfrac{1}{2}$, then you end up with a quadratic equation.

5. **Take the square root of both sides of the quadratic equation and solve for *x*.**

You need to consider both a positive and negative root. Then multiply both parts of the fraction by the denominator to rationalize the denominator.

$$\sqrt{\sin^2 x} = \pm\sqrt{\dfrac{1}{2}}$$
$$\sin x = \pm\dfrac{\sqrt{2}}{2}$$

Now, considering both solutions:

If $\sin x = \dfrac{\sqrt{2}}{2}$, then $x = \sin^{-1}\left(\sqrt{\dfrac{2}{2}}\right) = 45°,\ 135°,\ldots$ or $45° + 360°n,\ 135° + 360°n$.

If $\sin x = -\sqrt{\dfrac{2}{2}}$, then $x = \sin^{-1}\left(-\sqrt{\dfrac{2}{2}}\right) = 225°,\ 315°,\ldots$ or $225° + 360°n,\ 315° + 360°n$.

This third-degree trig equation has a whole slew of answers:

$x = 180°n$
$x = 45° + 360°n$
$x = 135° + 360°n$
$x = 225° + 360°n$
$x = 315° + 360°n$

You can combine these last four equations that are describing the solution for *x*. The equations all begin with multiples of 45 degrees and can be combined to read $x = 45° + 90°n$. This single equation generates all the same angles as the last four statements combined. How do you know you can simplify this way? Because the angles of 45, 135, 225, and 315 degrees are all 90 degrees apart in value. By starting with 45 and adding 90 over and over, you get all the listed angles, as well as the infinite number of their multiples.

The next example is a fourth-degree equation, but this one is quadratic-like, meaning that it factors like a quadratic trinomial with two binomial factors. This problem has the possibility of having a great number of solutions — or none.

Solve $2\cos^4 x - 9\cos^2 x + 4 = 0$ for the solutions that are between 0 and 2π.

1. **Factor the trinomial as the product of two binomials.**

$$\left(2\cos^2 x - 1\right)\left(\cos^2 x - 4\right) = 0$$

2. **Set each factor equal to 0.**

$$2\cos^2 x - 1 = 0 \text{ or } \cos^2 x - 4 = 0$$

3. **Solve for the function in each equation by getting the cosine terms alone on one side of the equation.**

$$2\cos^2 x - 1 = 0$$
$$2\cos^2 x = 1$$
$$\cos^2 x = \frac{1}{2}$$

$$\text{and } \cos^2 x - 4 = 0$$
$$\cos^2 x = 4$$

4. **Take the square root of each side of each equation.**

$$\sqrt{\cos^2 x} = \pm\sqrt{\frac{1}{2}}$$
$$\cos x = \pm\frac{\sqrt{2}}{2}$$

$$\text{and } \sqrt{\cos^2 x} = \pm\sqrt{4}$$
$$\cos x = \pm 2$$

5. **Solve for the values of *x* that satisfy the equations.**

If $\cos x = \frac{\sqrt{2}}{2}$, then $x = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}, \frac{7\pi}{4}$.

If $\cos x = -\frac{\sqrt{2}}{2}$, then $x = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}, \frac{5\pi}{4}$.

If $\cos x = \pm 2$, then you have a problem — that equation doesn't compute! The cosine function results in values between −1 and 1. (Find out more about the range of cosine in Chapter 8.) This factor doesn't give any new

solutions to the original problem. So the solutions of the equation are
$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$, all between 0 and 2π.

Factoring by grouping

The process of factoring by grouping works in very special cases; one case is when the original equation is the result of multiplying two binomials together that have some unrelated terms in them. You usually can apply this type of factoring when you're facing an even number of terms and can find common factors in different groups of them. The types of equations that you can solve by using grouping may look like $4\sin x \cos x - 2\sin x - 2\cos x + 1 = 0$ or $\sin^2 x \sec x + 2\sin^2 x = \sec x + 2$. In the first equation, the first two terms have an obvious common factor, $2\sin x$. The second two have no common factor other than 1, but to make grouping work, factor out -1.

Solve $4\sin x \cos x - 2\sin x - 2\cos x + 1 = 0$ for all the possible answers between 0 and 2π.

1. **Factor 2 sin *x* out of the first two terms and −1 out of the second two.**
   $2\sin x(2\cos x - 1) - 1(2\cos x - 1) = 0$

   Now you have two terms on the left, each with a factor of $2\cos x - 1$.

2. **Factor that common factor out of the two terms.**
   $(2\cos x - 1)(2\sin x - 1) = 0$

3. **Set the two factors equal to 0.**
   $2\cos x - 1 = 0 \qquad 2\sin x - 1 = 0$
   $2\cos x = 1 \quad$ or $\quad 2\sin x = 1$
   $\cos x = \frac{1}{2} \qquad\qquad \sin x = \frac{1}{2}$

4. **Solve for the values of *x* that satisfy the equation.**
   If $\cos x = \frac{1}{2}$, then $x = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3},\ \frac{5\pi}{3}$.

   If $\sin x = \frac{1}{2}$, then $x = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6},\ \frac{5\pi}{6}$.

So the solutions are: $x = \frac{\pi}{6}, \frac{\pi}{3},\ \frac{5\pi}{6}, \frac{5\pi}{3}$

This next example of grouping requires that you begin by moving the two terms on the right to the left. Another twist is that one of the resulting factors turns out to be a quadratic. How can math be much more fun than this?

Solve $\sin^2 x \sec x + 2\sin^2 x = \sec x + 2$ for all the angles between 0 and 360 degrees.

1. **Move the terms on the right to the left by subtracting them from both sides.**

$\sin^2 x \sec x + 2\sin^2 x - \sec x - 2 = 0$

2. **Factor $\sin^2 x$ out of the first two terms and -1 out of the second two.**

$\sin^2 x(\sec x + 2) - 1(\sec x + 2) = 0$

3. **Factor $\sec x + 2$ out of the two terms.**

$(\sec x + 2)(\sin^2 x - 1) = 0$

4. **Set the two factors equal to 0.**

$\sec x + 2 = 0, \ \sec x = -2$

$\sin^2 x - 1 = 0, \ \sin^2 x = 1, \ \sin x = \pm 1$

when you take the square root of both sides.

5. **Solve for the values of x that satisfy the equations.**

If $\sec x = -2$, then $x = \sec^{-1}(-2) = 120°, \ 240°$.

If $\sin x = 1$, then $x = \sin^{-1}(1) = 90°$.

If $\sin x = -1$, then $x = \sin^{-1}(-1) = 270°$.

So the solutions are $x = 90°, 120°, \ 240°, \ 270°$.

Using the Quadratic Formula

When quadratic equations factor, life is good. When they don't, you can still survive, thanks to that wonderful quadratic formula. In case you've forgotten the exact formula, here it is.

The quadratic formula says that if you have a quadratic equation in the form $ax^2 + bx + c = 0$, then its solutions are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

TIP

In trig, a trig function replaces the x or variable part of the quadratic formula. For example, find the solution of $\sin^2 x - 4\sin x - 1 = 0$ for all angles between 0 and 360 degrees. Instead of just xs, the variable terms are $\sin x$s.

1. **Identify the values of the a, b, and c in the formula.**

The values are $a = 1, \ b = -4$, and $c = -1$.

2. **Fill in the quadratic formula with these values and simplify.**

$$\sin x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16+4}}{2} = \frac{4 \pm \sqrt{20}}{2}$$

$$= \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$$

3. **Find approximate values for sin _x_ from the solved form.**

Using a calculator, $2 \pm \sqrt{5} \approx 2 \pm 2.236$. So, sin x is either about 4.236 or –0.236.

4. **Use the calculator to find approximate angles with these sines.**

If $\sin x = 4.236$, you get an impossible result. The value of the sine ranges from –1 to 1, so sin x can't have this value.

If $\sin x = -0.236$, then $x = \sin^{-1}(-0.236) \approx -14°$. You need to determine which angles between 0 degrees and 360 degrees have this sine.

The sine is negative in Quadrant III and Quadrant IV. Using the reference angle of 14 degrees, you have $180° + 14° = 194°$ and $360° - 14° = 346°$. (Refer to Chapter 8 for more on reference angles.) The two solutions are $194°$ and $346°$.

Incorporating Identities

Some trig equations contain more than one trig function. Others have mixtures of multiple angles and single angles with the same variable. Some examples of such equations include $3\cos^2 x = \sin^2 x,\ 2\sec x = \tan x + \cot x,$ and $\cos 2x + \cos x + 1 = 0.$ To get these equations into a more–manageable form so that you can use factoring or one of the other methods in this chapter to solve them, you call upon identities to substitute some or all of the terms (for more on basic trig identities, see Chapter 10).

For example, to solve $3\cos^2 x = \sin^2 x$ for all the angles between 0 and 2π, apply the Pythagorean identity.

1. **Replace sin²_x_ with its equivalent from the Pythagorean identity,**
$\sin^2 x + \cos^2 x = 1 - \cos^2 x$ **or** $\sin^2 x = 1 - \cos^2 x.$

$3\cos^2 x = 1 - \cos^2 x$

2. **Add $\cos^2 x$ to each side and simplify by dividing.**

$$4\cos^2 x = 1$$

$$\cos^2 x = \frac{1}{4}$$

3. **Take the square root of each side.**

$$\sqrt{\cos^2 x} = \pm\sqrt{\frac{1}{4}}$$

$$\cos x = \pm\frac{1}{2}$$

4. **Solve for the values of x that satisfy the equation.**

If $\cos x = \frac{1}{2}$, then $x = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3},\ \frac{5\pi}{3}$.

If $\cos x = -\frac{1}{2}$, then $x = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3},\ \frac{4\pi}{3}$.

In this next example, you begin with three different trig functions. A good tactic is to replace each function by using either a ratio identity or a reciprocal identity. This usually results in having just sines and cosines in your new version of the equation. Using these identities creates fractions, and fractions require common denominators. By the way, having fractions in trig equations is *good*, because the products that result from multiplying and making equivalent fractions are usually parts of identities that you can then substitute in to make the expression much simpler.

Solve $2\sec x = \tan x + \cot x$ for all the possible solutions in degrees.

1. **Replace each term with its respective reciprocal or ratio identity.**

$$2\left(\frac{1}{\cos x}\right) = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$\frac{2}{\cos x} = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

2. **Rewrite the fractions with the common denominator $\sin x \cos x$.**

Multiply each term by a fraction that equals 1, with either sine or cosine in both the numerator and denominator.

$$\frac{2}{\cos x} \cdot \frac{\sin x}{\sin x} = \frac{\sin x}{\cos x} \cdot \frac{\sin x}{\sin x} + \frac{\cos x}{\sin x} \cdot \frac{\cos x}{\cos x}$$

$$\frac{2\sin x}{\sin x \cos x} = \frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x}$$

3. **Add the two fractions on the right. Then, using the Pythagorean identity, replace the new numerator with 1.**

$$\frac{2\sin x}{\sin x \cos x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$$

$$\frac{2\sin x}{\sin x \cos x} = \frac{1}{\sin x \cos x}$$

4. **Set the equation equal to 0 by subtracting the right term from each side.**

Perform the subtraction to create a single fraction.

$$\frac{2\sin x - 1}{\sin x \cos x} = 0$$

5. **Set the numerator equal to 0.**

$$2\sin x - 1 = 0 \text{ or } 2\sin x = 1, \ \sin x = \frac{1}{2}$$

If the numerator is equal to 0, then the whole fraction is equal to 0. The denominator can't equal 0 — such a number doesn't exist.

6. **Solve for the values of *x* that satisfy the original equation.**

$$x = \sin^{-1}\left(\frac{1}{2}\right) = 30°, \ 150°,\ldots$$

$$x = 30° + 360°n \text{ or } x = 150° + 360°n$$

In the next example, two different angles are in play. One angle is twice the size of the other, so you use a double-angle identity to reduce the terms to only one angle. The trick is to choose the correct version of the cosine double-angle identity.

Solve $\cos 2x + \cos x + 1 = 0$ for x between 0 and 2π.

1. **Replace cos 2*x* with $2\cos^2 x - 1$.**

$$2\cos^2 x - 1 + \cos x + 1 = 0$$

This version of the cosine double-angle identity is preferable because the other trig function in the equation already has a cosine in it.

2. **Simplify the equation. Then factor out cos *x*.**

$$2\cos^2 x + \cos x = 0$$

$$\cos x (2\cos x + 1) = 0$$

3. **Set each factor equal to 0.**

$$\cos x = 0 \ \text{ or } \ 2\cos x + 1 = 0, \ 2\cos x = -1, \ \cos x = -\frac{1}{2}$$

4. **Solve for the values of _x_ that satisfy the original equation.**

If $\cos x = 0$, then $x = \cos^{-1}(0) = \dfrac{\pi}{2}, \ \dfrac{3\pi}{2}$.

If $\cos x = -\dfrac{1}{2}$, then $x = \cos^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}, \ \dfrac{4\pi}{3}$.

This last example looks deceptively simple. The catch is that you have to recognize a double–angle identity upfront and make a quick switch. This example is also a nice segue into the next section on equations with multiple–angle solutions.

Solve $\sin x \cos x = \dfrac{1}{2}$ for all the solutions between 0 and 360 degrees.

1. **Use the sine double-angle identity to create a substitution for the expression on the left.**

Starting out with the double-angle identity and multiplying each side by $\dfrac{1}{2}$, you get:

$$\sin 2x = 2\sin x \cos x$$
$$\frac{1}{2}\sin 2x = \sin x \cos x$$

2. **Replace the expression on the left of the original equation with its equivalent from the double-angle identity.**

$$\sin x \cos x = \frac{1}{2}$$
$$\frac{1}{2}\sin 2x = \frac{1}{2}$$

3. **Multiply each side of the equation by 2.**

$$2 \cdot \frac{1}{2}\sin 2x = \frac{1}{2} \cdot 2$$
$$\sin 2x = 1$$

4. **Rewrite the expression as an inverse function.**

$$2x = \sin^{-1}(1)$$

See Chapters 14 and 15 for more on inverse functions.

5. **Determine which angles within two rotations satisfy the expression.**

$$2x = \sin^{-1}(1) = 90°, \ 450°$$

You use two rotations because the coefficient of _x_ is 2.

6. **Divide each term by 2.**

$$\frac{2x}{2} = \frac{90°}{2}, \ \frac{450°}{2}$$
$$x = 45°, \ 225°$$

Notice that the resulting angles are between 0 and 360 degrees.

You can generalize the double-angle technique from the preceding example for other multiple-angle expressions.

Finding Multiple-Angle Solutions

Multiple-angle expressions are those where the angle measure is some multiple of a variable — for example, $2x$ or $3y$. In this section, I show you how to take these expressions apart and solve for all the additional solutions that are possible. Because the trig functions are *periodic* (meaning they repeat their patterns infinitely), the number of possibilities for solutions increases tremendously. The larger the multiplier, the more the possible solutions.

TRIG RULES

When solving a trig equation of the form $ax = f^{-1}(x)$ where you want the solution to be all the angles within one complete rotation, write out all the solutions within the number of complete rotations that a represents. Then divide each angle measure by a.

Problems that lend themselves to this technique include $2\sin^2 5x = 1$ and $\cos\left(\frac{1}{2}x\right) = -\frac{\sqrt{3}}{2}$. In the first example, I solve $2\sin^2 5x = 1$ for all the angles between 0 and 2π.

1. **Divide each side by 2; then take the square root of each side.**

$$\sin^2 5x = \frac{1}{2}$$

$$\sqrt{\sin^2 5x} = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$

$$\sin 5x = \pm\frac{\sqrt{2}}{2}$$

2. **Solve for 5x, which represents the angles that satisfy the equation within one rotation.**

If $\sin 5x = \frac{\sqrt{2}}{2}$, then $5x = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4},\ \frac{3\pi}{4}$.

If $\sin 5x = -\frac{\sqrt{2}}{2}$, then $5x = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{5\pi}{4},\ \frac{7\pi}{4}$.

3. **Extend the solutions to five rotations by adding 2π to each of the original angles four times.**

$$5x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \frac{17\pi}{4}, \frac{19\pi}{4}, \frac{25\pi}{4}, \frac{27\pi}{4}, \frac{33\pi}{4}, \frac{35\pi}{4}$$

and

$$5x = \frac{5\pi}{20}, \frac{7\pi}{20}, \frac{13\pi}{20}, \frac{15\pi}{20}, \frac{21\pi}{20}, \frac{23\pi}{20}, \frac{29\pi}{20}, \frac{31\pi}{20}, \frac{37\pi}{20}, \frac{39\pi}{20}$$

4. **Divide all the terms by 5 and simplify.**

$$x = \frac{\pi}{20}, \frac{3\pi}{20}, \frac{9\pi}{20}, \frac{11\pi}{20}, \frac{17\pi}{20}, \frac{19\pi}{20}, \frac{25\pi}{20}, \frac{27\pi}{20}, \frac{33\pi}{20}, \frac{35\pi}{20}$$

$$x = \frac{\pi}{20}, \frac{3\pi}{20}, \frac{9\pi}{20}, \frac{11\pi}{20}, \frac{17\pi}{20}, \frac{19\pi}{20}, \frac{5\pi}{4}, \frac{27\pi}{20}, \frac{33\pi}{20}, \frac{7\pi}{4}$$

and

$$x = \frac{5\pi}{20}, \frac{7\pi}{20}, \frac{13\pi}{20}, \frac{15\pi}{20}, \frac{21\pi}{20}, \frac{23\pi}{20}, \frac{29\pi}{20}, \frac{31\pi}{20}, \frac{37\pi}{20}, \frac{39\pi}{20}$$

$$x = \frac{\pi}{4}, \frac{7\pi}{20}, \frac{13\pi}{20}, \frac{3\pi}{4}, \frac{21\pi}{20}, \frac{23\pi}{20}, \frac{29\pi}{20}, \frac{31\pi}{20}, \frac{37\pi}{20}, \frac{39\pi}{20}$$

Notice that all 16 solutions are angles with measures less than 2π.

This next example has a proper-fraction multiplier rather than a multiplier greater than 1.

Solve $\cos\left(\frac{1}{2}x\right) = -\frac{\sqrt{3}}{2}$ for all the solutions between 0 and 360 degrees.

1. **Rewrite the equation as an inverse trig equation.**

$$\frac{1}{2}x = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$$

2. **Determine which angles satisfy the inverse equation within one full rotation.**

$$\frac{1}{2}x = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = 150°, \ 210°$$

3. **Multiply all the terms by 2.**
$x = 300°, \ 420°$

4. **Throw out the second angle, because its measure is greater than 360 degrees.**

The only solution is 300 degrees. When you replace the x in the original equation with this angle measure, you get a true statement.

Squaring Both Sides

When solving trig equations, you have so many choices of techniques to use for the solution. Many times, more than one method will work — although one method is usually quicker or easier than another. And then you'll come across a trig equation that defies your finest attempts. Two last–ditch efforts that you can use when solving trig equations are to square both sides of the equation or to multiply each term through by a trig function that you've carefully selected. I show you the first of these two methods here, and I show you the second method in the next section. Examples of equations that respond well to squaring both sides include $\sin x + \cos x = \sqrt{2}$ and $\cos x - \sqrt{3} \sin x = 1$. Note the radicals in the equations.

Solve $\sin x + \cos x = \sqrt{2}$ for all the possible angles in degrees.

1. **Square both sides of the equation.**

When squaring a binomial, be sure not to forget the middle term.

$$(\sin x + \cos x)^2 = (\sqrt{2})^2$$
$$\sin^2 x + 2\sin x \cos x + \cos^2 x = 2$$

2. **Use the Pythagorean identity to replace $\sin^2 x + \cos^2 x$ with the number 1.**

$$\sin^2 x + \cos^2 x + 2\sin x \cos x = 2$$
$$1 + 2\sin x \cos x = 2$$

3. **Subtract 1 from each side. Then replace the expression on the left using the sine double-angle formula.**

$$2\sin x \cos x = 1$$
$$\sin 2x = 1$$

4. **Solve for the value of 2x by using the inverse function. Then write a few angle solutions to determine a pattern.**

$$2x = \sin^{-1}(1) = 90°, \ 450°, \ 810°, \ldots$$

Because you're supposed to find all the possible solutions, you're not bound by only two rotations.

5. **Divide every term by 2.**

$$2x = 90°, \ 450°, \ 810°, \ldots$$
$$x = 45°, \ 225°, \ 405°, \ldots$$

6. **Check for extraneous solutions.**

   Because squaring both sides of the equation loses information about the signs, you may have introduced extraneous solutions. Plug in the values of x that you found to check:

   $\sin(45°) + \cos(45°) = \sqrt{2}$, so 45° is a solution.

   $\sin(225°) + \cos(225°) = -\sqrt{2}$, so 225° is not a solution.

7. **Write an expression for all the solutions.**

   $x = 45° + 360°n$

In the next example, you need to do a little shifting at first. To solve $\cos x - \sqrt{3}\sin x = 1$, get the term with the radical in it to one side of the equation by itself. Otherwise, when you square both sides, you end up with a radical factor in one of the terms. That situation isn't always bad, but dealing with it is usually a little more awkward than not.

Solve the equation $\cos x - \sqrt{3}\sin x = 1$ for all the possible angles from 0 to 360 degrees.

1. **Add the radical term to both sides and subtract 1 from both sides.**

   You get $\cos x - 1 = \sqrt{3}\sin x$.

2. **Square both sides.**

   $$(\cos x - 1)^2 = (\sqrt{3}\sin x)^2$$
   $$\cos^2 x - 2\cos x + 1 = 3\sin^2 x$$

3. **Replace $\sin^2 x$ with $1 - \cos^2 x$ from the Pythagorean identity.**

   Doing so creates an equation with terms that have all the same functions, $\cos x$, in them.

   $$\cos^2 x - 2\cos x + 1 = 3(1 - \cos^2 x)$$

4. **Simplify the equation by distributing the 3 on the right and then bringing all the terms to the left to set the equation equal to 0.**

   $$\cos^2 x - 2\cos x + 1 = 3 - 3\cos^2 x$$
   $$4\cos^2 x - 2\cos x - 2 = 0$$

5. **Divide every term by 2.**

   $$2\cos^2 x - \cos x - 1 = 0$$

6. **Factor the quadratic equation.**

   $$(2\cos x + 1)(\cos x - 1) = 0$$

7. **Set each factor equal to 0.**

$$2\cos x + 1 = 0$$
$$2\cos x = -1$$
$$\cos x = -\frac{1}{2}$$

$$\cos x - 1 = 0$$
or $\cos x = 1$

8. **Solve each equation for the value of *x*.**

If $\cos x = -\dfrac{1}{2}$, then $x = \cos^{-1}\left(-\dfrac{1}{2}\right) = 120°, \ 240°$.

When you check for extraneous solutions, you find out that plugging in $120°$ makes the original equation false. Only $240°$ is a solution.

If $\cos x = 1$, then $x = \cos^{-1}(1) = 0°, \ 360°$.

The angles 0 and 360 degrees have the same terminal side. You usually list just one of them: 0 degrees.

Multiplying Through

The technique of multiplying through a trig equation by a carefully selected function shouldn't be your first choice — or your second, third, or fourth choice. This method is usually a last resort. Not that the method is terribly hard; it just requires sitting back and looking at the equation, and magically coming up with the best function to multiply through by. You can find the best function by guess or by golly, but then, that would take all the fun out of it — you want to guess right the first time. Here's an example of an equation that this technique works well on.

Solve $2\sin x - \csc x = 1$ for all the solutions between 0 and 2π.

1. **Multiply each term by sin *x*.**

Why sin *x*? I chose that function because I could see that the products of the individual terms would be either different powers of sine or just a number. Notice that the product of csc *x* and its reciprocal, sin *x*, is 1.

$$2\sin x \cdot \sin x - \csc x \cdot \sin x = 1 \cdot \sin x$$
$$2\sin^2 x - 1 = \sin x$$

2. **Subtract sin *x* from each side to set the equation equal to 0.**

$$2\sin^2 x - \sin x - 1 = 0$$

3. **Factor the quadratic equation.**

$$(2\sin x + 1)(\sin x - 1) = 0$$

4. **Set each factor equal to 0.**

$$2\sin x + 1 = 0 \qquad \sin x - 1 = 0$$
$$\sin x = -\frac{1}{2} \quad \text{or} \quad \sin x = 1$$

5. **Solve for the values that satisfy the equations.**

If $\sin x = -\frac{1}{2}$, then $x = \sin^{-1}\left(-\frac{1}{2}\right) = \frac{7\pi}{6}, \ \frac{11\pi}{6}$.

If $\sin x = 1$, then $x = \sin^{-1}(1) = \frac{\pi}{2}$.

Solving with a Graphing Calculator

Some of the more–advanced graphing calculators can make short work of solving trig equations. A graphing calculator comes in very handy when the equation is complicated, has several different functions or angle multiples, or has fractional or decimal values that don't lend themselves to the traditional solving methods that I discuss throughout this chapter. For example, I prefer to use a graphing calculator to solve equations like $\cos 2x = 2\cos x$ and $\cos^2 x - 0.4 \sin x = 0.6$.

First, here's how to solve $\cos 2x = 2\cos x$ for all solutions between -2π and 2π.

1. **Put the cos 2x in the y menu (the graphing menu) of your calculator. Put the $2\cos x$ on the right as a second entry.**

$$y_1 = \cos 2x$$
$$y_2 = 2\cos x$$

2. **Set the window of your calculator to show the graphs.**

Set the x values from -2π to 2π. (Be sure that your calculator is set in the radian mode.) In decimal form, let $x = -6.5$ to 6.5 to give a little room on either side of the left and right ends.

Set the y values to go from -3 to 3. Doing so gives room above and below the graph. If you have an auto-fit capability, use it to make the graph fit automatically after you choose the x values you want the graph to encompass.

3. **Graph the two functions and see where they intersect (see Figure 16-1).**

4. **Use the intersect feature on the calculator to determine the solutions.**
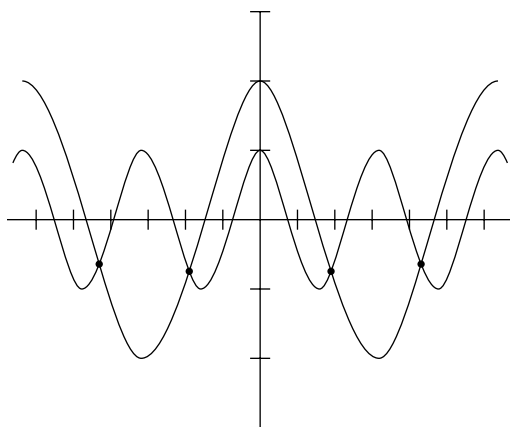
FIGURE 16-1:
The graphs of
$y = \cos 2x$
and $y = 2\cos x$.

The x-coordinates of the intersection points are the solutions (rounded to four decimal places): 1.9455 and 4.3377. These solutions are in radians — the π value is already multiplied through.

You can also find the solutions to the preceding example with a graphing calculator solver feature, but, usually, you still need to look at the graph anyway so you know how many solutions you're trying to find. The solver feature usually finds only one solution at a time, and you need to give it a hint to know where to find them.

This next example has decimals built in, so you probably can't factor it. You can solve it by using identities and writing it as a quadratic, and then using the quadratic formula. This calculator method gives you another option.

Solve $\cos^2 x - 0.4 \sin x = 0.6$ for all angles between $-\pi$ and π.

1. **Put $\cos^2 x - 0.4\sin x$ in the graphing y menu of your calculator. Put 0.6 as a second entry.**

   $y_1 = \cos^2 x - 0.4\sin x$
   $y_2 = 0.6$

2. **Set the window of your calculator to show the graphs.**

   Set the horizontal, x values from $-\pi$ to π. In decimal form, use $x = -3.2$ to 3.2 to give a little room on either side of the ends.

   Set the vertical, y values to go from -3 to 3. Doing so gives room above and below the graph. If you have an auto-fit capability, use it to make the graph fit automatically.

3. **Graph the two functions and see where they intersect (see Figure 16-2).**
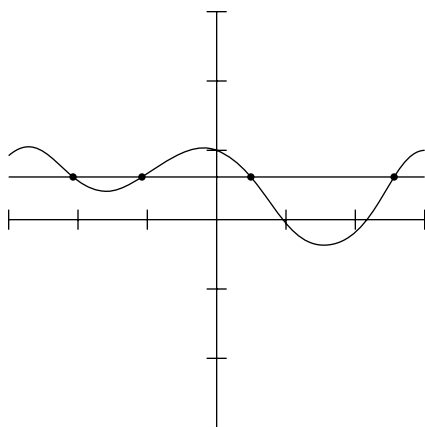
4. **Use the intersect feature on the calculator to determine the solutions.**

The *x*-coordinates of the intersection points are the solutions (rounded to four decimal places): $x = -2.0998,\ -1.0418,\ 0.4817,\ \text{and}\ 2.6598$. These solutions are in radians — the π value is already multiplied through.

Chapter **17**

Obeying the Laws and Applying Them

Triangles are very useful figures. Since humankind figured out how to keep records, people have documented the applications of triangles in mathematics and many other sciences and the arts. The right triangle gets the most use; Pythagoras saw to it that others recognized right triangles for the powerful polygons that they are. But *oblique triangles* (those that aren't acute triangles or right triangles) have their place, too. You can't always arrange to have a nice right triangle when you want it. Here's where oblique triangles and the laws of sines and cosines come into play.

The law of sines uses — believe it or not — the sines of a triangle's angles. With three carefully selected parts of the triangle (there are six to choose from), you can solve for the sizes of all the other parts. Of course, you have to obey the law, and the choices you can make are limited. That's where the law of cosines comes in to save the day. This law isn't as user-friendly, but it picks up where the law of sines falls short.

Trigonometry opens up all sorts of possibilities for solving area problems. By using the tools in this chapter, you won't find a triangle that you can't lick.

Describing the Parts of Triangles

When it comes to triangles, you'll find right triangles and bigger-than-right triangles, called *obtuse triangles.* Other classifications exist too, such as acute, equilateral, isosceles, and so on. But no matter what you call it, any triangle has exactly six parts: three angles and three sides. After you have information about the parts of a triangle, you can perform all sorts of computations and manipulations, using triangles to model situations and solve problems.

Standardizing the parts

Usually, when you name the parts of a triangle, you follow a system or pattern. Having this system helps you sort out the information, even when you don't have a picture of the triangle to help you. The most common system is to name the angles of the triangle with capital letters, usually A, B, and C, and name the sides opposite each of the angles with the lowercase letter that matches. Figure 17-1a shows an example.

TECHNICAL STUFF

Another common practice is to name the angles with Greek letters, such as α, β, and γ, and put those labels inside the triangle between the two sides forming the angle. Refer to Figure 17-1b. But in this chapter, I stick with the capital and lowercase letters, because it coordinates the angles and the sides.

FIGURE 17-1: Naming the parts of a triangle.

(a)   (b)
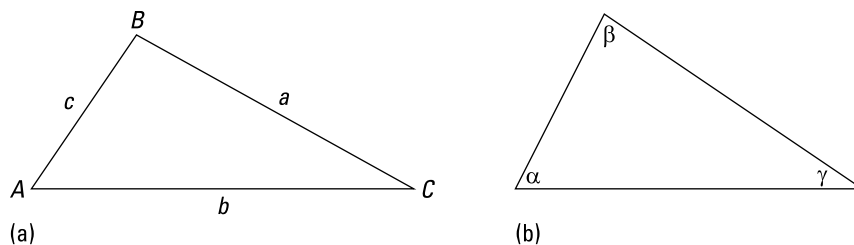
Determining a triangle

Even though every triangle has six parts, you only need to have information about or know the measures of three particular parts to determine the others. For example, if you know the measures of the three sides, then you know that the three angles are uniquely determined. You can't construct more than one shape and size of triangle from those three sides.

After you know the values for three carefully chosen pieces of a triangle, you can use any of the three different rules or laws that allow you to find the other three

parts of the triangle. I discuss all three laws in their own sections later in this chapter.

Finding the one and only

Several combinations of parts uniquely determine a triangle. You'll probably recognize these rules from geometry, when you did proofs. Here's a list of all of the combinations you can use here.

To *uniquely determine* a triangle (find only one possible shape and size), you need

REMEMBER

>> **SSS:** The measures of the three sides

>> **SAS:** The measures of two sides and the angle between them

>> **ASA:** The measures of two angles and the side between them

>> **AAS:** The measures of two angles and a side that does not lie between the angles

The last rule is actually just another version of the one directly before it. When you have two angles, you can determine the third, so the side lies between two known angles. Figure 17-2 shows these situations.
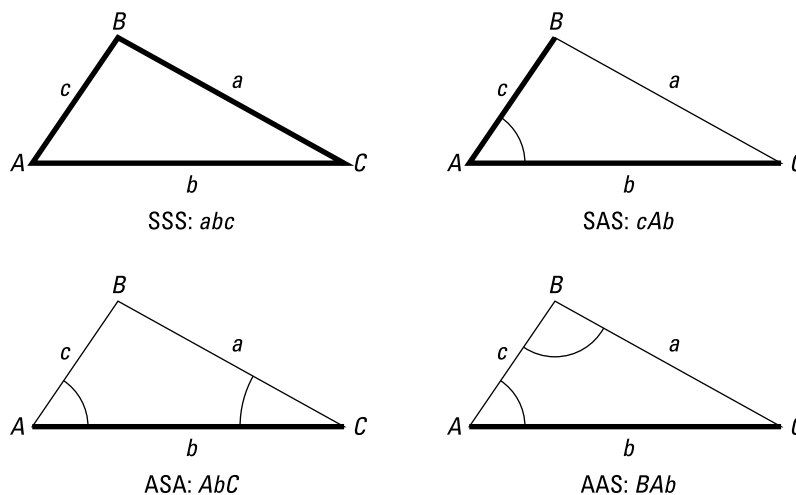
FIGURE 17-2:
Ways of determining the one and only triangle.

SSS: *abc*

SAS: *cAb*

ASA: *AbC*

AAS: *BAb*

You may have noticed that I didn't mention one combination — AAA, where all three angles are known. I left it out on purpose because, in such a case, all you can be sure of is that the two triangles are *similar* — they're the same shape but not necessarily the same size.

Dealing with the ambiguous case

Four situations allow you to uniquely determine a triangle, and I list them in the preceding section. One additional case can be helpful, even though you may end up with two different triangles instead of one unique triangle: SSA, the measures of two sides and an angle that *isn't* between them. This situation is a little tricky, because often, two different triangles are possible — which is why it's known as the *ambiguous case.* Sometimes, this case is still better than nothing, as long as you're aware that more than one triangle can exist. Figure 17-3 illustrates such a situation. In the two triangles, sides *a* and *c* and angle *A* are the same measure in each triangle. The angle measure *B* and the length of side *b*, however, aren't the same in both triangles.
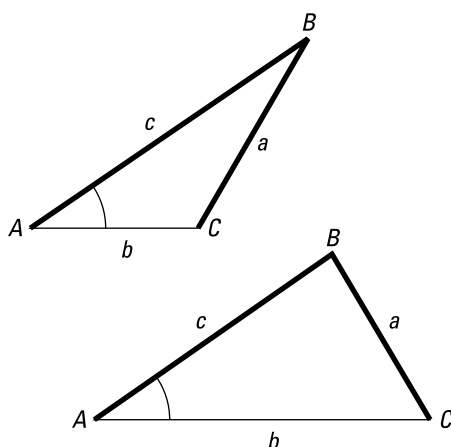
FIGURE 17-3:
The ambiguous
case.

SSA: *acA*

Following the Law of Sines

When you already have two angles, as in the case of ASA or AAS (see the preceding section), you can use the *law of sines* to find the measures of the other parts of the triangle*.* This law uses the ratios of the sides of a triangle and the sines of their opposite angles. The bigger a side, the bigger its opposite angle (and its sine). The longest side is always opposite the largest angle. Here's how it goes.

TRIG RULES

The law of sines for triangle *ABC* with sides *a*, *b*, and *c* opposite those angles, respectively, says

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{and} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

So, the law of sines says that in a single triangle, the ratio of each side to the sine of its angle is equal to the ratio of any other side to the sine of its angle. When working with the law of sines, you use two of the ratios at a time, setting them equal to one another to form a proportion.

TIP

For example, consider a triangle where side *a* is 86 inches long and angles *A* and *B* are 84 and 58 degrees, respectively. Figure 17-4 shows a picture of the triangle, and the following steps show you how to find the missing three parts.

FIGURE 17-4:
Finding the three missing parts of a triangle.
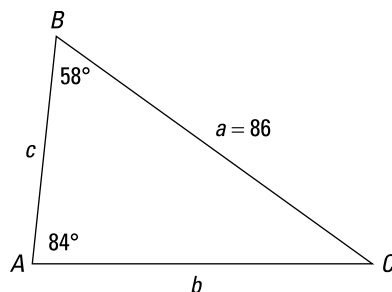
1. **Find the measure of angle *C*.**

   The sum of the measures of a triangle's angles is 180 degrees. So, find the sum of angles *A* and *B*, and subtract that sum from 180.

   $$180 - (84 + 58) = 180 - 142 = 38$$

   Angle *C* measures 38 degrees.

2. **Find the measure of side *b*.**

   Using the law of sines and the proportion $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$, fill in the values that you know.

   $$\frac{86}{\sin 84°} = \frac{b}{\sin 58°}$$

   Use the given values when writing a proportion, not those that you've determined yourself. That way, if you make an error, you can spot it more easily later.

   Use a calculator or computer to determine the values of the sines. Rounding to four places is sufficient.

   $$\frac{86}{0.9945} = \frac{b}{0.8480}$$

Multiply each side by 0.8480 to solve for the length *b*. Because the original measures are whole numbers, round this answer to the nearer whole number.

$$(0.8480)\frac{86}{0.9945} = \frac{b}{0.8480}(0.8480)$$
$$73.3331 = b$$

Side *b* measures about 73 inches.

3. **Find the measure of side *c*.**

Using the law of sines and the proportion $\frac{a}{\sin A} = \frac{c}{\sin C}$, fill in the values that you know.

$$\frac{86}{\sin 84°} = \frac{c}{\sin 38°}$$

Again, it's best to use the given values, not those that you determined. In this case, however, you have to use a computed value, the angle *C*.

Use a calculator to determine the values of the sines.

$$\frac{86}{0.9945} = \frac{c}{0.6157}$$

Multiply each side by 0.6157 to solve for the length *c*. Because the original measures were given as whole numbers, round this answer to the nearer whole number.

$$(0.6157)\frac{86}{0.9945} = \frac{c}{0.6157}(0.6157)$$
$$53.2430 = c$$

Side *c* measures about 53 inches.

Although knowing how to find the missing measures in an oblique triangle seems wonderful, you may wonder, "What's the point?" One major reason for solving triangles is so you can apply them to practical problems. For example, the question, "How tall is it?" seems to be a reasonable request.

Suppose a tree is growing on a hillside. The tree is completely vertical, but the hillside inclines at a 10-degree angle from the horizontal. Josh is standing 100 feet downhill from the tree. The angle of inclination from Josh's feet to the top of the tree is 32 degrees. How tall is the tree? First, take a look at a visual of the situation in Figure 17-5 and then review the steps that follow.

1. **Determine the triangle that you can use to solve the problem.**

You know that one side is 100 feet long, and you can determine two angles, so use the triangle that Figure 17-6 (a) shows.

FIGURE 17-5:
How tall is
the tree?

FIGURE 17-5:
How tall is
the tree?

FIGURE 17-6:
The triangle
needed to
calculate the
height of the tree,
and extending
the triangle to do
calculations.

(a)   (b)
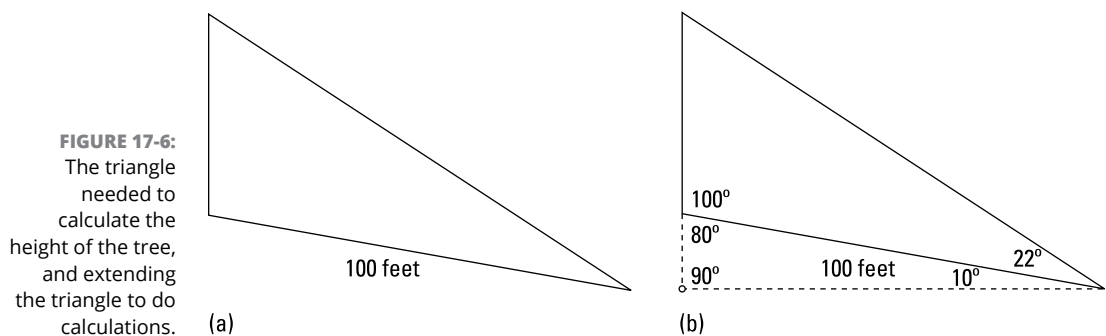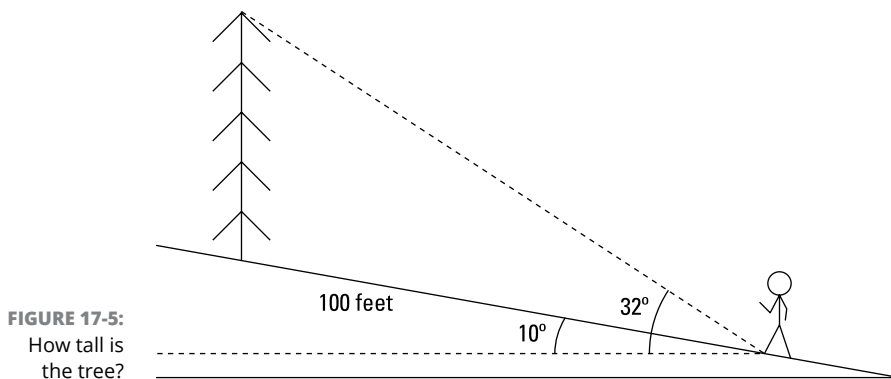
2. **Determine the two angles on either side of the base of the triangle.**

You determine the angle on the right of the 100-foot base by subtracting the hill's 10-degree inclination from the tree's 32-degree inclination: $32 - 10 = 22$ degrees. (These angles are also known as *angles of elevation,* which you can find out more about in Chapter 9.)

You extend the side representing the tree downward to form a right triangle with the hill as the hypotenuse. You then determine an angle of 80 degrees for the third angle in the right triangle. Subtracting the measures of the other two angles from 180, you get the 80-degree angle. This 80-degree angle is supplementary to the angle above it — see Figure 17-7 (b). *Supplementary angles* add up to 180, so the angle supplementary to the 80-degree angle is 100 degrees. Another way to find this 100-degree angle is to use the exterior-angle rule that follows.

The measure of an exterior angle of a triangle is equal to the sum of the two nonadjacent interior angles.

**TRIG
RULES**

3. **Calculate the measure of the third angle in the upper triangle.**

Adding the two base angles together and subtracting their sum from 180 degrees, you get $180 - (22 + 100) = 180 - 122 = 58$ degrees.

4. **Determine the height of the tree.**

The tree is the side opposite the angle measuring 22 degrees. Using the law of sines, you can write the following proportion:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{\text{tree height}}{\sin 22°} = \frac{100}{\sin 58°}$$

Solve for the height of the tree.

$$\frac{\text{tree height}}{\sin 22°} = \frac{100}{\sin 58°}$$
$$\frac{\text{tree height}}{0.3746} = \frac{100}{0.8480}$$
$$(\,0.3746\,)\frac{\text{tree height}}{0.3746} = \frac{100}{0.8480}(\,0.3746\,)$$
$$\text{tree height} = 44.1745$$

The tree is about 44 feet tall.

Continuing with the Law of Cosines

The *law of cosines* comes in handy when you have two or more sides — as in situations involving SSS and SAS — and need the measures of the other three parts. When you have two sides, you need the angle between them. If the angle isn't between the two sides, then you have the ambiguous case, SSA. Although such a situation isn't impossible, you must deal with it carefully. (See the section, "Determining a triangle," earlier in this chapter, for more on these cryptic notations.)

Defining the law of cosines

The law of cosines has three different versions that you can use depending on which parts of the triangle you have measures for. Notice the pattern: The squares of the three sides appear in the equations, along with the cosine of the angle opposite one of the sides — the side set equal to the rest of the stuff.

TRIG RULES

The law of cosines for triangle *ABC* with sides *a*, *b*, and *c* opposite those angles, respectively, says

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

In plain English, these equations say that the square of one side is equal to the squares of the other two sides, added together, minus twice the product of those two sides times the cosine of the angle opposite the side you're solving for. Whew!

Law of cosines for SAS

When you have two sides of a triangle and the angle between them, you can use the law of cosines to solve for the other three parts. Consider the triangle *ABC* where *a* is 15, *c* is 20, and angle *B* is 124 degrees. Figure 17-7 shows what this triangle looks like.
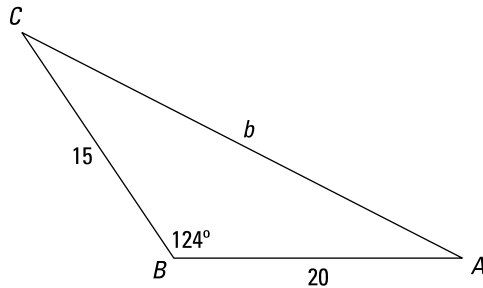
Now, to solve for the measure of the missing side and angles:

1. **Find the measure of the missing side by using the law of cosines.**

Use the law that solves for side *b*.

$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$= 15^2 + 20^2 - 2(15)(20) \cos 124°$$
$$= 225 + 400 - 600(-0.5592)$$
$$= 960.52$$

You end up with the value for b^2. Take the square root of each side and just use the positive value (because a negative length won't work here).

$$b^2 = 960.52$$
$$b = 30.992$$

The length of side *b* is about 31.

2. **Find the measure of one of the missing angles by using the law of cosines.**

Using the law that solves for a, fill in the values that you know.

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$15^2 = 31^2 + 20^2 - 2(31)(20)\cos A$$

Solve for cos A by simplifying, moving the other two terms to the left, and dividing by the coefficient.

$$225 = 961 + 400 - 1{,}240\cos A$$
$$-1{,}136 = -1{,}240\cos A$$
$$\frac{-1{,}136}{-1{,}240} = \cos A$$
$$0.9161 = \cos A$$

Using your calculator or computer to find angle A, you determine that $A = \cos^{-1}(0.9161) = 23.6375$, or about 24 degrees.

You can also switch to the law of sines to solve for this angle. Don't be afraid to mix and match when solving these triangles.

3. **Find the measure of the last angle.**

Determine angle B by adding the other two angle measures together and subtracting that sum from 180.

$$180 - (124 + 24) = 180 - 148 = 32$$

Angle B measures 32 degrees.

How about an application that uses this SAS portion of the law of cosines? Consider this situation: A friend wants to build a stadium in the shape of a regular pentagon (five sides, all the same length) that measures 920 feet on each side. How far is the center of the stadium from the corners? The left part of Figure 17-8 shows a picture of the stadium and the segment you're solving for.

You can divide the pentagon into five isosceles triangles. The base of each triangle is 920 feet, and the two sides are equal, so call them both a. (Refer to the right-hand picture in Figure 17-8.) Use the law of cosines to solve for a, because you can get the angle between those two congruent sides, plus you already know the length of the side opposite that angle.

1. **Determine the measure of the angle at the center of the pentagon.**

A circle has a total of 360 degrees. Divide that number by 5, and you find that the angle of each triangle at the center of the pentagon is 72 degrees.

FIGURE 17-8:
Pentagonal
stadium and an
inner triangle.
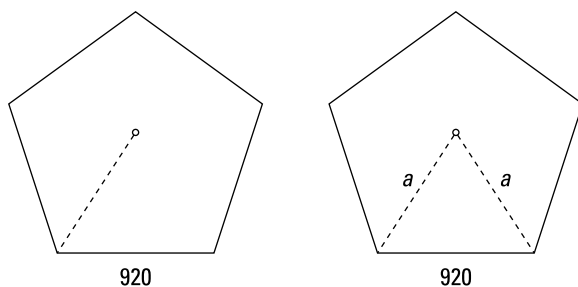
2. **Use the law of cosines, with the side measuring 920 feet being the side solved for.**

$$c^2 = a^2 + a^2 - 2aa\cos C$$
$$920^2 = 2a^2 - 2a^2\cos 72°$$

Because the other two sides are the same measure, write them both as *a* in the equation.

3. **Solve for the value of *a*.**

$$920^2 = 2a^2 - 2a^2\cos 72°$$
$$846,400 = 2a^2(1 - \cos 72°)$$
$$\frac{846,400}{1 - \cos 72°} = 2a^2$$
$$\frac{846,400}{1 - 0.3090} =$$
$$\frac{846,400}{0.6910} =$$
$$1,224,891.462 =$$
$$612,445.731 = a^2$$
$$782.5891 = a$$

The distance from the center to a corner is between 782 and 783 feet. Now your friend knows how much fencing it'll take to divide the stadium into five equal triangles.

Law of cosines for SSS

When you know the values for two or more sides of a triangle, you can use the law of cosines. In the following case, you know all three sides but none of the angles. Solve for the measures of the three angles in triangle *ABC*, which has sides where *a* is 7, *b* is 8, and *c* is 2.

As you can see in Figure 17-9, the triangle appears to have two acute angles and one obtuse angle, the obtuse angle being opposite the longest side.
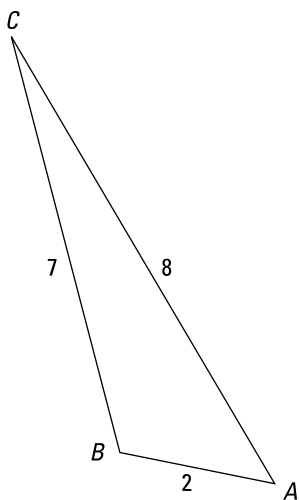
1. **Solve for the measure of angle *A*.**

   Using the law of cosines where side *a* is on the left of the equation, substitute the values that you know and simplify the equation.

   $$a^2 = b^2 + c^2 - 2bc\cos A$$
   $$7^2 = 8^2 + 2^2 - 2(8)(2)\cos A$$
   $$49 = 64 + 4 - 32\cos A$$
   $$-19 = -32\cos A$$
   $$\frac{-19}{-32} = \cos A$$
   $$0.5938 = \cos A$$

   Now use your calculator to find the measure of *A*.

   $$A = \cos^{-1}(0.5938) = 53.5729$$

   Angle *A* measures about 54 degrees.

2. **Solve for the measure of angle *B*.**

   Using the law of cosines where side *b* is on the left of the equation, input the values that you know and simplify the equation.

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$8^2 = 7^2 + 2^2 - 2(7)(2)\cos B$$
$$64 = 49 + 4 - 28\cos B$$
$$11 = -28\cos B$$
$$\frac{11}{-28} = \cos B$$
$$-0.3929 = \cos B$$

The negative cosine means that the angle is obtuse — its terminal side is in the second quadrant. When you use a calculator to find the measure of *B,* be sure you enter the negative sign.

$$B = \cos^{-1}(-0.3929) = 113.1351$$

Angle *B* measures about 113 degrees.

3. **Determine the measure of angle *C*.**

Because angle *A* measures 54 degrees and angle *B* measures 113 degrees, add them together and subtract the sum from 180 to get the measure of angle *C*.

$$180 - (54 + 113) = 180 - 167 = 13$$

Angle *C* measures only 13 degrees.

Being ambiguous

Many people are visual learners, and they solve problems better when using a picture. This characteristic will serve them well when it comes to solving triangles that are SSA, meaning that they know the measures of two sides and an angle that isn't between those sides. Drawing a picture helps explain why the situation may have more than one answer. When you use this setup in an actual application, the correct answer is usually pretty clear. First, I show you how to do one of these problems in general; then I show how it may actually play out in real life.

Find the missing parts of the triangle *ABC* that has sides *a* and *b* measuring 85 units and 93 units, respectively, and angle *A* measuring 61 degrees. Figure 17-10 presents the situation.

1. **Find the length of side *c* by using the law of cosines with *a* on the left-hand side of the equation.**

Use this form because after you input the known values, it's the only one that will have just one variable to solve for — even though that variable has two powers.

FIGURE 17-10:
The ambiguous
case — two
possible triangles.
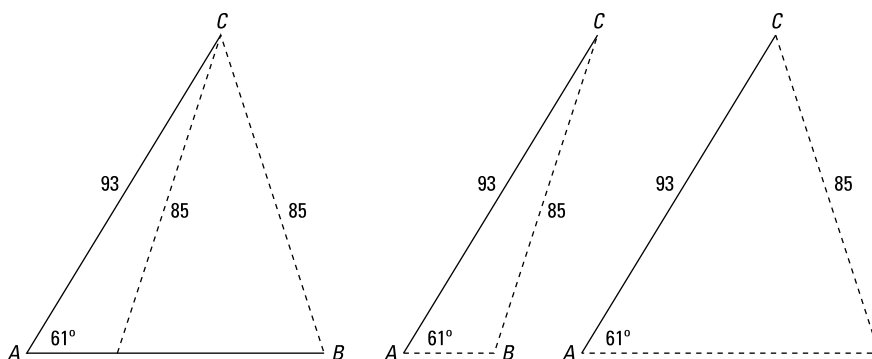
Enter the values into the law of cosines.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$85^2 = 93^2 + c^2 - 2(93)(c)\cos 61°$$

Simplify the equation by performing all the operations and getting the variables alone on the right side.

$$7{,}225 = 8{,}649 + c^2 - 186(c)(0.4848)$$
$$-1{,}424 = c^2 - 90.1728c$$

You end up with a quadratic equation.

Use the quadratic formula and a calculator to determine the solutions.

$$0 = c^2 - 90.1728c + 1{,}424$$
$$c = 69.7599 \text{ or } 20.4129$$

So c measures either 70 or 20.

2. **Let c measure 70, and find the measures of the other two angles.**

This time, take a departure from the law of cosines and use, instead, the law of sines.

Use angle A and side a, and pair the ratio with angle C and side c to get the following:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$
$$\frac{\sin 61°}{85} = \frac{\sin C}{70}$$

Now multiply each side by 70, and solve for the sine of C.

$$70 \cdot \frac{\sin 61°}{85} = \frac{\sin C}{\cancel{70}} \cdot \cancel{70}$$

$$70 \cdot \frac{0.8746}{85} = \sin C$$

$$0.7203 = \sin C$$

Solve for the angle with that sine.

$$C = \sin^{-1}(0.7203) = 46.0793$$

The measure of angle C is about 46 degrees.

If angle A is 61 degrees and angle C is 46 degrees, then angle B is 180 degrees minus the sum of A and C: $180 - (61 + 46) = 180 - 107 = 73$ degrees.

3. **Now let c measure 20, and find the measures of the other two angles.**

Go back to the law of cosines to do this part. You can compare the two methods — the one in this step and the one in Step 2 — to see which one you like better.

Use the law with c on the left-hand side of the equation to solve for the cosine of angle C:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$20^2 = 85^2 + 93^2 - 2(85)(93)\cos C$$

$$400 = 7,225 + 8,649 - 15,810\cos C$$

$$-15,474 = -15,810\cos C$$

$$\frac{-15,474}{-15,810} = \cos C$$

$$0.9787 = \cos C$$

Use your calculator to find the measure of angle C.

$$C = \cos^{-1}(0.9787) = 11.8468$$

Angle C measures about 12 degrees, which means that angle B is $180 - (61 + 12) = 180 - 73 = 107$ degrees.

The ambiguous case might cause a bit of confusion. Why would you want two answers? The following example may help clear up this mystery. You really *don't* want two answers. You just want the one that answers your question.

TWO WRONGS MAKE A RIGHT

When students are first introduced to fractions, they're often tempted to take some liberties with the rules that can get them into trouble. Imagine the frustration to the teacher and student alike when the student stumbles on one of the four fractions, with two digits in the numerator and denominator, where incorrect cancellation results in a correct answer. The four fractions where such a situation can occur are $\dfrac{64}{16}, \dfrac{98}{49}, \dfrac{95}{19},$ and $\dfrac{65}{26}$.

When a student mistakenly crosses out the two like digits, the result is actually the correct answer: $\dfrac{\cancel{6}4}{1\cancel{6}} = \dfrac{4}{1} = 4, \dfrac{\cancel{9}8}{4\cancel{9}} = \dfrac{8}{4} = 2, \dfrac{\cancel{9}5}{1\cancel{9}} = \dfrac{5}{1} = 5,$ and $\dfrac{\cancel{6}5}{2\cancel{6}} = \dfrac{5}{2}$. Thank goodness only four such fractions exist.

Slim and Jim are both sitting at the intersection of two roads, which forms a 50-degree angle. They leave the intersection at the same time — Slim in his old, slow, beat-up pickup truck, and Jim in his nifty-swifty Jeep. When Jim is 400 yards down the road, the two of them are 320 yards apart. How far has Slim driven at that point?

You definitely need a picture for this problem (see Figure 17-11).

FIGURE 17-11: Slim (*S*) and Jim (*J*) travel on two roads that make a 50-degree angle at their intersection (*I*).
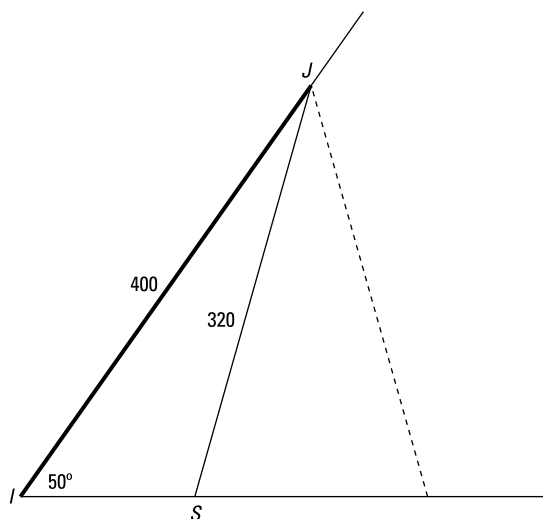
You can safely assume that Slim couldn't have gone farther than Jim in his old clunker — unless his truck had hidden powers. Figure out how far Slim drove, the distance from *I* to *S* (in this example, the distance is labelled *j* to be consistent with the triangle labels), by using the law of cosines. The side *s* is 400 yards, and angle *I* is 50 degrees.

1. **Write the law of cosines, and replace the letters with the values.**

$$i^2 = s^2 + j^2 - 2(s)(j)\cos I$$
$$320^2 = 400^2 + j^2 - 2(400)(j)\cos 50°$$
$$102,400 = 160,000 + j^2 - 800j(0.6428)$$
$$0 = 57,600 + j^2 - 514.24j$$

This equation simplifies to a quadratic equation with the variable *j*.

2. **Solve the quadratic equation.**

$$0 = j^2 - 514.24j + 57,600$$

Use a calculator and the quadratic formula, and you get two solutions: $x = 349.3734$ and $x = 164.8666$. Either answer gives you a distance smaller than the distance that Jim traveled. Refer to Figure 17-12, and choose the answer that appears to be correct.

Finding the Areas of Triangles

Finding the area of a triangle sounds relatively easy. Most grade-school children get plenty of chances to do just that. They're given a triangle and the length of the base and the height, or altitude, drawn to that base. Simple! Just plug those values into the formula, and you have it. But think about it: How many times do you have a triangular plot of land or a triangular sail for a boat and also the measure of the *altitude?*

What you find in this section is a formula for every occasion. Give me a triangle, and I can find the area. Although the base and altitude would be nice, I can also do the problem with the measures of the three sides. You have two sides and an included angle? Sure, I can do that. How about two angles and an included side? I have a formula for that, too.

Of course, if you don't have the measurements for one of these exact situations, you can go to the law of sines, cosines, or tangents to fill in the blanks and find the sides or angles that you need.

Finding area with base and height

The most basic formula for finding the area of a triangle occurs when you know the base and the height. The height is drawn perpendicular to the base up to the vertex opposite that base. Figure 17-12 shows you what I mean.
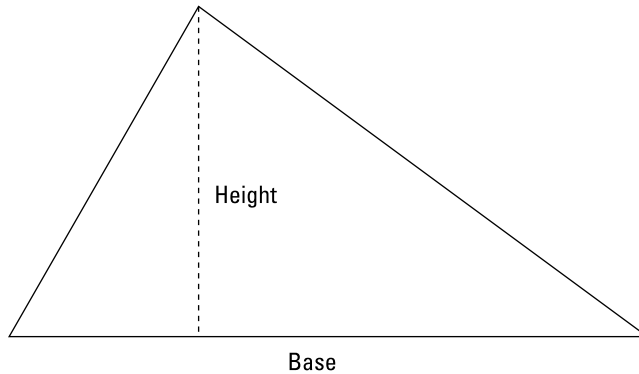
FIGURE 17-12:
A triangle with a
base and height
drawn.

The equation for the area, A, of a triangle with base b and height h is $A = \frac{1}{2}bh$.

REMEMBER

For example, to find the area of a triangle with a base measuring 12 inches and a height measuring 5 inches, input the values into the equation. You find that $A = \frac{1}{2}(12)(5) = 30$, or 30 square inches.

If the triangle happens to be a right triangle, then you're really in business. The base and height are the legs, or the two sides that are perpendicular to one another. Just find half of the base times the height. Here's an example.

Kirsten has a corner lot and wants to make a triangular garden where the two sidewalks meet. She has a 20-foot piece of border to go along the diagonal, or hypotenuse, of the triangle. She wants one side along the sidewalk to be 12 feet. How many square feet of garden will she have? Figure 17-13 illustrates the situation.

1. **Find the length of the other leg of the right triangle.**

Using the Pythagorean Theorem, and calling the missing length *x*, you get

$$x^2 + 12^2 = 20^2$$
$$x^2 + 144 = 400$$
$$x^2 = 400 - 144 = 256$$
$$x = \sqrt{256} = 16$$

The other side is 16 feet long.

20 feet
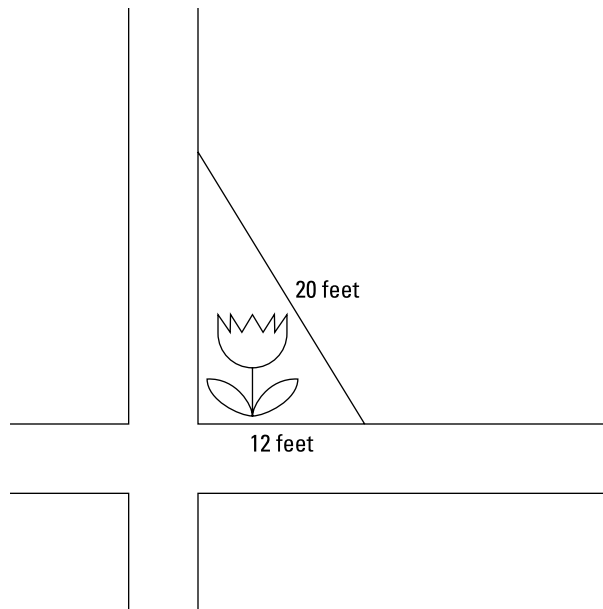
12 feet

2. **Find the area of the triangle.**

The base is 12 feet, and the height is 16 feet. Using the formula, you get

$$A = \frac{1}{2}bh = \frac{1}{2}(12)(16) = 96$$

The area is 96 square feet, which is a lot of garden to weed!

Finding area with three sides

Suppose that you have 240 yards of fencing, and you decide to build a triangular corral for your llama. Why triangular? You heard that llamas favor the shape, of course. You want the llama to have enough room to run around, so you need to know the area. What should the lengths of the triangle's sides be? You can solve this little problem by using Heron's formula for the area of a triangle.

REMEMBER

Heron's formula says that if a triangle *ABC* has sides of lengths *a*, *b*, and *c* opposite the respective angles, and the semi-perimeter, *s*, is half of the triangle's perimeter, then the area of the triangle is $A = \sqrt{s(s-a)(s-b)(s-c)}$.

In the problem of the fencing and the llama, you have many ways to make a triangular corral from 240 yards of fencing. Figure 17-14 shows a few of the possibilities. Notice that in each case, the lengths of the sides add up to 240. For the sake of this problem, don't worry about a gate.
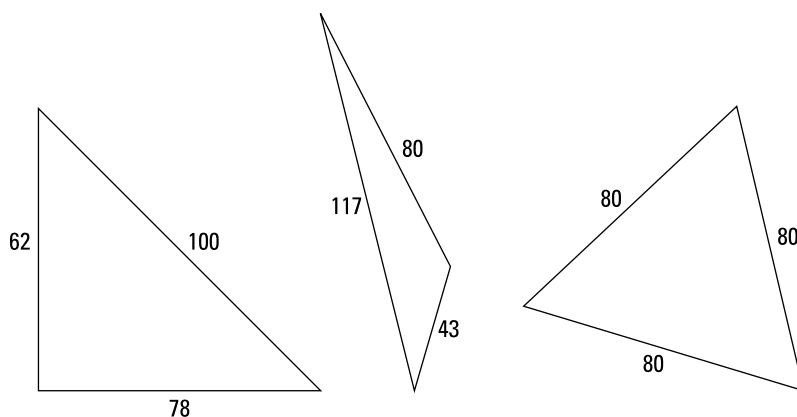
Which triangle has the greatest area? Obviously, one of them is a bit on the scrawny side, even though it uses up 240 yards of fencing, like the others. Here's how to compute the areas for the three triangles.

1. **Find the semi-perimeter, *s*, for each triangle.**

   Referring to Figure 17-14,

   - Left triangle: $\frac{1}{2}(62+100+78)=\frac{1}{2}(240)=120$

   - Center triangle: $\frac{1}{2}(117+80+43)=\frac{1}{2}(240)=120$

   - Right triangle: $\frac{1}{2}(80+80+80)=\frac{1}{2}(240)=120$

   Not surprisingly, all the semi-perimeters are the same, because all the perimeters are 240.

2. **Use Heron's formula to find each area.**

   Again, referring to Figure 17-14,

   - Left triangle: $A=\sqrt{120(120-62)(120-100)(120-78)}$
     $=\sqrt{120(58)(20)(42)}=2,417.933$

   - Center triangle: $A=\sqrt{120(120-117)(120-80)(120-43)}$
     $=\sqrt{120(3)(40)(77)}=1,052.996$

   - Right triangle: $A=\sqrt{120(120-80)(120-80)(120-80)}$
     $=\sqrt{120(40)(40)(40)}=2,771.281$

The triangle on the right has the greatest area. Of the shapes in Figure 17–14, that triangle is the best. But you may be wondering whether another shape gives more area than that one. The answer: no. With calculus, you can prove that an equilateral triangle gives you the greatest possible area with any amount of fencing. Without calculus, you just have to try a bunch of shapes to convince yourself (or trust me).

Finding area with SAS

When you know the lengths of two of a triangle's sides plus the measure of the angle between those sides, you can find the area of the triangle. This method requires a little trigonometry — you have to find the sine of the angle involved. But the formula is really straightforward.

TRIG RULES

If triangle *ABC* has sides measuring *a*, *b*, and *c* opposite the respective angles, you can find the area with one of these formulas:

$$A = \frac{1}{2}ab\sin C \qquad A = \frac{1}{2}bc\sin A \qquad A = \frac{1}{2}ac\sin B$$

For example, look at the 30-60-90 right triangle in Figure 17-15. I use this particular example because the numbers come out so nicely.
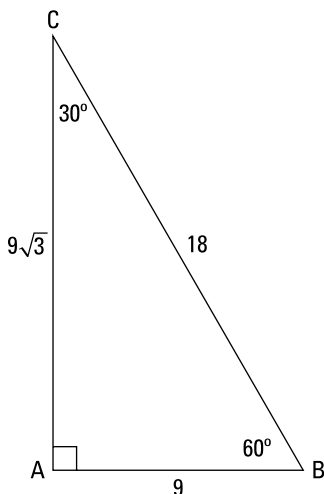
First, find the area by using angle *B* and the two sides forming it.

1. **Choose the correct version of the formula.**

   The formula that uses angle *B* is $A = \frac{1}{2}ac\sin B$.

2. **Find the sine of the angle.**

   $$\sin 60° = \frac{\sqrt{3}}{2}$$

3. **Substitute the values into the formula and simplify.**

   $$A = \frac{1}{2}\left(\cancel{18}^{\,9}\right)(9)\frac{\sqrt{3}}{2} = \frac{81\sqrt{3}}{2}$$

Now find the area by using angle C and the two sides forming it.

1. Choose the correct version of the formula.

The formula that uses angle C is $A = \frac{1}{2}ab \sin C$.

2. Find the sine of the angle.

$$\sin 30° = \frac{1}{2}$$

3. Substitute the values into the formula and simplify.

$$A = \frac{1}{2}\left(\cancel{18}^{\,9}\right)\left(9\sqrt{3}\,\right)\frac{1}{2} = \frac{81\sqrt{3}}{2}$$

Using the method involving angle A gives you the same result, of course. For a quick comparison, just use the formula to find the area, because you're dealing with a right triangle: $A = \frac{1}{2}bh$. These methods all produce the same result.

Finding area with ASA

As you probably suspected, when you have two angles and the side between them, you can find the area of a triangle. The formulas go as follows.

TRIG RULES

In triangle ABC, if the measures of the sides are a, b, and c opposite the respective angles, you can determine the area by using one of the following equations:

$$\text{Area} = \frac{a^2 \sin B \sin C}{2 \sin A} \qquad \text{Area} = \frac{b^2 \sin A \sin C}{2 \sin B} \qquad \text{Area} = \frac{c^2 \sin A \sin B}{2 \sin C}$$

These formulas are actually built from the formula for finding the area with SAS, with a little help from the law of sines. Here's how one of the formulas came to be.

1. Start with the SAS rule for area.

$$A = \frac{1}{2}ab \sin C$$

2. Write the law of sines involving angles *A* and *B*.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

3. Solve for *b* in the proportion.

$$\sin B \cdot \frac{a}{\sin A} = \frac{b}{\cancel{\sin B}} \cdot \cancel{\sin B}$$

$$\frac{\sin B \cdot a}{\sin A} = b$$

4. **Substitute the equivalent for *b* into the area formula in Step 1.**

$$A = \frac{1}{2}a\left(\frac{\sin B \cdot a}{\sin A}\right)\sin C$$

$$= \frac{a^2 \sin B \sin C}{2\sin A}$$

That's the first formula. Feel free to create the others yourself.

Now consider an example. Say you have a triangle with angle *A*, which is 45 degrees, and angle *B*, which is 55 degrees, and the side between them, *c*, equal to 10. Find the area.

1. **Choose the correct formula — the one with *c²* in it.**

$$A = \frac{c^2 \sin A \sin B}{2\sin C}$$

2. **Find the sines of the two given angles.**

The sine of 45 degrees equals 0.7071, and the sine of 55 degrees equals 0.8192.

3. **Find the sine of the third angle.**

Angle *C* measures $180 - (45 + 55)$, or $180 - 100$, which equals 80 degrees.

The sine of 80 degrees equals 0.9848.

4. **Substitute the values into the formula and solve.**

$$A = \frac{10^2 (0.7071)(0.8192)}{2(0.9848)} = 29.4098$$

The area is a little over 29 square units.

5

The Graphs of Trig Functions

Create the basic graphs of the six trig functions.

Use the basic graphs of sine and cosine to more easily graph cosecant and secant.

Perform transformations on graphs of trig functions to make them fit a particular situation.

Use trig functions to model periodic applications — things occurring over and over as time goes by.

Chapter **18**

Graphing Sine and Cosine

The graphs of the sine and cosine functions are very similar. If you look at them without a coordinate axis for reference, you can't tell them apart. They keep repeating the same values over and over — and the values, or outputs, are the same for the two functions. These two graphs are the most recognizable and useful for modeling real-life situations. The sine and cosine curves can represent anything tied to seasons — the weather, shopping, hunting, and daylight. The equations and graphs of the curves are helpful in describing what happens during those seasons. You also find the curves used in predator–prey scenarios and physical cycles.

The ABCs of Graphing

You can graph trig functions in a snap — well, maybe not that fast — but you can do it quickly and efficiently with just a few pointers. If you set up the axes properly and have a general understanding of the different functions' shapes, then you're in business.

Different kinds of values represent the two axes in trig graphs. The *x*-axis is in angle measures, and the *y*-axis is in plain old numbers. The *x*-axis is labeled in

either degrees or radians. Often, a graph represents the values from -2π to 2π to accommodate two complete cycles of the sine, cosine, secant, or cosecant functions (or four complete cycles of the tangent or cotangent functions). If the x-axis is labeled in degrees, it typically ranges from $-360°$ to $360°$, which is a wide number range. That range is in sharp contrast to the y-axis, which often just goes from -5 to 5. You'll find that radians — which are real numbers — are preferable when graphing trig functions. The y-axis is labeled in real numbers; how high and low the range extends depends on the particular function or variation of a function that you're graphing.

If you're using a graphing calculator, you need to be aware of what mode you're in when creating graphs. Otherwise, you'll get completely baffling results or none at all. For more on changing your calculator's mode, head back to Chapter 15. It just takes the press of a button or two, and you're in the right mood — oops, mode.

Waving at the Sine

The graph of the sine function is a nice, continuous wave that rolls along gently and keeps repeating itself. The domain, or x-values, of the sine function includes all angles in degrees or all real numbers in radians, so the curve has no breaks or holes. The range, or y-values, of the sine function consists of all the numbers between -1 and 1, including those two values. Figure 18-1 shows a graph of the sine function from about $-360°$ to $360°$ (or -2π to 2π).

Figure 18-1 shows two complete cycles of the sine curve — the curve goes through its routine twice on the graph. The x-axis is labeled in degrees, and the y-axis is labeled in integers. If you could see the sine curve forever in either direction, it wouldn't look any different. The curve repeats the same pattern over and over again, to infinity and beyond.
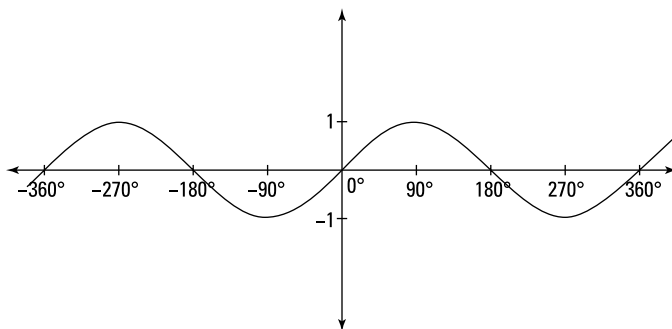
FIGURE 18-1:
The graph
of $y = \sin x$.

Describing amplitude and period

The sine function and any of its variations have two important characteristics: the amplitude and period of the curve. You can determine these characteristics by looking at either the graph of the function or its equation.

Gaining height with the amplitude

The amplitude of the sine function is the vertical distance from the middle value or line running horizontally through the graph up to the highest point. In other words, the amplitude is half the distance from the lowest value to the highest value. In the sine and cosine equations, the amplitude is the coefficient (multiplier) of the sine or cosine. For example, the amplitude of $y = \sin x$ is 1. To change the amplitude, multiply the sine function by a number. Take a look at Figure 18-2, which shows the graphs of $y = 3\sin x$ and $y = \frac{1}{2}\sin x$.
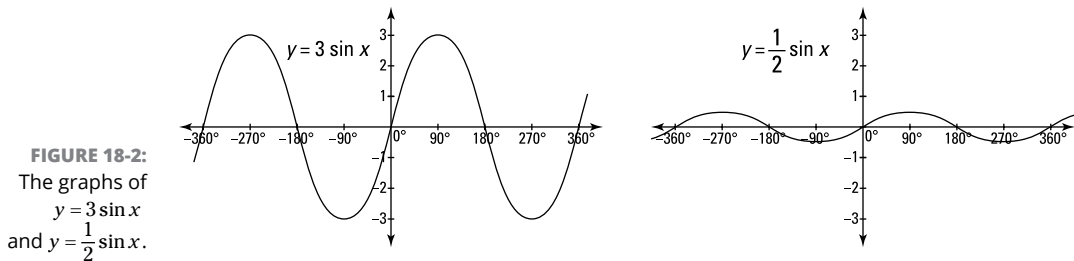
FIGURE 18-2:
The graphs of
$y = 3\sin x$
and $y = \frac{1}{2}\sin x$.

THE SOUND OF MUSIC

Sounds are created by vibrations. Tuning forks can produce pure tones when they vibrate, and sine waves can model those tones. A formula for a pure tone is $y = A\sin(2\pi ft)$, where A stands for amplitude (loudness), f stands for frequency (vibrations per second), and t is a unit of time. If a string, tuning fork, or something similar vibrates at the rate of 256 times per second, then you hear middle C. When you double the frequency of any pure tone, you go up one octave, so 512 vibrations per second gives you the C above middle C.

If you add waves of different frequencies and loudness together, you get more-interesting and complex tones. The string of a violin or the inside of an oboe, for example, can vibrate with more than one frequency at the same time.

As you can see, multiplying by a number greater than 1 makes the graph extend higher and lower. The amplitude of $y = 3\sin x$ is 3. Conversely, multiplying by a number smaller than 1 (but bigger than 0) makes the graph shrink in value — it doesn't go up or down as far. The amplitude of $y = \frac{1}{2}\sin x$ is $\frac{1}{2}$.

Punctuating with the period

The period of a function is the extent of input values it takes for the function to run through all the possible values and start all over again in the same place to repeat the process. In the case of the sine function, the period is 2π, or 360 degrees. Pick any place on the sine curve, follow the curve to the right or left, and 2π or 360 units from your starting point along the x-axis, the curve starts the same pattern over again.

Multiplying the angle variable, x, by a number changes the period of the sine function. If you multiply the angle variable by 3, such as in $y = \sin 3x$, then the curve will make three times as many completions in the usual amount of space. So, multiplying by 3 actually *reduces* the length of the period. In the case of $y = \sin \frac{1}{2}x$, only half the curve fits in the same space. So, a coefficient less than 1 increases the number of inputs that the function needs to complete a cycle. Figure 18-3 shows pictures of these two graphs. On the left, you have $y = \sin 3x$, making six complete cycles of the function where you usually find two. On the right, you have $y = \sin \frac{1}{2}x$, making only one complete cycle where you usually see two.

FIGURE 18-3: Graphs of $y = \sin 3x$ and $y = \sin \frac{1}{2}x$.
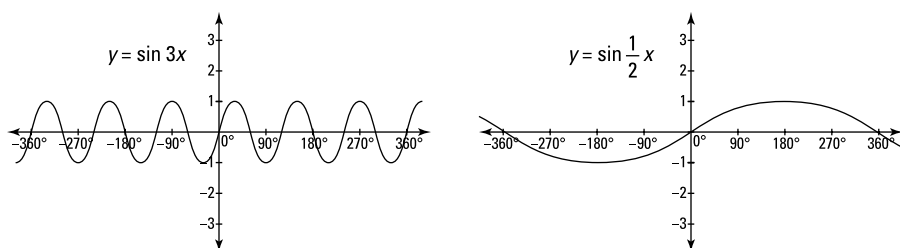
The location of the multiplier makes a big difference. Multiplying the sine function by 4 and its angle variable by 4 results in two completely different graphs. The graph of $y = 4\sin x$ is much higher than usual — the amplitude is greater than that of the standard sine function. The graph of $y = \sin 4x$ has an amplitude of 1, but the period is smaller and the curve is more scrunched together — it repeats over and over more quickly.

WARNING

Formalizing the sine equation

A general equation for the sine function is $y = A \sin B(x + C) + D$. The A and B are numbers that determine the amplitude and period of the basic sine function, respectively. The C and D create shifts in the starting and ending places and can even move the curve off the x–axis. (See the next section, "Translating the sine," for more on those movements.) When C and D are both equal to zero, you have the basic sine function $y = A \sin Bx$.

The graph of the function $y = A \sin Bx$ has an amplitude of A and a period of $\frac{2\pi}{B}$ (or, in degrees, $\frac{360}{B}$). The amplitude, A, is the distance measured from the y–value of a horizontal line drawn through the middle of the graph (or the average value) to the y–value of the highest point of the sine curve, and B is the number of times the sine curve repeats itself within 2π, or 360 degrees.

TRIG RULES

By keeping these two values in mind, you can quickly sketch the graph of this basic sine curve — or picture it in your head. For example, when graphing $y = 4 \sin 2x$, you follow these steps:

1. **Adjust for the amplitude.**

   The amplitude is 4, so the curve will extend up 4 units and down 4 units from the middle. To allow for some space above and below, set the y-axis to go from –5 to 5.

2. **Take into account the period.**

   The coefficient 2 on the x means that two complete graphs of the sine are within the space that usually houses only one.

3. **Graph the curve from -2π to 2π (see Figure 18-4).**

   You can see that the height of the graph goes from –4 to 4 and that four complete cycles are in the space that usually houses only two.

Translating the sine

Playing around with the amplitude and period of the sine curve can result in some interesting changes to the basic curve. That curve is still recognizable, though. You can see the rolling, smooth curve crossing back and forth over a middle line. In addition to those changes, you have two other options for altering the sine curve — shifting the curve up, down, or sideways. These shifts are called *translations* of the curve. These translations are accounted for in the more general equation for the sine: $y = A \sin B(x + C) + D$.
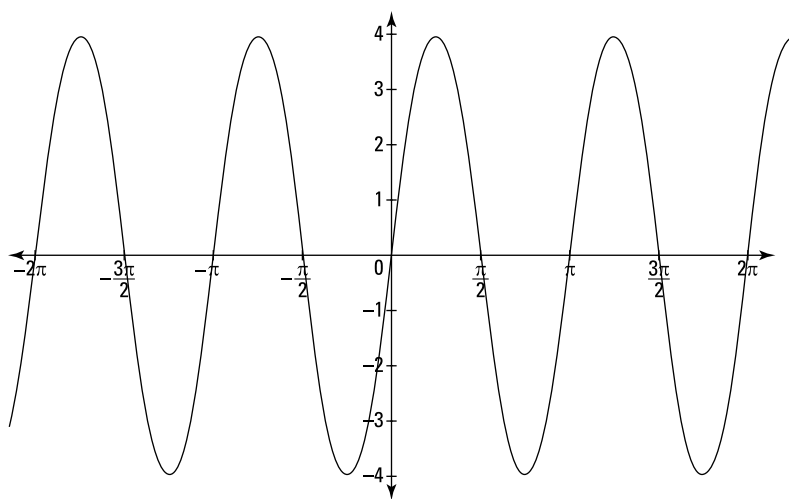
FIGURE 18-4:
The graph
of $y = 4\sin 2x$.

Sliding up or down

You can move a sine curve up or down by simply adding or subtracting a number from the equation of the curve. In terms of the equation, if D is positive, you move the curve upward that amount; if D is negative, it goes down. For example, the graph of $y = \sin x + 8$ moves the whole curve up 8 units, with the sine curve crossing back and forth over the line $y = 8$. On the other hand, the graph of $y = \sin x - 2$ slides everything down 2 units, with the curve crossing back and forth over the line $y = -2$. Figure 18-5 shows what the two graphs look like.
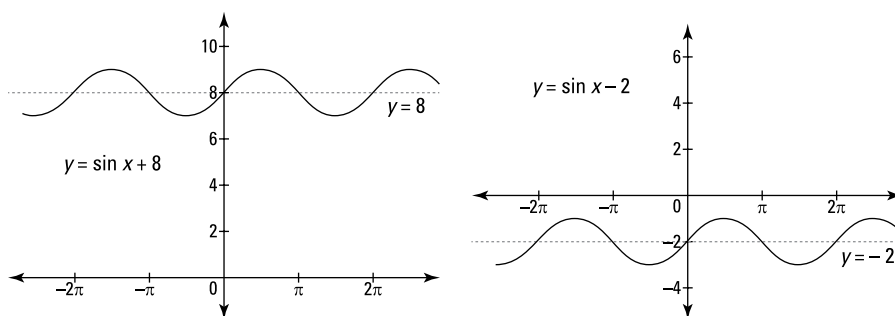
FIGURE 18-5:
The graphs of
$y = \sin x + 8$
and $y = \sin x - 2$.

As you can see, the basic shape of the sine curve is still recognizable — the curves are just shifted up or down on the coordinate plane.

Shifting left or right

By adding or subtracting a number from the angle (the variable) in a sine equation, you can move the curve to the left or right of its usual position. This is the C part of the general equation. This shift, or translation, relates the sine curve to the cosine curve. But the translation of the sine itself is important: Shifting the curve left or right can change the places that the curve crosses the x-axis or some other horizontal line. For example, the graph of $y = \sin\left(x + \dfrac{\pi}{4}\right)$ is the usual sine curve slid $\dfrac{\pi}{4}$ units to the left, and the graph of $y = \sin(x - \pi)$ slid π units to the right. The graph of $y = \sin(x + 3)$ shifts the curve 3 units to the left. Note that this is an integer, not an angle measure. Figure 18-6 shows the graphs of the original sine equation and these three shifted equations.
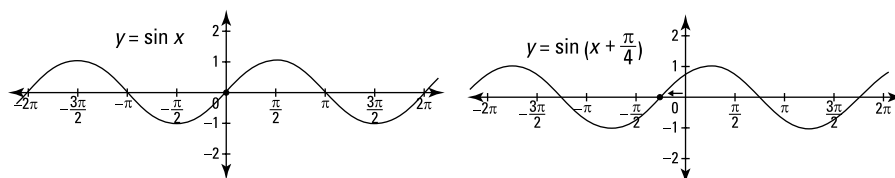
FIGURE 18-6:
Comparing the graphs of $y = \sin x$, $y = \sin\left(x + \dfrac{\pi}{4}\right)$, $y = \sin(x - \pi)$, and $y = \sin(x + 3)$.
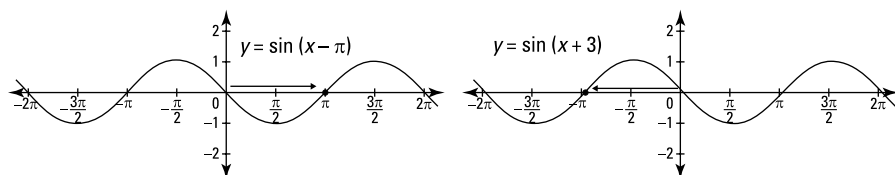
Take a look at the point marked on each graph in Figure 18-6. This point illustrates how an *intercept* (where the curve crosses an axis) shifts on the graph when you add or subtract a number from the angle variable.

⚠️ **WARNING** Note the difference between adding or subtracting a number to the function and adding or subtracting a number to the angle measure. These operations affect the curve differently, as you can see by comparing the curves in Figure 18-5 and Figure 18-6.

Graphing Cosine

The graph of the cosine function looks very much like that of the sine function. This quality is due to the fact that they're related by domain and range, as well as by several identities. An identity involving a shift explains the relationship best, because that shift can make the graph of the sine function look like the cosine function.

Comparing cosine to sine

The relationship between the sine and cosine graphs is that the cosine is the same as the sine shifted to the left by 90 degrees, or $\frac{\pi}{2}$. The equation that represents this relationship is $\cos x = \sin\left(x + \frac{\pi}{2}\right)$. Look at the graphs of the sine and cosine functions on the same coordinate axes, as shown in Figure 18-7. Each tick mark on the x-axis represents a multiple of $\frac{\pi}{2}$. The graph of the cosine is the darker curve; note how it's shifted to the left of the sine curve.
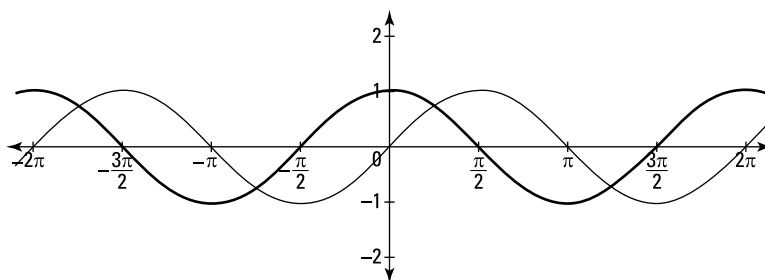
The graphs of the sine and cosine functions illustrate a property that exists for several pairings of the functions. This property is based on the right triangle and the two acute or complementary angles in a right triangle. The identities that arise from the triangle are called the *co-function identities*.

The co-function identities are

TRIG RULES

$\sin\theta = \cos(90° - \theta)$	$\csc\theta = \sec(90° - \theta)$
$\cos\theta = \sin(90° - \theta)$	$\sec\theta = \csc(90° - \theta)$
$\tan\theta = \cot(90° - \theta)$	$\cot\theta = \tan(90° - \theta)$

These identities show how the function values of the complementary angles in a right triangle are related. For example, $\cos\theta = \sin(90° - \theta)$ means that if θ is equal to 25 degrees, then $\cos 25° = \sin(90° - 25°) = \sin 65°$. This equation is a roundabout way of explaining why the graphs of sine and cosine are different by just a slide. You probably noticed that these co-function identities all use the difference of angles, but the slide of the sine function to the left was a sum. The shifted sine graph and the cosine graph are really equivalent — they become graphs of the same set of points.

Using properties to graph cosine

The cosine function has the same amplitude and period as the sine function: The amplitude is 1, and the period is 2π, or 360 degrees. The variations on the cosine work the same way as on the sine. If you want to change the amplitude, multiply the cosine function by a number. If you want to change the period, multiply or divide the angle variable by a number. To slide the whole curve up, down, right, or left, add or subtract a number from the whole function or the angle variable.

For the equation $y = 3\cos x$, the amplitude is 3, meaning the graph stretches up and down to 3 units. The graph of $y = \cos(x - 3)$ is shifted 3 units to the right. Notice how the intercept (0,1) has shifted to (3,1). Figure 18-8 shows you the graphs.

FIGURE 18-8:
The graph of $y = 3\cos x$ and $y = \cos(x - 3)$.
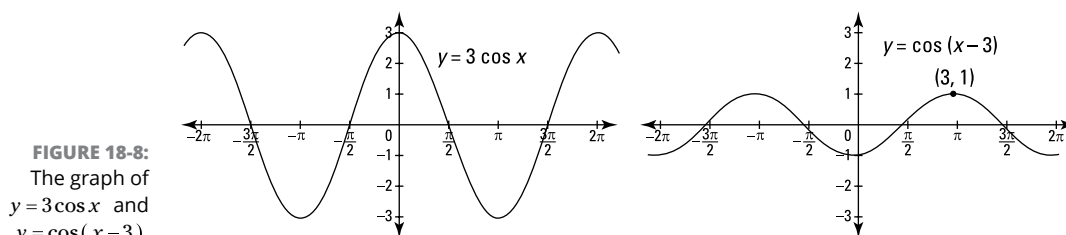
The graphs of sine and cosine are difficult to tell apart when they're shifted about. But that fact just shows how much those functions have in common, which can work to your advantage when you're applying them.

Applying the Sines of the Times

The sine curve and its co-function, cosine, are great for modeling situations that happen over and over again in a predictable fashion. Some examples include the weather, seasonal sales of goods, body temperature, the tide's height in a harbor,

average temperatures, and so on. In this section, I show you a few examples of how you can use these functions in practical situations. In each case, I point out how the graph and formula illustrate the amplitude, period, and any shifts (for more on those concepts, check out the section, "Waving at the Sine," earlier in this chapter).

Sunning yourself

San Diego, California, is a gorgeous part of the world. Whether it's summer or winter, you want to be there. But what if you're someone who likes long, sunny days? When is the best time to go there? Assume that the following formula gives you the number of hours of daylight in San Diego when you input any day of the year. Letting t be the day of the year (from 1 to 365), you can figure the number of hours of sunlight, H, with the equation $H(t) = 2.4 \sin(0.017t - 1.377) + 12$. Figure 18-9 shows the graph of this equation.

The amplitude of the sine curve is 2.4, which means that the number of daylight hours extends 2.4 hours above and below the average number of daylight hours. The average number of daylight hours is 12, which is the translation upward. You see the average daylight hours represented by the horizontal line. When you add or subtract the 2.4, you find that the hours of sunlight range from 14.4 to 9.6, depending on the time of year.

The period is $\dfrac{2\pi}{0.017} \approx 370$, which is a little longer than a year because of the rounding in the formula (and that helps when there's a leap year). The coefficient (multiplier) on the t in the function $H(t) = 2.4\sin(0.017t - 1.377) + 12$ means that 0.017 of the curve takes up the usual amount of space for one curve, 2π units. As you can see on the graph, the day with the most sunlight is June 27. You can determine that high point by using a graphing calculator that finds it for you — as well as the y-value of when t equals 14.4, or you can use calculus to solve it! Do you know, now, when you want to go to San Diego?

Averaging temperature

A relatively reasonable model for the average daily temperature in Peoria, Illinois, is $T(x) = 50 - 42\cos(0.017x - 0.534)$, where x is the day of the year starting with January 1 as Day 1. The $T(x)$ represents the temperatures in degrees Fahrenheit. The graph is in radian measure, because radians are real numbers, as opposed to degrees — you need numbers to count off the days. Figure 18-10 shows what the graph of the function looks like for the whole year.

The multiplier on the cosine function is 42, so the amplitude of the curve is 42. Don't worry about the negative sign in front of the 42. The curve goes upward and downward anyway, so the negative sign just makes it go downward and then upward, instead of the reverse.

The period is affected by the multiplier 0.017. The result of that multiplication is that only 0.017 of a cosine curve takes up the usual amount of space for an entire curve, which is 2π, or a little over 6 units. Because this graph is for a whole year, the curve has to spread out over 365 units, so that each of the horizontal units has just a little part of it.

The shift upward of 50 units is the middle or average temperature for the year. This is shown with the horizontal line at a height of 50. Add the amplitude of 42 to this number, and the average temperature gets up to 92 degrees; subtract the amplitude, and the average gets down to 8 degrees. Note that the curve starts a little to the right of the y-axis to account for when the seasons change. If you want more details on curve translations to the left, right, up, and down, go to Chapter 21.

What do you do with the graph? You can estimate when the highest and lowest temperatures occur and get an idea of the types of temperatures to expect if you move to Peoria, Illinois. Figure 18-11 shows the graph of the average temperatures with points for some days of the year and the average temperatures on those days. A graphing calculator is indispensable when graphing these figures and calculating values. You can either input x-values to find out what the y-values are at those points, or you can trace along the curve to get the measures.

Taking your temperature

The temperature of a person's body fluctuates during the day instead of staying at a normal 98.6 degrees. And actually, not everyone has a "normal" temperature. Lots of people run either hot or cold.

If you're one of the special people with a normal temperature, then your temperature goes up and down by about 1 degree each day. The formula $T(x) = \sin(x + 0.262) + 98.6$ may be a model of your temperature during a 24-hour

period. The variable *x* is the number of hours since midnight, so this equation uses a 24-hour clock. The temperatures are given in degrees Fahrenheit. The graph is in radians, so you can enter the numbers for the hours. Figure 18-12 shows what a graph of the temperatures may look like, noting a few times and temperatures. Now you see why your feet get cold in the wee hours of the morning.

FIGURE 18-12:
The body temperatures of a certain person over a 24-hour period.
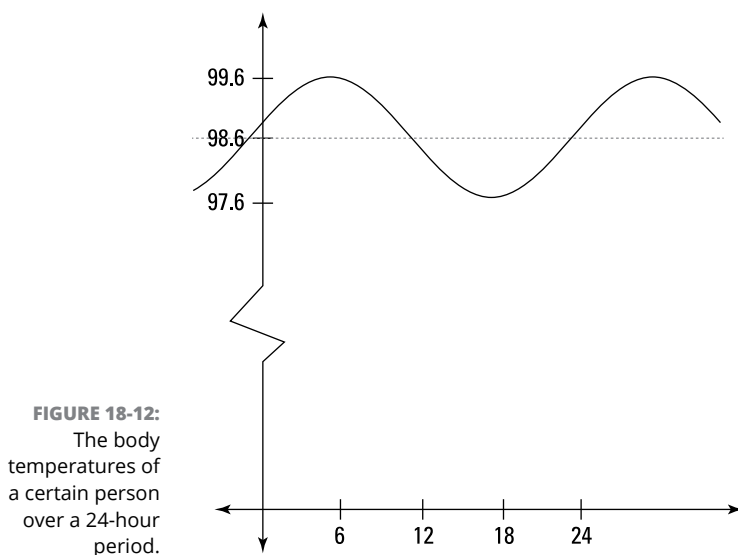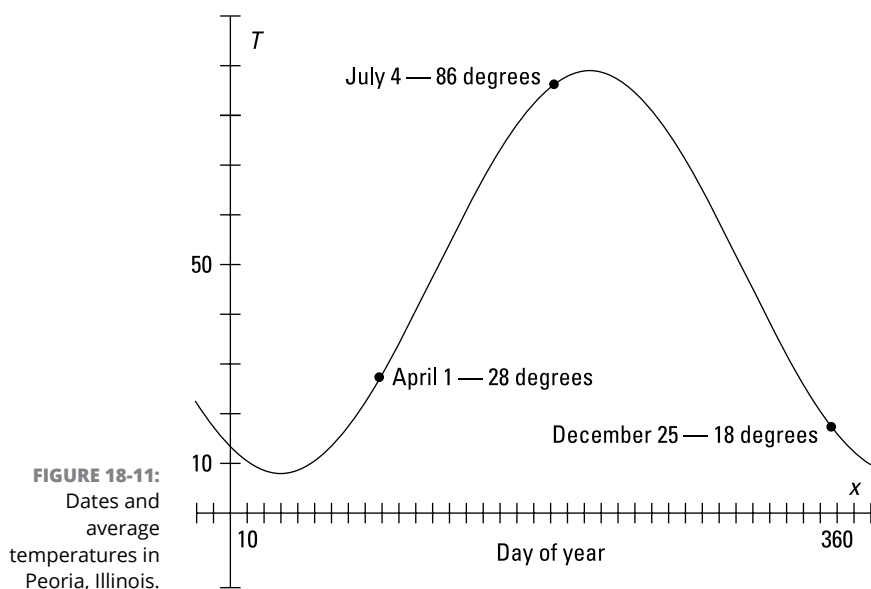
Making a goal

Even though people in many parts of the world play soccer year-round, certain times of the year show an increase in the sales of outdoor soccer shoes. Here's a model for the sales of pairs of shoes where N is in millions of pairs and m is the month of the year: $N(m) = 44 \sin(0.524m) + 70$. From the equation, you can tell that the average number of pairs sold is 70 million, which is the vertical shift upward. That number fluctuates between 26 million and 114 million, which you find by adding and subtracting the amplitude, 44, to and from the average. The period of this model is $\dfrac{2\pi}{0.524} \approx 11.99$, or 12 months.

Figure 18-13 shows a graph of this function. A graph like this one can help distributors and retailers with their plans for sales.
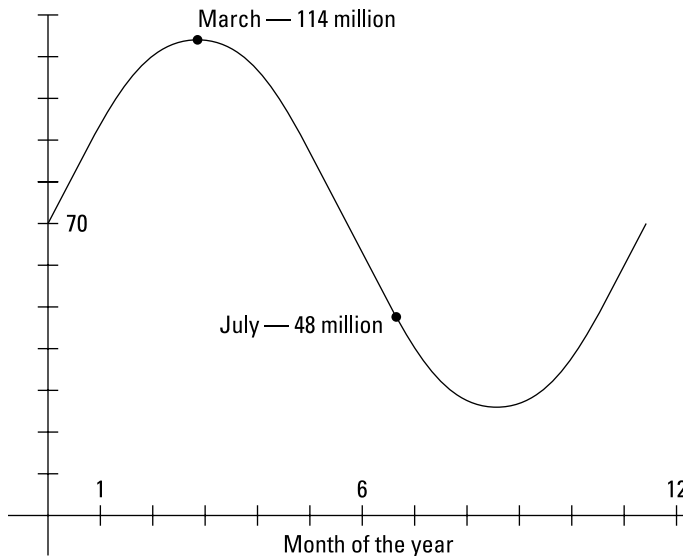
The sales of soccer shoes in millions of pairs of shoes.

Theorizing with biorhythms

Many years ago, the public showed great interest in a person's *biorhythms*, which are the physical, emotional, and intellectual cycles that a person experiences in life. Many people even wrote books about them. Some believe that these cycles affect how a person reacts to situations in their life. They even go so far as to say that the positions of these curves have influenced major decisions of famous movie stars and politicians.

What do biorhythms have to do with trigonometry? Everything! This biorhythm theory uses the sine curve. Supposedly, our life cycles start at birth and fluctuate like sine curves. The physical cycle is 23 days long, the emotional cycle is 28 days long, and the intellectual cycle is 33 days long. If you plot all these cycles on a graph, starting on the day you were born, you can see where these cycles are right now and what they'll look like in the future. Figure 18-14 shows a graph of the three biorhythm cycles starting on the day a person is born.

In Figure 18-14, you can see how the different cycles have different periods. Imagine these sine curves going on for years and years, crossing over the *x*-axis and over one another. Figure 18-15 shows some biorhythm cycles plotted for some imaginary person for some year in the month of March.
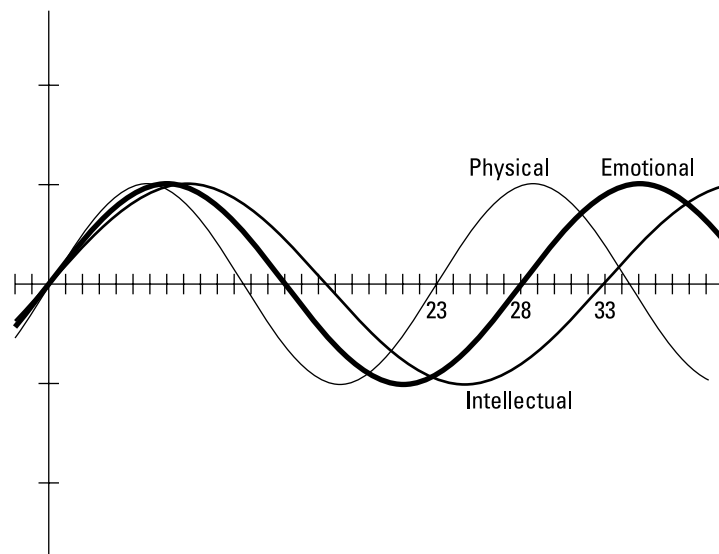
FIGURE 18-14:
The three biorhythm cycles, starting at birth.
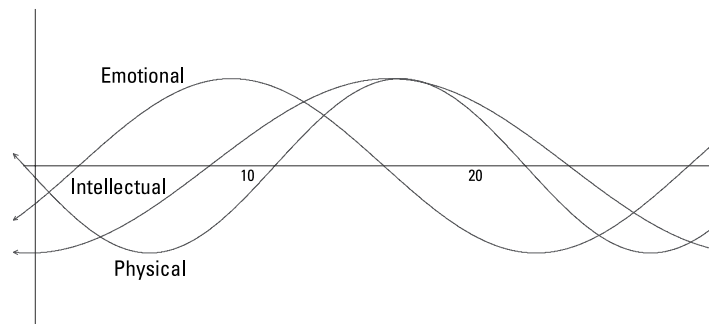
FIGURE 18-15:
John Doe's biorhythm cycles for March.

If you believe in the biorhythm theory that says these curves exist, you can see that on about the 13th of the month, all the cycles are above the *x*-axis, and after about the 25th, they're all below the *x*-axis. Supposedly, when a curve is above the *x*-axis, everything is bright and sunny — a person is in good health, emotionally fine, and very smart and with it. When the curve is below the *x*-axis, the person tends to be sick, depressed, and dull. In addition, the theory says that when the cycles cross from above to below the axis, or vice versa, those days are critical. A critical day is when upheaval and crises are possible. Such a day is a good time to stay in bed — if that's even safe. I guess there's no way to prove or disprove this theory, but it sure makes interesting use of the sine curve!

Chapter **19**

Graphing Tangent and Cotangent

T he tangent and cotangent functions have lots of similarities. You can write both functions in terms of sine and cosine, so they share the same function values in their ratios. One difference between tangent and cotangent is that they don't have function values in the same places for the *x*-values in their domain — they shift over by 90 degrees. Even though their domains (or *x*-values) are restricted, tangent and cotangent are the only trig functions with ranges (or *y*-values) that go all the way from negative infinity to positive infinity. The challenges in graphing tangent and cotangent are in dealing with the domain restrictions and *asymptotes* (dotted vertical lines used to determine the shape of a curve), as you see in this chapter.

Checking Out Tangent

The tangent function can be written as the ratio of the sine divided by the cosine: $\tan\theta = \frac{\sin\theta}{\cos\theta}$. (For more information on the tangent function, see Chapters 6 and 7.) The sine and cosine functions have values for every *x*-value, so no matter what number you put in for *x*, you'll get an answer. The only problem occurs when the cosine function is equal to 0, because a fraction can't have a 0 in the

denominator. So, wherever the cosine function is equal to 0, the graph of the tangent curve doesn't exist, and this is indicated with asymptotes. Another interesting property comes into play with the fact that the sine and cosine are both positive in the first quadrant and negative in the third quadrant. As a result, the tangent is positive in those two quadrants and negative in the other two, because either the sine or cosine is negative, but not both.

Determining the period

The sine and cosine functions have a period of 2π, or 360 degrees, which means that after every 2π, the function pattern starts all over again. In the case of the tangent function, though, the length of the period is only π — half as long as that of sine or cosine. The tangent function repeats its pattern over and over with twice the frequency as sine and cosine.

Assigning the asymptotes

An *asymptote* is a line that helps give direction and shape to a graph. This line isn't part of the function's graph; instead, it helps determine the shape of the curve by having the curve hug or get very close to the asymptote. Asymptotes are usually indicated with dashed lines. However, if you use your graphing calculator to graph a tangent curve, the asymptotes might appear as solid lines. Some calculators seemingly want to keep everything connected. And with other calculators, the asymptotes won't be shown at all. Just remember that the asymptote is just there for form.

The asymptotes for the graph of the tangent function occur regularly, each of them π, or 180 degrees, apart. They separate each piece of the tangent curve, or each complete cycle, from the next.

The equations of the tangent's asymptotes are all of the form $x = \frac{\pi}{2}(2n+1)$, where n is an integer. Under that stipulation for n, the expression $2n+1$ always results in an odd number. By replacing n with various integers, you get lines such as $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{5\pi}{2}$, $x = \frac{7\pi}{2}$, $x = -\frac{\pi}{2}$, $x = -\frac{3\pi}{2}$, $x = -\frac{5\pi}{2}$, and $x = -\frac{7\pi}{2}$.

The reason that asymptotes always occur at these odd multiples of $\frac{\pi}{2}$ is because those points are where the cosine function is equal to 0. As such, the domain of the tangent function includes all real numbers *except* the numbers that occur at these asymptotes.

Figure 19-1 shows what the tangent function looks like when graphed. The tangent values go infinitely high as the angle measure approaches 90 degrees,

270 degrees, and so on (as you move from left to right on the graph). The values go infinitely low as the angle measure approaches −90 degrees, −270 degrees, and so on (as you move from right to left on the graph).
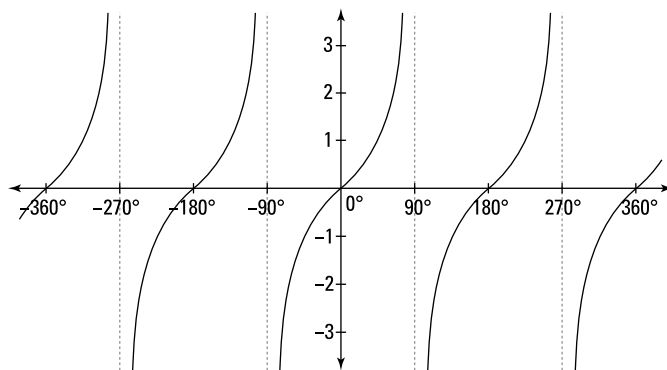
As you can see, the tangent function repeats its values over and over. One main difference between this function and the sine and cosine functions is that the tangent has all these breaks between the cycles. As you move from left to right, the tangent appears to go up to positive infinity. It actually disappears at the top of the graph and then picks up again at the bottom, where the values come from negative infinity. Graphing calculators and other graphing utilities may not show the graph disappearing at the top, so it's up to you to know what's actually happening, even though the picture may not look exactly that way.

REMEMBER Because some graphing calculators try to connect the tangent function to make it continuous across the screen, you could get a false impression of any curve with vertical asymptotes. One way to get rid of those extra lines is to change your calculator to the *dot* mode (as opposed to the *connected* mode). Most calculators have ways to change the settings (or mode) for things such as degrees and radians, dotted graphs and connected graphs, and floating and fixed decimals. These changes are usually easy to make — just see your calculator's manual for specific instructions. The hard part is remembering what setting you're in.

Fiddling with the tangent

You can alter the tangent function with multiplication, addition, and subtraction. In some cases, the effects are similar to those that occur when you alter the sine and cosine functions. Because these results aren't similar all the time, you should consider the alterations on a case-by-case basis.

Multiplying the tangent

You can multiply the tangent function by a number, but doing so doesn't affect the tangent function the way that it affects the sine function. Multiplying the sine by a number changes its amplitude, making the function include larger and smaller values. The tangent values, however, already go from negative infinity to positive infinity.

When you multiply the entire tangent function by a number, here's what happens:

>> **If you multiply by a number bigger than 1,** the graph of the function gets steeper more quickly.

>> **If you multiply by a fraction between 0 and 1,** the graph of the function gets flatter.

>> **If you multiply by a negative number,** the curve flips over the *x*-axis.

Figure 19-2 shows graphs of the basic tangent function ($y = \tan x$) and three multiples of the function to illustrate this property. Notice how the multiple of 3 makes the tangent curve steeper, whereas the multiple of 0.2 makes it flatten out. Both functions still have values that go from negative infinity to positive infinity, but the rate at which they get there changes. And a negative multiple flips the curve over the *x*-axis.
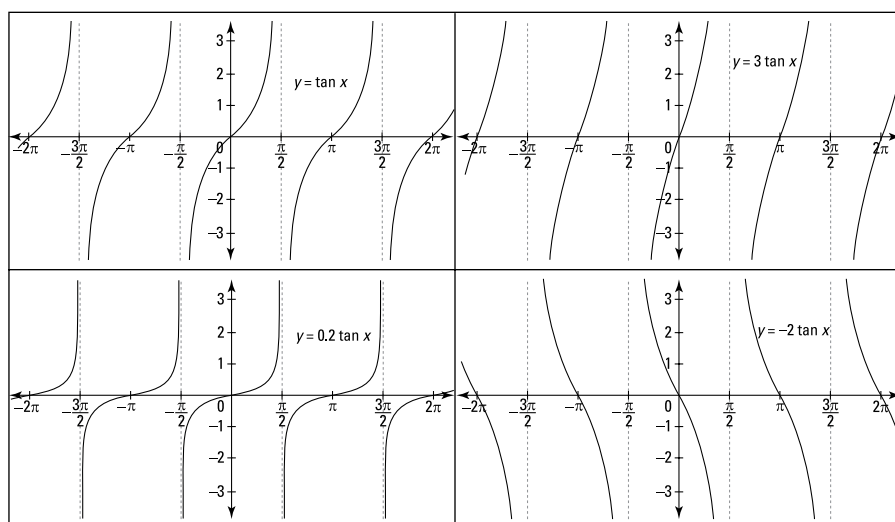
FIGURE 19-2:
The graphs of the tangent function and three multiples.

Multiplying the angle

Multiplying the angle variable in the tangent function has the same effect as it does with the sine and cosine functions. If the multiple is 2, as in $y = \tan 2x$, then the tangent function makes twice as many cycles in the usual amount of space. In other words, the period is $\frac{\pi}{2}$, which is one-half the tangent's usual period, π.

Because multiplying the angle variable of the tangent function mirrors the results of doing the same with the sine and cosine functions, I don't go into detail here — for more information, refer to Chapter 18.

Figure 19-3 shows a few graphs to illustrate the effect of multiplying the angle variable by a number greater than 1 and then by a number between 0 and 1.
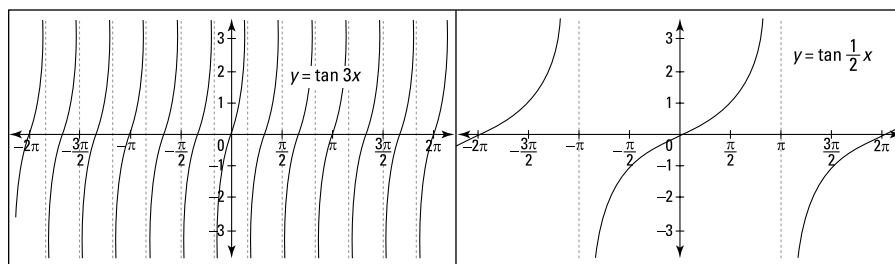
The graph of $y = \tan 3x$ has three times as many tangent curves as usual. The graph of $y = \tan \frac{1}{2}x$ has only half as many cycles — or it takes twice as long to complete one cycle.

Adding to tangent

Adding a number to the tangent function results in raising the curve on the graph by that amount. Likewise, subtracting a number drops the curve. Because the tangent function has values from negative infinity to positive infinity, adding to or subtracting from the function doesn't change what values the tangent has — it just changes where they happen. When you add or subtract, the *point of inflection* in the tangent curve (where the curve appears to flatten out a bit) shifts up or down. Figure 19-4 shows some graphs to illustrate this shift.

Adding or subtracting a number from the angle variable of the tangent function has the same effect as with the sine and cosine — it moves the curve to the left or right. The graph of $y = \tan(x+1)$ shifts one unit to the left, including the asymptotes, and the graph of $y = \tan\left(x - \frac{\pi}{4}\right)$ moves everything to the right by $\frac{\pi}{4}$. Figure 19-4 shows these shifted curves.

FIGURE 19-4: The tangent function raised, lowered, and shifted left and right.

If you have a tough time telling one graph of the tangent function from another, just look for the point of inflection of the tangent curve. The point of inflection is a good reference mark when looking at all these variations.

Confronting the Cotangent

The graphs of the tangent function lay the groundwork for the graphs of the cotangent. After all, they're cofunctions and reciprocals, and have all sorts of connections. The two graphs are similar in so many ways: They both have asymptotes crossing the graph at regular intervals, go from negative infinity to positive

infinity in value, and are affected by multiplying and adding. The biggest difference is in the direction the graphs are drawn. The values of the tangent function appear to *rise* as you read from left to right. The function goes upward, disappears off the graph, and then reappears down below to start all over again. The cotangent function does the opposite — it appears to *fall* when you read from left to right.

The asymptotes of the cotangent curve occur where the sine function equals 0, because $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$. Equations of the asymptotes are of the form $x = n\pi$, where n is an integer. Some examples of the asymptotes are $x = -3\pi$, $x = -2\pi$, $x = -\pi$, $x = 0$, $x = \pi$, $x = 2\pi$, and $x = 3\pi$. (For an explanation of asymptotes, refer to the section, "Assigning the asymptotes," earlier in this chapter.) Figure 19-5 shows the cotangent function graphed between -2π and 2π.

Like the other functions, cotangent repeats the same values over and over. You can apply the same types of variations to cotangent that you can to tangent (refer to the section, "Fiddling with the tangent," for the details). Figure 19-6 provides four examples of variations: multiplying the angle variable, adding to the function, adding to the angle variable, and a combination of subtracting from the function and subtracting from the angle variable.
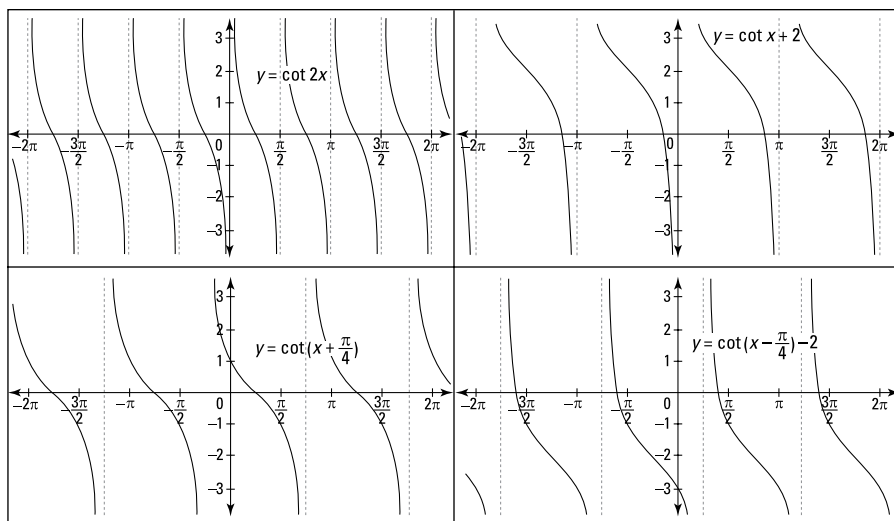
FIGURE 19-6: Variations on the graph of the cotangent function.

Chapter **20**

Graphing Two More Trig Functions

T he functions cosecant and secant have similarities to one another not only because they're the reciprocals of sine and cosine, but also because their graphs look very much alike. As you see in this chapter, the easiest way to sketch the graphs of these two functions is to relate them to the graphs of their reciprocals. Doing so helps determine the *asymptotes* (where the curve approaches infinity or negative infinity), turning points, and general shape of the curves.

Seeing the Cosecant for What It Is

The cosecant function is the *reciprocal* of the sine function (meaning, the cosecant equals the number 1 divided by the sine). Even though the sine function has a domain that includes every possible number, that characteristic can't be true of its reciprocal. Whenever the sine function is equal to 0, the cosecant function doesn't exist. That fact helps determine the asymptotes you use to graph the cosecant function.

Identifying the asymptotes

The domain of the cosecant function is any number except multiples of π, because those measures are where the sine function is equal to 0. You can use this situation to identify the asymptotes by simply writing equations that use multiples of π. The asymptotes of the cosecant function are of the form $x = n\pi$, where n is some integer. Some examples of the asymptote equations are $x = -3\pi$, $x = -2\pi$, $x = -\pi$, $x = 0$, $x = \pi$, $x = 2\pi$, $x = 3\pi$, and $x = 4\pi$.

Using the sine graph

One really efficient way of graphing the cosecant function is to first make a quick sketch of the sine function. With that sketch in place, you can draw the asymptotes through the *x-intercepts* (where the curve crosses the *x*-axis). These are the places where $\sin x = 0$. You can also use the maximum and minimum values on the sine function to locate the minimum and maximum points (known as *turning points*) of the cosecant function.

To graph $y = \csc x$:

1. **Sketch the graph of $y = \sin x$ from $-360°$ to $360°$ (-2π to 2π), as shown in Figure 20-1 (a).**

2. **Draw the vertical asymptotes through the *x*-intercepts, as Figure 20-1 (b) shows.**

3. **Draw $y = \csc x$ between the asymptotes and down to (and up to) the sine curve, as shown in Figure 20-2 (a).**

   The cosecant goes down to the top of the sine curve and up to the bottom of the sine curve. The sine and cosecant share those points where the *y*-values are 1 and –1.

   After using the asymptotes and reciprocal as guides to sketch the cosecant curve, you can erase those extra lines, leaving just $y = \csc x$. Figure 20-2 (b) shows what this function looks like all on its own.

The range of the cosecant function includes all values equal to or greater than 1 and all values equal to or less than –1. In Figure 20-2 (b), you can see that a gap in function values lies between 1 and –1. The cosecant curve, just like all the other trig functions, keeps repeating its pattern over and over.

(a)
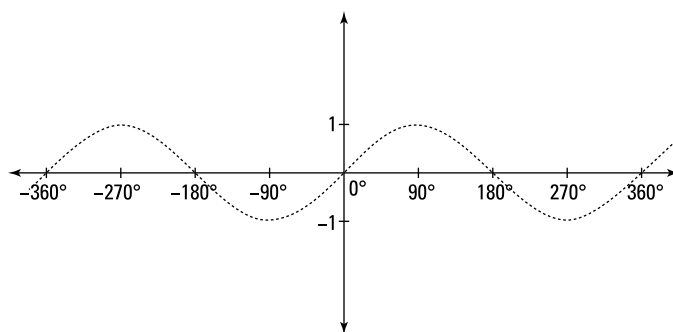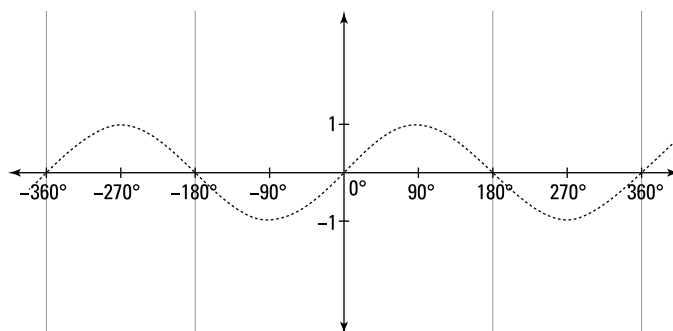
FIGURE 20-1:
Sketches of the
sine function and
the cosecant's
asymptotes.

(b)

Varying the cosecant

How can you make changes to the cosecant function? This function is affected by the same multiplication, addition, and subtraction principles that affect the other functions (check out Chapter 18 for more-detailed info).

Adding or subtracting a number to or from the *cosecant function* results in slides of the graph up or down. Adding or subtracting numbers to the *angle variable* slides the graph left or right. And now I get right to it and do two slides for the price of one, sliding the graph to the left by $\frac{\pi}{2}$ units and up by 1 unit. The equation of that graph is $y = \csc\left(x + \frac{\pi}{2}\right) + 1$. To find out why adding to the angle, x, moves the graph left, head on back to Chapter 18. Meanwhile, Figure 20-3 shows the graph of this equation.

You can see how the asymptotes have the same shifts as the curve — to the left and up. And I've drawn in the horizontal line to show you the vertical shift. The shape of the graph is pretty clear. And you can see the shift of the tops and bottoms of the curve — to the left and upward.

(a)
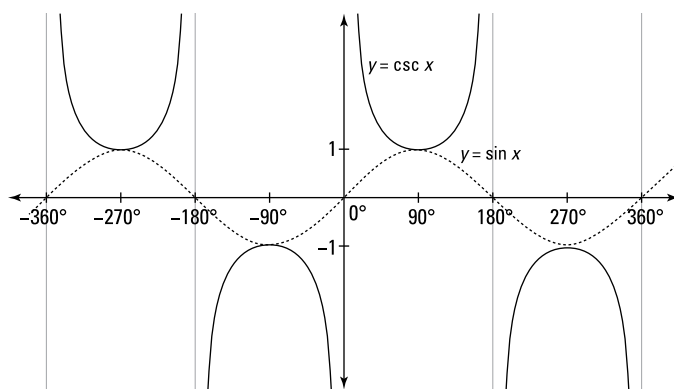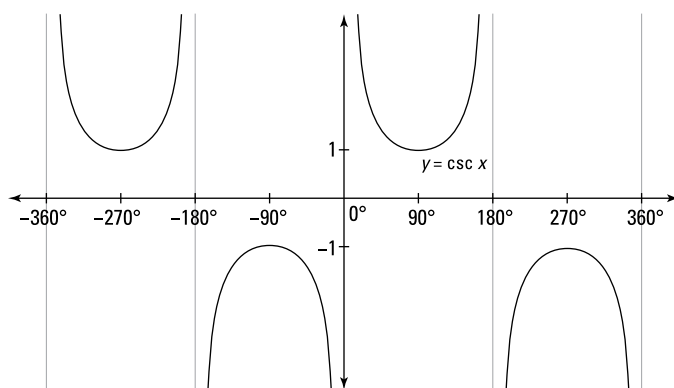
FIGURE 20-2:
Drawing the
cosecant curve by
using the sine as
a guide.

(b)
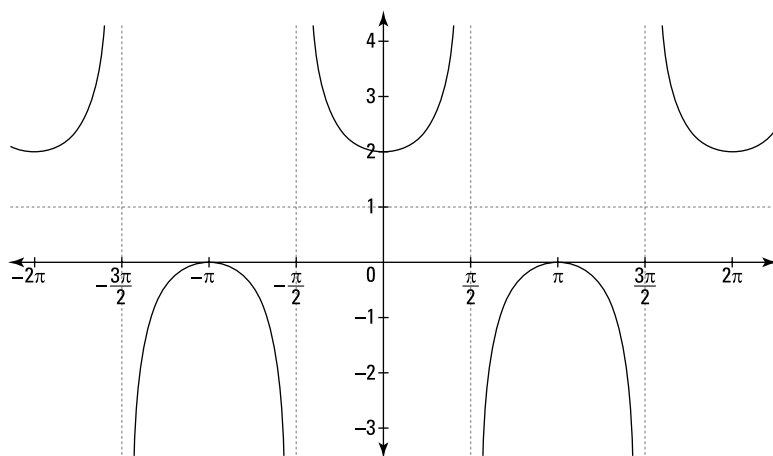
FIGURE 20-3:
The graph of
$y = \csc\left(x + \dfrac{\pi}{2}\right) + 1.$

Multiplying by a number changes the steepness and period of the cosecant function. If you multiply the function by $\frac{1}{4}$, the curve drops down from above and moves up from below to be $\frac{1}{4}$ unit from the x-axis. If you multiply the angle variable by 2, twice as much of the curve fits in the usual amount of horizontal space. Figure 20-4 shows both changes in the graph of $y = \frac{1}{4}\csc 2x$.
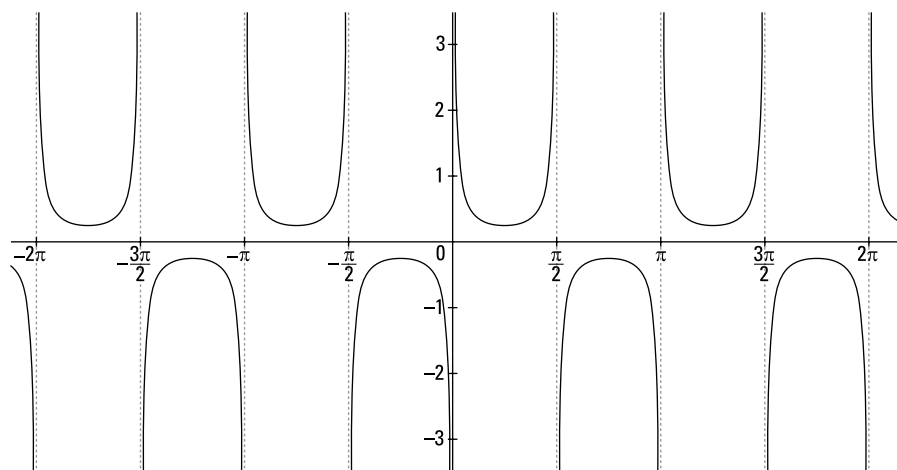
Unveiling the Secant

The techniques that you use to graph the secant curve parallel those that you use to graph the cosecant. First, identify the asymptotes by determining where the reciprocal of secant — cosine — is equal to 0. Then, sketch in that reciprocal, and you can determine the turning points and general shape of the secant graph.

Determining the asymptotes

Because the secant equals 1 divided by the cosine, the secant function is *undefined*, or doesn't exist, whenever the cosine function is equal to 0. You can write the equations of the asymptotes by setting y equal to those values where the cosine is equal to 0, so the asymptotes are $x = -\frac{7\pi}{2}$, $x = -\frac{5\pi}{2}$, $x = -\frac{3\pi}{2}$, $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{5\pi}{2}$, $x = \frac{7\pi}{2}$, and so on. Another way to express the equations of all the asymptotes is to write $x = \frac{(2n+1)\pi}{2}$, where n is some integer.

Sketching the graph of secant

To sketch the graph of the secant function, the easiest method is to use the graph of the cosine. Graph the cosine very lightly or with a dotted curve — the same as with the asymptotes. A lot of busywork is associated with this graph, but you just have to ignore all the extra stuff and zoom in on the graph that you want. To sketch the graph of the secant function:

1. **Sketch the graph of $y = \cos x$ from −360° to 360° (−2π to 2π), as shown in Figure 20-5 (a).**

2. **Draw the vertical asymptotes through the *x*-intercepts (where the curve crosses the *x*-axis), as Figure 20-5 (b) shows.**

3. **Draw $y = \sec x$ between the asymptotes and down to (and up to) the cosine curve, as shown in Figure 20-6 (a).**

   The secant goes down to the top of the cosine curve and up to the bottom of the cosine curve — where the cosine has a value of 1 and −1, respectively.
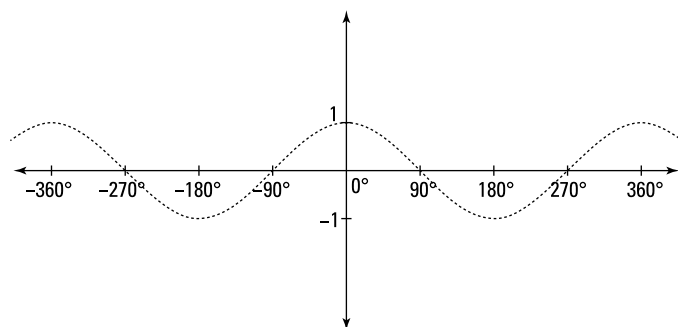
(a)
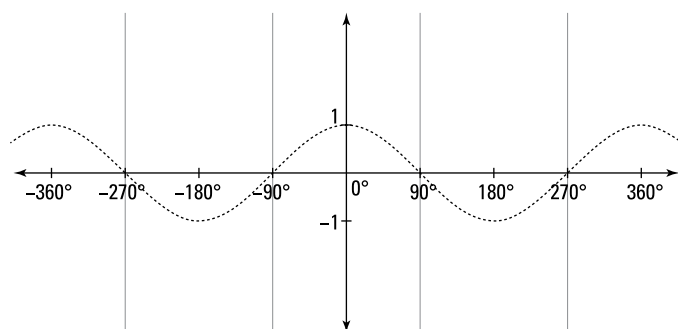
(b)

Figure 20-6 (b) shows you your secant curve.

(a)

(b)
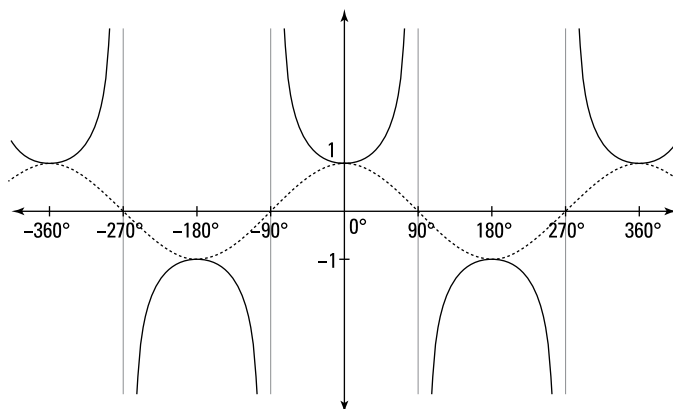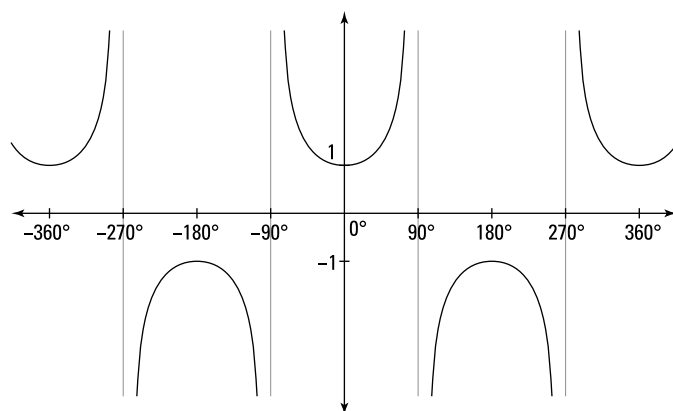
Fooling around with secant

The secant graph is different from the cosecant in several ways, but one of the most obvious ways is that the secant graph is symmetric about the y-axis. The secant is a mirror reflection over that axis. You can use this property to do something interesting to the graph.

The usual translations and multiplications affect the secant graph. If you multiply the function by −1 and add 2 to the function, as in the equation $y = -1\sec(x) + 2$, Figure 20-7 shows you how it's flipped over the x-axis and lifted by 2 units.
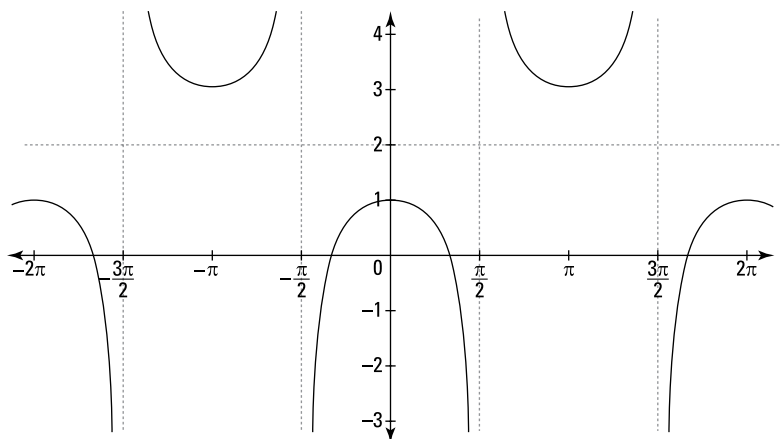
You probably noticed that the asymptotes don't seem to be different. They aren't — and they shouldn't be. By adding 2 to the function, you shift the graph 2 units upward. And the negative multiplier just reflects the curve over the *x*-axis.

NOW YOU SEE IT; NOW YOU DON'T

One optical illusion, called the Kanizsa triangle, causes the eye to perceive a white equilateral triangle where none is actually drawn. Here it is:
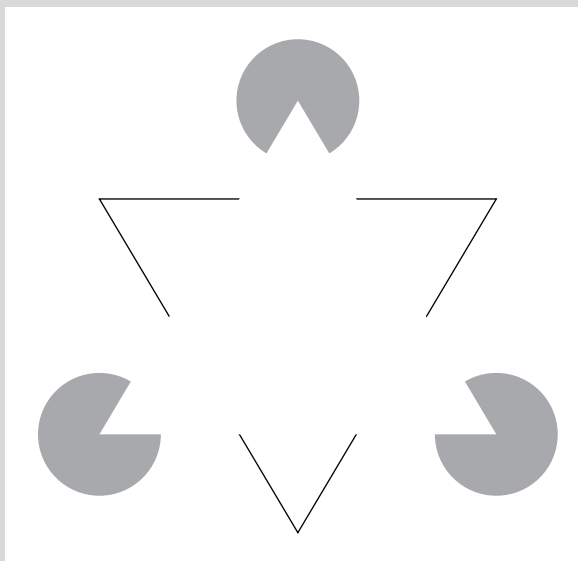
Laying Out the Inverse Functions

The six basic trig functions all have inverses. In Chapter 14, you find information on the notation used to indicate inverse functions, what their respective domains are, and how to use them.

The inverse trig functions — $y = \sin^{-1} x$, $y = \cos^{-1} x$, $y = \tan^{-1} x$, $y = \cot^{-1} x$, $y = \sec^{-1} x$, and $y = \csc^{-1} x$ — are useful when solving trigonometric equations or doing applications involving trigonometry. The graphs of the inverse trig functions are rather unique; inverse sine and inverse cosine are rather abrupt and disjointed, but inverse tangent and inverse cotangent seem to go on forever, within narrow confines. The reason you find these big differences is because of the range or outputs of the original functions. The range of sine and cosine is between –1 and 1, so the inverse function has inputs of just those values. The tangent and cotangent have infinite ranges — which is why their inverses have infinite domains.

Why in the world are the graphs of inverse functions of any importance? For the same reason that all pictures are important — for their visual impact. Especially in the world of trig functions, remembering the general shape of a function's graph goes a long way toward helping you remember more about the function values and using them effectively.

Graphing inverse sine and cosine

The first two graphs sort of go together — they have a common characteristic. The input values for both $y = \sin^{-1} x$ and $y = \cos^{-1} x$ are all the numbers from –1 to 1, including those numbers. The inputs are restricted to those values because they're the *output* values of the sine and cosine.

The output, or range, values for these two inverse functions are different. The range of $y = \sin^{-1} x$ consists of angles in the first and fourth quadrants.

In radians, the range is $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$. The range of $y = \cos^{-1} x$, on the other hand, consists of angles in the first and second quadrants, or angles from 0 to π.

Figure 20-8 shows what the graphs of inverse sine and cosine look like.

The points indicated on the graphs are at $x = -1$ and $x = 1$. These points are the extreme values of the inputs. The y-values represent the angle measures. If you want to find a point on either graph, just find some number between –1 and 1, and find the place on the graph corresponding to that x-value.
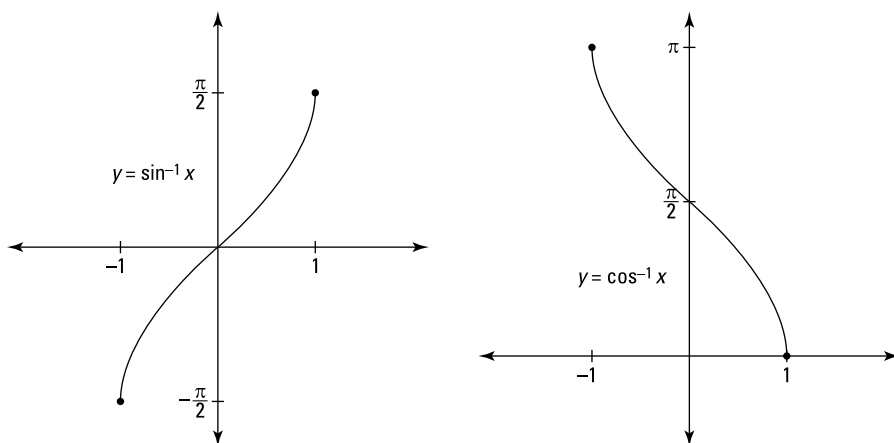
FIGURE 20-8:
The graphs of
$y = \sin^{-1} x$
and $y = \cos^{-1} x$.

Taking on inverse tangent and cotangent

The tangent and cotangent functions have restricted inputs — certain angles don't jibe with them. But their outputs go through all the real numbers. If you switch those two groups of numbers to fit the inverses of tangent and cotangent, you can say that the *inputs* go through all the real numbers, and the *outputs* are restricted. The graphs of these two inverse functions are quite interesting because they both involve two horizontal asymptotes. The asymptotes help with the shapes of the curves and emphasize the fact that some angles won't work with the functions.

The two horizontal asymptotes for the inverse tangent function are $y = -\dfrac{\pi}{2}$ and $y = \dfrac{\pi}{2}$, because the tangent function doesn't exist for those two angle measures. The tangent function isn't defined wherever the cosine is equal to 0. If you need to review the tangent function, go to Chapters 6 and 7. The graph of the inverse tangent has *x*-values from negative infinity to positive infinity, with all *y*-values between those two asymptotes.

The two horizontal asymptotes for the inverse cotangent function are $y = 0$ and $y = \pi$. As with the inverse tangent, the inverse cotangent function goes from negative infinity to positive infinity between the asymptotes. Check out both graphs in Figure 20-9.

The main differences between these two graphs is that the inverse tangent curve *rises* as you go from left to right, and the inverse cotangent curve *falls* as you go from left to right. Also, the horizontal asymptotes for inverse tangent capture the angle measures for the first and fourth quadrants; the horizontal asymptotes for inverse cotangent capture the first and second quadrants. The measures between these asymptotes are, of course, consistent with the ranges of the two inverse functions.
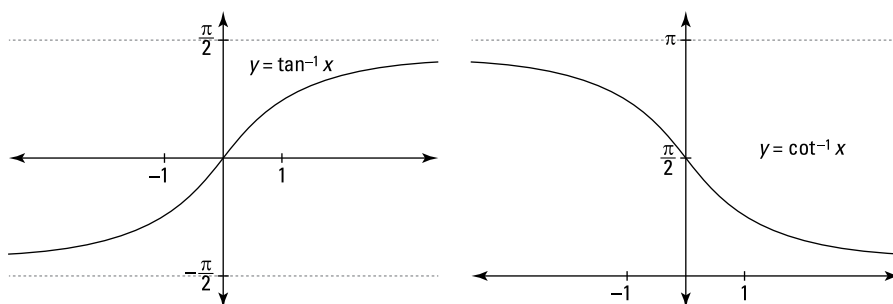
Crafting inverse secant and cosecant

The graphs of the inverse secant and inverse cosecant will take a little explaining. First, keep in mind that the secant and cosecant functions don't have any output values (y-values) between -1 and 1, so a wide-open space plops itself in the middle of their graphs. This idea translates into a wide-open space between the x-values -1 and 1 in the graphs of their inverses. Also, the graphs of secant and cosecant go infinitely high and infinitely low along the y-axis. So, the graphs of the inverses have a horizontal asymptote. All this talk probably seems like nonsense, so take a look at Figure 20-10, which shows the graphs.

FIGURE 20-10:
The graphs of
$y = \sec^{-1} x$
and $y = \csc^{-1} x$.
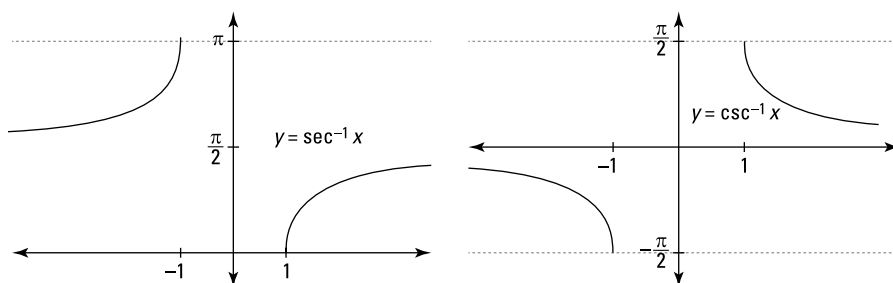
The graph of $y = \sec^{-1} x$ lies between 0 and π on the y-axis. All the output values are in the first and second quadrants. But a horizontal asymptote runs through the graph: the line $y = \dfrac{\pi}{2}$. The secant isn't defined at $\dfrac{\pi}{2}$, so its inverse won't have an output value there. The graph of the inverse secant goes from the point $(1,0)$ and moves upward, staying below the horizontal asymptote as the x-values go to positive infinity. It also comes from negative infinity along the x-axis above the horizontal asymptote, moving upward to the point $(-1, \pi)$.

The graph of $y = \csc^{-1} x$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, with a horizontal asymptote of $y = 0$. (The cosecant isn't defined at $x = 0$, so its inverse doesn't have an output value there.) The graph of inverse cosecant covers angle measures from the first and fourth quadrants. On the right, the graph goes from the point $\left(1, \frac{\pi}{2}\right)$ down toward the horizontal asymptote as the x-values go to positive infinity. On the left, the graph's x-values come from negative infinity, where they're just below the asymptote, and move down to the point $\left(-1, -\frac{\pi}{2}\right)$.

Chapter **21**

Topping Off Trig Graphs

The graphs of the trigonometric functions can take on many variations in their shapes and sizes. As wonderful as these graphs are just by themselves, they're even better and more useful when you adjust them to fit a particular situation. In Chapters 18, 19, and 20, I show you how to make the trig functions slide about by moving them up, down, left, and right. I also show you how to make them steeper and flatter. In this chapter, I complete the trig story with additional transformations, as well as the even-more-exciting possibilities that occur when you combine graphs. I start off with a basic template for a trig function and progress from that point.

The Basics of Trig Equations

You can identify all the different transformations that can be performed on a trig function from a certain form of the function's equation. First, check out the general equation and then consider some examples of what the specific equations may look like.

TRIG RULES

The general form for a trig equation is $y = Af[B(x+C)]+D$, where

» *f* represents the trig function.

» *A* represents the *amplitude,* or steepness.

- A positive *A* means that the graph is oriented as usual.

- A negative *A* means that the graph is flipped over a horizontal line.

» *B* determines the *period* of the graph (the length of the interval needed for the graph of the function to start repeating itself) using the formula $\frac{2\pi}{B}$ or $\frac{\pi}{B}$ $\left(\frac{360°}{B}\text{ or }\frac{180°}{B}\right)$, depending on the function's usual period.

» *C* determines a shift to the left or right.

» *D* determines a shift up or down.

Here are some examples of trig functions using this format:

$$y = -2\sin\left[6\left(x + \frac{\pi}{4}\right)\right] + 3$$
$$y = \frac{1}{2}\cos\left[\frac{1}{6}(x - \pi)\right] + 1$$
$$y = -\cot 4x + \frac{1}{2}$$

Each of the numbers changes the basic graph in a particular way.

In the graph of $y = -2\sin\left[6\left(x + \frac{\pi}{4}\right)\right] + 3$, you have a graph that has six complete cycles of the sine curve in the space where you would usually find one. The graph has a highest value of 5 (adding the 2 to the 3), is shifted to the left slightly, and goes downward where the sine curve usually goes upward. Does this seem a bit mysterious? If so, refer back to Chapter 18 for more details on sleuthing out the particulars.

Ready for another? In the graph of $y = \frac{1}{2}\cos\left[\frac{1}{6}(x - \pi)\right] + 1$, the graph seems a bit stunted, if you just look between -2π and 2π. That's because there's only $\frac{1}{6}$ of the usual cosine curve in the space where you usually find the whole curve.

This curve is also only half the usual height, and it's slid up by 1 unit and to the right by about 3. It would be very difficult to try to create the function equation if you had just the graph to go by.

The last one is quick and easy. The graph of $y = -\cot 4x + \frac{1}{2}$ has four complete cotangent cycles where you'd usually find one. It's also "flipped" over a horizontal axis, and it's raised by $\frac{1}{2}$ unit.

In the following sections, I give you some tools to work with when figuring out the graphs of these types of trig functions.

Flipping over a horizontal line

When you multiply a trig function by a negative number, all the output values are opposites. The positive values become negative, and the negative values become positive. The effect that this operation has on the graph is that it appears to have a reflection or flip over a horizontal line. For example, Figure 21-1 shows the graph of $y = -\sin x$ compared to $y = \sin x$.
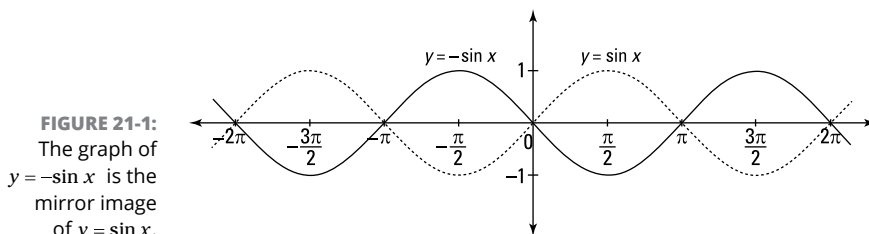
See how the two graphs compare? The original graph appears to be flipped over the x-axis.

Interpreting the equation

Each of the different letters of the general equation for a trig function has a purpose. Here's a more-detailed explanation of each part.

A is for amplitude

The letter A represents the amplitude of the sine or cosine function, and it affects the steepness or flatness of the graphs of any of the trig functions. If the absolute value (ignore the + or − sign) of A is some number greater than 1, then the graph is steeper than usual. If the absolute value of A is between 0 and 1, then the graph is flatter. The higher the number, the steeper the curve. The closer the number is to 0, the flatter the curve.

B is for becoming (the period)

The multiplier B affects the length of the graph's *period*, or how far it goes along the x-axis. The sine, cosine, cosecant, and secant all normally have a period of 2π (360°). The tangent and cotangent have a period of π (180°). If you divide the normal period of the function by the value of B, you get the length of the new, adjusted period. Another way to put it: B tells you how many complete cycles the curve will make in the space that usually has only one. If B is 2, the graph has two complete cycles where there's usually one.

C is for cruisin' left or right

The value of C changes the graph by moving the whole curve to the left or right of where it usually is. If you subtract C, the graph moves C units to the right. If you add C, it moves C units to the left.

D is for distancing yourself up or down

The value of D tells how far up or down the graph moves from its original position. A positive D moves the graph up, and a negative D moves it down. The value of D also represents the average or middle value of the sine and cosine curves and the middle of the open space of the secant and cosecant curves.

Graphing with the General Form

Now is the time to put all your knowledge to work and do some serious graphing. In the general equation for a trig function, $y = Af[B(x+C)]+D$, the letters A, B, C, and D all represent values, but they have to be in those exact places, and the equation has to be in that exact form. You need to factor, multiply, or manipulate the equation in other ways to get it in the general form if you want to use these values to figure out what the graph looks like.

More likely than not, you'd draw the graphs in this section with the help of a graphing calculator. The only problem with graphing calculators is that entering these complicated functions correctly is often a real challenge. You can't use brackets or braces to help keep the groupings straight. You're stuck with parentheses, which can get messy when you're dealing with a lot of them. And when you have to enter a fraction, you need the divide sign; just be sure to put the numerator in parentheses if it's negative or has more than one term. The main reason I provide the examples in this section is so you know what to expect. When you know how all these variations work, you're able to recognize when you have an error in your graphing calculator work, and so you can avoid that age-old saying, "Garbage in, garbage out."

The first example involves graphing $y = 3\sin\left[2\left(x - \frac{\pi}{4}\right)\right] + 1$.

1. **Determine the amplitude of the curve.**

The 3 represents the A, which is the amplitude of a sine curve. The function stretches 3 units above and below the middle. The 3 is positive, so the curve doesn't flip or reflect over a horizontal line.

2. Find the period of the function.

The 2 represents the *B*, which means that the curve makes two complete cycles in the amount of space where it usually has only one. Because the normal period of the sine function is 2π units, in this function the period is $\frac{2\pi}{2}$, or π units long.

3. Determine the shift left or right.

The $-\frac{\pi}{4}$ represents the value of *C*. Because *C* is negative, the shift is $\frac{\pi}{4}$ units to the right.

4. Find the shift up or down.

The last number, 1, is the value of *D*, which is a shift upward by 1 unit.

5. Input all the values to graph the equation, as shown in Figure 21-2.

$$y = 3\sin\left[2\left(x - \frac{\pi}{4}\right)\right] + 1$$
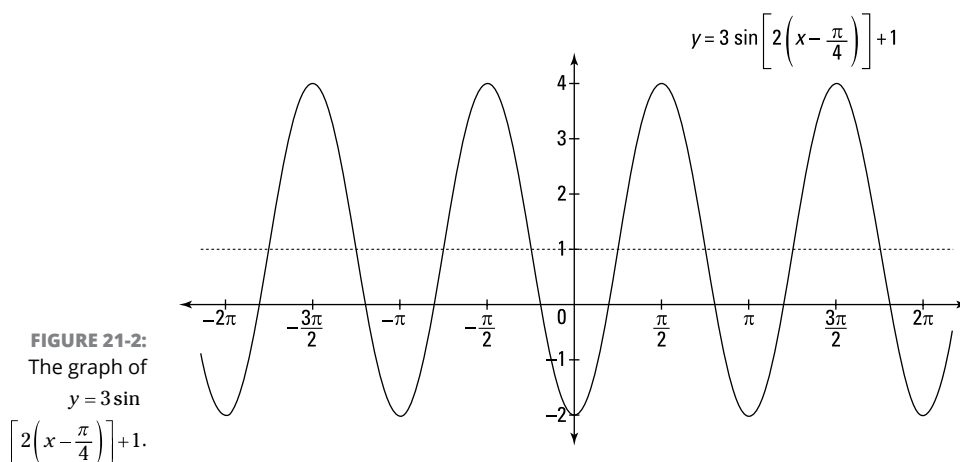
The graph of
$y = 3\sin$
$\left[2\left(x - \frac{\pi}{4}\right)\right] + 1.$

In Figure 21-2, I drew the line $y = 1$ to show the middle, so you can see the result of the vertical shift. The graph shown goes from about -2π to 2π on the *x*-axis, where you'd normally expect to find two complete cycles. Instead, the graph has four.

The next graph has a flip over a horizontal line. The curve doesn't flip over the *x*-axis because the graph is dropped down by 3 units. Instead, the flip is over the horizontal line $y = -3$. Without further ado, here's how to sketch the graph of $y = -2\cos\left[\frac{1}{3}(x + \pi)\right] - 3$.

1. **Determine the amplitude.**

   The -2 in front tells you two things. First, the amplitude is 2, or the curve is twice as high as usual. Second, the negative sign tells you that the whole cosine curve is flipped over a horizontal line. Where the curve usually goes up, it goes down, and vice versa.

2. **Find the period.**

   The multiplier of $\frac{1}{3}$ spreads the curve out quite a bit — only one-third as much curve is in the same amount of space that a 2π period (the period of the basic cosine function) usually has. In fact, the new period is $2\pi \div \frac{1}{3} = 2\pi \cdot 3 = 6\pi$.

3. **Determine the shift left or right.**

   The value of *C* is π, which is a little more than 3 units. The graph moves 3 units to the left, because *C* is positive.

4. **Find the vertical shift.**

   D represents the number –3, so the whole graph shifts down 3 units.

5. **Now use all these values to graph the curve from -4π to 4π, as Figure 21-3 shows.**

$$y = -2\cos\left[\frac{1}{3}(x+\pi)\right] - 3$$

FIGURE 21-3:
The graph of
$y = -2\cos$
$\left[\frac{1}{3}(x+\pi)\right] - 3$.
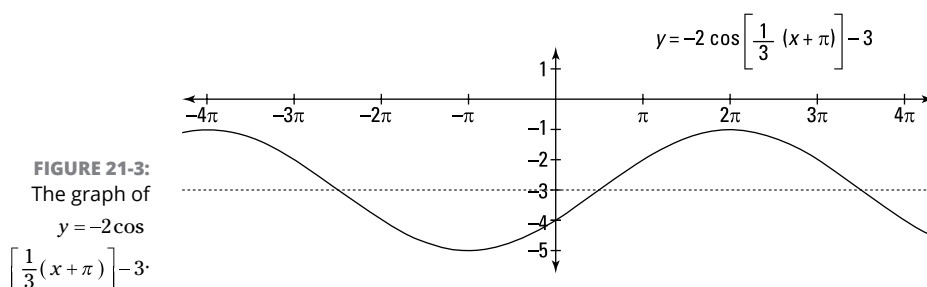
As you can see, the graph in Figure 21-3 goes from -4π to 4π, which is much more than is usually displayed. Going from -2π to 2π doesn't show a complete cycle. The period is 6π, so the graph needs more space. The dotted line is $y = -3$, which is the middle or average of the graph.

The graph in Figure 21-4 shows that you don't always have to have a value for one of the letters in the general form. Well, actually, each of the letters always has a value, but when that value is 1 or 0, it doesn't show up. Just know that the part of the equation with the value of 1 or 0 doesn't change the basic graph. In the graph of $y = -\tan 2x + \pi$, you find two situations where the original graph doesn't change.

THE SEED OF LIFE

A *compass* is an instrument that you can use to draw a circle. All you do is place the sharp, pointed end at the center of the circle and drag the pencil end around. One of the basic constructions that students can produce is a set of circles that intersect with one another in an interesting pattern. This pattern is formed by marking six equidistant points on the original circle, using the radius set on the compass from that circle. Then six new circles are formed using the six points as centers. When constructed correctly, those circles form what's called the *seed of life*, an arrangement that has ancient Egyptian beginnings. This pattern is also a part of 13th-century Italian art. Here's what it looks like:
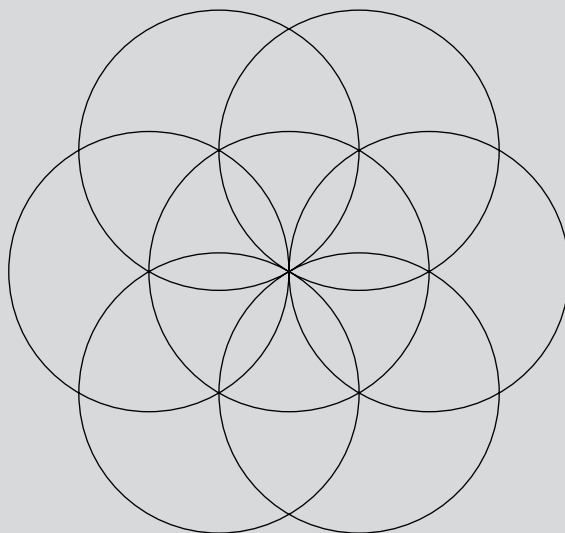
1. **Determine the steepness.**

   With the tangent function, I don't refer to the multiplier as *amplitude* because the tangent curve doesn't have a highest or lowest point, as the sine and cosine curves do. Any multiplier *A* affects the steepness. In this case, that steepness doesn't change, because the *A* is essentially a 1. Because the 1 is negative in this equation, the graph flips over a horizontal line.

2. **Find the period of the function.**

   The multiplier of 2 on the angle measure makes the period of this tangent curve equal to $\frac{\pi}{2}$, because the normal period of the tangent function is π, and you have to divide by 2 here. The graph makes twice as many cycles in the usual amount of space.

3. **Determine the shift left or right.**

   Here's another case where the graph doesn't change. The equation has no number in place of *C* — that value is actually 0. So, the graph doesn't shift left or right.

4. **Find the vertical shift.**

   The number π is the *D* value. That number is positive, so the graph shifts up π units, which is about 3 units.

5. **Graph the function from -2π to 2π, as Figure 21-4 shows.**

You can see that the graph in Figure 21-4 shows eight complete cycles of the tangent function. The reason it shows so many cycles is because the graph goes from -2π to 2π, and each cycle is only $\frac{\pi}{2}$ in length. The dotted line shows the horizontal shift of π.
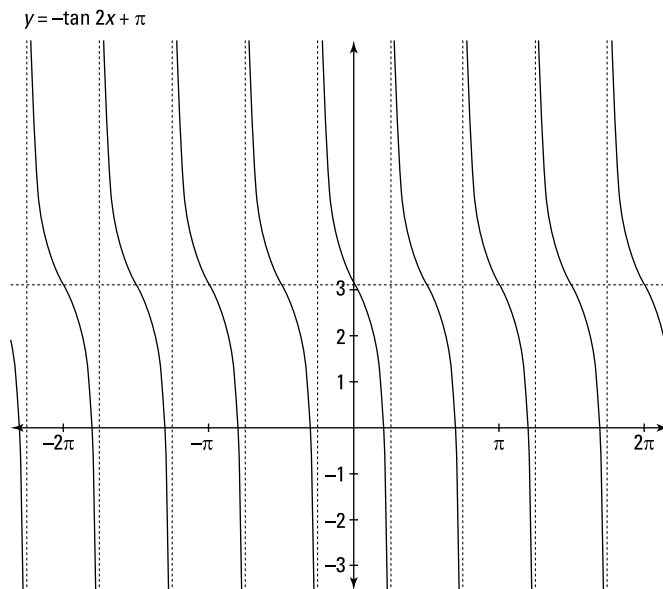
$y = -\tan 2x + \pi$

Adding and Subtracting Functions

Just when you thought this book couldn't get any better, I add yet another twist to the trigonometry picture. You can model many applications in physics and the cycles of nature with curves that you create by adding two trig functions together, subtracting one from another, or adding a trig function and some algebraic

function. When you add functions together, you can obtain the graph of this sum by taking each x-value from each function and finding the sum of the y-values that correspond to those x-values. Then you can plot the points with the x-values that you used and the y-values that you found. I show you a couple of examples, and because they get too messy very quickly, I bail and use a graphing calculator.

The function $y = x + \sin x$ is the sum of the sine function $y = \sin x$ and the algebraic function $y = x$. The algebraic function $y = x$ is a line that cuts diagonally through the third and first quadrants. The sine has y-values that go from -1 to 1, over and over again. Table 21-1 shows some of the separate functions' values and then the sum of those values.

TABLE 21-1   **The y-Values of $y = x + \sin x$.**

x	-2π	$-\pi$	-2	-1	$-\dfrac{\pi}{2}$	0	$\dfrac{\pi}{2}$	1	2	π	2π
$y = \sin x$	0	0	-0.909	-0.841	-1	0	1	0.841	0.909	0	0
$y = x + \sin x$	-2π	$-\pi$	-2.909	-1.841	-2.571	0	2.571	1.841	2.909	π	2π

You can get a better idea of how this addition works by looking at the graph. I used a graphing calculator to graph $y = x + \sin x$ from -2π to 2π (see the left side of Figure 21-5). And then, to give you an even better picture of what's going on, and because I think this curve is neat, I graphed it from -4π to 4π in Figure 21-5, on the right side.
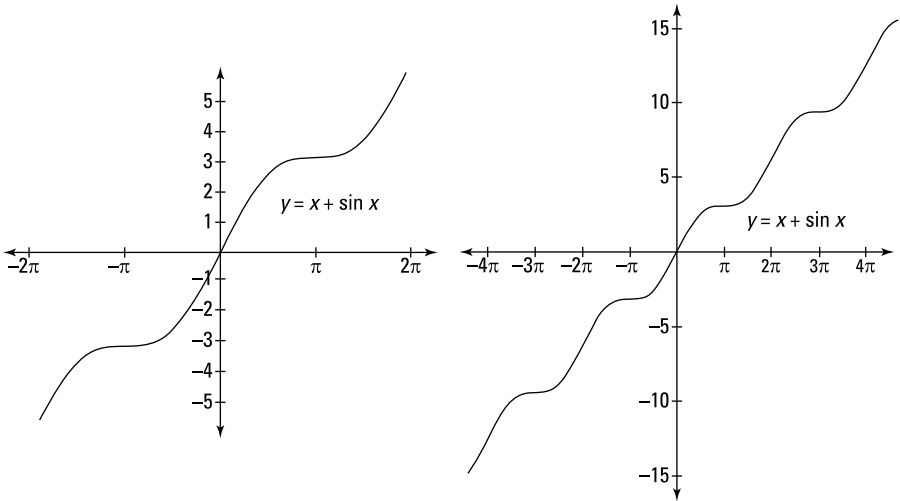
FIGURE 21-5:
The graph
of $y = x + \sin x$.

The next example shows what can happen when you subtract one trig function from another. Of course, I had to experiment with all sorts of different combinations of functions to make this graph come out especially interesting. You should try your hand at it, too. Here's my contribution, the graph of $y = 2\sin x - \cos 3x$. I subtracted the function values of each, one from the other, to produce the very nice curve in Figure 21-6, which I drew from -4π to 4π. Is it a heartbeat or a pretty design?
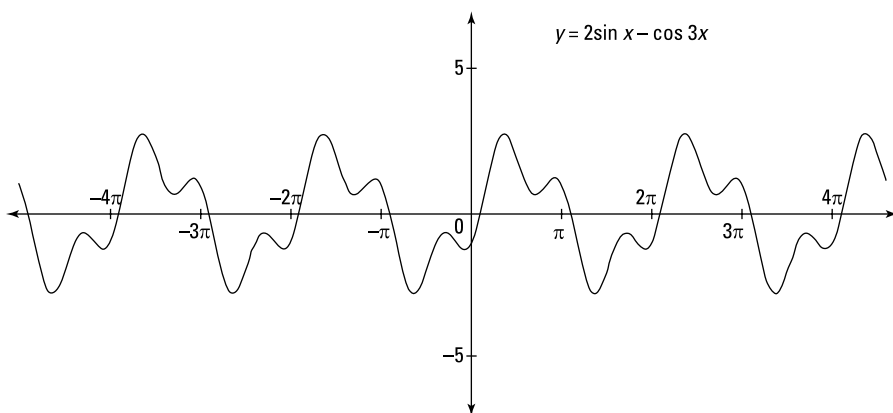
FIGURE 21-6:
The graph of
$y = 2\sin x - \cos 3x$.

Applying Yourself to the Task

The graphs of some of the trig functions that you can create by altering the functions or combining them are fun to look at. They may even be useful when you're preparing a special border or other artwork. But the practical uses of these graphs are what you consider in this section. A cardiologist looks at a graph of the heart's function and detects whether it's beating properly. The graphs of earthquake activity are of special interest to those hoping to predict the next one — with enough time to warn everyone.

Measuring the tide

Along the coast, the tides are of particular interest. The tides are affected by the gravitational pull of both the moon and the sun. The high tides and low tides follow a periodic pattern that you can model with the sine function. On a particular winter day, the high tide in Boston, Massachusetts, occurred at midnight. To determine the height of the water in the harbor, use the equation $H(t) = 4.8\sin\left[\dfrac{\pi}{6}(t+3)\right] + 5.1$, where t represents the number of hours since midnight.

1. **Input the value of *t* into the equation.**

   At midnight, the value of *t* is 0. Putting 0 in for *t* in the equation gives you

   $$H(0) = 4.8\sin\left[\frac{\pi}{6}(3)\right] + 5.1$$

   $$= 4.8\sin\frac{\pi}{2} + 5.1$$

   $$= 4.8(1) + 5.1 = 9.9$$

   The greatest value of sine occurs at $\frac{\pi}{2}$, so it makes sense that high tide would be when the formula uses the sine of that value.

2. **Determine the altitude.**

   The multiplier of 4.8 is the amplitude — how far above and below the middle value the graph goes. The tides go 4.8 feet above and below the average amount on this particular day. The number added on at the end, 5.1, is the average height for the tides. So, the tide goes up to 9.9 feet and down to 0.3 foot — wading depth.

3. **Find the period of the function.**

   The multiplier of $\frac{\pi}{6}$ affects the period. The period of the sine function is usually 2π. Divide 2π by $\frac{\pi}{6}$, and you get $2\pi \div \frac{\pi}{6} = 2\pi \cdot \frac{6}{\pi} = 12$. The period is 12 hours, so you know that the tides go through their entire cycle in 12 hours. The 3 added to the *t* represents a horizontal shift; that number determines what times of day the high tide and low tide occur. Figure 21-7 shows a graph of this function and the different stages of the tide at different times. By looking at the graph, you can plan your sailing and clam-digging activities.

Tracking the deer population

The graph in this section shows the population of a herd of deer, starting at the first of April and ending at the next April. New deer are born in the spring, so an increase in the herd size is expected. Predators take care of the weak deer — both young and old. And then you have to consider the weather; winter can be very hard on the population. Look at the graph in Figure 21-8 and see if it demonstrates what you'd expect. This cycle is the result of finding the sum of two different sine functions: $D(m) = 400 + 40\sin(0.524m) + 20\sin(1.047m)$ represents the population of the herd, where *m* is the number of months since April.

The herd experiences a high of about 450 deer in June and a low of about 350 deer in February. This model shows a herd that stays pretty close to being the same size year after year.

FIGURE 21-7:
The tides in
Boston on one
wintry day.

FIGURE 21-8:
The population of
a herd of deer
from one April to
the next.
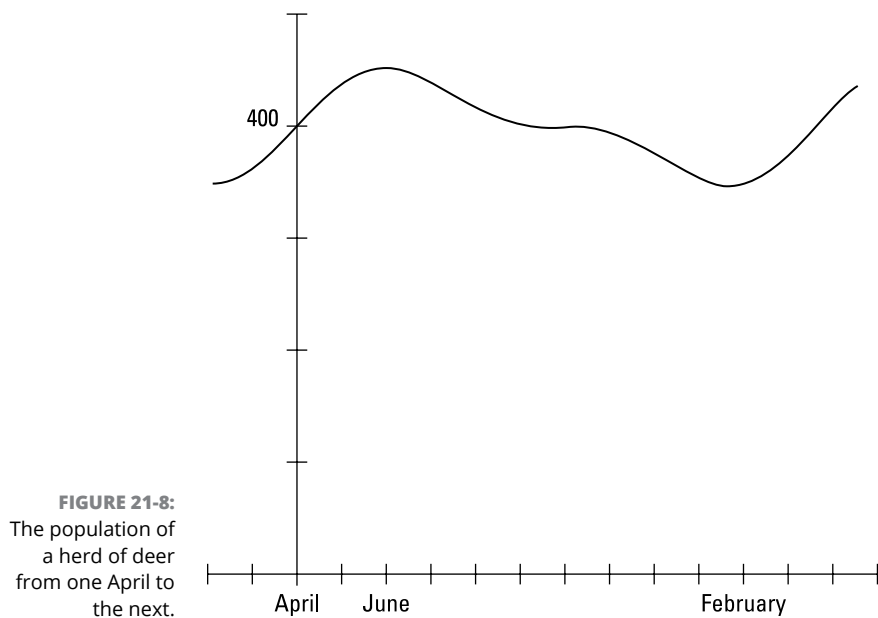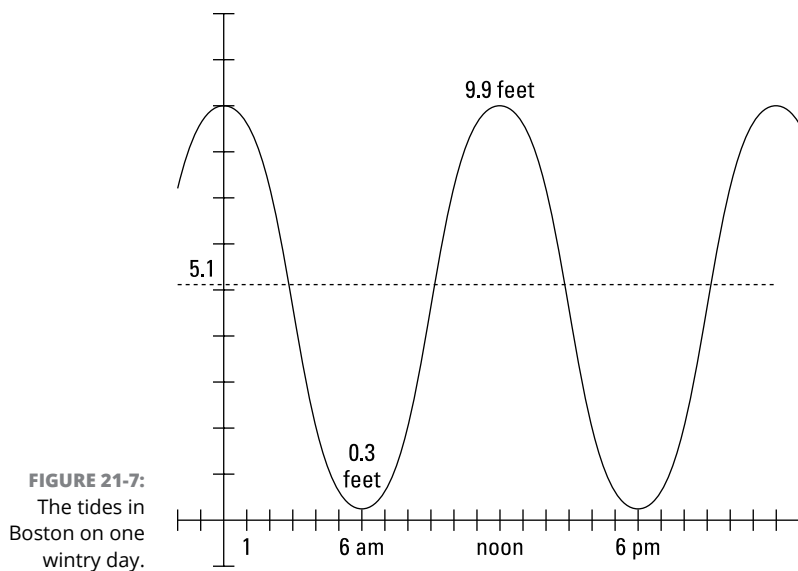
Measuring the movement of an object on a spring

In this section, a trig function proves useful in a model for an object attached to a spring. The same pattern doesn't occur over and over, as it does in the previous sections, but this is a great example of a trig function at work.

The equation $H(t) = 3(0.7)^t \cos 5t + 4$ represents the height of an object attached to a spring, where t is the amount of time that has passed — usually in seconds. The equation has a trig function multiplied by an exponential function. When you first release the spring, the object hits a height of about 7 feet. It jumps up and down, finally settling in at about 4 feet high, as shown in Figure 21-9.
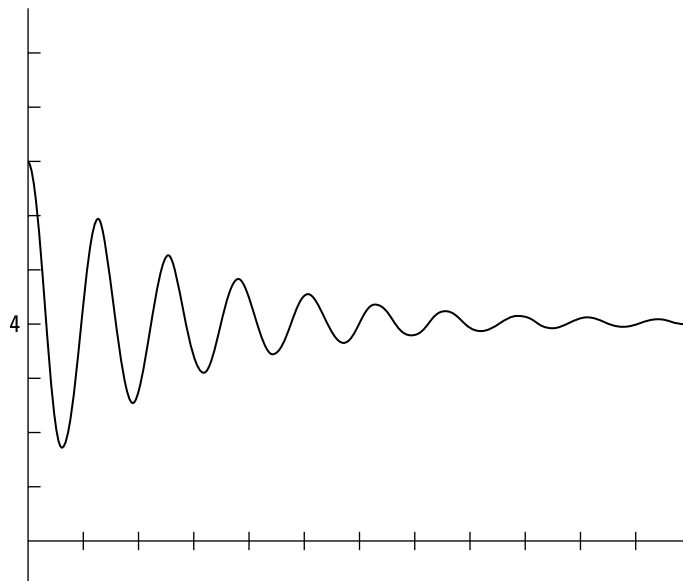
You can probably come up with a similar model to show how a bungee jumper goes up and down. So, you see, the trig functions have all sorts of applications — many of them very useful.

6

The Part of Tens

Chapter **22**

Ten Basic Identities . . . Plus Some Bonuses

A big advantage of trig expressions and equations is that you can adjust them in so many ways to suit your needs. The basic identities that I list in this chapter are the ones people use most frequently (and remember most often). Secant, cosecant, and cotangent are technically the three reciprocal functions, but you can write identities to show their reciprocals, too. You'll see other ratios and some extensions of the Pythagorean Theorem. And you'll also find some alternate notation and optional formats.

Reciprocal Identities

A *reciprocal identity* is just what the name implies: putting a 1 in the numerator of a fraction and then inserting what it takes to create a reciprocal in the denominator.

Reciprocating the sine

Take a look at the first reciprocal identity and its counterpart:

$$\sin\theta = \frac{1}{\csc\theta} \text{ and } \csc\theta = \frac{1}{\sin\theta}$$

An alternate way of writing these identities uses an exponent of -1 rather than a fraction:

$$\sin\theta = (\csc\theta)^{-1} \text{ and } \csc\theta = (\sin\theta)^{-1}$$

REMEMBER

Note that the exponents apply to the entire function. These are not the inverse functions: $\csc^{-1}\theta$ and $\sin^{-1}\theta$.

Checking in with the cosine

Next comes the second reciprocal identity and its counterpart.

$$\cos\theta = \frac{1}{\sec\theta} \text{ and } \sec\theta = \frac{1}{\cos\theta}$$

Again, another way of writing these is to use an exponent of -1. The parentheses are used to be sure you recognize that this is the reciprocal, not the inverse.

$$\cos\theta = (\sec\theta)^{-1} \text{ and } \sec\theta = (\cos\theta)^{-1}$$

Off on a tangent with its reciprocal

The tangent and its reciprocal at least have names that sound alike. The other two basic functions and their reciprocals (see the preceding equations) also seem to have names that are nicely related.

$$\tan\theta = \frac{1}{\cot\theta} \text{ and } \cot\theta = \frac{1}{\tan\theta}$$

And, to finish off the alternate notation:

$$\tan\theta = (\cot\theta)^{-1} \text{ and } \cot\theta = (\tan\theta)^{-1}$$

Ratio Identities

The *ratio identities* also involve fractions — just like the reciprocal identities. But the ratio identities have functions in both the numerator and denominator. The two ratio identities use the same two functions: sine and cosine. To avoid confusion, here's a helpful way to remember which ratio identity has the sine in the numerator. Tangent and sine have beginning letters that are very close in the alphabet, and cotangent and cosine have the same beginning letters. This train of thought helped me out in high school when I first saw these identities. See if it helps you here.

Creating the ratio identity for tangent

The ratio identity for the tangent has the sine in the numerator and cosine in the denominator.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

And, if you want to take the identity even further and have only the sine function appearing, you can use a version of the Pythagorean identity and write it as

$$\tan\theta = \frac{\sin\theta}{\pm\sqrt{1-\sin^2\theta}}$$

For more on Pythagorean identities, see the next section.

Making the cotangent a ratio identity

The cotangent ratio identity has cosine over sine.

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

And, just as with the tangent, you can revise the original ratio identity to contain just the cosine.

$$\cot\theta = \frac{\cos\theta}{\pm\sqrt{1-\cos^2\theta}}$$

Pythagorean Identity Plus

One of the most frequently used identities is the Pythagorean identity involving both the sine and the cosine. And two other Pythagorean identities can be created from the original — with a little substitution and algebra.

You find this basic Pythagorean identity and its full description in Chapter 10.

$$\sin^2\theta + \cos^2\theta = 1$$

This can also be written as $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$ when you want to perform a useful substitution.

You can create a second basic Pythagorean identity by dividing each term in the original identity by the square of the cosine function and simplifying.

$$\tan^2\theta + 1 = \sec^2\theta$$

This also has an alternate version: $\tan^2\theta = \sec^2\theta - 1$

And last but certainly not least, you get the third Pythagorean identity from the first one by dividing each of the terms in that identity by the square of the sine.

$$1 + \cot^2\theta = \csc^2\theta$$

Finishing up, you can also say that $\cot^2\theta = \csc^2\theta - 1$.

Opposite-Angle Identities

Opposite-angle identities allow you to change a function involving a negative angle into an equivalent involving a positive angle. The identities for sine and tangent seem to make the most sense. The identity involving cosine may look weird, but it's right!

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

Multiple-Angle Identities

And now for some more bonus identities. If you need an identity for the function $\sin 3x$ or $\cos 4x$, you can always create them yourself by applying an addition or double-angle formula. For example, $\sin 3x$ can be written as $\sin(2x + x)$ and $\cos 4x$ can be $\cos(2 \cdot 2x)$. Just to save you the trouble, I'm providing you a few of these special identities.

Going multiple with sine

These two identities were created by using $\sin(2x + x)$, $\sin 2x$, and $\sin(2 \cdot 2x)$.

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\sin 4\theta = 8\cos^3\theta\sin\theta - 4\cos\theta\sin\theta$$

Cosine cooperates

The identities involving the cosine of $3x$ and $4x$ have some similarities to those with the sine.

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$$

Tangent keeps its fractional origin

The multiple-angle formulas for tangent get rather involved.

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Chapter **23**

Ten Not-So-Basic Identities

I n Chapters 10 and 11, I cover the most frequently used identities at great length. Here are ten identities that don't appear in those chapters, because you won't use them all that often. A few are rather obscure. These identities don't lend themselves to memorization very well — you'll be better off just looking them up if you need them.

Product-to-Sum Identities

The product-to-sum identities look very much alike. You have to pay close attention to the subtle differences so that you can apply them correctly. Even though the product looks nice and compact, it's not always as easy to deal with in calculus computations — the sum or difference of two different angles is preferred.

The first identity has two angles, *A* and *B*. When you multiply the sine of one angle times the cosine of the other angle, you end up with one-half the sum of a sum identity and a difference identity. Whew!

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

This time, multiply the sines of both angles together, and the result equals one-half the difference between a sum identity and a difference identity:

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

This identity has a mix-and-match feel to it. Two different angles and two different functions are used. There seems to be something for everyone.

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

The last product-to-sum identity uses the cosines of two angles:

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

Just in case you think this is hocus-pocus or that I'm making these up, let me show you an example of one of these new identities. Using $A = 45$ degrees and $B = 30$ degrees and the identity $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$,

$$\cos 45° \sin 30° = \frac{1}{2}[\sin(45° + 30°) - \sin(45° - 30°)]$$

$$\cos 45° \sin 30° = \frac{1}{2}[\sin(75°) - \sin(15°)]$$

$$\frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{1}{2}\left[\frac{\sqrt{6} + \sqrt{2}}{4} - \frac{\sqrt{6} - \sqrt{2}}{4} \right]$$

Where did I get those values for the sine of 75 and 15 degrees? I found the sine of 75 degrees back in Chapter 11. For 15 degrees, I used the sine of the difference between 45 degrees and 30 degrees. Now, simplifying,

$$\frac{\sqrt{2}}{4} = \frac{1}{2}\left[\frac{\cancel{\sqrt{6}} + \sqrt{2} - \cancel{\sqrt{6}} + \sqrt{2}}{4} \right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}\sqrt{2}}{4} \right] = \frac{\sqrt{2}}{4}$$

Sum-to-Product Identities

The sum-to-product identities are useful for modeling what happens with sound frequencies. Think of two different tones represented by sine curves. Add them together, and they beat against each other with a warble — how much depends on their individual frequencies. The identities give a function modeling what's happening.

The first identity takes two different angles, *A* and *B*, and adds their sines together. The result: twice the product of the sine and cosine of two new angles that are created by halving the sum and difference of the angles. See for yourself:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

You can technically call this next identity a difference-to-product identity, although math gurus usually classify it with the sum-to-product identities. Of course, you can consider the difference to be a sum if you call it the sum of a sine and the opposite of another sine.

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

This next identity involves the sum of the cosines of two angles.

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

As you probably expect, the last sum-to-product identity has the difference of the cosines of two angles.

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

For a look at how you use these identities, I show you the difference of the cosines of angles $A = 60$ degrees and $B = 30$ degrees.

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos 60° - \cos 30° = -2\sin\left(\frac{60°+30°}{2}\right)\sin\left(\frac{60°-30°}{2}\right)$$

$$\cos 60° - \cos 30° = -2\sin(45°)\sin(15°)$$

$$\frac{1}{2} - \frac{\sqrt{3}}{2} = -2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$$

I picked up the sine of 15 degrees from the previous section. Simplifying,

$$\frac{1-\sqrt{3}}{2} = -\cancel{2}\left(\frac{\sqrt{2}}{\cancel{2}}\right)\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$$

$$= -\frac{\sqrt{2}\left(\sqrt{6}-\sqrt{2}\right)}{4} = -\frac{\sqrt{12}-\sqrt{4}}{4}$$

$$= -\frac{2\sqrt{3}-2}{4} = -\frac{\cancel{2}\left(\sqrt{3}-1\right)}{\cancel{4}\,^2}$$

$$= \frac{1-\sqrt{3}}{2}$$

Reduction Formula

The reduction formula reduces two trig functions into one. It's useful when studying the force of a spring, the position of a swinging pendulum, or the current in an electrical circuit. This formula takes the sum of two different functions with the common input x and changes the sum to a single function where the multiplier of the sine is the amplitude and the phase shift is θ. The a and b are the coordinates of some point on the terminal side of θ when it's in standard position:

$$a\sin x + b\cos x = \sqrt{a^2+b^2}\,\sin\left(x+\theta\right)$$

To get the sine and cosine of x, you can use simplified versions:

$$\sin x = \frac{b}{\sqrt{a^2+b^2}} \quad \text{and} \quad \cos x = \frac{a}{\sqrt{a^2+b^2}}$$

Mollweide's Equations

Karl Mollweide was an astronomy teacher. His work in astronomy and mathematics led to his discovery of identities that can take the measures of the sides of a triangle and relate them to an expression involving trig functions. Mollweide's equations involve all six parts of a triangle: the three angles, A, B, and C, and the three corresponding sides opposite those angles, a, b, and c.

$$\frac{a+b}{c} = \frac{\cos\left(\dfrac{A-B}{2}\right)}{\sin\left(\dfrac{C}{2}\right)} \quad \text{or} \quad \frac{a-b}{c} = \frac{\sin\left(\dfrac{A-B}{2}\right)}{\cos\left(\dfrac{C}{2}\right)}$$

Showing an example of one of these proportions, I use a 30–60–90 right triangle, with sides 1, $\sqrt{3}$, and 2, because the numbers are so nice. The side opposite angle $A = 30$ degrees measures 1. The side opposite angle $B = 60$ degrees measures $\sqrt{3}$. And the side opposite the right angle measures 2. Using the left–hand equation,

$$\frac{a+b}{c} = \frac{\cos\left(\dfrac{A-B}{2}\right)}{\sin\left(\dfrac{C}{2}\right)} = \frac{\cos\left(\dfrac{30° - 60°}{2}\right)}{\sin\left(\dfrac{90°}{2}\right)}$$

$$= \frac{\cos(-15°)}{\sin(45°)} = \frac{\cos(15°)}{\sin(45°)}$$

$$\frac{1+\sqrt{3}}{2} = \frac{\dfrac{\sqrt{6}+\sqrt{2}}{4}}{\dfrac{\sqrt{2}}{2}} = \frac{\sqrt{6}+\sqrt{2}}{\cancel{4}^{\,2}} \cdot \frac{\cancel{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{12}+\sqrt{4}}{2\sqrt{4}}$$

$$= \frac{2\sqrt{3}+2}{4} = \frac{\cancel{2}\left(\sqrt{3}+1\right)}{\cancel{4}^{\,2}} = \frac{1+\sqrt{3}}{2}$$

Appendix

Graphs and Function Values

The Six Basic Trigonometric Functions and Their Graphs

The six trig functions have reciprocals and inverses. You find here a quick reference to the different notation and where the functions occur.

Function	Abbrev.	Domain	Period	Reciprocal	Inverse
sine	sin	All reals	2π or $360°$	csc	$\sin^{-1} x$ or arcsin
cosine	cos	All reals	2π or $360°$	sec	$\cos^{-1} x$ or arccos
tangent	tan	$x \neq \frac{\pi}{2}(2n+1)$ $x \neq (2n+1) \cdot 90°$	π or $180°$	cot	$\tan^{-1} x$ or arctan
cotangent	cot	$x \neq n\pi$ $x \neq n \cdot 90°$	π or $180°$	tan	$\cot^{-1} x$ or arccot
secant	sec	$x \neq \frac{\pi}{2}(2n+1)$ $x \neq (2n+1) \cdot 90°$	2π or $360°$	cos	$\sec^{-1} x$ or arcsec
cosecant	csc	$x \neq n\pi$ $x \neq n \cdot 90°$	2π or $360°$	sin	$\csc^{-1} x$ or arccsc

Figure A-1 shows you the graphs of the basic functions.
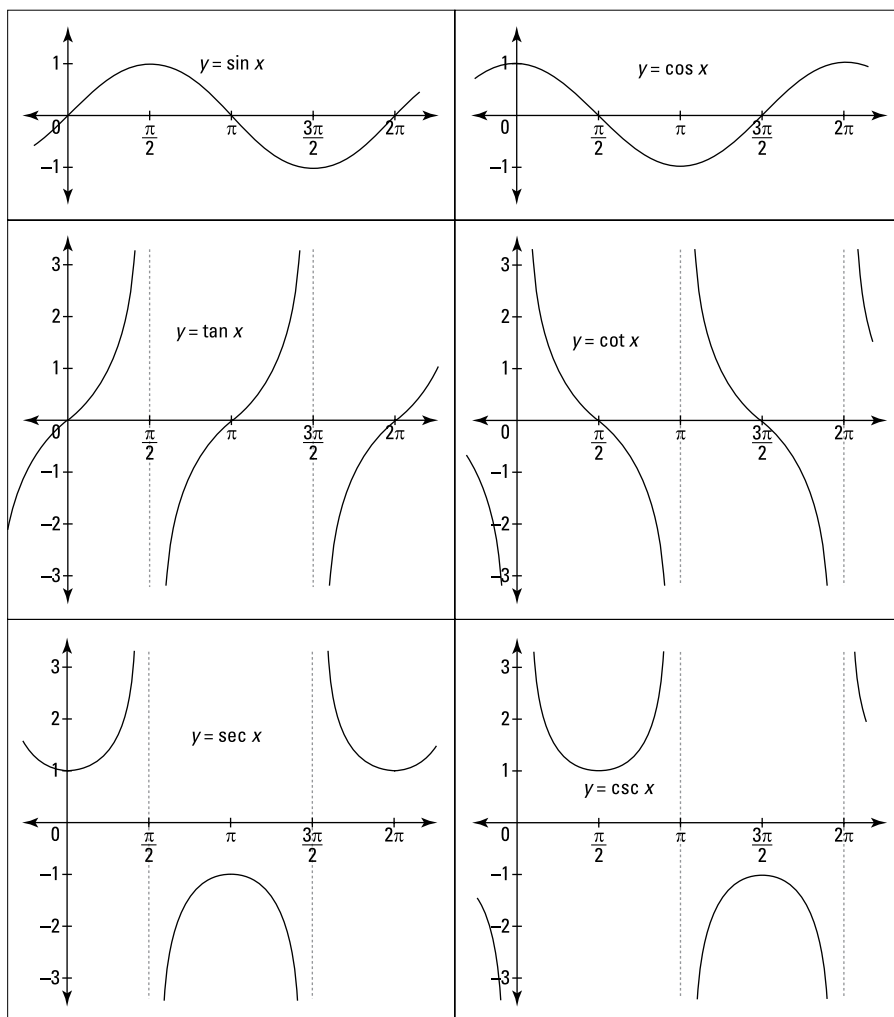
Trig Values of the Most Commonly Used Functions

Your calculator or computer can quickly give you the value of a particular trig function for a particular angle, but having the actual, unrounded version of the most commonly used functions is helpful and gives you exact values.

Deg.	0°	30°	45°	60°	90°	120°	135°	150°	180°
Rad.	0	$\dfrac{\pi}{6}$	$\dfrac{\pi}{4}$	$\dfrac{\pi}{3}$	$\dfrac{\pi}{2}$	$\dfrac{2\pi}{3}$	$\dfrac{3\pi}{4}$	$\dfrac{5\pi}{6}$	π
sin x	0	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{1}{2}$	0
cos x	1	$\dfrac{\sqrt{3}}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{1}{2}$	0	$-\dfrac{1}{2}$	$-\dfrac{\sqrt{2}}{2}$	$-\dfrac{\sqrt{3}}{2}$	-1
tan x	0	$\dfrac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Und.	$-\sqrt{3}$	-1	$-\dfrac{\sqrt{3}}{3}$	0
cot x	Und.	$\sqrt{3}$	1	$\dfrac{\sqrt{3}}{3}$	0	$-\dfrac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	Und.
sec x	1	$\dfrac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Und.	-2	$-\sqrt{2}$	$-\dfrac{2\sqrt{3}}{3}$	-1
csc x	Und.	2	$\sqrt{2}$	$\dfrac{2\sqrt{3}}{3}$	1	$\dfrac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Und.

Trig Values of the Most Commonly Used Functions (Continued)

This table is a continuation of the exact values of the trig function. These are the values found in the third and fourth quadrants.

Deg.	180°	210°	225°	240°	270°	300°	315°	330°	360°
Rad.	π	$\dfrac{7\pi}{6}$	$\dfrac{5\pi}{4}$	$\dfrac{4\pi}{3}$	$\dfrac{3\pi}{2}$	$\dfrac{5\pi}{3}$	$\dfrac{7\pi}{4}$	$\dfrac{11\pi}{6}$	2π
sin x	0	$-\dfrac{1}{2}$	$-\dfrac{\sqrt{2}}{2}$	$-\dfrac{\sqrt{3}}{2}$	-1	$-\dfrac{\sqrt{3}}{2}$	$-\dfrac{\sqrt{2}}{2}$	$-\dfrac{1}{2}$	0
cos x	-1	$-\dfrac{\sqrt{3}}{2}$	$-\dfrac{\sqrt{2}}{2}$	$-\dfrac{1}{2}$	0	$\dfrac{1}{2}$	$\dfrac{\sqrt{2}}{2}$	$\dfrac{\sqrt{3}}{2}$	1
tan x	0	$\dfrac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Und.	$-\sqrt{3}$	-1	$-\dfrac{\sqrt{3}}{3}$	0

cot x	Und.	$\sqrt{3}$	1	$\dfrac{\sqrt{3}}{3}$	0	$-\dfrac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	Und.
sec x	-1	$-\dfrac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	Und.	2	$\sqrt{2}$	$\dfrac{2\sqrt{3}}{3}$	1
csc x	Und.	-2	$-\sqrt{2}$	$-\dfrac{2\sqrt{3}}{3}$	-1	$-\dfrac{2\sqrt{3}}{3}$	$-\sqrt{2}$	-2	Und.

Calculator Usage

Calculators, computers, and even watches can provide the values of trig functions. But the trick is to enter the trig expression correctly. Here are some standard formats you can use.

The –1 exponents on the function names indicate the inverse function button.

Be sure your calculator is set in the correct mode: degree or radian.

REMEMBER

The following entries give you the function values of the six basic trig functions. Enter the function and angle in question.

Function	Calculator Entry
sine x	$\sin x$
cosine x	$\cos x$
tangent x	$\tan x$
cotangent x	$\tan x \ \wedge \ -1$
secant x	$\cos x \ \wedge \ -1$
cosecant x	$\sin x \ \wedge \ -1$

You are needing to determine the angle that has a particular value. The arcsine is interpreted as "the angle whose sin is" and the same for the other entries.

Function	Calculator Entry
arcsine x	$\sin^{-1} x$
arccosine x	$\cos^{-1} x$
arctangent x	$\tan^{-1} x$
arccotangent x	$\tan^{-1}\left(\dfrac{1}{x}\right)$
arcsecant x	$\cos^{-1}\left(\dfrac{1}{x}\right)$
arccosecant x	$\sin^{-1}\left(\dfrac{1}{x}\right)$

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About the Author

Mary Jane Sterling is also the author of *Algebra I For Dummies, Algebra II For Dummies, Math Word Problems For Dummies, Business Math For Dummies,* and *Linear Algebra For Dummies.* She taught junior high school and high school math for several years before beginning her 35-year tenure at Bradley University in Peoria, Illinois. In her current "retirement," Mary Jane still shares mathematical excursions with those of all ages — making math a fun experience. She and her husband, Ted, are enjoying spending time with their children and grandchildren, and traveling.

Dedication

I would like to dedicate this edition of *Trigonometry For Dummies* to Nu Chapter of Delta Kappa Gamma, Illinois State. Working with the members of this teachers' group for the past 25 years has allowed me to have wonderful interactions with members and participate in projects helping local schools and students. The friendships and fellowship have enriched my life immensely, just as the club's activities have enriched the community.

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